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THE APPROXIMATE SOLUTION OF RICCATI'S EQUATION*

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1. Introduction. Bessel functions of the zeroth order possess the property that their derivatives give Bessel functions of the first order; i.e.,

$$J_0'(x) = -J_1(x). \tag{1}$$

In this paper we make use of this fact in approximating the solution of the nonlinear differential equation, the *Riccati equation*,

$$L(y) \equiv y' - P(x)y - Q(x)y^2 = R(x), \tag{2}$$

by a sum $S_n(x)$ of n terms of Bessel functions of the zeroth order. We consider the existence and uniqueness of such a sum of Bessel functions $y = S_n(x)$ that satisfies the condition that $y_0 = S_n(x_0)$, x_0 in the interval $[0, 1]$ denoted by I, and which at the same time gives the *best approximation* to the solution of (2) in I in the sense that the integral

$$\int_0^1 | R(x) - L[S_n(x)] |^m \, dx, \qquad m > 0 \text{ and fixed} \tag{3}$$

is a minimum.

2. The Riccati differential equation. If $Q(x) \neq 0$ for x in I,[1] then, by reducing (2) to a second order linear differential equation, we can show that, if P, Q, R, and Q'/Q are analytic in I, there exists a unique continuous solution $y = y(x)$ such that for a given y_0 we have $y_0 = y(x_0)$, (see [2]). Throughout our work we shall assume that the coefficients in (2) are such that there exists a unique continuous solution $y = y(x)$ that satisfies $y_0 = y(x_0)$ and possesses a continuous first derivative and a second derivative of bounded variation.

3. The approximating sum. The following facts are known (see [1], [10]). First, the set $\{J_n(\lambda_i x)\}$ of nth order Bessel functions is an orthogonal set over I with respect to the weight function x, i.e.,

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[1] The interval [0, 1] has been chosen for I for simplicity. It could equally well have been chosen as $[0, c]$ for any $c > 0$.

$$\int_0^1 x J_n(\lambda_j\, x) J_n(\lambda_k\, x)\; dx = c_j\, \delta_{jk}\,, \tag{4}$$

where the δ_{jk} is the Kronecker delta and the c_j's are constants, if the set of eigenvalues $\{\lambda_i\}$ are the positive roots of one of the equations

$$\begin{aligned} &\text{(a)} \quad J_n(x) = 0 \quad \text{or} \\ &\text{(b)} \quad J_n{}'(x) = 0. \end{aligned} \tag{5}$$

We assume that $\lambda_i < \lambda_{i+1}$, $i = 1, 2, \cdots$. Next, the normalized Fourier-Bessel coefficients for the expansion of a function $f(x)$ in terms of the Bessel functions $\{J_n(\lambda_i x)\}$ are

$$B_m = \frac{\displaystyle\int_0^1 x f(x) J_n(\lambda_m\, x)\; dx}{\displaystyle\int_0^1 x [J_n(\lambda_m\, x)]^2\; dx}. \tag{6}$$

And finally, for both $J_0(x)$ and $J_1(x)$ the first positive eigenvalue is greater than 1, a fact we shall use later.

It is also known that λ_m is of the order of m, or $\lambda_m = O(m)$ (see [5]). That is, $\lambda_m/m < K$, a constant, for large enough m.

Moore [6] and Scherberg [8] have considered the degree of convergence of the general expression

$$f(x) = \sum_{m=1}^{\infty} B_m J_n(\lambda_m x), \quad n \geqq 0, \tag{7}$$

where the B_m's are given in (6). The *degree of convergence* of sums of Bessel functions to a given function $f(x)$ is defined to be the absolute value of the difference between the function and the first n terms of a series of Bessel functions that is known to converge to $f(x)$.

