THE HARMONIC ANALYSIS OF STATIONARY STOCHASTIC PROCESSES[1]

By

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(Received August 16, 1948)

CONTENTS

1. Introduction. Let $z(t)$ be a k-dimensional stochastic process depending on a continuous or integral-valued time parameter t and $(t_1, t_2, \dots, t_n)$ be a set of t-values, then the process is said to be *stationary* if the distribution of the set of random variables $\{z(t_1), z(t_2), \dots, z(t_n)\}$ is independent of all translations of the time.

[1] Most of the results in §§ 1–6 are obtained in author's previous paper, "*The harmonic analysis of stationary stochastic processes*" (Japanese) Kenkyū Hōkoku, Faculty of Science, Kyūsyū University, vol. 2, no. 1, 1947, which was received by the editors May 9, 1945. In preparation of this paper, the author has found, according to a paper by J. L. Doob ("*The elementary Gaussian processes*," Annals of Math. St., vol. XV, no. 3, 1944), that Theorem 13 is established in a paper by H. Cramér, published during the war and was inaccessible in Japan. The results in § 7 is published in Kōkyūroku, Institute of Mathematical Statistics, Department of Education, vol. 3 (1937).

The stationary stochastic process (s.s.p.) is a fundamental process in the theory of stochastic processes and plays an important rôle, among others, in the statistical analysis of phenomena changing in some kinds of equilibrium states. It has been studied by many authors, especially we shall mention the contributions due to A. Khintchine, E. Slutsky, H. Cramér, H. Wold.[2]

In the case of a continuous time parameter t one of fundamental properties required for the analysis of stochastic processes will be measurability, for which a necessary and sufficient condition has been given by A. Kolmogoroff.[3] For a one-dimensional s. s. p. $z(t)$ the measurability condition of Kolmogoroff can be stated that

$$P(|z(t+h)-z(t)|>\delta)\to 0, \quad h\to 0 \tag{1.1}$$

for every t and $\delta>0$. If, in particular, the second order moments of $z(t)$ exist, (1.1) is implied by the (mean-square) continuity:

$$E(|z(t+h)-z(t)|^2)\to 0, \quad h\to 0. \tag{1.1$'$}$$

In this paper, except in §9, we shall deal with the s.s.p.'s with vanishing mean values and finite second order moments, satisfying (1.1)$'$ in the case of a continuous parameter. Obviously this assumption means no essential restriction. In §§ 1–7 we shall treat s.s.p.'s depending on a continuous time parameter. It is, however, easy to modify the results obtained for the case of an integral-valued parameter. The processes considered in §§ 1–7 are two-dimensional

$$z(t)=(x(t),\ \ y(t))$$

for which is sometimes conveniently used the complex form

$$z(t)=x(t)+iy(t)$$

with real components $x(t)$ and $y(t)$. Higher dimensional cases may be treated in a similar manner. For any s. s. p. satisfying (1.1)$'$ there exist correlation functions

$$f(\tau)=E(x(t+\tau)x(t)),\ \ g(\tau)=E(y(t+\tau)y(t)),\ \ h(\tau)=E(x(t+\tau)y(t)),$$

which depend only on τ and are continuous. For one-dimensional s.s.p. Khintchine[4] was the first to introduce the *spectral distribution function* with respect to which the correlation function becomes a

[2] Khintchine [1], Slutsky [1], Cramér [2], Wold [1].

[3] Doob [2], Ambrose [1], Kawada [1].

[4] Khintchine [1].

Fourier-Stieltjes transform. H. Cramér[5] generalized his result to higher dimensional cases. In our case Cramér's theorem is stated as follows.

Theorem 1. *There exist functions $F(\lambda)$, $G(\lambda)$, and $H(\lambda)$ called the spectral distribution functions (s.d.f.'s) of the process, giving*

$$(1.2)\qquad f(\tau)=\int_{-\infty}^{\infty} e^{i\lambda\tau}\,dF(\lambda),\quad g(\tau)=\int_{-\infty}^{\infty} e^{i\lambda\tau}\,dG(\lambda)$$
$$h(\tau)=\int_{-\infty}^{\infty} e^{i\lambda\tau}\,dH(\lambda),$$

where $F(\lambda)$ and $G(\lambda)$ are non-decreasing bounded functions, and $H(\lambda)$ is a complex—valued function of bounded variation.

Since $f(\tau)$ and $g(\tau)$ are real-valued, a simple application of the inversion formula of Fourier-Stieltjes transforms shows that $F(b)-F(a)=F(-a)-F(-b)(0\leqq a<b)$, i.e. the increments of $F(\lambda)$ and $G(\lambda)$ are symmetric with respect to $\lambda=0$.

Now let us suppose that $F(\lambda)$ is decomposed, by the Lebesgue decomposition theorem, into a step function $F_1(\lambda)$, singular function $F_2(\lambda)$, and absolutely continuous function $F_3(\lambda)$, and similarly for G and H with corresponding subscripts, then the symmetric property for the increments of $F(\lambda)$ and $G(\lambda)$ implies that of their respective three components. In this connection H. Cramér[6] has given characteristic relations between the functions $F(\lambda)$, $G(\lambda)$, $H(\lambda)$, and their components to the following effect.

Theorem 2. *A necessary and sufficient condition that $F(\lambda)$, $G(\lambda)$ and $H(\lambda)$ with the properties in Theorem 1 should be s.d.f.'s of a two-dimensional s.s.p. is that the increments of $F(\lambda)$ and $G(\lambda)$ should be symmetric with respect to $\lambda=0$, $-\bar{H}(-\lambda)=H(\lambda)$, and*

$$(1.3)\qquad |\Delta H|^2\leqq \Delta F\cdot\Delta G,$$
$$|\Delta H_i|^2\leqq \Delta F_i\cdot\Delta G_i\qquad (i=1,2,3),$$

where ΔF etc. denote increments over some intervals.

Let $I_k=(a_k,b_k)$ $(k=1,2,\dots)$ be sets of non-overlapping internals and write $\Delta F=F(b_k)-F(a_k)$ etc., then by (1.3)

$$\sum_k|\Delta_k H|\leqq\sum(\Delta_k F\cdot\Delta_k G)^{\frac{1}{2}}\leqq(\sum\Delta_k F\cdot\sum\Delta_k G)^{\frac{1}{2}}$$

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5) Cramér, loc. cit.

6) Cramér, loc. cit.

whence we are led to

$$\int_A |dH(\lambda)| \leqq \left(\int_A dF(\lambda) \cdot \int_A dG(\lambda)\right)^{\frac{1}{2}} \tag{1.4}$$

for any Borel-set A, and similar relations for the components. (1.4) means that any set of measure zers with respect to $F(\lambda)$ is also of measure zero with respect to $H(\lambda)$. Hence by the Radon-Nikodym theorem

$$H(b)-H(a) = \int_a^b p(\lambda)\, dF(\lambda)$$

with a suitable function $p(\lambda)$ integrable with respect to $F(\lambda)$. Similar relations hold between $H(\lambda)$ and $G(\lambda)$, and their corresponding three components.

2. Slutsky's B_2—processes. Slutsky[7] introduced first the Fourier coefficients of an s.s.p. and obtained its B_2—component. Birkhoff's ergodic theorem assures the existence of the Fourier coefficients

$$\begin{aligned} \frac{1}{2}(A_\lambda + iB_\lambda) &= U_\lambda = \underset{t}{M}[x(t)e^{-i\lambda t}], \\ \frac{1}{2}(A_\lambda' + iB_\lambda') &= U_\lambda' = \underset{t}{M}[y(t)e^{-i\lambda t}], \\ c_\lambda &= U_\lambda + iU_\lambda', \end{aligned} \tag{2.1}$$

for every λ, where $\underset{t}{M}$ is the operator $\lim_{A\to\infty} A^{-1}\int_0^A \cdot\, dt$. Slutsky's results generalized to our two-dimensional case can be stated as follows.

Theorem 3.

(i) *A's are uncorrelated with B's;*

(ii) *U_μ is uncorrelated with U_λ and $\bar{U}_\lambda$, $|\lambda| \neq |\mu|$, and $E(U_\lambda) = 0$, $E(|U_\lambda|^2) = F(\lambda+0) - F(\lambda-0)$;*

(iii) *If S, S' are the sets of all discontinuity points of $F(\lambda)$ and $G(\lambda)$, then defining $z_1(t)$ by*

$$z_1(t) = \sum_{\lambda \in S+S'} c_\lambda e^{i\lambda t}, \tag{2.3}$$

which converges in probability, we get the decomposition $z(t) = z_1(t) + z'(t)$ with uncorrelated s.s.p.'s $z_1(t)$ and $z'(t)$ having respective s.d.f.'s $(F_1(\lambda), G_1(\lambda), H_1(\lambda))$, $(F'(\lambda), G'(\lambda), H'(\lambda))$ where we put $F'(\lambda) = F_2(\lambda)$

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[7] Slutsky, loc. cit.

$+F_3(\lambda)$. *$z_1(t)$ is a B_2-function with probability* 1:

Proof. Most of the statements can be proved in the same way as in the one-dimensional case. We have only to prove the stationarity property of $z_1(t)$ and $z'(t)$ and to calculate their correlations. Let us write the partial sums of the series defining $z(t)$ in the form

$$\begin{aligned}\sum_{|\lambda|<l} c_\lambda e^{i\lambda t} &= \lim_{A\to\infty} A^{-1}\int_0^A z(s)\,\Big(\sum_{|\lambda|<l} e^{i\lambda(t-s)}\Big)\,ds \\ &= \lim_{A\to\infty} A^{-1}\int_0^A z(s+t)\,K(s)\,ds \\ &= \lim_{A\to\infty}\ \text{st.}\lim_{A\to\infty}(nA)^{-1}\sum_{k=0}^{n-1} z\Big(\frac{k}{n}A+t\Big)K\Big(\frac{k}{n}A\Big),^{3)} \\ K(s) &= \sum_{|\lambda|<l} e^{-i\lambda s}.\end{aligned}$$

Since $\sum z((k/n)A+t)K((k/n)A)$ is an s.s.p. with the time parameter t, $z_1(t)$ is easily seen to be stationary. By the same reasoning the stationarity of $z'(t)$ follows from the relation

$$z'(t) = \text{st.}\lim_{l\to\infty}\ \lim_{A\to\infty} A^{-1}\int_0^A (z(t)-z(s+t))\,K(s)\,ds.$$

Next we pass on to calculation of the correlation functions. Let us put $z_1(t) = x_1(t)+iy_1(t)$ and $z'(t) = x'(t)+iy'(t)$, then

$$\begin{aligned}E(U_\lambda e^{i\lambda s} y(t)) &= \lim_{A\to\infty} E\Big(A^{-1}\int_0^A x(u)\,e^{i\lambda(s-u)}\,du\cdot y(t)\Big) \\ &= \lim_{A\to\infty}\Big(\int_{-\infty}^{\infty} e^{i(\lambda s-\mu t)}\,dH(\mu)\Big)\Big(A^{-1}\int_0^A e^{i(\mu-\lambda)u}\,du\Big) \\ &= \lrcorner_\lambda H e^{i\lambda(s-t)},\end{aligned} \tag{2.4}$$

where $\lrcorner_\lambda$ means the jump at λ. Since (1.3) means that the discontinuity points of $H(\lambda)$ are included in the common part of the sets of discontinuity points of $F(\lambda)$ and $G(\lambda)$, say S and S' respectively, (2.4) gives

$$E(x_1(s)\,y(t)) = \sum_{\lambda\in S} E(U_\lambda e^{i\lambda s} y\cdot y(t)) = \int_{-\infty}^{\infty} e^{i\lambda(s-t)}\,dH_1(\lambda). \tag{2.5}$$

From (2.5) it follows at once that

$$\begin{aligned}E(x_1(s)\,U_\lambda' e^{i\lambda t}) &= \lrcorner_{-\lambda} H\cdot e^{-i\lambda(s-t)}, \\ E(x_1(s)\ y_1(t)) &= \int_{-\infty}^{\infty} e^{i\lambda(s-t)}\,dH_1(\lambda).\end{aligned} \tag{2.6}$$

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[3] st. lim means the limit in probability.

Hence, making use of results in one-dimensional case, $z_1(t)$ has the s.d.f.'s $(F_1(\lambda), G_1(\lambda), H_1(\lambda))$. Writing

$$E(x'(s)y'(t)) = E\{(x(s)-x_1(s))(y(t)-y_1(t))\}$$

and using (2.5) and (2.6), it is easily shown that $z'(t)$ has the s.d.f.'s $(F'(\lambda), G'(\lambda), H'(\lambda))$. (2.5) and (2.6), together with the results known in the one-dimensional case, show that $z_1(t)$ and $z'(t)$ are uncorrelated.

3. Metrical transitivity. In this section we shall investigate the metrical transitivity of a normal s.s.p. The metrical transitivity of a normal s.s.p. is closely related to the continuity of its s.d.f.'s. According to the following theorem, $z'(t)$ in the decomposition in §2 becomes metrically transitive.

Theorem 4. *If the s.d.f.'s of a normal s.s.p. $z(t)$ are continuous, then the process is metrically transitive.*

Proof. Without loss of generality, we may suppose that $F(\infty) = F(\infty) = 1$ and $F(-\infty) = G(-\infty) = 0$. Let $\Phi(z_1, z_2, \ldots, z_n)$ be a measureable function of n complex variables $z_1, z_2, \ldots, z_n$ satisfying $E|\Phi(z(t_1), z(t_2), \ldots, z(t_n))| < \infty$ for any set of t-values $t_1, \ldots, t_n$. Then it is sufficient to prove that

$$\underset{t}{M}[\Phi\{z(t_1+t), z(t_2+t), \ldots, z(t_n+t)\}] = E[\Phi\{z(t_1), z(t_2), \ldots, z(t_n)\}]$$

In the following we shall discuss the simplest case $n = 1$, the general case being proved in the same way. First we suppose that Φ is positive on a bounded open set, zero outside the set, and has everywhere bounded second derivatives. Let us put

$$\Psi(x, y) = \sum_{m, n=-\infty}^{\infty} \Phi(x+mT, y+nT) = \Phi(x, y) + \Phi_T(x, y), \tag{3.1}$$
$$\Phi(z) = \Phi(x, y).$$

Then $\Psi(x, y)$ is periodic with period T having the Fourier expansion

$$\Psi(x, y) = \sum_{t=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} A_{s,t} \exp\{2\pi i(sx+ty)/T\}.$$

The existence of the bounded second derivatives yields

$$A_{s,t} = O(|st|^{-2})$$

which ensures the absolute convergence of the Fourier expansion.

Since $\Phi \geqq 0$, $\Psi \geqq 0$, $\Phi_T \geqq 0$, we have

$$E\{(A^{-1}\int_0^A \zeta(t)\,dt - E(\zeta))^2\}$$

$$= A^{-1}\int_0^A \left(1-\frac{|\tau|}{A}\right) E(\zeta(\tau)\,\zeta(0))\,d\tau - \{E(\zeta(0))\}^2$$

$$\leqq A^{-1}\int_{-A}^A \left(1-\frac{|\tau|}{A}\right) E\{\zeta^*(\tau)\zeta^*(0) + \zeta^*(\tau)\,\zeta_T(0) + \zeta_T(\tau)\,\zeta^*(0) + \zeta_T(\tau)\,\zeta(0)\}\,d\tau - (E(\zeta^*) - E(\zeta_T))^2, \tag{3.2}$$

where

$$\zeta(\tau) = \Phi(z(\tau)), \quad \zeta_T(\tau) = \Phi_T(z(\tau)), \quad \zeta^*(\tau) = \Psi(z(\tau)).$$

The second term of the last expression in (3.2) tends to zero, as $T\to\infty$, since it does not exceed $3KE(\zeta_T)$, where K is an upper bound of Φ, and $E(\zeta_T)\to 0$ by the bounded convergence of Φ_T to zero. On account of the absolute convergence of (3.1)

$$A^{-1}\int_{-A}^A \left(1-\frac{|\tau|}{A}\right) E(\zeta^*(\tau)\,\zeta^*(0))\,d\tau$$

$$= A^{-1}\int_{-A}^A \left(1-\frac{|\tau|}{A}\right)d\tau \cdot \sum_{s_1,t_1,s_2,t_2} A_{s_1,t_1} A_{s_2,t_2}$$

$$\cdot E\left\{\exp\left((s_1 x(t) + t_1 y(t) + s_2 x(0) + t_2 y(0))\frac{2\pi i}{T}\right)\right\} \tag{3.3}$$

$$= \sum_{s_1,t_1,s_2,t_2} A_{s_1,t_1} A_{s_2,t_2} A^{-1}\int_{-A}^A \left(1-\frac{|\tau|}{A}\right)$$

$$\cdot \exp\left(-\frac{1}{2}\left(\frac{2\pi}{T}\right)^2 Q(s,t)\right) d\tau,$$

$$Q(s,t) = s_1^2 + t_1^2 + s_2^2 + t_2^2 + 2s_1t_1H(\infty) + 2s_2t_2H(\infty) + 2s_1s_2f(\tau) + 2t_1t_2g(\tau) + 2s_1t_2\,h(\tau) + 2s_2t_1\,h(-\tau).$$

By means of the Taylor expansion of the exponential function

$$A^{-1}\int_{-A}^A \left(1-\frac{|\tau|}{A}\right) \exp\left(-\frac{1}{2}\left(\frac{2\pi}{T}\right)^2 Q(s,t)\right) d\tau$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{2\pi}{T}\right)^2 (s_1^2 + t_1^2 + s_2^2 + t_2^2 + 2s_1t_1\,H(\infty) + 2s_2t_2\,H(\infty))\right\}\left\{1 + O\left((A^{-1}\int_0^A f^2(\tau)d\tau)^{\frac{1}{2}} + (A^{-1}\int_0^A g^2(\tau)d\tau)^{\frac{1}{2}} + (A^{-1}\int_0^A |h(\tau)|^2 d\tau)^{\frac{1}{2}} + (A^{-1}\int_0^A |h(-\tau)|^2 d\tau)^{\frac{1}{2}}\right)\right\} \tag{3.4}$$

$$\to \exp\left\{-\frac{1}{2}\left(\frac{2\pi}{T}\right)^2(s_1^2+t_1^2+s_2^2+t_2^2+2s_1t_1 H(\infty)\right.$$

$$\left.+2s_2t_2 H(\infty))\right\}, \quad A\to\infty.$$

where we have used the fact that if $U(x)$ is continuous and of bounded variation, then

$$A^{-1}\int_0^A \left|\int_{-\infty}^{\infty} e^{ix\tau} dU(x)\right|^2 d\tau \to 0, \qquad A\to\infty.$$

Combining (3.3) and (3.4)

$$A^{-1}\int_{-A}^{A}\left(1-\frac{|\tau|}{A}\right)E(\zeta^*(\tau)\,\zeta^*(0))\,d\tau \to (E(\zeta))^2, \quad A\to\infty.$$

Hence by (3.2)

$$\lim_{A\to\infty} E\{(A^{-1}\int_0^A \Phi(z(t))\,dt - E(z))^2\} = 0 \tag{3.5}$$

To consider the general case we approximate Φ by a finite sum, $S(z)$, of these $\Phi(z)$ for which we have proved (3.5), such that

$$E(|\Phi(z(t)) - S(z(t))|) < \varepsilon.$$

Then we have

$$E\{|A^{-1}\int_0^A \zeta(t)\,dt - E(\zeta)|\} \leqq E\{|A^{-1}\int_0^A (\zeta(t)-S(t))\,dt|\}$$

$$+E\{|A^{-1}\int_0^A S(t)\,dt - E(S)|\} + |E(S)-E(\zeta)|,$$

where $S(t) \equiv S(z(t))$.

Since $S(t)$ satisfies (3.5)

$$\overline{\lim_{A\to\infty}}\, E\{|A^{-1}\int_0^A \zeta(t)\,dt - E(\zeta)|\} < 2\varepsilon;$$

whence, making $\varepsilon\to 0$, $A^{-1}\int_0^A \zeta(t)\,dt$ converges in mean to $E(\zeta)$. By Birkhoff's ergodic theorem we obtain

$$A^{-1}\int_0^A \Phi(z(t))\,dt \to E(\Phi(z)), \quad A\to\infty,$$

with probability 1, as was to be proved.

4. Decomposition of the process z′(t). In this section we shall establish a decomposition of $z'(t)$ into two s.s.p.'s whose properties are to be discussed in § 5.

Theorem 5. *$z'(t)$ is the sum of two uncorrelated s.s.p.'s $z_2(t)$,*

$z_3(t)$ with their spectral distribution functions $(F_2(\lambda), G_2(\lambda), H_2(\lambda))$ and $(F_3(\lambda), G_3(\lambda), H_3(\lambda))$. These two processes are uncorrelated with $z_1(t)$.

