Journal of Physics A: Mathematical and Theoretical

PAPER • **OPEN ACCESS**

On the geometry of the Titchmarsh counterexample

To cite this article: W P Schleich *et al* 2022 *J. Phys. A: Math. Theor.* **55** 484006

OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **55** (2022) 484006 (15pp)
https://doi.org/10.1088/1751-8121/aca5d5

On the geometry of the Titchmarsh counterexample

W P Schleich[1,2,*], **I Tkáčová**[3] **and H Maier**[4]

[1] Institut für Quantenphysik and Center for Integrated Quantum Science and Technology (IQST), Universität Ulm, Albert-Einstein-Allee 11, Ulm, D-89081, Germany
[2] Institute for Quantum Science and Engineering (IQSE), and Texas A&M AgriLife Research and Hagler Institute for Advanced Study, Texas A&M University, College Station, TX 77843-4242, United States of America
[3] Department of Physics, Faculty of Electrical Engineering and Computer Science, VSB-Technical University of Ostrava, 17. Listopadu 2172/15, Ostrava—Poruba 70833, Czech Republic
[4] Institut für Zahlentheorie und Wahrscheinlichkeitstheorie, Universität Ulm, D-89081 Ulm, Germany

E-mail: wolfgang.schleich@uni-ulm.de

Received 14 June 2022; revised 18 November 2022
Accepted for publication 24 November 2022
Published 9 December 2022

Abstract
We study the lines of constant phase corresponding to the ratio formed by the building blocks of the Titchmarsh counterexample, that is by two Dirichlet L-functions whose characters are the complex conjugate of each other. This ratio *on* the critical line is sensitive to zeros *off* the critical line.

Keywords: Riemann zeta function, Titchmarsh counterexample, lines of constant phase, lines of constant height

(Some figures may appear in colour only in the online journal)

1. Introduction

The verification of the Riemann Hypothesis, that is of the claim that the so-called non-trivial zeros of the Riemann zeta function ζ are *all* located *on* the critical line in the complex plane defined by the real part $1/2$ is a famous still unsolved problem in mathematics. In his seminal

* Author to whom any correspondence should be addressed.

book [1] Edward Charles Titchmarsh introduced a function ξ_T which shares many properties of the analytic continuation of ζ, that is the Riemann function ξ [5] and is known to have [2] zeros *off* the critical line. The appearance of these zeros has been explained [3] as a consequence of the universality [4] of the underlying Dirichlet L-functions. In the present article we point out a crucial difference between ξ and ξ_T which manifests itself in the lines of constant phase [5] of the ratios formed by the elements of the corresponding analytic continuations.

1.1. Formulation of the problem

For this purpose, we study a class of complex-valued functions F characterized by their representation as a superposition

$$F(s) = f(s) + f(1-s) \tag{1}$$

of a *single* complex-valued function f evaluated at s *and* at $1-s$ where $s \equiv \sigma + i\tau$.

Obviously, F satisfies the elementary functional equation

$$F(s) = F(1-s). \tag{2}$$

A zero s_0 of F appears when the ratio

$$g(s) \equiv \frac{f(s)}{f(1-s)} \tag{3}$$

assumes the value -1, that is

$$g(s_0) = -1. \tag{4}$$

This class includes but is not limited to an appropriately defined hyperbolic function c, the Riemann function ξ and the Titchmarsh counterexample ξ_T. In the present article we show that for c and ξ the absolute values of the corresponding ratios g_c and g_R along the critical line $s = 1/2 + i\tau$ are unity.

In contrast, for the Titchmarsh counterexample the ratio g_T on the critical line is real, and assumes all values from $-\infty$ to $+\infty$. This behavior originates from the fact that f_T associated with ξ_T displays zeros on the critical line, and the zeros of $f_T(s)$ and $f_T(1-s)$ are disjunct. As a result, poles and zeros appear in g_T.

In the case of two consecutive poles followed by two consecutive zeros, there must be two points, where the first derivative of g_T vanishes. Here, two lines of constant phase leave symmetrically the critical line and unite again later. Provided the interval formed by the values of g_T at the two points of vanishing derivative includes -1, there are two symmetrically located zeros of ξ_T.

We suspect that this dramatically different behavior of g_T versus g_R on the critical line may provide us with yet another perspective on the Riemann Hypothesis. Indeed, we have already followed [6–8] an approach based on the lines of constant phase of ξ and ξ_T rather than g_R and g_T.

Moreover, we emphasize that the Riemann zeta function plays a central role not only in mathematics but also in physics. Three examples suffice to illustrate this point. (i) Indeed, the distribution of eigenvalues of Gaussian unitary ensembles of random matrices is similar [9] to that of the non-trivial zeros of the Riemann zeta function. (ii) There exists an intimate

[5] Throughout our article we refer to the product ξ defined by equation (11) which is free of the pole and the trivial zeros of the Riemann zeta function ζ located at $s = 1$ and the negative even integers, respectively, as the *Riemann function*.

connection [10] between the inverted harmonic oscillator and the Mellin transform of ξ; and (iii) the long-standing Polya–Hilbert Hypothesis of finding a Hamiltonian whose eigenvalues are given by the zeros of ξ has recently been verified [11].

1.2. Outline

Our article is organized as follows: In section 2 we show that c, ξ, and ξ_T are elements of the class of functions defined by equation (1), and present explicit expressions for the corresponding building blocks f_c, f_R and f_T. Then in section 3 we demonstrate that for c and ξ the critical line is a line of constant *height* of g_c and g_R with value unity. In sharp contrast, g_T corresponding to ξ_T is real along the critical line and thus a line of constant *phase*. This distinct difference between c and ξ on one hand, and ξ_T on the other, is the deeper reason for our ability of identifying zeros of ξ_T located on and off the critical line by analyzing g_T on it. Indeed, we demonstrate in section 4 that g_T assumes zeros and poles on the critical line and the geometry of the lines of constant phase of g_T allows us to determine the location of these zeros. We conclude in section 5 by providing a brief summary and an outlook.

