# Some mean value results related to Hardy's function

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#### Abstract

Let  $\zeta(s)$  and Z(t) be the Riemann zeta function and Hardy's function respectively. We show asymptotic formulas for  $\int_0^T Z(t)\zeta(1/2+it)dt$  and  $\int_0^T Z^2(t)\zeta(1/2+it)dt$ . Furthermore we derive an upper bound for  $\int_0^T Z^3(t)\chi^\alpha(1/2+it)dt$  for  $-1/2<\alpha<1/2$ , where  $\chi(s)$  is the function which appears in the functional equation of the Riemann zeta function:  $\zeta(s)=\chi(s)\zeta(1-s)$ .

### 1 Introduction

Let Z(t) be Hardy's function defined by

$$Z(t) = \zeta(1/2 + it)\chi^{-1/2}(1/2 + it),$$

where as usual  $\zeta(s)$  is the Riemann zeta-function and  $\chi(s)$  is the gamma factor appearing in the functional equation of  $\zeta(s)$ :

(1.1) 
$$\zeta(s) = \chi(s)\zeta(1-s).$$

The explicit form of  $\chi(s)$  is

(1.2) 
$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)$$

and its asymptotic behavour is given by

(1.3) 
$$\chi(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{1/2 - \sigma - it} e^{i(t \pm \frac{\pi}{4})} \left(1 + O\left(\frac{1}{|t|}\right)\right)$$

for  $|t| \ge t_0 > 0$ , where  $t \pm \frac{\pi}{4} = t + \operatorname{sgn}(t) \frac{\pi}{4}$ . (See Ivić [2].)

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From (1.1), it follows that Z(t) is a real-valued even function for real t and  $|Z(t)| = |\zeta(1/2 + it)|$ . Therefore the zeros of  $\zeta(s)$  on the critical line  $\operatorname{Re} s = 1/2$  coincide with the real zeros of Z(t). Hardy proved that  $\int_0^T Z(t)dt \ll T^{7/8}$  and  $\int_0^T |Z(t)|dt > \frac{1}{2}T$ , from which he succeeded to show the infinity of the number of zeros of  $\zeta(s)$  on the critical line. (See [1, p. 51].)

However, in 2004, Ivić [3] proved that

$$\int_0^T Z(t)dt \ll T^{1/4+\varepsilon},$$

where  $\varepsilon$  is an arbitrary small positive number which needs not be the same at each occurrence. It shows that Z(t) changes sign quite often. Ivić's result was sharpened by Jutila [8, 9] and Korolev [11] independently. From  $Z(t)^2 = |\zeta(1/2 + it)|^2$ , we see that

$$\int_0^T Z^2(t)dt = T\log T + (2\gamma - 1 - \log 2\pi)T + O(T^{1/3 + \varepsilon}).$$

Since there is a lot of cancellation it is expected that the cubic power moment has an exponent less than 1. In fact, Ivić showed that

$$\int_{T}^{2T} Z^{3}(t)dt = 2\pi \sqrt{\frac{2}{3}} \sum_{\left(\frac{T}{2\pi}\right)^{3/2} \le n \le \left(\frac{T}{\pi}\right)^{3/2}} \frac{d_{3}(n)}{n^{1/6}} \cos\left(3\pi n^{2/3} + \frac{1}{8}\pi\right) + O(T^{3/4+\varepsilon})$$

and conjectured that

(1.4) 
$$\int_0^T Z^3(t)dt \ll T^{3/4+\varepsilon},$$

([5, Chapter 11]), but we only know that the left hand side  $\ll T(\log T)^{5/2}$  at present. Here  $d_3(n)$  denotes the number of triples  $(k_1, k_2, k_3)$  such that  $n = k_1 k_2 k_3, k_j \in \mathbb{Z}, k k_j > 0$ .

In this paper we shall prove the several mean values of the functions combined with Z(t) and  $\zeta(1/2+it)$ .

**Theorem 1.** For large T > 0, we have

$$\int_0^T Z(t)\zeta\left(\frac{1}{2} + it\right)dt = \frac{2\sqrt{2}\pi}{3}e^{\frac{\pi i}{8}}\left(\frac{T}{2\pi}\right)^{3/4}\left(\frac{1}{2}\log\frac{T}{2\pi} + 2\gamma - 2\log 2 - \frac{2}{3}\right) + O(T^{1/2}\log^2 T).$$

Ivić's conjecture (1.4) would follow from the bound of exponential sum

(1.5) 
$$\sum_{N \le n \le 2\sqrt{2}N} \frac{d_3(n)}{n^{1/6}} e^{3\pi i n^{2/3}} \ll N^{1/2+\varepsilon},$$

or, as Ivić noted [6, (1.6)], from

(1.6) 
$$\sum_{N \le n \le 2N} d_3(n) e^{3\pi i n^{2/3}} \ll N^{2/3 + \varepsilon}.$$

It seems that (1.6) (or (1.5)) is out of reach of the present method of exponential sums. However if we replace  $d_3(n)$  by d(n) (the divisor function  $d(n) = \sum_{n=d_1d_2} 1$ ), we can prove the following theorem in the frame of Theorem 1

**Theorem 2.** Let A be a parameter such that  $A \gg N^{-1/4}$ . Then we have

$$\sum_{N \le k \le 2\sqrt{2}N} \frac{d(k)}{k^{1/6}} e^{3\pi i (Ak)^{2/3}}$$

$$= \sqrt{3} A^{-4/3} \sum_{A^{4/3}N^{1/3} \le k \le \sqrt{2}A^{4/3}N^{1/3}} d(k) k^{1/2} e^{-\pi i (k/A)^2}$$

$$+ O(A^{-1/3}N^{1/2+\varepsilon}) + O(A^{1/3}N^{1/6} \log N) + O(A^{-1/9}N^{2/9+\varepsilon})$$

$$\ll A^{2/3}N^{1/2} \log N.$$

For another kind of mean value of Z(t) and  $\zeta(1/2+it)$  we have

**Theorem 3.** For large T > 0 we have

$$\int_0^T Z^2(t)\zeta(1/2+it)dt = T\left\{\frac{1}{2}\left(\log\frac{T}{2\pi}\right)^2 + a_1\log\frac{T}{2\pi} + a_2\right\} + O(T^{3/4+\varepsilon}),$$

where  $a_1 = 3\gamma - 1$ ,  $a_2 = 3\gamma_1 + 3\gamma^2 - 3\gamma + 1$ ,  $\gamma_j$  being the coefficients of Laurant expansion of  $\zeta(s)$  at s = 1 and  $\gamma = \gamma_0$  the Euler constant.

