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COMPLETE SETS OF BESSEL AND LEGENDRE FUNCTIONS

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1. Although the Legendre functions $P_\lambda(x)$ have been extensively studied for both integral and non-integral values of the index, little attention seems to have been paid to the problem of completeness of sets of such functions with non-integral indices. An exception to this occurs in the work of Hille [3], who showed that, for certain sets of indices which make the functions orthogonal, they are complete in $L^2(-1, 1)$.

A similar situation exists for sets of Bessel functions $\{J_\nu(\lambda_n x)\}_{n=1}^{\infty}$, where ν is fixed. The completeness of such sets has apparently been discussed only in the case where the λ_n are eigenvalues corresponding to boundary-value problems; here again the functions form an orthogonal set, with respect to a suitable weight function.

It is the purpose of this paper to establish completeness criteria for sets of Legendre and Bessel functions where the $\{\lambda_n\}$ are of a more general character. In each case two types of criteria are obtained. A criterion of one type states that the set in question is complete if an associated set of trigonometric functions is complete. Since the latter have been studied extensively by Paley and Wiener and Levinson [6], further conditions for completeness can be read off from their results. Note, however, that these authors call "closure" what we call "completeness." A criterion of the other type demands that the λ_n satisfy inequalities which state essentially that they do not grow too rapidly.

A set of functions $f_n(x)$ is said to be complete in a class of functions $F(a, b)$, or simply complete $F(a, b)$, if $\int_a^b f_n(x)g(x)\,dx = 0, n = 1, 2, \cdots$ and $g(x) \in F(a, b)$ together imply that $g(x)$ vanishes almost everywhere on (a, b). In general the functions $f_n(x)$ need not belong to $F(a, b)$; for example if $F = L^p$, the appropriate class for them is $L^{p'}$, $p' = p/(p - 1)$. L^∞ will denote the class of essentially bounded functions, C the set of continuous functions, on an interval (a, b). For a finite interval it is clear that completeness L^p carries with it completeness L^q for all $q \geqq p$.

The techniques employed are much the same for the two kinds of functions. We shall treat the Bessel functions first because they exemplify all the difficulties of the problems without the complication of analytical details which encumber the other case.

PART I. BESSEL FUNCTIONS

2. We shall investigate the completeness of sets of Bessel functions with respect to L^p spaces. The sets considered in the classical cases are orthogonal with respect to the weight function x on the interval $(0, 1)$. This suggests at least two possible approaches to the general problem. We may consider the completeness of the set $\{J_\nu(\lambda_n x)\}_{n=1}^{\infty}$ in classes of functions $f(x)$ for which $\int_0^1 |f(x)|^p x\,dx$ exists, or alternatively the set $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}_{n=1}^{\infty}$ with respect to the ordinary L^p spaces. The latter course is adopted here, and the reader who compares the results with the literature should take account of the extra factor $x^{\frac{1}{2}}$.

We restrict ourselves to indices with $\nu > -\frac{3}{2}$; for if $\nu \leqq -\frac{3}{2}$, $x^{\frac{1}{2}}J_\nu(x)$ is not integrable in $(0, 1)$. Most of the results concern the case $\nu > -\frac{1}{2}$. We exclude the case $\nu = -\frac{1}{2}$, when our functions reduce to sets of trigonometric functions, about which we have nothing new to say. In each case it is assumed that ν is fixed and n runs over the positive integers.

THEOREM 1. *If $\nu > -\frac{1}{2}$, the set $\{x^{\frac{1}{2}}J_\nu(\lambda_n x)\}$ is complete $L^p(0, 1)$, $1 \leqq p < \infty$, if for sufficiently large n*

$$0 < \lambda_n \leqq \pi\left(n + \tfrac{1}{4} + \tfrac{1}{2}\nu - \frac{1}{2p}\right). \tag{2.1}$$

The result remains true for $p = \infty$ if "$\leqq$" is replaced by "$<$" in (2.1).

THEOREM 2. *If $\nu > -\frac{1}{2}$, $0 \leqq \delta < 1$, and λ_n is the n^{th} positive zero of $J_{\nu+\delta}(x)$, then $\{x^{\frac{1}{2}}J_\nu(\lambda_n x)\}$ is complete $L^p(0, 1)$ for $p > 1/(1 - \delta)$.*

For $\delta = 0$, Theorem 2 is in the literature [9, p. 616]. By use of the identity

$$J_\nu(x) = x^{-\nu-1}\frac{d}{dx}\left(x^{\nu+1}J_{\nu+1}(x)\right) \tag{2.2}$$

it is easily verified that the conclusion of this theorem is false for $\delta = 1$. For $x^{\nu+\frac{1}{2}}$ clearly belongs to L^∞, and

$$\int_0^1 x^{\frac{1}{2}}J_\nu(\lambda_n x)x^{\nu+\frac{1}{2}}\,dx = \lambda_n^{-\nu-2}\int_0^{\lambda_n} u^{\nu+1}J_\nu(u)\,du$$

$$= \lambda_n^{-\nu-2}\int_0^{\lambda_n}\frac{d}{du}\left(u^{\mu+1}J_{\mu+1}(u)\right)du = \lambda_n^{-\nu-2}\left[\lambda_n^{\nu+1}J_{\nu+1}(\lambda_n)\right] = 0.$$

This suggests the addition of a function to our set to take care of $\delta = 1$. It turns out that adding one function will do even more.

THEOREM 3. *If $\nu > -\frac{1}{2}$, $0 \leqq \delta < 2$, and λ_n is the n^{th} positive zero of $J_{\nu+\delta}(x)$, then $\{x^{\nu+\frac{1}{2}}, x^{\frac{1}{2}}J_\nu(\lambda_n x)\}$ is complete $L^p(0, 1)$ for $p > 1/(2 - \delta)$.*

In case $\nu = 0$ Theorem 3, with $L^p(0, 1)$ replaced by $C(0, 1)$, and the factor $x^{\frac{1}{2}}$ removed, has been proved recently by Juncosa [5, p. 470]. With a little additional care, we can extend Theorem 3 to $-\frac{3}{2} < \nu < -\frac{1}{2}$. We need to have our functions in $L^{p'}$ if they are to be complete L^p; hence we assume $p > 1/(\nu + \frac{3}{2})$ for our next theorem. We also exclude $\nu = -1$, since $J_{-1}(x) = J_1(x)$ and Theorem 1 gives a sharper critieron.

THEOREM 4. *If $-\frac{3}{2} < \nu < -\frac{1}{2}$, $(\nu + \frac{3}{2})^{-1} < p < \infty$, $\nu \neq -1$, then the set $\{x^{\nu+\frac{1}{2}}, x^{\frac{1}{2}}J_\nu(\lambda_n x)\}$ is complete $L^p(0, 1)$ if*

$$0 < \lambda_n \leqq \pi\left(n + \tfrac{3}{4} + \tfrac{1}{2}\nu - \frac{1}{2p}\right) \tag{2.3}$$

and, in particular, if $0 \leqq \delta < 2$, $p > 1/(2-\delta)$, *and* λ_n *is the* n^{th} *positive zero of* $J_{\nu+\delta}(x)$.

In Theorem 2 we restricted δ to nonnegative values. By use of the following result we can remedy that deficiency.

THEOREM 5. *If* $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ *is complete* $L^\infty(0, 1)$, *then* $\{x^{\frac{1}{2}} J_{\nu+1}(\lambda_n x)\}$ *is complete* $L(0, 1)$, *provided* $\nu > -\frac{1}{2}$, $\lambda_n \neq 0$.

By combining Theorems 2 and 5 we have

THEOREM 6. *If* $\nu > \frac{1}{2}$ *and* λ_n *is the* n^{th} *positive zero of* $J_{\nu-\delta}(x)$, $0 < \delta \leqq 1$, *then* $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ *is complete* $L(0, 1)$.

For $\delta = 1$, a similar result has been proved by Juncosa [5, p. 469].

The last theorem of this part gives a completeness criterion of the other type.

