Semigroups of analytic functions in analysis and applications

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Dedicated to the blessed memory of
Andrei Aleksandrovich Gonchar

Semigroups of analytic functions in analysis and applications

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Abstract. This survey considers problems of analysis and certain related areas in which semigroups of analytic functions with respect to the operation of composition appear naturally. The main attention is devoted to holomorphic maps of a disk (or a half-plane) into itself. The role of fixed points is highlighted, both in the description of the structure of semigroups and in applications. Interconnections of the problem of fractional iteration with certain problems in the theory of random branching processes are pointed out, as well as with certain questions of non-commutative probability. The role of the infinitesimal description of semigroups of conformal maps in the development of the parametric method in the theory of univalent functions is shown.

Bibliography: 94 titles.

Keywords: one-parameter semigroup, infinitesimal generator, evolution family, evolution equation, fractional iterates, Koenigs function, fixed points.

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1. Introduction

Semigroups of analytic functions with respect to the operation of composition arise naturally in various questions of analysis and its applications. For example, the problem of fractional iteration, the study of which has a rich history and goes back to papers of Schröder [1] in 1871 and Koenigs [2] in 1884, can be regarded as the problem of embedding the positive-integer iterates into a one-parameter semigroup. In the theory of random branching processes, iterates of a probability generating function describe the dynamics of a Galton–Watson process, that is, a Markov branching process with discrete time. The problem of embedding iterates into a one-parameter semigroup of probability generating functions is equivalent to embedding a Galton–Watson process into a homogeneous Markov branching process with continuous time. The problem of embedding the reciprocal Cauchy transform of a probability distribution into a one-parameter semigroup arises in non-commutative probability theory in connection with analogues of the Lévy–Khinchin formula for infinitely divisible laws. The problem of embedding iterates of a holomorphic map of the unit disk into itself into a one-parameter semigroup also arises in the theory of composition operators.

Another direction of studies connected with semigroups of analytic functions has its origin in the classical 1923 paper [3] of Löwner. There he laid the foundations of the parametric method in geometric function theory and for the first time made progress in solving the coefficient problem stated as a conjecture by Bieberbach [4] in 1916. For decades this problem determined the direction of development of the theory of univalent functions and was solved by de Branges [5], [6] in 1984 with the use of the Löwner parametric method. The approach systematically developed by Löwner (see [7]–[10]) was based on representing an element of a semigroup as a composition of infinitesimal transformations, which in conceptual terms relates it to the theory of Lie groups.

In the present paper a complete analogue of the Berkson–Porta formula is established for the infinitesimal generator of a one-parameter semigroup of holomorphic maps of the unit disk into itself with given fixed points. The problem of embedding a holomorphic map into a one-parameter semigroup is solved in terms of the Koenigs function. We point out the connections of the embedding problem and the problem of fractional iterates with various questions in the theory of conformal maps, the theory of branching processes, and non-commutative probability. We establish the equivalence of the conditions of embeddability and infinite divisibility in the semigroup of probability generating functions.

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2. One-parameter semigroups and infinitesimal transformations

The problem of fractional iteration for a given function $f(z)$ consists in finding a family of functions $f^t(z)$, $t \geqslant 0$, satisfying the following conditions: $f^1(z) = f(z)$,

$f^0(z) = z$ and $f^{t+s}(z) = f^t \circ f^s(z)$ for all $s, t \geqslant 0$. This problem has a rich history and has been studied in various settings by many authors. The first studies concerned the local case, where the function f and its iterates were defined by power series only in a neighbourhood of a fixed point a, $f(a) = a$. Koenigs [2] already showed that if $|f'(a)|$ is neither 0 nor 1, then f can be embedded into a family of fractional iterates. Later the direction connected with entire and meromorphic functions was developed, and here the results turned out to be diametrically opposite to the local case. For example, Baker [11], and also Karlin and McGregor [12], showed that in this case embeddability is possible only for fractional linear functions under fairly general assumptions. We also point out the papers [13], [14] and the monographs [15], [16], which contain surveys of this topic and reflect the connection of the problem of fractional iteration with functional equations. Moreover, as noted by Hadamard [17], substantial progress was made in the investigation of the problem of fractional iteration when they began to regard a family of fractional iterates as a one-parameter semigroup, and to describe it by using the infinitesimal generator.

The connection of the problem of fractional iteration with the problem of embedding a Galton–Watson process into a homogeneous Markov branching process with continuous time, as well as certain problems in the theory of composition operators and the theory of conformal maps, stimulated the study of the case when the function and its iterates map the unit disk (or the upper half-plane) into itself. Here it is often required that all iterates inherit certain properties of the function itself.

Let $\mathfrak{P}$ be the set of all functions f holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and taking values in $\mathbb{D}$, that is, $f(\mathbb{D}) \subseteq \mathbb{D}$. Obviously, $\mathfrak{P}$ forms a topological semigroup with respect to the operation of composition and the topology of locally uniform convergence in $\mathbb{D}$. The role of the identity in this semigroup is played by the identity transformation $f(z) \equiv z$. Moreover, $\mathfrak{P}$ is a convex set in the linear topological space $\mathscr{H}(\mathbb{D})$ of holomorphic functions in $\mathbb{D}$ with the topology of locally uniform convergence.

Regarding $\mathbb{R}^+ = \{t \in \mathbb{R}\colon t \geqslant 0\}$ as an additive semigroup with the ordinary topology of the real numbers, we understand a one-parameter semigroup in $\mathfrak{P}$ to be a continuous homomorphism $t \mapsto f^t$ acting from $\mathbb{R}^+$ to $\mathfrak{P}$. This means that the family $\{f^t\}_{t \geqslant 0}$ of functions in $\mathfrak{P}$ satisfies the following conditions:

(i) $f^{t+s}(z) = f^t \circ f^s(z)$ for $s, t \geqslant 0$;

(ii) $f^t(z) \to z$ locally uniformly in $\mathbb{D}$ as $t \to 0$.

Note that for non-negative integer values of t we obtain ordinary iterates $f^n = f \circ f^{n-1}$, $n = 2, 3, \dots$, of the function $f = f^1$. Therefore, the elements of the family $\{f^t\}_{t \geqslant 0}$ are also called fractional iterates of f. Thus, the fractional iterates problem for $f \in \mathfrak{P}$ consists in the question of the existence of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ such that $f^1 = f$. One-parameter semigroups play an important role in the study of the structure of the semigroup $\mathfrak{P}$ itself and its subsemigroups.

As a rule, synthesis of algebraic and topological properties leads to fairly stringent constructions. This can be illustrated by the infinite differentiability with respect to t of the family of functions $f^t(z)$ of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$. Indeed, we fix an arbitrary $r \in (0, 1)$ and show that for any $z \in K_r =$

$\{\zeta \in \mathbb{C}\colon |z| \leqslant r\}$ the function $t \mapsto f^t(z)$ is infinitely differentiable in some neighbourhood of the point $t = 0$. Then the condition (i) extends the differentiability property to the entire half-axis $(0, \infty)$.

Let η be a function that is infinitely differentiable on $\mathbb{R}$, vanishes outside the interval $(0, 1)$, and satisfies the conditions $\eta(t) > 0$ for $t \in (0, 1)$ and

$$\int_{-\infty}^{\infty} \eta(t)\,dt = \int_0^1 \eta(t)\,dt = 1.$$

For $\delta > 0$ let η_δ be the function $\eta_\delta(t) = \eta(t/\delta)$. We now use the choice of $\delta > 0$ as follows. We fix $\rho \in (r, 1)$ and choose $\delta > 0$ in such a way that the following conditions hold for $t \in (0, \delta)$:

(a) $|f^t(z)| \leqslant \rho$ for $z \in K_r$;

(b) $\operatorname{Re}\{(f^t)'(z)\} > 1/2$ for $z \in K_\rho$.

This can be done, since $f^t(z) \to z$ and $(f^t)'(z) \to 1$ locally uniformly in $\mathbb{D}$ as $t \to 0$. We now define the function

$$F(z) = \int_{-\infty}^{\infty} \eta_\delta(t) f^t(z)\,dt = \int_0^\delta \eta_\delta(t) f^t(z)\,dt.$$

Obviously, F is holomorphic in $\mathbb{D}$. Moreover,

$$\operatorname{Re} F'(z) = \int_0^\delta \eta_\delta(t) \operatorname{Re}\{(f^t)'(z)\}\,dt \geqslant \frac{1}{2}\int_0^\delta \eta_\delta(t)\,dt = \frac{\delta}{2} > 0$$

for $z \in K_\rho$, and thus F is univalent in the disk K_ρ. We also introduce the function

$$\Phi(z, t) = F(f^t(z)),$$

$z \in \mathbb{D}$, $t \geqslant 0$. The obvious transformations

$$\Phi(z, t) = \int_{-\infty}^{\infty} \eta_\delta(s) f^s(f^t(z))\,ds = \int_{-\infty}^{\infty} \eta_\delta(s) f^{s+t}(z)\,ds = \int_{-\infty}^{\infty} \eta_\delta(\tau - t) f^\tau(z)\,d\tau$$

show that $\Phi(z, t)$ is infinitely differentiable with respect to t.

Next, by condition (a) the point $f^t(z)$ belongs to the disk K_ρ for $z \in K_r$ and $t \in [0, \delta]$. Therefore, for these t and z we can write

$$f^t(z) = F^{-1}(\Phi(z, t)),$$

whence the function $t \mapsto f^t(z)$ is infinitely differentiable in the interval $(0, \delta)$ for any $z \in K_r$. Furthermore,

$$\begin{aligned}\frac{\partial}{\partial t} f^t(z) &= -\frac{1}{F'(f^t(z))} \int_{-\infty}^{\infty} \eta_\delta'(\tau - t) f^\tau(z)\,d\tau \\ &= -\frac{1}{F'(f^t(z))} \int_{-\infty}^{\infty} \eta_\delta'(s) f^s(f^t(z))\,ds = v(f^t(z)),\end{aligned}$$

where

$$v(z) = -\frac{1}{F'(z)} \int_{-\infty}^{\infty} \eta_\delta' f^\tau(z)\,d\tau = \frac{\partial}{\partial t} f^t(z)\bigg|_{t=0}.$$

Since r was chosen arbitrarily in $(0,1)$, the function

$$v(z)=\frac{\partial}{\partial t}f^t(z)\bigg|_{t=0}=\lim_{t\to 0}\frac{f^t(z)-z}{t}$$

is defined and holomorphic in the whole unit disk $\mathbb{D}$. We call this function the *infinitesimal generator* of the one-parameter semigroup $t\mapsto f^t$, or an *infinitesimal transformation* of the semigroup $\mathfrak{P}$. The infinitesimal generator completely characterizes the one-parameter semigroup $t\mapsto f^t$ via the differential equation

$$\frac{\partial}{\partial t}f^t(z)=v(f^t(z))$$

and the initial condition $f^t(z)\big|_{t=0}=z$.

The form of the infinitesimal generator v of a one-parameter semigroup $t\mapsto f^t$ in $\mathfrak{P}$ was first given by Löwner [3] in the case where all the functions f^t fix the point $z=0$, that is, $f^t(0)=0$, $t\geqslant 0$. In this case $v(z)=-zp(z)$, where p is a holomorphic function in $\mathbb{D}$ with non-negative real part. This result can easily be carried over to the case where the functions f^t of a one-parameter semigroup $t\mapsto f^t$ fix an arbitrary point $a\in\mathbb{D}$. Indeed, in this case the functions $g^t=L\circ f^t\circ L^{-1}$, where

$$L(z)=\frac{z-a}{1-\bar{a}z}\,,$$

also form a one-parameter semigroup $t\mapsto g^t$ in $\mathfrak{P}$ and $g^t(0)=0$, $t\geqslant 0$. Furthermore, if

$$\frac{\partial}{\partial t}g^t(z)\bigg|_{t=0}=-zp(z),$$

then

$$\frac{\partial}{\partial t}f^t(z)\bigg|_{t=0}=\frac{1}{L'(z)}\frac{\partial}{\partial t}g^t(L(z))\bigg|_{t=0}=-\frac{L(z)}{L'(z)}p(L(z))=(a-z)(1-\bar{a}z)\frac{p(L(z))}{1-|a|^2}\,.$$

In other words, if a one-parameter semigroup $t\mapsto f^t$ in $\mathfrak{P}$ fixes a point $a\in\mathbb{D}$, then its infinitesimal generator has the form

$$v(z)=(a-z)(1-\bar{a}z)p(z),$$

where p is a holomorphic function in $\mathbb{D}$ with non-negative real part.

We observe that if $f\in\mathfrak{P}$ and $f(z)\not\equiv z$, then by the hyperbolic metric principle (see, for example, [18], [19]) there can be at most one fixed point in $\mathbb{D}$. On the other hand, a function $f\in\mathfrak{P}$ may have no fixed points at all in $\mathbb{D}$. In this connection an important role is played by the result established by Denjoy [20] and Wolff [21].

Theorem 1. *Suppose that $f\in\mathfrak{P}$ is not a fractional linear transformation of the unit disk $\mathbb{D}$ onto itself. Then there exists a unique point q, $|q|\leqslant 1$, such that the sequence of positive-integer iterates f^n, $n=1,2,\dots$, converges to q locally uniformly in $\mathbb{D}$. Furthermore, if $|q|=1$, then there exist the angular limits*

$$\lim_{z\to q}f(z)=f(q),\qquad \lim_{z\to q}f'(z)=f'(q)$$

and $f(q)=q$, $0<f'(q)\leqslant 1$.

In the literature, q is called the Denjoy–Wolff point of the function f. If $q \in \mathbb{D}$, then obviously $f(q) = q$ and q is the unique fixed point in $\mathbb{D}$, and $|f'(q)| \leqslant 1$ by the generalized Schwarz lemma. Thus, the Denjoy–Wolff point of a function f in $\mathfrak{P}$ is an attracting fixed point. Moreover, all the iterates of f (including fractional iterates if they exist) have one and the same Denjoy–Wolff point. In terms of this notion Berkson and Porta [22] gave a complete description of infinitesimal transformations of the semigroup $\mathfrak{P}$.

Theorem 2. *For a function v holomorphic in $\mathbb{D}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q, $|q| \leqslant 1$, it is necessary and sufficient that it admits a representation in the form*

$$v(z) = (q - z)(1 - \overline{q}z)p(z), \tag{1}$$

where p is a holomorphic function in $\mathbb{D}$ with non-negative real part.

Equation (1) is known in the literature as the Berkson–Porta formula. The question of how (1) is transformed when there exist other fixed points has been studied by many authors in various settings (see, for example, [23]–[25]). This question also arises in the theory of composition operators [26], [27].

Suppose that $f \in \mathfrak{P}$ has Denjoy–Wolff point q, $|q| \leqslant 1$, and f also fixes the points $a_1, \dots, a_n$. As already mentioned, $a_1, \dots, a_n$ must be on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and the condition that they are fixed points is understood in the sense of the angular limits

$$\lim_{z \to a_k} f(z) = a_k,$$

$k = 1, \dots, n$.

Theorem 3. *For a function v holomorphic in $\mathbb{D}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q, $|q| \leqslant 1$, and fixed points $a_1, \dots, a_n \in \mathbb{T}$ at which the functions f^t, $t > 0$, have finite angular derivatives, it is necessary and sufficient that it admits a representation in the form*

$$v(z) = \frac{(q - z)(1 - \overline{q}z)}{\displaystyle\sum_{k=1}^{n} \lambda_k \frac{1 + \overline{a}_k z}{1 - \overline{a}_k z} + p(z)}, \tag{2}$$

where $\lambda_k > 0$, $k = 1, \dots, n$, and p is a holomorphic function in $\mathbb{D}$ with non-negative real part.

Proof. First we observe that by results in [24] (see Theorem 1 and Lemma 2) a function v holomorphic in $\mathbb{D}$ is the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q, $|q| \leqslant 1$, and a fixed point $a \in \mathbb{T}$ at which the functions f^t, $t > 0$, have finite angular derivatives if and only if it admits a representation in the form

$$v(z) = \frac{(q - z)(1 - \overline{q}z)}{\lambda \dfrac{1 + \overline{a}z}{1 - \overline{a}z} + g(z)},$$

where $\lambda > 0$ and the function g is holomorphic in $\mathbb{D}$ and has non-negative real part.