We note that the integral of the left hand side has an asymptotic form. It may be interesting to compare with Ivić's conjecture (1.4).

As for another mean value, we shall prove the following

**Theorem 4.** Let  $\alpha$  be a real fixed constant such that  $-1/2 < \alpha < 1/2$ . Then we have

$$\int_{T}^{2T} Z^{3}(t) \chi^{\alpha}(1/2 + it) dt \ll \begin{cases} T^{1 - \frac{\alpha}{6} + \varepsilon} & \text{if } 0 \le \alpha < 1/2 \\ T^{1 + \frac{\alpha}{6} + \varepsilon} & \text{if } -1/2 < \alpha \le 0. \end{cases}$$

The cubic moment of Hardy's function corresponds to  $\alpha=0$ , but unfortunately this gives only  $O(T^{1+\varepsilon})$ .

#### 2 Some lemmas

**Lemma 1.** Suppose that f(x) and  $\varphi(x)$  are real-valued functions on the interval [a,b] which satisfy the conditions

- 1)  $f^{(4)}(x)$  and  $\varphi''(x)$  are continuous.
- 2) there exist numbers  $H, A, U, 0 < H, A < U, 0 < b a \le U$ , such that

$$A^{-1} \ll f''(x) \ll A^{-1}, \quad f^{(3)} \ll A^{-1}U^{-1}, \quad f^{(4)}(x) \ll A^{-1}U^{-2}$$
  
$$\varphi(x) \ll H, \quad \varphi'(x) \ll HU^{-1}, \quad \varphi''(x) \ll HU^{-2}.$$

3) f'(c) = 0 for some  $c, a \le c \le b$ . Then

$$\int_{a}^{b} \varphi(x) \exp(2\pi i f(x)) dx = \frac{1+i}{\sqrt{2}} \frac{\varphi(c) \exp(2\pi i f(c))}{\sqrt{f''(c)}} + O(HAU^{-1}) + O\left(H \min(|f'(a)|^{-1}, \sqrt{A})\right) + O\left(H \min(|f'(b)|^{-1}, \sqrt{A})\right).$$

This is Lemma 2 of Karatsuba-Voronin [10, p.71].

Remark 1. Here we give an important remark. As is noted in Ivić and Zhai [7], the proof actually shows that if there is no c which satisfies the condition 3, the term containing c does not appear in the right hand side. Moreover if c = a or c = b, then the main term is to be halved.

**Lemma 2.** For  $\frac{1}{2} \le \sigma < 1$  fixed,  $1 \ll x, y \ll t^k, s = \sigma + it, xy = (\frac{t}{2\pi})^k, t \ge t_0$  and  $k \ge 1$  a fixed integer, we have

$$\zeta^{k}(s) = \sum_{m=1}^{\infty} \rho\left(\frac{m}{x}\right) d_{k}(m) m^{-s} + \chi^{k}(s) \sum_{m=1}^{\infty} \rho\left(\frac{m}{y}\right) d_{k}(m) m^{s-1} + O(t^{k(1-\sigma)/3-1}) + O(t^{k(1/2-\sigma)-2} y^{\sigma} \log^{k-1} t).$$

Here  $\chi(s)$  is the function defined by (1.2) and  $\rho(u) (\geq 0)$  is a smooth function such that  $\rho(u) + \rho(1/u) = 1$  for u > 0 and  $\rho(u) = 0$  for  $u \geq 2$ .

This is Lemma 4 of [7]. See also [5, Theorem 4.16]. For the proof of Theorem 4 we need the following lemma.

**Lemma 3.** Let  $\alpha, \beta, \gamma$  be fixed real numbers such that  $\alpha(\alpha - 1)\beta\gamma \neq 0$  and let

$$S = \sum_{h=H+1}^{2H} \sum_{n=N+1}^{2N} \left| \sum_{M < m \le 2M} e \left( X \frac{m^{\alpha} h^{\beta} n^{\gamma}}{M^{\alpha} H^{\beta} N^{\gamma}} \right) \right|^{*},$$

where \* means that

$$\left| \sum_{N \le n \le N'} z_n \right|^* = \max_{N \le N_1 \le N_2 \le N'} \left| \sum_{n=N_1}^{N_2} z_n \right|.$$

Then we have

$$S \ll (HNM)^{1+\varepsilon} \left\{ \left( \frac{X}{HNM^2} \right)^{1/4} + \frac{1}{M^{1/2}} + \frac{1}{X} \right\}.$$

This is Theorem 3 of Robert and Sargos [12]. Note that  $e(x) := e^{2\pi ix}$ .

