Research Article

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Ruled surfaces generated by elliptic cylindrical curves in the isotropic space

https://doi.org/10.1515/gmj-2017-0044
Received January 23, 2015; accepted April 27, 2016

Abstract: In this paper we study the ruled surfaces generated by elliptic cylindrical curves in the isotropic 3-space $\mathbb{I}^3$. We classify such surfaces in $\mathbb{I}^3$ with constant curvature and satisfying an equation in terms of the components of the position vector field and the Laplacian operator. Several examples are given and illustrated by figures.

Keywords: Ruled surface, elliptic cylindrical curve, isotropic space, isotropic mean curvature, relative curvature, Laplacian operator

MSC 2010: 53A35, 53A40, 53B25

1 Introduction

In addition to Euclidean geometry on $\mathbb{R}^3$, there are other important geometries on $\mathbb{R}^3$ called *Laguerre geometry* and *isotropic geometry*. Laguerre geometry is the geometry of oriented planes and spheres in $\mathbb{R}^3$ and is related to several applied sciences, e.g., computer science and visualization [25, 28]. The isotropic geometry naturally appears when the properties of functions are geometrically visualized and interpreted via their graph surfaces [27].

One of the remarkable applications of isotropic geometry is pertinent to the image processing, see [23, 24]. Another application, given by Pottmann and Liu, is the study of discrete surfaces in the isotropic geometry with applications in architectural design [26].

More recently, Chen, Decu and Verstraelen [11] and Decu and Verstraelen [14] studied the production models in microeconomics via isotropic geometry. For more details of this research topic situated at the interplay between differential geometry and economics, see [4, 5, 8, 9, 33, 35].

Vrănceanu (1900–1979) studied in [34] the elliptic cylindrical curves from the viewpoint of the famous Levi-Civita and Fenchel theorems from surfaces theory. Crasmareanu [13] classified Tzitzeica elliptic and hyperbolic cylindrical curves via the solution of forced harmonic equations.

One of the targets in classical differential geometry is to classify the surfaces with constant curvature. In this respect we study the ruled surfaces generated by the elliptic cylindrical curves in isotropic spaces. In the present paper, we classify such ruled surfaces with constant curvature in isotropic spaces and satisfying an equation in terms of the components of the position vector field and the Laplacian operator.

This condition is natural, being related to the so-called *submanifolds of finite type*, introduced by Chen in 1983 [6, 7]. See also [3, 10, 12, 17, 18, 22, 32].

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2 Basics on isotropic spaces

For later use, we provide a brief review of curves and surfaces in isotropic spaces from [16, 29–31].

An isotropic space based on the following group G_6 of affine transformations (so-called *isotropic congruence transformations* or *i-motions*) is a Cayley–Klein space:

$$\mathbf{x}' = a + \mathbf{x}\cos\phi - \mathbf{y}\sin\phi,$$
$$\mathbf{y}' = b + \mathbf{x}\sin\phi - \mathbf{y}\cos\phi,$$
$$\mathbf{z}' = c + d\mathbf{x} + e\mathbf{y} + \mathbf{z}.$$

This means that i-motions are indeed composed of an Euclidean motion in the $\mathbf{xy}$-plane (i.e. translation and rotation) and an affine shear transformation in $\mathbf{z}$-direction.

In general, the following terminology is used for the isotropic spaces. Consider the points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The projection in the $\mathbf{z}$-direction onto $\mathbb{R}^2$, $(x_1, x_2, x_3) \mapsto (x_1, x_2, 0)$, is called the *top view*. In the sequel, many metric properties in isotropic geometry (invariants under G_6) are Euclidean invariants in the top view such as the isotropic distance, so-called i-distance. The *i-distance* of two points $\mathbf{x}$ and $\mathbf{y}$ is defined as the Euclidean distance of their top views, i.e.,

$$\|\mathbf{x} - \mathbf{y}\|_i = \sqrt{\sum_{j=1}^{2}(y_j - x_j)^2}.$$

Two points having the same top view are called *parallel points*. The i-metric is degenerate along the lines in the $\mathbf{z}$-direction, and such lines are called *isotropic* lines. The plane containing an isotropic line is called an *isotropic plane*. Therefore, an *isotropic* 3*-space* $\mathbb{I}^3$ is the product of the Euclidean 2-space $\mathbb{R}^2$ and an isotropic line with a degenerate parabolic distance metric.

Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{I}^3$ be an admissible curve (i.e. without isotropic tangents) parametrized by arc-length $s \in I$. In the coordinate form, it can be written as

$$\gamma(s) = (x(s), y(s), z(s)),$$

where x, y and z are smooth functions of one variable. Denote the first derivative with respect to s by one prime, the second derivative by two primes, etc. Then the curvature and torsion functions of γ are respectively defined by

$$\kappa(s) = x'(s)y''(s) - x''(s)y'(s)$$

and

$$\tau(s) = \frac{1}{\kappa(s)}\det(\gamma'(s), \gamma''(s), \gamma'''(s)), \quad \kappa(s) \neq 0$$

for all $s \in I$. Moreover, the associated trihedron of γ is given by

$$\mathbf{T}(s) = (x'(s), y'(s), z'(s)),$$
$$\mathbf{N}(s) = \frac{1}{\kappa(s)}(x''(s), y''(s), z''(s)),$$
$$\mathbf{B}(s) = (0, 0, 1).$$

In the sequel, the Frenet formulas of such vectors are

$$\mathbf{T}' = \kappa\mathbf{N}, \quad \mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \mathbf{B}' = 0.$$

Let M^2 be a surface immersed in $\mathbb{I}^3$ which has no isotropic tangent planes. Such a surface M^2 is said to be *admissible* and can be parametrized by

$$X : D \subseteq \mathbb{R}^2 \to \mathbb{I}^3 : (u_1, u_2) \mapsto (X_1(u_1, u_2), X_2(u_1, u_2), X_3(u_1, u_2)),$$

where X_1, X_2 and X_3 are smooth and real-valued functions on a domain $D \subseteq \mathbb{R}^2$. Denote by g the metric on M^2 induced from $\mathbb{I}^3$. The components of the first fundamental form of M^2 can be calculated via the induced metric g as follows:

$$g_{ij} = g(X_{u_i}, X_{u_j}), \quad X_{u_i} = \frac{\partial X}{\partial u_i}, \quad i, j \in \{1, 2\}.$$

The unit normal vector field of M^2 is completely isotropic, i.e. $(0, 0, 1)$. Also, the components of the second fundamental form are

$$h_{ij} = \frac{\det(X_{u_iu_j}, X_{u_1}, X_{u_2})}{\sqrt{g_{11}g_{22} - g_{12}^2}}, \quad X_{u_iu_j} = \frac{\partial^2 X}{\partial u_i \partial u_j}, \quad i, j \in \{1, 2\}.$$

Thus the *relative curvature* (the so-called *isotropic curvature* or *isotropic Gaussian curvature*) and the *isotropic mean curvature* are respectively defined by

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} \quad \text{and} \quad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\det(g_{ij})}.$$

If the induced metric g on M^2 is non-degenerate, then the Laplacian operator of M^2 with respect to g is calculated by

$$\Delta = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i}\Big(\sqrt{\det(g_{ij})}g^{ij}\frac{\partial}{\partial u_j}\Big), \tag{2.1}$$

where $(g^{ij}) = (g_{ij})^{-1}$.

3 Classification of elliptic cylindrical curves in $\mathbb{I}^3$

Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{I}^3$, $\gamma(s) = (x(s), y(s), z(s))$ be an admissible curve in $\mathbb{I}^3$. The curve γ is called *elliptic cylindrical* if it is of the form

$$\gamma(s) = (\cos s, \sin s, z(s)), \tag{3.1}$$

where $z(s)$ is a smooth function of one variable. Then its curvature and torsion functions are respectively

$$\kappa(s) = 1 \quad \text{and} \quad \tau(s) = z'(s) + z'''(s) \tag{3.2}$$

for all $s \in I \subseteq \mathbb{R}$. Note that all elliptic cylindrical curves in $\mathbb{I}^3$ are parameterized by the arclength s and have constant curvature 1.

