# On a Hilbert Space of Analytic Functions and an Associated Integral Transform
# Part I

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## 1. Introduction

(a) The states of a quantum mechanical system of $n$ degrees of freedom are usually described by functions either in configuration space (real variables $q_1, \cdots, q_n$) or in momentum space (real variables $p_1, \cdots, p_n$). Even in classical mechanics the complex combinations

$$\eta_k = 2^{-1/2}(q_k - ip_k), \qquad \xi_k = 2^{-1/2}(q_k + ip_k) \tag{1}$$

have proved useful. In quantum theory, these combinations are familiar from the treatment of the harmonic oscillator, and in addition they appear as creation and annihilation operators of Bose particles in field theory.

If $q_k$, $p_k$ are selfadjoint operators satisfying the canonical commutation rules

$$[q_k, p_l] = i\delta_{kl}, \quad [q_k, q_l] = 0, \quad [p_k, p_l] = 0$$

(with Planck's constant $h = 2\pi$), then it follows that

$$\xi_k = \eta_k^*, \qquad \eta_k = \xi_k^*, \tag{2}$$

$$[\xi_k, \eta_l] = \delta_{kl}, \quad [\xi_k, \xi_l] = 0, \quad [\eta_k, \eta_l] = 0. \tag{3}$$

As early as 1928,[1] Fock introduced the operator solution $\xi_k = \partial/\partial\eta_k$ of the commutation rule $[\xi_k, \eta_k] = 1$, in analogy to Schrödinger's solution $p_k = -i\partial/\partial q_k$ of the relation $[q_k, p_k] = i$, and applied it to quantum field theory.

(b) It is the purpose of the present paper to study in greater detail the function space $\mathfrak{F}_n$ on which Fock's solution is realized, and its connection with the conventional Hilbert space $\mathfrak{H}_n$ of square integrable functions $\psi(q)$.

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[1]Fock, V., *Verallgemeinerung und Lösung der Diracschen statistischen Gleichung.* Z. Physik, Vol. **49**, **1928**, pp. **339–357**. Focks' method has been further developed by Dirac. See, Dirac, P. A. M., *La seconde quantification*, Ann. Inst. H. Poincaré, Vol. **11**, **1949**, pp. **15–47**.

Since $\xi_k = \partial/\partial\eta_k$, the functions in $\mathfrak{F}_n$ depend only on the variables $\eta_k$, but do not explicitly depend on the $\xi_k$ (see equation (1)), i.e. they are *analytic* functions of the variables $\eta_k = 2^{-1/2}(q_k - ip_k)$. We denote the complex variables by $z_1, \cdots, z_n$, and $z = (z_1, \cdots, z_n)$ stands for a point in the complex $n$-dimensional space $C_n$. We shall also write $z = x+iy$, where $x$, $y$ are points in the real $n$-dimensional Euclidean space $R_n$. Similarly, $q = (q_1, \cdots, q_n) \in R_n$. To a certain extent $C_n$ is reminiscent of the real $2n$-dimensional phase space. The analogy, however, has severe limitations if applied to $\mathfrak{F}_n$, particularly in view of the built-in non-commutativity of $q_k$ and $p_k$.

The following two problems arise: 1) To find a positive real function $\rho_n(x, y)$ which defines the inner product in $\mathfrak{F}_n$

$$(f, g) = \int \overline{f(z)} g(z) \rho_n(x, y) d^n z \tag{4}$$

(where $d^n z$ is the $2n$-dimensional volume element $\prod dx_k \prod dy_k$)[2] such that the operators $z_k$ and $\partial/\partial z_k$ are adjoint. 2) To find an integral kernel $A_n(z, q)$ such that

$$f(z) = \int A_n(z, q) \psi(q) d^n q, \qquad d^n q = dq_1 \cdots dq_n, \tag{5}$$

is a unitary mapping of $\mathfrak{H}_n$ onto $\mathfrak{F}_n$ which properly relates the operators $\eta$, $\xi$ and $z$, $\partial/\partial z$ in the two Hilbert spaces.[3]

(c) DETERMINATION OF $\rho_n$. The defining equations for $\rho_n$ are

$$(z_k f, g) = \left(f, \frac{\partial g}{\partial z_k}\right), \qquad 1 \leqq k \leqq n, \tag{6}$$

which presumably hold for functions $f$, $g$ that do not grow too fast at infinity. From (4),

$$\left(f, \frac{\partial g}{\partial z_k}\right) = \int \frac{\partial}{\partial z_k} (\bar{f} g \rho_n) d^n z - \int \bar{f} g \frac{\partial \rho_n}{\partial z_k} d^n z.$$

(Since $f$ is analytic, $\partial \bar{f}/\partial z_k = 0$.) If, for the functions considered, the first integral vanishes, (6) reduces to

$$\int \bar{z}_k \bar{f} g \rho_n d^n z = -\int \bar{f} g \frac{\partial \rho_n}{\partial z_k} d^n z. \tag{6a}$$

This suggests[4] $\partial \rho_n/\partial z_k = -\bar{z}_k \rho_n$ or

[2]Unless the domain of integration is explicitly indicated all integrals extend over the whole range of the integration variables, i.e., $C_n$ for $z$, and $R_n$ for $q$.

[3]The following discussion is merely heuristic, in particular because the operators and their domains are not clearly defined. The precise definitions will be given in §§ 1–3.

[4]The question under what conditions these equations follow from (6a) will not be analyzed. For our purpose it is sufficient to know that the Gaussian in (7) satisfies the equation (6a).

$$\tfrac{1}{2}\left(\frac{\partial\rho_n}{\partial x_k}-i\frac{\partial\rho_n}{\partial y_k}\right)=-(x_k-iy_k)\rho_n, \qquad z_k=x_k+iy_k,$$

i.e.,

$$\frac{\partial\rho_n}{\partial x_k}=-2x_k\rho_n, \qquad \frac{\partial\rho_n}{\partial y_k}=-2y_k\rho_n.$$

Consequently,

$$\rho_n=c\exp\{-\bar z\cdot z\}, \tag{7}$$

where $\bar z\cdot z=\sum_k\bar z_kz_k=\sum_k(x_k^2+y_k^2)$, $c=$ const.

(d) DETERMINATION OF THE KERNEL $A_n$. The following conditions are to be satisfied. If the integral transform (5) maps $\psi$ into $f$, it maps $\eta_k\psi$ into $z_kf$, and $\xi_kf$ into $\partial f/\partial z_k$, where (by (1))

$$\eta_k=2^{-1/2}\left(q_k-\frac{\partial}{\partial q_k}\right), \qquad \xi_k=2^{-1/2}\left(q_k+\frac{\partial}{\partial q_k}\right).$$

It is again assumed that $\psi$ is sufficiently smooth and vanishes sufficiently fast at infinity. Since $\xi_k$ and $\eta_k$ are adjoint, we have

$$\int A_n(\eta_k\psi)d^nq=\int(\xi_kA_n)\psi\,d^nq=z_kf=\int z_kA_n\psi\,d^nq, \tag{8a}$$

$$\int A_n(\xi_k\psi)d^nq=\int(\eta_kA_n)\psi\,d^nq=\frac{\partial f}{\partial z_k}=\int\frac{\partial A_n}{\partial z_k}\psi\,d^nq. \tag{8b}$$

Hence we conclude that

$$z_kA_n=\xi_kA_n=2^{-1/2}\left(q_kA_n+\frac{\partial A_n}{\partial q_k}\right),$$

$$\frac{\partial A_n}{\partial z_k}=\eta_kA_n=2^{-1/2}\left(q_kA_n-\frac{\partial A_n}{\partial q_k}\right),$$

or

$$\frac{\partial A_n}{\partial q_k}=(2^{1/2}z_k-q_k)A_n, \qquad \frac{\partial A_n}{\partial z_k}=(2^{1/2}q_k-z_k)A_n, \tag{9}$$

$$A_n(z,q)=c'\exp\{-\tfrac{1}{2}(z^2+q^2)+2^{1/2}z\cdot q\}, \qquad c'=\text{const.}$$

(The constants, $c$, $c'$ in (7) and (9) will be chosen as $\pi^{-n}$ and $\pi^{-n/4}$, respectively.)

REMARK ON THE NOTATION. The elements of real or complex $n$-dimensional Euclidean space ($R_n$ or $C_n$) will be called points or vectors (synonymously); $a\cdot b=\sum_{k=1}^n a_kb_k$ is the scalar product of two vectors $a$, $b$, and $a^2=a\cdot a$.

(e) The determination of $\rho_n$ and of $A_n$ concludes the introductory heuristic discussion. The rest of the paper is concerned with a study of the Hilbert space $\mathfrak{F}_n$—defined by the inner product (4)—and the isomorphic mapping (5) of $\mathfrak{H}_n$ onto $\mathfrak{F}_n$, which are of intrinsic mathematical interest. In addition, the Hilbert space $\mathfrak{F}_n$ provides a useful tool for quantum-mechanical and for group theoretical problems.

Part I consists of three sections. In § 1, I survey the basic properties of $\mathfrak{F}_n$, § 2 is devoted to the mapping (5), in particular to the proof of its unitarity, and in § 3 various operators on $\mathfrak{F}_n$ are considered. The main advantage of $\mathfrak{F}_n$ seems to be the relative ease with which operators (and their domains) can be rigorously defined. This has to do with the strong correlation of local and global properties which is characteristic for analytic functions.

Part II and Part III will appear later. Part II is concerned with harmonic polynomials in $\mathfrak{F}_n$ and the corresponding decomposition of $\mathfrak{F}_n$. Part III utilizes $\mathfrak{F}_n$ for an analysis of the rotation group, centering about the Wigner- and Racah-coefficients. Its methods are closely related to the very interesting procedure used by Schwinger[5] in his treatment of the rotation group.

## 1. The Hilbert Space $\mathfrak{F}_n$ [6]

1a. *Preliminary remarks.* Before turning to the Hilbert space $\mathfrak{F}_n$, I list a somewhat odd collection of known theorems which will be frequently used in the sequel.

The first is entirely elementary.

A. *Let* $S = \sum_{k=1}^{\infty} b_k$ *be a series with non-negative real terms, let* $\gamma_{ki}$, $i = 1, 2, \cdots$, *be so chosen that* 1) $0 \leqq \gamma_{ki} \leqq 1$, 2) $\lim_{i\to\infty} \gamma_{ki} = 1$, *and set*

$$S_i = \sum_k \gamma_{ki} b_k .$$

*$S$ converges if and only if the $S_i$ are uniformly bounded, and in that case* $S = \lim S_i$.

An obvious variant applies if the $\gamma_k$ are functions $\gamma_k(\tau)$ of a continuous variable $\tau$.

The second theorem concerns Laplace integrals in $n$ variables. Consider a quadratic form

$$T(x) = \sum_{k,l} t_{kl} x_k x_l , \qquad t_{kl} = t_{lk},$$

[5] Schwinger, J., *On angular momentum,* New York Univ. Inst., Math. Sci., AEC Comp. Appl. Math. Center, Res. Rep. No. NYO-3071, 1952.