Now suppose that $t \mapsto f^t$ is a one-parameter semigroup in $\mathfrak{P}$ with Denjoy–Wolff point q and with fixed points $a_1, \ldots, a_n \in \mathbb{T}$ at which the functions f^t have finite angular derivatives, and let v be the infinitesimal generator of this one-parameter semigroup. By the observation made above, for every $k = 1, \ldots, n$ there exist $\alpha_k > 0$ and holomorphic functions g_k in $\mathbb{D}$ with non-negative real part such that the equations

$$v(z) = \frac{(q-z)(1-\overline{q}z)}{\alpha_k \dfrac{1+\overline{a}_k z}{1-\overline{a}_k z} + g_k(z)} \tag{3}$$

hold for $k = 1, \ldots, n$. On the other hand, the infinitesimal generator v admits the representation (1), which can be written in the form

$$v(z) = (q-z)(1-\overline{q}z)\frac{1}{g(z)}\,,$$

where g is also a holomorphic function in $\mathbb{D}$ with non-negative real part. But then this function g satisfies the equations

$$g(z) = \alpha_k \frac{1+\overline{a}_k z}{1-\overline{a}_k z} + g_k(z),$$

$k = 1, \ldots, n$. Summing these equations over k from 1 to n, we get that

$$g(z) = \frac{1}{n}\sum_{k=1}^{n}\left(\alpha_k \frac{1+\overline{a}_k z}{1-\overline{a}_k z} + g_k(z)\right).$$

Thus, v can be represented by the formula (2) if we set

$$\lambda_k = \frac{\alpha_k}{n}\,, \qquad p(z) = \frac{1}{n}\sum_{k=1}^{n} g_k(z).$$

Conversely, suppose that v has the form (2). We need to show that v is the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q and with fixed points $a_1, \ldots, a_n \in \mathbb{T}$ at which the functions f^t have finite angular derivatives. This follows from the observation made above and from the fact that for every $k = 1, \ldots, n$ the function v can be represented in the form (3) by setting

$$\alpha_k = \lambda_k, \qquad g_k(z) = g(z) + \sum_{j \neq k} \alpha_j \frac{1+\overline{a}_j z}{1-\overline{a}_j z}\,. \qquad \square$$

3. The embedding problem and the Koenigs function

As mentioned above, the problem of fractional iterates for a function f in $\mathfrak{P}$ consists in the possibility of embedding it into a one-parameter semigroup in $\mathfrak{P}$. Already in the first studies of the iterates problem (see, for example, [17]), the connection of this problem with the Abel and Schröder functional equations was discovered. This topic is elucidated in detail with an extensive bibliography in the monographs [15], [16], and not just for analytic functions.

Let $f \in \mathfrak{P}$ and let Φ be a solution of the Abel equation

$$\Phi(f(z)) = \Phi(z) + 1.$$

Then all the positive-integer iterates f^n, $n = 1, 2, \dots$, of the function f satisfy the equations

$$\Phi(f^n(z)) = \Phi(z) + n.$$

A similar situation is also observed for the Schröder equation

$$\Psi(f(z)) = \gamma\Psi(z),$$

where γ is some constant. Its solution Ψ and the iterates f^n, $n = 1, 2, \dots$, satisfy the relations

$$\Psi(f^n(z)) = \gamma^n\Psi(z).$$

In this connection, methods for solving the Abel and Schröder functional equations were studied, as well as the properties of solutions that guarantee the existence of fractional iterates of the function f. In [28] conditions for embeddability of $f \in \mathfrak{P}$ into a one-parameter semigroup were obtained in terms of solutions of the Abel and Schröder equations. We present two results from that paper.

Theorem 4. *Suppose that $f \in \mathfrak{P}$ is not a fractional linear transformation of the unit disk $\mathbb{D}$ onto itself, $f(0) = 0$, and $f'(0) = \gamma \neq 0$. Then f is embeddable into a one-parameter semigroup in $\mathfrak{P}$ if and only if there exists a solution F of the functional equation*

$$F \circ f(z) = \gamma F(z)$$

that is a holomorphic function in $\mathbb{D}$ and satisfies the condition

$$\frac{zF'(z)}{F(z)} = \frac{p(0)}{p(z)}\,,$$

where p is a holomorphic function in $\mathbb{D}$ with positive real part such that $e^{-p(0)} = \gamma$.

Note that under the hypotheses of this theorem $z = 0$ is the Denjoy–Wolff point of f, and in the case $f'(0) = 0$ the function f is not embeddable. The following result concerns the case of a boundary Denjoy–Wolff point [28].

Theorem 5. *Suppose that $f \in \mathfrak{P}$ is not a fractional linear transformation of $\mathbb{D}$ onto itself and $z = 1$ is its Denjoy–Wolff point. Then f is embeddable into a one-parameter semigroup in $\mathfrak{P}$ if and only if there exists a solution F of the functional equation*

$$F \circ f(z) = F(z) + 1$$

that is a holomorphic function in $\mathbb{D}$ satisfying the condition

$$\operatorname{Re}\{(1 - z)^2 F'(z)\} > 0$$

for $z \in \mathbb{D}$.

Now suppose that $f \in \mathfrak{P}$ is not a fractional linear transformation of $\mathbb{D}$ onto itself and has the Denjoy–Wolff point $q \in \mathbb{D}$. If $f'(q) = \gamma \neq 0$, then (see, for example, [29]) there exists the limit

$$\lim_{n\to\infty} \frac{f^n(z) - q}{\gamma^n} = K(z) \tag{4}$$

and it is a holomorphic function in $\mathbb{D}$ satisfying the conditions $K(q) = 0$ and $K'(q) = 1$. This function is a solution of the Schröder functional equation

$$K \circ f(z) = \gamma K(z),$$

and is called the Koenigs function. A parametric representation of the class of Koenigs functions (4) that correspond to embeddable functions f was recently obtained in [24].

Theorem 6. *For a function K holomorphic in $\mathbb{D}$ to be the Koenigs function (4) for some function f embeddable into a one-parameter semigroup in $\mathfrak{P}$ with the Denjoy–Wolff point $q \in \mathbb{D}$ it is necessary and sufficient that it admits a representation in the form*

$$K(z) = (z-q)\biggl(\frac{1-\overline{q}z}{1-|q|^2}\biggr)^{\sigma^2} \exp\biggl\{(1+\sigma^2)\int_{\mathbb{T}} \log\frac{1-\varkappa q}{1-\varkappa z}\,d\mu(\varkappa)\biggr\} \tag{5}$$

with some $\sigma = e^{i\theta}$, $-\pi/2 < \theta < \pi/2$, and some probability measure μ on the unit circle $\mathbb{T}$. Here the power function and the logarithm are understood as the continuous branches that take the respective values 1 and 0 at $z = q$.

We also observe that the function K defined by (5) is univalent and maps $\mathbb{D}$ onto a θ-spiral domain. Furthermore, the family of functions

$$f^t(z) = K^{-1}(e^{-\sigma t}K(z)),$$

$t \geqslant 0$, defines a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q. Thus, (5) defines the Koenigs functions of one-parameter semigroups in $\mathfrak{P}$ with Denjoy–Wolff point q in $\mathbb{D}$. The following result distinguishes those Koenigs functions that correspond to one-parameter semigroups with additional fixed points.

Theorem 7. *Let $q \in \mathbb{D}$ and let $a_1, \dots, a_n$ be pairwise distinct points on the unit circle $\mathbb{T}$. A function K holomorphic in $\mathbb{D}$ is the Koenigs function of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q and with fixed points $a_1, \dots, a_n$ at which the functions f^t, $t > 0$, have finite angular derivatives if and only if for some positive numbers $\lambda_1, \dots, \lambda_n$ and a non-negative γ satisfying the condition $\lambda_1 + \dots + \lambda_n + \gamma = 1$, some $\sigma = e^{i\theta}$ with $-\pi/2 < \theta < \pi/2$, and some probability measure μ on $\mathbb{T}$ the following equation holds*:

$$\begin{aligned} K(z) = (z-q)\biggl(\frac{1-\overline{q}z}{1-|q|^2}\biggr)^{\sigma^2} &\prod_{k=1}^{n}\biggl(\frac{1-\overline{a}_k q}{1-\overline{a}_k z}\biggr)^{\lambda_k(1+\sigma^2)} \\ &\times \exp\biggl\{\gamma(1+\sigma^2)\int_{\mathbb{T}} \log\frac{1-\varkappa q}{1-\varkappa z}\,d\mu(\varkappa)\biggr\}, \end{aligned} \tag{6}$$

where the power functions and the logarithm are understood as the continuous branches that take the respective values 1 and 0 at $z = q$.

Proof. Let $t \mapsto f^t$ be a one-parameter semigroup in $\mathfrak{P}$ with Denjoy–Wolff point $q \in \mathbb{D}$ and with fixed points $a_1, \dots, a_n \in \mathbb{T}$ at which the functions f^t have finite angular derivatives. Then by Theorem 3 its infinitesimal generator v can be represented in the form

$$v(z) = \frac{(q-z)(1-\overline{q}z)}{g(z)}\,,$$

where

$$g(z) = \sum_{k=1}^{n} \alpha_k \frac{1+\overline{a}_k z}{1-\overline{a}_k z} + p(z),$$

$\alpha_k > 0$, $k = 1, \dots, n$, and p is a holomorphic function in $\mathbb{D}$ with non-negative real part. If K is the Koenigs function corresponding to this one-parameter semigroup, then for all $t \geqslant 0$

$$K \circ f^t(z) = (f^t)'(q) K(z).$$

By differentiating these equations with respect to t we get that

$$K'(f^t(z)) v(f^t(z)) = v'(q) e^{t v'(q)} K(z).$$

Here we used the equation $(f^t)'(q) = e^{v'(q)t}$, which is obtained as a result of integrating the differential equation

$$\frac{d}{dt}(f^t)'(q) = v'(q)(f^t)'(q)$$

with the initial condition $(f^t)'(q)\big|_{t=0} = 1$. Setting $t = 0$ in the relation obtained above, we arrive at the equation

$$K'(z) v(z) = v'(q) K(z). \tag{7}$$

This equation can be regarded as a differential equation for the Koenigs function K, which has a unique solution with the initial conditions $K(q) = 0$, $K'(q) = 1$.

We now define the function K by the formula (6) with the parameters chosen as follows. The numbers λ_k, $k = 1, \dots, n$, and γ are defined by

$$\lambda_k = \frac{\alpha_k}{\operatorname{Re} g(q)} \, \frac{1-|q|^2}{|1-\overline{a}_k q|^2}\,, \qquad \gamma = 1 - \sum_{k=1}^{n} \lambda_k.$$

We note that

$$\sum_{k=1}^{n} \lambda_k = \frac{1}{\operatorname{Re} g(q)} \operatorname{Re}\biggl\{ \sum_{k=1}^{n} \alpha_k \frac{1+\overline{a}_k q}{1-\overline{a}_k q} \biggr\} = \frac{\operatorname{Re}\{g(q) - p(q)\}}{\operatorname{Re} g(q)} = 1 - \frac{\operatorname{Re} p(q)}{\operatorname{Re} g(q)},$$

and consequently

$$\gamma = \frac{\operatorname{Re} p(q)}{\operatorname{Re} g(q)} \geqslant 0.$$

Next, by the Riesz–Herglotz formula the function p can be represented in the form

$$p(z) = \lambda \int_{\mathbb{T}} \frac{1+\varkappa z}{1-\varkappa z} \, d\nu(\varkappa) + i\beta,$$

where ν is a probability measure on $\mathbb{T}$, $\lambda = \operatorname{Re} p(0) \geqslant 0$, and $\beta = \operatorname{Im} p(0)$. As the probability measure μ in (6) we take the measure

$$d\mu(\varkappa) = \frac{\operatorname{Re} p(0)}{\operatorname{Re} p(q)} \frac{1-|q|^2}{|1-\varkappa q|^2}\, d\nu(\varkappa)$$

transformed from ν. Obviously, μ is also a probability measure on $\mathbb{T}$. Finally, we set $\sigma^2 = \overline{g(q)}/g(q)$.

Thus, to complete the proof of the necessity of representation of the Koenigs function in the form (6) it is sufficient to show that the function K with the parameters defined above satisfies (7). Differentiating equation (6), we get that

$$\begin{aligned}\frac{K'(z)}{K(z)} &= \frac{1}{z-q} - \sigma^2 \frac{\overline{q}}{1-\overline{q}z} + (1+\sigma^2)\Biggl[\sum_{k=1}^{n} \lambda_k \frac{\overline{a}_k}{1-\overline{a}_k z} + \gamma \int_{\mathbb{T}} \frac{\varkappa}{1-\varkappa z}\, d\mu(\varkappa)\Biggr] \\ &= \frac{1-|q|^2}{(z-q)(1-\overline{q}z)} + (1+\sigma^2)\Biggl[\sum_{k=1}^{n} \lambda_k \biggl(\frac{\overline{a}_k}{1-\overline{a}_k z} - \frac{\overline{q}}{1-\overline{q}z}\biggr) \\ &\qquad + \gamma \int_{\mathbb{T}} \biggl(\frac{\varkappa}{1-\varkappa z} - \frac{\overline{q}}{1-\overline{q}z}\biggr) d\mu(\varkappa)\Biggr].\end{aligned}$$

This implies the equation

$$\begin{aligned}\frac{(z-q)(1-\overline{q}z)K'(z)}{(1-|q|^2)K(z)} &= 1 + (1+\sigma^2)\sum_{k=1}^{n} \lambda_k \frac{(\overline{a}_k-\overline{q})(z-q)}{(1-\overline{a}_k z)(1-|q|^2)} \\ &\qquad + (1+\sigma^2)\gamma \int_{\mathbb{T}} \frac{(\varkappa-\overline{q})(z-q)}{(1-\varkappa z)(1-|q|^2)}\, d\mu(\varkappa).\end{aligned}$$

In view of the choice of the parameters σ, γ, and λ_k, this equation can be transformed as follows:

$$\begin{aligned}\frac{(z-q)(1-\overline{q}z)K'(z)}{(1-|q|^2)K(z)} &= 1 + \frac{2}{g(q)}\sum_{k=1}^{n} \alpha_k \frac{(\overline{a}_k-\overline{q})(z-q)}{(1-\overline{a}_k z)|1-\overline{a}_k q|^2} \\ &\qquad + \frac{2\operatorname{Re} p(q)}{g(q)} \int_{\mathbb{T}} \frac{(\varkappa-\overline{q})(z-q)}{(1-|q|^2)(1-\varkappa z)}\, d\mu(\varkappa).\end{aligned}$$

Replacing the measure μ by ν and taking into account the connection between the functions p and g, we have

$$\begin{aligned}\frac{g(q)(z-q)(1-\overline{q}z)K'(z)}{(1-|q|^2)K(z)} &= p(q) + \sum_{k=1}^{n} \alpha_k \biggl(\frac{2(\overline{a}_k-\overline{q})(z-q)}{(1-\overline{a}_k z)|1-\overline{a}_k q|^2} + \frac{1+\overline{a}_k q}{1-\overline{a}_k q}\biggr) \\ &\qquad + 2\lambda \int_{\mathbb{T}} \frac{(\varkappa-\overline{q})(z-q)}{|1-\varkappa q|^2(1-\varkappa z)}\, d\nu(\varkappa).\end{aligned}$$

Next, since

$$\frac{2(\overline{a}_k-\overline{q})(z-q)}{(1-\overline{a}_k z)|1-\overline{a}_k q|^2} + \frac{1+\overline{a}_k q}{1-\overline{a}_k q} = \frac{1+\overline{a}_k z}{1-\overline{a}_k z}\,,$$

and

$$p(q)+2\lambda\int_{\mathbb{T}}\frac{(\varkappa-\overline{q})(z-q)}{|1-\varkappa q|^2(1-\varkappa z)}\,d\nu(\varkappa)=\lambda\int_{\mathbb{T}}\frac{1+\varkappa z}{1-\varkappa z}\,d\nu(\varkappa)+i\beta=p(z),$$

we arrive at the following equation:

$$\frac{K'(z)}{K(z)}=\frac{(1-|q|^2)g(z)}{g(q)(z-q)(1-\overline{q}z)}=\frac{v'(q)}{v(z)}\,.$$

This means that (7) holds, and the necessity is proved.

The sufficiency of representation of the Koenigs function by (6) was established in [24] (see Proposition 2). □

In the case of a boundary Denjoy–Wolff point $q\in\mathbb{T}$, a definition of the Koenigs function analogous to (4) is hindered by the fact that the corresponding limit function may turn out to be identically constant (see, for example, [29]). In [30], [31] other methods were studied for normalization of the sequence of iterates in order to get a limit function that is not identically constant in the case of a boundary Denjoy–Wolff point. Here, as a rule, the limit function is a solution of the Abel functional equation. It was shown in [24] that to every one-parameter semigroup $t\mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point $q\in\mathbb{T}$ there corresponds a unique function F that is a solution of the equation

$$F\circ f^t(z)=F(z)+t$$

for all $t\geqslant 0$ and that satisfies the condition $F(0)=0$. It is natural to call this function the *Koenigs function of the one-parameter semigroup* $t\mapsto f^t$ in the case of a boundary Denjoy–Wolff point. A parametric representation of the class of Koenigs functions corresponding to one-parameter semigroups with a given Denjoy–Wolff point $q\in\mathbb{T}$ was obtained in [24].

