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A NUMERICAL METHOD FOR DETERMINING THE EIGENVALUES AND EIGENFUNCTIONS OF ANALYTIC KERNELS*

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1. Introduction. This paper is concerned with a numerical method for determining the eigenfunctions and eigenvalues of the integral equation

$$\lambda\psi(x) = \int_a^b K(x, y)\psi(y)w(y)\,dy, \qquad -\infty < a \leqq x \leqq b < \infty, \tag{1}$$

when the kernel can be expanded in a double power series

$$K(x, y) = \sum_0^\infty a_{ij}x^iy^j, \qquad a_{00} = 1. \tag{2}$$

The development presented is purely formal without regard for the niceties of convergence or existence.

Our method stems from consideration of the modified equation

$$\lambda\psi(x) = \int_a^b K(cx, cy)\psi(y)w(y)\,dy \tag{3}$$

in which c is a parameter. We expand the eigenfunctions and eigenvalues of this equation in power series in c and show that the coefficients in these expansions can be determined by a simple (nonlinear) recurrence. The method can be considered as a perturbation scheme in which the given kernel $K(x, y)$ is regarded as a perturbation of the easily treated degenerate kernel $\hat{K}(x, y) \equiv 1$ which results when $c \to 0$.

Specifically, we write as solutions to (3),

$$\lambda_n = c^{2n}\sum_{i=0}^\infty \chi_i(n)c^i, \qquad \chi_0(n) \neq 0, \tag{4}$$

$$\psi_n(x) = \sum_{j=0}^\infty c^j\phi_j^{(n)}(x), \qquad n = 0, 1, 2, \cdots. \tag{5}$$

The assumed form for λ_n, i.e., $\lambda_n = O(c^{2n})$, serves to define our labeling scheme for the nth eigenvalue and eigenfunction. The choice made is justified in the Appendix. We further suppose that we can write

$$K(cx, cy) = \sum_{m=0}^\infty c^m \sum_{\mu,\nu=0}^m \kappa_{\mu\nu}^m P_\mu(x)P_\nu(y), \tag{6}$$

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where the functions $P_\mu(x)$ satisfy

$$(7) \qquad \int_a^b P_\mu(x)P_\nu(x)w(x)\,dx = \delta_{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, \cdots,$$

with $\delta_{\mu\nu}$ the usual Kronecker symbol.[1] Finally, we set

$$(8) \qquad \phi_j^{(n)}(x) = \sum_{k=0}^{\infty} B_k^j(n)P_k(x).$$

The important fact is that with the assumptions just made, the integral equation (3) is formally satisfied when the series (8) has *only a finite number of nonzero terms*. We exhibit a solution of the form

$$(9) \qquad \phi_j^{(n)}(x) = \sum_{k=-j}^{j} A_k^j(n)P_{n+k}(x).$$

For fixed n, the recurrence scheme already mentioned permits the calculation of $\chi_j(n)$, A_k^{j+k} and A_{-k}^{j+k}, $k = 0, 1, \cdots, p$, in terms of certain of the given data $\kappa_{\mu\nu}^m$ and the previously calculated quantities $\chi_i(n)$, A_k^{i+k}, A_{-k}^{i+k}, $k = 0, 1, \cdots, p$, $i = 0, 1, \cdots, j - 1$. No matrix eigenvalue problem needs to be solved. The recurrence scheme for the determination of the coefficients in the expansions of λ_n and ψ_n requires a single inversion of an $n \times n$ matrix formed from some of the κ's. Once this matrix has been inverted (and granted the convergence of the method presented here), λ_n and ψ_n can be computed to any desired degree of accuracy by elementary operations.

Although awkward for hand calculations and for theoretical considerations, the recurrence is easily handled by a digital computer. Section 4 discusses the results of a number of computations made with this scheme. The method is derived in §2. The recurrence is summarized and described in §3.

2. Derivation of basic formulas. We make the following definitions:

$$(10) \qquad \chi_j(n) = 0 \quad \text{for} \quad j < 0,$$

$$(11) \qquad P_\mu(x) = 0 \quad \text{for} \quad \mu < 0,$$

$$(12) \qquad A_k^j(n) = 0 \quad \text{if} \quad |k| > j \quad \text{or } n + k < 0,$$

$$(13) \qquad A_0^j(n) = \delta_{j0},$$

$$(14) \qquad \kappa_{\mu\nu}^m = 0 \quad \text{if} \quad m < 0, \quad \mu < 0, \quad \nu < 0, \quad \text{or} \quad \mu + \nu > m.$$

[1] In the case of usual interest to us, w is a nonnegative weight function and the $P_\mu(x)$ are the associated orthogonal polynomials with P_μ of degree μ. The form (6) for the expansion of $K(cx, cy)$ then follows at once.

Unless indicated otherwise, in what follows all summation indices range over all values for which the summands are not by definition zero.