In our work we shall consider the series

$$f(x) = \sum_{m=1}^{\infty} B_m J_0(\lambda_m x) \tag{8}$$

in the special case in which $n = 0$ in (5) and (7); i.e., we consider the case in which the λ_i's are the distinct positive roots of

$$-J_1(x) = J_0{}'(x) = 0 \tag{9}$$

and the B_m's are as given in (6) with $n = 0$. Thus we have that the sets $\{J_0(\lambda_i x)\}$ and $\{J_1(\lambda_i x)\}$ are both sets of orthogonal functions with respect to the weight function x and over the same interval [0, 1], and the corresponding sets of eigenvalues $\{\lambda_i\}$ are the same in each case.

4. Preliminary lemmas. In this section we state, but do not prove, two lemmas that are needed in our further discussion. The first of these lemmas is needed for the proof of the second. These lemmas and their proofs may be found in [6] and [8].

In the following discussion the order symbol O will be used; e.g., $O(1/\lambda_i)$ is such that $\lambda_i O(1/\lambda_i)$ is bounded for large λ_i or for large i, since $\lambda_i > 1$ and λ_i is of the order of i.

LEMMA 1. *If $F(x)$ is a function such that $F(x)/x$ has bounded variation for x in I, then*

$$\int_0^1 F(x)J_m(\lambda_n x)\,dx = O\left(\frac{1}{\lambda_n}\right)\Big/\lambda_n^{3/2}.$$

LEMMA 2. *For x in I let the function $f(x)$ be such that $f'(x)$ exists and has bounded variation. Then the general coefficient B_m in (7) is such that*

$$B_m = O\left(\frac{1}{\lambda_m}\right)\Big/\lambda_m^{3/2}.$$

5. Convergence of sums of Bessel functions to a given function. We now consider a theorem that establishes the degree of convergence of a sum $S_n(x)$ of *zeroth order* Bessel functions to a suitably restricted function $f(x)$ for x in I. An analogous theorem is then developed involving a sum $Q_n(x)$ of *first order* Bessel functions and $f'(x)$, the derivative of the function $f(x)$ given in the first of these theorems.

THEOREM 1. *If $f(x)$ is such that $f'(x)$ exists and has bounded variation for x in I, then*

$$| f(x) - S_n(x) | = O\left(\frac{1}{\lambda_n}\right)\Big/\lambda_n^{1/2},$$

where $S_n(x) = \sum_{m=1}^{n} B_m J_0(\lambda_m x)$, *and* $B_m = O(1/\lambda_m)/\lambda_m^{3/2}$.

Proof. From Lemma 2 the coefficient of the general term of $S_n(x)$ is such that $B_m = O(1/\lambda_m)/\lambda_m^{3/2}$, so that we have

$$\begin{aligned} f(x) - S_n(x) &= \sum_{m=1}^{\infty} B_m J_0(\lambda_m x) - \sum_{m=1}^{n} B_m J_0(\lambda_m x) \\ &= \sum_{m=n+1}^{\infty} B_m J_0(\lambda_m x) \\ &= \sum_{m=n+1}^{\infty} \left[O\left(\frac{1}{\lambda_m}\right)\Big/\lambda_m^{3/2}\right] J_0(\lambda_m x). \end{aligned}$$

However, $| J_0(t) | \leqq 1$ for all values of t so that

$$\begin{aligned} | f(x) - S_n(x) | &= \sum_{m=n+1}^{\infty} O\left(\frac{1}{\lambda_m}\right)\Big/\lambda_m^{3/2} \\ &= \sum_{m=n+1}^{\infty} O\left(\frac{1}{m}\right)\Big/ m^{3/2} = O\left(\frac{1}{m}\right)\sum_{m=n+1}^{\infty}\frac{1}{m^{3/2}}, \end{aligned}$$

since λ_m is of the order of m. But

$$\sum_{m=n+1}^{\infty} \frac{1}{m^{3/2}} \leqq \lim_{c\to\infty} \int_n^c \frac{1}{x^{3/2}}\,dx = 2n^{-1/2}.$$

Hence,

$$|f(x) - S_n(x)| = O\left(\frac{1}{n}\right) n^{-1/2} = O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2},$$

where again we make the interchange of n and λ_n as above. This concludes the proof of Theorem 1.

