STATISTICAL DATA ANALYSIS AND INFERENCE
Y. Dodge (Editor)

A VIEW OF HARMONIZABLE PROCESSES

M. M. Rao

Department of Mathematics
University of California, Riverside
Riverside, California 92521

This paper contains an account of harmonizable processes, the structure of their spectral domains in many cases, and certain results on sampling and statistical inference problems. Also some potential avenues for future research in the area are indicated.

1. INTRODUCTION

One of the motivations for the concept of harmonizability is to enlarge the applications of stationary processes while retaining the powerful Fourier analytic methods that play a key role in the latter studies. The flexibility afforded here may be explained directly as follows. If $\{X_n, -\infty < n < \infty\}$ is a sequence of (complex) random variables with means zero and covariances $r(m,n) = E(X_m \bar{X}_n)$, then it is (weakly or Khintchine) stationary if $r(m;n) = \tilde{r}(m-n)$. A simple example of such a process is an orthonormal sequence of $L^2(\Omega, \Sigma, P)$. If one considers a truncation of such a family $\{Y_n, -\infty < n < \infty\}$, where $Y_n = X_n$, $n > 0$, and $Y_n = 0$ for $n \leq 0$, then the new sequence is not stationary but turns out to be a harmonizable process. More generally, if $Y_n = TX_n$, $-\infty < n < \infty$, where T is any bounded linear transformation on $L^2(P)$ $(= L^2(\Omega, \Sigma, P))$, then the new process is always harmonizable. Thus it is advantageous to study this enlarged class, characterize and classify its subclasses from a theoretical point of view. It is equally useful to know the applications of the enlarged class. Thus the following is an account of the results on some recent work including a fair amount of some previously unpublished material on the subject. Here is a brief outline.

After introducing various classes of harmonizability in the next section, a detailed account of the spectral domain analysis of some of the classes is given in Section 3. This is useful since the emphasis in harmonizable as well as stationary classes is on the spectral (or frequency) domain so as to use the powerful methods of the classical Fourier analysis in contrast to the general time domain studies. Then Section 4 treats a problem of sampling the process in a way that allows a reconstruction of the process itself. A brief discussion is given in Section 5 on a computation of spectral functions from data, and an asymptotically unbiased estimator of the spectral bimeasure of a harmonizable family is included. The related class(KF) is also discussed there. Some statistical inference questions on harmonizable processes with low multiplicity are discussed in Section 6, and the final section contains a sketch of a few other asymptotic results (laws of large numbers) as well as certain indications of possible directions for future work.

2. HARMONIZABILITY AND RELATED CHARACTERIZATIONS

A first extension of stationarity was given by Loève in the late 1940's, and a readable account is included in his book [16], under the name "harmonizability." This turns out to be a special case of a general concept due to Bochner [3] called "V-boundedness," and also one of Rozanov's [25] which was again termed "harmonizable." The last two concepts can be shown to be equivalent for second order processes, and they extend Loève's concept. For this reason the Loève concept is now called strongly harmonizable, and the more general one, weakly harmonizable. These are stated precisely as follows:

2.1. Definition

A second order process $\{X_t, t \in G\}$ with mean $E(X_t) = 0$, and continuous covariance $r(s,t) = E(X_s\bar{X}_t)$, is *strongly harmonizable* if r can be expressed as: ($G = \mathbb{R}$ or $\mathbb{Z}$)

$$r(s,t) = \int_{\widehat{G}}\int_{\widehat{G}} e^{is\lambda - it\lambda'}\rho(d\lambda, d\lambda'), \qquad s,t \in G, \tag{1}$$

where $\rho(\cdot,\cdot)$ is a positive (semi-)definite function of finite Vitali variation on $\hat{G}\times\hat{G}$ with $\hat{G} = \mathbb{R}$ or $[-\pi,\pi)$ respectively. The process is *weakly harmonizable* if r has the representation (1) but that ρ satisfies the same conditions except that it has finite Fréchet variation instead of Vitali's variation.

To avoid misunderstanding let us recall the last three terms. A function $\rho : G \times G \to \mathbb{C}$ is positive semi-definite if $\rho(s,t) = \overline{\rho(t,s)}$ and for any $a_i \in \mathbb{C}$, $1 \le i \le n$, $n \ge 1$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{a}_j\rho(s_i,s_j) \ge 0, \qquad s_i \in G, \tag{2}$$

and positive definite if, moreover, $= 0$ holds only when all $a_i = 0$. Letting $(\Delta_{hk}\rho)(s,t) = \rho(s+h,t+k) - \rho(s,t+k) - \rho(s+h,t) + \rho(s,t)$, the increment, ρ is of Vitali ($=$ ordinary) variation finite if

$$\sup_{\pi,\pi'}\sum_{i=1}^{n}\sum_{j=1}^{n'} |(\Delta_{h_ik_j}\rho)(s_i,t_i)| < \infty \tag{3}$$

where $\pi : s_1 < s_2 < \cdots < s_{n+1}$, $\pi' : t_1 < t_2 < \cdots < t_{n'+1}$ from G ($= \mathbb{R}$ or $\mathbb{Z}$), and $0 \le h_i \le s_{i+1} - s_i$, $0 \le k_i \le t_{i+1} - t_i$; and ρ has Fréchet variation finite if (3) is replaced by

$$\sup_{\pi,\pi'}\{|\sum_{i=1}^{n}\sum_{j=1}^{n'} a_i\bar{b}_i(\Delta_{h_ik_j}\rho)(s_i,t_j)| : |a_i|\le 1, |b_j|\le 1\} < \infty. \tag{4}$$

It is clear that Vitali variation is not less than Fréchet's, and the converse implication is however false. Actually it can be shown that for ρ satisfying (2), (4) always holds but (3) does not, so that weak harmonizability is more general than its strong counterpart.

Since for the case of weak harmonizability ρ is not of finite Vitali variation, the integral in (1) cannot be taken in the usual Lebesgue sense, and a different definition is necessary. The appropriate concept is fortunately available from the work of M. Morse and W. Transue of middle 1940's, and thus (2) is interpreted as the MT-integral. But this is not an absolute integral in contrast to the Lebesgue concept, and the dominated convergence theorem is not valid for it. However, a slightly restricted version remedies this defect, and it is detailed in [6]. Even then the Jordan decomposition is not valid. But these problems seem unavoidable if a generalized concept such as (1) is to be employed for realistic applications. One may turn to the last reference for omitted details; but its results will be used below. See also [17].

If the process is stationary, then $r(s,t) = \tilde{r}(s-t)$ and $\rho(\cdot)$ of (1) is supported on the diagonal $s = t$ of $\hat{G}\times\hat{G}$, so that one has

$$r(s,t) = \int_{\widehat{G}} e^{i(s-t)\lambda}\rho(d\lambda), \qquad s,t\in G. \tag{5}$$

Here $\rho(\cdot)$ is a nonnegative, bounded, and nondecreasing function. It is called the spectral function of the process. In the more general harmonizable case also the possibly complex valued positive semi-definite $\rho(\cdot,\cdot)$ is again termed the *spectral (bi)measure* of the process. These functions reflect several properties of the covariance function $r(\cdot,\cdot)$, but have a better potential for applications than r, and a deeper analysis is possible by employing the finer Fourier techniques with them.