THEOREM 7. *If* $-\frac{1}{2} < \nu < \frac{1}{2}$, $1 < p < 2/(2\nu + 1)$, $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ *is complete* $L^p(0, 1)$ *if* $\{\cos \lambda_n(x)\}$ *is complete* $L^q(0, 1)$, $q = 2p/(2 - p - 2\nu p)$.

Similar but more complicated results can be obtained for $\nu > \frac{1}{2}$.

3. This section contains a number of lemmas. Let $\{\lambda_n\}$, $n = \pm 1, \pm 2, \cdots$ be a sequence of real numbers, none of which is zero, and let $\Lambda(t)$ denote the number of λ_n such that $|\lambda_n| \leqq t$.

LEMMA 3.1. *If*

$$\int_1^r t^{-1} \Lambda(t)\, dt > 2r - \alpha \log r - B, \qquad r > 1, \tag{3.2}$$

where B is a constant; if $H(z)$ *is an entire function such that*

$$|H(x + iy)| \leqq |y|^{-\gamma} \int_0^\pi h(t) s(t) e^{t|y|}\, dt \tag{3.3}$$

where $h(t) \in L^p(0,\pi)$, $1 \leqq p < \infty$, $s(t) = 1$ *or* $(\sin t)^{1/p'}$, $p' = p/(p-1)$, $\beta = \gamma + 1/p'$ *or* $\gamma + 2/p'$ *in the two cases,* $\alpha \leqq \beta$; *and* $H(\lambda_n) = 0$, *then* $H(z) \equiv 0$. *If* $p = \infty$ *the same conclusion holds if* $\alpha < \beta$.

This is a slight generalization of a result of Levinson [6, p. 7]. Since his proof requires rearrangement to give the desired result, we shall give a proof of the lemma here, but only in the case $1 < p < \infty$. The necessary modifications in the remaining cases are left to the reader.

We prove first that

$$\lim_{y \to \infty} \{\log |H(x + iy)| - \pi |y| + \beta \log |y|\} = -\infty. \tag{3.4}$$

Given a positive δ, choose a positive ϵ so that $\int_{\pi-\epsilon}^{\pi} \{h(t)\}^p\, dt < \delta^p$. We then have

$$|y|^\gamma |H(x+iy)| \leqq \left(\int_0^{\pi-\epsilon} \{h(t)\}^p\, dt\right)^{1/p} \left(\int_0^{\pi-\epsilon} \{s(t)\}^{p'} e^{p'|y|t}\, dt\right)^{1/p'}$$
$$+ \delta \left(\int_{\pi-\epsilon}^{\pi} \{s(t)\}^{p'} e^{p'|y|t}\, dt\right)^{1/p'} \leqq e^{\pi|y|} |y|^{-(1+\sigma)/p'} (A |y|^\sigma e^{-\epsilon|y|} + \delta),$$

where $\delta = 0$ when $s(t) = 1$, $\sigma = 1$ otherwise, and A is independent of y and δ. Thus

$$\text{(3.5)} \quad \log | H(x + iy) | - \pi | y | + \beta \log | y | \leqq \log (A | y |^{\sigma} e^{-\epsilon |y|} + \delta);$$

by choosing δ sufficiently small and then $| y |$ large enough, with a fixed ϵ, we can make the right-hand side of (3.5) arbitrarily small. This proves (3.4).

In particular, we have

$$\log | H(re^{i\theta}) | \leqq \pi r | \sin \theta | - \beta \log r - \beta \log | \sin \theta | + \text{constant}.$$

Suppose that $H(z) \not\equiv 0$. Then, by Jensen's theorem, if $n(r)$ denotes the number of zeros of $H(re^{i\theta})$ of absolute value at most r,

$$\text{(3.6)} \qquad \int_1^r t^{-1} n(t)\, dt \leqq \frac{1}{2\pi} \int_0^{2\pi} \log | H(re^{i\theta}) | \, d\theta + \text{constant}$$

$$\leqq 2r - \beta \log r + \text{constant}.$$

Also $\Lambda(t) \leqq n(t) = O(t)$ as $t \to \infty$. Let

$$F(z) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n},$$

the product converging because $\Lambda(t) = O(t)$. Then $\varphi(z) = H(z)/F(z)$ is entire. Let $n_1(r)$ denote the number of zeros of $\varphi(z)$ of absolute value at most r, so that $n_1(r) = n(r) - \Lambda(r)$. We then have, using (3.2) and (3.6),

$$\int_1^r \frac{n_1(u)}{u}\, du = \int_1^r \frac{n(u)}{u}\, du - \int_1^r \frac{\Lambda(u)}{u}\, du$$

$$\leqq (\alpha - \beta) \log r + \text{constant}.$$

Since $\alpha \leqq \beta$, the right side is bounded as $r \to \infty$, and consequently $n_1(r) \equiv 0$; that is, $\varphi(z)$ has no zeros. By the Hadamard theorem $H(z) = ae^{bz}F(z)$ for some constants a and b, so that, in particular,

$$\text{(3.7)} \qquad \log | H(iy) | = cy + \log | F(iy) | + \text{constant},$$

where c is a real constant.

On the other hand (cf. [6, p. 9]), using (3.2) we have

$$\log | F(iy) | = \frac{1}{2} \int_0^{\infty} \log \left(1 + \frac{y^2}{u^2}\right) d\Lambda(u)$$

$$= \int_0^{\infty} \frac{2y^2 u}{(u^2 + y^2)^2}\, du \int_0^u \frac{\Lambda(t)}{t}\, dt \geqq \pi | y | - \alpha \log | y | - \text{constant},$$

and so

$$\log | H(iy) | - \pi | y | + \alpha \log | y | \geqq cy - \text{constant}.$$

Since $\alpha \leqq \beta$ this contradicts (3.4); for let $y \to \pm \infty$, the sign being chosen the same as that of c if $c \neq 0$. Hence the assumption that $H(z) \not\equiv 0$ is untenable, and Lemma 3.1 is established.

LEMMA 3.8. *Hypothesis* (3.2) *is satisfied if* $\lambda_n = -\lambda_{-n}$ *and, for sufficiently large* n,

$$0 < \lambda_n \leqq n + \tfrac{1}{2}\alpha - \tfrac{1}{2}. \tag{3.8}$$

We may assume (3.8) for all n, since this changes (3.2) only by changing B. Let $\zeta = \frac{1}{2}(\alpha - 1)$. In the interval $k + \zeta \leqq u < k + 1 + \zeta$ we have $\Lambda(u) \geqq 2u$. Let n be defined by $n + \zeta \leqq x < n + 1 + \zeta$. Then

$$\int_0^x \frac{\Lambda(u)}{u}\,du \geqq 2\int_{1+\zeta}^{2+\zeta}\frac{du}{u} + 2\int_{2+\zeta}^{3+\zeta}\frac{2du}{u} + \cdots + 2\int_{n-1+\zeta}^{n+\zeta}\frac{n-1}{u}\,du$$
$$= 2\sum_{k=2}^{n-1} k \log\left(1 + \frac{1}{k+\zeta}\right).$$

Since $\log(1+t) \geqq t - t^2$ for $t \geqq 0$, we have

$$\int_0^x \frac{\Lambda(u)}{u}\,du \geqq 2\sum_{k=2}^{n-1} k\left\{\frac{1}{k+\zeta} - \frac{1}{2}\left(\frac{1}{k+\zeta}\right)^2\right\}$$
$$\geqq 2\sum_{k=2}^{n-1}\left\{1 - \frac{\zeta}{k+\zeta} - \frac{1}{2k}\right\}$$
$$= 2n - 2\zeta\log n - \log n + O(1)$$
$$= 2n - \alpha\log n + O(1)$$
$$= 2x - \alpha \log x + O(1).$$

LEMMA 3.9. *If* $f(x) \in L(0, 1)$, $\nu > -\frac{1}{2}$, *and*

$$g(u) = \int_u^1 \frac{x^{\frac{1}{2}-\nu} f(x)\,dx}{(x^2 - u^2)^{\frac{1}{2}-\nu}}, \tag{3.10}$$

then $g(u) = 0$ *almost everywhere implies that* $f(x) = 0$ *almost everywhere.*

If $-\frac{1}{2} < \nu < \frac{1}{2}$, we have

$$\int_y^\pi \frac{ug(u)\,du}{(u^2 - y^2)^{\frac{1}{2}+\nu}(x^2 - u^2)^{\frac{1}{2}-\nu}} = \int_y^\pi f(x)\,dx \int_y^x \frac{udu}{(u^2 - y^2)^{\frac{1}{2}+\nu}(x^2-u^2)^{\frac{1}{2}-\nu}}$$
$$= \frac{1}{2}\int_y^\pi f(x)\,dx \int_{y^2}^{x^2} \frac{dt}{(t - y^2)^{\frac{1}{2}+\nu}(x^2 - t)^{\frac{1}{2}-\nu}} = \frac{\pi}{2\sin(\nu + \frac{1}{2})\pi}\int_y^\pi f(x)\,dx,$$

and the conclusion follows.