Insert (4), (5), (6) and (9) into (3). There results

$$\sum_{i,j,k} c^{2n}\chi_i(n)c^i c^j A_k^{\,j}(n)P_{n+k}(x)$$
$$= \int_a^b dy w(y) \sum_{\substack{m,\mu,\nu\\ s,t}} c^m \kappa_{\mu\nu}^m P_\mu(x)P_\nu(y)c^s A_t^{\,s}(n)P_{n+t}(y).$$

From (7), the right side vanishes unless $\nu = n + t$, so that we have

$$\sum_{i,j,k} c^{2n+i+j}P_{n+k}(x)\chi_i(n)A_k^{\,j}(n) = \sum_{\substack{m,\mu\\ s,t}} c^{m+s}P_\mu(x)\kappa_{\mu,n+t}^m A_t^{\,s}.$$

In the left member, set $i = l - j - 2n$ and $k = \mu - n$ to introduce the indices l and μ. On the right replace m by $l - s$. There results

$$\sum_{l,\mu,j} c^l P_\mu(x)\chi_{l-j-2n}(n)A_{\mu-n}^{\,j}(n) = \sum_{\substack{l,s\\ \mu,t}} c^l P_\mu(x)\kappa_{\mu,n+t}^{l-s}A_t^{\,s}(n)$$

so that (3) will hold if

$$\sum_j \chi_{l-j-2n}(n)A_{\mu-n}^{\,j}(n) = \sum_{s,t} \kappa_{\mu,n+t}^{l-s}A_t^{\,s}(n), \quad l, \mu = 0, 1, 2, \cdots. \tag{15}$$

From (10) and (12) we conclude that the left-hand sum of (15) vanishes identically if $l < n + \mu$. Equations (12) and (14) yield the same conclusions for the right-hand sum of (15). Accordingly, we set $l = \mu + n + p$ and write

$$\begin{aligned}(16)\quad &\sum_j \chi_{\mu+p-j-n}(n)A_{\mu-n}^{\,j}(n)\\ &\qquad = \sum_{s,t} \kappa_{\mu,n+t}^{\mu+n+p-s}A_t^{\,s}(n), \quad \mu, p = 0, 1, 2, \cdots.\end{aligned}$$

We shall show that these equations serve to determine the $\chi_j(n)$ and the $A_k^{\,j}(n)$ from the initial value (13) and the $\kappa_{\mu\nu}^m$.

It is convenient now to define

$$\begin{aligned} r_j^{\,i}(n) &= A_{-j}^{\,i+j}(n),\\ R_i^{\,j}(n) &= A_j^{\,i+j}(n), \qquad\qquad i, j = 0, 1, 2, \cdots,\\ r_j^{\,i}(n) &= R_j^{\,i}(n) = 0 \quad \text{if} \quad i < 0 \quad \text{or} \quad j < 0.\end{aligned} \tag{17}$$

We also define

$$\begin{aligned} S_\mu^{\,p}(n) &= \sum_{s,t} \kappa_{\mu,n+t}^{\mu+n+p-s}A_t^{\,s}(n)\\ &= \sum_{k,l} \kappa_{\mu,n-l}^{\mu+n+p-l-k}A_{-l}^{\,k+l}(n)\end{aligned} \tag{18}$$

on setting $t = -l$ and $s = k + l$. The last inequality of (14) and the first inequality of (12) show that the summand is nonzero only for $0 \leqq k \leqq p$. The summand also vanishes unless $-k \leqq l \leqq n$. Using (17), we now have

$$S_\mu^{\,p}(n) = \sum_{k=0}^{p} \sum_{l=1}^{n} \kappa_{\mu,n-l}^{\mu+n+p-l-k} r_l^{\,k}(n) + \sum_{l=-p}^{0} \sum_{k=-l}^{p} \kappa_{\mu,n-l}^{\mu+n+p-l-k} A_{-l}^{l+k}(n)$$

or

$$S_\mu^{\,p}(n) = \sum_{k=0}^{p} g_{\mu p}^{k}(n) + G_\mu^{\,p-2}(n). \tag{19}$$

Here

$$g_{\mu p}^{k}(n) \equiv \sum_{l=1}^{n} \kappa_{\mu,n-l}^{\mu+n+p-l-k} r_l^{\,k}(n) \tag{20}$$

and

$$\begin{aligned} G_\mu^{\,p-2}(n) &= \sum_{l=-p}^{0} \sum_{k=-l}^{p} \kappa_{\mu,n-l}^{\mu+n+p-l-k} A_{-l}^{l+k}(n) \\ &= \sum_{i=0}^{p} \sum_{\nu=0}^{p-1} \kappa_{\mu,n+i}^{\mu+n+p-\nu} A_i^{\,\nu}(n), \end{aligned}$$

where to obtain the last line we have set $i = -l$ and $\nu = k + l$. Since $A_i^{\,\nu}(n)$ vanishes if $\nu < i$, the inner sum can start from $\nu = i$ and the outer sum can be terminated at $k = [p/2]$, where the bracket denotes *integer part of*. Finally, on setting $\nu = i + j$, we find

$$G_\mu^{\,p-2}(n) = \sum_{i=0}^{[p/2]} \sum_{j=0}^{p-2i} \kappa_{\mu,n+i}^{\mu+n+p-i-j} R_i^{\,j}(n)$$

on using the definition (17). We note that for $p = 0$ and $p = 1$,

$$G_\mu^{-2}(n) = \kappa_{\mu,n}^{\mu+n}, \qquad G_\mu^{-1}(n) = \kappa_{\mu,n}^{\mu+n+1}, \tag{21}$$

which follows from (17) and (13). For $p \geqq 2$, the $i = 0$ terms can be split off to yield

$$G_\mu^{\,p-2}(n) = \kappa_{\mu,n}^{\mu+n+p} + \sum_{i=1}^{[p/2]} \sum_{j=0}^{p-2i} \kappa_{\mu,n+i}^{\mu+n+p-i-j} R_i^{\,j}(n). \tag{22}$$

