Z. Wahrscheinlichkeitstheorie verw. Gebiete
53, 283–290 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete


# Counterexamples to Results of M.M. Rao

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**Summary.** Orlicz spaces and Prediction Operators in these spaces have been investigated by M.M. Rao in a number of well known papers. Since these papers are widely recognized as pioneering work in this field, it seems worth while to point out that some of his main results and many of his proofs are false. In particular M.M. Rao's result on strict convexity of Orlicz spaces and his proofs of convergence theorems for prediction sequences are false.

## 1. Introduction and Notations

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $E$ be a Banach space. $\Phi$: $\mathbb{R} \to [0, \infty)$ is called a Young's function ($Y$-function), if $\Phi$ is symmetric, convex, $\Phi(0)=0$, $\Phi(x)>0$ for $x \neq 0$. If $\Phi$ is a $Y$-function, let $L_\Phi(\Omega, \mathscr{A}, P;\ E)$ be the space of all equivalence classes of strongly measurable $E$-valued functions $X$ on $\Omega$, which satisfy:

$$N_\Phi(X)=\inf\left\{k>0\colon \int \Phi\left(\frac{\|X\|}{k}\right) dP \leqq \Phi(1)\right\}<\infty.$$

Then by the same arguments as in the case $E=\mathbb{R}$ (see: Luxembourg [5]) it is verified that $(L_\Phi, N_\Phi)$ is a Banach space. If $\mathscr{B} \subset \mathscr{A}$ is a $\sigma$-field, define:

$$L_\Phi(\mathscr{B})=\{X \in L_\Phi(\Omega, \mathscr{A}, P; E)\colon \text{there is a } \mathscr{B}\text{-measurable function in the class } X\}$$

*1.1 Definition.* Assume that for each $X \in L_\Phi(\Omega, \mathscr{A}, P; E)$ there is exactly one $X_0 \in L_\Phi(\mathscr{B})$, such that:

(1.2) $\quad N_\Phi(X-X_0)=\inf\{N_\Phi(X-Y)\colon Y \in L_\Phi(\mathscr{B})\}$.

Then define an operator $P_\Phi^{\mathscr{B}}$ on $L_\Phi$ by $P_\Phi^{\mathscr{B}} X=X_0$.

In [6–13] M.M. Rao obtains important results on the operators $P_\Phi^{\mathscr{B}}$ and the spaces $L_\Phi$. Repentantly, some of these results are false others are proven in a false way or turn out to be trivial.

0044-3719/80/0053/0283/$01.60

Section 2 of this paper discusses M.M. Rao's results in [12] on the existence and uniqueness of $X_0$ in (1.2). It turns out that Rao's existence proof is wrong unless $E$ is reflexive. His uniqueness result is not true in general.

The main problem investigated in [8] and [12] is the convergence of $P_{\Phi}^{\mathscr{B}_n}X$ for a sequence of $\sigma$-fields $\mathscr{B}_n$ with $\mathscr{B}_n \subset \mathscr{B}_{n+1}$ for $n \in \mathbb{N}$. The results are extensions of the convergence theorem of Ando and Amemiya for prediction sequences in $L_p$ $(1<p<\infty)$. The proof of M.M. Rao's a.e.-convergence theorem is not correct, for the generalizations of Lemma 1 of [1] stated by M.M. Rao (see Lemma 2 of [8] and Lemma 3.3 of [12]) are not true for general $\Phi$. The last statement of M.M. Rao in this affair is [13]. In this correction note M.M. Rao makes us believe that he always intended to state and prove the theorem under the additional condition that $L_\Phi$ has a certain property (*). (In [8] and [12] however, this property was stated as an immediate consequence of the definition of $P_{\Phi}^{\mathscr{B}}$; see: [8] p. 171 and [12] p. 135.) Repentantly, this additional condition is very restrictive. It turns out (see Th. 3.2 below) that $L_\Phi$ has this property only if $P$ assumes only finitely many values or $\Phi(x)=c|x|^p$ with $p>1$. In the first case the results of M.M. Rao are obvious, in the second case i.e. for $L_p$-spaces they are well known (see [1]). This seems to be at variance with M.M. Rao's claim that his results "are the best possible in the context of $L_\Phi$-spaces" (see [12] p. 130). Notice that the proof in [12] remains false even in the restricted case, because M.M. Rao's Lemma 2.6, which is "crucial for the work" is false (see Counterexample 3.4 below). We conclude our paper with a collection of some further errors of M.M. Rao, concerning Orlicz spaces.

