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Ornstein–Uhlenbeck operators and semigroups

V. I. Bogachev

Abstract. This survey gives an account of the state of the art of the theory of Ornstein–Uhlenbeck operators and semigroups. The domains of definition and the spectra of such operators are considered, along with related Sobolev classes with respect to Gaussian measures. Considerable attention is given to various functional inequalities involving such operators and semigroups. Generalized Mehler semigroups are briefly discussed. Major recent achievements are presented and remaining open problems are indicated.

Bibliography: 214 titles.

Keywords: Ornstein–Uhlenbeck operator, Ornstein–Uhlenbeck semigroup, Gaussian measure, Chebyshev–Hermite polynomial, Mehler formula.

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Introduction

This survey gives an account of the state of the art of the theory of Ornstein–Uhlenbeck operators and semigroups, including relatively recent achievements and open problems. Ornstein–Uhlenbeck random processes closely connected with the main objects are mentioned but not discussed in any detail, because this is a separate and substantial subject requiring its own survey. The main objects indicated, bearing the names of two prominent scholars of the 20th century, L. Ornstein (1880–1941) and G. Uhlenbeck (1900–1988) (for instance, see [53]), are among the most classical objects in analysis, differential equations, mathematical physics, and random processes. Like their close relatives — the Laplace operator Δ, the heat semigroup, and the Wiener process (Brownian motion) — they are distinguished by the elementary character of the basic concepts and the depth of problems connected with them. The Ornstein–Uhlenbeck semigroup is a rather rare example of a simple explicit expression for the solution of the multidimensional parabolic equation

$$\frac{\partial u}{\partial t} = Lu,$$

where L is a second-order operator (the Ornstein–Uhlenbeck operator, also called the Hermite operator), also having a simple form:

$$L\varphi(x) = \Delta\varphi(x) - \langle x, \nabla\varphi(x)\rangle. \tag{0.1}$$

The Ornstein–Uhlenbeck semigroup itself is defined (on various suitable classes of functions) by the formula

$$T_t f(x) = \int_{\mathbb{R}^d} f\bigl(e^{-t}x - \sqrt{1-e^{-2t}}\, y\bigr)\, \gamma(dy), \tag{0.2}$$

where γ is the standard Gaussian measure on $\mathbb{R}^d$, that is, the measure with density

$$\varrho(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$$

with respect to Lebesgue measure. It is quite remarkable that by this formula one can also define a semigroup on infinite-dimensional spaces with Gaussian measures without invoking Lebesgue measure, which does not exist in the infinite-dimensional case.

In infinite-dimensional analysis, the Gaussian measure often plays the role to which Lebesgue measure is allocated in the finite-dimensional case. Moreover, the Ornstein–Uhlenbeck operator has a number of features similar to those of the Laplace operator. For example, in place of the finite-dimensional integration by parts formula

$$\int_{\mathbb{R}^d} f\Delta g\, dx = -\int_{\mathbb{R}^d} \langle\nabla f, \nabla g\rangle\, dx$$

for smooth functions with compact support, the finite-dimensional formula

$$\int_{\mathbb{R}^d} fLg\, d\gamma = -\int_{\mathbb{R}^d} \langle\nabla f, \nabla g\rangle\, d\gamma \tag{0.3}$$

is used, which has precisely the same infinite-dimensional analogue, with gradients taking values in the Cameron–Martin space of the measure γ (this is discussed below). Of course, the formula (0.3) on $\mathbb{R}^d$ can be deduced from the previous formula by a straightforward calculation using the explicit expression for the density of γ, but (0.3) does not contain Lebesgue measure.

An extremely important aspect of the theory of Ornstein–Uhlenbeck semigroups is connected with the circumstance that the standard Gaussian measure γ is invariant with respect to the standard Ornstein–Uhlenbeck semigroup (the integrals of the functions $T_t f$ with respect to γ are constant) and satisfies the stationary Fokker–Planck–Kolmogorov equation

$$L^*\gamma = 0,$$

understood in the sense of the identity

$$\int_{\mathbb{R}^d} Lf\,d\gamma = 0, \qquad f \in C_0^\infty(\mathbb{R}^d),$$

with the Ornstein–Uhlenbeck operator L. In applications, analogous equations arise that are obtained by perturbations of the drift term $-x$ of this equation by non-linear terms. This subject is briefly developed in § 8.

The formula (0.1) defines the standard Ornstein–Uhlenbeck operator, but under the same name more general second-order operators of the form

$$L_{A,B}\varphi(x) = \sum_{i,j=1}^{d} \alpha^{ij}\,\partial_{x_i}\partial_{x_j}\varphi(x) - \sum_{i,j=1}^{d} \beta^{ij} x_j\,\partial_{x_i}\varphi(x) \tag{0.4}$$

are also considered, where $A = (\alpha^{ij})$ and $B = (\beta^{ij})$ are some constant matrices such that A is symmetric and non-negative definite. In the coordinate-free form we can write

$$L_{A,B}\varphi(x) = \operatorname{trace}\bigl(AD^2\varphi(x)\bigr) - \langle Bx, \nabla\varphi(x)\rangle. \tag{0.5}$$

In the case of diagonal matrices and a non-negative-definite matrix B we obtain

$$L_{A,B}\varphi(x) = \sum_{i=1}^{d} \alpha_i\,\partial_{x_i}^2\varphi(x) - \sum_{i=1}^{d} \beta_i x_i\,\partial_{x_i}\varphi(x), \tag{0.6}$$

where the α_i and $\beta_i \geqslant 0$ are all the eigenvalues of these matrices. If there are zeros among them, then a particular situation arises which is not considered here (the zero numbers β_i lead to the Laplace operator with respect to part of the variables and the zero numbers α_i give a first-order operator with respect to the corresponding variables). But if all these numbers are positive, then by a linear change of variables one can pass to the case $\alpha_i \equiv 1$, which gives an operator of the form

$$L_{I,B}\varphi(x) = \Delta\varphi(x) - \langle Bx, \nabla\varphi(x)\rangle \tag{0.7}$$

with a symmetric positive-definite linear operator B. However, for applications to linear stochastic equations

$$d\xi_t = dw_t - \frac{1}{2}B\xi_t\,dt$$

it is necessary to consider general linear operators B, which we shall not do, with the exception of some particular results. An operator $L_{A,B}$ of the form (0.6) is the generator of the semigroup defined by the formula

$$S_t f(x) = \int_{\mathbb{R}^d} f\bigl(e^{-tB}x - \sqrt{1-e^{-2tB}}\, y\bigr)\, \gamma_\sigma(dy), \tag{0.8}$$

where B is the diagonal matrix with entries β_i and γ_σ is the centred Gaussian measure on $\mathbb{R}^d$ with covariance $\sigma = AB^{-1}$, that is, the image of the standard Gaussian measure under the map $x \mapsto \sqrt{\sigma}\, x$ (here A is the diagonal matrix with entries α_i). As in the case of the standard Ornstein–Uhlenbeck operator, the term 'semigroup generator' requires some precision, since the semigroup defined by the indicated formula acts on different function spaces. General forms of types (0.5), (0.6), and (0.8) of operators and semigroups become more important when we pass to the infinite-dimensional case, where possibilities for changing variables are much more limited. Moreover, in many problems A and B are unbounded self-adjoint operators in the infinite-dimensional case (and often non-commuting). For instance, here one frequently encounters suitably interpreted operators of the form (0.5).

As happens for important classical objects, Ornstein–Uhlenbeck operators and semigroups are also encountered under other names. The Ornstein–Uhlenbeck operator is often called the Hermite operator (as well as the equation connected with it), and in the physics literature it is also called the particle number operator. The Ornstein–Uhlenbeck semigroup is also called the Mehler semigroup or the Hermite semigroup. Indeed, there are grounds for this. The equation with the operator of the form $D^2 - xD$ or $D^2 - 2xD$ is used in Hermite's paper [104] from 1864, where he (somewhat later than Chebyshev [46]) considered some properties of the important class of polynomials now called Chebyshev–Hermite polynomials or just Hermite polynomials. The latter is especially typical for authors outside Russia, since it was Hermite's paper from which foreign researchers learned about these polynomials. It should also be noted that Chebyshev's paper rather quickly became known, all the more so because the idea of constructing orthogonal polynomials with respect to a scalar product with an arbitrary weight was first proposed there (hence from a formal point of view virtually all orthogonal polynomials should have been called Chebyshev polynomials, though in the established terminology he is given credit only for his famous polynomials of best approximation on a closed interval and, together with Hermite, for the polynomials discussed below for a Gaussian weight). In 1866 the classical article by Mehler [145] appeared, in which an integral formula for the solution to the Cauchy problem with the Hermite operator was derived. This formula ((10) on p. 172 of [145]) has the form

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{F(y_1,\dots,y_\nu)}{(\sqrt{\pi}\,\sqrt{1-\varrho^2}\,)^\nu} e^{-Q}\, dy_1 \cdots dy_\nu, \qquad Q = \sum_{s=1}^{s=\nu} \biggl(\frac{\varrho x_s - y_s}{\sqrt{1-\varrho^2}}\biggr)^2.$$

On the next page of Mehler's paper it is explained that if we make the change of variables

$$y_1 = \varrho x_1 + \sqrt{1-\varrho^2}\, z_1, \quad \dots, \quad y_\nu = \varrho x_\nu + \sqrt{1-\varrho^2}\, z_\nu,$$

then the limit of the integral as $\varrho \to 1$ is $F(x_1,\dots,x_\nu)$. The integral obtained after such a change gives the now standard expression for the Ornstein–Uhlenbeck semigroup. It is further shown in Mehler's paper that the kernel

$$E(x,y)=\frac{1}{\sqrt{1-\varrho^2}}\exp\biggl(\frac{2\varrho xy-\varrho^2(x^2+y^2)}{1-\varrho^2}\biggr) \tag{0.9}$$

satisfies the parabolic equation

$$2\varrho\,\frac{\partial E}{\partial\varrho}-2x\,\frac{\partial E}{\partial x}+\frac{\partial^2 E}{\partial x^2}=0$$

with respect to the variable x. The factor 2ϱ is the difference from the equation for the Ornstein–Uhlenbeck semigroup and is connected with the use of ϱ^2.

These three papers have been cited by many investigators for already a century and a half. Closely related equations were considered in the dissertation of the St. Petersburg researcher Wera Lebedeff (1880–1970), who in 1906 defended it in Göttingen with Hilbert as advisor (see [156]). Upon marriage, she got the double surname of Myller-Lebedeff and from 1910 worked at the university in Iaşi in Romania (her husband Alexandru Myller was also a student of Hilbert in the same years, also worked at the university in Iaşi, and in 1944–1945 was its rector). Of later work involving topics like Hermite polynomials and Mehler formulae, we should mention the following papers of Hille, one of the founders of the theory of operator semigroups: [105] (which contains an extensive bibliography on Hermite polynomials, including works by Chebyshev, Hermite, Mehler, Lebedeff, and many others, and also Mehler's formula) and [106] and [108], where the terms 'Hermite operator' for D^2-2xD and 'Hermite equation' are used. In his well-known monograph [107] on operator semigroups Hille employed the term 'Hermite semigroups' for general semigroups defined by means of expansions in Hermite polynomials in the form

$$T(\xi)[f](t)=\sum_{n=1}^{\infty}e^{-\xi\lambda_n}f_nH_n(t), \qquad f(t)=\sum_{n=1}^{\infty}f_nH_n(t).$$

Although in a whole series of old papers (from the 19th and the first third of the 20th century) one encounters elliptic and parabolic equations with the Hermite operator, such equations were not yet an object of independent study, as was the case with the Laplace operator and the heat equation. However, we can mention the well-known physics paper [194] by Smoluchowski, in which (in connection with Brownian motion) he considered the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\beta\,\frac{\partial}{\partial x}(uf(x))$$

with the linear coefficient $\beta f(x)=-\gamma x$ (see the formula (50) on p. 588 of [194]). For the solution a formula was given which had already been derived by Mehler. In general it should be noted that elliptic equations satisfied by Chebyshev–Hermite polynomials have long been encountered in many physics papers on various problems. For example, they can be found in works of the founders of quantum mechanics (including Schrödinger [183], p. 515, and Fock [72], p. 72). In the study of harmonic oscillators the operator

$$N=\frac{d^2}{dx^2}-x^2$$

arises, which is also sometimes called the Hermite operator and which is unitarily equivalent to the Ornstein–Uhlenbeck operator (if the former is considered acting in L^2 with respect to Lebesgue measure and the latter acting in L^2 with respect to the Gaussian measure). A derivation of Mehler's formula for the corresponding integral kernel (0.9) is discussed in [212], which even presents three different approaches, including those of Hardy and Hille, but their relative non-triviality is explained by different original conditions: the question is not about the transformation (0.2) but about finding a generating function for the Chebyshev–Hermite polynomials.

The situation changed after publication in 1930 of the classical paper [202] by Uhlenbeck and Ornstein, where a model of Brownian motion with a drift was presented which soon came to be called the Ornstein–Uhlenbeck process. The second part of this work was published in 1945 by Wang and Uhlenbeck [209] already after the death of Ornstein. From the point of view of the theory of stochastic integral equations developed somewhat later, this process is described by the equation

$$d\xi_t = dw_t - \frac{1}{2}\xi_t \, dt,$$

understood in the sense of the integral identity (even without stochastic integrals)

$$\xi_t = \xi_0 + w_t - \frac{1}{2}\int_0^t \xi_s \, ds.$$

More precisely, Uhlenbeck and Ornstein were actually interested not in the process now called by their names, but in its primitive. Their goal was a model of stochastic motion in which trajectories would have finite velocities, which is not achieved in the case of the classical Brownian motion with almost surely non-differentiable trajectories. But why should we take for this the primitive of some new process and not the usual Wiener process? Some physical explanations can be seen in [202], [64], and [166], but it seems that an important factor is the existence of a stationary distribution for the Ornstein–Uhlenbeck process (this is the standard Gaussian measure). In the case of taking the primitive of the Wiener process we obtain a process with variance tending to infinity as time increases. In [111] the opinion was expressed that consideration of the Ornstein–Uhlenbeck process could be traced back to Laplace, who had studied a differential equation that is a Fokker–Planck equation for this process.

The term 'Ornstein–Uhlenbeck process' was already used by Doob [64]. In his well-known monograph [166] Nelson used the terms 'Ornstein–Uhlenbeck process' and 'Ornstein–Uhlenbeck theory' many times, and he also considered the corresponding semigroup and operator (its generator), but without giving them any names. I have not managed to clarify when the terms 'Ornstein–Uhlenbeck operator' and 'Ornstein–Uhlenbeck semigroup' were first used. This terminology perhaps emerged in the mid-1960s, in discussions of the Euclidean quantum field theory which was rapidly developing at the time. This argument was suggested to me by L. Gross. In any case, the terms were used explicitly in the papers and the dissertation of his Ph.D. student Piech in the early 1970s [176], and from the very beginning in the infinite-dimensional case (one of the earliest papers in this area in the infinite-dimensional case was Umemura's 1965 paper

[203], where the infinite-dimensional Ornstein–Uhlenbeck operator was called the 'infinite-dimensional Laplace operator'). Her papers (see also [175]) along with Gross's papers [95] and [96] greatly influenced the theory of Ornstein–Uhlenbeck semigroups in both infinite and finite dimension. I also note that in response to the question about the origin of this terminology, Piech mentioned the influence of the English translation of Dynkin's monograph [65], in which the connections between diffusion processes and generators of their transition semigroups were studied in detail (although the particular Ornstein–Uhlenbeck process was not considered there), after which the identification of processes and their generators became commonplace. Unfortunately, I did not ask Nelson the same question when I had the opportunity, since I did not even think about this at the time.

In the last two decades, the number of papers investigating or applying the Ornstein–Uhlenbeck semigroup has increased considerably, and moreover, not only in the infinite-dimensional case where it serves as a certain substitute for the heat semigroup and possesses remarkable properties independent of dimension, but also in the finite-dimensional case. A substantial factor that has helped draw the attention of many researchers to the Ornstein–Uhlenbeck semigroup and operator has been the development of the Malliavin calculus since the mid-1970s (see the surveys [137], [26], [112], [170], [188], [138]). By means of this semigroup one can study various fine functional inequalities and estimates (like the logarithmic Sobolev and isoperimetric inequalities and their generalizations), and one can use it as an effective tool for smoothing. Considerable attention has been given to the investigation of the spectral properties of its generator, the Ornstein–Uhlenbeck operator. Finally, a number of generalizations of this semigroup have appeared. A survey of these investigations is given below. Due to lack of space, the bibliography below does not include many works on the topic of this survey, although I tried to reflect in it, at least selectively, publications of most researchers working on related questions. I hope to prepare a more complete bibliography for a monograph in preparation.

The questions touched upon in this survey have been discussed with many colleagues. I am particularly indebted to L. Ambrosio, V. Barbu, G. Da Prato, D. Elworthy, W. Farris, L. Gross, Yu. G. Kondratiev, E. D. Kosov, A. Lunardi, G. Metafune, R. A. Minlos, J. van Neerven, Yu. A. Neretin, A. Piech, S. N. Popova, M. Röckner, B. Schmuland, A. V. Shaposhnikov, S. V. Shaposhnikov, and F.-Y. Wang.

1. Chebyshev–Hermite polynomials and basic properties of the Ornstein–Uhlenbeck semigroup

Many areas of mathematics and physics involve the Chebyshev–Hermite polynomials (sometimes called Hermite polynomials for brevity), which are defined by the equalities

$$H_0 = 1, \qquad H_k(t) = \frac{(-1)^k}{\sqrt{k!}} e^{t^2/2} \frac{d^k}{dt^k} e^{-t^2/2}, \quad k \geqslant 1.$$

These polynomials satisfy many interesting relations (sometimes rather unobvious), discovered over the century and a half of their investigations. For the questions we discuss, for example, the following relations will be useful:

$$H_k'(t) = \sqrt{k}\, H_{k-1}(t) = tH_k(t) - \sqrt{k+1}\, H_{k+1}(t).$$

A characteristic (up to a sign) property of these polynomials is that they are obtained by applying the standard Gram–Schmidt orthogonalization procedure to the sequence of powers $1, t, t^2, \ldots$ in the Hilbert space $L^2(\gamma)$ for the standard Gaussian measure on the real line. Moreover, the system of functions $\{H_k\}$ is an orthonormal basis in $L^2(\gamma)$. This is not completely obvious from the construction. One way of verifying the equality to zero of any element $g \in L^2(\gamma)$ orthogonal to all powers t^k is as follows. Consider the analytic function

$$f(z) = \int e^{izt} g(t)\, \gamma(dt).$$

Differentiating with respect to z, we get that $f^{(k)}(0) = 0$ for all k, whence $f = 0$. Thus, the function $g(t)e^{-t^2/2}$ has the identically zero Fourier transform, and hence $g = 0$.

We remark that the Chebyshev–Hermite polynomials are often introduced by means of orthogonalization with the Gaussian weight $\pi^{-1/2}e^{-x^2}$, which does of course lead to different functions, but the connection between these two variants is easily established by means of the change of variable $x = \sqrt{2}\, y$.

For the standard Gaussian measure $\gamma = \gamma_d$ on $\mathbb{R}^d$ there is an orthonormal basis in $L^2(\gamma)$ consisting of polynomials of the form

$$H_{k_1,\ldots,k_d}(x_1, \ldots, x_d) = H_{k_1}(x_1) \cdots H_{k_d}(x_d), \qquad k_i \geqslant 0.$$

It is convenient to group them according to the sums $k = k_1 + \cdots + k_d$. Of course, to every non-negative integer k there corresponds not just one polynomial, but finitely many pairwise orthogonal polynomials of degree k. The linear span of these polynomials for fixed k will be denoted by $\mathscr{X}_k$. The subspaces $\mathscr{X}_k$ are pairwise orthogonal and give the whole of $L^2(\gamma)$ in a direct sum:

$$L^2(\gamma) = \bigoplus_{k=0}^{\infty} \mathscr{X}_k.$$

This means that for the operators I_k of orthogonal projection onto $\mathscr{X}_k$ we have the orthogonal expansion

$$f = \sum_{k=0}^{\infty} I_k(f), \qquad f \in L^2(\gamma).$$

As in the case of general orthogonal bases, for every function $f \in L^2(\gamma)$ the indicated series converges in $L^2(\gamma)$, which in the one-dimensional case leads to the expansion into Chebyshev–Hermite polynomials convergent in $L^2(\gamma)$:

$$f = \sum_{k=0}^{\infty} a_k H_k, \qquad a_k = \int_{-\infty}^{+\infty} f H_k \, d\gamma.$$

This formal series can be considered also for all $f \in L^p(\gamma)$ with $p > 1$. Of course, for $p \geqslant 2$ we still have convergence in $L^2(\gamma)$, but, as shown by Pollard [179], for every $p \neq 2$ there exists a function $f \in L^p(\gamma)$ such that its corresponding series does not converge in $L^p(\gamma)$. In the case $p < 2$ there even exists a function $f \in L^p(\gamma)$ such that $\limsup_{k\to\infty} |a_k H_k(x)|^{1/k} = r > 1$ for each x (r does not depend on x).

Muckenhoupt [154] obtained sufficient conditions for convergence of expansions in Chebyshev–Hermite polynomials almost everywhere (see Corollary 2 in [154]; one should bear in mind that the orthogonalization there is taken with the weight $\pi^{-1/2}e^{-x^2}$). These conditions are of a rather restrictive nature, as is clear from Pollard's result. We recall that for classical Fourier series the situation is simpler (in the sense of the formulation of the final result, but the question itself was an open problem for half a century): convergence of the Fourier series with respect to trigonometric functions holds almost everywhere for all functions $f \in L^p[0, 2\pi]$ with $p > 1$.

Throughout, γ will denote the standard Gaussian measure on $\mathbb{R}^d$ if we are considering the finite-dimensional case, or the standard Gaussian measure on the space of sequences $\mathbb{R}^\infty$ (the countable power of the standard Gaussian measure on the real line). Special note will be made of cases where we are considering centred Gaussian measures on abstract locally convex spaces.

The norm on $L^p(\gamma)$ with $1 \leqslant p \leqslant \infty$ will be denoted by $\|f\|_p$.

The norm and inner product in $\mathbb{R}^d$ are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C_b^\infty(\mathbb{R}^d)$ be the set of bounded infinitely differentiable functions with bounded derivatives, and $C_0^\infty(\mathbb{R}^d)$ its subset of functions with compact support. The space of bounded continuous functions on $\mathbb{R}^d$ is denoted by $C_b(\mathbb{R}^d)$ and equipped with its usual norm $\|f\| = \sup_x |f(x)|$. For a non-negative Borel measure μ on $\mathbb{R}^d$, let $L^p_{\mathrm{loc}}(\mu)$ be the class of all μ-measurable functions f for which the function $|f|^p$ is integrable with respect to μ on every ball. As usual, we do not distinguish between equivalence classes and their representatives, but in some results we refer to versions of functions with certain properties.

The simplest properties of the Ornstein–Uhlenbeck semigroup are collected in the next theorem.