If $\nu \geqq \frac{1}{2}$, write

$$G(u) = g(u^{\frac{1}{2}}) = \int_{u^{\frac{1}{2}}}^1 x^{\frac{1}{2}-\nu}(x^2 - u)^{\nu-\frac{1}{2}} f(x)\,dx.$$

Let $k = [\nu - \frac{1}{2}] + 1$ if $\nu + \frac{1}{2}$ is not an integer, $k = \nu - \frac{1}{2}$ if $\nu + \frac{1}{2}$ is an integer. Then $G^{(k)}(u^2)$ is of the form already considered, and the conclusion follows from the first part of the proof.

4. To prove Theorem 1 we start from the representation [9, p. 48]

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos xt\, dt, \qquad \nu > -\tfrac{1}{2}. \tag{4.1}$$

Making the change of variable $xt = u$, we obtain

$$\begin{aligned} x^{\frac{1}{2}} J_\nu(\lambda x) &= C_\nu(\lambda) x^{\frac{1}{2}-\nu} \int_0^x (x^2-u^2)^{\nu-\frac{1}{2}} \cos \lambda u\, du, \\ C_\nu(\lambda) &= \lambda^\nu / 2^{\nu-1}\Gamma(\nu+\tfrac{1}{2})\Gamma(\tfrac{1}{2}). \end{aligned} \tag{4.2}$$

To show that the set $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ is complete in $L^p(0, 1)$ we have to show that

$$\int_0^1 f(x) x^{\frac{1}{2}} J_\nu(\lambda_n x)\, dx = 0 \qquad (n = 1, 2, \cdots) \tag{4.3}$$

with $f(x) \in L^p(0, 1)$ implies that $f(x)$ vanishes almost everywhere. Using (4.2) we obtain from (4.3)

$$C_\nu(\lambda_n) \int_0^1 g(u) \cos \lambda_n u\, du = 0 \qquad (n = 1, 2, \cdots) \tag{4.4}$$

where

$$g(u) = \int_u^1 \frac{x^{\frac{1}{2}-\nu} f(x)\, dx}{(x^2-u^2)^{\frac{1}{2}-\nu}}. \tag{4.5}$$

By Lemma 3.9 it is enough, in view of (4.5), to show that $g(u)$ vanishes almost everywhere. This we show to be a consequence of (2.1) and (4.4). For let

$$H(z) = \int_0^1 g(u) \cos zu\, du.$$

$H(\pi z)$ is an even entire function. Moreover,

$$\begin{aligned} |H(x+iy)| &\leqq \int_0^1 e^{u|y|}\, du \int_u^1 |f(t)|\,(t-u)^{\nu-\frac{1}{2}}\, dt \\ &= \int_0^1 |f(t)|\, dt \int_0^t e^{u|y|}(t-u)^{\nu-\frac{1}{2}}\, du \\ &= \int_0^1 e^{t|y|}\, |f(t)|\, dt \int_0^t e^{u|y|} u^{\nu-\frac{1}{2}}\, du \\ &< \Gamma(\nu+\tfrac{1}{2})\, |y|^{-\nu-\frac{1}{2}} \int_0^1 e^{t|y|}\, |f(t)|\, dt, \end{aligned} \tag{4.6}$$

so that (3.3) is satisfied with $\gamma = \nu + \frac{1}{2}$, $s(t) = 1$. Applying Lemmas 3.8 and 3.1, we see that $H(z) = 0$, and so $g(u) = 0$ almost everywhere. This completes the proof of Theorem 1.

Theorem 2 now follows because the n^{th} positive zero of $J_{\nu+\delta}(z)$ is [9, p. 506]

$$\lambda_n = (n + \tfrac{1}{2}\nu + \tfrac{1}{2}\delta - \tfrac{1}{4})\pi + O(1/n) \tag{4.7}$$

which for large n is less than $\pi(n + \frac{1}{2}(\nu + \frac{1}{2}) - 1/2p)$, provided $p > 1/(1 - \delta)$.

5. We now prove Theorems 3 and 4. From (4.1) and (2.2) it follows that

$$x^{\frac{1}{2}} J_\nu(\lambda x) = C_{\nu+1}(\lambda)\lambda^\nu x^{-\nu-\frac{1}{2}} \int_0^x (x^2 - u^2)^{\nu+\frac{1}{2}}\{(2\nu + 2)\cos\lambda u - \lambda u \sin\lambda u\}\, du.$$

Thus (4.3) leads to

$$\int_0^1 g(u)\{(2\nu + 2)\cos\lambda_n u - \lambda_n u \sin\lambda_n u\}\, du = 0 \qquad (n = 1, 2, \cdots),$$

where

$$g(u) = \int_u^1 x^{-\nu-\frac{1}{2}}(x^2 - u^2)^{\nu+\frac{1}{2}} f(x)\, dx. \tag{5.1}$$

Since $f(x)$ is orthogonal to $x^{\frac{1}{2}+\nu}$, $g(u)$ is also orthogonal to 1; for

$$\int_0^1 du \int_u^1 x^{-\nu-\frac{1}{2}}(x^2 - u^2)^{\nu+\frac{1}{2}} f(x)\, dx$$

$$= \int_0^1 x^{-\nu-\frac{1}{2}} f(x)\, dx \int_0^x (x^2 - u^2)^{\nu+\frac{1}{2}}\, du$$

$$= \int_0^1 x^{\frac{1}{2}+\nu} f(x)\, dx \int_0^1 (1 - t^2)^{\nu+\frac{1}{2}}\, dt.$$

Thus the entire function

$$G(z) = \int_0^1 g(u)\{(2\nu + 2)\cos zu + zu \sin zu\}\, du$$

has $G(\pm\lambda_n) = 0$ $(n = 1, 2, \cdots)$ and $G(0) = 0$.

Writing $g_1(u) = \int_u^1 g(t)\, dt$, we have

$$\int_0^1 g(u)\cos zu\, du = -z \int_0^1 g_1(u)\sin zu\, du,$$

since $g_1(0) = G(0)/(2\nu + 2) = 0$. Thus

$$G(z) = -z \int_0^1 \{g_1(u) + ug(u)\}\sin zu\, du. \tag{5.2}$$

Since in (5.1) $2x \geqq x + u \geqq x$, we have, if $\nu > -\frac{1}{2}$,

$$u\,|\,g(u)\,| \leqq 2u \int_u^1 x^{-1}(x - u)^{\nu+\frac{1}{2}}\,|\,f(x)\,|\, dx,$$

$$u\,|\,g(u)\,| \leqq \int_u^1 (x - u)^{\nu+\frac{1}{2}}\,|\,f(x)\,|\, dx, \tag{5.3}$$

while if $-\frac{3}{2} < \nu < -\frac{1}{2}$

$$u\,|\,g(u)\,| \leqq u \int_u^1 x^{-1}(x - u)^{\nu+\frac{1}{2}}\,|\,f(x)\,|\,dx$$

and we still have (5.3). Hence

$$\begin{aligned} |\,g_1(u)\,| &\leqq 2 \int_u^1 dt \int_t^1 x^{-1}(x - t)^{\nu+\frac{1}{2}}\,|\,f(x)\,|\,dx \\ &= 2 \int_u^1 x^{-1}\,|\,f(x)\,|\,dx \int_u^x (x - t)^{\nu+\frac{1}{2}}\,dt \\ &= \frac{2}{\nu + \frac{3}{2}} \int_u^1 x^{-1}(x - u)^{\nu+\frac{3}{2}}\,|\,f(x)\,|\,dx \\ &\leqq \frac{2}{\nu + \frac{3}{2}} \int_u^1 (x - u)^{\nu+\frac{1}{2}}\,|\,f(x)\,|\,dx. \end{aligned}$$

