For a continuously indexed process the above theorem is not immediately applicable since the index is not a countable (not even) linearly ordered set. However for many second order processes it can be reduced to the preceding case as shown below. The idea here is to replace the whole process by (linear) combinations of a fixed suitably chosen countable set of random variables for which Theorem 1 is applicable. [The general case will be treated after this one.] It is based on the Karhunen-Loève representation which we now describe.

Thus let $\{X(t), t \in I \subset \mathbb{R}\}$ be a second order (scalar) process with mean function $t \mapsto m(t) = E(X(t))$ and a continuous covariance function $(s,t) \mapsto r(s,t) = E[(X(s) - m(s))\overline{(X(t) - m(t))}]$. Then $r(s,t)$ is positive definite, and if I is a compact interval, assumed hereafter, by the classical Mercer theorem (cf., e.g., Riesz and Sz.-Nagy [1], p.245) since $\int_I \int_I |r(s,t)|^2 \, ds\, dt < \infty$, it can be represented by a uniformly convergent series: ('bar' for complex conjugate)

$$r(s,t) = \sum_{i=1}^{\infty} \frac{\psi_i(s)\overline{\psi}_i(t)}{\lambda_i}; \ \lambda_i > 0, \tag{14}$$

where the $\psi_i(\cdot)$ are continuous functions satisfying the integral equation

$$\psi(t) = \lambda \int_I r(s,t)\psi(s)\, ds, \tag{15}$$

and $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. Here the λ_i are the eigenvalues (counted according to their multiplicity) and ψ_i are the corresponding eigenfunctions of the "kernel" r, and $\{\psi_n, n \geq 1\}$ forms a complete orthonormal set in the Lebesgue space $L^2(I)$, with Lebesgue measure, satisfying (15). This classical result and its relation to the Hilbert-Schmidt theory of symmetric kernels is nicely treated in the above reference, and their properties are needed here. First we consider the case that $E(X(t)) = m(t) = 0$ so that the $X(t)$ are centered. Now define the random variables

$$\xi_n = \sqrt{\lambda_n} \int_I X(t)\bar{\psi}_n(t)\, dt, \ n \geq 1, \tag{16}$$

where the integral is obtained using Fubini's theorem, since $X(t,\omega)$ is jointly measurable ($r(\cdot,\cdot)$ being jointly continuous) in (t,ω). [Alternatively it may be regarded as a Bochner integral.] In any case, we get

$$E(\xi_n\bar{\xi}_m) = \sqrt{\lambda_m\lambda_n} \int_I \int_I E(X(s)\bar{X}(t))\psi_n(s)\bar{\psi}_m(t)\, ds\, dt$$

$$= \sqrt{\lambda_n \lambda_m} \int_I [\int_I r(s,t)\psi_n(s)\,ds]\bar{\psi}_m(t)\,dt$$
$$= \sqrt{\frac{\lambda_m}{\lambda_n}} \int_I \psi_n(t)\bar{\psi}_m(t)\,dt, \text{ by (15)},$$
$$= \delta_{nm}\sqrt{\frac{\lambda_m}{\lambda_n}}.$$

It follows (on expanding inner products) that $X_n(t) = \sum_{k=1}^n \xi_k \frac{\psi_k(t)}{\sqrt{\lambda_k}} \to X(t)$ in $L^2(P)$, by (14), where $X(t) = \sum_{n=1}^\infty \xi_n \frac{\psi_n(t)}{\sqrt{\lambda_n}}$, and conversely if $X(t)$ is given by this series, converging in mean, then $E(X(s)\overline{X}(t)) = \lim_n E(X_n(s)\overline{X}_n(t)) = \sum_{n=1}^\infty \frac{\psi_n(s)\bar{\psi}_n(t)}{\lambda_n}$ holds. If $E(X(t)) = m(t) \neq 0$, then the above argument applied to $Y(t) = X(t) - m(t)$ establishes the following classical *Karhunen-Loève* representation:

4. Proposition. *If $\{X(t), t \in I\}$ is a second order process with $E(X(t)) = m(t)$, and a continuous covariance function $r(\cdot,\cdot)$ on a compact interval I, then*

$$X(t) = m(t) + \sum_{n=1}^\infty \xi_n \frac{\bar{\psi}_n(t)}{\sqrt{\lambda_n}}, \quad t \in I, \tag{17}$$

holds uniformly in t, and the convergence is in $L^2(P)$ where the $\lambda_n > 0$ and ψ_n are the eigenvalues and the corresponding (complete orthonormal in $L^2(I)$) eigenfunctions of the kernel r, satisfying (15), and hence the ξ_n are orthonormal in $L_0^2(P)$, given by (16).

Let $\mathcal{F} = \sigma(X(t) \in I)$, $\mathcal{F}_\infty = \sigma(\xi_n, n \geq 1)$ be the σ-algebras generated by the random variables shown. Since each $X(t)$ is a linear combination of the ξ_n, by (17), it follows that the $X(t)$ are $\mathcal{F}_\infty$-measurable for each $t \in I$ so that $\mathcal{F} \subset \mathcal{F}_\infty$. On the other hand each ξ_n is $\mathcal{F}$-measurable, by (16), for $n \geq 1$, so that $\mathcal{F}_\infty \subset \mathcal{F}$ and hence $\mathcal{F} = \mathcal{F}_\infty \subset \Sigma$. If $\tilde{P} = P|\mathcal{F}$, it is then determined by the $X(t)$ as well as the ξ_n. Thus *using (17), we can transfer the testing problem for measures P and Q, (or $\tilde{P}, \tilde{Q}$ on $\mathcal{F} = \mathcal{F}_\infty$) to the* **sequence** $\{\xi_n, n \geq 1\}$, *to find the likelihood function* $f_\infty = \frac{d\tilde{Q}^c}{d\tilde{P}}$ by the approximation procedure of Theorem 1 with $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$. Consequently, if $f_n = \frac{d\tilde{Q}_n^c}{d\tilde{P}_n}$ where $\tilde{P}_n = \tilde{P}|\mathcal{F}_n$, $\tilde{Q}_n = \tilde{Q}|\mathcal{F}_n$, then $f_n \to f_\infty$ a.e. as in Theorem 1, and f_∞ is the desired likelihood function.

This method will now be illustrated to gain an insight into the type of calculations needed for some test problems.

5. Example. Let $\{X(t), t \in [0,1]\}$ be a Gaussian process with mean