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THE EXPANSION THEOREM FOR SINGULAR SELF-ADJOINT LINEAR DIFFERENTIAL OPERATORS*

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1. There has recently appeared three papers giving independently and almost simultaneously a much simplified approach to the expansion theorem for self-adjoint singular ordinary differential equations. Levitan [7] treats by means of a simple lemma the differential operator of arbitrary order but does not deal with the problem of uniqueness of the expansion. Yosida [11] and Levinson [5, 6] considered only the second order case but with uniqueness proved in the limit point case. All three accounts make important use of the Helly selection theorem for a set of functions of bounded variation.

Here it will be shown that using the lemma of Levitan [7] and a treatment rather similar to that used by Levinson [6] for the inverse transform theorem, it is possible to obtain a simple self-contained proof of uniqueness under broad conditions. In particular none of the apparatus of Hilbert space or the theory of singular integral equations will be required.

One of the results that follow, Theorem II, has been demonstrated by Coddington [2] with the use of Levitan's lemma and theorems from Hilbert space theory. Moreover Theorem III is implicit in Coddington's results because of the remarks which follow his Theorem 3 [2, p. 735].

The methods of the present paper can easily be carried over to singular self-adjoint systems using the formulations given by Bliss [1] for the non-singular case.

For problems of even order with separated boundary conditions the results of Weyl [10], Stone [8] and Titchmarsh [9] for the second order case have been generalized by Kodaira [3, 4].

2. Let L be the differential operator

$$L = p_0\left(\frac{d}{dt}\right)^n + p_1\left(\frac{d}{d_t}\right)^{n-1} + \cdots + p_n$$

where the p_j are complex-valued functions of t with $n - j$ continuous derivatives on the open interval (a, b), that is of class C_{n-j} on $a < t < b$. The cases $a = -\infty$, $b = \infty$, or both are allowed. Let $p_0(t) \neq 0$ on (a, b) and let L be identical with its adjoint operator defined by

$$(-1)^n\left(\frac{d}{dt}\right)^n(\bar{p}_0\cdot) + (-1)^{n-1}\left(\frac{d}{dt}\right)^{n-1}(\bar{p}_1\cdot) + \cdots + \bar{p}_n,$$

where the bar is used for complex conjugate.

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Let D denote the class of complex-valued functions ϕ which are of class C_{n-1} on (a, b) and such that $\phi^{(n-1)}$ is absolutely continuous on every closed interval contained in (a, b). Consider the boundary relations

$$U_i(x) = \sum_{j=1}^{n} (M_{ij}\, x^{(j-1)}\, (\alpha) + N_{ij}\, x^{(j-1)}\, (\beta) = 0, i = 1, \cdots , n$$

for some α and β where $a < \alpha < \beta < b$ and M_{ij} and N_{ij} are constants. The n equations $U_i(x) = 0$ will be denoted by $U(x) = 0$. The boundary problem on the closed interval $[\alpha, \beta]$

$$L(x) + \lambda x = 0, \qquad U(x) = 0,$$

where λ is a parameter is said to be self-adjoint if for any two functions x and y of class D for which $U(x) = 0$ and $U(y) = 0$,

$$\int_\alpha^\beta (L(x)\bar{y} - x\bar{L}(y))\, dt = 0.$$

For a self-adjoint boundary problem it is well known that there exists a unique set of characteristic values $\{\lambda_j\}$, with repetitions for multiple characteristic values, which are real, and which have no finite limit point. For each λ_j there is a characteristic function $\psi_j(t)$ of class C_n such that $L(\psi_j) + \lambda_j\psi_j = 0$, $U(\psi_j) = 0$. The characteristic functions are orthonormal so that

$$\int_\alpha^\beta \psi_j\bar{\psi}_k\, dt = \delta_j^k,$$

and any function $f(t)$ of class $\mathfrak{L}^2(\alpha, \beta)$ satisfies the completeness relationship

$$\int_\alpha^\beta |f(t)|^2\, dt = \sum_j \left| \int_\alpha^\beta f\bar{\psi}_j\, dt \right|^2. \tag{2.0}$$

Let c be a fixed point in (a, b) and suppose $\alpha < c < \beta$. Let $\phi_k(t, \lambda)$ be the solution of $L(x) + \lambda x = 0$ which satisfies the conditions

$$\phi_k^{(j-1)}\, (c, \lambda) = \delta_k^j, \qquad j, k = 1, \cdots , n.$$

Denote the interval $[\alpha, \beta]$ by γ. Clearly the $\phi_k(t, \lambda)$ are independent solutions. Thus there exist constants C_{jk} such that

$$\psi_j(t) = \sum_{k=1}^{n} C_{jk}\, \phi_k(t, \lambda_j). \tag{2.1}$$

Here λ_j also depends on γ or more accurately on the boundary conditions on γ, $U_\gamma(x) = 0$. The completeness relationship (2.0) becomes

$$\int_\alpha^\beta |f(t)|^2\, dt = \sum_j \left[\sum_{k,l=1}^{n} \bar{C}_{jk\gamma}\, C_{jl\gamma}\, g_k(\lambda_j)\bar{g}_l(\lambda_j) \right] \tag{2.2}$$

where

$$g_k(\lambda) = \int_\alpha^\beta f(t)\bar{\phi}_k(t, \lambda)\, dt. \tag{2.3}$$

Let the function $\rho_{kl\gamma}\, (\sigma)$ be constant except at $\sigma = \lambda_j$ where

$$\rho_{kl\gamma}(\lambda_j + 0) - \rho_{kl\gamma}(\lambda_j - 0) = \bar{C}_{jk\gamma}\, C_{jl\gamma}\,.$$

(The modification at a multiple characteristic root is clear.) Let $\rho_{kl\gamma}(\sigma) = \rho_{kl\gamma}(\sigma + 0)$ and $\rho_{kl\gamma}(0) = 0$. Thus the $\rho_{kl\gamma}(\sigma)$ are n^2 step functions and (2.2) can be written as

$$\int_\alpha^\beta |f|^2\, dt = \int_{-\infty}^{\infty} \sum_{k,l=1}^{n} g_k(\sigma)\bar{g}_l(\sigma)\, d\rho_{kl\gamma}(\sigma). \tag{2.4}$$

Let ρ_γ denote the matrix $(\rho_{kl\gamma})$. Clearly if $\sigma_2 > \sigma_1$ the matrix $\rho_\gamma(\sigma_2) - \rho_\gamma(\sigma_1)$ is Hermitian and positive semi-definite. The last is a consequence of $| \sum_{k=1}^{n} C_{jk\gamma}\, v_k |^2 \geqq 0$ for any vector $v = (v_1, \cdots, v_n)$.