Theorem 1.1. *The operators T_t form a strongly continuous contraction semigroup on $L^p(\gamma)$ for $p \in [1,\infty)$, that is,*

$$T_{t+s} = T_t \circ T_s, \quad \|T_t f\|_p \leqslant \|f\|_p, \quad \lim_{t\to 0} \|T_t f - f\|_p = 0 \quad \forall\, f \in L^p(\gamma). \tag{1.1}$$

In addition, the measure γ is invariant for T_t, that is,

$$\int T_t f\, d\gamma = \int f\, d\gamma \quad \forall\, f \in L^1(\gamma), \tag{1.2}$$

and, with the integral of f denoted by $I(f)$,

$$\lim_{t\to\infty} \|T_t f - I(f)\|_p = 0 \quad \forall\, f \in L^p(\gamma), \quad p \in [1,\infty). \tag{1.3}$$

Finally, on the space $L^2(\gamma)$ the operators T_t are self-adjoint and non-negative in the sense of quadratic forms, and on the spaces $L^p(\gamma)$ they are non-negative in the sense of ordered spaces, that is, they take non-negative functions to non-negative functions.

Verification of the properties (1.1)–(1.3) is completely elementary (the inequality in (1.1) follows from Hölder's inequality). The last relation in (1.1) is obvious for bounded continuous functions f, and then the estimate for the norm yields the same relation for all f in $L^p(\gamma)$. Similarly for (1.3). The self-adjointness and non-negativity of T_t on $L^2(\gamma)$ are obvious, as well as the non-negativity on $L^p(\gamma)$ in the sense of the pointwise comparison of functions.

Of course, for $p = \infty$ the estimate $\|T_t f\|_\infty \leqslant \|f\|_\infty$ is also true, but the third property (strong continuity) on $L^\infty(\gamma)$ fails even in the one-dimensional case and even for bounded continuous functions. To show this, it suffices to take a bounded Lipschitz function f for which $f(n) = 1$ and $f(e^{-1/n}n) = 0$. Such a function obviously exists, since $(1 - e^{-1/n})n \to 1$. For this function we have

$$\begin{aligned} |T_{1/n}f(x) - f(e^{-1/n}x)| &\leqslant \int_{\mathbb{R}} \bigl|f\bigl(e^{-1/n}x - \sqrt{1-e^{-2/n}}\,y\bigr) - f(e^{-1/n}x)\bigr|\,\gamma(dy) \\ &\leqslant C\sqrt{1-e^{-2/n}}\,, \end{aligned}$$

so the quantity $\|T_{1/n}f - f\|_\infty$ cannot tend to zero because $f(e^{-1/n}n) - f(n) = 1$.

The integral formula for $T_t f$ enables us to use well-known integral inequalities. For example, from Jensen's inequality for a convex function V we get that

$$V(T_t f) \leqslant T_t(V(f)).$$

In particular, for $f > 0$ we have

$$T_t \log f \leqslant \log T_t f \quad \text{and} \quad T_t(f \log f) \geqslant T_t f \log T_t f,$$

provided that these integrals exist.

Remark 1.2. By means of the formula

$$T_t^*\nu(B) = \int_{\mathbb{R}^d} T_t I_B(x)\,\nu(dx)$$

one can define the action of the 'adjoint' semigroup on bounded Borel measures. This action can be written as a composition of a homothety and a convolution, so for $t > 0$ the measure $T_t^*\nu$ is absolutely continuous. Hence, the measure $T_t^*\nu$ has a density $g_t \in L^1(\gamma)$ with respect to γ. Since $T_t^*\nu = T_{t-s}^*T_s^*\nu$ for $t > s$, we have $g_t = T_{t-s}g_s$. By (1.3) the measures $T_t^*\nu$ converge in variation to the measure $\nu(\mathbb{R}^d)\gamma$ as $t \to +\infty$. As $t \to 0$, these measures converge weakly (but not in variation) to ν.

Remark 1.3. We remark also that γ is the unique invariant probability measure for the semigroup $\{T_t\}_{t\geqslant 0}$, that is, if for some probability measure μ the integral of $T_t f$ equals the integral of f for all $f \in C_b(\mathbb{R}^d)$ and $t \geqslant 0$, then $\mu = \gamma$. This can be shown using different methods, in particular, it can be obtained from general results presented in Chap. 5 of [35], but we consider a proof that can be extended to the infinite-dimensional case. Let $l(x) = \langle x, v\rangle$ be a linear function on $\mathbb{R}^d$. We observe that

$$\begin{aligned} T_t \exp\bigl(il(x)\bigr) &= \exp\bigl(ie^{-t}l(x)\bigr) \int_{\mathbb{R}^d} \exp\bigl(-\sqrt{1-e^{-2t}}\,l(y)\bigr)\,\gamma(dy) \\ &= \exp\bigl(ie^{-t}l(x)\bigr) \exp\biggl(-(1-e^{-2t})\frac{|v|^2}{2}\biggr). \end{aligned}$$

The equality of the integrals of $\exp(il)$ and $T_t \exp(il)$ with respect to the measure μ means the following identity for the Fourier transform $\widetilde{\mu}$ of the measure μ, defined as the integral of $\exp(il)$:

$$\widetilde{\mu}(e^{-t}l) = \exp\biggl(-e^{-2t}\frac{|v|^2}{2}\biggr)\exp\biggl(\frac{|v|^2}{2}\biggr)\widetilde{\mu}(l).$$

Letting $t \to +\infty$, we conclude that $\widetilde{\mu}(l) = \exp(-|v|^2/2)$, that is, $\mu = \gamma$. Note that the same conclusion holds if the measure μ is invariant with respect to a single operator T_τ with $\tau > 0$, since in that case it is invariant with respect to the operators $T_{k\tau}$ for all $k \in \mathbb{N}$, so the reasoning above remains in force. It also shows that every (even signed) measure which is invariant with respect to the semigroup $\{T_t\}_{t\geqslant 0}$ coincides with γ up to a constant factor.

It follows from the general theory of continuous operator semigroups on Banach spaces (see, for instance, [59], [107]) that for every $p \in [1, \infty)$ the set

$$D_p(L) := \biggl\{ f \in L^p(\gamma)\colon \lim_{t\to 0}\frac{1}{t}(T_t f - f) \text{ exists in } L^p(\gamma)\biggr\}$$

is a dense linear subspace of $L^p(\gamma)$, and the linear operator with domain $D_p(L)$ given by

$$Lf := \lim_{t\to 0}\frac{1}{t}(T_t f - f)$$

is closed, that is, has a closed graph: if $f_n \in D_p(L)$, $f_n \to f$, and $Lf_n \to g$ in $L^p(\gamma)$, then $f \in D_p(L)$ and $Lf = g$. This operator is called the generator of the semigroup $\{T_t\}_{t\geqslant 0}$. In the case of the Ornstein–Uhlenbeck semigroup, L is called the Ornstein–Uhlenbeck operator. It is convenient to write the operators T_t in the form $T_t = \exp(tL)$, with L on $L^2(\gamma)$ having a non-positive quadratic form, that is, to indicate that the corresponding operator exponential $\exp(tL)$ coincides with T_t.

For different p the domains $D_p(L)$ are different, so from the formal point of view we have a continuum of operators, but for $r < p$ we obviously have $D_p(L) \subset D_r(L)$, and the restriction of L from $D_r(L)$ to $D_p(L)$ gives its values on the smaller subspace. Hence the operator L itself will not be equipped with any indices. We shall see below that L is given by the differential expression indicated above in (0.1), but using this expression requires some care, since the domains of definition of L go beyond the classes of smooth functions.

The simplest case of calculation of the domain of L is for $L^2(\gamma)$. To this end we observe the following fact, which is important in itself: the Chebyshev–Hermite polynomials are eigenfunctions for T_t, that is, for $d = 1$ we have

$$T_t H_k = e^{-kt} H_k,$$

and in the multidimensional case, where k is a multi-index $k = (k_1, \dots, k_d)$,

$$T_t H_{k_1,\dots,k_d} = e^{-(k_1+\dots+k_d)t} H_{k_1,\dots,k_d},$$

which can be written as

$$T_t H_k = e^{-|k|t} H_k, \qquad |k| = k_1 + \dots + k_d.$$

If k is a number in $0, 1, 2, \dots$, then

$$T_t f = e^{-kt} f, \qquad f \in \mathscr{X}_k.$$

Theorem 1.4. *The domain of L in $L^2(\gamma)$ for $d = 1$ is*

$$D_2(L) = \Big\{ f = \sum_{k=0}^{\infty} c_k H_k \colon \sum_{k=0}^{\infty} k^2 |c_k|^2 < \infty \Big\},$$

and

$$Lf = -\sum_{k=0}^{\infty} k c_k H_k.$$

In the multidimensional case

$$D_2(L) = \Big\{ f = \sum_{k=0}^{\infty} c_k f_k \colon f_k \in \mathscr{X}_k, \ \sum_{k=0}^{\infty} k^2 \|f_k\|_2^2 < \infty \Big\},$$

$$Lf = -\sum_{k=0}^{\infty} k f_k.$$

Proof. These assertions are easily verified by means of the formula

$$T_t f = \sum_{k=0}^{\infty} e^{-kt} c_k H_k$$

and an analogous formula in the multidimensional case. The specific feature of the Hilbert case enables us to easily determine the conditions for the existence of the limit of $t^{-1}(T_t f - f)$ in the norm. $\square$

The expressions presented are very convenient in many problems, but they do not exhibit the differential nature of L. In addition, it is not obvious from these expressions without additional investigation that the class $D_2(L)$ contains the set $C_b^\infty(\mathbb{R}^d)$ or at least $C_0^\infty(\mathbb{R}^d)$. However, the inclusion $C_0^\infty(\mathbb{R}^d) \subset D_p(L)$ for all $p < \infty$ can be verified directly along with an explicit calculation of Lf for $f \in C_0^\infty(\mathbb{R}^d)$. For this we employ the equality (for simplicity of expression, just in the one-dimensional case)

$$\begin{aligned} &f\big(e^{-t}x - \sqrt{1-e^{-2t}}\, y\big) - f(x) \\ &\qquad = \int_0^t f'\big(e^{-s}x - \sqrt{1-e^{-2s}}\, y\big)\big(-e^{-s}x - e^{-2s}(1-e^{-2s})^{-1/2} y\big)\, ds. \end{aligned}$$

After integration in y with respect to the measure γ we obtain two terms. The first term multiplied by t^{-1} tends in $L^p(\gamma)$ to $-xf'(x)$ as $t \to 0$, as can be easily shown by means of the Lebesgue theorem. The second term is transformed by integration by parts with respect to y into

$$\int_0^t \int f''\big(e^{-s}x - \sqrt{1-e^{-2s}}\, y\big) e^{-2s}\, ds\, \gamma(dy).$$

This expression with the factor t^{-1} tends to $f''(x)$ in $L^p(\gamma)$ as $t \to 0$. Of course, this remains in force for $f \in C_b^2(\mathbb{R}^d)$.

Thus, for functions $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have the expression (0.1) for $L\varphi$. This yields the following equality that will also be useful below:

$$T_t\varphi(x) - \varphi(x) = \int_0^t T_s L\varphi(x)\, ds \tag{1.4}$$

pointwise; here the function $L\varphi$ belongs to $C_0^\infty(\mathbb{R}^d)$ and the function $T_sL\varphi(x)$ is continuous in both arguments and bounded. Actually, this equality is true for all φ in the domain of L in $L^1(\gamma)$.

To describe the domains of the generators of the Ornstein–Uhlenbeck semigroup in the spaces $L^p(\gamma)$ more precisely, that is, of the corresponding Ornstein–Uhlenbeck operators, we need the Sobolev classes with respect to Gaussian measures. These classes will be considered in the next section.

The explicit expression for T_tf implies the useful relation

$$\nabla T_t f = e^{-t} T_t \nabla f, \tag{1.5}$$

which is frequently used in calculations.

We also note that the Ornstein–Uhlenbeck operator is connected with the divergence $\operatorname{div}_\gamma w$ of a vector field $w = (w^i)$ with respect to the measure γ, also denoted by the symbol δw and given by the equality

$$\operatorname{div}_\gamma w(x) := \delta w(x) := \operatorname{div}\, w(x) - \langle w(x), x\rangle = \sum_{i=1}^d \bigl(\partial_{x_i} w^i(x) - x_i w^i(x)\bigr). \tag{1.6}$$

The divergence $\operatorname{div}_\gamma w$ of the field w with respect to the measure γ plays a role analogous to the usual divergence $\operatorname{div} w$: in the case of a smooth function f one has the integration by parts formula

$$\int_{\mathbb{R}^d} \langle \nabla f, w\rangle\, d\gamma = -\int_{\mathbb{R}^d} f \operatorname{div}_\gamma w\, d\gamma,$$

which can be verified directly for f and w with bounded derivatives. By means of this divergence the Ornstein–Uhlenbeck operator can be expressed in the form

$$Lf = \operatorname{div}_\gamma \nabla f. \tag{1.7}$$

The simple equality (1.4) is the basis for deriving a number of rather unobvious integral inequalities and estimates for distributions of functions on spaces with Gaussian measures. Moreover, it becomes possible to extend such results to considerably more general situations, acting by analogy with the Ornstein–Uhlenbeck semigroup and operator. A thorough discussion can be found in the very informative book [120] by Ledoux; here we only give some typical examples.

Example 1.5. (i) For a function $f \geqslant 0$ such that f and $f\log f$ are integrable with respect to the measure γ, where we set $f(x)\log f(x) := 0$ if $f(x) = 0$, the entropy is defined by

$$\operatorname{Ent}_\gamma(f) := \int_{\mathbb{R}^d} f\log f\, d\gamma - \int_{\mathbb{R}^d} f\, d\gamma\ \log \int_{\mathbb{R}^d} f\, d\gamma.$$

With the aid of Jensen's inequality for the convex function $t\log t$, it is not difficult to verify that $\operatorname{Ent}_\gamma(f)\geqslant 0$. Under rather broad conditions on the function f one has the representation

$$\operatorname{Ent}_\gamma(f)=-\int_0^\infty \frac{d}{dt}\biggl(\int_{\mathbb{R}^d} T_tf\log T_tf\,d\gamma\biggr)\,dt. \tag{1.8}$$

It is very simple to prove this equality for functions f of class C_b^∞ bounded away from zero by a positive constant. In this case the expression under the integral sign can be written as

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} T_tf\log T_tf\,d\gamma&=\int_{\mathbb{R}^d} LT_tf\log T_tf\,d\gamma+\int_{\mathbb{R}^d} LT_tf\,d\gamma\\&=-\int_{\mathbb{R}^d}\frac{|\nabla T_tf|^2}{T_tf}\,d\gamma,\end{aligned}$$

where we have taken into account the formula (0.3), the symmetry of L in $L^2(\gamma)$, and the equality $L1=0$. This implies the representation

$$\operatorname{Ent}_\gamma(f)=\int_0^\infty\int_{\mathbb{R}^d}\frac{|\nabla T_tf|^2}{T_tf}\,d\gamma\,dt. \tag{1.9}$$

In § 5 (see Theorem 5.1) we shall show how to use this representation to derive the logarithmic Sobolev inequality.

(ii) Similarly, for functions $f,g\in C_b^\infty(\mathbb{R}^d)$ we obtain a representation for the covariance:

$$\operatorname{cov}_\gamma(f,g)=\int_{\mathbb{R}^d}(f-\mathsf{E}f)(g-\mathsf{E}g)\,d\gamma=\int_{\mathbb{R}^d} f(g-\mathsf{E}g)\,d\gamma,\qquad \mathsf{E}g=\int_{\mathbb{R}^d} g\,d\gamma.$$

Here we have

$$\operatorname{cov}_\gamma(f,g)=-\int_0^\infty\int_{\mathbb{R}^d} fLT_tg\,d\gamma\,dt=\int_0^\infty\int_{\mathbb{R}^d}\langle\nabla f,\nabla T_tg\rangle\,d\gamma.$$

In this formula we can use the equality $\nabla T_tg=e^{-t}T_t\nabla g$, which in view of the estimate $|T_t\nabla g|\leqslant T_t|\nabla g|$ and the equality of the integrals of $T_t|\nabla g|$ and $|\nabla g|$ implies the estimate

$$\operatorname{cov}_\gamma(f,g)\leqslant \operatorname{Lip}(f)\int_{\mathbb{R}^d}|\nabla g|\,d\gamma, \tag{1.10}$$

where $\operatorname{Lip}(f)$ is the Lipschitz constant of the function f. By passage to the limit this estimate extends to Lipschitz functions f and locally Sobolev functions g with gradients in $L^1(\gamma)$, that is, to functions in the Gaussian Sobolev class $W^{1,1}(\gamma)$ considered below; such a function fg is automatically integrable with respect to the measure γ (see (2.4)).

Remark 1.6. Note that the operator L with domain $D_2(L)$ is self-adjoint in $L^2(\gamma)$; on the linear span of the polynomials H_k it is essentially self-adjoint (its closure is self-adjoint). This can be readily derived from the diagonal form of L, but it can also be obtained as a corollary of a general theorem on the essential self-adjointness of the generator of a strongly continuous semigroup of self-adjoint operators, acting in a Hilbert space on a dense linear subspace contained in the domain of the generator and invariant with respect to the semigroup. The same reasoning gives the essential self-adjointness of L on $C_b^\infty(\mathbb{R}^d)$, and in the complex case on the linear span of the functions of the form $e^{il}\psi$, where l is a linear function and ψ is a polynomial. The class $C_0^\infty(\mathbb{R}^d)$ is not invariant with respect to the semigroup, but the operator L is essentially self-adjoint also on this class. This follows from the fact that $C_0^\infty(\mathbb{R}^d)$ is dense in the Sobolev space $W^{2,2}(\gamma)$ that coincides with $D_2(L)$ as indicated in § 3. Of course, it is also possible to give a direct proof by the well-known criterion for essential self-adjointness of symmetric operators that consists in verifying that the range of the operator $L - iI$ is dense. This can easily be done on the basis of the obvious denseness of the range of $L - iI$ on the space of polynomials.

There is an extensive literature devoted to diverse problems involving expansions in Chebyshev–Hermite functions or polynomials for the Ornstein–Uhlenbeck operator or for the closely related Hermite operator $\Delta - |x|^2$, estimates for norms of such polynomials, and other properties of eigenfunctions (see the book [201] and the papers [1], [47], [117], [118], [153]–[155], and [198], where additional references can be found).

A functional calculus for Ornstein–Uhlenbeck operators was discussed in [81]. In the recent paper [44] a number of results were obtained on generators of general diffusion semigroups, a particular case of which is the Ornstein–Uhlenbeck semigroup. These results are connected with functional calculus and convergence properties.

Some general properties of Ornstein–Uhlenbeck semigroups (in particular, nonsymmetric) connected with analyticity, generators, and their domains were investigated in [48], [49], [51], [92], [93], [123], [133], [159], and [200]. For applications it is useful to consider Ornstein–Uhlenbeck semigroups also on spaces of bounded continuous functions, although strong continuity is lacking on such spaces (see, for instance, [56], [163]). Here we do not discuss degenerate operators of Ornstein–Uhlenbeck type and the corresponding hypoellipticity problems (see, for instance,[173]).

The survey [205] discusses interesting connections with other semigroups connected with orthogonal polynomials.

2. Gaussian Sobolev classes

We recall that the usual Sobolev class (or space) $W^{p,r}(\mathbb{R}^d)$ on $\mathbb{R}^d$ with $p \in [1, \infty)$ and $r \in \mathbb{N}$ consists of all functions $f \in L^p(\mathbb{R}^d)$ such that the generalized partial derivatives $\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} f$ for all $k \leqslant r$ (derivatives in the sense of $\mathscr{D}'$) are given by functions in $L^p(\mathbb{R}^d)$. The space $W^{p,r}(\mathbb{R}^d)$ is a Banach space with respect to the natural Sobolev norm defined as the sum of the L^p-norms of the function itself and all its partial derivatives up to order r inclusive. The set $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{p,r}(\mathbb{R}^d)$, and hence without using generalized derivatives the space $W^{p,r}(\mathbb{R}^d)$ can be defined as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the Sobolev norm. To avoid confusion with the Gaussian Sobolev classes considered below, we do not introduce any notation for the Sobolev norms on $W^{p,r}(\mathbb{R}^d)$. It is readily verified that for all functions $\zeta \in C_0^\infty(\mathbb{R}^d)$ and $f \in W^{p,r}(\mathbb{R}^d)$ the product ζf belongs to $W^{p,r}(\mathbb{R}^d)$. Therefore, it is natural to introduce the local Sobolev class $W^{p,r}_{\mathrm{loc}}(\mathbb{R}^d)$

as the class of functions f on $\mathbb{R}^d$ such that ζf belongs to $W^{p,r}(\mathbb{R}^d)$ for all $\zeta \in C_0^\infty(\mathbb{R}^d)$. Of course, as in the case of L^p, we deal with equivalence classes rather than individual functions (throughout, this will not be specified).

The construction of Sobolev classes with respect to the Gaussian measure γ_d is completely analogous, and moreover, the second method (completion) turns out to be even more intuitive, although a description by means of generalized derivatives is also possible, as will be noted in § 4. First, for the measure $\gamma = \gamma_d$ (the index d will often be omitted, since on $\mathbb{R}^d$ we will consider only the standard Gaussian measure) we introduce the weighted Sobolev norms

$$\|f\|_{p,r} = \|f\|_{W^{p,r}(\gamma)} = \|f\|_{L^p(\gamma)} + \sum_{k\leqslant r}\biggl(\int_{\mathbb{R}^d}\biggl(\sum_{i_1,\dots,i_k\leqslant d}|\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}f|^2\biggr)^{p/2}d\gamma\biggr)^{1/p}$$

on the space $C_0^\infty(\mathbb{R}^d)$ or on $C_b^\infty(\mathbb{R}^d)$ (or on the space of polynomials). Next, we take the completion $W^{p,r}(\gamma)$ of the space obtained (it can be verified that all the three classes give the same completion, hence for definiteness we can assume that first the space $C_b^\infty(\mathbb{R}^d)$ is taken). An element $f \in W^{p,r}(\gamma)$ is a function in $L^p(\gamma)$, and the completion procedure associates with it functions $\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}f \in L^p(\gamma)$ that are its generalized derivatives in the sense of $\mathscr{D}'$, as one can easily verify. It is clear that $f \in W^{p,r}_{\mathrm{loc}}(\mathbb{R}^d)$, but the function f can fail to be in the class $W^{p,r}(\mathbb{R}^d)$ (the simplest example is a non-zero constant). One can verify that the classes $W^{p,r}(\gamma)$ admit the following characterization (the equality (2.2) follows from Meyer's result in § 3 about the equivalence of norms).

