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Critical point asymptotics for Gaussian random waves with densities of any Sobolev regularity

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A B S T R A C T

We consider Gaussian random monochromatic waves u on the plane depending on a real parameter s that is directly related to the regularity of its Fourier transform. Specifically, the Fourier transform of u is $f\,d\sigma$, where $d\sigma$ is the Hausdorff measure on the unit circle and the density f is a function on the circle that, roughly speaking, has exactly $s-\frac{1}{2}$ derivatives in L^2 almost surely. When $s=0$, one recovers the classical setting for random waves with a translation-invariant covariance kernel. The main thrust of this paper is to explore the connection between the regularity parameter s and the asymptotic behavior of the number $N(\nabla u, R)$ of critical points that are contained in the disk of radius $R \gg 1$. More precisely, we show that the expectation $\mathbb{E}N(\nabla u, R)$ grows like the area of the disk when the regularity is low enough ($s<\frac{3}{2}$) and like the diameter when the regularity is high enough ($s>\frac{5}{2}$), and that the corresponding exponent changes according to a linear interpolation law in the intermediate regime. The transitions occurring at the endpoint cases involve the square root of the logarithm of the radius. Interestingly, the highest asymptotic growth rate occurs only in the classical translation-invariant setting, $s=0$. A key step of the proof of this result is the obtention of precise asymptotic expansions for certain Neumann series of Bessel functions. When the regularity parameter is $s>5$, we show that in fact $N(\nabla u, R)$ grows

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like the diameter with probability 1, albeit the ratio is not a universal constant but a random variable.

1. Introduction

Nazarov and Sodin have developed some powerful techniques to derive asymptotic laws for the distribution of the zero set of smooth Gaussian functions of several variables [16,17]. Specifically, their theory applies to two different but related settings: the restriction to large balls of Gaussian functions on Euclidean space with translation-invariant covariance kernels and to Gaussian ensembles of high degree polynomials on the sphere or the torus with asymptotically translation-invariant kernels. In the first setting, a prime example arising in spectral theory is the study of Gaussian random monochromatic waves; in the second, that of random spherical harmonics of high frequency.

In this paper we are concerned with asymptotic laws for the number of critical points (i.e., the zeros of the gradient). We consider this question in the context of Gaussian random monochromatic waves on the plane, which are solutions to the Helmholtz equation on $\mathbb{R}^2$,

$$\Delta u + u = 0\,. \tag{1.1}$$

As is well known, the study of critical points is a central topic in spectral theory [22,23,14,6] (and, in general, in the geometric study of solutions to differential equations [21,1,2,9]), both in the deterministic and random settings. This is partly because they are very closely related to the geometry of the nodal components.

1.1. *Random densities with $s - \frac{1}{2}$ derivatives*

When u is polynomially bounded, the Helmholtz equation simply means that u is the Fourier transform of a distribution supported on the unit circle, which we identify with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ via the map

$$E(\phi) := (\cos\phi, \sin\phi)\,. \tag{1.2}$$

Solutions to the Helmholtz equation are necessarily analytic, but their Fourier transforms do not have any a priori regularity properties. There are some connections, though, between the regularity of the Fourier transform of u and the decay rate of u at infinity. Most important is the classical result of Herglotz ensuring that u has the sharp fall-off at infinity (which is as $|x|^{-\frac{1}{2}}$ in a space-averaged sense) if and only if one can write

$$u(x) = \int_{\mathbb{T}} e^{-ix\cdot E(\phi)}\, f(\phi)\, d\phi \tag{1.3}$$

with some square-integrable density f, and that in this case the norm $\|f\|_{L^2(\mathbb{T})}$ quantitatively captures the decay rate of u. For details and generalizations, see e.g. [10, Appendix A].

The main thrust of this paper is to understand the connection between the distribution of the critical points of u, defined as in (1.3), and the regularity of the density f. To this end, we consider the usual ansatz for random plane waves [7,20] and tweak it by introducing a real parameter $s \in \mathbb{R}$ to control the regularity of f:

$$u(x) := \sum_{l\neq 0} a_l\, |l|^{-s} e^{il\theta}\, J_l(r)\,. \tag{1.4}$$

Here the real and imaginary part of a_l are independent standard Gaussian random variables subject to the constraint $a_l = (-1)^l \overline{a_{-l}}$ (which makes u real valued), $(r, \theta) \in \mathbb{R}^+ \times \mathbb{T}$ are the polar coordinates. This is equivalent to taking the Gaussian random density

$$f(\phi) := \frac{1}{2\pi} \sum_{l\neq 0} i^l a_l |l|^{-s} e^{il\phi} \tag{1.5}$$

and then defining u through the formula (1.3), which must be understood in the sense of distributions.

Of course, the rationale behind this definition is that $\{|l|^{-s}e^{il\phi}\}_{l\neq 0}$ is an orthonormal basis of the Sobolev space $\dot{H}^s(\mathbb{T})$ of functions with zero mean and s derivatives in L^2, which reduces to the space of square-integrable functions of zero mean when $s = 0$. The covariance kernel of u is translation-invariant when $s = 0$, so the Nazarov–Sodin theory is applicable in this case (see Remark 4.2 for details), but this is not the case for nonzero s. One should note that the proofs work verbatim if one replaces the weight $|l|^{-s}$ by a more general expression such as

$$\sigma_l = \sigma_{-l} = |l|^{-s} + p_{-s-1}(l)\,, \tag{1.6}$$

where the function $p_{-s-1}(t)$ is an arbitrary classical symbol of order $-s-1$ (which does not necessarily vanish at 0). The resulting constants, however, depend on the specific sequence σ_l.

It is not hard to see that the parameter s describes the regularity of the density in the sense that f has exactly $s - \frac{1}{2}$ derivatives in L^2 almost surely, as measured using Sobolev or Besov spaces. Specifically, one can show that, for any $\delta > 0$,

$$f \in \left[H^{s-\frac{1}{2}-\delta}(\mathbb{T})\backslash H^{s-\frac{1}{2}}(\mathbb{T})\right] \cap \left[B^{s-\frac{1}{2}}_{2,\infty}(\mathbb{T})\backslash B^{s-\frac{1}{2}+\delta}_{2,\infty}(\mathbb{T})\right]$$

with probability 1; see Proposition 2.2 for details.

1.2. Statement of the main theorem

Our main result provides an asymptotic estimate for the growth of the expected number of critical points contained in a disk of large radius R, which we denote by

$$N(\nabla u, R) := \#\{x \in B_R : \nabla u(x) = 0\}\,,$$

as a function of the regularity parameter s. It is elementary that this quantity is an upper bound for the expected number of nodal components contained in B_R. With the usual ansatz for random plane waves, it is well known that $N(\nabla u, R)$ grows asymptotically like the area of the disk; more precisely [4], when $s = 0$ one has

$$\mathbb{E}N(\nabla u, R) \sim \kappa(0)\, R^2\,,$$

where $\kappa(0) := 1/(2\sqrt{3})$ and where the notation $q(R) \sim Q(R)$ means that the quotient $q(R)/Q(R)$ tends to 1 as $R \to \infty$.

We should mention from the onset that the effect of changing the regularity parameter s can be quite drastic, as one should not expect that the number of critical points grows like the area in all regularity regimes. To illustrate this, recall that, when $s = 0$, the Nazarov–Sodin theory ensures the number of nodal components of u contained in B_R grows as

$$N(u, R) \sim \nu_0 R^2$$

almost surely for some constant $\nu_0 > 0$. In contrast, the results proven in [10] show that

$$N(u, R) \sim \nu_\infty R$$

almost surely for $s > 4$, with $\nu_\infty := 1/\pi$. Understanding the asymptotic behavior of the number of nodal components in other regimes is an extremely challenging open problem. Consequently, our main objective in this paper is to analyze the intriguing transitions between distinct asymptotic regimes in the simpler case of critical points.

In the case of critical points, it is also natural to wonder about the asymptotic growth in the case of very negative regularities $s < 0$. Recall that, by the Faber–Krahn inequality, the number of nodal components of a solution to the Helmholtz equation contained in B_R is at most cR^2, where c is a universal constant. However, the number of critical points is not bounded a priori: in Appendix A we show that, given any continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, there exists a solution to the Helmholtz equation on $\mathbb{R}^2$ having at least $\rho(R)$ nondegenerate critical points in B_R, for all $R > 1$. Thus, one could in principle expect the average number of critical points in a large ball R to have a fast growth in R for small enough regularities.

Our main result provides a satisfactory, and quite intriguing, answer to both questions. It turns out that the growth of the expected number of critical points is like the square of the radius for $s < \frac{3}{2}$, linear for $s > \frac{5}{2}$, and the corresponding exponent changes according to a linear interpolation law in the intermediate regime $\frac{3}{2} < s < \frac{5}{2}$. The transitions occurring at the endpoint cases involve not only a power of the radius but also the square root of its logarithm. Furthermore, the highest asymptotic growth of the expected number of critical points is attained exactly for $s = 0$, that is, in the usual setting of random plane waves.

Theorem 1.1. *For any real s, the following statements hold:*

(i) *There exist explicit positive constants $\kappa(s), \tilde{\kappa}_{\frac{3}{2}}, \tilde{\kappa}_{\frac{5}{2}}$ such that the expected number of critical points of the Gaussian random function u satisfies*

$$\mathbb{E}N(\nabla u, R) \sim \begin{cases} \kappa(s)\, R^2 & \text{if} \quad s < \frac{3}{2}\,, \\ \tilde{\kappa}_{\frac{3}{2}} \frac{R^2}{\sqrt{\log R}} & \text{if} \quad s = \frac{3}{2}\,, \\ \kappa(s)\, R^{2-(s-\frac{3}{2})} & \text{if} \quad \frac{3}{2} < s < \frac{5}{2}\,, \\ \tilde{\kappa}_{\frac{5}{2}} R\sqrt{\log R} & \text{if} \quad s = \frac{5}{2}\,, \\ \kappa(s)\, R & \text{if} \quad s > \frac{5}{2}\,. \end{cases}$$

(ii) *In the region where the growth of $\mathbb{E}N(\nabla u, R)$ is volumetric, the constant $\kappa(s)$ depends continuously on s. More precisely, $\kappa(s)$ is a C^∞ function of $s \in (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}]$ but it is only Lipschitz at $s = \frac{1}{2}$. Furthermore, $\kappa(s)$ is strictly increasing on $(-\infty, 0)$, strictly decreasing on $(0, \frac{3}{2})$, and tends to 0 as $s \to -\infty$ and as $s \to \frac{3}{2}^-$. In the region $s \in (\frac{3}{2}, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$ the constant $\kappa(s)$ is also C^∞.*

Fig. 1 summarizes Theorem 1.1 in a more visual way. The fact that the highest asymptotic growth for the number of critical points occurs precisely in the translation-invariant case $s = 0$ is somewhat surprising. Naively one could expect that rougher density functions, which feature wilder oscillations, would exhibit more critical points. Theorem 1.1 shows that, strictly speaking, this is only the case for regularities $s > 0$.

1.3. Motivation

Although the study of geometric properties of smooth Gaussian random fields has attracted much attention in recent years, most of the research is concentrated on the study of stationary fields that are invariant under translations. Under mild technical assumptions, local quantities of these fields have constant densities, so in particular the expected number of critical points in a domain grows asymptotically as the area of the domain. Understanding the asymptotic behavior of geometric quantities for Gaussian fields that are not stationary and invariant under translations is usually very hard, and little information is available.

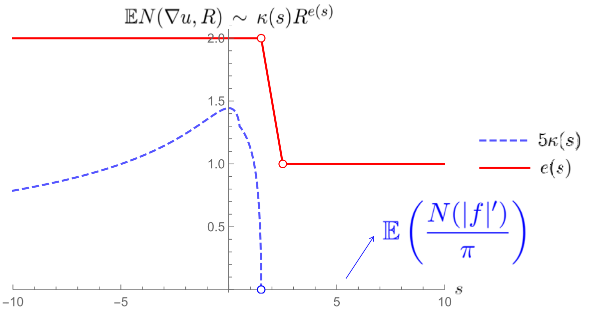

Fig. 1. Consider the asymptotic behavior of $\mathbb{E}N(\nabla u, R) \sim \kappa(s)R^{e(s)}$ proved in Theorem 1.1. In red, we have plotted the exponent $e(s)$ as a function of $s \in \mathbb{R}\backslash\{\frac{3}{2}, \frac{5}{2}\}$. Logarithmic effects appear at the endpoints $s = 3/2$ and $s = 5/2$. In blue, we have plotted $\kappa(s)$ in the region where the asymptotic growth is volumetric, $s < \frac{1}{2}$. The maximum of $\kappa(s)$ in this region is attained at $s = 0$ and that $\kappa(s)$ is not continuously differentiable at $s = 1/2$. The reader can find a plot of $\kappa(s)$ in the range $s \in (\frac{3}{2}, \frac{5}{2})$ in Fig. 3, cf. Section 4. Note that $\kappa(s) = \mathbb{E}N(|f|')/\pi$ by Theorem 1.3.

The random plane wave field, which is a prime example of a stationary, translation-invariant Gaussian field, describes a class of random solutions to the Helmholtz equation that is conjectured to be a universal model for high-energy eigenfunctions of the Laplacian. The ansatz (1.4) introduces a one-parameter generalization u of the random wave model (which is recovered in the case $s = 0$), which enjoys the property that the parameter measures the regularity of the density f of the Fourier transform of u. Furthermore, as the regularity of f is related to the decay properties of u at infinity, the case $s > 0$ can be particularly useful to describe a class of random solutions of the Helmholtz equation (which plays a major role in physics) tending to zero at infinity. This way, one can think of the random field u with $s > \frac{1}{2}$ as describing random waves on the plane in a scattering regime. It is intuitively clear that the typical behavior of a scattering wave, which falls off at infinity, cannot be that of a typical random monochromatic wave, which describes the local behavior (on small scales) of a typical high energy eigenfunction on a compact manifold and does not decay at infinity.

