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The Hasimoto Transformation and Integrable Flows on Curves

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Abstract. A simple formula for the differential of the Hasimoto transformation (relating vortex filament flow to the cubic Schrödinger equation) is presented.

As an idealization of the motion of a thin vortex tube in three dimensional hydrodyamics, one can consider the evolution of a twisted space curve γ governed by the equation of motion $\frac{\partial\gamma}{\partial t} = 2\kappa B$ (the "filament equation"), where κ is the curvature of γ and B is its binormal (see [1] and [7]).

Hasimoto introduced the transformation $\gamma \to \psi = \kappa e^{i\int_a^s \tau\, du}$, τ being the torsion of γ ([3]); this transforms the filament equation for γ into the cubic Schrödinger equation for ψ, which is known to be completely integrable. Thus, the curvature and torsion of soliton solutions of the filament flow can be explicitly computed using now standard techniques. The Frenet equations for the curves themselves can then be solved by transformation to Ricatti equations ([7]).

In this note a simple formula is presented for the differential of the Hasimoto transformation. It applies to an arbitrary arclength preserving variation vector field W along γ and may be regarded as a convenient general expression for the induced variation of curvature and torsion of γ. At the same time, it is indicative of the special structure which exists when W happens to be one of the infinite list of Hamiltonian vector fields commuting with the filament flow, and will play a key role in the authors' approach to the theory ([4]); here we limit ourselves to some illustrative examples (including, however, conclusions the authors have not seen stated in the literature).

The Differential Formula. For simplicity we assume everything to be C^∞. We consider the space of curves with nonvanishing curvature $\Gamma = \{\gamma : [a,b] \to R^3 \ \ni: \ \kappa \neq 0\}$, where we allow $a = -\infty$, $b = \infty$.

A tangent vector to Γ at γ is represented by a vector field $W = fT+gN+hB$ along γ; here f, g, h are functions on $[a,b]$, T is the unit tangent to γ, N the principal unit normal, and B the binormal. Using the Frenet equations $T' = \kappa N, N' = -\kappa T + \tau B, B' = -\tau N$ (the $'$ denotes derivative with respect to arclength parameter s), one can show that W must satisfy $f' = g\kappa$ in order to preserve unit speed parameterization infinitesimally.

For any vector field W along γ, one can always add on a tangential term such that the resulting vector field preserves arclength parameterization. For this purpose, we introduce the linear "normalization operator"

$$\mathcal{N}W = (\int^s g\kappa\, du)T + gN + hB \ ;$$

here we will be considering "geometric" vector fields - vector fields whose components are expressed in terms of κ, τ, and their derivatives - where the antiderivative is conveniently chosen so that it vanishes on curves with $\kappa, \tau \equiv 0$.

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Let $\Psi = \{\psi : [a,b] \to \mathbf{C}\}$. The Hasimoto transformation, $\mathcal{H} : \Gamma \to \Psi$, will be written

$$\mathcal{H}(\gamma) = \psi = \kappa\sigma \ ;$$

here and below we use the shorthand $\sigma(s) = e^{i\int_a^s \tau \, du}$. The differential of $\mathcal{H}$ can be expressed simply:

$$d\mathcal{H}(W) = < Z, -\mathcal{R}^2 W > \ + ic\psi.$$

Here, Z is the complex vector field along γ given by

$$Z = (N + iB)\sigma,$$

$\mathcal{R}$ is the linear "recursion operator"

$$\mathcal{R}W = -\mathcal{N}(T \times W'),$$

$\times$ means the usual cross product, and c is a real constant involving boundary terms. The bracket $<,>$ denotes the standard Hermitian inner product for complex vectors. The formula may be verified by straightforward computation; for sake of brevity, we omit the derivation. Introducing the operator $\mathcal{M}(W) \ = < Z, -\mathcal{R}^2 W >$, the formula can be written as

$$d\mathcal{H}(W) \ \equiv \ \mathcal{M}(W) \ (\ mod \ i\psi \ ).$$

The Filament Flow. We now apply our differential formula to the field $W = 2\kappa B$. Note that W satisfies the arclength preserving condition $f' = g\kappa$, so the formula for $d\mathcal{H}$ can be applied.

We compute

$$W' = -2\kappa\tau N + 2\kappa' B, \ \ T \times W' = -2\kappa' N - 2\kappa\tau B,$$

and

$$\mathcal{R}W = -\mathcal{N}(T \times W') = \kappa^2 T + 2\kappa' N + 2\kappa\tau B.$$

Continuing,

$$(\mathcal{R}W)' = (\kappa^3 + 2\kappa'' - 2\kappa\tau^2)N + 2(2\kappa'\tau + \kappa\tau')B,$$

$$T \times (\mathcal{R}W)' = -2(2\kappa'\tau + \kappa\tau')N + (\kappa^3 + 2\kappa'' - 2\kappa\tau^2)B,$$

$$\mathcal{R}^2 W = -\mathcal{N}(T \times (\mathcal{R}W)') = 2\kappa^2\tau T + 2(2\kappa'\tau + \kappa\tau')N - (\kappa^3 + 2\kappa'' - 2\kappa\tau^2)B,$$

and finally

$$\begin{aligned} \mathcal{M}(W) \ &= \ \sigma < (N + iB), -\mathcal{R}^2 W > \ = \\ &= \sigma[-2(2\kappa'\tau + \kappa\tau') + i(2\kappa'' + \kappa^3 - 2\kappa\tau^2)] \end{aligned}$$