Theorem 8. *For a function F holomorphic in $\mathbb{D}$ to be the Koenigs function of a one-parameter semigroup $t\mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point $q\in\mathbb{T}$ it is necessary and sufficient that it admits a representation in the form*

$$\begin{aligned}F(z)&=i\beta\frac{\overline{q}z}{1-\overline{q}z}+\lambda_1\frac{\overline{q}z}{(1-\overline{q}z)^2}\\&\qquad+\lambda_2\int_{\mathbb{T}\setminus\{\overline{q}\}}\biggl(\log\frac{1-\varkappa z}{1-\overline{q}z}+i\operatorname{Im}\{\varkappa q\}\frac{\overline{q}z}{1-\overline{q}z}\biggr)\frac{d\mu(\varkappa)}{1-\operatorname{Re}\{\varkappa q\}}\end{aligned}$$

with some $\beta\in\mathbb{R}$ and $\lambda_1,\lambda_2\geqslant 0$ and some probability measure μ on $\mathbb{T}\setminus\{\overline{q}\}$. Here the logarithm is understood as the continuous branch vanishing at $z=0$.

Note that convergence of the integral in the definition of the Koenigs function is ensured by the fact that

$$\lim_{\varkappa\to\overline{q}}\biggl\{\frac{1}{1-\operatorname{Re}\{\varkappa q\}}\biggl(\log\frac{1-\varkappa z}{1-\overline{q}z}+i\operatorname{Im}\{\varkappa q\}\frac{\overline{q}z}{1-\overline{q}z}\biggr)\biggr\}=\frac{\overline{q}z}{(1-\overline{q}z)^2}\,.$$

The problem of distinguishing those Koenigs functions F that correspond to one-parameter semigroups in $\mathfrak{P}$ with a given set of fixed points is solved in the following theorem.

Theorem 9. *Let q, $a_1, \dots, a_n$ be pairwise distinct points on the unit circle $\mathbb{T}$. A function F holomorphic in $\mathbb{D}$ is the Koenigs function of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ with Denjoy–Wolff point q and with fixed points $a_1, \dots, a_n$ at which the functions f^t, $t > 0$, have finite angular derivatives if and only if for some positive numbers $\lambda_1, \dots, \lambda_n$, some non-negative γ_1, γ_2, some $\beta \in \mathbb{R}$, and some probability measure μ on $\mathbb{T} \setminus \{\overline{q}\}$ the following equation holds*:

$$F(z) = i\beta \,\frac{\overline{q}z}{1-\overline{q}z} + \sum_{k=1}^{n} \lambda_k \log \frac{1-\overline{a}_k z}{1-\overline{q}z} + \gamma_1 \frac{\overline{q}z}{(1-\overline{q}z)^2} + \gamma_2 \int_{\mathbb{T}\setminus\{\overline{q}\}} \left(\log \frac{1-\varkappa z}{1-\overline{q}z} + i \operatorname{Im}\{\varkappa q\} \frac{\overline{q}z}{1-\overline{q}z} \right) \frac{d\mu(\varkappa)}{1-\operatorname{Re}\{\varkappa q\}}\,, \tag{8}$$

where the logarithms are understood as the continuous branches vanishing at $z = 0$.

Proof. Let $t \mapsto f^t$ be a one-parameter semigroup in $\mathfrak{P}$ with Denjoy–Wolff point $q \in \mathbb{T}$ and with fixed points $a_1, \dots, a_n$ at which the functions f^t have finite angular derivatives. Since $|q| = 1$, the infinitesimal generator v of this one-parameter semigroup can be represented in the form

$$v(z) = \frac{q(1-\overline{q}z)^2}{g(z)}\,,$$

where

$$g(z) = \sum_{k=1}^{n} \alpha_k \frac{1+\overline{a}_k z}{1-\overline{a}_k z} + p(z),$$

with $\alpha_k > 0$ for $k = 1, \dots, n$ and p a holomorphic function in $\mathbb{D}$ with non-negative real part. Let F be the Koenigs function of the one-parameter semigroup $t \mapsto f^t$, that is, the equation

$$F \circ f^t(z) = F(z) + t$$

holds for all $t \geqslant 0$. Differentiating this equation with respect to t and setting $t = 0$, we get that

$$F'(z) = \frac{1}{v(z)} = \frac{g(z)}{q(1-\overline{q}z)^2}\,.$$

Hence, since $\mathbb{D}$ is simply connected, the function F is uniquely determined in view of the condition $F(0) = 0$.

We now define F by (8) with parameters chosen as follows. By the Riesz–Herglotz theorem the function p can be represented in the form

$$p(z) = \int_{\mathbb{T}} \frac{1+\varkappa z}{1-\varkappa z}\, d\nu(\varkappa) + i\gamma,$$

where $\gamma \in \mathbb{R}$ and ν is a finite non-negative Borel measure on $\mathbb{T}$. We set

$$\lambda_k = \frac{\alpha_k}{1-\operatorname{Re}\{\overline{a}_k q\}}, \quad \beta = \gamma + \sum_{k=1}^{n} \lambda_k \operatorname{Im}\{\overline{a}_k q\}, \quad \gamma_1 = \nu(\{\overline{q}\}), \quad \gamma_2 = \nu(\mathbb{T}) - \gamma_1.$$

If $\gamma_2 = 0$, then the definition of μ does not matter. In the case $\gamma_2 > 0$ we define the probability measure μ on $\mathbb{T} \setminus \{\overline{q}\}$ by the equation

$$d\mu(\varkappa) = \frac{1}{\gamma_2}\, d\nu(\varkappa).$$

We now show that the function F defined with these parameters satisfies the condition $F'(z) = 1/v(z)$. This will prove the necessity of representation of the Koenigs function by (8). Straightforward calculations show that

$$\begin{aligned} F'(z) &= i\beta \frac{\overline{q}}{(1-\overline{q}z)^2} + \sum_{k=1}^{n} \lambda_k \frac{\overline{q} - \overline{a}_k}{(1-\overline{q}z)(1-\overline{a}_k z)} + \gamma_1 \frac{\overline{q}(1+\overline{q}z)}{(1-\overline{q}z)^3} \\ &\quad + \gamma_2 \int_{\mathbb{T}\setminus\{\overline{q}\}} \left(\frac{\overline{q} - \varkappa}{(1-\overline{q}z)(1-\varkappa z)} + i\,\mathrm{Im}\{\varkappa q\} \frac{\overline{q}}{(1-\overline{q}z)^2} \right) \frac{d\mu(\varkappa)}{1 - \mathrm{Re}\{\varkappa q\}} \\ &= \frac{\overline{q}}{(1-\overline{q}z)^2} \Bigg[i\beta + \sum_{k=1}^{n} \lambda_k \frac{(1-\overline{a}_k q)(1-\overline{q}z)}{1-\overline{a}_k z} + \gamma_1 \frac{1+\overline{q}z}{1-\overline{q}z} \\ &\quad + \gamma_2 \int_{\mathbb{T}\setminus\{\overline{q}\}} \left(\frac{(1-\varkappa q)(1-\overline{q}z)}{1-\varkappa z} + i\,\mathrm{Im}\{\varkappa q\} \right) \frac{d\mu(\varkappa)}{1-\mathrm{Re}\{\varkappa q\}} \Bigg]. \end{aligned}$$

Using the equations

$$\begin{aligned} \frac{(1-\overline{a}_k q)(1-\overline{q}z)}{1-\overline{a}_k z} &= (1 - \mathrm{Re}\{\overline{a}_k q\}) \frac{1+\overline{a}_k z}{1-\overline{a}_k z} - i\,\mathrm{Im}\{\overline{a}_k q\}, \\ \frac{(1-\varkappa q)(1-\overline{q}z)}{1-\varkappa z} &= (1 - \mathrm{Re}\{\varkappa q\}) \frac{1+\varkappa z}{1-\varkappa z} - i\,\mathrm{Im}\{\varkappa q\}, \end{aligned}$$

we can rewrite the representation obtained for $F'(z)$ in the form

$$\begin{aligned} F'(z) = \frac{\overline{q}}{(1-\overline{q}z)^2} \Bigg[& i\beta + \sum_{k=1}^{n} \lambda_k (1 - \mathrm{Re}\{\overline{a}_k q\}) \frac{1+\overline{a}_k z}{1-\overline{a}_k z} \\ & - i \sum_{k=1}^{n} \lambda_k \,\mathrm{Im}\{\overline{a}_k q\} + \gamma_1 \frac{1+\overline{q}z}{1-\overline{q}z} + \gamma_2 \int_{\mathbb{T}\setminus\{\overline{q}\}} \frac{1+\varkappa z}{1-\varkappa z}\, d\mu(\varkappa) \Bigg]. \end{aligned}$$

Taking into account the choice of the parameters, we now get that

$$\begin{aligned} F'(z) &= \frac{\overline{q}}{(1-\overline{q}z)^2} \left[i\gamma + \sum_{k=1}^{n} \alpha_k \frac{1+\overline{a}_k z}{1-\overline{a}_k z} + \int_{\mathbb{T}} \frac{1+\varkappa z}{1-\varkappa z}\, d\nu(\varkappa) \right] \\ &= \frac{\overline{q}}{(1-\overline{q}z)^2} \left[\sum_{k=1}^{n} \alpha_k \frac{1+\overline{a}_k z}{1-\overline{a}_k z} + p(z) \right] = \frac{1}{v(z)}\,. \end{aligned}$$

Thus, the necessity is proved.

The sufficiency of representation of the Koenigs function by (8) was established in [24] (see Proposition 3). □

The connection between the existence of fixed points of a one-parameter semigroup and the behaviour of the Koenigs function has been investigated by many authors. In this connection we point out the paper [32].

4. Evolution families and the evolution equation

From the viewpoint of dynamics the family of functions f^t, $t \geqslant 0$, of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}$ forms a semiflow. Furthermore, if v is the infinitesimal generator of this one-parameter semigroup, then the semiflow f^t, $t \geqslant 0$, is generated by the vector field $v(z)$, since $w = f^t(z)$ is a solution of the autonomous differential equation

$$\frac{dw}{dt} = v(w) \tag{9}$$

with the initial condition $w\big|_{t=0} = z$. In other words, the map $t \mapsto f^t(z)$ is an integral curve of the differential equation (9), and the map $z \mapsto f^t(z)$, $z \in \mathbb{D}$, effects a shift along the integral curves of this equation. As can be seen from the results in the preceding section, it is certainly not true that every function $f \in \mathfrak{P}$ can be embedded into a one-parameter semigroup. The natural replacement of the autonomous equation (9) by the non-autonomous equation

$$\frac{dw}{dt} = V(w,t), \tag{10}$$

where for every $t > 0$ the function $V(\,\cdot\,,t)$ is an infinitesimal transformation of $\mathfrak{P}$, must give rise to a wider class of embeddable functions f. Here we must pass from a one-parameter semigroup to a more general construction and specify conditions on the right-hand side of (10) that guarantee the existence and uniqueness of a solution of the Cauchy problem for this equation.

Definition 1. A two-parameter family $\{w_{t,s}\colon 0 \leqslant s \leqslant t \leqslant T\}$ in the semigroup $\mathfrak{P}$ is called an *evolution family* in $\mathfrak{P}$ if the following conditions hold:

(i) $w_{t,s}(z) = w_{t,\tau} \circ w_{\tau,s}(z)$ for $0 \leqslant s \leqslant \tau \leqslant t \leqslant T$,

(ii) $w_{t,s}(z) \to z$ locally uniformly in $\mathbb{D}$ as $(t-s) \to 0$.

Note that if $t \mapsto f^t$ is a one-parameter semigroup in $\mathfrak{P}$, then

$$\{w_{t,s} = f^{t-s}\colon 0 \leqslant s \leqslant t \leqslant T\}$$

is an evolution family in $\mathfrak{P}$. In contrast to the case of a one-parameter semigroup, the problem of differentiability of an evolution family is much more difficult to solve here. These questions were first considered by Löwner [3] for the semigroup $\mathfrak{L}$ of functions f holomorphic and univalent in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) > 0$, and $|f(z)| < 1$ for $z \in \mathbb{D}$. Obviously, a function f in $\mathfrak{L}$ has the origin as the Denjoy–Wolff point, and the infinitesimal transformations of $\mathfrak{L}$ are described by the formula $v(z) = -zp(z)$, where p is a holomorphic function in $\mathbb{D}$ with non-negative real part. Thus, the evolution equation (10) for $\mathfrak{L}$ takes the form

$$\frac{dw}{dt} = -wP(w,t), \tag{11}$$

where $P(\,\cdot\,,t)$, $0 \leqslant t \leqslant T$, is a family of holomorphic functions in $\mathbb{D}$ with non-negative real part.

Now let $\{w_{t,s}\colon 0 \leqslant s \leqslant t \leqslant T\}$ be an evolution family in the semigroup $\mathfrak{L}$. Then the family of functions $g(z,t) = w_{T,t}(z)$, $0 \leqslant t \leqslant T$, has the property that

$$g(z,t') = w_{T,t'}(z) = w_{T,t''} \circ w_{t'',t'}(z) = g\bigl(w_{t'',t'}(z), t''\bigr)$$

for $0 \leqslant t' \leqslant t'' \leqslant T$, that is, the domains $g(\mathbb{D}, t)$, $0 \leqslant t \leqslant T$, expand as t increases, and $g(\mathbb{D}, T) = \mathbb{D}$. Moreover, by Carathéodory's theorem and the condition (ii) this family of domains depends on t continuously in the sense of convergence to the kernel. Conversely, from a family of functions $g(z, t)$, $0 \leqslant t \leqslant T$, mapping the unit disk $\mathbb{D}$ to a family of domains with the properties listed above and normalized by the conditions $g(0, t) = 0$, $g'(0, t) > 0$, the evolution family $\{w_{t,s} \colon 0 \leqslant s \leqslant t \leqslant T\}$ can be uniquely reconstructed from the equation

$$w_{t,s}(z) = g^{-1}(g(z, s), t).$$

This observation makes it possible to construct evolution families in $\mathfrak{L}$. The case where the domains $g(\mathbb{D}, t)$ are obtained by removing from $\mathbb{D}$ the cut along a Jordan arc was studied in detail by Löwner [3]. Here the functions $g(z, t)$ were normalized by the condition

$$\frac{d}{dz} g(z, t)\bigg|_{z=0} = g'(0, t) = e^{t-T},$$

$0 \leqslant t \leqslant T = \log(1/\beta)$, where $\beta = g'(0, 0)$, and the corresponding evolution family satisfied the condition

$$w'_{t,s}(0) = e^{s-t},$$

$0 \leqslant s \leqslant t \leqslant T$. For this normalization the problem of differentiability with respect to t of the evolution family and the function $g(z, t)$ is easily solved by using the following lemma of Löwner [3].

Lemma 1. *Suppose that $f \in \mathfrak{L}$ and $f'(0) = \beta$. Then for all $z \in \mathbb{D}$*

$$|f(z) - z| \leqslant (1 - \beta)|z| \frac{1 + |z|}{1 - |z|}.$$

This lemma shows that a neighbourhood of the identity transformation (with respect to the topology of locally uniform convergence in $\mathbb{D}$) in the semigroup $\mathfrak{L}$ is described in terms of the quantity $f'(0)$. Moreover, Löwner showed that the family of functions $g(z, t)$, $0 \leqslant t \leqslant T$, satisfies the differential equation

$$\frac{\partial}{\partial t} g(z, t) = z P(z, t) \frac{\partial}{\partial z} g(z, t), \tag{12}$$

in which the function P has the form

$$P(z, t) = \frac{1 + \varkappa(t) z}{1 - \varkappa(t) z}, \tag{13}$$

where $\varkappa(t)$ is a continuous unimodular (that is, $|\varkappa(t)| \equiv 1$) function on $[0, T]$. The evolution family $\{w_{t,s} \colon 0 \leqslant s \leqslant t \leqslant T\}$ is then reproduced by the evolution equation (11) with the same function $P(z, t)$. More precisely, $w = w_{t,s}(z)$ is the solution of equation (11) with the initial condition $w\big|_{t=s} = z$, $0 \leqslant s \leqslant t \leqslant T$.

