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On the Function $S(T)$ in the Theory of the Riemann Zeta-Function

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The function $S(T)$ is the error term in the formula for the number of zeros of the Riemann zeta-function above the real axis and up to height T in the complex plane. We assume the Riemann hypothesis, and examine how well $S(T)$ can be approximated by a Dirichlet polynomial in the L^2 norm.

1. INTRODUCTION

Let $N(T)$ denote the number of zeros $\rho=\beta+i\gamma$ of the Riemann zeta-function $\zeta(s)$ for which $0<\gamma<T$, where T is not equal to any γ, and otherwise

$$N(T)=\lim_{\varepsilon\to 0} 1/2\{N(T+\varepsilon)+N(T-\varepsilon)\}.$$

As usual the zeros are counted with multiplicity. It follows from the functional equation and the argument principle that

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+\frac{7}{8}+O\left(\frac{1}{T}\right)+S(T). \tag{1.1}$$

Here the term $O(1/T)$ is continuous in T, and $S(T)$ is defined, for $T\neq\gamma$, by

$$S(T)=1/\pi \arg \zeta(1/2+iT), \tag{1.2}$$

where the argument is obtained by continuous variation along the horizontal line $\sigma+iT$ starting with the value zero at $\infty+iT$. If $T=\gamma$, then in agreement with $N(T)$ we define

$$S(T)=\lim_{\varepsilon\to 0} 1/2\{S(T+\varepsilon)+S(T-\varepsilon)\}.$$

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Concerning the size of $S(T)$, it is known that

$$S(T) \ll \log T, \qquad S(T) = \Omega \pm \{(\log T)^{1/3}/(\log\log T)^{7/3}\}, \tag{1.3}$$

due respectively to von Mangoldt (see [1]) in 1905 and Selberg [16] in 1946. Assuming the Riemann hypothesis (RH), the best results are

$$S(T) \ll \log T/\log\log T, \qquad S(T) = \Omega_{\pm}\{(\log T/\log\log T)^{1/2}\}, \tag{1.4}$$

due respectively to Littlewood [10] in 1924 and Montgomery [11, Theorem 13.17] (see also [13]) in 1971.

The next question to consider is the statistical behavior of $S(T)$, and here an essentially complete answer is known. In [16] Selberg obtained an asymptotic formula for the even moments of $S(T)$. His result is that

$$\int_0^T |S(t)|^{2k}\, dt = \frac{(2k)!}{k!(2\pi)^{2k}} T(\log\log T)^k + O(T(\log\log T)^{k-1/2}) \tag{1.5}$$

for $k = 1, 2, 3, \ldots$. A few years earlier Selberg [15] had obtained (1.5) on the RH, but with a better error term

$$O(T(\log\log T)^{k-1}).$$

Recently, Ghosh [2, 3] has obtained an asymptotic formula for *every* moment; namely

$$\int_0^T |S(t)|^{\lambda}\, dt \sim \frac{\Gamma\left(\dfrac{\lambda+1}{2}\right)}{\pi^{\lambda+1/2}} T(\log\log T)^{\lambda/2} \tag{1.6}$$

for any real number $\lambda > -1$. We should mention that the results (1.5) and (1.6) are actually special cases of what has been proved; both Selberg and Ghosh obtain their results for integrals over $[T, T+H]$, where $T^a \leqslant H \leqslant T$ and $a > \frac{1}{2}$ ($a > 0$ is acceptable on RH). As a consequence of (1.6), $|S(T)|$ is normally distributed around its average order $(\log\log T)^{1/2}$ (see [3]).

In proving (1.5), Selberg used an explicit formula to find a Dirichlet polynomial which approximates $S(T)$ closely in the L^q norm, with $q = 2k$. The approximation is good even when very few terms are taken in the Dirichlet polynomial, and therefore the $2k$th moment of the polynomial is asymptotic to the diagonal terms, which give the main term in (1.5). Specifically, Selberg proved, for $T^{\varepsilon/k} \leqslant x \leqslant T^{1/k}$ and any $\varepsilon > 0$,

$$\int_0^T \left| S(t) + \frac{1}{\pi} \sum_{p \leqslant x} \frac{\sin(t \log p)}{\sqrt{p}} \right|^{2k} dt \leqslant c(k)\, T, \tag{1.7}$$

where the sum is over prime numbers p, and $c(k)$ is a constant depending only on k. The constant $c(k)$ may be made explicit, Ghosh [2, Lemma 5] obtained, for absolute constants A and B, $c(k) = B(Ak)^{4k}$, when $x = T^{1/40k}$.

In this paper we examine the result (1.7) more closely in the mean square case when $k = 1$. Our results depend on the Riemann hypothesis, and use the techniques developed by Montgomery [12] in his work on pair correlation of zeros of $\zeta(s)$. Following Montgomery, we define

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0 < \gamma, \gamma' \leqslant T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'), \tag{1.8}$$

where $w(u) = 4/(4 + u^2)$, and $\alpha \geqslant 0$, $T \geqslant 2$. We will write this function $F(\alpha)$ when we do not want to emphasize the T dependence. We have $F(\alpha) = F(-\alpha)$, and $F(\alpha) \geqslant 0$ (see [7]). Our main result is

THEOREM 1. *Assume the Riemann hypothesis. For fixed* $0 < \beta \leqslant 1$, *and* $x = T^{\beta}$, *we have*

$$\int_0^T \left| S(t) + \frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} \right|^2 dt = \frac{T}{2\pi^2} \left(\log(1/\beta) + \int_1^{\infty} \frac{F(\alpha, T)}{\alpha^2} \, d\alpha \right) + o(T). \tag{1.9}$$

Also,

$$\int_0^T |S(t)|^2 \, dt = \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left(\int_1^{\infty} \frac{F(\alpha, T)}{\alpha^2} \, d\alpha + C + \sum_{m=2}^{\infty} \sum_p \left(\frac{1}{m} + \frac{1}{m^2} \right) \frac{1}{p^m} \right) + o(T). \tag{1.10}$$

Here C is Euler's constant, and $\Lambda(n) = \log p$ if $n = p^m$, for p a prime and $m \geqslant 1$, and $\Lambda(n) = 0$ otherwise. Here and throughout this paper p will denote a prime, and sums over p are over all primes. We will also use the convention that all summations involving $\Lambda(n)$ start at $n = 2$.

In order to examine the term

$$\int_1^{\infty} \frac{F(\alpha, T)}{\alpha^2} \, d\alpha,$$

we need to have information on averages of $F(\alpha, T)$. Montgomery proved [12], assuming RH, for fixed $0<\beta<1$,

$$\sum_{0<\gamma,\gamma'\leqslant T}\left[\frac{\sin\frac{\beta}{2}(\gamma-\gamma')\log T}{\frac{\beta}{2}(\gamma-\gamma')\log T}\right]^2 w(\gamma-\gamma')\sim\left(\frac{1}{\beta}+\frac{\beta}{3}\right)\frac{T}{2\pi}\log T \quad (1.11)$$

as $T\to\infty$, where w is the function in (1.8), and the terms with $\gamma=\gamma'$ are equal to 1. This also holds for $\beta=1$ (see comment following Eq. (4.6)). Using this result, we prove in Section 7 the following lemma, which we shall need to use occasionally.

LEMMA A. *Assume the Riemann hypothesis. For any* $\varepsilon>0$ *and* T *sufficiently large, we have*

$$\int_c^{c+1} F(\alpha, T)\,d\alpha<\tfrac{8}{3}+\varepsilon \quad (1.12)$$

and

$$\int_{c-1}^{c+1} F(\alpha, T)\,d\alpha>\tfrac{2}{3}-\varepsilon, \quad (1.13)$$

uniformly for any real number c *(*c *may be a function of* T*).*

As an immediate application of Lemma A, we have

$$\int_1^\infty\frac{F(\alpha, T)}{\alpha^2}\,d\alpha\leqslant\sum_{n=1}^\infty\frac{1}{n^2}\int_n^{n+1}F(\alpha, T)\,d\alpha\leqslant\frac{4\pi^2}{9}+\varepsilon\leqslant 4.4,$$

and

$$\int_1^\infty\frac{F(\alpha, T)}{\alpha^2}\,d\alpha\geqslant\sum_{n=1}^\infty\frac{1}{(2n+1)^2}\int_{2n-1}^{2n+1}F(\alpha, T)\,d\alpha\geqslant\frac{\pi^2}{12}-\frac{2}{3}-\varepsilon>0.15.$$

By arguing directly from (1.11), we obtain better bounds.