In order to keep our article self-contained we first briefly review in appendix A properties of Dirichlet L-functions Λ crucial for the main theme of our article. We then show that although in general their analytic continuation is *not* of the form, equation (1), the critical line is still a contour line of the corresponding ratio g_Λ with value unity. Since the *generalized* Riemann Hypothesis states that all zeros of Λ are also located on the critical line, this distinct property of g_T compared to g_c, g_R and g_Λ supports the validity of the Riemann Hypothesis.

1.3. Dedication

It is with great pleasure that we dedicate this article to Prof. Sir Michael Victor Berry on the occasion of his 80th birthday. We have chosen for our contribution the possibility of identifying zeros of the Titchmarsh counterexample *off* the critical line by the behavior of the ratio g_T of the building blocks of ξ_T *on* the critical line. We are confident that this topic might find his interest since in a very stimulating discussion at the 600th Heraeus Seminar at Bad Honnef in 2015 we have learned from him that Ernest Oliver Tuck [12, 13] has argued that his incompressibility function of ξ *on* the critical axis line is sensitive to zeros *off* the critical line. Unfortunately, he was wrong as shown [14] by Michael Berry and Pragya Shukla.

'Happy Birthday' and many more happy and healthy years with fun in science!

2. Elements of class of functions

Throughout our article we consider functions that satisfy the elementary functional equation, equation (2), and result from the superposition, equation (1) of a single function f evaluated at the two points s and $1-s$. In the present section we provide explicit expressions for f giving rise to the hyperbolic cosine, the Riemann function and the Titchmarsh counterexample.

Indeed, the most elementary example

$$f_c(s) \equiv \frac{1}{2}\mathrm{e}^{s-1/2} \tag{5}$$

leads us by equation (1) to the function

$$c(s) \equiv \cosh\left(s-\frac{1}{2}\right). \tag{6}$$

Moreover, for the choice

$$f_R(s) \equiv \frac{1}{2}\left[s(s-1)\gamma(s) + \frac{1}{2}\right] \tag{7}$$

with the integral transform

$$\gamma(s) \equiv \int_1^\infty dx\, \omega(x) x^{\frac{s}{2}-1} \tag{8}$$

of the Jacobi theta function

$$\omega(x) \equiv \sum_{n=1}^{\infty} \mathrm{e}^{-\pi n^2 x} \tag{9}$$

we arrive at the familiar analytic continuation

$$\xi(s) = \frac{1}{2}s(s-1)\left[\gamma(s) + \gamma(1-s)\right] + \frac{1}{2} \tag{10}$$

of the Riemann function

$$\xi(s) \equiv \pi^{-s/2}(s-1)\Gamma\left(\frac{s}{2}+1\right)\zeta(s), \tag{11}$$

where Γ and ζ denote the Gamma and the Riemann zeta function, respectively.

Finally, we consider the case

$$f_T(s) \equiv \frac{1}{2\,\cos\theta}\mathrm{e}^{-i\theta}\Lambda(s,\chi_1) \tag{12}$$

with the Dirichlet L-function $\Lambda = \Lambda(s,\chi_1)$ of complex-valued character χ_1 mod 5 where the parameter θ is defined by the condition

$$\tan(2\theta) = \frac{\sqrt{5}-1}{2}.$$

For a general introduction into and an overview over Dirichlet L-functions, we refer to [15, 16]. However, in appendix A we briefly summarize properties of Λ relevant to the present discussion.

Indeed, from equation (A10) we recall the functional equation

$$\Lambda(1-s,\chi_1) = \mathrm{e}^{2i\theta}\Lambda(s,\chi_2) \tag{13}$$

and thus arrive at the expression

$$f_T(1-s) = \frac{1}{2\cos\theta}\mathrm{e}^{i\theta}\Lambda(s,\chi_2)\,, \tag{14}$$

leading us by the superposition, equation (1), of $f_T(s)$ and $f_T(1-s)$ to the Titchmarsh counter-example

$$\xi_T(s) \equiv \frac{1}{2\,\cos\theta}\left[\mathrm{e}^{-i\theta}\Lambda(s,\chi_1) + \mathrm{e}^{i\theta}\Lambda(s,\chi_2)\right]. \tag{15}$$

Hence, the class of functions given by equation (1) includes all three examples.

3. Des Pudels Kern

The crucial difference between the functions c and ξ_R on the one hand, and ξ_T on the other, stands out most clearly when we consider for each of them the value $f(s^*)$ where the star indicates the complex conjugate. This operation tests the symmetry of f with respect to the real axis.

Indeed, we find for c and ξ_R the symmetry relations

$$f_c(s^*) = f_c^*(s) \tag{16}$$

and

$$f_R(s^*) = f_R^*(s)\,. \tag{17}$$

However, the dependence of f_T on the complex-valued character χ_1, and the presence of the phase factor $\exp(-i\theta)$, prevent a similar relation for the building block f_T of the Titchmarsh counterexample, that is

$$f_T(s^*) \neq f_T^*(s)\,. \tag{18}$$

In this section we first verify the identities, equations (16) and (17) as well as show the breakdown of this symmetry for f_T as expressed by equation (18). We then demonstrate that this distinct difference implies that the critical line is a line of constant *height* for the ratios g_c and g_R formed by f_c and f_R, but is a line of constant *phase* for g_T defined by f_T.

3.1. Confirmation and breakdown of symmetry with respect to real axis

We start by noting that the definition, equation (5), of f_c immediately implies the symmetry relation, equation (16).

A slightly more complicated argument verifies the corresponding property, equation (17), for ξ_R. Indeed, since the integration variable x in γ given by equation (8) is real the Jacobi theta function ω defined by equation (9) is real as well, and with the identity

$$x^{s^*/2} = \exp\left(\frac{s^*}{2}\ln x\right) = \left[\exp\left(\frac{s}{2}\ln x\right)\right]^* \tag{19}$$

we find

$$\gamma(s^*) = \gamma(s)^*. \tag{20}$$

Moreover, the polynomial $s(s-1)$ satisfies the relation $s^*(s^*-1) = [s(s-1)]^*$ which with the definition, equation (7), of f_R leads us to equation (17).

