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AN ELEMENTARY PROOF OF THE GROTHENDIECK INEQUALITY

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ABSTRACT. An elementary proof of the Grothendieck inequality is given.

Since its appearance in [**2**, pp. 59–64] and reformulation in [**3**, pp. 277–280], Grothendieck's fundamental inequality has enjoyed several restatements and proofs within various frameworks of analysis (detailed accounts of which appear in [**4**]). The purpose of this note is to give an elementary and self-contained proof of the inequality: the argument below, an adaptation of the proof given in [**1**], requires knowing only that the expectation of a product of independent random variables equals the product of their expectations.

Let $\mathbf{R}^{\mathbf{N}}$ denote the space of sequences of real numbers with finitely many nonzero terms. $\mathbf{R}^{\mathbf{N}}$ will be equipped with the usual inner product,

$$\langle x, y\rangle = \sum_n x(n)y(n), \qquad x, y \in \mathbf{R}^{\mathbf{N}},$$

and Euclidean norm,

$$\|x\| = \langle x, x\rangle^{1/2}, \qquad x \in \mathbf{R}^{\mathbf{N}}.$$

B will denote the unit ball in $\mathbf{R}^{\mathbf{N}}$, i.e. $B = \{x \in \mathbf{R}^{\mathbf{N}} : \|x\| \leq 1\}$.

THEOREM (GROTHENDIECK'S INEQUALITY). *Let $(a_{mn})_{m,n=1}^{\infty}$ be an array of complex numbers which satisfies*

$$\left|\sum_{m,n=1}^{N} a_{mn} s_m t_n\right| \leq \max_{1\leq m,n\leq N} |s_m|\,|t_n| \tag{1}$$

for all sequences of complex numbers $(s_m)_{m=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, and all integers $N \geq 1$. Then, for all sequences of vectors in $\mathbf{R}^{\mathbf{N}}$, $(x_m)_{m=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$,

$$\left|\sum_{m,n=1}^{N} a_{mn}\langle x_m, y_n\rangle\right| \leq K \max_{1\leq m,n\leq N} \|x_m\|\,\|y_n\| \tag{2}$$

for all $N \geq 1$ and some universal constant K.

To start, define a real-valued function on $\mathbf{R}^{\mathbf{N}} \times \mathbf{R}^{\mathbf{N}}$ by

$$A(x, y) = \prod_n (1 + x(n)y(n)), \qquad x, y \in \mathbf{R}^{\mathbf{N}}, \tag{3}$$

and estimate

$$|A(x, y)| \leq e^{\sum \ln(1+|x(n)y(n)|)} \leq e^{\sum |x(n)y(n)|} \leq e^{\|x\|\cdot\|y\|}. \tag{4}$$

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LEMMA 1. *Suppose $(a_{mn})_{m,n=1}^{\infty}$ satisfies the hypothesis of the theorem above. Then, for all sequences of vectors $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in the unit ball of $\mathbf{R}^{\mathbf{N}}$,*

$$\left|\sum_{m,n=1}^{N} a_{mn} A(x_m, y_n)\right| \leq e$$

for all $N \geq 1$.

PROOF. Let $(Z_n)_{n=1}^{\infty}$ be a sequence of independent real-valued random variables on some probability space so that

(5) $$\mathbf{E}(Z_n) = 0, \quad \mathbf{E}(Z_n^2) = 1, \quad \text{and} \quad |Z_n| = 1 \quad \text{a.s. for all } n.$$

($\mathbf{E}$ denotes expectation; $(Z_n)_{n=1}^{\infty}$ could be taken as the usual system of Rademacher functions.) Given $x \in \mathbf{R}^{\mathbf{N}}$, define a random variable

$$F(x) = \prod_n (1 + ix(n)Z_n) \qquad (i = \sqrt{-1}),$$

and estimate (by (4))

(6) $$|F(x)| \leq \left(\prod_n (1 + x(n)^2)\right)^{1/2} \leq e^{\|x\|^2/2} \quad \text{almost surely.}$$

For any $x, y \in \mathbf{R}^{\mathbf{N}}$,

$$\begin{aligned} \mathbf{E}(F(x)\overline{F(y)}) &= \prod_n \mathbf{E}(1 + ix(n)Z_n)(1 - iy(n)Z_n) \quad \text{(by independence)} \\ &= \prod_n (1 + x(n)y(n)) \quad \text{(by (5))} \\ &= A(x, y). \end{aligned}$$