Using (18), we see that the basic equation (15) now becomes

$$S_\mu^{\,p}(n) = \sum_j \chi_{\mu+p-j-n}(n) A_{\mu-n}^{j}(n), \qquad \mu, p = 0, 1, 2, \cdots. \tag{23}$$

We proceed to express the right member of this equation in an appropriate manner. For $\mu = 0, 1, 2, \cdots, n-1$, we have

$$(24)\quad \sum_j \chi_{\mu+p-j-n}(n) A^j_{\mu-n}(n) = \sum_{\nu=0}^{p-2(n-\mu)} \chi_{2(\mu-n)+p-\nu}(n) r^\nu_{n-\mu}(n) = w^{p-2(n-\mu)}_{n-\mu}(n),$$

where, to obtain the second sum, we have set $j = n - \mu + \nu$ and used (17). The right member is obtained from the definition

$$(25)\quad \begin{aligned} w_j^i(n) &= \sum_{\nu=0}^{i} \chi_{i-\nu}(n) r_j^\nu(n); \\ w_j^i(n) &\equiv 0, \qquad i < 0. \end{aligned}$$

In this notation, (23) gives

$$(26)\quad S_\mu^p(n) = w^{p-2(n-\mu)}_{n-\mu}(n), \quad \mu = 0, 1, \cdots, n-1, \quad p = 0, 1, 2, \cdots.$$

From (19), an equivalent form that will be useful is

$$(27)\quad \begin{aligned} g^p_{\mu p}(n) &= -\sum_{k=0}^{p-1} g^k_{\mu p}(n) - G_\mu^{p-2}(n) + w^{p-2(n-\mu)}_{n-\mu}(n), \\ &\mu = 0, 1, \cdots, n-1, \quad p = 0, 1, 2, \cdots. \end{aligned}$$

We also note that from (26), (25), (17), (13) and (12),

$$(28)\quad S_n^p(n) = \chi_p(n), \qquad p = 0, 1, 2, \cdots.$$

When $\mu = n, n+1, \cdots$, the right side of (23) can be written

$$\sum_j \chi_{\mu+p-j-n}(n) A^j_{\mu-n}(n) = \sum_{\nu=0}^{p} \chi_{p-\nu}(n) R^\nu_{\mu-n}(n),$$

where we have set $j = \mu - n + \nu$ and used (17). Now set $\mu - n = l$. The remainder of equations (23), i.e., those not already represented by (27), become

$$(29)\quad \begin{aligned} S^p_{n+l}(n) &= \chi_0(n) R_l^p(n) + \sum_{\nu=0}^{p-1} \chi_{p-\nu}(n) R_l^\nu(n), \\ &l, p = 0, 1, 2, \cdots. \end{aligned}$$

3. The recurrence scheme. The foregoing seemingly disconnected and randomly chosen formulas and definitions permit the systematic recursive calculation of the quantities $r_l^p(n)$, $R_l^p(n)$ and $\chi_p(n)$, $l = 0, 1, 2, \cdots$, from members of the set

$$Q_{p-1}(n) \equiv \left\{ r_j^i(n), R_j^i(n), \chi_i(n), \kappa^m_{\mu\nu} \,\middle|\, \begin{matrix} i = 0, 1, 2, \cdots, p-1, \\ j, \mu, \nu = 0, 1, 2, \cdots \end{matrix} \right\}.$$

To see this, let us suppose that the quantities in $Q_{p-1}(n)$ are known. Our notation has been chosen so that the auxiliary quantities $G_\mu^{\alpha}(n)$, $g_{\mu\nu}^{\alpha}(n)$ and $w_\mu^{\alpha}(n)$ depend only on quantities in $Q_\alpha(n)$, as can be verified from the definitions (21), (22), (20) and (25). For our induction then we can suppose that the quantities $G_\mu^{\alpha}(n)$, $g_{\mu\nu}^{\alpha}(n)$ and $w_\mu^{\alpha}(n)$ are known for $\alpha = 0, 1, 2, \cdots, p-1$.