## 2. Existence and Uniqueness of Best Approximants

In order to discuss the existence result of M.M. Rao, we specialize his Lemma 3.5 of [12] to the trivial case that $(\Omega, \mathscr{A}, P)$ is a one point probability space. Then Rao's Lemma yields:

(i) Let $E$ be a strictly convex $B$-space with the $RN$-property. Then: If $C \subset E$ is a nonempty closed convex subset, then for any $f_0 \in E$ the functional $F(\,.\,)$ defined by $F(f)=\|f-f_0\|$ assumes its minimum on the set $C$, whenever $C$ is weakly sequentially complete.

**2.1 Example.** Let $E=l_1=\{(a_i)\in\mathbb{R}^{\mathbb{N}}\colon \sum|a_i|<\infty\}$ and $\|(a_i)\|=\sum|a_i|+(\sum|a_i|^2)^{1/2}$. Then $(E, \|\ \|)$ is a Banach space and $\|\ \|$ is strictly convex and equivalent to the usual $\|\ \|_1$ on $l_1$. $E$ has the $RN$-property, since $(E, \|\ \|_1)$ is a separable dual space and the $RN$-property is an isomorphic invariant. However, every closed convex set $C \subset E$ is weakly sequentially complete, since $E$ is weakly sequentially complete (see Dunford, Schwartz [3] IV.8.6). Thus according to (i): For every nonempty closed convex set $C \subset E$ and every $f_0 \in E$ there is $f \in C$ such that $\|f-f_0\|=\inf\{\|f-g\|\colon g\in C\}$. According to Singer [14] p. 99 Cor. 2.4 this implies that $E$ is reflexive, which is obviously wrong.

Therefore the existence result of Lemma 3.5 in [12] is false. The proof of 3.5 is correct iff $E$ is reflexive, for every uniformly integrable set $H \subset L_1(E)$ is weakly sequentially compact iff $E$ is reflexive.

Lemma 2.2 of [12] gives the solution to the uniqueness problem in (1.2). M.M. Rao claimes:

(ii) $L_\Phi(\Omega, \mathscr{A}, P; E)$ is strictly convex, if $E$ is strictly convex and $\Phi$ is strictly convex, continously differentiable and $\Phi'(t) \to \infty$ for $t \to \infty$.

For the case $E = \mathbb{R}$ this was first stated in [6] Th. 4. The proof of Th. 4 yields another result that is explicitely stated in [10] p. 556, 557:

(iii) If $\Phi$ is strictly convex and continuously differentiable, then $\int \Phi\left(\frac{|X|}{N_\Phi(X)}\right) dP = \Phi(1)$ for all $X \in L_\Phi(\Omega, \mathscr{A}, P; \mathbb{R}) - \{0\}$.

The following theorem shows that M.M. Rao's assertions (ii) and (iii) are wrong even in the case $E = \mathbb{R}$ unless $\int \Phi(|X|)\, dP < \infty$ for all $X \in L_\Phi$. In [15] Sundarasan has shown that $L_\Phi(\Omega, \mathscr{A}, P; \mathbb{R})$ is isomorphic to a strictly convex space $L_{\Phi_1}(\Omega, \mathscr{A}, P; \mathbb{R})$, if $\Phi$ satisfies a so called $\Delta_2$-condition ($\Phi(2x) \leqq c\Phi(x)$ for all $x \geqq x_0$ with some $x_0 \geqq 0$ and $c > 0$). The proof of Sundarasan's Th. 1 obviously also works under the sole assumption that $\int \Phi(|X|)\, dP < \infty$ for all $X \in L_\Phi$. In [11] Th. 1 Rao tries to generalize Sundarasan's result to the case of an arbitrary Y-function $\Phi$. But the main tool in the proof is Th. 4 of [6], which is false whenever $\int \Phi(|X|)\, dP = \infty$ for some $X \in L_\Phi$. Thus Rao's proof is false, whenever his theorem is a nontrivial extension of Sundarasan's result.