Later Kufarev [33] and Pommerenke [34], [35] studied the question of differentiability of a family of conformal maps $g(z, t)$, $t \geqslant 0$, of $\mathbb{D}$ onto expanding domains in a more general setting. In particular, it was established that the appropriately normalized family of functions $g(z,t)$, $t \geqslant 0$, satisfies equation (12) with a function $P(z,t)$ that is measurable with respect to t and holomorphic with respect to z in $\mathbb{D}$ and has non-negative real part. In this connection, equations (11) and (12) with a function $P(z,t)$ of the form (13) are often associated in the literature with the name of Löwner, and they are called the Löwner–Kufarev equations in the case when $P(z,t)$ is measurable with respect to t and holomorphic with respect to z in $\mathbb{D}$ and has non-negative real part. In [36] it was shown that the family of functions $g(z,t)$ and the corresponding evolution family that are reproduced by equations (11), (12) with a function $P(z,t)$ of the form (13) do not have to consist of maps of the unit disk onto domains obtained by erasing a cut. This question again attracted attention (see, for example, [37]–[39]) in connection with the development of stochastic Löwner evolution, also known in the literature as SLE (see [40] and the literature cited therein).

The following result gives a fairly general existence theorem for the evolution equation (11) (see [41]).

Theorem 10. *Suppose that a complex-valued function $P(z,t)$ with non-negative real part is defined on $\mathbb{D}\times[0,T]$, $T>0$, and is holomorphic with respect to z, measurable with respect to t, and such that $|P(0,t)|$ is an integrable function on $[0,T]$. Then for any $z\in\mathbb{D}$ and $s\in[0,T)$ there exists a unique solution $w=w(t,z,s;P)$ of equation (11) that is absolutely continuous on $[s,T]$ and satisfies the initial condition $w\big|_{t=s}=z$. Furthermore, $w^P_{t,s}\colon z\mapsto w(t,z,s;P)$ realizes a conformal map of the unit disk $\mathbb{D}$ into itself fixing the origin.*

Thus, a function $P(z,t)$ satisfying the hypotheses of Theorem 10 defines an evolution family $\{w^P_{t,s}\colon 0\leqslant s\leqslant t\leqslant T\}$ in $\mathfrak{P}$. If we require in addition that the condition $\operatorname{Im}P(0,t)\equiv 0$ holds, then we obtain an evolution family in the semigroup $\mathfrak{L}$. Löwner showed that any $f\in\mathfrak{L}$ can be approximated in the topology of locally uniform convergence of functions holomorphic in $\mathbb{D}$ by the transformations $w^P_{T,0}$, $T\geqslant 0$, with functions $P(z,t)$ of the form (13). This result was given a definitive form thanks to investigations of Kufarev [33], Pommerenke [34], and Gutlyanskii [42]: every map f in $\mathfrak{L}$ can be represented in the form $f=w^P_{T,0}$ with some $T\geqslant 0$ and a function $P(z,t)$ that is measurable with respect to t on $[0,T]$ and holomorphic with respect to z in $\mathbb{D}$, has non-negative real part, and is such that $P(0,t)\equiv 1$. In other words, every function f of the semigroup $\mathfrak{L}$ can be embedded into some evolution family in $\mathfrak{L}$ that is reproduced by an evolution equation. Furthermore, $P(z,t)$ in the statement of Theorem 10 can be called the *infinitesimal generating function* of the evolution family $\{w^P_{t,s}\colon 0\leqslant s\leqslant t\leqslant T\}$.

The problem of a topology in the space of infinitesimal generating functions that corresponds to locally uniform convergence in $\mathbb{D}$ of evolution families was solved in [43], [44] (see also [41]). We present the corresponding definitions and some results.

Let D be a domain in the finite complex plane $\mathbb{C}$, and let $\mathscr{H}(D)$ be the linear space of all holomorphic functions in D endowed with the topology of locally uniform convergence in D. In fact, $\mathscr{H}(D)$ is a metrizable complete locally convex space, or a Fréchet space. Therefore, many basic results about compact convex sets (see, for example, [45], [46]) are valid in the space $\mathscr{H}(D)$. In particular, if $\mathscr{K}\subset\mathscr{H}(D)$ is a compact subset, then its closed convex hull $\overline{\operatorname{co}}\,\mathscr{K}$ is also compact, and by the Krein–Milman theorem,

$$\overline{\mathrm{co}}\,\mathscr{K} = \overline{\mathrm{co}}\,\mathrm{ext}\,\mathscr{K}, \qquad \mathrm{ext}\,\overline{\mathrm{co}}\,\mathscr{K} \subseteq \mathrm{ext}\,\mathscr{K} \subseteq \mathscr{K},$$

where $\mathrm{ext}\,\mathscr{K}$ is the set of extreme points of $\mathscr{K}$.

Next, let $I = [\alpha, \beta] \subset \mathbb{R}$ and let $\mathfrak{F}(D, I)$ be the set of complex-valued functions $F(z,t)$ on $D \times I$ that are holomorphic with respect to z, measurable with respect to t, and such that for any compact subset Δ of the domain D the function

$$M(t; F, \Delta) = \sup\{|F(z,t)| \colon z \in \Delta\}$$

is integrable on I. Obviously, $\mathfrak{F}(D, I)$ is a linear space over the field of complex numbers. Moreover, standard arguments using the Cauchy integral formula show that the operation of differentiation with respect to z transforms the space $\mathfrak{F}(D, I)$ into itself, and the integral $\int_I F(z,t)\,dt$ is a holomorphic function in D.

Definition 2. Let F, $F_n \in \mathfrak{F}(D, I)$, $n = 1, 2, \dots$. We say that the sequence $\{F_n\}$ converges weakly in $\mathfrak{F}(D, I)$ to the function F if

$$\int_I \eta(t) F_n(z,t)\,dt \to \int_I \eta(t) F(z,t)\,dt$$

locally uniformly in D as $n \to \infty$ for any bounded measurable function $\eta(t)$ on I.

The significance of this definition for evolution families is revealed in the following result [43], [41].

Theorem 11. *Suppose that complex-valued functions $P_n(z,t)$, $n = 1, 2, \dots$, are defined on $\mathbb{D} \times [0, T]$, $T > 0$, have non-negative real parts, and are holomorphic with respect to z, measurable with respect to t, and such that*

$$|P_n(z,t)| \leqslant M(t)$$

for almost all $t \in [0, T]$ and $n = 1, 2, \dots$, where $M(t)$ is an integrable function on $[0, T]$. Then $P_n \in \mathfrak{F}(\mathbb{D}, [0, T])$, $n = 1, 2, \dots$, and the following assertions are equivalent:

(a) *$\{P_n\}$ converges weakly in $\mathfrak{F}(\mathbb{D}, [0, T])$ to a function $P \in \mathfrak{F}(\mathbb{D}, [0, T])$;*

(b) *the function $P(z,t)$ satisfies the same requirements as the functions P_n, $n = 1, 2, \dots$, and $w_{t,s}^{P_n}(z) \to w_{t,s}^{P}(z)$ locally uniformly in $\mathbb{D}$ as $n \to \infty$ for all $s, t \in [0, T]$ with $s \leqslant t$.*

We point out some important properties of weak convergence in $\mathfrak{F}(D, I)$. In particular, the following compactness principle holds [43], [41].

Theorem 12. *Suppose that $F_n \in \mathfrak{F}(D, I)$, $n = 1, 2, \dots$, and for every compact subset $\Delta \subset D$ there exists a function $M_\Delta(t)$ that is integrable on I and such that*

$$|F_n(z,t)| \leqslant M_\Delta(t)$$

for almost all $t \in I$, $z \in \Delta$, and $n = 1, 2, \dots$. Then the sequence $\{F_n\}$ contains a subsequence converging weakly in $\mathfrak{F}(D, I)$. Moreover, the limit function $F(z,t)$ of any weakly converging subsequence of $\{F_n\}$ satisfies the condition

$$F(\,\cdot\,, t) \in \overline{\mathrm{co}}\,\{F_n(\,\cdot\,, t) \colon n = 1, 2, \dots\}$$

for almost all $t \in I$.

Let $\mathscr{K} \subset \mathscr{H}(D)$ be some set of holomorphic functions in D. We say that a function $F(z,t)$ in $\mathfrak{F}(D,I)$ is $\mathscr{K}$-valued if $F(\,\cdot\,,t) \in \mathscr{K}$ for almost all $t \in I$. In connection with this notion we can state an assertion that follows immediately from the preceding theorem.

Corollary 1. *Let $\mathscr{K}$ be a compact convex subset of the space $\mathscr{H}(D)$. Then the set of $\mathscr{K}$-valued functions is compact (more precisely, sequentially compact) with respect to weak convergence in $\mathfrak{F}(D,I)$.*

We say that a function $F(z,t)$ in $\mathfrak{F}(D,I)$ is a step function if there exist a finite partition $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ of the closed interval $I = [\alpha,\beta]$ and a set of functions $f_1,\dots,f_n$ in $\mathscr{H}(D)$ such that $F(z,t) \equiv f_k(z)$ for $t \in (t_{k-1},t_k]$, $k = 1,\dots,n$. The following result was established in [47] (see also [44], [41]).

Theorem 13. *Let $\mathscr{K}$ be a compact subset of $\mathscr{H}(D)$, and let $\mathscr{Q} = \operatorname{ext}\overline{\operatorname{co}}\,\mathscr{K}$ be the set of extreme points of its closed convex hull. Then for any $\mathscr{K}$-valued function $F(z,t)$ in $\mathfrak{F}(D,I)$ there exists a sequence of $\mathscr{Q}$-valued step functions $F_n(z,t)$, $n = 1,2,\dots$, such that $\{F_n\}$ converges weakly in $\mathfrak{F}(D,I)$ to F.*

Results on weak convergence of infinitesimal generating functions give rise naturally to an infinitesimal description of the semigroup $\mathfrak{L}$ and make it possible to better understand its structure. As already noted, Löwner showed that every function f in $\mathfrak{L}$ can be approximated in the topology of locally uniform convergence by the maps $w^P_{T,0}$ with some $T > 0$ and some generating function $P(z,t)$ of the form (13). To look at this result from a somewhat different viewpoint, we introduce the class $\mathscr{C}$ of functions p holomorphic in the unit disk $\mathbb{D}$ and satisfying the conditions $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$ and $p(0) = 1$. In the literature this class of functions is often called the Carathéodory class. We note that $\mathscr{C}$ is a convex compact subset of the space $\mathscr{H}(\mathbb{D})$, and the set of its extreme points has the form

$$\operatorname{ext}\mathscr{C} = \left\{p(z) = \frac{1+\varkappa z}{1-\varkappa z} \colon \varkappa \in \mathbb{T}\right\}.$$

In particular, the Riesz–Herglotz formula can be interpreted from the viewpoint of convex analysis as a realization of the Krein–Milman and Choquet theorems (see, for example, [48]). Thus, the maps $w^P_{T,0}$, $T > 0$, with $\operatorname{ext}\mathscr{C}$-valued infinitesimal generating functions $P(z,t)$ form a dense subset of $\mathfrak{L}$ with respect to locally uniform convergence in $\mathbb{D}$. We also note that $(w^P_{t,0})'(0) = e^{-t}$ in the case when $P(\,\cdot\,,t) \in \mathscr{C}$ for almost all t, as follows immediately from equation (11). Moreover, for any $\beta \in (0,1)$ the set

$$\{f \in \mathfrak{L} \colon f'(0) = \beta\}$$

is compact with respect to the topology of locally uniform convergence in $\mathbb{D}$. Further, it follows from Theorem 12 that the set of $\mathscr{C}$-valued functions in the space $\mathfrak{F}(\mathbb{D},[0,T])$ forms a compact set with respect to weak convergence. This fact and Theorem 11 show that for a fixed $T > 0$ the set of maps $w^P_{T,0}$ with $\mathscr{C}$-valued infinitesimal generating functions $P(z,t)$ also forms a compact subset of $\mathfrak{L}$. Summarizing the foregoing, we arrive at the following result.

Theorem 14. *For a function f holomorphic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = \beta > 0$ to belong to the semigroup $\mathfrak{L}$ it is necessary and sufficient that it admits a representation in the form*

$$f(z) = w^P_{T,0}(z),$$

where $T = -\log\beta$ and the function $P(z,t)$ is holomorphic with respect to z in $\mathbb{D}$, measurable with respect to t on $[0,T]$, and such that $P(\,\cdot\,,t) \in \mathscr{C}$ for almost all $t \in [0,T]$.

As can be seen from the preceding, the cone of infinitesimal transformations of $\mathfrak{L}$ is described by the formula $v(z) = -\alpha z p(z)$, where $\alpha \geqslant 0$ and $p \in \mathscr{C}$. From the viewpoint of an infinitesimal description of the semigroup, the most natural way of defining subsemigroups consists in restricting the cone of infinitesimal transformations. In this connection we introduce some definitions and notation.

Let $\mathscr{K}$ be some subset of the Carathéodory class $\mathscr{C}$. Let $\mathfrak{L}(\mathscr{K})$ denote the set of maps f in $\mathfrak{L}$ of the form $f(z) = w^P_{T,0}(z)$, where $T \geqslant 0$ and the infinitesimal generating function $P(z,t)$ satisfies the condition $P(\,\cdot\,,t) \in \mathscr{K}$ for almost all $t \in [0,T]$. By Theorem 14, $\mathfrak{L}(\mathscr{C}) = \mathfrak{L}$. We present two results on the structure of the semigroup $\mathfrak{L}(\mathscr{K})$.

Theorem 15 (see [47], [41]). *Let $\mathscr{K}$ be a closed convex proper subset of the class $\mathscr{C}$. Then $\mathfrak{L}(\mathscr{K})$ is a closed proper subsemigroup of $\mathfrak{L}$.*

By closedness of the subsemigroup $\mathfrak{L}(\mathscr{K})$ we mean the following property: if a sequence of functions f_n, $n = 1, 2, \dots$, in $\mathfrak{L}(\mathscr{K})$ converges locally uniformly in $\mathbb{D}$ to some function f in $\mathfrak{L}$, then the limit function f also belongs to $\mathfrak{L}(\mathscr{K})$.

Now let h be an arbitrary function in $\mathscr{C}$. If we set $P(z,t) \equiv h(z)$, then the family of maps

$$\phi^h_t(z) = w^P_{t,0}(z),$$

$t \geqslant 0$, can be regarded as the one-parameter semigroup $t \mapsto \phi^h_t$ in $\mathfrak{L}$ with the infinitesimal generator $v(z) = -zh(z)$. We state the following result in this notation.

Theorem 16 (see [47], [41]). *Let $\mathscr{K}$ be a closed convex subset of the class $\mathscr{C}$, and let $\mathscr{Q} = \operatorname{ext}\mathscr{K}$ be the set of its extreme points. Then the functions of the form*

$$\phi(z) = \phi^{h_1}_\tau \circ \dots \circ \phi^{h_n}_\tau(z),$$

where $\tau \geqslant 0$ and $h_k \in \mathscr{Q}$ $(k = 1, \dots, n;\, n = 1, 2, \dots)$, form a dense subset of $\mathfrak{L}(\mathscr{K})$ with respect to the topology of locally uniform convergence in $\mathbb{D}$.

In the case $\mathscr{K} = \mathscr{C}$ the subsemigroup $\mathfrak{L}(\mathscr{K})$ coincides with the whole semigroup $\mathfrak{L}$. In this case $\mathscr{Q}$ consists of the functions

$$g_\varkappa(z) = \frac{1 + \varkappa z}{1 - \varkappa z}\,,$$

$\varkappa \in \mathbb{T}$. Furthermore,

$$\phi^{g_\varkappa}_\tau(z) = K^{-1}_\varkappa(e^{-\tau} K_\varkappa(z)),$$

where

$$K_\varkappa(z) = \frac{z}{(1 + \varkappa z)^2}$$

is the Koebe function. Hence, it is evident that $\phi_\tau^{g_\varkappa}$ maps $\mathbb{D}$ onto the domain that is obtained from the unit disk by making a radial cut starting at the point $\overline{\varkappa}$ of the unit circle. The length of the cut depends on the quantity τ. In this special case the result of Theorem 16 coincides with the lemma in [5] that de Branges used in the original version of the proof of the Bieberbach conjecture.

Evolution equations and the question of embeddability into evolution families for semigroups of conformal maps of a strip and a half-plane were studied in [44], [49]. We also point out that evolution families of conformal maps and the Löwner–Kufarev equation are widely used in various applications (see, for example, [50], [51]).

5. The parametric method in the theory of univalent functions

By a univalent function in a domain $D \subset \mathbb{C}$ we mean an analytic function f mapping this domain injectively onto another domain of the complex plane, that is, $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$, $z_1, z_2 \in D$. These functions play an important role both from the viewpoint of the theory of conformal maps and from the viewpoint of applications. In particular, in applications univalency is often associated with physical realizability of a mathematical model. A central role in the theory of univalent functions is played by extremal problems. The conformal mapping problem itself can be regarded as an extremal problem. Moreover, many problems connected with the geometry of conformal mapping reduce to estimating functionals and finding the ranges of values of systems of functionals depending on the values of a univalent function and its derivatives at a given point. A number of results in this direction have come to be called distortion theorems, rotation theorems, growth theorems, and so on, in connection with their geometric meaning. It should also be noted that extremal problems in the theory of univalent functions do not fit in the framework of the classical calculus of variations, and for several decades the investigation of them has stimulated the emergence and refinement of a number of subtle special methods. We do not go into a detailed description of all the methods and results in the theory of univalent functions. Many surveys have been devoted to this (see, for example, [52]–[54]). We only point out some special features of several methods.

