Appendix C: Risk Neutrality in the Heston model

In order to understand the use of risk neutrality we first state the main result and then we prove it.

Main result

We start with the Heston dynamics under the physical measure P

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{P,1} \\
dV_t &= a^P(\bar{V}^P - V_t)dt + \eta\sqrt{V_t} dW_t^{P,2} \\
dW_t^{P,1} dW_t^{P,2} &= \rho dt
\end{aligned}
$$

and we seek to obtain a risk-neutral evolution where $E_t^Q(dS_t / S_t) = rdt$. As we show below, using the multidimensional Girsanov's theorem and making appropriate choices, the Heston dynamics under the risk-neutral measure Q can be expressed as

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^{Q,1} \\
dV_t &= a^Q(\bar{V}^Q - V_t)dt + \eta\sqrt{V_t} dW_t^{Q,2} \\
dW_t^{Q,1} dW_t^{Q,2} &= \rho dt
\end{aligned}
$$

where $a^Q = a^P + \gamma$, $\bar{V}^Q = \frac{a^P \bar{V}}{a^P + \gamma}$ and γ is a parameter linked to the price of volatility risk.

Therefore, the Heston dynamics under the risk-neutral measure exhibit a similar pattern to that of the physical measure, but with a variance process that is defined by the parameters a^Q and $\bar{V}^Q$ instead of a^P and $\bar{V}^P$. A remarkable feature is that a^Q and $\bar{V}^Q$ already incorporate the impact of the volatility risk premium γ. Consequently, when calibrating the risk-neutral model to market prices, we can directly solve for a^Q and $\bar{V}^Q$, and we will not need to estimate γ explicitly.

In section 3, for simplicity, we omitted the Q superscripts. However, it should be noted that the values for a and $\bar{V}$ that we used through the paper are the risk-neutral ones (i.e. a^Q and $\bar{V}^Q$), and not those under the physical measure. The use of risk-neutral dynamics is justified when all the risks related to holding options can be hedged away. Within the Heston model, there are two sources of uncertainty: the underlying asset movements and the volatility movements. The first risk source can be hedged away implementing a delta-hedging strategy in similar terms to those of the BSM framework.

However, in order to hedge the volatility risk, a liquid market for volatility related contracts is needed. Consequently, the use of risk-neutral pricing is conditioned by the assumption of perfect hedging. If hedging is not possible, we might need to go back to the dynamics under the physical measure, which requires different models and hypothesis in order to estimate the appropriate risk premiums and the corresponding real-world distribution.

Proof

We start again with the Heston dynamics under the physical measure

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{P,1} \\
dV_t &= a^P(\bar{V}^P - V_t)dt + \eta\sqrt{V_t}dW_t^{P,2} \quad \text{(C.1)} \\
dW_t^{P,1}dW_t^{P,2} &= \rho dt
\end{aligned}
$$

where the discounted underlying price is a martingale under P.

To obtain the risk-neutral dynamics we should find an equivalent martingale measure (EMM) where the process dS_t / S_t has a drift of rdt. To achieve this we perform a change of probability measure using Girsanov's theorem. In particular, we define a new EMM through the Radon-Nikodym derivative:

$$
\left.\frac{dQ}{dP}\right|_t = M_t
$$

where M_t is an exponential martingale of the form

$$
M_t = \exp\left\{\int_t^T C_s dW_s^{P,1} - \frac{1}{2}\int_t^T C_s^2 ds \quad + \int_t^T D_s dW_s^{P,2} - \frac{1}{2}\int_t^T D_s^2 ds\right\}
$$

and it is the solution of the SDE

$$
\frac{dM_t}{M_t} = C_t dW^{P,1} + D_t dW^{P,2}
$$

with initial value $M_0 = 1$.