**2.2 Theorem.** *Let $\Phi$ be a strictly convex Y-function. Then:*

(2.3) $(L_\Phi(\Omega, \mathscr{A}, P; \mathbb{R}), N_\Phi)$ *is strictly convex* $\Leftrightarrow \int \Phi(|X|)\, dP < \infty$ *for all* $X \in L_\Phi$.

(2.4) *If* $\int \Phi(|X|)\, dP = \infty$ *for some* $X \in L_\Phi$, *then there is* $Y \in L_\Phi - \{0\}$, $Y \geqq 0$, *such that* $\int \Phi\left(\frac{Y}{N_\Phi(Y)}\right) dP < \Phi(1)$.

*Proof.* (2.3) "$\Leftarrow$" is contained in the proof of Th. 1 in [15].

(2.4): We show first: If $(\Sigma, \mathscr{F}, \mu)$ is a finite measure space, $\Phi$ a Y-function and if there is $Y \in L_\Phi(\Sigma, \mathscr{F}, \mu; \mathbb{R})$ with $\int \Phi(|Y|)\, d\mu = \infty$, then for every $\varepsilon > 0$ there is $Y_\varepsilon \in L_\Phi(\Sigma, \mathscr{F}, \mu; \mathbb{R})$, $Y_\varepsilon \geqq 0$, such that: $\int \Phi(Y_\varepsilon)\, d\mu < \varepsilon$, $\int \Phi((1+\varepsilon) Y_\varepsilon)\, d\mu = \infty$.

Let $\varepsilon > 0$ be given. There is $Z_\varepsilon \in L_\Phi$, $Z_\varepsilon \geqq 0$, such that $\int \Phi(Z_\varepsilon)\, d\mu < \infty$ and $\int \Phi((1+\varepsilon) Z_\varepsilon)\, d\mu = \infty$. Then for $n \to \infty$ $\int \Phi(Z\, 1_{\{Z_\varepsilon > n\}})\, d\mu \to 0$. Therefore one can take $Y_\varepsilon = Z_\varepsilon 1_{\{Z_\varepsilon > n\}}$ with a sufficiently large $n \in \mathbb{N}$.

If there is $Y \in L_\Phi(\Omega, \mathscr{A}, P; \mathbb{R})$ such that $\int \Phi(|Y|)\, dP = \infty$, there is a partition of $\Omega$ into countably many disjoint $\Omega_n$, $n \in \mathbb{N}$, such that $\int_{\Omega_n} \Phi(|Y|)\, dP = \infty$ for $n \in \mathbb{N}$. Now apply the preceding remark to $(\Sigma, \mathscr{F}, \mu) = (\Omega_n, \mathscr{A} \cap \Omega_n, P|\mathscr{A} \cap \Omega_n)$ and choose functions $Y_n \in L_\Phi(\Omega_n, \mathscr{A} \cap \Omega_n, P|\mathscr{A} \cap \Omega_n; \mathbb{R})$, $Y_n \geqq 0$ such that: $\int_{\Omega_n} \Phi(Y_n)\, dP \leqq \varepsilon_n$ and $\int_{\Omega_n} \Phi((1+\varepsilon_n) Y_n)\, dP = \infty$ with $\varepsilon_n := 2^{-n-1}\Phi(1)$. Define $Y := \sum_{n \in \mathbb{N}} Y_n 1_{\Omega_n}$. Then $Y \in L_\Phi - \{0\}$, $Y \geqq 0$, $\int \Phi(Y)\, dP \leqq \frac{1}{2}\Phi(1)$ and $\int \Phi(aY)\, dP = \infty$ for all $a > 1$. Thus $N_\Phi(Y) = 1$ and this proves (2.4).