If $\nu > -\frac{1}{2}$, since $f(x) \in L$, these inequalities show that $|\,g_1(u) + ug(u)\,|$ is bounded; if $-\frac{3}{2} < \nu < -\frac{1}{2}$, we have assumed $p > 1/(\nu + \frac{3}{2})$ and then Hölder's inequality shows that $|\,g_1(u) + ug(u)\,|$ is still bounded. Consequently (5.2) shows that $G(z)/z \to 0$ as $z \to 0$, so that $G(z)/z^2$ is entire. If we now reason as in (4.6), with $\nu - \frac{1}{2}$ replaced by $\nu + \frac{1}{2}$, we find

$$\left|\frac{G(iy)}{(iy)^2}\right| \leqq \frac{A}{|\,y\,|^{\nu+\frac{3}{2}}} \int_0^1 e^{x|y|}\,|\,f(x)\,|\,dx,$$

where A depends only on ν. As in the proof of Theorem 1, then, we infer that $G(z) \equiv 0$ if (2.3) is satisfied. Hence, under (2.3), $g_1(u) + ug(u) = 0$ almost everywhere. Since $g_1'(u) = -g(u)$ for almost all u, $g'(u)$ exists almost everywhere and $-g(u) = -ug'(u) + g(u)$, $g(u) = au^2$. But since $g_1(0) = \frac{1}{3}a = 0$, $a = 0$, and so $g(u) = 0$ almost everywhere. By (5.1) and Lemma 3.9, this establishes Theorem 3 and the first part of Theorem 4. The second part follows in the same way as Theorem 2 follows from Theorem 1.

6. To prove Theorem 5 we must show that if $f(x) \in L(0, 1)$ and

$$I_n = \int_0^1 x^{\frac{1}{2}} J_{\nu+1}(\lambda_n x) f(x)\,dx = 0$$

then $f(x) \equiv 0$. We have

$$\begin{aligned} I_n &= \int_0^1 x^{\nu+1} J_{\nu+1}(\lambda_n x) x^{-\nu-\frac{1}{2}} f(x)\,dx \\ &= \int_0^1 x^{\nu+1} J_{\nu+1}(\lambda_n x)\,d \int_x^1 y^{-\nu-\frac{1}{2}} f(y)\,dy. \end{aligned}$$

Now integrate by parts. Since $f(x) \in L$ and $\nu > -\frac{1}{2}$ the integrated parts vanish, and by (2.2)

$$I_n = -\int_0^1 \left\{ \int_x^1 y^{-\nu-\frac{1}{2}} f(y)\, dy \right\} \frac{d}{dx} [x^{\nu+1} J_{\nu+1}(\lambda_n x)]\, dx$$
$$= -\lambda_n \int_0^1 \left\{ \int_x^1 y^{-\nu-\frac{1}{2}} f(y)\, dy \right\} x^{\nu+1} J_\nu(\lambda_n x)\, dx = 0.$$

By hypothesis $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ is complete L^∞; it therefore remains only to show that

$$x^{\nu+\frac{1}{2}} \int_x^1 y^{-\nu-\frac{1}{2}} f(y)\, dy$$

is bounded. This is obvious, since $f(y) \in L$.

This theorem combined with Theorem 2 gives Theorem 6.

7. We conclude Part I by proving Theorem 7. To do this we go back to (4.4) and (4.5), and observe that $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ is complete with respect to a given class F if $\{\cos \lambda_n x\}$ is complete with respect to the class of transforms (4.5) of that class. Now, by (4.5)

$$|g(u)| \leqq G(u) = \int_n^1 \frac{|f(x)|}{(u - u)^{\frac{1}{2}-\nu}}\, dx,$$

since $\nu < \frac{1}{2}$. $G(u)$ is, except for a multiplicative constant, the fractional integral of $f(x)$ of order $\alpha = \nu + \frac{1}{2}$. By theorems of Hardy and Littlewood [2, p. 290] $G(u)$, and hence $g(u)$, belongs to L^q, $q = p/(1 - \alpha p)$, if $1 < p < 1/\alpha$. Hence $\{x^{\frac{1}{2}} J_\nu(\lambda_n x)\}$ is complete L^p if $\{\cos \lambda_n x\}$ is complete L^q, with this value of q.

Part II. Legendre functions

8. In this part we shall consider the completeness of sets of Legendre functions in intervals $(a, 1)$, $a \geqq -1$. For subintervals of $(-1, 1)$ not containing 1, and for intervals not contained in $(-1, 1)$, the completeness properties of Legendre functions are rather different. An indication of the difficulties arising for intervals (a, b), $-1 < a < b < 1$, will be found in §12; the case of intervals outside $(-1, 1)$ will be discussed in a later paper.

We originally set out to prove that the completeness in $L^p(a, b)$ of $\{P_{\lambda_n}(x)\}_{n=1}^\infty$, $\Re(\lambda_n) \geqq -\frac{1}{2}$, is a consequence of the completeness of $\{\cos (\lambda_n + \frac{1}{2})x\}_{n=1}^\infty$ in L^p in a suitably related interval. However, we found that much more than this is true. Let Lip $\alpha(a, b)$ denote the class of functions $g(x)$ satisfying $|g(x + h) - g(x)| \leqq A\, |h|^\alpha$ for x and $x + h$ in (a, b). Then we have the following theorem.

Theorem 8. *A sufficient condition for the set $\{P_{\lambda_n}(x)\}_{n=1}^\infty$ to be complete $L^p(a, 1)$, where $-1 \leqq a < 1$, is that $\{\cos (\lambda_n + \frac{1}{2})x\}_{n=1}^\infty$ is complete $F(0, \cos^{-1} a)$, where*

if $p = 1$, then $F = L$ and $a > -1$;

if $1 < p \leqq 2$, then $F = L^r$ and $r < p/(2 - p)$;

if $2 < p < \infty$, *then* $F = \text{Lip}\,\alpha,\ \alpha \leqq (p-2)/(2p)$;

if $p = \infty$, *then* $F = \prod_{0<\alpha<\frac{1}{2}} \text{Lip}\,\alpha$.

If $s > t$, $L^t \supset L^s \supset \text{Lip}\,\alpha(0 < \alpha \leqq 1)$, and so a set complete L^t is necessarily complete L^s, and *a fortiori* in any Lip α. Hence we have in particular the following simpler, though weaker, theorem.

THEOREM 9. *A sufficient condition for the set* $\{P_{\lambda_n}(x)\}$ *to be complete* $L^p(a, 1)$, *where* $-1 \leqq a < 1$, *and* $a > -1$ *if* $p = 1$, *is that* $\{\cos(\lambda_n + \frac{1}{2})x\}$ *is complete* $L^p(0, \cos^{-1}a)$.

We observe that it is necessary to assume $a > -1$ if $p = 1$, since $P_\lambda(x)$ is unbounded in the neighborhood of $x = -1$ if λ is not an integer.

Since sufficient conditions for the completeness of $\{\cos(\lambda_n + \frac{1}{2})x\}$ can be read off from the corresponding conditions for sets $\{e^{i\lambda_n x}\}$, conditions on the λ_n implying the completeness of $\{P_{\lambda_n}(x)\}$ can easily be obtained. For example, it is an easy deduction from a theorem of Levinson [6, p. 6] that

(8.0) $\{\cos \lambda_n + \frac{1}{2})x\}$ *is complete* $L^r(0, \pi)$, $r \geqq 1$, *if* $0 < \lambda_n + \frac{1}{2} \leqq n - 1/(2r)$.