### 3 Proofs of Theorem 1 and 2

Proof of Theorem 1 Let T > 0 be a large number and put

(3.1) 
$$J = \int_{T}^{2T} Z(t)\zeta\left(\frac{1}{2} + it\right)dt.$$

By the definition of Z(t) and applying Lemma 2 we have

$$\begin{split} Z(t)\zeta\left(\frac{1}{2}+it\right) &= \zeta^2\left(\frac{1}{2}+it\right)\chi^{-1/2}\left(\frac{1}{2}+it\right) \\ &= \left(\sum_{k=1}^\infty \rho\left(\frac{k}{x}\right)\frac{d(k)}{k^{1/2+it}} + \chi^2\left(\frac{1}{2}+it\right)\sum_{k=1}^\infty \rho\left(\frac{k}{y}\right)\frac{d(k)}{k^{1/2-it}} \right. \\ &\quad + O\left(t^{-2/3}\right) + O\left(t^{-2}y^{1/2}\log t\right)\left(\chi^{-1/2}\left(\frac{1}{2}+it\right), \end{split}$$

where  $xy = (t/2\pi)^2$ . Substituting this expression to (3.1), we have

$$(3.2) J = J_1 + J_2 + O(T^{1/3}),$$

where

(3.3) 
$$J_1 = \sum_{k=1}^{\infty} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) k^{-it} \chi^{-1/2} \left(\frac{1}{2} + it\right) dt,$$

and

(3.4) 
$$J_2 = \sum_{k=1}^{\infty} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) k^{it} \chi^{3/2} \left(\frac{1}{2} + it\right) dt.$$

We take

$$x = 2\left(\frac{t}{2\pi}\right), \quad y = \frac{1}{2}\left(\frac{t}{2\pi}\right),$$

and put  $K = \frac{T}{\pi}$ . Then the ranges of k in the sums in (3.3) and (3.4) are in fact  $k \leq 4K$  and  $k \leq K$  respectively.

We first consider  $J_1$ . By (1.3), we find that

$$k^{-it}\chi^{-1/2}\left(\frac{1}{2} + it\right) = e^{-\frac{\pi i}{8}}e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log k^2)} + O(1/t),$$

therefore we have

$$J_1 = e^{-\frac{\pi i}{8}} \sum_{k < 4K} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log k^2)} dt + O(T^{1/2}\log T).$$

We evaluate the integral by applying Lemma 1 with  $\varphi(t) = \rho\left(k\left(\frac{\pi}{t}\right)\right)$ ,  $f(t) = \frac{1}{4\pi}(t\log\frac{t}{2\pi} - t - t\log k^2)$ . Note that  $\varphi(t)$  satisfies the conditions of Lemma 1 with H = 1, U = T. Since  $f'(t_0) = 0$  if and only if  $t_0 = 2\pi k^2$ , the main term of the integral appears for k such that

$$\left(\frac{T}{2\pi}\right)^{1/2} \le k \le \left(\frac{T}{\pi}\right)^{1/2}.$$

Hence we get

$$\begin{split} & \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log k^2)} dt \\ &= M(k) + O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{T}{\pi k^2})|}\right)\right), \end{split}$$

where

$$M(k) = e^{\frac{\pi i}{4}} \rho\left(\frac{1}{2k}\right) 2\sqrt{2\pi k} e^{-\pi i k^2} = 2\sqrt{2\pi} e^{\frac{\pi i}{4}} k(-1)^k$$

for k satisfying the condition (3.5) and 0 otherwise. This yields that

$$J_{1} = 2\sqrt{2\pi}e^{\frac{\pi i}{8}} \sum_{(\frac{T}{2\pi})^{1/2} \le k \le (\frac{T}{\pi})^{1/2}} ' (-1)^{k} d(k) k^{1/2}$$

$$+ \sum_{k \le 4K} \frac{d(k)}{k^{1/2}} O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^{2}})|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{T}{\pi k^{2}})|}\right)\right)$$

$$+ O(T^{1/2} \log T)$$

$$=: R_{0} + R_{1} + R_{2} + R_{3} + O(T^{1/2} \log T),$$

where  $\sum'$  means that the terms for  $k = (T/2\pi)^{1/2}$  and  $k = (T/\pi)^{1/2}$  are to be halved if they are integers. It is clear that  $R_1 \ll T^{1/2} \log T$ . To estimate  $R_2$ , we divide the sum into four parts:

$$\begin{split} \sum_{k \le 4K} &= \sum_{1 \le k < \frac{1}{2} (\frac{T}{2\pi})^{1/2}} + \sum_{\frac{1}{2} (\frac{T}{2\pi})^{1/2} \le k < (\frac{T}{2\pi})^{1/2}} + \sum_{(\frac{T}{2\pi})^{1/2} \le k \le 2 (\frac{T}{2\pi})^{1/2}} + \sum_{2 (\frac{T}{2\pi})^{1/2} < k \le 4K} \\ &=: S_1 + S_2 + S_3 + S_4. \end{split}$$

For  $S_1$ , since  $\min(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}) \ll \frac{1}{\log 4}$ , we have  $S_1 \ll T^{1/4} \log T$ . For  $S_4$ , we have the same upper bound for  $\min(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|})$ , hence we have

 $S_4 \ll T^{1/2} \log T$ . Now we consider  $S_2$ . We write  $k = [(\frac{T}{2\pi})^{1/2}] - j$  for k in this range and set  $S_2 = S_{2,1} + S_{2,2}$ , where  $S_{2,1}$  is the sum for j = 0, 1, 2 and  $S_{2,2}$  is the sum for  $j \geq 3$ . For  $S_{2,1}$  we adopt  $\min(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}) = \sqrt{T}$ , hence  $S_{2,1} \ll T^{1/4+\varepsilon}$ . For  $S_{2,1}$ , we have

$$\log \frac{(\frac{T}{2\pi})^{1/2}}{k} = \left| \log \frac{\left[ (\frac{T}{2\pi})^{1/2} \right] - j}{(\frac{T}{2\pi})^{1/2}} \right| \approx \frac{j}{(\frac{T}{2\pi})^{1/2}},$$

from which we get

$$S_{2,2} \ll \sum_{i} \frac{d(k)}{k^{1/2}} \frac{(\frac{T}{2\pi})^{1/2}}{j} \ll T^{1/4+\varepsilon}$$

Therefore  $S_2 \ll T^{1/4+\varepsilon}$ . It is the same for  $S_3$ . Taken together we have  $R_2 \ll T^{1/2} \log T$ . Similarly we have  $R_3 \ll T^{1/2} \log T$ .