Finally we address the case of ξ_T where according to the definition, equation (12), we find

$$f_T(s^*) = \frac{1}{2\,\cos\theta}\mathrm{e}^{-i\theta}\Lambda(s^*,\chi_1)\,, \tag{21}$$

which due to the dependence of Λ on χ_1 takes the form

$$f_T(s^*) = \frac{1}{2\,\cos\theta}\mathrm{e}^{-i\theta}\left[\Lambda(s,\chi_1^*)\right]^*\,. \tag{22}$$

When we recall the symmetry relation $\chi_1^* = \chi_2$, for the character χ_1 we obtain the expression

$$f_T(s^*) = \frac{1}{2\,\cos\theta}\mathrm{e}^{-i\theta}\left[\Lambda(s,\chi_2)\right]^* \tag{23}$$

which is obviously not identical to f_T^*. Two features of f_T prevent this identity: (i) the presence of the phase factor $\exp(-i\theta)$ in front of the Dirichlet function Λ, and (ii) the dependence of Λ on χ_1 which leads to the emergence of χ_2 rather than χ_1.

3.2. The critical line: line of constant height of ratios g_c and g_R

The symmetry relations, equations (16) and (17) for f_c and f_R have an immediate consequence on the behavior of the corresponding ratios g_c and g_R on the critical line $s = 1/2 + i\tau$. Indeed, for c and ξ_R we obtain the representations

$$g_c\left(\frac{1}{2}+i\tau\right) = \frac{f_c\left(\frac{1}{2}+i\tau\right)}{f_c\left(\frac{1}{2}-i\tau\right)} = \frac{f_c\left(\frac{1}{2}+i\tau\right)}{f_c\left(\frac{1}{2}+i\tau\right)^*} = \mathrm{e}^{i2\varphi_c(\tau)} \tag{24}$$

and analogously

$$g_R\left(\frac{1}{2}+i\tau\right) = \mathrm{e}^{i2\varphi_R(\tau)} \tag{25}$$

where $\varphi_c = \varphi_c(\tau)$ and $\varphi_R = \varphi_R(\tau)$ are the phases of f_c and f_R along the critical line.

Hence, we arrive at the identity

$$\left|g_c\left(\frac{1}{2}+i\tau\right)\right| = \left|g_R\left(\frac{1}{2}+i\tau\right)\right| = 1 \tag{26}$$

which indicates that along the critical line, g_c and g_R have lines of constant height with value unity.

According to equation (4) zeros of c and ξ_R occur on the critical line for the imaginary parts $\tau_c^{(k)}$ and $\tau_R^{(k)}$ where the phases φ_c and φ_R of g_c and g_R assume odd integer multiples of π, leading us to the condition

$$\varphi_c\left(\tau_c^{(k)}\right) = (2k+1)\frac{\pi}{2} \tag{27}$$

and

$$\varphi_R\left(\tau_R^{(k)}\right) = (2k+1)\frac{\pi}{2}. \tag{28}$$

Here k is an integer.

Since in the case of c the phase φ_c is just τ, we obtain the familiar explicit formula

$$\tau_c^{(k)} = (2k+1)\frac{\pi}{2} \tag{29}$$

for the imaginary parts $\tau_c^{(k)}$ of the zeros of c on the critical line.

Unfortunately, due to the more complicated expression for f_R given by equations (7)–(9) no analytic expression for $\varphi_R(\tau)$ is known. Despite this complication, the zeros of ξ_R on the critical line follow from a condition identical to that of c. Indeed, they are located where the phase lines of g_R with an odd integer multiple of π cross the critical line.

3.3. The critical line: line of constant phase of ratio g_T

We now turn to the case of ξ_T and study the ratio

$$g_T\left(\frac{1}{2}+i\tau\right) = \mathrm{e}^{-2i\theta}\frac{\Lambda\left(\frac{1}{2}+i\tau,\chi_1\right)}{\Lambda\left(\frac{1}{2}+i\tau,\chi_2\right)} \tag{30}$$

following from the definition, equation (3), of g with equations (12) and (14).

We note that on the critical line the functional equation, equation (A9), of $\Lambda(s,\chi_1)$ reads

$$\Lambda\left(\frac{1}{2}+i\tau,\chi_1\right) = \mathrm{e}^{2i\theta}\Lambda\left(\frac{1}{2}+i\tau,\chi_1\right)^* \tag{31}$$

and enforces the representation

$$\Lambda\left(\frac{1}{2}+i\tau,\chi_1\right)=\mathrm{e}^{i\theta}\lambda_1(\tau)\,, \tag{32}$$

where $\lambda_1=\lambda_1(\tau)$ is a real function which is not necessarily positive.

Similarly, we find from the functional equation, equation (A10), the expression

$$\Lambda\left(\frac{1}{2}+i\tau,\chi_2\right)=\mathrm{e}^{-i\theta}\lambda_2(\tau), \tag{33}$$

where λ_2 is a real function which is not necessarily positive. Since $\chi_1\neq\chi_2$, the two functions λ_1 and λ_2 are different.

When we substitute the representations equations (32) and (33) into the expression, equation (30), of g_T, we arrive at the relation

$$g_T\left(\frac{1}{2}+i\tau\right)=\frac{\lambda_1(\tau)}{\lambda_2(\tau)}\equiv\lambda(\tau). \tag{34}$$

Hence, on the critical line g_T is real, and according to equation (4) a zero $s_0=\frac{1}{2}+i\tau_0$ of ξ_T arises when

$$g_T\left(\frac{1}{2}+i\tau_0\right)=-1. \tag{35}$$

A comparison between the behavior of the ratios g_c, g_R and g_T on the critical line expressed by equations (24), (25) and (34), brings out the importance of the symmetry with respect to the real axis. Indeed, for functions such as c or ξ which enjoy this symmetry, the critical line is a line of constant *height* of g with $|g|=1$. In this case, the phase of g can assume any value as τ increases, and a zero of c or ξ arises for an odd integer multiple of π.

In contrast, a violation of this symmetry as displayed by ξ_T, leads to a situation where the ratio g_T is real along the critical line. When we allow for zeros or poles of g_T, the critical line is a line of piecewise constant phase with φ_T being an integer multiple of π. Zeros of ξ_T on the critical line arise at $\tau-$values where g_T is -1 as indicated by equation (35).