The first step in the recurrence comes from (20) and (27) which can be combined to yield

$$\sum_{l=1}^{n} \kappa_{\mu,n-l}^{\mu+n-l} r_l^{p}(n) = -\sum_{k=0}^{p-1} g_{\mu p}^{k}(n) - G_\mu^{p-2}(n) + w_{n-\mu}^{p-2(n-\mu)}(n), \qquad \mu = 0, 1, \cdots, n-1. \tag{30}$$

The right members of these n equations are known. We suppose that the determinant is different from zero and solve them to obtain $r_1^p(n)$, $r_2^p(n)$, $\cdots$, $r_n^p(n)$. From (20) now $g_{\mu p}^{p}$ can be found and then from (19) the $S_\mu^p(n)$ are known. Equation (29) in the form

$$R_l^p(n) = \frac{1}{\chi_0(n)}\left[S_{n+l}^{p}(n) - \sum_{\nu=0}^{p-1} \chi_{p-\nu}(n) R_l^{\nu}(n)\right] \tag{31}$$

permits calculation of the quantities $R_1^p(n)$, $R_2^p(n)$, $\cdots$. Finally the new χ value is found from (28):

$$\chi_p(n) = S_n^p(n); \tag{32}$$

and the induction is complete.

By (17), the r's and R's are a rearrangement of the A's. To obtain the solution ψ of (3) to terms of order c^M and less, we must determine all A's with superscript M or less. This requires the determination of

$$R_0^p, R_1^p, \cdots, R_{M-p}^p, r_1^p, r_2^p, \cdots, r_n^p, \qquad p = 0, 1, \cdots, M-1, \tag{33}$$

by the recurrence. Examination of this procedure in detail shows that only the coefficients $\kappa_{\mu\nu}^{m}$, $\mu, \nu = 0, 1, \cdots, m$, $m = 0, 1, 2, \cdots, M + 2n$, are involved.

As an illustration of the recurrence we write out $\psi_n(x)$ to terms of order c^2 and λ_n/c^{2n} to terms of order c. We have $M = 2$ and by (33) need R_0^0, R_1^0, R_2^0, R_0^1, R_1^1 as well as $r_1^i, \cdots, r_n^i$, $i = 1, 2$. For $p = 0$, (30) becomes

$$\sum_{l=1}^{n} \kappa_{\mu,n-l}^{\mu+n-l} r_l^{0}(n) = -G_\mu^{-2}(n) = -\kappa_{\mu,n}^{\mu+n}, \qquad \mu = 0, 1, \cdots, n-1. \tag{34}$$

We set $(b_{ij}(n)) = (\kappa_{i,n-j}^{i+n-j})^{-1}$ so that

$$r_i^0(n) = -\sum_{j=0}^{n-1} b_{ij}(n) \kappa_{j,n}^{j+n}, \qquad i = 1, 2, \cdots, n.$$

Then

$$g^0_{\mu\nu}(n) = \sum_{l=1}^{n} \kappa_{\mu,n-l}^{\mu+n+\nu-l} r_l^0(n), \qquad \mu = 0, 1, \cdots, n+2, \quad \nu = 0, 1, 2,$$

and by (19) and (21),

$$S_\mu^0(n) = g^0_{\mu 0}(n) + G_\mu^{-2}(n) = g^0_{\mu 0}(n) + \kappa_{\mu,n}^{\mu+n}.$$

Now from (32),

$$\chi_0(n) = S_n^0(n) = \sum_{l=1}^{n} \kappa_{n,n-l}^{2n-l} r_l^0(n) + \kappa_{n,n}^{2n}. \tag{35}$$

From (31),

$$R_l^0 = \frac{S_l^0(n)}{\chi_0(n)}, \qquad l = 0, 1, 2.$$

Now with $p = 1$ we return to (30):

$$\sum_{l=1}^{n} \kappa_{\mu,n-l}^{\mu+n-l} r_l^1(n) = -g^0_{\mu 1}(n) + \kappa_{\mu,n}^{\mu+n+1}, \qquad \mu = 0, 1, \cdots, n-1,$$

or

$$r_i^1(n) = -\sum_{j=0}^{n-1} b_{ij}(n)[-g^0_{jl}(n) + \kappa_{j,n}^{j+n+1}], \qquad i = 1, 2, \cdots, n.$$

Then

$$\begin{aligned} S_\mu^1(n) &= \sum_{k=0}^{1} g^k_{\mu 1}(n) + \kappa_{\mu,n}^{\mu+n+1} \\ &= \sum_{k=0}^{1} \sum_{l=1}^{n} \kappa^{\mu+n+1-l-k} r_l^k(n) + \kappa_{\mu,n}^{\mu+n+1}, \\ \chi_1(n) &= S_n^1(n), \\ R_l^1(n) &= \frac{S^1_{n+1}(n) - \chi_1(n) R_l^0(n)}{\chi_0(n)}, \qquad l = 0, 1. \end{aligned}$$

To terms of order c^2, the solution of (3) is

$$\begin{aligned} \psi_n(x) &= P_n(x) + c[A^1_{-1}(n) P_{n-1}(x) + A_1^1(n) P_{n+1}(x)] \\ &\quad + c^2[A^2_{-2}(n) P_{n-2}(x) + A^2_{-1}(n) P_{n-1}(x) \\ &\quad + A_1^2(n) P_{n+1}(x) + A_2^2(n) P_{n+2}(x)] \\ &= P_n(x) + c[r_1^0(n) P_{n-1}(x) + R_1^0(n) P_{n+1}(x)] \\ &\quad + c^2[r_2^0(n) P_{n-2}(x) + r_1^1(n) P_{n-1}(x) \\ &\quad + R_1^1(n) P_{n+1}(x) + R_2^0(n) P_{n+2}(x)], \\ \lambda_n &= c^{2n}[\chi_0(n) + c\chi_1(n)]. \end{aligned}$$

By the preceding formulas, the r's, R's and χ's are given in terms of $\kappa_{\mu\nu}^{m}$ with $m \leqq 2n + 2$.

