195. H. Whitney, "The singularities of a smooth n-manifold in (2n — 1)-space," Ann. Math., 45, 247-293 (1949).
196. H. Whitney, "The self-intersections of a smooth n-manifold in 2n-space," Ann. Math., 45, 220-246 (1949).
197. J. P. Wilson, "Some minimal imbeddings of homogeneous spaces," J. London Math. Soc., 1, No. 2, 335-340 (1969).
198. P. Wintgen, "Homotopien von Untermannigfaltigkeiten mit nicht ausgearteter zweiter Fundamentalform," Czechoslovak Math. J., 25, No. 3, 424-437 (1975).
199. P. Wintgen, "Über von höherer Ordnung reguläre Immersion," Math. Nachr., 85, 177-184 (1978).
200. Y.-C. Wong, "Sectional curvatures of Grassmann manifolds," Proc. Nat. Acad. Sci. U.S.A., 60, No. 1, 75-79 (1968).
201. W. Wu, "On the imbedding of orientable manifolds in a Euclidean space," Sci. Sinica, 12, No. 1, 25-33 (1963).
202. T. Yasui, "Note on the enumeration of embeddings of real projective spaces," Hiroshima Math. J., 3, No. 2, 409-418 (1973).
203. T. Yasui, "The enumeration of liftings in fibrations and the embedding problem. II," Hiroshima Math. J., 6, No. 3, 465-484 (1976).
204. T. Yasui, "The enumeration of liftings in fibrations and the embedding problem. I," Hiroshima Math. J., 6, No. 2, 227-255 (1976).

CERTAIN QUESTIONS OF LOBACHEVSKII GEOMETRY, CONNECTED WITH PHYSICS

S. B. Kadomtsev, É. G. Poznyak, and D. D. Sokolov

UDC 514.8

An historical survey of the development of Lobachevskii geometry as a typical representative of a geometry of negative curvature, Friedman cosmology, Lobachevskii geometry, and an interpretation of the velocity space in the special theory of relativity as a Lobachevskii space are presented.

1. Lobachevskii Geometry — A Typical Representative of a Geometry of Negative Curvature

Historically, Lobachevskii geometry appeared as the first non-Euclidean geometry realized to be such. This was precisely the turning point in the development of geometry from the still intuitively operating geometry of Euclid to modern geometry absorbing within itself both the concept of a curved Riemannian space as well as the algebraic-group ideas of Klein. It happens that this all-encompassing aspect was the principal one in the development of Lobachevskii geometry in the last century. Just as he apparently was recognized as the foremost among the three scholars tending the cradle of this science (Lobachevskii, Bolyai, Gauss), it is his name that the geometry validly bears. In particular, this is manifest in the complex evolution of Lobachevskii's views on the question of a geometry of the real world, which resulted in the statement on the possibility or impossibility of geometric constructions not having a direct relation to real space. We stress the profound nontriviality of recognizing the geometry constructed as a non-Euclidean one. Strictly speaking, the fact is that this geometry was not the first, but the second non-Euclidean geometry. Since the time of Greek (or, anyway, Hellenistic) antiquity another non-Euclidean geometry — spherical geometry — has been very well known. Even then the degree of its development was not principally different from the degree of development of Euclidean geometry. Apparently, the

Translated from Itogi Nauki i Tekhniki, Seriya Problemy Geometrii, Vol. 13, pp. 157-188, 1982.

0090-4104/84/2504-1331$07.50

origin of spherical geometry can be attributed to Eudoxus,* and by the end of the First Century A.D. there already was a systematic exposition of this science, due to Menelaus of Alexandria, which reached us in the Arabic translation by Thábit ibn Quarrah [99]. The popularity of this work is attested by the fact that the famous theorem of Menelaus originated in it. Even though spherical geometry could not play the role suitable to Lobachevskii geometry, it could be said that spherical geometry can in obvious fashion be realized globally on a surface in Euclidean space. The question of realizing Lobachevskii geometry on a surface in Euclidean space was first raised and partially solved by Beltrami [74]. He constructed a realization of a horocircle of the Lobachevskii plane on a surface of constant negative curvature. It should be noted that surfaces of constant negative curvature were studied earlier by Minding [100].

The all-encompassing value of the question on the possibility of realizing the whole Lobachevskii plane on a surface in Euclidean space was so great that in 1900 Hilbert [89] named it among the cardinal problems of mathematics. In 1901 he gave a negative answer to this question [90].

At the present time the role designated above for Lobachevskii geometry in science in the last century has already been exhausted. Lobachevskii geometry in the narrow sense of this word as a science, analogously to the geometry of Euclid's "Elements" and Menelaus' "Sphaerica," also, of course, has been exhausted as far as ideas are concerned. This does not signify that here it is impossible to prove individual, perhaps very difficult, theorems; moreover, similar studies have already constituted an extensive library. The thing is that these studies do not lie at the frontiers of mathematics. However, the very complex of ideas connected with the birth of this geometry and using its concepts continues to develop actively. The main attention was transferred from the study of Lobachevskii geometry as a closed theory to its investigation in connection with numerous adjoining problems of mathematics. Therefore, the role that Lobachevskii geometry now plays in science has changed. Now it is treated, foremost, as a typical manifold of negative curvature modelling the properties of other manifolds of negative curvature and being sufficiently simple to study. Spherical geometry is precisely such a natural model of manifolds of negative curvature. It seems that a similar approach in which different spaces are investigated with respect to their resemblance to the simplest spaces of constant curvature (or of maximum mobility) now occupies a central position in many branches of geometry. Furthermore, in those questions in which the metrics of constant curvature are not such typical manifolds, every advance is associated with immense difficulty even for a proper statement of the problem. It is worth noting that from this point of view the classical Euclidean space is of lesser interest since it does not model the properties of a wide circle of spaces.

The physical applications of this science, to which the present survey is devoted, are mainly connected precisely with the treatment of Lobachevskii geometry as a typical representative of spaces of negative curvature. The most traditional of these applications are those in the general theory of relativity which from a mathematical viewpoint is based on the geometry of curved spaces. Less widely known are the other applications — in nuclear physics, in the physics of elementary particles, etc. — which are connected, in particular, with the geometric interpretation of the nonlinear equations encountered in these sciences. Within the scope of this survey we do not intend to cover all the presently existing applications or to give a comprehensive list of the physics literature on these questions. We wish to acquaint the reader with a small number of examples, dear to us, which serve to clarify the general situation in which Lobachevskii geometry comes into contact with modern physics. It will be useful to us to make, where necessary, a comparison with situations arising in other spaces, in particular, spaces of positive curvature.

"As Aetius (II, 2, 2) informs us: σφαιροειδῆ τόν κόσμον ."
Added by translator: This Greek quotation says: "The world is spherical." (However, the Greek word "kosmos" here can also be translated as "universe"; only the context would clarify this.)

Aëtius of Amida in Mesopotamia was a Greek physician of the Sixth Century A.D. Eudoxus (c. 390-340 B.C.) was a celebrated mathematician and astronomer in Cnidus. Menelaus (c. 100 A.D.) of Alexandria wrote a textbook on spherical geometry (the Sphaerica) containing the earliest known theorems on spherical trigonometry. It survives today in an Arabic translation by Thábit ibn Quarrah (c. 836-901 A.D.), an Arab mathematician, physician, and philosopher.

First of all we consider the results related with the development of global Riemannian geometry. They are based on the fact that Lobachevskii geometry (both two-dimensional as well as multidimensional) models the exponential instability of geodesics on spaces of negative curvature. Analogously, a sphere models the appearance of conjugate points on spaces of positive curvature. An elementary analysis of this fact was carried out in detail in [3]. We remark that in the same place the reader can become acquainted with the numerous applications of the ideas of Lobachevskii geometry in hydrodynamics.

The properties of stability and instability of geodesics are studied, as is well known, with the aid of comparison theorems for the Jacobi equation [19]. The investigation of this equation leads to numerous estimates connecting the geometric characteristics of spaces of variable and of constant curvatures. These estimates are summarized in the famous Hadamard—Cartan theorems [86, 78] and in the sphere theorem [95]. According to the first one of them a complete simply connected Riemannian manifold of nonpositive sectional curvature is diffeomorphic with Euclidean space. A second one states that a complete simply connected Riemannian manifold whose sectional curvature varies by less than four times is homeomorphic to a sphere. A vast number of geometric inequalities, also going back to a considerable extent to the idea of comparison with a space of constant curvature, are presented in [14].

We stress that the possibility of separating two large classes of Riemannian spaces by the sign of their sectional curvatures is by far not something that is self-understandable. In order to illustrate this we turn to pseudo-Riemannian spaces. The first thing to pay attention to is the fact that the sign of the sectional curvature does not by itself determine the stability properties of the geodesics. For example, the two-dimensional Jacobi equation now has the form

$$y''+K(v,\ v)y=0,$$

where y is the Jacobian component of the field, perpendicular to the geodesic, v is the geodesic's tangent vector. Thus, the stability of the solutions is now determined by the sign of the product K(v, v), which can be different for one and the same sign of K, since the metric is sign-definite. Therefore, on a manifold of, say, a negative curvature there are simultaneously present both exponential instability for some geodesics as well as conjugate points on the others. Furthermore, the very concept of sectional curvature can be introduced now for not all two-dimensional directions. Indeed, the curvature in a direction σ spanned by linearly independent vectors v and w equals

$$K_\sigma=\frac{(R\,(v,\,w)\,w,\,v)}{(v,\,v)\,(w,\,w)-(v,\,w)^2}.$$

However, the Gramm determinant occurring in the denominator vanishes for isotropic two-dimensional directions (i.e., those on which a degenerate metric is induced). If in this case the numerator also vanishes, then the concept of sectional curvature can be preserved with the aid of continuation by continuity (see [36]).

The problem of metric and topological classification of spaces of constant curvature has attracted much attention. The problems arising in this connection can be looked up in [20, 21]. A detailed exposition of the results is in [121]. In general features the situation is that the problem was able to be reduced to the problem of picking out subgroups in a motion group of a specific class, but for a Lobachevskii space this group problem proved to be transcendentally complicated. Comparison theorems for non-simply connected spaces (see [14] regarding them) also have been examined. This circle of questions finds wide physical applications which we take up in Sec. 2.