Now let us assume that $z(s)$ satisfies the initial conditions

$$z(0) = z_0, \quad z'(0) = u_0, \quad z''(0) = v_0, \quad z'''(0) = w_0. \tag{3.3}$$

It follows from (3.2) that an elliptic cylindrical curve has constant torsion if the following ordinary differential equation (ODE) holds:

$$z'(s) + z'''(s) = \text{const.} \tag{3.4}$$

After solving ODE (3.4), we derive

$$z(s) = -v_0 \cos s - w_0 \sin s + (u_0 + w_0)s + (v_0 + z_0).$$

Therefore we come to the next result.

Theorem 3.1. *Let γ be an elliptic cylindrical curve in $\mathbb{I}^3$ with the initial conditions* (3.3). *Then its torsion is a constant function if and only if γ has the expression*

$$\gamma(s) = (\cos s, \sin s, -v_0 \cos s - w_0 \sin s + (u_0 + w_0)s + (v_0 + z_0)). \tag{3.5}$$

Note that an elliptic cylindrical curve in form (3.5) has constant curvature $\kappa = 1$ and constant torsion $\tau = u_0 + w_0$.

Example 3.2. Let us put $z(s) = \cos s + \sin s + 3s$, $s \in [0, 2\pi]$, satisfying $z_0 = 1$, $u_0 = 4$, $v_0 = -1$ and $w_0 = -1$. Then the elliptic cylindrical curve $\gamma = \gamma(s)$ has constant torsion $\tau = 3$ and we draw it as in Figure 1.

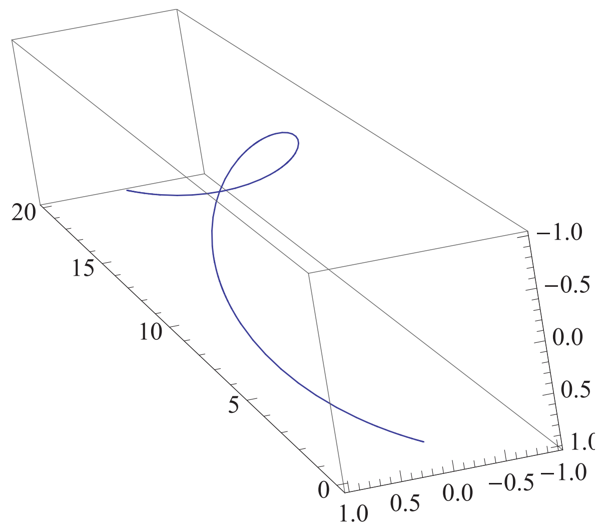

Figure 1: An elliptic cylindrical curve, $\kappa = 1$ and $\tau = 3$.

4 Ruled surfaces generated by elliptic cylindrical curves in $\mathbb{I}^3$

Let $\gamma = \gamma(s)$, $s \in I \subseteq \mathbb{R}$, be an elliptic cylindrical curve as in (3.1). The *tangent developable* (resp. the *principal normal surface* and the *binormal surface*) of the curve γ in $\mathbb{I}^3$ is defined to be a ruled surface along γ whose rulings are given by the tangent vectors (resp. the principal normals and the binormals) of γ, i.e.

$$X : I \times \mathbb{R} \to \mathbb{I}^3 : (s, t) \mapsto \gamma(s) + t\,\mathbf{T}(s), \quad I \subseteq \mathbb{R}, \tag{4.1}$$

$$X : I \times \mathbb{R} \to \mathbb{I}^3 : (s, t) \mapsto \gamma(s) + t\,\mathbf{N}(s), \quad I \subseteq \mathbb{R}, \tag{4.2}$$

$$X : I \times \mathbb{R} \to \mathbb{I}^3 : (s, t) \mapsto \gamma(s) + t\,\mathbf{B}(s), \quad I \subseteq \mathbb{R}, \tag{4.3}$$

where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is the Frenet trihedron of γ. See [1, 2, 15, 19–21]. We investigate the geometry of such ruled surfaces in the next three subsections.