It does not seem to be useful to write out explicit relations for the r's and R's in terms of the κ's alone. The expression for $\chi_0(n)$, however, is particularly simple. Write

$$d_{ij} = \kappa_{i,n-j}^{i+n-j},$$

which suppresses the dependence of the d's on n. Equations (34) and (35) are

$$\sum_{l=1}^{n} d_{\mu l} r_l^0(n) = -d_{\mu 0}, \qquad \mu = 0, 1, \cdots, n-1,$$

$$\chi_0 = \sum_{l=1}^{n} d_{nl} r_l^0(n) + d_{n0}.$$

One readily finds from these equations that

$$\chi_0(n) = (-1)^n \frac{\begin{vmatrix} d_{00} & \cdots & d_{0n} \\ \vdots & & \vdots \\ d_{n0} & \cdots & d_{nn} \end{vmatrix}}{\begin{vmatrix} d_{01} & \cdots & d_{0n} \\ \vdots & & \\ d_{n-1,1} & \cdots & d_{n-1,n} \end{vmatrix}}.$$

It is shown in the Appendix that this is in agreement with the result given by Fredholm theory.

4. Results of computations. Analysis of the convergence properties of the foregoing scheme appears to be difficult. To get some feel for its range of applicability, a computer program was written for the algorithm and some trial computations carried out. The recurrence simplifies somewhat for kernels having the symmetry $K(-x, -y) = K(x, y)$, and only this case was investigated numerically. The program used the values $w(y) \equiv 1$, $a = -1$, $b = 1$ so that the $P_\mu(x)$ were the suitably normalized Legendre polynomials. Three different kernels were investigated:

(i) *The prolate spheroidal wave functions.* The integral equation

$$\chi\psi(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} \psi(y)\, dy \tag{36}$$

has been much studied in the past [1]. The eigenfunctions are certain prolate spheroidal wave functions. Numerical tables of the ψ's and χ's have been published [2], [4], [5] and programs for computing these quantities with great accuracy are available. The recurrence described in this paper was tried for $K = \sin (x - y)/(x - y)$ for a number of different n and c

values. The ψ's computed by the recurrence were compared with the appropriate prolate spheroidal functions, and the λ's computed by the present scheme were compared with the quantity $\pi\chi/c$. The input to the recurrence program, the κ's defined by (6), are found to be

$$\kappa_{2\alpha,2\beta}^{2m} = \frac{(-1)^m}{(2m+1)!\sqrt{(2\alpha+1/2)(2\beta+1/2)}} \sum_{j=\alpha}^{m-\beta} \binom{2m}{2j} U_\alpha^{\ j} U_\beta^{\ m-j},$$

$$\kappa_{2\alpha+1,2\beta+1}^{2m} = \frac{(-1)^{m+1}}{(2m+1)!\sqrt{(2\alpha+3/2)(2\beta+3/2)}} \sum_{j=\alpha+1}^{m-\beta} \binom{2m}{2j-1} V_\alpha^{\ j-1} V_\beta^{\ m-j}$$

with

$$U_j^{\ n} = \frac{(4j+1)2^{2j}(2n)!(n+j)!}{(n-j)!(2n+2j+1)!},$$

$$V_j^{\ n} = \frac{(4j+3)2^{2j+1}(2n+1)!(n+j+1)!}{(n-j)!(2n+2j+3)!}.$$

For small c, the algorithm converged rapidly. As c was increased, the partial sum of (4) appeared to oscillate rather than approach a fixed value. Some feel for the behavior of the convergence is given from the following (n, c, s, t) symbols which give the first term number t at which the partial sum of (4) for λ_n appeared to stop changing in the sth significant figure: (0, .1, 8, 3), (0, .5, 8, 5), (0, 1, 8, 6) (0, 2, 8, 13) (0, 3, 3, 24),(1, .1, 8, 5), (1, .5, 8, 7), (1, 1, 8, 10) (1, 2, 2, 24) (4, .1, 8, 4) (4, 1, 8, 12), (7, 1, 8, 5), (7, 3, 8, 9), (7, 7, 5, 11). In all cases reported here, 24 terms of (4) were computed. When the partial sums settled down to 8 significant figures in 8 or fewer terms, agreement of the λ's computed here with the quantities $\pi\chi/c$ was to 6 or 7 places. When convergence took longer, the accuracy seemed to be less, especially for larger values of n. The ψ's behaved much like the λ's with regard to convergence and accuracy.