A detailed study of the intrinsic geometry of complete two-dimensional Riemannian manifolds with curvature bounded between two negative constants also is turned away from the idea of comparison with the Lobachevskii plane. First of all we note the result in [87] according to which such a metric is conformally equivalent to the Lobachevskii plane. This enables the delineation on it of domains comparable with known domains of the Lobachevskii plane, for instance, a horocycle [43]. The properties of horocycles bounding horocircles were studied in detail in [72] and turned out to be generally similar to the properties of a horocycle on the Lobachevskii plane.

A particularly influential role is played by the idea of comparing the properties of surfaces of variable and constant curvatures in geometry as a whole. We recall that a central place among the problems of this science is occupied by the question of realizing metrics of a prescribed class. In case of an affirmative answer to this question there naturally arises the problem of the arbitrariness of the realization and its degree of regularity for a guaranteed regularity of the metric, and under a further investigation, the problems of rigidity, stability, etc. Other classical problems of global geometry raise the question on the restoration of a surface, not from its metric but from its other characteristics, say, from the Gaussian or mean curvature.

The principal achievement in the realization problem in the middle of the present century was the justification of the definitive role of the sign of the curvature for the possibility of realizing the metric. Roughly speaking, metrics of positive curvature are well immersed into a Euclidean space, but metrics of negative curvature, in general, cannot be immersed. A singular place among the surfaces in a three-dimensional Euclidean space is occupied by a complete surface of constant positive curvature, viz., a sphere. Complete manifolds close in a specific sense to manifolds of constant positive curvature have precisely realizations similar to a sphere. For example, if the curvature of such a manifold is nonzero, then it is realized as an ovaloid. This realization is unique (say, in the class of general convex surfaces), while its regularity is in natural fashion determined by the regularity of the metric. When the condition that the curvature is nonzero is relaxed the resemblance to a sphere is lessened.

In general we can say that surfaces of positive curvature (or, somewhat more widely, convex surfaces) in Euclidean space form a natural class possessing the properties of a sphere. This fundamental conclusion was formulated by Aleksandrov in the prewar years during discussions on methods for solving the Weyl problem. It happens that precisely this transition from solving an important but restricted problem, such as Weyl's problem was, to the consideration of convex surfaces as a natural class was the boundary separating the initial stage of development of global geometry in the works of Hilbert, Minkowski, Christoffel, Weyl, Cohn-Vossen, and others, from its contemporary stage of development. At this stage the general theory of convex surfaces, covering all the basic aspects of their structure, was created in the classic works of Aleksandrov [1], Pogorelov [37], and their numerous students and successors.

A completely different picture is observed for surfaces of negative curvature. First of all, as Hilbert showed [90], among these surfaces there are no realizations of a complete metric of constant curvature, viz., the Lobachevskii plane. Efimov [22] showed that this fact is by no means accidental: complete metrics of nonzero negative curvature do not admit of a regular realization as a whole in Euclidean space. In other words, the least upper bound of the curvature is nonnegative on every complete surface in Euclidean space.

The fact that surfaces of negative curvature do not form a natural class does not at all signify that they do not have properties important and interesting to geometers. The meaning of this assertion is that the class of surfaces of negative curvature is partitioned into a number of subclasses within which considerable similarity is shown, but surfaces of different subclasses are very little like each other. These subclasses possess the properties of the different surfaces of negative curvature and realize certain narrow classes of metrics. Obviously, for the delineation of similar subclasses it is necessary to remove either the condition that the surface be complete or the condition that the curvature be nonzero, considering metrics in which the curvature tends to zero in a prescribed manner at infinity. In the first case Poznyak [42] proved that geodesic strips of metrics of nonzero negative curvature are realized by surfaces similar to "Minding spools," while Shikin [71] proved that horocyclic domains of such metrics also are realized, but now as surfaces similar to a pseudosphere. Verner (see [15]) investigated a class of surfaces bringing to mind a hyperboloid of one sheet and being different from a horn and a bowl. In recent times the attention of researchers has been drawn to the class of surfaces similar to hyperbolic paraboloids [55].

This picture stands out in greater relief if we turn to the theory of surfaces in the pseudo-Euclidean space $E_{(2,1)}^3$ (see [47]). In this case the natural classes are characterized not only by the sign of the Gaussian curvature but also by the sign of the determinant of the first quadratic form. A combination of these signs yields four classes. Two of them, viz., the class of surfaces of negative curvature with a positive-definite metric and the class of surfaces of positive curvature with an indefinite metric, prove to be natural classes. Both these natural classes possess the properties of the corresponding components of the unit sphere in the pseudo-Euclidean space. It is important to emphasize that the Monge—Ampere equations turn out to be elliptic for surfaces from the first of these classes and hyperbolic for the other. The surfaces of one of these classes are convex surfaces and of the other, saddle surfaces. These facts refute the conjecture that the delineation of natural classes of surfaces is connected with the type of the Monge—Ampere equations or with the convexity property.

The two other classes, viz., of surfaces of positive curvature with a positive-definite metric and of surfaces of negative curvature with an indefinite metric, are not natural classes. Estimates for the curvature, of the type of Efimov's estimates, are fulfilled for the surfaces of these classes. Both these estimates can be combined into the inequality

$$\inf_{\Phi} K \operatorname{sign} \Delta \geqslant 0,$$

where K is the Gaussian curvature and Δ is the determinant of the metric tensor of the intrinsically complete surface Φ with a nondegenerate metric. The proof of this estimate, somewhat falling outside the scope of Sokolov's paper [48], is presented in the Appendix.

An analysis of all the six classes mentioned of surfaces in Euclidean and pseudo-Euclidean spaces shows that the appearance of a natural class of surfaces is connected with the presence of a sphere's components (or of the whole sphere) in this class. We remark that according to the results of Shefel' [70] and Kadomtsev [25] an increase of the dimensions of the enveloping space does not change this situation in any way if some natural group conditions are fulfilled. Namely, if the realization of a sphere in a high-dimensional space is convex in some sense, then this realization is indeed three-dimensional. A realization of the Lobachevskii plane is possible in a high-dimensional space, but this realization does not have maximum mobility as a surface of the enveloping space. Similar phenomena are observed also in the pseudo-Euclidean space.

The noted role of spheres in the formation of natural classes of surfaces enables us to understand better why the construction of analogous natural classes of multidimensional surfaces causes such difficulties. A sphere cannot play the role of a typical realization for high-dimensional manifolds. Indeed, a sphere is a hypersurface, i.e., has the codimension 1, whereas the immersion of an n-dimensional manifold has, in general, a dimension not less than $n(n - 1)/2$, which coincides with unity only in the two-dimensional case. It is significant that in the multidimensional analogs of the Minkowski problem, where a similar noncorrespondence of dimensions does not arise, the questions of a multidimensional generalization of the two-dimensional results do not encounter such principal difficulties [38]. We remark as well that a sphere proves to be an atypical surface also in the study of spaces with a degenerate metric. Therefore, in particular, their study in a number of papers, among which it is necessary to note that by Artykbaev (see [4]), did not lead further to the delineation of natural classes of surfaces.

2. Friedmann Cosmology and Lobachevskii Geometry

Relativistic cosmology furnishes the most important and simple example of how Lobachevskii geometry is used in physics as a typical example of the geometry of Riemannian manifolds of negative curvature. The original observed fact on which contemporary relativistic cosmology is based was the very high degree of homogeneity and isotropy of the regions of the Universe accessible to observations. It is understood that we are dealing with homogeneity and isotropy in a very large — cosmological — scale. In the small scale surrounding us the universe is sharply inhomogeneous and anisotropic. For example, anisotropy in the scale of the Galaxy is well seen with the naked eye, i.e., in the sky we see a strip in which the stars are concentrated — the so-called Milky Way. Thus, stars are grouped into galaxies, galaxies into clusters of galaxies. These galactic clusters, as has been ascertained in most recent times, form a complicated cellular structure like a honeycomb with a typical size of 50-100 megaparsecs (1 megaparsec = 10^6 parsecs = $3 \cdot 10^{24}$ cm).

To the extent of the transition to structures of ever higher scale, the density of matter changes ever less, and, finally, on the scales many times the size of the cells in the cellular structure, it becomes approximately constant.

The representation on the homogeneity of the Universe in cosmological scales was formed in the 1930s on the basis of the study of the distribution of galaxies; however, the representation to a really exceptionally high degree of the Universe's isotropy was able to be done only after Penzias and Wilson [104], in the 1960s, inaugurated and investigated relic thermal radio-frequency emission. It turned out that it has a fantastic degree of isotropy —

by the available estimates the variation of its temperature in different directions is not greater than $\Delta T/T \sim 10^{-4}$. It is clear that we can verify the fact of the Universe's isotropy; however, if we do not assume that the earth is located at a specially selected "center" of the Universe (the so-called Copernicus principle), then homogeneity follows from isotropy.

Another fundamental observed foundation for cosmology is the fact of the expansion of the Universe. This fact can be established from the redness of the spectrum of remote galaxies. A certain quantity z, called the red shift, is a measure of this redness. We emphasize that the red shift of distant objects can be measured directly with high accuracy. This redness is explained by the doppler-effect due to the fact that the distant galaxies recede relative to each other. The velocity of this recession is proportional to the distance between the objects. Therefore, the red shift can be looked upon as a measure of the distance to the remote objects. For objects not too remote the distance is proportional to the red shift; farther out the dependence between them becomes more complicated. This dependency and the coefficient of proportionality are determined by the so-called Hubble constant whose absolute value is very poorly known. Below, all distances will be measured in units of z. For orientation we mention that galaxies up to $z = 0.2$–0.4 have been well investigated, while quasars are known up to $z = 3.5$.