As a result, we get

(3.6) 
$$J_1 = 2\sqrt{2\pi}e^{\frac{\pi i}{8}} \sum_{\left(\frac{T}{2\pi}\right)^{1/2} \le k \le \left(\frac{T}{\pi}\right)^{1/2}} {}'(-1)^k d(k)k^{1/2} + O(T^{1/2}\log T).$$

Next we consider  $J_2$ . Similarly to the case of  $J_1$ , we have by (1.3),

(3.7) 
$$J_2 = e^{\frac{3\pi i}{8}} \sum_{k \le K} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-\frac{3}{2}i(t\log\frac{t}{2\pi} - t - t\log k^{2/3})} dt + O(T^{1/2}\log T).$$

We apply Lemma 1 to the above integral with  $\varphi(t) = \rho(2k(2\pi/t))$  and  $f(t) = -\frac{3}{4\pi}(t\log\frac{t}{2\pi} - t - t\log k^{2/3})$ . In this case  $f'(t_0) = 0$  if and only if  $t_0 = 2\pi k^{2/3}$  and  $t_0$  is contained in the interval [T, 2T] if and only if  $(\frac{T}{2\pi})^{3/2} \le k \le (\frac{T}{\pi})^{3/2}$ . Since the range of the sum over k is  $1 \le k \le K$ , there are no such k, that is, the integral in (3.7) does not have the main term. Considering the error term by Lemma 1 we find that

$$J_2 \ll \sum_{k \leq K} \frac{d(k)}{k^{1/2}} \left( 1 + \min\left( (\sqrt{T}, \frac{1}{|\log \frac{T}{2\pi k^{2/3}}|}) + \min\left( \sqrt{T}, \frac{1}{|\log \frac{T}{\pi k^{2/3}}|} \right) \right)$$
  
=:  $R'_1 + R'_2 + R'_3$ .

We have clearly  $R_1' \ll T^{1/2} \log T$ . For  $R_2'$  and  $R_3'$  we note that  $|\log \frac{T}{k^{2/3}}| \gg 1$  since  $k \leq K$ , which implies that  $R_2', R_3' \ll T^{1/2} \log T$ . Hence

(3.8) 
$$J_2 \ll T^{1/2} \log T$$
.

From (3.2), (3.6) and (3.8), we get

$$J = 2\sqrt{2\pi}e^{\frac{\pi i}{8}} \sum_{(\frac{T}{2\pi})^{1/2} \le k \le (\frac{T}{\pi})^{1/2}} {(-1)^k d(k) k^{1/2} + O(T^{1/2} \log T)}.$$

Now dividing the interval [0,T] as  $\bigcup_j [T/2^j, T/2^{j-1}]$  and summing the above evaluations we see that

(3.9) 
$$\int_{0}^{T} Z(t)\zeta\left(\frac{1}{2} + it\right)dt = 2\sqrt{2\pi}e^{\frac{\pi i}{8}} \sum_{k \le (\frac{T}{2\pi})^{1/2}} (-1)^{k}d(k)k^{1/2} + O(T^{1/2}\log^{2}T).$$

It is known that for  $x \gg 1$ 

$$\sum_{k \le x} (-1)^k d(k) = \frac{x}{2} (\log x + 2\gamma - 1 - 2\log 2) + O(x^{1/3 + \varepsilon}),$$

(see e.g. Ivic [4]). By partial summation we get

$$\sum_{k \le x} (-1)^k d(k) k^{1/2} = \frac{1}{3} x^{3/2} \left( \log x + 2\gamma - 2\log 2 - \frac{2}{3} \right) + O(x^{5/6 + \varepsilon}).$$

Substituting this form in (3.9) we get

$$\int_0^T Z(t)\zeta\left(\frac{1}{2} + it\right)dt = \frac{2\sqrt{2}\pi}{3}e^{\frac{\pi i}{8}}\left(\frac{T}{2\pi}\right)^{3/4}\left(\frac{1}{2}\log\frac{T}{2\pi} + 2\gamma - 2\log 2 - \frac{2}{3}\right) + O(T^{1/2}\log^2 T).$$

This proves the assertion of Theorem 1.

Proof of Theorem 2. Let A be a parameter such that  $T^{-1/2} \ll A \ll T^{3/2}$ . We shall consider the integral

$$J_A = \int_T^{2T} Z(t)\zeta\left(\frac{1}{2} + it\right)A^{it}dt$$

by the same way as in the proof of Theorem 1. Applying Lemma 2 we get

$$(3.10) J_A = J_{A,1} + J_{A,2} + O(T^{1/3}),$$

where we put

(3.11) 
$$J_{A,1} = \int_{T}^{2T} \chi^{-1/2} \left(\frac{1}{2} + it\right) \sum_{k=1}^{\infty} \rho\left(\frac{k}{x}\right) \frac{d(k)}{k^{1/2 + it}} A^{it} dt$$

and

(3.12) 
$$J_{A,2} = \int_{T}^{2T} \chi^{3/2} \left( \frac{1}{2} + it \right) \sum_{k=1}^{\infty} \rho \left( \frac{k}{y} \right) \frac{d(k)}{k^{1/2 - it}} A^{it} dt,$$

where  $xy = (\frac{t}{2\pi})^2$ . We shall evaluate both  $J_{A,1}$  and  $J_{A,2}$  by taking two different choices of x and y. Hereafter we put

$$K_0 = \left(\frac{T}{\pi}\right)^{1/2}.$$

The case  $x = 8A(\frac{t}{2\pi})^{1/2}$  and  $y = \frac{1}{8A}(\frac{t}{2\pi})^{3/2}$ . The ranges of  $J_{A,1}$  and  $J_{A,2}$  are at most  $k \le 16AK_0$  and  $k \le \frac{1}{4A}K_0^3$ , respectively. By (1.3) and the trivial estimate for the error term we get

(3.13) 
$$J_{A,1} = e^{-\frac{\pi i}{8}} \sum_{k \le 16AK_0} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log(\frac{k}{A})^2)} dt + O\left(A^{1/2}T^{1/4 + \varepsilon}\right),$$