4. Zeros off the critical line

So far we have concentrated on the mechanism for the appearance of zeros of F *on* the critical line. Here the ratio g has played a central role. We now show that the behavior of g *on* the critical line, even allows us to identify zeros of F that are located *off* the critical line. We demonstrate this property using ξ_T which has such zeros.

For this purpose we first recall that the Dirichlet L-functions $\Lambda_1=\Lambda_1(s)\equiv\Lambda(s,\chi_1)$ and $\Lambda_2=\Lambda_2(s)\equiv\Lambda(s,\chi_2)$ have simple zeros [3, 6] on the critical line[6] as exemplified by figure 1. Hence, at a zero of Λ_2 the ratio g_T has a simple pole, and at a zero of Λ_1 a zero. Since poles are sources and zeros are sinks of phase lines [6] we find a flow pattern of g_T illustrated in figure 2.

The influence of the locations of the zeros and poles of g_T on the zeros of ξ_T shown on the left of figure 2 stands out most clearly when we consider for increasing τ the sequence of two zeros and two poles of g_T located on the critical line as depicted in the middle of figure 2. On the right we present g_T on the critical line where according to equation (34) g_T is real.

[6] We emphasize that for our argument the still unverified generalized Riemann Hypothesis, that is the claim that *all* zeros of Λ are on the critical line, is not of importance.

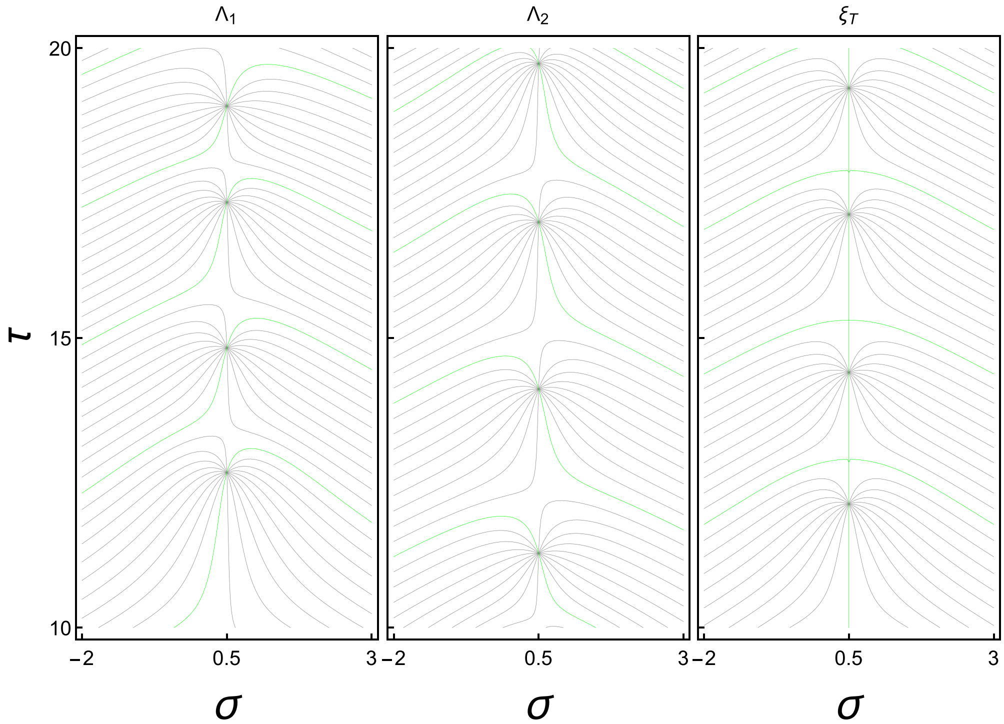

Figure 1. Lines of constant phase (grey lines) for the Dirichlet L-functions $\Lambda_1(s) \equiv \Lambda(s, \chi_1)$ (left) and $\Lambda_2 \equiv \Lambda(s, \chi_2)$ (middle) corresponding to the characters χ_1 and $\chi_2 \equiv \chi_1^*$ defined by equation (A6), compared and contrasted to the ones of the Titchmarsh counterexample ξ_T (right) in identical domains of the complex plane $s \equiv \sigma + i\tau$. The phase shifts of $+2\theta$ and -2θ in Λ_1 and Λ_2 with respect to the critical line $\sigma = 1/2$ are a result of the corresponding phases of the normalized Gauss sums, equation (A5), and the functional equations, equations (A9) and (A10). In contrast, ξ_T which is determined by a superposition, equation (15), of Λ_1 and Λ_2 such that it satisfies the functional equation, equation (2) does not display a phase shift but enjoys an antisymmetry of its phase.

Due to its simple pole, g_T has to either increase from the second zero in the middle of figure 2 to *plus* infinity and increase from *minus* infinity after the pole, or decrease from the second zero to *minus* infinity and decay from *plus* infinity after the pole. Only in the second scenario, the condition, equation (35) for a zero of ξ_T is satisfied for a value of τ before the pole as shown on the right of figure 2. Moreover, we note that after the second pole g_T increases from minus infinity through the top zero of g_T. Hence, there must be another zero of ξ_T between this pole and this zero of g_T.

So far we have explained the emergence of a zero of ξ_T *on* the critical line. We now discuss the mechanism underlying the appearance of a zero *off* the critical line.

For this purpose we consider a situation depicted in the middle of figure 3 where Λ_1 has two consecutive zeros which are not separated by a zero of Λ_2. As a consequence, g_T displays two adjacent simple zeros with a zero of the first derivative g_T' in between. This point τ_0' on the critical line marked by a green triangle in the middle of figure 3 is the starting point of *two* lines of constant phase shown in green that move away symmetrically from their point of birth into the complex plane, and return to the critical line at another point $\tilde{\tau}_0'$ on the critical line,