Theorem 2.1. *The equality*

$$W^{p,r}(\gamma) = \{f \in W^{p,r}_{\mathrm{loc}}(\mathbb{R}^d)\colon \|f\|_{W^{p,r}(\gamma)} < \infty\} \tag{2.1}$$

holds for all $p \in [1,\infty)$ and $r \in \mathbb{N}$. Moreover, for $p > 1$

$$W^{p,2}(\gamma) = \{f \in W^{p,2}_{\mathrm{loc}}(\mathbb{R}^d)\colon \Delta f - \langle x, \nabla f\rangle \in L^p(\gamma)\}. \tag{2.2}$$

If $f \in L^p(\gamma)$, $p > 1$, and $t > 0$, then the function $T_t f$ is in the classes $W^{p,r}(\gamma)$ for $r \geqslant 1$ and has an infinitely differentiable version (the latter assertion can be easily verified by means of the theorem on differentiating the integral with respect to a parameter, applied to the explicit expression for $T_t f(x + s_1e_1 + \dots + s_de_d)$). In Theorem 5.8 below we give an analogous assertion in the infinite-dimensional case.

In analogy to the class $BV(\mathbb{R}^d)$ of functions of bounded variation, consisting of functions $f \in L^1(\mathbb{R}^d)$ whose generalized partial derivatives $\partial_{x_i} f$ are bounded measures on $\mathbb{R}^d$ (not necessarily given by functions), we can introduce the class $BV(\gamma)$. There are several equivalent descriptions of this class (see [80]). The first one defines it as the set of functions $f \in L^1(\gamma)$ that are limits in $L^1(\gamma)$ of sequences of functions with uniformly bounded norms in $W^{1,1}(\gamma)$. The norm on $BV(\gamma)$ can be defined by the equality

$$\|f\|_{BV(\gamma)} = \inf\bigl\{C\colon \{f_n\} \subset W^{1,1}(\gamma),\ f_n \to f \text{ a.e.},\ \|f_n\|_{1,1} \leqslant C\bigr\}. \tag{2.3}$$

Another description, using generalized derivatives, is this: these are functions f in $L^1(\gamma)$ for which $f\sqrt{|\log|f||} \in L^1(\gamma)$, and $fe^{-|x|^2/2} \in BV(\mathbb{R}^d)$. The third description, following from results in [80], is given in the proposition below in terms of the Ornstein–Uhlenbeck semigroup.

Proposition 2.2. *A function $f \in L^1(\gamma)$ belongs to $BV(\gamma)$ precisely when $T_t f \in W^{1,1}(\gamma)$ for all $t > 0$ and*

$$\sup_{t>0} \int_{\mathbb{R}^d} |\nabla T_t f| \, d\gamma < \infty.$$

For the usual Sobolev spaces there are well-known embedding theorems according to which the a priori integrability of functions in $W^{p,r}(\mathbb{R}^d)$ is improved: for example, a function in $W^{1,1}(\mathbb{R}^d)$ actually belongs not only to $L^1(\mathbb{R}^d)$ but also to $L^{d/(d-1)}(\mathbb{R}^d)$. This precise effect does not hold for Gaussian Sobolev classes (on the whole space), but there is some improvement of integrability, although not of a power order but logarithmically. In § 5 we give precise formulations. Here we mention only one result in this direction.

Proposition 2.3. *There is a number C such that for every d and every $f \in W^{1,1}(\gamma)$ the inequalities*

$$\int_{\mathbb{R}^d} |x_i f(x)| \, \gamma(dx) \leqslant C \|f\|_{1,1} \tag{2.4}$$

hold for all i. The same inequalities are true also for all functions in $BV(\gamma)$ with the norm $\|f\|_{BV(\gamma)}$ instead of $\|f\|_{1,1}$.

Below we give more general and sharper estimates for embeddings in the Orlicz class.

From (0.3) we pass to the limit and obtain the identity

$$\int_{\mathbb{R}^d} f Lg \, d\gamma = -\int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \, d\gamma, \tag{2.5}$$

$$f \in W^{p,1}(\gamma), \quad g \in W^{q,2}(\gamma), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

For a function $f \in L^1(\gamma)$ with zero integral with respect to γ we introduce the Kantorovich norm $\|f\|_{\mathrm{K},\gamma}$ by

$$\|f\|_{\mathrm{K},\gamma} = \sup \biggl\{ \int_{\mathbb{R}^d} f g \, d\gamma \colon g \in \mathrm{Lip}_1 \biggr\}, \tag{2.6}$$

where Lip_1 is the class of functions that are Lipschitz with constant 1. The following inequality was obtained in [40].

Theorem 2.4. *Every function $f \in W^{1,1}(\gamma)$ with zero integral with respect to γ satisfies the two-sided estimate*

$$\frac{\|f\|^2_{L^1(\gamma)}}{2\|\nabla f\|_{L^1(\gamma)}} \leqslant \|f\|_{\mathrm{K},\gamma} \leqslant \|\nabla f\|_{L^1(\gamma)}. \tag{2.7}$$

Now we mention a result from [41], in which W_2 denotes the Kantorovich 2-metric defined for a pair of probability measures μ and σ with finite second moments by the equality

$$W_2^2(\mu, \nu) = \inf \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \sigma(dx \, dy),$$

where the infimum is taken over all probability measures σ on $\mathbb{R}^d\times\mathbb{R}^d$ with projections μ and ν to the factors.

Theorem 2.5. *For all $f\in W^{p,1}(\gamma)$*

$$\gamma(x\colon T_tf(x)\geqslant 2u)\leqslant \inf_{t>0}\biggl[\frac{K_p\arccos(e^{-t})}{u^p}\|\nabla f\|_p^p+\frac{1}{(e^{2t}-1)u\log u}W_2^2(f\cdot\gamma,\gamma)\biggr],$$

where $f\cdot\gamma$ is the measure with density f with respect to γ and

$$K_p^p=\frac{1}{\sqrt{2\pi}}\int |x|^pe^{-x^2/2}\,dx.$$

The proof is based on a lemma that is of independent interest.

Lemma 2.6. *Let $u\in W^{p,1}(\gamma)$. Then*

$$\|T_tu-u\|_p\leqslant K_pc_t\|\nabla u\|_p,$$

where

$$c_t=\int_0^t\frac{e^{-s}}{\sqrt{1-e^{-2s}}}\,ds=\arccos(e^{-t}). \tag{2.8}$$

Above we discussed the standard Sobolev classes with respect to Gaussian measures and corresponding in the infinite-dimensional case to the Cameron–Martin norm for defining norms of derivatives (that is, the standard norm on $\mathbb{R}^n$ for the standard Gaussian measure) and to the standard Ornstein–Uhlenbeck operator in the definition of Sobolev classes with its help. However, there are other norms and modifications of the Ornstein–Uhlenbeck operator, as is natural for applications to Ornstein–Uhlenbeck processes with different drifts (for example, see [49]–[51], [134], [185]).

We mention that in the last years there have been active studies of the Ornstein–Uhlenbeck semigroup on domains (see [10], [89], [99]) and Gaussian Sobolev classes on domains, which is beyond the scope of our survey.

3. Domains and spectra of Ornstein–Uhlenbeck operators

The domain of the generator of the Ornstein–Uhlenbeck semigroup in $L^2(\gamma)$ was calculated above. For a general space $L^p(\gamma)$ with $p\in(1,\infty)$ the following assertion is true.

Theorem 3.1. *Let $p\in(1,\infty)$. Then*

$$D_p(L)=W^{p,2}(\gamma)=\bigl\{f\in W^{p,2}_{\mathrm{loc}}(\mathbb{R}^d)\colon f,\Delta f-\langle x,\nabla f\rangle\in L^p(\gamma)\bigr\}.$$

Proof. We already know that $C_0^\infty(\mathbb{R}^d)\subset D_p(L)$ and that on $C_0^\infty(\mathbb{R}^d)$ the generator Lf is given by the indicated differential expression. Let $f\in D_p(L)$. Then for all $g\in C_0^\infty(\mathbb{R}^d)$ we have

$$\int Lf\,g\,d\gamma=\int fLg\,d\gamma$$

(here and below in the case of integration over the whole space we do not indicate the domain of integration), whence it follows that $\Delta f-\langle x,\nabla f\rangle$ as a distribution is given by the pointwise defined function $Lf \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. It is known from the theory of elliptic equations that then $f \in W^{p,2}_{\mathrm{loc}}(\mathbb{R}^d)$. From the equality (2.1) we obtain the desired inclusion $f \in W^{p,2}(\gamma)$.

Conversely, let $f \in W^{p,2}(\gamma)$. Then there exists a sequence of functions f_j of class $C_0^\infty(\mathbb{R}^d)$ convergent to f in $W^{p,2}(\gamma)$. By Meyer's inequalities presented below (see Theorem 3.3 for $r=2$) this gives the convergence $Lf_j \to Lf$ in $L^p(\gamma)$, which by the closedness of the generator and the inclusion $C_0^\infty(\mathbb{R}^d) \subset D_p(L)$ implies that $f \in D_p(L)$ and $Lf = \Delta f - \langle x, \nabla f\rangle$. It was already noted in Theorem 2.1 that $W^{p,2}(\gamma)$ coincides with the class of functions $f \in W^{p,2}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\Delta f - \langle x, \nabla f\rangle \in L^p(\gamma)$. $\square$

The domain $D_1(L)$ of the Ornstein–Uhlenbeck operator in $L^1(\gamma)$ has no such explicit description.

Theorem 3.2. *The set $D_1(L)$ consists of all $f \in L^1(\gamma)$ such that the distribution $\Delta f - \langle x, \nabla f\rangle \in \mathscr{D}'(\mathbb{R}^d)$ is given by a function in $L^1(\gamma)$. Moreover, $D_1(L)$ strictly contains $W^{1,2}(\gamma)$.*

Proof. Let $f \in D_1(L)$. For every function $g \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int gLf\,d\gamma = \lim_{t\to 0}\frac{1}{t}\int g(T_tf - f)\,d\gamma = \lim_{t\to 0}\frac{1}{t}\int f(T_tg - g)\,d\gamma = \int fLg\,d\gamma,$$

since the functions $t^{-1}(T_tg - g)$ converge pointwise to Lg and are uniformly bounded in view of (1.4) and the inclusion $Lg \in C_0^\infty(\mathbb{R}^d)$. The equality obtained means that the locally integrable function ϱLf, where ϱ is the standard Gaussian density, coincides as a distribution with the distribution

$$\Delta(f\varrho) - \operatorname{div}(f\varrho x) = \varrho\Delta f + f\Delta\varrho - df\varrho - \varrho\langle\nabla f, x\rangle - f\langle\nabla\varrho, x\rangle.$$

Since $\Delta\varrho(x) = d\varrho(x) - |x|^2\varrho(x)$ and $\nabla\varrho(x) = -\varrho(x)x$, on the right-hand side we obtain the same expression ϱLf, but with Lf in the sense of distributions.

Suppose now that $f \in L^1(\gamma)$ is a function such that the generated element $Lf \in \mathscr{D}'(\mathbb{R}^d)$ is given by a γ-integrable function (which will also be denoted by Lf). We verify that $f \in D_1(L)$. To do this we observe that

$$T_tf - f = \int_0^t T_sLf\,ds \tag{3.1}$$

in the sense of equality in $L^1(\gamma)$. Indeed, the right-hand side is defined by the continuity of the map $s \mapsto T_sLf$ with values in $L^1(\gamma)$. Hence, it suffices to establish that after multiplication by a function of class C_0^∞ both sides have equal integrals with respect to the measure γ. Let $g \in C_0^\infty(\mathbb{R}^d)$. Then

$$\int g(T_tf - f)\,d\gamma = \int f(T_tg - g)\,d\gamma,$$

$$\begin{aligned}\int g\Big(\int_0^t T_sLf\,ds\Big)\,d\gamma &= \int_0^t\int gT_sLf\,d\gamma\,ds = \int_0^t\int T_sgLf\,d\gamma\,ds \\ &= \int Lf\int_0^t T_sg\,ds\,d\gamma.\end{aligned}$$

Our assertion will be proved if we manage to move the operator L to the function ψ obtained by integrating $T_s g$ with respect to s over $[0,t]$, since for functions of class C_0^∞ the formula (3.1) to be proved is true. This function ψ belongs to the class $C_b^\infty(\mathbb{R}^d)$ but not to the class $C_0^\infty(\mathbb{R}^d)$, for which we are currently able to carry out such an operation by regarding Lf as an element of $\mathscr{D}(\mathbb{R}^d)$. However, in fact the equality

$$\int Lf\psi\, d\gamma = \int fL\psi\, d\gamma$$

is also true for functions $\psi \in C_b^\infty(\mathbb{R}^d)$ such that $L\psi$ and $|\nabla\psi|$ are bounded, and this is the case for our function ψ because of the boundedness of Lg and the equalities

$$LT_s g = T_s Lg, \qquad \nabla T_s g = e^{-s} T_s \nabla g.$$

Indeed, let us take a function $\zeta \in C_0^\infty(\mathbb{R}^d)$ equal to 1 on the unit ball and vanishing outside the doubled ball. Letting

$$\zeta_j(x) = \zeta\biggl(\frac{x}{j}\biggr),$$

we observe that

$$L(\zeta_j\psi) = \zeta_j L\psi + \psi L\zeta_j + 2\langle\nabla\psi, \nabla\zeta_j\rangle.$$

In the identity

$$\int Lf(\zeta_j\psi)\, d\gamma = \int fL(\zeta_j\psi)\, d\gamma$$

we can now pass to the limit as $j \to \infty$. By the Lebesgue dominated convergence theorem the left-hand side tends to the integral of $Lf\psi$ with respect to the measure γ. On the right-hand side, after substituting the indicated equality for $L(\zeta_j\psi)$, we obtain three terms. The term with $\zeta_j L\psi$ tends to the integral of $fL\psi$ with respect to γ. The remaining terms tend to zero as $j \to \infty$ by the Lebesgue theorem, since the functions $L\zeta_j$ and $|\nabla\zeta_j|$ tend pointwise to zero and for them we have the estimates $|\nabla\zeta_j| \leqslant j^{-1}\max_x|\nabla\zeta(x)|$ and

$$|L\zeta_j(x)| \leqslant Cj^{-2} + j^{-1}|x|\,\biggl|\nabla\zeta\biggl(\frac{x}{j}\biggr)\biggr|,$$

where the right-hand side is uniformly bounded in view of the equality

$$\psi\biggl(\frac{x}{j}\biggr) = 0 \quad \text{for } |x| \geqslant 2j.$$

Finally, the formula (3.1) implies that $f \in D_1(L)$, since by this formula and the equality $\lim_{s\to 0}\|T_s Lf - Lf\|_{L^1(\gamma)} = 0$ the functions $t^{-1}(T_t f - f) - Lf$ converge to zero with respect to the norm in $L^1(\gamma)$.

The inclusion $W^{1,2}(\gamma) \subset D_1(L)$ now follows from what we have proved. Indeed, if $f \in W^{1,2}(\gamma)$, then the Sobolev derivatives $\partial_{x_i}\partial_{x_j} f$ belong to $L^1(\gamma)$. It follows that $L^1(\gamma)$ also contains the functions $x_i\partial_{x_i} f$ (see Proposition 2.3). Thus, the function Lf as a distribution is given by an element of $L^1(\gamma)$.

However, the indicated inclusion is strict. This is seen from the following argument. It is known (see [58]) that there exists a Lebesgue integrable function f with bounded support in $\mathbb{R}^d$, $d > 1$, such that the distribution Δf is an ordinary integrable function, but the second partial derivatives $\partial^2_{x_i} f$ in the sense of distributions are not integrable functions (the function f belongs to the first Sobolev class, but not to the second). For the reader's convenience we take from [58] an example of such a function. Let

$$f(x) = \eta(|x|)V(|x|),$$

where $\eta \in C_0^\infty(\mathbb{R})$ has support in $[-1/3, 1/3]$, $\eta = 1$ on $[-1/4, 1/4]$, and

$$V(r) = \int_r^{1/2} \frac{s^{1-d}}{\log s}\, ds, \qquad 0 < r \leqslant 1.$$

For $0 < |x| < 1/4$ we have

$$\nabla f(x) = -\frac{x}{|x|^d \log|x|}\,, \quad \Delta f(x) = \frac{1}{|x|^d (\log|x|)^2}\,,$$

hence Δf (along with $|\nabla f|$) is an integrable function. It is important that the integrable function obtained (defined by the indicated formula away from the origin) coincides with Δf in the sense of distributions, that is, for every function $\varphi \in C_0^\infty$ the integral of $f\Delta\varphi$ coincides with the integral of the function $\varphi\Delta f$. Indeed, it is easy to see that $f \in W^{1,1}(\mathbb{R}^d)$. Therefore, it remains to compare the integrals of $-\langle \nabla f, \nabla\varphi\rangle$ and $\varphi\Delta f$. The simplest method is this: for a function φ vanishing in a neighbourhood of zero we can just integrate by parts. In the general case, in place of φ we can substitute the function $\eta_\varepsilon\varphi$, where η_ε is a smooth function vanishing in the ball of radius $\varepsilon > 0$ centred at zero, equal to 1 outside the doubled ball, and satisfying the estimate $|\nabla\eta_\varepsilon| \leqslant C\varepsilon^{-1}$ with some constant (it is clear that this can be achieved). The integrals of $-\eta_\varepsilon\langle \nabla f, \nabla\varphi\rangle$ and $\eta_\varepsilon\varphi\Delta f$ tend to the integrals we are interested in as $\varepsilon \to 0$, and the integral of $|\nabla\eta_\varepsilon|\,|\nabla f|$ tends to zero, since it is bounded by the integral of $C\varepsilon^{-1}|x|^{1-d}|\log|x|\,|^{-1}$ over the difference of the balls of radii ε and 2ε, which obviously tends to zero. The failure of integrability of the second derivatives in a neighbourhood of zero can also be verified by direct calculations which reduce to a consideration of the function $r^{-d}|\log r|^{-1}$ in a neighbourhood of zero, but it can also be seen from a finer version of the Sobolev embedding theorem for the Lorentz class and an explicit verification that the function $|\nabla f|$ does not belong to this class (see [58]). Note, however, that

$$|\nabla f| \in L^{d/(d-1)}(\mathbb{R}^d), \quad \text{that is,} \quad f \in W^{d/(d-1),1}(\mathbb{R}^d),$$

so that there is no contradiction with the usual embedding theorem.

Then $Lf = \Delta f - \langle f, x\rangle$ in the sense of distributions is also an integrable function with compact support. As shown above, $f \in D_1(L)$, although f does not belong to $W^{1,2}(\gamma)$, since it does not even belong to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$. $\square$

Below we shall need certain Orlicz classes of integrable functions. Let Φ be a strictly increasing convex function on $[0,\infty)$ with $\Phi(0) = 0$ such that Φ' increases to infinity. The Orlicz (or Orlicz–Luxemburg) norm $\|f\|_\Phi$ of a measurable function f on a space Ω with a measure μ is defined by the formula

$$\|f\|_\Phi = \inf\biggl\{\lambda > 0\colon \int_\Omega \Phi\biggl(\frac{|f|}{\lambda}\biggr)\,d\mu \leqslant 1\biggr\}.$$

The Orlicz space $L^\Phi(\mu)$ consists of all functions f with $\|f\|_\Phi < \infty$. For some Orlicz spaces there is a tradition to use a peculiar notation.

For example, the Orlicz space $L\log L$ consists of functions $f \in L^1(\gamma)$ such that

$$|f|\log|f| \in L^1(\gamma),$$

where for $f(x) = 0$ we set $|f(x)|\log|f(x)| := 0$. The Orlicz norm $\|g\|_{L\log L}$ is defined by the formula

$$\|g\|_{L\log L} = \inf\biggl\{\lambda > 0\colon \int_{\mathbb{R}^d}\int_0^{|g(x)|/\lambda} \log(1+t)\,dt\,\gamma(dx) \leqslant 1\biggr\}.$$

The Orlicz space $L\sqrt{\log L}$ consists of functions $f \in L^1(\gamma)$ such that

$$|f|\sqrt{|\log|f||} \in L^1(\gamma),$$

where, as above, we set $|f(x)|\sqrt{|\log|f(x)||} := 0$ if $f(x) = 0$. This space possesses a natural complete norm

$$\|f\|_{L\sqrt{\log L}} = \inf\biggl\{\alpha > 0\colon \int_{\mathbb{R}^d}\int_0^{|f(x)|/\alpha} \sqrt{\log(1+s)}\,ds\,\gamma(dx) < \infty\biggr\}.$$

The more general spaces $L^p\log^s L$ are defined similarly.

The description of the domains of the Ornstein–Uhlenbeck operator involves the remarkable inequalities of Meyer (see [150] and [151]), which establish the equivalence of norms on Gaussian Sobolev classes: the norms defined above by means of integral norms of derivatives are estimated from both sides by the norms generated by powers of the Ornstein–Uhlenbeck operator. The rth-order derivative of a function f will be denoted by $D^r f$. The gradient ∇f will be denoted also by Df for uniformity (however, sometimes one writes $\nabla^r f$ instead of $D^r f$). We recall that the Hilbert–Schmidt norm of the derivative $D^r f(x)$ is defined by

$$\|D^r f(x)\|_{\mathscr{H}_r} := \biggl(\sum_{1\leqslant i_j\leqslant d} |\partial_{x_{i_1}}\cdots\partial_{x_{i_r}} f(x)|^2\biggr)^{1/2}.$$

Theorem 3.3. *If $p \in (1,\infty)$ and $r \in \mathbb{N}$, then the space $W^{p,r}(\gamma)$ coincides with the space $H^{p,r}(\gamma) := (I-L)^{-r/2}(L^p(\gamma))$ and there exist numbers $m_{p,r}$ and $M_{p,r}$ independent of d such that*

$$m_{p,r}\|D^r f\|_{L^p(\gamma,\mathscr{H}_r)} \leqslant \|(I-L)^{r/2} f\|_{L^p(\gamma)} \leqslant M_{p,r}\bigl[\|D^r f\|_{L^p(\gamma,\mathscr{H}_r)} + \|f\|_{L^p(\gamma)}\bigr]. \quad (3.2)$$

The case $p = 1$ differs from $p > 1$. In [186] the following assertion was proved.

Theorem 3.4. *For every $\alpha > 0$ there exists a number $C(\alpha) > 0$ such that*

$$\|(\alpha I - L)^{1/2} f\|_1 \leqslant C(\alpha)(\sqrt{\alpha}\,\|Df\|_1 + \|f\|_{L\log L}),$$
$$\|Df\|_{L^1(\gamma,\mathscr{H}_1)} \leqslant C(\alpha)\|(\alpha I - L)^{1/2} f\|_{L\log L}.$$

Substantial generalizations of this result were obtained in [187] for broad Orlicz classes defined by convex functions Φ (see the definition above).

Theorem 3.5. *Let $\alpha\Phi'(t) \leqslant t\Phi''(t) \leqslant \beta\Phi'(t)$, where $1 < \alpha < \beta$ and the function Φ' is either convex or concave. Then there exist positive numbers C_1 and C_2 such that*

$$C_1(\|Df\|_\Phi + \|f\|_\Phi) \leqslant \|(I - L)f\|_\Phi \leqslant C_2(\|Df\|_\Phi + \|f\|_\Phi)$$

for all functions of class C_b^∞.