The behavior of ρ_γ is to be considered when $[\alpha, \beta]$ tends to (a, b). The M_{ij} and N_{ij} in the boundary condition may be constant in what follows or may depend on the interval $[\alpha, \beta]$ under consideration and therefore the boundary condition will be designated by $U_\gamma(x) = 0$. (The dependence of M_{ij} and N_{ij} in U_γ on $[\alpha, \beta]$ need not be continuous.)

Theorem I (Levitan [7]). *Given a set of intervals $\{\gamma\}$ tending to (a, b) and a self-adjoint boundary condition $U_\gamma(x) = 0$ associated with each interval of the set. Then the set $\{\gamma\}$ contains a sequence $\{\gamma_j\}$ such that for almost all σ_1 and σ_2*

$$\lim_{j\to\infty}[\rho_{\gamma_j}(\sigma_2) - \rho_{\gamma_j}(\sigma_1)] \tag{2.5}$$

exists. If the limit is denoted by the matrix $\rho(\sigma_2) - \rho(\sigma_1)$ then this matrix is Hermitian and positive semi-definite.

In what follows $\sum$ denotes $\sum_{j,k=1}^{n}$.

Corollary. *If $f(t) \in \mathfrak{L}^2(a, b)$ and if the integrals*

$$g_k(\sigma) = \int_a^b f(t)\bar{\phi}_k(t, \sigma)\, dt \tag{2.6}$$

are convergent then with ρ determined in Theorem I

$$\int_a^b |f(t)|^2\, dt = \int_{-\infty}^{\infty} \sum g_j(\sigma)\bar{g}_k(\sigma)\, d\rho_{jk}(\sigma). \tag{2.7}$$

Moreover if

$$f_A(t) = \int_{-A}^{A} \sum g_j(\sigma)\phi_k(t, \sigma)\, d\rho_{jk}(\sigma). \tag{2.8}$$

then on (a, b)

$$\text{l.i.m.}_{A\to\infty} f_A(t) = f(t). \tag{2.9}$$

(*Even if the integrals* (2.6) *do not converge,* if $a < \tilde{a} < \tilde{b} < b$, *and*

$$g_j(\sigma) = \int_{\tilde{a}}^{\tilde{b}} f(t)\bar{\phi}_j(t, \sigma)\, dt \tag{2.10}$$

then there exist $g_j(\sigma)$ measurable B (Borel) *such that as* $(\tilde{a}, \tilde{b}) \to (a, b)$

$$\int_{-\infty}^{\infty} \sum (g_j(\sigma) - \tilde{g}_j(\sigma))(\bar{g}_k(\sigma) - \bar{\tilde{g}}_k(\sigma))\, d\rho_{jk}(\sigma) \to 0.) \tag{2.11}$$

THEOREM II (Coddington [2]). *If neither equation*

$$L(x) \pm ix = 0$$

has a solution which is $\mathfrak{L}^2(a, b)$ *then the Hermitian positive semidefinite matrix* $\rho(\sigma)$ *satisfying* (2.7) *is unique* (*except for an additive constant matrix and the values of* $\rho(\sigma)$ *at point of discontinuity*) *and if* σ_1 *and* σ_2 *are points of continuity of* $\rho(\sigma)$ *then*

$$\rho_\gamma(\sigma_2) - \rho_\gamma(\sigma_1) \to \rho(\sigma_2) - \rho(\sigma_1)$$

as $\gamma \to (a, b)$ *irrespective of how the self adjoint boundary condition* U_γ *varies with* γ.

A case of great importance not covered in Theorem II is where the open interval (a, b) is replaced by the interval $[a, b)$, $-\infty < a \leqq t < b$, and where the subintervals $[\alpha, \beta]$ are replaced by $[a, \beta]$. It is now assumed that the coefficients $p_j(t)$ of L are of class C_{n-j} on $[a, b)$ and $p_0(t) \neq 0$ on $[a, b)$. Actually as will be seen in §6 these conditions on $p_j(t)$ can be replaced by much less stringent ones). Since $a = \alpha$ the dependence on γ is now simply a dependence on β and will be so indicated. It is assumed that the boundary condition $U_\beta(x) = 0$ contain m equations of the form

$$\sum_{j=1}^{n} M_{ij}\, x^{(j-1)}(a) = 0 \qquad i = 1, \cdots, m, \tag{2.12}$$

where the coefficients M_{ij} do not depend on β. Thus $N_{ij} = 0$ for $i = 1, \cdots m$. The m equations (2.12) will be denoted by $U^{(1)}(x) = 0$. The remaining $n - m$ equations of $U_\beta(x) = 0$ will be denoted by $U_\beta^{(2)}(x) = 0$. It follows from simple algebraic considerations that there exist $n - m$ independent solutions $\chi_k(t, \lambda)$ of $L(x) + \lambda x = 0$ which all satisfy (2.12), $U^{(1)}(\chi_k) = 0$, and which together with their $n - 1$ derivatives assume values at $t = a$ independent of λ. Any solution satisfying (2.12) is a linear combination of the χ_k. (The independence of the χ_k is equivalent to the statement that if for each k, $k = 1, \cdots$, $n - m$, $\chi_k^{(j-1)}(a, \lambda)$, $j = 1, \cdots, n$, is regarded as a vector with n components then the $n - m$ vectors span a space of dimension $n - m$.)

