The Riemann Hypothesis and Holomorphic Index in Complex Dynamics

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ABSTRACT
We present an interpretation of the Riemann hypothesis in terms of complex and topological dynamics. For example, the Riemann hypothesis is true and all zeros of the Riemann zeta function are simple if and only if a meromorphic function that is explicitly given in this note has no attracting fixed point. To obtain this, we use the holomorphic index (residue fixed point index) that characterizes the local properties of the fixed points in complex dynamics.

1. The Riemann zeta function

For $s \in \mathbb{C}$, the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

converges if $\operatorname{Re} s > 1$ and $\zeta(s)$ is analytic on the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$. It is also analytically continued to a holomorphic function on $\mathbb{C} - \{1\}$, and $s = 1$ is a simple pole. Hence, we regard $\zeta(s)$ as a meromorphic function (a holomorphic map) $\zeta : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This is *the Riemann zeta function* ([Titchmarsh 86, p.13]).

It is known that $\zeta(s) = 0$ when $s = -2, -4, -6, \ldots$. These zeros are called *trivial zeros* of the Riemann zeta function ([Titchmarsh 86, p. 30]). We say the other zeros are *non-trivial*.

The *Riemann hypothesis*, the most important conjecture on the Riemann zeta function, concerns the alignment of the non-trivial zeros of ζ:

The Riemann hypothesis. *All non-trivial zeros lie on the vertical line* $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$.

The line $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$ is called *the critical line*. It has been numerically verified that the first 10^{13} non-trivial zeros above the real axis lie on the critical line [Gourdon].

It has also been conjectured that every zero of ζ is simple. We refer to this conjecture as *the simplicity hypothesis*. (See [Rubinstein and Sarnak 94, p176] or [Mueller 83, p177] for example. For some related results and observations, see [Titchmarsh 86, Section 10.29, Section 14.34, Section 14.36].)

The aim of this note is to reformulate the Riemann hypothesis in terms of complex and topological dynamical systems. More precisely, we translate the locations of the non-trivial zeros into dynamical properties of the fixed points of a meromorphic function of the form

$$v_g(z) = z - \frac{g(z)}{z\, g'(z)},$$

where g is a meromorphic function on $\mathbb{C}$ that shares (non-trivial) zeros with ζ. For example, we set $g = \zeta$ or the *Riemann xi function*

$$\xi(z) := \frac{1}{2} z(1-z)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z).$$

The function $v_g(z)$ is carefully chosen such that
- If $g(\alpha) = 0$ then $v_g(\alpha) = \alpha$; and
- the *holomorphic index* (or the *residue fixed point index*; see Section 2) of v_g at α is α itself when α is a simple zero of $g(z)$.

See Section 3 for more details.

For a given meromorphic function $g : \mathbb{C} \to \widehat{\mathbb{C}}$, we say a fixed point α of g is *attracting* if $|g'(\alpha)| < 1$, *indifferent* if $|g'(\alpha)| = 1$, or *repelling* if $|g'(\alpha)| > 1$. We show the following translations of the Riemann hypothesis (plus the simplicity hypothesis):

Theorem 1. *The following conditions are equivalent:*
(a) *The Riemann hypothesis is true and every non-trivial zero of ζ is simple.*

(b) *Every non-trivial zero of ζ is an indifferent fixed point of the meromorphic function $\nu_\zeta(z) := z - \frac{\zeta(z)}{z\,\zeta'(z)}$.*

(c) *The meromorphic function ν_ζ above has no attracting fixed point.*

(d) *There is no topological disk D with $\overline{\nu_\zeta(D)} \subset D$.*

Note that (d) is a topological property of the map $\nu_\zeta : \mathbb{C} \to \widehat{\mathbb{C}}$ in contrast to the analytic (or geometric) nature of (a). We will present the proof and some variants of this theorem in Section 4.

Section 5 is devoted to some numerical observations and questions on the linearization problem. Section 6 is an appendix wherein we apply Newton's method to the Riemann zeta function. We also give a "semi-topological" criterion for the Riemann hypothesis in terms of the Newton map (Theorem 14).

Remarks.
- The following well-known facts are used in this note (see p.13 and p.45 of [Titchmarsh 86]):
  - The functional equation

  $$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s) \quad (1\text{-}1)$$

  implies that if α is a non-trivial zero of $\zeta(s)$, then so is $1 - \alpha$ and they have the same order.
  - Every non-trivial zero is located in the *critical strip*

  $$\mathcal{S} := \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}.$$

  Hence, the non-trivial zeros are symmetrically arrayed with respect to $s = 1/2$ in $\mathcal{S}$. In fact, there is another symmetry (which we do not use in this note) with respect to the real axis, which comes from the relation $\overline{\zeta(s)} = \zeta(\bar{s})$. By these properties, we will primarily consider the zeros that lie on the upper half of the critical strip $\mathcal{S}$.
- We can apply the method of this note to other zeta functions without extra effort. (See [Ivić 85, Section 1.8] for examples of zeta functions.) However, we need a functional equation, such as (∗), to obtain a result similar to Theorem 1. For example, it is valid for the Dirichlet L-functions.
- We used *Mathematica* 10.0 for all the numerical calculations.

2. Fixed points and holomorphic indices

2.1. Multiplier

Let g be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. We say that $\alpha \in \Omega$ is a *fixed point* of g with *multiplier* $\lambda \in \mathbb{C}$ if $g(\alpha) = \alpha$ and $g'(\alpha) = \lambda$. The multiplier λ is the primary
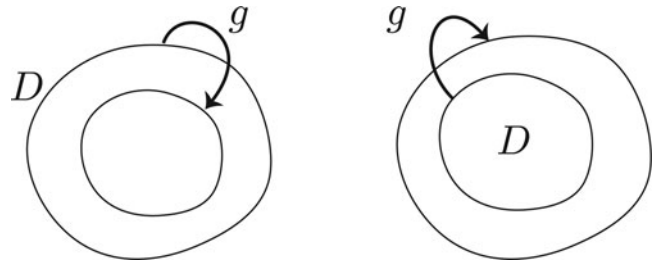

Figure 1. Topological disks that contain an attracting or a repelling fixed point.

factor that determines the local dynamics near α. We say the fixed point α is *attracting, indifferent,* or *repelling* according to whether $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$.