Corollary 3.6. *Let $p > 1$ and $\beta \geqslant 0$. Then the operator*

$$(I - L)^{-1}\colon L^p \log^{p\beta} L \to L^p \log^{p(\beta+1/2)} L$$

is continuous.

There are also other generalizations of the Meyer inequalities which are assertions about multipliers in Sobolev spaces. By the use of various tools (a functional calculus, Hilbert transforms, and so on) the boundedness of a number of operators on Sobolev spaces has been established. We mention the following result from [110]: the operator $D(I - L)^{-1/2}$ is bounded from the Orlicz space $L\log^{\alpha+3/2} L(\gamma)$ to the space $L\log^{\alpha} L(\gamma, H)$ for $\alpha \geqslant 0$; however, it remains unclear whether this operator is bounded from $L\log^{\alpha+1} L(\gamma)$ to the space $L\log^{\alpha} L(\gamma, H)$.

The so-called spectral multipliers are operators of the form $\varphi(L)$ with a certain function φ and they are defined on Chebyshev–Hermite polynomials by

$$\varphi(L)H_{k_1,\dots,k_n} = \varphi(k_1 + \cdots + k_n)H_{k_1,\dots,k_n}.$$

The problem is to extend the operator $\varphi(L)$ to some class of functions (like L^p or a Sobolev class) as a bounded operator with values in some class of functions. For example, Stein [197] proved that for the boundedness of such an operator on $L^p(\gamma)$ with $1 < p < \infty$ it suffices that the function φ be of the form

$$\varphi(\lambda) = \lambda \int_0^\infty \Phi(t) e^{-t\lambda}\, dt, \qquad \lambda \geqslant 0,$$

where the function $\Phi\colon (0,\infty) \to \mathbb{C}$ is bounded. It is shown in [54] that it suffices to have the following weaker condition: φ is analytic in a sector with some angle greater than $\pi|1/p - 1/2|$. The case $p = 1$ is considered in [114].

It is proved in [204] that the operator $D^\alpha(-L)^{-|\alpha|/2}$ with a multi-index α is continuous on the space $L^p(\gamma_d)$ for $1 < p < \infty$ (such operators are called Riesz transforms). Unlike Meyer's inequality, the norm of such an operator depends not only on p but also on the dimension d.

A number of authors have investigated the Littlewood–Paley function

$$g_\gamma(f)(x) = \biggl(\int_0^\infty |t\nabla T_t f(x)|^2 \,\frac{dt}{t}\biggr)^{1/2}$$

and its generalizations to higher-order derivatives (see [22] and [130]).

There are many papers devoted to Riesz transforms and Meyer-type inequalities (see [2], [50], [67], [74], [76], [83], [84], [97], [140], [141], [143], [158], [174], [178], [191], [204], [214], where additional references can be found).

Remark 3.7. In this survey we consider the Ornstein–Uhlenbeck semigroup on function spaces connected with the corresponding Gaussian measure, but it should be noted that in the finite-dimensional case it is also defined on the usual spaces $L^p(\mathbb{R}^d)$ with respect to Lebesgue measure. This follows from the fact that the operators T_t take L^∞ to L^∞ (as is true for all measures equivalent to Lebesgue measure), and also take $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, since

$$\int_{\mathbb{R}^d} |T_t f(x)|\,dx \leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(e^{-t}x+\sqrt{1-e^{-2t}}\,y)|\,\gamma(dy)\,dx = e^{dt}\int_{\mathbb{R}^d}|f(u)|\,du.$$

The case $p>1$ is treated similarly, but it also follows from an interpolation theorem. On the spaces $L^p(\mathbb{R}^d)$ with $p<\infty$ the semigroup $\{T_t\}_{t\geqslant 0}$ is continuous, which can be verified as in the case of the Gaussian measure, and therefore its generators have domains $\widetilde{D}_p(L)$ in $L^p(\mathbb{R}^d)$. By the boundedness of the Gaussian density we have the inclusions

$$\widetilde{D}_p(L)\subset D_p(L),$$

and moreover, on $\widetilde{D}_p(L)$ the action of L is given by the previous differential expression in the sense of distributions. In the case $p\in(1,\infty)$ the set $\widetilde{D}_p(L)$ coincides with the class of functions $f\in W^{p,2}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $Lf\in L^p(\mathbb{R}^d)$. The case $p=1$ is analogous to the case considered in Theorem 3.2.

We remark also that an analogous argument shows the boundedness of the operators T_t on weighted classes $L^p(\theta\,dx)$ with a locally integrable weight θ satisfying the condition $\theta(e^{-t}x)\leqslant K_t\theta(x)$ with some constants K_t.

Let us turn to spectra of Ornstein–Uhlenbeck operators acting in the complex spaces $L^p(\gamma)$. The spectrum consists of all complex numbers λ for which the operator $L-\lambda I$ does not have a bounded inverse. For points λ in the complement of the spectrum (called the resolvent set), $L-\lambda I$ maps $D_p(L)$ one-to-one onto $L^p(\gamma)$, and is continuous if $D_p(L)$ is equipped with the graph norm $\|f\|_p+\|Lf\|_p$. Furthermore, the operator $(L-\lambda I)^{-1}$ is continuous from $L^p(\gamma)$ to $L^p(\gamma)$. The difference from invertible bounded operators is that the range of the operator $(L-\lambda I)^{-1}$ is not the whole of the space $L^p(\gamma)$ but $D_p(L)$.

As we have seen, for $p=2$ the operator L has an orthonormal eigenbasis with eigenvalues $0,-1,-2,\dots$, and the dimension d influences only the multiplicities of these eigenvalues (in the infinite-dimensional case the kernel subspaces of non-zero eigenvalues are infinite-dimensional), so that these non-positive integers form the full spectrum. The answer is the same for all $p\in(1,\infty)$, since the eigenfunctions are Chebyshev–Hermite polynomials belonging to all the spaces $L^p(\gamma)$, and no new points of the spectrum appear, as can easily be verified (a more general result is given below in Theorem 3.10).

Theorem 3.8. *For all $p\in(1,\infty)$ the spectrum of the operator (L,D_p) acting in the complex space $L^p(\gamma)$ coincides with the set $\{0,-1,-2,\dots\}$ of non-positive integers.*

However, the case $p=1$ is special.

Theorem 3.9. *The spectrum of the operator* (L, D_1) *acting in the complex space* $L^1(\gamma)$ *coincides with the left half-plane* $\operatorname{Re} z \leqslant 0$, *and moreover, all the numbers with negative real part are eigenvalues.*

Consider a general operator $L_{A,B}$ of the form (0.5) under the additional assumption that the operator

$$Q := 2\int_0^\infty e^{-sB} A e^{-sB^*}\, ds$$

exists (this is equivalent to the integrability of the function $\operatorname{trace}(e^{-sB}Ae^{-sB^*})$ on $[0,\infty)$) and is non-degenerate. Then it is the covariance operator of a non-degenerate symmetric Gaussian measure μ.

The indicated operator $L_{A,B}$ is the generator of the strongly continuous semigroup $\{S_t\}_{t\geqslant 0}$ in each space $L^p(\mu)$ with $p \in [1,\infty)$, on a domain D_p depending on p, and it is given by the indicated differential expression on D_p in the sense of distributions. The measure μ is invariant with respect to the semigroup $\{S_t\}_{t\geqslant 0}$. This can be seen in different ways, for example, it suffices to verify that the integrals with respect to μ of functions of the form $L_{A,B}\exp(il)$ are zero, where l is a linear function (one can deduce from this that the integral of $S_t \exp(il)$ does not depend on t, which then enables us to establish the invariance of μ). Let l also denote the vector defining the functional l in the form

$$l(x) = \langle l, x\rangle.$$

In an orthonormal eigenbasis of the operator A we have

$$L_{A,B}\exp(il) = -\langle Al, l\rangle \exp(il) + i\langle B^*l, x\rangle \exp(il).$$

The integral of $\exp(il)$ with respect to μ equals $\exp(-\langle Ql, l\rangle/2)$, and the integral of $i\langle B^*l, x\rangle \exp(il)$ equals $\langle Ql, B^*l\rangle \exp(-\langle Ql, l\rangle/2)$, as is easily verified by differentiating the integral of $\exp(i\langle l + tB^*l, x\rangle)$ at zero. Thus, the desired identity reduces to a proof of the equality

$$\langle Al, l\rangle = \langle BQl, l\rangle. \tag{3.3}$$

If the operator B is symmetric and commutes with A, then this equality is obtained by the simple proof that the integral of $\exp(-2tB)$ over $[0,\infty)$ is $(2B)^{-1}$. In the general case one can see that the identity

$$Q = 2\int_0^t e^{-sB} A e^{-sB^*}\, ds + e^{-tB} Q e^{-tB^*} \tag{3.4}$$

holds, because

$$2\int_t^\infty e^{-sB} A e^{-sB^*}\, ds = 2\int_0^\infty e^{-(\tau+t)B} A e^{-(\tau+t)B^*}\, d\tau = e^{-tB} Q e^{-tB^*}.$$

Differentiating (3.4) at zero, we obtain the equality

$$A = \frac{1}{2}(BQ + QB^*), \tag{3.5}$$

which implies (3.3).

The existence of the operator Q is necessary and sufficient for the existence of an invariant measure of the semigroup $\{S_t\}_{t\geqslant 0}$ generated by the operator $L_{A,B}$ (see [57], § 6.2). Under our assumption of the non-degeneracy of A this is also equivalent to the property that the spectrum of B lies in the open right half-plane in $\mathbb{C}$. The semigroup itself can be defined without this condition on the space of bounded continuous (or bounded Borel) functions by a Mehler-type formula

$$S_t f(x) = (2\pi)^{-d/2}(\det Q_t)^{-1/2}\int_{\mathbb{R}^d} f(e^{-tB}x - y)\exp\biggl(-\frac{1}{2}\langle Q_t^{-1}y, y\rangle\biggr)\,dy,$$
$$Q_t = 2\int_0^t e^{-sB}Ae^{-sB^*}\,ds.$$

As already mentioned, the Ornstein–Uhlenbeck semigroup is not strongly continuous on these spaces.

The proof of the following general result is given in [148] (instead of invertibility of A, the even broader condition of invertibility of the operators Q_t is used there).

Theorem 3.10. *Under the indicated conditions on A and B, the spectrum of the operator (L, D_p) acting in the complex space $L^p(\mu)$ with $p \in (1,\infty)$ consists of its eigenvalues and equals*

$$\biggl\{-\sum_{j=1}^r k_j z_j \colon k_j \in \mathbb{N}\cup\{0\},\ z_1,\dots,z_r \textit{ are all eigenvalues of } B\biggr\}.$$

In addition, all eigenfunctions are polynomials.

For $A = B = I$ (the identity operator) we obtain the set of all non-positive integers.

It is shown in [148] that for $p = 1$ the spectrum coincides with the left half-plane and all numbers with negative real part are eigenvalues (of course, there are also non-polynomial eigenfunctions), as in Theorem 3.9.

Note an unobvious property of the spectrum in the previous theorem: it does not depend on the matrix A (with the exception of those properties which are needed to ensure the hypotheses of the theorem). This interesting circumstance is already seen in the following assertion (see [148], Lemma 3.3), which underlies the proof of Theorem 3.10.

Proposition 3.11. *Let $p \in (1,\infty)$. Under the conditions of the previous theorem, a number λ belongs to the spectrum of the operator (L, D_p) acting in the complex space $L^p(\mu)$ precisely when there exists a non-zero homogeneous polynomial ψ such that*

$$\langle Bx, D\psi(x)\rangle = \lambda\psi(x).$$

In [148] the multiplicities and indices of eigenvalues of the operator (L, D_p) are studied. In particular, the algebraic multiplicity of the eigenvalue λ equals

$$\sum_{n_1\lambda_1+\dots+n_r\lambda_r=\lambda}\ \prod_{j=1}^r \frac{(k_j+n_j-1)!}{n_j!\,(k_j-1)!}\,.$$

It should be emphasized that the results presented concern the spectrum of the Ornstein–Uhlenbeck operator acting in the spaces L^p with respect to the invariant Gaussian measure. The situation changes radically if we consider the same operator acting in L^p with respect to Lebesgue measure (see [147]).

The domains of the generator of the semigroup generated by $L_{A,B}$ acting in the spaces $L^p(\mu)$ are investigated in [149].

Theorem 3.12. *Under the conditions of Theorem 3.10 the domain of the generator of the Ornstein–Uhlenbeck semigroup $\{S_t\}_{t\geqslant 0}$ generated by $L_{A,B}$ acting in $L^p(\mu)$ coincides with the Sobolev class $W^{p,2}(\mu)$.*

In [149] the domain of the generator acting in $L^p(\mathbb{R}^d)$ is also described.

4. The infinite-dimensional case

Here we discuss infinite-dimensional analogues of some of the objects introduced above. In subsequent sections we also mention versions of presented results for the case of an infinite-dimensional space, and with rare exceptions the formulations do not change, as will be noted. Only at one place, where the infinite-dimensional analogue is so far an open question, will there be special mention of this.

Let X be a real Hausdorff locally convex space with topological dual space X^*. A non-negative measure μ on the Borel σ-algebra $\mathscr{B}(X)$ of the space X is called a Radon measure if, for every Borel set $B\subset X$, the value $\mu(B)$ equals the supremum of the values $\mu(K)$ over compact subsets $K\subset B$. A probability Radon measure γ on X is called a centred Gaussian measure if every continuous linear functional l on X is a centred Gaussian random variable on (X,γ). The latter means that the induced measure $\gamma\circ l^{-1}$ is either the Dirac measure at zero or has a density of the form $(2\pi\sigma)^{-1/2}\exp\bigl(-t^2/(2\sigma)\bigr)$. On Gaussian measures, see [25] and [26]. The norm on $L^p(\gamma)$ is denoted by $\|f\|_p$, as above in the finite-dimensional case. It is known that for a Radon Gaussian measure the spaces $L^p(\gamma)$ with $p<\infty$ are separable (for arbitrary Radon measures this is false).

In the case of an infinite-dimensional space an important role is played by the so-called Cameron–Martin space of the measure γ, defined as the set $H=H(\gamma)$ of all vectors $h\in X$ such that $\gamma_h\sim\gamma$, where $\gamma_h(B)=\gamma(B-h)$. If γ is the countable power of the standard Gaussian measure on the real line and is considered on the space $\mathbb{R}^\infty$ of all real sequences, then H is the standard Hilbert space l^2 (for the standard Gaussian measure on $\mathbb{R}^d$ the Cameron–Martin space is $\mathbb{R}^d$ itself). For a general Radon centred Gaussian measure, the Cameron–Martin space is also a separable Hilbert space (see [25], Theorem 3.2.7 and Proposition 2.4.6) with the inner product $\langle\,\cdot\,,\cdot\,\rangle_H$ and the norm $|\cdot|_H$ defined by

$$|h|_H=\sup\biggl\{l(h)\colon \int_X l^2\,d\gamma\leqslant 1,\ l\in X^*\biggr\}.$$

Let $\{l_i\}_{i=1}^\infty\subset X^*$ be an orthonormal basis in the closure X^*_γ of the set X^* in $L^2(\gamma)$. There is an orthonormal basis $\{e_i\}_{i=1}^\infty$ in H such that $l_i(e_j)=\delta_{i,j}$.

For every vector $h\in H$, there is a unique element $\widehat{h}\in X^*_\gamma$ such that

$$l(h)=\int_X l(x)\widehat{h}(x)\,\gamma(dx)\quad \forall\, l\in X^*.$$

According to the Cameron–Martin formula, for every $h \in H$ the shifted measure $\gamma(\cdot - h)$ has the density $\exp(\widehat{h} - |h|_H^2/2)$ with respect to the measure γ.

Let $\mathscr{F}\mathscr{C}^\infty(X)$ be the set of all functions φ on X of the form

$$\varphi(x) = \psi\bigl(l_1(x), \dots, l_n(x)\bigr), \quad \text{where} \quad \psi \in C_b^\infty(\mathbb{R}^n), \quad l_i \in X^*,$$

and let $\mathscr{F}\mathscr{C}_0^\infty(X)$ be its subclass of functions for which ψ can be chosen in $C_0^\infty(\mathbb{R}^n)$. We observe that $\mathscr{F}\mathscr{C}_0^\infty(X)$ is not a linear space (unlike $\mathscr{F}\mathscr{C}^\infty(X)$), since a non-zero function in $C_0^\infty(\mathbb{R}^n)$ does not have compact support as a function on $\mathbb{R}^{n+1}$. The class $\mathscr{F}\mathscr{C}^\infty(X)$ is dense in L^p for every Radon measure (if $p < \infty$).

Analogues of Chebyshev–Hermite polynomials of degree k on X are obtained by a simple substitution of elements of X_γ^* into finite-dimensional polynomials of degree k, that is, these are functions of the form

$$H_{k_1,\dots,k_n}(l_1, \dots, l_n), \qquad l_i \in X_\gamma^*.$$

If we take an orthonormal basis in X_γ^* as $\{l_j\}$, then the system of functions obtained (for all $k_j \geqslant 0$ and $n = 0, 1, 2, \dots$) will be an orthonormal basis in $L^2(\gamma)$. In the case of infinite-dimensional X_γ^* the space $\mathscr{X}_k$ equal to the closure of the linear span of the polynomials $H_{k_1,\dots,k_n}(l_1, \dots, l_n)$ with $k_1 + \dots + k_n = k$ is also infinite-dimensional for $k > 0$ ($\mathscr{X}_0$ consists of the constants). As in the finite-dimensional case, the spaces $\mathscr{X}_k$ are pairwise orthogonal and give an orthogonal decomposition

$$L^2(\gamma) = \bigoplus_{k=0}^{\infty} \mathscr{X}_k.$$

The operator of projection of $L^2(\gamma)$ onto $\mathscr{X}_k$ is denoted by I_k.

One similarly introduces the space $\mathscr{X}_k(E)$ of polynomial maps of degree k with values in a separable Hilbert space E: in the space $L^2(\gamma, E)$ of square-integrable E-valued maps we take the closure of the linear span of maps of the form $f_1 v_1 + \dots + f_n v_n$, where $f_i \in \mathscr{X}_k$ and $v_i \in E$.

It turns out that most of the results discussed can be carried over literally to the infinite-dimensional case thanks to a remarkable result of Tsirelson, who proved that every Radon centred Gaussian measure γ with an infinite-dimensional Cameron–Martin space is linearly isomorphic to the standard Gaussian measure γ_∞ on $\mathbb{R}^\infty$ (the countable power of the standard Gaussian measure on the real line). Thus, all infinite-dimensional Radon centred Gaussian measures are linearly isomorphic. A precise formulation is this: in the space X on which the measure γ is defined there exists a Borel linear subspace of measure 1 which can be mapped one-to-one by a Borel linear map T with a Borel inverse onto a Borel linear subspace $E \subset \mathbb{R}^\infty$ of measure 1 with respect to γ_∞ such that $\gamma \circ T^{-1} = \gamma_\infty$. Moreover, the operator T is an isometry of the Cameron–Martin spaces of these measures. A very important feature of Tsirelson's result is that the isomorphism T is defined constructively: if $\{l_n\} \subset X^*$ is an orthogonal basis in X_γ^*, then we can take $Tx = \bigl(l_n(x)\bigr)_{n=1}^\infty$ (this map is even continuous). The only object that is not given constructively is a subspace X_0 on which T is injective. Of course, such isomorphisms are seldom continuous in both directions, and for isomorphisms of general spaces one cannot always obtain continuity even in one direction. Also, it usually cannot be made one-to-one or Borel measurable on the whole space. For example, l^2 with a Gaussian measure cannot be transformed into the Wiener measure on $C[0,1]$ by a Borel linear map of the whole of l^2, since such a map must be continuous, but the Wiener measure on $C[0,1]$ vanishes on all continuously embedded Hilbert spaces. Further, one cannot embed a space of a larger algebraic dimension injectively into $\mathbb{R}^\infty$. A separable Banach space with a Gaussian measure having a dense Cameron–Martin subspace can be transformed by an injective continuous linear operator into the measure γ_∞, but one cannot obtain continuity of the inverse operator, since neighbourhoods of zero in $\mathbb{R}^\infty$ contain cylinders with finite-dimensional bases. Nevertheless, there are enough such isomorphisms for most of the results discussed here to be insensitive to our choice of the space and to reduce to the case of the measure γ_∞ on $\mathbb{R}^\infty$.

The Ornstein–Uhlenbeck semigroup is defined by the same formula (0.2) with an obvious change of the domain of integration:

$$T_t f(x) = \int_X f\bigl(e^{-t}x - \sqrt{1-e^{-2t}}\,y\bigr)\,\gamma(dy). \tag{4.1}$$

It can be verified similarly that this semigroup is strongly continuous and contracting on the whole of $L^p(\gamma)$ with $p<\infty$ and that all the other assertions of Theorem 1.1 are also true. However, this does not require a new proof, but can be obtained from the finite-dimensional case by using the fact that the set of cylindrical functions is dense.

On the space $L^2(\gamma)$ the operators T_t are symmetric, so the generator L is self-adjoint. In addition,

$$T_t\big|_{\mathscr{X}_k} = e^{-tk} I,$$

so for every choice of an orthonormal basis in each subspace $\mathscr{X}_k$ the operator T_k on $L^2(\gamma)$ becomes diagonal with the same eigenvalues e^{-tk} as in the finite-dimensional case (but for $k>0$ these eigenvalues have infinite multiplicity).

As above, the semigroup $\{T_t\}_{t\geqslant 0}$ has generators (L, D_p) acting in $L^p(\gamma)$, the operator L is called the Ornstein–Uhlenbeck operator, and on smooth cylindrical functions it is defined by essentially the same expression as on $\mathbb{R}^d$. For the measure γ_∞ this is literally the same expression, and in the general case the formula is this: if $\{e_n\}$ is an orthonormal basis in the Cameron–Martin space H and $\{\widehat{e}_n\}$ is the corresponding basis in X^*_γ, then for functions of the form $f=\psi(\widehat{e}_1,\dots,\widehat{e}_n)$ we have

$$Lf = \sum_{i=1}^{n} (\partial_{e_i}^2 f - \widehat{e}_i\,\partial_{e_i} f),$$

where ∂_h is the derivative along the vector h, that is,

$$\partial_h f(x) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}\,.$$

For a general cylindrical function of the form $f=\psi(l_1,\dots,l_n)$, where the functionals $l_i\in X^*$ are not linear combinations of the $\widehat{e}_i$, we have the series

$$Lf = \sum_{i=1}^{\infty} (\partial_{e_i}^2 f - \widehat{e}_i\,\partial_{e_i} f).$$

Convergence of this series requires some justification. Below we return to this question for more general functions in the domain of the generator (here a very typical peculiarity of the infinite-dimensional case arises), but for cylindrical functions the justification is not difficult. Indeed, we observe that

$$\partial_{e_i} f(x) = \sum_{j=1}^{n} \partial_{x_j}\psi(l_1,\ldots,l_n) l_j(e_i),$$
$$\partial_{e_i}^2 f(x) = \sum_{j,k=1}^{n} \partial_{x_k}\partial_{x_j}\psi(l_1,\ldots,l_n) l_j(e_i) l_k(e_i).$$

Thus, for fixed j and k we obtain two series with $l_j(e_i)l_k(e_i)$ and $\widehat{e}_i(x)l_j(e_i)$. The first series is numerical and converges to $(l_j, l_k)_{L^2(\gamma)}$, since

$$l_j(e_i) = (l_j, \widehat{e}_i)_{L^2(\gamma)},$$

where $\{\widehat{e}_i\}$ is a basis in X_γ^*. The second series for the same reason converges in $L^2(\gamma)$ to l_j, but the convergence actually holds in all of $L^p(\gamma)$, as is seen from the facts below on convergence of polynomials of fixed degree (in this case, of degree one). We draw attention to the fact that even in the simplest situation considered we are dealing with convergence of series in L^2 and not pointwise convergence. Of course, so far we checked only convergence of the series but not the equality of its sum to Lf. The latter can be verified as follows. First, whatever the concrete form of Lf, we observe that f belongs to the domain of L in all the spaces L^p by the finite-dimensional case. Again, by virtue of known facts for $\mathbb{R}^d$ the expression for Lf has the indicated form for functions of finitely many $\widehat{e}_i$. The established convergence of the series in all the spaces L^p yields the desired equality.