The characteristic functions associated with a self-adjoint boundary value problem must now be a linear combination of the χ_k. Thus to each $\lambda_j = \lambda_{j\beta}$ there corresponds constants $C_{jk\beta}$ such that

$$\sum_{k=1}^{n-m} C_{jk\beta}\, \chi_k(t, \lambda_j),$$

form an orthonormal system. Corresponding to the $n \times n$ matrix $\rho_\gamma(\sigma)$ of Theorem II there is now an $n - m \times n - m$ matrix $\rho_\beta(\sigma)$. Theorem I is readily shown to be valid for this case also. Instead of Theorem II there is now

THEOREM III (Coddington [2, p. 735]). *If neither equation* $L(x) \pm ix = 0$ *has a solution which is* $\mathfrak{L}^2(a, b)$ *and which satisfies the boundary condition* $U^{(1)}(x) = 0$, *then the* $n - m \times n - m$ *matrix* $\rho(\sigma)$ *is unique* (in the same sense as in Theorem II). *Moreover if* σ_1 *and* σ_2 *are points of continuity of* $\rho(\sigma)$ *then*

$$\rho_\beta(\sigma_2) - \rho_\beta(\sigma_1) \to \rho(\sigma_2) - \rho(\sigma_1)$$

as $\beta \to b$ independent of the $n - m$ equations $U_\beta^{(2)}(x) = 0$ which may vary with β. For any function f of $\mathfrak{L}^2(a, b)$

$$\int_a^b |f|^2\, dt = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n-m} g_j(\sigma)\bar{g}_k(\sigma)\, d\rho_{jk}(\sigma)$$

and the analogues of the other relations (2.6)–(2.11) *hold.* (Here $g_j(\sigma) = \int_a^b f(t)\bar{\chi}_j(t, \sigma)\, dt$.)

Theorem III includes as a special case the limit point case of Weyl [10] for the problem $n = 2$. For the case $n = 2m$ the Theorem overlaps with results of Kodiara [3, 4] who considers separated boundary conditions, m at a and m at b, but where otherwise a complete classification of all possible cases is obtained.

3. The functions $\rho_{jk\gamma}(\sigma)$ of Theorem I are all bounded independent of γ in the following sense.

LEMMA 3.1 (Levitan [7]). *There exists a function $M(u) < \infty$ such that for $|\sigma| \leqq u$,*

$$|\rho_{jk\gamma}(\sigma)| \leqq M(u). \tag{3.0}$$

The function $M(u)$ depends on L but not on γ or U_γ.

PROOF. From the definition of $\rho_{jk\gamma}(\sigma)$ and the Schwartz inequality it follows easily that

$$\left(\int_{-u}^{u} |d\rho_{jk\gamma}(\sigma)|\right)^2 \leqq \int_{-u}^{u} d\rho_{jj\gamma}(\sigma) \int_{-u}^{u} d\rho_{kk\gamma}(\sigma).$$

Thus to prove the lemma it suffices to show that

$$\sum_{j=1}^{n} \int_{-u}^{u} d\rho_{jj\gamma}(\sigma) \leqq M(u). \tag{3.1}$$

Let η be a fixed constant chosen so that $c + \eta < b$. It follows from the continuity in (t, λ) of $\phi_k^{(j-1)}(t, \lambda)$ as defined above (2.1) that for $|\sigma| \leqq u$ and $c \leqq t \leqq c + \eta$ there exists an A_1, depending on u such that

$$|\phi_k^{(j-1)}(t, \sigma)| \leqq A_1, \qquad j, k = 1, \cdots n.$$

From $L(\phi_k) + \sigma\phi_k = 0$ follows that from some A_2, $|\phi_k^{(n)}(t, \sigma)| \leqq A_2$. Using the mean value theorem and setting $A = \max(A_1, A_2)$

$$|\phi_k^{(j-1)}(t, \sigma) - \delta_k^j| \leqq A(t - c), c \leqq t \leqq c + \eta, \tag{3.2}$$

for $j, k = 1, \cdots n$, $|\sigma| \leqq u$.

Let $\tilde{f}(t)$ be chosen so that $\tilde{f}(t) \geqq 0$ and $\tilde{f}(t)$ is of class C_n on (a, b). Moreover let $\tilde{f}(t)$ vanish outside of the interval $(c, c + h)$ where $h = \min(\eta, 1/(10An^2))$, and

$$\int_c^{c+h} \tilde{f}\, dt = 1.$$

By (3.2) if $|\sigma| \leqq u$

$$\frac{3}{2} < \left| \int_c^{c+h} \bar{f}(t)\bar{\phi}_j^{(j-1)}(t, \sigma)\, dt \right| > \frac{1}{\sqrt{2}} \tag{3.3}$$

and

$$\left| \int_c^{c+h} \bar{f}(t)\bar{\phi}_k^{(j-1)}(t, \sigma)\, dt \right| < \frac{1}{4n^2}, \qquad k \neq j. \tag{3.4}$$

Clearly $\bar{f}$ depends on u. By (2.0) if $\alpha < c < c + h < \beta$,

$$\int_\alpha^\beta |\bar{f}^{(k-1)}(t)|^2\, dt = \sum_j \left| \int_\alpha^\beta \bar{f}^{(k-1)}\bar{\psi}_j\, dt \right|^2. \tag{3.5}$$

But

$$(-1)^{k-1} \int_\alpha^\beta \bar{f}^{(k-1)}\bar{\psi}_j\, dt = \int_\alpha^\beta \bar{f}\,\bar{\psi}_j^{(k-1)}\, dt = \sum_{l=1}^n C_{jl\gamma} \int_\alpha^\beta \bar{f}\,\bar{\phi}_l^{(k-1)}(t, \lambda_{j\gamma})\, dt.$$

Using (3.3) and (3.4) it follows that

$$\begin{aligned} \left| \int_\alpha^\beta \bar{f}^{(k-1)}\bar{\psi}_j\, dt \right|^2 &\geqq \tfrac{1}{2} |C_{jk\gamma}|^2 - \frac{3}{8n^2}\left(\sum_{l=1}^n |C_{jl\gamma}|\right)^2 \\ &\geqq \tfrac{1}{2} |C_{jk\gamma}|^2 - \frac{3}{8n}\sum_{l=1}^n |C_{jl\gamma}|^2. \end{aligned}$$

