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A New Property of Reproducing Kernels for Classical Orthogonal Polynomials*

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We exhibit a second-order differential operator commuting with the reproducing kernel $\sum_{n=0}^{T} \phi_n(\lambda)\,\phi_n(\mu)/h_n$ each time that $\{\phi_n(\lambda)\}$ is one of the classical orthogonal polynomials: Jacobi, Laguerre, Hermite and Bessel. This is the analog of a known property in the study of time and band-limited signals.

INTRODUCTION

Let $w(\lambda)$ be a positive weight function in the open interval (a, b) for which all the moments

$$\int_a^b \lambda^n w(\lambda)\, d\lambda, \qquad n = 0, 1, 2,...,$$

are finite.

Let $\phi_n(\lambda)$ be a sequence of polynomials, $n = 0, 1, 2,...$, with degree $\phi_n(\lambda) = n$, satisfying

$$\int_a^b \phi_n(\lambda)\, \phi_m(\lambda)\, w(\lambda)\, d\lambda = h_n \delta_{n,m}.$$

Consider now two Hilbert spaces: The space of all real-valued sequences c_n, $n = 0, 1, 2,...$, with $\sum_0^\infty c_n^2 < \infty$ denoted here by $l^2(Z^+)$ and the space of all measurable real-valued functions $f(\lambda)$, $\lambda \in (a, b)$, satisfying $\int_a^b f^2(\lambda)\, w(\lambda)\, d\lambda < \infty$. Denote this space by $L^2((a, b), w(\lambda)\, d\lambda) \equiv L^2(w)$.

It is now clear that the map F

$$(c_n)_0^\infty \xrightarrow{F} \sum_0^\infty c_n \frac{\phi_n(\lambda)}{\sqrt{h_n}}$$

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is an isometry of $l^2(Z^+)$ into $L^2(w)$. If polynomials are dense in $L^2(w)$, this map is unitary with an inverse F^{-1} given by

$$f(\lambda) \xrightarrow{F^{-1}} \int_a^b f(\lambda) \frac{\phi_n(\lambda)}{\sqrt{h_n}} w(\lambda)\, d\lambda.$$

We are calling our map F to remind ourselves of the usual Fourier transform. Here $Z^+ = \{0, 1, 2, \ldots\}$ takes up the role of "physical space" and (a, b) the role of "frequency space."

The stage is set to introduce the operators of "time limiting" and "band limiting:"

Time limiting, at level T, acts on $l^2(Z^+)$ by simply setting equal to zero all the components with index larger than T. This operator will be denoted by χ_T.

Band limiting, at level Ω, acts on $L^2(w)$ by multiplication by the characteristic function of the interval (a, Ω), $\Omega \leqslant b$. This operator will be denoted by χ_Ω.

Now we can introduce the main characters of our story: The operators performing time–band–time and band–time–band limitation:

$$S_1 \equiv \chi_T F^{-1} \chi_\Omega F \chi_T$$

and

$$S_2 \equiv \chi_\Omega F \chi_T F^{-1} \chi_\Omega.$$

These two operators are quite related, and there is natural "duality" between them. Indeed, if one considers the operator

$$E\colon l^2(Z^+) \to L^2(w)$$

defined as

$$E = \chi_\Omega F \chi_T,$$

one gets

$$E^* = \chi_T F^{-1} \chi_\Omega$$

and finally

$$S_1 = E^* E$$

and

$$S_2 = E E^*.$$

Therefore S_1 and S_2 are the selfadjoint operators whose eigenvectors v_i and w_i give the "singular vectors" of E and E^*. The nonzero eigenvalues of S_1 are among the eigenvalues λ_i of S_2 and one has

$$E = \sum_{\lambda_i \neq 0} \sqrt{\lambda_i}\, w_i v_i^T,$$

$$E^* = \sum_i \sqrt{\lambda_i}\, v_i w_i^T.$$

The first operator is just a finite dimensional matrix given by

$$(S_1)_{m,n} \equiv \int_a^\Omega \frac{\phi_m(\lambda)\,\phi_n(\phi)\,w(\lambda)\,d\lambda}{\sqrt{h_m h_n}}, \qquad 0 \leqslant m, n \leqslant T.$$