A close relation exists between harmonizability and stationarity. This and related properties are given by the following fundamental characterizations which are also used in some applications.

2.2. **Theorem**

Let $\{X_t, t\in G\}\subset L_0^2(P)$ $(= L_0^2(\Omega,\Sigma,P)$ of centered random variables) be weakly harmonizable. Then there exists a probability space $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$ [possibly an enlarged one than the original (Ω,Σ,P)], an orthogonal projection Q on $L_0^2(\tilde{P})$ onto $L_0^2(P)$, and a stationary process $\{Y_t, t\in G\}\subset L_0^2(\tilde{P})$ $(\supset L_0^2(P))$ such that $X_t = QY_t, t\in G$.

Conversely, if $\{Y_t, t \in G\} \subset L_0^2(P)$ is any stationary or (weakly) harmonizable process and T is any bounded linear operator on $L_0^2(P)$, then $X_t = TY_t, t \in G$, is weakly (but not necessarily strongly) harmonizable.

By this result one notes that the Y_n-sequence of the introduction with T as the orthogonal projection onto the closed linear span of $\{X_n, n > 0\}$, denoted $\overline{\text{sp}}\{X_n, n > 0\}$, is weakly harmonizable. It can be shown that the Y-process is not strongly harmonizable. This is an important motivation for the general study. Also the above theorem implies that the class of weakly harmonizable processes is closed under (bounded) linear transformations. The proof of this result in the forward direction depends on a relatively deep theorem of Grothendieck while the reverse direction is a consequence of the following representation theorem and a few properties of the Dunford and Schwartz theory of integration relative to vector measures.

2.3. **Theorem**

Let $\{X_t, t \in G\}$ be a second order process whose covariance function is bounded and continuous. Then the process is weakly harmonizable iff there is a σ-additive function Z on the Borel σ-algebra $\mathcal{B}(\hat{G})$ of $\hat{G}$ into $L_0^2(P)$ such that

$$\text{(i)} \quad E(Z(A)\bar{Z}(B)) = \rho(A, B) \quad \left(= \int_A \int_B \rho(d\lambda, d\lambda') \right) \tag{6}$$

$$\text{(ii)} \qquad X_t = \int_{\widehat{G}} e^{it\lambda} Z(d\lambda), \quad t \in G, \tag{7}$$

where the integral in (7) is in the Dunford-Schwartz sense and ρ is as in Definition 1. In particular, the X_t-process is strongly harmonizable iff ρ of (6) is of bounded Vitali variation, and stationary iff $Z(\cdot)$ has orthogonal values (i.e., $E(Z(A)\bar{Z}(B)) = 0$ for $A \cap B = \emptyset$). In all cases the representation (7) holds, and X_t is uniformly continuous on compacts of G.

A detailed proof of the above results can be obtained, for instance, from [21] where related references to the works of others may be found. Let us present an example of a weakly harmonizable process. Its verification can be seen from the same reference.

Example. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ be a contractive linear operator. If A^* is its adjoint and $x_0 \in \mathcal{H}$ is any element, define

$$Y_n = A^n x_0, \; n \geq 0; \text{ and } = (A^*)^n x_0 \text{ if } n < 0. \tag{8}$$

Then $\{Y_n, -\infty{<}n{<}\infty\}$ is weakly harmonizable.

In case A is unitary in (8), the corresponding Y-process is stationary. This illustration actually shows a deep relation between the harmonizability study and the Hilbert space operator theory. In the continuous parameter case $\{A^n, n \geq 0\}$ is replaced by a weakly continuous contraction semi-group of operators there.

Another useful weak harmonizability characterization will be presented before turning to an analysis of the spectral properties. Abstracting the characterization of continuous positive semi-definite (point) functions on G as Fourier transforms of positive bounded measures, Bochner introduced the fundamental concept of "V-boundedness" which enables a similar characterization of signed (and more generally vector) measures from their Fourier transforms. Its $L_0^2(P)$-version is relevant here. Thus a measurable mapping (or process) $X : G \to L_0^2(P)$ is called ***V-bounded*** if the following two conditions are met:

(i) $\|X(t)\|_2^2 = E(|X_t|^2) \leq M < \infty,\ t \in G$,

(ii) $\|\int_G X(t)\overline{\varphi(t)}dt\|_2 \leq C \sup_{\lambda \in \widehat{G}} |\hat{\varphi}(\lambda)|,\ \varphi \in L^1(G)$,

where $\hat{\varphi}$ is the Fourier transform of the Lebesgue integrable φ on G and the (vector) integral in (ii) is in the Bochner sense, $0 < C < \infty$ being an absolute constant.

With this concept, the desired result is given by:

2.4. **Theorem**

Let $\{X_t, t \in G\} \subset L_0^2(P)$ be a process. Then it is weakly harmonizable iff it is V-bounded and has a continuous covariance.

Such a compact statement is not possible for the strongly harmonizable case. Several related important results on V-bounded and other processes are in [3]. This concept is useful even when the index set G is a noncommutative locally compact group. An extension of the above theorem for unimodular G and related results with references and necessary details can be found in [24]. (See also [30].)

3. SPECTRAL DOMAINS OF A CLASS OF HARMONIZABLE PROCESSES

Let $\{X_t, t \in G\}$ be a harmonizable process whose covariance function r admits the representation (1) with ρ as its spectral measure. Here ρ is taken to be positive definite so that for each pair of bounded Borel functions $f, g : \hat{G} \to \mathbb{C}$, one may define

$$(f, g) = \int_{\widehat{G}}\int_{\widehat{G}} f(\lambda)\overline{g(\lambda')}\rho(d\lambda, d\lambda') \tag{9}$$

and $(\cdot,\cdot)$ is then an inner product on $B(\hat{G}, \mathbb{C})$, the space of bounded complex Borel functions. Here the integral is in the MT-sense when the process is weakly harmonizable and in Lebesgue's sense if the process is strongly harmonizable. Define

$$\mathcal{L}^2(\rho) = \{f \in \mathbb{C}^{\hat{G}} : f\text{-Borel}, (f, f) = \|f\|^2 < \infty\}. \tag{10}$$

Then $\mathcal{L}^2(\rho)$ is called the *spectral domain space* of the process. In what follows the strongly harmonizable case is considered, so that the integral in (9) is in the Lebesgue sense and the standard computations can be used. The more general case needs the MT-integral and a theorem of Grothendieck, already noted; but the special case clarifies the analytical problems involved for $\mathcal{L}^2(\rho)$.