2.2. Topological characterization

Attracting and repelling fixed points of holomorphic mappings have purely topological characterizations (cf. Milnor [Milnor 06, Section 8]):

Proposition 2 (Topological characterization of fixed points). *Let g be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. The function g has an attracting (respectively, repelling) fixed point if and only if there exists a topological disk $D \subset \Omega$ such that $\overline{g(D)} \subset D$ (respectively, $g|_D$ is injective and $\overline{D} \subset g(D) \subset \Omega$).*

See Figure 1. The condition that g is holomorphic is essential. For example, if we only assume that g is C^∞, any kind of fixed point may exist in a disk D with $\overline{g(D)} \subset D$.

Proof. If a topological disk D in Ω satisfies $\overline{g(D)} \subset D$, we may observe the map $g : D \to g(D) \subset D$ via the Riemann map, and it is sufficient to consider the case of $D = \mathbb{D}$, the unit disk. By the Schwarz–Pick theorem (see [Ahlfors 10, Section 1]) the map is strictly contracting with respect to the distance $d(z, w) := |z - w|/|1 - \bar{z}w|$ (or the hyperbolic distance) on $\mathbb{D}$, and it must have an attracting fixed point.

The converse is easy, and the repelling case is analogous. □

2.3. Holomorphic index

Let α be a fixed point of a holomorphic function $g : \Omega \to \mathbb{C}$. We define the *holomorphic index* (or *residue fixed point index*) of α by

$$\iota(g, \alpha) := \frac{1}{2\pi i} \int_C \frac{1}{z - g(z)} \, dz,$$

where C is a small circle around α in the counterclockwise direction. The holomorphic index is mostly determined by the multiplier:
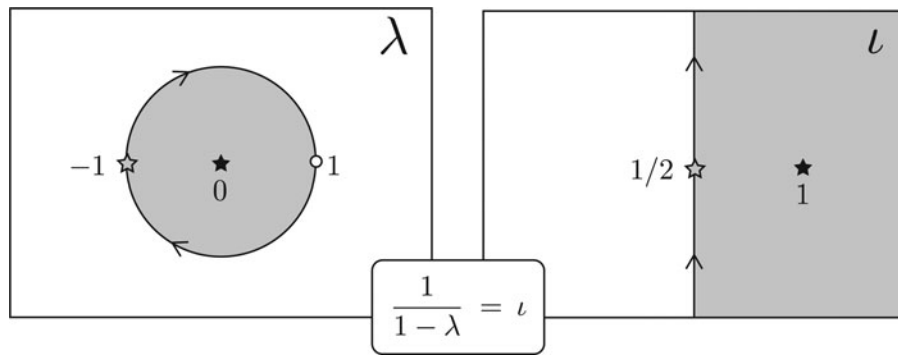
Figure 2. Relation between multipliers and holomorphic indices.

Proposition 3. *If the multiplier $\lambda := g'(\alpha)$ is not 1, then we have $\iota(g, \alpha) = \frac{1}{1-\lambda}$.*

See [Milnor 06, Lem.12.2] for the proof.

Remark. Any complex number K can be the holomorphic index of a fixed point of multiplier 1. For example, the polynomial $g(z) = z - z^2 + Kz^3$ has a fixed point at zero with $g'(0) = 1$ and $\iota(g, 0) = K$.

Since the Möbius transformation $\lambda \mapsto \frac{1}{1-\lambda} = \iota$ sends the unit disk to the half-plane $\{\iota \in \mathbb{C} : \operatorname{Re} \iota > 1/2\}$, fixed points are classified as follows (Figure 2):

Proposition 4 (Classification by index). *Suppose that the multiplier $\lambda = g'(\alpha)$ is not 1. Then α is attracting, indifferent, or repelling if and only if the holomorphic index $\iota = \iota(g, \alpha)$ satisfies $\operatorname{Re} \iota > 1/2$, $= 1/2$, or $< 1/2$, respectively.*

Note that the "critical line" in the ι-plane corresponds to indifferent fixed points whose multipliers are not 1.

3. The nu function

Let $g : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function. (We regard such a meromorphic function as a holomorphic map onto the Riemann sphere.) We define the *nu function $\nu_g : \mathbb{C} \to \widehat{\mathbb{C}}$ of g by*
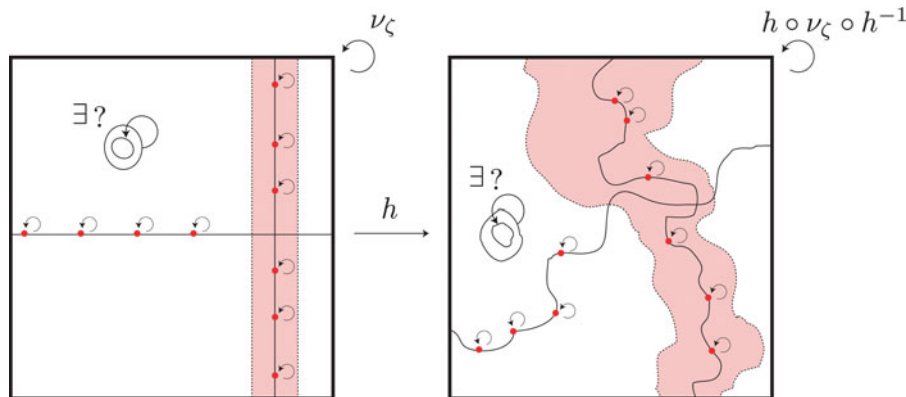
$$\nu_g(z) := z - \frac{g(z)}{zg'(z)}.$$

This is also a non-constant meromorphic function on $\mathbb{C}$ unless $g(z) = g_m(z) := C(1 - \frac{1}{mz})^m$ for some $C \in \mathbb{C} - \{0\}$ and $m \in \mathbb{Z} - \{0\}$.

