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STOCHASTIC PROCESSES AS CURVES IN HILBERT SPACE

HARALD CRAMÉR

1. In this paper the theory of spectral multiplicity in a separable Hilbert space will be applied to the study of stochastic processes $x(t)$, where $x(t)$ is a complex-valued random variable with a finite second-order moment, while the parameter t may take any real values.

For an account of multiplicity theory we may refer to Chapter 7 of Stone's book [13] which deals with the case of a separable space. The treatment of the subject found e.g. in the books by Halmos [6] and Nakano [10] is mainly concerned with the more general and considerably more intricate case of a non-separable space.

Our considerations will apply to certain classes of curves in a purely abstract Hilbert space, and it is only a question of terminology when, throughout this paper, we confine ourselves to that particular realization of Hilbert space which has proved useful in probability theory.

We finally observe that all our statements may be directly generalized to the case of vector processes of the form $\{x_1(t), \cdots, x_n(t)\}$.

2. In Sections 2—4 we shall now introduce some basic definitions and some auxiliary concepts.

Consider the set of all complex-valued random variables x defined on a fixed probability space and satisfying the relations

$$\mathbf{E}\{x\} = 0, \qquad \mathbf{E}\{|x|^2\} < \infty.$$

Two variables x and y will be regarded as identical if

$$\mathbf{E}\{|x-y|^2\} = 0.$$

The set of all these variables forms a Hilbert space H, if the inner product is defined in the usual way:

$$(x, y) = \mathbf{E}\{x\bar{y}\}.$$

Convergence of sequences of random variables will always be understood as strong convergence in the topology thus introduced, i.e. as convergence in quadratic mean according to probability terminology.

If for every real t a random variable $x(t) \in H$ is given, the set of variables $x(t)$ may be regarded as a stochastic process with continuous time t or, alternatively, as a curve C in the Hilbert space H. It is well known that various properties of stochastic processes have been studied by regarding them as curves in H (cf., e.g., [7], [8], [11], [12], [2], [3]). We shall give in this paper some further applications of this point of view.

We define certain subspaces of H associated with the $x(t)$ curve or process by writing

$$H(x) = S\{x(u),\ -\infty < u < \infty\},$$
$$H(x, t) = S\{x(u),\ u \leqq t\},$$
$$H(x, -\infty) = \bigcap_t H(x, t).$$

Here we denote by $S\{\cdots\}$ the subspace of H spanned by the random variables indicated between the brackets.

Evidently $H(x)$ is the smallest subspace of H which contains the whole curve C generated by the $x(t)$ process, while $H(x, t)$ is the smallest subspace containing the "arc" of C formed by all points $x(u)$ with $u \leqq t$. If the parameter t is interpreted as time, $H(x, t)$ will perresent the "past and present" of the $x(t)$ process from the point of view of the instant t, while $H(x, -\infty)$ represents the "infinitely remote past" of the process.

For the processes $y(t), z(t), \cdots$ we use in the sequel the analogous notations $H(y, t)$, $H(z, t), \cdots$ for subspaces defined in the corresponding way.

In the sequel we shall only consider stochastic processes $x(t) \in H$ which are assumed to satisfy the following conditions (A) and (B):

(A) The subspace $H(x, -\infty)$ contains only the zero element of H.

(B) For all t the limits $x(t \pm 0)$ exist and $x(t-0) = x(t)$.

The condition (A) implies that $x(t)$ is a *regular*, or *purely non-deterministic* process (cf. [3]). From (B) it follows, as shown in [3], that the space $H(x)$ is separable, and that the $x(t)$ curve has at most an enumerable number of discontinuities. The condition $x(t-0) = x(t)$ is not essential, and is introduced here only in order to avoid some trivial complications.