Thus if $\{X_t, t \in G\}$ is strongly harmonizable, with $G = \mathbb{R}$ or $\mathbb{Z}$, let ρ be its spectral measure on $\hat{G} \times \hat{G}$ which is of bounded (Vitali) variation. Then the hermitian property of ρ implies that $|\rho|(A, B) = |\rho|(B, A)$ where $\rho(\cdot, \cdot)$ denotes its variation measure on $\mathcal{B}(\hat{G}) \otimes \mathcal{B}(\hat{G})$ the product Borel σ-algebra of $\hat{G} \times \hat{G}$. Define $\beta : \mathcal{B}(\hat{G}) \to \mathbb{R}^+$ as:

$$\beta(A) = \tfrac{1}{2}\{|\rho|(A, \hat{G}) + |\rho|(\hat{G}, A)\} = |\rho|(A, \hat{G}), \ A \in \mathcal{B}(\hat{G}). \tag{11}$$

Consider the Lebesgue space $L^2(\rho)$ of ρ-square integrable scalar functions on $\hat{G} \times \hat{G}$, i.e., $f \in L^2(\rho)$ iff

$$\|f\|_2^2 = \int_{\widehat{G}\times\widehat{G}} |f(s,t)|^2 |\rho|(ds, dt) < \infty.$$

Then $L^2(\rho)$ is a complete normed linear space under the norm $\|\cdot\|_2$ above (cf. [12], III.3.1 and IV.3.6). Since $|\rho|(\hat{G} \times \hat{G}) < \infty$, a function $\varphi : \hat{G} \to \mathbb{C}$, $\mathcal{B}(\hat{G})$-measurable, can be identified with $\tilde{\varphi} = \varphi \times 1 : \hat{G} \times \hat{G} \to \mathbb{C}$ and it belongs to $L^2(\rho)$ iff $|\varphi|^2$ is integrable relative to β of (11). In fact,

$$\int_{\widehat{G}\times\widehat{G}} |\tilde{\varphi}|^2 |\rho|(ds, dt) = \int_{\widehat{G}} |\varphi(s)|^2 \beta(ds). \tag{12}$$

If one considers the subclass of $L^2(\rho)$, of elements such as φ above (i.e., of functions of one variable only), then

$$\int_{\widehat{G}}\int_{\widehat{G}} \varphi(s)\overline{\varphi(t)}\rho(ds, dt)$$

exists iff

$$\int_{\widehat{G}}\int_{\widehat{G}} |\varphi(s)\overline{\varphi(t)}||\rho|(ds, dt) < \infty,$$

and this is true iff $\varphi \in L^2(\rho)$. Thus in our case of positive definite ρ, $\mathcal{L}^2(\rho)$ is defined also for $\varphi \in L^2(\rho)$. Then $\mathcal{L}^2(\rho) \subset L^2(\rho)$, since (φ, φ) in (9) is undefined otherwise. But by (12) such functions are a subclass of $L^2(\beta) \subset L^2(\rho)$, so that $\mathcal{L}^2(\rho) \subset L^2(\beta)$.

To use this relation, consider for $f \in L^1(\beta)$ and $B \in \mathcal{B}(\hat{G})$, the identity:

$$\int_B f(\lambda)\beta(d\lambda) = \frac{1}{2}\int_B\int_B f(s)|\rho|(ds, dt) + \frac{1}{2}\int_B\int_B f(t)|\rho|(ds, dt).$$

If $f \in L^2(\beta)$, then one has with (9)

$$
\begin{aligned}
0 \le B(f,f) &= \int_{\widehat{G}}\int_{\widehat{G}} f(s)\bar{f}(t)\rho(ds,dt) \le \int_{\widehat{G}}\int_{\widehat{G}} |f(s)||\bar{f}(t)||\rho|(ds,dt) \\
&\le \frac{1}{2}\int_{\widehat{G}}\int_{\widehat{G}} [|f(s)|^2 + |f(t)|^2]|\rho|(ds,dt) \\
&= \int_{\widehat{G}} |f(s)|^2 \beta(ds).
\end{aligned} \tag{13}
$$

Thus for f, g of the above type

$$
\begin{aligned}
|B(f,g)| &\le \int_{\widehat{G}} |f(s)\overline{g(t)}|\beta(ds), \quad \text{by (13)} \\
&\le \|f\|_{2,\beta}\|g\|_{2,\beta}\,,
\end{aligned} \tag{14}
$$

by the CBS inequality. Hence the sesquilinear form $B(\cdot,\cdot)$ on $L^2(\beta) \times L^2(\beta)$ is positive definite, bounded, and so has a unique extension onto the latter space with the same properties. Denoting this by the same symbol, one has by a classical representation theorem (cf. [26], p. 63)

$$
B(f,g) = (Af,g) = \int_{\widehat{G}} (Af)(s)\overline{g(s)}\beta(ds), \tag{15}
$$

for a unique $A : L^2(\beta) \to L^2(\beta)$. Further the positive definiteness of $B(\cdot,\cdot)$ is equivalent to that of A. So by (14) and (15), A is a positive definite bounded operator and so has a unique square root $A^{\frac{1}{2}}$ with the same properties. Thus our positive definiteness hypothesis of ρ translates to that of $A^{\frac{1}{2}}$ in the Hilbert space $L^2(\beta)$ on which it acts. But this means (see the classic [26], Def. 2.14, p. 56) $(A^{\frac{1}{2}})^{-1}$ exists and is bounded, and this is the form that is used in the work below. It may be noted that the latter is actually a consequence of the positive definiteness of A on $L^2(\beta)$. This can be "justified" from the following computation.

The positive definiteness of A on $L^2(\beta)$ implies the existence of a constant $C > 0$ such that $(Af,f) \ge C\|f\|^2_{2,\beta}$, $f \in L^2(\beta)$.

For, by the homogeneity of this expression, one can take $\|f\|_2 = 1$. Let $T = \|A\|$, the operator bound. Then $0 < T < \infty$ since the trivial case that $A = 0$ can be excluded. If $\{E(\lambda), 0 \le \lambda \le T\}$ is the resolution of the identity of A, let $\mu_f(\cdot) = (E(\cdot)f,f)$. Then $0 \le \mu_f(\cdot) \le T$ is an increasing left continuous function with $\mu_f(T) = 1$, i.e., a distribution function. By the classical spectral theorem (cf. [26], Thm. 5.9), one has

$$
(Af,f) = \int_0^T \lambda d\mu_f(\lambda).
$$

Suppose now that no such $C > 0$ in the positive definiteness definition exists. Then $\inf\{(Af, f) : \|f\|_{2,\beta} = 1\} = 0$, so there is a sequence f_n, $\|f_n\|_{2,\beta} = 1$, and $\lim_n (Af_n, f_n) = 0$. If $\{\mu_{f_n}, n \geq 1\}$ is the corresponding set of distribution functions, by the Helly selection principle there exists a subsequence $\mu_{f_{n_k}} \to \mu_0$, a nondecreasing function, at all continuity points of the latter. Since the domain here is a compact interval, it follows that

$$\lim_{k\to\infty} \mu_{f_{n_k}}(0) = 0 = \mu_0(0) \quad \text{and} \quad \lim_{k\to\infty} \mu_{f_{n_k}}(T) = 1 = \mu_0(T).$$