One of the basic objects of study in the theory of univalent functions is the class $\mathscr{S}$ of functions f that are holomorphic and univalent in $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The essential non-linearity of this object creates the main difficulties in constructing an effective calculus of variations on it. The first results in the theory of univalent functions were obtained by the area method, which goes back to papers of Grönwall, Bieberbach, and Faber and is based on the simple geometric principle of non-negativity of area, which is expressed, as a rule, in terms of the coefficients of an expansion of the function itself or of a related analytic expression with respect to some orthonormal system. A fairly comprehensive idea of this method and its applications can be obtained from the monographs [55], [56]. It was by the area method that a sharp estimate for the second coefficient of the Taylor expansion of a function of class $\mathscr{S}$ was first obtained, and this prompted Bieberbach to state in 1916 his conjecture that $|c_n| \leqslant n$ for any function $f(z) = z+c_2z^2+\cdots$ of class $\mathscr{S}$. For a long time this conjecture determined the direction of development of the theory of univalent functions. The first progress in settling the conjecture was made only in 1923 by Löwner [3], who confirmed it for $n = 3$ and laid the foundations for a new method which subsequently became known as the parametric method. This was the method that de Branges used [5], [6] to obtain a complete solution for the Bieberbach conjecture in 1984.

The idea of the variational method is closer to the classical calculus of variations and is connected with a description of the local structure of classes of univalent functions in the space of all holomorphic functions. The class $\mathscr{S}$ is a compact subset of the linear space $\mathscr{H}(\mathbb{D})$ with a metrizable topology. Variation of a function $f \in \mathscr{S}$ (or the variational formula) consists in choosing directions in $\mathscr{H}(\mathbb{D})$ such that a shift along them from f leads to functions whose distance to the class $\mathscr{S}$ is a quantity of higher order of smallness than the length of the shift. The known methods differ by the way of choosing such directions: the variational-geometric method of Lavrent'ev [57], the method of boundary variations of Schiffer [58], and the Schiffer–Goluzin method of interior variations [52]. Variation of functions of class $\mathscr{S}$ gives rise to variation of Gâteaux differentiable functionals and to necessary conditions for a local extremum which are expressed in the form of a differential equation for extremal functions. For example, analysis of the Schiffer–Goluzin equation obtained by the method of interior variations yields qualitative information about extremals. But carrying the solution of the problem to completion is often hampered by the complexity of the equation obtained and by the presence of unknown parameters in it. Moreover, from the viewpoint of the general calculus of variations the Schiffer–Goluzin equation is only a necessary condition for an extremum and does not even distinguish local maxima and minima. In the classical calculus of variations the question of distinguishing local maxima and minima is settled by using the second variation. In problems in the theory of univalent functions the situation is substantially complicated by the fact that the space $\mathscr{H}(\mathbb{D})$ is infinite-dimensional and by difficulties connected with the study of the structure of the class $\mathscr{S}$ as a subset of this space. Nevertheless, the fruitfulness of considering the second variation was demonstrated in [59].

The method of extremal metrics involves analysing differential-geometric singularities of extremal conformal maps. In the papers of Grötzsch and Teichmüller this method was based on inequalities connecting length and area. An important role in its development was played by the notion of the extremal length (or modulus) of a family of curves, introduced by Ahlfors and Beurling. Application of this method is based on establishing a connection between the problem under consideration in the theory of univalent functions and a certain extremal-metric problem. Here an extremal metric appears as a solution of the corresponding extremal-metric problem and is generated by a certain quadratic differential. It is via a quadratic differential that there is a connection between the method of extremal metrics and the variational method, which mutually complement each other in a number of studies. One can get an idea about the method of extremal metrics from the monograph [60] and the survey [54].

The geometric aspect of the method of extremal metrics makes it possible to combine it also with the method of symmetrization. The idea of symmetrization arose in the investigation of problems in geometry and mathematical physics. Due to geometric and physical interpretations of analytic functions and conformal maps, symmetrization found applications in the theory of univalent functions (see, for example, [60]–[63]). The current state of the method of symmetrization and related questions can be found in the recent monograph [64].

In many respects the difficulties of constructing an effective calculus of variations on classes of univalent functions are due to the essential non-linearity of the object. The sum of two univalent functions is not necessarily univalent. On the other hand, the operation of composition (if it is defined) does not violate univalency. It is this fact to which Löwner points when singling out the semigroup $\mathfrak{L}$. As noted above, Löwner solved the problem of an infinitesimal description of $\mathfrak{L}$ approximatively. Furthermore, he showed that the limit

$$f(z)=\lim_{t\to\infty} e^t w^P_{t,0}(z) \tag{14}$$

exists and is a function f in the class $\mathscr{S}$ for any infinitesimal generating function $P(z,t)$ of the form (13) on $[0,\infty)$. Moreover, the functions f thus obtained form a dense subset $\mathscr{S}_L$ of $\mathscr{S}$. Consequently, the sharp estimates for functionals continuous with respect to the topology of locally uniform convergence in $\mathbb{D}$ coincide on the classes $\mathscr{S}$ and $\mathscr{S}_L$. This circumstance lies at the basis of the classical version of the parametric method. By expressing the Taylor coefficients of a function f of class $\mathscr{S}_L$ in terms of the unimodular function $\varkappa(t)$ in (13), Löwner confirmed the Bieberbach conjecture for the first time in the case $n=3$.

New possibilities for the parametric method were opened by Goluzin's paper [65], in which he found an unexpected application of it to the rotation problem. This problem also arose from Bieberbach's paper [66], where he established the estimate

$$|\arg f'(z_0)|\leqslant 2\log\frac{1+|z_0|}{1-|z_0|}$$

for functions $f\in\mathscr{S}$, where the branch of the argument vanishing at $z_0=0$ is considered. We can assume without loss of generality that $z_0=r$, $0<r<1$. Geometrically, the quantity $\arg f'(r)$ expresses the angle through which the tangent rotates as the point moves along the image of the closed interval $[0,r]$ under mapping by the function f. The inequality obtained by Bieberbach proved to be unattainable in the class $\mathscr{S}$. The rotation problem involves the question of sharp inequalities for $\arg f'(r)$. Using the parametric method for such problems for the first time, Goluzin obtained the estimates

$$\arg f'(r)\leqslant\begin{cases} 4\arcsin r & \text{for}\quad 0<r\leqslant\dfrac{1}{\sqrt{2}}\,,\\ \pi+\log\dfrac{r^2}{1-r^2} & \text{for}\quad \dfrac{1}{\sqrt{2}}<r<1. \end{cases}$$

The sharpness of the inequality for $r\leqslant 1/\sqrt{2}$ was established by producing an extremal function. The problem of sharpness of the inequalities for $r>1/\sqrt{2}$ was first solved by Bazilevich [67]. Later Goluzin used the parametric method to derive the main results in the theory of univalent functions in a uniform way. Various aspects of this method and its applications are reflected in many papers (see, for example, [19], [35], [52], [63], [68], [69]). Here it has often been noted that the parametric method is good for obtaining sharp estimates but does not give information about the uniqueness of extremals or their description.

The difficulties arising in solving the problem of uniqueness of extremals in the framework of the parametric method were due first and foremost to the fact that the extremal problem was being considered on the class $\mathscr{S}_L$. It followed from results in [34], [42] that if in (14) the infinitesimal generating function $P(z,t)$ of the special form (13) is replaced by an arbitrary measurable $\mathscr{C}$-valued function, then a parametric representation of the entire class $\mathscr{S}$ is obtained. This result can be stated in the following form (see, for example, [35]).

Theorem 17. *For a function f holomorphic in $\mathbb{D}$ to belong to the class $\mathscr{S}$ it is necessary and sufficient that it can be represented in the form (14) with some function $P(z,t)$ that is holomorphic with respect to z in $\mathbb{D}$, measurable with respect to t in $[0,\infty)$, and such that $P(\,\cdot\,,t)\in\mathscr{C}$ for almost all $t\in[0,\infty)$.*

As mentioned in the preceding section, restriction of the cone of infinitesimal transformations of the semigroup $\mathfrak{L}$ leads to subsemigroups $\mathfrak{L}(\mathscr{K})$. Similarly, restriction of the set of infinitesimal generating functions $P(z,t)$ in (14) results in singling out subclasses of univalent functions. More precisely, let $\mathscr{K}$ be some subset of the Carathéodory class $\mathscr{C}$. Let $\mathscr{S}(\mathscr{K})$ denote the class of functions f that can be represented by (14) with some function $P(z,t)$ holomorphic with respect to z in $\mathbb{D}$, measurable with respect to t on $[0,\infty)$, and such that $P(\,\cdot\,,t)\in\mathscr{K}$ for almost all $t\in[0,\infty)$. Concretization of the subset $\mathscr{K}\subset\mathscr{C}$ gives rise to various known classes of univalent functions. The structure of these classes was studied in [47], [41]. In particular, it turned out that the closure of the classes $\mathscr{S}(\mathscr{K})$ is described by the same structure.

Theorem 18. *Let $\mathscr{K}$ be some subset of the class $\mathscr{C}$. Then the closure $\overline{\mathscr{S}(\mathscr{K})}$ of the class $\mathscr{S}(\mathscr{K})$ in the topology of locally uniform convergence in $\mathbb{D}$ coincides with the class $\mathscr{S}(\overline{\mathrm{co}}\,\mathscr{K})$, that is,*

$$\overline{\mathscr{S}(\mathscr{K})}=\mathscr{S}(\overline{\mathrm{co}}\,\mathscr{K}),$$

where $\overline{\mathrm{co}}\,\mathscr{K}$ is the closed convex hull of the set $\mathscr{K}$.

As mentioned above, the first obstacle in the analysis of the question of uniqueness of extremals was the fact that the classical version of the parametric method was based on a parametric representation only of a dense subset $\mathscr{S}_L$ of the class $\mathscr{S}$. A parametric representation of the class of all univalent functions made it possible to solve the problem of uniqueness of extremals in some special cases. However, difficulties of a different nature appear in many problems in the theory of univalent functions. This is connected with the fact that one and the same function f of class $\mathscr{S}$ can be obtained by the formula (14) with essentially different infinitesimal generating functions $P(z,t)$. It turned out to be possible to overcome these difficulties by using the semigroup structure of the maps $w^P_{t,s}$ in [70]. The development of this approach also made it possible to obtain a uniqueness theorem for boundary functions of the system

$$I(f)=(I_1(f),\dots,I_4(f))$$

of real-valued functionals defined on the class $\mathscr{S}$ by the formulae

$$I_1(f) + iI_2(f) = \log \frac{f(z_0)}{z_0}, \qquad I_3(f) + iI_4(f) = \log \frac{z_0 f'(z_0)}{f(z_0)},$$

where z_0 is a given point of the unit disk $\mathbb{D}$, and the logarithms are understood as the continuous branches vanishing at $z_0 = 0$.

Many intrinsic problems in the theory of univalent functions are included in the study of the ranges of values of systems of functionals depending on the values of a univalent function and its derivative [71]. A wide range of problems including distortion and rotation theorems reduce to the study of the set

$$D(z_0) = \{I(f) \colon f \in \mathscr{S}\}.$$

In particular, the rotation theorem consists in a sharp estimate for the sum $I_2(f) + I_4(f)$.

Since $\mathscr{S}$ is compact and the functionals $I_1, \dots, I_4$ are continuous with respect to the topology of locally uniform convergence in $\mathbb{D}$, it follows that $D(z_0)$ is a closed bounded subset of $\mathbb{R}^4$. A function f in $\mathscr{S}$ is called a boundary function with respect to the system I if the point $I(f)$ belongs to the boundary of the set $D(z_0)$. Information about the boundary of $D(z_0)$ makes it possible to obtain estimates for concrete functionals depending on the values of a function and its derivative in the class $\mathscr{S}$, and the extremals of these functionals are boundary functions. A description of the boundary of $D(z_0)$ in a closed form was first given in [72]. Later a parametrization of the system I was obtained in [42] which made it possible to prove the convexity of the domain $D(z_0)$ and to obtain a simpler description of it. In [73], [74] the uniqueness of boundary functions was established and a description of them was given. In particular, the following result holds.

Theorem 19. *Let $z_0 \in \mathbb{D}$, $z_0 \neq 0$. Then $D(z_0)$ is a closed convex bounded set in $\mathbb{R}^4$ each boundary point of which is due to a unique function of class $\mathscr{S}$.*

The full statement of this result (see [73], [74]) also contains a description of the form of the boundary functions. The presence of rectilinear line segments on the boundary of $D(z_0)$ is the reason for the non-uniqueness of extremals in concrete problems on estimates for functionals. In particular, this explains the non-uniqueness of extremals in the rotation theorem for $1/\sqrt{2} < |z_0| < 1$. In this case the extremal functions form a one-parameter family. Every function in this family contributes a point of a line segment on the boundary of $D(z_0)$. The semigroup properties of extremal evolution families made it possible to determine the structure of the extremals and obtain a complete description of them.

6. Semigroups of probability generating functions and branching processes

The goal of this section is to show the connections between the dynamics of holomorphic maps and some questions in the theory of branching random processes. We hold to the terminology of the well-known monographs [75]–[77], where one can also find the requisite information about branching processes.

6.1. The embedding problem. As a rule, branching processes describe the development of a population of particles of a single type which, independently of one another, reproduce other particles of the same type. Particles may mean animals, bacteria, neutrons in chain reactions, and so on. The simplest model of a branching process with discrete time is a so-called Galton–Watson process. It is assumed that at the initial moment of time the population consists of a single particle which in the next generation turns into k particles with probability $p_k \geqslant 0$, $k = 0, 1, \dots$, $\sum_{k=0}^{\infty} p_k = 1$. The particles in the first generation, independently of one another, each form the second generation with the same probability law, and so on. Let $\xi_0 = 1$, $\xi_1, \xi_2, \dots$ denote the number of particles in the zeroth, first, second, ... generation, respectively. Then $\xi_0, \xi_1, \xi_2, \dots$ form a homogeneous Markov chain. The initial data of this process is assumed to be the probability distribution $\mathsf{P}(\xi_1 = k) = p_k$, $k = 0, 1, 2, \dots$. An important tool in the theory of branching processes is given by probability generating functions. In particular, a Galton–Watson process is completely determined by the probability generating function of the random variable ξ_1:

$$f(z) = \mathsf{E} z^{\xi_1} = \sum_{k=0}^{\infty} p_k z^k.$$

The assumptions made about the process imply that the generating function of the random variable ξ_n expressing the size of the nth generation turns out to be the nth iterate of the function f, that is,

$$\mathsf{E} z^{\xi_n} = f^n(z) = f \circ \dots \circ f(z).$$

Thus, the probability generating function f can be regarded as the initial data of the process, and the dynamics of the process is described by positive-integer iterates of f.

The possibility of embedding the Galton–Watson process with generating function f into a homogeneous Markov branching process with continuous time is equivalent (see [75]) to the existence of fractional iterates of f, or to the possibility of embedding this function into a one-parameter semigroup $t \mapsto f^t$. Here it is required that all the iterates f^t, $t \geqslant 0$, are also probability generating functions. We introduce some definitions and state this problem in a somewhat more precise form.

Let $\mathfrak{P}^+$ denote the set of all probability generating functions f, that is, those that can be represented as the sum of a power series $f(z) = \sum_{k=0}^{\infty} p_k z^k$ with $p_k \geqslant 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Obviously, a probability generating function f is holomorphic in the unit disk $\mathbb{D}$ and continuous in its closure $\overline{\mathbb{D}}$. Moreover, $|f(z)| \leqslant 1$ for $z \in \mathbb{D}$, and if f and g are two probability generating functions, then the composition $h = f \circ g$ is also a probability generating function. Thus, $\mathfrak{P}^+$ is a semigroup with respect to the operation of composition. If we exclude from $\mathfrak{P}^+$ the function $f(z) \equiv 1$, which corresponds to the degenerate random variable taking the value zero with probability 1, then $\mathfrak{P}^+$ becomes a proper subsemigroup of $\mathfrak{P}$. As in [78], we make the concept of embeddability more precise by using the following definition.

Definition 3. A probability generating function f is said to be embeddable if there exists a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}^+$ such that $f^1 = f$. The class of all embeddable probability generating functions is denoted by $\mathscr{E}(\mathfrak{P}^+)$.