For $s \neq 0$, the Gaussian random field u is not translation-invariant, so extracting precise information about the distribution of nodal sets for the case $s \neq 0$ is extremely hard, as there are essentially no tools one can resort to. The case of the expected number of critical points, which we consider in this paper, is doable because one can start off from the Kac–Rice formula. This way, the analysis eventually boils to understanding how high frequency effects determine the asymptotic behavior of series involving Bessel functions.

1.4. Some ideas about the proof

Let us now discuss how the aforementioned strategy is implemented in the proof of Theorem 1.1. As we have already mentioned, the asymptotic analysis of $N(\nabla u, R)$ hinges on the celebrated Kac–Rice counting formula, which, under suitable technical hypotheses, expresses the expected number of zeros of a random field (in this case, the gradient ∇u) has in terms of a multivariate integral. As is well known, this formula has been used profusely in the literature [11,17,4,5], and in particular lies at the heart of the computation of $\mathbb{E}N(\nabla u, R)$ for $s = 0$ and of the finer asymptotics bounds for the expected number of extrema and saddle points and for higher order correlations obtained in [4] also in the translation-invariant case $s = 0$.

The coefficients that appear in the Kac–Rice integral formula involve, via the variance matrix of ∇u, weighted series of Bessel functions of the form

$$\mathcal{J}_{s,m,m'}(r) := \sum_{l=1}^{\infty} l^{-2s} J_{l+m}(r)\, J_{l+m'}(r)\,, \tag{1.7}$$

where m and m' are certain integers. $\mathcal{J}_{s,m,m'}$ is sometimes called in the literature a second type Neumann series. It is clear that the way each term $J_{l+m}(r)\, J_{l+m'}(r)$ contributes to the sum for $r \gg 1$ and $l \gg 1$ will depend on whether the "angular frequency" l is much larger than r, much smaller than r, or roughly of the same size; moreover, the effect of each group of angular frequencies will have a different relative weight in the sum depending on the power s appearing in l^{-2s}. More precisely, a key step of the proof is to establish the following technical result, which controls the asymptotic behavior of $\mathcal{J}_{s,m,m'}(r)$:

Lemma 1.2. *For any pair of nonnegative integers m, m' and any real s, the large-r asymptotic behavior of $\mathcal{J}_{s,m,m'}$ is*

$$\begin{aligned}
&\mathcal{J}_{s,m,m'}(r) = c^1_{s,m-m'}\, r^{-2s} + o(r^{-2s}) && \text{if} \quad s < \tfrac{1}{2}\,,\\
&\mathcal{J}_{s,m,m'}(r) = c^2_{m-m'} \frac{\log r}{r} + O(r^{-1}) && \text{if} \quad s = \tfrac{1}{2} \text{ and } m - m' \text{ is even}\,,\\
&\mathcal{J}_{s,m,m'}(r) = \frac{c^3_{m-m'} - c^4 \sin(2r - c^7_{m+m'})}{r} + o(r^{-1}) && \text{if} \quad s = \tfrac{1}{2} \text{ and } m - m' \text{ is odd}\,,\\
&\mathcal{J}_{s,m,m'}(r) = \frac{c^5_{s,m-m'} - c^6_s \sin(2r - c^7_{m+m'})}{r} + o(r^{-1}) && \text{if} \quad s > \tfrac{1}{2}
\end{aligned}$$

with some explicit constants that will be defined later on.

Ultimately, the different asymptotic regimes that the expectation of $N(\nabla u, R)$ can exhibit can be traced back to the asymptotic behavior of functions of the form (1.7). One should note that, in general, the highly oscillatory nature of summands in (1.7) makes the analysis of the asymptotic behavior of $\mathcal{J}_{s,m,m'}(r)$ rather subtle. An exception to this general fact is precisely the case $s = 0$, where all the associated series can be computed exactly using that the covariance kernel of u is translation-invariant (or, equivalently, the addition formula for Bessel functions); this makes it much easier to analyze the corresponding asymptotic behavior of $\mathbb{E}N(\nabla u, R)$. To illustrate this fact, in the very short Appendix B we carry out the analysis of the translation invariant case $s = 0$.

1.5. Almost sure asymptotics for high regularity random densities

In the particular case of smooth enough density functions, one can use the methods of our previous paper [10] to understand the asymptotic behavior of the number of critical points (not only of its expectation value) in greater detail. Specifically, one can prove the following:

Theorem 1.3. *If $s > 5$,*

$$N(\nabla u, R) \sim \frac{N(|f|')}{\pi} R$$

with probability 1*. In particular, $N(\nabla u, R)$ grows linearly almost surely.*

Here the random variable $N(|f|') := \#\{\phi \in \mathbb{T} : |f(\phi)|' = 0\}$ (which is at least 2 almost surely) denotes the number of critical points of the (non-Gaussian) random function $|f|$. In particular, the asymptotic growth of $N(\nabla u, R)$ is linear with probability 1, albeit the ratio is not a universal constant but a random variable. In view of Theorem 1.1, a consequence of this asymptotic formula is an explicit formula for the expectation $\mathbb{E}N(|f|')$ when $s > 5$.

Finally, we want to emphasize that our study of the asymptotic behavior of critical points of Gaussian random monochromatic waves in terms of the regularity of their Fourier transform is completely different from the use of frequency-dependent weights considered by Rivera in the context of random Gaussian fields on compact manifolds [19]. Indeed, Rivera's central result is that the Nazarov–Sodin asymptotics still holds, with different constants, for series of the form

$$F(x) := \sum_{0<\lambda_k \leqslant L} \lambda_k^{-\frac{s}{2}} \, a_k \, e_k(x) \,,$$

where $L \gg 1$, (e_k, λ_k) are the eigenfunctions and eigenvalues of the Laplacian on a compact n-manifold, $s < \frac{n}{2}$ and $a_l \sim \mathcal{N}(0,1)$. The random Gaussian field F does not yield to monochromatic waves in the high-frequency limit (after the standard localization procedure) because it involves a linear combination of eigenfunctions with eigenvalues in the frequency window $[0, L]$. The behavior when $s > \frac{n}{2}$ is not explored (it depends on the geometry of the manifold), and in the limit case $s = \frac{n}{2}$ Rivera obtains a correction term of logarithmic type with respect to the Nazarov–Sodin asymptotics. In contrast, we consider Gaussian random monochromatic waves on $\mathbb{R}^2$ of the form (1.4), and compute their expected number of critical points for any real number s; we also obtain a logarithmic correction at the breakdown of the Nazarov–Sodin regime (when $s = \frac{3}{2}$ in our case).

1.6. Organization of the paper

In Section 2, we start by showing the relation between the parameter s and the regularity of the random function u. Sections 3, 4 and 5 are respectively devoted to the proofs of Lemma 1.2 and Theorems 1.1 and 1.3. We have divided each of these sections into a number of subsections to emphasize the main ideas of each proof. The paper concludes with two Appendices. In Appendix A, we construct solutions to the Helmholtz equation on the plane for which the number of nondegenerate critical points contained in B_R grows arbitrarily fast as $R \to \infty$. In Appendix B, we revisit the translation-invariant case ($s = 0$) and explain the key simplifications that appear in this extremely important case.

2. Almost sure regularity of the random density function

Our objective in the section is to show that, with probability 1, the Gaussian random function f, defined in (1.5), has exactly $s - \frac{1}{2}$ derivatives in L^2, measured using suitable Sobolev or Besov spaces.

To prove the main result we will need the following version of the strong law of large numbers for sequences of random variables that are labeled by two integers:

Lemma 2.1. *Let $\{K_N\}_{N=1}^{\infty}$ be a sequence of positive integers such that*

$$\liminf_{M\to\infty} \frac{K_M}{\sum_{N=1}^{M} K_N} > 0\,.$$

If $\{b_{N,k} : 1 \leqslant k \leqslant K_N,\ N \geqslant 1\}$ are i.i.d. random variables with mean μ, then

$$\lim_{N\to\infty} \frac{1}{K_N} \sum_{k=1}^{K_N} b_{N,k} = \mu$$

almost surely.

Proof. The strong law of large numbers ensures that

$$S_M := \frac{1}{Q_M} \sum_{N=1}^{M} \sum_{k=1}^{K_N} b_{N,k} - \mu \tag{2.1}$$

converges to 0 almost surely as $M \to \infty$, with $Q_M := \sum_{N=1}^{M} K_N$. Thus, from the identity

$$S_M = \frac{Q_{M-1}}{Q_M} S_{M-1} + \frac{K_M}{Q_M} \left(\frac{1}{K_M} \sum_{k=1}^{K_M} b_{M,k} - \mu \right)$$

and the fact that $Q_{M-1}/Q_M \leqslant 1$ we obtain

$$\limsup_{M\to\infty}\left|\frac{1}{K_M}\sum_{k=1}^{K_M} b_{M,k}-\mu\right| \leqslant \frac{\lim_{M\to\infty}(|S_M|+|S_{M-1}|)}{\liminf_{M\to\infty}\frac{K_M}{Q_M}}=0$$

almost surely. Notice that we have used the assumption $\liminf_{M\to\infty}\frac{K_M}{Q_M}>0$. The lemma then follows. $\square$

We are now ready to prove the main result of this section. Here and in what follows, we shall use the notation $q\approx Q$ or $q\lesssim Q$ when there exists a constant C (independent of the large parameter under consideration) such that $Q/C\leqslant q\leqslant CQ$ or $q\leqslant CQ$, respectively.

Proposition 2.2. *For each $\delta>0$, the Gaussian random function (1.5) satisfies*

$$f\in\left[H^{s-\frac12-\delta}(\mathbb{T})\backslash H^{s-\frac12}(\mathbb{T})\right]\cap\left[B^{s-\frac12}_{2,\infty}(\mathbb{T})\backslash B^{s-\frac12+\delta}_{2,\infty}(\mathbb{T})\right]$$

almost surely.

Proof. Let us recall that the $H^\sigma(\mathbb{T})$ norm of the function f defined in (1.5) is

$$\|f\|^2_{H^\sigma(\mathbb{T})}=\sum_{l=-\infty}^{\infty}|a_l|^2 l^{2\sigma-2s}\,.$$

To analyze this quantity, consider the set of integers $\Lambda_N:=\{l:2^{N-1}\leqslant l<2^N\}$ and the subsequences

$$\sum_{l=-(2^M-1)}^{2^M-1}|a_l|^2l^{2\sigma-2s}=|a_0|^2+2\sum_{N=1}^{M}\sum_{l\in\Lambda_N}l^{2\sigma-2s}|a_l|^2\approx|a_0|^2+\sum_{N=1}^{M}2^{N(2\sigma-2s)}\sum_{l\in\Lambda_N}|a_l|^2\,.$$

Since $|\Lambda_N|\approx 2^N$,

$$\frac{|\Lambda_M|}{\sum_{N=1}^M|\Lambda_N|}\approx\frac{2^M}{2^{M+1}}=\frac12$$

is bounded away from zero. Hence one can apply Lemma 2.1 to infer that

$$\frac{1}{|\Lambda_N|}\sum_{l\in\Lambda_N}|a_l|^2\to 1$$

almost surely as $N\to\infty$. Therefore, with probability 1,

$$\sum_{l=-(2^M-1)}^{2^M-1}|a_l|^2l^{2\sigma-2s}\approx|a_0|^2+\sum_{N=1}^{M}2^{N(2\sigma-2s+1)}\frac{1}{|\Lambda_N|}\sum_{l\in\Lambda_N}|a_l|^2\approx|a_0|^2+\sum_{N=1}^{M}2^{N(2\sigma-2s+1)}\,.$$

This shows that, with probability 1, $\|f\|_{H^\sigma(\mathbb{T})} < \infty$ if and only if $\sigma < s - \frac{1}{2}$.

The estimate for the Besov norm follows from an analogous reasoning using that

$$\|f\|^2_{B^\sigma_{2,\infty}(\mathbb{T})} = \sup_{1\leqslant N<\infty} \sum_{l\in\Lambda_N} l^{2\sigma-2s}|a_l|^2 . \quad \square$$

Remark 2.3. The result and the proof remain valid in higher dimensions with minor modifications. Specifically, let $\{Y_{lm} : 1 \leqslant m \leqslant d_l,\ 0 \leqslant l < \infty\}$ be an orthonormal basis of spherical harmonics on the unit $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, with $\Delta_{\mathbb{S}^{n-1}} Y_{lm} + l(l+n-2)Y_{lm} = 0$. Consider the Gaussian random function

$$f(x) := \sum_{l=1}^{\infty} \sum_{m=1}^{d_l} l^{-s} a_{lm} Y_{lm}(x) ,$$

where a_{lm} are independent standard Gaussian variables and $s \in \mathbb{R}$. Then

$$f \in \left[H^{s-\frac{n-1}{2}-\delta}(\mathbb{S}^{n-1})\backslash H^{s-\frac{n-1}{2}}(\mathbb{S}^{n-1})\right] \cap \left[B^{s-\frac{n-1}{2}}_{2,\infty}(\mathbb{S}^{n-1})\backslash B^{s-\frac{n-1}{2}+\delta}_{2,\infty}(\mathbb{S}^{n-1})\right]$$

almost surely.

To spell out the details, the proof in higher dimension starts with the formula

$$\|f\|^2_{H^\sigma(\mathbb{S}^{n-1})} := \sum_{l=1}^{\infty} \sum_{m=1}^{d_l} |a_{lm}|^2 l^{2\sigma-2s} .$$

Since $d_l = c_n l^{n-2} + O(l^{n-3})$, the set

$$\Lambda_N := \{(l,m) : 2^{N-1} \leqslant l < 2^N,\ 1 \leqslant m \leqslant d_l\}$$

satisfies $|\Lambda_N| \approx 2^{N(n-1)}$. Lemma 2.1 then ensures

$$\frac{1}{|\Lambda_N|} \sum_{(l,m)\in\Lambda_N} |a_{lm}|^2$$

converges to 1 almost surely as $N \to \infty$, and the result follows from the same argument as above. Obviously, the result also remains valid if one replaces the weight l^s by another quantity $w_l \approx l^s$.