As t increases through real values, the subspaces $H(x, t)$ will obviously form a never decreasing family. For a fixed finite t the union of all $H(x, u)$ with $u < t$ may not be a closed set, but if we define $H(x, t-0)$ as the closure of this union, it is easily proved that we always have $H(x, t-0) = H(x, t)$. Similarly, the union of the $H(x, u)$ for all real u may not be closed, but if $H(x, +\infty)$ is defined as its closure, we shall have $H(x, +\infty) = H(x)$.

Suppose that a certain time point t is such that for all $h > 0$ we have $H(x, t-h) \neq H(x, t+h)$. Every time interval $t-h < u < t+h$ will then contain at least one $x(u)$ not included in $H(x, t-h)$. This may be expressed by saying that the process receives a new impluse, or an *innovation*, during the interval $(t-h, t+h)$ for every $h > 0$. The set of all time points t with this property will be called the *innovation spectrum* of the $x(t)$ process.

3. Suppose that we are given a stochastic process $x(t)$ satisfying the conditions (A) and (B). By P_t we shall denote the projection operator in the Hilbert space $H(x)$ with range $H(x, t)$. It then follows from the properties of the $H(x, t)$ family given above that we have

$$P_t \leqq P_u \text{ for } t < u,$$
$$(1) \qquad P_{t-0} = P_t \text{ for all } t,$$
$$P_{-\infty} = 0,\ P_{+\infty} = 1,$$

where 0 and 1 denote respectively the zero and the identity operator in $H(x)$.

It follows that the P_t form a *spectral family* of projections or a *resolution of the identity* according to Hilbert space terminology. As we have seen, the projections P_t are, for all real t, uniquely determined by the given $x(t)$ process. We shall return to the properties of the P_t family in Section 5.

For any random variable $z \in H(x)$ with $\mathbf{E}\{|z|^2\} = 1$ we now define a stochastic process by writing

$$z(t) = P_t z.$$

It is then readily seen that $z(t)$ will be a process with orthogonal increments satisfying (A) and (B). Writing

$$F_z(t) = \mathbf{E}\{|z(t)|^2\},$$

it follows that $F_z(t)$ is, for any fixed z, a distribution function of t such that $F_z(t-0) = F_z(t)$ for all t.

The points of increase of $z(t)$ coincide with the points of increase of $F_z(t)$ and form a subset of the innovation spectrum of $x(t)$, and, accordingly, we shall denote $z(t)$ as a *partial innovation process* associated with $x(t)$. The space $H(z, t)$ is spanned by the random variables $z(u)$ with $u \leqq t$, and it is known (cf., e.g., [5], p. 425—428) that $H(z, t)$ is identical with the set of all random variables of the form

$$\int_{-\infty}^{t} g(t, u) dz(u),$$

where $g(t, u)$ is a non-random function such that the integral

$$\int_{-\infty}^{t} |g(t, u)|^2 dF_z(u)$$

is convergent.

4. Consider now the class Q of all distribution functions $F(t)$ determined such as to be continuous to the left for all t. We introduce a partial ordering in Q by saying that F_1 is *superior* to F_2, and writing $F_1 \succ F_2$, whenever F_2 is absolutely continuous with respect to F_1. If $F_1 \succ F_2$ and $F_2 \succ F_1$, we say that F_1 and F_2 are *equivalent*.

The set of all distribution functions equivalent to a given $F(t)$ forms an *equivalence class* R. A partial ordering is introduced in the set of all equivalence classes in the obvious way by writing $R_1 \succ R_2$ when the corresponding relation holds for any $F_1 \in R_1$ and $F_2 \in R_2$. A point t is called a point of increase for the equivalence class R whenever t is a point of increase for any $F \in R$.

In the sequel we shall be concerned with never increasing sequences of equivalence classes:

$$R_1 \succ R_2 \succ \cdots \succ R_N. \tag{2}$$

The number N of elements in a sequence of this form, which may be finite or infinite, will be called the *total multiplicity* of the sequence. We shall also define a *multiplicity function* $N(t)$ of the sequence (2) by writing for any real t

$N(t) =$ the number of those R_n in (2) for which t is a point of increase.