[6] The content of this section is closely related to S. Bergman's work on Hilbert space methods in the theory of analytic functions. See, for example, Bergman, S., *The Kernel Function and Conformal Mapping,* Mathematical Surveys No. 5, Amer. Math. Soc., New York, 1950.

and a linear form $b \cdot x = \sum_k b_k x_k$ in $n$ real variables, with complex coefficients $t_{kl}$ and $b_k$, and set

$$L(T, b) = \int_{R_n} \exp\{-T(x)+2b \cdot x\} d^n x. \tag{1.1}$$

B. *The integral $L(T, b)$ is absolutely convergent if and only if the real part of $T$ is positive definite. Then* det $T$ (*the determinant of* $T$) $\neq 0$, *and*

$$L(T, b) = \pi^{n/2}(\det T)^{-1/2} \exp\{T^{-1}(b)\}. \tag{1.1a}$$

Here, $T^{-1}$ is the quadratic form constructed with the inverse matrix, and the square root of det $T$ may be defined as follows. Decompose $T$ into its real and imaginary parts, $T = T'+iT''$, consider $T(\alpha) = T'+i\alpha T''$ for a real parameter $\alpha$, $0 \leqq \alpha \leqq 1$, and construct $(\det T)^{-1/2}$ by analytic continuation of $(\det T(\alpha))^{-1/2}$ starting with a positive value for $\alpha = 0$. If $n = 1$, and $T = tx^2$, this amounts to requiring that $t^{-1/2}$ has a *positive real part.*

REMARK. The computation of $L$ in (1.1a) may be considerably simplified by an appropriate, real or complex, linear transformation of the variables $x$. Let $x' = Wx$, i.e. $x'_k = \sum w_{kl} x_l$, so that $T(x) = T'(x')$, and $b \cdot x = b' \cdot x'$. Then $T^{-1}(b) = T'^{-1}(b')$, but det $T =$ (det $W)^2$ det $T'$.

C. The third and last theorem to be mentioned deals with integrals of the form

$$F(z) = \int_D f(z, \tau) d^k \tau.$$

$D$ is a measurable set in $R_k$, $z = (z_1, \cdots, z_m)$ a point in an open set $E$ of $C_m$. *Assume that, for every $z$ in a neighborhood $N$ ($|z_j - b_j| < \rho_j$) of the point $b$ in $E$, $f$ is analytic in $z$ for every $\tau$, measurable in $\tau$, and*

$$|f(z, \tau)| \leqq \eta(\tau), \qquad |z_j - b_j| < \rho_j,$$

*where $\eta$ is summable over $D$. Then $F(z)$ is analytic in $N$, and its partial derivatives are obtained by differentiating under the sign of integration, the resulting integrals being summable.*

It follows in particular that the power series expansion of $F$ in $(z_j - b_j)$ may be obtained by expanding $f$ in a power series and interchanging summation and integration. No further convergence proof is needed. In many of the later applications the construction of an appropriate $\eta(\tau)$ is quite straightforward and will not be carried out explicitly.

## 1b. *Basic properties of $\mathfrak{F}_n$.*

DEFINITION.[7] The elements of $\mathfrak{F}_n$ are entire analytic functions $f(z)$,

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[7]The Hilbert space defined here has already been used by I. E. Segal for a representation of the quantum mechanical canonical operators. (Lectures at the Summer Seminar on Applied Mathematics, 1960, Boulder, Colorado.)

$z = (z_1, \cdots, z_n) = x+iy$. The inner product is defined by

$$(1.2) \qquad (f, g) = \int \overline{f(z)} g(z) d\mu_n(z), \qquad d\mu_n(z) = \rho_n d^n x d^n y, \qquad \rho_n = \pi^{-n} \exp\{-\bar{z} \cdot z\}.$$

The integral extends over $C_n$. $f$ belongs to $\mathfrak{F}_n$ if $(f, f) < \infty$. Its norm is $||f|| = \sqrt{(f, f)}$. Strong convergence in Hilbert space will be denoted by a double arrow, e.g. $f_j \Rightarrow g$, or equivalently by $g = \mathrm{Lim}\, f_j$, point wise convergence by a single arrow, $f_j(z) \to g(z)$, or by $g(z) = \lim f_j(z)$.

Let $f$ be an entire function with the power series

$$(1.3) \qquad f(z) = \sum_{m_i} \alpha_{m_1 m_2 \cdots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}.$$

It will be convenient to use a shorthand notation and to write $m$ for the sequence $(m_1, m_2, \cdots, m_n)$ of non-negative inters, $\alpha_{[m]}$ for the expansion coefficient, and $z^{[m]}$ for the product $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. We shall also write $[m!]$ for $m_1!\, m_2! \cdots m_n!$, and set $|m| = m_1+m_2+ \cdots +m_n$. Thus

$$(1.3a) \qquad f(z) = \sum_m \alpha_{[m]} z^{[m]}$$

To express the inner product in terms of the expansion coefficients, we first compute

$$M(\sigma) = \int_{|z_k| \leqq \sigma} |f(z)|^2 d\mu_n(z), \qquad 0 < \sigma < \infty.$$

Then $(f, f) = \lim_{\sigma \to \infty} M(\sigma)$. Inserting (1.3a) and using polar coordinates $z_k = r_k e^{i\phi_k}$, we find

$$M(\sigma) = \sum_{m, m'} \bar{\alpha}_{[m]} \alpha_{[m']} \theta_{mm'}(\sigma),$$

$$\theta_{mm'}(\sigma) = \prod_k \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} e^{i(m'_k - m_k)\phi} d\phi \int_0^\sigma e^{-r^2} r^{m_k + m'_k + 1} dr \right\}.$$

It follows that $\theta_{mm'}(\sigma) = 0$ if $m \neq m'$, and that

$$\theta_{mm}(\sigma) = \gamma_{[m]}(\sigma) \prod_k m_k! = \gamma_{[m]}(\sigma)[m!], \qquad 0 < \gamma_{[m]}(\sigma) < 1,$$

where $\lim_{\sigma \to \infty} \gamma_{[m]}(\sigma) = 1$. Hence

$$M(\sigma) = \sum_m [m!] |\alpha_{[m]}|^2 \gamma_{[m]}(\sigma)$$

and, by A in § 1a,

$$(1.4) \qquad (f, f) = \sum_m [m!] |\alpha_{[m]}|^2$$

i.e., either both sides of (1.4) are infinite, or both sides are finite and equal.

Every set of coefficients for which the sum in (1.4) converges defines an entire function $f \in \mathfrak{F}_n$. By linearity we obtain for the inner product of two functions $f$, $g$

$$(1.5)\qquad (f, g) = \sum_m [m!]\overline{\alpha_{[m]}}\beta_{[m]}, \qquad g(z) = \sum \beta_{[m]} z^{[m]}.$$

The simplest *orthonormal* set of vectors in $\mathfrak{F}_n$ is given by

$$(1.6)\qquad u_{[m]}(z) = \frac{z^{[m]}}{\sqrt{[m!]}} = \prod_k \frac{z_k^{m_k}}{\sqrt{m_k!}}.$$

For any function $f \in \mathfrak{F}_n$, $(u_{[m]}, f) = [m!]^{1/2}\alpha_{[m]}$, so that (1.4) expresses the completeness of the system $u_{[m]}$.

Let $\mathfrak{P}_l$ be the subspace of homogeneous polynomials of order $l$. Then the $\mathfrak{P}_l$ are mutually orthogonal, and $\mathfrak{F}_n = \sum_{l=0}^{\infty} \mathfrak{P}_l$.

AN INEQUALITY. From (1.3a) we obtain, by Schwarz' inequality,

$$|f(z)|^2 \leqq (\sum |\alpha_{[m]} z^{[m]}|)^2 \leqq (\sum [m!]|\alpha_{[m]}|^2) \cdot \left(\sum \frac{|z^{[m]}|^2}{[m!]}\right).$$

The last sum equals $\exp\{\sum_k \bar{z}_k z_k\} = e^{\bar{z}\cdot z}$. Thus by (1.4),

$$(1.7)\qquad |f(z)| \leqq e^{\bar{z}\cdot z/2}\,||f||.$$

Similarly,

$$(1.7a)\qquad \left|\frac{\partial f}{\partial z_k}\right|^2 \leqq (1+\bar{z}_k z_k) e^{\bar{z}\cdot z}\,||f||^2.$$

**1c.** ***Principal vectors, and the reproducing kernel.*** A relation of the form

$$(1.7b)\qquad |f(z)| \leqq \omega(z)\,||f||$$

is quite typical for a Hilbert space of analytic functions, and the main conclusions to be drawn do not depend on the specific form of $\omega$.

First of all, *strong* convergence in $\mathfrak{F}_n$ implies *pointwise* convergence, because

$$|f(z) - g(z)| \leqq \omega(z)\,||f - g||$$

for any $f, g \in \mathfrak{F}_n$, and by (1.7) the convergence is uniform on any compact set.

Secondly, for a fixed $a$ in $C_n$, the mapping $f \to f(a)$ defines a bounded linear functional. It is necessarily of the form

$$(1.8)\qquad f(a) = (\boldsymbol{e}_a, f)$$

with a uniquely defined $\boldsymbol{e}_a \in \mathfrak{F}_n$. Conversely, (1.8) implies (1.7b), with

$\omega(a) = \|\mathbf{e}_a\|$. The vectors $\mathbf{e}_a$ will be called the *principal vectors* of $\mathfrak{F}_n$. In many ways they resemble a continuous set of orthonormal vectors. In particular,

$$(f, g) = \int (f, \mathbf{e}_a)(\mathbf{e}_a, g) d\mu_n(a). \tag{1.8a}$$

The $\mathbf{e}_a$ are complete, i.e. their finite linear combinations are dense in $\mathfrak{F}_n$, because the only vector orthogonal to all of them is $f = 0$. In integral form, (1.8) reads

$$f(w) = \int \mathscr{K}(w, z) f(z) d\mu_n(z), \qquad \mathscr{K}(w, z) = \overline{\mathbf{e}_w(z)}, \tag{1.9}$$

where the "*reproducing kernel*" $\mathscr{K}$ is—apart from an insignificant difference in the notation—S. Bergman's kernel function for $\mathfrak{F}_n$. Note that, by the definition (1.8), $\mathbf{e}_w(z) = (\mathbf{e}_z, \mathbf{e}_w)$. Thus

$$\mathscr{K}(w, z) = \overline{\mathscr{K}(z, w)} = (\mathbf{e}_w, \mathbf{e}_z). \tag{1.9a}$$

$\mathscr{K}$ is analytic in $w$ and $\bar{z}$. (It is the analog of the $n$-dimensional delta function $\delta(q-q')$ for the customary Hilbert space of quantum mechanics.)

In terms of any complete orthonormal set $v_1, v_2, \cdots$

$$\mathbf{e}_a = \lim_{k\to\infty} \sum_{h=1}^{k} (v_h, \mathbf{e}_a) v_h = \lim_{k\to\infty} \sum_{h=1}^{k} \overline{v_h(a)} v_h$$

(by (1.8)). Since strong convergence implies pointwise convergence,

$$\mathbf{e}_a(z) = \sum_h \overline{v_h(a)} v_h(z) \tag{1.9b}$$

irrespective of the choice of the $v_h$.