3. Asymptotics for weighted Bessel series

In this section we shall prove Lemma 1.2. In view of the well-known asymptotics

$$J_l(r) = \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \cos\left(r - \frac{(2l+1)\pi}{4}\right) + O(r^{-1})$$

for Bessel functions, it is easy to check that the series

$$\mathcal{J}_{s,m,m'}(r) := \sum_{l=1}^{\infty} l^{-2s} J_{l+m'}(r)\, J_{l+m}(r)\,. \tag{3.1}$$

is locally uniformly convergent by the standard bound [18, (10.14.4)]

$$|J_l(r)| \leqslant \frac{r^l}{2^l l!}\,.$$

We are interested in the effect of the parameters $s \in \mathbb{R}$ and $m', m \in \mathbb{Z}$.

In view of the well-known integral representation formula [18, (10.9.2)] for Bessel functions of integer order,

$$J_l(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir \sin x - ilx}\, dx\,,$$

one can write

$$\mathcal{J}_{s,m,m'}(r) = \frac{1}{4\pi^2} \sum_{l=1}^{\infty} l^{-2s} g_{\lambda_l}(r). \tag{3.2}$$

Here we have set $\lambda_l := l/r$,

$$g_\lambda(r) := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{ir\varphi_\lambda(x,y) - i(m'x - my)}\, dx\, dy\,,$$

and the phase function is

$$\varphi_\lambda(x,y) := \lambda(y-x) + \sin x - \sin y\,.$$

Notice that we have used that J_l is real valued, and hence $J_l = \overline{J_l}$.

A straightforward application of the stationary phase formula [13, Theorem 7.7.5] gives the following asymptotic formula for g_λ. Here and in what follows, we will use the notation

$$f(\lambda) := \sqrt{1-\lambda^2} - \lambda \arccos \lambda\,, \qquad \mu := m + m'\,, \qquad \nu := m - m'\,.$$

Also, we will use the notation $O_p(r^{-k})$ to emphasize that a certain quantity of order r^{-k} is not bounded uniformly with respect to the parameter p.

Lemma 3.1. *Suppose that $\lambda \neq 1$. For $r \gg 1$, one then has*

$$g_\lambda(r) = \frac{4\pi\left[\cos\left(\nu \arccos\lambda\right) + \sin\left(2rf(\lambda) - \mu \arccos\lambda\right)\right]}{r|1-\lambda^2|^{1/2}} + O_\lambda(r^{-2}),$$

where the error term is not bounded uniformly for large λ or for λ close to 1.

Proof. For $\lambda \neq 1$, the phase function $\varphi_\lambda(x, y)$ has four critical points

$$\{(x_i, y_i)\}_{i=1}^4 := \{(\pm \arccos\lambda, \pm \arccos\lambda)\}$$

with the same Hessian:

$$\nabla^2 \varphi_\lambda(x_i, y_i) = \begin{pmatrix} \mp\sqrt{1-\lambda^2} & 0 \\ 0 & \pm\sqrt{1-\lambda^2} \end{pmatrix}.$$

The stationary phase method [13, Theorem 7.7.5] then yields

$$\begin{aligned} g_\lambda(r) &= \frac{2\pi}{r} \sum_{i=1}^4 e^{\frac{1}{4} i\pi\sigma_i} e^{i(my_i - x_i m')} e^{ir\varphi_\lambda(x_i, y_i)} \frac{1}{|\det \nabla^2 \varphi_\lambda(x_i, y_i)|} + O_\lambda(r^{-2}) \\ &= \frac{4\pi\left[\cos\left(\nu \arccos\lambda\right) + \sin\left(2rf(\lambda) - \mu \arccos\lambda\right)\right]}{r|1-\lambda^2|^{1/2}} + O_\lambda(r^{-2}), \end{aligned}$$

as claimed. In this formula, σ_i is the signature of the matrix $\nabla^2 \varphi_\lambda(x_i, y_i)$. $\square$

Therefore, the asymptotic analysis of $g_\lambda(r)$ becomes problematic when λ is close to 1 (because in this case the phase function presents degenerate or "almost degenerate" critical points) and when λ is large (because the error terms are not uniformly bounded in this case). Consequently, we will fix a small parameter $\delta > 0$ and consider smooth cutoff functions $[0, \infty) \to [0, 1]$ such that

$$\chi_{\mathrm{sm}}(\lambda) := \begin{cases} 0 & \text{if } \lambda > 1 - \delta, \\ 1 & \text{if } \lambda < 1 - 2\delta, \end{cases}$$

$$\chi_{\mathrm{lar}}(\lambda) := \begin{cases} 0 & \text{if } \lambda < 1 + \delta, \\ 1 & \text{if } \lambda > 1 + 2\delta, \end{cases}$$

$$\chi_{\mathrm{med}}(\lambda) := 1 - \chi_{\mathrm{sm}}(\lambda) - \chi_{\mathrm{lar}}(\lambda).$$

We can then split $\mathcal{J}_{s,m,m'}(r)$ as

$$\mathcal{J}_{s,m,m'}(r) = \frac{1}{4\pi^2}\left(\mathrm{I} + \mathrm{II} + \mathrm{III}\right)$$

with

$$\mathrm{I} := \sum_{l=1}^{\infty} \chi_{\mathrm{sm}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r)\,, \qquad \mathrm{II} := \sum_{l=1}^{\infty} \chi_{\mathrm{med}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r)\,,$$
$$\mathrm{III} := \sum_{l=1}^{\infty} \chi_{\mathrm{lar}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r)\,.$$

Note that I only involves frequencies smaller than $(1-\delta)r$, II involves frequencies close to 1 (more precisely, in the interval $(1-2\delta)r < l < (1+2\delta)r$), and III involves frequencies larger than $(1+\delta)r$.

3.1. The small frequency region

In view of the asymptotic expansion for $g_\lambda(r)$ proved in Lemma 3.1, it is natural to consider the closely related quantities

$$\mathrm{I}' := \frac{4\pi}{r} \sum_{l=1}^{\infty} \chi_{\mathrm{sm}}(\lambda_l)\, \lambda_l^{-2s} \frac{\cos(\nu \arccos \lambda_l)}{(1-\lambda_l^2)^{1/2}}\,,$$
$$\mathrm{I}'' := \frac{4\pi}{r} \sum_{l=1}^{\infty} \chi_{\mathrm{sm}}(\lambda_l)\, \lambda_l^{-2s} \frac{\sin(2r\, f(\lambda_l) - \mu \arccos \lambda_l)}{(1-\lambda_l^2)^{1/2}}\,.$$

Lemma 3.1 obviously implies

$$\mathrm{I} = r^{-2s}(\mathrm{I}' + \mathrm{I}'') + O_\delta(r^{-2s-1})\,. \tag{3.3}$$

Let us start by analyzing the large r behavior of I′ when $s \leqslant \frac{1}{2}$:

Lemma 3.2. *For $r \gg 1$ and some $\eta > 0$ depending on s,*

$$\mathrm{I}' = \begin{cases} \frac{4\pi^2 2^{2s-1}\Gamma(1-2s)}{\Gamma(1-s-\frac{\nu}{2})\Gamma(1-s+\frac{\nu}{2})} + O(\delta^{\frac{1}{2}}) + O_\delta(r^{-\eta}) & \text{if } s < \frac{1}{2}, \\ 4\pi \cos\left(\frac{\pi\nu}{2}\right) \log r + O_\delta(1) & \text{if } s = \frac{1}{2} \text{ and } \nu \text{ is even,} \\ 2\pi^2 \sin\left(\frac{\pi}{2}|\nu|\right) + O_\delta(r^{-1}) + O(\delta^{\frac{1}{2}}) & \text{if } s = \frac{1}{2} \text{ and } \nu \text{ is odd.} \end{cases}$$

Proof. Let us start with the case $s \leqslant 0$. The basic observation here is that, as the function

$$h(\lambda) := 4\pi\, \chi_{\mathrm{sm}}(\lambda) \frac{\lambda^{-2s}}{\sqrt{1-\lambda^2}} \cos(\nu \arccos \lambda)$$

is Hölder continuous,

$$\frac{1}{r}\sum_{l=1}^{(1-\delta)r} h(\lambda_l) = \int_0^{1-\delta} h(\lambda)\,d\lambda + O_\delta(r^{-1})$$

by standard results about the convergence of Riemann sums for integrands of bounded variation. If $s \leqslant 0$, the result then follows from the formula

$$\int_0^1 \frac{\lambda^{-2s}\cos(\nu\arccos\lambda)}{\sqrt{1-\lambda^2}}\,d\lambda = \frac{\pi 2^{2s-1}\Gamma(1-2s)}{\Gamma\left(1-s-\frac{\nu}{2}\right)\Gamma\left(1-s+\frac{\nu}{2}\right)} \tag{3.4}$$

and the estimate $\arcsin 1 - \arcsin(1-\delta) = O(\delta^{1/2})$.

For $s \in (0, \frac{1}{2})$, the integrand is an unbounded function in L^1_{loc}, so the argument does not apply. Let us take a small constant ε such that, for simplicity of notation, εr is an integer, and write

$$\mathrm{I}' = \frac{1}{r}\sum_{l=1}^{\varepsilon r-1} h(\lambda_l) + \frac{1}{r}\sum_{l=\varepsilon r}^{(1-\delta)r} h(\lambda_l) =: \mathrm{I}_1' + \mathrm{I}_2'\,.$$

Obviously, as $|h(\lambda)| \approx \lambda^{-2s}$ for small λ, and $\int_{l/r}^{(l+1)/r}\lambda^{-2s}d\lambda \approx r^{-1}\lambda_l^{-2s}$, we conclude that

$$\left|\mathrm{I}_1' - \int_0^\varepsilon h(\lambda)\,d\lambda\right| \lesssim \varepsilon^{2-2s} + r^{-1+2s}\,.$$

To estimate I_2', we use that

$$\mathrm{I}_2' - \int_\varepsilon^{1-\delta} h(\lambda)\,d\lambda = \sum_{l=\varepsilon r}^{(1-\delta)r}\int_{(l-1)/r}^{l/r}[h(\lambda_l)-h(\lambda)]\,d\lambda = \frac{1}{r}\sum_{l=\varepsilon r}^{(1-\delta)r}\frac{h'(\lambda_l^*)}{r}$$

for some $\lambda_l^* \in (\frac{l-1}{r}, \frac{l}{r})$. Therefore, as $|h'(\lambda)| \lesssim \lambda^{-2s-1}$,

$$\left|\mathrm{I}_2' - \int_\varepsilon^{1-\delta} h(\lambda)\,d\lambda\right| \lesssim \frac{\varepsilon^{-1-2s}}{r}\,,$$

where the constant in $\lesssim$ depends on δ.

Putting together the estimates for I_1' and I_2' with $\varepsilon \approx r^{-\frac{1}{2}}$, we obtain

$$\mathrm{I}' = \int_0^{1-\delta} h(\lambda)\,d\lambda + O_\delta(r^{s-\frac{1}{2}}) = \int_0^1 h(\lambda)\,d\lambda + O_\delta(r^{s-\frac{1}{2}}) + O(\delta^{1/2})\,.$$

Using again the formula (3.4), this proves the lemma when $s \in (0, \frac{1}{2})$.

Let us now pass to the case $s = \frac{1}{2}$. We start by assuming that the integer ν is odd, so that $\cos\left(\frac{\pi\nu}{2}\right) = 0$. Since

$$\cos(\nu \arccos \lambda_l) = \cos\frac{\pi\nu}{2} + \lambda_l \nu \sin\frac{\pi\nu}{2} + O(\lambda_l^2)\,, \tag{3.5}$$

it turns out that the corresponding integrand is differentiable at $\lambda = 0$ in this case, so the same arguments as in the case $s < 0$ show

$$\sum_{l=1}^{(1-\delta)r} \frac{\chi_{\mathrm{sm}}(\lambda_l)}{\lambda_l r} \frac{4\pi\cos(\nu\arccos\lambda_l)}{(1-\lambda_l^2)^{1/2}} = 4\pi\int_0^{1-\delta} \frac{\chi_{\mathrm{sm}}(\lambda)\cos(\nu\arccos\lambda)}{\lambda\sqrt{1-\lambda^2}}\, d\lambda + O_\delta(r^{-1})\,.$$

The result then follows from the formula

$$\int_0^1 \frac{\cos(\nu\arccos\lambda)}{\lambda\sqrt{1-\lambda^2}}\, d\lambda = \frac{\pi}{2}\sin\left(\frac{\pi}{2}|\nu|\right)\,.$$

To conclude, consider the case when $s = \frac{1}{2}$ and ν is even. Obviously, by (3.5),

$$\frac{4\pi}{r}\left|\sum_{l=1}^{(1-\delta)r} \chi_{\mathrm{sm}}(\lambda_l)\left(\frac{\cos(\nu\arccos\lambda_l)}{\lambda_l(1-\lambda_l^2)^{1/2}} - \frac{\cos\frac{\pi\nu}{2}}{\lambda_l}\right)\right| \lesssim \frac{1}{r}\sum_{l=1}^{(1-\delta)r}\lambda_l \lesssim 1\,,$$

where the constant in $\lesssim$ depends on δ. The leading contribution of this sum is therefore given by the harmonic series, which satisfies

$$\sum_{l=1}^{(1-\delta)r} \frac{\chi_{\mathrm{sm}}(\lambda_l)}{r\lambda_l} = \sum_{l=1}^{r/2}\frac{1}{l} + \sum_{l=\frac{r}{2}+1}^{(1-\delta)r}\frac{\chi_{\mathrm{sm}}(\lambda_l)}{l} = \log r + O(1)\,.$$

This completes the proof of the lemma. $\square$

Now we pass to analyzing the contribution of the second term, I''. As this term is somewhat oscillating due to the presence of the large parameter r in the argument of a sine, it makes sense to expect this term should be subdominant.