(ii) *A degenerate kernel.* The kernel

$$K(cx, cy) = A + D \cos cx \cos cy + c^2Exy$$

with $A + D = 1$ has eigenvalues and eigenfunctions

$$\begin{aligned}
&\lambda_0 = a + b, \quad \lambda_1 = \tfrac{2}{3} Ec^2, \quad \lambda_2 = a - b,\\
&\lambda_n = 0 \quad \text{for} \quad n > 2,\\
&\psi_0 = 2A \frac{\sin c}{c} + (\lambda_0 - 2A) \cos cx,\\
&\psi_1 = x,\\
&\psi_2 = 2A \frac{\sin c}{c} + (\lambda_2 - 2A) \cos cx.
\end{aligned} \tag{37}$$

Here

$$a = A + \frac{D}{2}\left(1 + \frac{\sin 2c}{2c}\right),$$

$$b = \sqrt{\left[A - \frac{D}{2}\left(1 + \frac{\sin 2c}{2c}\right)\right]^2 + 4AD\left(\frac{\sin c}{c}\right)^2}.$$

One finds $\kappa_{00}^{0} = 2$, $\kappa_{01}^{1} = 2B/\sqrt{3}$, $\kappa_{11}^{2} = 2E/3$, and all other nonzero κ's are given by

$$\kappa_{2\alpha,2\beta}^{2j} = D\sqrt{(2\alpha + \tfrac{1}{2})(2\beta + \tfrac{1}{2})}\, 2^{2\alpha+2\beta+2}(-1)^j$$

$$\cdot \sum_{n=0}^{j-\alpha-\beta} \frac{(\alpha + j - n - \beta)!(2\beta + n)!}{n!(j - \alpha - \beta - n)!(2\alpha + 2j - 2\beta - 2n + 1)!(4\beta + 2n + 1)!}.$$

The recurrence algorithm was computed for this kernel with $A = .75 - i$, $D = .25 + i$ and $E = 3.0$. Results were compared with the exact solutions (37). For λ_1 and ψ_1, the present algortihm gives exact values for all c in this degenerate case. For $n = 0, 2, 4$, the recurrence was carried out for 24 terms. At $c = .75$, λ_0 and λ_2 were correct to 7 figures after 11 terms; at $c = 1.25$, they were correct to 5 figures after 24 terms; at $c = 1.5$, there was no convergence. The ψ's behaved in a quite similar manner. Some mystery surrounds the computation of λ_4. Convergence was similar to that obtained for λ_0 and λ_2, no change in the 7th figure after about 11 terms for $c = .75$, apparent convergence to 4 or 5 figures at $c = 1.25$ after 24 terms, no convergence at $c = 1.5$ after 24 terms. When convergence was obtained, it was to a complex number of magnitude 10^{-12}, whereas the true value of λ_4 is zero. It is likely that the computed number is the result of the accumulated roundoff error from the many steps of the algorithm. On the computer used, the smallest nonzero number that can be handled is 10^{-38}.

(iii) *The bandpass kernel.* Solutions of

$$\frac{\pi\lambda}{bc}\psi(x) = \int_{-1}^{1} \frac{\sin cb(x - y)}{cb(x - y)} \cos ac(x - y)\psi(y)\, dy$$

were investigated by the present scheme. Interest was centered on finding those values of $A = ac$ and $B = bc$ for which degeneracy of eigenvalues occurred. Such degeneracy has been studied by Morrison [3]. For fixed values of a and b, λ_0, λ_1 and λ_2 were plotted as functions of c. Points of crossings of the curves were sought. As in the other cases studied, the method worked very well for small c, but failed to converge (in 24 terms) as c became large. Some typical values where convergence began to fail are: for $a = .15$, $b = .1$, $c = 10$ for λ_0, $c = 17$ for λ_1, $c = 10.5$ for λ_2; for $a = .25$, $b = .1$, $c = 7.5$ for λ_0, $c = 10$ for λ_1, and $c = 7$ for λ_2; for $a = .4$, $b = .1$, $c = 2$ for λ_0, $c = 3.5$ for λ_1, $c = 2$ for λ_2. The computations permitted finding the locus in the $(A\text{-}B)$-plane where $\lambda_0 = \lambda_1$. The locus agreed with that described by Morrison [3]. Convergence difficulties by this method prevented finding parameter values for which $\lambda_1 = \lambda_2$.

As we summarize the very brief experience to date, it would seem that the method works well for kernels that are close to the degenerate kernel $\kappa \equiv 1$ in the sense that they do not oscillate in the range of integration. Oscillations seem to cause convergence difficulties. Much more computing experience is needed, however, to fully explore the value of this method. Theoretical investigation of the range of validity of the expansions would be welcomed.

Appendix. Fredholm development. We are concerned with the equation

$$\psi(x) = \nu \int_a^b K(cx, cy)\psi(y)w(y)\,dy \tag{A1}$$

for small values of c. This is (3) with λ replaced by $1/\nu$. We assume

$$K(cx, cy) = \sum_{\alpha,\beta} c^{\alpha+\beta} a_{\alpha\beta} x^{\alpha} y^{\beta}. \tag{A2}$$

The Fredholm determinant $\mathfrak{D}(\nu)$, whose zeros are the characteristic values of K, is given by

$$\mathfrak{D}(\nu) = 1 + \sum_{n=1}^{\infty} (-1)^n d_n \nu^n, \tag{A3}$$

where

$$d_n = \frac{1}{n!} \int_a^b dx_1 \cdots \int_a^b dx_n \,|\, K(cx_1, cx_j)w(x_j) \,|.$$