$N(t)$, like N, may be finite or infinite, and the total multiplicity N will obviously satisfy the relation

$$N = \sup N(t),$$

where t runs through all real values.

If, in particular, we have $N(t) = 0$ for all t in some closed interval $[a, b]$, all functions $F(t)$ belonging to any of the equivalence classes in the sequence (2) will be constant throughout $[a, b]$.

5. To any $x(t)$ satisfying the conditions (A) and (B) there corresponds according to Section **3** a uniquely determined spectral family of projections P_t satisfying (1). It then follows from the theory of spectral multiplicity in a separable Hilbert space ([13], Chapter 7) that to the same $x(t)$ there corresponds a uniquely determined, never increasing sequence (2) of equivalence classes, having the following properties:

If N is the total multiplicity of the sequence (2), it is possible to find N orthonormal random variables $z_1, \cdots, z_N \in H(x)$ such that the corresponding processes with orthogonal increments defined in Section **3** satisfy the relations

$$\begin{aligned} &F_{z_n}(t) \in R_n, \\ &H(z_m, t) \perp H(z_n, t), \qquad m \neq n, \\ &H(x, t) = \sum_1^N H(z_n, t), \end{aligned} \tag{3}$$

where the last sum denotes the vector sum of the mutually orthogonal subspaces involved.

Now $x(t)$ is always an element of $H(x, t)$ and from Section **3** we then obtain the following theorem, previously given in somewhat less precise form in [2] and [3].

Theorem 1. *To any stochastic process* $x(t)$ *satisfying* (A) *and* (B) *there corresponds a uniquely determined sequence* (2) *of equivalence classes such that* $x(t)$ *can be represented in the form*

$$x(t) = \sum_1^N \int_{-\infty}^t g_n(t, u) dz_n(u), \tag{4}$$

where the $z_n(u)$ *are mutually orthogonal processes with orthogonal increments satisfying* (3). *The* $g_n(t, u)$ *are non-random functions such that*

$$\sum_1^N \int_{-\infty}^t |g_n(t, u)|^2 dF_{z_n}(u) < \infty.$$

The number N, *which is called the total spectral multiplicity of the* $x(t)$ *process, is the uniquely determined number of elements in* (2) *and may be finite or infinite. No representation of the form* (4) *with these properties exists for any smaller value of* N.

The sequence (2) corresponding to a given $x(t)$ process will be said to determine the *spectral type* of the process.

The relation (4) gives a linear representation of $x(t)$ in terms of past and present innovation elements $dz_n(u)$. The *total innovation process* associated with $x(t)$ is an N-dimensional vector process $\{z_1(t), \cdots, z_N(t)\}$ where, as before, N may be finite or infinite.

It is interesting to compare this with the situation in the case of a regular process with *discrete* time ([2], Theorem 1) where a similar representation always holds with $N = 1$.

Also in the particular case of a *stationary* process with continuous time, satisfying (A) and (B), it follows from well-known theorems that we have $N = N(t) = 1$ for all t, and that the only element in the corresponding sequence (2) may be represented by any absolutely continuous distribution function $F(t)$ having an everywhere positive density function.

6. The best linear least squares prediction of $x(t+h)$ in terms of all $x(u)$ with $u \leqq t$ is obtained from (4) in the form

$$P_t x(t+h) = \sum_1^N \int_{-\infty}^t g_n(t+h, u) dz_n(u).$$

The error involved in this prediction is

$$x(t+h) - P_t x(t+h) = \sum_1^N \int_t^{t+h} g_n(t+h, u) dz_n(u). \tag{5}$$

Now consider the multiplicity function $N(t)$ associated with the sequence (2), as defined in Section 3. Suppose that in the closed interval $t \leqq u \leqq t+h$ we have $N(u) \leqq N_1 < N$. Then all terms with $n > N_1$ in the second member of (5) will reduce to zero, so that the innovation entering into the process during $[t, t+h]$ will only be of dimensionality N_1. Speaking somewhat loosely, we may say that the multiplicity function $N(t)$ determines for every t the dimensionality of the innovation element $\{dz_1(t), \cdots\}$.