Proof. Let $w_1(t)$ and $w_2(t)$ be continuous in $(-\infty, \infty)$ and put

(4.1) $$x_1(t)=\int_{a_1}^{b_1} x(s)w_1(t-s)\,ds, \quad x_2(t)=\int_{a_2}^{b_2} x(s)\,w_2(t-s)\,ds,$$

then we obtain the relation

$$E(x_1(t_1)\,x_2(t_2))=\int_{a_1}^{b_1}\int_{a_2}^{b_2} w_1(t_1-s)\,w_2(t_2-t)\,E(x(s)x(t))\,ds\,dt$$

(4.2) $$=\int_{-\infty}^{\infty} e^{i\lambda(t_1-t_2)}dF(\lambda)\left(\int_{a_1-t_1}^{b_1-t_1} w_1(-s)e^{i\lambda s}ds\int_{a_2-t_2}^{b_2-t_2} w_2(-t)e^{-i\lambda t}dt\right)$$

which holds for any one-dimensional s.s.p. $x(t)$ with the *s.d.f.* $F(\lambda)$ and proves to be useful in the following considerations. Since $F_2(\lambda)$ and $G_2(\lambda)$ are singular with symmetric increments with respect to $\lambda=0$, there is a set $\mathfrak{E}$ of λ which is also symmetic and satisfies

(4.3) $$\int_{C\mathfrak{E}} dF_2(\lambda)=\int_{C\mathfrak{E}} dG_2(\lambda)=0, \quad |\mathfrak{E}|=0.^{9)}$$

Obviously the set $\mathfrak{E}+S+S'$ is also symmetric and enables us to choose an open set $M=M_\delta$ depending on $\delta>0$ such that

$$\mathfrak{E}+S+S'\subset M, \quad |M|<\delta.$$

Let us put

$$K(a,\delta;\,x)=1 \quad \text{for} \quad x\in CM\cdot(-a,\,a).$$
$$=0 \quad \text{otherwise},$$

and define its Fourier transform $k(a,\delta;\,x)$ and the function $w(a,\delta;\,x)$

$$k(a,\delta;\,x)=(2\pi)^{-1}\int_{-\infty}^{\infty} K(a,\delta;\,s)\,e^{-isx}\,ds,$$
$$w(a,\delta;\,x)=\left(1-\frac{|x|}{A}\right)k(a,\delta;\,x),$$

where it is to be noted that $k(a,\delta;\,x)$ is a real-valued function of x. We shall introduce an s.s.p. $z(A,\,a,\,\delta;\,t)$ with a time parameter t

(4.4) $$z(A,\,a,\,\delta;\,t)=\int_{-A}^{A} \dot{z}(s)\,w(a,\delta;\,t-s)\,ds,$$

then it can be shown that the process defined in (4.4) converges

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9) $|\mathfrak{E}|$ denotes the Lebesgue measure of $\mathfrak{E}$.

in mean to a definite s.s.p. In the following, for the sake of simplicity, we prove this only for the real part $x(t)$. First we observe that the s.s.p. $x(A, a, \delta; t)$ defined in (4.4) is real. (4.2) gives us

$$E(x(A, a, \delta; t)x(A', x, \delta; t+\tau)) = \int_{-\infty}^{\infty} e^{-i\lambda\tau} dF(\lambda)\int_{-A-t}^{A-t}\left(1-\frac{|s|}{A}\right)$$

$$\cdot k(a, \delta; -s)e^{i\lambda s}ds\int_{-A'-t-\tau}^{A'-t-\tau}\left(1-\frac{|t|}{A'}\right)k(a, \delta; -t)e^{-i\lambda t}dt$$

$$=\int_{-\infty}^{\infty} e^{-i\lambda\tau} dF(\lambda)\left((2\pi)^{-1}\int_{-A}^{A}\left(1-\frac{|s|}{A}\right)e^{i\lambda s}ds\int_{-\infty}^{\infty} K(a, \delta; x)e^{isx}dx+O\left(\frac{1}{A'}\right)\right)$$

$$\cdot\left((2\pi)^{-1}\int_{-A'}^{A'}\left(1-\frac{|t|}{A'}\right)e^{-ist}dt\int_{-\infty}^{\infty} K(a, \delta; x)e^{ixt}dx+O\left(\frac{1}{A}\right)\right)$$

$$\to\int_{-\infty}^{\infty} K^2(a, \delta; x)e^{-i\lambda\tau}dF_3(\lambda) = \int_{N_a} e^{-i\lambda\tau}dF_3(\lambda);\ A, A'\to\infty,$$

where $N_a = CM\cdot(-a, a)$. By means of this relation, it is easily shown that there exists an s.s.p. $x(a, \delta; t)$ such that

$$x(a, \delta; t) = \underset{A\to\infty}{\text{l.i.m.}}\ x(A, a, \delta; t)$$

for every fixed t and

$$E(x(a, \delta; t)x(a, \delta; t+\tau)) = \int_{N_a} e^{-i\lambda\tau}dF_3(\lambda)$$

In a similar manner we get

$$E\{(x(a, \delta; t)-x(a, \delta; t+\tau))^2\} = \left(\int_{N_a}+\int_{N_{a'}}-2\int_{N_aN_{a'}}\right)dF_3(\lambda)$$

$$\to 0;\ a, a'\to\infty.$$

For every t $x(a, \delta; t)$ converges in mean to an s.s.p. $x(\delta, t)$

$$x(\delta; t) = \underset{a\to\infty}{\text{l.i.m.}}\ x(a, \delta; t)$$

and

$$E(x(\delta; t)x(\delta; t+\tau)) = \int_{CM} e^{-i\lambda t}dF_3(\lambda) \tag{4.5}$$

An elementary calculation gives us

$$E\{(x(\delta; t)-x(\delta'; t))^2\} = \left(\int_{CM_\delta}+\int_{CM_{\delta'}}-2\int_{CM_\delta\cdot CM_{\delta'}}\right)dF_3(\lambda)$$

$$\to 0;\ \delta, \delta'\to 0.$$

whence $x(\delta;\ t)$ converges in mean, for every fixed t, to an s.s.p. $x_3(t)$. Making $\delta\rightarrow 0$ in (4.5) we can easily show that the s.d.f. of $x_3(t)$ is $F_3(\lambda)$. An s.s.p. $y_3(t)$ with the s.d.f. $G_3(\lambda)$ can be obtained from $y(t)$ in the same way. Writing $z_3(t)=x_3(t)+iy_3(t)$, $z_2(t)=z'(t)-z_3(t)$ we get the required decomposition

$$z'(t)=z_2(t)+z_3(t).$$

Now we shall prove that $z_1(t)$, $z_2(t)$, $z_3(t)$ are mutually uncorrelated. (2.1) and (4.4) give

$$E(x_3(s)U_\lambda)=\lim_{\delta\to 0}\lim_{a\to\infty}\lim_{A\to\infty}\int_{-\infty}^{\infty}e^{i\mu s}dF(\mu)\int_{-A}^{A}w(a,\delta;\ -t)e^{i\mu t}dt \left(A^{-1}\int_0^A e^{iu(\lambda-\mu)}du\right). \tag{4.6}$$

Now

$$\int_{-A}^{A}w\cdot e^{i\mu t}dt\rightarrow 0 \quad \text{for} \quad \mu\bar{\varepsilon}N_a,$$

and, since $\lambda\varepsilon S$

$$A^{-1}\int_0^A e^{iu(\lambda-\mu)}du\rightarrow 0 \quad \text{for} \quad \mu\varepsilon N_a.$$

The integrands in (4.6) being uniformly bounded with respect to μ the right-hand side of (4.6) tends to zero on making $A\rightarrow\infty$. Hence it follows that $E(x_3(s)x_1(t))=0$ for every s and t, together with similar relations between remaining parts of $z_1(t)$ and $z_3(t)$. Hence the s.s.p.'s $z_1(t)$ and $z_3(t)$ are uncorrelated. Using (4.4) it is easy to show that

$$E(x_3(s)x(t))=\int_{-\infty}^{\infty}e^{i\lambda(s-t)}dF_3(\lambda), \tag{4.7}$$

whence by writing $E\{x_3(s)x_2(t)\}=E\{x_3(s)x(t)-x_3(s)x_1(t)-x_3(s)x_3(t)\}$ $x_3(s)$ and $x_2(t)$ are seen to be uncorrelated, and similarly for the other parts of $z_3(s)$ and $z_2(t)$. Thus the s.s.p.'s $z_2(t)$ and $z_3(t)$ are uncorrelated. Since, as is shown in §2, $z_1(t)$ and $z'(t)$ are uncorrelated, $z_1(t)$ and $z_2(t)$ are uncorrelated.

It remains to show that $z_2(t)$ and $z_3(t)$ have their respective s.d.f.'s $F_2(\lambda)$, $G_2(\lambda)$, $H_2(\lambda)$, and $F_3(\lambda)$, $G_3(\lambda)$, $H_3(\lambda)$. (1.4) gives us

$$\int_{CM}|dH_k(\lambda)|=0,\quad k=1,\ 2.$$

By this relation, reasoning as in the foregoing, we have

$$
(4.8)\qquad
\begin{aligned}
E(x_3(s)y_3(t)) &= \lim_{\delta\to 0}\lim_{a\to\infty}\int_{N_a} e^{i\lambda(s-t)}\,dH_3(\lambda)\\
&= \int_{-\infty}^{\infty} e^{i\lambda(s-t)}\,dH_3(\lambda),
\end{aligned}
$$

as was to be proved. Writing $z_2(t) = z(t)-z_1(t)-z_3(t)$ and noting that $z_2(t)$ and $z_3(t)$ are uncorrelated, it is easily shown that $z_3(t)$ has the s.d.f.'s $(F_3(\lambda),\ G_3(\lambda),\ H_3(\lambda))$.
The decomposition of $z(t)$ thus established is unique. We have the following theorem.

Theorem 6. *If an s.s.p. $z(t)$ with its s.d.f.'s $(F(\lambda),\ G(\lambda),\ H(\lambda))$ is represented as the sum of s.s.p's $z_1^*(t)$, $z_2^*(t)$, $z_3^*(t)$ with their respective s.d.f.'s $(F_1(\lambda), G_1(\lambda), H_1(\lambda))$, $(F_2(\lambda), G_2(\lambda), H_2(\lambda))$, $(F_3(\lambda), G_3(\lambda), H_3(\lambda)$, then $z_1^*(t)$, $z_2^*(t)$, $z_3^*(t)$ coincide for every t, with probability* 1, *with $z_1(t)$, $z_2(t)$, $z_3(t)$ obtained in Theorem* 3 *and Theorem* 5.

Proof. According to Theorem 3, the Fourier coefficients for $z_2^*(t)$, $z_3^*(t)$ all vanish and $z_1^*(t)$, $z_2(t)$ have common Fourier coefficients. Hence the s.s.p.'s $z_1^*(t)$ coincides with $z_1(t)$. By means of the proof of Theorem 5

$$
z_3(t) = \underset{\delta\to 0}{\text{l.i.m.}}\ \underset{a\to\infty}{\text{l.i.m.}}\ \underset{A\to\infty}{\text{l.i.m.}}\int_{-A}^{A} z_3^*(s)w(a,\ \delta\,;\ t-s)\,ds
$$

for every t with probability 1, and again by the method for the proof of Theorem 5 it is easily shown that $z_3(t)-z_3^*(t)$ has the identically vanishing s.d.f.'s, i.e. $z_3^*(t)$ coincides with $z_3(t)$. Hence $z_1^*(t)$ coincides with $z_1(t)$.

5. Mixing properties of components. Let $\Phi_i(z_1,\ z_2,\ \dots,\ z_n)$ $(i = 1,\ 2)$ be bounded measurable functions of an arbitary number of complex variables $z_1,\ z_2,\ \dots,\ z_n$. A stochastic process $z(t)$ is said to be strongly mixing if

$$
(5.1)\qquad
\begin{aligned}
\Psi(t) = E\{\Phi_1(z(t_1),\ z(t_2),\ \dots,\ z(t_n))\Phi_2(z(t_1'+t),\ z(t_2'+t)\dots,\ z(t_n'+t))\}\\
-E\{\Phi_1(z(t_1),\ z(t_2),\ \dots,\ z(t_n))\}\cdot E\{\Phi_2(z(t_1'),\\
z(t_2'),\ \dots,\ z(t_n'))\}\to 0,\ \text{as}\ t\to\infty\,;
\end{aligned}
$$

and it is said to be weakly mixing, if

$$
(5.2)\qquad \underset{t}{M}[\Psi^2(t)] = 0.
$$

According to the following theorem, $z'(t)$ is weakly mixing

Theorem 7. *A normal s.s.p. $z(t)$ with continuous s.d.f.'s is* *weakly mixing.*

Proof. We shall consider a special case when $n=1$ and Φ_1, Φ_2 are periodic functions with period T having absolutely convergent Fourier expansions

$$\left.\begin{matrix}\Phi_1(z)\\ \Phi_2(z)\end{matrix}\right\} = \sum_{m,n} \left\{\begin{matrix}A_{m,n}\\ B_{m,n}\end{matrix}\right. \times \exp(2\pi i(mx+ny)/T),\ z=x+iy. \tag{5.3}$$

Let us put $\Phi_i(z(t)) = \zeta_i(t)$ then (5.3) yields

$$E(\zeta_1(t)\zeta_2(0)) = \sum_{k,l,m,n} A_{k,l} B_{m,n} \exp\{-\frac{1}{2}\frac{4\pi^2}{T^2}(k^2+l^2+n^2+2klh(0)$$
$$+2mnh(0)+2kmf(t)+2lng(t)+2knh(t)+2lmh(-t)\}$$

As in the proof of theorem 4, using the Taylor expansion of the exponential function

$$\begin{aligned} m = M[E(\zeta_1(t)\zeta_2(0))] &= \sum A_{k,l} B_{m,n} \\ &\cdot\exp\left\{-\frac{1}{2}\frac{4\pi^2}{T^2}(k^2+l^2+m^2+n^2+2klh(0)+2mnh(0))\right\} \\ &= \mu_1\cdot\mu_2, \end{aligned} \tag{5.4}$$

where $\mu_1 = E(\zeta_1(0))$, $\mu_2 = E(\zeta_2(0))$. Similarly

$$m' = M[(E(\zeta_1(t)\zeta_2(0)))^2] = \mu_1^2\cdot\mu_2^2. \tag{5.5}$$

(5.4), (5.5) give us

$$M[\Psi^2(t)] = m'+\mu_1^2\cdot\mu_2^2-2m\mu_1\cdot\mu_2 = 0.$$

The general case is treated by the same device as in the proof of Theorem 4.

The s.s.p. $z(t)$ is strongly mixing. Indeed we have the following theorem.

Theorem 8.[10] *A normal s.s.p. $z(t)$ with absolutely continuous s.d.f.'s is strongly mixing.*

Proof. Arguing as in the proof of Theorem 4, it is sufficient to establish the relation (5.1) for those special functions which have been considered in the proof of Theorem 7. The Riemann–Lebesgue theorem leads us to the relations

$$f(t)\to 0,\quad g(t)\to 0,\quad h(t)\to 0,\quad \text{as } |t|\to\infty. \tag{5.6}$$

Hence by (5.4) we have

[10] Precisely, the absolute continuity of $F(\lambda)$, $G(\lambda)$ is sufficient, since $H(\lambda)$ becomes necessarily absolutely continuous.

$$E(\zeta_1(t)\zeta_2(0)) \rightarrow \mu_1,\ \mu_2, \quad t \rightarrow \infty,$$

as was to be proved.

Theorem 9. *Let $z(t)$ be a normal s.s.p.*

(i) *A necessary and sufficient condition that $z(t)$ should be weakly mixing is that $F(\lambda)$ and $G(\lambda)$ should be continuous.*

(ii) *A necessary and sufficient condition that $z(t)$ should be strongly mixing is that the relation* (5.6) *should hold.*[11]

Proof of (i). If $F(\lambda)$ and $G(\lambda)$ are continuous, $z(t)$ is weakly mixing, since all the s.d.f.'s are then continuous with the help of (1.4).

Conversely, if $z(t)$ is weakly mixing, then (5.2) must hold with special functions $\Phi_1(z) = \Phi_2(z) = x$, i.e.

$$M[|f(t)|^2] = \sum_{\lambda \varepsilon S} |\Delta_\lambda F|^2 = 0,$$

whence $F(\lambda)$ must be continuous, and similary for $G(\lambda)$.

Proof of (ii) As in the proof of (i), (5.1) with special functions gives at once the proof.

It should be noted that in Theorem 9 necessary and sufficient conditions for two kinds of mixing types are given in different forms, one depends on the continuity of the s.d.f.'s and the other on an asymptotic behaviour of the correlation functions. In this connection we shall prove the following two theorems.

Theorem 10. *There exists a normal s.s.p. with a singular s.d.f. $F(\lambda)$.*

Proof. The proof of this theorem follows immediately from a well-known property of the Fourier-Stieltjes moments of a singular distribution. According to A. C. Schaeffer's[12] generalization of a theorem due to Littlewood, Wiener and Wintner, there is a non-decreasing singular function $U(x)$ such that

$$C_u = \int_{-\pi}^{\pi} e^{iux} d\,U(x) = O(r(u)/\sqrt{u}), \ u \rightarrow \infty,$$

for any $r(u)$, which may tend to ∞ as slowly as we please. Let

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[11] The second half of this theorem was first established by K. Itō [3], [4], and later independently by the author in [1].

[12] A. C. Schaeffer [1].

$$(5.7)\qquad \begin{aligned} F(x) &= U(x), & -\pi \leqq x \leqq \pi; \\ &= U(-\pi), & x \leqq -\pi; \\ &= U(\pi), & x \geqq \pi \end{aligned}$$

then by Khintchine-Kolmogoroff's theorem, for which a proof will be given later, there exists a normal s.s.p. $x(t)$ having $F(\lambda)$ as its s.d.f. According to Theorem 9, the s.s.p. $x(t)$ is strongly mixing, although its s.d.f. $F(\lambda)$ is singular.

Theorem 11. *There exists a normal s.s.p. $x(t)$, which is not strongly mixing, but weakly mixing.*

Proof. Let E be the ternary set of Cantor constructed over the interval $(-\pi, \pi)$ and define a non-decreasing continuous function $U(x)$ constant on every contiguous interval to E but not constant as a whole. As is well-known $U(x)$ is a singular function and its Fourier-Stieltjes coefficients C_n do not vanish as $n \to \infty$[13]. According to theorem 8, a normal s.s.p. $x(t)$ with the s.d.f. $F(\lambda)$ given by (5.7), whose existence is a consequence of Khintchine-Kolmogoroff's theorem, is not strongly mixing but weakly mixing.

The investigations of Paley and Wiener on random functions provide us with examples of normal s.s.p's. They define a fundamental stochastic process $\psi(x, \omega)$ by the formal series

$$(5.8)\qquad \psi(x) = \Psi(x, \omega) \sim x(-\log \omega_1)^{\frac{1}{2}} e^{2\pi i \omega_2} + \sum_1^{\infty} \frac{e^{inx}}{in} (-\log \omega_{4n-1})^{\frac{1}{2}} e^{2\pi i \alpha_{4n}} + \sum_1^{\infty} \frac{e^{-inx}}{-in} (-\log \omega_{4n+1})^{\frac{1}{2}} e^{2\pi i \alpha_{4n+2}},$$

where ω's are independent random variables, each uniformly distributed in $(0, 1)$. $\psi(x)$ considered as a stochastic process depending on a time parameter x becomes a normal process, called a homogeneous differential process[14]. The increments $\psi(x_2)-\psi(x_1)$, $\dots$, $\psi(x_n)-\psi(x_{n-1})$ on non-overlapping intervals (x_1, x_2), $\dots$, (x_{n-1}, x_n) are independent normal variables with vanishing means and variances $2\pi(x_2-x_1)$ etc. Let $F(x) \in L^2$ in $(-\pi, \pi)$, then we can define an integral[15]

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13) A. Zygmund [1].

14) Paley and Wiener [1], p. 156, Doob [1]. Recently K. Itō has introduced a more general and ingeneous definition of an integral with respect to $\psi(x)$ useful as a tool of studying the Markoff process.

15) Paley and Wiener, loc. cit.

$$\int_{-\pi}^{\pi} F(x)\, d\psi(x).$$

Let $T^t(-\infty < t < \infty)$ be a group of unitrary transformation in the space L^2 satisfying $T^t T^u = T^{t+u}$ for every t and u, and put

$$z(t) = \int_{-\pi}^{\pi} T^t F(x)\, d\psi(x), \tag{5.9}$$

then, as is easily shown, $z(t)$ is a normal s.s.p. with the mean value zero and s.d.f.'s

$$\begin{aligned} F(\lambda) = G(\lambda) &= \frac{\pi}{2}\{\|E_\lambda F\|^2 - \|E_{-\lambda} F\|^2 + \|F\|^2\}, \\ H(\lambda) &= \frac{\pi i}{2}\{\|E_\lambda F\|^2 + \|E_{-\lambda} F\|^2 - \|F\|^2\}, \end{aligned} \tag{5.10}$$

where E_λ is the resolution of the identity. Now $z(t)$ being expressed in the form given by (5.9), some of the properties of the $z_1(t)$ process follow easily from known properties of the operators T^t. By fundamental properties of the integral with respect to $\psi(x)$ we can show that if $F_n(x)$ converges in mean to $G(x)$, then the random variable

$$\int_{-\pi}^{\pi} F_n(x)\, d\psi(x)$$

converges in mean to the random variable expressible as the integral of $G(x)$ with respect to $\psi(x)$.