Now we show:

(2.5) If $Y$ is as in (2.4), then there is an $\varepsilon > 0$, such that $\{a \in \mathbb{R}: N_\Phi(Y-a) = \inf_{b \in \mathbb{R}} N_\Phi(Y-b)\} \supset [0, \varepsilon]$.

Let $Y$ be as in (2.4). Let $a \in \mathbb{R}$ be arbitrary and $0 < k < N_\Phi(Y)$. Choose $m$ with $k < m < N_\Phi(Y)$. Then:

$$\frac{Y}{m} = \frac{k}{m}\,\frac{Y-a}{k} + \left(1 - \frac{k}{m}\right)\frac{a}{m(1-k/m)}$$

Convexity of $\Phi$ yields:

$$\Phi\left(\frac{Y}{m}\right) \leqq \frac{k}{m} \Phi\left(\frac{Y-a}{k}\right) + \left(1-\frac{k}{m}\right) \Phi\left(\frac{a}{m(1-k/m)}\right)$$

$$\infty = \int \Phi\left(\frac{Y}{m}\right) dP \leqq \frac{k}{m} \int \Phi\left(\frac{Y-a}{k}\right) dP + \left(1-\frac{k}{m}\right) \Phi\left(\frac{a}{m(1-k/m)}\right)$$

Thus $\int \Phi\left(\frac{Y-a}{k}\right) dP = \infty$ for all $k \in (0, N_\Phi(Y))$, $a \in \mathbb{R}$ and therefore $N_\Phi(Y - a) \geqq N_\Phi(Y)$ for all $a \in \mathbb{R}$. Since $\Phi$ is continuous and $\int \Phi\left(\frac{Y}{N_\Phi(Y)}\right) < \Phi(1)$ we obtain for sufficiently small $a \geqq 0$:

$$\int \Phi\left(\frac{Y-a}{N_\Phi(Y)}\right) dP \leqq \Phi\left(\frac{a}{N_\Phi(Y)}\right) + \int \Phi\left(\frac{Y}{N_\Phi(Y)}\right) dP \leqq \Phi(1).$$

Thus there is $\varepsilon > 0$, such that $N_\Phi(Y-a) = N_\Phi(Y)$ for $a \in [0, \varepsilon]$.

(2.3) "$\Rightarrow$" follows now immediately from (2.4) and (2.5).

## 3. Convergence of $P_\Phi^{\mathscr{B}_n} X$ for an Increasing Sequence $\mathscr{B}_n$ of Sub-$\sigma$-fields of $\mathscr{A}$

Since the main errors of M.M. Rao's theorems on this subject persist for $E = \mathbb{R}$, we restrict ourselves to this case. In [13] M.M. Rao made an additional assumption under which his proofs are allegedly correct, the $(*)$-property of $L_\Phi$.

**3.1 Definition.** Assume that $P_\Phi^{\mathscr{B}}$ is uniquely defined for every $\sigma$-field $\mathscr{B} \subset \mathscr{A}$. $L_\Phi$ has the $(*)$-property if $P_\Phi^{\mathscr{B}}(XY) = Y P_\Phi^{\mathscr{B}} X$ for $X \in L_\Phi$ and bounded $\mathscr{B}$-measurable $Y$ and every $\sigma$-field $\mathscr{B} \subset \mathscr{A}$.

The $L_\Phi$-spaces with this property will be described in the following theorem.