We shall next show that $f'(x)$ can be approximated by a sum of *first order* Bessel functions; i.e., we shall consider a series of the form

$$(10) \qquad f'(x) = \sum_{m=1}^{\infty} B_m J_1(\lambda_m x),$$

which is possible by (7).

Again, by using the results of Lemma 2, if $f''(x)$ exists and has bounded variation, the coefficient for the general term of the expansion of $f'(x)$ as a sum of first order Bessel functions is such that

$$B_m = O\left(\frac{1}{\lambda_m}\right) \Big/ \lambda_m^{3/2}.$$

Thus, by defining $Q_n(x)$ as the sum of the first n terms in (10), we have that

$$\begin{aligned} f'(x) - Q_n(x) &= \sum_{m=1}^{\infty} B_m J_1(\lambda_m x) - \sum_{m=1}^{n} B_m J_1(\lambda_m x) \\ &= \sum_{m=n+1}^{\infty} B_m J_1(\lambda_m x) = \sum_{m=n+1}^{\infty} \left[O\left(\frac{1}{\lambda_m}\right) \Big/ \lambda_m^{3/2}\right] J_1(\lambda_m x). \end{aligned}$$

Using the same argument as in the proof of Theorem 1, it follows that

$$|f'(x) - Q_n(x)| = O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2}.$$

Hence, we may state the following theorem:

THEOREM 2. *If $f'(x)$ is such that $f''(x)$ exists and is of bounded variation for x in I, then*

$$|f'(x) - Q_n(x)| = O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2},$$

where $Q_n(x) = \sum_{m=1}^{n} B_m J_1(\lambda_m x)$, and $B_m = O(1/\lambda_m)/\lambda_m^{3/2}$.

6. Approximation of $f(x)$ and $f'(x)$ by sums of Bessel functions. We now consider the degree of convergence of sums of Bessel functions to a given function $f(x)$ and the derivative of these sums as they converge to $f'(x)$.

THEOREM 3. *If $f(x)$ and $f'(x)$ are such that $f''(x)$ exists and has bounded variation in I, then there exists a sum of zeroth order Bessel functions such that*

$$|f^{(k)}(x) - S_n^{(k)}(x)| = O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2}, \qquad k = 0, 1.$$

Proof. Assume $f'(x)$ is a function of x as described by Theorem 2 and that $Q_n(x)$ is a sum of Bessel functions

$$Q_n(x) = \sum_{m=1}^{n} B_m J_1(\lambda_m x).$$

Then from Theorem 2 we have that

$$|f'(x) - Q_n(x)| = O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2}.$$

Next define $R_n(x)$ by

$$R_n(x) = f(0) + \int_0^x Q_n(t)\,dt. \tag{11}$$

That is, $R_n(x)$ is a sum of zeroth order Bessel functions. From (11) we can write

$$\begin{aligned} f(x) - R_n(x) &= f(x) - f(0) - \int_0^x Q_n(t)\,dt \\ &= \int_0^x f'(t)\,dt - \int_0^x Q_n(t)\,dt = \int_0^x [f'(t) - Q_n(t)]\,dt. \end{aligned}$$

Therefore

$$\begin{aligned} |f(x) - R_n(x)| &= \left| \int_0^x [f'(t) - Q_n(t)]\,dt \right| \\ &= (1 - 0) O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2}, \end{aligned}$$

and

$$|f'(x) - R_n'(x)| = O\left(\frac{1}{\lambda_n}\right) \Big/ \lambda_n^{1/2},$$

since $R_n'(x) = Q_n(x)$ from (11). From these two equations we see that $f(x)$ can be uniformly approximated by a sum of zeroth order Bessel functions such that $f'(x)$ is, at the same time, uniformly approximated by the derivative of this sum, and the theorem is proved.

7. Existence and uniqueness of minimizing sums. The proof of existence of sums of zeroth order Bessel functions that will minimize (3) can be shown in a manner analogous to the discussion of the least mth power approximation by Jackson [3] and McEwen [4] who used other types of functions; this proof is omitted here.