It is not difficult to comprehend that the Universe is homogeneous and isotropic only in a certain reference frame, namely, that in which relic emission is at rest (on the average). In other words, homogeneity and isotropy fix a convenient naturally selected frame of reference in space—time. From the fact of the Universe's expansion it follows (see [49], for instance) that the space—time of the real Universe is diffeomorphic to the direct product $M^3 \times R^1$, where R^1 is the time axis and M^3 is a space section. If the homogeneity and isotropy of the real Universe were ideal, then this would signify that the space—time's metric can be reduced to the form

$$ds^2 = c^2dt - a^2(t)\,d\Sigma^2, \tag{1}$$

where $d\Sigma^2$ is a space form of constant curvature, being the metric of the space section, while the function $a(t)$ describes the temporal evolution of the Universe. The real Universe is homogeneous and isotropic only approximately, but since Schur's theorem is in general ill-posed in Hadamard's sense, as Gribkov established [18], i.e., a space with sectional curvature changing little with respect to direction can strongly differ from a space of constant curvature, we have no postulate additionally that the metric of the real Universe is in some sense close to metric (1).

The concrete form of function $a(t)$ is found with the aid of the Einstein equations. This was done already in the 1920s by the Soviet mathematician Friedmann [84]. Therefore, metric (1) is called the Friedmann model of the Universe. It is curious that neither Friedmann nor Einstein, who at this time were also seeking homogeneous and isotropic models of the Universe, still did not have reliable information on the fact that the real Universe indeed is homogeneous and isotropic. At the time it still was not at all clear that the series of nebulae observed were galaxies similar to the Galaxy of the Milky Way (a remark by I. S. Shklovskii).

The Friedmann model with a fixed value of the Hubble constant is determined by the value of the mean density of the Universe. Here three qualitatively different variants are possible: the mean density ρ can be greater than, equal to, or less than a certain quantity called the critical density ρ_c. Here the curvature of the space form $d\Sigma^2$ will be, respectively, greater than, equal to, or less than zero. In all three cases the Universe came into existence from some singular state, while in one of the cases also ends its evolution in a singular state. In our time there apparently is no necessity of stressing that the presence of singular states signifies not the act of creation and annihilation of the Universe but the restriction on the methods of describing the Universe within the framework of the general theory of relativity.

Thus, if the mean density of the Universe is less than critical, then our Universe can be described by Lobachevskii geometry. However, what is the Universe's density in fact? The answer to this question requires a tedious and very difficult calculation of the masses of the luminous objects. The performing of this calculation is tied to the name of the Dutch astronomer Oort [101]. It turned out that the mean density of the luminous matter in the Universe is roughly (1/20)-th of the critical density ($\rho/\rho_c = 1/20$). However, the heavy particles, viz., the baryons of which the luminous objects mainly consist, comprise only a small percentage of all the particles in the Universe. The number of particles — photons — in relic emission exceeds by roughly 10^8 times the number of baryons. It is precisely this fact to bear in mind when speaking of the hot Universe. Since photons do not have a rest mass, this circumstance changes the estimate of the mean density by little. However, roughly there are as many neutrinos in the Universe. Up to recently it was reckoned that this particle also has zero rest mass; but most recently a group of Soviet physicists [97] established, as a result of very tedious and subtle measurements, that apparently a neutrino has a nonzero rest mass, and, moreover, this mass can suffice to make the Universe's mean density close to critical and, possibly, even greater than it.

What really is the radius of curvature of the space section of the real Universe? This question was first posed by Lobachevskii himself in connection with the question on which of the geometries best describes the real Universe. By investigating the parallaxes of the closest stars he showed that the radius of curvature of the Universe is much greater than the typical distance to the nearest stars, equal to several parsecs. Contemporary estimates of the radius of curvature depend upon the value of the Universe's mean density. However, if the Universe's density is much less than critical, then the radius of curvature, measured in units of red shift, ceases to depend upon the mean density. To be precise, as is not difficult to verify (see [50], for example),

$$R = e - 1 \approx 1{,}7.$$

In other words, if $z \ll 1.7$, then the geometry of the space can be taken to be Euclidean, while for $z \geqslant 1.7$, the geometry of the space is essentially non-Euclidean.

A more detailed description of the distribution and evolution of matter in the Universe assumes a variation of the curvature from point to point and with respect to directions. In other words, it is necessary to examine models close to the Friedmann one. This is done in the so-called perturbation theory in which it is assumed that the variations of density $\delta\rho/\rho$, of velocity, and of other physical quantities are small. On the strength of the Einstein equations the Universe's temporal evolution also is close to the evolution of Friedmann's model and it can be studied by perturbation theory methods. The matter is more subtle with the space structure of such perturbed models. The thing is that small relative variations of density do not at all necessarily correspond to small relative variations of curvature K; roughly speaking, $\delta K/K \sim \delta\rho/(\rho - \rho_c)$, i.e., in order for $\delta K/K$ to be small, a smallness of quantities of the type $\delta\rho/(\rho - \rho_c)$ is necessary. In other words, if the density of the Universe is close to critical, then a small perturbation in it can strongly change the space structure of the Universe. Things are not restricted even by the fact that a model with a space of negative curvature can be made into a model with a space of positive curvature. For example, a space section can be made diffeomorphic to the direct product of a sphere and a straight line. The fulfillment of the condition $\delta\rho \ll (\rho - \rho_c)$ is not as harmless as it seems at first glance. The thing is that at the early stages in the evolution of the Universe, when the process of development of the inhomogeneities started, the Universe's density, as is well known, was close to critical.

A space form of constant curvature can have a highly diverse topological and geometric structure (for one and the same curvature). It is commonly known that from a space of zero curvature we can splice tori and other more complex manifolds by means of identifying points of a cylinder. It is clear that similar splicings are possible also in spaces of positive and negative curvature. In connection with physics Clifford and Klein were the first to turn their attention to this possibility. Their interest was in the so-called gravitational paradox arising when considering an unbounded space uniformly filled with matter within the framework of Newton's theory of gravity. Here, as is easy to be convinced, it turns out that the gravitational potential becomes infinite. Historically the gravitational paradox was resolved within the framework of the general theory of relativity; next, under a retrospective analysis of Newton's theory of gravity from the point of view of the general theory of relativity it was clear (see [23]) that the gravitational paradox can be resolved also within the framework of Newton's theory itself if instead of the directly unobservable potential we examine the actually observable second derivatives of the potential. Clifford and Klein tried to resolve this paradox by an entirely different method — they assumed that the real (Newtonian!) space has the topology of, say, the torus in which there simply is no infinity. Physics did not develop along this path, but the work of Clifford and Klein resulted in the formulation of a scientific direction investigating space forms of constant curvature. In connection with this, those of them which are closed are often called Clifford—Klein space forms.

The first stage in the study of space forms of constant curvature was completed in the Thirties and was connected with the name of Hopf. The main achievement of this stage was the algebraization of the problem: it was shown that space forms of constant curvature are factor spaces of universal coverings (i.e., spheres in Euclidean space and Lobachevskii space) by those discrete subgroups of the motion group which do not have fixed points. In the subsequent developments efforts were concentrated on seeking such subgroups. The results of these researches, intensively pursued by various mathematicians, have been summarized in [121]. To be precise, a topological and geometric classification was made of space forms of constant curvature, not fully complete but sufficiently so for applications of the theory of space forms of zero curvature. To the contrary, the problem of classifying space forms of negative curvature in the general setting is exceedingly difficult and still has not been solved.

As a matter of fact, right after the construction of the general theory of relativity it was noted that locally isometric models of space—time may be topologically different. The first to turn their attention to this were Einstein [83] and Klein [94]. However, up to the Seventies the question of what really is the Universe's space structure was analyzed only in infrequent isolated papers. Among these we should particularly note the paper [102] by the Hungarian astronomer Paal who tried to explain certain singularities in the distribution of quasars by a topological structure of the space section. Although at the present time it is clear that these observational singularities are connected apparently with complely different effects (observational selection), nevertheless Paal's paper was the first wherein the problem of the global structure of the space section was considered from the point of view of concrete data of astrophysics.

What do we know today about the structure of the space section of the Universe? It is clear that, remaining within the framework of the general theory of relativity, this structure can be determined only from observations: the Einstein equations are equally well satisfied by any variants of the global structure of metric $d\Sigma^2$. In order to make the needed comparison with observations it is necessary first of all to construct such geometric characteristics of the space form, or, as we say in cosmology, of the spliced universes, which could be compared with the observations. The fact of the matter is that the usual topological and geometric classification of space forms of constant curvature, because of its algebraic character, does not contain similar quantities; it is clear that arbitrarily diverse stars can be seen in a telescope, but never a free cyclic group.

The parameters characterizing the structure of spliced universes and admitting of comparison with observations were constructed by Sokolov and Shvartsman [53]. To construct these parameters it is necessary first to introduce the concept of so-called ghosts. On the space form $d\Sigma^2$ we consider a point b corresponding to an observer. On the universal covering of the space form $d\Sigma^2$ there exists a class $\{B_i\}$ of selected points corresponding to the observer. One of them (it makes no difference which) is named the original B of observer b, while the ones remaining are called the observer's ghosts. Now on $d\Sigma^2$ we consider some point a. We take it that some luminous object is located at a. By observation we see a portrait of the universal covering; in other words, we perceive the light rays taking different paths from the one object a as rays emitted by different sources A_i disposed at equivalent points of the universal covering M. The source A which is nearest to the original B of the observer is called the original of the object, while the remaining sources are called ghosts of the source. Clearly, we can also reckon that to one object there will correspond several originals of the object, located at different distances from the original of the observer. Then, as the original we shall arbitrarily take one of these sources (the set of similar points on the universal covering has measure zero). Now we denote the collection of all originals of the sources (arising as a ranges $d\Sigma^2$) by H(B), where point B fixes the position of the observer's original, and we call this set the domain of originals. It is evident that the domain of originals is a fundamental domain. Now it is easy to construct the parameters we need, which we agree to call the splicing parameters. To be precise, the least upper bound of the distances from point B to the boundary of the domain of originals is called the maximum splicing parameter and the greatest lower bound of these distances is called the minimum splicing parameter.