**3.2 Theorem.** *Assume that $\Phi$ is strictly convex, differentiable (then $\Phi'$ is continuous and $\Phi'(0) = 0$, since $\Phi$ is a Y-function) and $\int \Phi(|X|)\, dP < \infty$ for all $X \in L_\Phi$. Then $P_\Phi^{\mathscr{B}}$ is uniquely defined for every $\sigma$-field $\mathscr{B} \subset \mathscr{A}$ (see Landers and Rogge [4]). Assume furthermore that $L_\Phi$ has the $(*)$-property. Then at least one of the two following statements is true:*

(a) *$\Phi(x) = c|x|^p$ for some $p > 1$, $c > 0$.*

(b) *$P$ attains only finitely many values.*

*Proof.* Assume that (b) is not fulfilled. Additionally one may assume $\Phi'(1) = 1$, for $\Phi'(1) > 0$ and $(L_\Phi, N_\Phi)$ does not change when $\Phi$ is multiplied by a constant. Consider the following sets:

$$T = \{u > 0 \colon \exists\, v(u) > 0\ \forall\, x \geqq 0 \quad \Phi'(x) = u \Phi'(v(u) x)\}$$

and for $L > 0$:

$$T_L = \{u \geqq 1 \colon \exists\, v(u) > 0\ \forall\, x \in [0, L] \quad \Phi'(x) = u \Phi'(v(u) x)\}.$$

Let $L>0$ be given. Since (b) is not fulfilled one may choose four disjoint sets $A_i$, $i=1,\dots 4$ in $\mathscr{A}$, such that $a_i=P(A_i)>0$ for $i=1,\dots 4$, $a_3 \leqq a_4$, $a_3+a_4<\frac{\Phi(1)}{\Phi(2L)}$ and $\{P(B)\colon B\in\mathscr{A},\ B\subset\Omega-\bigcup_{i=1}^{4} A_i\}$ is an infinite set. There is a sequence $(B_k)_{k\in\mathbb{N}}$ in $\mathscr{A}$ with $B_k\subset\Omega-\bigcup_{i=1}^{4} A_i$, such that: $b_k=P(B_k)>0$, $b_k<\frac{\Phi(1)}{\Phi(2L)}-(a_3+a_4)$, $b_k>b_{k+1}$ for $k\in\mathbb{N}$ and $b_k \xrightarrow[k\in\mathbb{N}]{} 0$. Let now $k\in\mathbb{N}$ be fixed. Let $\mathscr{B}=\sigma\{A_1\cup A_2, A_3\cup A_4\cup B_k\}$. For $a\geqq 0$ let $X_a\in L_\Phi$ be defined by:

$$X_a := \begin{cases} -a & \text{on } A_1 \\ a & \text{on } A_2 \\ -1 & \text{on } A_3 \\ 1 & \text{on } A_4\cup B_k \\ 0 & \text{elsewhere} \end{cases}$$

It is easy to see that $a\to\delta(a)=\inf\{N_\Phi(X_a-Y)\colon Y\in L_\Phi(\mathscr{B})\}$ is a continuous function and $\lim_{a\to\infty}\delta(a)=\infty$. Since $\int\Phi(|X_0 2L|)\,dP=(a_3+a_4+b_k)\,\Phi(2L)<\Phi(1)$ follows $\delta(0)\leqq N_\Phi(X_0)\leqq\frac{1}{2L}$. Therefore: $\{\delta(a)\colon a\geqq 0\}\supset\left[\frac{1}{2L},\infty\right)$. For $a\geqq 0$ there are uniquely determined $b(a)$, $c(a)$ such that:

$$P_\Phi^{\mathscr{B}} X_a = \begin{cases} b(a) & \text{on } A_1\cup A_2 \\ c(a) & \text{on } A_3\cup A_4\cup B_k \\ 0 & \text{elsewhere} \end{cases}$$

It is clear that always $c(a)\in[-1,1]$. Now define a function $f_a\colon[-1,1]\to\mathbb{R}$ by

$$f_a(t)=a_3\,\Phi\left(\frac{t+1}{\delta(a)}\right)+(a_4+b_k)\,\Phi\left(\frac{1-t}{\delta(a)}\right)$$

$f_a$ has an absolute minimum at $t=c(a)$. Using the differential calculus one obtains: $c(a)\in(-1,1)$ since $f_a'(-1)<0$, $f_a'(1)>0$ and $f_a'(c(a))=0$. This gives the following formula:

$$a_3\,\Phi'\left(\frac{c(a)+1}{\delta(a)}\right)=(a_4+b_k)\,\Phi'\left(\frac{1-c(a)}{\delta(a)}\right)$$