In other words, a probability generating function f belongs to the class $\mathscr{E}(\mathfrak{P}^+)$ if and only if it defines a Galton–Watson process admitting an embedding into a homogeneous Markov branching process with continuous time. The class of embeddable probability generating functions was first considered in [79], where the problem of a description of it was posed. In that paper it was proved that if $f(0) = 0$ and f is an entire function, then it is not embeddable, and the following result was established.

Theorem 20. *Suppose that $f(z) = \sum_{k=0}^{\infty} p_k z^k$ is a probability generating function with $p_0 = 0$ and $0 < p_1 < 1$. Then $f \in \mathscr{E}(\mathfrak{P}^+)$ if and only if the numbers b_n, $n = 2, 3, \dots$, defined by the recurrence relations*

$$b_n = \frac{1}{p_1 - p_1^n} \sum_{j=1}^{n-1} b_j [\beta_{nj} - (n-j+1) p_{n-j+1}]$$

are non-negative, where $b_1 = \log p_1$ and β_{nj} is the coefficient of z^n in the power series expansion of $[f(z)]^j$.

Further progress in solving the embedding problem was made in [78].

Theorem 21. *Suppose that $f \in \mathfrak{P}^+$ is holomorphic at the point $z = 1$, $f'(1) > 1$, $f(0) = 0$, and $f'(0) > 0$. Suppose also that f is holomorphic in $\mathbb{C} \setminus S$, where S is a countable closed subset of the extended complex plane. Then $f \in \mathscr{E}(\mathfrak{P}^+)$ if and only if f is a fractional linear function.*

Thus, if a probability generating function f is meromorphic in $\mathbb{C}$, then it is not embeddable, except for the case of a fractional linear function. Moreover, the following necessary conditions for embeddability were obtained in [78] (see also [77]).

Theorem 22. *Suppose that $f \in \mathfrak{P}^+$ and q is the smallest non-negative root of the equation $f(x) = x$. Suppose also that $f^{(\mathrm{iv})}(q) < \infty$ (this condition is meaningful only in the case $q = 1$). Then the following assertions hold:*

(a) *if $3(f''(q))^2 - 2f'(q)f'''(q) > 0$, then $f \notin \mathscr{E}(\mathfrak{P}^+)$;*

(b) *if $3(f''(q))^2 - 2f'(q)f'''(q) = 0$, $q > 0$, and $f \in \mathscr{E}(\mathfrak{P}^+)$, then f is a fractional linear function.*

The question of a description of the class $\mathscr{E}(\mathfrak{P}^+)$ was raised in the monographs [75], [77] and many other papers in connection with the fact that the analytic apparatus for studying homogeneous Markov branching processes with continuous time is considerably more extensive than the means for studying Galton–Watson processes. Criteria and simple necessary conditions for embeddability were obtained in [80], [81]. In particular, the following general embeddability criterion holds [81].

Theorem 23. *Suppose that $f \in \mathfrak{P}^+$ and $f(z) \not\equiv 1$. Then $f \in \mathscr{E}(\mathfrak{P}^+)$ if and only if there exists a solution v of the equation*

$$v \circ f(z) = v(z) f'(z)$$

which is a holomorphic function in $\mathbb{D}$ satisfying the conditions $v'(0) = -1$ and $v^{(n)}(0) \geqslant 0$ for $n = 2, 3, \dots$.

Before stating other results in the papers mentioned above, we make some remarks and introduce the requisite notation.

Let f be a probability generating function. The smallest non-negative root q of the equation $f(x) = x$ has important meaning, both probabilistic and dynamical. First of all, q is the probability of extinction of the Galton–Watson process with generating function f. From the viewpoint of the dynamics of a holomorphic map, q is the Denjoy–Wolff point of f, that is, $f^n(z) \to q$ locally uniformly in $\mathbb{D}$ as $n \to \infty$. We also observe that for any probability generating function f the Denjoy–Wolff point q belongs to the closed interval $[0, 1]$, since $0 \leqslant f^n(0) < 1$ for all $n = 1, 2, \dots$ and $q = \lim_{n\to\infty} f^n(0)$.

Let $\mathfrak{P}^+[q]$, $0 \leqslant q \leqslant 1$, denote the set of functions f in $\mathfrak{P}^+$ for which q is the Denjoy–Wolff point, that is, the corresponding Galton–Watson process has extinction probability q. Obviously, $\mathfrak{P}^+[q]$ is a subsemigroup of the semigroup $\mathfrak{P}^+$.

Another method of classifying Galton–Watson processes is connected with the mathematical expectation of the number of descendants of a single particle in the next generation. This quantity is expressed in terms of the generating function f as the angular derivative $f'(1)$. If $f'(1) < 1$, then the corresponding process is said to be subcritical; the set of functions f in $\mathfrak{P}^+$ satisfying this condition is denoted by $\mathfrak{P}^+_{\rm sub}$. The set $\mathfrak{P}^+_{\rm c}$ of functions f in $\mathfrak{P}^+$ that satisfy the condition $f'(1) = 1$ and correspond to critical processes is defined similarly. Let $\mathfrak{P}^+_{\rm sup}$ denote the set of probability generating functions f satisfying the condition $f'(1) > 1$ (or $f'(1) = +\infty$) and corresponding to supercritical processes. The subsets thus defined are also subsemigroups of $\mathfrak{P}^+$. Moreover,

$$\mathfrak{P}^+[1] = \mathfrak{P}^+_{\rm c} \cup \mathfrak{P}^+_{\rm sub}, \qquad \mathfrak{P}^+_{\rm sup} = \bigcup_{0\leqslant q<1} \mathfrak{P}^+[q].$$

We assume that the identity element, that is, the identity transformation $f(z) \equiv z$, belongs to all the subsemigroups defined above. Moreover, their structure as convex subsets of the space $\mathscr{H}(\mathbb{D})$ is reflected in the following two results [80].

Theorem 24. *A function f holomorphic in $\mathbb{D}$ belongs to $\mathfrak{P}^+_{\rm c}$ (respectively, to $\mathfrak{P}^+_{\rm sub}$) if and only if it can be represented in the form*

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{\lambda_k}{k}(z^k - 1),$$

where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$ (respectively, $\sum_{k=1}^{\infty} \lambda_k < 1$).

Theorem 25. *A function f holomorphic in $\mathbb{D}$ belongs to $\mathfrak{P}^+[q]$, $0 \leqslant q < 1$, if and only if it can be represented in the form*

$$f(z) = q + (1-q) \sum_{k=1}^{\infty} \lambda_k \frac{z^k - q^k}{1 - q^k}\,,$$

where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

If a process is subcritical, then the extinction probability q is equal to 1, and $q \in [0, 1)$ in the case of a supercritical process. For these processes the embeddability condition has the following form [80].

Theorem 26. *A non-critical Galton–Watson process with extinction probability q and generating function f admits an embedding into a homogeneous Markov branching process with continuous time if and only if $f'(z) \neq 0$ for $z \in \mathbb{D}$ and there exists a locally uniform limit*

$$\lim_{n\to\infty} \frac{q - f^n(z)}{(f^n)'(z)} = u(z)$$

in $\mathbb{D}$ satisfying the conditions $u^{(k)}(0) \geqslant 0$ for $k = 2, 3, \dots$.

In the critical case the following result holds [80].

Theorem 27. *A critical Galton–Watson process with generating function f admits an embedding into a homogeneous Markov branching process with continuous time if and only if $f'(z) \neq 0$ for $z \in \mathbb{D}$ and there exists a locally uniform limit*

$$\lim_{n\to\infty} \frac{(f^n)'(0)}{(f^n)'(z)} = u(z)$$

in $\mathbb{D}$ satisfying the conditions $u(f(0)) = f'(0)$ and $u^{(k)}(0) \geqslant 0$ for $k = 2, 3, \dots$.

We now present several necessary conditions for embeddability. The following assertion concerns the probability generating functions corresponding to supercritical processes. When the extinction probability is equal to zero, the result is stated in terms of the initial probabilities. In the case $0 < q < 1$ a necessary condition for embeddability is stated in terms of the derivatives of the probability generating function at the point $z = q$, as in the corresponding result of Karlin and McGregor. The difference lies in the fact that it is possible to lower the maximal order of derivatives involved from the third to the second [80].

Theorem 28. *Suppose that $f \in \mathfrak{P}^+[q]$, $0 \leqslant q < 1$, and $f(z) \not\equiv z$. Then the following assertions hold*:

(a) *if*

$$\frac{f''(q)}{f'(q)(1 - f'(q))} > \frac{2}{1-q},$$

then $f \notin \mathscr{E}(\mathfrak{P}^+)$;

(b) *if*

$$\frac{f''(q)}{f'(q)(1 - f'(q))} = \frac{2}{1-q}$$

and $f \in \mathscr{E}(\mathfrak{P}^+)$, then f is a fractional linear function.

For $q = 0$ the result takes the following form.

Corollary 2. *Suppose that $f(z) = p_1 z + p_2 z^2 + \cdots$ is a probability generating function with $p_1 \neq 1$. Then the following assertions hold*:

(a) *if the point (p_1, p_2) does not belong to the set*

$$\{(x, y) \in \mathbb{R}^2 \colon 0 \leqslant y \leqslant x(1 - x),\ 0 \leqslant x \leqslant 1\},$$

then $f \notin \mathscr{E}(\mathfrak{P}^+)$;

(b) *if $p_2 = p_1(1 - p_1)$ and $f \in \mathscr{E}(\mathfrak{P}^+)$, then f is a fractional linear function.*

In the case $f(0) \neq 0$ it is also possible to obtain necessary conditions for embeddability that are stated in terms of the initial probabilities [81].

Theorem 29. *Suppose that* $f(z) = \sum_{k=0}^{\infty} p_k z^k \not\equiv z$ *belongs to the semigroup* $\mathfrak{P}^+[q]$, $0 < q \leqslant 1$. *Then the following assertions hold*:

(a) *if the point* (p_0, p_1) *does not belong to the set*

$$\left\{(x,y) \in \mathbb{R}^2\colon \frac{1}{q}(q-x)(1-x) \leqslant y < \frac{1}{q}(q-x),\ 0 \leqslant x < q\right\},$$

then $f \notin \mathscr{E}(\mathfrak{P}^+)$;

(b) *if* $p_1 = (q-p_0)(1-p_0)/q$ *and* $f \in \mathscr{E}(\mathfrak{P}^+)$, *then* f *is a fractional linear function.*

Conditions for embeddability of a probability generating function into a one-parameter semigroup in terms of the Koenigs function were obtained in [82].

6.2. Infinitesimal transformations and infinite divisibility in the semigroup of probability generating functions. When describing the cone of infinitesimal transformations of the semigroup $\mathfrak{P}^+$, one has to overcome two difficulties. First, the subsemigroup $\mathfrak{P}^+$ is not a closed subset of the semigroup $\mathfrak{P}$ in the topology of the space $\mathscr{H}(\mathbb{D})$. Second, when reproducing a one-parameter semigroup from the infinitesimal generator, one has to be sure about the non-negativity of the coefficients in the Taylor expansion of the functions in the one-parameter semigroup.

In overcoming the first difficulty we consider the closure $\overline{\mathfrak{P}^+}$ of $\mathfrak{P}^+$ in the topology of locally uniform convergence in $\mathbb{D}$. Obviously, a function f holomorphic in $\mathbb{D}$ belongs to $\overline{\mathfrak{P}^+}$ if and only if in its Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ the coefficients a_k satisfy the conditions $a_k \geqslant 0$ and $\sum_{k=0}^{\infty} a_k \leqslant 1$. Thus, it is natural to pose the question of infinitesimal transformations of the semigroup $\overline{\mathfrak{P}^+}$.

First we observe that any function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\overline{\mathfrak{P}^+}$ satisfies the inequality

$$|f(z) - z| \leqslant 2(1 - f'(0)), \qquad z \in \mathbb{D}. \tag{15}$$

Indeed,

$$\begin{aligned} |f(z) - z| &= \left|\sum_{k=0}^{\infty} a_k z^k - z\right| = \left|\sum_{k=2}^{\infty} a_k (z^k - z) + a_0(1-z) + (f(1)-1)z\right| \\ &\leqslant (1 - f(1)) + 2a_0 + 2\sum_{k=2}^{\infty} a_k = (1 - a_1) + (f(1) - a_1) \leqslant 2(1 - a_1), \end{aligned}$$

which is equivalent to (15).

Theorem 30. *For a function* v *holomorphic in* $\mathbb{D}$ *to be the infinitesimal generator of a one-parameter semigroup* $t \mapsto f^t$ *in* $\overline{\mathfrak{P}^+}$ *it is necessary and sufficient that it admits a representation in the form*

$$v(z) = \alpha(h(z) - z), \tag{16}$$

where $\alpha \geqslant 0$ *and the function* h *is holomorphic in* $\mathbb{D}$ *and has an expansion* $h(z) = b_0 + \sum_{k=2}^{\infty} b_k z^k$ *with* $b_0, b_k \geqslant 0$ *and* $b_0 + \sum_{k=2}^{\infty} b_k \leqslant 1$.

Proof. Suppose that $t \mapsto f^t$ is a one-parameter semigroup in $\overline{\mathfrak{P}^+}$ and $f^t(z) = \sum_{k=0}^{\infty} a_k(t) z^k$. Let us determine the form of its infinitesimal generator

$$v(z) = \lim_{t \to 0} \frac{f^t(z) - z}{t} = \sum_{k=0}^{\infty} c_k z^k.$$

Since the family of functions $f^t(z)$, $t \geqslant 0$, is infinitely differentiable with respect to t and $f^t(z) \to z$ locally uniformly in $\mathbb{D}$ as $t \to 0$, it follows that $a_k(t)$, $k = 0, 1, 2, \dots$, are also infinitely differentiable with respect to t and $a_k(t) \to 0$ as $t \to 0$ for $k \neq 1$, while $a_1(t) \to 1$ as $t \to 0$. Consequently, $c_1 \leqslant 0$ and $c_k \geqslant 0$ for $k \neq 1$. Moreover, by passing to the limit in the inequality

$$\frac{1}{t}\left(\sum_{k=0}^{n} a_k(t) - 1\right) \leqslant 0$$

as $t \to 0$ we get that $c_0 + \sum_{k=2}^{n} c_k \leqslant -c_1$, $n = 3, 4, \dots$. Hence, the equality $c_1 = 0$ implies that $v(z) \equiv 0$ and $f^t(z) \equiv z$ for all $t > 0$. Therefore, we consider the case $c_1 < 0$. Then $c_0 + \sum_{k=2}^{\infty} c_k \leqslant -c_1$, and the infinitesimal generator v can be represented in the form (16) with $\alpha = -c_1$ and

$$h(z) = -\frac{c_0}{c_1} - \sum_{k=2}^{\infty} \frac{c_k}{c_1} z^k.$$

The necessity of representation of the infinitesimal generator in the form (16) is proved.

We now prove the sufficiency of the representation (16). Here we can assume without loss of generality that $\alpha = 1$, since if $v(z)$ is the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$, then $\alpha v(z)$ is the infinitesimal generator of the one-parameter semigroup $t \mapsto f^{\alpha t}$.

