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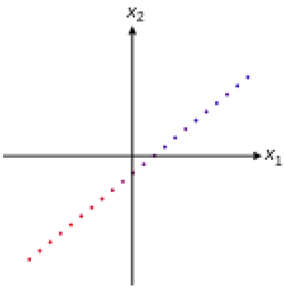
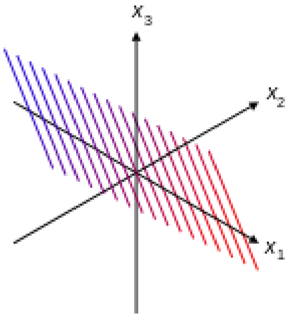
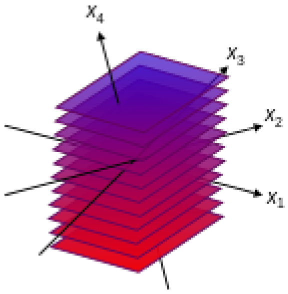
Level set



Points at constant slices of $x_2 = f(x_1)$.

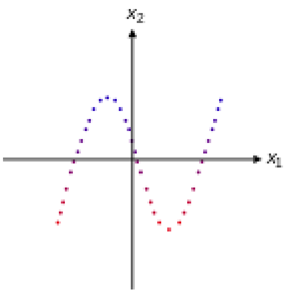


Lines at constant slices of $x_3 = f(x_1, x_2)$.

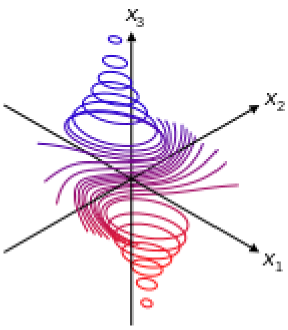


Planes at constant slices of $x_4 = f(x_1, x_2, x_3)$.

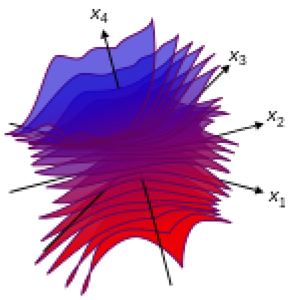
$(n - 1)$-dimensional level sets for functions of the form $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $a_1, a_2, \ldots, a_n$ are constants, in $(n + 1)$-dimensional Euclidean space, for $n = 1, 2, 3$.



Points at constant slices of $x_2 = f(x_1)$.



Contour curves at constant slices of $x_3 = f(x_1, x_2)$.



Curved surfaces at constant slices of $x_4 = f(x_1, x_2, x_3)$.

$(n - 1)$-dimensional level sets of non-linear functions $f(x_1, x_2, \ldots, x_n)$ in $(n + 1)$-dimensional Euclidean space, for $n = 1, 2, 3$.

In mathematics, a **level set** of a real-valued function *f* of *n* real variables is a set where the function takes on a given constant value *c*, that is:

$$L_c(f) = \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = c\}\ ,$$

When the number of independent variables is two, a level set is called a **level curve**, also known as *contour line* or *isoline*; so a level curve is the set of all real-valued solutions of an equation in two variables x_1 and x_2. When $n = 3$, a level set is called a **level surface** (or *isosurface*); so a level surface is the set of all real-valued roots of an equation in three variables x_1, x_2 and x_3. For higher values of *n*, the level set is a **level hypersurface**, the set of all real-valued roots of an equation in $n > 3$ variables.

A level set is a special case of a fiber.

Contents

Alternative names

Level sets show up in many applications, often under different names. For example, an implicit curve is a level curve, which is considered independently of its neighbor curves, emphasizing that such a curve is defined by an implicit equation. Analogously, a level surface is sometimes called an implicit surface or an isosurface.

The name isocontour is also used, which means a contour of equal height. In various application areas, isocontours have received specific names, which indicate often the nature of the values of the considered function, such as isobar, isotherm, isogon, isochrone, isoquant and indifference curve.

Intersections of a co-ordinate function's level surfaces with a trefoil knot. Red curves are closest to the viewer, while yellow curves are farthest.