We note some consequences of this formula. First, differentiating $\psi = \kappa\sigma$, one gets $\psi' = \sigma(\kappa' + i\kappa\tau)$, and $\psi'' = \sigma[(\kappa'' - \kappa\tau^2) + i(2\kappa'\tau + \kappa\tau')]$. One easily concludes that the filament flow $\gamma_t = W$ induces a flow on ψ satisfying

$$-i\psi_t = -id\mathcal{H}(W) = 2\psi'' + |\psi|^2\psi + A(t)\psi,$$

where $A(t)$ is some real time-dependent function; this is a form of the cubic Schrödinger equation (see [7]).

Next, observing that $d\mathcal{H}(W) = \sigma(\kappa_t + i\kappa\int_a^s \tau_t \, du)$, one can also read off the following formulas for the evolution of κ and τ themselves from the expression listed above for $\mathcal{M}(W)$:

$$\kappa_t = -2(2\kappa'\tau + \kappa\tau') = \frac{-2}{\kappa}(\kappa^2\tau)',$$

$$\tau_t = (\frac{1}{\kappa}(2\kappa'' + \kappa^3 - 2\kappa\tau^2))'.$$

The first equation implies $(\kappa^2)_t = -4(\kappa^2\tau)'$, hence (with appropriate boundary conditions) the total squared curvature $H_2(\gamma) = \int_a^b \kappa^2 ds$ is a constant of motion. The second equation implies that the total (negative) torsion $H_1(\gamma) = \int_a^b -\tau \, ds$ is also a constant of motion. As the notation suggests, H_1 should actually be thought of as the "first" constant of motion of the filament flow, although its existence is not mentioned in the references cited earlier.

We note in passing that the three vectorfields $W = 2\kappa B$, $\mathcal{R}W$, and $\mathcal{R}^2W$ are the Hamiltonian vectorfields associated to $H_0 = 2\int_a^b |\frac{\partial\gamma}{\partial u}| du$ (twice the arclength of γ), H_1, and H_2 respectively (details will be given in [4]). Computations similar to those above show that the flow induced on ψ by $\mathcal{R}W$ satisfies the modified KdV equation.

Killing Fields. Here we wish to discuss a different role played by these same vector fields — namely as *Killing fields.* We call an arclength preserving vector field W along γ *Killing* if the induced variations of κ and τ are both zero. A dimension argument shows (see [5]) that in this case W extends to a Killing field on R^3, that is, an infinitesimal generator of a one-parameter group of isometries of R^3. We thus have the following

LEMMA. *If W is a unit speed, locally arclength preserving vectorfield along γ such that $\mathcal{M}(W) = \lambda i\psi$ for some $\lambda \in R$, then W extends to a Killing field on R^3.*

Now suppose γ is an *elastic curve*, that is, any finite length of γ is a critical point of H_2 subject to first-order boundary conditions and possibly an arclength constraint $H_0 = c$. As shown in [5], γ satisfies the Euler equations $2\kappa'' + \kappa^3 - 2\kappa\tau^2 - \lambda\kappa = 0$ and $(\kappa^2\tau)' = 0$. But then $\mathcal{M}(2\kappa B) = \lambda i\psi$, so the lemma implies γ yields a *congruence solution* of the filament flow, i.e., γ moves without changing shape, only position (the planar case of this result was observed in [2]). With just a little more work, a converse to this result can be obtained: the most general congruence solution γ is "associated elastic"; that is, γ has the same curvature as that of an elastic curve, but the torsion may differ by a constant. We are indebted to U. Pinkall for calling our attention to this result (indeed, this was the starting point for our investigation).

Next, let γ again be an elastic curve; thus $\kappa^2\tau = c$, for some constant c, and so the Euler equations for elastica can be written as $\mathcal{R}^2(W) = 2cT - \frac{\lambda}{2}W$, where again $W = 2\kappa B$. Consider this time the vector field $Y = \mathcal{R}(W) - \lambda T$. We have $\mathcal{R}^2(Y) = \mathcal{R}(\mathcal{R}^2(W) - \lambda\mathcal{R}(T)) = \mathcal{R}((2cT - \frac{\lambda}{2}W) - \lambda(\frac{-W}{2})) = \mathcal{R}(2cT) = -cW$. Therefore, $\mathcal{M}(Y) = 2ci\psi$, so Y is another Killing field for γ. As a consequence, any elastic curve is also an initial condition for a congruence solution for the H_1 flow, since $\mathcal{R}(2\kappa B)$ differs from a Killing field by a tangential (reparameterizing) term.

Finally, the fact that we have now produced two Killing fields for an elastic curve is noteworthy for another reason. As shown in [5], given these two fields, one cannot only read off much qualitative information about these congruence solutions, but one can integrate the Frenet equations explicitly. Indeed, precisely these two Killing fields were used in [6] to completely classify the *closed* elastic curves in R^3, determine their knot types, etc. What is significant here is that these fields play dual roles as Hamiltonian vectorfields corresponding to H_0 and H_1 and as Killing fields associated with H_2. The Ricatti equation approach mentioned earlier appears not to capture this simple and useful relationship.

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