Since we are working with EMMs, the expectation of a given stochastic process Z under the new measure Q can be computed as

$$
E_t^Q(Z) = E_t^P(M_t Z)
$$

Therefore, if we consider the expectation of infinitesimal increments

$$
E_t^Q(dZ) = E_t^P\left(\frac{M_t + dMt}{Mt}dZ\right) = E_t^P\left[\left(1 + \frac{dMt}{Mt}\right)dZ\right] = E_t^P dZ + E_t^P(C_t dW^{P,1} + D_t dW^{P,2})dZ
$$

Using the equation above, we can compute the drift and volatility for the process dS_t / S_t under Q

$$
\begin{aligned}
E_t^Q\left(\frac{dS_t}{S_t}\right) &= E_t^P\left(\frac{dS_t}{S_t}\right) + E_t^P\left((C_t dW^{P,1} + D_t dW^{P,2})\frac{dS_t}{S_t}\right) \\
&= E_t^P(\mu dt + \sqrt{V_t}dW_t^{P,1}) + E_t^P\left((C_t dW^{P,1} + D_t dW^{P,2})(\mu dt + \sqrt{V_t}dW_t^{P,1})\right) \\
&= E_t^P(\mu dt) + E_t^P(\sqrt{V_t}dW_t^{P,1}) + E_t^P(C_t \mu dW^{P,1} dt) + E_t^P(C_t\sqrt{V_t}(dW_t^{P,1})^2) + \\
&\quad + E_t^P(D_t \mu dW^{P,2} dt) + E_t^P(D_t\sqrt{V_t}dW^{P,2}dW_t^{P,1}) \\
&= \mu dt + 0 + 0 + C_t\sqrt{V_t}dt + 0 + \rho D_t\sqrt{V_t}dt \\
&= (\mu + C_t\sqrt{V_t} + D_t\sqrt{V_t}\rho)dt
\end{aligned}
$$

$$
\begin{aligned}
E_t^Q\left(\frac{dS_t}{S_t}\right)^2 &= E_t^P\left[\left(1 + C_t dW^{P,1} + D_t dW^{P,2}\right)\left(\mu dt + \sqrt{V_t}dW_t^{P,1}\right)\right]^2 \\
&= E_t^P\left[V_t(dW_t^{P,1})^2\right] \\
&= V_t dt
\end{aligned}
$$

where we expanded the initial expressions and we used the fact that Weiner processes are distributed as $N(0,\sqrt{t})$ and, consequently, $E(dW_t^{P,1}dW_t^{P,2}) = \rho dt$. We also used the basic rules of stochastic calculus $E(dW_t) = 0$; $E(dW_t dt) = 0$; $E(dt^2) = 0$ and $E\left[(dW_t)^2\right] = dt$.

Similarly, the drift and volatility for dV_t can be computed as

$$
\begin{aligned}
E_t^Q\left(dV_t\right) &= E_t^P\left(dV_t\right) + E_t^P\left((C_t dW^{P,1} + D_t dW^{P,2})dV_t\right) \\
&= E_t^P(a^P(\bar{V}^P - V_t)dt) + E_t^P\left((C_t dW^{P,1} + D_t dW^{P,2})\left[a^P(\bar{V}^P - V_t)dt + \eta\sqrt{V_t}dW_t^{P,2}\right]\right) \\
&= a^P(\bar{V}^P - V_t)dt + E_t^P\left[C_t\eta\sqrt{V_t}dW^{P,1}dW^{P,2}\right] + E_t^P\left[D_t\eta\sqrt{V_t}(dW^{P,2})^2\right] \\
&= \left[a^P(\bar{V}^P - V_t) + \rho C_t\eta\sqrt{V_t} + D_t\eta\sqrt{V_t}\right]dt \\
E_t^Q\left(dV_t\right)^2 &= E_t^P\left[\left(1 + C_t dW^{P,1} + D_t dW^{P,2}\right)\left(a^P(\bar{V}^P - V_t)dt + \eta\sqrt{V_t}dW_t^{P,2}\right)\right]^2 \\
&= E_t^P\left[\eta^2 V_t(dW_t^{P,2})^2\right] \\
&= \eta^2 V_t dt
\end{aligned}
$$