Since $X_a=X_0$ on $A_3\cup A_4\cup B_k$ the $(*)$-property of $L_\Phi$ implies that $c(a)$ does not depend on $a$. Therefore there is $c\in(-1,1)$, such that:

$$a_3\,\Phi'\left(\frac{c+1}{\delta(a)}\right)=(a_4+b_k)\,\Phi'\left(\frac{1-c}{\delta(a)}\right)\qquad\text{for all } a\geqq 0.$$

As $a_3<a_4+b_k$ we have $c>0$. Since $\{\delta(a)\colon a\geqq 0\}\supset\left[\frac{1}{2L},\infty\right)$, we obtain $a_3\,\Phi'(x(c+1))=(a_4+b_k)\,\Phi'(x(1-c))$ for all $x\in[0,2L]$ and hence:

$$\Phi'(x)=a_3^{-1}(a_4+b_k)\,\Phi'\left(\frac{1-c}{1+c}\,x\right)\qquad\text{for all } x\in[0,2L].$$

Set $u_k = a_3^{-1}(a_4 + b_k)$ for $k \in \mathbb{N}$. We have proved that for every $k \in \mathbb{N}$ there is $v_k > 0$ such that

$$\Phi'(x) = u_k \Phi'(v_k x) \qquad \text{for all } x \in [0, 2L].$$

Since $u_k \xrightarrow[k\in\mathbb{N}]{} a_3^{-1} a_4 > 0$, $\Phi'(1) = 1$ and $(\Phi')^{-1}$ is continuous, we have $v_k \xrightarrow[k\in\mathbb{N}]{} (\Phi')^{-1}(a_4^{-1} a_3) > 0$. Therefore we have $v_k v_{k+1}^{-1} \leqq 2$ for sufficiently large $k$. Thus for sufficiently large $k \in \mathbb{N}$ and all $x \in [0, L]$ we obtain, applying the above relation twice:

$$\Phi'(v_k v_{k+1}^{-1} x) = u_{k+1} \Phi'(v_k x) = u_{k+1} u_k^{-1} \Phi'(x)$$

Therefore $u_k u_{k+1}^{-1} \in T_L$ for all sufficiently large $k \in \mathbb{N}$. Thus 1 is an accumulation point of $T_L$, and since $T_L$ is a semigroup with respect to multiplication and a closed subset of $[1, \infty)$, follows $T_L = [1, \infty)$. As this holds for every $L > 0$ and $T$ is a group with respect to multiplication, we have $T = (0, \infty)$. Thus:

$$\forall u > 0 \ \exists v > 0 \ \forall x > 0 \qquad \Phi'(x) = u \Phi'(vx)$$

If $y > 0$ is given, apply the above relation with $u = \dfrac{1}{\Phi'(y)}$. Since $\Phi'(1) = 1$ and $\Phi'$ is strictly increasing follows $v = y$. Thus we obtain:

$$\forall x > 0 \ \forall y > 0 \qquad \Phi'(xy) = \Phi'(x) \Phi'(y).$$

According to Dieudonné [2] this implies $\Phi'(x) = x^s$ for $x > 0$ with some $s \in \mathbb{R}$, and $s > 0$, since $\Phi'$ is strictly increasing.

**3.3 Remark.** Even if $P$ is purely atomic with finitely many atoms, the $L_\Phi$-spaces have not necessarily the (*)-property. The proof of 3.2 shows that for strictly convex differentiable $\Phi$, even in the case of 4-point probability spaces, $\Phi'$ must fulfill several equations of the form $\Phi'(x) = u\Phi'(vx)$ $x \in [0, L]$ with $u, v, L > 0$, whenever $L_\Phi$ has the (*)-property.