THEOREM 2. *Assume the Riemann hypothesis. For any* $\varepsilon>0$ *and* T *sufficiently large, we have*

$$2/3-\varepsilon<\int_1^\infty\frac{F(\alpha, T)}{\alpha^2}\,d\alpha<2. \quad (1.14)$$

The constants in Theorem 2 could be improved on somewhat by more complicated arguments, but it appears difficult to prove they may be replaced with an asymptotic relation as $T \to \infty$. However, there are grounds for conjecturing that there is a limiting behavior. Montgomery has conjectured, on number-theoretic grounds, that

$$F(\alpha, T) = 1 + o(1) \qquad \text{uniformly for } 1 \leqslant \alpha \leqslant M, \tag{1.15}$$

for any fixed M. This conjecture implies the well-known pair correlation conjecture for zeros of $\zeta(s)$. Taking M to be an integer, we have by (1.12) and (1.15),

$$\int_1^\infty \frac{F(\alpha, T)}{\alpha^2}\, d\alpha = \int_1^M \frac{1+o(1)}{\alpha^2}\, d\alpha + O\left(\sum_{n=M}^{\infty} \frac{1}{n^2} \int_n^{n+1} F(\alpha, T)\, d\alpha \right)$$
$$= 1 + o(1) + O\left(\frac{1}{M}\right).$$

Since M may be taken as large as we please, we are led to the following:

Conjecture. We have, as $T \to \infty$,

$$\int_1^\infty \frac{F(\alpha, T)}{\alpha^2}\, d\alpha = 1 + o(1). \tag{1.16}$$

As a corollary of Theorem 1, we obtain a lower bound for the constant $c(k)$ of (1.7):

COROLLARY. *Assume the Riemann hypothesis. We have, for $x = T^\beta$, fixed β with $0 < \beta \leqslant 1$, $k \geqslant 1$, and T sufficiently large,*

$$\int_0^T \left| S(t) + \frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} \right|^{2k} dt > T\left(\frac{1}{20} \log \frac{1}{\beta}\right)^k. \tag{1.17}$$

The proof is immediate by applying Hölder's inequality to (1.9). Letting $\beta = 1/k$, we obtain $c(k) > (A \log k)^k$, where A is an absolute constant. The corollary places a limitation on how well $S(t)$ can be approximated in L^q norm by the Dirichlet polynomial

$$-\frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n}.$$

It is not hard to prove, using a lemma of Titchmarsh (see (5.10)), that under reasonable conditions this Dirichlet polynomial is the one which best approximates $S(t)$ in the L^2 norm. Therefore, the lower bound in (1.17) holds for any short Dirichlet polynomial used to approximate $S(t)$.

2. An Approximate Formula for $S(t)$

Our starting point is an explicit formula of Montgomery [12], which depends on the Riemann hypothesis. For $x \geqslant 1$, $s = \sigma + it$, we have

$$
\begin{aligned}
(2\sigma - 1)\sum_{\gamma} \frac{x^{i(\gamma - t)}}{(\sigma - 1/2)^2 + (t-\gamma)^2} &= x^{\sigma - 1/2}\frac{\zeta'}{\zeta}(\sigma + it) \\
&- x^{1/2-\sigma}\frac{\zeta'}{\zeta}(1 - \sigma + it) + \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{it}}\left(\frac{x^{\sigma - 1/2}}{n^{\sigma}} - \frac{x^{1/2-\sigma}}{n^{1-\sigma}}\right) \\
&+ x^{1/2 - it}\left(\frac{1 - 2\sigma}{(\sigma - it)(1 - \sigma - it)}\right) \\
&+ x^{-1/2-it}\sum_{n=1}^{\infty} \frac{x^{-2n}(2\sigma - 1)}{(2n + 1 - \sigma + it)(2n + \sigma + it)},
\end{aligned}
\tag{2.1}
$$

provides $s \neq 1$, $s \neq 1/2 + i\gamma$, $s \neq -2n$. This formula is obtained from the well-known unconditional formula [9]

$$
\frac{\zeta'}{\zeta}(s) = -\sum_{n \leqslant x} \frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho - s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{(2n+s)},
\tag{2.2}
$$

where $x > 1$, $x \neq p^n$, $s \neq 1$, $s \neq -2n$, $s \neq \rho$. Equation (2.1) follows from (2.2) by assuming RH so that $\rho = 1/2 + i\gamma$ and combining (2.2) appropriately when $s = \sigma + it$ and $s = 1 - \sigma + it$. We next note, for $t \geqslant 1$,

$$
\operatorname{Im}\frac{\zeta'}{\zeta}(1 - \sigma + it) = \operatorname{Im}\frac{\zeta'}{\zeta}(\sigma + it) + O\left(\frac{|\sigma - 1/2|}{t}\right).
\tag{2.3}
$$

To see this, we use the functional equation in the form

$$
\zeta(s) = \chi(s)\,\zeta(1-s),
$$

where

$$
\chi(s) = \frac{\pi^{s-1/2}\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}.
$$

Hence

$$
\operatorname{Im}\frac{\zeta'}{\zeta}(s) = -\operatorname{Im}\frac{\zeta'}{\zeta}(1-s) + \operatorname{Im}\frac{\chi'}{\chi}(s).
$$

By the reflection principle

$$
\operatorname{Im}\frac{\zeta'}{\zeta}(1-s) = -\operatorname{Im}\frac{\zeta'}{\zeta}(1-\bar{s}),
$$

and (2.3) follows from the estimate

$$\operatorname{Im}\frac{\chi'}{\chi}(s) \ll \frac{|\sigma-1/2|}{t} \qquad (t \geqslant 1), \tag{2.4}$$

which may be easily proved from the partial fraction formula for the logarithmic derivative of the gamma function.

We now take imaginary parts of (2.1) and apply (2.3) to obtain, for $t \geqslant 1$, $t \neq \gamma$,

$$\begin{aligned}
&(x^{\sigma-1/2} - x^{1/2-\sigma}) \operatorname{Im}\frac{\zeta'}{\zeta}(\sigma+it) \\
&\quad = \sum_{n \leqslant x} \Lambda(n) \sin(t \log n) \left(\frac{x^{\sigma-1/2}}{n^{\sigma}} - \frac{x^{1/2-\sigma}}{n^{1-\sigma}}\right) \\
&\qquad - x^{1/2} \operatorname{Im}\left(\frac{x^{-it}(1-2\sigma)}{(\sigma-it)(1-\sigma-it)}\right) \\
&\qquad - \sum_{\gamma} \sin((t-\gamma)\log x) \frac{2(\sigma-1/2)}{(\sigma-1/2)^2+(t-\gamma)^2} \\
&\qquad + O\left(\frac{x^{-5/2}\,|\sigma-1/2|}{t}\right) + O\left(\frac{x^{1/2-\sigma}\,|\sigma-1/2|}{t}\right);
\end{aligned} \tag{2.5}$$

here the second to last error term is obtained from

$$\begin{aligned}
&\left| x^{-1/2-it} \sum_{n=1}^{\infty} \frac{x^{-2n}(2\sigma-1)}{(2n+1-\sigma+it)(2n+\sigma+it)} \right| \\
&\qquad \ll x^{-5/2}\,|\sigma-1/2| \sum_{n=1}^{\infty} \frac{1}{|2n+it+1-\sigma|\;|2n+it+\sigma|},
\end{aligned}$$

and, since the sum is unchanged by replacing σ by $1-\sigma$, we may suppose $\sigma \geqslant \frac{1}{2}$ and conclude

$$\begin{aligned}
&\ll x^{-5/2}(\sigma-1/2) \sum_{n=1}^{\infty} \frac{1}{|2n+it+1-\sigma|^2} \\
&\ll \frac{x^{-5/2}(\sigma-1/2)}{t}.
\end{aligned}$$