The Sobolev classes $W^{p,r}(\gamma)$ are now introduced as the completions of $\mathscr{FC}^\infty(X)$ with respect to the Sobolev norms

$$\|f\|_{p,r} = \|f\|_{W^{p,r}(\gamma)} = \|f\|_{L^p(\gamma)} + \sum_{k\leqslant r}\biggl(\int_X \biggl(\sum_{i_j\geqslant 1} |\partial_{e_{i_1}}\cdots\partial_{e_{i_k}} f|^2\biggr)^{p/2} d\gamma\biggr)^{1/p},$$

where $\{e_i\}$ is an orthonormal basis in H. This expression can be rewritten in a more invariant form without using bases. To do this we recall that the Hilbert–Schmidt norm of a bounded linear operator A on H is defined by

$$\|A\|_{\mathscr{H}} = \biggl(\sum_{n=1}^{\infty}\langle Ae_n, Ae_n\rangle_H\biggr)^{1/2}.$$

If $\|A\|_{\mathscr{H}} < \infty$, then A is called a Hilbert–Schmidt operator. It is readily verified that the quantity $\|A\|_{\mathscr{H}}$ is actually independent of our choice of a basis. The space $\mathscr{H}$ of Hilbert–Schmidt operators itself becomes a separable Hilbert space with the inner product

$$(A,B)_{\mathscr{H}} = \sum_{n=1}^{\infty}\langle Ae_n, Be_n\rangle_H,$$

which also does not depend on the basis. It is clear that

$$2(A,B)_{\mathscr{H}} = \|A+B\|_{\mathscr{H}}^2 - \|A\|_{\mathscr{H}}^2 - \|B\|_{\mathscr{H}}^2.$$

The space of Hilbert–Schmidt operators from H to a separable Hilbert space E is defined similarly. It is equipped with the norm

$$\|A\|_{\mathscr{H}} = \biggl(\sum_{n=1}^{\infty}(Ae_n, Ae_n)_E\biggr)^{1/2}$$

and with the corresponding inner product, and is denoted by the symbol $\mathscr{H}(H,E)$. Let

$$\mathscr{H}_0 = \mathbb{R}, \quad \mathscr{H}_1 = H, \quad \mathscr{H}_{n+1} = \mathscr{H}(H, \mathscr{H}_n).$$

The spaces $\mathscr{X}_k$ belong to all the classes $W^{p,r}(\gamma)$ (this follows from the results below, and for $p=2$ is easily verified directly by means of the equality $T_t I_k = e^{-tk} I_k$).

In addition to partial derivatives along vectors in H, one can introduce the gradient along H and higher-order derivatives along H. The gradient D_H along H is defined by

$$\langle D_H f(x), h\rangle_H = \partial_h f(x).$$

Thus, this is just the gradient at zero of the function

$$h \mapsto f(x+h)$$

on the Hilbert space H for a fixed element x. Hence, with x we associate the vector $D_H f(x) \in H$, which yields a map from X to H. We can also differentiate this map along H, that is, we can again consider the map

$$h \mapsto D_H f(x+h)$$

for fixed x. Its derivative is an operator $D_H^2 f(x)$ on H. For a smooth cylindrical function f this is a Hilbert–Schmidt operator. Therefore, $D_H^2 f$ takes values in the Hilbert space $\mathscr{H}_2$ of Hilbert–Schmidt operators. This enables us to continue taking derivatives along H inductively, remaining all the time in the framework of Hilbert space-valued maps with values in the spaces $\mathscr{H}_n$. Consequently, the original function takes values in $\mathbb{R}$, its first derivative along H takes values in H, the second derivative takes values in $\mathscr{H}_2$, and so on.

Hence for $r=2$

$$\|f\|_{p,2} = \|f\|_{W^{p,2}(\gamma)} = \|f\|_{L^p(\gamma)} + \biggl(\int_X |D_H f|_H^p \, d\gamma\biggr)^{1/p} + \biggl(\int_X \|D_H^2 f\|_{\mathscr{H}}^p \, d\gamma\biggr)^{1/p}.$$

As a result of completion, elements $f \in W^{p,r}(\gamma)$ have Sobolev derivatives $D_H^k f \in L^p(\gamma, \mathscr{H}_k)$, obtained as the limits of the usual derivatives of smooth cylindrical functions. These Sobolev derivatives are characterized by integration by parts formulae. For example, if $r=1$, then for all $\varphi \in \mathscr{FC}^\infty(X)$ and $h \in H$ we have the equality

$$\int_X \partial_h \varphi(x) f(x)\,\gamma(dx) = -\int_X \varphi(x)\langle D_H f(x), h\rangle_H\,\gamma(dx) + \int_X \varphi(x) f(x)\widehat{h}(x)\,\gamma(dx).$$

This identity can be taken as the definition of $D_H f$: for $f \in L^p(\gamma)$ the existence of a map $D_H f \in L^p(\gamma, H)$ satisfying this identity means that $f \in W^{p,1}(\gamma)$ and the indicated map is the Sobolev derivative of f along H. For $r > 1$ the situation is analogous. In the case $p = 1$ there is a minor subtlety: it is necessary to assume in addition the integrability of the product $f\widehat{h}$, which follows from the relation $f \in W^{1,1}(\gamma)$, but not from the relation $f \in L^1(\gamma)$.

The definition of the Sobolev classes $W^{p,1}(\gamma, E)$ of maps with values in a separable Hilbert space E is completely analogous, and then the classes $W^{p,r}(\gamma)$ can be defined inductively:

$$f \in W^{p,r+1}(\gamma) \quad \text{if } f \in W^{p,r}(\gamma) \text{ and } D_H^r f \in W^{p,1}(\gamma, \mathscr{H}_r).$$

Unlike in the finite-dimensional case, no kind of Sobolev differentiability implies continuity (except for cylindrical functions obeying the usual embedding theorems). For example, elements $f \in X_\gamma^*$ can fail to have continuous modifications (say, on $\mathbb{R}^\infty$ only functionals of finitely many variables are continuous), even if they belong to all Sobolev classes. Typical examples of functionals in X_γ^* without continuous modifications are the function $\sum_{n=1}^\infty n^{-1}x_n$ on $\mathbb{R}^\infty$ and the stochastic integral

$$\int_0^1 h(t)\, dw_t$$

with respect to a Wiener trajectory on $C[0,1]$ with the Wiener measure, where for h we take a continuous function with unbounded variation.

Sobolev classes of Hilbert space-valued maps are useful, in particular, in that they give natural ranges of values for derivatives. For example, the operator

$$D_H \colon W^{p,r}(\gamma) \to W^{p,r-1}(\gamma, H)$$

is continuous. Moreover, the very useful divergence operator

$$\delta = \operatorname{div}_\gamma \colon W^{p,1}(\gamma, H) \to L^p(\gamma)$$

defined by the formula

$$\int_X f\delta v\, d\gamma = -\int_X \langle v, D_H f\rangle_H\, d\gamma, \qquad f \in \mathscr{F}\mathscr{C}^\infty(X), \tag{4.2}$$

is also continuous. In fact, the divergence operator δ is adjoint to the operator $-D_H$ from $W^{p',2}(\gamma)$ to $W^{p',1}(\gamma, H)$, but of course it is necessary to show that it takes values in $L^p(\gamma)$, and not just in the dual of $W^{p',2}(\gamma)$, which is $W^{p,-2}(\gamma)$ (see [25], [26], [188]). We introduce the continuous operator

$$\delta \colon W^{p,r}(\gamma, H) \to W^{p,r-1}(\gamma)$$

similarly. If the vector field $v \in W^{p,1}(\gamma, H)$ is written by means of an orthonormal basis $\{e_n\}$ in H in the form

$$v = \sum_{n=1}^\infty v_n e_n,$$

then $v_n \in W^{p,1}(\gamma)$ and the partial sums $\sum_{n=1}^{N} v_n e_n$ converge to v in $W^{p,1}(\gamma, H)$. Consequently,

$$\delta v = \sum_{n=1}^{\infty} [\partial_{e_n} v_n - \widehat{e}_n v_n] \tag{4.3}$$

in the sense of convergence in $L^p(\gamma)$. As in the finite-dimensional case (see (1.7)), we have the equality

$$Lf = \delta D_H f, \qquad f \in W^{p,2}(\gamma),$$

following from the fact that after multiplication by a smooth cylindrical function φ and integration with respect to the measure γ, we obtain on both sides the integral of $-\langle D_H\varphi, D_H f\rangle_H$, because the integral of φLF equals the integral of $fL\varphi$. Since $D_H f \in W^{p,1}(\gamma, H)$, this equality implies the equality

$$Lf = \sum_{n=1}^{\infty} [\partial_{e_n}^2 f - \widehat{e}_n\, \partial_{e_n} f]$$

where the series converges in $L^p(\gamma)$. In particular, for the standard Gaussian measure on $\mathbb{R}^\infty$ we obtain the same formula as in the finite-dimensional case:

$$Lf = \sum_{n=1}^{\infty} [\partial_{x_n}^2 f - x_n\, \partial_{x_n} f].$$

If $f \in W^{p,r}(\gamma)$ for some $r > 2$, then this series converges in $W^{p,r-2}(\gamma)$, since $D_H f \in W^{p,r-1}(\gamma, H)$. However, there is a substantial difference from the finite-dimensional case: either or both of the two series with terms $\partial_{x_n}^2 f$ and $x_n\, \partial_{x_n} f$ can fail to converge. For example, for the function

$$f(x) = \sum_{n=1}^{\infty} n^{-1}(x_n^2 - 1),$$

which belongs to all the classes $W^{p,r}(\gamma)$, we have

$$\partial_{x_n}^2 f = 2n^{-1}.$$

The class $BV(\gamma)$ is introduced in a special way. There are several equivalent definitions. For example, as on $\mathbb{R}^d$, one can define it as the subset of $L^1(\gamma)$ consisting of the functions f for which there exists a sequence of functions in $W^{1,1}(\gamma)$ that is bounded in the norm of $W^{1,1}(\gamma)$ and converges to f almost everywhere. The norm on $BV(\gamma)$ is defined by (2.3). This is equivalent to the existence of an H-valued vector measure Λ of bounded variation such that

$$\int_X \partial_h \varphi f \, d\gamma = \int_X \varphi f \widehat{h} \, d\gamma - \int_X \varphi \, d\langle h, \Lambda\rangle_H \quad \forall\, \varphi \in \mathscr{F}\mathscr{C}^\infty(X), \quad h \in H.$$

Most of the equalities and inequalities for Ornstein–Uhlenbeck operators and semigroups and gradients known in the finite-dimensional case for smooth functions and independent of dimension extend to the infinite-dimensional case by relatively simple passage to limits. For example, (2.5) takes the form

$$\int_X f Lg \, d\gamma = -\int_X \langle Df, Dg \rangle_H \, d\gamma, \tag{4.4}$$
$$f \in W^{p,1}(\gamma), \quad g \in W^{q,2}(\gamma), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

which will not differ at all from the finite-dimensional formula if we omit the indication of the domain of integration and the index H in the notation for the inner product in the Cameron–Martin space and write D instead of ∇ in the case of $\mathbb{R}^d$. The formula (4.4) is one of the most important infinite-dimensional formulae for integration by parts.

Theorem 4.1. *In the infinite-dimensional case*, *Proposition* 2.2 *and Theorems* 3.3–3.5 *remain valid with the same constants.*

As in the finite-dimensional case we have the equality

$$D_H T_t f = e^{-t} T_t D_H f.$$

Note that in the infinite-dimensional case one can also define the action of T_t^* on bounded Borel measures by the formula

$$T_t^* \nu(B) = \int_X T_t I_B(x) \, \nu(dx),$$

but if the measure ν is not absolutely continuous with respect to γ, then the measure $T_t^*\nu$ can be singular with respect to γ for all t. For example, if ν is the Dirac measure at zero, then $T_t^*\nu$ is the image of γ under the homothety with coefficient $(1-e^{-2t})^{1/2}$, but all such measures are mutually singular with γ in the infinite-dimensional case. And if ν is given by a density f with respect to γ, then the measure $T_t^*\nu$ is given by the density $T_t f$, and hence the measures $T_t^*\nu$ converge in variation to $\nu(X)\gamma$ as $t \to +\infty$ by the infinite-dimensional analogue of (1.3).

Theorem 2.5 is also valid in the infinite-dimensional case if we write $|x-y|_H$ instead of $|x-y|$ in the definition of the Kantorovich metric W_2.

The estimate (1.10) for Lipschitz functions f is replaced by the estimate

$$\operatorname{cov}_\gamma(f,g) \leqslant C(f) \int_X |D_H g|_H \, d\gamma, \qquad f \in W^{1,1}(\gamma), \tag{4.5}$$

for Borel functions f on X that are Lipschitz along the Cameron–Martin space H, that is, that satisfy the condition

$$|f(x+h) - f(x)| \leqslant C(f)|h|_H, \qquad x \in X, \quad h \in H.$$

For such a function f the function fg is automatically integrable with respect to the measure γ for all $g \in W^{1,1}(\gamma)$. This can be shown in various ways. For example, one can show that in the case $C(f) = 1$ the function $\exp(cf^2)$ is integrable if $c < 1/2$ and the function g belongs to the Orlicz class $L\sqrt{\log L}$. However, one can derive this directly from the finite-dimensional case by passing to the limit and using

Fatou's theorem. To do this it suffices to observe that $|f|$ and $|g|$ satisfy the same conditions as f and g, so it suffices to prove the uniform boundedness of the integrals of $T_t f T_t g$ for non-negative functions, which easily reduces to finite-dimensional approximations.

The classes BV on infinite-dimensional spaces have been actively investigated in recent years (see [5], [80], [27], and references therein).

The spectrum of the operator L acting in $L^p(\gamma)$ for $p \in (1, \infty)$ consists as above of all non-positive integer eigenvalues (for non-zero numbers the kernel subspaces are now infinite-dimensional), but the more general case of infinite-dimensional analogues of the operators $L_{A,B}$ in (0.5) involves some peculiarities (see [161]).

5. Functional inequalities

A direction that has been intensively developing in the last two decades can be briefly described as the investigation of functional inequalities for operator semigroups and their generators. Model examples of generators for which many inequalities of this sort have been obtained are the Laplacian and the Ornstein–Uhlenbeck operator. In this section we briefly discuss two very important properties of the Ornstein–Uhlenbeck semigroup and its generator, expressed by inequalities (the so-called hypercontractivity inequality and the logarithmic Sobolev inequality); moreover, these two properties turn out to be equivalent. In passing we consider some useful weaker inequalities, including the Poincaré inequality, and also some other inequalities connected with embedding theorems and estimates for distribution functions. A number of authors contributed to the discovery of these properties. Nash's paper [157] from 1958 is the earliest paper known to the author where the Poincaré inequality for Gaussian measures was given explicitly with gradients (of course, being written in terms of Hermite expansions, it becomes trivial). The beginning of the modern intensive investigations of the logarithmic Sobolev inequality was laid by Gross's paper [96] (where the inequality was explicitly proved with gradients by two methods), which became very popular. Later a derivation of it from the hypercontractivity inequality for the semigroup was found. In turn, the property of hypercontractivity of the semigroup, that is, a certain increasing of the order of integrability of a function, has been studied since the beginning of the 1960s by Nelson (see [164]–[168]) and later by many other researchers (see [91], [70], [189], and also references in [7] and [12]). In Stam's paper [196] from 1959 an inequality was proved in the one-dimensional case that is equivalent to the logarithmic Sobolev inequality and has the form

$$\frac{1}{2\pi e} \int \frac{|\varrho'(x)|^2}{\varrho(x)}\, dx \, \exp\biggl(-2 \int \varrho(x) \log \varrho(x)\, dx\biggr) \geqslant 1$$

for probability densities ϱ with respect to Lebesgue measure, that is, is expressed in terms of Fisher information and Shannon entropy. It was noted in [196] that this inequality had been communicated by de Bruijn. However, passage to the logarithmic Sobolev inequality with the Gaussian measure is not achieved by a simple change of the function, but requires additional arguments (indicated much later in [19]; see also [7], Chap. 10), so the equivalence of the Stam estimate and the logarithmic Sobolev inequality for the Gaussian measure was realized much later than the appearance of the main works on hypercontractivity and Gross's paper. A multidimensional analogue of Stam's inequality, unlike the logarithmic Sobolev inequality for the Gaussian measure, contains the dimension, so it has no direct infinite-dimensional version. The cited paper of Stam became well-known, and at present there are about 150 citations of it in the MathSciNet database, where only publications over the last 20 years are taken into account. Among relatively recent papers in this area we mention [52], [79], and [90]. In the recent monograph [15] by Bakry, Gentil, and Ledoux these problems are discussed from a general point of view and an extensive bibliography is given.

Let γ be a Radon centred Gaussian measure on a locally convex space X. For example, one can assume that this is the countable power of the standard Gaussian measure on the real line (or even simply the standard Gaussian measure on $\mathbb{R}^d$).

Theorem 5.1. *The following logarithmic Sobolev inequality holds for every function $f \in W^{2,1}(\gamma)$:*

$$\int_X f^2 \log|f|\,d\gamma \leqslant \int_X |D_H f|_H^2\,d\gamma + \frac{1}{2}\biggl(\int_X f^2\,d\gamma\biggr)\log\biggl(\int_X f^2\,d\gamma\biggr). \tag{5.1}$$

One of the many known ways of proving the logarithmic Sobolev inequality is based on the representation (1.9). It clearly suffices to prove the inequality for smooth functions f on $\mathbb{R}^d$ with the standard Gaussian measure such that $0 < c_1 \leqslant f \leqslant c_2$ with some constants. Using (1.5) and the estimate

$$|T_t\nabla f|^2 \leqslant T_t f T_t\biggl(\frac{|\nabla f|^2}{f}\biggr),$$

which follows from the Cauchy–Bunyakovskii inequality in the integral representation for T_t, we arrive at the inequality

$$\mathrm{Ent}_\gamma(f) \leqslant \int_0^\infty e^{-2t}\biggl(\int_{\mathbb{R}^d} T_t\biggl(\frac{|\nabla f|^2}{f}\biggr)\,d\gamma\biggr)\,dt = \frac{1}{2}\int_{\mathbb{R}^d}\frac{|\nabla f|^2}{f}\,d\gamma.$$

It remains to take f^2 instead of f.

For $p \geqslant 2$, we obtain from (5.1) the inequality

$$\int_X |f|^p \log\biggl(\frac{|f|}{\|f\|_p}\biggr)\,d\gamma \leqslant \frac{p}{2}\int_X |f|^{p-2}|D_H f|_H^2\,d\gamma, \tag{5.2}$$

where for $f \in W^{p,2}(\gamma)$ such that $f \geqslant 0$ the right-hand side equals the integral of the function $\dfrac{p}{2(p-1)}f^{p-1}Lf$.

The logarithmic Sobolev inequality is equivalent to the hypercontractivity property.

Theorem 5.2. *The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geqslant 0}$ is hypercontractive, that is, for all $p > 1$ and $q > 1$*

$$\|T_t f\|_q \leqslant \|f\|_p$$

for all $t > 0$ such that $e^{2t} \geqslant (q-1)/(p-1)$.

Corollary 5.3. *Let $p \geqslant 2$. Then the operator $I_k\colon f \mapsto I_k(f)$ from $L^2(\gamma)$ to $L^p(\gamma)$ is bounded and*

$$\|I_k(f)\|_p \leqslant (p-1)^{k/2}\|f\|_2. \tag{5.3}$$

Furthermore, for $p \in (1,\infty)$ the operators I_k are bounded on $L^p(\gamma)$ and

$$\|I_k\|_{\mathscr{L}(L^p(\gamma))} \leqslant (M-1)^{k/2}, \tag{5.4}$$

where $M = \max\{p, p/(p-1)\}$.

Corollary 5.4. *Let $f \in \mathscr{X}_k$. For every $\alpha \in (0, k/(2e))$ the inequality*

$$\gamma\bigl(x\colon |f(x)| \geqslant t\|f\|_2\bigr) \leqslant c(\alpha,k)\exp(-\alpha t^{2/k})$$

holds, where $c(\alpha,k) = \exp(\alpha) + k/(k-2e\alpha)$.

Corollary 5.5. *The spaces $\mathscr{X}_k$ are closed with respect to convergence in measure. Moreover, every sequence in $\bigoplus_{k=0}^m \mathscr{X}_k$ that is convergent in measure converges in $L^p(\gamma)$ for every $p \in [1,\infty)$. The norms in $L^p(\gamma)$ with $p \in [1,\infty)$ are equivalent on each space $\bigoplus_{k=0}^m \mathscr{X}_k$. In addition, for each $p > 0$ the topology on $\bigoplus_{k=0}^m \mathscr{X}_k$ induced by the metric in $L^p(\gamma)$ coincides with the topology of convergence in measure. Finally, for $q > p > 1$,*

$$\|f\|_p \leqslant \|f\|_q \leqslant \left(\frac{q-1}{p-1}\right)^{k/2}\|f\|_p \quad \forall\, f \in \mathscr{X}_k. \tag{5.5}$$

The same is true for the spaces $\mathscr{X}_k(E)$ of maps with values in a separable Hilbert space E.

Actually, for convergence in $L^p(\gamma)$ of a sequence in $\bigoplus_{k=0}^m \mathscr{X}_k$ it suffices that it converge in measure on some set of positive measure (see [28], Theorem 2.2).

From Corollaries 5.3 and 5.5 it follows that the operator I_k extends to a bounded operator $I_k\colon L^r(\gamma) \to \mathscr{X}_k$ even for $r \in (1,2)$, and moreover,

$$\|I_k f\|_p \leqslant (p-1)^{k/2}(r-1)^{-k}\|f\|_r.$$

The Poincaré inequality for Gaussian measures asserts the following.