Therefore, for any $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$ and all $N \geq 1$,

$$\begin{aligned} \left|\sum_{m,n=1}^{N} a_{mn} A(x_m, y_n)\right| &= \left|\sum_{m,n=1}^{N} a_{mn} \mathbf{E}(F(x_m)\overline{F(y_n)})\right| \\ &\leq \mathbf{E}\left|\sum_{m,n=1}^{N} a_{mn} F(x_m)\overline{F(y_n)}\right| \\ &\leq e \quad \text{(by (1) and (6)).} \quad \text{Q.E.D.} \end{aligned}$$

Next, expand the product on the right-hand side of (3):

(7) $$A(x, y) = 1 + \langle x, y\rangle + \cdots + \sum_{n_1 > \cdots > n_J} x(n_1)\cdots x(n_J)y(n_1)\cdots y(n_J) + \cdots.$$

Let $\{E_J\}_{J=2}^{\infty}$ be an infinite partition of the natural numbers $\mathbf{N}$, so that each $E_J \subset \mathbf{N}$ is infinite. Let W_J be the J-dimensional wedge in $\mathbf{N}^J$ given by

$$W_J = \{(n_1, \ldots, n_J) \in \mathbf{N}^J : n_1 > \cdots > n_J\},$$

and set up a one-to-one correspondence between E_J and W_J, $J \geq 2$:

$$n \in E_J \leftrightarrow (n_1, \ldots, n_J) \in W_J.$$

Given an arbitrary $x \in B$, define a vector $\phi(x) = (\phi(x)(n))_{n\in\mathbf{N}}$ in $\mathbf{R}^{\mathbf{N}}$ by

$$\phi(x)(n) = x(n_1)\cdots x(n_J), \qquad n \in E_J,\ J = 2, \dots,$$

and estimate

$$(8) \qquad \|\phi(x)\| = \left(\sum_{J=2}^{\infty}\sum_{n\in E_J}(x(n_1)\cdots x(n_J))^2\right)^{1/2} \leq \left(\sum_{J=2}^{\infty}\frac{1}{J!}\left(\sum_{n\in\mathbf{N}}x(n)^2\right)^J\right)^{1/2} \leq (e-2)^{1/2} \equiv \delta < 1.$$

Write

$$\phi_\delta(x) = \phi(x)/\delta, \qquad x \in B,$$

and, by the estimate above, note that ϕ_δ is a map from B into B. Solving for $\langle x, y\rangle$ in (7), $x, y \in B$, we obtain

$$(9) \qquad \langle x, y\rangle = A(x, y) - 1 - \delta^2\langle\phi_\delta(x), \phi_\delta(y)\rangle.$$

Therefore, applying (9) recursively, we obtain for each $J > 0$

$$(10) \qquad \langle x, y\rangle = \sum_{j=0}^{J}(-\delta^2)^j[A(\phi_\delta^j(x), \phi_\delta^j(y)) - 1] + (-\delta^2)^{J+1}\langle\phi_\delta^{J+1}(x), \phi_\delta^{J+1}(y)\rangle$$

(ϕ_δ^j denotes the jth iterate of ϕ_δ). Finally, letting $J \to \infty$ in (10), we deduce

LEMMA 2. *For all $x, y \in B$*

$$\langle x, y\rangle = \sum_{j=0}^{\infty}(-\delta^2)^j[A(\phi_\delta^j(x), \phi_\delta^j(y)) - 1].$$

PROOF OF GROTHENDIECK'S INEQUALITY. It suffices to establish (2) for $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$. By Lemmas 1 and 2, we estimate

$$\left|\sum_{m,n=1}^{N} a_{mn}\langle x_m, y_n\rangle\right| \leq \sum_{j=0}^{\infty}\delta^{2j}\left|\sum_{m,n=1}^{N} a_{mn}[A(\phi_\delta^j(x_m), \phi_\delta^j(y_n)) - 1]\right| \leq \sum_{j=0}^{\infty}\delta^{2j}(e+1) = (e+1)/(3-e). \quad \text{Q.E.D.}$$

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