Using the set $u_{[m]}$ in $\mathfrak{F}_n$, one finds

$$\begin{aligned} \mathbf{e}_a(z) &= \sum_m \prod_k \frac{(\bar{a}_k z_k)^{m_k}}{m_k!} = e^{\bar{a}\cdot z}, \\ \mathscr{K}(w, z) &= e^{w\cdot\bar{z}}. \end{aligned} \tag{1.10}$$

1d. *Bounded linear operators on* $\mathfrak{F}_n$. Let $L$ be a bounded linear operator on $\mathfrak{F}_n$, and $L^*$ its adjoint. With the help of the principal vectors, $L$ may be represented as an integral transform. For any $f \in \mathfrak{F}_n$,

$$(Lf)(w) = (\mathbf{e}_w, Lf) = (L^*\mathbf{e}_w, f),$$

so that

$$(Lf)(w) = \int L(w, z) f(z) d\mu_n(z), \tag{1.11}$$

where $L(w, z) = \overline{(L^* \boldsymbol{e}_w)(z)} = \overline{(\boldsymbol{e}_z, L^* \boldsymbol{e}_w)} = \overline{(L\boldsymbol{e}_z, \boldsymbol{e}_w)}$, or

$$L(w, z) = (\boldsymbol{e}_w, L\boldsymbol{e}_z) = (L\boldsymbol{e}_z)(w). \tag{1.11a}$$

$L(w, z)$ is analytic in $w$ and $\bar{z}$. If $L = 1$, then $L(w, z) = \mathscr{K}(w, z)$. The two integrals

$$\int |L(w, z)|^2 d\mu_n(z) \quad \text{and} \quad \int |L(z, w)|^2 d\mu_n(z) \tag{1.11b}$$

are finite for every $w$. If $M = L^*$, then

$$M(w, z) = \overline{L(z, w)}, \tag{1.12}$$

and if $N = ML$, for any two bounded operators $L$ and $M$, then

$$N(w, w') = \int M(w, z) L(z, w') d\mu_n(z). \tag{1.12a}$$

$L$ is unitary if and only if $LL^* = L^*L = 1$, i.e.

$$\int L(w, z)\overline{L(w', z)} d\mu_n(z) = \int \overline{L(z, w)} L(z, w') d\mu_n(z) = \mathscr{K}(w, w'). \tag{1.13}$$

So far we have started from a given operator $L$ and have constructed the associated integral kernel $L(w, z)$. We turn now to the converse problem, to determine $L$ from a given kernel. The following may be asserted:

THEOREM 1.1. *Let $\kappa$ be a positive constant, and $h_a \in \mathfrak{F}_n$ a set of vectors (defined for every $a \in C_n$) satisfying the following condition: For every finite set $a_\nu \in C_n$, $\nu = 1, \cdots, k$, and every set of complex constants $\gamma_\nu$*

$$\|\sum_{\nu=1}^{k} \gamma_\nu h_{a_\nu}\| \leqq \kappa \|\sum_{\nu=1}^{k} \gamma_\nu \boldsymbol{e}_{a_\nu}\|. \tag{1.14}$$

*Then there exists a uniquely defined bounded operator $L$ on $\mathfrak{F}_n$ (with bound $\leqq \kappa$) such that $L\boldsymbol{e}_a = h_a$, and hence $L(w, z) = h_z(w)$.*

Proof: The uniqueness of $L$ is clear, because we must have $L\boldsymbol{e}_a = h_a$, and

$$L(\sum_{\nu=1}^{k} \gamma_\nu \boldsymbol{e}_{a_\nu}) = \sum_{\nu=1}^{k} \gamma_\nu h_{a_\nu}, \tag{1.14a}$$

where we may assume $a_\nu \neq a_\mu$ if $\nu \neq \mu$. Now every finite set of distinct principal vectors $\boldsymbol{e}_{a_\nu}$ is linearly independent. Consequently, (1.14a) defines $L$ unambiguously for all finite linear combinations of principal vectors, and, by (1.14), $L$ is bounded on this set. Since this set is dense in $\mathfrak{F}_n$, $L$ is uniquely extended to all $f \in \mathfrak{F}_n$ by closure.

The boundedness condition (1.14) is clearly satisfied if $(h_a, h_b) = (\boldsymbol{e}_a, \boldsymbol{e}_b)$ in which case $\kappa = 1$ and $L$ is isometric.

1e. *Decomposition of* $\mathfrak{F}_n$. To every decomposition of $n$ into the sum of two positive integers, $n = n'+n''$, corresponds a decomposition of $\mathfrak{F}_n$ into the product

$$\mathfrak{F}_n = \mathfrak{F}_{n'} \otimes \mathfrak{F}_{n''}. \tag{1.15}$$

In fact, if we set $z' = (z_1, \cdots, z_{n'})$, $z'' = (z_{n'+1}, \cdots, z_n)$, then $d\mu_n(z) = d\mu_{n'}(z')d\mu_{n''}(z'')$ (see (1.2)). Similarly, the kernel $\mathscr{K}$ is the product of two kernels $\mathscr{K}'$, $\mathscr{K}''$, and the functions $\boldsymbol{e}_a(z)$ as well as the orthonormal vectors $u_{[m]}$ are decomposed accordingly. The decomposition (1.15) may be continued. Specifically,

$$\mathfrak{F}_n = \mathfrak{F}_1 \otimes \mathfrak{F}_1 \otimes \cdots \otimes \mathfrak{F}_1 \qquad (n \text{ factors}). \tag{1.15a}$$

1f. *Characteristic sets.*

DEFINITION. A pointset $S_n \subset C_n$ is called characteristic if

$$f(a) = 0 \quad \text{for all} \quad a \in S_n, \qquad f \in \mathfrak{F}_n,$$

implies that $f = 0$.

If $S_n$ is characteristic, then the vectors $\boldsymbol{e}_a$, $a \in S_n$, are complete in $\mathfrak{F}_n$, because $(\boldsymbol{e}_a, f) = f(a) = 0$ implies that $f = 0$.

It is easy to construct such sets. In view of the decomposition (1.15a) one may take $S_n$ as a Cartesian product $S_n = S_1' \times S_1'' \times \cdots \times S_1^{(n)}$, where each $S_1^{(k)}$ is characteristic in $C_1$.

Examples for a characteristic $S_1$ are: 1) Any infinite sequence of points in the complex plane which converge to a finite limit. 2) Any infinite sequence $a_\nu$, $a_\nu \neq 0$, such that

$$\sigma = \sum_{\nu=1}^{\infty} |a_\nu|^{-2-\eta} = \infty$$

for some positive $\eta$. (Since $f \in \mathfrak{F}_1$ is an entire function of order $\leqq 2$, $f(a_\nu) = 0$ and $\sigma = \infty$ imply $f = 0$.)

WEAK AND STRONG CONVERGENCE IN $\mathfrak{F}_n$. If a sequence $g_\nu$ in $\mathfrak{F}_n$ converges weakly to $g$, it follows from (1.8) that, for every $a$, $g_\nu(a)$ converges to $g(a)$. In terms of characteristic sets, the following sufficient condition may be derived:

THEOREM 1.2. *Let $g_\nu$ be a sequence of elements in $\mathfrak{F}_n$ which satisfies the following conditions*: 1) $\|g_\nu\| < \gamma$, *for some positive* $\gamma$. 2) *For every point $a$ of some characteristic set $S_n$, $g_\nu(a)$ is convergent. Then the sequence $g_\nu$ has a weak limit $g \in \mathfrak{F}_n$, and, on any compact set in $C_n$, $g_\nu(z)$ converges uniformly to $g(z)$. If, in addition,* $\lim \|g_\nu\| = \|g\|$, *the sequence is strongly convergent.*

The proof is rather straightforward and will be omitted.

## 1g. *The classes* $\mathfrak{G}_\lambda$.

DEFINITION. An entire analytic function $f(z)$ belongs to the class $\mathfrak{G}_\lambda$, $0 < \lambda < 1$, if

$$(1.16) \qquad |f(z)| \leqq \gamma e^{(1/2)\lambda^2 \bar{z}\cdot z}, \quad \text{for all } z \in C_n,$$

for a suitable positive $\gamma$.

$\mathfrak{G}_\lambda \subset \mathfrak{F}_n$, because the inequality (1.16) implies the convergence of $(f, f)$ (see (1.2)).

Define, for any $f$, the function $f_\lambda$ by

$$(1.17) \qquad f_\lambda(z) = f(\lambda z), \qquad 0 < \lambda < 1.$$

If $f \in \mathfrak{F}_n$, then (by (1.7)) $f_\lambda \in \mathfrak{G}_\lambda$. From (1.4),

$$(f_\lambda, f_\lambda) = \sum_m [m!]\lambda^{2|m|}|\alpha_{[m]}|^2, \qquad |m| = m_1 + \cdots + m_n.$$

We conclude therefore (see A in § 1a): $f$ belongs to $\mathfrak{F}_n$ if and only if all $f_\lambda \in \mathfrak{F}_n$, $0 < \lambda < 1$, and their norms $\|f_\lambda\|$ are uniformly bounded. (This will prove a useful criterion.)

Furthermore $\|f - f_\lambda\|^2 = \sum_m [m!](1-\lambda^{|m|})^2|\alpha_{[m]}|^2$. Hence, if $f \in \mathfrak{F}_n$, then $f_\lambda \Rightarrow f$ as $\lambda \to 1$.

## 1h. *Evaluation of some integrals.*

In order not to interrupt the following discussion, I insert here the evaluation of some integrals which will frequently occur. Let

$$(1.18) \qquad I_n(\gamma, \delta; a, b) = \int e^{(1/2)\gamma z^2 + a\cdot z} e^{(1/2)\delta \bar{z}^2 + b\cdot \bar{z}} d\mu_n(z),$$

where $\gamma$, $\delta$ are complex constants, and $a, b \in C_n$. This is an integral in the $2n$ real variables $x$, $y$ of the form (1.1). Since $I_n = \prod_{k=1}^n I_1(\gamma, \delta; a_k, b_k)$, the computation is straightforward. It leads to the following result:

1) $I_n$ is absolutely convergent if and only if

$$(1.18a) \qquad |\gamma + \delta|^2 < 4.$$

2) If (1.18a) holds, then

$$(1.18b) \qquad I_n = (1-\gamma\delta)^{-n/2} \exp\left\{\frac{\delta a^2 + \gamma b^2 + 2a\cdot b}{2(1-\gamma\delta)}\right\},$$

$$(1.18c) \qquad (1-\gamma\delta)^{-1/2} \text{ has positive real part.}$$

(From (1.18a), $\mathscr{Re}(1-\gamma\delta) = 1 - \frac{1}{4}|\gamma+\delta|^2 + \frac{1}{4}|\gamma-\delta|^2 > 0$.) Set $\gamma = \delta$, $a = b$. By (1.18a), $e^{(1/2)\gamma z^2 + a\cdot z} \in \mathfrak{F}_n$ if and only if $|\gamma| < 1$.