Neumann's ergodic theorem states

$$A^{-1}\int_0^A T^t F(x)\, e^{-i\lambda t}\, dt$$

convarges in mean to $(\Delta_\lambda E) F(x)$, as $A \to \infty$, where we put $\Delta_\lambda E = E_{\lambda+0} - E_{\lambda-0}$. Hence the random nariable

$$A^{-1}\int_0^A z(t)\, e^{-i\lambda t}\, dt = \int_{-\pi}^{\pi} \left(A^{-1}\int_0^A T^t F(x)\, e^{-i\lambda t} dt\right) d\psi(x)$$

converges in mean to

$$c_\lambda = \int_{-\pi}^{\pi} (\Delta_\lambda E)\, F(x)\, d\psi(x),$$

the Fourier coefficient of the s.s.p. $z(t)$. The orthogonality of $F(x)$ and $G(x)$ implies the independence of $\int F d\psi$ and $\int G d\psi$. As is well-known $(\Delta_\lambda E)F$ and $(\Delta_\mu E)F$ are orthogonal, so that c_λ and c_μ and independent variables. In the general case treated in §2, c_λ are $c_{-\lambda}$ were not necessarily independent. Those special circumstances in the present case are easily seen to depend on the fact that in the present case the s.d.f.'s are represented in the special forms given by (5.10). Possibility of obtaining $z_1(t)$ from $z(t)$ is now obvious from the known fact that every $F \varepsilon L^2$ is represented as follows:

$$F = F_1 + F_2, \; F_2 = \sum (\Delta_\lambda E) F, \; T^t(\Delta_\lambda E) F = e^{i\lambda t}(\Delta_\lambda E) F.$$

Indeed we have

$$z_1(t) = \int_{-\pi}^{\pi} T^t F^2 \, d\varphi = \sum e^{i\lambda t} \int_{-\pi}^{\pi} (\Delta_\lambda E) F \, d\varphi,$$

$$z'(t) = \int_{-\pi}^{\pi} T^t F \, d\varphi.$$

Then, the s.d.f.'s of $z'(t)$, $(\pi/2)\{\|E_\lambda F_1\|^2 + \|E_{-\lambda} F_1\|^2 - \|F_1\|^2\}$ etc., now being continuous, $z'(t)$ becomes weakly mixing by Theorem 7.[16]

After Paley and Wiener we can consider an expression similar to (5.9) for a group of unitary operators defined on $L^2(-, \infty)$. For this purpose we have only to note that a simple transformation defines a homogeneous normal differential process $\Psi(x)$ in $(-\infty, \infty)$ with the mean value 0 and, for its increment on (a, b), the variance $2\pi(b-a)$.[17] Of these groups, the most important may be that which is obtained by all translations of the real axis i.e. the group of operators $T^t f(x) = f(x+t)$ In this case we have

$$(T^t f, f) = \int_{-\infty}^{\infty} f(x+t) \bar{f}(x) \, dx = \int_{-\infty}^{\infty} |g(\lambda)|^2 e^{i\lambda t} \, d\lambda,$$

where $g(\lambda)$ is the Fourier transform

$$g(\lambda) = \text{l.i.m.}\, (2\pi)^{-1/2} \int_{-A}^{A} f(x) e^{-i\lambda x} \, dx.$$

hence

$$\|E_\lambda f\|^2 = \int_{-\infty}^{\lambda} |g(x)|^2 dx$$

is absolutely continuous. Thus the s.s.p. defined by

$$z(t) = \int_{-\infty}^{\infty} f(x+t) \, d\Psi(x) \tag{5.11}$$

is strongly mixing. Conversely any normal s.s.p. $\zeta(t)$ having an absolutely continuous s.d.f. is shown to be represented by the integral (5.11) in the sense that for any set of t-values $t_1, t_2, \ldots, t_n$ and $2n$-dimensional Borel set A we have

[16] This generalizes the result due to Paley and Wiener, loc. cit., pp. 166–168.

[17] Paley and Wiener, loc. cit., p. 169.

$$P\{(\zeta(t_1), \zeta(t_2), \ldots, \zeta(t_n))\varepsilon A\} = P\{((z(t_1), z(t_2), \ldots, z(t_n))\varepsilon A\}$$

for a suitable choice of the function $f(x)$ in (5.11).

K. Itō[17] has proved that a necessary and sufficient condition that a one-dimensional normal stationary process should be a simple Markoff process is that its correlation funition should become $e^{-k|\tau|}$ with some $k \geqq 0$. In this case

$$|g(x)|^2 = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-k|\tau|} e^{-i\tau x} d\tau = \frac{1}{\pi} \frac{k}{k^2+x^2}.$$

If we assume

$$g(x) = \left(\frac{k}{\pi(k^2+x^2)}\right)^{1/2}$$

Then

$$f(\xi) = \frac{\sqrt{2k}}{\pi}\int_0^{\infty} \frac{\cos x\xi}{(k^2+x^2)^{1/2}} dx = \frac{\sqrt{2k}}{\pi} K_0(k|\xi|),$$

and the required Markoff process is given by

$$x(t) = \frac{\sqrt{2k}}{\pi}\int_{-\infty}^{\infty} K_0(k|\xi+t|)\, d(\Psi_1/\sqrt{\pi}),$$

where Ψ_1 denotes the real part of Ψ. On the other hand, if we put

$$g(x) = \sqrt{\frac{k}{\pi}}\left(\frac{k}{k^2+x^2} + i\frac{x}{k^2+x^2}\right),$$

then

$$f(x) = \sqrt{2k}e^{-kx}(x \geqq 0); \quad f(x) = 0 \quad (x<0).$$

and the Markoff process is given by

$$x(t) = \int_{-\infty}^{t} \sqrt{2k}\, e^{-(t-x)} d[-\Psi_1(x)/\sqrt{\pi}]$$

i.e. the form established by K. Itō[18]

Let $F(\lambda)$ be the s.d.f. of a one-dimensional s.s.p. $x(t)$, and let it be absolutely continuous. Let us put

$$f(x) = \underset{A\to\infty}{\text{l.i.m.}}\, (2\pi)^{-1/2}\int_{-A}^{A} \sqrt{F'(\lambda)}\, c(\lambda)\, e^{i\lambda x} d\lambda,$$

and define an s.s.p. $\xi(t)$

$$\xi(t) = \int_{-\infty}^{\infty} f(x+t)\, d(\Psi_1/\sqrt{\pi}),$$

---

18) K. Itō [5].

where $c(\lambda)$ is an arbitary measurable function such that $\bar{c}(\lambda) = c(-\lambda)$, $|c(\lambda)| = 1$. Then $\xi(t)$ has the s.d.f. $F(\lambda)$.

The homogeneous normal differential process proves to be useful in the classical theory of Brownian motions. In the following, let us denote by $\psi(x)$ a real homogeneous normal differential process, which is, for example, obtained by taking the real part of $\Psi(x)$. Let its mean value be zero and the variance of the increment in (a, b) be $b-a$. A fundamental principle used in the following application is that the joint distribution of the variables $\int f(x)d\psi$ and $\int g(x)d\psi$ is normal with vanishing mean values and second order moments

$$\int_{-\infty}^{\infty} f^2(x)dx, \quad \int_{-\infty}^{\infty} g^2(x)dx, \quad \int_{-\infty}^{\infty} f(x)\, g(x)dx.$$

A fundamental equation which has been used by many authors is due to Langevin:

$$m\frac{du}{dt} = -fu + F(t), \quad u = dx/dt, \tag{5.13}$$

where u is the velocity, m the mass of the particles, f the frictional coefficient, $F(t)$ an irregular force exerted on the particles. An assumption frequently used by the authors to solve the equation is that[19] the correlation between $F(t_1)$ and $F(t_2)$ displays a sharp maximum when $t_1 = t_2$ and rapidly decreases with $|t_1 - t_2|^{-1}$. We shall treat the equation from a different standpoint, which consist in regarding $F(t)$ as a formal derivative of a homogeneous normal differential process[20]. Since the homogeneous normal differential process is nowhere differentiale with respect to its time parameter, this may seem absurd. But it will not be difficult to see that

$$\int_0^t F(s)\, ds$$

tends to a homogeneous normal differential process under the former assumption on the correlation, and hence we are justified to use our present assumption.

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[19] G. E. Uhlenbeck and L. S. Ornstein [1], L. S. Ornstein and W. R. van Wijk [1].

[20] I stated this interpretation of classical statistical problems in phisics in my paper [1]. Recently, in preparation of this paper, I have found that a more general method had been developped by Doob, in his paper referred to in 1), as an application of the theory of normal stationary processes in several-dimensional spaces.

Under the initial condition $u(0)=u_0$, (5.13) gives

$$u(t)=u_0e^{-\beta t}+e^{-\beta t}\int_0^t e^{\beta x}d\psi(x), \tag{5.14}$$

where $\beta=f/m$ and

$$\psi(t)=\int_0^t (F(x)/m)\,dx$$

is regarded as a homogeneous normal differential process. Let the variance of the increment of $\psi(t)$ in the interval (a, b) be $c(b-a)$, then the equipartition principle of energy gives $c=2\beta kT/m$. Hence the distribution of $u(t)$ is determined by the second order moment of the second term of (5.14). The distribution is given by

$$\left(\frac{m}{2\pi kT(1-e^{-2\beta t})}\right)^{1/2}\exp\left(\frac{-m}{2kT}\frac{(u-u_0e^{-\beta t})^2}{(1-e^{-2\beta t})}\right).$$

If we put (5.14) in the form

$$u(t)=\int_{-\infty}^t e^{-(t-x)}d\psi(x)+o(1),\ \text{as}\ t\to\infty,$$

we may conclude that, except for a vanishing term, $u(t)$ is a normal stationary Markoff process, and therefore strongly mixing. The distribution of the velocity finally becomes, after a mixing process, a Maxwellian distribution

$$\left(\frac{m}{2\pi kT}\right)^{1/2}\exp\left(-\frac{mu^2}{2kT}\right),$$

whatever the inital distribution may be.

The same method enables us to discuss the displacements of the particle, which is given, under the initial condition $x(0)=x_0$, by

$$s=x(t)-x_0=\frac{u_0}{\beta}(1-e^{-\beta t})-\frac{1}{\beta}e^{-\beta t}\int_0^t e^{-\beta x}d\psi(x)+\frac{1}{\beta}\psi(t).$$

We can easily obtain the distribution of $x(t)$ applying the method of calculating the moments of the variables $\int fd\psi$, $\int gd\psi$.

6. Spectral Analysis of s.s.p.'s. Our starting point is the formula (4.2). Given any even non-negative function $\varphi(\lambda)$ integrable with respect to $F(\lambda)$, we can construct from a one-dimensional s.s.p. $x(t)$ an s.s.p. with the s.d.f.

$$F^*(\lambda)=\int_{-\infty}^{\lambda}\varphi(s)\,dF(s).$$

For this purpose, let us choose a function $G(s)$ which involves some parameters and satisfies

$$\lim \int G(-s)e^{i\lambda s}ds = \sqrt{\varphi(\lambda)}, \tag{6.1}$$

where the limit is to be taken with respect to the parameters involved and the range of the integration may depend on these parameters. Then the s.s.p. given by

$$x^*(t) = \lim \int x(s)\, G(t-s)\, ds, \tag{6.2}$$

with similar specifications for the limit and integration as in (6.1), is an s.s.p. with the s.d.f. $F^*(\lambda)$. It was by this principle that s.s.p.'s $z_1(t)$ and $z_3(t)$ were obtained from $z(t)$. In the following, we shall state some examples for this method.

Let μ be a discontinuity point of $F(\lambda)$, and let

$$\begin{aligned} \varphi(\lambda) &= 1 \quad \text{for} \quad \lambda = \pm\mu, \\ &= 0 \quad \text{for} \quad \lambda \neq \pm\mu, \end{aligned}$$

then the s.d.f. $F^*(\lambda)$ is a step-function with jumps $\Delta_\mu F$ at $\lambda = \pm\mu$ and the correlation function $(e^{-i\mu\tau} + e^{i\mu\tau})\Delta_\mu F = 2\Delta_\mu F \cos \mu\tau$. In this case we may take, as a function $G(s)$ in (6.1), the function

$$G(s) = A^{-1}(e^{-i\mu s} + e^{i\mu s})$$

and as the range of the integration, the interval $(0, A)$. Substituting this form of $G(s)$ in (6.2) we have the s.s.p. $x^*(t) = A_\mu \cos \mu t + B_\mu \sin \mu t$, i.e. periodic term of $z_1(t)$.

On the other hand, if we make use of a $G(x)$ depending on a parameter $\varepsilon > 0$ which tends to 0

$$G(s) = \frac{2}{\pi} \cos \mu s \frac{1-\cos \varepsilon s}{\varepsilon s^2},$$

then

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} G(-s)\, e^{i\lambda s}\, ds = \begin{cases} 1 & (\lambda = \pm\mu), \\ 0 & (\lambda \neq \pm\mu), \end{cases}$$

whence this $G(s)$ gives also the same s.d.f. as before. According to (6.2) the corresponding s.s.p. may be given by

$$x^*(t) = \underset{\substack{a\to\infty \\ b\to-\infty \\ \varepsilon\to 0}}{\text{l.i.m.}} \int_a^b x(s) \frac{2}{\pi} \cos \mu(t-s) \frac{1-\cos \varepsilon(t-s)}{\varepsilon (t-s)^2}\, ds.$$

Since $M[|x(t)|]$ exists with probability 1, the last integral is written in the form

$$x^*(t) = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} x(t+s)\frac{2}{\pi}\cos \mu s \frac{\sin^2 \varepsilon s/2}{\varepsilon s^2/2}\, ds.$$

By an application of a special Tauberian theorem of Wiener[21], the right-hand side of the equation is equal to

$$\lim_{A\to\infty} 2A^{-1}\int_0^A x(t+s)\cos \mu s\, ds,$$

which is another expression for the periodic term.

Theorem 12. *For every* λ *and* t *there exist*

$$\xi(t, \lambda) = \underset{A\to\infty}{\text{l.i.m.}}\ \pi^{-1}\int_{-A}^{A} x(t+s)\frac{\sin \lambda s}{s}\, ds,\text{[22]} \tag{6.3}$$

$$\xi_c(t, \lambda) = \underset{A\to\infty}{\text{l.i.m.}}\ \pi^{-1}\int_{-A}^{A} x(t+s)\frac{\cos \lambda s - 1}{s}\, ds,$$

and

$$\xi(t, \lambda\pm 0) = \underset{\mu\to\lambda\pm 0}{\text{l.i.m.}}\ \xi(t, \mu), \quad \xi_c(t, \lambda\pm 0) = \underset{\mu\to\lambda\pm 0}{\text{l.i.m.}}\ \xi_c(t, \mu). \tag{6.4}$$

It holds that

$$\xi(t, +0) = \frac{1}{2}A_0, \quad \xi_c(t, +0) = 0, \tag{6.5}$$

$$\begin{aligned} \xi(t, \lambda+0) - \xi(t, \lambda-0) &= A_\lambda \cos \lambda t + B_\lambda \sin \lambda t, \\ \xi_c(t, \lambda+0) - \xi_c(t, \lambda-0) &= -B_\lambda \cos \lambda t + A_\lambda \sin \lambda t, \quad \lambda \neq 0. \end{aligned} \tag{6.6}$$

Let us put

$$x(t, \lambda) = \xi(t, \lambda+0) \quad (\lambda\neq 0); = 0 \quad (\lambda = 0),$$

and

$$x_c(t, \lambda) = \xi_c(t, \lambda+0) \quad (\lambda\neq 0); = 0 \quad (\lambda = 0).$$

Then

$$E\{(x(t, d) - x(t, c))(x(t, b) - x(t, a))\} = 0 \tag{6.7$_1$}$$

for every t *and* a, b, c, d, *such that* $0 \leqq a < b \leqq c < d < \infty$, *and similarly for* $x_c(t, \lambda)$;

$$\begin{aligned} E\{(x(t, \lambda') - x(t, \lambda))^2\} &= E\{(x_c(t, \lambda') - x_c(t, \lambda))^2\} \\ &= 2(F(\lambda'+0) - F(\lambda+0)), \quad 0 < \lambda < \lambda'; \end{aligned}$$

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21) Wiener [1].

22) l.i.m. denotes the mean-square limit of the random variable.

$$(6.7)_2 \qquad E(x^2(t,\lambda)) = F(\lambda+0)-F(-\lambda-0),\ E(x^2_c(t,\lambda)) = 2(F(\lambda+0)-F(+0)),\quad \lambda>0;$$

$$E(x(s,\lambda)x_c(t,\lambda')) = -2\int_0^\lambda \sin\mu(s-t)\,dF(\mu),\ \lambda<\lambda'.$$

There exist

$$(6.8) \qquad x(t,\infty) = \underset{\mu\to\infty}{\text{l.i.m.}}\ x(t,\mu),\ x_c(t) = x_c(t,\infty) = \underset{\mu\to\infty}{\text{l.i.m.}}\ x_c(t,\mu)$$

for every fixed t, *and*

$$(6.9) \qquad x(t,\infty) = x(t),$$

with probability 1.

If in particular, $x(t)$ is a normal s.s.p., $x(t,\lambda)$ and $x_c(t,\lambda)$ are differential normal processes depending on a parameter λ, $0\leq\lambda<\infty$, for every fixed t, and "l.i.m." in (6.4), (6.8) can be replaced, for every sequence of μ-values, by "lim".

Proof. First we observe that

$$\varphi_\lambda(\mu) = \lim_{A\to\infty}\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}\cos\mu u\,du = \begin{cases} 0 & (|\mu|>\lambda), \\ 1/2 & (\mu=\lambda), \\ 1 & (|\mu|<\lambda), \\ 1/2 & (\mu=-\lambda), \end{cases}$$

$$(6.10) \qquad \psi_\lambda(\mu) = \lim_{A\to\infty}\pi^{-1}\int_{-A}^{A}\frac{\cos\lambda u-1}{u}\sin\mu u\,du = \begin{cases} 0 & (|\mu|>\lambda \text{ or } \mu=0), \\ -1/2 & (\mu=\lambda), \\ -1 & (0<\mu<\lambda), \\ 1 & (-\lambda<\mu<\lambda), \\ 1/2 & (\mu=-\lambda). \end{cases}$$

Whence, by making use of (4.1) and (4.2) the limits in (6.3) exist.