Now, for $t \neq \gamma$, we have

$$S(t) = -\frac{1}{\pi}\int_{1/2}^{\infty} \operatorname{Im}\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma. \tag{2.6}$$

We now assume $\sigma > \frac{1}{2}$. Dividing (2.5) by $(x^{\sigma-1/2} - x^{1/2-\sigma})$ and integrating, we obtain, for $x \geqslant 4$, $t \geqslant 1$, $t \neq \gamma$,

$$S(t)=\frac{-1}{\pi}\sum_{n\leqslant x}\Lambda(n)\sin(t\log n)$$
$$\times\int_{1/2}^{\infty}\left(\frac{x^{\sigma-1/2}}{n^{\sigma}}-\frac{x^{1/2-\sigma}}{n^{1-\sigma}}\right)\frac{d\sigma}{2\sinh((\sigma-1/2)\log x)}$$
$$+\frac{x^{1/2}}{\pi}\operatorname{Im}\left(x^{-it}\int_{1/2}^{\infty}\left(\frac{1-2\sigma}{(\sigma-it)(1-\sigma-it)}\right)\frac{d\sigma}{2\sinh((\sigma-1/2)\log x)}\right)$$
$$+\frac{1}{\pi}\sum_{\gamma}\sin((t-\gamma)\log x)\int_{1/2}^{\infty}\left(\frac{(\sigma-1/2)}{(\sigma-1/2)^2+(t-\gamma)^2}\right)\frac{d\sigma}{\sinh((\sigma-1/2)\log x)}$$
$$+O\left(\frac{1}{t}\int_{1/2}^{\infty}(x^{-5/2}+x^{1/2-\sigma})\frac{\sigma-1/2}{\sinh((\sigma-1/2)\log x)}\,d\sigma\right). \tag{2.7}$$

The error term is, on letting $u=(\sigma-1/2)\log x$,

$$\ll\frac{1}{t\log^2 x}\int_0^{\infty}(x^{-5/2}+e^{-u})\frac{u}{\sinh u}\,du$$
$$\ll\frac{1}{t\log^2 x}.$$

Next, with the same substitution, we have

$$\int_{1/2}^{\infty}\left(\frac{x^{\sigma-1/2}}{n^{\sigma}}-\frac{x^{1/2-\sigma}}{n^{1-\sigma}}\right)\frac{d\sigma}{2\sinh((\sigma-1/2)\log x)}$$
$$=\frac{1}{n^{1/2}\log x}\int_0^{\infty}\frac{\sinh\left(\left(1-\dfrac{\log n}{\log x}\right)u\right)}{\sinh u}\,du.$$

This last integral is known [6, p. 344],

$$\int_0^{\infty}\frac{\sinh au}{\sinh bu}\,du=\frac{\pi}{2b}\tan\frac{\pi a}{2b},\qquad b>|a|. \tag{2.8}$$

Hence our last expression is

$$=\frac{\pi}{2n^{1/2}\log x}\tan\left(\frac{\pi}{2}\left(1-\frac{\log n}{\log x}\right)\right)$$
$$=\frac{\pi}{2n^{1/2}\log x}\cot\left(\frac{\pi}{2}\frac{\log n}{\log x}\right)$$
$$=\frac{1}{n^{1/2}\log n}f\left(\frac{\log n}{\log x}\right),$$

where $f(u)=(\pi/2)\,u\cot((\pi/2)\,u)$.

Making the same change of variables in the remaining integrals in (2.7), and gathering our results, we have now proved

LEMMA 1. *Assume the Riemann hypothesis. For $t \geqslant 1$, $t \neq \gamma$, $x \geqslant 4$, we have*

$$S(t) = -\frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} f\left(\frac{\log n}{\log x}\right) - x^{1/2} g(x, t)$$
$$+ \frac{1}{\pi} \sum_{\gamma} \sin((t-\gamma) \log x) \int_0^{\infty} \frac{u}{u^2 + ((t-\gamma) \log x)^2} \frac{du}{\sinh u}$$
$$+ O\left(\frac{1}{t \log^2 x}\right), \tag{2.9}$$

where

$$f(u) = \frac{\pi}{2} u \cot\left(\frac{\pi}{2} u\right), \tag{2.10}$$

and

$$g(x, t) = \frac{1}{\pi} \operatorname{Im} \int_{1/2}^{\infty} \frac{x^{-it}}{(((1/2 - it) \log x)^2 - u^2)} \frac{u}{\sinh u}\, du. \tag{2.11}$$

The weight $f(u)$ decreases "smoothly" from 1 to 0, and this is the main difference between (2.9) and earlier formulas for $S(t)$. For example, Selberg's approximate formula for $S(t)$ has

$$f(u) = e^{-2u} \min(1, 2(1-u))$$

in the case of the RH [15], and this weight has a derivative with a jump discontinuity. A disadvantage of our formula over Selberg's is that while the dependence on the zeros is explicit, we do not obtain a good pointwise estimate for the sum over zeros. It is possible to obtain a form of (2.9) which involves a general weight $f(u)$ and its Fourier transform. Certain assumptions must be made to get the same type of error terms.

The term $-x^{1/2} g(x, t)$ is small for $x \leqslant t$. In fact, from (2.11) we have

$$g(x, t) \ll \frac{1}{t^2 \log^2 x}, \tag{2.12}$$

and therefore this term may be absorbed into the error term when $x \leqslant t$. For $x > t$ this term is significant, and acts to cancel the main contribution of the Dirichlet polynomial (this will be shown in Section 4).

3. The Mean Square of the Sum over Zeros

In this section we will evaluate

$$R=\int_1^T \left|\frac{1}{\pi}\sum_\gamma \sin((t-\gamma)\log x)\int_0^\infty \frac{u}{u^2+((t-\gamma)\log x)^2}\frac{du}{\sinh u}\right|^2 dt; \qquad (3.1)$$

the result being a sum over differences $\gamma-\gamma'$ of zeros in $0<\gamma,\ \gamma'\leqslant T$.

Let

$$h(v)=\sin v\int_0^\infty \frac{u}{u^2+v^2}\frac{du}{\sinh u}. \qquad (3.2)$$

We note

$$h(v)\ll \min\left(1,\frac{1}{v^2}\right), \qquad (3.3)$$

from which it follows, since there are $\ll \log t$ zeros in $[t, t+1]$, that

$$\sum_\gamma |h((t-\gamma)\log x)|\ll \log(|t|+2) \qquad (x\geqslant 4). \qquad (3.4)$$

In particular we see the series is absolutely convergent. Hence

$$R=\frac{1}{\pi^2}\sum_\gamma\sum_{\gamma'}\int_1^T h((t-\gamma)\log x)\,h((t-\gamma')\log x)\,dt. \qquad (3.5)$$

We now employ an argument due to Montgomery [12, p. 187]. Using (3.3) we find, for $1\leqslant t\leqslant T$,

$$\sum_{\substack{\gamma\\ \gamma\notin[0,T]}} |h((t-\gamma)\log x)|\ll\left(\frac{1}{T+1}+\frac{1}{T-t+1}\right)\log T. \qquad (3.6)$$

We now argue the terms $\gamma\notin[0,T]$ contribute $\ll\log^3 T$ to R, which may be seen by taking the sum $\sum_{\gamma,\gamma\notin[0,T]}\sum_{\gamma'}$ inside the integral and using (3.4) and (3.6). Therefore, we may restrict the sum to terms $\gamma, \gamma'\in[0,T]$ with an error $\ll\log^3 T$. By a similar argument, we may extend the range of integration for these terms to the whole line $(-\infty,\infty)$ with an error $\ll\log^2 T$. We conclude

$$R=\frac{1}{\pi^2}\sum_{0<\gamma,\gamma'\leqslant T}\int_{-\infty}^{\infty} h((t-\gamma)\log x)\,h((t-\gamma')\log x)\,dt+O(\log^3 T). \qquad (3.7)$$