Thus

$$\sum_{|\lambda_{j\gamma}| \leqq u} \left| \int_\alpha^\beta \bar{f}^{(k-1)}\bar{\psi}_j\, dt \right|^2 \geqq \frac{1}{2}\int_{-u}^u d\rho_{kk\gamma}(\sigma) - \frac{3}{8n}\int_{-u}^u \sum_{l=1}^n d\rho_{ll\gamma}(\sigma).$$

Using this in (3.5)

$$\int_\alpha^\beta |\bar{f}^{(k-1)}|^2\, dt > \frac{1}{2}\int_{-u}^u d\rho_{kk\gamma}(\sigma) - \frac{3}{8n}\int_{-u}^u \sum_{l=1}^n d\rho_{ll\gamma}(\sigma).$$

Summing the above for $1 \leqq k \leqq n$ there follows

$$\sum_{k=1}^n \int_\alpha^\beta |\bar{f}^{(k-1)}|^2\, dt > \frac{1}{8}\sum_{k=1}^n \int_{-u}^u d\rho_{kk\gamma}(\sigma).$$

Since the left side is independent of γ this proves (3.1).

Proof of Theorem I. The proof is an immediate consequence of the above lemma and the Helly selection theorem. The Hermitian, positive semi-definiteness of $\rho(\sigma_2) - \rho(\sigma_1)$ follows from that of $\rho_{\gamma_j}(\sigma_2) - \rho_{\gamma_j}(\sigma_1)$.

To prove the Plancheral relationship (2.7) consider first a function $f(t)$ of class C_n on (a, b) and vanishing identically outside of some closed interval contained in (a, b). By (2.4) if α is near enough to a and β to b then

$$\int_a^b |f(t)|^2\, dt = \int_{-\infty}^\infty \sum g_j(\sigma)\bar{g}_k(\sigma)\, d\rho_{jk\gamma}(\sigma). \tag{3.6}$$

Since $f(t)$ vanishes near a and b and

$$g_j(\sigma) = \int_a^b f(t)\bar{\phi}_j(t, \sigma)\, dt,$$

it follows that

$$\int_a^b L(f(t))\bar{\phi}_j(t, \sigma)\, dt = \int_a^b f(t)\bar{L}(\phi_j(t, \sigma))\, dt$$

$$= -\sigma \int_a^b f(t)\bar{\phi}_j(t, \sigma)\, dt = -\sigma g_j(\sigma).$$

thus

$$\int_a^b | L(f) |^2\, dt = \int_{-\infty}^{\infty} \sum \sigma^2 g_j(\sigma) g_k(\sigma)\, d\rho_{jk\gamma}(\sigma).$$

From the above it follows that for any $A > 0$,

$$\frac{1}{A^2} \int_a^b | L(f) |^2\, dt > \left(\int_{-\infty}^{-A} + \int_A^{\infty}\right) \sum g_j(\sigma)\bar{g}_k(\sigma)\, d\rho_{jk\gamma}(\sigma).$$

Thus the convergence of the integral on the right of (3.6) is uniform in γ. Letting γ take on the values γ_j and letting $j \to \infty$ it follows from (3.6) that

$$\int_a^b | f(t) |^2\, dt = \int_{-\infty}^{\infty} \sum g_j(\sigma) g_k(\sigma)\, d\rho_{jk}(\sigma). \tag{3.7}$$

The extension of the result to any $f(t) \in \mathfrak{L}^2(a, b)$ can now be obtained. However here it suffices to take the case of functions which vanish outside of a closed interval contained in (a, b).

It will be proved that for such a function $f(t)$.

$$f(t) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} \sum g_j(\sigma)\phi_k(t, \sigma)\, d\rho_{jk}(\sigma). \tag{3.8}$$

Since functions in $\mathfrak{L}^2$ can be approximated in $\mathfrak{L}^2$ by a sequence of functions $\{f_{(j)}\}$ of class C_n as above with a difference tending to zero as $j \to \infty$, it follows that (3.7) applied to the difference $f_{(l)} - f_{(m)}$ as $l, m \to \infty$ yields the result (3.7). (For any f in $\mathfrak{L}^2(a, b)$ this then yields (2.10), (2.11) and (2.7).)

In what follows

$$\int_{-\infty}^{\infty} \sum G_j(\sigma) H_k(\sigma)\, d\rho_{jk}(\sigma)$$

will be denoted by

$$\int_{-\infty}^{\infty} GH\, d\rho.$$

If f and F vanish outside of some closed interval contained in (a, b) and (3.7) is applied to $f + i^l F$, $l = 0, 1, 2, 3$, then there results

$$\int_a^b f\bar{F}\, dt = \int_{-\infty}^{\infty} g\bar{G}\, d\rho. \tag{3.9}$$

where g_j and G_j are the transforms of f and F. Let

$$f_A(t) = \int_{-A}^{A} \sum g_j \phi_k \, d\rho_{jk}\, .$$

Then multiplying the above by $\bar{F}$ and integrating there follows

$$\int_a^b f_A \bar{F}\, dt = \int_{-A}^{A} g\bar{G}\, d\rho.$$

Thus subtracting from (3.9)

$$\int_a^b (f - f_A)\bar{F}\, dt = \left(\int_{-\infty}^{-A} + \int_A^{\infty}\right) g\bar{G}\, d\rho.$$

or

$$\begin{aligned}\left| \int_a^b (f - f_A)\bar{F}\, dt \right|^2 &\leqq \left(\int_{-\infty}^{-A} + \int_A^{\infty}\right) g\bar{g}\, d\rho \int_{-\infty}^{\infty} G\bar{G}\, d\rho \\ &= \left(\int_{-\infty}^{-A} + \int_A^{\infty}\right) g\bar{g}\, d\rho \int_a^b |\, F\,|^2\, dt.\end{aligned}$$

Let $F = f - f_A$ over some interval $[\alpha, \beta]$ of (a, b) and zero otherwise. Then the above yields

$$\int_\alpha^\beta |\, f - f_A\,|^2\, dt \leqq \left(\int_{-\infty}^{-A} + \int_A^{\infty}\right) g\bar{g}\, d\rho.$$

Since the right side is independent of α and β

$$\int_a^b |f - f_A\,|^2\, dt \leqq \left(\int_{-\infty}^{-A} + \int_A^{\infty}\right) g\bar{g}\, d\rho$$

which proves (3.8), that is

$$f(t) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} \sum g_j\, \phi_k\, d\rho_{jk}\, . \tag{3.10}$$

(The results (2.9) for any $f \in \mathfrak{L}^2(a, b)$ is proved in the same way.)

