Spectral Properties of Real Parts of Weighted Shift Operators

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1. Introduction. Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space with fixed orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$. Suppose that $C = \int \lambda dE\lambda$ is a bounded self-adjoint operator with the following matrix representation with respect to the given basis:

$$
C = \frac{1}{2}\begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & 0 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & 0 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & 0 & a_4 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}. \tag{1.1}
$$

The operator C is a special case of the Jacobi matrix discussed by Stone [5]. It is the aim of this paper to use the structural properties of C as a Jacobi matrix to analyze its spectral properties.

Since real parts of unitarily equivalent operators are unitarily equivalent, it will always be assumed that the weight sequence $\{a_n\}$ in (1.1) is non-negative. Note that if $a_n > 0$ for all n, then ϕ_1 is a cyclic vector.

2. The eigenvalue problem. This section develops a technique for analyzing the point spectrum of C under the assumption the weight sequence is strictly positive. Appropriate examples will follow. The first three lemmas essentially appear in Stone [5].

Lemma 1. *Each basis element ϕ_n can be expressed in the form $P_n(C)\phi_1$, where $P_n(C)$ denotes a polynomial in C.*

Proof. Clearly $P_1(C) \equiv I$ and $C\phi_1 = (a_1/2)\phi_2$ so that $P_2(C) = (2/a_1)C$. For $k \geq 2$, $C\phi_k = (1/2)[a_{k-1}\phi_{k-1} + a_k\phi_{k+1}]$. Hence $\phi_{k+1} =$

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$(1/a_k)[2C\phi_k - a_{k-1}\phi_{k-1}]$ and $P_{k+1}(C) = (1/a_k)[2CP_k(C) - a_{k-1}P_{k-1}(C)]$. The result follows by induction.

Lemma 2. *If $C = \int \lambda dE\lambda$, then the polynomials $\{P_n(\lambda)\}_{n=1}^{\infty}$ are orthonormal with respect to the measure μ defined for every Borel subset β of the real line by $\mu(\beta) = \|E(\beta)\phi_1\|^2$. That is,*

$$\int_{\mathrm{sp}(C)} P_m(\lambda)P_n(\lambda)\, d\|E_\lambda \phi_1\|^2 = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

where $\mathrm{sp}(C)$ *denotes the spectrum of* C.

Proof. It is sufficient to note that

$$\int_{\mathrm{sp}(C)} P_n(\lambda)P_m(\lambda) d\|E_\lambda \phi_1\|^2 = \langle P_m(C)\phi_1, P_n(C)\phi_1\rangle = \langle \phi_m, \phi_n\rangle .$$

For future reference note that the polynomials $\{P_n(\lambda)\}_{n=1}^{\infty}$ satisfy the following:

$$(2.1)\qquad \begin{aligned} &P_1(\lambda) \equiv 1 \qquad P_2(\lambda) = \frac{2\lambda}{a_1} \\ &P_n(\lambda) = \frac{2\lambda P_{n-1}(\lambda) - a_{n-2}P_{n-2}(\lambda)}{a_{n-1}} \qquad (n \geq 3). \end{aligned}$$

Lemma 3. *If $Cx = \lambda x$, where $x = \sum_{n=1}^{\infty} x_n \phi_n$, then $x_n = P_n(\lambda)x_1$. The real number λ is an eigenvalue of C if and only if $\sum_{n=1}^{\infty} |P_n(\lambda)|^2 < \infty$.*

Proof. If $x = (x_1,x_2,x_3,\ldots,)$ then $Cx = (1/2)(a_1 x_2, a_1 x_1 + a_2 x_3, a_2 x_2 + a_3 x_4,\ldots,)$. The result follows by induction.

Define the operator J as follows:

$$(2.2)\qquad J = \frac{1}{2i}\begin{bmatrix} 0 & -a_1 & 0 & 0 & \cdots \\ a_1 & 0 & -a_2 & 0 & \cdots \\ 0 & a_2 & 0 & -a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

It is easily verified that $CJ - JC = (i/2)K$ where K is a diagonal operator such that $K\phi_1 = a_1^2\phi_1$ and for $n > 1$, $K\phi_n = (-a_{n-1}^2 + a_n^2)\phi_n$. The operator K plays a significant role in the results that follow.