Theorem 5.6. *The Poincaré inequality*

$$\int_X \left(f - \int_X f\,d\gamma\right)^2 d\gamma \leqslant \int_X |D_H f|_H^2\,d\gamma \tag{5.6}$$

holds for all $f \in W^{2,1}(\gamma)$. In addition, if $p \geqslant 1$, then for all $f \in W^{p,1}(\gamma)$

$$\int_X \left|f - \int_X f\,d\gamma\right|^p d\gamma \leqslant \left(\frac{\pi}{2}\right)^p M_p \int_X |D_H f|_H^p\,d\gamma, \tag{5.7}$$

where M_p is the moment of order p of the standard Gaussian measure on the real line.

It was shown in [102] that the hypercontractivity of the Ornstein–Uhlenbeck semigroup is equivalent to the inequality

$$\biggl(\int_{\mathbb{R}^d} \exp(e^{2t}T_t f)\,d\gamma_d\biggr)^{e^{-2t}} \leqslant \int_{\mathbb{R}^d} e^f\,d\gamma_d$$

for all $t>0$ and all functions $f\in L^1(\gamma_d)$ such that $e^f\in L^1(\gamma_d)$.

At present various generalizations and modifications of the logarithmic Sobolev inequality and the Poincaré inequality have been obtained for generators of semigroups and measures in broad classes, of which a very special case is that of the Ornstein–Uhlenbeck operator and the Gaussian measure. This direction has been considerably influenced by the paper [14] of Bakry and Emery, the ideas in which have been developed by many authors. It was quite often that functional inequalities discovered in the general case were new even for the special Gaussian case under discussion. For example, in [42] (see also [20]) there is a discussion of two-sided pointwise estimates of the form

$$\frac{e^{2t}-1}{2}\Phi''(T_t f)\Gamma(T_t f) \leqslant \mathrm{Ent}^{\Phi}_{T_t}(f) \leqslant \frac{1-e^{-2t}}{2}T_t[\Phi''(f)\Gamma(f)],$$

where Φ is a smooth strictly convex function on some interval I such that the function $-1/\Phi''$ is convex, f is a smooth function with values in the interval I, $\Gamma(f)=|\nabla f|^2$, and

$$\mathrm{Ent}^{\Phi}_{T_t}(f) = T_t\Phi(f) - \Phi(T_t f).$$

Letting $t\to+\infty$, one can obtain generalizations of the logarithmic Sobolev inequality and the Poincaré inequality from such inequalities for a suitable choice of Φ (and taking into account that $T_t f$ tends to the integral of f).

There are papers in which estimates are constructed on the basis of expansions with higher-order derivatives (see, for example, [119] and the references given there), a typical particular case being the equality for variances

$$\int f^2\,d\gamma - \biggl(\int f\,d\gamma\biggr)^2 = \sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\int |D^k f|^2\,d\gamma \\ - \frac{(-1)^{n-1}}{(n-1)!}\int_0^\infty 2e^{-2nt}\int |T_t D^n f|^2\,d\gamma\,dt,$$

and also analogous equalities for compositions $\Phi(f)$.

Remark 5.7. From the point of view of operators acting in a Hilbert space, the Ornstein–Uhlenbeck operator is completely characterized by the property that it possesses an orthonormal eigenbasis $\{e_n\}_{n\geqslant 0}$ with eigenvalues $-n$ of multiplicity d_n (equal to 1 in the one-dimensional case). However, it actually has a much richer structure due to the possibility to consider it acting in the spaces L^p. Here for $p<\infty$ it is the generator of a Markov operator semigroup, that is, a semigroup of operators taking non-negative functions to non-negative functions and 1 to 1. Furthermore, the operators T_t of the Ornstein–Uhlenbeck semigroup are not only Markov but, for $t>0$, also ergodic: in L^2 the only eigenfunctions with eigenvalue 1 are constants. The number 1 is an isolated point of the spectrum of these operators; this phenomenon is called the existence of a 'spectral gap'). In the well-known paper [189] the question was posed as to whether the existence of a spectral gap follows from the additional property of a symmetric ergodic Markov operator T which is called 'hyperboundedness' and is expressed by the inclusion

$$T(L^2) \subset L^p \quad \text{for some } p > 2.$$

Only recently a positive answer to this question was given in [152], and Wang [208] strengthened this result by showing that for an ergodic Markov operator P on $L^2(\mu)$ (not necessarily symmetric) the existence of a spectral gap for the symmetrized operator $(P+P^*)/2$ is equivalent to the property that

$$\lim_{R\to+\infty} \sup_{f\geqslant 0,\ \|f\|_2\leqslant 1} \|f(Pf-R)^+\|_1 < 1$$

(for a hyperbounded operator this limit is zero). However, we saw above that the Ornstein–Uhlenbeck semigroup satisfies not only the inclusion $T_t(L^2) \subset L^p$ but also a sharp estimate for the norm of the embedding, which is not included in the hyperboundedness property. The latter circumstance is important: it is shown in Proposition 11 of [152] that for every $K \geqslant 2$ and $\varepsilon > 0$ there exists a self-adjoint ergodic Markov operator M on $L^2(\mu)$ with some measure μ that has a spectral gap of size ε (that is, the distance from 1 to the rest of the spectrum is at least ε) such that

$$\|M\|^4_{\mathscr{L}(L^2(\mu),L^4(\mu))} = K.$$

The estimate $K \geqslant 2$ is not by chance: it was shown earlier in [206] that for $K < 2$ there is an a priori estimate of the size of the spectral gap.

We also mention the following fact (see, for example, Proposition 5.4.8 in [25], where the case $p \geqslant 2$ was considered, or [68], p. 75).

Theorem 5.8. *Let $p > 1$ and $f \in L^p(\gamma)$. Then $T_t f \in W^{p,n}(\gamma)$ for all $t > 0$ and $n \geqslant 1$, the function $h \mapsto T_t f(x+h)$ is infinitely Fréchet differentiable on H for almost every x, and*

$$\partial_h T_t f(x) = \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}\, y)\widehat{h}(y)\, \gamma(dy),$$

$$\int_X |D_H^n f(x)|^p_{\mathscr{H}_n}\, \gamma(dx) \leqslant C_{p,n} \frac{e^{-nt}}{(1-e^{-2t})^{n/2}} \int_X |f(x)|^p\, \gamma(dx),$$

where $C_{p,n}$ depends only on p and n. Moreover, by the hypercontractivity property $T_t f \in W^{q,n}(\gamma)$ for all $q < 1 + (p-1)e^{2t}$. Therefore, for fixed $q > 1$ and $n \geqslant 1$ the inclusion $T_t f \in W^{q,n}(\gamma)$ holds for all sufficiently large t. Analogous assertions are true for maps with values in a separable Hilbert space.

For $p = 1$ this is false, as we shall see below. The first equality in Theorem 5.8 implies that if $|f(x)| \leqslant 1$, then

$$|D_H f(x)| \leqslant \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\,.$$

In the infinite-dimensional case the function $T_t f$ can fail to have a continuous version (this happens, for example, for a measurable linear function f without continuous versions).

To consider embeddings of the spaces $W^{1,1}(\gamma)$ and $BV(\gamma)$, we need the aforementioned Orlicz space $L\sqrt{\log L}$. For the function

$$w(x) = \int_0^x (e^{s^2} - 1)\, ds$$

one also introduces the Orlicz space of functions with finite norm

$$\|g\|_w = \inf\left\{\alpha > 0\colon \int w\left(\frac{|g|}{\alpha}\right) d\gamma \leqslant 1\right\}.$$

Let

$$u(x) = \int_0^x \bigl(\log(1+s)\bigr)^{1/2}\, ds.$$

Young's inequality implies (see [80]) the inequality

$$\|fg\|_1 \leqslant \bigl(\|u(|f|)\|_1 + 1\bigr)\|g\|_w. \tag{5.8}$$

We present the following result from [80], with a proof.

Theorem 5.9. *There is a number $C_1 > 0$ independent of d such that the inequality*

$$\|f\|_{L\sqrt{\log L}} \leqslant C_1\|f\|_{BV(\gamma)} \tag{5.9}$$

holds for all $f \in BV(\gamma)$, where $\gamma = \gamma_d$. In particular, this is true for all $f \in W^{1,1}(\gamma)$. These assertions are also valid in the infinite-dimensional case.

Proof. Let Φ be the standard Gaussian distribution function and

$$\Psi(s) = \Phi'(\Phi^{-1}(s)), \qquad 0 < s < 1.$$

It is straightforward to verify that

$$\lim_{s\to 0} \frac{\Psi(s)}{s\sqrt{-2\log s}} = 1.$$

There is a number $\delta \in (0, 1/(e-1))$ such that

$$\Psi(s) \geqslant x\sqrt{\log\left(1 + \frac{1}{s}\right)} \qquad \forall\, s \in (0, \delta].$$

Let $f \in C_b^\infty(\mathbb{R}^d)$ and $\|f\|_{1,1} \leqslant 1/\sqrt{\log(1+1/\delta)}$. By the familiar Gaussian isoperimetric inequality we have the estimate

$$\int_{\mathbb{R}^d} |\nabla f|\, d\gamma_d \geqslant \int_0^\infty \Psi\bigl(\gamma_d(x\colon |f(x)| \geqslant s)\bigr)\, ds.$$

If $s \geqslant 1/\delta$, then

$$\gamma_d\bigl(x\colon |f(x)| \geqslant s\bigr) \leqslant \frac{\|f\|_1}{s} \leqslant \frac{1}{s} \leqslant \delta,$$

and also

$$\Psi\big(\gamma_d(x\colon |f(x)|\geqslant s)\big)\geqslant \gamma_d(x\colon |f(x)|\geqslant s)\sqrt{\log\biggl(1+\frac{1}{\gamma_d(x\colon |f(x)|\geqslant s)}\biggr)}\,.$$

Therefore,

$$\begin{aligned}
1&\geqslant \sqrt{\log\biggl(1+\frac{1}{\delta}\biggr)}\,\|f\|_{1,1}\geqslant \sqrt{\log\biggl(1+\frac{1}{\delta}\biggr)}\,\|f\|_1+\|\nabla f\|_1\\
&\geqslant \sqrt{\log\biggl(1+\frac{1}{\delta}\biggr)}\int_0^\infty \gamma_d(|f|\geqslant s)\,ds+\int_{1/\delta}^\infty \gamma_d(|f|\geqslant s)\sqrt{\log(1+s)}\,ds\\
&\geqslant \int_0^\infty \gamma_d(|f|\geqslant s)\sqrt{\log(1+s)}\,ds\\
&=\int_{\mathbb{R}^d}\int_0^{|f(x)|}\sqrt{\log(1+s)}\,ds\,\gamma_d(dx).
\end{aligned}$$

Consequently, $\|f\|_{L\sqrt{\log L}}\leqslant 1$. Thus, the constant C_1 in the formulation is completely determined by the behaviour of the function Φ. The infinite-dimensional case follows from the finite-dimensional case. $\square$

Corollary 5.10. *For all $f\in BV(\gamma)$, where $\gamma=\gamma_d$, the inequality*

$$\gamma(x\colon |f(x)|>r)\leqslant C_2\|f\|_{BV(\gamma)}\frac{1}{r\sqrt{\log r}}\,,\qquad r>0,$$

holds, where C_2 is some constant (independent of d). Hence this inequality is also true in the infinite-dimensional case.

Proof. By Chebyshev's inequality, for $R>0$ we have

$$\gamma_d\bigl(x\colon |f(x)|\sqrt{|f(x)|}>R\bigr)\leqslant \frac{J}{R}\,,$$

where J is the integral of $|f|\sqrt{|f|}$ with respect to the measure γ_d. This integral can be estimated with some constant in terms of $\|f\|_{L\sqrt{\log L}}$, and then in terms of $\|f\|_{BV(\gamma_d)}$ by the previous theorem. Taking $R=r\sqrt{\log r}$, we obtain the desired inequality because of the strict monotonicity of the function $r\sqrt{\log r}$ on $(1,\infty)$. $\square$

The proof of the following theorem was given in [80], Proposition 3.6. It employs the following notation in the case of the measure $\gamma=\gamma_d$:

$$V(f):=\sup\biggl\{\int_{\mathbb{R}^d} f\,\mathrm{div}_\gamma w\,d\gamma\colon w\in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d),\ |w(x)|\leqslant 1\biggr\},$$

where the divergence $\mathrm{div}_\gamma w$ is defined by (1.6).

In the infinite-dimensional case the quantity $V(f)$ is introduced similarly: we consider finite-dimensional vector fields w taking values in the Cameron–Martin space H and having the form

$$w=w_1h_1+\dots+w_nh_n,$$

where $w_i \in \mathscr{F}\mathscr{C}^\infty(X)$, $h_i \in H$, $|w(x)|_H \leqslant 1$, and the divergence $\delta w = \operatorname{div}_\gamma w$ is defined as explained in § 4.

For functions f in the class $W^{1,1}(\gamma)$ or $BV(\gamma)$ the quantity $V(f)$ is finite.

Theorem 5.11. *Let $f \in L\sqrt{\log L}$. Then $T_t f \in W^{1,1}(\gamma)$ for all $t > 0$, and*

$$\lim_{t\to 0} \|T_t f - f\|_{L\sqrt{\log L}} = 0.$$

If $f \in BV(\gamma)$, then

$$V(T_t f) \leqslant e^{-t} V(f), \qquad \lim_{t\to 0} V(T_t f) = V(f).$$

We mention the following result from Proposition 3.5 in [80], and we give a proof in order to estimate the corresponding constant.

Proposition 5.12. *For every $t > 0$, every function $f \in C_b^\infty(\mathbb{R}^d)$ satisfies the inequality*

$$\int_{\mathbb{R}^d} |\nabla T_t f|\, d\gamma \leqslant C(t)\|f\|_{L\sqrt{\log L}},$$

where

$$C(t) = 2\frac{e^{-t}}{(1-e^{-2t})^{1/2}}\,.$$

Thus, the operator ∇T_t extends to a bounded operator from the Orlicz space $L\sqrt{\log L}$ to the space $L^1(\gamma, \mathbb{R}^d)$. The same is true in the infinite-dimensional case, and the operator $D_H T_t$ extends to a bounded operator from $L\sqrt{\log L}$ to $L^1(\gamma, H)$.

Proof. Let $f \in C_b^\infty(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, $|h| = 1$. From the equality

$$\partial_h f(x) = \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \int_{\mathbb{R}^d} f\bigl(e^{-t}x + \sqrt{1-e^{-2t}}\, y\bigr)\langle h, y\rangle\, \gamma(dy)$$

and (5.8) we get that

$$|\partial_h f(x)| \leqslant \frac{e^{-t}}{(1-e^{-2t})^{1/2}}\bigl(\bigl\|u\circ|f|(e^{-t}x + \sqrt{1-e^{-2t}}\,\cdot)\bigr\|_1 + 1\bigr)\|l_h\|_w,$$

where $l_h(x) = \langle x, h\rangle$. It suffices to estimate the quantity $\|l_h\|_w$ in the one-dimensional case for $h = 1$. In this case it does not exceed 4, since for $\alpha = 4$ the corresponding integral is not greater than 1. Consequently,

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla f|\, d\gamma &\leqslant \frac{e^{-t}}{(1-e^{-2t})^{1/2}}\biggl(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u\circ|f|\bigl(e^{-t}x + \sqrt{1-e^{-2t}}\, y\bigr)\, \gamma(dx)\, \gamma(dy) + 1\biggr) \\ &= \frac{e^{-t}}{(1-e^{-2t})^{1/2}}\biggl(\int_{\mathbb{R}^d} u(|f|)\, d\gamma + 1\biggr) \leqslant 2\frac{e^{-t}}{(1-e^{-2t})^{1/2}}\,,\end{aligned}$$

as required. The infinite-dimensional case follows directly from the finite-dimensional case. $\square$

Note that one of the ways to see that the operators T_t do not take $L^1(\gamma)$ to $W^{1,1}(\gamma)$ or $BV(\gamma)$ is to consider the functions

$$f_\alpha(x) = \exp\biggl(\alpha x - \frac{\alpha^2}{2}\biggr)$$

of unit norm in $L^1(\gamma)$. For them, $\|T_t f_\alpha\|_{1,1} \geqslant \alpha e^{-t}$. Hence even on the linear span of these functions the norm of the operator T_t from $L^1(\gamma)$ to $W^{1,1}(\gamma)$ is infinite. With the aid of the same functions one can verify that the image of $L^1(\gamma)$ under T_t does not belong to the Orlicz space $L\sqrt{\log L}$, which contains $BV(\gamma)$, because the integrals of the functions $T_t f_\alpha |\log T_t f_\alpha|^{1/2}$ with respect to the measure γ tend to infinity as $\alpha \to +\infty$ (observe that $T_t f_\alpha = f_{e^{-t}\alpha}$).

For every non-negative function $f \in L^1(\gamma_d)$, the Chebyshev inequality and the equality of the integrals of f and $T_t f$ with respect to γ_d imply the estimate

$$\gamma_d\bigl(x\colon T_t f(x) > r\bigr) \leqslant \frac{1}{r}\|f\|_{L^1(\gamma)}.$$

In particular, if the integral of f is 1, then the right-hand side equals $1/r$. This estimate can be slightly improved by putting on the right-hand side a factor $\alpha(t)$ tending to zero at infinity. It turns out that it is possible to find an optimal factor of this sort. The following fact was established in [124] (which strengthens some earlier results from [17] and [66]).

Theorem 5.13. *There is a constant C such that for all $d \geqslant 1$, $r > 1$, and $t > 0$ the inequality*

$$\gamma_d(x\colon T_t f(x) > r) \leqslant C \max\biggl\{1, \frac{1}{t}\biggr\} \frac{1}{r\sqrt{\log r}} \tag{5.10}$$

holds for every probability density f with respect to the measure γ_d. Thus, this estimate holds with the same constant for Gaussian measures on infinite-dimensional spaces.

In the proof of the theorem the properties of the Ornstein–Uhlenbeck semigroup are used only to derive the following lemma.

Lemma 5.14. *Let f be a probability density with respect to γ_d. Then*

$$D^2 \log T_t f(x) \geqslant -\frac{1}{2t} I,$$

where the second derivative is understood in the sense of distributions and the inequality holds pointwise for bounded functions f.

Proof. Suppose first that the function f is bounded (but its integral can differ from 1). We use the representation

$$T_t f(x) = g_{1-s} * f(\sqrt{s}\, x), \qquad s = e^{-2t},$$

where g_{1-s} is the density of the centred Gaussian measure with covariance $(1-s)I$. For every unit vector $v \in \mathbb{R}^d$ we find by direct differentiation that

$$\begin{aligned} \partial_v T_t f(x) &= -\frac{\sqrt{s}}{1-s}\, \frac{1}{(2\pi(1-s))^{d/2}} \int \langle \sqrt{s}x - y, v\rangle \exp\biggl(-\frac{(\sqrt{s}x - y)^2}{2(1-s)}\biggr) f(y)\, dy, \\ \partial_v^2 T_t f(x) &= -\frac{s}{1-s}\, T_t f(x) + \frac{s}{(1-s)^2}\, \psi * f(\sqrt{s}\, x), \end{aligned}$$

where $\psi(x) = \langle x, v\rangle^2 g_{1-s}(x)$. Hence,

$$\begin{aligned}\partial_v^2 \log T_t f(x) &= \frac{\partial_v^2 T_t f(x)}{T_t f(x)} - \frac{|\partial_v T_t f(x)|^2}{(T_t f(x))^2} \\ &= -\frac{s}{1-s} + \frac{s(1-s)^{-2} T_t f(x) \psi * f(\sqrt{s}\, x) - |\partial_v T_t f(x)|^2}{(T_t f(x))^2}\,.\end{aligned}$$

By the Cauchy–Bunyakovskii inequality the second term is non-negative. It remains to observe that $s/(1-s) \leqslant 1/(2t)$ since $2te^{-2t} \leqslant 1-e^{-2t}$. Thus, the inequality to be proved is pointwise true for the smooth functions $T_t \min\{f, N\}(x)$. Letting $N \to \infty$, we obtain the desired inequality in the sense of distributions. □

The main estimate from [124] is as follows.

Theorem 5.15. *There is a constant C such that for all $d \geqslant 1$ and $r > 1$, if f is a bounded smooth positive probability density with respect to the measure γ_d and*

$$D^2 \log f(x) \geqslant -\beta I$$

pointwise for some number $\beta > 0$, then

$$\gamma_d\bigl(x\colon f(x) > r\bigr) \leqslant C \max\{1, \beta\} \frac{1}{r\sqrt{\log r}}\,. \tag{5.11}$$

A consideration of the exponential functions $f_\alpha(x) = \exp(\alpha x - \alpha^2/2)$ shows that the estimates obtained are sharp in the part concerning dependence on r.

Theorem 5.15 is deduced in [124] from the following assertion.

Theorem 5.16. *Suppose that a random vector ξ with values in $\mathbb{R}^d$ has a distribution density f with respect to the measure γ_d and f satisfies the conditions of the previous theorem. Then*

$$\mathsf{P}\bigl(f(\xi) \in (r, er]\bigr) \leqslant C \max\{1, \beta\} \frac{1}{\sqrt{\log r}}\,. \tag{5.12}$$

We explain how to get (5.11) from (5.12). Let η be a random vector on the same probability space as ξ, with the standard Gaussian distribution in $\mathbb{R}^d$. Then the following relations hold, in which E denotes the expectation:

$$\begin{aligned}\mathsf{P}\bigl(f(\eta) > r\bigr) &= \sum_{k=0}^\infty \mathsf{P}\bigl(f(\eta) \in (e^k r, e^{k+1} r]\bigr) \\ &\leqslant \sum_{k=0}^\infty (e^k r)^{-1} \mathsf{E}\bigl(f(\eta) I_{f(\eta)\in(e^k r, e^{k+1} r]}\bigr) \\ &= \sum_{k=0}^\infty (e^k r)^{-1} \mathsf{P}\bigl(f(\xi) \in (e^k r, e^{k+1} r]\bigr) \\ &\leqslant \sum_{k=0}^\infty (e^k r)^{-1} C \max\{\beta, 1\} \frac{1}{\sqrt{\log(e^k r)}} \\ &\leqslant C \frac{e}{e-1} \frac{1}{r} \frac{\max\{\beta, 1\}}{\sqrt{\log r}}\,.\end{aligned}$$

However, the proof of Theorem 5.16 is highly non-trivial and is accomplished in [124] by methods of stochastic analysis, with the aid of Itô's formula and Girsanov's theorem. It would be interesting and useful to find an analytic proof of this result.

If the operators T_t were mapping $L^1(\gamma_d)$ to $W^{1,1}(\gamma_d)$ (which is not the case), then the estimate (5.10) would follow from the embedding of $W^{1,1}(\gamma_d)$ in the Orlicz space (see Theorem 5.9).

