

The Wigner function, as outlined, serves as a pivotal tool for analyzing non-stationary processes, especially when considering the evolution of spectral properties over time. In the context of a process that is mildly non-stationary—where the process exhibits translational invariance in terms of waveform shape but demonstrates variation in scale—this function can be instrumental in capturing and quantifying the dynamics of such changes.

Given a time series  $x[t]$  with its non-stationary auto-covariance function  $C_x(t_1, t_2) = \langle (x[t_1] - \mu[t_1])(x[t_2] - \mu[t_2])^* \rangle$ , where  $\mu(t)$  represents the time-dependent mean and  $*$  denotes the complex conjugate, the process involves first identifying the time-lagged auto-correlation by considering the average time  $t = \frac{t_1+t_2}{2}$  and time lag  $\tau = t_1 - t_2$ .

To analyze the non-stationary correlation that slowly grows over time, indicating changes in the scale of waveform based on interval distances, we utilize the Wigner function defined as:

$$W_x(t, f) = \int_{-\infty}^{\infty} C_x\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-2\pi i \tau f} d\tau.$$

This definition leverages the Fourier transform of the lagged auto-correlation function, effectively transitioning from a time-domain representation to a time-frequency domain representation, where  $f$  denotes frequency. For a mean-zero time series, this simplifies to:

$$W_x(t, f) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-2\pi i \tau f} d\tau.$$

In the scenario where the scale of waveform changes over time, the Wigner function's time-frequency representation will reveal how the spectral density, or energy distribution across frequencies, evolves. For mildly non-stationary processes, the gradual change in waveform scale can be traced as a function of time within the Wigner distribution, thus providing a comprehensive view of the process's dynamic behavior in both time and frequency domains.

The key advantage here is that, unlike traditional Fourier analysis which assumes stationarity, the Wigner function accommodates non-stationarity, allowing for the examination of how specific frequencies' contributions to the process change over time. This is particularly useful in identifying and characterizing the non-stationary correlation's evolution, providing insights into the underlying dynamics of the process.

By analyzing the Wigner function across different time intervals, one can detect variations in the spectral density that correspond to the slowly growing non-stationary correlation, thereby capturing the essence of how the process's characteristics evolve. This method is not only pertinent for theoretical analysis but also for practical applications where understanding the temporal evolution of spectral properties is crucial.

Indeed, the evolutionary or locally stationary process paradigm inherently accounts for changes in the correlation structure that depend on the interval over which the spectrum is calculated. This adaptability is due to the framework's

flexibility in handling variations in the statistical properties of the process over time, including the length of the observation window.

When you observe that the correlation function depends on the length of the interval, this observation aligns with the characteristic behavior of evolutionary processes. The framework accommodates variations in statistical properties over different intervals, capturing how these properties change with time and, implicitly, with the length of the interval over which measurements are taken or analyzed.

In evolutionary processes, the dependence on both the time difference ( $\tau$ ) and the central or average time ( $t$ ) of the interval effectively accounts for the interval's length. This is because the average time  $t$  reflects the position of the interval within the timeline, while the difference  $\tau$  captures the interval's length, providing a comprehensive view of how the process's statistical properties vary over time and across different intervals.

Therefore, there might not be a need for additional scaling by the length of the window unless there are specific characteristics of your process or analysis objectives that are not captured by this framework. The evolutionary process paradigm, by considering the interval's position (through  $t$ ) and length (through  $\tau$ ), naturally incorporates the influence of the interval length on the correlation function and the spectrum calculated over it.