Lemma 4. *If x is an eigenvector for C, then $\langle Kx,x\rangle = 0$.*

Proof. Suppose that x is an eigenvector corresponding to the eigenvalue λ. Note that $CJ - JC = (C - \lambda I)J - J(C - \lambda I) = -(i/2)K$.

Lemma 3 and Lemma 4 yield a necessary condition for λ to be an eigenvalue for C.

Proposition 1. *If $Cx = \lambda x$, then*

$$\langle Kx,x\rangle = a_1^2\, P_1^2(\lambda) + \sum_{n=2}^{\infty} (-a_{n-1}^2 + a_n^2)\, P_n^2(\lambda) = 0. \tag{2.3}$$

It will be shown below that the condition is not sufficient. Nevertheless it does lead to some interesting results. Furthermore, the following is true:

Lemma 5. *Let $S_N(\lambda) = a_1^2\, P_1^2(\lambda) + \sum_{n=2}^{N} (-a_{n-1}^2 + a_n^2)\, P_n^2(\lambda)$. Then*

$$\begin{aligned} S_N(\lambda) &= a_{N-1}^2\, P_{N-1}^2(\lambda) - 2\lambda a_{N-1}\, P_{N-1}(\lambda)P_N(\lambda) + a_N^2\, P_N^2(\lambda) \\ &= [a_{N-1}P_{N-1}(\lambda) - \lambda P_N(\lambda)]^2 + (a_N^2 - \lambda^2)P_N^2(\lambda). \end{aligned}$$

Proof. Since $P_2(\lambda) = 2\lambda/a_1$,

$$\begin{aligned} S_2(\lambda) &= a_1^2\, P_1^2(\lambda) - a_1^2\, P_2^2(\lambda) + a_2^2\, P_2^2(\lambda) \\ &= a_1^2\, P_1^2(\lambda) - 2\lambda a_1\, P_1(\lambda)\, P_2(\lambda) + a_2^2\, P_2^2(\lambda). \end{aligned}$$

Assume now that the result has been verified for $N = 2, \ldots, k$ and recall that $a_k\, P_{k+1}(\lambda) = 2\lambda P_k(\lambda) - a_{k-1}\, P_{k-1}(\lambda)$. Then

$$\begin{aligned} S_{k+1}(\lambda) &= a_{k-1}^2\, P_{k-1}^2(\lambda) - 2\lambda a_{k-1}P_{k-1}(\lambda)P_k(\lambda) + a_k^2\, P_k^2(\lambda) \\ &\quad + (-a_k^2 + a_{k+1}^2)P_{k+1}^2(\lambda) \\ &= a_{k-1}^2\, P_{k-1}^2(\lambda) - 2\lambda a_{k-1}P_{k-1}(\lambda)\, P_k(\lambda) + a_k^2\, P_k^2(\lambda) \\ &\quad - [2\lambda P_k(\lambda) - a_{k-1}\, P_{k-1}(\lambda)]^2 + a_{k+1}^2\, P_{k+1}^2(\lambda) \\ &= a_k^2\, P_k^2(\lambda) - 4\lambda^2\, P_k^2(\lambda) + 2\lambda a_{k-1}\, P_{k-1}(\lambda)\, P_k(\lambda) + a_{k+1}^2\, P_{k+1}^2(\lambda) \\ &= a_k^2\, P_k^2(\lambda) - 2\lambda P_k(\lambda)\, a_k\, P_{k+1}(\lambda) + a_{k+1}^2\, P_{k+1}^2(\lambda). \end{aligned}$$

The above results will now be used to analyze the following examples.

Example 1. If the sequence $\{a_n\}$ monotonically increases to $\alpha \neq 0$ the operator T, defined by $T\phi_n = a_n\phi_{n+1}$, is hyponormal and $\mathrm{sp}(T) =$

$\{\lambda : |\lambda| \le \alpha\}$. The spectrum of C, the real part of T, is $[-\alpha,\alpha]$, and it follows immediately from (2.3) that C has no eigenvalues.