If, in particular, $N(u) = 0$ for $t \leqq u \leqq t+h$, it follows that the process does not receive any innovation at all during this interval. Accordingly in this case the whole second member of (5) reduces to zero, so that exact prediction is possible over the interval considered.

7. We now introduce the correlation function of the $x(t)$ process:

$$r(s, t) = \mathbf{E}\{x(s)\overline{x(t)}\}.$$

As before we assume that all stochastic processes considered satisfy the conditions (A) and (B). We proceed to prove the following theorem, which shows that the spectral type of a process is uniquely determined by the correlation function.

Theorem 2. *Let $x(t)$ and $y(t)$ be two processes satisfying* (A) *and* (B) *and having the same correlation function $r(s, t)$. The sequences of equivalence classes, which correspond to $x(t)$ and $y(t)$ in the way described in Theorem* 1, *are then identical.*

$x(t)$ and $y(t)$ define two curves situated, respectively, in the spaces $H(x)$ and $H(y)$. We now define a transformation V from the x-curve to the y-curve by writing

$$Vx(t) = y(t),$$

and extend this definition by linearity to the linear manifold in $H(x)$ determined by all points $x(t)$. It is readily seen that this definition is unique, and that the transformation is isometric. It follows, in fact, from the equality of the correlation functions that any linear relation $\sum c_n x(t_n) = 0$ implies and is implied by the corresponding relation $\sum c_n y(t_n) = 0$, which shows that the transformation is unique, while the isometry follows from the identity

$$r(s, t) = \big(x(s), x(t)\big) = \big(Vx(s), Vx(t)\big).$$

The transformation can now be extended to an isometric transformation V defined in the whole space $H(x)$. If we consider the restriction of V to $H(x, t)$, it is immediately seen that we have for all t

$$VH(x, t) = H(y, t).$$

Denoting by $P_t^{(x)}$ and $P_t^{(y)}$ the spectral families of projections corresponding, respectively, to $x(t)$ and $y(t)$, we then obtain

$$VP_t^{(x)}V^{-1} = P_t^{(y)}.$$

Thus the two spectral families are isometrically equivalent, and the assertion of the theorem now follows directly from Hilbert space theory. In the particular case when $H(x) = H(y)$, the transformation V will be unitary.

On the other hand, two processes with isometrically equivalent spectral families do not necessarily have the same correlation function. In other words, *the correlation function is not uniquely determined by the spectral type.*

In order to see this, it is enough to consider the two processes $x(t)$ and $y(t) = f(t)x(t)$, where $f(t)$ is a non-random function such that $0 < m < |f(t)| < M$ for all t. It is clear that $H(x, t) = H(y, t)$ for all t, while the correlation functions differ by the factor $f(s)\overline{f(t)}$.

8. In this section it will be shown that we can always find a stochastic process possessing any given spectral type. We shall even prove the more precise statement contained in the following theorem.

Theorem 3. *Suppose that a sequence of equivalence classes of the form* (2) *is given. Then there exists a harmonizable process*

$$x(t) = \int_{-\infty}^{\infty} e^{it\lambda}\,dy(\lambda)$$

which has the spectral type defined by the given sequence.

Comparing this statement with the final remark in Section 5, it will be seen how restricted the class of stationary processes is in comparison with the class of harmonizable processes.

In order to prove the theorem we denote by $A_1, A_2, \cdots$ a sequence of disjoint sets of real points such that the measure of every A_n is positive in any non-vanishing interval.[1] If $\alpha_n(v)$ is the characteristic function of A_n, we thus have

$$\int_a^a \alpha_n(v)dv > 0$$

for all n and for any real $a < b$.