4.1 Tangent developables of elliptic cylindrical curves in $\mathbb{I}^3$

From (3.1) and (4.1), a tangent developable M^2 of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ is of the form

$$X(s, t) = (\cos s - t \sin s, \sin s + t \cos s, z(s) + tz'(s)).$$

The components of the first and the second fundamental form are respectively calculated as follows:

$$g_{11} = 1 + t^2, \quad g_{12} = g_{22} = 1; \tag{4.4}$$

and

$$h_{11} = -t(z' + z'''), \quad h_{12} = h_{22} = 0. \tag{4.5}$$

Thus we have the following result.

Theorem 4.1. *Let M^2 be a tangent developable of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying the initial conditions* (3.3). *Then the following properties hold:*
(i) *M^2 always has a vanishing relative curvature.*
(ii) *If M^2 has a constant isotropic mean curvature, then it is isotropic minimal and has the form*

$$X(s, t) = (\cos s - t \sin s, \sin s + t \cos s, (-v_0 + tu_0) \cos s + (u_0 + tv_0) \sin s + (v_0 + z_0)). \tag{4.6}$$

Proof. It is easy to see from (4.5) that $h_{11}h_{22} - h_{12} = 0$, i.e., M^2 always has a vanishing relative curvature. Now, let us assume that M^2 has constant isotropic mean curvature H. Hence we get from (4.4) and (4.5) that

$$H = -\frac{z' + z'''}{2t} \tag{4.7}$$

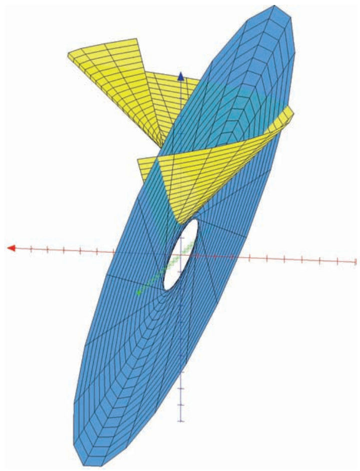

Figure 2: Isotropic minimal (blue) and non-minimal (yellow) tangent developable ruled surfaces.

for constant H. Then we can rewrite (4.7) as

$$-2Ht = z' + z'''. \tag{4.8}$$

The left-hand side of (4.8) is a function of t while the right-hand side is a function of s. This is only possible when $H \equiv 0$ and

$$z' + z''' = 0. \tag{4.9}$$

After solving ODE (4.9) by considering the initial conditions (3.3), we obtain

$$z(s) = -v_0 \cos s + u_0 \sin s + (v_0 + z_0),$$

which gives the proof. □

Remark 4.2. Note that a tangent developable of an elliptic cylindrical curve with zero torsion in $\mathbb{I}^3$ is always isotropic minimal.

Example 4.3. Let M^2 be a tangent developable of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying the initial conditions (3.3). For $u_0 = 1$ and $v_0 = w_0 = z_0 = 0$, M^2 is parametrized by

$$X(s,t) = (\cos s - t \sin s, \sin s + t \cos s, s + t), \quad (s,t) \in [0, 2\pi] \times [0, 2\pi].$$

Thus it has nonzero isotropic mean curvature $H = -\frac{1}{2t}$ and can be drawn as the yellow shape in Figure 2.

Example 4.4. Let M^2 be an isotropic minimal tangent developable of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying the initial conditions (3.3). Taking $u_0 = 1$, $v_0 = 2$ and $z_0 = -2$ in (4.6), it can be parametrized by

$$X(s,t) = (\cos s - t \sin s, \sin s + t \cos s, (-2 + t) \cos s + (1 + 2t) \sin s), \quad (s,t) \in [0, 2\pi] \times [0, 2\pi].$$

It is the blue shape in Figure 2.

Now, let us consider the tangent developable M^2 of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying an equation in terms of the position vector field $X = (X_1, X_2, X_3)$ and the Laplacian operator Δ with respect to the induced metric. The next result gives a classification for the tangent developables satisfying the equation $\Delta X_i = \lambda_i X_i$, where λ_i are real constants for all $i = 1, 2, 3$.

Theorem 4.5. *Let M^2 be a tangent developable of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ fulfilling the equations $\Delta X_i = \lambda_i X_i$, where λ_i are real constants for all $i = 1, 2, 3$ and Δ is the Laplacian operator of M^2. Then M^2 is isotropic minimal and parametrized by* (4.6).