LEMMA 3.2 (Levinson [6]). *Let $G_j(\sigma)$, $j = 1, \cdots n$, be continuous functions of σ on $(-\infty, \infty)$ and let*

$$\int_{-\infty}^{\infty} G\bar{G}\, d\rho(\sigma) < \infty. \tag{3.11}$$

Let

$$P_A(t) = \int_{-A}^{A} \sum G_j\, \phi_k\, d\rho_{jk}(\sigma). \tag{3.12}$$

Then $\text{l.i.m.}_{A\to\infty} P_A(t) = P(t)$ *exists and*

$$\int_a^b |\, P(t)\,|^2 \leqq \int_{-\infty}^{\infty} G\bar{G}\, d\rho. \tag{3.13}$$

PROOF. Let $f(t)$ vanish outside of some closed interval contained in (a, b) and let its transform be given by $g_j(\sigma)$. Then multiplying (3.12) by $\bar{f}(t)$ and integrating there results

$$\int_a^b P_A(t)\bar{f}(t)\,dt = \int_{-A}^{A} G\bar{g}\,d\rho.$$

Let $B > A$. Then

$$\int_a^b (P_B - P_A)\bar{f}\,dt = \left(\int_{-B}^{-A} + \int_A^B\right) G\bar{g}\,d\rho.$$

From this follows

$$\left|\int_a^b (P_B - P_A)\bar{f}\,dt\right|^2 \leqq \left(\int_{-B}^{-A} + \int_A^B\right) G\bar{G}\,d\rho \left(\int_{-B}^{-A} + \int_A^B\right) g\bar{g}\,d\rho$$
$$\leqq \left(\int_{-B}^{-A} + \int_A^B\right) G\bar{G}\,d\rho \int_a^b |f|^2\,dt$$

by use of (3.6). Let $f = P_B - P_A$ over an interval $[\alpha, \beta]$ and vanish outside. Then the above yields

$$\int_\alpha^\beta |P_B - P_A|^2\,dt \leqq \left(\int_{-B}^{-A} + \int_A^B\right) G\bar{G}\,d\rho.$$

Since the right side is independent of $[\alpha, \beta]$ the inequality persists with $\alpha = a$ and $\beta = b$ and the lemma is proved. Taking $A = 0$ there follows readily (3.13).

4. PROOF OF THEOREM 2. Suppose there exists two distinct matrices, $\rho(\sigma)$ and $\tilde{\rho}(\sigma)$. Let f vanish outside of some closed interval contained in (a, b) and have transform $g_j(\sigma)$. Let

$$F_A(t) = \int_{-A}^{A} \sum g_j(\sigma)\phi_k(t, \sigma)\,d\rho_{jk}(\sigma)/(i - \sigma) \tag{4.1}$$

and $\tilde{F}_A(t)$ be defined with $\rho(\sigma)$ replaced by $\tilde{\rho}(\sigma)$. By Lemma 3.2 there exist

$$\text{l.i.m.}_{A\to\infty} F_A(t) = F(t), \qquad \text{l.i.m.}_{A\to\infty} \tilde{F}_A(t) = \tilde{F}(t). \tag{4.2}$$

Let

$$f_A(t) = \int_{-A}^{A} \sum g_j(\sigma)\phi_k(t, \sigma)\,d\rho_{jk} \tag{4.3}$$

and

$$\tilde{f}_A(t) = \int_{-A}^{A} \sum g_j(\sigma)\,\phi_k(t, \sigma)\,d\tilde{\rho}_{jk}\,.$$

By (3.10)

$$\text{l.i.m.}_{A\to\infty} f_A(t) = \text{l.i.m.}_{A\to\infty} \tilde{f}_A(t) = f(t). \tag{4.4}$$

Clearly by (4.1) since $L(\phi_j) = -\sigma\phi_j$,

$$L(F_A - \tilde{F}_A) + i(F_A - \tilde{F}_A) = f_A - \tilde{f}_A\,.$$

By the variation of constants fromula this implies

$$(4.5) \qquad F_A - \tilde{F}_A = \sum_{j=1}^{n} \phi_j(t, i) \int_c^t \theta_j(s)[f_A(s) - \tilde{f}_A(s)]\, ds + \sum_{j=1}^{n} C_j(A)\phi_j(t, i)$$

where $\theta_j(s)$ are continuous functions of s and $C_j(A)$ are constants that depend on A. Letting $A \to \infty$ in (4.5) and using (4.2) and (4.4) there results because of the independence of $\phi_j(t, i)$that the $C_j(A)$ are uniformly bounded and moreover tend to limiting values as A $\to \infty$. Thus

$$F(t) - \tilde{F}(t) = \textstyle\sum_{j=1}^{n} C_j(\infty)\phi_j(t, i).$$

Since F and $\tilde{F}$ are $\mathfrak{L}^2(a, b)$ and no solution of $L(x) + ix = 0$ is $\mathfrak{L}^2(a, b)$ it follows that all $C_j(\infty)$ above vanish. Thus

$$(4.6) \qquad F(t) = \tilde{F}(t).$$

If

$$F_{A2}(t) = \int_{-A}^{A} \frac{\sum g_j\, \phi_k\, d\rho_{jk}}{(i - \sigma)^2}$$

and $\tilde{F}_{A2}$ is defined similarly it follows as before that

$$L(F_{A2} - \tilde{F}_{A2}) + i(F_{A2} - \tilde{F}_{A2}) = F_A - \tilde{F}_A\,.$$

The same argument as before then leads with the aid of (4.6) to

$$\text{l.i.m.}_{A\to\infty}\, (F_{A2}(t) - \tilde{F}_{A2}(t)) = 0.$$

In this way it can be seen that for all integer $r \geqq 0$,

$$(4.7) \qquad \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \frac{\sum g_j(\sigma)\phi_k(t, \sigma)\, d(\rho_{jk} - \tilde{\rho}_{jk})}{(i - \sigma)^r} = 0.$$