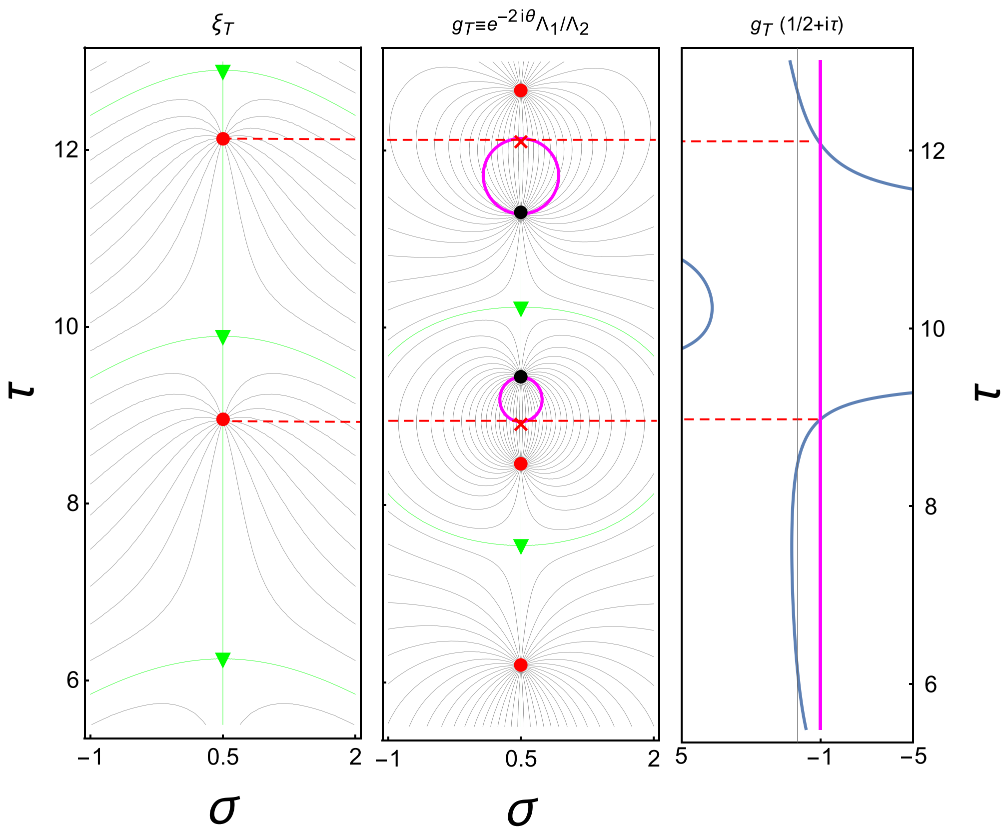

Figure 2. Origin of the zeros of the Titchmarsh counterexample ξ_T *on* the critical line explained by the behavior of the ratio g_T formed by the two Dirichlet L-functions Λ_1 and Λ_2 together with the phase factor $\exp(-2i\theta)$ *on* the critical line. In the interval of $\tau-$values shown in the left picture the two zeros (red dots) of ξ_T are located on the critical line, and the lines of constant phase (grey curves) terminating in them are separated by separatrices (green curves). Each separatrix merges with the critical line in a point (green triangles) where the first derivative of ξ_T vanishes. The ratio g_T displayed (middle) by its lines of constant phase (grey curves) shows for increasing values of τ on the critical line two zeros (red dots) of g_T followed by two poles (black dots) and another zero (red dot). At τ-values where the blue curve (right) representing g_T on the critical axis, crosses the magenta solid line indicating the value -1, there are zeros of ξ_T. Indeed, the magenta solid lines (middle) which indicate the value -1 of g_T, pass through the critical line at points marked by crosses which are the zeros of ξ_T. Due to the pairing of two consecutive zeros and two consecutive poles, g_T has to display vanishing first derivatives on the critical line as indicated for g_T by green triangles. Here two phase lines first emerge and then terminate forming a loop. However, no zeros of ξ_T exist on this loop since the values of g_T in the points of vanishing first derivative are both positive, and the interval given by them does not include -1, as shown by the blue curve on the right.

now located between two adjacent zeros of Λ_2, that is two adjacent poles of g_T. In this case we have the possibility of zeros of ξ_T located *off* the critical line since along this line the phase of the critical line is preserved.