Lemma 3.3. *There exists some $\eta > 0$, depending on s, such that*

$$\mathrm{I}'' = \begin{cases} O_\delta(r^{-\eta}) & \text{if } s < \frac{1}{2}\,,\\ -4\pi\log 2\,\sin\left(2r - \frac{\pi\mu}{2}\right) + O_\delta(r^{-\eta}) & \text{if } s = \frac{1}{2}\,. \end{cases}$$

Proof. We start with the case $s < \frac{1}{2}$. Let $\beta \in (0, 1)$ be some constant that we will specify later and write

$$\mathrm{I}'' = \mathrm{Im}\left(\frac{1}{r}\sum_{l=1}^{\lfloor r^\beta \rfloor} h(\lambda_l)\, e^{i2rf(\lambda_l)} + \frac{1}{r}\sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} h(\lambda_l)\, e^{i2rf(\lambda_l)}\right) =: \mathrm{Im}(\mathrm{I}_1'' + \mathrm{I}_2'')\,,$$

with $h(\lambda) := 4\pi\chi_{\mathrm{sm}}(\lambda)\lambda^{-2s}e^{-i\mu \arccos\lambda}(1-\lambda^2)^{-\frac{1}{2}}$. As $s < \frac{1}{2}$, the first term can be easily estimated as

$$|\mathrm{I}_1''| \lesssim \frac{1}{r}\sum_{l=1}^{\lfloor r^\beta \rfloor} \lambda_l^{-2s} \lesssim r^{-(1-2s)(1-\beta)}\,.$$

By hypothesis, the RHS is $r^{-\eta}$ for some $\eta > 0$.

To estimate I_2'', decompose the interval $(\lceil r^\beta \rceil, (1-\delta)r]$ as the union of N disjoint intervals of the form $(l_n, l_n + \Lambda_n]$. We assume that l_n are integers and that the lengths of the intervals satisfy $\Lambda_n \approx r^\gamma$ for some $\gamma \in (0, \beta)$. This implies that $N \approx r^{1-\gamma}$.

The basic idea is that, with this choice of the scales, one can expect that the function h will be approximately constant in each interval but the phase of the complex exponential will oscillate rapidly. This will lead to cancellations. To make this idea precise, suppose that $\lambda - \lambda_{l_n} \in (0, \Lambda_n/r)$ and write

$$f(\lambda) =: f(\lambda_{l_n}) - (\lambda - \lambda_{l_n})\arccos(\lambda_{l_n}) + R_n(\lambda)\,, \tag{3.6}$$

where the function $R_n(\lambda)$ plays the role of an error term. Differentiating this identity with respect to λ, and noticing that $f'(\lambda) = -\arccos\lambda$, one immediately obtains that the bound $|R_n'(\lambda)| \lesssim |\lambda - \lambda_n|$ holds uniformly in n. As a consequence of this, setting $L := r(\lambda - \lambda_{l_n})$, one infers that

$$\left|\frac{d}{d\lambda}\left(h(\lambda)e^{i2rR_n(\lambda)}\right)\right| \leqslant |h'(\lambda)| + |h(\lambda)2rR_n'(\lambda)| \lesssim r^{(\beta-1)(\alpha_1-1)} + r^{(\beta-1)\alpha_0}L$$

where

$$\alpha_0 := \min\{0, -2s\}\,, \qquad \alpha_1 := \min\{1, -2s\}\,.$$

As usual, the constant in $\lesssim$ depends on δ.

By the mean value theorem, observing that $R_n(\lambda_{l_n}) = 0$, one then has from Equation (3.6) that

$$\begin{aligned}&\left|\sum_{l=l_n+1}^{l_n+\Lambda_n}\left(h(\lambda_l)e^{i2rf(\lambda_l)} - h(\lambda_{l_n})e^{i2rf(\lambda_{l_n})}e^{i2f'(\lambda_{l_n})L}\right)\right|\\ &\qquad \lesssim r^{(\beta-1)(\alpha_1-1)+2\gamma-1} + r^{(\beta-1)\alpha_0+3\gamma-1}\,,\end{aligned}$$

with $\lesssim$ depending on δ. As the implicit constants are uniform in n and there are $N \approx r^{1-\gamma}$ intervals, this implies

$$\mathrm{I}_2'' = \frac{1}{r}\sum_{n=1}^{N} h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} \sum_{L=0}^{\Lambda_n} e^{i2f'(\lambda_{l_n})L} + O_\delta(r^{(\beta-1)(\alpha_1-1)+\gamma-1} + r^{(\beta-1)\alpha_0+2\gamma-1}) .$$

The leading contribution is therefore

$$\begin{aligned}\frac{1}{r}\sum_{n=1}^{N} h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} \sum_{L=0}^{\Lambda_n} e^{i2f'(\lambda_{l_n})L} &= \frac{1}{r}\sum_{n=1}^{N} h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} \frac{1-e^{-2i\arccos(\lambda_{l_n})(r^\gamma+1)}}{1+e^{-2i\arcsin(\lambda_{l_n})}} \\ &\lesssim r^{(\beta-1)\alpha_0-\gamma} ,\end{aligned}$$

the constant in $\lesssim$ depending on δ. Note that the denominator is bounded from below because $\lambda < 1-\delta$. Thus, choosing $\gamma \in (0, \frac{1}{2})$ and β sufficiently close to 1 (depending on γ and s), we conclude that

$$|\mathrm{I}_2''| \lesssim r^{-\eta'}$$

for some $\eta' > 0$.

Let us now pass to the case $s = \frac{1}{2}$. Arguing as above, one can pick some β close to, but smaller than, 1 such that

$$\sum_{l=\lceil r^\beta \rceil}^{(1-\delta^-)r} \frac{\chi_{\mathrm{sm}}(\lambda_l)\sin\left(2rf(\lambda_l) - \mu\arccos\lambda_l\right)}{l(1-\lambda_l^2)^{1/2}} = O_\delta(r^{-\eta})$$

for some $\eta > 0$. For the sum going from $l=1$ to $\lfloor r^\beta \rfloor$, we can disregard the $(1-\lambda_l^2)^{1/2}$ term because

$$\left|\sum_{l=1}^{\lfloor r^\beta \rfloor}\left[\frac{\sin\left(2rf(\lambda_l) - \mu\arccos\lambda_l\right)}{l(1-\lambda_l^2)^{1/2}} - \frac{\sin\left(2rf(\lambda_l) - \mu\arccos\lambda_l\right)}{l}\right]\right| \lesssim \sum_{l=1}^{\lfloor r^\beta \rfloor}\frac{\lambda_l}{r} \lesssim r^{-2+2\beta} .$$

The identity

$$\begin{aligned}\sin\left(2rf(\lambda_l) - \mu\arccos\lambda_l\right) &= \sin\left(2r - \frac{\pi\mu}{2}\right)\cos\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right) \\ &\quad + \cos\left(2r - \frac{\pi\mu}{2}\right)\sin\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)\end{aligned}$$

enables us to write

$$\begin{aligned}&\sum_{l=1}^{\lfloor r^\beta \rfloor}\frac{\sin\left(2rf(\lambda_l) - \mu\arccos\lambda_l\right)}{l} \\ &\qquad = \sin\left(2r - \frac{\pi\mu}{2}\right)\sum_{l=1}^{\lfloor r^\beta \rfloor}\frac{\cos\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)}{l}\end{aligned}$$

$$+\cos\left(2r-\frac{\pi\mu}{2}\right)\sum_{l=1}^{\lfloor r^\beta\rfloor}\frac{\sin\left(2r(f(\lambda_l)-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_l\right)\right)}{l}.$$

The asymptotic expansions

$$f(\lambda)-1=-\frac{\pi\lambda}{2}+O(\lambda^2),\quad \frac{\pi}{2}-\arccos\lambda=\lambda+O(\lambda^2)$$

ensure that

$$2r(f(\lambda_l)-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_l\right)=-\pi l+rO(\lambda^2).$$

The quantity $rO(\lambda^2)$ is of order $r^{2\beta'-1}$ whenever $l<r^{\beta'}$. Fixing some $\beta'\in(0,\frac{1}{2})$, we therefore have

$$\begin{aligned}\sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\frac{\cos\left(2r(f(\lambda_l)-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_l\right)\right)}{l}&=\sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\left(\frac{\cos(\pi l)}{l}+\frac{r^2O(\lambda_l^4)}{l}\right)\\&=-\log 2+O(r^{-\min\{\beta',2-4\beta'\}}).\end{aligned}$$

Here we have used that

$$\sum_{l=1}^{L}\frac{\cos(\pi l)}{l}=-\log 2+O(L^{-1}).$$

Similarly,

$$\sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\frac{\sin\left(2r(f(\lambda_l)-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_l\right)\right)}{l}=\sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\frac{rO(\lambda_l^2)}{l}=O(r^{2\beta'-1}).$$

It only remains to consider the sum from $\lceil r^{\beta'}\rceil$ to $\lfloor r^\beta\rfloor$, where we can also assume that $\chi_{\mathrm{sm}}(\lambda_l)=1$. To this end, we define the function

$$Q:=\sum_{l=\lceil r^{\beta'}\rceil}^{\lfloor r^\beta\rfloor}\frac{e^{i\left(2r(f(\lambda_l)-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_l\right)\right)}}{l}=:\sum_{l=\lceil r^{\beta'}\rceil}^{\lfloor r^\beta\rfloor}\frac{e^{-i(\pi l+\varphi(\lambda_l,r))}}{l}.$$

To show this sum goes to zero as $r\to\infty$, we are going to exploit the cancellations of consecutive terms. For this, let us define

$$\begin{aligned}\Delta_{2k}&:=\varphi(\lambda_{2k+1},r)-\varphi(\lambda_{2k},r)\\&=2r(f(\lambda_{2k})-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_{2k}\right)\\&\quad-\left[2r(f(\lambda_{2k+1})-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_{2k+1}\right)\right]-\pi.\end{aligned}$$

More explicitly,

$$\Delta_{2k} = 2\sqrt{r^2 - 4k^2} - 2\sqrt{r^2 - (2k+1)^2} - (4k+\mu)\arccos\left(\frac{2k}{r}\right) + (4k+\mu+2)\arccos\left(\frac{2k+1}{r}\right) - \pi.$$

By the mean value theorem, there exists some $\lambda_* \in (2kr^{-1}, (2k+1)r^{-1})$ such that

$$|\Delta_{2k}| \leqslant \left|\pi r - 2\arccos\lambda_* + \frac{\mu}{r}(1-\lambda_*^2)^{-1/2}\right| r^{-1} \lesssim \frac{l}{r}$$

for $\lceil r^{\beta'}\rceil < l < \lfloor r^{\beta}\rfloor$. This enables us to estimate Q as

$$\begin{aligned}|Q| &= \left|\sum_{k=\lceil r^{\beta'}\rceil/2}^{\lfloor r^{\beta}\rfloor/2} e^{i2r(f(\lambda_{2k})-1)+\mu(\frac{\pi}{2}-\arccos\lambda_{2k})}\left(\frac{1}{2k} - \frac{e^{-i\Delta_{2k}}}{2k+1}\right)\right| \\ &\lesssim \sum_{k=\lceil r^{\beta'}\rceil/2}^{\lfloor r^{\beta}\rfloor/2}\left(\frac{1}{k^2}+\frac{1}{r}\right) \lesssim r^{-\beta'} + r^{\beta-1}. \quad \square\end{aligned}$$

Let us finally consider the case $s > \frac{1}{2}$:

Lemma 3.4. *If $s > \frac{1}{2}$, there exists some $\eta > 0$ depending on s such that*

$$\mathrm{I} = \frac{1}{\pi r}\zeta(2s)\left(\cos\frac{\pi\nu}{2} - \left(2^{1-2s}-1\right)\sin\frac{\pi\mu - 4r}{2}\right) + O_\delta(r^{-1-\eta}).$$

Here ζ is the Riemann's zeta function.

Proof. Let us use again the integral formula for Bessel functions to write

$$J_{l+m'}(r)J_{l+m}(r) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{ir(\sin x - \sin y)}\, e^{-i((l+m')x-(l+m)y)}\,dx\,dy.$$

Applying the stationary phase argument [13, Theorem 7.7.5] with phase function $\sin x - \sin y$ and amplitude $e^{-i((l+m')x-(l+m)y)}$, one readily obtains the asymptotic expansion

$$J_{l+m'}(r)J_{l+m}(r) = \frac{\cos\left(\frac{1}{2}\pi\nu\right) - \sin\left(\frac{1}{2}\left(2\pi l + \pi\mu - 4r\right)\right)}{\pi r} + R_l(r),$$

where the error term satisfies the pointwise bound

$$|R_l(r)| \lesssim \frac{l^4}{r^2}.$$

Now, pick some $\beta \in (0, \frac{1}{4})$ and write

$$\mathrm{I} = \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} J_{l+m'}(r) J_{l+m}(r) + \sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} \chi_{\mathrm{sm}}(\lambda_l) l^{-2s} J_{l+m'}(r) J_{l+m}(r) =: \mathrm{I}_1 + \mathrm{I}_2 \,.$$

Then

$$\begin{aligned}\mathrm{I}_1 &= \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} \left[\frac{\cos\left(\frac{1}{2}\pi\nu\right) - \sin\left(\frac{1}{2}\left(2\pi l + \pi\mu - 4r\right)\right)}{\pi r} + R_l(r) \right] \\ &=: \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} \frac{\cos\left(\frac{1}{2}\pi\nu\right) - \sin\left(\frac{1}{2}\left(2\pi l + \pi\mu - 4r\right)\right)}{\pi r} + \mathcal{R} \,,\end{aligned}$$

where the error term is bounded as

$$|\mathcal{R}| = \left| \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} R_l(r) \right| \lesssim \frac{1}{r^2} \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{4-2s} \lesssim r^{-2}(1 + r^{\beta(5-2s)}) \,.$$

This decay is smaller than r^{-1} if $\beta < \frac{1}{4}$. Expanding the sine, the above series can be computed in closed form in terms of the zeta function:

$$\mathrm{I}_1 = \frac{1}{\pi r}\zeta(2s) \left[\cos\left(\frac{1}{2}\pi\nu\right) - \left(2^{1-2s} - 1\right) \sin\left(\frac{1}{2}\left(\pi\mu - 4r\right)\right) \right] + O(r^{-2} + r^{\beta(5-2s)-2}) \,.$$

To control the remaining term, we use that $s > \frac{1}{2}$ and the bound for g_λ proved in Lemma 3.1 to write

$$|\mathrm{I}_2| \lesssim \left| \sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} \chi_{\mathrm{sm}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r) \right| \lesssim \frac{1}{r} \sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} l^{-2s} \leqslant \frac{1}{r} \sum_{l=\lceil r^\beta \rceil}^{\infty} l^{-2s} \lesssim r^{-\beta(2s-1)-1} \,.$$

As usual, the constant in $\lesssim$ depends on δ. The lemma then follows. □

3.2. Intermediate frequency region

Our next goal is to derive bounds for the term

$$\mathrm{II} = \sum_{l=\lceil (1-2\delta)r \rceil}^{\lfloor (1+2\delta)r \rfloor} \chi_{\mathrm{med}}(\lambda_l)\, l^{-2s} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{il(y-x)} e^{ir(\sin x - \sin y)} e^{i(my - m'x)}\, dx\, dy \,.$$

The difficulty here is that one cannot apply the standard stationary phase method as we did above because the critical points of the phase function

$$\varphi_l(x,y) := \lambda(y-x) + \sin x - \sin y$$

are either degenerate or not uniformly non-degenerate. The main result is the following:

Lemma 3.5. *For any real s and all large enough r (depending on δ),*

$$|\mathrm{II}| \leqslant C\delta^{\frac{1}{2}}\, r^{-2s}\,,$$

where C is independent of δ.