Example 2. If the sequence $\{a_n\}$ monotonically decreases to $\alpha \neq 0$, then

$$(2.4)\quad C = \frac{\alpha}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & a_1-\alpha & 0 & 0 & \dots \\ a_1-\alpha & 0 & a_2-\alpha & 0 & \dots \\ 0 & a_2-\alpha & 0 & a_3-\alpha & \dots \\ 0 & 0 & a_3-\alpha & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

The first operator on the right has a continuous spectrum covering the interval $[-\alpha,\alpha]$, and the second operator is compact. Therefore, by Weyl's Theorem, $\mathrm{sp}(C) \supset [-\alpha,\alpha]$ and any points of the spectrum of C outside this interval must be eigenvalues [2, Problem 143]. Furthermore, C has no eigenvalues in $[-\alpha,\alpha]$. For suppose $\lambda \in [-\alpha,\alpha]$ is an eigenvalue. By Lemma 3, $\sum P_n^2(\lambda) < \infty$. Let $P_{N_0}^2(\lambda) = \max\limits_n P_n^2(\lambda)$ and choose N_1 such that $n > N_1$ implies that $P_n^2(\lambda) < (1/2)P_{N_0}^2(\lambda)$. By Proposition 1 and Lemma 5

$$\begin{aligned}\langle Kx,x\rangle &= S_{N_0}(\lambda) + \sum_{N_0+1}^{\infty} (-a_{n-1}^2 + a_n^2)\, P_n^2(\lambda) \\ &\ge (a_{N_0}^2 - \lambda^2)\, P_{N_0}^2 + \sum_{N_0+1}^{N_1} (-a_{n-1}^2 + a_n^2)\, P_{N_0}^2(\lambda) \\ &\quad + \frac{1}{2}\sum_{N_1+1}^{\infty} (-a_{n-1}^2 + a_n^2)\, P_{N_0}^2(\lambda) \\ &\ge \frac{1}{2}(a_{N_1}^2 - \lambda^2)\, P_{N_0}^2(\lambda) + \frac{1}{2}(\alpha^2 - \lambda^2)\, P_{N_0}^2(\lambda).\end{aligned}$$

If $\lambda^2 < \alpha^2$ or if $\lambda^2 = \alpha^2$ and $a_{N_1}^2 \neq \alpha^2$, then $\langle Kx,x\rangle > 0$ which leads to a contradiction. If $\lambda^2 = \alpha^2$ and $a_{N_1}^2 = \alpha^2$, then $a_n^2 = \alpha^2$ for $n > N_1$. This special case is discussed in Example 3 below.

Example 3. Let $a_1, \dots, a_{N-1}$ be arbitrary positive real numbers and assume that $a_n = \alpha > 0$ for $n \ge N$. Then (2.3) reduces to the equation

$$S_N(\lambda) = a_1^2\, P_1^2(\lambda) + \sum_{n=2}^{N} (-a_{n-1}^2 + a_n^2)\, P_n^2(\lambda) = 0.$$

Since P_n is a polynomial of degree $(n - 1)$ it follows immediately that C can have at most $(2N - 2)$ eigenvalues. Furthermore, by Lemma 5,

$$S_N(\lambda) = [a_{N-1} P_{N-1}(\lambda) - \lambda P_N(\lambda)]^2 + (a_N^2 - \lambda^2) P_N^2(\lambda).$$