Let $A_n(z, q) = \pi^{-n/4} \exp\{-\frac{1}{2}(z^2+q^2) + 2^{1/2} z\cdot q\}$, and set

(1.19) $$J_n(\alpha, \beta; p, q) = \int A_n(\alpha z, p) A_n(\bar{\beta}\bar{z}, \bar{q}) d\mu_n(z),$$

where $\alpha$, $\beta$ are complex constants, and $p, q \in C_n$. Then

$$J_n(\alpha, \beta; p, q) = \pi^{-n/2} \exp\{-\tfrac{1}{2}(p^2+\bar{q}^2)\} I_n(-\alpha^2, -\beta^2; 2^{1/2}\alpha p, 2^{1/2}\beta q).$$

From (1.18) we find:

1) $J_n$ is absolutely convergent if and only if

(1.19a) $$|\alpha^2+\beta^2| < 2.$$

2) $J_n$ depends only on the combination $\kappa = \alpha\bar{\beta}$ and may be expressed in the form

(1.19b) $$J_n(\alpha, \beta; p, q) = \sigma_n(\alpha\bar{\beta}, p, q),$$

(1.19c) $$\sigma_n(\kappa, p, q) = [\pi(1-\kappa^2)]^{-n/2} \exp\left\{-\frac{1}{4}\left[\frac{1-\kappa}{1+\kappa}(p+\bar{q})^2 + \frac{1+\kappa}{1-\kappa}(p-\bar{q})^2\right]\right\},$$

(1.19d) $(1-\kappa^2)^{-1/2}$ has positive real part.

By (1.19a), $A_n(\alpha z, q) \in \mathfrak{F}_n$ if and only if $|\alpha| < 1$.

## 2. The Mapping $A_n$ of $\mathfrak{H}_n$ onto $\mathfrak{F}_n$

This section is devoted to an analysis of the integral transform (5) and its inverse, and to the proof that it defines a unitary mapping of $\mathfrak{H}_n$ onto $\mathfrak{F}_n$.

$\mathfrak{H}_n$ is the Hilbert space $L_2(R_n)$ based on the inner product

$$(\psi_1, \psi_2) = \int \overline{\psi_1(q)} \psi_2(q) d^n q.$$

2a. *The kernel* $A_n(z, q)$. We note first that, for fixed $z$, the kernel

(2.1) $$A_n(z, q) = \pi^{-n/4} \exp\{-\tfrac{1}{2}(z^2+q^2) + 2^{1/2} z \cdot q\}$$

belongs to $\mathfrak{H}_n$. One readily verifies that

(2.2) $$\int A_n(z, q) \overline{A_n(w, q)} d^n q = e^{z \cdot \bar{w}},$$

which will prove important.

The transformation $f = \boldsymbol{A}_n \psi$ is defined by

(2.3) $$f(z) = (\boldsymbol{A}_n \psi)(z) = \int A_n(z, q) \psi(q) d^n q$$

for any $\psi \in \mathfrak{H}_n$. Since $A_n \in \mathfrak{H}_n$, the integral is always defined, and by Schwarz' inequality (set $w = z$ in (2.2))

$$(2.4) \qquad |f(z)| \leqq e^{\bar{z}\cdot z/2}\|\psi\|.$$

Moreover, $f(z)$ is analytic, as follows from C in § 1a. Since

$$|2^{1/2}z\cdot q| \leqq 2\bar{z}\cdot z+\tfrac{1}{4}q^2,$$

$|A_n|$ is majorized by $\pi^{-n/4}\exp(\frac{5}{2}\alpha^2-\frac{1}{4}q^2)$ for $\bar{z}\cdot z \leqq \alpha^2$. From (2.4) we conclude that $f_\lambda \in \mathfrak{G}_\lambda \subset \mathfrak{F}_n$ for $0<\lambda<1$ (see § 1g). It remains to show that $f$ itself belongs to $\mathfrak{F}_n$.

2b. *The class* $\mathfrak{C}$. We first restrict $\psi$ (in (2.3)) to the class $\mathfrak{C}$ of continuous functions with compact support, which is, of course, dense in $\mathfrak{H}_n$. Let $\psi \in \mathfrak{C}$, and assume $\psi=0$ outside the sphere $D_r$. ($D_r$ contains the points $q^2<r^2<\infty$.) Consider the integral

$$F_\lambda = (f_\lambda, f_\lambda) = \int |f(\lambda z)|^2 d\mu_n(z), \qquad 0<\lambda<1.$$

Inserting (2.3), we have

$$F_\lambda = \int \overline{A_n(\lambda z, q)}A_n(\lambda z, p)\overline{\psi(q)}\psi(p)d\mu_n(z)d^nq d^np.$$

This integral is absolutely convergent because the $p$- and $q$-integrations extend only over $D_r$, and the integral over $z$ is absolutely convergent (see (1.19a), for $\alpha=\beta=\lambda$). Carrying out the $z$-integration, we obtain (by (1.19c))

$$(2.5) \qquad F_\lambda = \int \sigma_n(\lambda^2, p, q)\overline{\psi(q)}\psi(p)d^np d^nq.$$

It will be convenient to write $\sigma_n$ in the form

$$\sigma_n(\lambda^2, p, q) = \{(1+\varepsilon^2)^n e^{-\varepsilon^2 s^2}\}\{(2\varepsilon\pi^{1/2})^{-n}e^{-t^2/\varepsilon^2}\},$$

$$(2.6) \qquad \varepsilon = \left(\frac{1-\lambda^2}{1+\lambda^2}\right)^{1/2}, \qquad s=\tfrac{1}{2}(p+q), \qquad t=\tfrac{1}{2}(p-q).$$

(As $\lambda \to 1$, $\varepsilon \to 0$, and $\sigma_n$ approaches the $n$-dimensional delta-function $\delta(p-q)$. This is the main point of the argument.)

Introduce the variables $s$, $t$, and insert (2.6) in the integral $F_\lambda$. (Note that $s$, $t$ range only over $D_r$.) Then

$$F_\lambda = (1+\varepsilon^2)^n\int_{D_r} e^{-\varepsilon^2 s^2}N_\varepsilon(s)d^ns,$$

$$N_\varepsilon(s) = (\varepsilon\pi^{1/2})^{-n}\int e^{-t^2/\varepsilon^2}\overline{\psi(s-t)}\psi(s+t)d^nt$$

$$= \pi^{-n/2}\int e^{-(t')^2}\overline{\psi(s-\varepsilon t')}\psi(s+\varepsilon t')d^nt'.$$

On $D_r$, $N_\varepsilon(s)$ converges uniformly to $|\psi(s)|^2$ as $\varepsilon \to 0$, so that

$$\lim_{\lambda\to 1} F_\lambda = \lim_{\lambda\to 1} ||f_\lambda||^2 = \int |\psi(s)|^2 d^n s = ||\psi||^2.$$

It follows that the $||f_\lambda||$ are uniformly bounded. Hence $f \in \mathfrak{F}_n$ (see § 1g), and $||f||^2 = \lim ||f_\lambda||^2 = ||\psi||^2$. Thus $\boldsymbol{A}_n$ is isometric on $\mathfrak{C}$.

2c. *Isometry on* $\mathfrak{H}_n$. Let $\psi_0$ be an element of $\mathfrak{H}_n$. There exists a sequence $\psi_j \in \mathfrak{C}$ which strongly converges to $\psi_0$. Set $f_0 = \boldsymbol{A}_n\psi_0$, and $f_j = \boldsymbol{A}_n\psi_j$. It follows from the isometry of $\boldsymbol{A}_n$ on $C$ that

$$||f_i - f_j|| = ||\boldsymbol{A}_n(\psi_i - \psi_j)|| = ||\psi_i - \psi_j||.$$

Hence $f_j$ is a strongly convergent sequence in $\mathfrak{F}_n$, with limit $g$, say. For every $z$, $g(z) = \lim f_j(z)$. From the inequality (2.4), we conclude that

$$|f_0(z) - f_j(z)| \leqq e^{\bar{z}\cdot z/2} ||\psi_0 - \psi_j||,$$

thus $f_0 = g$. Consequently,

$$||f_0|| = \lim ||f_j|| = \lim ||\psi_j|| = ||\psi_0||.$$

This establishes the isometry of $\boldsymbol{A}_n$ on all of $\mathfrak{H}_n$.

2d. *Unitarity of* $\boldsymbol{A}_n$. To prove the unitarity of $\boldsymbol{A}_n$, it is sufficient to show that its range is dense in $\mathfrak{F}_n$. Define the function $\chi_a \in \mathfrak{H}_n$ by

$$\chi_a(q) = \overline{A_n(a, q)}, \qquad a \in C_n. \tag{2.7}$$

Setting $w = a$ in (2.2), we may write (see (1.10))

$$\boldsymbol{e}_a(z) = \int A_n(z, q)\chi_a(q) d^n q,$$

or

$$\boldsymbol{e}_a = \boldsymbol{A}_n \chi_a. \tag{2.8}$$

Since the principal vectors $\boldsymbol{e}_a$ are complete in $\mathfrak{F}_n$, this concludes the proof of

THEOREM 2.1. *$f = \boldsymbol{A}_n\psi$ is a unitary mapping of $\mathfrak{H}_n$ onto $\mathfrak{F}_n$.*

COROLLARY. *Let $S_n$ be a characteristic set of vectors $a$ (see § 1f). Since $\boldsymbol{e}_a$, $a \in S_n$, are complete in $\mathfrak{F}_n$, it follows from (2.8) that the corresponding $\chi_a$, $a \in S_n$, are complete in $\mathfrak{H}_n$. Similarly, other facts proved about $\boldsymbol{e}_a$ may be "translated" into corresponding assertions about the $\chi_a$ in $\mathfrak{H}_n$.*

2e. *The orthonormal set* $\phi_{[m]}$. To find the functions $\phi_{[m]}$ which are mapped into $u_{[m]}$, we may utilize (2.2), which we rewrite (set $b = \bar{w}$) as

(2.2a) $$e^{b\cdot z} = \int A_n(z, q)A_n(b, q)d^n q.$$

Take first $n = 1$. Then

$$u_m = \frac{z^m}{\sqrt{m!}} = \frac{1}{\sqrt{m!}}\frac{\partial^m}{\partial b^m} e^{b\cdot z}|_{b=0}.$$

In (2.2a), we may interchange integration and differentiation with respect to $b$. Hence

(2.9) $$\phi_m(q) = \frac{1}{\sqrt{m!}}\frac{\partial^m}{\partial b^m} A_1(b, q)|_{b=0}.$$

Inserting $A_1$ and setting $b = 2^{1/2}\gamma$,

$$\phi_m(q) = [2^m m!\sqrt{\pi}]^{-1/2} e^{-q^2/2}\frac{\partial^m}{\partial \gamma^m}(e^{2\gamma q-\gamma^2})|_{\gamma=0}.$$

Since $e^{2\gamma q-\gamma^2}$ is the well-known generating function of the Hermite polynomials $H_m(q)$, we find the normalized Hermite functions

(2.9a) $$\phi_m(q) = [2^m m!\sqrt{\pi}]^{-1/2} e^{-q^2/2} H_m(q).$$

From (2.9), replacing $b$ by $z$, one obtains the convergent expression

(2.10) $$A_1(z, q) = \sum u_m(z)\phi_m(q).$$

If (2.10) is used for both functions $A_1$ in the integral $J_1$ (eq. (1.19)) for real $p$ and $q$, one finds

$$\sigma_1(\kappa, p, q) = \sum_m \kappa^m \phi_m(p)\phi_m(q)$$

which is equivalent to Mehler's formula,[8] valid for $|\kappa| < 1$.