Let $a=(\gamma-\gamma')\log x$. A change of variables gives

$$R=-\frac{1}{\pi^2\log x}\sum_{0<\gamma,\gamma'\leqslant T} h*h(a)+O(\log^3 T), \tag{3.8}$$

where $*$ is the convolution defined by

$$f*g(w)=\int_{-\infty}^{\infty} f(w-u)\,g(u)\,du.$$

By (3.3) $h(v)\in L^1$ and therefore the convolution is well defined. To evaluate $h*h(a)$, we use the fact that, for $f, g\in L^1$,

$$\widehat{f*g}=\hat{f}\hat{g},$$

where $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(w)=\int_{-\infty}^{\infty} f(u)\,e(-wu)\,du, \qquad e(u)=e^{2\pi iu}.$$

When $\hat{f}\in L^1$, we also have the inversion formula

$$f(u)=\int_{-\infty}^{\infty}\hat{f}(w)\,e(wu)\,dw.$$

Now

$$\widehat{h*h}(a)=(\hat{h}(a))^2, \tag{3.9}$$

and an easy calculation using the well-knowing integral

$$\int_0^{\infty}\frac{\cos(\beta x)}{b^2+u^2}\,du=\frac{\pi}{2\,|b|}\,e^{-|\beta b|},$$

gives

$$\hat{h}(a)=\frac{\pi}{2i}\int_0^{\infty}\frac{1}{\sinh u}\left(-e^{-u|1+2\pi a|}+e^{-u|1-2\pi a|}\right)du.$$

Using the result [6, p. 356]

$$\int_0^{\infty} e^{-u}\,\frac{\sinh bu}{\sinh u}\,du=\frac{1}{b}-\frac{\pi}{2}\cot\left(\frac{\pi}{2}b\right),$$

we conclude

$$\hat{h}(a)=\begin{cases} -i\left(\dfrac{1}{2a}-\dfrac{\pi^2}{2}\cot(\pi^2 a)\right), & |2\pi a|\leqslant 1 \\ -\dfrac{i}{2a}, & |2\pi a|>1. \end{cases} \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), we have now proved

LEMMA 2. *For $x\geqslant 4$, $T\geqslant 2$, and R defined in* (3.1), *we have*

$$R=\frac{1}{\pi^2\log x}\sum_{0<\gamma,\gamma'\leqslant T}\hat{k}((\gamma-\gamma')\log x)+O(\log^3 T), \tag{3.11}$$

where

$$k(u)=\begin{cases} \left(\dfrac{1}{2u}-\dfrac{\pi^2}{2}\cot(\pi^2 u)\right)^2 & \textit{for} \quad |u|\leqslant\dfrac{1}{2\pi} \\ \dfrac{1}{4u^2} & \textit{for} \quad |u|>\dfrac{1}{2\pi}. \end{cases} \tag{3.12}$$

4. RELATION WITH $F(\alpha, T)$

We may evaluate sums over $\gamma-\gamma'$ in terms of the function $F(\alpha, T)$ defined in (1.8). We write $F(\alpha, T)=F(\alpha)$ throughout this section. On multiplying (1.8) by a suitable function $r(\alpha)$ and integrating we obtain

$$\sum_{0<\gamma,\gamma'\leqslant T}\hat{r}\left((\gamma-\gamma')\frac{\log T}{2\pi}\right)w(\gamma-\gamma')=\frac{T}{2\pi}\log T\int_{-\infty}^{\infty}F(\alpha)\,r(\alpha)\,d\alpha$$
$$=\left(\frac{T}{2\pi}\log T\right)F*r(0), \tag{4.1}$$

where $\hat{r}$ is the Fourier transform of r, as defined in Section 3. The weight $w(\gamma-\gamma')$ may usually be dropped with an acceptable error, and our first step is to prove this for the function $k(u)$ defined in (3.12). The result we obtain is

LEMMA 3. *Let $\beta=\log x/\log T$ (i.e., $x=T^{\beta}$). Then for $\beta>0$ we have*

$$R=\frac{T}{(2\pi^2\beta)^2}\int_{-\infty}^{\infty}F(\alpha)\,k(\alpha/2\pi\beta)\,d\alpha+O\left(\frac{T}{\beta^3\log T}\right)+O(\log^3 T). \tag{4.2}$$

Proof. We first note that

$$\hat{k}(y) \ll \min(1, 1/y^2). \tag{4.3}$$

To see this, note

$$|\hat{k}(y)| \leqslant \hat{k}(0) = \int_{-\infty}^{\infty} k(u)\, du \leqslant 2\pi,$$

and also, on integrating by parts,

$$\begin{aligned} |\hat{k}(y)| &= \frac{1}{\pi y} \left| \int_0^{\infty} k'(u)\, e(-uy)\, du \right| \\ &\ll \frac{1}{\pi y} \left(O\left(\frac{k'\left(\frac{1}{2\pi}+0\right) - k'\left(\frac{1}{2\pi}-0\right)}{y} \right) + \frac{1}{y} \int_0^{\infty} |k''(u)|\, du \right) \\ &\ll \frac{1}{y^2}. \end{aligned}$$

We now claim that

$$\sum_{0<\gamma,\gamma' \leqslant T} \hat{k}((\gamma-\gamma')\log x) = \sum_{0<\gamma,\gamma' \leqslant T} \hat{k}((\gamma-\gamma')\log x)\, w(\gamma-\gamma') + O\left(\frac{T}{\beta^2}\right); \tag{4.4}$$

for the difference of the two sums above is

$$\sum_{0<\gamma,\gamma' \leqslant T} \hat{k}((\gamma-\gamma')\log x)\, \frac{(\gamma-\gamma')^2}{4+(\gamma-\gamma')^2},$$

which by (4.3) is

$$\begin{aligned} &\ll \frac{1}{\log^2 x} \sum_{0<\gamma,\gamma' \leqslant T} \frac{1}{4+(\gamma-\gamma')^2} \\ &\ll \frac{1}{\log^2 x} \sum_{0<\gamma' \leqslant T} \left(\sum_{\gamma} \frac{1}{4+(\gamma-\gamma')^2} \right) \\ &\ll \frac{1}{\log^2 x} \sum_{0<\gamma' \leqslant T} \log \gamma' \ll \frac{T \log^2 T}{\log^2 x} = \frac{T}{\beta^2}. \end{aligned}$$

We have used again the fact that there are $\ll \log t$ zeros in $[t, t+1]$.

We could now apply (4.1) to evaluate the sum on the right in (4.4), but is is easier to proceed directly,

$$\begin{aligned}
&\sum_{0<\gamma,\gamma'\leqslant T} \hat{k}((\gamma-\gamma')\log x)\, w(\gamma-\gamma') \\
&\quad = \int_{-\infty}^{\infty} k(u) \sum_{0<\gamma,\gamma'\leqslant T} e(-u(\gamma-\gamma')\log x)\, w(\gamma-\gamma')\, du \\
&\quad = \frac{T}{2\pi}\log T \int_{-\infty}^{\infty} F(2\pi u\beta)\, k(u)\, du \\
&\quad = \frac{T\log T}{(2\pi)^2\beta}\int_{-\infty}^{\infty} F(\alpha)\, k(\alpha/2\pi\beta)\, d\alpha. \qquad (4.5)
\end{aligned}$$

Lemma 3 follows from Lemma 2, (4.4), and (4.5).