We further take in each equivalence class R_n appearing in the given sequence (2) a distribution function $F_n(t) \in R_n$. Obviously we can choose the functions $F_1, \cdots, F_N$ so that the integrals

$$k_n^2 = \int_{-\infty}^{\infty} e^{t^2} dF_n(t) \tag{6}$$

converge for all n. We then have $1 \leqq k_n < \infty$. Assuming that the basic probability field is not too restricted, we can then find N mutually orthogonal stochastic processes $z_1(t), \cdots, z_N(t)$ with orthogonal increments such that

$$F_{z_n}(t) = \mathbf{E}\{|z_n(t)|^2\} = F_n(t).$$

We now introduce the following definition:

$$g_n(t, u) = \begin{cases} \dfrac{1}{nk_n} e^{-t} \displaystyle\int_u^t (t-v)\alpha_n(v)dv, & u < t, \\ 0, & u \geqq t, \end{cases} \tag{7}$$

and

$$x_n(t) = \int_{-\infty}^{t} g_n(t, u)dz_n(u), \tag{8}$$

$$x(t) = \sum_1^N x_n(t).$$

We then have for $u < t$

$$0 < g_n(t, u) < \frac{1}{nk_n} e^{-t}(t-u)^2,$$

and hence by (6),

$$\begin{aligned} \mathbf{E}\{|x_n(t)|^2\} &< \frac{1}{n^2 k_n^2} e^{-2t} \int_{-\infty}^{\infty} (t-u)^4 dF_n(u) \\ &\leqq \frac{8}{n^2 k_n^2} e^{-2t} \int_{-\infty}^{\infty} (t^4+u^4) dF_n(u) \leqq \frac{8(t^4+k_n^2)}{n^2 k_n^2}, \end{aligned}$$

so that the series for $x(t)$ converges in quadratic mean if $N = \infty$. (We note that the $x_n(t)$, like the $z_n(t)$, are mutually orthogonal.)

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[1] The use of the sets A_n for the construction of processes with given multiplicity properties goes back to a correspondence between Professor Kolmogorov and the present author (cf. [4]). A simple way of constructing the A_n is the following: Let $1 < n_1 < n_2 < \cdots$ be positive integers such that $\sum_1^\infty 1/n_k$ converges. Almost every real x then has a unique expansion $x = r_0 + \sum_1^\infty r_k/(n_1 \cdots n_k)$, where the r_k are integers and $0 \leqq r_k < n_k$ for $k \geqq 1$. If A_n is the set of those x for which the number of zeros among the r_k with $k \geqq 1$ is finite and of the form $2^n(2p+1)$ where p is a non-negative integer, then the sequence $A_1, A_2, \cdots$ has the required properties.

We now proceed to prove a) that the $x(t)$ process defined by (8) has the given spectral type, and b) that it is harmonizable.

It follows from the construction of the $z_n(t)$ and from (8) that we have

$$F_{z_n}(t) \in R_n,$$
$$H(z_m, t) \perp H(z_n, t), \qquad m \neq n,$$
$$H(x, t) \subset \sum_1^N H(z_n, t).$$

If we can show that the sign of equality holds in the last relation, the relations (3) will be satisfied and it then follows that the $x(t)$ process defined by (8) has the given spectral type. In order to prove this it is sufficient to show that we have

$$z_n(t) \in H(x, t)$$

for all n and t.

We have

$$e^t x(t) = \sum_1^N \int_{-\infty}^t g_n(t, u) dz_n(u) = \sum_1^N \frac{1}{nk_n} \int_{-\infty}^t (t-u)\alpha_n(u) z_n(u) du.$$

It is shown without difficulty that the derivative in q.m. of this random function exists for all t and has the expression

$$\frac{d}{dt}(e^t x(t)) = \sum_1^N \frac{1}{nk_n} \int_{-\infty}^t \alpha_n(u) z_n(u) du, \tag{9}$$

where the last sum converges in q.m. We now want to show that for almost all t (Lebesgue measure) we may differentiate once more in q. m., and so obtain

$$\frac{d^2}{dt^2}(e^t x(t)) = \sum_1^N \frac{1}{nk_n} \alpha_n(t) z_n(t). \tag{10}$$