Let us write ξ_1 and $\xi_{1,c}$, instead of ξ and ξ_c, in (6.3) with x replaced by x_1, then

$$\pi^{-1}\int_{-A}^{A}x_1(t+u)\frac{\sin\lambda u}{u}du = \frac{1}{2}A_0\cdot\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}du + \sum_\mu \pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}(A_\mu\cos\mu(u+t)+B_\mu\sin\mu(u+t))\,du,$$

whence

$$E(\pi^{-1}\int_{-A}^{A} x_1(t+u)\frac{\sin\lambda\mu}{u}du-\frac{1}{2}A_0\cdot\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}du$$

$$-\sum_{\mu\leqq\lambda}\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}(A_\mu\cos\mu(u+t)+B_\mu\sin\mu(u+t))\,du)^2$$

$$=\sum_{\mu>\lambda}E(A^2{}_\mu)(\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}\cos\mu(u+t)\,du)^2$$

$$+\sum_{\mu>\lambda}E(B_\mu^2)(\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}\sin\mu(u+t)\,du)^2$$

$$=\sum_{\mu>\lambda}2\Delta_\mu F\{(\pi^{-1}\int_{-A}^{A})^2+(\pi^{-1}\int_{-A}^{A})^2\}\to 0,\quad A\to\infty,$$

by (6.10). Hence, again by (6 10), we have

$$\xi_1(t,\lambda)=\underset{A\to\infty}{\text{l.i.m.}}\Big(\frac{1}{2}A_0\cdot\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}du$$

$$+\sum_{\mu\leqq\lambda}\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}(A_\mu\cos\mu(u+t)+B_\mu\sin\mu(u+t))\,du\Big) \tag{6.11}$$

$$=\frac{A_0}{2}+\sum_{\mu<\lambda}(A_\mu\cos\mu t+B_\mu\sin\mu t)+\frac{1}{2}(A_\lambda\cos\lambda t+B_\lambda\sin\lambda t).$$

In a similar way we obtain

$$\xi_{1,c}(t,\lambda)=\sum_{\mu<\lambda}(-B_\mu\cos\mu t+A_\mu\sin\mu t) \tag{6.12}$$

$$+\frac{1}{2}(B_\lambda\cos\lambda t+B_\lambda\sin\lambda t).$$

In view of (4.1), (4.2), (6.10), we have

$$E(\xi(s,\lambda)\xi(t,\lambda')=\int_{-\infty}^{\infty}e^{i\mu(s-t)}\,dF(\mu)\Big(\lim_{A\to\infty}\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}e^{i\mu u}du\Big)$$

$$\Big(\lim_{A\to\infty}\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda' u}{u}e^{-i\mu u}\,du\Big)=\int_{-\infty}^{\infty}e^{i\mu(s-t)}\varphi_\lambda(\mu)\,\varphi'_\lambda(\mu)\,dF(\mu), \tag{6.13}$$

whence

$$E\{(\xi(t,\lambda_1)-\xi(t,\lambda_2))^2\}=\int_{-\infty}^{\infty}\varphi_{\lambda_1}{}^2(\mu)dF(\mu)+\int_{-\infty}^{\infty}\varphi_{\lambda_2}{}^2(\mu)\,dF(\mu) \tag{6.14}$$

$$-2\int_{-\infty}^{\infty}\varphi_{\lambda_1}\varphi_{\lambda_2}\,dF(\mu),$$

and similarly for $\xi_c(t, \lambda)$:

$$(6.13') \qquad E(\xi_c(s, \lambda)\xi_c(t, \lambda')) = \int_{-\infty}^{\infty} e^{i\mu(s-t)} \psi_\lambda(\mu)\psi_{\lambda'}(\mu)\, dF(\mu),$$

$$(6.14') \qquad E\{(\xi(t, \lambda_1)-\xi(t, \lambda_2))^2\} = \int_{-\infty}^{\infty} \psi_{\lambda_1}{}^2(\mu)\, dF(\mu) + \int_{-\infty}^{\infty} \psi_{\lambda_2}{}^2(\mu)\, dF(\mu) - 2\int_{-\infty}^{\infty} \psi_{\lambda_1}(\mu)\psi_{\lambda_2}(\mu)\, dF(\mu).$$

As $\lambda_1, \lambda_2 \to \lambda \pm 0$ the right-hand sides of (6.14), (6.14′) tends to 0, so that the limits in (6.4) exist. When λ is a point of continuity, the right-hand members of (6.14), (6.14′) tend to 0 as $\lambda_1 \to \lambda + 0$, $\lambda_2 \to \lambda - 0$. This implies that there exist

$$(6.15) \qquad \xi(t, \lambda \pm 0) = \xi(t, \lambda), \quad \xi_c(t, \lambda \pm 0) = \xi_c(t, \lambda),$$

with probability 1, for every continuity point of $F(\lambda)$. Now let $\xi'(t, \lambda)$ and $\xi'_c(t, \lambda)$ be stochastic processes defined by (6.3) with $x(t)$ replaced by $x'(t)$ then, since the s.d.f. $F'(\lambda)$ of $x'(t)$ is continuous, (6.15) gives

$$\xi(t, \lambda+0) - \xi(t, \lambda-0) = \xi_1(t, \lambda+0) - \xi_1(t, \lambda-0) + \xi'(t, \lambda+0) - \xi'(t, \lambda-0) = \xi_1(t, \lambda+0) - \xi_1(t, \lambda-0),$$

and similarly

$$\xi_c(t, \lambda+0) - \xi_c(t, \lambda-0) = \xi_{1,c}(t, \lambda+0) - \xi_{1,c}(t, \lambda-0).$$

The last two relations combined with (6.11), (6.12) give (6.5) and (6.6).

The existence of the limits in (6.8) is obvious by making $\lambda_1 \to \infty$, $\lambda_2 \to \infty$ in (6.14), (6.14). (6.9) follows by making use of the first part of (6.10).

Since

$$\lim_{\lambda \to \mu+0} \varphi_\lambda(x) = \begin{cases} 1 & (|x| \leqq \mu), \\ 0 & (|x| > \mu), \end{cases}$$

and

$$\lim_{\lambda \to \mu+0} \psi_\lambda(x) = \begin{cases} -1 & (0 < x \leqq \mu), \\ 0 & (x = 0 \text{ or } |x| > \mu), \\ 1 & (-\mu \leqq x < 0), \end{cases}$$

letting $\lambda \to \mu + 0$, $\lambda' \to \mu' + 0$ in (6.13), (6.13′)

$$E(x(s, \mu)x(t, \mu')) = \int_{-\mu}^{\mu} \cos \mu(s-t)\, dF(\mu), \quad \mu \leqq \mu';$$

$$(6.16) \qquad E(x_c(s, \mu)x_c(t, \mu')) = 2\int_{+0}^{\mu} \cos (s-t)\, dF(\mu), \quad \mu \leqq \mu'.$$

Hence

$$E\{(x(s,\lambda')-x(s,\lambda))(x(t,\lambda')-x(t,\lambda))\} = 2\int_{\lambda+0}^{\lambda'}\cos\mu(s-t)\,dF(\mu).$$

(6.17) $$E\{(x_c(s,\lambda')-x_c(s,\lambda))(x_c(t,\lambda')-x_c(t,\lambda))\} = 2\int_{\lambda+0}^{\lambda'}\cos\mu(s-t)\,dF(\mu),\quad 0<\lambda\leqq\lambda';$$

and

$$E\{(x(t,\lambda')-x(t,\lambda))^2\} = E\{(x_c(t,\lambda')-x_c(t,\lambda))^2\} = 2\,(F(\lambda'+0)-F(\lambda+0)),\ 0<\lambda\leqq\lambda'.$$

$$E(x^2(t,\mu)) = F(\mu+0)-F(-\mu-0),\ E(x_c^2(t,\mu)) = 2\,(F(\mu+0)-F(+0)).$$

again by (6.16) we have

(6.18) $$E\{(x(s,\lambda_2)-x(s,\lambda_1))(x(t,\lambda_4)-x(t,\lambda_3))\} = E\{(x_c(s,\lambda_2)-x_c(s,\lambda_1))\,(x_c(t,\lambda_4)-x_c(t,\lambda_3))\} = 0,\quad \lambda_1<\lambda_2\leqq\lambda_3<\lambda_4,$$

whence if, in particular, the s.s.p. $x(t)$ is normal, $x(t,\lambda)$ and $x_c(t,\lambda)$ are normal differential processes depending on a parameter λ in $(0,\infty)$, for every fixed t.

By (4.1) and (4.2)

(6.19) $$E(x(s,\lambda)\,x_c(t,\lambda')) = \int_{-\infty}^{\infty} e^{i\mu(s-t)}\Big(\lim_{A\to\infty}\pi^{-1}\int_{-A}^{A}\frac{\sin\lambda u}{u}\cos\mu u\,du\Big)\cdot\Big(\lim_{A\to\infty}-i\pi^{-1}\int_{-A}^{A}\frac{\cos\lambda v-1}{v}\sin\mu v\,dv\Big)\,dF(\mu) = -2\int_0^{\lambda}\sin\mu(-t)\,dF(\mu),\quad \lambda\leqq\lambda',$$

whence $x(t,\lambda)$ and $x_c(t,\lambda')$ are uncorrelated and

$$E(x(s)x_c(t)) = -2\int_0^{\infty}\sin\lambda(s-t)\,dF(\lambda).$$

(6.19) gives

(6.20) $$E\{x(s,\lambda')-x(s,\lambda))(x_c(t,\lambda')-x_c(t,\lambda))\} = -2\int_{\lambda+0}^{\lambda'}\sin\mu(s-t)\,dF(\mu),\quad \lambda\leqq\lambda'.$$

Thus the theorem has been completely proved.

Thus prepared, we shall proceed to the spectral analysis of a one-dimensional $x(t)$ process.

Let $f(\lambda)$ $(0\leqq\lambda<\infty)$ be a bounded continuous function, then the series

$$S_{\varDelta} = \sum_{i=1}^{\infty} f(\lambda'_i)(x(t, \lambda_i) - x(t, \lambda_{i-1})), \quad \lambda_{i-1} \leqq \lambda'_i \leqq \lambda_i$$

converges in mean for every division $\varDelta$ of $(-\infty, \infty)$:

$$0 = \lambda_0 < \lambda_1 < \cdots\cdots, \quad \delta = \max_i (\lambda_i - \lambda_{i-1}),$$

and, as $\delta \to 0$, $S_{\varDelta}$ converges in mean to a definite random variable independent of the division $\varDelta$. This limit is called the integral with respect to $x(t, \lambda)$ and we shall use the ordinary notation

$$\underset{\delta \to 0}{\text{l.i.m.}}\ S_{\varDelta} = \int_0^\infty f(\lambda) d_\lambda x(t, \lambda).$$

We shall establish fundamental formulas for such an integral. (6.17), (6.18) give us

$$E\left(\int_0^\infty f(\lambda) d_\lambda x(s, \lambda) \cdot \int_0^\infty g(\lambda) d_\lambda x(t, \lambda)\right)$$

$$= \lim_{\delta \to 0} E\left[\left\{\sum_{i=0}^{\infty} f(\lambda_i)(x(s, \lambda_i) - x(s, \lambda_{i-1}))\right\} \cdot \left\{\sum_{i=0}^{\infty} g(\lambda_i)(x(t, \lambda_i) - x(t, \lambda_{i-1}))\right\}\right]$$

$$= \lim_{\delta \to 0}\left(f(\lambda_1)\, g(\lambda_1)\int_{-\lambda_1}^{\lambda_1} \cos \mu(s-t)\, dF(\mu) + \sum_{i=2}^{\infty} f(\lambda_i)\, g(\lambda_i) \cdot 2\int_{\lambda_{i-1}+0}^{\lambda_i} \cos \mu(s-t)\, dF(\mu)\right)$$

$$= \int_{-\infty}^{\infty} f(\lambda) g(\lambda) \cos \lambda(s-t)\, dF(\lambda),$$

where $f(-\lambda) = f(\lambda)$, $g(-\lambda) = g(\lambda)$. Similarly

$$E\left(\int_0^\infty f(\lambda)\, d_\lambda x_c(s, \lambda) \cdot \int_0^\infty g(\lambda) d_\lambda x_c(t, \lambda)\right) = 2\int_{+0}^{\infty} f(\lambda)\, g(\lambda) \cos \lambda(s-t)\, dF(\lambda),$$

(6.21)′
$$E\left(\int_0^\infty f(\lambda) d_\lambda x(s, \lambda) \cdot \int_0^\infty g(\lambda) d_\lambda x_c(t, \lambda)\right) = -2\int_0^\infty f(\lambda) g(\lambda) \sin \lambda(s-t)\, dF(\lambda).$$

Using these formulas we can prove the following theorem.

Theorem 13. *Any one-dimensional s.s.p. $x(t)$ is expressed in the form*

$$(6.22)\qquad x(t)=\int_0^\infty \cos\lambda t\, d_\lambda x(0,\lambda)-\int_0^\infty \sin\lambda t\, d_\lambda x_c(0,\lambda)$$
$$=\int_{-\infty}^\infty e^{i\lambda t}\, d_\lambda S(0,\lambda)$$

and it holds that

$$(6.23)\qquad x(s+t)=\int_0^\infty \cos\lambda s\, d_\lambda x(t,\lambda)-\int_0^\infty \sin\lambda s\, d_\lambda x_c(t,\lambda)$$
$$=\int_{-\infty}^\infty e^{i\lambda s}\, d_\lambda S(t,\lambda),$$

where

$$(6.24)\qquad \begin{gathered} S(t,\lambda)=\underset{\substack{A\to\infty\\ \delta\to 0}}{\text{l.i.m.}}\ (2\pi)^{-1}\int_{-A}^{A} x(t+u)\frac{e^{-iu(\lambda+\delta)}-1}{-iu}\,du, \\ \lambda\neq 0;\quad =0,\quad \lambda=0; \\ 2S(t,\lambda)=x(t,\lambda)+ix_c(t,\lambda). \end{gathered}$$

Proof. In view of

$$x(s+t,\infty)=x(s+t),\ i.e.\ x(s+t)=\int_0^\infty d_\lambda x(s+t,\lambda),$$

and (6.21), (6.21), we obtain

$$\begin{aligned} &E\left\{\left(x(s+t)-\int_0^\infty \cos\lambda s\, d_\lambda x(t,\lambda)+\int_0^\infty \sin\lambda s\, d_\lambda x_c(t,\lambda)\right)^2\right\} \\ &=F(\infty)-F(-\infty)+E\left\{\left(\int_0^\infty \cos\lambda s\, d_\lambda x(t,\lambda)\right)^2\right\} \\ &+E\left\{\left(\int_0^\infty \sin\lambda s\, d_\lambda x_c(t,\lambda)\right)^2\right\}-2E\left\{\left(\int_0^\infty \cos\lambda s\, d_\lambda x(t,\lambda)\right)\right. \\ &\cdot\left.\left(\int_0^\infty \sin\lambda s\, d_\lambda x_c(t,\lambda)\right)\right\}-2E\left((x(s+t)\int_0^\infty \cos\lambda s\, d_\lambda x(t,\lambda)\right. \\ &\left.-x(s+t)\int_0^\infty \sin\lambda s\, d_\lambda x_c(t,\lambda)\right)=F(\infty)-F(-\infty) \\ &-\int_{-\infty}^\infty(\cos^2\lambda s+\sin^2\lambda s)\,dF(\lambda)=0, \end{aligned}$$

which provès (6.22) and (6.23). (6.24) is an obvious consequence from (6.23).

Similarly $x_c(t)$ is written in the form

$$x_c(t)=\int_0^\infty \sin\lambda t\, d_\lambda x(0,\lambda)+\int_0^\infty \cos\lambda t\, d_\lambda x_c(0,\lambda).$$

If we denote by $x'_c(t)$ the s.s.p. defined by the second formula in (6.3) with $x(t)$ replaced by $x'(t)$, then obviously we obtain $x_c(t)=x_{1c}(t)+x'_c(t)$. Sinĉe, as is obvious from (6.16), the s.d.f. of $x'_c(t)$, $F'(\lambda)$, is continuous, the B_2—process of $x_c(t)$ is the same as what we may obtain from $x_{1c}(t)$. But, from (6.12) we obtain

$$x_{1c}(t) = \lim_{\lambda\to\infty} x_c(t, \lambda) = \sum(A_\lambda \sin \lambda t - B_\lambda \cos \lambda t),$$

which is just the form conjugate to the series defining $x(t)$ and itself a B_2—process. Hence the B_2—process of $x_c(t)$ is the process $x_{1c}(t)$.

It is to be noted that if λ_1, and λ_2 are continuity points of $F(\lambda)$, then (6.24) gives

$$S(0, \lambda_2) - S(0, \lambda_1) = \underset{A\to\infty}{\text{l.i.m.}}\,(2\pi)^{-1}\int_{-A}^{A} \frac{e^{-i\lambda_2 u} - e^{-i\lambda_1 u}}{-iu}du \cdot \int_{-\infty}^{\infty} e^{i\lambda u}\, dS(0, \lambda),$$

which is similar to the well-known inversion formula for Fourier-Stieltjes transforms.

Similar arguments applied to an s.s.p., $x(n)$, $n = 0, \pm 1, \pm 2, \dots$, depending on an integral-valued parameter enable us to write it in the form

$$x(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, dS(\lambda),\ n = 0,\ \pm 1,\ \dots,$$

(6.22′)
$$S(-\lambda) = \overline{S(\lambda)},\ E(|S(b)-S(a)|^2) = (2\pi)^{-1}(F(b)-F(a)),\quad 0 \leq a < b \leqq \pi.$$

As an immediate consequence of the spectral analysis of an s.s.p. $x(t)$, it follows that given any $\varepsilon > 0$ and $\delta > 0$ we can find superposed harmonics $T(t) = \sum c_\nu e^{i\lambda_\nu t}$ such that

$$P(|x(t) - T(t)| > S) < \varepsilon$$

for every value of t.

It is well-known that if the s.d.f. $F(\lambda)$ satisfies the condition

$$\int_{-\infty}^{\infty} \lambda^2\, dF(\lambda) < \infty,$$

then $x(t)$ is absolutely continuous, with probability 1. The differentiated process $x'(t)$ is an s.s.p. and is given by

$$x'(t) = \int_{-\infty}^{\infty} i\lambda\, e^{i\lambda t}\, dS(\lambda).$$

It's correlation function is

$$E(x'(t+\tau)x'(t)) = \int_{-\infty}^{\infty} \lambda^2 \cos \lambda t\, dF(\lambda) = -f''(\tau).$$

The following theorem provides us with a sufficient condition for an analiticity of an s.s.p.

Theorem 14. *If the s.d.f. of an s.s.p., $x(t)$, satisfies*

$$\int_0^\infty e^{\varepsilon|\lambda|}\, dF(\lambda) < \infty \tag{6.25}$$

for some $\varepsilon > 0$, then there exists a random function $\xi(z)$, $z = t + iy$, regular in the strip $|y| < \varepsilon/2$ and $\xi(z) = x(t)$ for $y = 0$, with probabilty 1.

Proof. Consider the random function

$$\xi_0(z) = \int_{-\infty}^\infty e^{iz\lambda}\, dS(0, \lambda), \quad |z| < \varepsilon/2,$$

the existence of the integral being assured by (6.25). By (6.25) we obtain

$$\int_0^\infty ((\varepsilon\lambda)^{2\nu}/(2\nu)\,!)\, dF(\lambda) \leqq \int_0^\infty e^{\varepsilon|\lambda|}\, dF(\lambda),$$

whence

$$\int_0^\infty \lambda^{2\nu}\, dF(\lambda) = O(\varepsilon^{-2\nu}\cdot(2\nu)\,!),$$

$$P\left\{\left|\frac{1}{\nu\,!}\int_{-\infty}^\infty \lambda^\nu\, dS(\lambda)\right| > \left(\frac{2}{\varepsilon'}\right)^\nu\right\} = O\left(\left(\frac{\varepsilon'}{2}\right)^{2\nu}\cdot\varepsilon^{-2\nu}\cdot\frac{(2\nu)\,!}{(\nu\,!)^2}\right) = o(\varepsilon'/\varepsilon)^{2\nu},$$

Hence, for any $\eta > 0$ we have

$$P\left(\sum_{\nu=n}^\infty \max_{|z|\leqq\rho}\left|\frac{(iz)^\nu}{\nu\,!}\int_{-\infty}^\infty \lambda^\nu dS(0, \lambda)\right| < \sum_{\nu=n}^\infty\left(\frac{2\rho}{\varepsilon'}\right)^\nu\right) > 1$$

$$- o\left(\sum_{\nu=n}^\infty\left(\frac{\varepsilon'}{\varepsilon}\right)^{2\nu}\right) > 1 - \eta, \quad 2\rho < \varepsilon' < \varepsilon,$$

where $n > N(\eta)$, $N(\eta)$ a sufficiently large integer. Hence

$$\sum_{\nu=0}^\infty \frac{(iz)^\nu}{\nu\,!}\int_{-\infty}^\infty \lambda^\nu\, dS(0, \lambda) \tag{6.26}$$

converges uniformly for $|z| < \varepsilon'/2$; whence $\xi_0(z)$ is a regular function in $|z| < \varepsilon/2$ with probability 1. Let a be a real number in the circle $|z| < \varepsilon/2$ and consider the function

$$\xi_a(z) = \int_{-\infty}^\infty e^{i(z-a)\lambda}\, dS(a, \lambda).$$

Then $\xi_a(z)$ is regular in the circle $|z - a| < \varepsilon/2$ with probability 1. According to (6.23), $\xi_0(z) = \xi_a(z)$ in a t-interval common to the circles $|z-a|<\varepsilon/2$, $|z|<\varepsilon/2$, i.e. $\xi_a(z)$ is an analytic continuation of $\xi_0(z)$. The set of all elements $\xi_a(z)$ with rational a defines an analytic function $\xi(z)$ regular in the strip $|y|<\varepsilon/2$. $\xi(z)$ obviously satisfies the required conditions.

7. Integral representation of normal processes. In §5 we explained that certain integrals with respect to a homogeneous normal differential process should lead us to interesting classes of s.s.p.'s. In this section we shall deal with a converse problem. Let $x(t)$ be a normal stochastic process, with the vanishing mean values and finite second order moments, satisfying

$$E\{(x(t+h)-x(t))^2\}\to 0, \quad h\to 0. \tag{7.1}$$

We have the following theorem.