Using the Taylor and Laurent series, it is not difficult to check:

Proposition 5 (Fixed points of ν_g). *Suppose that $\alpha \neq 0$. Then, α is a fixed point of ν_g if and only if α is a zero or a pole of g. Moreover,*
- *if α is a zero of g of order $m \geq 1$, then $\nu_g'(\alpha) = 1 - \frac{1}{m\alpha}$ and $\iota(\nu_g, \alpha) = m\alpha$;*
- *if α is a pole of g of order $m \geq 1$, then $\nu_g'(\alpha) = 1 + \frac{1}{m\alpha}$ and $\iota(\nu_g, \alpha) = -m\alpha$.*

If 0 is a zero or a pole of g of order $m \geq 1$, then $\nu_g(0) = 1/m$ or $-1/m$, respectively. In particular, 0 is not a fixed point of ν_g.

An immediate corollary is the following:

Corollary 6. *The function ν_g has no fixed point with multiplier 1.*

Figure 3. Topological deformation of the dynamics of ν_ζ. By Theorem 10, we can still state the Riemann hypothesis and the simplicity hypothesis in the deformed dynamics.

Hence we can always apply Proposition 4 to the fixed points of ν_g.

Remark. There is another way to calculate the holomorphic index. (This is actually how the author found the function ν_g.) Suppose that α is a zero of g of order m. A variant of the argument principle ([Ahlfors 79, p.153]) yields that for any holomorphic function ϕ defined near α, we have

$$\frac{1}{2\pi i} \int_C \phi(z) \frac{g'(z)}{g(z)} \, dz = m\phi(\alpha),$$

where C is a small circle around α in the counterclockwise direction. Set $\phi(z) := z$. Then the equality above is equivalent to

$$\frac{1}{2\pi i} \int_C \frac{1}{z - \nu_g(z)} \, dz = m\alpha.$$

The same argument is also valid when α is a pole.

Example. Consider a rational function $g(z) = \frac{(z+1)(z-1/2)^2}{z^3(z-1)}$. By Propositions 4 and 5, the zeros -1 and $1/2$ of g are repelling and attracting fixed points of ν_g, respectively. The pole 1 is a repelling fixed point, however, the other pole 0 is not a fixed point of ν_g.

4. The Riemann hypothesis

Let us consider the nu function ν_ζ of the Riemann zeta function, and prove Theorem 1.

4.1. Trivial zeros and the pole

It is known that the trivial zeros $\alpha = -2, -4, \ldots$ of ζ are all simple [Titchmarsh 86, p30]. By Propositions 4 and 5, their holomorphic indices are α themselves and they are all repelling fixed points of ν_ζ. Similarly, the unique pole $z = 1$ of ζ is simple and it is a repelling fixed point of ν_ζ. Hence, we have the following:

Proposition 7. *Every fixed point of ν_ζ off the critical strip $\mathcal{S}$ is repelling.*

4.2. Non-trivial zeros

Let α be a non-trivial zero of order $m \geq 1$ in the critical strip $\mathcal{S}$. By Proposition 5, α is a fixed point of ν_ζ with multiplier $\lambda := 1 - 1/(m\alpha) \neq 1$, and its holomorphic index is $\iota := m\alpha$.

If the Riemann hypothesis holds, then $\operatorname{Re} \iota = m\operatorname{Re}\alpha = m/2$. Thus $\operatorname{Re} \iota = 1/2$ if $m = 1$ and $\operatorname{Re}\iota \geq 1$ if $m \geq 2$. By Proposition 4, we have the following:

Proposition 8. *Under the Riemann hypothesis, any fixed point α of ν_ζ in $\mathcal{S}$ is a zero of ζ that lies on the critical line. Moreover,*

- *If α is a simple zero of ζ, then α is an indifferent fixed point of ν_ζ with multiplier $\neq 1$.*
- *If α is a multiple zero of ζ, then α is an attracting fixed point of ν_ζ.*
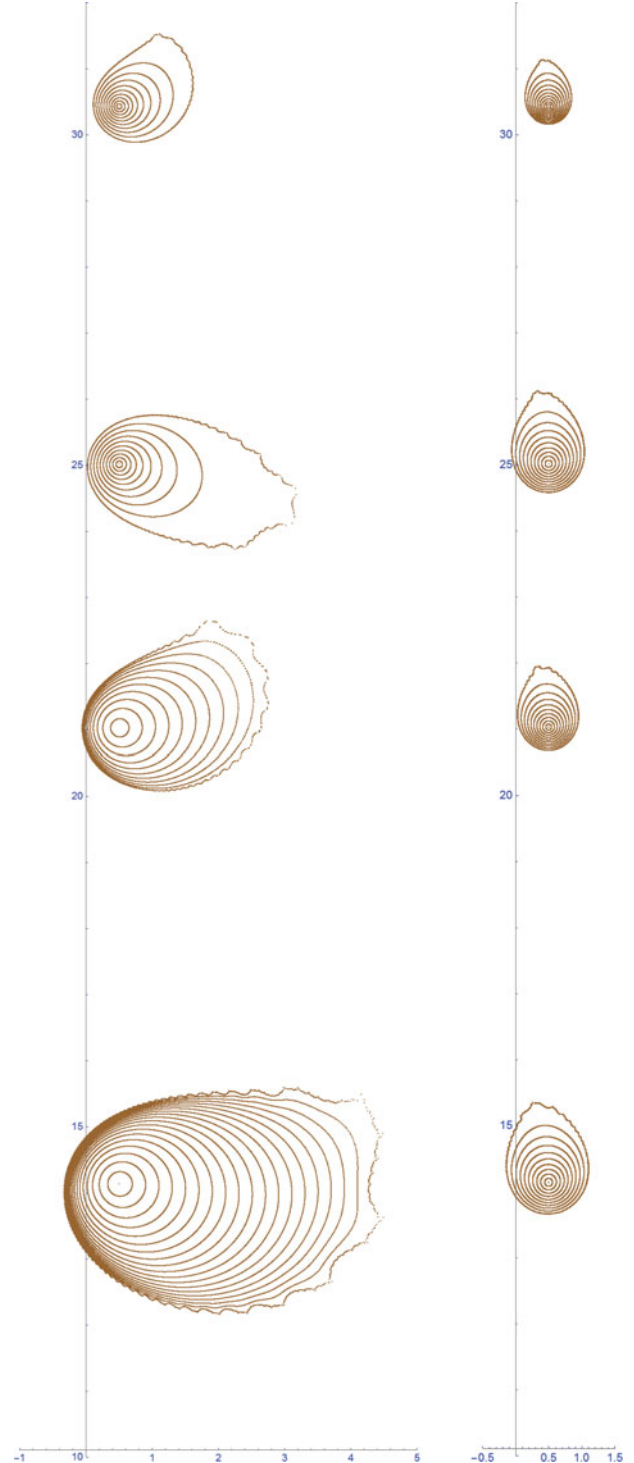
Figure 4. First four Siegel disks (?) of ν_ζ (left) and ν_ξ (right).