In order to prove this we must show that the random variable

$$W = \sum_1^N \frac{1}{nk_n} \left(\frac{1}{h} \int_t^{t+h} \alpha_n(u) z_n(u) du - \alpha_n(t) z_n(t) \right)$$

converges to zero in q. m. for almost all t as $h \to 0$. We have $W = W_1 + W_2$, where

$$W_1 = \sum_1^N \frac{1}{nk_n h} \int_t^{t+h} \alpha_n(u) (z_n(u) - z_n(t)) du,$$

$$W_2 = \sum_1^N \frac{1}{nk_n} z_n(t) \left(\frac{1}{h} \int_t^{t+h} \alpha_n(u) du - \alpha_n(t) \right).$$

Now both W_1 and W_2 are sums of mutually orthogonal random variables and we have

$$\mathbf{E}|W_1|^2 = \sum_1^N \frac{1}{n^2 k_n^2 h^2} \int_t^{t+h}\int_t^{t+h} \alpha_n(u)\alpha_n(v)[F_n(\min(u, v)) - F_n(t)]\,du\,dv$$
$$\leqq \sum_1^N \frac{2}{n^2 h^2}\int_t^{t+h} (t+h-u)[F_n(u)-F_n(t)]du \leqq 2\sum_1^N \frac{F_n(t+h)-F_n(t)}{n^2}$$

and

$$\mathbf{E}|W_2|^2 = \sum_1^N \frac{1}{n^2 k_n^2} F_n(t)\left[\frac{1}{h}\int_t^{t+h} \alpha_n(u)du - \alpha_n(t)\right]^2$$
$$\leqq \sum_1^N \frac{1}{n^2}\left[\frac{1}{h}\int_t^{t+h}\alpha_n(u)du - \alpha_n(t)\right]^2.$$

However, all the F_n are continuous almost everywhere, and it follows that W_1 tends to zero in q. m. for almost all t. On the other hand, the metric density of any A_n exists almost everywhere and is equal to $\alpha_n(t)$ so that W_2 tends to zero in q. m. almost everywhere. Thus we have shown that (10) holds for almost all t.

Let now m be a given integer, $1 \leqq m \leqq N$. The sets A_n being disjoint, it then follows from (10) that for almost all $t \in A_m$

$$\frac{d^2}{dt^2}(e^t x(t)) = \frac{1}{mk_m} z_m(t).$$

The first member of the last relation is evidently an element of $H(x, t)$, so that we have $z_m(t) \in H(x, t)$ for almost all $t \in A_m$. Now A_m is of positive measure in every non-vanishing interval, while $z_m(t)$ is by definition everywhere continuous to the left in q. m. Thus $z_m(t) \in H(x, t)$ for all t and all $m = 1, \cdots, N$, and according to the above this proves that $x(t)$ has the given spectral type.

In order to prove also that $x(t)$ is harmonizable we introduce the Fourier transform $h_n(\lambda, u)$ of $g_n(t, u)$ with respect to t. From (7) we obtain

$$h_n(\lambda, u) = \int_{-\infty}^{\infty} g_n(t, u)e^{-it\lambda}dt = \frac{1}{nk_n}\int_u^{\infty} e^{-t(1+i\lambda)}dt\int_u^t (t-v)\alpha_n(v)dv.$$