Also omitted is the proof of the uniqueness of a minimizing sum for the case when $m > 1$. The usual proof by contradiction can be used. Two sums that make (3) a minimum are assumed, and by considering their arithmetic mean one is led to the required contradiction (see [3]).

8. The approximate solution of Riccati's equation. Before we arrive at the principal theorem of the paper, we state a lemma that is a special case of a theorem due to Oberg [7]. His theorem for sums of orthonormal functions says that if

$$S_n(x) = \sum_{i=0}^{n} a_i \Phi_i(x)$$

is a sum of orthonormal functions, and if w_n is the maximum of $| S_n(x) |$ for all x in the interval $[a, b]$, then

$$| dS_n(x)/dx | \leqq M(b - a)^{1/2} \lambda_n w_n ,$$

where M is a constant and λ_n is a variable which increases with n. Also, this theorem is similar to one developed by Stein [9] in approximating the solutions of integro-differential equations.

LEMMA 3. *If $g_n(x)$ is a sum of n terms of zeroth order Bessel functions that satisfies a boundary condition such as $y_0 = g_n(x_0)$, if $P'(x)$ and $Q'(x)$ of* (2) *are bounded for x in I, and further if w is the maximum for $| g_n(x) |$ for x in this interval, then*

(a) $| L[g_n(x)] | \leqq A\lambda_n w^2$ *and* (b) $| dL[g_n(x)]/dx | \leqq B\lambda_n^2 w^2$

for all x in I, where A and B are positive constants not depending on n, and L is defined by (2).

The proof of this theorem follows that of Oberg [7] and is not included.

We are now ready to consider the approximate solution of the Riccati equation, with the boundary condition $y_0 = y(x_0)$, by a sum of zeroth order Bessel functions. That is, we want to find the conditions on the sum $S_n(x)$ which satisfies $y_0 = S_n(x_0)$ and which approximates the solution $y(x)$ such that the integral in (3) is a minimum; the sum $S_n(x)$ is called the *minimizing sum* for $y(x)$ and is the *best approximation* for fixed m in (3).

Let $g_n(x)$ be an arbitrary sum of zeroth order Bessel functions that satisfies the boundary condition $y_0 = g_n(x_0)$ and is such that

$$| y^{(k)}(x) - g_n^{(k)}(x) | \leqq \eta, \qquad k = 0, 1, \quad \eta < 1. \tag{12}$$

Note that η depends on n; however, it is not so indicated in the following discussion. The statement in (12) is possible by Theorem 3 and is true for all values of x in I, where $y(x)$ is the solution of the Riccati equation. Let $r_n(x) = y(x) - g_n(x)$, and let $Q_n(x) = S_n(x) - g_n(x)$, where $S_n(x)$ is the sum of zeroth order Bessel functions which satisfies the boundary condition and which makes

$$v_n = \int_0^1 | R(x) - L[S_n(x)] |^m \, dx, \qquad m > 0 \quad \text{and fixed,} \tag{13}$$

a minimum. It is possible that $Q_n(x) = 0$, since $S_n(x)$ and $g_n(x)$ both satisfy the boundary condition, and 0 is a sum of zeroth order Bessel functions trivially; note that the boundary condition for $Q_n(x)$ is $Q_n(x_0) = 0$. Thus using (13) and (2) we can write

$$\begin{aligned} v_n &= \int_0^1 | R(x) - L[S_n(x)] |^m \, dx = \int_0^1 |L(y) - L(S_n) |^m \, dx \\ &= \int_0^1 | L(y - S_n) - 2QS_n(y - S_n) |^m \, dx \\ &= \int_0^1 | L(r_n - Q_n) - 2QS_n(r_n - Q_n) |^m \, dx \\ &\leqq \int_0^1 | L(r_n) - 2QS_n \, r_n |^m \, dx, \end{aligned} \tag{14}$$

assuming $Q_n(x) = 0$ as a possibility. The final step in (14) is permissible since $S_n(x)$ is the minimizing sum for $y(x)$, and the value of the integral is no less for any other sum. Then since

$$L(r_n) = r_n' - P(x)r_n - Q(x)r_n^2,$$

we have that

$$\begin{aligned} | L(r_n) | &\leqq | r_n' | + | P | | r_n | + | Q | | r_n^2 | \\ &\leqq \eta + M_1\eta + M_2\eta^2 \leqq N\eta, \end{aligned} \tag{15}$$

where M_1 and M_2 are the upper bounds for $| P(x) |$ and $| Q(x) |$ over I, and N is a constant.