It can now be shown that the eigenvalues of C lie outside the interval $[-\alpha,\alpha]$. For if $\lambda \in (-\alpha,\alpha)$ and $S_N(\lambda) = 0$, then $P_N(\lambda) = 0$, and so $P_{N-1}(\lambda) = 0$. If $N = 2$ this contradicts the fact that $P_1(\lambda) \equiv 1$. If $N > 2$, $a_{N-1} P_N(\lambda) = 2\lambda P_{N-1}(\lambda) - a_{N-2} P_{N-2}(\lambda)$ implies that $P_{N-2}(\lambda) = 0$. A similar argument shows that $P_{N-3}(\lambda) = \dots = P_1(\lambda) = 0$ which again leads to a contradiction. If $\lambda = \alpha$, then $S_N(\lambda) = 0$ implies that $a_{N-1} P_{N-1}(\alpha) = \alpha P_N(\alpha)$. An induction argument shows that for $k = 1, 2, \dots, P_{N+k}(\alpha) = P_{N+k-1}(\alpha)$. For $P_{N+1}(\alpha) = (1/\alpha)[2\alpha P_N(\alpha) - a_{N-1} P_{N-1}(\alpha)] = P_N(\alpha)$. If $P_{N+k}(\alpha) = P_{N+k-1}(\alpha)$, then $P_{N+k+1}(\alpha) = (1/\alpha)[2\alpha P_{N+k}(\alpha) - P_{N+k-1}(\alpha)] = P_{N+k}(\alpha)$. Hence $\sum_{n=1}^{\infty} P_n^2(\alpha)$ diverges unless $P_N(\alpha) = 0$. But $P_N(\alpha) = 0$ implies $P_{N-1}(\alpha) = 0$, and hence it follows, as above, that $P_{N-2}(\alpha) = \dots = P_1(\alpha) = 0$. Finally, Lemma 3 and the observation that $|P_n(\lambda)| = |P_n(-\lambda)|$ for all n imply that $\lambda = -\alpha$ is also not an eigenvalue. All eigenvalues of C must therefore lie outside the interval $[-\alpha,\alpha]$.

All of the above examples were analyzed using Proposition 1 and Lemma 5. Unfortunately (2.3) gives a necessary but not a sufficient condition for λ to be an eigenvalue for C.

Example 4. Assume that for $k = 1, 2, \dots, a_{2k-1}^2 = 1$ and $a_{2k}^2 = 1 + 1/k$. Consider $\lambda = 0$. By (2.1)

$$P_n^2(0) = \frac{a_{n-2}^2}{a_{n-1}^2} P_{n-2}^2(0).$$

Since $P_2(0) = 0$ it follows that $P_{2k}(0) = 0$ for each k. Moreover, $P_1(0) = 1$ and a little computation shows that $P_{2k-1}^2(0) = 1/k$. Clearly $\sum_{n=1}^{\infty} |P_n(0)|^2$ diverges so that $\lambda = 0$ cannot be an eigenvalue. Yet by Lemma 5, $S_N(0) = a_{N-1}^2 P_{N-1}^2(0) + a_N^2 P_N^2(0)$. Hence $\{S_N(0)\}$ converges to zero and (2.3) holds.

3. The absolutely continuous part of C. If $C = \int \lambda dE_\lambda$, let $\mathscr{H}_a(C)$ denote the set of elements x in $\mathscr{H}$ for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ. It is known that $\mathscr{H}_a(C)$ is a subspace which reduces C. The restriction of C to $\mathscr{H}_a(C)$ is called the absolutely continuous part of C, and in case $\mathscr{H}_a(C) = \mathscr{H}$ the operator C is said to be absolutely continuous.

The aim now is to identify the subspaces of absolute continuity for the operators discussed in the first three examples above. Toward this end consider the following result due to Putnam [3, Theorem 2.2.4].

Theorem 1. *If A and B are self-adjoint with $AB - BA = iD$, $D \geq 0$ or $D \leq 0$, then $\mathscr{H}_a(A)$ contains the smallest subspace reducing A and containing the range of D.*

It follows immediately from the above theorem and the commutator equation $CJ - JC = (i/2)K$ that the operator C of Example 1 is absolutely continuous.

Theorem 2. *If the sequence $\{a_n\}$ decreases monotonically to $\alpha \neq 0$ then $\mathscr{H}_a(C) \supset E[-\alpha,\alpha]\mathscr{H}$.*

The proof of Theorem 2 requires the following lemma which is contained in the proof of Theorem 1.