Theorem 15. *Any normal process satisfying (7.1) is expressed in the form*

$$x(t)=\pi^{-1}\int_{-\pi}^{\pi} K(t,\ x)\ d_x\psi(x,\ \omega) \tag{7.2}$$

for every value of t, with probabilty 1, *where* $K(t, x)\varepsilon L^2(-\pi, \pi)$ *for every value of t and satisfies*

$$\int_{-\pi}^{\pi}\{K(t+h,\ x)-K(t,\ x)\}^2\,dx\to 0, \quad h\to 0,$$

and $\psi(x)(-\pi\leqq x\leqq\pi)$ *is a homogeneous normal differential process.*

Proof. First we remember that $x(t, \omega)$ can be regarded as a curve depending on a parameter t. For if $\{r_n\}$ is the set of all rational numbers, the set $\{x(r_n, \omega)\}$ determines a closed linear manifold $\mathfrak{M}$, i.e. a Hilbert space in which $x(t, \omega)$ is imbeded, since $E\{(x(t)-x(r_n))^2\}\to 0$, as $r_n\to t$. Consider a complete orthogonal system $(a_0(\omega), a_1(\omega), b_1(\omega), \ldots)$ of $\mathfrak{M}$ such that $E(a_0^2)=2\pi$, $E(a_n^2)=E(b_n^2)=\pi$, $n=1, 2, \ldots$. Let us expand $x(t, \omega)$ by means of the orthogonal system:

$$\begin{gathered} x(t,\ \omega)\sim\frac{c_0(t)}{2}\,a_0(\omega)+\sum_{n=1}^{\infty}(c_n(t)\,a_n(\omega)+d_n(t)\,b_n(\omega)), \\ \left.\begin{matrix} c_n(t) \\ d_n(t) \end{matrix}\right\}=\pi^{-1}\left\{\begin{matrix} E(x(t,\ \omega)\,a_n(\omega)) \\ E(x(t,\ \omega)\,b_n(\omega)). \end{matrix}\right. \end{gathered} \tag{7.3}$$

Let us consider an isometric transformation between $\mathfrak{M}$ and $L^2(-\pi, \pi)$ induced by the correspondence between two othogonal sets $(a_0(\omega); a_1(\omega), b_1(\omega), \ldots)$ and $(1, \cos x, \sin x, \ldots)$, and let $K(t, x)$ be the image of $x(t, \omega)$, then

$$K(t, x) \sim \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} (c_n(t) \cos nx + d_n(t) \sin nx). \tag{7.4}$$

Since $(a_0, a_1, b_1, \ldots)$ is a set of normally distributed independent variables, it is easily shown that the formal series

$$\psi(x, \omega) \sim \frac{x}{2} a_0(\omega) + \sum_{1}^{\infty} \frac{-b_n(\omega) \cos nx + a_n(\omega) \sin nx}{n} \tag{7.5}$$

defines a homogeneous normal differential process[23)]

$$\begin{aligned} &E\{(\psi(b)-\psi(a))(\psi(d)-\psi(c))\} \\ &= \frac{\pi}{2}(b-a)(d-c) + \pi \sum_{n=1}^{\infty} n^{-2}(\cos nb - \cos na)(\cos nd - \cos nc) \\ &\qquad + \pi \sum_{n=1}^{\infty} n^{-2}(\sin nb - \sin na)(\sin nd - \sin nc) \\ &= \begin{cases} 0 & (\pi \leqq a < b \leqq c < d \leqq \pi), \\ \pi^2(b-a) & (a = c, \quad b = d). \end{cases} \end{aligned}$$

The relation

$$E\left(\pi^{-1}\int_{-\pi}^{\pi} f(x)\, d\psi \cdot \pi^{-1}\int_{-\pi}^{\pi} g(x)\, d\psi\right) = \int_{-\pi}^{\pi} f \cdot g\, dx$$

enables us to determine the Fourier-Stieltjes coefficients with respect to $\psi(x)$. In fact

$$\begin{aligned} &E\left(\pi^{-1}\int_{-\pi}^{\pi} \cos nx\, d\psi(x) - a_n\right)^2 \\ &= \int_{-\pi}^{\pi} \cos^2 nx\, dx - 2\pi^{-1} E\left(a_n(\omega)\int_{-\pi}^{\pi} \cos nx \cdot d\psi(x, \omega)\right) + E(a_n^2), \end{aligned} \tag{7.6}$$

and

$$E\left(a_n \cdot \int_{-\pi}^{\pi} \cos nx\, d\psi(x)\right) = \lim_{\delta \to 0} \sum_{j=1}^{\nu} E(a_n \cos nx_j\, \Delta_j x),$$

where $-\pi = x_0 < x_1 < \ldots < x_\nu = \pi$, $\delta = \max_j (x_j - x_{j-1})$, $\Delta_j \psi = \psi(x_j) - \psi(x_{j-1})$. Since from (7.5) we get

$$E(a_n \Delta_j \psi) = \pi n^{-1}(\sin nx_j - \sin nx_{j-1}),$$

it holds that

$$E\left(a_n \int_{-\pi}^{\pi} \cos nx\, d\psi\right) = \pi n^{-1} \int_{-\pi}^{\pi} \cos nx\, d(\sin nx) = \pi^2.$$

Hence the right-hand side of (7.6) vanishes. Hence

23) Paley and Wiener [1], p. 147.

$$\pi^{-1}\int_{-\pi}^{\pi}\cos nx\,d\psi(x)=a_n,\quad \pi^{-1}\int_{-\pi}^{\pi}\sin nx\,d\psi(x)=b_n. \tag{7.7}$$

Now

$$K(t,x)=\underset{\substack{(x)\\N\to\infty}}{\text{l.i.m.}}\left(\frac{c_0(t)}{2}+\sum_{n=1}^{N}(c_n(t)\cos nx+d_n(t)\sin nx)\right),$$

whence by (7.7)

$$\pi^{-1}\int_{-\pi}^{\pi}K(t,x)\,d\psi(x)=\underset{\substack{(x)\\N\to\infty}}{\text{l.i.m.}}\ \pi^{-1}\int_{-\pi}^{\pi}\Big[\frac{c_0(t)}{2}$$
$$+\sum_{n=1}^{N}(c_n(t)\cos nx+d_n(t)\sin nx)\Big]d\psi(x)$$
$$=\underset{\substack{(\omega)\\N\to\infty}}{\text{l.i.m.}}\left[\frac{c_0(t)}{2}a_0(\omega)+\sum_{n=1}^{N}(c_n(t)a_n(\omega)+d_n(t)b_n(\omega))\right]=x(t,\omega).$$

If, in particular, $x(t)$ is a normal s.s.p. we can prove the theorem.

Theorem 16. *If $x(t)$ is a normal s.s.p. with the vanishing first moment and finite second moment satisfying (7.1), then we can write $K(t,x)$ as follows*

$$K(t,x)=T^tF(x),\quad K(0,x)=F(x),$$

where T^t is a group of unitary trnsformations in $L^2(-\pi,\pi)$, $T^tT^s=T^{t+s}$ for every s and t.

Proof. First we shall introduce a group of unitary transformations S^t in $\mathfrak{M}$. By $\sum_n=\sum_n\xi\cdot x(\tau)$ we denote a finite sum of $x(\tau_{n,\nu})$, $\sum_\nu \xi_{n,\nu}\,x(\tau_{n,\nu})$, depending on an integral-valued parameter n. If $\|f-\sum_n\|\to 0$, $n\to\infty$, $f\varepsilon\mathfrak{M}$, for a set of sums $\{\sum_n\}$, then $\|\sum_m\xi\cdot x(\tau+t)-\sum_n\xi\cdot x(\tau+t)\|=\|\sum_m\xi\cdot x(\tau)-\sum_n\xi\cdot x(\tau)\|\to 0$ $(m, n\to\infty)$. Hence we can write $S^tf=\lim_{n\to\infty}\sum_n\xi\cdot x(\tau+t)$. S^tf is independent of the choice of approximating sums $\sum_n$. In fact, for any other approximating sequence of sums $\sum'_n$ to f, a rearrangement of $\sum_n$ and $\sum'_n$ yields an approximating sum to f. Thus a one-parameter family of transformations, $S^t(-\infty<t<\infty)$, has been defined in $\mathfrak{M}$, such that $S^tf\varepsilon\mathfrak{M}$ for every $f\varepsilon\mathfrak{M}\,(-\infty<t<\infty)$, i.e. the domain and range of the transformations S^t are the same $\mathfrak{M}$. Further, if $\sum_n$ and $\sum'_n$ are approximating sums of f and g of $\mathfrak{M}$, the stationarity of the process $x(t)$ yields

$$(S^t f,\ S^t g) = \lim_{n\to\infty} E\{(\textstyle\sum_n \xi \cdot x(\tau+t))(\sum_n' \xi' \cdot x(\tau'+t))\}$$
$$= \lim_{n\to\infty} E\{(\textstyle\sum_n \xi \cdot x(\tau))(\sum_n' \xi' \cdot x(\tau'))\} = (f,\ g),$$

i.e. S^t is unitary. The relation $S^s S^t = S^{s+t}$ is an immediate consequence from the definition of S^t. To prove the continuity of S^t, let us expand f into the series

$$f \sim \frac{c_0}{2} a_0 + \sum_{n=1}^{\infty} (c_n a_n + d_n b_n).$$

Since the complete orthogonal set $(a_0, a_1, \ldots)$ can be regarded as a set of linear forms $\sum_\nu$ it holds that $\| S^h a_j - a_j \| \to 0$ etc. $(h \to 0)$, whence for any $\varepsilon > 0$

$$\|S^h f - f\| = \| \frac{c_0}{2} S^h a_0 + \sum_{n=1}^{N} (c_u S^h a_n + d_n S^h b_n) - \frac{c_0}{2} a_0$$
$$- \sum_{n=1}^{N} (c_n a_n + d_n b_n)\| + 2 \|\sum_{N+1}^{\infty} (c_n a_n + d_n b_n)\| < \varepsilon,$$

if N is sufficiently large and $|h|$ is sufficiently small. This means that S^t is continuous. The required transformations T^t in $L^2(-\pi, \pi)$ are obtained by the correspondence between $\mathfrak{M}$ and L^2. T^t now satisfies the required conditions. Hence we have proved the theorem.

Another expression for a normal s.s.p. is obtained from Theorem 10 applying the transformation due to Wiener:

$$x(t) = \int_{-\infty}^{\infty} T^t f(x) \cdot d\Psi(x).$$

The above method leads us to a proof of the Khintchine-Kolmogoroff[24] theorem on the existence of the s.s.p. with an assigned s.d.f.

Theorem 17. *Let $F(\lambda)$ be a non-decreasing bounded function, then there exists a normal s.s.p. with the s.d.f. $F(\lambda)$.*[25]

Proof. Let $\{\varphi_1(\lambda),\ \varphi_2(\lambda),\ \ldots\}$ be a complete orthomormal set in the space L^2 of all $f(\lambda)$ such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\, dF(\lambda) < \infty$$

and let $\{\alpha_1,\ \alpha_2,\ \ldots\}$ be a set of normally distributed independent

24) Doob [1].

25) A different proof has been given by K. Itō [1].

variables with the mean value 0 and variance 1. For every $f \varepsilon L^2$, let us consider a corresponding random variable $\psi(f)$ defined by

$$\psi(f) \sim \sum c_n d_n, \quad c_n = \int_{-\infty}^{\infty} f(\lambda)\, \varphi_n(\lambda)\, dF(\lambda).$$

In fact, since $\sum c^2_n = \|f\|^2$, the formal series converges, with probability 1, to a normal random variable. We have

$$E(\psi(f)\psi(g)) = \sum c_n d_n = (f, g). \tag{7.8}$$

Taking

$$f(\lambda) = 1, \qquad -\infty < \lambda \leqq t;$$
$$= 0, \qquad \lambda > t,$$

we obtain a random variable

$$\psi(t) = \sum a_n \int_{-\infty}^{t} \varphi_n \, dF(\lambda), \tag{7.9}$$

which is, as is easily shown by (7.8), a normal differential process such that $E\{(\psi(b)-\psi(a))^2\} = F(b)-F(a)$. Next we shall define another differential process $\psi^*(t)$ by means of (7.9) with α's replaced by β's which are independent of α's, and write

$$x(t) = \int_{-\infty}^{\infty} \cos \lambda t \, d\psi(\lambda) + \int_{-\infty}^{\infty} \sin \lambda t \, d\psi^*(\lambda),$$

then obviously $x(t)$ is a normal process satisfying

$$E\{(x(t)x(t+\tau))\} = \int_{-\infty}^{\infty} \cos \lambda t \cos \lambda(t+\tau)\, dF(\lambda)$$
$$+ \int_{-\infty}^{\infty} \sin \lambda t \sin \lambda(t+\tau)\, dF(\lambda) = \int_{-\infty}^{\infty} \cos \lambda\tau \, dF(\lambda).$$

Hence $x(t)$ is a normal s.s.p. and satisfies the required conditions.

8. **Stochastic interpolation.** In may applications it is sometimes necessary to predict a quantity $x(t)$ changing with time, on the basis of a succession of observations $x(t_1), x(t_2), \ldots, x(t_n)$. A method frequently used for such a prediction consists in finding a sum $S_n = S(t_1, \ldots, t_n; t) = \sum_j a_{j^n} x(t_j)$ which minimizes

$$E((x(t) - \sum a_j x(t_j))^2). \tag{8.1}$$

Let us denote the minimized value by $\sigma_n{}^2 = \sigma^2(t_1, \ldots, t_u; t)$. Suppose that the mean value of $x(t)$ is 0 and the variance 1, then the minizing condition is given by

$$\sum_{j=1}^{n} a_j \mu_{ij} = \mu_i (i = 1, 2, \ldots, n), \tag{8.2}$$
$$\mu_i = E(x(t)x(t_i)), \; \mu_{ij} = E(x(t_i)x(t_j)).$$

If, in particular, the process $x(t)$ is normal, the conditional distribution of $x(t)$ for given $x(t_1), \ldots, x(t_n)$ is also normal with the mean value S_n and the standard deviation σ_n^2. Kolmogoroff[26] has given an important result for the greatest lower bound of σ_n in the case of a stationary $x(t)$. In this section we shall prove the Kolmogoroff theorem and discuss some related problems.

Suppose that S_n depends only on the variables corresponding to the t-values nearest to t. Then we have three cases:

(a) backward extrapolation, $t<t_1<\ldots<t_n$;
(b) forward extrapolation, $t_1<\ldots<t_n<t$;
(c) interpolation, $t_{i-1}<t<t_i$.

In each of these cases, (8.2) is to be satisfied by the respective set of values

(i) $a_2 = a_3 = \ldots = a_n = 0$;
(ii) $a_1 = a_2 = \ldots = a_{n-1} = 0$;
(iii) $a_j = 0 \ (j \neq i-1, i)$.

Let $f(\tau) = E\{x(t)x(t+\tau)\}$, then (8.2) for (i), (ii) gives

$$f(s)f(t) = f(s+t)(s \geqq 0,\ t \geqq 0);$$

whence, in view of $|f(s)| \leqq 1$ and $f(-s) = f(s)$, we obtain

$$f(s) = e^{-k|s|}, \quad k \geqq 0. \tag{8.3}$$

(8.2) for (iii) gives

$$\begin{vmatrix} f(s) & 1 & f(s+t) \\ f(t) & f(s+t) & 1 \\ f(t+u) & f(s+t+u) & f(u) \end{vmatrix} = 0 \quad (s,\ t,\ u \geqq 0). \tag{8.4}$$

If there are any points of discontinuity of $F(\lambda)$, say λ_ν, forming the average $\underset{u}{M}$, with respect to u, of (8.4) multiplied by $e^{-i\lambda_\nu u}$, we obtain

$$f(s)(f(s+t)-e^{i\lambda_\nu(s+t)})-f(t)(1-f(s+t)e^{i\lambda_\nu(s+t)}) \\ +e^{i\lambda_\nu t}(1-f^2(s+t)) = 0.$$

The operation $\underset{t}{M}$ applied to the above equation multiplied by $e^{-i\lambda_\nu t}$ then gives

$$f(s)(\delta_\nu e^{i\lambda_\nu s}-e^{i\lambda_\nu s})-(\delta_\nu - e^{i\lambda_\nu s}\sum_\nu \delta_\nu^2 e^{i\lambda_\nu s})+(1-\sum_\nu \delta_\nu^2) = 0,$$

where $\delta_\nu = F(\lambda_\nu+0)-F(\lambda_\nu-0)$. Applying the operation $\underset{s}{M}$ to this

[26] Kolmogoroff [1].

relation multiplied by $e^{-i(\lambda_\nu+\lambda'_\nu)s}$, where λ'_ν is any discontinuity point of $F(\lambda)$ other than λ_ν, it follows that

$$\delta_\nu+\delta_\nu'=1$$

with δ'_ν, the jump of $F(\lambda)$ at $\lambda=\lambda_\nu'$. This is impossible, since the existence of jump δ_ν implies at the same time a jump at $\lambda=-\lambda_\nu$. Thus there is at most only one point of discontinuity of $F(\lambda)$ in $(0, \infty)$. Except for this case, $F(\lambda)$ is continuous and $\underset{s}{M}$ applied to (8.4) gives $f(t+u)=f(t)f(u)$, whence we have again

$$f(t)=e^{-k|t|} \qquad (k\geqq 0).$$

Hence the case (c) gives us two possibilities that $F(\lambda)$ is a step-function with only one point of discontinuity in $(0, \infty)$ or $f(s)$ is given by (8.3). Conversely these two cases satisfy (8.2) with (iii).

Let $t_1<t_2< \dots <t_n<t$ be a set of t-values. The process is said to be a Markoff process of order h when

$$F(x(t_1), \dots, x(t_n); x(t))=F(x(t_{n-h+1}), \dots, x(t_n); x(t)),$$

and it is said to be a simple Markoff process when the last equation is satisfied for $h=1$.

Now let $x(t)$ be a normal Markoff process, then

$$E(x(t_1), \dots, x(t_n); x(t))=E(x(t_n); x(t))^{27)}.$$

According to a relation between the conditional mean value and the sum S_n, the results for the case (b) applied to the last equation lead us to the conclusion that the correlation function should be written in the form given by (8.3). Conversely if the correlation function is given by (8.3), then, by the results established for (b), $S(t_1, \dots, t_n; t)=S(t_n; t)$ and $\sigma(t_1, \dots, t_n; t)=\sigma(t_n; t)$. Hence $F(x(t_1), \dots, x(t_n); x(t))$ and $F(x(t_n); x(t))$ are normal distributions with the common mean value and standard deviation, therefore they are identical.

Thus we can state the theorem.

Theorem 18. *A necessany and sufficient condition that every extrapolating sum for $x(t)$ should depend only on the variable corresponding to the t-value nearest to t is that the correlation function*

27) $E(x_1, \dots, x_n; x)$, $F(x_1, \dots, x_n; x)$ mean the conditional mean value and distribution for given $x_1, \dots, x_n$.

should be $f(t) = e^{-k|t|} (k \geqq 0)$.

A necessary and sufficient condition that every interpolating sum should depend only on the variables corresponding to the t-values nearest to t is that the correlation function should be either $f(t) = e^{-k|t|}$ $(k \geqq 0)$ *or* $\cos \lambda t$.

A necessary and sufficient condition that a normal s.s.p. should be a simple Markoff process is that the correlation function should be exponential as indicated above.[28]

The case of an integral-valued parameter is treated in a similar way.

Next we shall consider a normal Markoff process of order 2. In this case we obtain the condition for minimizing (8.1)

$$(8.5) \qquad \begin{vmatrix} f(s) & 1 & f(t) \\ f(s+t) & f(t) & 1 \\ f(s+t+u) & f(t+u) & f(u) \end{vmatrix} = 0 \quad (s,\ t,\ u \geqq 0).$$

From this equation we can readily conclude as before that the s.d.f. is continuous except for a case when it is a step-function with only one point of discontinuity in $(0, \infty)$. Taking $t = u = 1$ in (8.5), we get a difference equation. In view of boundedness of $f(t)$, the solution of the difference equation must become

$$f(t) = e^{-\lambda_1 t}\pi_1(t) + e^{-\lambda_2 t}\pi_2(t),$$

where $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$ and π's are composed of certain periodic functions. Substituting this in (8.5), it holds that

$$(8.5') \qquad \begin{vmatrix} K(s) & 1 & e^{-\lambda_2 t}K(t) \\ e^{-\lambda_2 t}K(s+t) & e^{-\lambda_2 t}K(t) & 1 \\ e^{-\lambda_2 t}K(s+t+u) & e^{-\lambda_2 t}K(t+u) & K(u) \end{vmatrix} = 0,$$

where $K(t) = \pi_1(t) + e^{-(\lambda_1-\lambda_2)t}\pi_2(t)$. In the following, we shall only consider the case when $\lambda_1 > \lambda_2 > 0$, the remaining case being treated in the same way. For this purpose we make use of the fact that given a bounded function $p(t)$ and a u.a.p. function $q(t)$ satisfying

$$e^{-\lambda t}p(t) + q(t) = 0 \qquad (\lambda > 0,\ t > 0),$$

then $p(t)$, $q(t)$ must identically vanish. As a simple application of this principle to (6.5)′ we get

$$K(s)(\pi_1(t)K(u) - \pi_1(t+u)) - K(u)\pi_1(s+t) + \pi_1(s+t+u) = 0.$$

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[28] This theorem is known, see K. Itō [5], Y. Kawada [2].