Table 1. Continued fraction expansions for some zeros of ζ.

n	$\mathrm{Im}\,\gamma_n$	θ_n	$[a_1, a_2, \ldots, a_{25}, a_{26}, \ldots, a_{50}]$
1	14.1347	0.0112552	[88,1,5,1,1,2,2,5,2,15,2,4,2,4,1,9,1,1,5,2,10,1,1,5,1, 2,7,100,9,2,2,3,2,5,1,8,179,23,1,1,35,1,3,1,2,8,7,34,4,1]
2	21.0220	0.00756943	[132,9,14,1,1,1,2,1,52,1,9,3,4,1,1,1,1,2,2,3,2,1,10,1,1, 1,9,1,1,6,5,1,5,3,1,5,2,6,1,135,1,1,5,1,2,3,2,9,1,3]
3	25.0109	0.00636259	[157,5,1,12,3,1,1,1,1,2,11,1,29,5,1,4,1,1,3,5,14,1,3,1,2, 3,6,14,4,1,41,4,1,1,7,4,1,3,21,1,4,3,1,2,2,1,16,1,2,3]
4	30.4249	0.00523061	[191,5,2,15,3,2,2,7,2,1,2,46,2,1,1,6,1,4,2,2,4,1,6,1,1, 2,5,1,8,1,2,2,5,1,4,39,3,19,5,2,9,1,1876,2,12,1,4,4,1,6]
10	49.7738	0.00319746	[312,1,2,1,48,1,1,4,1,3,1,2,5,1,21,1,4,1,3,2,1,1,8,1,6, 9,1,2,3,2,1,2,1,1,1,4,1,1,1,5,2,1,3,126,1,24,3,2,29,5]
10^2	236.524	0.00067289	[1486, 7,1,4,1,2,1,53,2,8,1,4,6,3,1,3,1,13,3,1,7,2,18,3,1, 1,1,1,1,2,1,210,3,4,1,3,1,2,2,8,1,1,7,2,2,2,1,12,2,2]
10^3	1419.42	0.000112127	[8918,2,48,220,1,15,1,1,6,1,1,6,2,4,149,1,15,3,6,1,4,1,1,64,11, 1,1,13,11,2,4,2,3,2,4,1,2,1,1,2,3,46,1,11,3,1,18,1,6,4]
10^4	9877.78	0.0000161124	[62063,1,15,2,1,2,1,8,1,2,6,2,1,2,2,1,5,1,1,186,3,5,4,14,9, 1,12,1,12,1,6,4,22,1,3,15,1,2,1,14,1,2,2,12,3,1,1,1,45,2]

In particular, if the simplicity hypothesis also holds, then all non-trivial zeros of ζ are indifferent fixed points of ν_ζ.

Hence, (a) implies (b) in Theorem 1. Now we show the converse.

Proposition 9. *If the fixed points of ν_ζ in the critical strip $\mathcal{S}$ are all indifferent, then both the Riemann hypothesis and the simplicity hypothesis are true.*

Proof. Let α be an indifferent fixed point of ν_ζ in the critical strip $\mathcal{S}$. Since ζ has no pole in $\mathcal{S}$, α is a zero of ζ of some order $m \geq 1$, and the holomorphic index is $\iota(\nu_\zeta, \alpha) = m\alpha$ by Proposition 5. By the functional equation (∗), the point $1 - \alpha$ is also a zero of ζ of order m contained in $\mathcal{S}$. and the holomorphic index is $\iota(\nu_\zeta, 1 - \alpha) = m(1 - \alpha)$.

By assumption, both α and $1 - \alpha$ are indifferent fixed points of ν_ζ. Hence by Proposition 4, the real parts of $\iota(\nu_\zeta, \alpha) = m\alpha$ and $\iota(\nu_\zeta, 1 - \alpha) = m(1 - \alpha)$ are both $1/2$. This happens only if $m = 1$ and $\mathrm{Re}\,\alpha = 1/2$. □

Let us finish the proof of Theorem 1:

Proof of Theorem 1. The equivalence of (a) and (b) is shown by Proposition 8 and Proposition 9 above. Condition (b) implies (c) by Proposition 7.

Suppose that (c) holds. Then any fixed point α of ν_ζ in the critical strip $\mathcal{S}$ is repelling or indifferent, and it is also a zero of ζ of some order m by Proposition 5. Hence the holomorphic index $\iota(\nu_\zeta, \alpha) = m\alpha$ satisfies $\mathrm{Re}\,m\alpha \in (0, 1/2]$. By the functional equation (∗), the same holds for $1 - \alpha \in \mathcal{S}$ and the index $\iota(\nu_\zeta, 1 - \alpha) = m(1 - \alpha)$ satisfies $\mathrm{Re}\,m(1 - \alpha) \in (0, 1/2]$. This implies that $m = 1$ and $\mathrm{Re}\,\alpha = 1/2$, and we conclude that (c) implies (a).