Hence by the Helly-Bray lemma (cf. [16], p. 180)

$$0 = \lim_{k\to\infty} (Af_{n_k}, f_{n_k}) = \lim_{k\to\infty} \int_0^T \lambda d\mu_{f_{n_k}}(\lambda) = \int_0^T \lambda d\mu_0(\lambda). \tag{16}$$

Also using integration by parts appropriately, one has with (16)

$$T\mu_0(T) = \lim_{k\to\infty} T\mu_{f_{n_k}}(T) = \lim_{k\to\infty} \int_0^T \mu_{f_{n_k}}(\lambda) d\lambda = \int_0^T \mu_0(\lambda) d\lambda. \tag{17}$$

It follows from (17) that

$$0 \leq \int_0^T [\mu_0(T) - \mu_0(\lambda)] d\lambda = 0, \tag{18}$$

and since μ_0 is increasing, this implies $\mu_0(\lambda) = \mu_0(T) = 1$, for almost all λ. But then for each $\varepsilon > 0$, a continuity point of μ_0, using (16) one has

$$0 = \int_0^T \lambda d\mu_0(\lambda) \geq \int_\varepsilon^T \lambda d\mu_0(\lambda) > \varepsilon\mu_0(\varepsilon) = \varepsilon > 0.$$

This contradiction shows that the initial assumption is false and there is a $C > 0$ such that $(Af, f) \geq C\|f\|_{2,\beta}^2$.

Thus resuming the analysis of (15), it follows that $(A^{\frac{1}{2}})^{-1}$ exists as a bounded operator on $L^2(\beta)$, and $\|f\|^2 = (Af, f) = \|A^{\frac{1}{2}} f\|_{2,\beta}^2$ defines another norm on $\mathcal{L}^2(\rho)$. If $\{f_n, n \geq 1\}$ in $\mathcal{L}^2(\rho)$ is a Cauchy sequence, then $\|f_n - f_m\| = \|A^{\frac{1}{2}}(f_n - f_m)\|_{2,\beta} \to 0$, as $n, m \to \infty$. Since $L^2(\beta)$ is complete, there is a $g \in L^2(\beta)$ such that $A^{\frac{1}{2}} f_n \to g$ in $L^2(\beta)$. Define $f = (A^{\frac{1}{2}})^{-1} g$. This is unambiguous by the preceding analysis, and

$$\|f\| = \|A^{\frac{1}{2}} f\|_{2,\beta} = \|g\|_{2,\beta} < \infty. \tag{19}$$

Hence $f \in \mathcal{L}^2(\rho)$ and

$$\|f_n - f\| = \|A^{\frac{1}{2}}(f_n - f)\|_{2,\beta} = \|A^{\frac{1}{2}} f_n - g\|_{2,\beta} \to 0,$$

as $n \to \infty$. So f is the limit in $\mathcal{L}^2(\rho)$ of the Cauchy sequence $\{f_n, n \geq 1\}$. This establishes the following since the linearity of $\mathcal{L}^2(\rho)$ is evident:

3.1. Theorem

The spectral domain space $\mathcal{L}^2(\rho)$ of a strongly harmonizable process (with a positive definite spectral measure ρ) is a complete inner product space.

A natural question here is this: Can the positive definiteness of ρ be relaxed to positive semi-definiteness? The stronger hypothesis was needed to establish (19) in using the fact that the domain of $(A^{\frac{1}{2}})^{-1}$ contains g so that f is well defined. Suppose that the positive semi-definite A of (15) has a closed range. This was automatic in the previous case since A^{-1} exists and is bounded there, and it will be shown that the theorem is true with essentially the same proof if the range $\mathcal{R}_A$ is closed.

Indeed by ([2], Thm. 2, p. 475), $A = P_{\mathcal{R}_A}\tilde{A}$ where $P_{\mathcal{R}_A}$ is the orthogonal projection onto the closed range $\mathcal{R}_A$ of A, and $\tilde{A}$ is a bounded operator with a bounded inverse on $L^2(\beta)$ (cf. also the discussion in [2] on p. 476 about the dimensional equivalence of $\mathcal{R}_A^{\perp}$ and $\mathcal{N}_A$, the null space of A). If $A^+ = P_{\mathcal{N}_A^{\perp}}\tilde{A}^{-1}$, then A^+ is a generalized inverse of A and $AA^+ = P_{\mathcal{R}_A}$. These facts allow us to modify the above proof to get the same conclusion.

Thus if $\{f_n, n \geq 1\} \subset \mathcal{L}^2(\rho)$ is a Cauchy sequence as before, with $g_n = A^{\frac{1}{2}}f_n$, $\|f_n - f_m\| = \|g_n - g_m\|_{2,\beta} \to 0$, let g be the limit in $L^2(\beta)$. Since by the current hypothesis $\mathcal{R}_{A^{1/2}}$ is also closed, let B^+ be the generalized inverse of $A^{\frac{1}{2}}$, and then $g \in \mathcal{R}_{A^{1/2}}$. So let $f = B^+g$. Since the domain of B^+ is $L^2(\beta)$, this is well defined, and

$$\|f\| = \|A^{\frac{1}{2}}(B^+g)\|_{2,\beta} = \|P_{\mathcal{R}_{A^{1/2}}}g\|_{2,\beta} \leq \|g\|_{2,\beta} < \infty,$$

since $P_{\mathcal{R}_{A^{1/2}}}$, being an orthogonal projection on $L^2(\beta)$, is a contraction. So $f \in \mathcal{L}^2(\rho)$. Also $A^{\frac{1}{2}} = P_{\mathcal{R}_{A^{1/2}}}A^{\frac{1}{2}}$, and hence

$$\begin{aligned}\|f_n - f\| &= \|A^{\frac{1}{2}}(f_n - f)\|_{2,\beta} = \|P_{\mathcal{R}_{A^{1/2}}}A^{\frac{1}{2}}f_n - A^{\frac{1}{2}}B^+g\|_{2,\beta} \\ &= \|P_{\mathcal{R}_{A^{1/2}}}(A^{\frac{1}{2}}f_n - g)\|_{2,\beta}, \text{ since } P_{\mathcal{R}_{A^{1/2}}} = A^{\frac{1}{2}}B^+, \\ &\leq \|g_n - g\|_{2,\beta} \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence $\mathcal{L}^2(\rho)$ is again complete, and this can be stated as:

3.2. Proposition

Let the spectral function ρ of the strongly harmonizable $\{X_t, t \in G\}$ be positive semi-definite and suppose that its induced operator A (of (15)) has a closed range. [Conditions for this to occur are given below.] *Then $\mathcal{L}^2(\rho)$ is again a complete (semi-)*

inner product space, so that if $\mathcal{N} = \{f \in \mathcal{L}^2(\rho) : \|f\| = 0\}$ *and* $\|\tilde{f}\|' = \|f\|$ *for* $\tilde{f} = f + \mathcal{N}$, *then* $(\mathcal{L}^2(\rho)/\mathcal{N}, \|\cdot\|')$ *is a Hilbert space.*

A sufficient condition that insures the closedness of $\mathcal{R}_B$ for a bounded operator B is as follows. If $\{f_n, n \geq 1\} \subset \mathcal{R}_B$ is any weakly convergent sequence with limit f in $L^2(\beta)$ (i.e., $(f_n, h) \to (f, h)$ for all $h \in L^2(\beta)$), then $f \in \mathcal{R}_B$. Thus $\mathcal{R}_B$ is weakly sequentially complete. In fact, since β is a Radon measure, the space $L^2(\beta)$ is separable here and this implies, by ([12], V.7.16), that $\mathcal{R}_B$ is weakly closed. But then by ([12], V.3.13), $\mathcal{R}_B$ is also strongly closed. Whether $\mathcal{R}_A$ in our special case (of semi-definite ρ) is always closed, has not been settled.

Regarding the preceding two propositions the following complements are in order and they should be useful in applications.

A. The spectral domain $\mathcal{L}^2(\rho)$ in (10) is defined as a subset of $L^2(\rho)$. In case ρ has infinite Vitali variation, then ρ does not necessarily determine a signed (or even a semi-bounded) measure. Then one has to resort to the MT-integral, and a deeper analysis shows (with a theorem of Grothendieck) that there is a suitable Radon measure β on $\hat{G}$ and then $\mathcal{L}^2(\rho)$ is a subset of $L^2(\beta)$. From this point on the argument proceeds as above. Suppose now that $G = \mathbb{R}$ and ρ has a finite Vitali variation only on each finite domain of the plane $\mathbb{R}^2$, so that the variation measure $|\rho|(\cdot,\cdot)$ determines a σ-finite Radon measure. In this case the preceding analysis shows only that $\mathcal{L}_E^2(\rho) = \{f \in L^2(\rho) : f\big|_A = 0$ for $A \subset (E \times E)^c\}$ is complete for $E \subset \mathbb{R}$, a finite domain. If $\mathcal{L}^2(\rho)$ is defined as $\cup\{\mathcal{L}_E^2(\rho) : E \subset \mathbb{R}$, a finite domain$\}$, then one cannot conclude that $\mathcal{L}^2(\rho)$ is complete, although it is a (semi-) inner product space. This general case was originally considered by Cramér [7]. Now define the inner product as

$$(f, g) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)\overline{g(t)}\rho(ds, dt) \tag{20}$$

for all compactly supported f, g in $B(\mathbb{R}, \mathbb{C})$. [Since $\tilde{f}(s,t) = f(s) \cdot 1$ need not be in $L^2(\rho)$, the preceding analysis leading to Theorem 3.1 and Proposition 3.2 breaks down, although $L^2(\rho)$ is complete by [12], III.3.6.] In this general case, Cramér [7] suggests that $\mathcal{L}^2(\rho)$ be considered as an inner product space with (20), in its own right and then use $\Lambda^2(\rho)$, the completion of $\mathcal{L}^2(\rho)$, as the spectral domain space. Note that since $\Lambda^2(\rho) \subset L^2(\rho)$ this completed space still consists of functions and not some ideal elements. Thus for most of the applications, with $\Lambda^2(\rho)$, the analysis can be carried out with an occasional additional argument on the limit elements. This implies that the possible noncompleteness of $\mathcal{L}^2(\rho)$ is not an impediment for applications, as indicated in [7]. However, one can show that the positive definiteness of ρ in (1) necessarily has finite Fréchet variation. Consequently, with the MT-integration and the Grothendieck result noted before, one has $\mathcal{L}^2(\rho) \subset L^2(\beta)$ and the completeness follows as above, and no additional argument is needed. Thus the usual applications such as prediction, filtering, and multiplicity theories (cf. Sec. 6 on the latter, and also [6]) can be carried out satisfactorily, with either approach.

B. A special case of Theorem 3.1 was established earlier by Truong-van ([27], Thm. 6). In fact this is given for strongly harmonizable processes of the form $X_t = AY_t$, $t \in G$, where A is a bounded invertible operator on $L_0^2(P)$ onto itself, and the Y_t-process is stationary. The result was given in the multidimensional case. However, the latter extension is not too difficult. The basic idea of Proposition 3.2 already appeared in a tentative form in the thesis of Kelsh ([15], pp. 65–69).

C. The multidimensional case with a different proof, for the weakly harmonizable case, was given in [6], under the same hypothesis. When the latter work was completed, unfortunately, the authors had not seen [27] and so no reference was made to it. The discussion on positive (semi-) definiteness given above should be helpful in the general case also, since not everybody seems to follow the classical Stone [26] definition of these concepts. It should also be noted that positive definiteness for point functions (and kernels) takes a somewhat different appearance than that for the bilinear (or sesquilinear) forms. An extended discussion is given here to draw some attention to these distinctions.

4. SAMPLING PROBLEMS

When the index is $\mathbb{R}$, sampling the process at appropriately chosen time points to base inference is a nontrivial matter since observing at equidistant intervals (the so-called periodic sampling) can introduce an "aliasing" effect, i.e., the sample paths of *different* processes may coincide at these points. This problem exists even for the stationary processes. Thus to avoid this difficulty, the time points of observation must be chosen with care. The following result presents a solution of this problem for a class of harmonizable processes.

Theorem 4.1.

Let $X : \mathbb{R} \to L_0^2(P)$ be a weakly harmonizable process with its spectral function vanishing off a set of diameter $\alpha_0 < \infty$, in $\mathbb{R}^2$. Then for each $\alpha > \alpha_0$, the processes $X_{n,\alpha} : \mathbb{R} \to L_0^2(P)$ defined by

$$X_{n,\alpha}(t) = \sum_{k=-n}^{n} X\left(\frac{k\pi}{\alpha}\right) \frac{\sin(\alpha t - k\pi)\sin^q(pt - \frac{kp\pi}{\alpha})}{(\alpha t - k\pi)p^q(t - \frac{k\pi}{\alpha})}, \tag{21}$$

where $q \geq 0$ is an integer and $0 < p < (\alpha - \alpha_0)/q$, satisfy

$$\|X(t) - X_{n,\alpha}(t)\|_2^2 \leq C_0 L_q(t) p^{-q}(\alpha - \alpha_0 - pq)^{-1}(\alpha/n)^{q+1}. \tag{22}$$

Here $0 < C_0 < \infty$ is an absolute constant and $L_q(t)$ is given by

$$L_q(t) = 2(1 - e^{-\pi})^{-1}(\tfrac{2}{\pi})^{q+1}|\sin \alpha t|. \tag{23}$$

This useful form with an error bound on the right side of (22) is obtained with an approximation of $e^{it\lambda}$ by suitable trigonometric polynomials. However the method of proof is different from the stationary or strongly harmonizable cases, even though the approximations given by (21) have the same form in all cases implying a certain "robustness." This is established by a specialization of [5], and the original motivation as well as the error estimation follow the work in [19]. If $q = 0$ in (21)–(23), the result reduces, in the stationary case, to the classical Kotel'nikov-Shanon formula. Further details will be omitted here.