We shall evaluate the integral by Lemma 1. Let  $f(t) = \frac{1}{4\pi}(t\log\frac{t}{2\pi} - t - t\log(\frac{k}{A})^2)$ . Then  $f'(t_0) = 0$  if and only if  $t_0 = 2\pi(\frac{k}{A})^2$  and  $T \le t_0 \le 2T$  if and only if

(3.14) 
$$A\left(\frac{T}{2\pi}\right)^{1/2} \le k \le A\left(\frac{T}{\pi}\right)^{1/2}.$$

We see that all k satisfing (3.14) are contained in the range  $k \leq 16AK_0$ . Therefore the integral in (3.13) has a main term which is given by

$$M_A(k) = e^{\frac{\pi i}{4}} \rho\left(\frac{1}{8}\right) 2\sqrt{2\pi} \frac{k}{A} e^{-\pi i(k/A)^2}$$

for  $A(\frac{T}{2\pi})^{1/2} \le k \le A(\frac{T}{\pi})^{1/2}$  and  $M_A(k) = 0$  otherwise. We note that  $\rho(1/8) = 1$  in the above formula. It follows from Lemma 1 and (3.13) that

$$J_{A,1} = e^{-\frac{\pi i}{8}} \sum_{A(\frac{T}{2\pi})^{1/2} \le k \le A(\frac{T}{\pi})^{1/2}} \frac{d(k)}{k^{1/2}} M_A(k)$$

$$+ \sum_{k \le 4AK_0} \frac{d(k)}{k^{1/2}} O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{(T/2\pi)^{1/2}}{k/A})|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{(T/2\pi)^{1/2}}{k/A})|}\right)\right)$$

$$+ O(A^{1/2} T^{1/4 + \varepsilon}).$$

Similarly to the proof of Theorem 1, we see that the contributions from the O-terms are bounded by  $O(A^{1/2}T^{1/4+\varepsilon} + A^{-1/2}T^{1/4+\varepsilon})$ . Hence we get

(3.15) 
$$J_{A,1} = e^{\frac{\pi i}{8}} \frac{2\sqrt{2}\pi}{A} \sum_{A(\frac{T}{2\pi})^{\frac{1}{2}} \le k \le A(\frac{T}{\pi})^{\frac{1}{2}}} d(k)k^{1/2}e^{-\pi i(k/A)^2} + O(A^{1/2}T^{1/4+\varepsilon}) + O(A^{-1/2}T^{1/4+\varepsilon}).$$

Next we consider  $J_{A,2}$ . Similarly to  $J_{A,1}$  we have

$$J_{A,2} = e^{\frac{3\pi i}{8}} \sum_{k \le \frac{1}{4A} K_0^3} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-\frac{3}{2}i(t\log\frac{t}{2\pi} - t - t\log(Ak)^{2/3})} dt + O(A^{-1/2}T^{3/4 + \varepsilon}).$$

If we put  $f(t) = -\frac{3}{4\pi}(t\log\frac{t}{2\pi} - t - t\log(Ak)^{2/3})$  this time,  $f'(t_0) = 0$  if and only if  $t_0 = 2\pi(Ak)^{2/3}$  and so  $T \le t_0 \le 2T$  if and only if

$$(3.16) \qquad \frac{1}{A} \left(\frac{T}{2\pi}\right)^{3/2} \le k \le \frac{1}{A} \left(\frac{T}{\pi}\right)^{3/2}.$$

Since K runs over  $1 \le k \le \frac{1}{4A}K_0^3$  there is no main term in the integral of  $J_{A,2}$ . Hence by Lemma 1, we get similarly that

(3.17) 
$$J_{A,2} \ll \sum_{k \leq \frac{1}{4A} K_0^3} \frac{d(k)}{k^{1/2}} \left( 1 + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{(T/2\pi)^{3/2}}{Ak})|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{(T/\pi)^{3/2}}{Ak})|}\right) \right)$$
$$\ll A^{-1/2} T^{3/4 + \varepsilon} + A^{1/2} T^{-1/4 + \varepsilon}.$$

From (3.10), (3.15) and (3.17), we obtain

(3.18) 
$$J_{A} = e^{\frac{\pi i}{8}} \frac{2\sqrt{2}\pi}{A} \sum_{A(\frac{T}{2\pi})^{\frac{1}{2}} \le k \le A(\frac{T}{\pi})^{\frac{1}{2}}} d(k)k^{1/2}e^{-\pi i(k/A)^{2}} + O(A^{1/2}T^{1/4+\varepsilon}) + O(A^{-1/2}T^{3/4+\varepsilon}) + O(T^{1/3}).$$

The case  $x = \frac{A}{4}(\frac{t}{2\pi})^{1/2}$  and  $y = \frac{4}{A}(\frac{t}{2\pi})^{3/2}$ . In this choice of x and y, the sums in (3.11) and (3.12) are actually over  $k \leq \frac{1}{2}AK_0$  and  $k \leq \frac{8}{A}K_0^3$  respectively. Thus

$$J_{A,1} = e^{-\frac{\pi i}{8}} \sum_{k \le \frac{A}{2}K_0} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log(\frac{k}{A})^2)} dt$$
$$+ O\left(A^{1/2}T^{1/4 + \varepsilon}\right)$$

and

$$J_{A,2} = e^{\frac{3\pi i}{8}} \sum_{k \le \frac{8}{A} K_0^3} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-\frac{3}{2}i(t\log\frac{t}{2\pi} - t - t\log(Ak)^{2/3})} dt + O(A^{-1/2}T^{3/4 + \varepsilon}).$$

As for  $J_{A,1}$ , the integral has a main term if and only if k satisfies (3.14). Since k runs over  $1 \le k \le \frac{A}{2}K_0$ , there are no such k. The contribution from the error term of the integral is the same as in the previous case since the range of the sum has the same order, hence we get