The equivalence of (c) and (d) comes from Proposition 2. □

4.3. A more topological version

Note again that condition (d) of Theorem 1 is a purely topological condition for the function ν_ζ. This means that the condition is preserved for any continuous function that are topologically conjugate to ν_ζ. More precisely, we have:

Theorem 10. *Conditions* (a) - (d) *of Theorem 1 are equivalent to:*
(e) *For any homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $h(\infty) = \infty$, the continuous function $\nu_{\zeta,h} := h \circ \nu_\zeta \circ h^{-1}$ has no topological disk D with $\overline{\nu_{\zeta,h}(D)} \subset D$.*

The proof is a routine. We may regard the dynamics of the map $\nu_{\zeta,h}$ as a topological deformation of the original dynamics of ν_ζ (Figure 3). Note that the critical line may not be a "line" any more when it is mapped by a homeomorphism. (That may even have a positive area!)

4.4. Variants

Next we consider ν_g for $g = \xi$, the *Riemann xi function* presented in the first section. It is known that $\xi : \mathbb{C} \to \mathbb{C}$ is an entire function whose zeros are exactly the non-trivial zeros of the Riemann zeta function. Moreover, we have the functional equation $\xi(z) = \xi(1 - z)$. See [Titchmarsh 86, p.16] for example.

Following the same argument as in the proof of Theorem 1, we have its variant in terms of ξ:

Theorem 11. *The following conditions are equivalent to conditions* (a) - (e) *stated above :*
(b') *All the fixed points of the meromorphic function $\nu_\xi(z) := z - \frac{\xi(z)}{z\,\xi'(z)}$ are indifferent.*
(c') *The meromorphic function ν_ξ above has no attracting fixed point.*
(d') *There is no topological disk D with $\overline{\nu_\xi(D)} \subset D$.*

The proof is simpler because ξ has neither trivial zeros nor poles.

Remark. To have an entire function it is enough to consider the function

$$\eta(z) := (z - 1)\zeta(z).$$

In this case, $\eta(z)$ and $\zeta(z)$ share all the zeros and one can state a theorem similar to Theorem 1. (We will apply Newton's method to this η in Section 6.)

5. Linearization problem

Under the Riemann hypothesis and the simplicity hypothesis, each non-trivial zero of ζ is of the form $\alpha = 1/2 + \gamma i \ (\gamma \in \mathbb{R})$, and α is an indifferent fixed point of ν_ζ with $\nu'_\zeta(\alpha) = 1 - 1/\alpha$ (Theorem 1 and Proposition 5). In this section, we present some brief observations on these fixed points. (This section is addressed to readers familiar with complex dynamics. See [Milnor 06, Section 10, Section 11] and references therein for more details on this subject.)

5.1. Non-trivial zeros and rotation number

We first point out the following easy fact:

Proposition 12. *Under the Riemann and simplicity hypotheses, non-trivial zero $\alpha = 1/2 + \gamma i \ (\gamma \in \mathbb{R})$ is an indifferent fixed point of ν_ζ with multiplier $e^{2\pi i\theta} \ (\theta \in \mathbb{R})$,*

where the values γ and θ are related by $\gamma = \frac{1}{2 \tan \pi\theta}$, or equivalently, $\theta = \frac{1}{\pi} \arctan \frac{1}{2\gamma}$.

Such a θ is called the *rotation number* of the indifferent fixed point α. We say α is *linearizable* if the local dynamics near α is conjugate to the rigid rotation $w \mapsto e^{2\pi i\theta}w$. A *Siegel disk* is the maximal domain where such a conjugacy exists. Now it is natural to ask:

Linearization problem. *Can θ be a rational number? Is α linearizable? That is, can ν_ζ have a Siegel disk?*

It is known that for Lebesgue almost every $\theta \in [0, 1) - \mathbb{Q}$, the irrationally indifferent fixed points with multiplier $e^{2\pi i\theta}$ are all linearizable. However, there is a dense subset of rotation numbers (consisting of some irrationals and all rationals in $[0, 1)$) that always gives non-linearizable indifferent fixed points. The sufficient conditions for (non)-linearizability are often described in terms of the continued fraction expansion of the rotation number $\theta \in [0, 1)$, which we denote by

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} =: [a_1, \ a_2, \ a_3, \ \ldots]$$

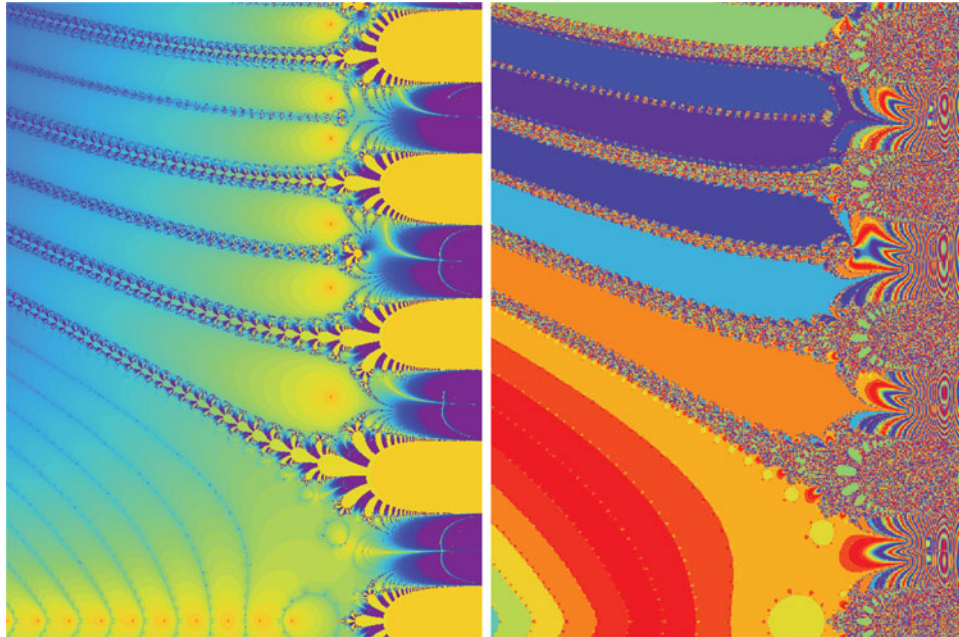where $a_1, \ a_2, \ a_3, \ \ldots$ are all in $\mathbb{N}$.