Thus, suppose that $h(z) = b_0 + \sum_{k=2}^{\infty} b_k z^k$, where $b_0, b_k \geqslant 0$, $b_0 + \sum_{k=2}^{\infty} b_k \leqslant 1$, and $v(z) = h(z) - z$. For $\lambda \in [0, 1]$ we introduce the function

$$\varphi_\lambda(z) = z + \lambda v(z) = \lambda h(z) + (1 - \lambda) z.$$

Obviously, $\varphi_\lambda \in \overline{\mathfrak{P}^+}$. Considering $\mathbb{R}^+$ as an additive semigroup, we single out in it the subsemigroups $I_1 \subset I_2 \subset \cdots$, where I_n consists of the numbers of the form $t = k \cdot 2^{-n}$, $k = 0, 1, 2, \dots$, and we put $I = \bigcup_{n=1}^{\infty} I_n$. For each $n = 1, 2, \dots$ we define a function $F_n \colon \mathbb{D} \times I_n \to \mathbb{C}$ by

$$F_n(z, k \cdot 2^{-n}) = \varphi_{2^{-n}} \circ \dots \circ \varphi_{2^{-n}}(z) = \varphi_{2^{-n}}^{k}(z),$$

$k = 0, 1, 2, \dots$. In other words, $F_n(\,\cdot\,, k \cdot 2^{-n})$ is the k-fold iterate of the function $\varphi_{2^{-n}}$. The map $t \mapsto F_n(\,\cdot\,, t)$ is a homomorphism acting from I_n to $\overline{\mathfrak{P}^+}$, that is,

$$F_n(z, s + t) = F_n(\,\cdot\,, s) \circ F_n(z, t) \tag{17}$$

for all $s, t \in I_n$. We also observe that for any $t \in I$ all the functions $F_n(\,\cdot\,, t)$ are defined starting from some index. Using Cantor's diagonal method, we find a subsequence $\{F_{n_j}\}$ such that $F_{n_j}(z,t) \to F(z,t)$ locally uniformly in $\mathbb{D}$ as $j \to \infty$ for all $t \in I$. If we fix $t > 0$ in I, then $t \in I_n$ starting from some index. Furthermore, in the representation $t = k\cdot 2^{-n}$ the number k tends to infinity as $n \to \infty$. Therefore, the inequality

$$F_n'(0,t) = \varphi_{2^{-n}}'\bigl(\varphi_{2^{-n}}^{k-1}(0)\bigr)\cdot\varphi_{2^{-n}}'\bigl(\varphi_{2^{-n}}^{k-2}(0)\bigr)\cdot\varphi_{2^{-n}}'(0) \geqslant \bigl(\varphi_{2^{-n}}'(0)\bigr)^k = \left(1-\frac{t}{k}\right)^k$$

implies that $F'(0,t) \geqslant e^{-t}$ for all $t \in I$. Here $F' = \partial F/\partial z$. Passing to the limit in (17) with respect to the subsequence n_j, we arrive at the relation

$$F(z,s+t) = F(\,\cdot\,,s)\circ F(z,t) \tag{18}$$

for all $s,t \in I$. Moreover, using the inequality (15), we get that for all $s < t$ in I

$$\begin{aligned}|F(z,t)-F(z,s)| &= |F(\,\cdot\,,t-s)\circ F(z,s)-F(z,s)|\\ &\leqslant 2\bigl(1-F'(0,t-s)\bigr) \leqslant 2(1-e^{s-t}).\end{aligned}$$

Hence, the function $t \mapsto F(z,t)$ is continuous on I uniformly with respect to $z \in \mathbb{D}$. Then it is possible to extend F to $\mathbb{D}\times\mathbb{R}^+$ by continuity. Obviously, $F(\,\cdot\,,t) \in \overline{\mathfrak{P}^+}$ for all $t \in \mathbb{R}^+$. The continuity of the operation of composition also implies that equation (18) holds for all $s,t \in \mathbb{R}^+$. Thus, $t \mapsto F(\,\cdot\,,t)$ is a one-parameter semigroup in $\overline{\mathfrak{P}^+}$. We now determine its infinitesimal generator.

For convenience we define functions F_n on $\mathbb{D}\times\mathbb{R}^+$ by $F_n(z,t) = F_n(z,k\cdot 2^{-n})$ for $k\cdot 2^{-n} \leqslant t < (k+1)\cdot 2^{-n}$, $k = 0,1,2,\dots$. Then for $t \in I_n$ we can write the equation

$$F_n(z,t)-z = \int_0^t v\circ F_n(z,s)\,ds. \tag{19}$$

Indeed, if $t = k\cdot 2^{-n}$, then

$$\begin{aligned}F_n(z,t)-z = F_n(z,k\cdot 2^{-n})-z &= \sum_{j=1}^k \bigl(F_n(z,j\cdot 2^{-n})-F_n(z,(j-1)\cdot 2^{-n})\bigr)\\ &= \sum_{j=1}^k \bigl(\varphi_{2^{-n}}\circ F_n(z,(j-1)\cdot 2^{-n})-F_n(z,(j-1)\cdot 2^{-n})\bigr)\\ &= 2^{-n}\sum_{j=1}^k v\circ F_n(z,(j-1)\cdot 2^{-n}) = \int_0^t v\circ F_n(z,s)\,ds.\end{aligned}$$

Passing to the limit in (19) with respect to the subsequence $\{n_j\}$, we arrive at the equation

$$F(z,t)-z = \int_0^t v\circ F(z,s)\,ds$$

for all $t \in I$. This extends by continuity to all $t \in \mathbb{R}^+$. Then

$$\lim_{t\to 0}\frac{F(z,t)-z}{t} = \lim_{t\to 0}\frac{1}{t}\int_0^t v\circ F(z,s)\,ds = v(z).$$

Consequently, $v(z) = h(z)-z$ is the infinitesimal generator of the one-parameter semigroup $t \mapsto F(\,\cdot\,,t)$. $\square$

A general description of the cone of infinitesimal transformations of the semigroup $\overline{\mathfrak{P}^+}$ can be given in more detail depending on the Denjoy–Wolff point q. First we observe that for the closures $\overline{\mathfrak{P}^+_{\mathrm{c}}}$ and $\overline{\mathfrak{P}^+_{\mathrm{sub}}}$ of the respective semigroups $\mathfrak{P}^+_{\mathrm{c}}$ and $\mathfrak{P}^+_{\mathrm{sub}}$ Theorem 24 gives us that $\overline{\mathfrak{P}^+_{\mathrm{c}}} = \overline{\mathfrak{P}^+_{\mathrm{sub}}} = \mathfrak{P}^+[1]$. It also follows from Theorems 24 and 25 that $\overline{\mathfrak{P}^+_{\mathrm{sup}}} = \overline{\mathfrak{P}^+}$. Moreover, a function f holomorphic in $\mathbb{D}$ belongs to $\overline{\mathfrak{P}^+[q]}$, $0 \leqslant q < 1$, if and only if it admits the representation in Theorem 25 with $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k \leqslant 1$. Thus, the following results hold (see [80]).

Theorem 31. *For a function v holomorphic in $\mathbb{D}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\overline{\mathfrak{P}^+[q]}$, $0 \leqslant q < 1$, it is necessary and sufficient that it admits a representation in the form*

$$v(z) = \alpha\biggl[q - z + (1-q)\sum_{k=2}^{\infty} \lambda_k \frac{z^k - q^k}{1 - q^k}\biggr],$$

where $\alpha \geqslant 0$, $\lambda_k \geqslant 0$, and $\sum_{k=2}^{\infty} \lambda_k \leqslant 1$.

We also observe that if $t \mapsto f^t$ is a one-parameter semigroup in $\mathfrak{P}^+[q]$, then $\sum_{k=2}^{\infty} \lambda_k = 1$ in the representation of the infinitesimal generator. The case $q = 1$ is described by the following theorem.

Theorem 32. *For a function v holomorphic in $\mathbb{D}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}^+_{\mathrm{c}}$ (respectively, in $\mathfrak{P}^+_{\mathrm{sub}}$) it is necessary and sufficient that it admits a representation in the form*

$$v(z) = \alpha\biggl[1 - z + \sum_{k=2}^{\infty} \frac{\lambda_k}{k}(z^k - 1)\biggr],$$

where $\alpha \geqslant 0$, $\lambda_k \geqslant 0$, and $\sum_{k=2}^{\infty} \lambda_k = 1$ (respectively, $\sum_{k=2}^{\infty} \lambda_k < 1$).

Along with the class of embeddable probability generating functions, it seems natural to introduce also the following definition.

Definition 4. We say that a probability generating function f is infinitely divisible in the semigroup $\mathfrak{P}^+$ if for every $n = 2, 3, \dots$ there exists f_n in $\mathfrak{P}^+$ such that

$$f(z) = f_n \circ \dots \circ f_n(z) = f_n^n(z), \tag{20}$$

that is, f is the n-fold iterate of the function f_n.

If ξ is an integer-valued random variable with probability generating function f, then the condition of infinite divisibility of f in $\mathfrak{P}^+$ can be stated as a condition on ξ in terms of sums of independent random variables [81]. This definition was introduced by analogy with the classical notion of infinitely divisible probability distributions. An important feature in the description of infinitely divisible distributions turned out to be the fact that they are embeddable into one-parameter semigroups of measures with respect to convolution. In other words, the notions of embeddability and infinite divisibility coincide in the semigroup of probability measures on $\mathbb{R}$ with the operation of convolution.

Theorem 33. *Suppose that a probability generating function $f(z) \not\equiv 1$ is infinitely divisible in the semigroup $\mathfrak{P}^+$. Then f admits an embedding into a one-parameter semigroup in $\mathfrak{P}^+$, that is, $f \in \mathscr{E}(\mathfrak{P}^+)$.*

Proof. Suppose that a probability generating function $f(z) = \sum_{k=0}^{\infty} p_k z^k$ is infinitely divisible in $\mathfrak{P}^+$. We assert that there exists a one-parameter semigroup $t \mapsto g^t$ in $\mathfrak{P}^+$ such that $f = g^1$. We distinguish two cases in the proof.

1) First suppose that $f \in \mathfrak{P}^+[q]$ and $0 \leqslant q < 1$. The case $f(z) \equiv z$ is trivial, and therefore in what follows we assume that $f(z) \not\equiv z$. Next, since $z = q$ is a fixed point for the function f and $p_k \geqslant 0$, it follows that $0 \leqslant f'(q) \leqslant 1$. The equality $f'(q) = 1$ cannot be attained, since this would imply the identity $f(z) \equiv z$. We assert that the equality $f'(q) = 0$ also cannot be attained. Indeed, in the case $q > 0$ the equality $f'(q) = \sum_{k=1}^{\infty} k p_k q^k = 0$ implies the identity $f'(z) \equiv 0$, which implies that $f(z) \equiv 1$, contradicting the hypotheses of the theorem. It remains to consider the case $q = 0$, that is, $p_0 = 0$ and $p_1 = f'(0) = 0$. But then in the representation (20) the function f_n also satisfies the conditions $f_n(0) = f_n'(0) = 0$, and the Taylor series expansion of this function will start at least from z^2, and the expansion of its nth iterate f_n^n from the power z^{2n}. This would mean that $p_0 = p_1 = \cdots = p_{2n} = 0$. Since the representation (20) is possible for any $n = 2, 3, \ldots$, the assumption $f(0) = f'(0) = 0$ made above would imply the identity $f(z) \equiv 0$, which is impossible. Thus, $0 < f'(q) < 1$.

For every $n = 2, 3, \ldots$ we define

$$\alpha_n = n(1 - f_n'(q)), \qquad v_n(z) = \frac{f_n(z) - z}{1 - f_n'(q)},$$

where f_n is from the representation (20). Since $f_n'(q) = (f'(q))^{1/n}$, we have

$$\lim_{n\to\infty} \alpha_n = -\log f'(q).$$

Next, we observe that if $\varphi \in \mathfrak{P}^+[q]$ and $0 < \varphi'(q) < 1$, then by Theorem 25 φ can be represented in the form

$$\varphi(z) = q + (1-q) \sum_{k=1}^{\infty} \lambda_k \frac{z^k - q^k}{1 - q^k},$$

where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$. Furthermore, $\lambda_1 = \varphi'(0) \leqslant \varphi'(q) < 1$ and

$$\frac{\varphi(z) - z}{1 - \varphi'(q)} = \frac{\psi(z) - z}{1 - \psi'(q)},$$

where

$$\psi(z) = q + (1-q) \sum_{k=2}^{\infty} \lambda_k^* \frac{z^k - q^k}{1 - q^k}, \qquad \lambda_k^* = \frac{\lambda_k}{1 - \lambda_1},$$

that is, $\psi \in \mathfrak{P}^+[q]$ and $\psi'(0) = 0$. Thus, for $n = 2, 3, \ldots$ we get that

$$n(f_n(z) - z) = \alpha_n v_n(z) = \alpha_n \frac{h_n(z) - z}{1 - h_n'(q)},$$

where $h_n \in \mathfrak{P}^+[q]$ and $h_n'(0) = 0$. By the compactness principle we can choose a subsequence $\{n_j\}$ in such a way that $h_{n_j}(z) \to h(z)$ locally uniformly in $\mathbb{D}$ as $j \to \infty$. Obviously, the limit function h belongs to $\overline{\mathfrak{P}^+[q]}$ and $h'(0) = 0$. We assert that $h'(q) < 1$. Indeed, the representation of the semigroup $\overline{\mathfrak{P}^+[q]}$ implies that

$$h(z) = q + (1-q)\sum_{k=2}^{\infty} \lambda_k \frac{z^k - q^k}{1-q^k}\,,$$

where $\lambda_k \geqslant 0$ and $\sum_{k=2}^{\infty} \lambda_k \leqslant 1$. But then

$$h'(q) = (1-q)\sum_{k=2}^{\infty} \lambda_k \frac{kq^{k-1}}{1-q^k} \leqslant (1-q)\frac{2q}{1-q^2} = \frac{2q}{1+q} < 1.$$

Here we used the fact that the sequence $\{kq^{k-1}/(1-q^k)\}$ is monotonically decreasing. This follows from the inequalities

$$\frac{kq^{k-1}}{1-q^k} - \frac{(k+1)q^k}{1-q^{k+1}} = \frac{q^{k-1}(k-(k+1)q+q^{k+1})}{(1-q^k)(1-q^{k+1})} \geqslant 0,$$

$k = 2, 3, \dots,\ 0 \leqslant q < 1$.

Thus,

$$\lim_{j\to\infty} n_j(f_{n_j}(z) - z) = \alpha(h(z) - z),$$

where

$$\alpha = -\frac{\log f'(q)}{1-h'(q)} > 0.$$

We note that $v(z) = \alpha(h(z)-z)$ is the infinitesimal generator of some one-parameter semigroup $t \mapsto g^t$ in $\overline{\mathfrak{P}^+[q]}$. Let us show that $f = g^1$. Differentiating both sides of the equation

$$\frac{\partial}{\partial t} g^t(z) = v(g^t(z))$$

with respect to z, we get that

$$\frac{\partial}{\partial t}\log\bigl((g^t)'(z)\bigr) = v'(g^t(z)),$$

where the logarithm is understood as the branch taking a real value at $z = q$. Integration with respect to t with use of the condition $(g^t)'(z)\big|_{t=0} \equiv 1$ yields

$$\log[(g^t)'(z)] = \int_0^t v'(g^s(z))\,ds.$$

Equality of the real parts can be written in the form

$$\log|(g^t)'(z)| = \int_0^t \operatorname{Re}\{v'(g^s(z))\}\,ds. \tag{21}$$

Since $v'(q) = \log f'(q) < 0$, there is an $r \in (0,1)$ such that $\operatorname{Re} v'(z) < 0$ in the non-Euclidean disk

$$Q = \left\{z \in \mathbb{D}\colon \left|\frac{z-q}{1-qz}\right| \leqslant r\right\}.$$

By the hyperbolic metric principle, $g^t(Q) \subseteq Q$ for all $t \geqslant 0$. Moreover, equation (21) and the definition of Q imply that $|(g^t)'(z)| \leqslant 1$ for all $t \geqslant 0$ and $z \in Q$. But then

$$|g^t(z') - g^t(z'')| = \left| \int_{[z',z'']} (g^t)'(z)\,dz \right| \leqslant |z' - z''|$$

for any z', z'' in Q and $t \geqslant 0$. We now fix an arbitrary $z \in Q$ and carry out the following estimates:

$$\begin{aligned}
|f(z) - g^1(z)| &= |f_n^n(z) - g^1(z)| \\
&\leqslant |f_n \circ f_n^{n-1}(z) - g^{1/n} \circ f_n^{n-1}(z)| + |g^{1/n} \circ f_n^{n-1}(z) - g^{1/n} \circ g^{(n-1)/n}(z)| \\
&\leqslant |f_n \circ f_n^{n-1}(z) - g^{1/n} \circ f_n^{n-1}(z)| + |f_n^{n-1}(z) - g^{(n-1)/n}(z)| \\
&\leqslant |f_n \circ f_n^{n-1}(z) - g^{1/n} \circ f_n^{n-1}(z)| + |f_n \circ f_n^{n-2}(z) - g^{1/n} \circ f_n^{n-2}(z)| \\
&\qquad + |g^{1/n} \circ f_n^{n-2}(z) - g^{1/n} \circ g^{(n-2)/n}(z)| \leqslant \cdots \\
&\leqslant \sum_{k=0}^{n-1} |f_n \circ f_n^k(z) - g^{1/n} \circ f_n^k(z)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\max_{z \in Q} |f(z) - g^1(z)| &\leqslant n \max_{z \in Q} |f_n(z) - g^{1/n}(z)| \\
&= \max_{z \in Q} |n(f_n(z) - z) - n(g^{1/n}(z) - z)| \\
&\leqslant \max_{z \in Q} |n(f_n(z) - z) - v(z)| + \max_{z \in Q} |n(g^{1/n}(z) - z) - v(z)|.
\end{aligned}$$

It follows from the above that the right-hand side of the last inequality tends to zero as n tends to infinity over the subsequence $\{n_j\}$. Consequently, $f(z) = g^1(z)$ for $z \in Q$. By the uniqueness theorem for analytic functions we see that $f(z) \equiv g^1(z)$, that is, $f \in \mathscr{E}(\mathfrak{P}^+)$.