5. ESTIMATION OF SPECTRAL FUNCTIONS

The importance of spectral functions for an analysis of harmonizable processes is clear from Section 3. Consequently it is useful to have a formula for the calculation of the spectral function from the observed data. Using Theorem 2.2 on the dilation and the classical results on stationary processes, this may be obtained.

Thus if r is the covariance function of a harmonizable process $\{X_t, t \in G\}$ with spectral function F, then it can be given by the following:

$$F(A_1, A_2) = \lim_{0 \leq S_1, S_2 \to \infty} \int_{-S_1}^{S_1} \int_{-S_2}^{S_2} \frac{e^{-i\lambda_1' u} - e^{-i\lambda_1 u}}{-iu} \cdot \frac{e^{i\lambda_2' v} - e^{i\lambda_2 v}}{iv} r(u, v)\,du\,dv, \tag{24}$$

where $A_i = (\lambda_i, \lambda_i')$, $\lambda_i \leq \lambda_i'$, and $F(\lambda_i \pm, \lambda_i' \pm) = F(\lambda_i, \lambda_i')$, $i = 1, 2$. Corresponding formula in the discrete parameter is similarly obtainable.

Since $r(\cdot, \cdot)$ is not generally known to the experimenter, an estimator of F is needed. For this, observe that (24) can be expressed as

$$F(A_1, A_2) = \lim_{0 \leq S_1, S_2 \to \infty} E(Y_{S_1, \lambda_1, \lambda_1'} \bar{Y}_{S_2, \lambda_2, \lambda_2'})$$

where

$$Y_{S, \lambda, \lambda'} = \int_{I_{\lambda, \lambda'}} \Big(\int_{D_S} e^{ius} X_s\, ds \Big) du$$

with $D_S = (-S, S)$, $I_{\lambda, \lambda'} = (\lambda, \lambda')$. This motivates the estimator

$$\widehat{F}_{S_1, S_2}(A_1, A_2) = \int_{A_1} \int_{A_2} \tilde{Y}_{S_1}(u) \bar{\tilde{Y}}_{S_2}(v)\,du\,dv, \tag{25}$$

where

$$\tilde{Y}_S(u) = \int_{D_S} e^{ius} X_s ds.$$

Thus $\widehat{F}$ is computable from the observed segment of the process on D_S. It can be shown that when A_1, A_2 are continuity intervals of F, as in (24), then $\widehat{F}_{S_1,S_2}(A_1, A_2)$ is asymptotically unbiased. This is an easy consequence of known results, e.g. from ([21], Thm. 8.2). When is $\widehat{F}_{S_1,S_2}$ consistent? Conditions similar to those in [18] on higher moments or a use of convergence factors, called "spectral windows" (cf. [29], p. 259ff), can be used. Details are not yet worked out.

There is also interest in finding, if possible, a nonnegative spectral function associated with harmonizable processes. Since it is known that each harmonizable process is of Karhunen class (cf. e.g. [6], p. 59) and the latter has nonnegative spectral functions, one may follow this route. However, in the representation of the covariance as

$$r(s,t) = \int_{\widehat{G}}\int_{\widehat{G}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda') = \int_{\widehat{G}} g(s,\lambda)\overline{g(t,\lambda)} H(d\lambda),$$

one does not have enough knowledge on $\{g(s,\cdot), s \in G\} \subset L^2(\widehat{G}, H)$, and this method does not necessarily give the desired solution. The alternative is a subclass of harmonizable processes, isolated by Kampé de Feriet and Frenkiel [13], hereafter called *class*(KF). This class was also studied independently by Rozanov [25], and by Parzen [18] under the name "asymptotic stationarity." This will be briefly discussed here.

Let $\{X_t, t \in \mathbb{R}\}$ be a mean continuous process from $L_0^2(P)$, with covariance r. It is then of class(KF) if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T r(s, s+h) ds = \tilde{r}(h), \qquad h \in \mathbb{R}, \tag{26}$$

exists. One replaces the integral by a sum in the discrete case. It can be verified that $\tilde{r}(\cdot)$ is a stationary covariance, so that by the Bochner-Riesz theorem (cf. [16], p. 209)

$$\tilde{r}(h) = \int_{\mathbb{R}} e^{ih\lambda} \mu(d\lambda), \tag{27}$$

for almost all (Lebesgue measure) h in $\mathbb{R}$, with μ as a bounded nonnegative measure, called the *associated spectral measure* of the process. Moreover, as Rozanov [25] noted, all strongly harmonizable processes belong to it. Recently Dehay [9] has shown that most weakly harmonizable processes, namely those whose spectral bimeasures ρ determine σ-finite Radon measures, belong to this class. However, this class does not exhaust the weakly harmonizable class since it is known (cf. e.g. [22], p. 297) that there exist discrete parameter weakly harmonizable processes which are not in class(KF). On the other hand, there are nonharmonizable (e.g. periodically correlated [14]) processes in class(KF). Thus it is a useful family to analyze both for its spectral properties and for estimation problems. Some interesting remarks on this class in [25] and an elaboration in [6] may be useful in considering further work on these and related future studies.

6. INFERENCE ON PROCESSES OF LOW MULTIPLICITY

It was already noted that each harmonizable process is of Karhunen class. The latter consists of families $\{X_t, t \in T\} \subset L_0^2(P)$ which are representable as

$$X_t = \int_S g(t,\lambda) Z(d\lambda), \qquad t \in T, \tag{28}$$

where $Z(\cdot)$ is a process of orthogonal increments (or orthogonally scattered vector measure on S into $L_0^2(P)$) and $g(t,\cdot) : S \to \mathbb{C}$ is a measurable function in $L^2(S,F)$, with $F(A) = E(|Z(A)|^2)$, $A \in \mathcal{B}(S)$, a σ-algebra of S (cf. e.g. [4], p. 473). Here $Z(\cdot)$ has desirable properties, but $g(t,\cdot)$ can have a very complicated structure. In fact Cramér has shown that there exist even strongly harmonizable processes of any prescribed multiplicity N, an integer, satisfying $1 \le N \le \infty$ (cf. [8], p. 11). On the other hand, for a harmonizable process, its mean square derivative of order k exists if the spectral measure has k (mixed) moments. This translates to these processes, when represented as (28), as: $g(\cdot,\lambda)$ is k-times differentiable and these derivatives belong to $L^2(S,F)$. To take advantage of this property it is desirable to make a preliminary simplification by decomposing the X_t-process into the purely nondeterministic and deterministic parts (called the Wold decomposition) and concentrate on the former, since the deterministic part does not play any significant role in either the prediction or inference studies. In the purely nondeterministic case there is the Cramér-Hida theory which includes the corresponding weakly harmonizable class of arbitrary multiplicity N. In fact, if $\{X_t, t \in \mathbb{R}\} \subset L_0^2(P)$ is a (mean) left continuous with right limits process at each $t \in \mathbb{R}$, then it can be represented as