Lemma 6. *If C and J are self-adjoint with $C = \int \lambda dE_\lambda$ and if $CJ - JC = (i/2)K$ then for any interval Δ and any $x \in \mathscr{H}$*

$$|\langle KE(\Delta)x, E(\Delta)x\rangle| \leq 2|\Delta|\ \|J\|\ \|E(\Delta)x\|^2.$$

Proof of Theorem 2. Assume initially that $a_n = \alpha$ for all $n \geq N$ and that Δ is an interval contained in $[-\alpha + 1/k, \alpha - 1/k]$. Then $E(\Delta)\phi_1 = \sum_{n=1}^{\infty} \langle E(\Delta)\phi_1,\phi_n\rangle \phi_n$ and

$$\langle KE(\Delta)\phi_1,E(\Delta)\phi_1\rangle = a_1^2|\langle E(\Delta)\phi_1,\phi_1\rangle|^2 + \sum_{n=2}^{\infty}(-a_{n-1}^2 + a_n^2)|\langle E(\Delta)\phi_1,\phi_n\rangle|^2$$

$$= a_1^2|\langle E(\Delta)\phi_1,\phi_1\rangle|^2 + \sum_{n=2}^{N}(-a_{n-1}^2 + a_n^2)\left|\int_\Delta P_n\,d\|E_\lambda\phi_1\|^2\right|^2$$

$$\geq a_1^2\|E(\Delta)\phi_1\|^4 + \|E(\Delta)\phi_1\|^2\sum_{n=2}^{N}(-a_{n-1}^2 + a_n^2)\int_\Delta P_n^2\,d\|E_\lambda\phi_1\|^2.$$

If $\int_\Delta P_{N_0}^2\,d\|E_\lambda\phi_1\|^2 = \max_{1,\ldots,N}\int_\Delta P_n^2 d\|E_\lambda\phi_1\|^2$, then

$$\begin{aligned}\langle KE(\Delta)\phi_1,E(\Delta)\phi_1\rangle &\geq \|E(\Delta)\phi_1\|^2\left[\int_\Delta S_{N_0}d\|E_\lambda\phi_1\|^2\right.\\ &\qquad\left.+\sum_{N_0+1}^{N}(-a_{n-1}^2 + a_n^2)\int_\Delta P_{N_0}^2 d\|E_\lambda\phi_1\|^2\right]\\ &\geq \|E(\Delta)\phi_1\|^2\left[\int_\Delta (a_{N_0}^2 - \lambda^2)P_{N_0}^2\,d\|E_\lambda\phi_1\|^2\right.\\ &\qquad\left.+\sum_{N_0+1}^{N}(-a_{n-1}^2 + a_n^2)\int_\Delta P_{N_0}^2\,d\|E_\lambda\phi_1\|^2\right]\\ &\geq \frac{\alpha}{k}\|E(\Delta)\phi_1\|^4.\end{aligned}$$

This result and the previous lemma imply that $\|E(\Delta)\phi_1\|^2 \leq M_k|\Delta|$ where $M_k = (2k/\alpha)\|J\|$.

To establish the theorem, approximate $C = \int \lambda dE_\lambda$ by the sequence $\{C_N\}$ obtained from C by setting $a_n = \alpha$ for $n \geq N$. Let $C_N = \int \lambda dE_\lambda^{(N)}$. As shown above $\|E^{(N)}(\Delta)\phi_1\|^2 \leq M_k\ |\Delta|$ for any interval $\Delta \subset [-\alpha + 1/k,\ \alpha - 1/k]$. Since $\|C_N - C\| \to 0$ and C has no eigenvalues in $[-\alpha,\alpha]$ it follows from [1, Corollary 3, page 923] that $\|E^{(N)}(\Delta)\phi_1\|^2 \to \|E(\Delta)\phi_1\|^2$ and hence that $\|E(\Delta)\phi_1\|^2 \leq M_k|\Delta|$.