Proof. The Laplacian operator Δ of M^2 with respect to the induced metric is calculated by taking $u_1 = s$ and $u_2 = t$ in (2.1) as

$$\Delta = \frac{1}{t^2}\left(\frac{1}{t}\frac{\partial}{\partial s} + \frac{t^2 - 1}{t}\frac{\partial}{\partial t} - 2\frac{\partial^2}{\partial s \partial t} + \frac{\partial^2}{\partial s^2} + (t^2 + 1)\frac{\partial^2}{\partial t^2}\right). \tag{4.10}$$

Let us put $X(s,t) = (X_1(s,t), X_2(s,t), X_3(s,t))$, where

$$X_1(s,t) = \cos s - t\sin s, \quad X_2(s,t) = \sin s + t\cos s, \quad X_3(s,t) = z(s) + tz'(s). \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\Delta X_1 = 0, \quad \Delta X_2 = 0, \quad \Delta X_3 = \frac{z' + z'''}{t}. \tag{4.12}$$

It is easily seen from (4.12) and $\Delta X_i = \lambda_i X_i$ for all $i = 1, 2, 3$ that $\lambda_1 = \lambda_2 = 0$ and

$$z'(s) + z'''(s) = \lambda_3(tz(s) + t^2 z'(s)). \tag{4.13}$$

By taking the partial derivative of (4.13) with respect to t, we derive

$$-\lambda_3 \frac{1}{2t} = \lambda_3 \frac{z'(s)}{z(s)},$$

which implies that $\lambda_3 = 0$. From (4.12) it follows that

$$z' + z''' = 0.$$

From (4.7), one obtains that M^2 is isotropic minimal and parametrized by (4.6); the proof of the theorem is completed. □

4.2 Principal normal surfaces of elliptic cylindrical curves in $\mathbb{I}^3$

By (3.1) and (4.2), the principal normal surface M^2 of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ has the expression

$$X(s,t) = ((1-t)\cos s, (1-t)\sin s, z(s) + tz''(s)).$$

The components of the first and the second fundamental form are respectively

$$g_{11} = (1-t)^2, \quad g_{12} = 0, \quad g_{22} = 1; \tag{4.14}$$

and

$$h_{11} = t(z'' + z^{(\mathrm{iv})}), \quad h_{12} = \frac{z' + z'''}{1-t}, \quad h_{22} = 0. \tag{4.15}$$

Theorem 4.6. *Let M^2 be a principal normal surface of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying the initial conditions* (3.3). *Then the following properties hold:*

(i) *If M^2 has constant relative curvature K, then $K \equiv 0$ and M^2 has the form*

$$X(s,t) = ((1-t)\cos s, (1-t)\sin s, -v_0(1-t)\cos s + u_0(1-t)\sin s + (v_0 + z_0)). \tag{4.16}$$

(ii) *If M^2 has constant isotropic mean curvature H, then $H \equiv 0$ and M^2 has the parametrization*

$$\begin{aligned} X(s,t) = \big((1-t)\cos s, (1-t)\sin s, &-v_0(1-t)\cos s - w_0(1-t)\sin s \\ &+ (u_0 + w_0)s + +(v_0 + z_0)\big). \end{aligned} \tag{4.17}$$

Proof. (i) Assume that M^2 has constant relative curvature K. From (4.14) and (4.15), we get

$$K = -\frac{(z' + z''')^2}{(1-t)^4}.$$

This implies that $K \equiv 0$ and

$$z' + z''' = 0. \tag{4.18}$$

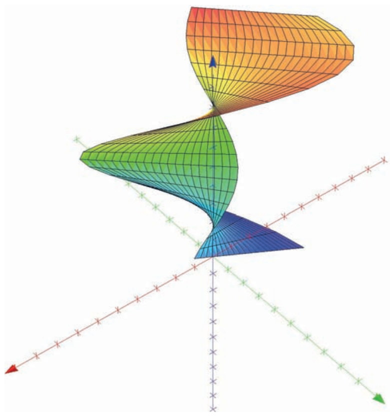

Figure 3: The principal normal surface of an elliptic cylindrical curve, $K = -\frac{4}{(1-t)^4}$, $H = 0$.