Let

$$\Gamma_k(s, \sigma) = \int_c^{c+s} \phi_k(t, \sigma)\, dt.$$

Then integrating (4.7) from c to $c + s$ there follows

$$(4.8) \qquad \int_{-\infty}^{\infty} \frac{\sum g_j(\sigma)\Gamma_k(s, \sigma)\, d(\rho_{jk}(\sigma) - \tilde{\rho}_{jk}(\sigma))}{(i - \sigma)^r} = 0.$$

From the definition of Γ_j it follows that for fixed s, $\bar{\Gamma}_j$ is the transform of the function which is 1 in $(c, c + s)$. Thus

$$\int_{-\infty}^{\infty} \Gamma\bar{\Gamma}\, d\rho < \infty.$$

A similar result is true for g and by the Schwartz inequality it follows that (4.8) converges uniformly in r.

Consider now

$$H(s, \lambda) = \int_{-\infty}^{\infty} \frac{\sum g_j \Gamma_k \, d[\rho_{jk} - \tilde{\rho}_{jk}]}{\lambda - \sigma} \tag{4.9}$$

for $|\lambda - i| < \frac{1}{2}$. Because

$$\frac{1}{\lambda - \sigma} = \frac{1}{i - \sigma + (\lambda - i)} = \frac{1}{i - \sigma} - \frac{\lambda - i}{(i - \sigma)^2} + \frac{(\lambda - i)^2}{(i - \sigma)^3} - \cdots$$

it follows from (4.8) that $H(s, \lambda) = 0$ for $|\lambda - i| < \frac{1}{2}$. But clearly $H(s, \lambda)$ is analytic in the upper half λ-plane. Thus

$$H(s, \lambda) = 0, \qquad \mathcal{I}(\lambda) > 0.$$

Because $L(x) - ix = 0$ has no solutions in $\mathfrak{L}^2(a, b)$ a similar result holds for λ in the lower half plane, i.e. for λ in (4.9) replaced by $\bar{\lambda}$. If $\lambda = u + iv$, $v > 0$, then since

$$\frac{1}{\bar{\lambda} - \sigma} - \frac{1}{\lambda - \sigma} = \frac{2iv}{(u - \sigma)^2 + v^2}$$

it follows that

$$v \int_{-\infty}^{\infty} \frac{\sum g_j \Gamma_k \, d(\rho_{jk} - \tilde{\rho}_{jk})}{(u - \sigma)^2 + v^2} = 0. \tag{4.10}$$

If u_1 and u_2 are points of continuity of $\rho(\sigma)$ and $\tilde{\rho}(\sigma)$ and (4.10) is integrated with respect to u from u_1 to u_2 with v held fixed and then $v \to 0$ it follows easily that

$$\int_{u_1}^{u_2} \sum g_j(\sigma) \Gamma_k(s, \sigma) \, d(\rho_{jk} - \tilde{\rho}_{jk}) = 0.$$

If the above is differentiated with respect to s then

$$\int_{u_1}^{u_2} \sum g_j(\sigma) \phi_k(s, \sigma) \, d(\rho_{jk} - \tilde{\rho}_{jk}) = 0. \tag{4.11}$$

Let $f(t)$ be taken as 1 in $(c, c + \tau)$ and 0 otherwise. Then $g_j(\sigma)$ becomes $g_j(\sigma, \tau)$. If (4.11) is differentiated with respect to τ there results

$$\int_{u_1}^{u_2} \sum \bar{\phi}_j(\tau, \sigma) \phi_k(s, \sigma) \, d(\rho_{jk} - \tilde{\rho}_{jk}) = 0. \tag{4.12}$$

Now let $1 \leqq p, q \leqq n$ and differentiate $p - 1$ times with respect to τ and $q - 1$ times with respect to s and then put $s = \tau = c$. There follows

$$\int_{u_1}^{u_2} \sum \delta_j^p \delta_k^q \, d(\rho_{jk} - \tilde{\rho}_{jk}) = 0$$

or

$$\int_{u_1}^{u_2} d(\rho_{pq}(\sigma) - \tilde{\rho}_{pq}(\sigma)) = 0, \qquad p, q = 1, \cdots, n.$$

Thus

$$\rho(u_2) - \rho(u_1) = \tilde{\rho}(u_2) - \tilde{\rho}(u_1)$$

and Theorem II is proved.

5. Before proving Theorem III it is necessary to show how Theorem I remains valid for this case. The m conditions at $x = a$ denoted by $U^{(1)}(x) = 0$, to which, are adjoint $n - m$ conditions $U_\beta^{(2)}(x) = 0$ so that the n conditions are selfadjoint leads in the case of an interval $[a, \beta]$ where $a < \beta < b$ to the matrix step function $\rho_\beta(\sigma)$ as has already been stated. The matrix $\rho_\beta(\sigma)$ is an $n - m \times n - m$ matrix.

To be more specific about the way $\chi_k(t, \lambda)$ are chosen let $R = (r_{jk})$ denote a non-singular $n \times n$ matrix with its first m rows $r_{jk} = M_{jk}$; $j = 1, \cdots, m, k = 1, \cdots n$, where M_{jk} are the coefficients in $U^{(1)}$. Let the constant vectors $v^{(1)}, \cdots, v^{(n-m)}$ be the solutions of

$$Rv^{(k)} = e_{m+k}, \qquad k = 1, \cdots, n - m, \tag{5.1}$$

where e_{m+k} is the column vector with all elements zero except for the element in the row $m + k$ which is 1. If the components of $v^{(k)}$ are given by $v_j^{(k)}$ then a suitable choice of χ_k is

$$\chi_k(t, \lambda) = \textstyle\sum_{j=1}^n v_j^{(k)} \phi_j(t, \lambda), \qquad k = 1, \cdots, n - m.$$