Proof. Since

$$\varphi_l(x,y) = (1-\lambda)(x-y) - \frac{1}{6}(x^3-y^3) + O(x^5) + O(y^5)\,,$$

when $1-2\delta \leqslant \lambda \leqslant 1+2\delta$ and $\delta \ll 1$, an elementary calculation shows that

$$|\nabla\varphi_l(x,y)| \geqslant c$$

whenever $|x|+|y| > 100\,\delta^{1/2}$, where c is a positive constant that depends on δ. Therefore, take some $\chi(t)$ be a smooth nonnegative function that is equal to 1 for $|t| < 100\,\delta^{1/2}$ and 0 for $|t| > 200\,\delta^{1/2}$. The non-stationary phase lemma then shows that

$$\mathrm{II}' := \sum_{l=\lceil(1-2\delta)r\rceil}^{\lfloor(1+2\delta)r\rfloor} \chi_{\mathrm{med}}(\lambda_l)\,\lambda_l^{-2s} \int_{\mathbb{R}^2} e^{il(y-x)} e^{ir(\sin x - \sin y)} e^{i(my-m'x)}\,\chi(x)\,\chi(y)\,dx\,dy$$

coincides with II modulo an exponentially small error. More precisely,

$$\left|\mathrm{II} - r^{-2s}\mathrm{II}'\right| < C_{\delta,N}\, r^{-N}$$

for any N and some constant depending on N and δ.

To estimate II', let us start by defining $z := y-x$ and writing

$$\mathrm{II}' = \sum_{l=r(1-2\delta)}^{r(1+2\delta)} \chi_{\mathrm{med}}(\lambda_l)\,\lambda_l^{-2s} \int_{\mathbb{R}^2} e^{ilz} e^{ir(\sin(y-z)-\sin y)} e^{i((m-m')y+m'z)} \chi(y-z)\,\chi(y)\,dy\,dz\,.$$

A first step is to consider the sum

$$S(r,z) := \frac{1}{r} \sum_{l=r(1-2\delta)}^{r(1+2\delta)} \chi_{\mathrm{med}}(\lambda_l)\,\lambda_l^{-2s} e^{ilz}$$

and to relate it to its continuous counterpart

$$F(r,z) := \int_{-\infty}^{\infty} \chi_{\rm med}(\lambda)\, \lambda^{-2s} e^{irz\lambda} d\lambda\,.$$

Note that it is not a priori obvious that $F(r,z)$ converges to $S(r,z)$ as $r \to \infty$ because, intuitively speaking, the sum is formally obtained by discretizing the integral with a "grid" of length $1/r$, and $r \gg 1$ is precisely the frequency at which the integrand oscillates.

We proceed as follows. Firstly, write

$$S(r,z) - F(r,z) = \sum_{l=r(1-2\delta)}^{r(1+2\delta)} \int_{\lambda_l}^{\lambda_l+\frac{1}{r}} \Big[\lambda_l^{-2s}\chi_{\rm med}(\lambda_l)\Big(\frac{e^{ir\lambda_l z}}{r} - e^{ir\lambda z}\Big) \\ + (\chi_{\rm med}(\lambda_l)\lambda_l^{-2s} - \chi_{\rm med}(\lambda)\lambda^{-2s})e^{ir\lambda z}\Big]\, d\lambda$$

and note that

$$\frac{e^{ilz}}{r} - \int_{\lambda_l}^{\lambda_l+\frac{1}{r}} e^{i\lambda r z}\, d\lambda = h(z)\, \frac{e^{ilz}}{r}$$

with

$$h(z) := \frac{ie^{iz} + z - i}{z}\,.$$

The function h is smooth at the origin; in fact, $h(z) = O(z)$. As moreover

$$|\chi_{\rm med}(\lambda_l)\lambda_l^{-2s} - \chi_{\rm med}(\lambda)\lambda^{-2s}| \lesssim \frac{\delta^{-1}}{r} \tag{3.7}$$

if $\lambda \in [\lambda_l, \lambda_l + \frac{1}{r}]$ and $|\lambda - 1| < 2\delta$, one obtains that the error

$$R(r,z) := S(r,z) - F(r,z) - h(z)S(r,z)$$

is bounded as

$$|R(r,z)| \leqslant \frac{C}{r}\,,$$

with C a constant independent of z and δ.

Since z will eventually be small, the fact that

$$S(r,z) = \frac{F(r,z) + R(r,z)}{1 - h(z)}$$

shows in which sense $S(r,z)$ and $F(r,z)$ are related. The reader can check that, had we argued as in (3.7), we would have obtained an error estimate of the form Cz, which is useless for our purposes.

One can thus write

$$\begin{aligned}\mathrm{II}' = r \int_{\mathbb{R}^3} \chi_{\mathrm{med}}(\lambda)\, \lambda^{-2s} e^{ir(\lambda z + \sin(y-z) - \sin y)} e^{i((m-m')y + m'z)} \frac{\chi(y-z)\chi(y)}{1-h(z)}\, d\lambda\, dz\, dy \\ + r \int_{\mathbb{R}^2} e^{ir(\sin(y-z) - \sin y)} e^{i((m-m')y + m'z)} R(r,z) \frac{\chi(y-z)\chi(y)}{1-h(z)}\, dz\, dy \\ =: \mathrm{II}'_1 + \mathrm{II}'_2\,.\end{aligned}$$

The bound for $R(r,z)$ and the fact that $\chi(t)$ is supported in $|t| < 200\,\delta^{1/2}$ immediately implies

$$|\mathrm{II}'_2| \leqslant C\delta\,,$$

where the constant does not depend on δ.

To analyze II'_1, one cannot directly apply the stationary phase formula to the integral over $\mathbb{R}^3$ because the critical set of the phase has dimension 1. Instead, let us define

$$H(r,y) := r \int_{\mathbb{R}^2} e^{ir(\lambda z + \sin(y-z))} \chi_{\mathrm{med}}(\lambda)\, \lambda^{-2s} e^{im'z} \frac{\chi(y-z)}{1-h(z)} d\lambda\, dz\,.$$

Then, the phase function $\varphi_y(\lambda, z) := \lambda z + \sin(y-z)$ has a unique critical point in the support of the integrand, $(\lambda^*, z^*) := (\cos y, 0)$, and its Hessian is

$$\nabla^2 \varphi_y(\lambda^*, z^*) = \begin{pmatrix} 0 & 1 \\ 1 & -\sin(y) \end{pmatrix}.$$

The stationary phase formula [13, Theorem 7.7.6] then ensures that, if r is large enough (depending on δ)

$$|H(r,y)| \leqslant C$$

with a constant independent of δ. Plugging this estimate into II'_1 and using again that $\chi(t)$ is supported in $|t| < 200\,\delta^{1/2}$, one finds

$$|\mathrm{II}'_1| \leqslant \int_{-\infty}^{\infty} \chi(y)\, |H(r,y)|\, dy \leqslant C\delta^{\frac{1}{2}}$$

with a constant independent of δ. Putting all the estimates together, the lemma is proven. $\square$

3.3. Large frequency region

The last lemma of this section shows that the contribution of the large frequencies is exponentially small:

Lemma 3.6. *For any N, $|\mathrm{III}| \lesssim r^{-N}$ for all large enough r (depending on δ).*

Proof. Let us now use l as the large parameter in the formula for $g_{\lambda_l}(r)$, which amounts to writing

$$g_{\lambda_l}(r) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{il\tilde{\varphi}_{\lambda_l}(x,y)} e^{-i(m'x-my)}\, dx\, dy$$

with

$$\tilde{\varphi}_\lambda(x,y) := y - x + \frac{\sin x - \sin y}{\lambda}\,.$$

If $\lambda > 1+\delta$, it is clear that

$$|\nabla\tilde{\varphi}_\lambda(x,y)| \geqslant c_\delta$$

for all $x, y \in [-\pi,\pi]$, where c_δ is a positive constant that only depends on δ. Therefore, the non-stationary phase lemma [13, Theorem 7.7.1] ensures that $g_{\lambda_l}(r)$ is an exponentially small function of l, meaning that for any N' there exists a constant C (depending on δ and N') such that

$$|g_{\lambda_l}(r)| < C|l|^{-N'}\,.$$

This immediately implies that

$$|\mathrm{III}| \lesssim \sum_{l=(1+\delta)r}^{\infty} l^{-2s}|g_{\lambda_l}(r)| \lesssim r^{-N}$$

for any N, as claimed. □

3.4. Asymptotics for series with derivatives of Bessel functions

The results we have derived above readily yield the asymptotic bounds for weighted sums of Bessel functions that we will crucially need in the next section. Specifically, Lemma 1.2 follows immediately by adding the estimates derived in the previous subsections and letting $\delta \to 0^+$. The explicit constants in the lemma are:

$$c^1_{s,\nu} := \frac{2^{2s-1}\Gamma(1-2s)}{\Gamma(1-s-\frac{\nu}{2})\Gamma(1-s+\frac{\nu}{2})}\,, \qquad c^4 := \frac{\log 2}{\pi}\,,$$

$$c^2_\nu := \pi^{-1}\cos\left(\frac{\pi\nu}{2}\right), \qquad c^5_{s,\nu} := \pi^{-1}\zeta(2s)\cos\left(\frac{\pi\nu}{2}\right),$$

$$c^3_\nu := 2^{-1}\sin\left(\frac{\pi|\nu|}{2}\right), \qquad c^6_{s,\nu} := \pi^{-1}\zeta(2s)(1-2^{1-2s})\,,$$

$$c^7_\mu := \frac{\pi\mu}{2}\,.$$

One should observe that, to estimate the expected number of critical points of the random monochromatic wave (1.4), we will also need asymptotic information about series with derivatives of Bessel functions. This follows easily as a byproduct of Lemma 1.2 using the well-known recurrence relations

$$J_l'(r) = \frac{J_{l-1}(r) - J_{l+1}(r)}{2}\,, \qquad J_l''(r) = \frac{J_{l+2}(r) + J_{l-2}(r) - 2J_l(r)}{4}\,.$$

In the following lengthly corollary of Lemma 1.2 we record the asymptotic formulas that we will need later on:

Corollary 3.7. *The following estimates hold:*

$$\sum_{l=1}^\infty l^{-2s} J_l(r)^2 = \begin{cases} \frac{2^{2s-1}\Gamma(1-2s) r^{-2s}}{\Gamma(1-s)^2} + o(r^{-2s}) & \text{if } s<\frac12,\\ \frac{\log r}{\pi r} + o(r^{-1}) & \text{if } s=\frac12,\\ \frac{\zeta(2s)((2^{1-2s}-1)\sin 2r+1)}{\pi r} + o(r^{-1}) & \text{if } s>\frac12,\end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l(r)J_l'(r) = \begin{cases} o(r^{-2s}) & \text{if } s<\frac12,\\ O(r^{-1}) & \text{if } s=\frac12,\\ \frac{(2^{1-2s}-1)\cos(2r)\zeta(2s)}{\pi r} + o(r^{-1}) & \text{if } s>\frac12,\end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l'(r)^2 = \begin{cases} \frac{\Gamma(\frac12-s) r^{-2s}}{4\sqrt{\pi}\Gamma(2-s)} + o(r^{-2s}) & \text{if } s<\frac12,\\ \frac{\log r}{\pi r} + O(r^{-1}) & \text{if } s=\frac12,\\ \frac{\zeta(2s)(1-(2^{1-2s}-1)\sin 2r)}{\pi r} + o(r^{-1}) & \text{if } s>\frac12,\end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l(r)J_l''(r) = \begin{cases} -\frac{\Gamma(\frac12-s) r^{-2s}}{4\sqrt{\pi}\Gamma(2-s)} + o(r^{-2s}) & \text{if } s<\frac12,\\ -\frac{\log r}{\pi r} + O(r^{-1}) & \text{if } s=\frac12,\\ -\frac{\zeta(2s)((2^{1-2s}-1)\sin 2r+1)}{\pi r} + o(r^{-1}) & \text{if } s>\frac12,\end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l'(r)J_l''(r) = \begin{cases} o(r^{-2s}) & \text{if } s<\frac12,\\ O(r^{-1}) & \text{if } s=\frac12,\\ -\frac{(2^{1-2s}-1)\cos(2r)\zeta(2s)}{\pi r} + o(r^{-1}) & \text{if } s>\frac12,\end{cases}$$

$$\sum_{l=1}^{\infty} l^{-2s} J_l''(r)^2 = \begin{cases} \frac{3\ 2^{2s-5}(2-2s)(4-2s)\Gamma(1-2s)r^{-2s}}{\Gamma(3-s)^2} + o(r^{-2s}) & \text{if } s < \frac{1}{2}, \\ \frac{\log r}{\pi r} + O(r^{-1}) & \text{if } s = \frac{1}{2}, \\ \frac{\zeta(2s)((2^{1-2s}-1)\sin(2r)+1)}{\pi r} + o(r^{-1}) & \text{if } s > \frac{1}{2}. \end{cases}$$

4. Proof of Theorem 1.1

We are now ready to present the proof of the main theorem, which will consist of a number of steps. Recall that we defined the random function u as

$$u := \sum_l a_l\, \sigma_l\, e^{il\theta}\, J_l(r)\,, \qquad \sigma_l := \begin{cases} |l|^{-s} & \text{if } l \neq 0\,, \\ 0 & \text{if } l = 0\,. \end{cases} \tag{4.1}$$

It will be apparent from the proof that the argument remains valid for much more general choices of σ_l, for example of the form (1.6). Of course, the value of the constants $\kappa(s)$, $\tilde{\kappa}_{\frac{3}{2}}$, $\tilde{\kappa}_{\frac{5}{2}}$ one gets depends on the specific choice of σ_l.