The double integral is absolutely convergent and we have

$$h_n(\lambda, u) = \frac{1}{nk_n}\int_u^{\infty}\alpha_n(v)dv\int_v^{\infty}(t-v)e^{-t(1+i\lambda)}dt$$
$$= \frac{1}{nk_n(1+i\lambda)^2}\int_u^{\infty}\alpha_n(v)e^{-v(1+i\lambda)}dv,$$

$$|h_n(\lambda, u)| < \frac{e^{-u}}{nk_n(1+\lambda^2)}. \tag{11}$$

Thus $h_n(\lambda, u)$ is, for any fixed u, absolutely integrable with respect to λ. On the other hand, it follows from (7) that $g_n(t, u)$ is everywhere continuous, so that we have the inverse Fourier formula

$$g_n(t, u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_n(\lambda, u)e^{it\lambda}d\lambda. \tag{12}$$

Now the correlation functions of $x_n(t)$ and $x(t)$ are, by (7) and (8),

$$r_n(s, t) = \mathbf{E}\{x_n(s)\overline{x_n(t)}\} = \int_{-\infty}^{\infty} g_n(s, u)\overline{g_n(t, u)}dF_n(u),$$

$$r(s, t) = \mathbf{E}\{x(s)\overline{x(t)}\} = \sum_1^N r^n(s, t).$$

Replacing the g_n by their expressions according to (12) we obtain

$$r_n(s, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda - t\mu)}d\lambda\, d\mu \int_{-\infty}^{\infty} h_n(\lambda, u)\overline{h_n(\mu, u)}dF_n(u),$$

the inversion of the order of integration being justified by absolute convergence according to (6) and (11).

If we write

$$c_n(\lambda, \mu) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} h_n(\lambda, u)\overline{h_n(\mu, u)}dF_n(u),$$

$$C_n(\lambda, \mu) = \int_{-\infty}^{\lambda}\int_{-\infty}^{\mu} c_n(\rho, \sigma)d\rho\, d\sigma,$$

it follows from well-known criteria (cf., e.g., [9], p. 466—469) that $C_n(\lambda, \mu)$ is a correlation function. Further $C_n(\lambda, \mu)$ is of bounded variation over the whole (λ, μ)-plane, its variation being bounded by the expression

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |c_n(\lambda,\mu)|\, d\lambda\, d\mu < \frac{1}{(2\pi)^2 n^2 k_n^2}\int_{-\infty}^{\infty}\frac{d\lambda}{1+\lambda^2}\int_{-\infty}^{\infty}\frac{d\mu}{1+\mu^2}\int_{-\infty}^{\infty} e^{-2u}dF_n(u)$$

$$< \frac{1}{4n^2k_n^2}\int_{-\infty}^{\infty} e^{1+u^2}dF_n(u) < \frac{1}{n^2}\cdot$$

obtained from (6) and (11).

It now follows that we have

$$r_n(s, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda - t\mu)}d_{\lambda,\mu}C_n(\lambda, \mu),$$

$$r(s, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda - t\mu)}d_{\lambda,\mu}C(\lambda, \mu),$$

where

$$C(\lambda, \mu) = \sum_1^N C_n(\lambda, \mu)$$

is a correlation function which, according to the above, is of bounded variation over the whole (λ, μ)-plane. Hence we may conclude (cf. [1], and [9], p. 476) that $x(t)$ is a harmonizable process

$$x(t) = \int_{-\infty}^{\infty} e^{it\lambda}dy(\lambda),$$

where $y(\lambda)$ has the correlation function $C(\lambda, \mu)$. We note that $x(t)$ is a regular process and is everywhere continuous in quadratic mean. The proof is completed.

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STOCHASTIC PROCESSES AS CURVES IN HILBERT SPACE

HARALD CRAMÉR *(STOCKHOLM)*

(Summary)

Regular complex-valued random processes $x(t)$ with finite moments of second order are studied by methods of Hilbert space geometry. A representation formula (4) is given for the process $x(t)$ in terms of "past and present innovations". The number N is called the complete spectral multiplicity of the process $x(t)$ and is the smallest number for which such a representation exists. It is shown that the multiplicity of $x(t)$ is uniquely determined by the corresponding correlation function and that one can always find a harmonizing process $x(t)$ which has the multiplicity prescribed in advance.