Also since $S_n(x)$ satisfies the conditions of Lemma 3, then $| S_n(x) | \leqq w$, ($w = \max \{| S_n(x) |, | g_n(x) |, | Q_n(x) |: x \text{ in } I\}$), and we have from (12), (14), and (15) that

$$\begin{aligned} v_n &\leqq \int_0^1 | L(r_n) - 2QS_n \, r_n |^m \, dx \leqq \int_0^1 (| L(r_n) | + 2 \, | QS_n \, r_n |)^m \, dx \\ &\leqq \int_0^1 (N\eta + 2M_2 \, w\eta)^m \, dx \leqq N_1^m \eta^m, \end{aligned} \tag{16}$$

where N_1 is a constant.

Let x_1 be the point in I where $|L[Q_n]|$ reaches its maximum, and denote this maximum by $A\lambda_n w^2$, A a constant. Then by the mean value theorem we have that

$$|L[Q_n(x)] - L[Q_n(x_1)]| \leqq |x - x_1| \, |dL[Q_n(X)]/dx|$$

for $|X - x_1| < |x - x_1|$. Let us choose an interval about x_1 such that

$$|x - x_1| \leqq A/2B\lambda_n,$$

or the part of the interval that is contained in [0, 1], if x_1 is nearer to 0 or 1 by less that $A/2B\lambda_n$, where A and B are constants. Then upon applying conclusion (b) of Lemma 3 we have that

$$|L[Q_n(x)] - L[Q_n(x_1)]| \leqq \tfrac{1}{2}A\lambda_n w^2.$$

Hence,

$$|L[Q_n(x)]| \geqq \tfrac{1}{2}A\lambda_n w^2$$

for x in I.

Now by (15) $|L(r_n)| \leqq N\eta$. Thus if n is chosen large enough so that $\lambda_n > 16M_2/A$ and if η is chosen so that

$$\eta \leqq \left(\frac{A\lambda_n - 16M_2}{16M_2}\right) w,$$

and recalling that $\eta < 1$, then either $N\eta \leqq \frac{1}{4}A\lambda_n w^2$ or $N\eta \geqq \frac{1}{4}A\lambda_n w^2$.

First, by assuming that $N\eta \leqq \frac{1}{4}A\lambda_n w^2$, we have that

$$\begin{aligned}
|L(y) - L(S_n)| &= |L(r_n + g_n) - L(Q_n + g_n)| \\
&= |r_n' + g_n' - Pr_n - Pg_n - Qr_n^2 - Qg_n^2 - 2Qr_ng_n \\
&\quad - (Q_n' + g_n' - PQ_n - Pg_n - QQ_n^2 - Qg_n^2 - 2QQ_ng_n)| \\
&= |L(r_n) - L(Q_n) - 2Qg_n(r_n - Q_n)| \\
&\geqq |L(Q_n)| - |L(r_n)| - 2|Q|\,|g_n|\,|r_n - Q_n| \\
&\geqq \tfrac{1}{2}A\lambda_n w^2 - \tfrac{1}{4}A\lambda_n w^2 - 2M_2 w(\eta + w) \geqq \tfrac{1}{8}A\lambda_n w^2.
\end{aligned}$$