4.1. A Kac–Rice formula

Our first objective is to derive an explicit, if hard to analyze, Kac–Rice type formula for the expected number of critical points of the Gaussian random function u.

In this subsection, we shall denote by

$$Du(r,\theta) := \begin{pmatrix} \partial_\theta u(r,\theta) \\ \partial_r u(r,\theta) \end{pmatrix}, \qquad D^2u(r,\theta) := \begin{pmatrix} \partial_{\theta\theta} u(r,\theta) & \partial_{r\theta} u(r,\theta) \\ \partial_{r\theta} u(r,\theta) & \partial_{rr} u(r,\theta) \end{pmatrix}$$

the derivative and Hessian of u in polar coordinates. To apply the Kac–Rice expectation formula, let us start by showing that $Du(r,\theta)$ has a non-degenerate distribution:

Lemma 4.1. *The variance of the Gaussian random variable $Du(r,\theta)$ is*

$$\operatorname{Var}[Du(r,\theta)] = \begin{pmatrix} 4\sum_{l=1}^{\infty} l^{2-2s} J_l(r)^2 & 0 \\ 0 & 4\sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2 \end{pmatrix} =: \begin{pmatrix} \widetilde{\Sigma}_{11}(r) & 0 \\ 0 & \widetilde{\Sigma}_{22}(r) \end{pmatrix}.$$

Proof. To compute the matrix

$$\operatorname{Var}[Du(r,\theta)] := \mathbb{E}[Du(r,\theta) \otimes Du(r,\theta)]\,,$$

recall the expression (4.1) for $u(r,\theta)$ and take advantage of the fact that $u(r,\theta)$ is real valued to write

$$\mathbb{E}[\partial_r u(r,\theta)^2] = \mathbb{E}[\partial_r u(r,\theta)\, \partial_r \overline{u(r,\theta)}] = \sum_{l\neq 0} \sum_{l'\neq 0} \mathbb{E}(a_l \overline{a_{l'}})\, |l|^{-s} |l'|^{-s} e^{i(l-l')\theta} J_l'(r)\, J_{l'}'(r)\,.$$

By the definition of the random variables a_l,

$$\mathbb{E}(a_l \overline{a_{l'}}) = 2\delta_{l,l'} \,,$$

so one obtains

$$\mathbb{E}[\partial_r u(r,\theta)^2] = 4 \sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2$$

The same argument yields

$$\begin{aligned}
\mathbb{E}[\partial_r u(r,\theta)\, \partial_\theta u(r,\theta)] &= \mathbb{E}[\partial_\theta u(r,\theta)\, \partial_r \overline{u(r,\theta)}] \\
&= \sum_{l\neq 0} \sum_{l'\neq 0} \mathbb{E}(a_l \overline{a_{l'}})\, il|l|^{-s}|l'|^{-s} e^{i(l-l')\theta} J_l(r)\, J_{l'}'(r) \\
&= 2i \sum_{l\neq 0} l|l|^{-2s} J_l(r)\, J_l'(r) = 0
\end{aligned}$$

by parity, and

$$\mathbb{E}[\partial_\theta u(r,\theta)^2] = 4 \sum_{l=1}^{\infty} l^{2-2s} J_l(r)^2 \,.$$

This easily implies that $\mathrm{Var}[Du(r,\theta)]$ is a strictly positive matrix for all (r,θ). □

Remark 4.2. The same computation as above shows that the covariance kernel of the random function (4.1) is

$$K(r,\theta;r',\theta') := \mathbb{E}[u(r,\theta)\, u(r',\theta')] = 4 \sum_{l=1}^{\infty} l^{-2s} J_l(r) J_l(r') \cos[l(\theta-\theta')] \,.$$

The covariance kernel is therefore invariant under rotations but, in general, not under translation. An exception to this general fact is the case $s = 0$. Indeed, it is well known that the covariance kernel of

$$\widetilde{u} := u + \sqrt{2} a_0\, J_0(r).$$

is $\widetilde{K}(x;x') = 2J_0(|x-x'|)$ by Graf's Addition Theorem. The corresponding spectral measure in this case is the Hausdorff measure on the unit circle. Observe that $\widetilde{u}$ will give the same asymptotics as u for $s = 0$ because, as we saw in Lemma 1.2, for $s = 0$ the series of Bessel functions is asymptotically of order 1 but the term $J_0(r)^2$ decays like r^{-1}. By Lemma 4.3, their covariances Σ_{ij} are then asymptotically equivalent. Note we have chosen to omit the term $l = 0$ in u for simplicity, especially when this term contributes to the asymptotic expansion (that is, for $s > \frac{1}{2}$ in Lemma 1.2).

Lemma 4.3. *The expected value of the number of critical points of the random monochromatic wave* (1.4) *is*

$$\mathbb{E}N(\nabla u, R) = \int_0^R \int_{\mathbb{R}^3} \frac{\left| z_1^2 \Sigma_{13}(r) - z_2^2 \Sigma_{22}(r) + z_3 z_1 \sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \right|}{(2\pi)^{\frac{3}{2}} \sqrt{\widetilde{\Sigma}_{11}(r)\widetilde{\Sigma}_{22}(r)}} e^{-\frac{1}{2}|z|^2} \, dz \, dr \,,$$

where

$$\Sigma_{11}(r) := 4\sum_{l=1}^{\infty} l^{4-2s} J_l(r)^2 - \frac{4\left(\sum_{l=1}^{\infty} l^{2-2s} J_l(r) J_l'(r)\right)^2}{\sum_{l=0}^{\infty} l^{-2s} J_l'(r)^2} \,,$$

$$\Sigma_{13}(r) := 4\sum_{l=1}^{\infty} (-1) l^{2-2s} J_l(r) J_l''(r) + \frac{4\sum_{l=1}^{\infty} l^{2-2s} J_l(r) J_l'(r) \sum_{l=1}^{\infty} l^{-2s} J_l'(r) J_l''(r)}{\sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2} \,,$$

$$\Sigma_{22}(r) := 4\sum_{l=1}^{\infty} l^{2-2s} J_l'(r)^2 - \frac{4\left(\sum_{l=1}^{\infty} l^{2-2s} J_l(r) J_l'(r)\right)^2}{\sum_{l=1}^{\infty} l^{2-2s} J_l(r)^2} \,,$$

$$\Sigma_{33}(r) := 4\sum_{l=1}^{\infty} l^{-2s} J_l''(r)^2 - \frac{4\left(\sum_{l=1}^{\infty} l^{-2s} J_l'(r) J_l''(r)\right)^2}{\sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2} \,.$$

Proof. As $Du(r,\theta)$ is a non-degenerate Gaussian random variable by Lemma 4.1, the Kac–Rice integral formula in polar coordinates [3, Proposition 6.6] ensures that

$$\mathbb{E}\left(N(\nabla u, R)\right) = \int_{B(R)} \mathbb{E}\left\{ |\det D^2 u(r,\theta)| \;\middle|\; Du(r,\theta) = 0 \right\} \rho_{Du(r,\theta)}(0) \, dr \, d\theta \tag{4.2}$$

where $\rho_{Du(r,\theta)} : \mathbb{R}^2 \to [0,\infty)$ denotes the probability distribution function of the $\mathbb{R}^2$-valued random variable $Du(r,\theta)$.

Next, let us reduce the computation of the conditional expectation to that of an ordinary expectation by introducing a new random variable $\zeta(r,\theta)$. Just like $D^2u(r,\theta)$, $\zeta(r,\theta)$ will take values in the space of 2×2 symmetric matrices, which we shall henceforth identify with $\mathbb{R}^3$ by labeling the matrix components of a symmetric matrix as

$$\zeta =: \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix} . \tag{4.3}$$

Specifically, let us set

$$\zeta(r,\theta) := D^2 u(r,\theta) - B(r,\theta) Du(r,\theta) \,, \tag{4.4}$$

where the linear operator $B(r,\theta)$ (which we can regard as a 3×2 matrix after identifying $D^2u(r,\theta)$ with a 3-component vector) is chosen so that the covariance matrix of $Du(r,\theta)$ and $\zeta(r,\theta)$ is 0:

$$B(r,\theta) := \mathbb{E}(D^2u(r,\theta)\otimes Du(r,\theta))\big[\mathbb{E}(Du(r,\theta)\otimes Du(r,\theta))\big]^{-1}$$

Indeed, one can plug (4.4) in the formula for $\mathbb{E}(\zeta(r,\theta)\otimes Du(r,\theta))$ and check that

$$\mathbb{E}(\zeta(r,\theta)\otimes Du(r,\theta)) = 0\,.$$

As $Du(r,\theta)$ and $\zeta(r,\theta)$ are jointly a Gaussian vector with zero mean, this condition ensures that they are independent random variables. This enables us to write the above conditional expectation as

$$\begin{aligned}\mathbb{E}\big\{|\det D^2u(r,\theta)|\;\big|\; Du(r,\theta)=0\big\} &= \mathbb{E}\big\{|\det[\zeta(r,\theta)+B(r,\theta)Du(r,\theta)]|\;\big|\; Du(r,\theta)=0\big\}\\ &= \mathbb{E}|\det\zeta(r,\theta)|\,.\end{aligned}$$

Let now us compute the covariance matrix of $\zeta(r,\theta)$. Since the variance matrix of $Du(r,\theta)$ is independent of θ, let us simply write $\operatorname{Var} Du(r)$, and similarly with other rotation-invariant quantities. One then has

$$\operatorname{Var}\zeta(r) = \operatorname{Var} D^2u(r) - \operatorname{Cov}(D^2u,Du)(r)\cdot \operatorname{Var} Du(r)^{-1}\cdot \operatorname{Cov}(D^2u,Du)(r)^\top \tag{4.5}$$

Arguing as in Lemma 4.1 and using that we have identified $D^2u(r,\theta)$ with a 3-component vector, one finds that

$$\operatorname{Var} D^2u(r) := \mathbb{E}[D^2u(r,\theta)\otimes D^2u(r,\theta)]$$

is given by the 3×3 matrix

$$\begin{aligned}&\operatorname{Var} D^2u(r)\\ &= \begin{pmatrix} 4\sum_{l=1}^\infty l^{4-2s}J_l(r)^2 & 0 & -4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l''(r)\\ 0 & 4\sum_{l=1}^\infty l^{2-2s}J_l'(r)^2 & 0\\ -4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l''(r) & 0 & 4\sum_{l=1}^\infty l^{-2s}J_l''(r)^2\end{pmatrix}.\end{aligned}$$

Similarly,

$$\operatorname{Cov}(D^2u,Du)(r) = \begin{pmatrix} 0 & -4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l'(r)\\ 4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l'(r) & 0\\ 0 & 4\sum_{l=1}^\infty l^{2-2s}J_l'(r)J_l''(r)\end{pmatrix} \tag{4.6}$$

Combining these formulas, we derive that

$$\Sigma(r) := \operatorname{Var}\zeta(r,\theta) = \begin{pmatrix} \Sigma_{11}(r) & 0 & \Sigma_{13}(r) \\ 0 & \Sigma_{22}(r) & 0 \\ \Sigma_{13}(r) & 0 & \Sigma_{33}(r) \end{pmatrix}, \tag{4.7}$$

where $\Sigma_{jk}(r)$ are defined as in the statement of the lemma.

Let us now consider the Cholesky decomposition of this matrix:

$$\Sigma(r) = M(r)^\top M(r)\,,$$

where the matrix $M(r)$ is given by

$$M(r) := \begin{pmatrix} \sqrt{\Sigma_{11}(r)} & 0 & \frac{\Sigma_{13}(r)}{\sqrt{\Sigma_{11}(r)}} \\ 0 & \sqrt{\Sigma_{22}(r)} & 0 \\ 0 & 0 & \sqrt{\Sigma_{33}(r) - \frac{\Sigma_{13}(r)^2}{\Sigma_{11}(r)}} \end{pmatrix}.$$

As the matrix $\Sigma(r)$ is positive definite and $\zeta(r,\theta)$ is a Gaussian random variable with zero mean and variance $\Sigma(r)$, one then infers that the 3-component random variable

$$Z(r,\theta) := \zeta(r,\theta)^\top M(r)^{-1}$$

is Gaussian, has zero mean and its variance matrix is the identity. It is thus straightforward that

$$\begin{aligned} \mathbb{E}|\det\zeta(r,\theta)| &= \int_{\mathbb{R}^3} \left|y_1y_3 - y_2^2\right| \rho_{\zeta(r,\theta)}(y)\,dy \\ &= \int_{\mathbb{R}^3} \left|z_1^2\Sigma_{13}(r) - z_2^2\Sigma_{22}(r) + z_3z_1\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2}\right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{\frac{3}{2}}}\,dz\,, \end{aligned}$$

where

$$\rho_{\zeta(r,\theta)}(y) := \frac{\exp\Big(-\frac{1}{2}y\cdot\Sigma^{-1}y\Big)}{(2\pi)^{3/2}(\det\Sigma(r))^{1/2}}$$

is the probability density distribution of the random variable $\zeta(r,\theta)$ and we have used the change of variables

$$y_1 =: \sqrt{\Sigma_{11}(r)}z_1\,, \qquad y_2 =: \sqrt{\Sigma_{22}(r)}z_2\,, \qquad y_3 =: \frac{\Sigma_{13}(r)}{\sqrt{\Sigma_{11}(r)}}z_1 + \sqrt{\Sigma_{33}(r) - \frac{\Sigma_{13}(r)^2}{\Sigma_{11}(r)}}z_3\,.$$

and the fact that the Jacobian determinant is $\det M(r) = (\det\Sigma(r))^{\frac{1}{2}}$. The lemma follows using that the probabability density function of the Gaussian random variable $Du(r,\theta)$ is

$$\rho_{Du(r,\theta)}(0) = \frac{1}{2\pi\sqrt{\widetilde{\Sigma}_{11}(r)\widetilde{\Sigma}_{22}(r)}} \tag{4.8}$$

as a consequence of the formula for $\operatorname{Var} Du(r,\theta)$ computed in Lemma 4.1 and of the fact that the density function of an $\mathbb{R}^k$-valued Gaussian random variable Y with zero mean and variance matrix Σ is

$$\rho_Y(y) := (2\pi)^{-\frac{k}{2}}(\det\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}y\cdot\Sigma^{-1}y}\,. \quad \square$$

4.2. Some technical lemmas

In the next subsections, we will discuss the behavior of the formula for the expected number of critical points that we have computed in Lemma 4.3 above. The analysis will strongly depend on the value of the parameter s. In the computations, we will use several technical lemmas repeatedly, often without further mention.