(3.19) 
$$J_{A,1} \ll A^{1/2} T^{1/4+\varepsilon} + A^{-1/2} T^{1/4+\varepsilon}.$$

On the other hand, the integral of  $J_{A,2}$  has a main term if and only if k satisfies (3.16), and in fact all k are in the range  $k \leq \frac{8}{A}K_0^3$ . Hence by Lemma 1,  $J_{A,2}$  has the following form:

$$J_{A,2} = e^{\frac{3\pi i}{8}} \sum_{\frac{1}{A}(\frac{T}{2\pi})^{3/2} \le k \le \frac{1}{A}(\frac{T}{\pi})^{3/2}} \frac{d(k)}{k^{1/2}} \widetilde{M}_A(k)$$

$$+ \sum_{k \le \frac{8}{A}K_0^3} \frac{d(k)}{k^{1/2}} O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log(\frac{(T/2\pi)^{3/2}}{Ak})|}\right)\right)$$

$$+ \min(\sqrt{T}, \frac{1}{|\log(\frac{(T/\pi)^{3/2}}{Ak})|}\right)$$

$$+ O(A^{-1/2}T^{3/4+\varepsilon}),$$

where

$$\widetilde{M}_A(k) = e^{-\frac{\pi i}{4}} \rho\left(\frac{1}{4}\right) \frac{2\sqrt{2}\pi}{\sqrt{3}} (Ak)^{1/3} e^{3\pi i (Ak)^{2/3}}$$

for  $\frac{1}{A}(\frac{T}{2\pi})^{3/2} \le k \le \frac{1}{A}(\frac{T}{\pi})^{3/2}$  and 0 otherwise. We see that the contribution from the O-term is the same as the previous case, therefore

(3.20) 
$$J_{A,2} = e^{\frac{\pi i}{8}} \frac{2\sqrt{2}\pi}{\sqrt{3}} A^{1/3} \sum_{\substack{\frac{1}{A}(\frac{T}{2\pi})^{3/2} \le k \le \frac{1}{A}(\frac{T}{\pi})^{3/2} \\ + O(A^{-1/2}T^{3/4+\varepsilon}) + O(A^{1/2}T^{-1/4+\varepsilon})} \frac{d(k)}{k^{1/6}} e^{3\pi i (Ak)^{2/3}}$$

From (3.19) and (3.20) we obtain that

(3.21) 
$$J_{A} = e^{\frac{\pi i}{8}} \frac{2\sqrt{2\pi}}{\sqrt{3}} A^{1/3} \sum_{\substack{\frac{1}{A}(\frac{T}{2\pi})^{3/2} \le k \le \frac{1}{A}(\frac{T}{\pi})^{3/2}}} \frac{d(k)}{k^{1/6}} e^{3\pi i (Ak)^{2/3}} + O(A^{-1/2}T^{3/4+\varepsilon}) + O(A^{1/2}T^{1/4+\varepsilon}) + O(T^{1/3}).$$

Now we have two expressions of  $J_A$ : (3.18) and (3.21). Comparing these expressions we obtain

(3.22) 
$$\sum_{\frac{1}{A}(\frac{T}{2\pi})^{3/2} \le k \le \frac{1}{A}(\frac{T}{\pi})^{3/2}} \frac{d(k)}{k^{1/6}} e^{3\pi i (Ak)^{2/3}}$$

$$= \sqrt{3}A^{-4/3} \sum_{A(\frac{T}{2\pi})^{\frac{1}{2}} \le k \le A(\frac{T}{\pi})^{\frac{1}{2}}} d(k)k^{1/2}e^{-\pi i (k/A)^2}$$

$$+ O(A^{-5/6}T^{3/4+\varepsilon}) + O(A^{1/6}T^{1/4+\varepsilon}) + O(A^{-1/3}T^{1/3+\varepsilon})$$

$$\ll A^{1/6}T^{3/4} \log T,$$

where the last inequality is obtained by the trivial estimate. In (3.22), we take  $T = 2\pi (AN)^{2/3}$ . Then (3.22) is transformed to

$$\begin{split} \sum_{N \leq k \leq 2\sqrt{2}N} \frac{d(k)}{k^{1/6}} e^{3\pi i (Ak)^{2/3}} \\ &= \sqrt{3} A^{-4/3} \sum_{A^{4/3}N^{1/3} \leq k \leq \sqrt{2} A^{4/3} N^{1/3}} d(k) k^{1/2} e^{-\pi i (k/A)^2} \\ &+ O(A^{-1/3} N^{1/2 + \varepsilon}) + O(A^{1/3} N^{1/6 + \varepsilon}) + O(A^{-1/9} N^{2/9 + \varepsilon}) \\ &\ll A^{2/3} N^{1/2} \log N \end{split}$$

for  $A \gg N^{-1/4}$ . This proves the assertion of Theorem 2.

#### 4 Proof of Theorem 3

The method is similar to the previous cases, but we shall write the necessary points for the sake of completeness. Let T be a large number. We put

$$I = \int_{T}^{2T} Z^{2}(t)\zeta\left(\frac{1}{2} + it\right)dt.$$

By the definition of Hardy's function and Lemma 2, we have

$$(4.1)$$

$$Z^{2}(t)\zeta\left(\frac{1}{2}+it\right) = \zeta^{3}\left(\frac{1}{2}+it\right)\chi^{-1}\left(\frac{1}{2}+it\right)$$

$$= \chi^{-1}\left(\frac{1}{2}+it\right)\sum_{k=1}^{\infty}\rho\left(\frac{k}{x}\right)\frac{d_{3}(k)}{k^{1/2+it}} + \chi^{2}\left(\frac{1}{2}+it\right)\sum_{k=1}^{\infty}\rho\left(\frac{k}{y}\right)\frac{d_{3}(k)}{k^{1/2-it}} + O\left(t^{-1/2}\right) + O\left(t^{-2}y^{1/2}\log^{2}t\right),$$

where  $xy = (\frac{t}{2\pi})^3$ .