The generalization to $n > 1$ is immediate, viz.,

(2.9b) $$\phi_m(q) = \prod_{k=1}^{n} \phi_{m_k}(q_k),$$

(2.10b) $$A_n(z, q) = \sum_m u_{[m]}(z)\phi_{[m]}(q).$$

(The completeness and orthonormality of the $\phi_{[m]}$ requires no proof, since $u_{[m]} = \boldsymbol{A}_n\phi_{[m]}$, and the $u_{[m]}$ are complete and orthonormal in $\mathfrak{F}_n$.)

REMARK. If $\mathfrak{H}_n$ is decomposed into the product $\mathfrak{H}_n = \mathfrak{H}_{n'} \otimes \mathfrak{H}_{n''}$ corresponding to the decomposition $\mathfrak{F}_n = \mathfrak{F}_{n'} \otimes \mathfrak{F}_{n''}$ (see § 1e), then $\boldsymbol{A}_n = \boldsymbol{A}_{n'} \otimes \boldsymbol{A}_{n''}$.

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[8] *Higher Transcendental Functions*, Vol. 2, Bateman Manuscript Project, McGraw-Hill, New York, 1953, p. 194, equ. (22).

2f. *The inverse operator* $\boldsymbol{A}_n^{-1}$. The existence of $\boldsymbol{A}_n^{-1}$ is implied by Theorem 2.1. The equation (2.10b) suggests the relation $\boldsymbol{A}_n^{-1}f = \boldsymbol{W}_n f$, where

$$(2.11) \qquad (\boldsymbol{W}_n f)(q) = \int \overline{A_n(z, q)} f(z) d\mu_n(z).$$

For fixed $q$, however, $A_n$ does not belong to $\mathfrak{F}_n$ (see the last sentence of § 1), and the integral need not converge. But, if $f \in \mathfrak{G}_\lambda$ (see (1.16)), the integral converges absolutely. Set $z = x+iy$. Then, from (1.16),

$$|A_n f(z)|\rho_n \leqq \gamma\pi^{-5n/4} \exp\left\{-\left(\frac{3-\lambda^2}{2} x^2 + \frac{1-\lambda^2}{2} y^2 + \tfrac{1}{2}q^2\right) + 2^{1/2}x \cdot q\right\}.$$

Hence, for $\psi = \boldsymbol{W}_n f$,

$$(2.12) \qquad |\psi(q)| \leqq \frac{2^n\gamma\pi^{-n/4}}{[(3-\lambda^2)(1-\lambda^2)]^{n/2}} \exp\left\{-\frac{1-\lambda^2}{2(3-\lambda^2)} q^2\right\}.$$

Thus, $\psi \in \mathfrak{H}_n$, and, in addition, $\psi(q)$ is an entire analytic function as is easily proved.

We show next: If $f \in \mathfrak{G}_\lambda$, then $\boldsymbol{A}_n^{-1}f = \boldsymbol{W}_n f$. It suffices to prove that $\boldsymbol{A}_n(\boldsymbol{W}_n f) = f$. Let $g = \boldsymbol{A}_n(\boldsymbol{W}_n f)$, so that, from (2.11),

$$(2.13) \qquad g(w) = \int A_n(w, q)\overline{A_n(z, q)} f(z) d^n q \, d\mu_n(z).$$

Note that, with $w = u+iv$, $z = x+iy$,

$$\rho_n |A_n(w, q) A_n(z, q) f(z)| \leqq \text{const.} \exp\{\tfrac{1}{2}(v^2-u^2) + 2^{1/2}u \cdot q\} e^{-T},$$
$$T = \tfrac{1}{2}(1-\lambda^2)(x^2+y^2) + (q^2+x^2+2^{1/2}q \cdot x);$$

$T$ is positive definite. Consequently, (2.13) is absolutely convergent. Integrating over $q$, we obtain, from (2.2) and (1.9),

$$g(w) = \int e^{w \cdot \bar{z}} f(z) d\mu_n(z) = f(w),$$

i.e., $g = f$.

The construction of $\boldsymbol{A}_n^{-1}$ may be completed as follows. For any $f \in \mathfrak{F}_n$, $f = \mathrm{Lim}_{\lambda\to 1} f_\lambda$, hence $\boldsymbol{A}_n^{-1}f = \mathrm{Lim}\, \boldsymbol{A}_n^{-1} f_\lambda$. Since $f_\lambda \in \mathfrak{G}_\lambda$, $\boldsymbol{A}_n^{-1}f_\lambda = \boldsymbol{W}_n f_\lambda$. Thus we obtain the explicit expression

$$(2.14) \qquad (\boldsymbol{A}_n^{-1}f)(q) = \operatorname*{Lim}_{\lambda\to 1} \int \overline{A_n(z, q)} f(\lambda z) d\mu_n(z).$$

Without proof, I mention another version:

$$(2.15) \qquad (\boldsymbol{A}_n^{-1}f)(q) = \operatorname*{Lim}_{\sigma\to\infty} \int_{|z_n|\leqq\sigma} \overline{A_n(z, q)} f(z) d\mu_n(z).$$

REMARK. If the variables $q' = 2^{1/2}q$ are introduced in the equation (2.3), then $f_1(z) = 2^{n/2}\pi^{n/4}e^{z^2/2}f(z)$ is the $n$-dimensional (two-sided) Laplace transform of $e^{-q'^2/4}\psi(2^{-1/2}q')$. This fact may be used to derive various relations for the elements of $\mathfrak{F}_n$. In particular, one obtains quite different inversion formulae (involving, for example, an integration only over the imaginary parts of $z_k$).

2g. The equation (2.10) shows that a generating function for the Hermite functions appears as the integral kernel of a unitary mapping. It is worth noting that a similar interpretation may be given to other classical generating functions. The following is an interesting example.

Let $\gamma$ be a real positive number, and consider the Hilbert spaces $\mathfrak{F}_\gamma$ and $\mathfrak{R}_\gamma$.

1) $\mathfrak{F}_\gamma$ consists of analytic functions $f(z)$ of one complex variable regular on $B$, the unit disk, $|z| < 1$. The inner product is

$$(f_1, f_2) = \int_B \overline{f_1(z)} f_2(z) d\mu_\gamma(z),$$

$$d\mu_\gamma(z) = \left(\frac{\gamma}{\pi}\right)(1-\bar{z}z)^{\gamma-1}\,dx\,dy.$$

2) $\mathfrak{R}_\gamma$ consists of measurable functions $\psi(q)$ of a real variable $q$, $0 \leqq q < \infty$, and is defined by the inner product

$$(\psi_1, \psi_2) = \int_0^\infty \overline{\psi_1(q)}\psi_2(q) d\nu_\gamma(q),$$

$$d\nu_\gamma(q) = \left(\frac{q^\gamma}{\Gamma(\gamma+1)}\right) dq.$$

If $f(z) = \sum \alpha_m z^m \in \mathfrak{F}_\gamma$, then

$$\|f\|^2 = \sum_m \binom{\gamma+m}{m}^{-1} |\alpha_m|^2.$$

It follows that the functions

$$u_m(z) = \binom{\gamma+m}{m}^{1/2} z^m$$

form a complete orthornormal system in $\mathfrak{F}_\gamma$. For the reproducing kernel, one finds $\mathscr{K}(w, z) = (1-w\bar{z})^{-(\gamma+1)}$.

A unitary mapping, $f = A\psi$, of $\mathfrak{R}_\gamma$ onto $\mathfrak{F}_\gamma$ is defined by

$$f(z) = \int_0^\infty A(z, q)\psi(q) d\nu_\gamma(q)$$

$$A(z, q) = \frac{\exp\{-q(1+z)/2(1-z)\}q}{(1-z)^{\gamma+1}}.$$

The inverse mapping is

$$\psi(q) = (A^{-1}f)(q) = \lim_{\sigma\to 1}\int_{|z|<\sigma} \overline{A(z, q)} f(z) d\mu_\gamma(z), \qquad 0 < \sigma < 1.$$

Now

$$A(z, q) = e^{-q/2}\sum_{m=0}^\infty z^m L_m^\gamma(q),$$

where $L_m^\gamma$ are Laguerre polynomials.[9] On $\mathfrak{R}_\gamma$, the functions

[9] op. cit., p. 189, equ. (17).

$$A_m^\gamma(q) = \binom{\gamma+m}{m}^{-1/2} e^{-q/2} L_m^\gamma(q)$$

are complete and orthonormal.[10] Thus

$$A(z, q) = \sum_{m=0}^{\infty} u_m(z) A_m^\gamma(q),$$

in analogy to (2.10).

In Part II we shall again encounter the Hilbert spaces $\mathfrak{K}_\gamma$ and the functions $A_m^\gamma$, but the associated $\mathfrak{F}_\gamma$ (and hence the transform $A$) will be different.

## 3. Operators on $\mathfrak{F}_n$ and $\mathfrak{H}_n$

The mapping $A_n$ establishes a unitary isomorphism between the linear operators on $\mathfrak{F}_n$ and those on $\mathfrak{H}_n$, namely,

(3.1) $$M = A_n^{-1} L A_n,$$

where $L$ is an operator on $\mathfrak{F}_n$, and $M$ the corresponding operator on $\mathfrak{H}_n$. The domains $\mathfrak{D}(L)$ and $\mathfrak{D}(M)$ are related by $\mathfrak{D}(M) = A_n^{-1}\mathfrak{D}(L)$. Frequently this isomorphism will be indicated by writing

(3.1a) $$M = \hat{L} \quad \text{or} \quad L = \tilde{M}.$$

In this section I shall consider various classes of operators which are easily analyzed on $\mathfrak{F}_n$, and in a number of cases translate the results into the language of $\mathfrak{H}_n$ (by the isomorphism (3.1)). My aim is primarily to illustrate the methods and results of the preceding sections.

3a. *The group G.* Most operators to be considered are closely related to the group $G$ of inhomogeneous unitary transformations of $C_n$ into itself, viz.

(3.2) $$z' = g(z) = c + Uz,$$

where $c \in C_n$, and $U$ is a linear unitary transformation. The elements $g$ of $G$ will be denoted by

(3.3) $$g = (c, U).$$

The product of $g$ with an element $g' = (c', U')$ is

(3.3a) $$g'g = (c' + U'c, U'U).$$

The unit element is $(0, 1)$, and the inverse of $g$ is

(3.3b) $$g^{-1} = (c, U)^{-1} = (-U^{-1}c, U^{-1}).$$

[10]op. cit., p. 188, equ. (2).

THE OPERATORS $V_g$. We first define $V$ for homogeneous unitary transformations $z' = Uz$ by

$$(3.4) \qquad (V_U f)(z) = f(U^{-1}z).$$

Clearly, $V_{U'}V_U = V_{U'U}$. $V_U$ is unitary on $\mathfrak{F}_n$, since it has an inverse, $(V_U)^{-1} = V_{U^{-1}}$, and it preserves inner products because the measure $d\mu_n(z)$ is invariant under the mapping $z' = Uz$.