The second operator acts in $L^2((a, \Omega), w(\lambda)\,d\lambda)$ by means of the integral kernel

$$S_2(\lambda, \mu) \equiv \sum_{n=0}^{T} \frac{\phi_n(\lambda)\,\phi_n(\mu)}{h_n}.$$

Consider now the problem of finding the eigenfunctions of S_1 or S_2. For arbitrary T, Ω there is no hope of doing this analytically, and one has to resort to numerical methods. Since S_1 is a full matrix and S_2 a very nonlocal operator, this is not an easy problem.

The purpose of this note is to show that an analog of a property uncovered—and put to use—in the case of the Fourier transform by Slepian, Landau and Pollak [1–5] holds here, at least if we are in the case of the classical orthogonal families: Jacobi, Laguerre, Hermite, or (the less well known) Bessel polynomials.

The special case of the Gegenbauer polynomials has been discussed earlier in connection with an extension of the work of Slepian *et al.* to some nonabelian situations [6]. For a somehow different aspect, see [7].

The Slepian, Landau, Pollak Property

The *property* in question is the following one:

(a) For each T, Ω there exists a symmetric tridiagonal matrix $\tilde{S}_1$, with simple spectrum, commuting with S_1.

(b) For each T, Ω there exists a selfadjoint differential operator $\tilde{S}_2$, with simple spectrum, commuting with S_2.

Since we will actually produce these "local" objects explicitly, one could then look for their eigenfunctions using (the very last stages of) an algorithm like $Q-R$. This will produce the eigenfunctions of S_1 and S_2.

As a consequence of the "duality" remarks made earlier, it follows easily that $\tilde{S}_1$ mentioned in (a) can be taken to be the matrix of the operator $\chi_T F^{-1}\tilde{S}_2 F\chi_T$ written in the basis $\phi_n(\lambda)$, $n=0,\ldots,T$. On account of this "duality" between $\tilde{S}_1$ and $\tilde{S}_2$, we will just produce $\tilde{S}_2$ and prove that it commutes with S_2. Part (a) *will then follow.*

This "duality" has been used before in [5] and was used in a very explicit form in Section 5 of [6]: If one sets $m=0$, one obtains the Legendre case. Notice the first diagonal above the main diagonal has all its elements nonzero, guaranteeing a simple spectrum.

A Theorem of Bochner

We do not *really* understand why the property of Slepian, Landau and Pollak holds, but we think this paper brings us closer to the root of this "accident." Indeed, consider the following situation: Let $\phi_n(\lambda)$, $n=0,1,2,\ldots$, be a sequence of (not necessarily orthogonal) polynomials with degree $\phi_n=n$, and assume that there exist functions $p_0(\lambda)$, $p_1(\lambda)$, $p_2(\lambda)$ and constants μ_n such that

$$p_0(\lambda)\frac{d^2}{d\lambda^2}\phi_n(\lambda)+p_1((\lambda)\frac{d}{d\lambda}\phi_n(\lambda)+p_2(\lambda)\,\phi_n(\lambda)=\mu_n\phi_n(\lambda),$$
$$n=0,1,2,\ldots. \quad (1)$$

Bochner [8] proved that this can happen only for very special choices of families $\phi_n(\lambda)$: The classical orthogonal polynomials of Jacobi, Laguerre or Hermite; the less well known Bessel polynomials; or, finally, the powers λ^n. We exclude the powers λ^n, and "keep" the Bessel polynomials, although the latter turn out to be orthogonal with respect to a measure on the unit circle instead of on the real line. Moreover, the moment functional $\mathscr{L}$ which orthogonalizes these polynomials is not positive (see Favard [9]). Strictly speaking, we just observe below that the Bessel polynomials satisfy relation (5′), and ignore in this case the meaning of (5).