Figure A.1. Left: the Julia set of N_ζ in $\{\text{Re } z \in [-20, 10], \ \text{Im } z \in [-1, 39]\}$. (It is symmetric with respect to the real axis.) The orange dots are disks that are close to the zeros of ζ. More bluish colors indicate the points that require more iteration to get close to the zeros. Right: the same region in different colors. Colors distinguish the zeros to converge.

5.2. Numerical observation

Table 1 lists the continued fractions up to the 50th term for some zeros calculated by built-in functions `ZetaZero` and `ContinuedFraction` of *Mathematica* 10.0. (Here, $1/2 + \gamma_n i$ ($\gamma_n > 0$) is the nth non-trivial zero of ζ from below and $\gamma_n = 1/(2 \tan \pi \theta_n)$. Note that $\theta_n \to 0$ as $\gamma_n \to \infty$.)
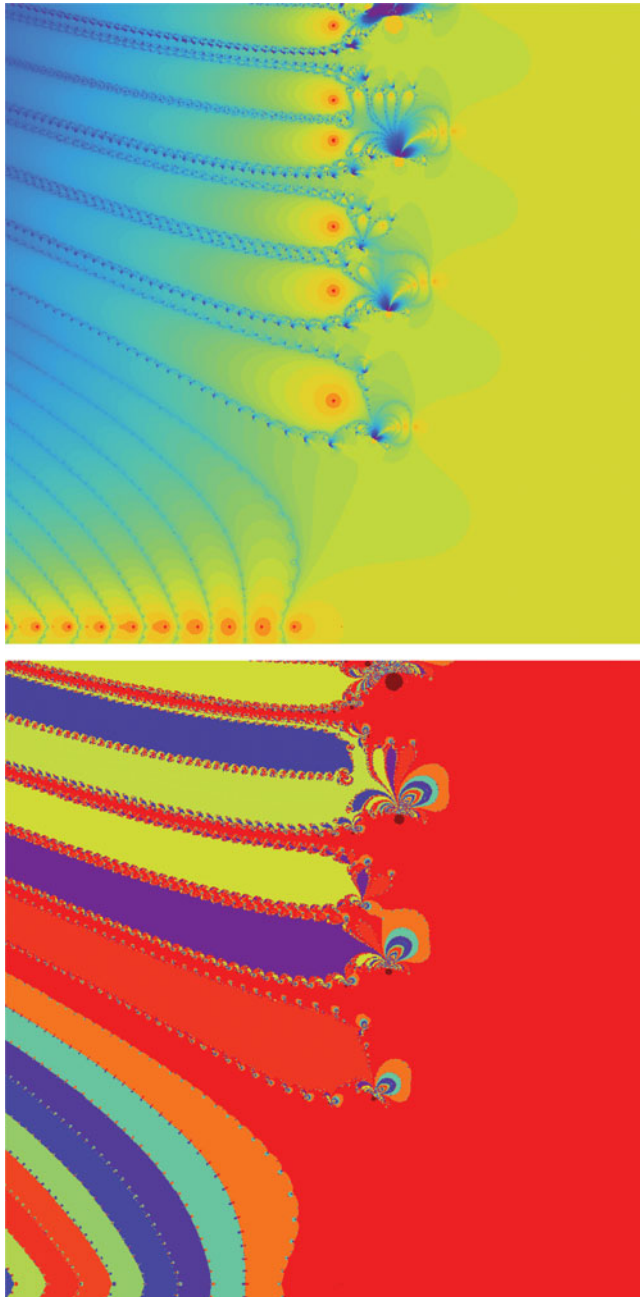

Figure A.2. The Julia set of N_η in $\{\mathrm{Re}\, z \in [-20, 20],\, \mathrm{Im}\, z \in [-1, 39]\}$, drawn in the same colors as in Figure A.1. Probably because η is entire, the dynamics of N_η is simpler than that of N_ζ. It is known that for any transcendental entire function g and its zeros, their immediate basins (the connected components of the Fatou set of N_g that contain the zeros) are simply connected and unbounded ([Mayer and Schleicher 06]).

Figure 4 shows some orbits near the first four non-trivial zeros of ζ in the dynamics of ν_ζ and ν_ξ.

Appendix: Newton's method

There are many root finding algorithms, but the most popular one is *Newton's method*. Let us apply it to the Riemann zeta function and its variants. The aim of this appendix is to describe it in terms of the holomorphic index. (See [Schleicher 08] for an intriguing investigation
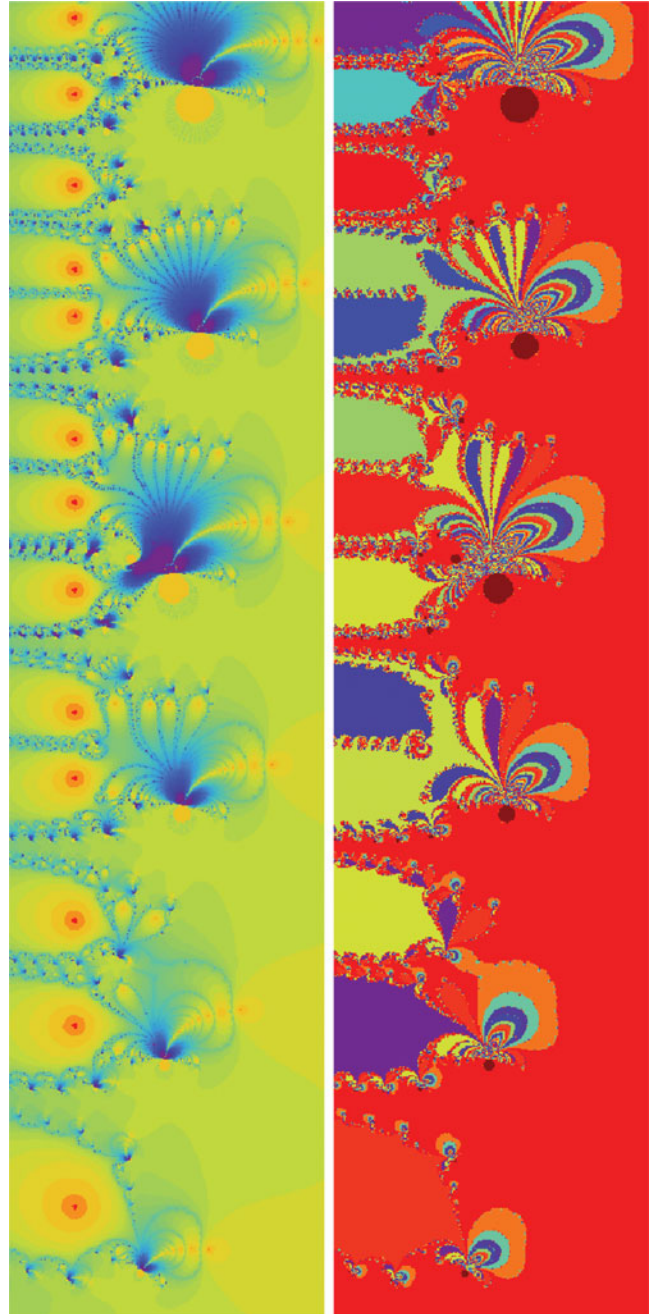