It is important to stress that the effects being examined of the complex structure of the space section are not the only cause which can lead to the appearance of ghosts. Already at the initial stage in the development of the general theory of relativity it was realized that ghosts can appear in the model of a universe with a spherical space section when light is able to run around the sphere's great circle several times. Actually, in the Friedmann model this can happen only when expansion is replaced by contraction; however, more complex models with the so-called cosmological constant can be constructed, in which ghosts of this form may be observed today [105]. Such ghosts are easily distinguished from the ghosts generated by a splicing — they will be located in the diametrically opposite directions on the celestial sphere. In spite of the fact that searches for such ghosts were undertaken (see [106], for example), the results proved negative [110]. Of much greater interest are the ghosts arising because of local inhomogeneities of the Universe. The thing is that a galaxy or some other object located on the path of light propagation acts on it like a lens. This effect is called a gravitational lens. The action of a gravitational lens can result in the emergence of an additional image of the remote source. The effect can actually be observed: two remote quasars with very similar characteristics were discovered, very closely disposed on the celestial sphere, and, moreover, a galaxy has been detected lying on the light propagation path from these sources, whose mass is sufficient for the creation of the necessary effect. A distinguishing feature of these ghosts is exactly the presence of the object, viz., the gravitational lens, as well as the small possible number of images and their extreme proximity to each other.

One would think that it is easy to organize searches for ghosts in any concrete model of the spliced universe. For this it is sufficient to choose a few sources easily accessible for observation and to look for their ghosts at places predicted in advance (clearly, it will in fact be necessary to look over the possible values of several parameters describing the splicing). However, the matter is complicated by the fact that from the different images of one and the same object the light comes to the observer at different times. On the other hand, all the objects bright and well accessible to observation evolve very rapidly (on a cosmological measure) and we simply are unable to identify one source of ghosts from another, possibly already merging source.

Within the scope of this article we shall not dwell on concrete estimates of the splicing parameters from available observations. We merely present the result: the maximum splicing parameter is not less than 400 megaparsecs ($z \sim 0.1$) [53]; the minimum splicing parameter is estimated from below by a quantity of the same order, but this estimate cannot be obtained by ghost seeking methods and other methods are needed, which we shall survey below. It is important, however, to emphasize right away that nothing in contemporary observational data contradicts the fact that the universe is spliced, say, in the form of a torus with a distance between the spliced planes of the order of $z \sim 0.5$-1 [44]. This fact signifies, in particular, that the oft-encountered assertion that the negativeness of the sign of the space section's curvature implies that the universe is spatially unbounded is based on a misunderstanding.

Thus, we return to methods of estimating the minimum splicing parameter. It is difficult to estimate it with the aid of seeking ghosts simply because the maximum splicing parameter is the distance after which there are no originals, while the minimum is the distance up to which there are no ghosts, while it is easier to recognize some outstanding original than to be convinced that there are not a small number of visible ghosts in the sky. However, here the fact comes to our aid that the splicings violate the global homogeneity and isotropy of the universe. Indeed, in the majority of spliced universes the domain of originals and the splicing parameters depend upon the observer's position. In certain particularly simple flat spliced universes (cylinders and tori) there is no such dependency, but the domain of originals does not pass into itself under a rotation around the observer's position (simply speaking, it is not a ball). There is only one preferred spliced universe — a projective space with a positive curvature — for which the splicing does not violate homogeneity and isotropy (the first to pay attention to this was Zel'dovich [122]. It is not difficult to prove this fact by noting that the fundamental group of the spliced universe remains invariant in the motion group of the universal covering under the actions of rotations and translations only when its consists of unities and reflections.

Global inhomogeneity and anisotropy of a spliced universe is reflected by the spectrum of the Laplace operator on this manifold, and next also on the perturbation portrait in the spliced universe. It is natural to think that for a fixed wavelength of the perturbation in the original spectrum of perturbations from which the clustering of galaxies was subsequently formed, all possible spatial modes are represented roughly alike. For example, in the simplest case of a global uniform flat three-dimensional torus the perturbation spectrum is discrete and, moreover, its wave vectors have the form

$$\left(\frac{2\pi m}{a}, \frac{2\pi n}{b}, \frac{2\pi p}{c}\right);\ m,\ n,\ p=0, \pm 1, \pm 2, \ldots,$$

where a, b, c are the dimensions of the torus; the observable picture of the distribution of clusterings of galaxies will correspondingly be modulated and strict periodicity in the cellular structure must be observed. Clearly, this effect will be noted only if the perturbation's wavelength is commensurate with the dimensions of the torus. An analogous picture will emerge also in the more complex models of the spliced universe. Therefore, it is reasonable to take it that the lower bound on the minimum splicing parameter is the characteristic size of the cellular structure (since we know nothing about the existence of some exact periodicity of it).

This picture can acquire essentially new features in a globally inhomogeneous universe, for instance, in a horn (of a toroidal section, spliced from a Lobachevskii space). The thing is that if in a neighborhood of the Earth the distance between the spliced planes is much greater than the characteristic distance between the galactic clusters, then equally we can find domains wherein all these distances are equal and the distance between the planes being identified begins to be less than the distance between the clusters; such a domain is called a G-domain. As Sokolov and Starobinskii [52] showed, the formations of galactic clusters and, it seems, of galaxies themselves do not take place in G-domains of such type. On the celestial sphere such a domain will appear as a dark spot. We could think for a while that because of the negligible angular dimensions of an object with linear dimensions equal to the typical distance between galactic clusters there is no hope of seeing such a small detail in the distribution of the galaxies. However, as a matter of fact, the ghosts of this object lie in immediate proximity to the original, and, moreover, they form a horoball on the universal covering, which is seen at the angle

$$\sin \varphi/2 = \frac{1}{z+1},$$

where z is the red shift by which we see the closest point of the boundary of the G-domain. For other types of splicing, G-domains are possible in which galaxies not groupable into clusters will be formed.

Another observational appearance of G-domains was noted by Sokolov and Starobinskii [51]. Since points of G-domains have many ghosts whose red shift differs little from the red shift of the original, it turns out that if the time of stable activity of a quasar is of the order of 10^6-10^7 years, while $z \sim 1$-3, then in the G-domain we observe 5-20 ghosts of this quasar, with practically the same red shift. In other words, if such a small cluster of quasars of roughly the same type were to be detected, then it would make sense to analyze it for the purpose of seeking splicings. An important distinctive peculiarity of this effect, as compared with the effect of the gravitational lens, is that we must observe many quasars at different stages of development. The case particularly favorable in this sense is the one when the quasar arises in immediate proximity of the G-domain's boundary in a horn with a toroidal section. There, under favorable conditions, several hundred ghosts of the quasar could be observed! The name G-domain, i.e., the domain of ghosts (from the German Geist — ghost), is connected precisely with this effect of the multiple appearance of ghosts.

The disconsoling situation outlined above on the possibility of an observational study of the Universe's structure compels us once again to return to the question on the possibility of a theoretical prediction of some topological structure or other of the space section. This question cannot be answered within the framework of the standard general theory of relativity in which some splicing or other is prescribed in the boundary conditions upon coming out of the quantum—gravitational stage of evolution of the Universe close to a singularity. Further, this global structure now cannot change. However, it is possible that the quantum theory of gravitation rejects some, and perhaps all, possible spliced universes except universal coverings. In the classical domain let the space section be a flat three-dimensional torus. Then in the quantum domain we apparently need to examine a superspace over the direct product of the three-dimensional torus by the time-axis, as well as a quantum—gravitational Feynman integral (see [56]) on this superspace. It should be noted that no consistent quantum theory of gravitation exists at the present time and that not only is this integral impossible to compute but also it is by far not always clear how to write it.

Therefore, we restrict ourselves only to the case of the powdered state equation, without discussing the question of its applicability in such a situation. We take it that in quantum theory a saddle-point of the Feynman integral corresponds to classical space—time. However, in the case at hand this integral is determined not only by local parameters — the components of the metric tensor — but also by three global parameters — the dimensions of the torus. Clearly, the desired saddle-point must be determined with regard to these parameters too. But it is easy to verify that the presence of the global parameters causes the absence of a saddle-point in the integral being examined. Indeed, at points that are saddle-points with respect to the local parameters the action is proportional to the manifold's volume, which is a monotone function of the parameters. For the usual universal covering there are no such global parameters, and the difficulty indicated does not arise. Thus, the case being considered is apparently forbidden by the quantum gravitation laws. Unfortunately, similar continuous families of diffeomorphic but nonisometric spliced universes do not altogether exhaust their diversity: more complex spliced universes are characterized by a discrete collection of parameters and it is impossible to apply to them directly the arguments presented.

One would think that the class of spliced universes being examined could be extended still further by including among them the factor spaces which have fundamental groups with fixed points. At these fixed points the space will be structured just as at a cone's vertex. Once again this does not lead to tangible observational consequences beyond these conic points. However, in this case the conic point itself should be looked upon as a point in space—time, and matter placed at it so as to satisfy the Einstein equations. The necessary calculations cannot be performed by the methods of the usual general theory of relativity since at the conic points space—time is not regular, and we can encounter a true singularity which, however, may have nothing in common with the quantum—gravitational singularities with whose aid the initial stages of the Universe's evolution must be described. An analysis of these irregularities by the methods of Aleksandrov's theory of surfaces of bounded curvature was made in [24]. It turns out that according to the Einstein equations matter with the most unrealistic properties (negative anisotropic motion, etc.) must be concentrated at the conic singularities. However, the impossibility of similar singularities is most brilliantly proved by considering quantum effects in their neighborhood, where, as Starobinskii [54] showed, there must be observed a vast polarization of vacuum, tending to infinity as the singularity is approached. In the usual spliced universes there also exists a certain polarization of vacuum, but it is unobservable because of its negligibly small magnitude.

