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A perfect memory makes the continuous Newton method look ahead

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Abstract

Hauser and Nedić (2005 *SIAM J. Optim.* **15** 915) have pointed out an intriguing property of a perturbed flow line generated by the continuous Newton method: it returns to the unperturbed one once the perturbation ceases to exist. We show that this feature is a direct consequence of the phase being constant along any Newton trajectory, that is, once a phase always that phase.

Keywords: continuous Newton method, Newton–Raphson method, Newton flow, qualitative analysis

Many examples demonstrate a deep connection between mathematics and physics. The most striking one relates [1] the statistics of the nearest neighbor separations of the non-trivial zeros of the Riemann zeta function [2] analyzed most prominently by Hugh Montgomery in the field of number theory, to the energy eigenvalue distributions of random matrices studied by Freeman Dyson. Indeed, random matrices play a central role in nuclear physics and quantum chaos. In the same spirit is the recent constructive proof [3] of the Hilbert–Pólya conjecture stating that the imaginary parts of these zeros are eigenvalues of a Hamiltonian. A more elementary example for the cross fertilization of mathematics and physics is the Newton flow [4, 5] which is intimately linked to the lines of constant phase of a complex-valued function. In the present note we demonstrate that this feature is at the very heart of the ability of the Newton method to 'look ahead' [6].

The Newton flow, that is the continuous Newton method [4, 5] is an efficient tool to find the zeros of a complex-valued function F and even yields [7, 8] insight into the distribution of the non-trivial zeros of the Riemann zeta function [2]. The power of this technique originates from the fact that the trajectories of the Newton flow are the lines of constant phase of F.

Reference [6] analyzed the influence of a perturbation δF on the Newton flow when δF acts in an open domain of the complex plane which does not include the zero of F. It is not surprising that there the perturbed Newton trajectories deviate from the unperturbed ones. However, what is surprising is the result that each trajectory leaves the region of perturbation at the same point as its unperturbed counterpart. We now show that this property is a consequence of the fact that along *any* Newton trajectory the phase of F is constant.

For this purpose we first recall that the flow lines

$$\dot{z}(t) = -\frac{F(z(t))}{F'(z(t))}, \tag{1}$$

where dot and prime denote derivatives with respect to the real-valued parameter t and the complex-valued argument z of F, respectively, yields the lines of constant phase of F. Indeed, the implicit definition

$$F(z(t)) = F(z(0))\mathrm{e}^{-t} \tag{2}$$

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of the trajectory $z(t)$ following from equation (1) shows that the phase of F at the initial point $z(0)$ is preserved along the *whole* trajectory till it terminates at a zero of F.

For the sake of simplicity we now assume that F has a simple zero[6]. In this case every phase out of the interval of 0 and 2π corresponds to one Newton trajectory and occurs only once. We focus on a single trajectory and choose the open domain D where δF acts such that the trajectory crosses it only once. In this region δF deflects and deforms the flow line according to the relation

$$\tilde{F}(\tilde{z}(\mathrm{t})) = (F + \delta F)(\tilde{z}(\mathrm{t})) = \tilde{F}(\tilde{z}(0))\mathrm{e}^{-t} = F(z(0))\mathrm{e}^{-t}, \tag{3}$$

rather than equation (2).

However, even along the perturbed trajectory $\tilde{z}(t)$ given by equation (3) the phase is identical to the one of $z(t)$ since it is determined by F evaluated at $z(0)$ which is outside of D. It is for this reason that $\tilde{z}(t)$ has to merge into $z(t)$ at the point where (i) $\delta F = 0$, and (ii) the phase of F is identical to that at the starting point. This location is obviously where $z(t)$ exits D. Therefore, $\tilde{z}(t)$ and $z(t)$ leave the domain of perturbation at the same point with the same slope.

We emphasize that in this argument the fact that F only displays a single zero is crucial. One might suspect that the situation is fundamentally different when F displays *two* distinct zeros since in this case there exist *two* trajectories that carry the same phase. When we again assume that two such flow lines cross the domain of perturbation only once we find four points on the boundary of D with identical phases: Two correspond to the entrance and two to the exits of the lines. Now a perturbation could force the two trajectories to interchange their exit points resulting in a crossing of the two perturbed trajectories. However, this phenomenon is only possible if the flow lines are separatrices [7]. Hence, two trajectories of identical phase which are not separatrices clearly demonstrate that even with two zeros the perturbed trajectories have to leave the domain of perturbation at the same points where the unperturbed ones exit.

In summary, a Newton trajectory always returns to the unperturbed one as soon as the perturbation ceases to exist. In [6] this remarkable feature has been summarized by the observation that the continuous Newton method can 'look ahead'. However, in view of our constant-phase argument we argue that the continuous Newton method enjoys a 'perfect memory'. Despite of perturbations the trajectories always remember and maintain the phase imprinted on them by the starting point.

[6] The Newton flows of higher order zeros differ from the simple ones not in their *shapes* in the complex plane but in the *speed* at which the flow lines approach the zeros.

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