Figure A.3. Details of the Julia set of N_η in $\{\mathrm{Re}\, z \in [-2, 10],\, \mathrm{Im}\, z \in [10, 60]\}$ ("Heads of Chickens").
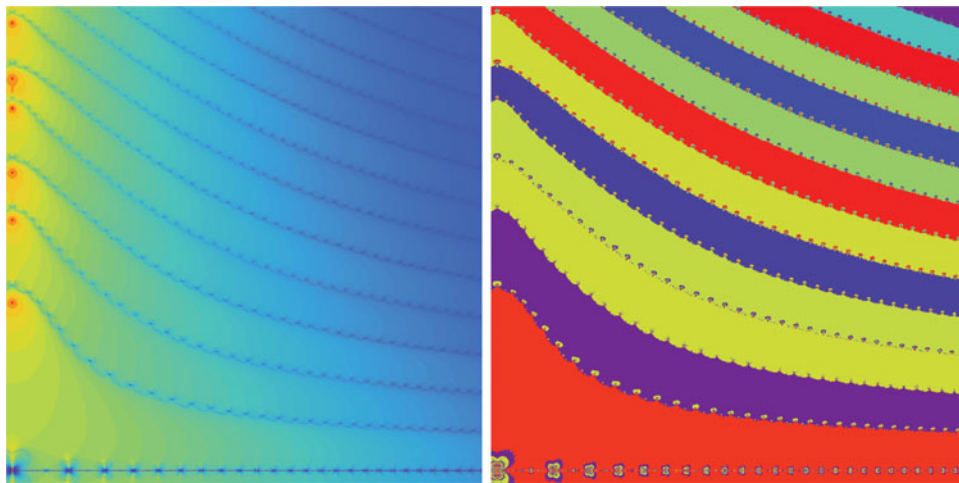
Figure A.4. Details of the Julia set of N_ξ in $\{\operatorname{Re} z \in [0, 40],\ \operatorname{Im} z \in [-1, 39]\}$. (It is symmetric with respect to the real axis and the critical line.) The dynamics seems surprisingly simple. Compare with the case of the hyperbolic cosine in Figure A.6.

on how we can efficiently detect all the zeros of ζ by Newton's method.)

A.1. Newton maps and fixed points

Let $g : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function. We define its *Newton map* $N_g : \mathbb{C} \to \widehat{\mathbb{C}}$ by

$$N_g(z) = z - \frac{g(z)}{g'(z)},$$

which is again meromorphic. (See [Bergweiler 99, Section 6].) Here is a version of Proposition 5 (and Corollary 6) for N_g:

Proposition 13 (Fixed points of N_g). *A given point $\alpha \in \mathbb{C}$ is a fixed point of N_g if and only if α is a zero or a pole of g. Moreover,*

- *if α is a zero of g of order $m \geq 1$, then α is an attracting fixed point of multiplier $N_g'(\alpha) = 1 - 1/m$ and its index is $\iota(N_g, \alpha) = m$.*
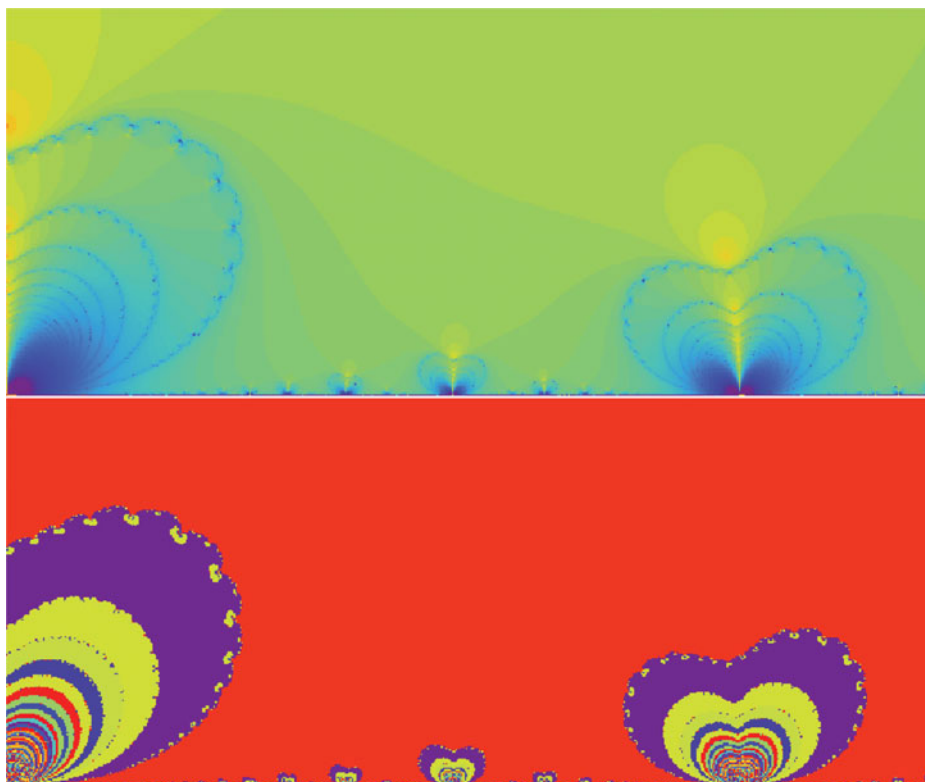