Here the bars denote an $n \times n$ determinant and we indicate the structure of the matrix by exhibiting the element in the ith row and jth column. On using (A2), this becomes

$$d_n = \frac{1}{n!} \int_a^b dx_1 \cdots \int_a^b dx_n \,\Big|\, \sum_{\alpha_i,\beta_i} c^{\alpha_i+\beta_i} a_{\alpha_i\beta_i} x_i^{\alpha_i} x_j^{\beta_i} w(x_j) \,\Big|,$$

where we write α_i and β_i for the summation indices in the ith row of the array. The usual rules for manipulating determinants now yield

$$\begin{aligned} d_n = \frac{1}{n!} \sum_{\alpha_1,\cdots,\alpha_n} \sum_{\beta_1,\cdots,\beta_n} \exp\left((\log c) \sum_{\nu=1}^{n} (\alpha_\nu + \beta_\nu)\right) \prod_{\mu=1}^{n} a_{\alpha_\mu\beta_\mu} \\ \cdot \int_a^b dx_1 \cdots \int_a^b dx_n \prod_{\sigma=1}^{n} x_\sigma^{\alpha_\sigma} \,|\, x_j^{\beta_i} w(x_j) |. \end{aligned} \tag{A4}$$

But

$$\prod_{\sigma=1}^{n} x_\sigma^{\alpha_\sigma} \mid x_j^{\beta_i} w(x_j) \mid = \mid x_j^{\alpha_j+\beta_i} w(w_j) \mid \tag{A5}$$

so that the integration indicated in (A4) can be done column by column inside the determinant (A5). Writing

$$h_\gamma = \int_a^b x^\gamma w(x)\, dx, \tag{A6}$$

we have

$$d_n = \frac{1}{n!} \sum_{\alpha_1,\cdots,\alpha_n} \sum_{\beta_1,\cdots,\beta_n} \exp\left((\log c) \sum_{\nu=1}^{n} (\alpha_\nu + \beta_\nu)\right) \prod_{\mu=1}^{n} a_{\alpha_\mu \beta_\mu} \mid h_{\alpha_j+\beta_i} \mid . \tag{A7}$$

Sum this equation over all $n!$ ways of relabeling the dummy indices $\beta_1, \cdots, \beta_n$ to obtain

$$n!\, d_n = \frac{1}{n!} \sum_{\alpha_1,\cdots,\alpha_n} \sum_{\beta_1,\cdots,\beta_n} \exp\left((\log c) \sum_{\nu=1}^{n} (\alpha_\nu + \beta_\nu)\right) \mid h_{\alpha_j+\beta_i} \mid \mid a_{\alpha_i+\beta_j} \mid .$$

Here we have used the fact that $\mid h_{\alpha_j+\beta_i} \mid$ is invariant under an even permutation of the β's and changes sign under an odd permutation. Noting now that both determinants change signs under odd permutations of the α's or β's and that both vanish if two α's or two β's are equal, we can finally write

$$d_n = \sum_{\alpha_1,\cdots,\alpha_n}{}' \sum_{\beta_1,\cdots,\beta_n}{}' \exp\left((\log c) \sum_{\nu=1}^{n} (\alpha_\nu + \beta_\nu)\right) \mid h_{\alpha_i+\beta_j} \mid \mid a_{\alpha_i+\beta_j} \mid , \tag{A8}$$

where the sums are over all distinct ordered n-tuples $0 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and all distinct ordered n-tuples $0 \leqq \beta_1 < \beta_2 < \cdots < \beta_n$.

Now set

$$B_n \begin{pmatrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{pmatrix} = A_n \begin{pmatrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{pmatrix} H_n \begin{pmatrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{pmatrix},$$

where $A_n \begin{pmatrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{pmatrix}$ is the determinant formed from rows $i_1, i_2, \cdots, i_n$ and columns $j_1, j_2, \cdots, j_n$ of the array $A = (a_{ij})$, and $H_n \begin{pmatrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{pmatrix}$ is formed similarly from $H = (h_{i+j})$. Then from (A8),

$$\begin{aligned} d_n &= c^{n(n-1)} \left[B_n \begin{pmatrix} 0, 1, \cdots, n-1 \\ 0, 1, \cdots, n-1 \end{pmatrix} + c \left\{ B \begin{pmatrix} 0, 1, \cdots, n-2, n \\ 0, 1, \cdots, n-2, n-1 \end{pmatrix} \right.\right. \\ &\qquad \left.\left. + B \begin{pmatrix} 0, 1, \cdots, n-2, n-1 \\ 0, 1, \cdots, n-2, n \end{pmatrix} \right\} + \cdots \right] \\ &= c^{n(n-1)} \sum_{j=0}^{\infty} E_{nj}\, c^j, \end{aligned}$$

and from (A3),

$$\mathfrak{D}(\nu) = 1 + \sum_{n=1}^{\infty} (-1)^n \nu^n c^{n(n-1)} \sum_{j=0}^{\infty} E_{nj} c^j.$$