Since $\{P_{\lambda_n}(x)\}$ is complete $L^p(-1, 1)$, $1 \leqq p < 2$, if $\{\cos(\lambda_n + \frac{1}{2})x\}$ is complete L^r, $r < p/(2-p)$, we see that $\{P_{\lambda_n}(x)\}$ is complete $L^p(-1, 1)$, $1 \leqq p < 2$, if

$$-\tfrac{1}{2} < \lambda_n \leqq n + A,\ A < -1/p.$$

This condition is slightly more restrictive than necessary; for $p > 2$ the use of Theorem 8 with existing results on completeness of trigonometric functions is still less satisfactory. In §13 we shall prove the following result for the interval $(-1, 1)$.

THEOREM 10. *If* $1 \leqq p < \infty$, $\{P_{\lambda_n}(x)\}$ *is complete* $L^p(-1, 1)$ *if the* λ_n *are real and*

$$-\tfrac{1}{2} < \lambda_n \leqq n - \frac{1}{p}. \tag{8.1}$$

If $p = \infty$, *the result remains true if* (8.1) *is replaced by* $-\frac{1}{2} < \lambda_n < n$.

In particular, we see that, if $\lambda_n = n - 1$, our theorem icludes the completeness of $\{P_n(x)\}_{n=0}^{\infty}$ in any L^p. Theorem 10 also proves the completeness of the sets considered by Hille, not only in L^2, but also in any L^p, $p \geqq 1$. For these systems [3, pp. 56–61], λ_n runs through the roots of

$$\Psi(x) + \Psi(-1 - x) = K, \tag{8.2}$$

where K is a real constant and $\Psi(x) = \Gamma'(x+1)/\Gamma(x+1)$. As Hille shows, there is one real root in each interval $(n, n+1)$, $n = 0, 1, \pm 2, \cdots$; either two real roots separated by $-\frac{1}{2}$ in $(-1, 0)$, or two conjugate imaginary roots of real part $-\frac{1}{2}$; and no other roots. Hille proves that the set $\{P_{\lambda_n}(x)\}$ is complete in $L^2(-1, 1)$ if λ_1 is the root of (8.2) in $(-\frac{1}{2}, 0)$ or one of the complex roots, and λ_n is the root in $(n-2, n-1)$, $n = 1, 2, \cdots$; since $P_\lambda(x) = P_{-1-\lambda}(x)$, the other roots contribute no additional functions. Since changing one λ_n

in a sequence to another number different from all the remaining λ_n preserves the completeness of a set $\{P_{\lambda_n}(x)\}$, we may assume $-\frac{1}{2} < \lambda_1 < 0$. Then since $\lambda_n < n - 1$, (8.1) is satisfied and Hille's result follows from Theorem 10.

For $a > -1$, Theorems 8 and 9 contain new information even for integral λ_n.

We have also investigated the extent to which Theorem 9 can be reversed; the following theorem is a partial converse.

THEOREM 11. *A sufficient condition for the set* $\{\sin(\lambda_n + \frac{1}{2})x\}$, $\Re(\lambda_n) > -\frac{1}{2}$, *to be complete* $L^p(0, \cos^{-1}a)$, *where* $-1 \leqq a < 1$, $1 \leqq p < \infty$, *and* $a > -1$ *if* $p = 1$, *is that* $\{P_{\lambda_n}(x)\}$ *is complete* $L^s(a, 1)$, *where* $s < 2p$; *if* $p = \infty$, $s = \infty$.

In particular, $\{\sin(\lambda_n + \frac{1}{2})x\}$ is complete L^p if $\{P_{\lambda_n}(x)\}$ is complete L^p.

The completeness of $\{\sin(\lambda_n + \frac{1}{2})x\}$ implies that of $\{1, \cos(\lambda_n + \frac{1}{2})x\}$, but not necessarily that of $\{\cos(\lambda_n + \frac{1}{2})x\}$; for example, let $\lambda_n = n - \frac{1}{2}$ $(n = 1, 2, \cdots)$.

It is natural to ask what completeness property is implied for a trigonometric system by the completeness of $\{P_{\lambda_n}(x)\}$ in an L^p class with $p < 2$, since Theorem 11 does not use the full force of the hypothesis that the set is complete in such a class. At least a partial answer is provided by Theorem 12.

THEOREM 12. *If* $\{P_{\lambda_n}(x)\}_{n=1}^{\infty}$ *is complete* $L^p(a, 1)$, $-1 \leqq a < 1$, $1 < p \leqq 2$, *then* $\{\sin(\lambda_n + \frac{1}{2})x\}_{n=1}^{\infty}$ *is complete in the class* L^p_α *of functions* $g(x)$ *such that* $(\sin x)^\alpha g(x) \in L^p(0, \cos^{-1}a)$, *where* $\alpha < (p-1)/p$.

9. In this section we collect a number of lemmas.

LEMMA 9.1. [4, p. 267]. *For* $-1 < x \leqq 1$,

$$P_\lambda(x) = \frac{2^{\frac{1}{2}}}{\pi}\int_0^{\cos^{-1}x} \frac{\cos(\lambda + \frac{1}{2})\varphi}{(\cos\varphi - x)^{\frac{1}{2}}}\,d\varphi.$$

LEMMA 9.2. *For* $-1 < x \leqq 1$ *and* $\Re(\lambda) \geqq -\frac{1}{2}$,

$$|P_\lambda(x)| \leqq A(\lambda)\{1 + |\log(1 + x)|\}, \tag{9.3}$$

where $A(\lambda)$ *depends only on* λ.

If λ is an integer, $P_\lambda(x)$ is a polynomial and (9.3) is certainly true. If λ is not an integer, we have from Lemma 9.1

$$|P_\lambda(x)| \leqq \frac{2^{\frac{1}{2}}}{\pi}\exp\{\pi|\Im(\lambda)|\}\int_0^{\cos^{-1}x}\frac{d\varphi}{(\cos\varphi - x)^{\frac{1}{2}}} = \frac{A(\lambda)}{\pi}\int_x^1 \frac{du}{(u - x)^{\frac{1}{2}}(1 - u^2)^{\frac{1}{2}}}.$$

If $x \geqq 0$, then,

$$|P_\lambda(x)| \leqq \frac{A(\lambda)}{\pi}\int_x^1 \frac{du}{(u - x)^{\frac{1}{2}}(1 - u)^{\frac{1}{2}}} = A(\lambda); \tag{9.4}$$

if $-1 < x < 0$,

$$\begin{aligned}|P_\lambda(x)| &\leqq \frac{A(\lambda)}{\pi}\int_x^0 \frac{du}{(u + |x|)^{\frac{1}{2}}(1 + u)^{\frac{1}{2}}} + \frac{A(\lambda)}{\pi}\int_0^1 \frac{du}{(u + |x|)^{\frac{1}{2}}(1 - u)^{\frac{1}{2}}} \\ &\leqq \frac{A(\lambda)}{\pi}\int_x^0 \frac{du}{\{u^2 + u(1 - x) - x\}^{\frac{1}{2}}} + \frac{A(\lambda)}{\pi}\int_0^1 \frac{du}{(1 - u^2)^{\frac{1}{2}}} \\ &= \pi^{-1}A(\lambda)\,|\log(1 + x)| + \tfrac{1}{2}A(\lambda).\end{aligned}$$

This inequality and (9.4) together imply (9.3).

LEMMA 9.5. *If* $-1 \leqq a < 1, f(x) \in L^p(a, 1), 1 \leqq p \leqq \infty, p > 1$ *if* $a = -1$, *and*

$$G(\varphi) = \int_a^{\cos\varphi} \frac{f(x)\,dx}{(\cos\varphi - x)^{\frac{1}{2}}}, \tag{9.6}$$

then $G(\varphi)$ *belongs to*

L *if* $p = 1$;

L^r, $r < p/(2-p)$, *if* $1 < p \leqq 2$;

Lip α, $\alpha \leqq (p-2)/(2p)$, *if* $2 < p < \infty$;

Lip α *for every* $\alpha < \frac{1}{2}$ *if* $p = \infty$.