Figure A.5. Details of the Julia set of N_ξ in $\{\operatorname{Re} z \in [0.5, 6.5],\ \operatorname{Im} z \in [0, 2.5]\}$.
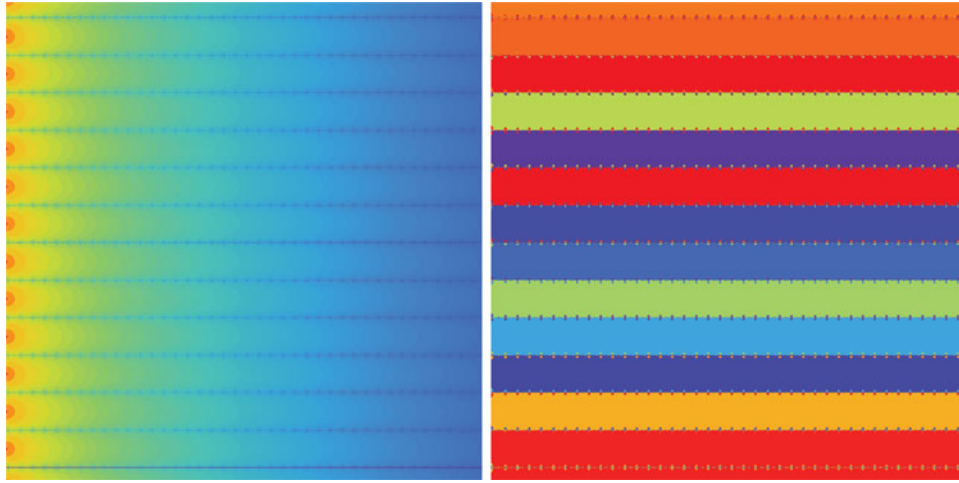
Figure A.6. The Julia set of the Newton map $N_{\cosh}$ of the hyperbolic cosine $\cosh z$ in $\{\operatorname{Re} z \in [0, 40],\ \operatorname{Im} z \in [-1, 39]\}$. Does the Julia set of $N_{\cosh}$ have the same topology as that of N_ξ? Do they belong to the same deformation space?

- *if α is a pole of g of order $m \geq 1$, then α is a repelling fixed point of multiplier $N_g'(\alpha) = 1 + 1/m$ and its index is $\iota(N_g, \alpha) = -m$.*

In particular, N_g has no fixed point of multiplier 1.

The proof is similar to that of Proposition 5 and is left to the reader.

Figure A.7. Details of the Julia set of the Newton map $N_{\cosh}$ in $\{\operatorname{Re} z \in [0, 1.5],\ \operatorname{Im} z \in [0, 3.5]\}$.

A.2. Newton's method

The idea of Newton's method is to use the attracting fixed points of N_g to detect the zeros of g. More precisely, by taking an initial value z_0 sufficiently close to a zero α, the sequence $\{N_g^n(z_0)\}_{n \geq 0}$ converges rapidly to the attracting fixed point α.

A.3. Holomorphic index and the argument principle

For the fixed point α of N_g its holomorphic index is

$$\iota(N_g, \alpha) = \frac{1}{2\pi i} \int_C \frac{1}{z - N_g(z)}\, dz = \frac{1}{2\pi i} \int_C \frac{g'(z)}{g(z)}\, dz$$

where C is a small circle around α in the counterclockwise direction. This is exactly the argument principle applied to g.

A.4. The Riemann hypothesis

When we apply Newton's method to the Riemann zeta function, all zeros of ζ are converted to attracting fixed points of $N_\zeta(z) = z - \zeta(z)/\zeta'(z)$. By Proposition 2, we have:

Theorem 14. *The Riemann hypothesis is true if and only if there is no topological disk D contained in the strip $\mathcal{S}' = \{z \in \mathbb{C} : 1/2 < \operatorname{Re} z < 1\}$ satisfying $\overline{N_\zeta(D)} \subset D$.*

A.5. Some pictures

It is easier to draw pictures of the chaotic locus (*the Julia set*) of Newton's map N_ζ than of v_ζ. In Figures A.1.–A.7., we present some pictures of the chaotic loci for the Newton maps of $\zeta(z)$, $\eta(z) := (z-1)\zeta(z)$ (this is an entire function), $\xi(z)$ and $\cosh(z)$ with comments.
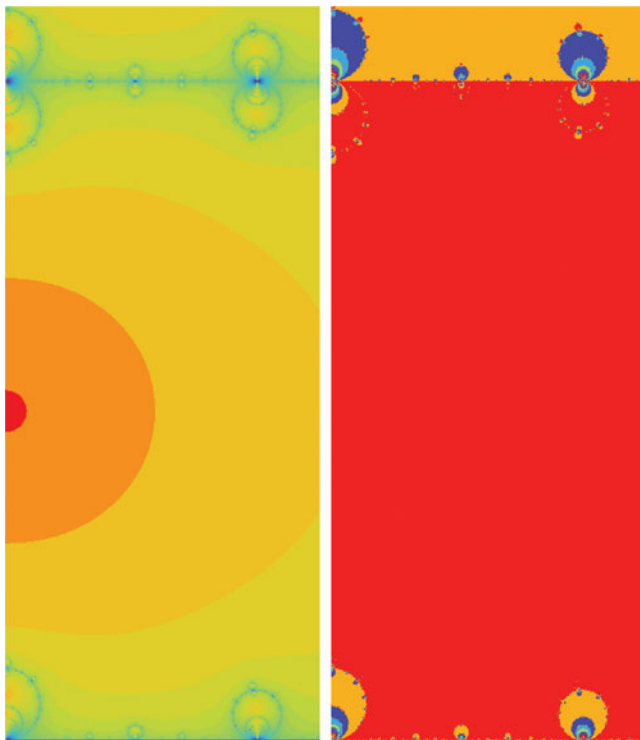
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