We now make use of Montgomery's theorem on $F(\alpha)$ [12], which states that, assuming RH,

$$F(\alpha)=\alpha+o(1)+T^{-2\alpha}\log T(1+o(1)) \qquad \text{as} \quad T\to\infty, \qquad (4.6)$$

uniformly for $0\leqslant\alpha\leqslant 1$. (Actually the theorem in [12] is proved for $0\leqslant\alpha<1$, with the statement that $\alpha=1$ may be obtained with more work. This has been done in [4], and uses a sieve upper bound for prime twins.) We will also use the previously mentioned elementary results

$$F(\alpha)=F(-\alpha), \qquad F(\alpha)\geqslant 0. \qquad (4.7)$$

LEMMA 4. *Assuming the Riemann hypothesis, we have, for fixed* $0<\beta\leqslant 1$, *where* $\beta=\log x/\log T$,

$$R=\frac{T}{2\pi^2}\left(1-\frac{\pi^2}{8}+\log\frac{\pi}{2}+\int_1^{\infty}\frac{F(\alpha)}{\alpha^2}d\alpha-\log\beta\right)+o(T). \qquad (4.8)$$

Proof. We have, by (3.12), (4.6), and (4.7),

$$\begin{aligned}
\int_{-\infty}^{\infty} F(\alpha)\, k(\alpha/2\pi\beta)\, d\alpha &= 2\left(\int_0^{\beta}+\int_{\beta}^{1}+\int_1^{\infty}\right) F(\alpha)\, k(\alpha/2\pi\beta)\, d\alpha \\
&= 2\int_0^{\beta}(\alpha+o(1)+T^{-2\alpha}\log T(1+o(1))) \\
&\quad \times\left(\frac{\pi\beta}{\alpha}-\frac{\pi^2}{2}\cot\left(\frac{\pi\alpha}{2\beta}\right)\right)^2 d\alpha \\
&\quad + 2\int_{\beta}^{1}(\alpha+o(1))\left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha+2\int_1^{\infty}F(\alpha)\left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha.
\end{aligned}$$

In the first integral the term $T^{-2\alpha}\log T(1+o(1))$ contributes $o(1)$ when evaluated since $k(u)$ is continuous and $k(0)=0$. All the remaining integrals are elementary to evaluate, and Lemma 4 now follows from Lemma 3.

For $\beta>1$ we also may obtain an expression for R, however, the dependence on β becomes more complicated. While we only need Lemma 4 in proving Theorem 1, it is interesting to see how R behaves as $\beta\to\infty$. We use the notation $f\cong g$ which means $f\ll g$ and $f\gg g$.

LEMMA 5. *Assume the Riemann hypothesis. For $1\leqslant\beta\leqslant T/\log^4 T$, we have*

$$R\cong T/\beta. \tag{4.9}$$

This result shows the term $x^{1/2}g(x,t)$ in Lemma 1 becomes significant on average for $x>t$.

Proof. By Lemma 3 it suffices to prove

$$\int_0^\infty F(\alpha)\,k(\alpha/2\pi\beta)\,d\alpha\cong\beta,$$

since $F(u)$ and $k(u)$ are nonnegative even functions. To obtain a lower bound, we note by (3.12) and Lemma A

$$\begin{aligned}\int_0^\infty F(\alpha)\,k(\alpha/2\pi\beta)\,d\alpha &\geqslant\int_\beta^\infty F(\alpha)\left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha\\ &\geqslant\pi^2\beta^2\sum_{n=0}^\infty\frac{1}{(\beta+2n+2)^2}\int_{\beta+2n}^{\beta+2n+2}F(\alpha)\,d\alpha\\ &\gg\beta^2\sum_{n=1}^\infty\frac{1}{(\beta+n)^2}\\ &\gg\beta.\end{aligned}$$

Similarly, an upper bound is obtained using Lemma A and the trivial estimate $k(u)\ll u^2$:

$$\begin{aligned}\int_0^\infty F(\alpha)\,k(\alpha/2\pi\beta)\,d\alpha &\ll\int_0^\beta F(\alpha)\left(\frac{\alpha}{2\pi\beta}\right)^2 d\alpha+\pi^2\beta^2\int_\beta^\infty\frac{F(\alpha)}{\alpha^2}\,d\alpha\\ &\ll\sum_{0\leqslant n\leqslant\beta+1}\int_n^{n+1}F(\alpha)\,d\alpha+\beta^2\sum_{n\geqslant\beta-1}\frac{1}{n^2}\int_n^{n+1}F(\alpha)\,d\alpha\\ &\ll\beta.\end{aligned}$$

5. The Mean Value of the Dirichlet Series

In order to prove Theorem 1, we need to compute the mean value of the Dirichlet series in Lemma 1, and also the cross term obtained from multiplying $S(t)$ with this series. We therefore define

$$G(T)=\int_1^T \left|\frac{1}{\pi}\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{1/2}}\frac{\sin(t\log n)}{\log n}f\left(\frac{\log n}{\log x}\right)\right|^2 dt \tag{5.1}$$

and

$$H(T)=\frac{2}{\pi}\int_1^T S(t)\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{1/2}}\frac{\sin(t\log n)}{\log n}f\left(\frac{\log n}{\log x}\right)dt, \tag{5.2}$$

where f is the function defined in Eq. (2.10). The evaluation of $G(T)$ and $H(T)$ is routine, and with a little care one could obtain asymptotic formula for $0<\beta\leqslant 1$, where $x=T^{\beta}$. However, we only need asymptotic formulas for some fixed $\beta>0$, and thus for simplicity we obtain formulas only for $0<\beta<1/2$, which is more than sufficient in proving Theorem 1.

Lemma 6. *We have, for $x=T^{\beta}$ and $0<\beta<\frac{1}{2}$ fixed,*

$$G(T)=\frac{T}{2\pi^2}\left(\log\log T+\log\beta+1-\frac{\pi^2}{8}-\log\frac{\pi}{2}\right.$$
$$\left.+C+\sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m}+\frac{1}{m^2}\right)\frac{1}{p^m}\right)+o(T) \tag{5.3}$$

(C is Euler's constant) and assuming the Riemann hypothesis,

$$H(T)=-\frac{T}{\pi^2}\left(\log\log T+\log\beta-\log\frac{\pi}{2}+C\right.$$
$$\left.+\sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m}+\frac{1}{m^2}\right)\frac{1}{p^m}\right)+o(T). \tag{5.4}$$

Proof of (5.3). We have

$$G(T)=\frac{1}{\pi^2}\sum_{n\leqslant x}\frac{\Lambda^2(n)f^2(\log n/\log x)}{n\log^2 n}\int_1^T\sin^2(t\log n)\,dt$$
$$+O\left(\sum_{n\leqslant x}\sum_{\substack{m\leqslant x\\ n\neq m}}\frac{\Lambda(n)\,\Lambda(m)}{(nm)^{1/2}\log n\log m\,|\log m/n|}\right)$$

$$= \frac{1}{\pi^2}\left(\frac{T}{2} + O(1)\right) \sum_{n \leqslant x} \frac{\Lambda^2(n) f^2(\log n/\log x)}{n \log^2 n}$$
$$+ O\left(x\left(\sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}}\right)^2\right),$$

since $|\log(m/n)| > A/n > A/x$, and therefore we conclude by the prime number theorem

$$G(T) = \frac{T}{2\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n) f^2(\log n/\log x)}{n \log^2 n} + O(x^2). \tag{5.5}$$

Next, the sum in (5.5) is

$$= \sum_{p \leqslant x} \frac{1}{p} f^2\left(\frac{\log p}{\log x}\right) + \sum_{m=2}^{\infty} \sum_{p^m \leqslant x} \frac{1}{m^2 p^m} f^2\left(\frac{m \log p}{\log x}\right),$$

and since the second sum converges uniformly for $x \geqslant 4$ and $f^2(m \log p/\log x) \to 1$ as $x \to \infty$ for any fixed p^m, we conclude

$$G(T) = \frac{T}{2\pi^2}\left(\sum_{p \leqslant x} \frac{1}{p} f^2\left(\frac{\log p}{\log x}\right) + \sum_{m=2}^{\infty} \sum_{p} \frac{1}{m^2 p^m} + o(1)\right). \tag{5.6}$$

We now define

$$T(u) = \sum_{2 \leqslant p \leqslant u} \frac{1}{p}, \tag{5.7}$$

and have [8, p. 22]

$$T(u) = \log \log u + C + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) + r(u), \tag{5.8}$$

where $r(u) \ll 1/\log u$, and the sum over primes on the right is equal to

$$\sum_{m=2}^{\infty} \sum_{p} \frac{1}{mp^m}.$$

Returning to the first sum in (5.6), we have

$$\sum_{p \leqslant x} \frac{1}{p} f^2\left(\frac{\log p}{\log x}\right) = \int_{2-0}^{x} f^2\left(\frac{\log u}{\log x}\right) dT(u)$$
$$= \int_{2-0}^{x} f^2\left(\frac{\log u}{\log x}\right) \frac{du}{u \log u} + \int_{2-0}^{x} f^2\left(\frac{\log u}{\log x}\right) dr(u)$$
$$= I_1 + I_2, \qquad \text{say.}$$

The integral I_1 is elementary to evaluate, the result obtained is

$$I_1 = \log\log x - \log\log 2 - \pi^2/8 + 1 - \log \pi/2 + O(1/\log x).$$

Now $f(u)$ is continuous in $[0, 1]$, $f(0) = 1$, and $f'(u) \ll 1$; therefore an integration by parts gives, together with (5.8),

$$\begin{aligned} I_2 &= -f^2\left(\frac{\log 2}{\log x}\right) r(2-0) + O\left(\frac{\log\log x}{\log x}\right) \\ &= \log\log 2 + C + \sum_{m=2}^{x} \sum_p \frac{1}{mp^m} + o(1). \end{aligned}$$

Collecting these results proves (5.3).