If $p = 1$, set $\cos\varphi = u$ in (9.6). Then

$$\begin{aligned}
\int_0^{\cos^{-1}a} |G(\varphi)|\,d\varphi &= \int_a^1 \frac{1}{(1-u^2)^{\frac{1}{2}}}\left|\int_a^u \frac{f(x)\,dx}{(u-x)^{\frac{1}{2}}}\right| du \\
&\leqq \frac{1}{(1+a)^{\frac{1}{2}}}\int_a^1 \frac{du}{(1-u)^{\frac{1}{2}}}\int_a^u \frac{|f(x)|\,dx}{(u-x)^{\frac{1}{2}}} \\
&= \frac{1}{(1+a)^{\frac{1}{2}}}\int_a^1 |f(x)|\,dx \int_x^1 \frac{du}{(1-u)^{\frac{1}{2}}(u-x)^{\frac{1}{2}}} \\
&= \frac{\pi}{(1-a)^{\frac{1}{2}}}\int_a^1 |f(x)|\,dx < \infty,
\end{aligned}$$

since $a > -1$.

If $1 < p < 2$, we have

$$\int_0^{\cos^{-1}a} |G(\varphi)|^r\,d\varphi = \int_a^1 \frac{1}{(1-u^2)^{\frac{1}{2}}}\left|\int_a^u \frac{f(x)\,dx}{(u-x)^{\frac{1}{2}}}\right|^r du.$$

Now if $f(x) \in L^p$, $1 < p < 2$, by a theorem of Hardy and Littlewood [2, p. 290; 11, p. 227] we have

$$\int_a^u \frac{f(x)\,dx}{(u-x)^{\frac{1}{2}}} \in L^q, \qquad q = \frac{2p}{2-p},$$

and if $r < p/(2-p)$,

$$\int_0^{\cos^{-1}a} |G(\varphi)|^r\,d\varphi \leqq \left\{\int_a^1 \left|\int_a^u \frac{f(x)\,dx}{(u-x)^{\frac{1}{2}}}\right|^q du\right\}^{r/q} \left\{\int_a^1 \frac{du}{(1-u^2)^{\frac{1}{2}q/(q-r)}}\right\}^{(q-r)/q}$$

and the right side is finite since

$$\frac{q}{2(q-r)} = \frac{p}{2-p}\Big/\left(\frac{2p}{2-p} - r\right) < 1.$$

If $f(x) \epsilon L^2$, the same argument shows that $G(\varphi)$ belongs to every L^p, $p < \infty$.

If $p > 2$, we have

$$G(\cos^{-1} u) = \int_a^u \frac{f(x)\, dx}{(u - x)^{\frac{1}{2}}},$$

and by another theorem of Hardy and Littlewood [11, p. 227], $G(\cos^{-1}u) \epsilon$ Lip α where $\alpha = (p - 2)/(2p)$. That is,

$$| G(\cos^{-1}u) - G(\cos^{-1}(u - h)) | \leqq Ah^{\alpha} \quad (0 \leqq u - h < u \leqq \cos^{-1}a).$$

Then when $a \leqq u < u + \delta < 1$,

$$\begin{aligned} | G(u + \delta) - G(u) | &\leqq A \,| \cos u - \cos (u + \delta) |^{\alpha} \\ &\leqq A\delta^{\alpha}\{\sin (u + \delta')\}^{\alpha}, \qquad 0 < \delta' < \delta, \\ &\leqq A\delta^{\alpha}. \end{aligned}$$

LEMMA 9.7. *If* $-1 \leqq a < 1, f(x) \epsilon L^p(0, \cos^{-1}a)$, $1 \leqq p \leqq \infty$, *and*

$$H(u) = \int_{\cos^{-1} u}^{\cos^{-1} a} \frac{f(x)\, dx}{(u - \cos x)^{\frac{1}{2}}} \qquad (a \leqq u \leqq 1),$$

then $H(u) \epsilon L^r(a, 1)$ *for* $r < 2p$; *if* $p = \infty$, $H(u) \epsilon L^{\infty}$.

If $p = \infty$, $| f(x) | \leqq M$ almost everywhere, and so

$$| H(u) | \leqq \frac{M}{(1 - a)^{\frac{1}{2}}} \int_a^u \frac{ds}{(u - s)^{\frac{1}{2}}(1 + s)^{\frac{1}{2}}} \leqq \frac{\pi M}{(1 - a)^{\frac{1}{2}}};$$

hence $H(u) \epsilon L^{\infty}$.

If $1 \leqq p < \infty$, we have

$$\begin{aligned} H(u) &= \int_a^u \frac{f(\cos^{-1} s)\, ds}{(1 - s^2)^{\frac{1}{2}}(u - s)^{\frac{1}{2}}} \\ &= \int_a^u \frac{g(s)\, ds}{(u - s)^{\frac{1}{2}}(1 - s^2)^{(p-1)/(2p)}}, \end{aligned} \tag{9.8}$$

where $g(s) = f(\cos^{-1}s)(1 - s^2)^{-1/(2p)}$. We have $g(s) \epsilon L^p$, $g(s)(1 - s^2)^{-(p-1)/(2p)} \epsilon$ L^q if $q < 2p/(1 + p)$ (note that $1 < q < 2$). Then by the theorem of Hardy and Littlewood used in Lemma 9.5, $H(u) \epsilon L^r$ if $r = 2q/(2 - q) < 2p$.

LEMMA 9.9. *If* $1 < p \leqq 2$ *and* $(\sin x)^{\alpha}f(x) \epsilon L^p$ *for an* α *such that* $\alpha < (p - 1)/p$, *then the function* $H(u)$ *of Lemma* 9.7 *belongs to* L^p.

It follows from a theorem of Hardy and Littlewood [1, 2, p. 258, Theorem 359] that if $l > 0, m > 0$, and $p > 1$,

$$\begin{aligned} &\int_a^1 (1 - s^2)^{(p-1)(1-l-m)} \left| \int_a^s \frac{F(u)\, du}{(s - u)^{\frac{1}{2}}} \right|^p ds \\ &\qquad \leqq K \int_a^1 (1 - u^2)^{(1-l)(p-1)} \,| F(u) |^p\, du \int_a^1 (1 - u^2)^{(1-m)(p-1)-p/2}\, du. \end{aligned}$$

We take $l + m = 1$, and $m < p/(2p - 2)$, so that $(1 - m)(p - 1) - \frac{1}{2}p > -1$. Then, with a new K,

$$(9.10) \qquad \int_a^1 \left| \int_a^s \frac{F(u)\,du}{(s-u)^{\frac{1}{2}}} \right|^p ds \leqq K \int_a^1 |F(u)|^p (1 - u^2)^{(1-l)(p-1)}\,du.$$

We are interested in the case $1 < p \leqq 2$; in this case, $p/(2p - 2) \geqq 1$, so that any positive l and m satisfying $l + m = 1$ can be used. Hence if $r < (p - 1)/p$,

$$(9.11) \qquad \int_a^1 \left| \int_0^s \frac{F(u)\,du}{(s-u)^{\frac{1}{2}}} \right|^p ds \leqq K \int_a^1 |F(u)(1 - u^2)^r|^p\,du.$$

We now apply (9.11) to $F(u) = g(u)(1 - u^2)^{(p-1)/(2p)}$, where $g(u)$ is defined as in the proof of Lemma 9.7. If $f(u)(\sin u)^\alpha \in L^p$, $(1 - s^2)^{\alpha/2}g(s) \in L^p$, and so $F(u)(1 - u^2)^r \in L^p$ if

$$r + \frac{1}{2p} - \tfrac{1}{2} \geqq \frac{\alpha}{2},$$

or

$$r \geqq \frac{\alpha p + p - 1}{2p}.$$

An r satisfying this inequality can also satisfy $r < (p - 1)/p$ if

$$\frac{\alpha p + p - 1}{2p} < \frac{p-1}{p},$$

or $\alpha < (p - 1)/p$.