If β is a Borel subset of $[-\alpha + 1/k,\ \alpha - 1/k]$ of Lebesgue measure zero, then for any $\varepsilon > 0$ there exists a disjoint sequence of intervals $\{\Delta_j\}$ such that $\Delta_j \subset [-\alpha + 1/k,\ \alpha - 1/k]$, $\beta \subset \cup \Delta_j$ and $\sum |\Delta_j| < \varepsilon$. Since $\|E(\beta)\phi_1\|^2 \leq \sum_j \|E(\Delta_j)\phi_1\|^2 \leq M_k\ \varepsilon$, it follows that $\|E(\beta)\phi_1\|^2 = 0$. If $\beta \subset (-\alpha,\alpha)$ then $\beta = \bigcup_k \beta_k$ where $\beta_k = \beta \cap [-\alpha + 1/k,\ \alpha - 1/k]$. Therefore $\|E(\beta)\phi_1\|^2 = \lim_{k\to\infty} \|E(\beta_k)\phi_1\|^2 = 0$. Since the end points are not eigenvalues the same result holds for $\beta \subset [-\alpha,\alpha]$. Hence $E\,[-\alpha,\alpha]\phi_1 \in \mathscr{H}_a(C)$ and $E\,[-\alpha,\alpha]\mathscr{H} \subset \mathscr{H}_a(C)$ as was to be shown.

Corollary. *If the sequence $\{a_n\}$ decreases monotonically to $\alpha \neq 0$ and* $\mathrm{sp}(C) = [-\alpha,\alpha]$, *then C is absolutely continuous.*

It is perhaps worth noting that convergence of the sequence $\{a_n\}$ to $\alpha \neq 0$ is not sufficient to guarantee that $E[-\alpha,\alpha]\mathscr{H} \subset \mathscr{H}_a(C)$. In fact, if $a_{2K-1} = 1$ $(k = 1,2,\ldots)$ and $a_{2k} = 1 + 1/k$ $(k = 1,2,\ldots)$, then $\{a_n\}$ converges to 1. An argument similar to that used in Example 4, however, shows that $\lambda = 0$ is an eigenvalue.

The final result completes the spectral analysis of the operator described in Example 3.

Theorem 3. *If the first $N - 1$ weights are non-zero and $a_n = \alpha$, $\alpha \neq 0$, for $n \geq N$, then $\mathscr{H}_a(C) \supset E\,[-\alpha,\alpha]\mathscr{H}$.*

The proof again depends on Lemma 6. Lack of monotonicity leads to certain modifications of the previous argument. The roots of the polynomials $P_1, \ldots P_N$ are used to subdivide $[-\alpha,\alpha]$ into subintervals such that in each subinterval $P_n \geq 0$ or $P_n \leq 0$ for $n = 1, \ldots, N$. If $[r,s]$ is such a subinterval the following lemmas hold.

Lemma 7. *For $n = 2, \ldots, N$ and $\Delta \subset [r,s]$*

$$\int_\Delta 2\lambda P_{n-1}\,d\mu \int_\Delta a_{n-1} P_n\,d\mu \leq \int_\Delta 2\lambda P_n\,d\mu \int_\Delta a_{n-1} P_{n-1}\,d\mu.$$

Proof. If $n = 2$, then

$$\int_\Delta 2\lambda P_1\, d\mu \int_\Delta a_1 P_2\, d\mu = \left|\int_\Delta 2\lambda d\mu\right|^2$$
$$\leq \int_\Delta 4\lambda^2 d\mu \int_\Delta P_1\, d\mu$$
$$= \int_\Delta 2\lambda P_2\, d\mu \int_\Delta a_1 P_1\, d\mu.$$

Now assume that the result holds for $n = 2, \ldots, k$ $(k < N)$ and recall that $a_k P_{k+1}(\lambda) = 2\lambda P_k(\lambda) - a_{k-1} P_{k-1}(\lambda)$. Then

$$\int_\Delta 2\lambda P_k\, d\mu \int_\Delta a_k P_{k+1}\, d\mu = \left|\int_\Delta 2\lambda P_k\, d\mu\right|^2 - \int_\Delta 2\lambda P_k\, d\mu \int_\Delta a_{k-1} P_{k-1}\, d\mu$$

and

$$\int_\Delta 2\lambda P_{k+1}\, d\mu \int_\Delta a_k P_k\, d\mu = \int_\Delta 4\lambda^2 P_k\, d\mu \int_\Delta P_k\, d\mu$$
$$- \int_\Delta 2\lambda a_{k-1} P_{k-1}\, d\mu \int_\Delta P_k\, d\mu.$$