Again by the same principle applied twice to the last equation we have

$$\pi_2(u)\pi_1(t)\pi_2(s) = 0 \quad (s,\ t,\ u \geqq 0),$$

whence either $\pi_1 = 0$ or $\pi_2 = 0$. Let, for example, $\pi_2 = 0$ and replace this in (8.5), then we obtain (8.5)′ with K replaced by π_1. The relation thus obtained finally yields $\pi(s)\pi(t) = \pi(s+u)$, whence $\pi(t) = 1$. Hence we have again $f(t) = e^{-k|t|}\ (k \geqq 0)$. According to Theorem 18 $x(t)$ then becomes a normal simple Markoff process. In other words, if $x(t)$ is a Markoff process of order 2, it reduces to the form $A \cos \lambda t + B \sin \lambda t$ except for the case when it is essentially a simple Markoff process. The case of Markoff processes of higher orders will be discussed in a similar manner.

Let $x(n)$ $(n = 0,\ \pm 1,\ \dots)$ be a s.s.p. depending on a integral-valued parameter, then Kolmogoroff's theorem on interpolation is stated as follows.

Theorem 19. *Let* $S(\pm 1,\ \dots,\ \pm n;\ 0) = \sum_{k=-n}^{n}{}' a_{n,k} x_k$ *be the minimizing sum*[29] *for* $x(0)$, *then there exists*

$$\sigma = \lim_{n\to\infty} \sigma(\pm 1,\ \dots,\ \pm n;\ 0)$$

and

$$\sigma^2 = \frac{1}{\pi^{-1}\int_0^\pi s^{-1}(x)\,dx}, \quad s(x) = F'(x). \tag{8.6}$$

when $\int_0^\pi s^{-1}(x)\,dx = \infty$, *the right-hand side of* (8.6) *stands for* 0.

Proof. We have

$$\sigma^2_n = E\big((x_0 - \textstyle\sum' a_{n,k} x_k)^2\big) = (2\pi)^{-1}\displaystyle\int_{-\pi}^{\pi} \Big|\, 1 - \sum_{-n}^{n}{}' a_{n,k} e^{ikx} \Big|^2 dF(x).$$

Since $\sigma^2_n \geqq \sigma^2_{n+1}$, $\sigma = \lim_{n\to\infty} \sigma_n$ exists. Making use of the projection of $x(0)$ to the closed linear manifold determined by $x(k)\,(k = \pm 1,\ \pm 2,\ \dots)$, it is readily seen that $\sum' a_{n,k} e^{ikx}$ comverges in mean, with exponent 2, with respect to the distribution $dF(x)$. The minimizing condition (8.2) is given by

$$\pi^{-1}\int_0^\pi (1 - \textstyle\sum' a_{n,k} \cos kx) \cos jx\, dF(x) = 0, \tag{8.7}$$

$$\pi^{-1}\int_0^\pi (\textstyle\sum' a_{n,k} \sin kx) \sin jx\, dF(x) = 0,$$

29) "′" means that the term for $k = 0$ is omitted.

$$j = \pm 1, \pm 2, \ldots, \pm n.$$

The second equation gives

$$\int_0^\pi (\textstyle\sum' a_{n,k} \sin kx)^2 \, dF(x) = 0;$$

whence by the first equation of (8.7)

$$\begin{aligned}\sigma^2_n &= \pi^{-1}\int_0^\pi (1-\textstyle\sum' a_{n,k} \cos kx)^2 \, dF(x) \\ &= \pi^{-1}\int_0^\pi (1-\textstyle\sum' a_{n,k} \cos kx) \, dF(x).\end{aligned}$$

Letting $n \to \infty$

$$\sigma^2 = \pi^{-1}\int_0^\pi p^2(x)\, dF(x) = \pi^{-1}\int_0^\pi p(x)\, dF(x). \tag{8.8}$$

and

$$\pi^{-1}\int_0^\pi p(x) \cos jx \, dF(x) = 0, \quad j = \pm 1, \quad \ldots, \quad \pm n, \tag{8.9}$$

where

$$p(x) = \underset{n\to\infty}{\text{l.i.m.}}\, (1-\textstyle\sum' a_{n,k} \cos kx).$$

Consider the function

$$\begin{aligned} g(x) &= 1 \quad \text{if} \quad |x| \leqq u, \\ &= 0 \quad \text{if} \quad u < |x| \leqq \pi, \ 0 < u < \pi, \end{aligned}$$

and let its Fourier coefficients be $a_0 = 2u/\pi$, a_1, a_2, ... Adding (8.8) and (8.9) multiplied respectively by $a_0/2$ and $a_j (j = 1, 2, \ldots)$, we obtain

$$\sigma^2 u = \int_0^u p(x)\, dF(x). \tag{8.10}$$

Differentiation of (8.10) with respect to u yields

$$\sigma^2 = p(u)\, s(u) \tag{8.11}$$

for almost all u. Hence, when $s(u) = 0$ on some set of u of positive measure, (8.11) gives us $\sigma = 0$. Next we consider the case when $s(u) > 0$ almost every where. By (8.8) and (8.10)

$$\sigma^2 \pi^{-1}\int_0^\pi p(x)\, dx = \pi^{-1}\int_0^\pi p(x)\, d\left(\int_0^x p(u)\, dF(u)\right) = \sigma^2.$$

Consequently,

$$\int_0^\pi p(x)\, dx = \pi.$$

Provided that $\sigma \neq 0$; whence by (8.11), if

$$\int_0^\pi s^{-1}(u)\, du < \infty,$$

it follows that $\sigma = 0$. But, when.

$$\int_0^\pi s^{-1}(u)\,du < \infty,$$

we have

$$\int_0^\pi |p_n(x)-p(x)|\,dx = \int_0^\pi |p_n(x)-p(x)|\sqrt{s(x)}\cdot\frac{dx}{\sqrt{s(x)}}$$

$$\leqq\left(\int_0^\pi |p_n(x)-p(x)|^2\,dF(x)\right)^{\frac{1}{2}}\left(\int_0^\pi s^{-1}(x)\,dx\right)^{\frac{1}{2}}\to 0, \text{ as } n\to\infty.$$

This implies

$$\int_0^\pi p(x)\,dx = \lim_{n\to\infty}\int_0^n p_n(x)\,dx = \pi.$$

Hence, again by (8.11)

$$\sigma^2\int_0^\pi s^{-1}(u)\,du = \pi,$$

which completes the proof.

Kolmogoroff's theorem on extrapolation is stated as follows.

Theorem 20. *Let* $S(-n, \ldots, -k-1, -k;\ 0) = \sum_{i=k}^{n} a_{n,i}\,x(-i)$ *be the minimizing sum extrapolating* $x(0)$, *where* $k>0$ *is an integer. Then*

$$\sigma_k = \lim_{n\to\infty}\sigma(-n, \ldots, -k-1, -k;\ 0)$$

exists and

$$\sigma^2_k = (1+b^2_1+\ldots+b^2_k)\exp\left(\pi^{-1}\int_0^\pi \log s(u)\,du\right), \tag{8.12}$$

where b_ν *are determined by*

$$\begin{aligned} &\log s(x)\sim a_0+a_1\cos x+a_2\cos 2x+\ldots, \\ &\exp\left(\frac{1}{2}(a_1 r+a_2 r^2+\ldots)\right) = 1+b_1 r+b_2 r^2+\ldots. \end{aligned} \tag{8.13}$$

When

$$\int_0^\pi |\log s(u)|\,du = \infty,$$

the right-hand side of (8.12) *stands for* 0.

Proof. The proof is based on known results in the theory of orthogonal polynomials on the unit circle $|z|=1$ with respect to a weight function.[30] Let $\pi_{n,k}(z) = z^n + a_{n-k}z^{n-k}+\ldots+a_0$ be an

[30] Detailed discussions on this subject are given in G. Szegö [1], to which we owe the notations and terminologies used in the present section.

arbitrary polynomial in z with the highest term z^n and vanishing terms of degrees from $n-1$ to $n-k+1$. Let us denote by $\sigma^2_{n,k}$ the greatest lower bound of $(2\pi)^{-1}\int_{-\pi}^{\pi} |\pi_{n,k}(z)|^2 dF(\theta)$ $(z = e^{i\theta})$ when $\pi_{n,k}(z)$ ranges over all polynomials of the type just stated. Then $\sigma_{n,k}(F)$ is non-increasing as $n\to\infty$ and there exits $\sigma_k(F) = \lim_{n\to\infty} \sigma_{n,k}(F)$. Now

$$E\{(S(-n, \ldots, -k-1, -k; 0)-x(0))^2\}$$
$$= (2\pi)^{-1}\int_{-\pi}^{\pi} |1-a_{n,k}e^{-ik\lambda}-\ldots-a_{n,n}e^{-in\lambda}|^2 dF(\lambda)$$
$$= (2\pi)^{-1}\int_{-\pi}^{\pi} |z^n-a_{n,k}z^{n-k}-\ldots-a_{n,n}|^2 dF(\lambda),\ z=e^{i\lambda}.$$

Hence

$$\sigma^2(-n, \ldots, -k; 0) = \sigma^2_{n,k}(F), \quad \sigma^2_k = \sigma^2_k(F)^{31)}.$$

If F and G are arbitrary non-decreasing functions in $(-\pi, \pi)$ which satisfy $\Delta F \leqq \Delta G$ for every increment Δ, then by definition $\sigma_{n,k}$ satisfies

$$\sigma_{n,k}(F) \leqq \sigma_{n,k}(G).$$

First we consider the case $k=1$. Let us write

$$\Phi(x) = (2\pi)^{-1}\int_{-\pi}^{x} s(u)\,du.$$

Then, taking into account of the fact that $\Delta\Phi \leqq \Delta F$ we have

$$\sigma^2(\Phi) \leqq \sigma^2(F) \leqq \sigma^2_n(F). \tag{8.14}$$

If $T(\theta)$ is a non-negative cosine trigonometrical polynomial of degree n with real coefficients, there corresponds a polynomial $P(z)$ of the same degree as $T(\theta)$ such that $P(0)>0$, $P(z) \neq 0$ for $|z|<1$, $T(\theta) = |P(e^{i\theta})|^2$ [32]. Let us denote by $\mathfrak{G}(T)$ the geometrical mean of $T(\theta)$. Obviously

$$\mathfrak{G}(T) = \exp((2\pi)^{-1}\int_{-\pi}^{\pi} \log T(\theta)\,d\theta) = (P(0))^2.$$

Let $P^*(z)$ be the polynomial reciprocal to $P(z)$. Then, since $(\bar{P}(0))^{-1}P^*(z)$ is a polynomial $\pi_n(z)$, we have

$$\sigma^2_n(F) \leqq (2\pi)^{-1}\int_{-\pi}^{\pi} |(\bar{P}(0))^{-1}P^*(z)|^2 dF(\theta),\ z = e^{i\theta};$$

whence

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31) When $k=1$ we omit the subscript.

32) G. Szegö, loc. cit., § 1.2 and § 12.3.

$$(8.15)\qquad \sigma^2_n(F) \leqq (\mathfrak{G}(T))^{-1}(2\pi)^{-1}\int_{-\pi}^{\pi} T(\theta)\, dF(\theta).$$

Since an even continuous function can be approximated by certain sequence of cosine polynomials, (8.15) holds for any non-negative B-measurable even function $T(\theta)$ such that

$$(2\pi)^{-1}\int_{-\pi}^{\pi} T(\theta)\, d\theta < \infty.$$

Let A be a B-measurable set such that

$$(8.16)\qquad \int_{CA} d\varDelta(\theta) = 0, \quad |A| = 0,$$

where $\varDelta(\theta) = F(\theta) - \Phi(\theta)$, and introduce a function $T(\theta) = U(\theta) \cdot S(\theta)$ with an arbitrary B-measurable $U(\theta)$ and $S(\theta)$ such that

$$(8.17)\qquad \begin{aligned} S(\theta) &= 0 \quad \text{if} \quad \theta \varepsilon A, \\ &= 1 \quad \text{if} \quad \theta \varepsilon CA. \end{aligned}$$

Combining (8.14), (8.15), (8.16), (8.17) and the well-known fact that $\sigma^2(\Phi) = \mathfrak{G}(s)$ we obtain

$$(8.18)\qquad \mathfrak{G}(s) \leqq \sigma^2(F) \leqq (\mathfrak{G}(U))^{-1}(2\pi)^{-1}\int_{-\pi}^{\pi} U(\theta)\, s(\theta)\, d\theta$$

for an arbitrary B-measurable even $U(\theta)$. Hence[33]

$$(8.18)\qquad \sigma^2 = \sigma^2(F) = \exp\left(\pi^{-1}\int_0^{\pi} \log s(\theta)\, d\theta\right),$$

provided that $\log s(\theta)$ is integrable. When $\int_0^{\pi} |\log s(\theta)|\, d\theta = \infty$ we put

$$\begin{aligned} s_n(\theta) &= s(\theta) \quad \text{if} \quad s(\theta) > 1/n, \\ &= 1/n \quad \text{if} \quad s(\theta) \leqq 1/n, \end{aligned}$$

and define $G_n(\theta)$,

$$G_n(\theta) = \varDelta(\theta) + \int_{-\pi}^{\theta} s_n(x)\, dx.$$

Since $\varDelta F \leqq \varDelta G_n$, we have

$$\sigma^2(F) \leqq \sigma^2(G_n) = \mathfrak{G}(s_n(\theta)).$$

Making $n \to \infty$, since $\mathfrak{G}(s_n) \to 0$, we have $\sigma^2 = \sigma^2(F) = 0$, which together with (8.18), proves the theorem for $k = 1$.

Next we shall consider the case $k = 2$. The proof for the general case is obtained by a similar argument. To prove the theorem for this case, we shall first establish the relation that $\sigma_2^2(\Phi) = \sigma_2^2(F)$. Obviously

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[33] G. Szegö, loc. cit., § 12.3.

$$\sigma_{n,2}^{2} = \sigma^2(-n, \ldots, -2;\ 0) = (2\pi)^{-1}\int_{-\pi}^{\pi}(p^2_n(x)+q^2_n(x))\,dF(x), \tag{8.19}$$

where

$$p_n(x) = 1-\sum_{k=2}^{n} a_{n,k}\cos kx,\quad q_n = \sum_{k=2}^{n} a_{n,k}\sin kx.$$

The minimizing condition is given by

$$(2\pi)^{-1}\int_{-\pi}^{\pi}(p_n(x)\cos kx+q_n(x)\sin kx)\,dF(x) = 0,\quad k = 2,\ \ldots,\ n. \tag{8.20}$$

Since $1-\sum_{k=2}^{n} a_{n,k}\, e^{-ikx}$ converges in mean as in the proof of Theorem 19, there exist the limits $p(x) = \text{l.i.m.}\ p_n(x)$, $q(x) = \text{l.i.m.}\ q_n(x)$. Combining (8.19) and (8.20)

$$\sigma_{n,2}^{2} = (2\pi)^{-1}\int_{-\pi}^{\pi} p_n(x)\,dF(x). \tag{8.21}$$

Hence making $n\to\infty$ in (8.20) and (8.21)

$$\begin{gathered}\sigma_2^2 = (2\pi)^{-1}\int_{-\pi}^{\pi} p_n(x)\,dF(x),\\ (2\pi)^{-1}\int_{-\pi}^{\pi}(p(x)\cos kx+q(x)\sin kx)\,dF(x) = 0,\ k = 2, 3, \ldots.\end{gathered} \tag{8.22}$$

Now let us put

$$\begin{gathered}\pi^{-1}\int_{-\pi}^{\pi}(p(x)\cos x+q(x)\sin x)\,dF(x) = c,\\ U(x) = \int_{-\pi}^{x} p(u)\,dF(u)-c\sin x,\quad V(x) = \int_{-\pi}^{x} q(u)\,dF(u),\end{gathered} \tag{8.23}$$

then (8.22) and (8.23) give

$$\pi^{-1}\int_{-\pi}^{\pi}\cos kx\,dU(x) = -\pi^{-1}\int_{-\pi}^{\pi}\sin kx\,dV(x),\quad k = 1, 2, \ldots. \tag{8.24}$$

Whence, the conjugate series of the Fourier-Stieltjes series of $U(x)$ is at the same time the Fourier-Stieltijes series of $-V(x)$. Consequently, by a theorem of *F.* and *M.* Riesz in the theory of Fourier series[34], $U(x)$ and $V(x)$ are absolutely continuous. Hence

$$c\sin x+U(x) = \int_{-\pi}^{x} p(u)\,s(u)\,du,\quad V(x) = \int_{-\pi}^{x} q(u)\,s(u)\,du,$$

and

$$\sigma_2^2 = (2\pi)^{-1}\int_{-\pi}^{\pi} p(u)\,s(u)\,du. \tag{8.22'}$$

Now let us write p'_n, p' etc. instead of p_n, p etc. when the weight

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[34] A. Zygmund [1], § 7.5.

function $F(x)$ is replaced by $\Phi(x) = \int_{-\pi}^{x} s(u)\,du$. Then by (8.20) with $F(x)$ replated by Φ, we obtain

$$(2\pi)^{-1}\int_{-\pi}^{\pi} (p'_n(x)\,p(x) + q'_n(x)\,q(x))\,s(x)\,dx$$
$$= (2\pi)^{-1}\int_{-\pi}^{\pi} p'_n(x)\,s(x)\,dx = \sigma_{2,2}^2(\Phi);$$

whence letting $n \to \infty$

(8.25) $$(2\pi)^{-1}\int_{-\pi}^{\pi} (p'(x)\,p(x) + q'(x)\,q(x))\,s(x)\,dx = \sigma_2^2(\Phi).$$

Also, if $m \geqq n$

$$(2\pi)^{-1}\int_{-\pi}^{\pi} (p_m(x)\,p'_n(x) + q_m(x)\,q'_n(x))\,dF(x) = (2\pi)^{-1}\int_{-\pi}^{\pi} p_m(x)\,dF(x).$$

Whence letting $m \to \infty$

$$(2\pi)^{-1}\int_{-\pi}^{\pi} (p(x)\,p'_n(x) + q(x)\,q'_n(x))\,s(x)\,dx = \sigma_2^2.$$

Finally making $n \to \infty$ in this equation, we obtain

(8.26) $$\sigma_2^2(\Phi) = \sigma_2^2,$$

since the left-hand side of the last equation tends to (8.25) for $n \to \infty$.

If we orthogonalize the system

$$\{(s(\theta))^{1/2} z^n\}, \quad z = e^{i\theta}, \ n = 0, 1, \dots,$$

we obtain a system of polynomials $\{\varphi_n(\theta)\}$, $z = e^{i\theta}$, $n = 0, 1, \dots$; which is called the orthonormal set of polynomials on the unit circle with respect to the weight function $\Phi(\theta)$. Let $\kappa_n > 0$ and c_{n-1} be the coefficients of z^n and z^{n-1} in $\varphi_n(z)$, then any $\pi_{n,2}(z)$ is represented by the set $\varphi_0(z), \varphi_1(z), \dots, \varphi_n(z)$:

$$\pi_{n,2}(z) = \kappa_n^{-1}\varphi_n(z) - c_{n-1}\kappa_n^{-1}\kappa_{n-1}^{-1}\varphi_{n-1}(z) - \sum_{j=0}^{n-2}\lambda_j\varphi_j(z),$$

where λ_j are real constants. Since $\{\varphi_n(z)\}$ satisfy

$$(2\pi)^{-1}\int_{-\pi}^{\pi} \varphi_m(z)\,\overline{\varphi_n(z)}\,s(\theta)\,d\theta = \delta_{m,n}, \quad z = e^{i\theta},$$

the minimized value of $(2\pi)^{-1}\int_{-\pi}^{\pi} |\pi_{n,2}(z)|^2 s(\theta)\,d\theta$ is attained by making $\lambda_j = 0$, i.e. by $\pi_{n,2}(z) = \kappa_n^{-1}\varphi_n(z) - c_{n-1}\kappa_{n-1}^{-1}\kappa_n^{-1}\varphi_{n-1}(z)$. Hence

(8.27) $$\sigma_{n,2}^2(\Phi) = \kappa_n^{-2} + c_{n-1}^2\kappa_n^{-2}\kappa_{n-1}^{-2}.$$

As a fundamental result in the theory of orthogonal polynomials we obtain

(8.28)
$$x_{n}^{-2} \to \mathfrak{G}(s), \quad n \to \infty,$$
$$c_{n-1}^{2} \to b_{1}^{2}\{\mathfrak{G}(s)\}^{-1}, \quad n \to \infty,$$

where b_1 is the constant mentioned in Theorem 16. The first of the relations in (8.28) is known. To establish the second, we make use of an important property of the polynomials[35]:

$$\varphi_n(z) \cong z^n\{\bar{D}(z^{-1})\}^{-1} \quad \text{uniformly for} \quad |z| \geqq R > 1,$$

provided that $\log s(x)$ is integrable, where

$$D(z) = \exp\{(2\pi)^{-1}\int_{-\pi}^{\pi} \frac{1}{2} \log s(t) \frac{1+ze^{-it}}{1-ze^{-it}}\, dt\}, \quad |z| < 1.$$

Let the Fourier series of $\log s(\theta)$ be

$$\log s(\theta) \sim \frac{1}{2} a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \dots,$$

Then an easy calculation shows that

(8.29)
$$D(z) = \exp\left\{\frac{1}{2}\left(\frac{a_0}{2} + a_1 z + \dots\right)\right\}, \quad |z| < 1.$$

If $\varphi_n^*(z)$ is the reciprocal polynomial of $\varphi_n(z)$

$$\varphi_n^*(z) = z^n \bar{\varphi}_n(z^{-1}) \cong \{D(z)\}^{-1} \text{ uniformly in } |z| < R^{-1} < 1.$$

But, since all the coefficients of $\varphi_n(z)$, $\varphi_n^*(z)$ are real, we obtain

$$r c_{n-1} = (2\pi)^{-1}\int_{-\pi}^{\pi} \varphi_n^*(z)\, e^{-i\theta}\, d\theta, \quad z = re^{i\theta}, \quad 0 < r < 1,$$
$$\cong (2\pi)^{-1}\int_{-\pi}^{\pi} \{D(z)\}^{-1} e^{-i\theta}\, d\theta = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{-(a_0/4)}$$
$$\cdot (1 + b_1 z + b_2 z^2 +) e^{-i\theta}\, d\theta = \{\mathfrak{G}(s)\}^{-1/2} b_1 r,$$

which proves the required relations. Now (8.26), (8.27) (8.28) give us (8.12) for $k = 2$, provided that $\log s(\theta)$ is integrable.