$$X_t = \sum_{n=1}^{N} \int_{-\infty}^{t} g_n(t,\lambda) Z_n(d\lambda) \tag{29}$$

where $E(Z_n(A)\overline{Z_m}(B)) = \delta_{mn} F_n(A \cap B)$, $\sum_{n=1}^{N} \int_{-\infty}^{t} |g_n(t,\lambda)|^2 dF_n(\lambda) < \infty$, and $F_1 \succ F_2 \succ \cdots \succ F_n$ (i.e., F_n dominates F_{n+1}). Moreover, N is the minimal integer for which (29) holds; it is called the *multiplicity* of the process. Acomparison of (28), with $T = S = \mathbb{R}$, and

(29) illustrates the complicated nature of g_n's, called the ***response functions*** corresponding to the ***innovation elements*** Z_n. (See [8] about proofs of these assertions.) An interesting result in this study is that if each g_n and $\frac{\partial g_n}{\partial t}$ are bounded and continuous for each $\lambda \leq t$, $g_n(t,t) = 1$, and F_n has a density f_n $(= \frac{dF_n}{d\lambda}) \not\equiv 0$, which may have at most a finite number of discontinuities in any finite interval of $\mathbb{R}$, then necessarily the multiplicity $N = 1$. Thus

$$X_t = \int_{-\infty}^{t} g(t,\lambda) Z(d\lambda), \tag{30}$$

and $g(t,\lambda) = 0$ for $\lambda > 0$ (cf. [8], Thm. 5.1). In what follows only these processes will be discussed and a signal detection problem from a noisy output is outlined. Note that although stationary processes always have multiplicity one, there exist many nonstationary processes of multiplicity one, so that even this family is quite general.

Let $Y_t = S_t + X_t$, $t \in T$, be an additive model (cf. [23] on such problems) describing the observation process Y_t consisting of a signal process S_t and a noise process X_t both of which are of type (30), each with multiplicity one. In particular, suppose X_t is given by (30), but S_t is of the form

$$S_t = \int_{-\infty}^{t} g(t,\lambda)\xi_\lambda d\mu(\lambda), \tag{31}$$

where ξ_t is a measurable process all of whose sample paths are μ-integrable. Hence the observed Y_t is representable as (here μ is a Radon measure on $\mathbb{R}$):

$$Y_t = \int^{t} g(t,\lambda)\xi_\lambda d\mu(\lambda) + \int^{t} g(t,\lambda) dZ(\lambda). \tag{32}$$

Then the desired testing or detection problem is the hypothesis $H_0 : S_t = 0$ (no signal) *vs.* the alternative $H_1 : S_t \neq 0$. If the probability measures determined by the observation and noise processes are denoted P_Y and P_X, then the detection problem is nontrivial when $P_Y << P_X$ (the P_Y measure is dominated by P_X or $P_Y \prec P_X$). The likelihood ratio $f(w) = \frac{dP_Y}{dP_X}(w)$ can then be used, with the Neyman-Pearson-Grenander theorem, to distinguish H_1 from H_0 with a prescribed acceptance level of risk (cf. [20], p. 200ff). However, to get concrete results one has to assume more. For instance, interesting work was outlined by Baker and Gualtierotti [1], if the $Z(\cdot)$ process in (32) is Brownian motion and $\xi_t d\mu(t)$ in (31) is replaced by $\eta(t)$ which is a solution of

$$d\eta(t) = g(\eta(t))dt + dZ(t), \tag{33}$$

a special Itô-type stochastic differential equation so that $\eta(t)$ is a diffusion process. The calculation of likelihood ratios present challenging problems even with these simplifying assumptions (cf. e.g. [20], §4.4).

7. SOME COMPLEMENTS AND DIRECTIONS

If $\{X_t, t \in I\!R\}$ is harmonizable, then from the dilation property (cf. Theorem 2.2) and the classical results on stationary processes one can establish the following:

$$\frac{1}{\tau}\int_0^\tau X_t dt \to Z(0), \text{ in } L_0^2(P), \tau \to \infty, \tag{34}$$

and similarly

$$\frac{1}{2\tau}\int_{-\tau}^\tau X_t dt \to Z(0), \text{ in } L_0^2(P). \tag{35}$$

Here $Z(\cdot)$ is the representing measure of the process. Thus the X_t-process obeys the weak law of large numbers (WLLN) iff $Z(\cdot)$ does not change the origin, i.e., $Z(0) = 0$. However, the corresponding strong law needs a different set of techniques. Extending the earlier results for the strongly harmonizable processes by V. F. Gaposhkin and others, Dehay [10] has obtained the following result. For the SLLN not only should $Z(\cdot)$ not charge '0' but it should be "diffuse" at '0' which implies that the spectral measure F associated with $Z(\cdot)$ must satisfy $F(I_n, I_n) \to 0$ as $n \to \infty$ for the symmetric intervals I_n descending to 0, as $n \to \infty$. To state the result precisely let $\sigma(X : \tau) = \frac{1}{2\tau}\int_{-\tau}^{\tau} X_s ds$, $X_s = \int_{I\!R} e^{is\lambda} Z(d\lambda)$. Then both the SLLN and the speed of convergence obtained in [10] can be stated as:

7.1. **Theorem**

If $\{X_t, t \in I\!R\}$ is weakly harmonizable, $E(X_t) = 0$, with $Z(\cdot)$ as its representing measure, then the process obeys the SLLN iff there exists a $p \geq 2$ such that $\lim_{n\to\infty} Z(I_n) = 0$ a.e., where $I_n = (-p^{-n}, p^{-n})$. Moreover, if there is a nondecreasing function $g : I\!R^+ \to I\!R^+$ with the following two properties:

(a) for each integer $p \geq 2, q_0 \geq 1$, $1 < a < \sqrt{p}$, one has $g^2(p^{q+1}) \leq a g^2(p^s)$, $s = p^q$, $q \geq q_0$ integers, and

(b) if β is the Grothendieck measure on $(I\!R, \mathcal{B}(I\!R))$ dominating the spectral measure F of the process [this will be the β of (13) in that case], and β has the property that for some $u_0 > 0$,

$$\int_{0<|u|<u_0} [(\log\log|u|^{-1})g(|u|^{-1})]^2 \beta(du) < \infty,$$

then $\lim_{\tau\to\infty} g(\tau)[\sigma(X : \tau) - Z(0)] = 0$, a.e.