In [52], some results on Sobolev inequalities were obtained in the general form

$$\|f\|_X \leqslant C\|\nabla f\|_Y,$$

where X and Y are some Banach spaces with so-called rearrangement invariant norms. Many earlier known inequalities can be represented in such a form.

We mention an interesting sharpening obtained in [16] (see also [13]) of the inequality (5.6) for the standard Gaussian measure on $\mathbb{R}^d$:

$$\int_{\mathbb{R}^d}\biggl(f-\int_{\mathbb{R}^d} f\,d\gamma\biggr)^2 d\gamma \leqslant \int_{\mathbb{R}^d}|\nabla f|^2\,d\gamma - \frac{1}{2d}\biggl(\int_{\mathbb{R}^d}\Delta f\,d\gamma\biggr)^2, \qquad f\in C_0^\infty(\mathbb{R}^d).$$

For functions $f\in W^{2,1}(\gamma_d)$ with zero integral with respect to γ_d this can be written as

$$\int_{\mathbb{R}^d}|f|^2\,d\gamma_d \leqslant \int_{\mathbb{R}^d}|\nabla f|^2\,d\gamma_d - \frac{1}{2d}\biggl(\int_{\mathbb{R}^d}|x|^2 f(x)\,\gamma_d(dx)\biggr)^2.$$

In [101] there is an interesting inequality for probability measures on $\mathbb{R}^d$ with densities of the form $e^{-\Phi}$, where Φ is a convex function; a particular case is the estimate given above. There is also an estimate in the opposite direction:

$$\int_{\mathbb{R}^d}\biggl(f-\int_{\mathbb{R}^d} f\,d\gamma\biggr)^2 d\gamma \geqslant \biggl|\int_{\mathbb{R}^d}\nabla f\,d\gamma\biggr|^2 + \frac{1}{2d}\biggl(\int_{\mathbb{R}^d}\Delta f\,d\gamma\biggr)^2, \qquad f\in C_0^\infty(\mathbb{R}^d),$$

where on the right-hand side we have a vector integral of ∇f (see [101]).

We also mention the papers [24], [71], and [122] (where there are additional references) connected with estimating the so-called log-Sobolev deficit

$$\delta_{\mathrm{LS}}(f) = \frac{1}{2}\int_{\mathbb{R}^d}\frac{|\nabla f|^2}{f}\,d\gamma_d - \mathrm{Ent}_{\gamma_d}(f).$$

For Borel sets E and F in $\mathbb{R}^d$ the following estimate is obtained in [162] with some universal constant C:

$$\biggl(\int_F |T_t(fI_E)|^2\,d\gamma_d\biggr)^{1/2} \leqslant C\frac{t}{\mathrm{dist}(E,F)}\exp\biggl(-\frac{\mathrm{dist}(E,F)^2}{2t}\biggr)\biggl(\int_E |f|^2\,d\gamma_d\biggr)^{1/2}.$$

For results connected with such estimates, see [6].

Another important class of functional inequalities for semigroups is connected with versions of Harnack's inequality (see Wang's book [207]). We mention an inequality of Harnack type due to Wang in the particular case of the Ornstein–Uhlenbeck semigroup we are discussing.

Theorem 5.17. *Let $f \in L^p(\gamma_d)$. If $p > 1$, then*

$$|T_t f(y)|^p \leqslant T_t|f|^p(x) \exp\biggl(\frac{1}{2}\,\frac{p}{p-1}\,\frac{|x-y|^2}{e^{2t}-1}\biggr), \qquad x, y \in \mathbb{R}^d.$$

If $0 < p < 1$ and $f \geqslant 0$, then

$$(T_t f(y))^p \geqslant T_t f^p(x) \exp\biggl(\frac{1}{2}\,\frac{p}{p-1}\,\frac{|x-y|^2}{e^{2t}-1}\biggr), \qquad x, y \in \mathbb{R}^d.$$

Proof. Let

$$\varrho_{x,y}(z) := \exp\biggl(\sqrt{\frac{1}{e^{2t}-1}}\,\langle y-x, z\rangle - \frac{1}{2}\,\frac{1}{e^{2t}-1}\,|y-x|^2\biggr).$$

For $p > 1$, we use a change of variables and Hölder's inequality to obtain

$$T_t f(y) = \int_{\mathbb{R}^d} f\bigl(e^{-t}x + \sqrt{1-e^{-2t}}\,z\bigr)\,\varrho_{x,y}(z)\,\gamma(dz) \leqslant [T_t|f|^p(x)]^{1/p}\|\varrho_{x,y}\|_q,$$

whence the desired estimate follows. If $0 < p < 1$, then we set $\alpha := 1/p$ and $g := f^p$. The estimate proved gives us the inequality

$$(T_t g(y))^\alpha \leqslant T_t g^\alpha(x) \exp\biggl(\frac{1}{2}\,\frac{\alpha}{\alpha-1}\,\frac{|x-y|^2}{e^{2t}-1}\biggr),$$

which easily implies the desired estimate. $\square$

For $f > 0$ this yields the logarithmic inequality

$$T_t \log f(x) \leqslant \log T_t f(y) + \frac{1}{2}\frac{1}{e^{2t}-1}|x-y|^2.$$

The logarithmic Sobolev inequalities, Poincaré inequalities, and hypercontractivity properties connected with Gaussian measures have been considered in many papers (see [7], [11], [12], [18], [43], [55], [62], [77], [90], [121]). It was shown in [213] that the hypercontractivity property for the Ornstein–Uhlenbeck semigroup is equivalent to certain estimates in $L^p(\mathbb{R}^d)$ for the heat equation semigroup.

6. Fractional Sobolev classes and Nikolskii–Besov classes connected with the Ornstein–Uhlenbeck semigroup

By means of the Ornstein–Uhlenbeck semigroup one can introduce fractional Sobolev classes. Here we mention several approaches based on powers of the Ornstein–Uhlenbeck operator, interpolation, and 'fractional integration by parts formulae'. This direction is already sufficiently developed and merits a separate survey (see [45], [85]–[88], [127]–[129], [131], and [177], where there is an extensive bibliography).

Let γ be an arbitrary centred Gaussian measure, for example, the standard Gaussian measure on $\mathbb{R}^d$. For $p \geqslant 1$, $f \in L^p(\gamma)$, and $\alpha > 0$ we set

$$V_\alpha(f) := \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}e^{-t}T_t f\,dt = (I-L)^{-\alpha/2}f,$$

where Γ denotes the Euler gamma-function. The scale of Sobolev classes $H^{p,\alpha}(\gamma)$ with $p>1$ and $\alpha>0$ is defined as follows:

$$H^{p,\alpha}(\gamma) := V_\alpha(L^p(\gamma)), \qquad \|f\|_{H^{p,\alpha}(\gamma)} := \|V_\alpha^{-1} f\|_p.$$

Now for $r<0$ we define $H^{p,r}(\gamma)$ as the dual space of $H^{p',-r}(\gamma)$, where $p'=p/(p-1)$.

It is known that $H^{p,\alpha}(\gamma)=W^{p,\alpha}(\gamma)$ for $p>1$ and $\alpha\in\mathbb{N}$.

One can verify that the family of operators V_α supplemented by the operator $V_0=I$ is a strongly continuous semigroup on $L^p(\gamma)$.

We note a useful inequality connected with norms in $H^{p,\alpha}(\gamma)$ (see [188], Proposition 4.10).

Proposition 6.1. *Let $\alpha<\beta<\gamma$. There is a number $c(\alpha,\beta,\gamma)$ such that*

$$\|(I-L)^\beta f\|_p \leqslant c(\alpha,\beta,\gamma)\|(I-L)^\alpha f\|_p^{(\gamma-\beta)/(\gamma-\alpha)}\|(I-L)^\gamma f\|_p^{(\beta-\alpha)/(\gamma-\alpha)}$$

for $p>1$ and all smooth cylindrical functions (hence also for all functions for which the indicated norms are finite).

By the multiplicative inequality in Proposition 6.1 it can be proved (see [188], p. 92) that for every $k\in\mathbb{N}$ and $\varepsilon>0$ there exists a sufficiently large number M such that

$$\|D_H^k f\|_p \leqslant \varepsilon\|D_H^{k+1} f\|_p + M\|f\|_p, \qquad f\in W^{p,k+1}(\gamma).$$

For $f\in L^p(\gamma)$ we set

$$K_t(f)=\inf\bigl\{\|f_1\|_p + t\|f_2\|_{W^{p,1}(\gamma)}\colon f=f_1+f_2,\ f_1\in L^p(\gamma),\ f_2\in W^{p,1}(\gamma)\bigr\}$$

and for $\alpha\in(0,1)$ we consider the class $\mathscr{E}^{p,\alpha}(\gamma)$ of functions with finite norm

$$\|f\|_{\mathscr{E}^{p,\alpha}(\gamma)} := \biggl(\int_0^\infty |t^{-\alpha}K_t(f)|^p t^{-1}\,dt\biggr)^{1/p}.$$

Similarly, using the same method of interpolation between the spaces $H^{p,k}(\gamma)$ and $H^{p,k+1}(\gamma)$ with integer k, one can define the classes $\mathscr{E}^{p,\alpha}(\gamma)$ for all real α.

As was shown by Watanabe [211] (who used an equivalent interpolation method), for all $p>1$, $\alpha\in\mathbb{R}$, and $\varepsilon>0$ we have the continuous embeddings

$$\mathscr{E}^{p,\alpha+\varepsilon}(\gamma)\subset H^{p,\alpha}(\gamma)\subset \mathscr{E}^{p,\alpha-\varepsilon}(\gamma).$$

This also gives the embeddings

$$H^{p,\alpha+\varepsilon}(\gamma)\subset \mathscr{E}^{p,\alpha}(\gamma)\subset H^{p,\alpha-\varepsilon}(\gamma).$$

We now mention a number of results from the recent papers [30]–[32], [115], and [116], where some analogues of Nikolskii and Nikolskii–Besov classes connected with the Ornstein–Uhlenbeck semigroup were considered.

We shall use the function c_t introduced in (2.8). Note that

$$c_t\leqslant (2t)^{1/2} \quad\text{and}\quad \lim_{t\to+\infty} c_t=\frac{\pi}{2}\,.$$

Recall (see (1.6)) that for a map $\Phi = (\Phi_i) \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the divergence with respect to the measure γ is defined by $\operatorname{div}_\gamma \Phi = \sum_{i=1}^d (\partial_{x_i}\Phi_i - x_i\Phi_i)$.

The next definition and the related theorem are stated for greater clarity for the case of the standard Gaussian measure γ_d on $\mathbb{R}^d$, but then they will be extended to the infinite-dimensional case upon taking into account their independence of dimension.

Definition 6.2. Let $\alpha \in (0,1]$, $p \in [1,\infty)$, and $\gamma = \gamma_d$. A function $f \in L^p(\gamma)$ belongs to the Gaussian Nikolskii–Besov class $B_p^\alpha(\gamma)$ if there exists a number C such that for every map $\Phi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$

$$\int_{\mathbb{R}^d} f \operatorname{div}_\gamma \Phi \, d\gamma \leqslant C \|\Phi\|_q^\alpha \| \operatorname{div}_\gamma \Phi\|_q^{1-\alpha},$$

where $1/p + 1/q = 1$. Let $V_\gamma^{p,\alpha}(f)$ be the infimum of such numbers C.

The next definition is given at once for the infinite-dimensional case.

Definition 6.3. For a function $f \in L^p(\gamma)$, let

$$U_\gamma^{p,\alpha}(f) := \sup_{t>0} t^{(1-\alpha)/2} \big\| |D_H T_t f|_H \big\|_p.$$

We recall (see Theorem 5.8) that for all $f \in L^p(\gamma)$ and $p > 1$

$$T_t f \in W^{p,1}(\gamma).$$

In the case $p = 1$ let $U_\gamma^{1,\alpha}(f) = \infty$ if $T_t f \notin W^{1,1}(\gamma)$ for some $t > 0$.

Theorem 6.4. *For any $f \in B_p^\alpha(\gamma)$ and $p \in [1,\infty)$*

$$\|f - T_t f\|_p \leqslant 2^{1-\alpha} C(p)^\alpha c_t^\alpha V_\gamma^{p,\alpha}(f),$$

where c_t is defined in (2.8) *and*

$$C(p) := \left((2\pi)^{-1/2} \int_{\mathbb{R}} |s|^p e^{-s^2/2} \, ds \right)^{1/p}. \tag{6.1}$$

Passing to the limit as $t \to +\infty$ in the previous theorem, we obtain the following inequality of Poincaré type. Let $\mathbb{E} f$ denote the integral of f with respect to γ.

Corollary 6.5. *For any $f \in B_p^\alpha(\gamma)$ and $p \in [1,\infty)$*

$$\|f - \mathbb{E} f\|_p \leqslant 2^{1-2\alpha} \pi^\alpha C(p)^\alpha V_\gamma^{p,\alpha}(f).$$

We now give an analogue of the fractional Hardy–Landau–Littlewood inequality with respect to the Gaussian measure, generalizing a similar estimate in [41] for $\alpha = 1$. The classical Hardy–Landau–Littlewood inequality with Lebesgue measure on the real line states that

$$\|\varphi'\|_{L^1(\mathbb{R})}^2 \leqslant 2 \|\varphi\|_{L^1(\mathbb{R})} \|\varphi''\|_{L^1(\mathbb{R})}.$$

This can be written as

$$\|f\|^2_{L^1(\mathbb{R})} \leqslant 2\|f'\|_{L^1(\mathbb{R})}\|f\|_{\mathrm{K}},$$

where for functions f with zero integral with respect to Lebesgue measure $\|f\|_{\mathrm{K}}$ is the Kantorovich norm, defined as the supremum over 1-Lipschitz functions g of the integrals of fg with respect to Lebesgue measure (as in (2.6), but with Lebesgue measure instead of the Gaussian measure).

Theorem 6.6. *For every function $f \in B_1^\alpha(\gamma)$ with zero integral with respect to $\gamma = \gamma_d$*

$$\|f\|_1 \leqslant 3(V_\gamma^{1,\alpha}(f))^{1/(1+\alpha)}\|f\|^{\alpha/(1+\alpha)}_{\mathrm{K},\gamma_d}.$$

For $\alpha = 1$ we obtain an estimate for functions in the class $BV(\gamma)$ or $W^{1,1}(\gamma)$.

The Gaussian Nikolskii–Besov classes have the characterization below in terms of the behaviour of the Ornstein–Uhlenbeck semigroup near zero.

Theorem 6.7. *If $f \in L^p(\gamma)$ and $p \in (1,\infty)$, then $V_\gamma^{p,\alpha}(f) < \infty$ precisely when $U_\gamma^{p,\alpha}(f) < \infty$. Moreover,*

$$U_\gamma^{p,\alpha}(f) \leqslant C(q)^{1-\alpha}V_\gamma^{p,\alpha}(f) \quad \textit{and} \quad V_\gamma^{p,\alpha}(f) \leqslant (4C(p)\alpha^{-1}+1)U_\gamma^{p,\alpha}(f),$$

where $1/p + 1/q = 1$ and $C(p)$ is defined by (6.1).

The implication $U_\gamma^{p,\alpha}(f) < \infty \Rightarrow V_\gamma^{p,\alpha}(f) < \infty$ is also true for $p = 1$.

We now present infinite-dimensional analogues of these results. Modifications are needed only for the results that use divergence and the Kantorovich norm. Let $\mathscr{FC}^\infty(X,H)$ be the class of all maps $\Phi\colon X \to H$ of the form

$$\Phi(x) = \sum_{i=1}^n \Psi_i(g_1(x),\dots,g_n(x))h_i, \tag{6.2}$$

where $\Psi_i \in C_b^\infty(\mathbb{R}^n)$, $g_i \in X^*$, and $h_i \in H$. Let $\mathscr{FC}_0^\infty(X,H)$ be the subset of this class consisting of maps for which the Ψ_i can be chosen to have compact support. In the representation (6.2) we can always take the vectors h_i to be orthogonal in H and the functionals g_i to be orthogonal in X_γ^* so that $g_i(h_j) = \delta_{ij}$. We recall that for such vector fields the γ-divergence is defined (see (4.2) and (4.3)).

If we take an orthonormal basis $\{e_i\}$ in H of the form $e_i = \widehat{l_i}$, where $l_i \in X^*$ (see § 4), then for a map $\Phi \in \mathscr{FC}^\infty(X,H)$ of the form

$$\Phi(x) = \sum_{i=1}^n \Psi_i\bigl(l_1(x),\dots,l_n(x)\bigr)e_i$$

we have

$$\operatorname{div}_\gamma \Phi(x) = \sum_{j=1}^n \bigl[\partial_{x_j}\Psi_j\bigl(l_1(x),\dots,l_n(x)\bigr) - l_j(x)\Psi_j\bigl(l_1(x),\dots,l_n(x)\bigr)\bigr].$$

For every map $\Phi \in \mathscr{FC}_0^\infty(X,H)$ its divergence $\operatorname{div}_\gamma \Phi$ is a bounded function.

By analogy with (2.6), the Kantorovich norm associated with the measure γ is defined by

$$\|f\|_{\mathrm{K},\gamma} := \sup\biggl\{\int_X \varphi f\,d\gamma\colon \varphi\in\mathscr{F}\mathscr{C}^\infty(X),\ |D_H\varphi|_H\leqslant 1\biggr\}$$

on functions $f\in L^1(\gamma)$ with zero integral with respect to γ for which this norm is finite. This is in fact the restriction of the Kantorovich norm generated by the subspace H to the space of signed measures with zero value on X with respect to which all H-Lipschitz functions are integrable (about such norms, see, for instance, [29]). By the formula

$$\|f\|_{\mathrm{K},\gamma} := \|f-\mathbb{E}f\|_{\mathrm{K},\gamma} + |\mathbb{E}f|$$

the Kantorovich norm extends naturally to the space of all γ-integrable functions f such that $\|f-\mathbb{E}f\|_{\mathrm{K},\gamma}<\infty$, where $\mathbb{E}f$ is the integral of f with respect to the measure γ.

It is known (see [25]) that for every function φ with $|D_H\varphi|_H\leqslant 1$ and zero integral the function $\exp(|\varphi|^2/4)$ is γ-integrable, and its integral is bounded by some universal constant. Therefore, $\|f\|_{\mathrm{K},\gamma}<\infty$ under the condition that $f\sqrt{|\log|f||}\in L^1(\gamma)$. Hence, this norm is finite on the Sobolev space $W^{1,1}(\gamma)$ and, more generally, on $BV(\gamma)$. Applying the inequality (4.5), we obtain the estimate

$$\|f\|_{\mathrm{K},\gamma}\leqslant \bigl\|\,|D_Hf|_H\bigr\|_{L^1(\gamma)}.$$

Definition 6.8. Let $\alpha\in(0,1]$ and $p\in[1,\infty)$. A function $f\in L^p(\gamma)$ belongs to the Gaussian Nikolskii–Besov class $B_p^\alpha(\gamma)$ if there is a number C such that for every map $\Phi\in\mathscr{F}\mathscr{C}_0^\infty(X,H)$ we have

$$\int_X f\operatorname{div}_\gamma\Phi\,d\gamma\leqslant C\|\Phi\|_q^\alpha\|\operatorname{div}_\gamma\Phi\|_q^{1-\alpha},$$

where $1/p+1/q=1$. Let $V_\gamma^{p,\alpha}(f)$ be the infimum of such numbers C.

The quantity $U_\gamma^{p,\alpha}(f)$ was already introduced in Definition 6.3 in the infinite-dimensional case.

The quantities $V_\gamma^{p,\alpha}(f)$ and $U_\gamma^{p,\alpha}(f)$ do not change after adding constants to f, hence they can be evaluated on functions with zero integral with respect to the measure γ.

Theorem 6.9. *For every function $f\in B_p^\alpha(\gamma)$ with $p\in[1,\infty)$*

$$\|f-T_tf\|_p\leqslant 2^{1-\alpha}C(p)^\alpha c_t^\alpha V_\gamma^{p,\alpha}(f),$$

where c_t is defined by (2.8) and $C(p)$ is defined by (6.1).

Corollary 6.10. *For every function $f\in B_p^\alpha(\gamma)$ with $p\in[1,\infty)$*

$$\|f-\mathbb{E}f\|_p\leqslant 2^{1-2\alpha}\pi^\alpha C(p)^\alpha V_\gamma^{p,\alpha}(f).$$

Theorem 6.11. *For every function $f\in B_1^\alpha(\gamma)$ with zero integral with respect to γ*

$$\|f\|_1\leqslant 3[V_\gamma^{1,\alpha}(f)]^{1/(1+\alpha)}\|f\|_{\mathrm{K},\gamma}^{\alpha/(1+\alpha)}.$$

Theorem 6.12. *A function $f \in L^p(\gamma)$ with $p \in (1,\infty)$ belongs to the Gaussian Nikolskii–Besov class $B_p^\alpha(\gamma)$ if and only if $U_\gamma^{p,\alpha}(f) < \infty$. Moreover,*

$$U_\gamma^{p,\alpha}(f) \leqslant C(q)^{1-\alpha} V_\gamma^{p,\alpha}(f) \quad \text{and} \quad V_\gamma^{p,\alpha}(f) \leqslant (2C(p)+1) U_\gamma^{p,\alpha}(f),$$

where $1/p + 1/q = 1$ and $C(p)$ is defined by (6.1).

The classes $B_p^\alpha(\gamma)$ can be compared with the scales of Gaussian fractional Sobolev classes defined above (there are also connections with the scales in [169], which we do not consider here; in [169] one can find additional references).

Theorem 6.13. *Let $\alpha \in (0,1)$ and $p \in (1,\infty)$. For every $\beta < \alpha$*

$$H^{p,\alpha}(\gamma) \subset B_p^\alpha(\gamma) \subset \mathscr{E}^{p,\beta}(\gamma).$$

Moreover, there exist numbers $C_1 = C_1(p,\alpha,\beta)$ and $C_2 = C_2(p,\alpha)$, depending only on the indicated parameters, such that

$$\|f\|_{\mathscr{E}^{p,\beta}(\gamma)} \leqslant C_1 \|f\|_p \max\bigl\{1, [V_\gamma^{p,\alpha}(f)]^{\beta/\alpha} \|f\|_p^{-\beta/\alpha}\bigr\},$$
$$V_\gamma^{p,\alpha}(f) \leqslant C_2 \|f\|_{H^{p,\alpha}(\gamma)}.$$

Therefore, the continuous embeddings

$$H^{p,\alpha}(\gamma) \subset B_p^\alpha(\gamma) \subset H^{p,\alpha-\varepsilon}(\gamma)$$

hold for all $\varepsilon > 0$.

Kosov [115], [116] introduced a full scale of Besov classes with respect to Gaussian measures.

Definition 6.14. Let $f \in L^p(\gamma)$, and let

$$\omega_{\gamma,p}(f,\varepsilon) := \sup\biggl\{\int_{\mathbb{R}^n} \operatorname{div}_\gamma \Phi f \, d\gamma \colon \Phi \in \mathscr{F}\mathscr{C}_0^\infty(X,H),$$
$$\|\operatorname{div}_\gamma \Phi\|_{p/(p-1)} \leqslant 1,\ \|\Phi\|_{p/(p-1)} \leqslant \varepsilon\biggr\}.$$

The function $\omega_{\gamma,p}(f,\,\cdot\,)$ is continuous, concave, and increasing on $(0,\infty)$.