Proof of (5.4). We have

$$H(T) = \frac{2}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2} \log n} f\left(\frac{\log n}{\log x}\right) \int_1^T S(t) \sin(t \log n)\, dt. \tag{5.9}$$

To evaluate the integral, we quote Lemma γ of [17]: assuming the RH, for $n \geqslant 2$,

$$\int_1^T S(t) \sin(t \log n)\, dt = -\frac{T}{2\pi} \frac{\Lambda(n)}{n^{1/2} \log n} + O(n^{3/2} \log T). \tag{5.10}$$

We prove a more precise version of this result in the next section (see (6.3)). Applying (5.10), we have

$$H(T) = -\frac{T}{\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2} f\left(\frac{\log n}{\log x}\right) + O(x^{2+\varepsilon}),$$

and (5.4) follows on evaluating this sum in the same way as the nearly identical sum in (5.5).

6. Proof of Theorem 1

Suppose $x = T^\beta$ and β is a fixed positive number less than $\frac{1}{2}$. By Lemma 1 and (2.12) we have, for $t \geqslant 1$ and $t \neq \gamma$,

$$\begin{aligned} &S(t) + \frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} f\left(\frac{\log n}{\log x}\right) \\ &= \frac{1}{\pi} \sum_\gamma \sin((t-\gamma)\log x) \int_0^x \frac{u}{u^2 + ((t-\gamma)\log x)^2} \frac{du}{\sinh u} \\ &\quad + O\left(\frac{1}{t \log^2 x}\right). \end{aligned}$$

Since the above formula holds except on a countable set of points, we have on squaring both sides and integrating from 1 to T,

$$\int_1^T (S(t))^2\,dt + H(T) + G(T) = R + O(T^{1/2}), \tag{6.1}$$

where the error term is obtained by the Cauchy–Schwarz inequality since $R \ll T$. The lower limit of integration may be replaced by zero since

$$\int_0^1 (S(t))^2\,dt \ll 1.$$

Lemmas 4 and 6 now give (1.10).

It remains to prove (1.9). We have

$$\begin{aligned}\int_0^T &\left| S(t) + \frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} \right|^2 dt \\ &= \int_0^T (S(t))^2\,dt + \frac{2}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2} \log n} \int_0^T S(t) \sin(t \log n)\,dt \\ &\quad + \frac{1}{\pi^2} \int_0^T \left| \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} \right|^2 dt \\ &= \int_0^T (S(t))^2\,dt + J(T) + K(T), \qquad \text{say.}\end{aligned} \tag{6.2}$$

The first term on the right is known since we have proved (1.10). Equation (1.9) will follow once $J(T)$ and $K(T)$ are evaluated, and this may be done in the same way as $G(T)$ and $H(T)$ were evaluated in the last section. However, since we want (1.9) to hold for $0 < \beta \leqslant 1$ (instead of $0 < \beta < \frac{1}{2}$), more care must be taken with the error terms.

Consider first $J(T)$. We replace (5.10) with the following result, assuming RH and for $n \geqslant 2$,

$$\begin{aligned}\int_0^T S(t) \sin(t \log n)\,dt = -\frac{T}{2\pi} \frac{\Lambda(n)}{n^{1/2} \log n} &+ O(n^{1/2} \log\log n) \\ &+ O\left(\frac{n^{1/2} \log T}{\log n}\right).\end{aligned} \tag{6.3}$$

This only requires minor changes in Titchmarsh's argument. Consider

$$\int_C \log \zeta(s)\, n^s\,ds,$$

where C is taken around the rectangle with corners at $\frac{1}{2}$, $1+1/\log n$, $1+1/\log n+iT$, $1/2+iT$, and suitable identations to exclude the singularities of $\log \zeta(s)$. Hence, on letting the radius of the identations go to zero, and assuming the RH,

$$\begin{aligned} i\int_0^T &\log \zeta(1/2+it)\, n^{1/2+it} dt \\ &= \int_{1/2}^{1+1/\log n} \log \zeta(\sigma)\, n^{\sigma}\, d\sigma + i\int_0^T \log \zeta(1+1/\log n+it)\, n^{1+1/\log n+it}\, dt \\ &\quad - \int_{1/2}^{1+1/\log n} \log \zeta(\sigma+iT)\, n^{\sigma+iT}\, d\sigma \\ &= I_1+I_2+I_3. \end{aligned}$$

First,

$$I_1 \ll n^{1+1/\log n} \int_{1/2}^{1+1/\log n} |\log|\sigma-1||\, d\sigma \ll n.$$

Next, by [18, Theorem 9.6(B)],

$$\log \zeta(s) = \sum_{|t-\gamma|\leqslant 1} \log(s-\rho) + O(\log(|t|+2)), \qquad -1\leqslant\sigma\leqslant 2, \tag{6.4}$$

where $|\mathrm{Im}[\log(s-\rho)]| < \pi$, and hence on RH,

$$|\log \zeta(\sigma+iT)| \ll \log T \log \frac{1}{\sigma-1/2}, \tag{6.5}$$

for $\frac{1}{2}<\sigma\leqslant\frac{5}{4}$, and $T\geqslant 2$. We thus have

$$I_3 \ll \log T \int_{1/2}^{1+1/\log n} \log \frac{1}{\sigma-1/2}\, n^{\sigma}\, d\sigma \ll \frac{n\log T}{\log n}.$$

Finally, using the Dirichlet series for $\log \zeta(s)$,

$$\begin{aligned} I_2 &= i\sum_{m=2}^{\infty} \frac{\Lambda(m)\, n^{1+1/\log n}}{m^{1+1/\log n} \log m} \int_0^T \left(\frac{n}{m}\right)^{it} dt \\ &= iT\frac{\Lambda(n)}{\log n} + O\Bigg(\sum_{\substack{m=2\\ n\neq m}}^{\infty} \frac{\Lambda(m)\, n}{m^{1+1/\log n} \log m\, |\log m/n|}\Bigg). \end{aligned}$$

The sum in the error term is

$$\begin{aligned}
&\ll n \sum_{m<n/2} \frac{\Lambda(m)}{m \log m} + \frac{1}{\log n} \sum_{\substack{m \\ 1 \leqslant |m-n| \leqslant n/2}} \frac{\Lambda(m)}{|\log m/n|} \\
&\quad + n \sum_{3n/2<m} \frac{\Lambda(m)}{m^{1+1/\log n} \log m} \\
&\ll n \log \log n,
\end{aligned}$$

where we have used the prime number theorem and the estimate [5]

$$\sum_{\substack{m \\ 1 \leqslant |m-n| \leqslant n/2}} \frac{\Lambda(m)}{|\log m/n|} \ll n \log n \log \log n. \tag{6.6}$$

We conclude

$$\int_0^T \log \zeta(1/2+it)\, n^{1/2+it}\, dt = \frac{\Lambda(n)\, T}{\log n} + O(n \log \log n) + O\left(\frac{n \log T}{\log n}\right).$$

Similarly, on considering

$$\int_C \log \zeta(s)\, n^{-s}\, ds,$$

we obtain

$$\int_0^T \log \zeta(1/2+it)\, n^{-1/2-it}\, dt \ll n^{-1/2} \log T.$$

Combining these results, taking imaginary parts, and applying (1.2) proves (6.3).