Because at $t = a$, $\phi_j^{(l-1)}(a, \lambda) = \delta_j^l$ it follows from this definition that

$$\chi_k^{(j-1)}(a, \lambda) = v_j^{(k)}. \tag{5.2}$$

Thus it follows easily from (5.1) that the $\chi_k(t, \lambda)$ satisfy $U^{(1)}(\chi_k) = 0$. Moreover

$$\textstyle\sum_{j=1}^n r_{m+l,j} v_j^{(k)} = \delta_k^l, \qquad l, k = 1, \cdots, n - m. \tag{5.3}$$

In proving the analogue of Lemma 3.1, $c = a$, and h is not chosen as it was there. The choice of h is left open for the present. The function $\tilde{f}(t)$ is of class C_n, etc. just as before. It is clear that given any $\varepsilon > 0$ and $u > 0$, h can be chosen small enough so that

$$\left| \int_a^{a+h} \tilde{f}\, \bar{\chi}_k^{(j-1)}(t, \sigma)\, dt - \bar{v}_j^{(k)} \right| \leqq \varepsilon, \qquad | \sigma | \leqq u. \tag{5.4}$$

If now the completeness relationship (2.0) is used for

$$\textstyle\sum_{j=1}^n (-1)^{j-1} \bar{r}_{m+l,j} \tilde{f}^{(j-1)}(t)$$

and the characteristic functions of $L(x) + \lambda x =$, $U_\beta(x) = 0$ are represented by

$$\psi_j(t) = \textstyle\sum_{l=1}^{n-m} C_{jl} \chi_l(t, \lambda_{j\beta}),$$

then proceeding much as in the proof of Lemma 3.1, and taking ε small enough, there results for $| \sigma | \leqq u$,

$$\sum_{l=1}^{n-m} \int_a^{a+b} \left| \sum_{j=1}^n (-1)^{j-1} \bar{r}_{m+l,j}\, \tilde{f}^{(j-1)}(t) \right|^2 dt > \frac{1}{8} \sum_{j=1}^{n-m} \int_{-u}^{u} d\rho_{jj\beta}(\sigma)$$

which proves the result. The analogue of Theorem I, which here yields at least one limiting matrix $\rho(\sigma)$ from $\rho_\beta(\sigma)$ as $\beta \to b$, is immediate.

The proof of Theorem III is very much like that of Theorem II up to the analogue of (4.12) which now would be

$$\int_{u_1}^{u_2} \sum_{k=1}^{n-m} \bar{\chi}_j(\gamma, \sigma)\chi_k(s, \sigma)\, d(\rho_{jk}(\sigma) - \tilde{\rho}_{jk}(\sigma)) = 0.$$

Differentiating $p - 1$ times with respect to τ and $q - 1$ times with respect to s and setting $s = \tau = a$ there results from (5.2)

$$\sum_{j,k=1}^{n-m} \bar{v}_p^{(j)}\, v_q^{(k)} \int_{u_1}^{u_2} d(\rho_{jk}(\sigma) - \tilde{\rho}_{jk}(\sigma)) = 0.$$

Using (5.3) this leads readily to

$$\sum_{j,k=1}^{n-m} \delta_j^J \delta_k^K \int_{u_1}^{u_2} d(\rho_{jk} - \tilde{\rho}_{jk}) = 0, \quad J, K = 1, \cdots, n - m,$$

or

$$\int_{u_1}^{u_2} d(\rho_{JK} - \tilde{\rho}_{JK}) = 0, \qquad J, K = 1, \cdots, n - m,$$

which proves Theorem III.

6. An examination of the proof of Theorem III discloses that the restrictions on $p_j(t)$ over $[a, b]$ are used only to establish certain properties of $\chi_k(t, \lambda)$. If it is assumed, as in Theorem II, that the restrictions on $p_j(t)$ hold only on (a, b), the conclusion of Theorem III is still valid if the $\chi_k(t, \lambda)$ have reasonable enough behavior to allow the argument of Theorem III to be carried out.

Rather than formulate a theorem here several examples will be given to show the scope of these remarks. Consider

$$\frac{d^2x}{dt^2} + \left(\lambda - \frac{\nu^2 - 1/4}{t^2}\right) x = 0, \qquad \nu \geqq 0, \tag{6.1}$$

on the interval $(0, \beta)$ for some fixed β. Let

$$\chi(t, \lambda) = \Gamma(\nu + 1)2^{\nu} t^{\frac{1}{2}} J_\nu(t\lambda^{\frac{1}{2}})/\lambda^{\frac{1}{2}}$$

where J_ν is the Bessel function. Then

$$\lim_{t\to 0} t^{-\frac{1}{2}-\nu}\chi(t, \lambda) = 1. \tag{6.2}$$

If the roots of $\chi(\beta, \lambda) = 0$ are $\lambda_{j\beta}$ it follows, by methods used for the non-singular case, that the functions $\chi(t, \lambda_{j\beta})$ are complete on $(0, \beta)$. Thus there exists a monotone non-decreasing step function $\rho_\beta(\sigma)$ such that for $f(t) \in \mathfrak{L}^2(0, \beta)$

$$\begin{aligned} \int_0^\beta |\, f(t)\,|^2\, dt &= \int_{-\infty}^{\infty} |\, g(\sigma)\,|^2\, d\rho_\beta(\sigma) \\ g(\sigma) &= \int_0^\beta f(t)\bar{\chi}(t, \sigma)\, dt. \end{aligned} \tag{6.3}$$

Consider now the case where $\beta \to \infty$. Using $\tilde{f}(t) \geqq 0$ of class C_2 on $(0, h)$, vanishing identically nearly 0 and h and

$$\int_0^h \tilde{f}(t)t^{\frac{1}{2}+\nu}\, dt = 1$$

it follows easily from the completeness relation (6.3) applied to $\tilde{f}(t)$ that Theorem I is valid for $\rho_\beta(\sigma)$ and thus at least one limiting function $\rho(\sigma)$ exists.

The argument of Theorem III may also be used. It leads directly to

$$\int_{u_1}^{u_2} \bar{\chi}(\tau, \sigma)\chi(s, \sigma)\, d(\rho(\sigma) - \tilde{\rho}(\sigma)) = 0.$$

Multiplying the above by $s^{-\frac{1}{2}-\nu}$ and $\tau^{-\frac{1}{2}-\nu}$ and letting $s \to 0$, $\tau \to 0$, there results

$$\int_{u_1}^{u_2} d(\rho(\sigma) - \tilde{\rho}(\sigma)) = 0$$

which proves the uniqueness of $\rho(\sigma)$ for this case. Thus the well known uniqueness of $\rho(\sigma)$ in the case of the Bessel transform theorem is obtained.