Note that use is made of the fact that

$$|r_n - Q_n| \leqq |r_n| + |Q_n| \leqq \eta + w,$$

since Q_n satisfies the conditions of Lemma 3. Therefore

$$\begin{aligned}
v_n &= \int_0^1 |L(y) - L(S_n)|^m \, dx \\
&\geqq \int_{x_1 - A/2B\lambda_n}^{x_1 + A/2B\lambda_n} |L(y) - L(S_n)|^m \, dx
\end{aligned}$$

$$\geqq \left[\frac{A}{B\lambda_n}\right]\left[\frac{A\lambda_n w^2}{8}\right]^m,$$

or

$$w^2 \leqq \left[\frac{8}{A\lambda_n}\right]\left[\frac{Bv_n \lambda_n}{A}\right]^{1/m}.$$

In the contrary case if we assume that $N\eta \geqq \frac{1}{4}A\lambda_n w^2$, then

$$w^2 \leqq \frac{4N\eta}{A\lambda_n}$$

directly. Thus in either event we have that

$$w^2 \leqq \frac{8}{A\lambda_n}\left(\frac{Bv_n \lambda_n}{A}\right)^{1/m} + \frac{4N\eta}{A\lambda_n} \leqq \frac{8}{A\lambda_n}\left(\frac{B}{A}\right)^{1/m} N_1 \eta \lambda_n^{1/m} + \frac{4N\eta}{A\lambda_n}$$

by application of (16). Finally, since $\lambda_n > 1$, we can write

$$w^2 \leqq C_1^2 \lambda_n^{1/m} \eta, \tag{17}$$

where C_1 is a constant. Using the fact that

$$| r_n^{(k)} - Q_n^{(k)} | \leqq | r_n^{(k)} | + | Q_n^{(k)} |, \qquad k = 0, 1,$$

then from (12) and (17) and the fact that $| Q_n^{(k)}(x) | \leqq w$, we have that

$$| r_n^{(k)} - Q_n^{(k)} | \leqq \eta + C_1 \lambda_n^{1/2m} \eta^{1/2} \leqq C \lambda_n^{1/2m} \eta^{1/2} = \lambda_n^{1/2m} \epsilon_n ,$$

where $\epsilon_n = C\sqrt{\eta}$ and C is a constant. Recall from (12) that η depends on n. But $r_n^{(k)} - Q_n^{(k)} = y^{(k)} - S_n^{(k)}$ and therefore

$$| y^{(k)} - S_n^{(k)} | \leqq \lambda_n^{1/2m} \epsilon_n , \qquad k = 0, 1.$$

We can thus state the following theorem.

THEOREM 4. *If $y(x)$ is the unique continuous solution of the Riccati equation in I which satisfies the boundary condition $y_0 = y(x_0)$ and has a continuous first derivative and a second derivative of bounded variation, and if $y = S_n(x)$ is a sum of n terms of zeroth order Bessel functions such that $y_0 = S_n(x_0)$ and also such that $S_n(x)$ makes (3) a minimum for fixed m, a sufficient condition for the uniform convergence of $S_n(x)$ to $y(x)$ and $S_n'(x)$ to $y'(x)$ throughout I is that*

$$\lim_{n \to \infty} \lambda_n^{1/2m} \epsilon_n = 0,$$

where

$$\epsilon_n = C\sqrt{\eta}, \qquad \eta \leqq \min\left\{1, \left(\frac{A\lambda_n - 16M_2}{16M_2}\right) w\right\},$$

and where η is given by (12).

9. Conclusion. We have shown in Theorem 3 that a suitably restricted function $f(x)$ can be uniformly approximated by a sum $S_n(x)$ of zeroth order Bessel functions and at the same time its derivative $f'(x)$ can be uniformly approximated by $S_n'(x)$. In Theorem 4 are given the conditions for convergence of $S_n(x)$ and $S_n'(x)$ to $y(x)$ and $y'(x)$, the solution of Riccati's equation and its derivative respectively, as the number of terms in $S_n(x)$ increases without bound. By the analogues of Jackson's work [3], the minimizing sum $S_n(x)$ exists and is unique. Thus $S_n(x)$ is the *best approximation* to the solution of (2) in the sense given by (3), and the solution of (2) can be expanded directly in a series of zeroth order Bessel functions under the assumptions of Theorem 4.

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