3. Velocity Space in the Special Theory of Relativity Is a Lobachevskii Space

In Sec. 1 we remarked that a Lobachevskii space can be realized as one of the components of the unit sphere in a pseudo-Euclidean space. This means that the orthogonal motion group in the special theory of relativity is the motion group in the Lobachevskii space. From the viewpoint of physics this fact can be interpreted as follows: the velocity space in the special theory of relativity is a Lobachevskii space. As an illustration it is useful to consider an analog of the velocity space in classical mechanics. The space—time of classical mechanics is the so-called Galilean space E_g with coordinates x, y, z, t, in which two metrics are specified:

$$\rho_1(p_1, p_2) = |t_1 - t_2|,$$
$$\rho_2(p_1, p_2) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

if $\rho_1 = 0$. The velocity space in this geometry, i.e., its unit sphere, has the equation $|t| = 1$ (it is unconnected!). The motion group of this space — the Galilean group — consists of all possible motions of the Euclidean space E^3. In a certain (precisely defined) sense it also is the limit of the Lorentz group as $v/c \to 0$. The Galilean group contains as a subgroup an orthogonal group of the three-dimensional space but, of course, does not reduce to it. In the analysis of the interpretation of classical mechanics with the aid of Galilean space—time it is necessary to remember that such an approach is in retrospection a presentation of classical mechanics in a system, unnatural to it, of concepts of relativity theory. To a certain extent an analogous retrospection is possible also when analyzing earlier physical theories. For example, we can try to set forth Aristotelian physics in terms of the theory of space—time (as done, for instance, by Penrose [103]). It is clear that the farther back we go from the present time the less we can retain similar retrospections of the inherent features of corresponding physical theories. However, similar investigations generate interesting geometric constructions. In this connection we note an interesting series of papers by Artykbaev on the theory of surfaces in Galilean space (see [4], for example). He proved that the geometry of Galilean space furnishes an adequate apparatus for the investigation of the Dirichlet problem for elliptic equations of the Monge—Ampere type in nonconvex regions.

Before we pass on to a detailed description of the velocity space in relativity theory, we discourse briefly on the history of the question. The first steps in forming the corresponding look at mechanics were already taken by Lobachevskii himself [31], as well as (in a somewhat more concrete form) by De-Tilly [81, 82], Genocchi [85], and Schering [108, 109], in connection with the elucidation of the possibility of constructing mechanics in Lobachevskii space. Relying on the well-known analogy between the kinematics and statics of a rigid body in Euclidean space and trying to carry it over into the mechanics of a non-Euclidean space, they ran into the necessity of constructing a new theory of vectors, suitable for non-Euclidean spaces.

A further development of this circle of ideas, with a bringing in of the tools of projective geometry, was effected by Lindemann [96], Clifford [79], Cox [80], and Buchheim [77]. Besides the usual vectors (point pairs) they examined pairs of planes and pairs of straight lines as elements of vector theory. In accordance with this, complex numbers with two identity elements were introduced in vector algebra. This theory was brought to its logical conclusion by Kotel'nikov [28]. He discovered the so-called principle of correspondence which enabled the study of the complex objects of this theory to be reduced to the study of a considerably simpler model, viz., a sheaf of vectors in Euclidean space.

Einstein's discovery of the special theory of relativity gave these investigations a new direction — the determination of the connection between the geometries of the velocity space in special theory of relativity and of the Lobachevskii space. It seems that the first one to turn his attention to this connection was Sommerfeld [111] in 1909, and next, independently, Varicak [120], Herglotz [88], and Robb [107].

Later, in 1914, there appeared the Russian translation [27] of Klein's paper, stimulating the works of the Kazan school of geometers. Klein proved that the Lorentz group is isomorphic with the motion group of Lobachevskii space. In 1923 Kotel'nikov [30] completely established the connection between Lobachevskii geometry and the special theory of relativity. Representing the velocity of a particle as an infinitely distant point in space—time, Kotel'nikov introduced a projective velocity space. In this space the velocity of a particle with zero rest mass (a photon) lies on the absolute, the velocity of a particle with positive rest mass lies inside the absolute, while the velocity of a particle with an imaginary rest mass (such particles are now called tachyons) lies outside the absolute. Here the inner region of the absolute is a Lobachevskii space with a characteristic constant c equal to the velocity of light.

Interest in these investigations arose anew only in the mid-1950s in connection with the appearance of the book [57] by Fock* and of the paper [58] by Chernikov. In the Fock book mentioned, the introduction of the Lobachevskii velocity space is based on the Einstein—Poincaré formulas for the relative velocity of particles and on Beltrami's model of Lobachevskii geometry. In contrast to Fock, Chernikov's construction is based on the introduction of a fiber space.

Let us explain the basic idea of Chernikov's construction. The space—time of relativity theory is a smooth manifold in which smooth curves correspond to the paths of the particles. Thus, at each point $x \in M^4$ the particles' velocity space is a projective three-dimensional space $P^3(x)$. The space, however, of all velocities of the particles is a seven-dimensional space with base M^4 and fiber $P^3(x)$. Here the differentials of the coordinates on manifold M^4 are homogeneous coordinates in $P^3(x)$. In fiber $P^3(x)$ we prescribe a metric $g_{ij}dx^i dx^j$ with signature $(+, -, -, -)$. For ordinary particles, $g_{ij}dx^i dx^j > 0$. This domain is a Lobachevskii space with a characteristic constant c equal to the velocity of light. The velocities of particles with zero rest mass (photons) are determined by the condition $g_{ij}dx^i dx^j = 0$, while of particles with imaginary rest mass (tachyons), by the condition $g_{ij}dx^i dx^j < 0$.

*Translator's note: Strictly speaking this Russian name should be transliterated as "Fok"; however, it has become so widely known in English as "Fock" that I have used the latter here.

A more detailed development of the theory of velocity space can be found in Chernikov [58-68]. At the present time this approach is now systematically used by a number of physicists (see [9, 26, 45, 46]).

Comparatively recently, the study of cosmic rays showed that at velocities close to that of light the Lorentz relations may be violated. The reason for this phenomenon is the following. Close to gravitating masses (the solar system, for instance) cosmic rays may be treated as test particles. In other words, their natural gravity field can be neglected in comparison with the gravity field of the gravitating masses. In return, at a distance from the gravitating masses, conversely, we can neglect the external gravity field — it is small in comparison with the natural field of the particles. Here the natural field, of course, can depend on the direction of the particles' velocity, which leads to an anisotropy of the velocity space. By the same token the space becomes a Finsler space.

The geometric aspects of such a local anisotropy were repeatedly discussed in the literature [11, 91, 93, 98, 116-118]; however, the question of making concrete the type of Finsler spaces suitable for describing this effect remained open. This question was resolved by Asanov [5-8] by studying the properties of the Finsler space $F_N\{\rho_A{}^m\}$ and proving that when $N=4$ the structure of this space is completely consistent with relativity theory.

In view of the above, the question on the connection between the motion groups of Lobachevskii and Euclidean spaces is of interest. A natural way to state this question is to ask about the possibility of isometric immersions of Lobachevskii spaces into a Euclidean space, under which some motion group of Lobachevskii space can be induced by motions of the enveloping Euclidean space. The first to tackle this question was, apparently, Bieberbach [76]. He constructed an isometric immersion of the Lobachevskii plane into Hilbert space, under which any intrinsic motion of the Lobachevskii plane is realized with the aid of a corresponding motion of the Hilbert space (examples of such immersions into E^3 of metrics of zero and of constant positive curvature are the plane and the sphere). Kadomtsev [25] studied these questions further. He proved the following two statements:

1) The Lobachevskii plane does not admit of C^0-immersions into any finite-dimensional Euclidean space as a surface of revolution with a pole (with a fixed point); here such an immersion of a part of the Lobachevskii plane now proves possible into E^4.

2) The Lobachevskii plane does not admit of C^0-immersions into any finite-dimensional Euclidean space as a surface admitting of motions on itself along any geodesic of itself* (a circular cylinder can serve as an example of such a surface of zero curvature in E^3).

In the same paper these statements were generalized to the case of immersions of the Lobachevskii space Λ^k into E^n.

We remark that a central place in the proof of these propositions is the proof of a certain auxiliary assertion which, incidently, is of independent interest. Its substance is as follows. Let L be a curve of class C^0 in Euclidean space, having the following property: for any two points of curve L there exist mutually congruent neighborhoods of these points (examples of such curves in E^3 are a straight line, a circle, and a helix). Then, the curve L is analytic.

Kadomtsev has obtained the following generalization of the two statements cited. The Lobachevskii plane cannot be immersed into any finite-dimensional Euclidean space, in class C^0, such that some continuous motion group can be realized with the aid of motions of the enveloping space. Let us explain briefly the idea of this proof. We assume to the contrary that the immersion indicated is possible. Then the metric of the corresponding surface can be led to the form

$$dl^2=ds^2+H^2(s)\,dt^2,$$

where the coordinate lines of t are simultaneously the paths of some one-parameter motion group of the Lobachevskii plane and of some one-parameter motion group of the enveloping Euclidean space. Allowing for the form of the motions of the enveloping space, we can obtain the following bound for the function H(s): $|H(s)|\leqslant Cs^2$ for sufficiently large s. On the other hand, the function H(s) obviously grows exponentially. The contradiction obtained proves the assertion.

*Translator's note: In English we would most likely say: "as a surface with parallel transport."