As another example consider the case

$$\chi(t, \lambda) = \Gamma(-\gamma + 1)2^{-\gamma}t^{\frac{1}{2}}J_{-\gamma}(t\lambda^{\frac{1}{2}})/\lambda^{\frac{1}{2}}$$

where $0 < \gamma < 1$. Here the above argument can be carried through with obvious changes.

If the equation

$$\frac{d^2x}{dt^2} + (\lambda - q(t))x = 0$$

is considered where $q(t)$ is real and

$$\int_0^1 t \mid q(t) \mid dt + \int_1^\infty \mid q(t) \mid dt < \infty$$

and if $x = \chi(t, \lambda)$ is the solution which satisfies $\chi(0, \lambda) = 0$, $\chi'(0, \lambda) = 1$ then it can be verified that the argument of Theorem III yields a unique $\rho(\sigma)$ over $(-\infty, \infty)$ so that $\rho_\beta(\sigma) \to \rho(\sigma)$ as $\beta \to \infty$ whatever the boundary condition at $t = \beta$ so long as the problem is self-adjoint.

Added September 26, 1953. The inverse tranform theorem requires some slight modification from that in [6] for $n = 2$.

It follows by Lemma 3.2 that given the matrix $\rho(\sigma)$ of Theorem II on $(-\infty, \infty)$ and continuous (or indeed Borel measureable functions) $G_j(\sigma)$ such that

$$\int_{-\infty}^{\infty} \sum G_j(\sigma)\bar{G}_k(\sigma)\, d\rho_{jk}(\sigma) < \infty$$

then

$$f(t) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \sum G_j(\sigma)\phi_k(t, \sigma)\, d\rho_{jk}(\sigma) \tag{7.0}$$

exists. A similar result holds for the χ case of Theorem III. The above result follows from Theorem I and makes no use of the uniqueness of $\rho(\sigma)$. However if the hypothesis of Theorem II holds then $\rho(\sigma)$ is unique and if $g_j(\sigma)$ is defined for $f(t)$ given in (7.0) as in the corollary to Theorem I then it will be shown that

$$\int_{-\infty}^{\infty} (G - g)(\overline{G - g})\, d\rho = 0, \tag{7.1}$$

that is G and g are equivalent in the metric of $\rho(\sigma)$. Let

$$h_A(t) = \int_{-A}^{A} \sum (G_j - g_j)\phi_k(t, \sigma)\, d\rho_{jk}(\sigma).$$

Clearly from the definition of $g_j(\sigma)$, $\text{l.i.m.}_{A\to\infty} h_A(t) = 0$.
Let

$$F_A(t) = \int_{-A}^{A} \frac{\sum (G_j - g_j)\phi_k\, d\rho_{jk}}{i - \sigma}.$$

Then it follows easily that

$$L(F_A) + iF_A = h_A .$$

Because the hypothesis of Theorem III and because $\text{l.i.m.}_{A\to\infty} h_A = 0$ it follows as in the argument following (4.5) that $\text{l.i.m.}_{A\to\infty} F_A(t) = 0$.

Repetitions lead to

$$\underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \frac{\sum (G_j - g_j)\phi_k\, d\rho_{jk}}{(i - \sigma)^r} = 0 \qquad r = 0, 1, 2, \cdots .$$

Much as in the proof of Theorem II it now follows that

$$\int_{u_1}^{u_2} \sum_{j=1}^{n} (G_j - g_j)\, d\rho_{jk}(\sigma) = 0 \qquad k = 1, \cdots, n.$$

Because u_1 and u_2 are any points of continuity of $\rho(\sigma)$ it follows that for any Borel measureable functions $Q_k(\sigma)$ of class $\mathfrak{L}^2(\rho)$,

$$\int_{-\infty}^{\infty} (G - g)\bar{Q}\, d\rho = 0.$$

Since the function $Q_k(\sigma)$ can be taken as $G_k - g_k$ over any finite interval, the conclusion (7.1) follows.

Thus any given G_j which determines a function f is equivalent to the transform g_j of f in the sense of (7.1).

A similar result holds for the case of Theorem III.

Added in Proof: See also the paper of Coddington on the Green's matrix in the current Canadian Journal of Mathematics.

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REFERENCES

1. G. A. BLISS, *A boundary value problem for a system of ordinary differential equations of the first order.* Trans. Amer. Math. Soc., Vol. 28 (1926) pp. 561–584.
2. E. A. CODDINGTON, *On the spectral representation of ordinary self-adjoint differential operators*, Proc. Nat. Acad. Sci., U. S. A., Vol. 38 (1952) pp. 732–737.

3. K. Kodaira, *Eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices*, Amer. J. Math., Vol. 71 (1949) pp. 921–945).
4. K. Kodaira, *On ordinary differential equations of any even order and the corresponding eigen function expansions*, Amer. J. Math., Vol. 72 (1950) pp. 502–544.
5. N. Levinson, *A simplified proof of the expansion theorem for singular second order linear differential equations*, Duke Math. J., Vol. 18 (1951) pp. 55–71.
6. N. Levinson, *Addendum to above*, Duke Math. J., Vol. 18 (1951) pp. 719–722.
7. B. M. Levitan, *Proof of a theorem on eigenfunction expansions for self-adjoint differential equations*, C. R. Doklady Acad. Sci., USSR, Vol. 73 (1950), pp. 651–654.
8. M. H. Stone, Linear Transformations in Hilbert Space, Amer. Math. Soc. Colloquim Publication (1932).
9. E. C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, Oxford (1946).
10. H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehorigen entwicklungen willkurlicher Funktionen*, Math. Ann., Vol. 68 (1910) pp 220–269.
11. K. Yosida, *On Titchmarsh-Kodaira's formula concerning Weyl-Stone's eigenfunction expansion*, Nagoya Math. J., Vol. 1 (1950) pp. 49–58.