The remaining three lemmas are variants of the well known solution of Abel's integral equation [10, p. 229].

LEMMA 9.12. *If* $-1 \leqq a \leqq 1$ *and*

$$G(\varphi) = \int_a^{\cos\varphi} \frac{f(x)\,dx}{(\cos\varphi - x)^{\frac{1}{2}}} \qquad (0 \leqq \varphi \leqq \cos^{-1} a)$$

where $f(x) \in L(a, 1)$, *then*

$$\int_a^x f(t)\,dt = \frac{1}{\pi} \int_{\cos^{-1} x}^{\cos^{-1} a} \frac{G(\varphi)\sin\varphi\,d\varphi}{(x - \cos\varphi)^{\frac{1}{2}}}, \qquad a \leqq x \leqq 1.$$

We have formally

$$\begin{aligned} \int_{\cos^{-1} x}^{\cos^{-1} a} \frac{G(\varphi)\sin\varphi\,d\varphi}{(x-\cos\varphi)^{\frac{1}{2}}} &= \int_a^x \frac{du}{(x-u)^{\frac{1}{2}}} \int_a^u \frac{f(t)\,dt}{(u-t)^{\frac{1}{2}}} \\ &= \int_a^x f(t)\,dt \int_t^u \frac{du}{(x-u)^{\frac{1}{2}}(u-t)^{\frac{1}{2}}} \\ &= \pi \int_a^x f(t)\,dt. \end{aligned}$$

Since $f(x) \in L$, the iterated integral is absolutely convergent, and so the change of order of integration is legitimate.

In the same way we can prove the next two lemmas.

LEMMA 9.13. *If* $f(\varphi) \in L(0, \pi)$, *and*

$$P(x) = \frac{1}{\pi}\int_0^{\cos^{-1} x} \frac{f(\varphi)\, d\varphi}{(\cos\varphi - x)^{\frac{1}{2}}}$$

then

$$\int_0^x f(\varphi)\, d\varphi = \int_{\cos x}^1 \frac{P(u)\, du}{(u - \cos x)^{\frac{1}{2}}}, \qquad 0 \leqq x \leqq \pi.$$

LEMMA 9.14. *If* $-1 \leqq a \leqq 1, f(x) \in L(0, \cos^{-1} a)$, *and*

$$F(u) = \int_{\cos^{-1} u}^{\cos^{-1} a} \frac{f(x)\, dx}{(u - \cos x)^{\frac{1}{2}}},$$

then

$$\int_{\cos^{-1} x}^{\cos^{-1} a} f(t)\, dt = \frac{1}{\pi}\int_a^{\cos x} \frac{F(u)\, du}{(\cos x - u)^{\frac{1}{2}}}, \qquad a \leqq x \leqq 1.$$

10. We now prove Theorem 8. Let $-1 \leqq a < 1$, and let $p > 1$ if $a = -1$, $p \geqq 1$ otherwise. Let $f(x) \in L^p(a, 1)$. We consider

$$H(\lambda) = \int_a^1 P_\lambda(x) f(x)\, dx. \tag{10.1}$$

We have to show that $H(\lambda_n) = 0$ $(n = 1, 2, \cdots)$ implies $f(x) = 0$ almost everywhere, provided that the set $\{\cos \lambda_n + \frac{1}{2})x\}_{n=1}^{\infty}$ is complete in the class specified in Theorem 8.

We have

$$\begin{aligned} H(\lambda) &= \frac{2^{\frac{1}{2}}}{\pi}\int_a^1 f(x)\, dx \int_0^{\cos^{-1} x} \frac{\cos(\lambda + \frac{1}{2})\varphi}{(\cos\varphi - x)^{\frac{1}{2}}}\, d\varphi, \\ &= \frac{2^{\frac{1}{2}}}{\pi}\int_0^{\cos^{-1} a} G(\varphi) \cos(\lambda + \tfrac{1}{2})\varphi\, d\varphi, \end{aligned} \tag{10.2}$$

where

$$G(\varphi) = \int_a^{\cos\varphi} \frac{f(x)\, dx}{(\cos\varphi - x)^{\frac{1}{2}}}. \tag{10.3}$$

The change of order of integration is legitimate because, by Lemma 9.1, the iterated integral in (10.2) is dominated by

$$\begin{aligned} &\frac{2^{\frac{1}{2}}}{\pi} \exp\{\pi \,|\, \Im(\lambda) \,|\} \int_a^1 |\, f(x) \,|\, dx \int_0^{\cos^{-1} x} \frac{d\varphi}{(\cos\varphi - x)^{\frac{1}{2}}} \\ &= \exp\{\pi \,|\, \Im(\lambda) \,|\} \int_a^1 |\, f(x) \,|\, P_{-\frac{1}{2}}(x)\, dx \\ &\leqq A \int_a^1 |\, f(x) \,|\, \{1 + |\log(1 + x)|\}\, dx. \end{aligned}$$

By Lemma 9.5, $G(\varphi)$ belongs to the class specified in Theorem 8. Since $\{\cos(\lambda_n + \frac{1}{2})\varphi\}$ is assumed complete in the appropriate class, from $H(\lambda_n) = 0$ $(n = 1, 2, \cdots)$, we infer $G(\varphi) = 0$ almost everywhere.

By Lemma 9.12, (10.3) implies that

$$\int_a^x f(t)\, dt = \frac{1}{\pi} \int_{\cos^{-1} x}^{\cos^{-1} a} \frac{G(\varphi) \sin \varphi\, d\varphi}{(x - \cos \varphi)^{\frac{1}{2}}};$$

hence $f(x) = 0$ almost everywhere if $G(\varphi) = 0$ almost everywhere.

11. To establish Theorem 11, we start from the representation of Lemma 9.1,

$$P_\lambda(x) = \frac{2^{\frac{1}{2}}}{\pi} \int_0^{\cos^{-1} x} \frac{\cos (\lambda + \frac{1}{2})\varphi\, d\varphi}{(\cos \varphi - x)^{\frac{1}{2}}}.$$

By Lemma 9.13, this gives us

$$\frac{\sin (\lambda + \frac{1}{2})x}{\lambda + \frac{1}{2}} = \int_0^x \cos (\lambda + \tfrac{1}{2})\varphi\, d\varphi = \frac{1}{2^{\frac{1}{2}}} \int_{\cos x}^1 \frac{P_\lambda(u)}{(u - \cos x)^{\frac{1}{2}}}, \qquad \Re(\lambda) > -\tfrac{1}{2}.$$

Now let $0 < b \leqq \pi$, $f(x) \in L^p(0, b)$, and consider

$$\begin{aligned} K(\lambda) &= \frac{1}{\lambda + \frac{1}{2}} \int_0^b f(x) \sin (\lambda + \tfrac{1}{2})x\, dx \\ &= \frac{1}{2^{\frac{1}{2}}\pi(\lambda + \frac{1}{2})} \int_0^b f(x)\, dx \int_{\cos x}^1 \frac{P_\lambda(u)}{(u - \cos x)^{\frac{1}{2}}}. \end{aligned}$$

Formally,

$$\begin{aligned} K(\lambda) &= \frac{1}{2^{\frac{1}{2}}\pi(\lambda + \frac{1}{2})} \int_{\cos b}^1 P_\lambda(u)\, du \int_{\cos^{-1} u}^b \frac{f(x)\, dx}{(u - \cos x)^{\frac{1}{2}}} \\ &= \frac{1}{2^{\frac{1}{2}}\pi(\lambda + \frac{1}{2})} \int_{\cos b}^1 P_\lambda(u) F(u)\, du. \end{aligned} \tag{11.1}$$

Let $a = \cos b$. Then

$$F(u) = \int_{\cos^{-1} u}^{\cos^{-1} a} \frac{\cdot f(x)\, dx}{(u - \cos x)^{\frac{1}{2}}} = \int_a^u \frac{f(\cos^{-1} s)\, ds}{(u - s)^{\frac{1}{2}}(1 - s^2)^{\frac{1}{2}}}.$$

Let

$$H(u) = \int_a^u \frac{|f(\cos^{-1} s)|\, ds}{(u - s)^{\frac{1}{2}}(1 - s^2)^{\frac{1}{2}}}. \tag{11.2}$$

By Lemma 9.7, $H(u) \in L^r(a, 1)$ for $r < 2p$. Using Lemma 9.2, we see that the integral in (11.1) is absolutely convergent, and so the change of order of integration is legitimate; also $F(u) \in L^r(a, 1)$, $r < 2p$. Hence if $\{P_{\lambda_n}(x)\}$ is complete $L^r(\cos b, 1)$ and $K(\lambda_n) = 0$ $(n = 1, 2, \cdots; \Re(\lambda_n) > -\frac{1}{2})$ we have $F(u) = 0$ almost everywhere, and hence (Lemma 9.14) $f(x) = 0$ almost everywhere.