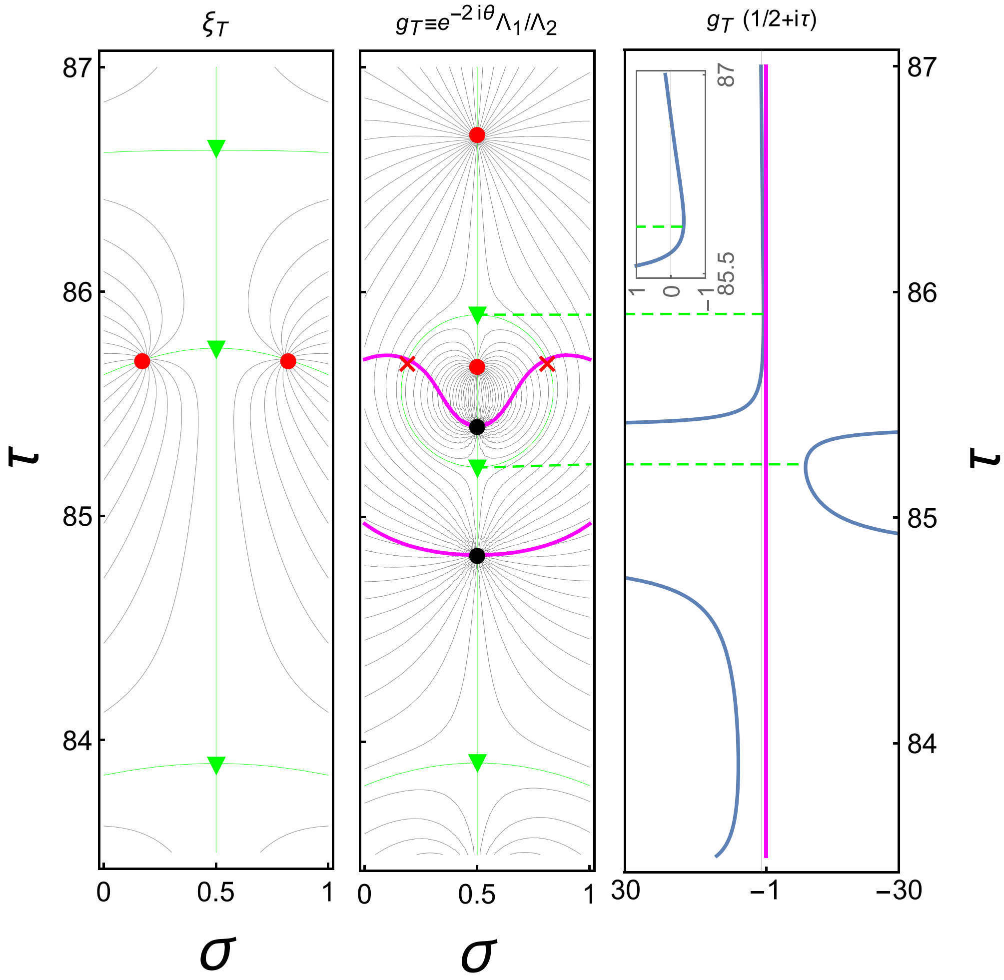

Figure 3. Origin of the zeros of the Titchmarsh counterexample ξ_T *off* the critical line explained by the behavior of g_T *on* the critical line. In the interval of $\tau-$values shown on the left, the two zeros (red dots) of ξ_T located off the critical line arise from *three* consecutive points (green triangles) where the first derivative vanishes. The ratio g_T displayed in the middle for the same domain of the complex plane exhibits two consecutive poles and two consecutive zeros, which lead to a vanishing of the first derivative of g_T (green triangles) between the poles, and between the zeros. Indeed, g_T on the critical line shown on the right by blue curves displays at these points vanishing first derivatives as indicated by the green dashed horizontal lines. The inset enlarges the behavior of g_T in one of these neighborhoods. In complete analogy to figure 2 a loop-shaped phase line appears, but now the interval formed by the values of g_T at the points of the vanishing first derivative includes -1. As a result, there must be *two* symmetrically located zeros of ξ_T *off* the critical line. This property is confirmed by the two crossings of the solid magenta line indicating the value -1 of g_T, and the phase loop marked by the two crosses.

However, we also need the ratio of g_T to assume the value -1 and this requirement is not always satisfied. In figure 2 we show such an example of a loop-shaped phase line, but it does not lead to a zero off the axis.

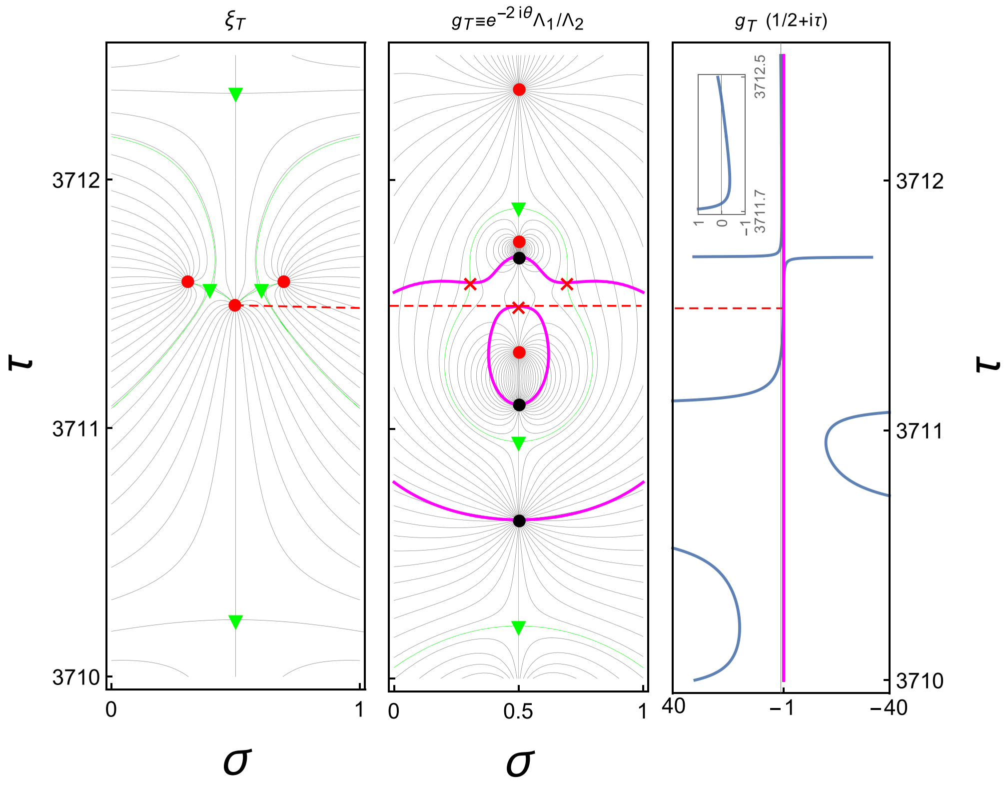

Figure 4. Origin of the zeros of the Titchmarsh counterexample ξ_T *off* the critical line, explained by the behavior of g_T *on* the critical line. In the domain of the complex plane shown in the left picture ξ_T displays three zeros (red dots), two of which are located *off* the critical line. In this case, two points of vanishing derivatives (green triangles) exist off the critical line as well. The flows of phase lines approaching the zero on the critical line from the right are divided by two separatrices collecting the flow toward the zero off the axis. Due to the antisymmetry of ξ_T in the phase, the same behavior occurs for the flows left of the critical line. In this case g_T (middle) displays two consecutive poles, a zero followed by a pole, and two consecutive zeros. Again a closed phase line appears which, due to the zero and the pole caught between the two pairs of poles and zeros, is in the shape of a pear. Indeed, two phase lines emerge from the point (green triangle) between the two poles where the first derivative of g_T vanishes and meet again in the next point of a vanishing derivative (green triangle) caught between the two zeros. The interval formed by the values of g_T on the critical line (right blue lines) at the points of the vanishing first derivative includes -1. As a result, there must be two symmetrically located zeros of ξ_T *off* the critical line. Moreover, due to the pole and the zero of g_T caught between the two poles and the two zeros an additional zero of ξ_T arises *on* the critical axis as demonstrated by the blue curve crossing the solid magenta line in the right picture. The emergence of *three* zeros of ξ_T is also apparent in the center figure by the crossing of the solid magenta lines with the green pear-shaped loop, and the critical line indicated by crosses.