The definition of $V_c$ for a pure translation, $z' = c+z$, is more involved, viz.

$$(3.5) \qquad (V_c f)(z) = e^{\bar{c}\cdot(z-c/2)} f(z-c).$$

Setting $f_1 = V_c f$, we have

$$|f_1(z)|^2 e^{-\bar{z}\cdot z} = |f(z-c)|^2 e^{-(\bar{z}-\bar{c})\cdot(z-c)}.$$

Inserting this in the definition (1.2) of the inner product, one immediately deduces the isometry of $V_c$. Unitarity follows from the relation $V_c V_{-c} = V_{-c}V_c = 1$.

For an arbitrary element $g$ of $G$, we set

$$(3.6) \qquad V_g = V_c V_U, \qquad g = (c, U).$$

The unitarity of $V_g$ follows from that of its factors. From (3.4) and (3.5),

$$(3.6\text{a}) \qquad (V_g f)(z) = e^{\bar{c}\cdot(z-c/2)} f(U^{-1}(z-c)) = e^{\bar{c}\cdot(z-c/2)} f(g^{-1}(z)).$$

For the principal vectors $\boldsymbol{e}_a$, one obtains

$$(3.6\text{b}) \qquad V_U \boldsymbol{e}_a = \boldsymbol{e}_{Ua}, \qquad V_c \boldsymbol{e}_a = e^{-c\cdot(\bar{a}+\bar{c}/2)} \boldsymbol{e}_{a+c},$$

$$(3.6\text{c}) \qquad V_g \boldsymbol{e}_a = \exp\{\tfrac{1}{2}\bar{c}\cdot c - \overline{g(a)}\cdot c\}\, \boldsymbol{e}_{g(a)}.$$

The operators $V_g$ define a "ray representation" (or representation up to a factor) of the group $G$. One obtains from (3.6a) and (3.3a)

$$(3.7) \qquad V_{g'}V_g = e^{i\nu(g',g)} V_{g'g}, \qquad \nu(g', g) = \mathscr{I}m\,\{\bar{c}'\cdot(U'c)\}.$$

Note that $V_g V_{g^{-1}} = 1$. Special cases of interest are

$$(3.7\text{a}) \qquad V_{c'}V_c = e^{(1/2)(\bar{c}'\cdot c - c'\cdot\bar{c})} V_{c'+c}, \qquad V_{c'}V_c V_{-c'} = e^{\bar{c}'\cdot c - c'\cdot\bar{c}} V_c,$$

$$(3.7\text{b}) \qquad V_U V_c V_{U^{-1}} = V_{Uc}.$$

The operators $V_g$ are strongly continuous in $g$, i.e. for fixed $f$, $V_g f$ is strongly continuous in the components of $c$ and the matrix elements of $U$. By Theorem 1.2 (in § 1f) this follows from the obvious facts that, for fixed $z$, (3.6a) is continuous in $g$, and that $\|V_g f\| = \|f\|$.

We turn now to a more detailed discussion of some subgroups of $G$.

3b. *Homogeneous transformations.* If, in (3.4), $U$ is a real orthogonal transformation, say $O$, then the corresponding $\hat{V}_O$ in $\mathfrak{H}_n$ (see (3.1a)) is given by

$$(\hat{V}_O\psi)(q) = \psi(O^{-1}q), \qquad \psi \in \mathfrak{H}_n. \tag{3.8}$$

This follows by standard arguments from the invariance of the kernel $A_n(z, q)$ under a simultaneous orthogonal transformation of $z$ and $q$, and from the orthogonal invariance of the measure $d^nq$.

THE OPERATORS $W_\tau$ AND $\hat{W}_\tau$. Restrict $U$ to the one-parameter subgroup $U = e^{i\tau} \cdot 1$, $\tau$ real, and denote the corresponding $V_U$ by $W_\tau$. (If $n = 1$, this subgroup coincides with the group of all unitary transformations.) Clearly

$$W_\tau W_{\tau'} = W_{\tau+\tau'}, \qquad \hat{W}_\tau \hat{W}_{\tau'} = \hat{W}_{\tau+\tau'}. \tag{3.9}$$

$U = e^{i\tau} \cdot 1$ is real orthogonal if $e^{i\tau} = \pm 1$, and by (3.8)

$$\hat{W}_0 = 1, \qquad \hat{W}_\pi = P, \tag{3.9a}$$

where $(P\psi)(q) = \psi(-q)$ ($P$ is the parity operator of quantum mechanics). In all other cases ($e^{2i\tau} \neq 1$) $\hat{W}_\tau$ may be determined in analogy with the procedures used in § 2. Consider first a function $\psi (\in \mathfrak{H}_n)$ which vanishes outside the cube $Q_\alpha$, $|q_k| < \alpha$, and let $\psi_1 = W_\tau \psi$. With $f = A_n\psi$, $g = W_\tau f$ (i.e., $g(z) = f(e^{-i\tau}z)$), we have

$$\psi_1 = \operatorname*{Lim}_{\lambda \to 1} \psi_{1\lambda}, \quad \psi_{1\lambda} = A_n^{-1} g_\lambda,$$

and $\psi_{1\lambda}(q)$ is the absolutely converging integral

$$\begin{aligned}\psi_{1\lambda}(q) &= \int A_n(\bar{z}, q) A_n(\lambda e^{-i\tau}z, q')\psi(q')d\mu_n(z)d^nq' \\ &= \int \sigma_n(\lambda e^{-i\tau}, q', q)\psi(q')d^nq' \qquad \text{(by (1.19b))}.\end{aligned}$$

On $Q_\alpha$ the kernel of the last integral converges uniformly to $\sigma_n(e^{-i\tau}, q', q)$ as $\lambda \to 1$. Thus

$$\psi_1(q) = \lim_{\lambda \to 1} \psi_{1\lambda}(q) = \int_{Q_\alpha} \sigma_n(e^{-i\tau}, q', q)\psi(q')d^nq'.$$

By (1.19d) the square root in $\sigma_n$ has positive real part. Writing $\tau$ in one of the forms ($k$ integral!)

$$\tau = 2k\pi + \varepsilon\theta, \qquad \varepsilon = \pm 1, \quad 0 < \theta < \pi,$$

we have $1 - e^{-2i\tau} = 2 \sin \theta e^{i\varepsilon(\pi/2-\tau)}$. Hence, if $e^{2i\tau} \neq 1$,

$$\sigma_n(e^{-i\tau}, q', q) = \frac{\exp\{-in\varepsilon(\frac{1}{4}\pi - \frac{1}{2}\theta)\}}{(2\pi\,|\sin \tau|)^{n/2}} \exp\left\{i \cot \tau \,\frac{q^2 + q'^2}{2} - i\frac{q \cdot q'}{\sin \tau}\right\}. \tag{3.10}$$

For any $\psi \in \mathfrak{H}_n$, set $\psi_\alpha(q) = \psi(q)$ if $q \in Q_\alpha$, and $\psi_\alpha(q) = 0$ if $q \notin Q_\alpha$. Then $\psi = \mathrm{Lim}_{\alpha\to\infty}\psi_\alpha$, and $\hat{W}_\tau\psi = \mathrm{Lim}\, \hat{W}_\tau\psi_\alpha$, i.e.

$$(\hat{W}_\tau\psi)(q) = \lim_{\alpha\to\infty}\int_{Q_\alpha} \sigma_n(e^{-i\tau}, q', q)\psi(q')d^nq'. \tag{3.10a}$$

For $\tau = \frac{1}{2}\pi$, $\sigma_n = (2\pi)^{-n/2}\exp\{-iq\cdot q'\}$, so that $F = \hat{W}_{\pi/2}$ is the Fourier transform. As a by-product, we thus obtain the basic results of the $L_2$-theory of the Fourier transform. $F$ is unitary, $F^2 = P$, and $F^4 = 1$. The inverse, $F^{-1} = \hat{W}_{-\pi/2}$, is constructed by substituting $-i$ for $i$. In addition, $F$ appears imbedded in a strongly continuous one-parameter group of unitary operators $\hat{W}_\tau$.[11]

For the orthonormal functions $u_{[m]}$ in $\mathfrak{F}_n$,

$$W_\tau u_{[m]} = e^{-i|m|\tau}u_{[m]}.$$

This implies

$$\hat{W}_\tau\phi_{[m]} = e^{-i|m|\tau}\phi_{[m]} \tag{3.10b}$$

for the Hermite functions defined by (2.9b).

3c. *Translations.* The operators $V_c$ which correspond to pure translations are closely related to the quantum mechanical canonical operators $p, q$. Let

$$\begin{aligned} c &= 2^{-1/2}(\alpha+i\beta), \qquad \alpha, \beta \in R_n, \\ T_{\alpha,\beta} &= A_n^{-1}V_cA_n. \end{aligned} \tag{3.11}$$

Then

$$(T_{\alpha,\beta}\psi)(q) = e^{-i\beta\cdot(q-\alpha/2)}\psi(q-\alpha). \tag{3.11a}$$

To prove (3.11a), let $f = A_n\psi$, $f_1 = V_c\psi$, $\psi_1 = T_{\alpha,\beta}\psi$. By (3.5),

$$\begin{aligned} f_1(z) &= \int e^{\bar{c}\cdot(z-c/2)}A_n(z-c, q)\psi(q)d^nq \\ &= \int e^{\bar{c}\cdot(z-c/2)}A_n(z-c, q)e^{i\beta\cdot(q+\alpha/2)}\psi_1(q+\alpha)d^nq. \end{aligned}$$

The assertion follows from the identity

$$e^{\bar{c}\cdot(z-c/2)}A_n(z-c, q)e^{i\beta\cdot(q+\alpha/2)} = A_n(z, q+\alpha)$$

which is easily verified and which implies $f_1 = A_n\psi_1$.

For $c = 2^{-1/2}(\alpha+i\beta)$, $c' = 2^{-1/2}(\gamma+i\delta)$, we find from (3.7a)

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[11]The problem of imbedding the Fourier transform in a one-parameter group has been considered by Condon, E. U., *Immersion of the Fourier transform in a continuous group of functional transformations*, Proc. Nat. Acad. Sci. U. S. A., Vol. 23, 1937, pp. 158–164, who constructed the operators $\hat{W}_\tau$ in $H_1$.

$$T_{\gamma,\delta}T_{\alpha,\beta} = e^{i\nu}T_{\gamma+\alpha,\,\delta+\beta}, \qquad \nu = \tfrac{1}{2}(\beta\cdot\gamma-\alpha\cdot\delta) \tag{3.12}$$

This relation is equivalent to that which H. Weyl substituted for the canonical commutation rules, and it is the starting point for Von Neumann's celebrated proof of the uniqueness of the canonical operators.[12]

For fixed $c$ and real $\tau$ the operators $V_{\tau c}$ (or $T_{\tau\alpha,\tau\beta}$) form a one-parameter group, because $V_{\tau c}V_{\tau' c} = V_{(\tau+\tau')c}$ (by (3.7a)). Except for the trivial case $c = 0$, we may assume—without loss of generality—that $\bar{c}\cdot c = 1$ or, equivalently, $\alpha^2+\beta^2 = 2$. By (3.7b) any two of these groups (defined, say, by $c$ and $c'$) are related by a unitary isomorphism since there exists a unitary transformation $U$ such that $c' = Uc$.