Since differentiability and strict convexity of $\Phi$ is assumed in [12], Rao's a.e.-convergence theorem 3.4 (3) of [12] covers only the case of $L_p$-spaces with $p > 1$, and some $L_\Phi$-spaces, where $P$ attains only finitely many values so that 3.4 (3) is trivial. Even for the cases covered by Rao's theorem the proof remains false and so is the proof of 3.4 (2). According to page 134 Lemma 2.6 is "crucial for the work of section 3", and it is indeed his main tool. This Lemma however is false. This is a part of the statement of 2.6:

(iv) Let $\Phi$ be a $Y$-function such that $\Phi'(x) \to \infty$ as $x \to \infty$. Let $\sup\{\Phi'(x)/x : 0 \leqq x < \infty\} < \infty$. Then there is a constant $c > 0$, such that for $x' \neq 0$ and any $x$ one has:

$$\Phi(x') \geqq \Phi(x) + \Phi'(x)(x - x') + c\Phi(x - x')$$

The following example shows that (iv) is false.

**3.4 Example.** Define a strictly convex differentiable $Y$-function $\Phi$ as follows:

$$\Phi(x)=\tfrac{1}{2}x^2 \quad 0\leqq x\leqq 1$$

$$\Phi(x)=\Phi(n)+(x-n)\,\Phi'(n)+(x-n)^2\frac{1}{2n} \qquad n\leqq x\leqq n+1$$

$$\Phi(x)=\Phi(-x) \quad x\leqq 0$$

Then $\Phi'(n+1)=1+\frac{1}{2}+\ldots+\frac{1}{n}$ for $n\in\mathbb{N}$. Furthermore $\Phi'(x)/x$ is continuous on $(0,\infty)$, $\lim_{x\to 0}\Phi'(x)/x=1$ and $\Phi'(x)\leqq n$ for $x\in[n,n+1]$. Therefore $\sup\{\Phi'(x)/x: x\geqq 0\}<\infty$. (iv) implies: There exists $c>0$, such that:

$$\Phi(x')\geqq\Phi(x)+\Phi'(x)(x'-x)+c\,\Phi(x-x') \qquad \text{for all } x,x'\in\mathbb{R}.$$

Setting $x=n$ and $x'=n+1$ one obtains:

$$\Phi(n+1)\geqq\Phi(n)+\Phi'(n)+c\,\Phi(1) \qquad \text{for all } n\in\mathbb{N}.$$

This gives $\frac{1}{2n}\geqq c\,\Phi(1)$ for all $n\in\mathbb{N}$, which is impossible, since $c>0$, $\Phi(1)>0$.

Some further errors of M.M. Rao:

(i) For $E=\mathbb{R}$ 3.4 (4) of [12] implies that $P_\Phi^{\mathscr{B}}X\in L_\Phi(\mathscr{B})$ is determined by $\int\Phi(|X-P_\Phi^{\mathscr{B}}X|)\,dP=\inf_{Y\in L_\Phi(\mathscr{B})}\int\Phi(|X-Y|)\,dP$ (see Landers, Rogge [4], remark 31). But this would imply $P_\Phi^{\mathscr{B}}(1_B X)=1_B P_\Phi^{\mathscr{B}}X$ for $B\in\mathscr{B}$, and this can only be true for those $L_\Phi$-spaces, which are described in Th. 3.2 above.

(ii) In order to apply Th. 3.6 of [12] in the way Rao does in the proof of 3.8, he must know that $\bigcup L_\Phi(\mathscr{B}_n)$ is dense in $L_\Phi(\mathscr{B}_\infty)$; this is not true in general if $\Phi$ does not fulfill a $\Delta_2$-condition.

(iii) On page 135 of [12] M.M. Rao mentions a result that is taken from [9]. He claims: "$P_\Phi^{\mathscr{B}}$ is the usual conditional expectation (if and) only if (when $E$ is a uniformly convex $B$-space) $L_\Phi(E)$ is $L_2(E)$". Of course this cannot be true. The conditional expectation does not change, if an equivalent norm on $E$ is introduced, whereas $P_\Phi^{\mathscr{B}}$ depends crucially on the norm, even for $\Phi(x)=x^2$.

These notes are a part of the authors diploma thesis written under the guidance of Prof. D. Landers.

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Received January 15, 1980