We can understand this result when we compare $g_T(1/2+i\tau_0')$ and, $g_T(1/2+i\tilde{\tau}_0')$. Indeed only, if the interval $[g_T(1/2+i\tau_0'), g_T(1/2+i\tilde{\tau}_0')]$ includes -1, we have two symmetrically located zeros off the axis.

The right panel of figure 2 shows that -1 is not included in the interval, whereas figures 3 and 4 depict situations where it is. Moreover, the zero of ξ_T on the critical line in figure 4 is a consequence of the same scenario as in figure 2. Indeed, on the critical line g_T decreases for increasing τ after the second pole from plus infinity to minus infinity at the second pole as shown by the blue line in the right panel of figure 4. Hence, g_T must assume the value -1 leading to a zero of ξ_T on the critical line.

5. Conclusions and outlook

In conclusion, the zeros of the Titchmarsh counterexample ξ_T originate from the superposition of two Dirichlet L-functions. Since their zeros on the critical line are disjunct, the ratio of the two functions exhibit zeros and poles giving rise to closed lines of constant phase.

Moreover, this ratio is real along the critical line, which thus is a line of constant *phase*. We emphasize that this feature is in sharp contrast to the ratios corresponding to the hyperbolic cosine c and the Riemann function ξ. Here the critical line is a line of constant *height*.

In appendix A we show that for a single Dirichlet L-function which can also be represented as a superposition of two contributions, and is expected to obey the generalized Riemann Hypothesis, the critical line is also a line of constant *height* of the corresponding ratio. We suspect that this property common to c, ξ and Λ could be intimately related to the validity of the Riemann Hypothesis.

It has often been argued that it is connected to the existence of an Euler product. Indeed, ξ and Λ enjoy such representations. However, due to ξ_T being a *superposition* of two Dirichlet L-functions no Euler product can be found for ξ_T.

To establish a connection between the critical line being a line of constant *height* of the ratio of the building blocks of the analytic continuation, and the existence of an Euler product is a fascinating and challenging problem. However, this topic is only one of many that come to mind in the context of studying the lines of constant phase of this ratio.

Indeed, so far we have concentrated mainly on the behavior of these ratios on the critical line, but it is worthwhile to study the lines of constant phase of g_R in the *complete* complex plane. Our analysis has clearly demonstrated that zeros of ξ must be located *on* the critical line because there $|g_R| = 1$ and phase lines corresponding to an odd integer multiple of π cross the critical line. But do zeros off the critical line as in the Titchmarsh counterexample exist also for ξ?

A violation of the Riemann Hypothesis would require an additional line of constant height of g_R with $|g_R| = 1$ *and* at least one additional phase line that crosses it with a phase given by an odd integer multiple of π. Needless to say, we do not have a conclusive argument in favor or against it, but note the identity

$$g_R(s = 0) = g_R(s = 1) = 1 \tag{36}$$

following from equations (3) and (7).

Hence, apart from the critical line there must be additional lines of constant height with $|g_R| = 1$. However, it is well-known that there are no zeros off the critical line close to the origin of the complex plane. Hence, there cannot be a phase line of integer multiple of π. Nevertheless, the question arises: how do the lines of constant phase of g_R look in this domain?

Moreover, there could be other contour lines with $|g_R| = 1$ away from the immediate neighborhood of the origin. However, the existence of such a line would require a pole of g_R created either by a zero or a pole of f_R. Obviously f_R is free of poles, but what about zeros?

Finally, we note from appendix A that the analytic continuation of Λ given by equation (A2) is free of the off-set and the quadratic polynomial appearing in the corresponding expression, equation (10), for ξ and giving rise to the complication of additional lines of constant height as expressed by equation (36). Hence, we expect the behavior of g_Λ in the complete complex plane to be much simpler than g_R. Moreover, in this case g_Λ is solely determined by the integral transform $\gamma = \gamma(s,\chi)$ which according to equation (A3) is free of a pole. Hence, an equivalent formulation of the generalized Riemann Hypothesis might be: The function γ is free of zeros.

Unfortunately these questions go beyond the scope of the present article and have to be postponed to a future publication.

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

We thank P C Abbott, M B Kim, H L Montgomery, J W Neuberger and M Zimmermann for many fruitful discussions and M E N Tschaffon for technical assistance. W P S is most grateful to Texas A & M University for a Faculty Fellowship at the Hagler Institute for Advanced Study at Texas A& M University and to Texas A& M AgriLife for the support of this work. The research of the IQST is financially supported by the Ministry of Science, Research and Arts Baden-Württemberg. I T is grateful for the financial support from the Technical University of Ostrava, Grant No. SP2018/44.

Appendix A. Dirichlet L-functions

Throughout our article, we focus on functions which satisfy the elementary representation, equation (1). For this class we show in the main body of our paper that in contrast to ξ_T the functions c and ξ enjoy ratios g_c and g_R formed by the building blocks of the corresponding analytic continuations, where the critical line is a line of constant height with $|g| = 1$.

It is well-known [15, 16] that Dirichlet L-functions do not obey equation (1) but a slightly more complicated relation. Nevertheless, the critical line is still a line of constant height of the corresponding ratio g_Λ as we verify now.

For this purpose, we first briefly summarize properties of Dirichlet L-functions central to our article. Here, for the sake of brevity we only state the results but do not derive them. We then turn to the discussion of g_Λ on the critical line.

A.1. A brief review

The definition of a Dirichlet L-function Λ of character χ mod q reads [15, 16]

$$\Lambda(s,\chi) \equiv \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\kappa}{2}\right) L(s,\chi), \tag{A1}$$

where

$$L(s,\chi) \equiv \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

denotes the Dirichlet L-series and the parameter κ assumes the values $\kappa = 0$ or $\kappa = 1$.

Moreover, Λ enjoys the analytic continuation

$$\Lambda(s,\chi) = \gamma(s,\chi) + \mathrm{e}^{i\beta(\chi)}\gamma(1-s,\chi^*) \tag{A2}$$

where

$$\gamma(s,\chi) \equiv \left(\frac{q}{\pi}\right)^{-\kappa/2} \int_1^\infty dx\, \omega(x,\chi) x^{\frac{s+\kappa}{2}-1} \tag{A3}$$

with

$$\omega(x,\chi) \equiv \sum_{n=1}^{\infty} \chi(n) n^\kappa \mathrm{e}^{-\pi n^2 x/q} \tag{A4}$$

are the generalizations of the corresponding expressions equations (8) and (9) for the integral transform γ of the Jacobi theta function ω in ξ to Λ.