By Stone's theorem, the group $V_{\tau c}$ is generated by a uniquely defined self-adjoint operator $L_c$ such that

$$V_{\tau c} = \exp\{-i\tau L_c\}, \quad T_{\tau\alpha,\tau\beta} = \exp\{-i\tau\hat{L}_{\alpha,\beta}\}, \tag{3.13}$$

where $\hat{L}_{\alpha,\beta} = A_n^{-1}L_cA_n$.

DETERMINATION OF $L_c$. $L_c$ is defined by

$$L_cf = \operatorname*{Lim}_{\tau\to 0} i\tau^{-1}(V_{\tau c}-1)f, \qquad \tau \neq 0, \tag{3.13a}$$

whenever this limit exists. Write for a given $f \in \mathfrak{F}_n$

$$f(z,\tau) = (V_{\tau c}f)(z) = e^{\tau\bar{c}\cdot(z-\tau c/2)}f(z-\tau c). \tag{3.13b}$$

If $h = L_cf$, then

$$h(z) = i\left.\frac{\partial f(z,\tau)}{\partial\tau}\right|_{\tau=0} = i(\Lambda_cf)(z), \tag{3.14}$$

$$(\Lambda_cf)(z) = (\bar{c}\cdot z)f(z)-c\cdot\nabla f(z), \qquad c\cdot\nabla f = \sum_k c_k\frac{\partial f}{\partial z_k}. \tag{3.14a}$$

Hence $f$ belongs to the domain $\mathfrak{D}(L_c)$ only if $\Lambda_cf \in \mathfrak{F}_n$.

Conversely, assume $h = i\Lambda_cf \in \mathfrak{F}_n$. It is easy to verify (from (3.13b)) that $\partial f(z,\tau)/\partial\tau = -i(V_{\tau c}h)(z)$. Hence

$$i\tau^{-1}(V_{\tau c}f-f) = \tau^{-1}\int_0^\tau V_{\tau' c}h\,d\tau' = k_\tau. \tag{3.14b}$$

This equation holds for every $z$, but may also be interpreted as a vector equation in Hilbert space. Now

$$k_\tau-h = \tau^{-1}\int_0^\tau (V_{\tau' c}-1)h\,d\tau'.$$

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[12]von Neumann, J., *Die Eindeutigkeit der Schrödingerschen Operatoren*, Math. Ann., Vol. 104, 1931, pp. 570–578.

It follows from the strong continuity of $V_{\tau' c}$ (in $\tau'$) that $\|k_\tau - h\| \to 0$ as $\tau \to 0$, and hence $L_c f = h = i\Lambda_c f$. To sum up: $f \in \mathfrak{D}(L_c)$ if and only if $\Lambda_c f \in \mathfrak{F}_n$, and then $L_c f = i\Lambda_c f$.

The differential operator $i\Lambda_c$ is linear in the components of $\alpha$ and $\beta$, and we write

$$(3.15) \qquad L_c = \alpha \cdot \tilde{p} + \beta \cdot \tilde{q}, \qquad \hat{L}_{\alpha,\beta} = \alpha \cdot p + \beta \cdot q,$$

so that, by (3.13) (for $\tau = 1$)

$$(3.15a) \qquad V_c = \exp\{-i(\alpha \cdot \tilde{p} + \beta \cdot \tilde{q})\}.$$

If $\alpha$, $\beta$ have the components $\alpha_l = \delta_{lk}$, $\beta_l = 0$, then $L_c = \tilde{p}_k$; if $\alpha_l = 0$, $\beta_l = \delta_{lk}$, then $L_c = \tilde{q}_k$, and

$$(3.15b) \qquad \tilde{q}_k f = 2^{-1/2}\left(z_k + \frac{\partial}{\partial z_k}\right) f, \quad \tilde{p}_k f = 2^{-1/2} i \left(z_k - \frac{\partial}{\partial z_k}\right) f,$$

consistent with the relations from which we started in the introduction.

While the differential operator $i\Lambda_c$ is linear in the components of $\alpha$ and $\beta$, the same is not true for $L_c$ as an operator on the Hilbert space $\mathfrak{F}_n$ because its domain extends, in general, beyond the intersection of the domains of those $\tilde{p}_k$, $\tilde{q}_k$ which appear in (3.15). Strictly speaking, the equation (3.15) merely introduces a notation which is occasionally convenient.

The isomorphism (3.7b) extends to the generators $L_c$. If $U = i \cdot 1$, then $V_U = W_{\pi/2}$, and $\hat{W}_{\pi/2}$ is the Fourier transform $F$ (see § 3b), and $Uc = ic = 2^{-1/2}(\beta - i\alpha)$. In particular, one obtains

$$(3.15c) \qquad F q_k F^{-1} = -p_k, \qquad F p_k F^{-1} = q_k,$$

the well-known relations which connect configuration and momentum space.

That the operator $q_k$ corresponds to the multiplication of $\psi$ by $q_k$ follows immediately from the consideration of the relevant group of transformations $\psi(q) \to e^{-i\tau q_k}\psi(q)$. The simplest definition of $p_k$ is then obtained from (3.15c), but from the equations (8a), (8b) of the introduction one concludes directly for a wide class of functions that $p_k = -i\partial/\partial q_k$—whenever the partial integrations used in their derivation can be justified.

It is not difficult to show that it suffices to define any $L_c$ (by (3.14)) for all polynomials in $\mathfrak{F}_n$, and that $L_c$ on $\mathfrak{D}(L_c)$ is then obtained by closure.

## 3d. *The operators $z_k$ and $\partial/\partial z_k$.*

Define the operators $Z_k$ and $Y_k$ on $\mathfrak{F}_n$ as follows:

$$(3.16a) \qquad (Z_k f)(z) = z_k f(z) \quad \text{if} \quad z_k f \in \mathfrak{F}_n,$$

$$(3.16b) \qquad (Y_k f)(z) = \frac{\partial f(z)}{\partial z_k} \quad \text{if} \quad \frac{\partial f}{\partial z_k} \in \mathfrak{F}_n.$$

THEOREM 3.1. *1) $Z_k$ and $Y_k$ are closed. 2) $\mathfrak{D}(Z_k) = \mathfrak{D}(Y_k)$. 3) $Z_k^* = Y_k$, and $Y_k^* = Z_k$. 4) $\mathfrak{D}(Z_k) = \mathfrak{D}(\tilde{p}_k) \cap \mathfrak{D}(\tilde{q}_k)$.*

Proof: For the sake of simplicity, we set $k = 1$, and we introduce the following abbreviation: If $m = (m_1, m_2, \cdots, m_n)$, then $m' = (1+m_1, m_2, \cdots, m_n)$. 1) Let $f_j \Rightarrow g$, $z_k f_j \Rightarrow h$. Then, for every $z$, $h(z) = \lim z_k f_j(z) = z_k g(z)$. Let next $f_j \Rightarrow g$, $\partial f_j/\partial z_k \Rightarrow h$. Then, for every $z$, $h(z) = \lim \partial f_j/\partial z_k$, and $\partial g/\partial z_k = \lim \partial f_j/\partial z_k$ (by (1.7a)), hence $h = \partial g/\partial z_k$. 2) Let

$$f = \sum_m \alpha_{[m]} z^{[m]} \in \mathfrak{F}_n.$$

Then

$$||z_k f||^2 = \sum_m (1+m_k)[m!]|\alpha_{[m]}|^2,$$

and

$$\left\|\frac{\partial f}{\partial z_k}\right\|^2 = \sum_m m_k[m!]|\alpha_{[m]}|^2.$$

Hence

$$(3.17) \qquad ||z_k f||^2 = ||f||^2 + \left\|\frac{\partial f}{\partial z_k}\right\|^2$$

In (3.17) either both sides are infinite, or they have the same finite value, which proves $\mathfrak{D}(Z_k) = \mathfrak{D}(Y_k)$. 3) Let

$$g = \sum \beta_{[m]} z^{[m]}, \quad h = \sum \gamma_{[m]} z^{[m]} = Z_1^* g.$$

Set $f = z^{[m]}$. Then $(f, h) = (z_1 f, g)$ implies that $\gamma_{[m]} = (1+m_1)\beta_{[m']}$, i.e. $h = \partial g/\partial z_1$. Hence $Z_1^* \subset Y_1$. Conversely, let

$$f = \sum \alpha_{[m]} z^{[m]} \in \mathfrak{D}(Z_1), \qquad g = \sum \beta_{[m]} z^{[m]} \in \mathfrak{D}(Y_1).$$

Then $(z_1 f, g) = (f, \partial g/\partial z_1)$ because both sides of the equation are equal to $\sum_m [m'!]\overline{\alpha_{[m]}}\beta_{[m']}$. This proves $Y_1 \subset Z_1^*$, and thus $Z_1^* = Y_1$. $Y_1^* = Z_1$ follows immediately. 4) If $f \in \mathfrak{D}(Z_k)$, then by 2) both $z_k f$ and $\partial f/\partial z_k \in \mathfrak{F}_n$, and hence both $\tilde{q}_k f$ and $\tilde{p}_k f$ are defined (by (3.15b)). The converse also follows from (3.15b). —

If $f \in \mathfrak{D}(Z_k)$, one deduces easily from (3.15b) that

$$(3.17b) \qquad ||\tilde{q}_k f||^2 + ||\tilde{p}_k f||^2 = ||Z_k f||^2 + ||Y_k f||^2.$$

$Z_k$ and $Y_k$ have all the properties of the conventional creation and annihilation operators (for bosons) in quantum field theory.[13] In addition, the

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[13]See Friedrichs, K. O., *Mathematical Aspects of the Quantum Theory of Fields*, Interscience Publishers, New York, 1953.

functions $u_{[m]}$ are the basic vectors which are usually employed, and $m_1$, $m_2, \cdots$ are the occupation numbers. In particular,

$$Y_k^* Y_k u_{[m]} = z_k \frac{\partial u_{[m]}}{\partial z_k} = m_k u_{[m]}, \tag{3.18a}$$

$$\frac{Z_1^{m_1} Z_2^{m_2} \cdots Z_n^{m_n}}{[m_1! m_2! \cdots m_n!]^{1/2}} u_0 = u_{[m]}. \tag{3.18b}$$

Notice that the principal vectors $\boldsymbol{e}_a$ are eigenvectors of $Y_k$ since $Y_k \boldsymbol{e}_a = \bar{a}_k \boldsymbol{e}_a$.

3e. *The harmonic oscillator.* On $\mathfrak{H}_n$, the Hamiltonian of $n$ identical uncoupled oscillators reads, upon subtraction of the zero-point energy, in suitable units

$$H = \tfrac{1}{2} \sum_{k=1}^{n} (p_k^2 + q_k^2 - 1) = \sum_{k=1}^{n} \eta_k \xi_k$$

(see equation (1) of the introduction). As to the domain of $H$, one may first define it for sufficiently regular functions $\psi$, with $p_k = -i\partial/\partial q_k$, and then pass to the closure. Taking, for example, finite linear combinations of the $\phi_{[m]}$ (see (2.9b)), one obtains

$$\tilde{H} = \boldsymbol{A}_n H \boldsymbol{A}_n^{-1} = \sum_k z_k \frac{\partial}{\partial z_k}$$

for all polynomials in $\mathfrak{F}_n$, and the closure is evidently

$$\tilde{H} f = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}$$

whenever the right-hand side belongs to $\mathfrak{F}_n$. This reduces the analysis of $\tilde{H}$ to a triviality. If $f = \sum_m \alpha_{[m]} z^{[m]}$, then $\tilde{H} f = \sum_m |m| \alpha_{[m]} z^{[m]}$. The eigenvalues are the non-negative integers, $l$, and the eigenfunctions are homogeneous polynomials of order $l$. On $H_n$ the corresponding eigenfunctions are linear combinations of the $\phi_{[m]}$ with $l = |m| = m_1 + m_2 + \cdots + m_n$.