We take  $x = 2(\frac{t}{2\pi})^{3/2}$  and  $y = \frac{1}{2}(\frac{t}{2\pi})^{3/2}$  in (4.1) and put  $K_3 = (T/\pi)^{3/2}$ . Then the ranges of k in the above two sums are at most  $k \leq 4K_3$  and  $k \leq K_3$ , respectively. Hence

$$(4.2) I = \sum_{k \le 4K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) k^{-it} \chi^{-1} \left(\frac{1}{2} + it\right) dt$$

$$+ \sum_{k \le K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) k^{it} \chi^2 \left(\frac{1}{2} + it\right) dt + O(T^{1/2})$$

$$=: I_1 + I_2 + O(T^{1/2}).$$

As for  $I_2$ , using (1.3), we get

$$I_2 = e^{\frac{\pi i}{2}} \sum_{k \le K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-2i(t \log \frac{t}{2\pi} - t - t \log \sqrt{k})} dt + O(T^{3/4} \log^2 T).$$

As previously, we apply Lemma 1 to the above integral with  $\varphi(t) = \rho \left(2k \left(\frac{2\pi}{t}\right)^{3/2}\right)$  and  $f(t) = -\frac{1}{\pi}(t\log\frac{t}{2\pi} - t - t\log\sqrt{k})$ . We see that  $f'(t_0) = 0$  if and only if  $t_0 = 2\pi\sqrt{k}$ , and this  $t_0$  is contained in the interval [T, 2T] if and only if

$$\left(\frac{T}{2\pi}\right)^2 \le k \le \left(\frac{T}{\pi}\right)^2.$$

Since k runs over the range  $1 \le k \le K_3$ , there is no k which satisfies (4.3), hence the main term does not appear in this integral. On the other hand, the error term of this integral is given by  $1 + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{2\pi\sqrt{k}}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{2\pi\sqrt{k}}|}\right) \ll 1$ , hence we get

(4.4) 
$$I_2 \ll \sum_{k \le K_3} \frac{d_3(k)}{k^{1/2}} \ll T^{3/4} \log^2 T.$$

Next we treat  $I_1$ . By (1.3) again, we have

$$I_1 = e^{-\frac{\pi i}{4}} \sum_{k \le 4K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{i(t\log\frac{t}{2\pi} - t - t\log k)} dt + O(T^{3/4}\log^2 T).$$

In this case  $\varphi(t) = \rho(k(\frac{2\pi}{t})^{3/2}/2)$  and  $f(t) = \frac{1}{2\pi}(t\log\frac{t}{2\pi} - t - t\log k)$ . We see that  $f'(t_0) = 0$  if and only if  $t_0 = 2\pi k$  and this  $t_0$  is contained in [T, 2T] if and only if

$$\frac{T}{2\pi} \le k \le \frac{T}{\pi}.$$

Hence we have

$$\int_{T}^{2T} \rho\left(\frac{k}{x}\right) e^{i(t\log\frac{t}{2\pi} - t - t\log k)} dt$$

$$= M(k) + O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{2\pi k}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{\pi k}|}\right)\right),$$

where M(k) is the main term given by

$$M(k) = e^{\frac{\pi i}{4}} \rho \left(\frac{1}{2\sqrt{k}}\right) (2\pi t_0)^{1/2} e^{-2\pi i k} = 2\pi e^{\frac{\pi i}{4}} k^{1/2}$$

for k such that  $\frac{T}{2\pi} \leq k \leq \frac{T}{\pi}$  and 0 otherwise. Therefore we get

(4.5)

$$\begin{split} I_1 &= 2\pi \sum_{\frac{T}{2\pi} \le k \le \frac{T}{\pi}}' d_3(k) \\ &+ \sum_{k \le 4K_3} \frac{d_3(k)}{k^{1/2}} \left( 1 + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{2\pi k}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{\pi k}|}\right) \right) \\ &= 2\pi \sum_{\frac{T}{2\pi} \le k \le \frac{T}{\pi}}' d_3(k) + O(T^{3/4} \log^2 T). \end{split}$$

Here we can get the last O-term by the same way as previously. Combining (4.2), (4.4) and (4.5), we obtain

$$I = 2\pi \sum_{\frac{T}{2\pi} \le k \le \frac{T}{\pi}}' d_3(k) + O(T^{3/4} \log^2 T).$$

Now dividing the interval [0,T] as  $\cup_j[2^jT,2^{j-1}T]$  we obtain that

$$\int_0^T Z^2(t)\zeta\left(\frac{1}{2} + it\right)dt = 2\pi \sum_{k \le \frac{T}{2\pi}} d_3(k) + O(T^{3/4}\log^3 T).$$

Theorem 3 follows from the well-known formula:

$$\sum_{n \le x} d_3(n) = x \left( \frac{1}{2} \log^2 x + (3\gamma - 1) \log x + 3\gamma_1 + 3\gamma^2 - 3\gamma + 1 \right) + O(x^{1/2}),$$

where  $\gamma_j$  is the coefficients of Laurant expansion of  $\zeta(s)$  at s=1.

## 5 Proof of Theorem 4

Let

$$I = \int_{T}^{2T} Z^{3} \left(\frac{1}{2} + it\right) \chi^{\alpha} \left(\frac{1}{2} + it\right) dt,$$

where  $\alpha$  is a fixed constant such that  $-1/2 < \alpha < 1/2$ . By the definition of Z(t) and Lemma 2 we have

$$I = I_1 + I_2 + O(T^{1/2}),$$

where

(5.1) 
$$I_1 = \sum_{k=1}^{\infty} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) k^{-it} \chi^{\alpha - \frac{3}{2}} \left(\frac{1}{2} + it\right) dt$$

and

(5.2) 
$$I_2 = \sum_{k=1}^{\infty} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) k^{it} \chi^{\alpha + \frac{3}{2}} \left(\frac{1}{2} + it\right) dt,$$

where  $xy = (\frac{t}{2\pi})^3$ . The evaluations of these integrals are the same as before, so we only sketch the outline of these evaluations.