4. Lobachevskii Geometry and Nonlinear Equations of Mathematical Physics

From the point of view of physics hyperbolic equations describe first of all various wave processes. In particular, nonlinear hyperbolic equations describe the propagation of nonlinear waves, drawing intent attention in the most diverse areas of physics. In this class is the equation bearing the somewhat unusual name the "sine-Gordon equation":

$$z_{tt}-z_{xx}=\sin z.$$

In relativistic quantum mechanics the wave function of a free particle of mass m and zero spin can be described by the so-called Klein—Gordon equation

$$w_{tt}-w_{xx}=m^2w.$$

The developing nonlinear field theory required nonlinear analogs of the Klein—Gordon equation; moreover, the simplest successful version consisted in replacing the right-hand side m^2w by $\sin w$. By analogy with the name "Klein—Gordon equation" arose the name "sine-Gordon equation." At that time it went unnoticed that the sine-Gordon equation was well known in geometry and had been for a long time. Already in 1878 in a report by the outstanding Russian mathematician Chebyshev "On the cutting out of garments" [119] there was considered the question of nets on surfaces, in which in any quadrangle of the net the opposite sides have a like length. If lines of this net are taken as coordinate lines on the surface, then the line elements of the surface take the form

$$ds^2=du^2+2\cos z\,du\,dv+dv^2,$$

where z is the angle between the net's lines, while u and v are arc lengths of the coordinate lines. Chebyshev indicated a differential equation connecting the curvature K of this surface and the angle z. In modern notation it has the form

$$z_{uv}=-K\sin z.$$

For a surface of constant negative (say, —1) curvature this equation reduces by a simple substitution to the sine-Gordon equation. In 1900 Hilbert [90] showed that the Chebyshev net on a surface of constant negative curvature is formed by families of asymptotic lines. We recall that asymptotic lines are lines on a surface of negative curvature the tangent vectors of which form at the origin the second quadratic form of the surface. In the asymptotic coordinates u and v this form is particularly simple:

$$\mathrm{II}=2M\,du\,dv.$$

It is obvious that the net angle z on a regular surface must satisfy the condition

$$0<z<\pi, \tag{2}$$

which signifies that two families of asymptotic lines indeed do exist on a saddle surface. It turns out that the converse is true as well: from a given regular solution of the sine-Gordon equation, satisfying condition (2), we can construct a regular surface of constant negative curvature. In this same paper Hilbert showed that on the whole surface there does not exist a regular solution of the sine-Gordon equation, satisfying the geometric regularity condition (2). This was the famous proof of the theorem on the nonrealizability of the Lobachevskii plane in a Euclidean space.

Beginning with Minding [100] and Beltrami [74] many concrete surfaces of constant negative curvature were found, realizing a part of the Lobachevskii plane. In particular, Bäcklund gave a method for constructing from a known surface of constant negative curvature a new surface of the same kind. Bianchi [75] systematically developed the theory of these simple solutions of the sine-Gordon equation. Afterwards it turned out that many of these concrete solutions of the sine-Gordon equation have completely specific physical sense and describe the propagation of solitary waves, the so-called solitons. In modern physics the tool of finding soliton solutions, developed by Bianchi and Bäcklund, finds the widest application. The survey article [35] acquaints us with it. We remark that the solution

$$z=4\,\mathrm{arctg}\,e^{u+v}$$

corresponds to one soliton. We direct the reader's attention to the fact that this solution corresponds to the union of both the regular components of the pseudosphere which globally is not a regular surface. Thus the question arises on whether a nonregular surface can be compared with each regular solution of the sine-Gordon equation just as the pseudosphere corresponds to the soliton. An approach to answering this question was indicated by Gribkov [17] who heeded the fact that asymptotic lines on surfaces of revolution of constant negative curvature remain, under a transition of edges and spikes, regular space curves, the concept of angle between which retains its meaning. Later Poznyak [41] gave a general method for associating a solution of the sine-Gordon equation with a surface in Euclidean space.

Surfaces in pseudo-Euclidean space enable us to give a geometric interpretation also to equations resulting from the sine-Gordon equation by complex substitutions (or, if desired, an interpretation of the complex solutions of this equation) of the type

$$w_{uv}=\operatorname{sh} w,$$
$$w_{uu}+w_{vv}=\sin w,$$
$$w_{uu}+w_{vv}=\operatorname{sh} w.$$

The first one of them corresponds to the angle between the asymptotic lines on a surface of constant positive curvature +1 with an indefinite metric. The geometric interpretation of elliptic equations must be somewhat more refined: they must be interpreted with the aid of convex surfaces, on which there are no asymptotic lines. These lines have to be introduced artificially by introducing complex coordinates on the surface. This method is analogous to the introduction of imaginary time in relativity theory, which permits us to analyze in certain cases a space with an indefinite metric not unlike a space with a positive-definite metric.

Returning again to the sine-Gordon equation, we note that Bianchi began to examine various boundary-value problems for this equation. We can acquaint ourselves with the contemporary state of this question from the survey article [39] by Poznyak, which gives as well a number of negative results, and also from his other article [40]. The multidimensional geometry of equations of the sine-Gordon type was developed by Aminov [2].

The authors thank R. F. Galeevaya, Ya. A. Smorodinskii, and S. F. Shandarin for useful discussions.

APPENDIX

THEOREM. Let Φ be an intrinsically complete surface of class C^2 with an indefinite metric of sign-constant curvature K in a pseudo-Euclidean space. Then surface Φ is extrinsically complete.

Proof. First of all we note that all anisotropic geodesics of surface Φ will be extrinsically complete curves. This is an obvious consequence of the inverse triangle inequality. Let us now assume that our theorem is false and that surface Φ^* is incomplete. Let us consider its completion $[\Phi^*]$ in E^3 and let P^* be a point of Φ^*. We surround point P^* with a closed ball $V_{P^*}(R)$ and we let $\Omega^*(P^*, R)$ be the connected component of set $[\Phi^*]$, lying in this ball and containing point P^*. We increase the ball's radius until points of set $[\Phi^*]\setminus\Phi$ appear in the set $\Omega^*(P^*, R)$. By a small shift of point P^* we easily achieve that precisely one point of set $[\Phi^*]\setminus\Phi$ appears on the boundary of set $\Omega^*(P^*, R)$. We denote this point by Q^*. Now our task will be to construct an anisotropic geodesic resting against the point Q corresponding to point Q^*. The implementation of this task is somewhat complicated by the fact that on a pseudo-Riemannian manifold geodesic completeness does not ensure geodesic connectivity. This, in its own turn, is related to the fact a quantity $K(v, v)$, where v is the geodesic's unit tangent vector, occurs in the Jacobi equation for the variations of the geodesics, instead of the curvature. Therefore, for some geodesics we can observe both exponential instability and a coming together of geodesics on one and the same manifold. From the conjugate points there can arise points which cannot be connected with the initial point by a geodesic. However, for such points, which can be joined by a curve on which $K(v, v) < 0$, the property of geodesic connectivity if preserved, as is easily seen, since the condition of scattering of shortest lines operates and the boundary-value problem for the Jacobi equation is always uniquely solvable (we took advantage of the sign-constancy of the curvature).

We now consider the set $\Omega(P^*, R)$ corresponding in the pseudo-Euclidean space to the set $\Omega^*(P^*, R)$. Two possibilities can present themselves: either in any neighborhood of point Q there exist points of this set which can be joined with Q by a curve such that the inequality $K(v, v) < 0$ is fulfilled on it, or such points are not found. Consider the first possibility. We fix one of the points indicated and we call it M. From the other points we form a sequence M_n converging to point Q. Obviously, we can achieve that points M and M_n also can be joined by curves on which $K(v, v) < 0$. We now join points M_n and M by geodesic segments γ_n. It is not difficult to choose the points M_n in such a way that the sequence γ_n converges to an anisotropic geodesic issuing from M and, clearly, resting against Q. To do this it is enough, as an example, to choose points either on the boundary of set $\Omega(P^*, R)$ or on isotropic lines issuing from point Q (also, of course, these lines themselves, being geodesics in the two-dimensional case, are used with difficulty since we have not proved their extrinsic completeness). It now remains for us to recall that anisotropic geodesics of surface Φ must be extrinsically complete, but this condition is clearly violated here.

We now turn to the second possibility. It is easy to see that now Q^* is an isolated point of set $[\Phi^*]\setminus\Phi$ and surface Φ has a spike at it, so that the coordinate z, treated as a function of point P on Φ^*, has an extremum at Q. We take it that the metric of the enveloping space has the form $ds^2 = dx^2 + dy^2 - dz^2$. Consider a point P_0 close to Q and from it issue a geodesic with a negative square of the tangent vector. On this geodesic the function z(P) cannot have local internal extrema; therefore, it rests against point Q, and once again we arrive at a contradiction with the extrinsic completeness of anisotropic geodesics. The theorem is proved.

We remark that in proving this theorem we did not even require geodesic completeness but, in general, the somewhat weaker conditions of completeness of the anisotropic geodesics. From the theorem proved follows, in particular, also the extrinsic completeness of an intrinsically complete tube with an indefinite metric of positive curvature. In this case the extrinsic completeness simply signifies that the tube is a surface going off to infinity, projecting onto the whole z axis.

LITERATURE CITED

1. A. D. Aleksandrov, Intrinsic Geometry of Convex Surfaces [in Russian], Fizmatgiz, Moscow—Leningrad (1948).
2. Yu. A. Aminov, "Isometric immersions of domains of n-dimensional Lobachevskii space in a (2n — 1)-dimensional Euclidean space," Mat. Sb., 111, No. 3, 402-433 (1980).
3. V. I. Arnold, Mathematical Methods of Classical Mechanics, Graduate Texts in Math., Vol. 60, Springer-Verlag, Berlin—Heidelberg—New York (1978).
4. A. Artykbaev, "Reconstruction of convex surfaces from their extrinsic curvature in spaces with projective metrics," Dokl. Akad. Nauk Uzb. SSR, No. 10, 6-7 (1976).
5. G. S. Asanov, "Finsler geometry of locally anisotropic space—time," in: Sixth All-Union Geometry Conference on Problems of Geometry (Rept. Abstr.) [in Russian], Vilnius Gos. Ped. Inst., Vilnius (1975), pp. 19-21.
6. G. S. Asanov, "Effects of locally anisotropic space—time defined by a Finsler structure," in: Problems of the Theory of Gravitation and Elementary Particles [in Russian], K. P. Stanyukovich (ed.), No. 8, Atomizdat, Moscow (1977), pp. 37-43.
7. G. S. Asanov "Gravitational field in a Finsler space based on the volume concept," Vestn. Mosk. Univ., Fiz., Astron., 17, No. 3, 288-296 (1976).
8. G. S. Asanov, "Finsler space with an algebraic metric defined by a field of frames," J. Sov. Math., 13, No. 5, 588-600 (1980).
9. N. M. Atakshiev, R. M. Mir-Kasimov, and Sh. M. Nagiev, "Equation of quasipotential type for the relativistic amplitude of scattering in a covariant Hamiltonian formulation of field theory on a light cone," Izv. Akad. Nauk Azer. SSR, No. 2, 15-22 (1977).
10. I. N. Bernshtein and V. F. Shvartsman, "On the connection between the size of the Universe and its curvature," Zh. Eksp. Teor. Fiz., 79, No. 5, 1617-1628 (1980).
11. G. Yu. Bogoslovskii, "Special relativistic theory of anisotropic space—time," in: Current Problems in the Physics of Elementary Particles [in Russian], Ob"edin. Inst. Yad. Issled., PI 2-8529, Dubna (1975), pp. 39-72.
12. A. N. Bogush, I. S. Kurochkin, and V. A. Fedorov, "Lobachevskii space vectors and relativistic kinematics," Dokl. Akad. Nauk SSSR, 236, No. 1, 58-60 (1977).
13. Yu. D. Burago and V. A. Zalgaller, "Convex sets in Riemannian spaces of nonnegative curvature," Usp. Mat. Nauk, 32, No. 3, 3-56 (1977).
14. Yu. D. Burago and V. A. Zalgaller, Geometric Inequalities [in Russian], Nauka, Leningrad (1980).
15. A. L. Verner, "Topological structure of complete surfaces of nonpositive curvature with a one-to-one spherical mapping," Vestn. Leningr. Univ., 20, No. 7, 16-29 (1965).