Hence,

$$\int_\Delta 2\lambda P_{k+1}\, d\mu \int_\Delta a_k P_k\, d\mu - \int_\Delta 2\lambda P_k\, d\mu \int_\Delta a_k P_{k+1}\, d\mu$$
$$= \int_\Delta 4\lambda^2 P_k\, d\mu \int_\Delta P_k\, d\mu - \left|\int_\Delta 2\lambda P_k\, d\mu\right|^2$$
$$+ \int_\Delta 2\lambda P_k\, d\mu \int_\Delta a_{k-1} P_{k-1}\, d\mu - \int_\Delta 2\lambda a_{k-1} P_{k-1}\, d\mu \int_\Delta P_k\, d\mu.$$

On $[r,s]$ $P_k \geq 0$ or $P_k \leq 0$. In either case $|\int_\Delta 2\lambda P_k\, d\mu|^2 \leq \int_\Delta 4\lambda^2 P_k\, d\mu \int_\Delta P_k\, d\mu$, and the result follows by induction.

Lemma 8. *For $k = 2, \ldots, N$ and $\Delta \subset [r,s]$*

$$a_1^2 \left|\int_\Delta P_1\, d\mu\right|^2 + \sum_{n=2}^{k} (-a_{n-1}^2 + a_n^2) \left|\int_\Delta P_n\, d\mu\right|^2$$
$$\geq a_{k-1}^2 \left|\int_\Delta P_{k-1}\, d\mu\right|^2 - \int_\Delta 2\lambda P_{k-1}\, d\mu \int_\Delta a_{k-1} P_k\, d\mu + a_k^2 \left|\int_\Delta P_k\, d\mu\right|^2.$$

Proof. If $k = 2$, then

$$
\begin{aligned}
a_1^2 &\left|\int_\Delta P_1\,d\mu\right|^2 + (-a_1^2 + a_2^2)\left|\int_\Delta P_2\,d\mu\right|^2 \\
&= a_1^2\left|\int_\Delta P_1\,d\mu\right|^2 - \left|\int_\Delta 2\lambda\,d\mu\right|^2 + a_2^2\left|\int_\Delta P_2\,d\mu\right|^2 \\
&= a_1^2\left|\int_\Delta P_1\,d\mu\right|^2 - \int_\Delta 2\lambda P_1\,d\mu \int_\Delta a_1 P_2\,d\mu \\
&\quad + a_2^2\left|\int_\Delta P_2\,d\mu\right|^2.
\end{aligned}
$$

Assume that the result holds for $k < N$ and consider $k + 1$. Then

$$
\begin{aligned}
a_1^2 &\left|\int_\Delta P_1\,d\mu\right|^2 + \sum_{n=2}^{k+1}(-a_{n-1}^2 + a_n^2)\left|\int_\Delta P_n\,d\mu\right|^2 \\
&\geq a_{k-1}^2\left|\int_\Delta P_{k-1}\,d\mu\right|^2 - \int_\Delta 2\lambda P_{k-1}\,d\mu \int_\Delta a_{k-1}P_k\,d\mu \\
&\quad + a_k^2\left|\int_\Delta P_k\,d\mu\right|^2 + (-a_k^2 + a_{k+1}^2)\left|\int_\Delta P_{k+1}\,d\mu\right|^2 \\
&= a_{k-1}^2\left|\int_\Delta P_{k-1}\,d\mu\right|^2 - \int_\Delta 2\lambda P_{k-1}\,d\mu \int_\Delta a_{k-1}P_k\,d\mu \\
&\quad + a_k^2\left|\int_\Delta P_k\,d\mu\right|^2 - \left|\int_\Delta [2\lambda P_k - a_{k-1}P_{k-1}]\,d\mu\right|^2 \\
&\quad + a_{k+1}^2\left|\int_\Delta P_{k+1}\,d\mu\right|^2 \\
&= -\int_\Delta 2\lambda P_{k-1}\,d\mu \int_\Delta a_{k-1}P_k\,d\mu \\
&\quad + \int_\Delta 2\lambda P_k\,d\mu \int_\Delta a_{k-1}P_{k-1}\,d\mu + a_k^2\left|\int_\Delta P_k\,d\mu\right|^2 \\
&\quad - \int_\Delta 2\lambda P_k\,d\mu \int_\Delta a_k P_{k+1}\,d\mu + a_{k+1}^2\left|\int_\Delta P_{k+1}\,d\mu\right|^2 \\
&\geq a_k^2\left|\int_\Delta P_k\,d\mu\right|^2 - \int_\Delta 2\lambda P_k\,d\mu \int_\Delta a_k P_{k+1}\,d\mu
\end{aligned}
$$