We now apply (6.3)

$$\begin{aligned}
J(T) = &-\frac{T}{\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + O\left(\sum_{n \leqslant x} \frac{\Lambda(n) \log \log n}{\log n}\right) \\
&+ O\left(\log T \sum_{n \leqslant x} \frac{\Lambda(n)}{\log^2 n}\right) \\
= &-\frac{T}{\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + O\left(\frac{x \log \log x}{\log x}\right) + O\left(\frac{x \log T}{\log^2 x}\right),
\end{aligned}$$

and hence, with $x = T^{\beta}$, $0 < \beta \leqslant 1$ fixed, and RH,

$$J(T) = -\frac{T}{\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + o(T). \tag{6.7}$$

We next evaluate $K(T)$. We have first

$$K(T) = \frac{1}{\pi^2} \sum\sum_{n,m \leqslant x} \frac{\Lambda(n)\,\Lambda(m)}{n^{1/2} \log n\, m^{1/2} \log m} \int_0^T \sin(t \log n) \sin(t \log m)\, dt.$$

Since

$$\int_0^T \sin^2(t \log n)\, dt = \frac{T}{2} - \frac{\sin(2T \log n)}{4 \log n} = \frac{T}{2} + O(1)$$

and

$$\begin{aligned}\int_0^T \sin(t \log n) \sin(t \log m)\, dt &= \frac{1}{2}\left(\frac{\sin(T \log m/n)}{\log m/n} - \frac{\sin(T \log mn)}{\log mn}\right)\\ &= \frac{1}{2} \operatorname{Im}\left(\frac{(m/n)^{iT}}{\log m - \log n}\right) + O(1),\end{aligned}$$

we have

$$\begin{aligned}K(T) &= \frac{T}{2\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + O\left(\sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n}\right)\\ &\quad + O\left(\left|\sum\sum_{\substack{n,m \leqslant x\\ m \neq n}} \left(\frac{\Lambda(m)\, m^{iT}}{m^{1/2} \log m}\right)\overline{\left(\frac{\Lambda(n)\, n^{iT}}{n^{1/2} \log n}\right)}\Big/ \log m - \log n\right|\right)\\ &\quad + O\left(\left(\sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2} \log n}\right)^2\right)\\ &= \frac{T}{2\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + O(E_1) + O(E_2) + O(E_3),\end{aligned}$$

say. Now $E_1 \ll \log \log x$, $E_3 \ll x/\log^2 x$, and therefore these are $o(T)$ for $0 < \beta \leqslant 1$. To estimate E_2, we use the generalized Hilbert inequality of Montgomery and Vaughan [14]: for real numbers $\lambda_1, \lambda_2, \ldots, \lambda_R$; complex numbers $u_1, u_2, \ldots, u_R$; and

$$\delta_r = \min_s |\lambda_r - \lambda_s|,$$

$$\left|\sum_r \sum_{\substack{s\\ r \neq s}} \frac{u_r \overline{u_s}}{\lambda_r - \lambda_s}\right| \leqslant \frac{3\pi}{2} \sum_r |u_r|^2 \delta_r^{-1}. \tag{6.8}$$

Now, with $\lambda_n = \log n$ we have

$$\delta_n = \min_m |\log n - \log m| = \log \frac{n+1}{n} \gg \frac{1}{n},$$

and hence

$$\begin{aligned} E_2 &\ll \sum_{n \leqslant x} \left| \frac{\Lambda(n)\, n^{iT}}{n^{1/2} \log n} \right|^2 n \\ &\ll \sum_{n \leqslant x} \frac{\Lambda^2(n)}{\log^2 n} \\ &\ll \frac{x}{\log x} = o(T), \end{aligned}$$

for $0 < \beta \leqslant 1$. We conclude, for $0 < \beta \leqslant 1$,

$$K(T) = \frac{T}{2\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + o(T). \tag{6.9}$$

We now complete the proof of (1.9). By (6.2), (6.7), and (6.9), we have, for $0 < \beta \leqslant 1$,

$$\begin{aligned} &\int_0^T \left| S(t) + \frac{1}{\pi} \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} \right|^2 dt \\ &\quad = \int_0^T (S(t))^2\, dt - \frac{T}{2\pi^2} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} + o(T). \end{aligned} \tag{6.10}$$

By (5.7) and (5.8),

$$\begin{aligned} \sum_{n \leqslant x} \frac{\Lambda^2(n)}{n \log^2 n} &= \sum_{p \leqslant x} \frac{1}{p} + \sum_{m=2}^{\infty} \sum_{p^m \leqslant x} \frac{1}{m^2 p^m} \\ &= T(x) + \sum_{m=2}^{\infty} \sum_{p} \frac{1}{m^2 p^m} + o(1) \\ &= \log \log x + C + \sum_{m=2}^{\infty} \sum_{p} \left(\frac{1}{m} + \frac{1}{m^2} \right) \frac{1}{p^m} + o(1). \end{aligned}$$

Taking $x = T^\beta$, $0 < \beta \leqslant 1$, and substituting (1.10) and the above result into (6.10) proves (1.9).

7. Proof of Theorem 2 and Lemma A

We write $F(\alpha, T) = F(\alpha)$ throughout this section. We first prove Lemma A. By the definition of $F(\alpha)$ in (1.8),

$$\int_{c-1}^{c+1} F(\alpha)(1-|\alpha-c|)\,d\alpha$$
$$= \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} T^{ic(\gamma-\gamma')} \left[\frac{\sin\frac{(\gamma-\gamma')}{2}\log T}{\frac{(\gamma-\gamma')}{2}\log T}\right]^2 w(\gamma-\gamma').$$

Here c is any real number, and c may depend on T if we wish. Trivially

$$\tfrac{1}{2}\int_{c-1/2}^{c+1/2} F(\alpha)\,d\alpha \leqslant \int_{c-1}^{c+1} F(\alpha)(1-|\alpha-c|)\,d\alpha \leqslant \int_{c-1}^{c+1} F(\alpha)\,d\alpha.$$

We now apply (1.11) with $\beta = 1$,

$$\int_{c-1/2}^{c+1/2} F(\alpha)\,d\alpha \leqslant 2\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} \left[\frac{\sin\frac{(\gamma-\gamma')}{2}\log T}{\frac{(\gamma-\gamma')}{2}\log T}\right]^2 w(\gamma-\gamma')$$
$$\leqslant \frac{8}{3} + \varepsilon,$$

and (m_γ = multiplicity of zero $\frac{1}{2} + i\gamma$)

$$\int_{c-1}^{c+1} F(\alpha)\,d\alpha \geqslant \left(\frac{T}{2\pi}\log T\right)^{-1} \left(\sum_{0<\gamma\leqslant T} m_\gamma \right.$$
$$\left. + \sum_{\substack{0<\gamma,\gamma'\leqslant T\\ \gamma\neq\gamma'}} T^{ic(\gamma-\gamma')} \left[\frac{\sin\frac{(\gamma-\gamma')}{2}\log T}{\frac{(\gamma-\gamma')}{2}\log T}\right]^2 w(\gamma-\gamma')\right)$$
$$\geqslant \left(\frac{T}{2\pi}\log T\right)^{-1} \left(2\sum_{0<\gamma\leqslant T} m_\gamma \right.$$
$$\left. - \sum_{0<\gamma,\gamma'\leqslant T} \left[\frac{\sin\frac{(\gamma-\gamma')}{2}\log T}{\frac{(\gamma-\gamma')}{2}\log T}\right]^2 w(\gamma-\gamma')\right)$$

$$\geqslant 2\left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma\leqslant T}1-\left(\frac{4}{3}+\varepsilon\right)$$
$$\geqslant 2-\frac{4}{3}-\varepsilon=\frac{2}{3}-\varepsilon,$$

which proves lemma A.