16. D. Hilbert, The Foundations of Geometry, The Open Court Publ. Co., La Salle, IL (1950).
17. I. V. Gribkov, "Construction of certain regular solutions of the "sine-Gordon" equation with the aid of surfaces of constant negative curvature," Vestn. Mosk. Univ., Ser. I, Mat., Mekh., No. 4, 78-83 (1977).
18. I. V. Gribkov, "Is Schur's theorem well-posed?," Dokl. Akad. Nauk SSSR, 252, No. 6, 1304-1307 (1980).
19. D. Gromoll, W. Klingenberg, and W. Meyer, "Riemannsche geometrie in Großen," Lect. Notes Math., 55, Springer-Verlag, Berlin—Heidelberg—New York (1968).
20. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, Modern Geometry. Methods and Applications [in Russian], Nauka, Moscow (1979).
21. N. V. Efimov, Higher Geometry [in Russian], 4th ed. rev. aug., Fizmatgiz, Moscow (1961).
22. N. V. Efimov, "Appearance of singularities on surfaces of negative curvature," Mat. Sb., 64, No. 2, 286-320 (1964).
23. Ya. B. Zel'dovich and I. D. Novikov, Structure and Evolution of the Universe [in Russian], Nauka, Moscow (1975).
24. Ya. B. Zel'dovich, D. D. Sokolov, and A. A. Starobinskii, "Certain questions of global geometry in the general theory of relativity," All-Union Conf. Non-Euclid. Geom.: 150 Years of Lobachevskii Geometry, Kazan', 1976 [in Russian], Moscow (1977), pp. 122-132.
25. S. B. Kadomtsev, "Impossibility of some special isometric immersions of Lobachevskii spaces," Mat. Sb., 107, No. 2, 175-198 (1978).
26. V. G. Kadyshevskii, "Quantum field theory with a non-Euclidean space of relative momenta," Mater. Ob''edin. Inst. Yad. Issled., Dubna (1970).
27. F. Klein, "On the geometric foundations of the Lorentz group," in: New Ideas in Mathematics [in Russian], No. 5, St. Peterburg (1914), pp. 144-174. [Supplied by translator: This is the Russian translation of: F. Klein, "Über die geometrischen Grundlagen der Lorentzgruppe," Jahresber. Deutsch. Math.-Verein., Bd. 19 (1910). It can be found also in: R. Fricke and A. Ostrowski (eds.), Felix Klein: Gesammelte Mathematische Abhandlungen, Erster Band, Verlag von Julius Springer, Berlin (1921), pp. 533-552; reprinted, Springer-Verlag, Berlin—Heidelberg—New York (1973).]
28. A. P. Kotel'nikov, "Projective theory of vectors," Izv. Kazansk. Fiz.-Mat. Obshch., 2nd. ser., VIII and IX (1892).
29. A. P. Kotel'nikov, "Spiral calculation," Uch. Zap. Kazansk. Univ., (1895).
30. A. P. Kotel'nikov, "Principle of relativity and Lobachevskii geometry," in: In Memorial Lobatchevskii, Vol. 2, Kazan (1927), pp. 37-66.
31. N. I. Lobachevskii, Collected Works [in Russian], 5 Volumes, V. F. Kagan (ed.), Vol. 2, Gostekhizdat, Moscow—Leningrad (1949).
32. G. P. Matvievskaya, "Theory of parallel lines in Antiquity and in the Middle Ages," All-Union Conf. Non-Euclid. Geom.: 150 Years of Lobachevskii Geometry, Kazan, 1976 [in Russian], Moscow (1977), pp. 88-95.
33. S. P. Olovyanishnikov, "On the bending of infinite convex surfaces," Mat. Sb., 18, No. 3, 429-440 (1946).
34. W. Pauli, Theory of Relativity, Pergamon Press, London (1958). [Supplied by translator: This was originally an article by Pauli: "Relativitätstheorie," Encyklopädie der mathematischen Wissenschaften, Vol. V19, B. G. Teubner, Leipzig (1921).]
35. E. N. Pelinovskii, "The Bäcklund transform," Izv. Vyssh. Uchebn. Zaved., Radiofiz. 1, 19-36 (1976).
36. A. Z. Petrov, Einstein Spaces, Pergamon Press Ltd., Oxford—Edinbergh—New York (1969).
37. A. V. Pogorelov, Extrinsic Geometry of Convex Surfaces, Amer. Math. Soc., Providence, RI (1973).
38. A. V. Pogorelov, The Multidimensional Minkowski Problem, V. H. Winston and Sons, Washington, DC; Halsted Press, New York—Toronto—London (1978).
39. É. G. Poznyak, "Geometric investigations connected with the equation $z_{xy} = \sin z$," J. Sov. Math., 13, No. 5, 677-686 (1980).
40. É. G. Poznyak, "Certain new results on isometric immersions of parts of the Lobachevskii plane into E^3," All-Union Conf. Non-Euclid. Geom.: 150 Years of Lobachevskii Geometry, Kazan, 1976 [in Russian], Moscow (1977), pp. 73-78.
41. É. G. Poznyak, "Geometric interpretation of regular solutions of the equation $z_{xy} = \sin z$," Differents. Uravn., 15, No. 7, 1332-1336 (1979).
42. É. G. Poznyak, "On the regular global realization of two-dimensional metrics of negative curvature," Dokl. Akad. Nauk SSSR, 170, No. 4, 786-789 (1966).

43. É. G. Poznyak and E. V. Shikin, "Surfaces of negative curvature," J. Sov. Math., 5, No. 6, 865-887 (1976).
44. A. A. Ruzmaikin and D. D. Sokolov, "Inhomogeneous large-scale magnetic field and global structure of the Universe," Astrofizika, 13, No. 1, 95-102 (1977).
45. N. B. Skachkov and I. L. Solovtsov, "One-boson exchange potentials in relativistic configuration space," Yad. Fiz., 26, No. 4, 695-702 (1977).
46. Ya. A. Smorodinskii, "Kinematics of collisions in a geometric setting," in: Questions of the Physics of Elementary Particles [in Russian], Izd. Akad. Nauk Arm. SSR (1963), pp. 242-271.
47. D. D. Sokolov, "Surfaces in a pseudo-Euclidean space," J. Sov. Math., 17, No. 1, 1676-1683 (1981).
48. D. D. Sokolov, "On convex surfaces with indefinite metric," Usp. Mat. Nauk, 33, No. 4, 227-228 (1978).
49. D. D. Sokolov, "Certain properties of pseudo-Riemannian manifolds and their applications to the mathematical problems of cosmology," Ukr. Geom. Sb., 15, No. 1, 104-112 (1974).
50. D. D. Sokolov, "Large-scale magnetic field and global structure of the Universe," Pis'ma Astron. Zh., 1, No. 3, 3-5 (1975).
51. D. D. Sokolov and A. A. Starobinskii, "Globally inhomogeneous 'spliced' universes," Astron. Zh., 52, No. 5, 1041-1048 (1975).
52. D. D. Sokolov and A. A. Starobinskii, "G-domains in spliced universes and their observational appearances," in: Relativistic Astrophysics. Cosmology. Gravitational Experiment [in Russian], Inst. Fiz. Akad. Nauk BSSR, Minsk (1976), pp. 105-107.
53. D. D. Sokolov and V. F. Shvartsman, "An estimate of the size of the Universe from a topological point of view," Zh. Eksp. Teor. Fiz., 66, No. 2, 412-420 (1974).
54. A. A. Starobinskii, "Influence of space—time topology on the polarization of vacuum in quantum fields," in: Classical and Quantum Theory of Gravitation [in Russian], Inst. Fiz. Akad. Nauk BSSR, Minsk (1976), pp. 110-111.
55. L. V. Ten, "Rigidity of complete surfaces of negative curvature that coincide with a hyperbolic paraboloid outside a compact region," Usp. Mat. Nauk, 35, No. 6, 175-176 (1980).
56. R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York (1965).
57. V. A. Fok (Fock), The Theory of Space, Time, and Gravitation, Pergamon Press, New York—London (1959).
58. N. A. Chernikov, "Decay of a particle and combination of particles in images of velocity space," Nauchn. Dokl. Vyssh. Shkoly, Fiz.-Mat. Nauk, No. 2, 158-161 (1958).
59. N. A. Chernikov, "Relativistic Maxwell—Boltzmann distribution and the integral form of the conservation laws," Dokl. Akad. Nauk SSSR, 144, 544-547 (1962).
60. N. A. Chernikov, "Kinetic equation for a relativistic gas in an arbitrary gravitational field," Dokl. Akad. Nauk SSSR, 144, 89-92 (1962).
61. N. A. Chernikov, "Applications of Lobachevskii geometry in the special theory of relativity," Int. Winter School Theor. Phys. [in Russian], Ob''edin. Inst. Yad. Issled., Dubna (1964).
62. N. A. Chernikov, Lectures on Lobachevskii Geometry and Relativity Theory [in Russian], Novosibirsk State Univ. (1965).
63. N. A. Chernikov, "Stochastic motion of relativistic particles," Preprint (Inst. Teor. Fiz., 68-44), Kiev (1968).
64. N. A. Chernikov, "Lobachevskii geometry and relativistic mechanics," Fiz. Elementar. Chastits At. Yad., 4, No. 3, 773-810 (1973).
65. N. A. Chernikov, "Introduction to Lobachevskii geometry in mechanics," in: Classical and Quantum Theory of Gravitation [in Russian], Inst. Fiz. Akad. Nauk BSSR, Minsk (1976), pp. 47-53.
66. N. A. Chernikov, "Lobachevskii geometry as a physical science," Preprint (Ob''edin. Inst. Yad. Issled., R2-10251), Dubna (1976).
67. N. A. Chernikov, "Introduction to Lobachevskii geometry in mechanics," Preprint (Ob''edin. Inst. Yad. Issled., R2-9620), Dubna (1976).
68. N. A. Chernikov, "Lobachevskii geometry as a physical science," All-Union Conf. Non-Euclid. Geom.: 150 Years of Lobachevskii Geometry, Kazan, 1976 [in Russian], Moscow (1977), pp. 146-153.
69. L. Schwartz, Méthodes Mathématiques pour les Sciences Physiques, Hermann et Cie., Paris (1961).