Now, in order select the desired EMM, we impose the restriction

$$
E_t^Q\left(\frac{dS_t}{S_t}\right) = rdt
$$

which gives us the equation $(\mu + C_t\sqrt{V_t} + D_t\sqrt{V_t}\rho)dt = rdt$. Rearranging terms we obtain the following relationship, which defines the market price of risk

$$
C_t + \rho D_t = -\frac{\mu - r}{\sqrt{V_t}}
$$

Additionally, we need to set the drift for the volatility process. In this case, an appropriate choice is

$$E_t^Q\left(dV_t\right)=\left[a^P(\bar{V}^P-V_t)-\gamma V_t\right]dt$$

where γ is a parameter related to the price of volatility risk. This constraint gives us the equation $[a^P(\bar{V}^P-V_t)+\rho C_t\eta\sqrt{V_t}+D_t\eta\sqrt{V_t}]dt=[a^P(\bar{V}^P-V_t)-\gamma V_t]dt$, which defines the price of volatility risk

$$\rho C_t+D_t=-\frac{\gamma\sqrt{V_t}}{\eta}$$

Considering the properties of EMMs, the multidimensional Girsanov´s theorem tells us that the Weiner processes under the new measure Q are

$$W_t^{Q,1}=W_t^{P,1}+\frac{\mu-r}{\sqrt{V_t}}t$$

$$W_t^{Q,2}=W_t^{P,2}+\frac{\gamma\sqrt{V_t}}{\eta}t$$

Therefore, rearranging terms and substituting on the initial dynamics we get

$$\begin{aligned}
dS_t &= \mu S_t dt+\sqrt{V_t}S_t dW_t^{P,1}\\
&= \mu S_t dt+\sqrt{V_t}S_t d\left(W_t^{Q,1}-\frac{\mu-r}{\sqrt{V_t}}t\right)\\
&= \mu S_t dt+\sqrt{V_t}S_t dW_t^{Q,1}-\sqrt{V_t}S_t\frac{\mu-r}{\sqrt{V_t}}dt\\
&= rS_t dt+\sqrt{V_t}S_t dW_t^{Q,1}
\end{aligned}$$

$$\begin{aligned}
dV_t &= a^P(\bar{V}^P-V_t)dt+\eta\sqrt{V_t}dW_t^{P,2}\\
&= a^P(\bar{V}^P-V_t)dt+\eta\sqrt{V_t}d\left(W_t^{Q,2}-\frac{\gamma\sqrt{V_t}}{\eta}t\right)\\
&= a^P(\bar{V}^P-V_t)dt+\eta\sqrt{V_t}dW_t^{Q,2}-\eta\sqrt{V_t}\frac{\gamma\sqrt{V_t}}{\eta}dt\\
&= \left(a^P\bar{V}^P-a^P V_t-\gamma V_t\right)dt+\eta\sqrt{V_t}dW_t^{Q,2}
\end{aligned}$$

and if we introduce the notation $a^Q=a^P+\gamma$ and $\bar{V}^Q=\frac{a^P\bar{V}}{a^P+\gamma}$, the process dV_t becomes

$$dV_t=a^Q(\bar{V}^Q-V_t)dt+\eta\sqrt{V_t}dW_t^{Q,2}$$

Finally, the correlation condition $dW_t^{P,1} dW_t^{P,2} = \rho dt$ is equivalent to require $E(dW_t^{P,1} dW_t^{P,2}) = \rho dt$. And considering the relationship between the Weiner processes under the physical and the risk-neutral measure we get

$$\begin{aligned}
\rho dt &= E(dW_t^{P,1} dW_t^{P,2}) \\
&= E\left[d\left(W_t^{Q,1} + \frac{\mu - r}{\sqrt{V_t}} t \right) d\left(W_t^{Q,2} + \frac{\gamma \sqrt{V_t}}{\eta} t \right) \right] \\
&= E\left(dW_t^{Q,1} dW_t^{Q,2} \right)
\end{aligned}$$

where we have used again the stochastic calculus rules $E(dW_t) = 0$; $E(dW_t dt) = 0$ and $E(dt^2) = 0$.