We now turn to the proof of Theorem 2. Define, for $T\geqslant 2$,

$$G(\alpha)=\left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma,\gamma'\leqslant T}\left[\frac{\sin\frac{\alpha}{2}(\gamma-\gamma')\log T}{\frac{\alpha}{2}(\gamma-\gamma')\log T}\right]^2 w(\gamma-\gamma'),\quad (7.1)$$

where $w(u)=4/4+u^2$. If we wish, we may delete the factor $w(\gamma-\gamma')$ with an error $o(1)$ by the same argument used to derive (4.4). By (1.11) we have on RH, for $0<\alpha\leqslant 1$,

$$G(\alpha)\sim 1/\alpha+\alpha/3.\quad (7.2)$$

For $\alpha>1$, we can only obtain upper and lower bounds for $G(\alpha)$ on RH:

LEMMA 7. *Assume the Riemann hypothesis. We have, for any $\varepsilon>0$ and T sufficiently large,*

$$G(\alpha)\geqslant 1-\varepsilon\qquad \textit{for all } \alpha,\quad (7.3)$$

and

$$G(\alpha)\leqslant\alpha-4+9/\alpha-14/(3\alpha^2)+\varepsilon\qquad \textit{for}\quad 1\leqslant\alpha\leqslant 2.\quad (7.4)$$

Also, for all α we have

$$G(n\alpha)\leqslant G(\alpha),\qquad n=1,2,3,\dots.\quad (7.5)$$

Proof. We have trivially

$$G(\alpha)\geqslant\left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma\leqslant T}m_\gamma\geqslant\left(\frac{T}{2\pi}\log T\right)^{-1}N(T)\geqslant 1-\varepsilon$$

by (1.1), for T sufficiently large. This proves (7.3). Next, we have $|\sin(nx)|\leqslant|n\sin(x)|$, for all integers n, which is easily proved by induction.

Using this inequality in (7.1) proves (7.5). Before proving (7.4), we first obtain a weaker upper bound from (7.2) and (7.5). We have $G(n\beta) \leqslant G(\beta)$, and on taking $(n-1)/n \leqslant \beta \leqslant 1$ we have $G(n\beta) \leqslant 1/\beta + \beta/3 + o(1)$. Now, letting $\alpha = n\beta$ we obtain, for any $\varepsilon > 0$ and T sufficiently large,

$$G(\alpha) \leqslant n/\alpha + \alpha/3n + \varepsilon \qquad \text{for} \quad n-1 \leqslant \alpha \leqslant n, \quad n = 1, 2, 3, \dots \tag{7.6}$$

This shows in particular that $G(\alpha) \leqslant 4/3 + \varepsilon$ if α is sufficiently large. We now turn to the proof of (7.4). We have, for $1 \leqslant \alpha \leqslant 2$, by (4.6),

$$\begin{aligned} G(\alpha) &= \frac{2}{\alpha^2} \int_0^\alpha (\alpha - |\beta|)\, F(\beta)\, d\beta \\ &= \frac{1}{\alpha} + \frac{2}{\alpha^2} \int_0^1 (\alpha - \beta)\, \beta\, d\beta + \frac{2}{\alpha^2} \int_1^\alpha (\alpha - \beta)\, F(\beta)\, d\beta + o(1) \\ &= \frac{2}{\alpha} - \frac{2}{3\alpha^2} + \frac{2}{\alpha^2} I(\alpha) + o(1), \end{aligned} \tag{7.7}$$

where

$$I(\alpha) = \int_1^\alpha (\alpha - \beta)\, F(\beta)\, d\beta. \tag{7.8}$$

We now obtain an upper bound for $I(\alpha)$ in the same way we obtained the upper bound in Lemma A,

$$\begin{aligned} I(\alpha) &+ \int_{2-\alpha}^1 (\alpha + \beta - 2)\, F(\beta)\, d\beta \\ &= \int_{2-\alpha}^\alpha ((\alpha - 1) - |\beta - 1|)\, F(\beta)\, d\beta \\ &= \int_{1-\alpha}^{\alpha - 1} ((\alpha - 1) - |\beta|))\, F(\beta + 1)\, d\beta \\ &= (\alpha - 1)^2 \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0 < \gamma, \gamma' \leqslant T} T^{i(\gamma - \gamma')} \\ &\quad \times \left[\frac{\sin \dfrac{\alpha - 1}{2} (\gamma - \gamma') \log T}{\dfrac{\alpha - 1}{2} (\gamma - \gamma') \log T} \right]^2 w(\gamma - \gamma'). \end{aligned}$$

Applying (4.6) and (7.2), for $1 \leqslant \alpha \leqslant 2$, we have

$$
\begin{aligned}
I(\alpha) \leqslant & -\int_{2-\alpha}^{1} (\alpha+\beta-2)\,\beta\, d\beta + o(1) \\
& + (\alpha-1)^2 \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma' \leqslant T} \\
& \times \left[\frac{\sin \frac{\alpha-1}{2}(\gamma-\gamma')\log T}{\frac{\alpha-1}{2}(\gamma-\gamma')\log T}\right]^2 w(\gamma-\gamma') \\
\leqslant & \left(\frac{\alpha^3}{6} - \alpha^2 + \frac{3}{2}\alpha - \frac{2}{3}\right) + \left((\alpha-1) + \frac{(\alpha-1)^3}{3}\right) + o(1) \\
\leqslant & \frac{\alpha^3}{2} + 2\alpha^2 + \frac{7}{2}\alpha - 2 + o(1),
\end{aligned}
$$

and (7.4) follows from (7.7) and the above bound.

We now relate

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2}\, d\alpha$$

to $G(\alpha)$.

LEMMA 8. *Assume the Riemann hypothesis. Then we have, as $T \to \infty$,*

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2}\, d\alpha = 3\int_1^\infty \frac{G(\alpha)}{\alpha^2}\, d\alpha - \frac{7}{3} + o(1). \tag{7.9}$$

Proof. Consider the function $I(\alpha)$ defined in Eq. (7.8). We have

$$I'(\beta) = \int_1^\beta F(\alpha)\, d\alpha, \qquad I''(\beta) = F(\beta),$$

and on integrating by parts twice

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2}\, d\alpha = \int_1^\infty \frac{I''(\alpha)}{\alpha^2}\, d\alpha = 6\int_1^\infty \frac{I(\alpha)}{\alpha^4}\, d\alpha, \tag{7.10}$$

where we have used Lemma A to show the $\alpha = \infty$ terms vanish. Now

$$\int_0^\beta F(\alpha)(\beta-\alpha)\, d\alpha = \frac{\beta^2}{2} G(\beta),$$

and by (4.6)

$$\int_0^1 F(\alpha)(\beta-\alpha)\,d\alpha=\frac{\beta}{2}+\int_0^1 \alpha(\beta-\alpha)\,d\alpha+o(1)$$
$$=\beta-1/3+o(1);$$

which shows, for $\beta\geqslant 1$, on RH,

$$I(\beta)=\frac{\beta^2}{2}G(\beta)-\beta+1/3+o(1). \tag{7.11}$$

Substituting this result into (7.10) proves Lemma 8.

We now prove Theorem 2. Substituting the lower bound (7.3) for $G(\alpha)$ into Lemma 8 proves the lower bound in the theorem. To obtain the upper bound, denote the upper bound for $G(\alpha)$ given in (7.4) by

$$h(\alpha)=\alpha-4+9/\alpha-14/(3\alpha^2). \tag{7.12}$$

We see $h(1)=h(2)=\frac{4}{3}$, and it is easy to check $h'(\alpha)$ has one root in the interval $[1,2]$ at $\alpha=1.258517...$, and $h(1.258517...)=1.463412....$ We conclude, by (7.5),

$$G(\alpha)\leqslant 1.464 \qquad \text{for all } \alpha\geqslant 1. \tag{7.13}$$

We now obtain the upper bound in Theorem 2 by applying (7.4), (7.13) and Lemma 8,

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2}\,d\alpha\leqslant 3\int_1^2 \frac{h(\alpha)}{\alpha^2}\,d\alpha+3\int_2^\infty \frac{1.464}{\alpha^2}\,d\alpha-7/3+o(1)$$
$$=3(\log 2+1/72)+1.5(1.464)-7/3+o(1)$$
$$=1.9837...<2.$$

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