Examples

Consider the 2-dimensional Euclidean distance:

$$d(x,y)=\sqrt{x^2+y^2}$$

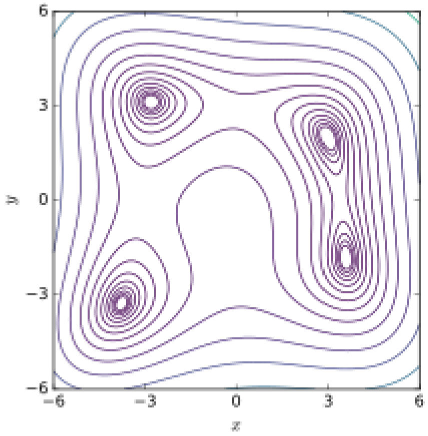
A level set $L_r(d)$ of this function consists of those points that lie at a distance of r from the origin, that make a circle. For example, $(3,4)\in L_5(d)$, because $d(3,4)=5$. Geometrically, this means that the point $(3,4)$ lies on the circle of radius 5 centered at the origin. More generally, a sphere in a metric space (M,m) with radius r centered at $x\in M$ can be defined as the level set $L_r(y\mapsto m(x,y))$.

A second example is the plot of Himmelblau's function shown in the figure to the right. Each curve shown is a level curve of the function, and they are spaced logarithmically: if a curve represents L_x, the curve directly "within" represents $L_{x/10}$, and the curve directly "outside" represents L_{10x}.

Level sets versus the gradient

> **Theorem:** If the function *f* is differentiable, the gradient of *f* at a point is either zero, or perpendicular to the level set of *f* at that point.

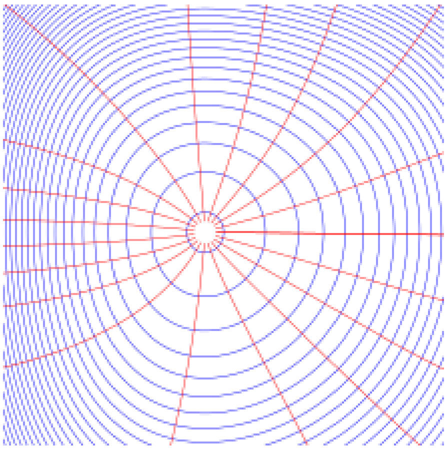
To understand what this means, imagine that two hikers are at the same location on a mountain. One of them is bold, and he decides to go in the direction where the slope is steepest. The other one is more cautious; he does not want to either climb or descend, choosing a path which will keep him at the same height. In our analogy, the above theorem says that the two hikers will depart in directions perpendicular to each other.



Log-spaced level curve plot of Himmelblau's function[1]

A consequence of this theorem (and its proof) is that if f is differentiable, a level set is a hypersurface and a manifold outside the critical points of f. At a critical point, a level set may be reduced to a point (for example at a local extremum of f) or may have a singularity such as a self-intersection point or a cusp.

Sublevel and superlevel sets



Consider a function *f* whose graph looks like a hill. The blue curves are the level sets; the red curves follow the direction of the gradient. The cautious hiker follows the blue paths; the bold hiker follows the red paths. Note that blue and red paths always cross at right angles.

A set of the form

$$L_c^-(f) = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) \leq c\}$$

is called a **sublevel set** of *f* (or, alternatively, a **lower level set** or **trench** of *f*). A **strict sublevel** set of *f* is

$$\{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) < c\}$$

Similarly

$$L_c^+(f) = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) \geq c\}$$

is called a **superlevel set** of *f* (or, alternatively, an **upper level set** of *f*). And a **strict superlevel set** of *f* is

$$\{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) > c\}$$

Sublevel sets are important in minimization theory. By Weierstrass's theorem, the boundness of some non-empty sublevel set and the lower-semicontinuity of the function implies that a function attains its minimum. The convexity of all the sublevel sets characterizes quasiconvex functions.[2]

See also

- Epigraph
- Level-set method
- Level set (data structures)

References

1. Simionescu, P.A. (2011). "Some Advancements to Visualizing Constrained Functions and Inequalities of Two Variables". *Journal of Computing and Information Science in Engineering*. **11** (1). doi:10.1115/1.3570770 (https://doi.org/10.1115%2F1.3570770).

2. Kiwiel, Krzysztof C. (2001). "Convergence and efficiency of subgradient methods for quasiconvex minimization". *Mathematical Programming, Series A*. Berlin, Heidelberg: Springer. **90** (1): 1–25. doi:10.1007/PL00011414 (https://doi.org/10.1007%2FPL00011414). ISSN 0025-5610 (https://www.worldcat.org/issn/0025-5610). MR 1819784 (https://www.ams.org/mathscinet-getitem?mr=1819784). S2CID 10043417 (https://api.semanticscholar.org/CorpusID:10043417).