Of course, if $\phi_n(\lambda)$ has property (1), then $\phi_n(\lambda)\equiv\phi_n(c+d\lambda)$ has the same property. Bochner proved that these are *all possible families* satisfying (1).

While we have not proved that properties (a), (b) mentioned above hold *only* for orthogonal polynomials of the form

$$\phi_n(c+d\lambda)$$

with ϕ_n Jacobi, Laguerre, Hermite or Bessel, we have a good number of (counter) examples to support this *conjecture.*

A good reference for Bessel polynomials, already introduced by Bochner, is [10]. Another indication of the role of a property similar to (1) is given in [11] (see especially the last paragraph). The theorem of Bochner has an analog for "discrete" orthogonal polynomials, discovered by Lesky [12]: One finds the Poisson–Charlier, Meixner, Krawtchouk and Hahn polynomials. In [13] Perlstadt shows that an analog of the property discussed here holds in all these cases.

Some Properties of the Classical Orthogonal Polynomials

Recall that for any family of orthogonal polynomials with

$$\int_a^b \phi_n^2(\lambda)\, w(\lambda)\, d\lambda = h_n$$

and

$$\phi_n(\lambda) = k_n \lambda^n + \cdots$$

one has the following properties.

1. *Christoffel–Darboux Identity*

$$\sum_{n=0}^{T} \frac{\phi_n(\lambda)\,\phi_n(\mu)}{h_n} = \frac{k_T}{k_{T+1}} \frac{\phi_{T+1}(\lambda)\,\phi_T(\mu) - \phi_T(\lambda)\,\phi_{T+1}(\mu)}{h_T(\lambda-\mu)}. \tag{2}$$

For this and other properties of orthogonal polynomials, we follow the notation in [14].

For the cases of interest here we have:

Jacobi $$h_n = \frac{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1},$$

$$k_n = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \frac{1}{2^n} \frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+n+1)};$$

Laguerre $h_n = \Gamma(1+\alpha)\binom{n+\alpha}{n}, \qquad k_n = \frac{(-1)^n}{n!};$

Hermite $h_n = \pi^{1/2}\, 2^n n!, \qquad k_n = 2^n;$

Bessel $h_n = \frac{(-1)^{n+1}\, bn!}{(2n+a-1)(n+a-2)^{(n-1)}}, \qquad k_n = \frac{(2n+a-2)^{(n)}}{b^n}.$

Here

$$a^{(n)} \equiv \frac{\Gamma(a+1)}{\Gamma(a-n+1)}.$$

Notice the sign in front of h_n for the Bessel case.

We need two more properties, which *do not* hold for arbitrary orthogonal polynomials.

2. *Second-Order Differential Equation*

For the classical orthogonal polynomials we have

$$\frac{1}{\rho(\lambda)} \frac{d}{d\lambda} \left(p(\lambda) \frac{d}{d\lambda} \phi_n(\lambda) \right) = \mu_n \phi_n(\lambda) \tag{3}$$

for an appropriate choice of $\rho(\lambda)$, $p(\lambda)$ and μ_n. Explicitly, we get

Jacobi $\quad \rho(\lambda) = (1-\lambda)^\alpha (1+\lambda)^\beta, \quad p(\lambda) = (1-\lambda^2)\rho(\lambda),$
$\quad \mu_n = -n(n+\alpha+\beta+1), \quad \alpha > -1, \quad \beta > -1;$

Laguerre $\quad \rho(\lambda) = e^{-\lambda}\lambda^\alpha, \quad p(\lambda) = e^{-\lambda}\lambda^{\alpha+1}, \quad \mu_n = -n, \quad \alpha > -1;$

Hermite $\quad \rho(\lambda) = e^{-\lambda^2}, \quad p(\lambda) = e^{-\lambda^2}, \quad \mu_n = -2n;$