The remaining case when

$$\int_{-\pi}^{\pi} |\log s(\theta)|\, d\theta = \infty$$

is proved by the same device as in the corresponding part of the proof of Theorem 19.

9. The sinusoidal limit theorem. A equence of stationary processes $\{x_n(t)\}$, t being continuous or integral-valued, is said to be asymptotically superposed harmomies when

(9.1)
$$\text{st.} \lim_{n\to\infty}\{x_n(t) - \sum_{k=1}^{m} (A_{n,k} \cos \lambda_k t + B_{n,k} \sin \lambda_k t)\} = 0,$$

[35] G. Szegö, loc. cit., § 12.1.

for every fixed t, and some $A_{n,k}$, $B_{n,k}$ such that

$$E(A_{n,k}^2) = E(B_{n,k}^2) = \delta_k, \quad E(A_{n,k} B_{n,k}) = 0, \tag{9.2}$$

where δ_k, $\lambda_k (k = 1, 2, \ldots, m)$ are fixed numbers.

The sinusoidal limit theorem of Slutsky[36] states that a sequence of stationary processes depending on a integral-valued prameter is asymptotic, under suitable conditions imposed on correlation coefficients, to a sinusoidal curve. It is, however, convenient to state the theorem in terms of the s.d.f. to generalize it to the case of a continuous parameter.

Theooem 21. *Let t be a continuous or integral-valued parameter, and let the s.d.f. $F_n(\theta)$ of an s.s.p. $x_n(t)$ tend to a step-function $F(\theta)$ with jumps δ_k at $\theta = \lambda_k (k = 1, 2, \ldots, m)$, then, $x_n(t)$ is asymptotic to superposed harmonics.*

Proof. Using the representations (7.22) and (7.22)′ we obtain

$$\begin{aligned} x_n(t) = \Bigg(\int_0^{(\lambda_1+\lambda_2)/2} + \int_{(\lambda_1+\lambda_2)/2}^{(\lambda_2+\lambda_3)/2} + \ldots \\ \ldots + \int_{(\lambda_m+\lambda_{m-1})/2}^{l}\Bigg) (\cos \lambda t \, dx_n(0, \lambda) - \sin \lambda t \, dx_{cn}(0, \lambda)), \end{aligned} \tag{9.3}$$

where l indicates π or $+\infty$ according as t is continuous or discrete. Let us put

$$A_{n,1} = \frac{\sqrt{\delta_1}}{\sqrt{F_n((\lambda_1+\lambda_2)/2) - F_n(0)}} \{x_n(0, (\lambda_1+\lambda_2)/2) - x_n(0, 0)\},$$

$$A_{n,2} = \frac{\sqrt{\delta_2}}{\sqrt{F_n((\lambda_2+\lambda_3)/2) - F_n((\lambda_1+\lambda_2)/2)}} \{x_n(0, (\lambda_2+\lambda_3)/2) - x_n(0, (\lambda_2+\lambda_1)/2)\},$$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$$A_{n,m} = \frac{\sqrt{\delta_m}}{\sqrt{1 - F_n((\lambda_{m-1}+\lambda_m)/2)}} \{x_n(0, l) - x_n(0, (\lambda_{m-1}+\lambda_m)/2)\},$$

and similarly for $B_{n,k}$. Since $F_n((\lambda_1+\lambda_2)/2) - F_n(0) \to \delta_1$ for $n \to \infty$ according to the hypothesis that $F_n \to F$ we have

$$\begin{aligned} & E\left\{\left(\int_0^{(\lambda_1+\lambda_2)/2} \cos \lambda t \, dx_n(0, \lambda) - A_{n,1} \cos \lambda_1 t\right)^2\right\} \\ & = \int_0^{(\lambda_1+\lambda_2)/2} \Big(\cos \lambda t - \frac{\sqrt{\delta_1}}{\sqrt{F_n((\lambda+\lambda_2)/2) - F_n(0)}} \\ & \qquad \cdot \cos \lambda_1 t\Big)^2 dF_n(\lambda) \to 0, \quad n \to \infty, \end{aligned} \tag{9.4}$$

[36] Slutsky [2].

and similarly for the other $A_{n,k}$ and $B_{n,k}$. Obviously $A_{n,k}$, $B_{n,k}$ satisfy (9.2). (9.1) follows at once from (9.3) and (9.4).

Slutsky applied the sinusoidal limit theorem to a sequence of moving averages constructed from uncorrelated random variables and Romanovsky generalized this to the case of stationary random variables[37].

Given an s.s.p. $x(t)$ depending on an integral-valued parameter t, we form a new sequence $\{x_n(t)\}$ $(n = 1, 2, \ldots)$ such that

$$x^{(1)}(t) = x(t)+x(t-1)+\ldots +x(t-s+1),$$
$$x^{(2)}(t) = x^{(1)}(t)+x^{(1)}(t-1)+\ldots +x^{(1)}(t-s+1),$$
$$x^{(k)}(t) = x^{(k-1)}(t)+x^{(k-1)}(t-1)+\ldots +x^{(k-1)}(t-s+1),$$
$$x_n(t) = \Delta^l x^{(k)}(t) = \Delta^{l-1}x^{(k)}(t)-\Delta^{l-1}x^{(k)}(t-1),$$
$$l = l_n, \quad k = k_n, \quad r = l/k,$$

where $s>0$ is a fixed integer, k, l tend to ∞ such that $r \to \alpha \neq 1$. Then (7.22) gives

$$x_n(t) = \int_{-\pi}^{\pi} (1-e^{-i\theta})^{l-k}(1-e^{is\theta})^k e^{i\theta t}\, dS(0, \theta).$$

Whence the s.d.f., say $F_n(\lambda)$, of $x_n(t)/\sqrt{E(x^2_n)}$ is given by

$$F_n(\lambda) = \int_{-\pi}^{\lambda} f_n(\theta)\, dF(\theta) \Big/ \int_{-\pi}^{\pi} f_n(\theta)\, dF(\theta),$$
$$f_n(\theta) = (\sin \theta/2)^{2(l-k)} (\sin s\theta/2)^{2k}.$$

Since $\sin^{2(\alpha-1)}\theta/2 \sin^2 s\theta/2$ $(0 \leqq \theta \leqq \pi)$ attains its maximum at one point, say $\theta = \lambda_0$, if $F(\lambda_0+\varepsilon)-F(\lambda_0-\varepsilon)>0$ for any $\varepsilon>0$, it is readily seen that $F_n(\theta)$ tends to a step-function[38] with jums at $\theta = \pm\lambda_0$. Hence $x_n(t)/\sqrt{E(x^2_n)}$ is asymptotically a sinusoidal curve.

10. Random addition of functions. Certain kinds of physical problems lead us to a study of the fluctuation of quantities caused by a successive occurrence of events or pulses. In many cases the fluctuation at an instant t is expressed by means of a function $f(t)$, $f(t) = 0$ for $t<0$, in the form[39]

(10.1) $$\theta_{t_0}(t) = \sum_i f(t-\tau_i)$$

with τ_i, the time of occurronce of the events, ranging in the

[37] Romanovsky [1], [2].
[38] Romanovsky [1].
[39] Fowler [1], § 20.71, Rowland [1], [2].

summation from t_0, at which the fluctuation starts, to t. Let $\nu(t)$ be the total number of events in (t_0, t). In equilibrium states, the occurrence of the events is frequently considered homogeneous in time with the average number $N\Delta t$, in the interval $(t, t+\Delta t)$; and independent in non-overlapping time intervals. Under these circumstances we are led to the Poisson distribution

$$(10.2)\qquad P(\Delta\nu(t) = n) = \frac{(N\Delta t)^n e^{-N\Delta t}}{n!} \quad (n = 0, 1, \ldots).$$

The distribution of $\theta_{t_0}(t)$ for large t is equivalent to that obtained by letting $t_0 \to -\infty$ and, as is shown in the following, it is stationary with respect to t.

Theorem 22. (i) *If*

$$\int_0^\infty |f(t)|\,dt < \infty,$$

$\theta_{t_0}(t)$ converges, as $t_0 \to -\infty$, to a strongly mixing s.s.p. $\theta(t)$ in probability law such that

$$E(\theta(t)) = \int_0^\infty f(t)\,dt.$$

(ii) *If*

$$\int_0^\infty f^2(t)\,dt < \infty$$

$\theta_{t_0}(t) - E(\theta_{t_0}(t))$ converges, in probability law, as $t_0 \to -\infty$, to a strongly mixing s.s.p. with the mean value 0 *and correlation function*

$$\rho(\tau) = \int_0^\infty f(t+|\tau|) f(t)\,dt.$$

Proof. If $h(x)$ is a continuous function in (a, b) the integral

$$x = \int_a^b h(s)\,d\nu(s)$$

exists with probability 1 and defines a random variable x. Since the logarithm of the characteristic function of $\nu(t+\Delta t) - \nu(t)$ is $N(e^{iz}-1)\Delta t$ we obtain

$$(10.3)\qquad \log E(e^{izx}) = N\int_a^b (\exp(izh(u)) - 1)\,du.$$

But, it is easily seen that $\theta_{t_0}(t)$ is expressed as follows

$$(10.4)\qquad \theta_{t_0}(t) = \int_{t_0}^t f(t-\tau)\,d\nu(\tau).$$

Hence, given real numbers $t_0 < t_1 < \ldots < t_k < t'_1 < \ldots < t'_k$, $z_1, z_2, \ldots, z_k$,

$z'_1, \dots, z'_k$; $t_{\nu+1}-t_\nu=\delta_\nu$, $t'_{\nu+1}-t'_\nu=\delta'_\nu$ $(\nu=1, \dots, k-1)$, $t'_1-t_k=\omega$; we have, in view of (10.3) and the fact that $f(t)=0$ for $t<0$, the relation

$$\psi_{t_0}(z, t) = \log E\{\exp(i\sum_{\nu=1}^{k} z_\nu \theta_{t_0}(t_\nu))\}$$
$$= N\int_{t_0}^{\infty}\{\exp(i\sum_{\nu=1}^{k} z_\nu f(t_\nu-\tau))-1\}d\tau.$$

From the restriction imposed on $f(t)$ the limit

$$\psi(z, t) = \lim_{t_0\to-\infty}\psi_{t_0}(z,t) = \int_{-\infty}^{\infty} e(z, \tau)\, d\tau$$

exists uniformly in $z_1, \dots, z_k$ belonging to any finite range, where

$$e(z, \tau) = \exp(i\sum_{\nu=1}^{k} z_\nu f(\delta_1+\dots+\delta_{\nu-1}+\tau))-1, \quad \delta_0 = 0.$$

Since $\psi(z, t)$ only depends on the differences δ_ν, $\theta_{t_0}(t)$ converges in probability law to an s.s.p. $\theta(t)$ such that

$$\psi(z, t) = \log E\{\exp(i\sum_{\nu=1}^{k} z_\nu \theta(t_\nu))\}.$$

Further, by the result just obtained

$$\psi(z, t; z', t') = \log E\{\exp(i\sum_{\nu=1}^{k} z_\nu\theta(t_\nu)+i\sum_{\nu=1}^{k} z'_\nu\theta(t'_\nu))\}$$
$$=\int_0^{\infty}\{\exp(i\sum_{\nu=1}^{k} z_\nu f(\delta_1+\cdots+\delta_{\nu-1}+\tau)+i\sum_{\nu=1}^{k} z'_\nu f(\delta_1+\cdots$$
$$+\delta_{k-1}+\omega+\delta'_1+\cdots+\delta'_{\nu-1}+\tau))-1\}d\tau$$
$$+\int_0^{\delta_1}\{\exp(i\sum_{\nu=2}^{k} z_\nu f(\delta_2+\dots+\delta_{\nu-1}+\tau)+i\sum_{\nu=2}^{k} z_\nu' f(\delta_2+\dots$$
$$+\delta_{k-1}+\omega+\delta_1'+\dots+\delta_{\nu-1}'+\tau))-1\}d\tau+\dots\dots$$
$$+\int_0^{\delta_{k-1}}\{\exp(i z_k f(\tau)+i\sum_{\nu=1}^{k} z'_\nu f(\omega+\delta'_1+\dots$$
$$+\delta'_{\nu-1}+\tau))-1\}d\tau+\psi(z', t')-\int_{\omega}^{\infty} e(z', \tau)\, d\tau.$$

The first term of the last expression is equal to

$$\int_0^{\infty} e(z,\tau)\, d\tau+O(\int_{\omega}^{\infty}|f(\delta_1+\cdots+\delta_{k-1}+\tau)|\,d\tau+\cdots$$
$$+\int_{\omega}^{\infty}|f(\delta_1+\cdots+\delta_{k-1}+\delta'_1+\cdots+\delta'_{k-1}+\tau)|\,d\tau)$$
$$\to\int_0^{\infty} e(z, \tau)\, d\tau, \quad \omega\to\infty,$$

and similarly for the other first k integrals in the expression.

Whence we can easily deduce the $\psi(z, t; z', t') \to \psi(z, t) + \psi(z', t')$ as $\omega \to \infty$, i.e.

$$E\{\exp(i \sum_{\nu=1}^{k} z_\nu \theta(t_\nu) + i \sum_{\nu=1}^{k} z'_\nu \theta(t'_\nu + \omega))\}$$
$$\longrightarrow E\{\exp(i \sum_{\nu=1}^{k} z_\nu \theta(t_\nu))\} \cdot E\{\exp(i \sum_{\nu=1}^{k} z'_\nu \theta(t'_\nu))\},$$

which means that the s.s.p. $\theta(t)$ is strongly mixing. The coefficient of z in $\psi(z, t)$ expanded in powers of z gives the mean value of $\theta(t)$. Thus we have proved the first part of the theorem.

The remaining part can be proved in a similar manner.

As is well-known, if a series of independent random variables converges in probability law, then it does also with probability 1. Hence, by this principle applied to (10.1), $\theta_{t_0}(t)$ converges with probability 1, for every t, to $\theta(t)$.

Concerning the distribution of $\theta_t(t)$ we get the following theorem.

Theorem 23. (i) *If $f(x)$ is continuous and satisfies the following conditions*

(a) $$\int_0^\infty f^2(x)\,dx = \infty;$$

(b) $$\rho(\tau) = \lim_{\omega\to\infty} \int_0^\omega f(x) f(x + |\tau|)\,dx / \sigma(\omega)\sigma(\omega+\tau)$$

exists for every τ, with $\sigma^2(\omega) = \int_0^\omega f^2\,dx;$

(c) $\eta_\varepsilon(\omega) = \sigma^{-2}(\omega) \int_S f^2(x)\,dx \to 0,$ as $\omega \to \infty$, for *every fixed $\varepsilon > 0$, where $S = E_x[|f(x)| > \varepsilon\sigma(\omega),\ 0 \leqq x \leqq \omega]$; then*

$$\xi_{t_0}(t) = \left(\theta_{t_0}(t) - N\int_0^{t-t_0} f(x)\,dx\right) \Big/ \left(N\int_0^{t-t_0} f^2(x)\,dx\right)^{1/2}$$

converges, as $t_0 \to -\infty$, in probability law to a normal s.s.p. with the mean value 0 *and correlation function* $\rho(\tau)$.

(ii) *If $f(x)$ is continuous and satisfies the conditions*

$$\int_0^\infty f^2(x)\,dx < \infty, \quad \int_0^\infty |f(x)|\,dx < \infty$$

then

$$\xi(t) = N^{-1/2}\left(\theta(t) - N\int_0^\infty f(t)\,dt\right)$$

converges, as $N \to \infty$, in probability law to a normal s.s.p. with the mean value 0 *and correlation function*

$$\rho(\tau) = \int_0^\infty f(t)f(t+|\tau|)\,dt.$$

Proof of (i). Given arbitrary real numbers $t_1, t_2, \ldots, t_k$, it is sufficient to show that

$$\psi(z_1, \ldots, z_k) = \log E\{\exp(i\sum_{\nu=1}^{k} z_\nu \xi_{t_0}(t_\nu))\} \to -\frac{1}{2}\sum_{\mu,\nu=1}^{k} z_\mu z_\nu \rho(t_\mu - t_\nu),$$

as $t_0 \to -\infty$, uniformly in the real numbers $z_1, \ldots, z_k$ belonging to an arbitrary finite domain $|z_1| < a, \ldots, |z_k| < a$. We may suppose without loss of generality that $N=1$. To prove the theorem for $k=2$, we observe that

$$\begin{aligned}\psi(z_1, z_2) &= \log E\{\exp(iz_1\xi_{t_0}(t_1) + iz_2\xi_{t_0}(t_2))\}\\ &= \int_0^\omega \{\exp(iK(z_1, z_2)) - 1 - iK(z_1, z_2)\}\,dx + \\ &\quad + \int_0^\tau \{\exp(iz_2 f(x)/\sigma(\omega)) - 1 - iz_2 f(x)/\sigma(\omega)\}\,dx\end{aligned}$$

where

$$K(z_1, z_2) = z_1 f(x)/\sigma(\omega) + z_2 f(x+\tau)/\sigma(\omega+\tau),$$

and $\omega = t_1 - t_0$, $\tau = t_2 - t_1$. The absolute value of the second integral does not exceed

$$O(\sigma^{-2}(\omega)\int_0^\tau f^2(x)\,dx),$$

which tends to 0, as $\omega \to \infty$, by (a). To estimate the first integral, we divide it into four parts

$$\sum_{n=1}^{4}\left(\int_{D_n} e^{iK} - |D_n| - i\int_{D_n} K\,dx\right) = \sum_{n=1}^{4} I_n, \text{ say},$$

in accordance with the division of the interval $(0, \omega)$ into D_n such that

$$\begin{aligned}&D_1 = E_x[|f(x)| < \varepsilon\sigma_1, \quad |f(x+\tau)| < \varepsilon\sigma_2],\\ &D_2 = E_x[|f(x)| > \varepsilon\sigma_1, \quad |f(x+\tau)| < \varepsilon\sigma_2],\\ (10.5)\qquad &D_3 = E_x[|f(x)| < \varepsilon\sigma_1, \quad |f(x+\tau)| > \varepsilon\sigma_2],\\ &D_4 = E_x[|f(x)| > \varepsilon\sigma_1, \quad |f(x+\tau)| > \varepsilon\sigma_2],\end{aligned}$$

where $\sigma_1 = \sigma(\omega)$, $\sigma_2 = \sigma(\omega+\tau)$. By means of the Taylor expansion of the exponential function and (10.5) we obtain

$$(10.6)\qquad I_1 = -\frac{1}{2}\int_{D_1} K^2\,dx + O\left(\int_{D_1} K^3\,dx\right) = -\frac{1}{2}\int_{D_1} K^2\,dx + O(\varepsilon),$$

where, as in the following, O depends only on a. By (c) and (10.5)

$$I_2=\int_{D_2}\{1+iz_1\sigma_1^{-1}f(x)+O(\sigma_1^{-2}f^2(x))\}\{1+iz_2\sigma_2^{-2}f(x+\tau)$$

$$-\frac{1}{2}\sigma_2^{-1}z_2^2f^2(x+\tau)+O(\varepsilon\sigma_2^{-2}f^2(x+\tau))\}\,dx-|D_2|-i\int_{D_2}Kdx$$

$$=-z_1z_2\sigma_1^{-1}\sigma_2^{-1}\int_{D_2}f(x)f(x+\tau)\,dx-\frac{1}{2}z_2^2\sigma_2^{-2}\int_{D_2}f^2(x+\tau)\,dx$$

$$+O(\sigma_1^{-1}\sigma_2^{-2}\int_{D_2}f(x)f^2(x+\tau)\,dx)$$

$$+O(\varepsilon\sigma_2^{-2}\int_{D_2}|1+iz_1\sigma_1^{-1}f(x)|f^2(x+\tau)\,dx)+O(\sigma_1^{-2}\int_{D_2}f^2(x)\,dx) \tag{10.7}$$

$$=-z_1z_2\sigma_1^{-1}\sigma_2^{-1}\int_{D_2}f(x)f(x+\tau)\,dx-\frac{1}{2}z_2^2\sigma_2^{-2}\int_{D_2+D_3+D_4}f^2(x+\tau)\,dx$$

$$+O(\varepsilon)+O(\gamma_\varepsilon(\omega+\tau)).$$

Similarly

$$I_3=-z_1z_2\sigma_1^{-1}\sigma_2^{-1}\int_{D_3}f(x)f(x+\tau)\,dx-\frac{1}{2}z_1^2\sigma_1^{-2}\int_{D_2+D_3+D_4}f^2(x)\,dx \tag{10.8}$$

$$+O(\varepsilon)+O(\gamma_\varepsilon(\omega+\tau)),$$

and

$$I_4=O(\sigma_1^{-2}\int_{D_4}f^2(x)\,dx+\sigma_2^{-2}\int_{D_4}f^2(x+\tau)\,dx)=o(1). \tag{10.9}$$

Combining (10.6), (10.7), (10.8), (10.9)

$$\psi(z_1,z_2)=-\frac{1}{2}(z_1^2+2\rho(\tau)z_1z_2+z_2^2)+O(\varepsilon)+o(1),$$

which proves the first part of the theorem for $k=2$, since ε can be chosen as small as we please. The general case is treated in a similar manner.