Now let

$$\nu_k = \frac{1}{c^{2k}} \sum_{r=0}^{\infty} L_r\, c^r.$$

Then

$$\mathfrak{D}(\nu_k) = \sum_{n=0}^{\infty} c^{n^2-n-2kn} \sum_{j=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} (-1)^n E_{nj}\, c^j \prod_{\mu=1}^{n} c^{r_\mu} L_{r_\mu}.$$

For small c, the dominant behavior of this expression is obtained from the terms for which $f(n) = n^2 - n - 2kn$ is a minimum. This occurs for $n = k$ and $n = k + 1$ and we find

$$\mathfrak{D}(\nu_k) = [(-1)^k L_0^k E_{k,0} + (-1)^{k+1} L_0^{k+1} E_{k+1,0}] c^{-k(k+1)} + O(c^{-k(k+1)+1}).$$

For ν_k to be a characteristic value, then, we must have

$$L_0 = \frac{E_{k,0}}{E_{k+1,0}},$$

or recalling that $\lambda = 1/\nu$, we have

$$\lambda_k = \frac{E_{k+1,0}}{E_{k,0}} c^{2k} + O(c^{2k+1}).$$

In terms of determinants and our expansion (4), this is

$$\text{(A9)} \qquad \chi_0(n) = \frac{|\, a_{ij}\,|_n \,|\, h_{i+j}\,|_n}{|\, a_{ij}\,|_{n-1} \,|\, h_{i+j}\,|_{n-1}},$$

where the row and column index in each determinant goes from zero to the subscript written outside the determinant.

We now show that (A9) and (37) agree when P_μ is a polynomial of degree μ. Comparison of (6) and (A2) shows that

$$\sum_{k=0}^{m} a_{k,m-k} x^k y^{m-k} = \sum_{\mu,\nu} \kappa_{\mu\nu}^m P_\mu(x) P_\nu(y),$$

so that

$$\kappa_{\mu\nu}^m = \sum_{k=0}^{m} a_{k,m-k} \int_a^b x^k P_\mu(x) w(x)\, dx \int_a^b y^{m-k} P_\nu(y) w(y)\, dy$$

from (7). In particular, from (36),

$$d_{ij} = \kappa_{i,n-j}^{i+n-j} = \sum_{k=0}^{i+n-j} a_{k,i+n-j-k} \int_a^b x^k P_i(x) w(x)\, dx \int_a^b y^{i+n-j-k} P_{n-j}(y) w(y)\, dy.$$

But

$$\int_a^b x^\alpha P_\mu(x) w(x)\, dx = 0 \quad \text{if} \quad \alpha < \mu,$$

so that only the term $k = i$ does not vanish in this sum. We have therefore

$$d_{ij} = a_{i,n-j} b_i b_{n-j}\,, \tag{A10}$$

where

$$b_\alpha = \int_a^b x^\alpha P_\alpha(x) w(x)\, dx \neq 0. \tag{A11}$$

In view of (A10), equation (37) now becomes

$$\chi_0(n) = (-1)^n \frac{\begin{vmatrix} a_{0n} & \cdots & a_{00} \\ \vdots & & \vdots \\ a_{nn} & \cdots & a_{n0} \end{vmatrix} b_0^2 \cdots b_n^2}{\begin{vmatrix} a_{0n-1} & \cdots & a_{00} \\ \vdots & & \vdots \\ a_{n-1,n-1} & \cdots & a_{n-1,0} \end{vmatrix} b_0^2 \cdots b_{n-1}^2} = b_n^2 \frac{|\, a_{ij} \,|_n}{|\, a_{ij} \,|_{n-1}}.$$

Comparison with (A9) shows our task will be complete if we prove that

$$b_n^2 = \frac{|\, h_{i+j} \,|_n}{|\, h_{i+j} \,|_{n-1}}. \tag{A12}$$

But this follows readily from the fact that

$$P_n(x) = \sum_{\alpha=0}^{n} c_\alpha x^\alpha, \qquad c_n \neq 0,$$

is orthogonal to x^β for $\beta = 0, 1, \cdots, n - 1$. Using the definition (A6), we see that this statement can be written

$$\sum_{\alpha=0}^{n} c_\alpha h_{\alpha+\beta} = 0, \qquad \beta = 0, 1, \cdots, n - 1,$$

while from (A11)

$$\sum_{\alpha=0}^{n} c_\alpha h_{\alpha+n} = b_n\,.$$

These $n + 1$ equations yield

$$c_n = b_n \frac{|\, h_{i+j} \,|_{n-1}}{|\, h_{i+j} \,|_n}. \tag{A13}$$

Now

$$x^n = b_nP_n(x) + e_{n-1}P_{n-1}(x) + \cdots + e_0P_0(x)$$
$$= b_n[c_nx^n + \cdots] + e_{n-1}P_{n-1}(x) + \cdots + e_0P_0(x).$$

On comparing the coefficient of x^n on both sides, we find

$$1 = b_n\, c_n = b_n^{\ 2}\, \frac{|\, h_{i+j}\, |_{n-1}}{|\, h_{i+j}\, |_n},$$

where the last equality follows from (A13). This equation is (A12), however, and our proof is now complete.

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