Bessel $\quad \rho(\lambda) = e^{-\beta/\lambda}\lambda^{\alpha-2}, \quad p(\lambda) = e^{-\beta/\lambda}\lambda^\alpha, \quad \mu_n = n(n+\alpha-1),$
$\quad \alpha \neq 0, -1, -2, \ldots, \quad b \neq 0.$

3. *Diferentiation Formula*

For the classical orthogonal polynomials we have

$$\frac{p(\lambda)}{\rho(\lambda)} \frac{d}{d\lambda} \phi_n(\lambda) = \alpha_n \phi_n(\lambda) + \lambda \beta_n \phi_n(\lambda) + \gamma_{n-1} \phi_{n-1}(\lambda) \tag{4}$$

for an appropriate choice of α_n, β_n, γ_{n-1}. Explicitly, we have

Jacobi $\quad \alpha_n = \dfrac{n(\alpha-\beta)}{\alpha+\beta+2n}, \quad \beta_n = -n, \quad \gamma_{n-1} = \dfrac{2(n+\alpha)(n+\beta)}{(\alpha+\beta+2n)};$

Laguerre $\quad \alpha_n = n, \quad \beta_n = 0, \quad \gamma_{n-1} = -(n+\alpha);$

Hermite $\quad \alpha_n = 0, \quad \beta_n = 0, \quad \gamma_{n-1} = 2n;$

Bessel $\quad \alpha_n = \dfrac{-\beta n}{(2n+\alpha-2)}, \quad \beta_n = n, \quad \gamma_{n-1} = \dfrac{\beta n}{(2n+\alpha-2)}.$

PROOF OF PROPERTY (b)

Consider the second-order differential operator

$$D \equiv D_\lambda \equiv \frac{1}{\rho(\lambda)} \frac{d}{d\lambda} \left((\lambda - \Omega)\, p(\lambda) \frac{d}{d\lambda} \right) + A\lambda$$

with A a constant. Sometimes we use D_λ to stress that D acts on the variable λ.

The property

$$DS_2 = S_2 D \tag{5}$$

on $L^2((a, \Omega), w)$ is equivalent to

$$D_\lambda S_2(\lambda, \mu) = D_\mu S_2(\lambda, \mu) \tag{5$'$}$$

and the pair of relations, coming from the integration by parts.

$$(\lambda - \Omega)\, p(\lambda) \frac{d}{d\lambda} S_2(\mu, \lambda) f(\lambda) \Big|_a^\Omega = 0 \tag{5$''$}$$

and

$$(\lambda - \Omega)\, p(\lambda)\, S_2(\mu, \lambda) \frac{d}{d\lambda} f(\lambda) \Big|_a^\Omega = 0. \tag{5$'''$}$$

Observing that the factor

$$(\lambda - \Omega)\, p(\lambda)$$

vanishes at $\lambda = a$ and $\lambda = \Omega$ for the Jacobi, Laguerre and Hermite cases, one can, with a bit of care, conclude the $DS_2 = S_2 D$ is equivalent to (5$'$). From now on we restrict our attention to (5$'$).

Since we have

$$D_\lambda = (\lambda - \Omega) \frac{1}{\rho(\lambda)} \frac{d}{d\lambda} \left(p(\lambda) \frac{d}{d\lambda} \right) + \frac{p(\lambda)}{\rho(\lambda)} \frac{d}{d\lambda} + A\lambda$$

it follows, upon using (3) and (4), that

$$\begin{aligned}(D_\lambda - D_\mu)\, S_2(\lambda, \mu) = (\lambda - \mu) \Bigg[&\sum_0^T (\mu_n + A + \beta_n) \frac{\phi_n(\lambda)\, \phi_n(\mu)}{h_n} \\ &- \sum_1^T \gamma_{n-1} \frac{\phi_n(\lambda)\, \phi_{n-1}(\mu) - \phi_{n-1}(\lambda)\, \phi_n(\mu)}{\lambda - \mu} \Bigg].\end{aligned}$$