The second half of the theorem follows easily by making use of the expression $\psi(z_1, z_2, \ldots)$ for $\xi(t)$.

If, in particular, $f(x)$ is a u.a.p. function

$$f(x)\sim\sum a_\lambda\cos\lambda x,$$

then $\rho(\tau)$ of (i) is given by

$$\rho(\tau)=\sum a_\lambda^2\cos\lambda\tau/\sum a_\lambda^2,$$

and the limit process is a B_2- function with probability 1, such that

$$x(t)=\sum(A_\lambda\cos\lambda t+B_\lambda\sin\lambda t),$$

where A_λ, B_λ are normally distributed independent variables with the mean value 0 and

$$E(A_\lambda^2) = E(B_\lambda^2) = a_\lambda^2/\sum a_\lambda^2.$$

If $f(x)$ is absolutely integrable over $(0, \infty)$, the s.s.p. $\theta(t)$ considered in (i) of Theorem 22 is strongly mixing. Consequently the law of large numbers holds:

$$A^{-1}\int_0^A \theta(t)\,dt \to \int_0^\infty f(x)\,dx, \quad A \to \infty,$$

with probability 1. In this connection, we shall prove the following theorem.

Theorem 24. *Let $f(t)$ be continuous and absolutely integrable over $(0, \infty)$, then*

$$P\Big(a < A^{-1/2}\Big(\int_0^A \theta(t)\,dt - A\int_0^\infty f(t)\,dt\Big) < b\Big) \to (2\pi)^{1/2}\sigma^{-1}\int_a^b e^{-u^2/2\sigma^2}\,du,$$

$$\sigma^2 = \Big(\int_0^\infty f(x)\,dx\Big)^2, \quad \textit{as } A \to \infty.$$

Proof. Let us put

$$x = A^{-1/2}\Big(\int_0^A \theta(t)\,dt - A\int_0^\infty f(t)\,dt\Big),$$

then, since

$$\log E\{\exp(iz\int_0^A \theta(t)\,dt)\} = \int_{-\infty}^\infty \exp\Big(iz\int_0^A f(t-\tau)\,dt\Big) - 1)\,d\tau$$

and

$$\int_{-\infty}^\infty d\tau\int_0^A f(t-\tau)\,dt = A\int_0^\infty f(t)\,dt,$$

we have

$$\log E(e^{izx}) = -\frac{z^2}{2}A^{-1}\int_{-\infty}^\infty \Big(\int_0^A f(t-\tau)\,dt\Big)^2\,d\tau \tag{10.10}$$
$$+ O\Big(A^{-2/3}\Big|\int_{-\infty}^\infty \Big(\int_0^A f(t-\tau)\,dt\Big)^3\,d\tau\Big|\Big).$$

The existence of the integrals on the right-hand side in (10.10) will be obvious from the following arguments. Let

$$g(u) = \int_{-\infty}^\infty f(u+\tau)f(\tau)\,d\tau,$$

then $g(u)$ is absolutely integrable over $(-\infty, \infty)$, and we have

$$A^{-1}\int_{-\infty}^\infty \Big(\int_0^A f(t-\tau)\,dt\Big)^2\,d\tau = A^{-1}\int_0^A\int_0^A ds\,dt\int_{-\infty}^\infty f(s-\tau)f(t-\tau)\,d\tau$$

$$= \int_{-A}^A \Big(1 - \frac{|t|}{A}\Big) g(t)\,dt \to \int_{-\infty}^\infty g(t)\,dt = \Big(\int_{-\infty}^\infty f(t)\,dt\Big)^2.$$

Whence, also

$$A^{-3/2}\left|\int_{-\infty}^{\infty}\left(\int_0^A f(t-\tau)\,dt\right)^3 d\tau\right| \doteq O\left(A^{-1/2}A^{-1}\int_{-\infty}^{\infty}\left(\int_0^A f(t-\tau)\,dt\right)^2 d\tau\right)$$
$$= O(A^{-1/2}).$$

Hence, by (10.10)

$$\log E(e^{izx}) \to -\frac{z^2}{2}\left(\int_{-\infty}^{\infty} f(t)^2\right), \quad A\to\infty,$$

uniformly for z in every bounded interval. This proves the theorem.

In the preceding considerations we have considered only one kind of event as the source of the fluctuation. We can easily generalize the method to the case of several types of events, say types $u, v, \ldots$. Let $\Delta_t\,\Delta_u\,\Delta_v \ldots \nu(t; u, v, \ldots)$ be the total number of events in the interval $(t, t+\Delta t)$ belonging to the region $(u, u+\Delta u; v, v+\Delta v; \ldots)$, and let it be distributed according to the Poisson distribution with the mean value $\Delta_t\,\Delta_u\,\Delta_v \ldots N(t; u, v, \cdots)$. Further, in accordance with the preceding considerations, we assume that $\Delta_t\,\Delta_u\,\Delta_v \ldots \nu(t; u, v, \ldots)$ are independent vairables when corresponding intervals of $t, u, \ldots$ are non-overlapping. If one event of the type $(u, v, \ldots)$ gives rise to an addition of the function $f(t; u, v, \ldots)$, the fluctuation at an instant t is given by[40]

$$\theta(t) = \int\cdots\int_{t_0}^{t} f(t-\tau; u, v, \ldots)\, d_t d_u d_v \ldots \nu(t; u, v, \ldots),$$

where $\int\cdots$ indicates integrations with respect to $u, v, \cdots$, taken over all possible values of these variables. As in the case of a single event we obtain

(10.11) $$\log E\{\exp(iz\theta(t))\} = \int\cdots\int_{t_0}^{t}\{\exp(izf(t-\tau; u, v, \ldots))-1\} \cdot d_t d_u \ldots N(t; u, \ldots),$$

(10.12) $$E(\theta(t)) = \int\cdots\int_{t_0}^{t} f(t-\tau; u, \ldots)\, d_t d_u \ldots N(t; u, \ldots),$$

(10.13) $$E\{(\theta(t)-E(\theta(t)))^2\} = \int\cdots\int_{t_0}^{t} f^2(t-\tau; u, \ldots) \cdot d_t d_u \ldots N(t; u, \ldots).$$

When the events are homogeneous in time $\Delta_t\,\Delta_u \ldots N(t; u, \ldots)$ reduces to the form $\Delta t\,\Delta_u\,\Delta_v \ldots N(u, v, \ldots)$. The fluctuation due to events occuring in the form of showers turns out to be a special case of the present considerations. Indeed when the events in $(t,$

40) Rowland [1].

$t+\Delta t$) consist of a single event, a pair of events, ... with respective probabilities $\lambda_1 \Delta t$, $\lambda_2 \Delta t$, ..., it is sufficient for our purpose to take $N(u, ...) = N(u)$, $N(u)$ being a step-function with jumps $\lambda_1, \lambda_2, ...$ at $u = 1, 2, ...$, and $f(t; u, ...) = uf(t)$. Then by (10.12) and (10.13) the mean value and variance of $\theta(t)$ are given by[41]

$$(\lambda_1+2\lambda_2+\ldots)\int_0^{t-t_0} f(t)\,dt,$$

and

$$(\lambda_1+2^2\lambda_2+\cdots)\int_0^{t-t_0} f^2(t)\,dt.$$

Now we shall apply these results to the torsional oscillation of a mirror suspended in a gas. According to Gerlach's experiment, the curves of the deflection of the mirror are different in appearance at different pressures, though their mean-square deviations have the same value. A theoretical treatment of this problem has been giben first by G. E. Uhlenbeck and S. Goudsmit[42]. In the following we shall mention it in connection with a stationary process. The deflection of the mirror after an impact, at $t=0$, of a molecule of the gas is given by solving the equation

$$I\frac{d^2\varphi}{dt^2}+r\frac{d\varphi}{dt}+D\varphi=0, \tag{10.14}$$

under the condition $\varphi(0)$, $\dot{\varphi}(0)=c$, where c is the initial angular velocity due to the bombardment of a molecule. If the normal component of the momentum of the molecule which strikes the surface element Δo of the mirror, at a distance x from its axis of rotation, is g, then by the conservation of angular momentum

$$c\int x^2\,dm = 2gx,$$

where the integral is to be taken over the surface of the mirror with respect to the mass of its elementary portion. Whence the solution of (10.14) is given by

$$f(t;\,o,g) = \frac{2gx}{\mu I}e^{-\lambda t}\sin \mu x,\ \lambda = \frac{1}{2}\,r/I,$$

$$\mu^2 = D/I-\frac{r^2}{4}/I^2.$$

41) J. M. Whittaker [1].

42) Uhlenbeck and Goudsmit [1], Fowler, loc. cit., §§ 20.7, 71, 72, 73.

Uhlenbeck and Goudsmit estimated the frictional coefficient by means of the Maxwellian distribution

$$r = 2m\bar{c}pOx^2/kT,$$

where $\bar{c}$ is the mean velocity, m the mass of the molecule, O the surface area of the mirror, in the radius of gyration. The events considered in the foregoing now consist of bombardments which are specified by o and g. Again by Maxwellian distribution, the number of molecules which strike the portion Δo, in a time-interval $(t, t+\Delta t)$, with a momentum lying between g and $g+\Delta g$ and normal to the portion, is gived by[43]

$$n(t; g, o)\,\Delta t\,\Delta g\,\Delta o = \frac{N}{V}\frac{|g|/m}{(2\pi kmT)^{1/2}}\,e^{-g^2/2mkT}\,\Delta o\,\Delta g\,\Delta t$$

with the total number of molecules N and the total volume V. In the present case the last expression gives $\Delta_t\Delta_g\Delta_o N(t; g, o)$, the average number of events of the type $(g, g+\Delta g; \Delta o)$ in an time-interval $(t, t+\Delta t)$. The occurrence of the events is homogeneous in time. Independence between successive impacts, which is justified if the gas is dilute, leads us to the Poisson distribution for $\Delta_t\Delta_g\Delta_o\nu(t; g, o)$. Thus prepared we are ready to give an expression which corresponds to (10.11)

$$\log E\{\exp(iz\theta(t))\}$$
$$=\int_{-\infty}^{\infty} dg\int do\int_{-\infty}^{t}\{\exp(iz\frac{2gx}{\mu I}e^{-\lambda(t-\tau)}\sin\mu(t-\tau)-1\}n(t; g, o)\,d\tau.$$

An easy modification of Theorem 22 shows that $\theta(t)$ is a strongly mixing s.s.p. Since $\int x\,d\,o = 0$, the mean value of $\theta(t)$ is zero. Hence by the ergodic theorem

$$\underset{t}{M}[\theta(t)] = 0$$

with probability 1. The correlation function is given by

$$\rho(\tau) =\int_{-\infty}^{\infty} dg\int do\int_0^{\infty}\frac{4g^2x^2}{\mu^2 I^2}e^{-\lambda s}\sin\mu s\,e^{-\lambda(s+\tau)}\sin\mu(s+\tau)$$

(10.15) $$\cdot\, n(s; g, o)\,ds$$

$$=\frac{4}{I^2}\frac{N}{Vm}\frac{N}{\sigma\sqrt{2\pi}}\left(4\sigma^4\int_0^{\infty}ue^{-u}du\right)\left(\int x^2\,do\right)$$

[43] Fowler, loc. cit., p. 776.

$$\left(\frac{e^{-\tau}}{\mu^2}\int_0^\infty e^{-2\lambda s}\frac{\cos\mu\tau-\cos(2\mu s+\mu\tau)}{2}ds\right),\quad \sigma=\sqrt{mkT}.$$

Hence easy calculations give us

$$\frac{1}{2}D\rho(\tau)=\frac{kT}{2}(\cos\mu\tau+\frac{\lambda}{\mu}\sin\mu\tau)e^{-\lambda\tau}, \tag{10.16}$$

where we have used the well-known formulas $\bar{c}=(8kT/\pi m)^{1/2}$, $pV=NkT$ and the expression for r already stated. As established in §6, the correlation function of the process $\dot\theta(t)$ is given by $-\rho''(\tau)$. Hence

$$E(\dot\theta(t)\dot\theta(t+\tau))=\frac{kT}{I}e^{-\lambda\tau}(\cos\mu\tau-\frac{\lambda}{\mu}\sin\mu\tau). \tag{10.12}$$

Taking $\tau=0$ in (10.16), (1017) we obtain the equipartition principle of energy

$$\frac{1}{2}D\underset{t}{M}[\theta^2(t)]=\frac{1}{2}I\underset{t}{M}[\dot\theta^2(t)]=\frac{kT}{2}.$$

If p is small, then r and hence λ is small. Hence if the pressure becomes very low $\rho(\tau)$ tends to $\frac{kT}{D}\cos\sqrt{\frac{D}{I}}\tau$ and the sinusoidal limit theorem enables us to conclude that

$$P(|\theta(t)-A\cos\sqrt{D/I}\,t-B\sin\sqrt{D/I}\,t|>\delta)<\varepsilon,$$
$$E(A^2)=E(B^2)=\frac{kT}{D},\ E(AB)=0,$$

for every t.

11. The law of the iterated logarithm for normal s.s.p. As is shown in §§ 3—5, the law of large numbers for normal s.s.p's is closely related to the continuity of the s.d.f.'s. In this connection we shall prove the following theorem.

Theorem 25. *Let $x_1, x_2, \dots,$ be a normal s.s.p. having the common mean value* 0, *vaaiance* 1, *and s.d.f. $F(x)$ such that*

$$F(b)-F(a)=(2\pi)^{-1}\int_a^b g(x)\,dx\quad (b>a\geqq 0),$$
$$g(x)\geqq 0,\ g(-x)=g(x),$$
$$\sqrt{g(x)}=\alpha+o\left(\frac{\log\log 1/|x|}{\log 1/|x|}\right),\ as\,|x|\to 0,$$

for some constant $\alpha\geqq 0$.

Then we have

$$P\left(\limsup_{n\to\infty}\frac{|x_1+x_2+\dots+x_n|}{\sqrt{2n\log\log n}}=a\right)=1.$$

Proof. Since x_n is given by

$$x_n=\int_{-\pi}^{\pi}e^{inx}\sqrt{g(x)}\,dS(x),$$

$$S(-x)=-\bar{S}(x),\ E(|S(b)-S(a)|^2)=(2\pi)^{-1}(b-a),$$

we obtain

$$\sum_{\nu=1}^{n}x_\nu=\sum_{\nu=1}^{n}y_\nu+\int_{-\pi}^{\pi}p_n(x)f(x)\,dS(x),$$

where

$$y_\nu=a\int_{-\pi}^{\pi}e^{i\nu x}\,dS(x),\ p_n(x)=e^{ix}\frac{1-e^{inx}}{1-e^{ix}},$$

$$f(x)=\sqrt{g(x)}-a.$$

The variables y_n being independently and equally distributed, the law of the iterated logarithm holds:[44]

$$P\left(\limsup_{n\to\infty}\frac{|y_1+\cdots+y_n|}{\sqrt{2n\log\log n}}=1\right)=1.$$

Hence it is sufficient for our purpose to prove that

(11.1) $$P\left(\lim_{n\to\infty}\frac{1}{\sqrt{n\log\log n}}\int_{-\pi}^{\pi}p_n(x)f(x)\,dS(x)=0\right)=1.$$

Now let us put

$$p_n=P\left(\frac{1}{\sqrt{n\log\log n}}\int_{-\pi}^{\pi}p_n(x)f(x)\,dS(x)>\delta\right),$$

then

$$p_n=\frac{2}{\sigma_n\sqrt{2\pi}}\int_\delta^\infty e^{-t^2/2\sigma_n^2}dt\sim\sqrt{\frac{2}{\pi}}\frac{\sigma_n}{\delta}e^{-\delta^2/2\sigma_n^2},$$

where

$$\sigma^2_n=(\pi n\log\log n)^{-1}\int_0^\pi\left(\frac{\sin nx/2}{\sin x/2}\right)^2f^2(x)\,dx=(\pi n\log\log n)^{-1}$$

(11.2) $$\left(\int_0^{1/n}+\int_{1/n}^{\pi}\right)\left(\frac{\sin nx/2}{\sin x/2}\right)^2f^2(x)\,dx$$

$$=o\left(\frac{1}{\log n}\right)+O\left((n\log\log n)^{-1}\int_{1/n}^{\pi}\frac{f^2(x)}{x^2}dx\right),$$

44) Hartman and Wintner [1].

and

$$(n \log \log n)^{-1}\int_{1/n}^{\pi} \frac{f^2(x)}{x^2}\,dx = (n \log \log n)^{-1}\left(\int_{1/n}^{\varepsilon} + \int_{\varepsilon}^{\pi}\right)\frac{f^2(x)}{x^2}dx$$

$$(11.3) \qquad = o\left(\frac{1}{\log n}\right) + (n \log \log n)^{-1}\int_{1/n}^{\varepsilon} \frac{f^2(x)}{x^{\eta}\,x^{2-\eta}}\,dx$$

$$= o\left(\frac{1}{\log n}\right) + o\left((n \log \log n)^{-1} n^{\eta} \frac{\log \log n}{\log n}\, n^{1-\eta}\right)$$

$$= o\left(\frac{1}{\log n}\right).$$

On combining (11.2) and (11.3), for every $\delta > 0$, we have

$$p_n = O(n^{-k}),\ k > 1,$$

consequently

$$\sum_{n=1}^{\infty} p_n < \infty.$$

This proves (11.1), and hence the theorem.

Let us consider the familiar example of the normal Markoff process x_k, in which the correlation function becomes $R(k) = \rho^{|k|}$, $k = 0, \pm 1, \pm 2, \ldots, |\rho| \leqq 1$. Since, when $\rho = 1$, and -1, x_k can be respectively put in the form $x_k = x$, or $(-1)^k x$, where x denotes a normal random variable, we are mainly interested in the case $|\rho| < 1$. In this case $g(x)$ can be expressed in an absolutely convergent Fourier series:

$$g(x) = \sum_{-\infty}^{\infty} \rho^{|n|} e^{inx}, \quad \rho^{|n|} = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{inx} g(x)\,dx,$$

and we have

$$\sqrt{g(x)} = \sqrt{g(0)}\sqrt{1-(4/g(0))\sum_{1}^{\infty} \rho^n \sin^2(nx/2)}$$

$$= \sqrt{g(0)} + O(|x|^2),\ as\ |x| \to 0,\ g(0) = (1+\rho)/(1-\rho).$$

By means of Theorem 25, we now conclude that

$$P\left(\limsup_{n\to\infty} \frac{|x_1 + x_2 + \ldots + x_n|}{\sqrt{2n \log \log n}} = \sqrt{\frac{1+\rho}{1-\rho}}\right) = 1.$$

This relation formally holds for the cases $\rho = 1$, and -1.

In conclusion, I wish to express my warmest appreciation of the helpful criticism and encouragement I have received from Prof. T. Kitagawa in the preparation of this paper.

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