The space $\mathfrak{F}_n$ is particularly convenient for the analysis of the symmetries of $\tilde{H}$—for example, in the nuclear shell model—because the transformations $V_U$, which leave $\tilde{H}$ invariant, are so easily expressed on $\mathfrak{F}_n$.

Note that $\exp\{-it\tilde{H}\} = W_t$, and $\exp\{-itH\} = \hat{W}_t$ (see (3.9), (3.10)).

3f. *Linear canonical transformations.* In what follows, we take $n = 1$. Consider a linear canonical transformation (as defined in Hamiltonian mechanics), (3.19) $$\tilde{q}' = \kappa_{11}\tilde{q}+\kappa_{12}\tilde{p}-\rho_1\cdot 1, \qquad \tilde{p}' = \kappa_{21}\tilde{q}+\kappa_{22}\tilde{p}-\rho_2\cdot 1,$$

(3.19a) $$\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21} = 1,$$

where $\kappa_{ij}$ and $\rho_j$ are real constants. If $\tilde{q}$, $\tilde{p}$ are the operators introduced in § 3c, then $\tilde{q}'$, $\tilde{p}'$ satisfy the canonical commutation rules. Formally,

(3.20) $$\alpha\tilde{p}'+\beta\tilde{q}' = \alpha'\tilde{p}+\beta'\tilde{q}-\theta,$$

(3.20a) $$\alpha' = \kappa_{22}\alpha+\kappa_{12}\beta, \qquad \beta' = \kappa_{21}\alpha+\kappa_{11}\beta, \qquad \theta = \rho_2\alpha+\rho_1\beta.$$

To make the meaning of these transformations precise, it is best to define the corresponding unitary operators ($c = 2^{-1/2}(\alpha+i\beta)$, $c' = 2^{-1/2}(\alpha'+i\beta')$)

(3.21) $$V'_c = \exp\{-i(\alpha\tilde{p}'+\beta\tilde{q}')\} = e^{i\theta}\exp\{-i(\alpha'\tilde{p}+\beta'\tilde{q})\} = e^{\sigma c-\bar{\sigma}\bar{c}}V_{c'},$$

where

(3.22) (a) $c' = \lambda c+\mu\bar{c}$, (b) $\bar{\lambda}\lambda-\bar{\mu}\mu = 1$, (c) $\sigma = 2^{-1/2}(\rho_1+i\rho_2)$,

(3.23) $$\lambda = \tfrac{1}{2}(\kappa_{11}+\kappa_{22}+i(\kappa_{21}-\kappa_{12})), \quad \mu = \tfrac{1}{2}(\kappa_{22}-\kappa_{11}+i(\kappa_{21}+\kappa_{12})).$$

Finally, we may also introduce the transforms $Z' = 2^{-1/2}(\tilde{q}'-i\tilde{p}')$, $Y' = 2^{-1/2}(\tilde{q}'+i\tilde{p}')$ of the operators $Y$ and $Z$ (see § 3d). From (3.19) and (3.15b),

(3.24) $$Z' = \bar{\lambda}Z-\mu Y-\bar{\sigma}\cdot 1, \qquad Y' = \lambda Y-\bar{\mu}Z-\sigma\cdot 1.$$

In terms of the operators $V'_c$ the canonical character of the transformation (3.19) is defined by the invariance of the relation (3.7a)—or, equivalently, (3.12). It follows indeed from (3.21) that

$$V'_{c_2}V'_{c_1} = e^{(1/2)(\bar{c}_2c_1-\bar{c}_1c_2)}V'_{c_1+c_2}.$$

As before, the one-parameter group $V'_{\tau c}$ defines the generator $L'_c = \alpha\tilde{p}'+\beta\tilde{q}'$.

We shall now construct a unitary operator $S$ such that, for all $c$,

(3.25) (a) $V'_c = SV_cS^{-1}$, (b) $L'_c = SL_cS^{-1}$.

Obviously the second equation (3.25) is a consequence of the first one. (The existence of $S$ follows from Von Neumann's theorem,[12] but this fact will not be used.)

As was shown in § 1d, it suffices to determine

(3.26) $$h_a = Se_a, \qquad h_a(z) = S(z, a),$$

for all $a$. 1) Since $||e_0|| = 1$, and $Ye_0 = de_0/dz = 0$, we must have $||h_0|| = 1$, and $Y'h_0 = 0$. By (3.24), $Y'h_0 = 0$ has the solution

(3.27) $$h_0 = \eta k, \qquad \eta = \text{const.}, \qquad k(z) = \exp\left\{\frac{\bar{\mu}}{2\lambda}z^2+\frac{\sigma}{\lambda}z\right\}.$$

From (1.18b), using (3.22b), we obtain

$$\text{(3.27a)} \qquad ||k||^2 = |\lambda| e^{\omega}, \qquad \omega = \bar{\sigma}\sigma + \tfrac{1}{2}\left(\frac{\mu}{\lambda}\,\sigma^2 + \frac{\bar{\mu}}{\bar{\lambda}}\,\bar{\sigma}^2\right),$$

This determines $\eta$ up to a factor of modulus 1:

$$\text{(3.27b)} \qquad |\eta| = |\lambda|^{-1/2} e^{-\omega/2}.$$

2) From $\boldsymbol{e}_a = e^{\bar{a}a/2} V_a \boldsymbol{e}_0$ (see (3.6b)) we conclude that

$$h_a = e^{\bar{a}a/2} V'_a h_0 = \exp\{\tfrac{1}{2}\bar{a}a + \sigma a - \bar{\sigma}\bar{a}\} V_{a'} h_0.$$

Using the definition (3.5), and $a' = \lambda a + \mu \bar{a}$, we find the explicit expression

$$\text{(3.27c)} \qquad h_a(z) = \eta \exp\left\{\frac{\bar{\mu}}{2\lambda}\, z^2 + \frac{z}{\lambda}\,(\sigma + \bar{a}) - \frac{\mu \bar{a}^2}{2\lambda} - \frac{\bar{a}}{\lambda}\,(\sigma\mu + \bar{\sigma}\lambda)\right\},$$

which determines $S$.

It remains to verify (a) the unitarity of $S$, (b) the validity of (3.25a). To verify (a), insert $S(z, w) = h_w(z)$ in the two integrals (1.13) which are evaluated by using (1.18b). To verify (b), it is sufficient to show—by straightforward computation—that, for all $c$ and $a$,

$$S V_c \boldsymbol{e}_a = V'_c S \boldsymbol{e}_a = V'_c h_a.$$

If $\kappa_{ij} = \delta_{ij}$ (and hence $\lambda = 1$, $\mu = 0$), we have, apart from an irrelevant phase factor, $S = V_{\bar{\sigma}}$, as may be directly verified from (3.7a).

A similar analysis may be carried out for arbitrary $n$. It is, of course, more elaborate, but presents no difficulties.

3g. *An application to field quantization.* I shall briefly sketch a simple application of the preceding results. Let $Z_j$, $Y_j$, $j = 1, 2, \cdots$, be an infinite family $\mathscr{F}$ of creation and annihilation operators of standard type (see § 3d) which are, as usual, represented on a Hilbert space $\mathfrak{H}$ spanned by the orthonormal set of vectors $u_{m_1 m_2 \cdots m_r}$ (see (3.18b)), where $r$ is unbounded. Define a new family $\mathscr{F}'$ on $\mathfrak{H}$ by

$$\text{(3.28)} \qquad Z'_j = \bar{\lambda}_j Z_j - \mu_j Y_j - \bar{\sigma}_j \cdot 1, \qquad Y'_j = \lambda_j Y_j - \bar{\mu}_j Z_j - \sigma_j \cdot 1,$$

$$\text{(3.28a)} \qquad \bar{\lambda}_j \lambda_j - \bar{\mu}_j \mu_j = 1$$

in analogy to (3.24). As is known, the family $\mathscr{F}'$ is unitarily equivalent to $\mathscr{F}$ if $\mathfrak{H}$ contains a vector

$$v = \sum_{m_1, m_2, \cdots} \zeta_{m_1 m_2 \cdots m_r} u_{m_1 m_2 \cdots m_r}$$

such that

$$(3.29) \qquad Y'_j v = 0 \qquad \text{for all } j.$$

(Then $v$ describes a state in which all occupation numbers referring to $\mathscr{F}'$ are zero.) The problem is to find a criterion for the existence of such a vector.

The equations (3.29) determine $v$ uniquely, apart from an inessential scalar factor. One obtains

$$(3.29\text{a}) \qquad \zeta_{m_1 m_2 \cdots m_r} = \gamma^{(1)}_{m_1} \gamma^{(2)}_{m_2} \cdots \gamma^{(r)}_{m_r}, \qquad \gamma^{(j)}_0 = 1 \text{ for all } j,$$

where the coefficients $\gamma^{(j)}_m$ may be characterized as follows: Let

$$k_j(z) = \sum_{m=0}^{\infty} \frac{\gamma^{(j)}_m}{\sqrt{m!}} z^m,$$

where $z$ denotes a *single* complex variable. Then

$$\lambda_j \frac{dk_j}{dz} = (\bar{\mu}_j z + \sigma_j) k_j,$$

and $k_j(0) = 1$, so that $k_j$ is given by (3.27) ($\lambda_j$, $\mu_j$, $\sigma_j$ being substituted for $\lambda$, $\mu$, $\sigma$) and is therefore a function in $\mathfrak{F}_1$. Furthermore, by (3.27a),

$$(3.29\text{b}) \qquad ||k_j||^2 = \sum_{m=0}^{\infty} |\gamma^{(j)}_m|^2 = |\lambda_j| e^{\omega_j} \geqq 1.$$

One may conclude from (3.29b) that

$$(3.29\text{c}) \qquad ||v||^2 = \sum_{m_1, m_2, \cdots} |\zeta_{m_1 m_2 \cdots m_r}|^2 = \prod_{j=1}^{\infty} ||k_j||^2.$$

Consequently $v$ is a vector in $\mathfrak{H}$ when the infinite product in the last equation converges. Note that $|\lambda_j|^2 = 1 + |\mu_j|^2$, and that, by (3.27a),

$$0 \leqq \left(1 - \left|\frac{\mu_j}{\lambda_j}\right|\right) |\sigma_j|^2 \leqq \omega_j \leqq 2|\sigma_j|^2.$$

It follows that $||v||$ is finite and the family $\mathscr{F}'$ is equivalent to $\mathscr{F}$ if and only if the following two conditions are satisfied:

$$1) \quad \sum_{j=1}^{\infty} |\mu_j|^2 < \infty, \qquad 2) \quad \sum_{j=1}^{\infty} |\sigma_j|^2 < \infty.$$

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