Assume that  $0 \le \alpha < \frac{1}{2}$ . We take  $x = 2(\frac{t}{2\pi})^{1/2}$  and  $y = \frac{1}{2}(\frac{t}{2\pi})^{1/2}$  and put  $K_4 = (\frac{T}{\pi})^{3/2}$ . Then k in the summations in (5.1) and (5.2) at most run over  $1 \le k \le 4K_4$  and  $1 \le k \le K_4$  respectively.

We shall treat  $I_1$  first. By (1.3), we see that the integral in (5.1) becomes

$$e^{\frac{\pi i}{4}(\alpha - \frac{3}{2})} \int_{T}^{2T} \rho\left(\frac{k}{x}\right) e^{(\frac{3}{2} - \alpha)i(t\log\frac{t}{2\pi} - t - t\log k^{\frac{1}{3/2} - \alpha})} dt + O(1).$$

The main term of the integral above appears only when

$$\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\alpha} \le k \le \left(\frac{T}{\pi}\right)^{\frac{3}{2}-\alpha},$$

in which case it is given by

$$M_{\alpha}(k) = e^{\frac{\pi i}{4}} \rho \left( k^{\frac{-2\alpha}{3-2\alpha}}/2 \right) \frac{2\pi}{\sqrt{3/2 - \alpha}} \ k^{\frac{1}{3-2\alpha}} \ e^{-(\frac{3}{2} - \alpha)ik^{\frac{1}{3/2 - \alpha}}}.$$

Computing the error term by Lemma 1, we get

(5.3)

$$\begin{split} I_1 &= e^{\frac{\pi i}{4}(\alpha - \frac{1}{2})} \frac{2\pi}{\sqrt{3/2 - \alpha}} \sum_{(\frac{T}{2\pi})^{3/2 - \alpha} \leq k \leq (\frac{T}{\pi})^{3/2 - \alpha}} \frac{d_3(k)}{k^{1/2}} k^{\frac{1}{3 - 2\alpha}} e^{-(\frac{3}{2} - \alpha)ik^{1/(3/2 - \alpha)}} \\ &+ \sum_{k \leq K_4} \frac{d_3(k)}{k^{1/2}} O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log\frac{(T/2\pi)^{3/2 - \alpha}}{k}|}\right)\right) \\ &+ \min\left(\sqrt{T}, \frac{1}{|\log\frac{(T/\pi)^{3/2 - \alpha}}{k}|}\right)\right). \end{split}$$

Just in the same way in the previous cases, we can see easily that the above O-term is estimated as  $O(T^{3/4} \log^2 T)$ .

On the other hand, for  $I_2$ , the main term does not appear from the integral by the assumption  $0 \le \alpha < 1/2$  and the sum over k is estimated as  $O(T^{3/4} \log^2 T)$ .

Now it remains to evaluate the sum over k in (5.3). Let

$$S = \sum_{\left(\frac{T}{2\pi}\right)^{3/2 - \alpha} \le k \le \left(\frac{T}{2\pi}\right)^{3/2 - \alpha}} \frac{d_3(k)}{k^{1/2}} k^{\frac{1}{3 - 2\alpha}} e^{-\left(\frac{3}{2} - \alpha\right)ik^{\frac{1}{3/2 - \alpha}}}.$$

By partial summation we may have (5.4)

$$S \ll T^{\frac{\alpha}{2} - \frac{1}{4}} \max_{(\frac{T}{2\pi})^{3/2 - \alpha} \le T' \le (\frac{T}{\pi})^{3/2 - \alpha}} \left| \sum_{(\frac{T}{2\pi})^{3/2 - \alpha} \le k \le T'} d_3(k) e^{-(3/2 - \alpha)ik^{\frac{1}{3/2 - \alpha}}} \right|.$$

Considering the definition of  $d_3(k)$ , it is reduced to the estimate of the sum of the form

$$S_1 := \sum_{T_1 \le k_1 k_2 k_3 \le 2T_1} e^{2\pi i c(k_1 k_2 k_3)^{\delta}},$$

where  $\delta = \frac{1}{3/2-\alpha}$ , c is a real constant and  $(\frac{T}{2\pi})^{3/2-\alpha} \leq T_1 \leq \frac{1}{2}(\frac{T}{\pi})^{3/2-\alpha}$ . Since  $\delta \neq 0, 1$  we can apply Lemma 3 (the theorem of Robert and Sargos). Divide the interval  $[T_1, 2T_1]$  into  $O(\log^3 T)$  subintervals of the form  $[H, 2H] \times [N, 2N] \times [M, 2M]$ . By symmetry of  $k_j$ , we can assume that M is the largest, hence  $M \gg T_1^{1/3}$ . Now applying Lemma 3 to the sum  $S_1$  by taking  $X = (HNM)^{\delta} \times T_1^{\delta}$ , we find that

$$(5.5) S_1 \ll T_1^{1+\varepsilon} \left(T_1^{(\delta-\frac{4}{3})/4} + T_1^{-1/6} + T_1^{-\delta}\right) \ll T_1^{2/3+\delta/4+\varepsilon}.$$

Here the last inequality follows from the assumption  $0 \le \alpha < 1/2$ . By (5.4), (5.5) and  $T_1 \asymp T^{3/2-\alpha}$ ,  $\delta = \frac{1}{3/2-\alpha}$  we find that

$$S \ll T^{1-\frac{\alpha}{6}+\varepsilon}$$
.

This proves the assertion in the case  $0 \le \alpha < 1/2$ .

In the case  $-1/2 < \alpha \le 0$ , we take  $x = \frac{1}{2} (\frac{t}{2\pi})^{3/2}$  and  $y = 2(\frac{t}{2\pi})^{3/2}$ . Then the main term arises from the integral corresponding  $I_2$  and the assertion is proved similarly. We omit the details in this case.

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