Definition 6.15. Let $\alpha \in (0,1]$, $p \in [1,\infty)$, and $\theta \in [1,\infty]$. The Gaussian Besov class $B_{p,\theta}^\alpha(\gamma)$ consists of all functions $f \in L^p(\gamma)$ such that

$$V_\gamma^{p,\theta,\alpha}(f) := \biggl(\int_0^\infty \bigl[r^{-\alpha}\omega_{\gamma,p}(f,r)\bigr]^\theta r^{-1}\,dr\biggr)^{1/\theta} < \infty.$$

For $f \in L^p(\gamma)$ and $p \in [1,\infty)$ also let

$$a_{\gamma,p}(f,t) := \biggl(\iint \bigl|f\bigl(e^{-t}x + \sqrt{1-e^{-2t}}\,y\bigr) - f(x)\bigr|^p \,\gamma(dx)\,\gamma(dy)\biggr)^{1/p}$$

and

$$A_\gamma^{p,\theta,\alpha}(f) := \biggl(\int_0^\infty \bigl[t^{-\alpha/2}a_{\gamma,p}(f,t)\bigr]^\theta t^{-1}\,dt\biggr)^{1/\theta}.$$

We observe that

$$\|f - T_t f\|_p \leqslant a_{\gamma,p}(f,t).$$

Now we present the two main results of [115] and [116].

Theorem 6.16. *The inequality*

$$A_\gamma^{p,\theta,\alpha}(f) \leqslant 2^{1-\alpha+1/\theta} C(p)^\alpha V_\gamma^{p,\theta,\alpha}(f)$$

holds for every function $f \in B^\alpha_{p,\theta}(\gamma)$ with $p \in [1,\infty)$. In addition, for $p \in (1,\infty)$ the converse assertion is true: if for a function $f \in L^p(\gamma)$ the quantity $A_\gamma^{p,\theta,\alpha}(f)$ is finite, then $f \in B^\alpha_{p,\theta}(\gamma)$ and

$$V_\gamma^{p,\theta,\alpha}(f) \leqslant 2^{-1/\theta}\biggl(1 + C\biggl(\frac{p}{p-1}\biggr)\biggr) A_\gamma^{p,\theta,\alpha}(f).$$

Theorem 6.17. *Let $p \in (1,\infty)$. For every function $f \in B^\alpha_{p,\theta}(\gamma)$ and every $\beta \in (0,\alpha)$ the function $|f|\,\bigl|\log|f|\,\bigr|^{\beta/2}$ belongs to $L^p(\gamma)$. Moreover, there exists a number $C = C(p,\theta,\alpha,\beta)$ depending only on the parameters p, θ, α, and β such that*

$$\biggl(\int_X |f|^p \biggl|\log\frac{|f|}{\|f\|_p}\biggr|^{p\beta/2} d\gamma\biggr)^{1/p} \leqslant C\bigl(\|f\|_p + V_\gamma^{p,\theta,\alpha}(f)\bigr).$$

In recent years, there have also been investigations of Gaussian analogues of the BMO and Hardy classes (see, for example, [135], [139], and [180]).

7. The maximal function

It has already been explained above that the continuity of the Ornstein–Uhlenbeck semigroup means convergence of $T_t f$ to f with respect to the norm of the corresponding space as $t \to 0$. However, in the case of integral norms this convergence does not imply convergence almost everywhere. The question of the latter turns out to be rather delicate. Let $f \in L^1(\gamma)$. In the infinite-dimensional case it remains an open question whether it is true that

$$\lim_{t\to 0} T_t f(x) = f(x) \quad \text{a.e.} \tag{7.1}$$

This problem is connected with the estimates

$$\gamma\Bigl(x\colon \sup_{t>0} |T_t f(x)| > R\Bigr) \leqslant cR^{-1} \tag{7.2}$$

for large R, that is, with the so-called weak (1-1)-estimates for the maximal function

$$Mf(x) := \sup_{t>0} |T_t f(x)|.$$

We recall that an operator T on the space $L^p(\gamma)$ with values in the space of measurable functions is of weak type (p-p) if the inequality

$$\gamma(x\colon |Tf(x)| > \lambda) \leqslant C\lambda^{-1/p}\|f\|_p, \qquad f \in L^p(\gamma), \quad \lambda > 0,$$

holds for some number C.

In the finite-dimensional case (7.1) is true, but is far from obvious. For $d = 1$ this was proved in [153], and the following general result was obtained in [190] (see also another proof in [146]).

Theorem 7.1. *The relation (7.1) holds for all $f \in L^1(\gamma_d)$. In addition, for every d there exists a number $c = c_d$ such that (7.2) holds.*

It is an open question whether the best possible numbers c_d are uniformly bounded.

In the infinite-dimensional case (7.1) holds if $f \in L^p(\gamma)$ with $p > 1$. This follows from a more general theorem due to Stein for diffusion semigroups (see [197], p. 70), where it is shown in addition that if $f \in L^p(\gamma)$, then for every $t > 0$ there is a version $\widetilde{T_t f}$ of the function $T_t f(\,\cdot\,)$ such that for every fixed x the function $t \mapsto \widetilde{T_t f}(x)$ is real analytic on $(0, \infty)$.

Theorem 7.2. *Let $p > 1$. Then there is a $C_p > 0$ such that for all $f \in L^p(\gamma)$*

$$\|Mf\|_p \leqslant C_p \|f\|_p.$$

Moreover,

$$\lim_{t \to 0} \widetilde{T_t f}(x) = f(x) \quad a.e.$$

If the function f is Borel measurable, then for $\widetilde{T_t f}(x)$ we can take $T_t f(x)$ for those x where the corresponding integral exists and also the function $LT_s f(x)$ of s is integrable on compact intervals in $(0, +\infty)$, and for all other points we set $T_t f(x) = 0$. In particular, if f is a bounded Borel function, then the desired version can be defined pointwise to be the function $T_t f(x)$ for almost all x, simultaneously for all t (but not always for all x; for example, if B is a Borel set, then $T_t I_B(0) = \gamma((1 - e^{-2t})^{-1/2} B)$ is not always continuous; see [25], Example 2.7.7). This can be seen from the equality

$$T_t f(x) = T_{1/k} f(x) + \int_{1/k}^{t} LT_s f(x)\, ds, \tag{7.3}$$

which is true for almost every x for all $k \in \mathbb{N}$ and $t \geqslant 1/k$. Indeed, Theorem 5.8 shows that $T_s f \in W^{2,2}(\gamma)$ and the function $LT_s f(x)$ is integrable on $X \times [1/k, t]$ for all $k \in \mathbb{N}$ and $t \geqslant 1/k$. Hence, the right-hand side of (7.3) is defined and continuous in t for almost all x. Consequently , every bounded continuous function belongs to the class $\mathscr{F}$ of bounded Borel functions f such that (7.3) is true for almost every x, for all $k \in \mathbb{N}$ and all $t \geqslant 1/k$ at once. It is clear that $\mathscr{F}$ is a linear space. If uniformly bounded functions $f_n \in \mathscr{F}$ converge pointwise to a function f, then $f \in \mathscr{F}$. Indeed, then the functions $(x, s) \mapsto LT_s f_n(x)$ on $X \times [1/k, T]$ for all $T > 1/k$ converge to the function $(x, s) \mapsto LT_s f(x)$ in $L^2(\gamma \otimes ds)$. Therefore, we can pass to a subsequence convergent almost everywhere. Then, for every fixed t, the integrals of $LT_s f_n(x)$ with respect to s in $[1/k, t]$ converge to the integral of $LT_s f_n(x)$ for almost all x, since by convergence in $L^2(\gamma \otimes ds)$ there exists a point $t_1 > t$ for which the integrals of $|LT_s f_n(x)|^2$ over $[1/k, t_1]$ converge to the integral of $|LT_s f_n(x)|^2$ for almost every x. This gives the uniform integrability of the functions $s \mapsto LT_s f_n(x)$

on $[1/k, t]$ and hence convergence of their integrals. By the monotone class theorem the set $\mathscr{F}$ contains all bounded Borel functions if X is a Souslin space. Everything reduces to this case due to the existence of a Souslin support of a Radon Gaussian measure. The version given by (7.3) is continuous with respect to t and is analytic with respect to t by the aforementioned theorem of Stein. By yet another passage to the limit the established property extends easily to unbounded Borel functions in $L^p(\gamma)$, provided that we define $T_t f(x)$ to be zero for points x at which the corresponding integral does not exist. For a proof we have to consider a non-negative function f and the bounded functions $f_n = \min\{f, n\}$.

As was shown in [26], Example 8.4.3,

$$\lim_{t\to+\infty} T_t f(x) = \int_X f(x)\,\gamma(dx) \quad \text{for } \gamma\text{-almost all } x.$$

A similar problem for another maximal function was negatively solved in [4], where it was shown that if C_d is the smallest number such that

$$\gamma\bigl(x\colon M_d f(x) > R\bigr) \leqslant C_d R^{-1}$$

for every function $f \in L^1(\gamma_d)$, where γ_d is the standard Gaussian measure on $\mathbb{R}^d$, and if we take the maximal function

$$M_d f(x) := \sup_{r>0} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)|\,\gamma_d(dy),$$

then $C_d \to \infty$ as $d \to \infty$.

Additional results and references relating to maximal functions for the Ornstein–Uhlenbeck semigroup can be found in [22], [23], [75], [82], [98], [100], [135], [136], [142], [144], [146], [192], [193], [199], and [210].

8. Perturbations of Ornstein–Uhlenbeck operators

A considerable number of papers have been devoted to the study of operators obtained by perturbing linear drifts of Ornstein–Uhlenbeck operators, that is, having the form

$$L_v f = Lf + \langle v, \nabla f\rangle$$

for some vector field v. Of course, in the finite-dimensional case every second-order operator with the Laplacian in the main part can be written in this form, but specific features of many problems are connected with additional conditions on the extra field $v(x)$ as compared to $-x$. For example, interesting problems arise even for bounded perturbations v for which the drift as a whole remains unbounded. In the infinite-dimensional case some additional special features are connected with the requirement that $v(x)$ must be an element of the Cameron–Martin subspace H. Since the element $-x$ almost surely does not belong to H in the case of infinite-dimensional H, the whole drift coefficient $b(x) = -x + v(x)$ acquires a very special structure.

For the new operator L_v a whole series of questions arises: in which space should it be considered, whether it generates a semigroup, the properties of this semigroup, the properties of its generator, whether the semigroup has invariant measures, the properties of such measures, and so on. There are also questions about the corresponding diffusion processes and stochastic equations, but we do not touch upon them here. We present a number of basic facts known about the operators L_v, and then mention some open questions of a not too special character. The following result from [38] gives an answer to a question posed by Shigekawa in [184].

Theorem 8.1. *Let $v\colon X \to H$ be a Borel map such that there exists a Borel probability measure μ for which*

$$|v|_H \in L^2(\mu), \qquad l(v) \in L^2(\mu) \quad \textit{for any } l \in X^*,$$

and the equality

$$L_v^*\mu = 0$$

holds in the sense of the identity

$$\int_X L_v f\, d\mu = 0 \quad \forall\, f \in \mathscr{F}\mathscr{C}^\infty(X). \tag{8.1}$$

Then the measure μ is absolutely continuous with respect to γ, and there exists a function $\psi \in W^{2,1}(\gamma)$ such that $\mu = \psi^2 \cdot \gamma$. Moreover,

$$\int_X |D_H\psi|_H^2\, d\gamma \leqslant \frac{1}{4}\int_X |v|_H^2\, d\mu.$$

The condition $l(v) \in L^2(\mu)$ is not required if equation (8.1) is defined via the class $\mathscr{F}\mathscr{C}_0^\infty(X)$.

Further results in this direction were obtained in [33] and [109]. So far there are no analogous results with estimates in L^p (see, however, the remark at the end of this section).

In the finite-dimensional case the following fact is known (see [34] or [35], § 1.5).

Theorem 8.2. *Let $b\colon \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field and let μ be a Borel measure on $\mathbb{R}^d$ with respect to which the function $|b|$ is locally integrable, that is, $|b| \in L^1_{\mathrm{loc}}(|\mu|)$. Suppose that μ satisfies the equation*

$$\Delta\mu - \operatorname{div}(b\mu) = 0 \tag{8.2}$$

in the sense of the identity

$$\int_{\mathbb{R}^d} [\Delta\varphi(x) + \langle b(x), \nabla\varphi(x)\rangle]\, \mu(dx) = 0 \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^d).$$

Then the measure μ is absolutely continuous with respect to Lebesgue measure. If $b(x) = -x$, then μ coincides with γ up to a constant factor.

It follows from this theorem that if $b(x) = -x + v(x)$, where the function $|v|$ is locally integrable with respect to μ and μ satisfies (8.2), then μ is absolutely continuous with respect to the Gaussian measure γ. Of course, in the infinite-dimensional case a precise analogue of this assertion even with a globally integrable field v is impossible, since even for $b(x) = -2x$ we obtain as a solution a Gaussian measure mutually singular with γ. But if v takes values in H, it was a long-standing open question as to whether just the integrability of $|v|_H$ with respect to the solution μ is sufficient to guarantee the absolute continuity of μ with respect to γ. A positive answer to this question has recently been given in a paper in preparation by the author with A.V. Shaposhnikov and S.V. Shaposhnikov, where it is shown that for $f = d\mu/d\gamma$ the function $f(\log f)^\alpha$ is γ-integrable for all $\alpha < 1/4$. It is not clear whether this bound on α is sharp. It was shown in [37] that in the finite-dimensional case the density of a probability solution can fail to belong to the class $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$; however, as stated in Theorem 8.2, this density always exists.

Various problems connected with perturbations of drifts of Ornstein–Uhlenbeck operators are considered in [3], [63], [73], [94], [171], [172], and [195]. There are also many papers on perturbations of Ornstein–Uhlenbeck operators by scalar potentials (see, for example, [21]).

9. Mehler-type semigroups

The Ornstein–Uhlenbeck semigroup has the following structure: the function $T_t f$ is a substitution of a linear operator into the convolution of the function f with the image of the measure γ under another linear operator. This can be written in the form

$$T_t f = (f * \gamma \circ S_t^{-1}) \circ A_t, \quad \text{where} \quad S_t x = \sqrt{1-e^{-2t}}\, x, \quad A_t x = e^{-t} x,$$

and can also be rewritten in terms of the action on measures. This leads to the idea of considering more general convolution semigroups of measures of the form

$$\mu_{t+s} = \mu_s * (\mu_t \circ T_s^{-1}), \qquad t, s \geqslant 0, \tag{9.1}$$

where $\{\mu_t\}_{t\geqslant 0}$ is a family of Borel probability measures on a separable Banach space X (or on a more general locally convex space), $\{T_t\}_{t\geqslant 0}$ is a semigroup of linear operators on X (in typical examples, a strongly continuous semigroup). In this case the given family of measures is called a $\{T_t\}$-convolution semigroup. For $T_t \equiv I$ we obtain the usual semigroup of measures with the relation

$$\mu_{t+s} = \mu_s * \mu_t.$$

On bounded Borel functions we obtain a Markov semigroup of operators

$$P_t f(x) = \int_X f(T_t x + y)\, \mu_t(dy),$$

which is called a generalized Mehler semigroup. It is shown in Lemma 2.1 of [39] that if $\{T_t\}_{t\geqslant 0}$ is a strongly continuous semigroup of operators on a separable Banach space, and $\{\mu_t\}$ is a $\{T_t\}$-convolution semigroup of measures such that the map $t \mapsto \mu_t$ is continuous in the weak topology on the space of measures, then the function $P_t f(x)$ is continuous with respect to (t, x) for every bounded continuous function f.

We mention the following result from [182]. Recall that a probability measure μ is said to be infinitely divisible if, for every n, it can be represented as the n-fold convolution of some probability measure.

Proposition 9.1. *Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous operator semigroup on a separable Banach space. Then the measures μ_t in a $\{T_t\}$-convolution semigroup are infinitely divisible.*

Even if $T_t \equiv I$, the measures in a convolution semigroup satisfying the identity $\mu_{t+s} = \mu_t * \mu_s$ need not be continuous with respect to t in the weak topology. As an example we can take the Dirac measures $\mu_t = \delta_{a(t)}$, where $a(t)$ is some discontinuous solution to the equation $a(t+s) = a(t) + a(s)$ (such a solution is constructed by means of a Hamel basis in the space $\mathbb{R}$ over the field of rational numbers). In [182] and [61] the question was considered of the continuity and absolute continuity of the functions $t \mapsto \widetilde{\mu_t}$ for the Fourier transforms of measures in a $\{T_t\}$-convolution semigroup. We recall that the Fourier transform of a measure μ on a locally convex space X is the complex function

$$\widetilde{\mu}(l) = \int_X e^{il}\,d\mu, \qquad l \in X^*,$$

on the topological dual space. Note that in terms of Fourier transforms the equality (9.1) is written in the form

$$\widetilde{\mu_{t+s}}(l) = \widetilde{\mu_s}(l)\widetilde{\mu_t}(T_s^* l).$$

As shown in [39], if for every $l \in X^*$ there exists a finite derivative

$$\lambda(l) = -\frac{d}{dt}\widetilde{\mu_t}(l)\bigg|_{t=0},$$

then

$$\widetilde{\mu_t}(l) = \exp\biggl(-\int_0^t \lambda(T_s^* l)\,ds\biggr).$$

However, the condition of differentiability at zero is not always fulfilled even in the absence of Dirac components (see examples in [61]).

According to the well-known Lévy–Khintchine theorem, the Fourier transform of an infinitely divisible measure μ on a separable Banach space has the form

$$\widetilde{\mu} = \exp(-\psi(l)), \qquad \psi(l) = -l(b) + \frac{1}{2}R(l,l) - \int K(l,x)\,M(dx),$$

where $b \in X$ is a constant vector, $R(l,l)$ is the covariance of a centred Gaussian measure γ,

$$K(l,x) = e^{il(x)} - 1 - il(x)I_U(x), \qquad U \text{ is the unit ball},$$

and M is the so-called Lévy measure on $X \setminus \{0\}$. Thus, the measure μ is expressed as the convolution of three components:

$$\mu = \delta_b * \gamma * \Lambda, \tag{9.2}$$

where δ_b is the Dirac measure at the point b, γ is a centred Gaussian measure, and the measure Λ can be chosen so that no Gaussian component can be extracted from it. Then the indicated decomposition will be uniquely determined. The measure Λ is called the purely jump component. It is straightforward to verify that for a $\{T_t\}$-convolution semigroup $\{\mu_t\}$ the Gaussian components γ_t of the measures μ_t also form a $\{T_t\}$-convolution semigroup. However, for the shifts b_t and the jump components Λ_t this can be false. It is shown in [182] that for all $l \in X^*$ the functions $t \mapsto \widetilde{\gamma_t}(l)$ and $t \mapsto \widetilde{\Lambda_t}(l)$ are continuous. Thus, the lack of continuity of $\widetilde{\mu_t}$ can only be caused by the Dirac component δ_{b_t} (as noted above, its discontinuity is possible if there are no additional conditions). In [61] the situation is considered where the component with the shift b_t is absent and the Lévy measures M_t in the decomposition (9.2) for μ_t are such that the measures $\min\{\|x\|, \|x\|^2\}\, M_t(dx)$ are finite. In this case, according to Theorem 2.3 in [61], the functions $t \mapsto \widetilde{\mu_t}(l)$ are absolutely continuous on intervals $[0, T]$, although the derivative at zero does not always exist.

In [39] the following fact is proved, which is useful for constructing Mehler-type semigroups on infinite-dimensional spaces.

Theorem 9.2. *Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous semigroup of operators on a separable Hilbert space H. Suppose that $\|T_{t_0}\|_{\mathscr{L}(H)} < 1$ for some $t_0 \geqslant 0$. Then H can be embedded by means of an injective Hilbert–Schmidt operator with a dense range into a separable Hilbert space E such that $\{T_t\}_{t\geqslant 0}$ extends to a strongly continuous contraction semigroup $\{T_t^E\}_{t\geqslant 0}$ on E.*

Moreover, a finite collection of commuting semigroups satisfying the stated condition can be simultaneously extended to such a space.

Finally, if the condition $\|T_{t_0}\|_{\mathscr{L}(H)} < 1$ is omitted, then the indicated extension of the semigroup still exists, but it need not be contracting. One can guarantee that H will be embedded by means of a Hilbert–Schmidt operator not only in E but also in the domain of the generator of the semigroup $\{T_t^E\}_{t\geqslant 0}$ on E.

In [39] extensions of semigroups were used to construct generalized Mehler semigroups in the following theorem. We recall that a cylindrical probability measure on a Hilbert space H is an additive set function ν on the algebra of cylinders in H (pre-images of Borel sets under continuous linear maps to finite-dimensional spaces) for which the finite-dimensional projections are countably additive, that is, one has countably additive functions

$$B \mapsto \nu(P^{-1}(B)), \qquad B \in \mathscr{B}(\mathbb{R}^n),$$

where P is a continuous operator from H to $\mathbb{R}^n$. The Fourier transform $\widetilde{\nu}$ of the measure ν arises naturally, and the value of it on a vector y is defined as the integral of $\exp(i(x, y))$ with respect to ν, understood as the integral of $\exp(it)$ with respect to the measure on the real line that is the image of ν under the functional $x \mapsto (x, y)$. Such a measure can fail to be countably additive on the full algebra of cylindrical sets, as is shown by the example of the 'standard cylindrical Gaussian measure' on H, which has the Fourier transform $\exp(-|y|^2/2)$. Its finite-dimensional projections under orthogonal projection operators are standard Gaussian measures on the ranges of the projection operators.

Theorem 9.3. *Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous semigroup of operators on a separable Hilbert space H, and let $\{\mu_t\}_{t\geqslant 0}$ be a family of cylindrical probability measures on H with continuous Fourier transforms satisfying the relation*

$$\mu_{t+s} = (\mu_t \circ T_s^{-1}) * \mu_s,$$

where $\mu_t \circ T_s^{-1}$ is the image of the measure μ_t under the map T_s. Let E be the Hilbert space in the last assertion of the previous theorem. Then the measures μ_t admit countably additive extensions to E, denoted by μ_t^E, and the family $\{p_t\}_{t\geqslant 0}$ defined on bounded Borel functions on E by

$$p_t f(x) := \int f(T_t^E x - y)\, \mu_t(dy) = (\mu_t^E * f)(T_t^E x), \qquad x \in E,$$

is a generalized Mehler semigroup on E.

The generator of such a semigroup is not always a differential operator, but turns out to be a pseudodifferential operator (see [36]).

Mehler-type semigroups are considered in many papers (see [8], [9], [39], [60], [61], [69], [78], [103], [113], [125], [126], [132], [160], [181], [182]).

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