The phase factor $\exp[i\beta(\chi)]$ in equation (A2) originates from the normalized Gauss sum

$$G(\chi) \equiv i^{-\kappa} \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) \mathrm{e}^{2\pi i n/q} = \mathrm{e}^{i\beta(\chi)} \tag{A5}$$

with $|G| = 1$ and makes equation (A2) different from the superposition, equation (1).

For the character χ_1 mod 5 with the values

$$\chi_1(1) = 1, \quad \chi_1(2) = i, \quad \chi_1(3) = -i, \quad \chi_1(4) = -1, \quad \chi_1(5) = 0 \tag{A6}$$

we find from the definition, equation (A5), of the normalized Gauss sum the phase

$$\beta(\chi_1) = 2\theta\,,$$

where θ is given implicitly by the relation

$$\tan(2\theta) = \frac{\sqrt{5}-1}{2}\,. \tag{A7}$$

Since $\beta(\chi^*) = -\beta(\chi)$ we obtain for $\Lambda(s,\chi_2)$ with $\chi_2 \equiv \chi_1^*$ the phase

$$\beta(\chi_2) = -2\theta.$$

The analytic continuation of Λ given by equation (A2) immediately leads us to the functional equation

$$\Lambda(s,\chi) = \mathrm{e}^{i\beta(\chi)}\Lambda(1-s,\chi^*)\,, \tag{A8}$$

which for the two building blocks of the Titchmarsh counterexample defined by equation (15) takes the form

$$\Lambda(s,\chi_1) = \mathrm{e}^{2i\theta}\Lambda(1-s,\chi_2) \tag{A9}$$

and

$$\Lambda(s,\chi_2) = \mathrm{e}^{-2i\theta}\Lambda(1-s,\chi_1)\,, \tag{A10}$$

where θ is given by equation (A7).

A.2. Critical line is line of constant height of g_Λ

Finally, we turn to the ratio

$$g_\Lambda(s,\chi) \equiv \mathrm{e}^{-i\beta(\chi)} \frac{\gamma(s,\chi)}{\gamma(1-s,\chi^*)}$$

defined in analogy to equation (3) and following from the analytic continuation, equation (A2), of Λ which on the critical line reads

$$g_\Lambda\left(\frac{1}{2}+i\tau,\chi\right) = \mathrm{e}^{-i\beta(\chi)} \frac{\gamma\left(\frac{1}{2}+i\tau,\chi\right)}{\gamma\left(\frac{1}{2}+i\tau,\chi\right)^*}.$$

As a result, we arrive at the expression

$$g_\Lambda\left(\frac{1}{2}+i\tau,\chi\right) = \mathrm{e}^{-i\beta(\chi)}\mathrm{e}^{2i\delta_\Lambda(\tau;\chi)} \tag{A11}$$

where $\delta_\Lambda = \delta_\Lambda(\tau;\chi)$ is the phase of the integral transform $\gamma\left(\frac{1}{2}+i\tau,\chi\right)$ of the generalized theta function $\omega(x,\chi)$ on the critical line.

Hence, the critical line is indeed a contour line of g_Λ with $\left|g_\Lambda\left(\frac{1}{2}+i\tau,\chi\right)\right| = 1$.

ORCID iD

W P Schleich https://orcid.org/0000-0002-9693-8882

References

[1] Titchmarsh E C 1951 *The Theory of the Riemann Zeta-Function* (Oxford: Clarendon)
[2] Spira R 1994 Some zeros of the Titchmarsh counterexample *Math. Comput.* **63** 747–8
[3] Karatsuba A A and Voronin S M 1992 *The Riemann Zeta-Function* (Berlin: de Gruyter)
[4] Voronin S M 1975 Theorem on the "universality" of the Riemann zeta-function *Izv. Akad. Nauk SSSR Ser. Mat.* **39** 475–86
Voronin S M *Math. USSR Izv.* **9** 443–5 (Engl. transl.)
[5] Schleich W P, Tkáčová I and Happ L 2022 Insights into complex functions *(Lecture Notes in Physics* vol 1000) (Heidelberg: Springer)
[6] Neuberger J W, Feiler C, Maier H and Schleich W P 2014 Newton flow of the Riemann zeta function: separatrices control the appearance of zeros *New J. Phys.* **16** 103023
[7] Neuberger J W, Feiler C, Maier H and Schleich W P 2015 The Riemann hypothesis illuminated by the Newton flow of ζ *Phys. Scr.* **90** 108015
[8] Schleich W P, Bezdeková I, Kim M B, Abbott P C, Maier H, Montgomery H L and Neuberger J W 2018 Equivalent formulations of the Riemann hypothesis based on lines of constant phase *Phys. Scr.* **93** 065201
[9] Mitchell G E, Richter A and Weidenmüller H A 2010 Random matrices and chaos in nuclear physics: nuclear reactions *Rev. Mod. Phys.* **82** 2845
[10] Twamley J and Milburn G J 2006 The quantum Mellin transform *New J. Phys.* **8** 328
[11] Bender C M, Brody D C and Müller M P 2017 Hamiltonian for the zeros of the Riemann zeta function *Phys. Rev. Lett.* **118** 130201
[12] Tuck E O 2007 Riemann's hypothesis *Notes for a Mathematics Coll. (University of Adelaide 31 August 2007)* Updated (*5 October 2007*) With Additional Computations
[13] Tuck E O 2008 Smallest minima of $1-z(t)z''(t)/(z'(t))^2$ preprint
[14] Berry M V and Shukla P 2008 Tuck's incompressibility function: statistics for zeta zeros and eigenvalues *J. Phys. A: Math. Theor.* **41** 385202
[15] Iwaniec H and Kowalski E 2003 *Analytic Number Theory* (Providence, RI: American Mathematical Society)
[16] Ellison W J and Ellison F 1985 *Prime Numbers* (New York: Wiley)