By solving ODE (4.18) with initial conditions (3.3), we derive

$$z(s) = -v_0 \cos s + u_0 \sin s + (v_0 + z_0),$$

which gives statement (i) of the theorem.

(ii) If M^2 has constant isotropic mean curvature H, then using (4.14) and (4.15), we find

$$H = \frac{t}{2(1-t)^2}(z'' + z^{(\mathrm{iv})}). \tag{4.19}$$

We deduce from (4.19) that $H \equiv 0$ and then

$$z'' + z^{(\mathrm{iv})} = 0. \tag{4.20}$$

By solving ODE (4.20) with initial conditions (3.3), we obtain

$$z(s) = -v_0 \cos s - w_0 \sin s + (u_0 + w_0)s + (v_0 + z_0). \qquad \square$$

Remark 4.7. Note that a principal normal surface of an elliptic cylindrical curve with zero torsion (resp. nonzero constant torsion) in $\mathbb{I}^3$ always has a vanishing relative curvature (resp. vanishing isotropic mean curvature).

Example 4.8. Let M^2 be a principal normal surface of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying the initial conditions (3.3). Taking $u_0 = 2.5$, $v_0 = w_0 = -0.5$ and $z_0 = 0.5$ in (4.17), we obtain that M^2 has non-vanishing relative curvature $K = -\frac{4}{(1-t)^4}$, is isotropic minimal and parametrized by

$$X(s,t) = ((1-t)\cos s, (1-t)\sin s, 0.5(1-t)(\cos s + \sin s) + 2s),$$

where $(s,t) \in [0, 2\pi] \times [0, 2\pi]$. Moreover, the base curve γ of M^2 has constant torsion 2. An example of such a ruled surface M^2 is shown in Figure 3.

Theorem 4.9. *Let M^2 be the principal normal surface of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$ satisfying the equations $\Delta X_i = \lambda_i X_i$, where λ_i are real constants for all $i = 1, 2, 3$ and Δ is the Laplace operator of M^2. Then M^2 is isotropic minimal and parametrized by* (4.17).

Proof. Assume that M^2 is the principal normal surface of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$. Then the Laplace operator Δ of M^2 is given by

$$\Delta = \frac{1}{1-t}\left(\frac{1}{1-t}\frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial t} + (1-t)\frac{\partial^2}{\partial t^2}\right). \tag{4.21}$$

The surface M^2 can be parametrized as $X(s,t) = (X_1(s,t), X_2(s,t), X_3(s,t))$, where

$$X_1(s,t) = (1-t)\cos s, \quad X_2(s,t) = (1-t)\sin s, \quad X_3(s,t) = z(s) + tz''(s). \tag{4.22}$$

It follows from (4.21) and (4.22) that

$$\Delta X_1 = 0, \quad \Delta X_2 = 0, \quad \Delta X_3 = \frac{t}{(1-t)^2}(z'' + z^{(\mathrm{iv})}). \tag{4.23}$$

We deduce from (4.23) and from $\Delta X_i = \lambda_i X_i$ for all $i = 1, 2, 3$ that $\lambda_1 = \lambda_2 = 0$ and

$$(z'' + z^{(\mathrm{iv})}) = \lambda_3 \frac{(1-t)^2}{t}(z(s) + tz''(s)). \tag{4.24}$$

After taking the partial derivative of (4.24) with respect to t, we derive $\lambda_3 = 0$ and

$$z'' + z^{(\mathrm{iv})} = 0.$$

This concludes (see (4.19)) the proof of the theorem. □

4.3 Binormal surfaces of elliptic cylindrical curves in $\mathbb{I}^3$

By (3.1) and (4.3), a binormal surface M^2 of an elliptic cylindrical curve $\gamma = \gamma(s)$ in $\mathbb{I}^3$, satisfying the initial conditions (3.3), has the form

$$X(s,t) = (\cos s, \sin s, z(s)) + t(0, 0, 1).$$

Note that M^2 is indeed some kind of the generalized cylinder in $\mathbb{I}^3$ and is non-admissible. Instead of the non-admissible binormal surface, we consider the surface

$$X(s,t) = t(\cos s, \sin s, z(s)) + (0, 0, 1), \tag{4.25}$$

which is a generalized cone with the vertex at the end point of the binormal vector of $\gamma = \gamma(s)$ in $\mathbb{I}^3$. Hence the components of the first and the second fundamental form are respectively

$$g_{11} = t^2, \quad g_{12} = 0, \quad g_{22} = 1; \tag{4.26}$$

and

$$h_{11} = -t(z'' + z), \quad h_{12} = h_{22} = 0. \tag{4.27}$$

Equality (4.26) implies that the metric on M^2 induced from $\mathbb{I}^3$ is given by $g = t^2 ds^2 + dt^2$.