$$+ a_{k+1}^2 \left| \int_\Delta P_{k+1}\, d\mu \right|^2 .$$

Proof of Theorem 3. Since $a_n = \alpha$ for $n \geq N$

$$\langle KE(\Delta)\phi_1, E(\Delta)\phi_1 \rangle = a_1^2 \left| \int_\Delta P_1\, d\mu \right|^2 + \sum_{n=2}^{N} (-a_{n-1}^2 + a_n^2) \left| \int_\Delta P_n\, d\mu \right|^2 .$$

Using the zeros of the polynomials $P_1, \ldots, P_N$, subdivide $[-\alpha,\alpha]$ into subintervals so that Lemma 7 and Lemma 8 hold. If $[r,s]$ is such a subinterval it is clearly enough to restrict attention to Borel subsets of $[r,s]$.

Choose k sufficiently large so that $[r + 1/k,\ s - 1/k]$ is a subinterval of $[r,s]$. Then for any interval $\Delta \subset [r + 1/k,\ s - 1/k]$,

$$\left| 2 \int_\Delta \lambda P_{N-1}\, d\mu \right| \left| \int_\Delta a_{N-1} P_N\, d\mu \right|$$
$$\leq \left(\alpha - \frac{1}{k}\right)^2 \left| \int_\Delta P_N\, d\mu \right|^2 + a_{N-1}^2 \left| \int_\Delta P_{N-1}\, d\mu \right|^2 .$$

Hence, by Lemma 8,

$$a_1^2 \left| \int_\Delta P_1\, d\mu \right|^2 + \sum_{n=2}^{N} (-a_{n-1}^2 + a_n^2) \left| \int_\Delta P_n\, d\mu \right|^2$$
$$\geq \left[\alpha^2 - \left(\alpha - \frac{1}{k}\right)^2 \right] \left| \int_\Delta P_N\, d\mu \right|^2$$
$$\geq \frac{\alpha}{k} \left| \int_\Delta P_N\, d\mu \right|^2$$
$$\geq \frac{\alpha m}{k} \|E(\Delta)\phi_1\|^4,$$

where m denotes the minimum of $|P_N|$ on $[r + 1/k,\ s - 1/k]$. This result and Lemma 6 imply that

$$\frac{\alpha m}{k} \|E(\Delta)\phi_1\|^4 \leq |\langle KE(\Delta)\phi_1,\ E(\Delta)\phi_1 \rangle| \leq 2\, |\Delta|\, \|J\| \|E(\Delta)\phi_1\|^2,$$

and hence that $\|E(\Delta)\phi_1\|^2 \leq (2k|\Delta|\, \|J\|)/(\alpha m)$. Given this inequality the proof is concluded by an argument similar to that used in the proof of Theorem 2.

Remark. Finally, it is interesting to note that if M denotes multiplication by λ on $L^2[-1,1]$, and if $\{\phi_n\}$ is the basis obtained by orthonormalizing

$\{1,\lambda,\lambda^2,\ldots,\}$, then the matrix of M with respect to $\{\phi_n\}$ is of the form (1.1) with the weight sequence monotonically decreasing to 1.

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