The details of this result involving several estimates for using the Borel-Cantelli lemma suitably, may be found in [10]. It is also well known that each weakly harmonizable process (or a field, meaning the index set $I\!R$ is replaced by $I\!R^k$, for example) can be approximated pointwise, in mean, by a sequence of strongly harmonizable ones, and then the metric approximation property of Hilbert space allows one to conclude that the convergence is uniform on compact subsets of the index set. This result has been refined, with even a more elementary argument, by Dehay and Moché [11]. They have shown that the sequential closure on compacta of the set of all processes $X : t \mapsto X(t) = \int_{\mathbb{R}^k} e^{i\langle t,\lambda\rangle} Y(\lambda) d\lambda$, where $Y : \mathbb{R}^k \to L_0^2(P)$ is (Bochner) integrable, contains the weakly harmonizable family as a proper subset. A natural problem here is to characterize the weakly harmonizable class by restricting the strongly harmonizable processes further. By strengthening the topology in this approximation procedure, Marc Mehlman has obtained such a result for his dissertation, at UCR, dealing with a finer structure of harmonizable classes.

There are several interesting directions along which a study on these processes can progress. Extensions of these results to the harmonizable random fields when the index G is not necessarily abelian are possible. Also from the point of view of inference, it appears useful to study such processes if the index set is a hypergroup (i.e., an object generalizing the double coset space of a locally compact group). For some indications of promising avenues in both these directions and for related references on the subject, one may see [24]. The finer aspects of the subject involving consistency and asymptotic distributions of estimators such as $\widehat{F}$ of (25), as well as the mean and covariance functions, seem to demand a serious effort even when the index is $\mathbb{R}$ or $\mathbb{Z}$. From the prediction and filtering points of view, much work remains to be done. A comprehensive treatment of the available results on these problems is recently given by Yaglom ([29], especially see Chapter IV). Most of this work with many concrete examples given in [29] would serve as guide posts for studying the corresponding problems for harmonizable random fields. In fact, each result on stationary processes, such as those initially pioneered by Wiener [28], motivates a study for the harmonizable case although the latter usually needs a different set of tools even when the final formula has a similar appearance in many cases.

Acknowledgement. This work was supported in part by the ONR Contract No. N00014-84-k-0356 (mod 00006).

REFERENCES

1. C. R. Baker and A. F. Gualtierotti (1984), "Detection of non-Gaussian signals in Gaussian noise," *in: Statistical Signal Processing*, Marcel Dekker Inc., New York, 107–116.

2. F. J. Beutler (1965), "The operator theory of the pseudo-inverse-II: unbounded operators with arbitrary range," *J. Math. Anal. Appl.* **10**, 471–493.

3. S. Bochner (1956), "Stationarity, boundedness, almost periodicity of random valued functions," *Proc. Third Berkeley Symp. Math. Statist. & Prob.* **2**, 7–27.

4. S. Cambanis (1975), "The measurability of a stochastic process of second order and its linear space," *Proc. Amer. Math. Soc.* **47**, 467–475.

5. D. K. Chang and M. M. Rao (1983), "Bimeasures and sampling theorems for weakly harmonizable processes," *Stochastic Anal. Appl.* **1**, 21–55.

6. ________ (1986), "Bimeasures and nonstationary processes," *in: Real and Stochastic Analysis*, Wiley, New York, 7–118.

7. H. Cramér (1951), "A contribution to the theory of stochastic processes," *Proc. Second Berkeley Symp. Math. Statist. & Prob.*, 329–339.

8. ________ (1971), *Structural and Statistical Problems For A Class of Stochastic Processes*, Princeton Univ. Press, New Jersey.

9. D. Dehay (1987), "On a class of asymptotically stationary harmonizable processes," *J. Multivar. Anal.* **22**, 251–257.

10. ________ (1987), "Strong law of large numbers for weakly harmonizable processes," *Stochastic Proc. Appl.* **24**, 259–267.

11. D. Dehay and R. Moché (1986), "Strongly harmonizable approximations of bounded continuous random fields," *Stochastic Proc. Appl.* **23**, 327–331.

12. N. Dunford and J. T. Schwartz (1958), *Linear Operators, Part I: General Theory*, Wiley-Interscience, New York.

13. J. Kampé de Feriet et F. N. Frenkiel (1959) "Estimation de la corrélation d'une fonction aléatoire nonstationaire," *C. R. Acad. Sci., Ser. B*, **249**, 348–351.

14. ________ (1962), "Correlation and spectra for nonstationary random functions," *Math Comp.* **16**, 1–21.

15. J. P. Kelsh (1978), *Linear Analysis of Harmonizable Time Series*," Ph.D Dissertation, Univ. of Calif., Riverside.

16. Loève, M. (1955), *Probability Theory*, Van Nostrand, Princeton, New Jersey.

17. R. Moché (1985), *Introduction aux Processus Harmonisables*, Lecture Notes, UER Math., Université des Sciences et Tech. de Lille.

18. E. Parzen (1962), "Spectral analysis of asymptotically stationary time series," *Bull. Inst. Int'l Statist.* **39**(2), 87–103.

19. Z. A. Piranashvili (1967), "On the problem of interpolation of random processes," *Theor. Prob. Appl.* **12**, 647–659.

20. M. M. Rao (1981), *Foundations of Stochastic Analysis*, Academic Press, New York.

21. ________ (1982), "Harmonizable processes: structure theory," *L'Enseign. Math.* **28**, 295–351.

22. ________ (1985), "Harmonizable, Cramér, and Karhunen classes of processes," *in: Handbook of Statistics, Vol 5*, North-Holland, Amsterdam, 279–310.

23. _______ (1989), "Harmonizable signal extraction, filtering and sampling," *in: Topics in NonGaussian Signal Processes*, Springer-Verlag, New York, 98–117.

24. _______ (1989), "Bimeasures and harmonizable Processes," *in: Probability Theory on Groups–IX*, Lect. Notes in Math., Springer-Verlag, Berlin (to appear).

25. Yu. A. Rozanov (1959), "Spectral analysis of abstract functions," *Theor. Prob. Appl.* **4**, 271–287.

26. M. H. Stone (1932), *Linear Transformations in Hilbert Space*, Amer. Math. Soc., Providence, Rhode Island.

27. B. Truong-van (1981), "Une généralisation du théorème de Kolmogorov-Aronszajn processus V-bornés q-dimensionnels: domaine spectral et dilations stationnaires," *Ann. Inst. Hénri Poincaré* **17**, 31–49.

28. N. Wiener (1949), *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*, Wiley, New York.

29. A. M. Yaglom (1987), *Correlation Theory of Stationary and Related Random Functions*, Vols I & II, Springer-Verlag, New York.

30. K. Ylinen (1986), "Random fields on noncommutative locally compact groups," *in: Probability Measures on Groups—VIII*, Lect. Notes in Math. No. 1210, Springer-Verlag, Berlin, 365–386.