70. S. Z. Shefel', "Completely regular isometric immersions into Euclidean space," Sib. Mat. Zh., 11, No. 2, 442-460 (1970).
71. E. V. Shikin, "On global isometric immersion into R^3 of certain metrics of nonpositive curvature," Dokl. Akad. Nauk SSSR, 215, No. 1, 61-63 (1974).
72. E. V. Shikin, "On the regularity of horocyclic coordinates," Mat. Zametki, 17, No. 3, 475-484 (1975).
73. P. A. Shirokov, "N. I. Lobachevskii as creator of a new geometric system (dedicated to the 150th anniversary of his birth)," Appendix in the book: N. I. Lobachevskii, Selected Papers on Geometry [in Russian], Kazan (1966).
74. E. Beltrami, "Sull superficie di rotazione che serve di tipo alle superficie pseudospheriche," Giorn. Math., 18, opere II (1872).
75. L. Bianchi, Lezioni di Geometria Differentiale, Vol. 1, Parte 2, Bologna (1927).
76. L. Bieberbach, "Eine singularitätenfreie Fläche konstanter negativer Krümmung im Hilbertschen Raum," Comment. Math. Helv., 4, 248-255 (1932).
77. A. Buchheim, "A memoir on biquaternions," Am. J. Math., 7, 293-326 (1885).
78. E. Cartan, La Géométrie des Espaces de Riemann, Mémorial des Sciences Mathématiques, Fascicule IX, Gauthiers-Villars et Cie., Paris (1925), 59 pp.
79. W. K. Clifford, "Preliminary sketch of biquaternions," Proc. London Math. Soc., 4, 381-395 (1873).
80. H. Cox, "On the applications of quaternions and Grassmann's Ausdehnungslehre to different kinds of uniform spaces," Trans. Cambridge Philos. Soc., Vol. IV, Pt. IV, 194-196 (1882).
81. J. De-Tilly, Etudes de Mécanique Abstraite, Mémoires Couronnés et Autres Mémoires, Publ. Acad. Belg., 21 (1870).
82. J. De-Tilly, "Rapport sur la 'Lettre a M. Quelet'," Bull. Acad. Belg., 36, No. 2 (1873).
83. A. Einstein, "Der Energiesatz in der allgemeine Relativitätstheorie," Sitzungsber. Preuss. Akad. Wiss., 1, 448-459 (1918).
84. A. Friedmann, "Über der Krümmung des Raumes," Z. Phys., 10, 377 (1922).
85. A. Genocchi, "Interna ad una dimostrazione di Daviet de Foncenex," Atti. Accad. Sci. Torino, 4(1869).
86. J. Hadamard, "Les surfaces a courbures opposées," J. Math. Pures Appl., Sér. 4, 5, 27-73 (1898).
87. P. Hartman and A. Wintner, "On the asymptotic curves of a surface," Am. J. Math., 73, No. 1, 149-172 (1951).
88. G. Herglotz, "Geometrie und relativistische Dynamik," Ann. Phys., 31, 404 (1910).
89. D. Hilbert, "Mathematische Probleme," Vortrag Gehalten auf dem Internationalen Math. Kongress zu Paris, 1900, Gött. Nachr. (1900), pp. 253-297.
90. D. Hilbert, "Ueber Flächen von constanter Gaussschen Krümmung," Trans. Am. Math. Soc., 2, 87-99 (1901).
91. J. I. Horváth, "New geometrical methods of the theory of physical fields," Nuovo Cimento, Ser. 10, 9, No. 2, Suppl. 2, 444-496 (1958).
92. J. I. Horváth and A. Moór, "Entwicklung einer einheitlichen Feldtheorie begründet auf die Finslersche Geometrie," Z. Phys., 131, 544-570 (1952).
93. J. I. Horváth and A. Moór, "Entwicklung einer Feldtheorie begründet auf einen allgemeinen metrischen Linienelementraum. I, II," Proc. Kon. Ned. Akad. Wetensch., A58, No. 4, 421-429, No. 5, 581-587 (1955).
94. F. Klein, "Über die Integralform der Erhaltungssätze und die Theorie de räumlich-geschlossenen Welt," Ges. Wiss. Gött., Math.-Phys. Kl., 394 (1918). [Supplied by translator: See my note to [27]; this paper is on pp. 586-612 of the collected works cited there.]
95. W. Klingenberg, "Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung," Comment. Math. Helv., 35, No. 1, 47-57 (1961).
96. F. Lindemann, "Über unendlich kleine Bewegungen und uber Kraftsysteme bei allgemeiner projectivischer Massbestimmung," Math. Ann., 7, 56-144 (1874).
97. V. A. Lubimov, E. G. Novikov, V. Z. Nozik, E. F. Tretyakov, and V. S. Kosik, "An estimate of the ν_e mass from the β-spectrum of tritium in the valine molecule," Phys. Lett., 94B, No. 2, 266-268 (1980).
98. M. Matsumoto, "On Einstein's gravitational field equation in a tangent Riemannian space of a Finsler space," Repts. Math. Phys., 8, No. 1, 103-108 (1975).
99. M. Krause (ed.), Menelaos, Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abu Nasr Mansur b'Ali b'Iraq mit Untersuchungen zur Geschlichte des Textes bei den islamischen Mathematiker, Berlin (1936).

100. F. Minding, "Über die Biegung krummer Flachen," J. Reine Angew. Math., 18, 365-368 (1938).
101. J. H. Oort, "The formation of galaxies and the origin of the high-velocity hydrogen," Astron. Appl., 7, 381-404 (1970).
102. G. Paal, "Structure of space—time and quasar's distribution," Acta Phys. Acad. Sci. Hung., 30, No. 1, 51-54 (1971).
103. R. Penrose, "Structure of space—time," in: Battelle Rencontres: 1967 Lectures in Mathematics and Physics, C. M. DeWitt and J. A. Wheeler (eds.), W. A. Benjamin, New York—Amsterdam (1968), pp. 121-235.
104. A. A. Penzias and R. W. Wilson, "A measurement of excess antenna temperature at 4080 Mcycles/sec," Astrophys. J., 142, No. 1, 419-421 (1965).
105. V. Petrosian, E. Salpeter, and P. Szekeres, "Quasistellar objects in universe with non-zero cosmological constant," Astrophys. J., 147, No. 3, 1222-1226 (1967).
106. D. E. Piper, "Radio sources opposite quasers as a cosmological test," Nature, 219, No. 5156, 846-847 (1968).
107. A. A. Robb, Optical Geometry of Motion: A New View of the Theory of Relativity, Cambridge Univ. Press (1911).
108. E. Schering, "Die Schwerkraft in Gaussschen Raum," Nachr. der Königl. Ges. Wiss. (1870).
109. E. Schering, "Die Schwerkraft in mehrfach ausgedehnten Gaussschen und Riemannschen Raumen," Nachr. der Königl. Ges. Wiss. (1873).
110. J. E. Solheim, "Radio sources in opposite directions as a cosmological test," Nature, 217, No. 5223, 41-43 (1968).
111. A. Sommerfeld, "Zur relativistische Dynamik," Phys. Z., 10, 826-829 (1909).
112. E. Study, "Eine neur Darstellung der Kräften der Mechanik durch geometrische Figuren," Ber. Math.-Phys. Klasse der Königl. Sächs. Ges. Wiss., Leipzig, 51 (1899).
113. E. Study, "Die Geometrie der Dynamen," Jahresber. Dtsch. Math. Ver., 8 (1900).
114. E. Study, Geometrie der Dynamen, Leipzig (1901).
115. E. Study, Über Nicht-Euklidische und Liniengeometrie, Festschrift der Philosophische Facultät zu Greiswald (1900).
116. Y. Takano, "Variation principle in Finsler space," Lett. Nuovo Cimento, 11, No. 10, 486-490 (1974).
117. Y. Takano, "Gravitational field in Finsler space," Lett. Nuovo Cimento, 10, No. 17, 747-750 (1974).
118. Y. Takano, "Theory of fields in Finsler spaces. I," Prog. Theor. Phys., 40, No. 5, 1159-1180 (1968).
119. P. Chebyshev (Tchebychef), "Sur la coupe des vetements," Assoc. Franc., Oeuvres (1878).
120. V. Varicak, "Geometrische Untersuchungen der relativistische Dynamik," Z. Phys., 11, No. 93, 287-586 (1910).
121. J. A. Wolf, Spaces of Constant Curvature, McGraw-Hill, New York (1967).
122. Ya. B. Zel'dovich, "Structure (topology) of the Universe," Comments on Astrophysics and Space Physics, 5, No. 6, 169-173 (1973).