Now using (2) we can express the second summation on the right-hand side as

$$\sum_{n=1}^{T} \gamma_{n-1} \frac{h_{n-1}}{h_n} \frac{k_n}{k_{n-1}} \sum_{\nu=0}^{n-1} \frac{\phi_\nu(\lambda)\,\phi_\nu(\mu)}{h_\nu}.$$

Exchanging the order of summation above gives

$$\sum_{\nu=0}^{T-1} \sum_{n=\nu+1}^{T} \left(\gamma_{n-1} \frac{h_{n-1}}{h_n} \frac{k_n}{k_{n-1}}\right) \frac{\phi_\nu(\lambda)\,\phi_\nu(\mu)}{h_\nu}$$

and thus the vanishing of $(D_\lambda - D_\mu)\, S_2(\lambda, \mu)$ is equivalent to the relations

$$\mu_T + \beta_T + A = 0, \tag{6}$$

$$\sum_{n=\nu+1}^{T} \gamma_{n-1} \frac{h_{n-1}}{h_n} \frac{k_n}{k_{n-1}} = \mu_\nu + A + \beta_\nu, \qquad 0 \leqslant \nu \leqslant T-1. \tag{6$'$}$$

To prove that (6) and (6′) are satisfied for the classical orthogonal polynomials *we will prove* that

$$(\mu_{r-1} - \mu_r) + (\beta_{r-1} - \beta_r) = \gamma_{r-1} \frac{h_{r-1}}{h_r} \frac{k_r}{k_{r-1}}, \qquad r = 1, \ldots, T. \tag{7}$$

If we now pick A so that (6) is satisfied and add up the (*yet to be proved*) relations (7) for $r = \nu + 1, \nu + 2, \ldots, T$ we get a nice telescoping sum giving

$$(\mu_\nu - \mu_T) + (\beta_\nu - \beta_T) = \sum_{r=\nu+1}^{T} \gamma_{r-1} \frac{h_{r-1}}{h_r} \frac{k_r}{k_{r-1}}.$$

But this is just (6′) once A has been picked as indicated above.

All that remains is to prove (7) for each one of the four families. This can be done case by case, and it is completely straightforward. We illustrate with the Bessel case:

$$\mu_r = r(r + \alpha - 1), \qquad \beta_r = r, \qquad \gamma_{n-1} = \frac{\beta n}{(2n + \alpha - 2)},$$

$$\frac{h_{r-1}}{h_r} = -\frac{(2r + \alpha - 1)(r + \alpha - 2)}{r(2r + \alpha - 3)},$$

$$\frac{k_r}{k_{r-1}} = \frac{(2r + \alpha - 2)(2r + \alpha - 3)}{\beta(r + \alpha - 2)}.$$

Then the left-hand side of (7) becomes

$$(r-1)(r+\alpha-2)-r(r-\alpha-1)+(r-1)-r=-(2r+\alpha-1).$$

The right-hand side of (7) gives

$$\begin{aligned}&-\frac{\beta r}{(2r+\alpha-2)}\,\frac{(2r+\alpha-1)(r+\alpha-2)}{r(2r+\alpha-3)}\,\frac{(2r+\alpha-2)(2r+\alpha-3)}{\beta(r+\alpha-2)}\\&\quad=-(2r+\alpha-1)\end{aligned}$$

and (7) is now proved.

In summary, given T and Ω, one can take for $\tilde{S}_2$ in (b) the operator D just constructed. We have

$$\tilde{S}_2=\frac{1}{\rho(\lambda)}\frac{d}{d\lambda}\left((\lambda-\Omega)\,p(\lambda)\frac{d}{d\lambda}\right)+A\lambda$$

with the following prescription for A (see (6)):

Jacobi	$A=T(T+\alpha+\beta+2)$,
Laguerre	$A=T$,
Hermite	$A=2T$,
Bessel	$A=-T(T+\alpha)$.

$\tilde{S}_2$ as given above is a singular Sturm–Liouville operator in (a,Ω) with a simple discrete spectrum.

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