Lemma 4.4. *Given constants of the form $a_{jk}(r) = \tilde{a}_{jk}(r) + \varepsilon_{jk}(r)$, with $1 \leqslant j,k \leqslant m$,*

$$\begin{aligned}&\int_{\mathbb{R}^m}\Bigg|\sum_{1\leqslant j,k\leqslant m} a_{jk}(r)z_jz_k\Bigg|e^{-\frac{1}{2}|z|^2}\,dz\\&\quad=\int_{\mathbb{R}^m}\Bigg|\sum_{1\leqslant j,k\leqslant m} \tilde{a}_{jk}(r)z_jz_k\Bigg|e^{-\frac{1}{2}|z|^2}\,dz + O\Big(\max_{1\leqslant j,k\leqslant m}|\varepsilon_{jk}(r)|\Big)\,.\end{aligned}$$

Proof. It stems from the elementary estimate

$$\left|\Bigg|\sum_{1\leqslant j,k\leqslant m} a_{jk}(r)z_jz_k\Bigg| - \Bigg|\sum_{1\leqslant j,k\leqslant m} \tilde{a}_{jk}(r)z_jz_k\Bigg|\right| \lesssim |z|^2\max_{1\leqslant j,k\leqslant m}|\varepsilon_{jk}(r)|\,. \quad \square$$

Lemma 4.5. *Let $q : [1,\infty) \to (0,\infty)$ be a continuous function with $\int_1^\infty q(r)\,dr = \infty$. Then, for $r \gg 1$ and any fixed r_0,*

$$\int_{r_0}^{r} o(q(r'))\,dr' = o\left(\int_{r_0}^{r} q(r')\,dr'\right)\,.$$

Proof. Consider any $\varepsilon > 0$ and assume, without any loss of generality, that $o(q(r')) \geqslant 0$. By definition, there is some R_ε such that $o(q(r)) \leqslant \varepsilon q(r)$ for all $r > R_\varepsilon$. Now set $Q(r) := \int_{r_0}^{r} q(r')\,dr'$ and write

$$\frac{\int_{r_0}^{r} o(q(r'))\,dr'}{Q(r)} = \frac{\int_{r_0}^{R_\varepsilon} o(q(r'))\,dr'}{Q(r)} + \frac{\int_{R_\varepsilon}^{r} o(q(r'))\,dr'}{Q(r)}$$

$$\leqslant \frac{C_\varepsilon}{Q(r)} + \frac{\varepsilon \int_{R_\varepsilon}^r q(r')\,dr'}{Q(r)} = o(1) + \varepsilon$$

as $r \to \infty$, since $Q(r) \to \infty$. Letting $\varepsilon \to 0$, the result follows. $\square$

The following lemma will be very useful in the analysis of the asymptotic behavior of the number of critical points of u:

Lemma 4.6. *Consider a positive smooth π-periodic function P and constants $a \geqslant 0$ and $b \in \mathbb{R}$. If $a = 0$, we also assume that $b \geqslant 0$. Then, for $R \gg 1$,*

$$\int_\pi^R r^a (\log r)^b\, P(r)\,dr \sim \frac{R^{a+1}(\log R)^b}{\pi(a+1)} \int_0^\pi P(r)\,dr\,.$$

Proof. Let us define $J := \lfloor R/\pi \rfloor$ and write $R = J\pi + R_1$, with $0 \leqslant R_1 < \pi$. We can then write

$$\int_\pi^R r^a (\log r)^b\, P(r)\,dr = \sum_{j=1}^{J-1} \int_{\pi j}^{\pi(j+1)} r^a (\log r)^b\, P(r)\,dr + \int_{\pi J}^{\pi J + R_1} r^a (\log r)^b\, P(r)\,dr\,.$$

The second term is obviously bounded as

$$\left| \int_{\pi J}^{\pi J + R_1} r^a (\log r)^b\, P(r)\,dr \right| \lesssim R^a (\log R)^b$$

To estimate the first term, let

$$B := \int_0^\pi P(r)\,dr\,.$$

As the function $r^a(\log r)^b$ is increasing for large enough r, we have

$$B(\pi j)^a [\log(\pi j)]^b \leqslant \int_{\pi j}^{\pi(j+1)} r^a (\log r)^b\, P(r)\,dr \leqslant B[\pi(j+1)]^a[\log(\pi(j+1))]^b$$

if j is larger that a certain integer $J_{a,b}$. With $\eta = 0, 1$, we can use the following asymptotic formula, which is an easy consequence of the Euler-Maclaurin formula,

$$\sum_{j=J_{a,b}}^{J-1} [\pi(j+\eta)]^a [\log(\pi(j+\eta))]^b \sim \frac{\pi^a (J+\eta-1)^{a+1}[\log(\pi(J+\eta-1))]^b}{a+1} \sim \frac{R^{a+1}(\log R)^b}{\pi(a+1)}$$

to derive the formula of the statement. Here we have used that $\pi J = R + O(1)$ and that the integral over $r \in [\pi, \pi J_{a,b}]$ is obviously bounded independently of R. $\square$

Before discussing the behavior of $\mathbb{E}N(\nabla u, R)$ in the different regularity regimes, one should note that the integral appearing in Lemma 4.3 is remarkably hard to analyze. We will be able to obtain much more convenient integral representations by means of the following lemma:

Lemma 4.7. *Let A, B, C be real constants. Then*

$$\int_{\mathbb{R}^3} \left|Az_1^2 + Bz_2^2 + 2Cz_1z_3\right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz = \frac{2}{\pi} \int_0^\infty \frac{1 - a(t)\cos\frac{1}{2}\Phi(t)}{t^2}\, dt\,,$$

where

$$\Phi(t) := \arg\left((1 - 2iBt)\left(1 - 2iAt + 4C^2t^2\right)\right)\,,$$
$$a(t) := (1 + 4B^2t^2)^{-\frac{1}{4}}\left[(1 + 4C^2t^2)^2 + 4A^2t^2\right]^{-\frac{1}{4}}\,.$$

Proof. Defining the matrix

$$M := \begin{pmatrix} A & 0 & C \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix},$$

one can write the above integral as

$$Q := \int_{\mathbb{R}^3} \left|Az_1^2 + Bz_2^2 + 2Cz_1z_3\right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz = \int_{\mathbb{R}^3} |z \cdot Mz| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz\,.$$

The results about Gaussian integrals involving an absolute value function derived in [15, Theorem 2.1] therefore ensure that

$$Q = \frac{2}{\pi} \int_0^\infty \left[1 - \frac{\det(I - 2itM)^{-\frac{1}{2}} + \det(I + 2itM)^{-\frac{1}{2}}}{2}\right] \frac{dt}{t^2}\,.$$

Now a straightforward computation yields the formula in the statement. $\square$

4.3. The case $s<\frac{1}{2}$

We are ready to compute the asymptotics for the number of critical points when $s < \frac{1}{2}$:

Lemma 4.8. *If* $s < \frac{1}{2}$,

$$\lim_{R\to\infty} \frac{\mathbb{E}N(\nabla u, R)}{R^2} = \kappa(s)$$

with

$$\kappa(s) := \frac{1}{2}\frac{1}{\sqrt{2-s}} \int_{\mathbb{R}^3} \left| \sqrt{\frac{1-2s}{8-4s}}\left(z_1^2 - z_2^2\right) + z_1 z_3 \right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz\,. \tag{4.9}$$

Proof. Let us compute the matrix $\Sigma(r)$. From Equation (4.7) and the asymptotic formulas for sums of Bessel functions recorded in Corollary 3.7, it follows that

$$\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)\,,$$

where the leading contribution is

$$\Sigma^0(r) := \begin{pmatrix} \dfrac{2^{2s-3}\Gamma(5-2s)r^{4-2s}}{\Gamma(3-s)^2} & 0 & \dfrac{\Gamma\left(\frac{3}{2}-s\right)r^{2-2s}}{\sqrt{\pi}\Gamma(3-s)} \\ 0 & \dfrac{\Gamma\left(\frac{3}{2}-s\right)r^{2-2s}}{\sqrt{\pi}\Gamma(3-s)} & 0 \\ \dfrac{\Gamma\left(\frac{3}{2}-s\right)r^{2-2s}}{\sqrt{\pi}\Gamma(3-s)} & 0 & \dfrac{3\Gamma\left(\frac{1}{2}-s\right)r^{-2s}}{2\sqrt{\pi}\Gamma(3-s)} \end{pmatrix}$$

and the error is bounded as

$$R_{jk}(r) = o(1)\Sigma^0_{jk}(r)\,.$$

Here and in what follows, $o(1)$ denotes a quantity that tends to zero as $r \to \infty$.

Let us define

$$I(r,z) := \left| z_1^2\Sigma_{13}(r) - z_2^2\Sigma_{22}(r) + z_3 z_1\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \right| \tag{4.10}$$

and note that, by the formula for $\Sigma(r)$ and the asymptotics for weighted sums of Bessel functions presented in Corollary 3.7,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim r^{2-2s}\pi^{-1/4}2^{s-\frac{1}{2}}(2-s)\left(\frac{\Gamma\left(\frac{1}{2}-s\right)\Gamma(3-2s)}{\Gamma(3-s)^3}\right)^{1/2}.$$

Likewise, the quantity

$$\sigma(r) := \widetilde{\Sigma}_{11}(r)\widetilde{\Sigma}_{22}(r) \tag{4.11}$$

satisfies the asymptotic bound

$$\sigma(r) \sim \frac{2\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{3}{2}-s\right)}{\pi\Gamma\left(2-s\right)^2} r^{2-4s}\,.$$

Finally, the integral

$$\mathcal{I}(r) := \frac{1}{2\pi\sqrt{\sigma(r)}} \int_{\mathbb{R}^3} I(z,r) \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}} dz \tag{4.12}$$

can be then estimated, as a consequence of Lemmas 4.4 and 4.7 and of the preceding asymptotic bounds, as

$$\mathcal{I}(r) \sim \frac{\kappa(s)}{\pi} r\,,$$

where $\kappa(s)$ is defined as in the statement. Thus, the integral formula in Lemma 4.3 ensures that

$$\mathbb{E}N(\nabla u, R) \sim 2\int_0^R \kappa(s) r\, dr = \kappa(s)\, R^2\,. \quad \square$$

In the next lemma, we analyze the behavior of the positive constant $\kappa(s)$ (which is written simply as $\kappa(s)$ in the statement of Theorem 1.1), for $s < \frac{1}{2}$. The key idea is to obtain an easier characterization of this constant as a one-dimensional integral. Interestingly, the global maximum of $\kappa(s)$ is attained at $s = 0$, that is, in the classical case of random waves with a translation-invariant covariance kernel. In Fig. 2 we have plotted $\kappa(s)$ for the first region of $s < 1/2$ using the next lemma.