2) Let us now consider the case $f \in \mathfrak{P}^+[1]$. As in the preceding case, we assume that $f(z) \not\equiv z$. We again denote by f_n the functions in the representation (20). For them, as well as for the function f, the point $z = 1$ is the Denjoy–Wolff point, that is, $f_n \in \mathfrak{P}^+[1]$, $n = 2, 3, \dots$. Consequently, $f_n(x) > x$ for $0 \leqslant x < 1$. We also point out that $0 < f(0) = a < 1$, since $f(z) \not\equiv 1$ and $f(z) \not\equiv z$. Moreover,

$$a = f_n^n(0) = f_n(f_n^{n-1}(0)) > f_n^{n-1}(0) > \cdots > f_n(0).$$

For each $n = 2, 3, \dots$ let

$$\alpha_n = n(1 - f_n'(a)), \qquad v_n(z) = \frac{f_n(z) - z}{1 - f_n'(a)}\,.$$

Since

$$f'(0) = (f_n^n)'(0) = f_n'(f_n^{n-1}(0)) \cdots f_n'(0) \leqslant (f_n'(a))^n,$$

it follows that $(f'(0))^{1/n} \leqslant f_n'(a)$ and

$$\alpha_n = n(1 - f_n'(a)) \leqslant n\bigl(1 - (f'(0))^{1/n}\bigr).$$

Noting that $0 < f'(0) < 1$, we see that the numerical sequence $\{\alpha_n\}$ is bounded. Further, by Theorem 24 and arguments similar to those above we obtain the representation

$$v_n(z) = \frac{h_n(z) - z}{1 - h'_n(a)}\,,$$

where $h_n \in \mathfrak{P}^+[1]$ and $h'_n(0) = 0$, $n = 2, 3, \dots$. Since $|h'_n(z)| \leqslant h'_n(1) \leqslant 1$,

$$|h_n(z') - h_n(z'')| \leqslant |z' - z''|$$

for any $z', z'' \in \mathbb{D}$ and $n = 2, 3, \dots$. This means that the sequence of functions h_n, $n = 2, 3, \dots$, is equicontinuous in the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. Using the Arzelà theorem and the boundedness of the sequence $\{\alpha_n\}$, we can choose a subsequence $\{n_j\}$ in such a way that $\alpha_{n_j} \to \alpha$ and $h_{n_j}(z) \to h(z)$ uniformly on $\overline{\mathbb{D}}$ as $j \to \infty$. By the Weierstrass theorem the limit function h is analytic in the unit disk and belongs to the semigroup $\mathfrak{P}^+[1]$. Moreover, the condition $h'(0) = 0$ also holds. It is evident from the representation

$$h(z) = 1 + \sum_{k=2}^{\infty} \frac{\lambda_k}{k}(z^k - 1),$$

where $\lambda_k \geqslant 0$ and $\sum_{k=2}^{\infty} \lambda_k \leqslant 1$, that

$$h'(a) = \sum_{k=2}^{\infty} \lambda_k a^{k-1} \leqslant a\; < 1.$$

Thus, we have the uniform limit in $\overline{\mathbb{D}}$

$$\lim_{j\to\infty} n_j(f_{n_j}(z) - z) = \lim_{j\to\infty} \alpha_{n_j} v_{n_j}(z) = \frac{\alpha}{1 - h'(a)}(h(z) - z) = v(z).$$

The limit function v is the infinitesimal generator of a one-parameter semigroup $t \mapsto g^t$ in $\mathfrak{P}^+[1]$.

It remains to show that $f = g^1$. Since the functions $g^{1/n}$ play the same role for g^1 as the functions f_n for f, we can use the arguments above, which imply that the hypotheses of the Arzelà theorem hold for the sequence of functions $n(g^{1/n}(z) - z)$, $n = 2, 3, \dots$, in $\overline{\mathbb{D}}$. Any subsequence of this sequence converging uniformly in $\overline{\mathbb{D}}$ must have as a limit the infinitesimal generator

$$v(z) = \lim_{t\to 0} \frac{g^t(z) - z}{t}\,.$$

In a way similar to that in the preceding case we get that

$$\max_{z\in\overline{\mathbb{D}}} |f(z) - g^1(z)| \leqslant \max_{z\in\overline{\mathbb{D}}} |n(f_n(z) - z) - v(z)| + \max_{z\in\overline{\mathbb{D}}} |n(g^{1/n}(z) - z) - v(z)|.$$

Passing to the limit with respect to the corresponding subsequence, we arrive at the equality $f(z) \equiv g^1(z)$. $\square$

The description of infinitesimal transformations of semigroups of probability generating functions has proved to be useful for studying problems in the theory of homogeneous Markov branching processes with continuous time. In particular, estimates for the distribution function of the extinction moment of a process were obtained in [83]. A complete description of the limit distributions for a subcritical process was given in [84]. The notion of an evolution family of probability generating functions made it possible to obtain certain analytic properties of time-inhomogeneous Markov branching processes [85].

7. The reciprocal Cauchy transform and non-commutative probability

In this section we indicate another direction that is being intensively developed lately and in which semigroups of analytic functions also appear naturally.

The notion of independence plays a central role in the classical probability theory. In essence, it is this notion that distinguishes probability theory from general measure theory. The most important distributions and random processes are defined in terms of independence. The basic analytic apparatus in classical probability theory is the method of characteristic functions. It is based on the Fourier transform

$$h_\xi(u) = \int_{\mathbb{R}} e^{iux}\, d\mu(x),$$

where ξ is a random variable with probability distribution μ. The decisive factor here is that the Fourier transform takes a convolution of probability measures into the product of the characteristic functions. A convolution of probability measures is involved in a sum of independent random variables: if μ and ν are the probability measures corresponding to the distributions of independent random variables ξ and η, then the sum $\xi+\eta$ has probability distribution $\mu * \nu$. This circumstance is used in the proof of the central limit theorem. The notion of convolution is connected with such important constructions as infinite divisibility, one-parameter semigroups of distributions, and processes with independent increments. The question of a description of infinitely divisible laws can be resolved in terms of characteristic functions. A random variable ξ is said to be infinitely divisible if for any positive integer n there exist independent identically distributed random variables $\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,n}$ such that

$$\xi = \xi_{n,1} + \xi_{n,2} + \dots + \xi_{n,n}.$$

The term 'infinite divisibility' can be extended to a probability distribution μ as follows: for every positive integer n there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n * \mu_n * \dots * \mu_n}_{n \text{ times}}.$$

If

$$h(u) = \int_{\mathbb{R}} e^{iux}\, d\mu(x), \qquad h_n(u) = \int_{\mathbb{R}} e^{iux}\, d\mu_n(x),$$

then $h(u) = (h_n(u))^n$ and the characteristic function h corresponds to an infinitely divisible distribution if and only if there exists a one-parameter family of characteristic functions $\{h_t\}_{t\geqslant 0}$ such that

$$h_0(u) \equiv 1, \quad h_1(u) = h(u), \quad h_{t+s}(u) = h_t(u)h_s(u).$$

In other words, h embeds into a one-parameter multiplicative semigroup of characteristic functions. The Lévy–Khinchin formula amounts to a description of such one-parameter semigroups (see, for example, [86]).

One of the features of the quantum and non-commutative probability theory is the fact that they have not one but several definitions of independence [87]. In quantum probability the commutative algebra of random variables is replaced by the non-commutative algebra $\mathscr{A}$ of operators acting in a Hilbert space $\mathscr{H}$. The role of a probability measure is played by a unit vector $\xi \in \mathscr{H}$, which is called a state. The probability distribution μ of a quantum random variable $X \in \mathscr{A}$ is defined in terms of the functional calculus in spectral theory:

$$\langle \xi, f(X)\xi \rangle = \int_{\sigma(X)} f(z)\, d\mu(z),$$

where $\sigma(X)$ is the spectrum of the operator X. The measure μ is also called the spectral measure associated with the vector ξ, and $\langle \xi, f(X)\xi \rangle$ can be interpreted as the mathematical expectation $\mathsf{E} f(X)$. The role of real random variables is played by selfadjoint operators X, and their probability distributions μ have supports on $\mathbb{R}$. Probability measures with non-compact supports correspond to unbounded operators. If X is a positive operator, then $\sigma(X) \subset \mathbb{R}^+ = \{x \in \mathbb{R}\colon x \geqslant 0\}$, and $\sigma(X) \subset \mathbb{T} = \{\varkappa \in \mathbb{C}\colon |\varkappa| = 1\}$ in the case of a unitary operator X.

Let $\mathscr{M}(\mathbb{R})$ (respectively, $\mathscr{M}(\mathbb{R}^+)$, $\mathscr{M}(\mathbb{T})$) denote the class of all probability measures on $\mathbb{R}$ (respectively, on $\mathbb{R}^+$, $\mathbb{T}$). Let $\mathscr{M}_0^2(\mathbb{R})$ denote the class of probability measures μ on $\mathbb{R}$ that have zero mean and finite variance, that is,

$$\int_{\mathbb{R}} x\, d\mu(x) = 0, \qquad \int_{\mathbb{R}} x^2\, d\mu(x) < \infty.$$

Various notions of independence in non-commutative probability theory lead to new operations which are non-commutative analogues of convolution in classes of probability measures. Instead of the Fourier transform, an analogous role is played in non-commutative probability by the so-called reciprocal Cauchy transform. By means of this and related transformations, certain convolutions of measures are transformed into compositions of the analytic functions that are the images of the measures under these transformations. As a result, semigroups of analytic functions and the problem of an infinitesimal description of them appear in a natural way. We now briefly describe some of these semigroups, without touching upon the algebraic nature of the corresponding types of independence.

If μ is a probability measure on $\mathbb{R}$, then its Cauchy transform is understood to be the analytic function G_μ defined in the upper half-plane $\mathbb{U}$ by the formula

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(u)}{z-u}\,, \qquad z \in \mathbb{U}.$$

Note that μ is uniquely determined by its Cauchy transform G_μ via the inversion formula

$$\mu(B) = -\frac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_B \operatorname{Im} G_\mu(x+i\varepsilon)\, dx,$$

where $B \subset \mathbb{R}$ is a Borel set such that $\mu(\partial B) = 0$ (see, for example, [88]). The probabilistic properties of a distribution are more adequately reflected by the so-called reciprocal Cauchy transform

$$F_\mu(z) = \frac{1}{G_\mu(z)},$$

which is a holomorphic map of the upper half-plane $\mathbb{U}$ into itself. If $\mu = \delta_0$ is the measure with unit mass concentrated at zero, then $F_\mu(z) \equiv z$, the identity transformation. Let

$$\mathfrak{F} = \{f = F_\mu \colon \mu \in \mathscr{M}(\mathbb{R})\}$$

denote the class of holomorphic transformations of $\mathbb{U}$ into itself that are the reciprocal Cauchy transforms of probability measures μ on $\mathbb{R}$. The subset $\mathfrak{F}_0^2 \subset \mathfrak{F}$ consists of the reciprocal Cauchy transforms of the probability measures $\mu \in \mathscr{M}_0^2(\mathbb{R})$. A functional-theoretic description of the classes $\mathfrak{F}$ and $\mathfrak{F}_0^2$ was given in [89]. In particular, a function f holomorphic in $\mathbb{U}$ and taking values in $\mathbb{U}$ belongs to the class $\mathfrak{F}$ if and only if $f(z) \to \infty$ as $z \to \infty$ and its angular derivative at infinity is equal to 1. This means that $\mathfrak{F}$ is closed under the operation of composition, that is, it is a semigroup. Further, a holomorphic function $f\colon \mathbb{U} \to \mathbb{U}$ belongs to $\mathfrak{F}_0^2$ if and only if there exists a finite positive measure ν on $\mathbb{R}$ such that

$$f(z) = z + \int_{\mathbb{R}} \frac{d\nu(u)}{u - z}, \qquad z \in \mathbb{U}.$$

This representation, that is, the class $\mathfrak{F}_0^2$, appears as a result of singling out the subset of functions with hydrodynamical normalization at infinity in the Nevanlinna (or Pick) class [49]. It was also established in [49] that $\mathfrak{F}_0^2$ is closed under the operation of composition. Thus, $\mathfrak{F}_0^2$ is a subsemigroup of the semigroup $\mathfrak{F}$.

Muraki [90] introduced one of the notions of independence in non-commutative probability which takes a corresponding convolution of probability distributions into the composition of the reciprocal Cauchy transforms.

Definition 5. Let μ and ν be two probability measures on $\mathbb{R}$ with reciprocal Cauchy transforms F_μ and F_ν, respectively. Then the *additive monotone convolution* $\lambda = \mu \triangleright \nu$ of μ and ν is defined as the unique probability measure λ on $\mathbb{R}$ with the reciprocal Cauchy transform $F_\lambda = F_\mu \circ F_\nu$.

In connection with analogues of Lévy processes and the Lévy–Khinchin formula on the basis of the additive monotone convolution [91], a construction of a one-parameter semigroup in $\mathscr{M}(\mathbb{R})$ (or $\mathscr{M}_0^2(\mathbb{R})$) with respect to the additive monotone convolution has emerged which via the reciprocal Cauchy transform is taken into a composition one-parameter semigroup in $\mathfrak{F}$ (or $\mathfrak{F}_0^2$). A description of the infinitesimal transformations of the semigroup $\mathfrak{F}_0^2$ (in the context of holomorphic maps of the upper half-plane into itself with hydrodynamic normalization at infinity) was obtained in [49].

Theorem 34. *For a function v holomorphic in $\mathbb{U}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{F}$ it is necessary and sufficient that it admits a representation in the form*

$$v(z) = \alpha + \int_{\mathbb{R}} \frac{1+uz}{u-z}\, d\nu(u),$$

where $\alpha \in \mathbb{R}$ and ν is a finite non-negative Borel measure on $\mathbb{R}$.

Theorem 35. *For a function v holomorphic in $\mathbb{U}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{F}_0^2$ it is necessary and sufficient that it admits a representation in the form*

$$v(z) = \int_{\mathbb{R}} \frac{d\nu(u)}{u-z}\,,$$

where ν is a finite non-negative Borel measure on $\mathbb{R}$.

If μ is a probability measure on $\mathbb{R}^+$, then instead of the Cauchy transform we can define the function

$$\psi_\mu(z) = \int_{\mathbb{R}^+} \frac{uz}{1-uz}\, d\mu(u),$$

which is an analytic function in $\mathbb{S} = \mathbb{C} \setminus \mathbb{R}^+$ and does not take the value -1. The function ψ_μ is called the generating function of moments of the measure μ. In view of these properties of ψ_μ, one can also define the function

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)}\,,$$

which is holomorphic in $\mathbb{S}$ and satisfies $\eta_\mu(\mathbb{S}) \subset \mathbb{S}$. Let $\mathfrak{T}$ denote the class

$$\mathfrak{T} = \{f = \eta_\mu \colon \mu \in \mathscr{M}(\mathbb{R}^+)\}$$

of holomorphic maps of the cut plane $\mathbb{S}$ into itself that are the η-transforms of probability measures μ on $\mathbb{R}^+$. An intrinsic description of this class was obtained in [92]. A holomorphic function $f \colon \mathbb{S} \to \mathbb{S}$ belongs to $\mathfrak{T}$ if and only if

$$(\operatorname{Im} z)(\operatorname{Im} f(z)) \geqslant 0$$

for $z \in \mathbb{S}$ and $\lim_{x \nearrow 0} f(x) = 0$. It is evident from this description that $\mathfrak{T}$ is a semigroup with respect to the operation of composition. The notion of a monotone convolution on the class $\mathscr{M}(\mathbb{R}^+)$ of measures was introduced by Bercovici [93]. It is defined in terms of the transform η_μ as follows.

Definition 6. Let μ and ν be two probability measures on $\mathbb{R}^+$ with transforms η_μ and η_ν, respectively. Then the *multiplicative monotone convolution* of μ and ν is defined as the unique probability measure $\lambda = \mu \circlearrowright \nu$ on $\mathbb{R}^+$ with the transform $\eta_\lambda = \eta_\mu \circ \eta_\nu$.

Thus, the study of the multiplicative monotone convolution carries over to the semigroup $\mathfrak{T}$ of analytic functions. A simple transformation of the results in [44] with use of the fact that $z = 0$ is a boundary fixed point yields the following theorem.

Theorem 36. *For a function v holomorphic in $\mathbb{S}$ to be the infinitesimal generator of a one-parameter semigroup $t \mapsto f^t$ in $\mathfrak{T}$ it is necessary and sufficient that it admits a representation in the form $v(z) = zh(z)$, where h is holomorphic in $\mathbb{S}$,*

$$(\operatorname{Im} z)(\operatorname{Im} h(z)) \geqslant 0$$

for $z \in \mathbb{S}$, and the integral

$$\int_0 \frac{du}{uh(-u)}$$

diverges.

We also mention that the multiplicative monotone convolution on the class $\mathscr{M}(\mathbb{T})$ of measures was studied in [94].

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