Remark 4.10. By using (4.26) and (4.27), it follows that M^2 has vanishing relative curvature and isotropic mean curvature $H = -\frac{z''+z}{t}$. This implies that if H is a constant, then it must be identically zero.

On the other hand, the Laplace operator of M^2 is

$$\Delta = \frac{1}{t^2}\frac{\partial^2}{\partial s^2} + \frac{1}{t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}. \tag{4.28}$$

Assume that the binormal surface M^2 satisfies $\Delta X_i = \lambda_i X_i$ for the real constants λ_i, $i = 1, 2, 3$. It is easily seen from (4.25) and (4.28) that $\lambda_1 = \lambda_2 = 0$ and

$$z'' + z = \lambda_3 t^2 z,$$

which implies that $\lambda_3 = 0$ and $z(s) = z_0 \cos s + u_0 \sin s$.

Therefore we have the next result.

Theorem 4.11. *Let M^2 be a generalized cone on an elliptic cylindrical curve $\gamma = \gamma(s)$ with vertex at the end point of the binormal vector of γ in $\mathbb{I}^3$ satisfying the initial conditions* (3.3). *If M^2 satisfies the equation $\Delta X_i = \lambda_i X_i$ for the real constants λ_i, $i = 1, 2, 3$, then it is isotropic minimal and has the form*

$$X(s,t) = (0, 0, 1) + t(\cos s, \sin s, z_0 \cos s + u_0 \sin s). \tag{4.29}$$

Example 4.12. Let M^2 be the isotropic minimal generalized cone given by (4.29). Taking $z_0 = u_0 = 1$ and $(s,t) \in [0, 2\pi] \times [0, 2\pi]$ in (4.29), we draw it in green in Figure 4. Further, by putting $z(s) = \frac{s-2}{2}$ in (4.25), M^2 is shown as the brown shape in Figure 4.

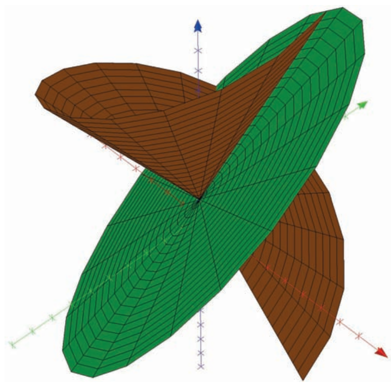

Figure 4: Isotropic minimal (green) and non-minimal cone (brown).

5 Conclusions

The results obtained in the above sections are summarized in Table 1 which classifies the ruled surfaces generated by an elliptic cylindrical curve in $\mathbb{I}^3$.

Properties	Tangent developable	Principal normal surface	A generalized cone on an elliptic cylindrical curve
Nonzero constant relative curvature	No	No	No
Null relative curvature	All	Has the form (4.16)	All
Nonzero constant isotropic mean curvature	No	No	No
Isotropic minimal	Has the form (4.6)	Has the form (4.17)	Has the form (4.29)
$\Delta X_i = \lambda_i X_i$ for real constants λ_i, $i = 1, 2, 3$	Only minimal ones	Only minimal ones	Only minimal ones

Table 1: Classification of the ruled surfaces generated by the elliptic cylindrical curves.

Acknowledgment: The second author would like to thank Professor Udo Simon for being her host Professor at Technical University of Berlin and his kind hospitality. Figure 1 in this paper is prepared with Wolfram Mathematica 7.0 and all other drawings with VisuMath (www.visumath.be).

Funding: The second author's research was supported by the DAAD Fellowship for University Professors and Researchers A/14/02337 at Technical University of Berlin, December 2014–January 2015.

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