Thus the completeness of $\{P_{\lambda_n}(x)\}$ in $(\cos b, 1)$ implies that of $\{\sin(\lambda_n + \frac{1}{2})x\}$ in $(0, b)$.

To prove Theorem 12 we proceed in the same way, using Lemma 9.9 to infer that $H(u)$ and $F(u)$ belong to L^p.

12. Our theorems break down for intervals (a, b), $-1 < a < b < 1$, as we shall show by an example. We could of course obtain sufficient conditions for the completeness of $\{P_{\lambda_n}(x)\}$ in such an interval either by regarding it as a subinterval of $(a, 1)$ or by studying $\int_a^b P_\lambda(x)f(x)dx$ by methods analogous to those used for trigonometric systems [cf. §13]. It should be noted that the asymptotic formulas for Legendre functions [4, p. 299] cannot be expected to be useful, since a relation of the form $g_n(x) = f_n(x) + \epsilon_n(x)$, $\lim_{n\to\infty} \epsilon_n(x) = 0$, does not necessarily imply that $\{g_n(x)\}$ is complete if $\{f_n(x)\}$ is complete. For example, let $\{h_n(x)\}_{n=0}^{\infty}$ be a complete set, arbitrary except that $h_0(x) = 1$. Let $f_0(x) = 1$, $f_n(x) = 1 + \delta_n h_n(x)$ $(n = 1, 2, \cdots)$, where $\{\delta_n\}$ is any sequence of constants. Then $\{f_n(x)\}$ is a complete set, but $\{f_n(x) - \delta_n h_n(x)\}$ is not.

We now show that $\{\cos(2n + \frac{1}{2})x\}_{n=0}^{\infty}$ is complete in $L^p(\frac{1}{2}\pi - c, \frac{1}{2}\pi + c)$, $0 < c < \pi/2$, while $\{P_{2n}(x)\}_{n=0}^{\infty}$ is not complete in $(\cos(\frac{1}{2}\pi + c), \cos(\frac{1}{2}\pi - c) = (-\sin c, \sin c)$. In fact, the second statement is immediate since $P_{2n}(x)$ is an even function. On the other hand, let $f(t) \in L^p(\frac{1}{2}\pi - c, \frac{1}{2}\pi + c)$ be such that

$$\int_{\frac{1}{2}\pi - c}^{\frac{1}{2}\pi + c} f(t)\cos(2n + \tfrac{1}{2})t\,dt = 0 \qquad (n = 0, 1, 2, \cdots).$$

Then

$$\int_{-c}^{c} f(u + \tfrac{1}{2}\pi)\cos\{2n + \tfrac{1}{2}\pi\}\,u = 0 \qquad (n = 0, 1, 2, \cdots).$$

Consider the function

$$F(z) = \int_{-c}^{c} f(u + \tfrac{1}{2}\pi)\cos\{zu + \tfrac{1}{4}\pi\}\,du.$$

We verify easily that $G(z) = F(2z + \frac{1}{2})$ is entire and of exponential type $2c < \pi$, and satisfies $G(n) = 0$ $(n = 0, 1, 2, \cdots)$. Hence by Carlson's theorem [8, p. 186] $G(z) \equiv 0$, $F(z) \equiv 0$,

$$\int_{-c}^{c} f(u + \tfrac{1}{2}\pi)\cos zu\,du \equiv \int_{-c}^{c} f(u + \tfrac{1}{2}\pi)\sin zu\,du;$$

since the first integral is even and the second is odd, both are zero, and consequently $f(u + \frac{1}{2}\pi) = 0$ almost everywhere in $-c < u < c$. Thus $f(t) = 0$ almost everywhere in $\frac{1}{2}\pi - c < t < \frac{1}{2}\pi + c$, and so $\{\cos(2n + \frac{1}{2})x\}$ is complete in that interval.

13. In this section we establish Theorem 10 for $1 \leqq p \leqq \infty$; the modifications necessary for $p = \infty$ are left to the reader. Take $a = -1$; from §10 we see that we have to show that

$$G(\varphi) = \int_{-1}^{\cos\varphi} \frac{f(x)\,dx}{(\cos\varphi - x)^{\frac{1}{2}}}, \qquad f(x) \in L^p, \tag{13.1}$$

$$H(z) = \frac{2^{\frac{1}{2}}}{\pi} \int_0^{\pi} G(\varphi) \cos z\varphi\, d\varphi, \tag{13.2}$$

$$H(\lambda_n + \tfrac{1}{2}) = 0 \qquad (n = 1, 2, \cdots), \tag{13.3}$$

and

$$-\tfrac{1}{2} < \lambda_n \leqq n - \frac{1}{p} \tag{13.4}$$

together imply that $H(z) \equiv 0$.

Now $H(z)$ is an entire function of exponential type (at most) π. We have

$$\begin{aligned}
\frac{\pi}{2^{\frac{1}{2}}} \mid H(iy) \mid &\leqq \int_0^{\pi} e^{\varphi|y|}\, d\varphi \int_{-1}^{\cos x} \frac{\mid f(x) \mid dx}{(\cos\varphi - x)^{\frac{1}{2}}} \\
&= \int_{-1}^{1} \mid f(x) \mid dx \int_0^{\cos^{-1} x} \frac{e^{\varphi|y|}}{(\cos\varphi - x)^{\frac{1}{2}}}\, d\varphi \\
&= \int_{-1}^{1} \mid f(x) \mid dx \int_x^{1} \frac{e^{|y|\cos^{-1} u}\, du}{(1 - u^2)^{\frac{1}{2}}(u - x)^{\frac{1}{2}}} \\
&\leqq \int_{-1}^{1} \mid f(x) \mid e^{|y| \cos^{-1} x}\, dx \int_x^{1} \frac{du}{(1 - u)^{\frac{1}{2}}(u - x)^{\frac{1}{2}}} \\
&= \pi \int_{-1}^{1} \mid f(x) \mid e^{|y|\cos^{-1} x}\, dx \\
&= \int_0^{\pi} h(u) e^{u|y|}\, du,
\end{aligned}$$

where $h(u) = f(\cos u) \sin u$. We have, since $p \geqq 1$,

$$\begin{aligned}
\int_0^{\pi} \mid h(u) \mid^p du &= \int_0^{\pi} \mid f(\cos u) \mid^p (\sin u)^{p-1} \sin u\, du \\
&\leqq \int_0^{\pi} \mid f(\cos u) \mid^p \sin u\, du = \int_{-1}^{1} \mid f(x) \mid^p dx.
\end{aligned}$$

Hence $h(u) \in L^p$. We now apply Lemmas 3.8 and 3.1, taking $\gamma = 0$, $s(t) = (\sin t)^{1/p'}$, and hence $\alpha = 2/p' = 2 - 2/p$. We conclude that $H(z) \equiv 0$ if

$$0 < \lambda_n + \tfrac{1}{2} \leqq n + \tfrac{1}{2} - \frac{1}{p}.$$

This completes the proof of Theorem 10.

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