Lemma 4.9. *The function $\kappa(s)$ is smooth, strictly increasing on $s \in (-\infty, 0)$, and strictly decreasing on $(0, \frac{1}{2})$. Furthermore,*

$$\lim_{s\to\frac{1}{2}^-} \kappa(s) = \sqrt{\frac{2}{3}}\frac{1}{\pi}\,, \qquad \lim_{s\to-\infty} \kappa(s) = 0\,.$$

Proof. The limiting values can be computed directly from the formula for $\kappa(s)$. Indeed, the (somewhat surprising) fact that $\kappa(s) \to 0$ as $s \to -\infty$ is obvious in view of Equation (4.9), and as is the limit

$$\lim_{s\to\frac{1}{2}^-} \kappa(s) = \int_{\mathbb{R}^3} \frac{|z_1 z_3|}{\sqrt{6}} \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz = \sqrt{\frac{2}{3}}\frac{1}{\pi}\,.$$

To analyze the behavior of $\kappa(s)$ for intermediate values of s, we use Lemma 4.7 to rewrite (4.9) as

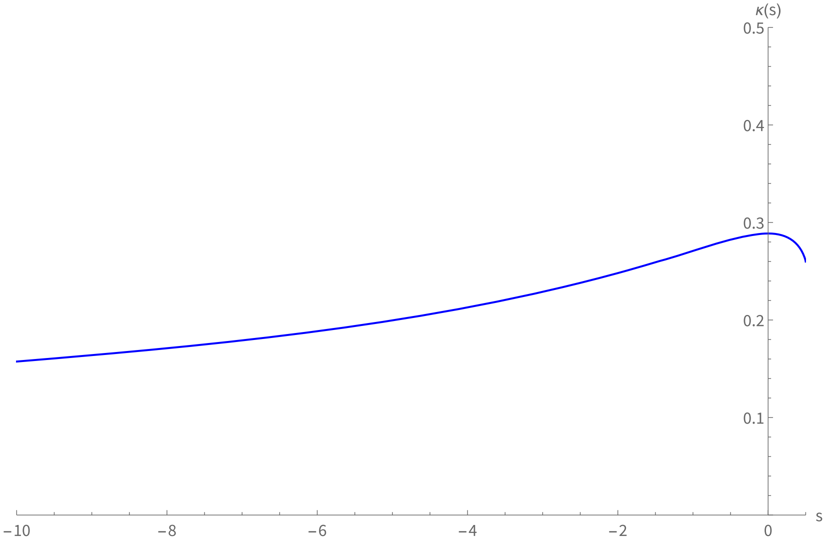

Fig. 2. Plot of the function $\kappa(s)$ on the region $s < 1/2$.

$$\kappa(s) = \frac{2}{\pi}\int_0^\infty \frac{1 - a(s,t)\cos\frac{1}{2}\Phi(s,t)}{t^2}\,dt$$

with

$$a(s,t) := \frac{\sqrt{2}(4-2s)}{\left[(1-2s)t^6 + \left(8(2-s)^2 + 6(1-s)t^2\right)^2\right]^{1/4}},$$

$$\Phi(s,t) := \arg\left(4 + \frac{2t^2\left(-6s + i\sqrt{1-2s}\,t + 6\right)}{(4-2s)^2}\right).$$

Note that

$$\partial_s a(s,t) = 4s\frac{3t^2\left(16(2-s)^2 + t^4 + 12(1-s)t^2\right)}{2\sqrt{2}\left((1-2s)t^6 + \left(8(2-s)^2 + 6(1-s)t^2\right)^2\right)^{5/4}},$$

$$\partial_s \tan\Phi(s,t) = -4s\frac{3t^3\left(-4s + t^2 + 8\right)}{2\sqrt{1-2s}\left(8(2-s)^2 + 6(1-s)t^2\right)^2}$$

because

$$\Phi(s,t) = \arctan\left(\frac{\sqrt{1-2s}\,t^3}{8(2-s)^2 + 6(1-s)t^2}\right) = \arctan\tan\Phi(s,t).$$

Using that the polynomials appearing on the numerators are all positive for $t > 0$ and $s < \frac{1}{2}$, it follows that $\kappa'(s)/s < 0$ for all $s \in (-\infty, 0) \cup (0, \frac{1}{2})$. The result then follows. $\square$

Remark 4.10. In the case $s = 0$, where $\kappa(s)$ attains its maximum, we recover the well-known asymptotic formula (see Appendix B) for the expected number of critical points:

$$\kappa(0) = \int\limits_{\mathbb{R}^3} \frac{\left|z_1^2 + 2\sqrt{2}z_3 z_1 - z_2^2\right|}{8} \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}} dz = \frac{1}{2\sqrt{3}} = 0,2886\ldots$$

where we have used that for $s = 0$ the integral above becomes

$$\frac{2}{\pi} \int\limits_0^\infty \frac{1 - \frac{2}{\sqrt{-\frac{it^3}{2}+3t^2+16}} - \frac{2}{\sqrt{\frac{it^3}{2}+3t^2+16}}}{t^2} \, dt = \frac{1}{2\sqrt{3}}.$$

4.4. The case $\boldsymbol{s = \frac{1}{2}}$

We shall next show that, in spite of the appearance of logarithmic terms in the formulas, the asymptotic behavior in the case $s = \frac{1}{2}$ coincides with the limit as $s \to \frac{1}{2}^+$ of the formula derived in Lemma 4.8.

Lemma 4.11. *For* $s = \frac{1}{2}$,

$$\mathbb{E}N(\nabla u, R) \sim \sqrt{\frac{2}{3}} \frac{1}{\pi} R^2 \,.$$

Proof. From Equation (4.7) and Corollary 3.7, we infer that in the case $s = \frac{1}{2}$, we can write

$$\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$$

where

$$\Sigma^0(r) = \begin{pmatrix} \frac{8r^3}{3} & 0 & \frac{4r}{3} \\ 0 & \frac{4r}{3} & 0 \\ \frac{4r}{3} & 0 & \frac{4\log r}{r} \end{pmatrix}$$

and the error is bounded as $\mathcal{R}_{ij}(r) = \Sigma^0_{ij}(r)\, o(1)$. Therefore,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim \frac{4}{3\pi} r\sqrt{6\log r}\,.$$

Likewise, the function $\sigma(r)$ defined in (4.11) satisfies

$$\sigma(r) \sim \frac{16\log r}{\pi^2}.$$

Plugging these formulas in (4.12), we obtain

$$\mathcal{I}(r) \sim \frac{r \int_{\mathbb{R}^3} |z_1 z_3| e^{-\frac{1}{2}|z|^2} dz}{\sqrt{6}\pi(2\pi)^{3/2}} = \sqrt{\frac{2}{3}} \frac{r}{\pi^2} . \quad \square$$

4.5. *The case* $\frac{1}{2} < s < \frac{3}{2}$

We shall next show that, in the regime $\frac{1}{2} < s < \frac{3}{2}$, the expected number of critical points contained in a large disk also grows like the area. The associated proportionality constant, which we denote by $\kappa(s)$, turns out to be smooth on $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$ but only continuous at $s = \frac{1}{2}$.

Lemma 4.12. *For* $\frac{1}{2} < s < \frac{3}{2}$, *then* $\mathbb{E} N(\nabla u, R) \sim \kappa(s) R^2$ *with*

$$\kappa(s) := \frac{1}{\pi} \sqrt{\frac{3-2s}{4-2s}} .$$

Proof. By Equation (4.7) and Corollary 3.7, $\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$ with

$$\Sigma^0(r) = \begin{pmatrix} \frac{2^{2s-3} r^{4-2s} \Gamma(5-2s)}{\Gamma(3-s)^2} & 0 & \frac{r^{2-2s}\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\Gamma(3-s)} \\ 0 & \frac{r^{2-2s}\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\Gamma(3-s)} & 0 \\ \frac{r^{2-2s}\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\Gamma(3-s)} & 0 & \frac{4^{2-s}\left(4^s-1\right)\zeta(2s)}{\pi r\left(\left(4^s-2\right)\sin(2r)+4^s\right)} \end{pmatrix}$$

and $\mathcal{R}_{ij} = \Sigma^0_{ij}(r)\, o(1)$. Therefore, as $4-4s < 3-2s$,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim \sqrt{\frac{2}{\pi}} \sqrt{\frac{\left(4^s-1\right) r^{3-2s} \zeta(2s)\Gamma(5-2s)}{\Gamma(3-s)^2 \left(\left(4^s-2\right)\sin(2r)+4^s\right)}} .$$

Similarly, and using the same notation as in the last two subsections,

$$\sigma(r) \sim \frac{4 r^{1-2s} \zeta(2s)\Gamma(3-2s)\left(\left(4^s-2\right)\sin(2r)+4^s\right)}{\pi\Gamma(2-s)^2} .$$

One can then plug these formulas in (4.12) to find

$$\mathcal{I}(r) \sim \frac{r}{\pi\left(1+\left(1-2^{1-2s}\right)\sin 2r\right)} \sqrt{\frac{2^{-2s}\left(1-2^{-2s}\right)(3-2s)}{(4-2s)}} \int_{\mathbb{R}^3} |z_1 z_3| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}} dz .$$

As $2^{1-2s} < 1$, this immediately implies

$$\mathbb{E}N(\nabla u, R) \sim \frac{4}{\pi}\sqrt{\frac{2^{-2s}\left(1-2^{-2s}\right)(3-2s)}{(4-2s)}} \int_0^R \frac{r}{1+\left(1-2^{1-2s}\right)\sin 2r}\, dr\,.$$

As

$$\int_0^\pi \frac{1}{1+b\sin 2r}\, dr = \frac{\pi}{\sqrt{1-b^2}} \tag{4.13}$$

for all $|b|<1$, the formula of the statement now follows using Lemma 4.6. □

Remark 4.13. It follows from Lemmas 4.8, 4.11 and 4.12 that $\kappa(s) \in C^\infty((-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}])$, and that $\kappa(s)$ is Lipschitz at $s=\frac{1}{2}$ but not C^1. It also follows that

$$\lim_{s\to-\infty} \kappa(s) = \lim_{s\to\frac{3}{2}^-} \kappa(s) = 0\,.$$

4.6. The case $\boldsymbol{s=\frac{3}{2}}$

Here we shall see that the expected number of critical points contained in a ball of large radius does not grow like the area of the ball any longer:

Lemma 4.14. *If* $s=\frac{3}{2}$,

$$\mathbb{E}N(\nabla u, R) \sim \frac{1}{\pi}\frac{R^2}{\sqrt{\log R}}\,.$$

Proof. The argument is essentially as before. Using Corollary 3.7 and Equation (4.7), one can write $\Sigma(r) = \sigma^0(r) + \mathcal{R}(r)$, with

$$\Sigma^0(r) := \frac{1}{\pi}\begin{pmatrix} 4r & 0 & \frac{4\log r}{r} \\ 0 & \frac{4\log r}{r} & 0 \\ \frac{4\log r}{r} & 0 & \frac{7\zeta(3)}{4r+3r\sin 2r} \end{pmatrix}$$

and $\mathcal{R}_{ij} = \Sigma^0_{ij}(r)\, o(r^0)$. Hence, keeping track of the errors using Lemmas 4.4-4.5 as before,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim \frac{2}{\pi}\sqrt{\frac{7\zeta(3)}{3\sin 2r+4}}\,,$$

$$\sigma(r) \sim \frac{4\zeta(3)\log r(3\sin 2r+4)}{\pi^2 r^2}\,.$$

This readily implies

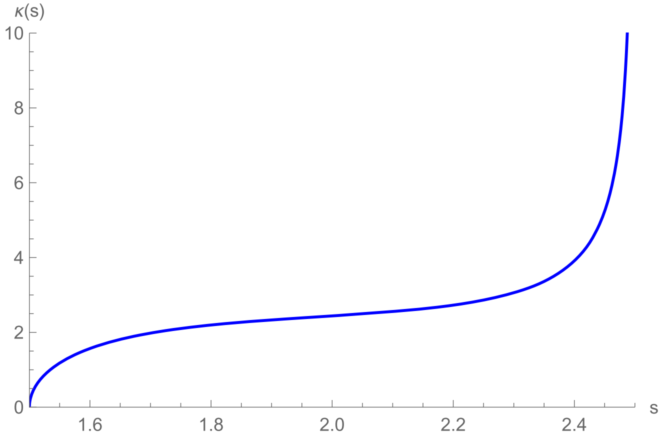

Fig. 3. Plot of the function $\kappa(s)$ on the region $3/2 < s < 5/2$.

$$\mathcal{I}(r) \sim \frac{r}{\sqrt{\log r}} \frac{\sqrt{7}}{\pi^2\,(3\sin 2r + 4)}\,,$$

so Lemma 4.3 ensures that the expected number of critical points satisfies

$$\mathbb{E}N(\nabla u, R) \sim \frac{2\sqrt{7}}{\pi^2} \int_\pi^R \frac{1}{4+3\sin 2r} \frac{r}{\sqrt{\log r}} dr\,.$$

The asymptotic behavior of this integral is

$$\int_\pi^R \frac{1}{4+3\sin 2r} \frac{r}{\sqrt{\log r}} dr \sim \frac{R^2}{2\pi\sqrt{\log R}} \int_0^\pi \frac{1}{4+3\sin 2r} dr = \frac{R^2}{2\sqrt{7\log R}}\,.$$

by Lemma 4.6, so the result follows. □

4.7. *The case* $\frac{3}{2}<s<\frac{5}{2}$

The analysis of the large R asymptotics presents no new difficulties:

Lemma 4.15. *For* $\frac{3}{2} < s < \frac{5}{2}$, $\mathbb{E}N(\nabla u, R) \sim \kappa(s) R^{\frac{7}{2}-s}$ *with*

$$\kappa(s) := -\frac{2^{2s+\frac{1}{2}} r^{\frac{5}{2}-s} \sqrt{\frac{(4^s-1)\Gamma(5-2s)}{\zeta(2s-2)}}}{\pi^{3/2}(7-2s)\Gamma(3-s)} \int_0^\pi \frac{dr}{\left((4^s-2)\sin(2r)+4^s\right)\sqrt{4^s-(4^s-8)\sin(2r)}}\,.$$

See Fig. 3.

Proof. Arguing as before, one finds that $\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$ with

$$\Sigma^0(r) = \frac{1}{\pi}\begin{pmatrix} \frac{\pi 2^{2s-3}\Gamma(5-2s) r^{4-2s}}{\Gamma(3-s)^2} & 0 & \frac{2^{3-2s}\zeta(2s-2)(2^{3-2s}-3\sin 2r-5)}{r((2^{1-2s}-1)\sin 2r-1)} \\ 0 & -\frac{2^{6-2s}(2^{2-2s}-1)\zeta(2s-2)}{(2^{3-2s}-1)r\sin 2r+r} & 0 \\ \frac{2^{3-2s}\zeta(2s-2)(2^{3-2s}-3\sin 2r-5)}{r((2^{1-2s}-1)\sin 2r-1)} & 0 & \frac{2^{4-2s}(2^{-2s}-1)\zeta(2s)}{r((2^{1-2s}-1)\sin 2r-1)} \end{pmatrix}$$

and $\mathcal{R}_{ij} = \Sigma^0_{ij}(r) o(1)$. This readily leads to the expression

$$\mathcal{I}(r) \sim \frac{\left(2^{2s-\frac{1}{2}} r^{\frac{5}{2}-s}\right)}{\sqrt{\pi}\Gamma(3-s)\left((4^s-2)\sin(2r)+4^s\right)}\sqrt{\frac{(4^s-1)\,\Gamma(5-2s)}{\zeta(2s-2)\left(4^s-(4^s-8)\sin(2r)\right)}}\,,$$

which implies

$$\mathbb{E}N(\nabla u, R) \sim \frac{4\left(2^{2s-\frac{1}{2}}\right)}{\sqrt{\pi}\Gamma(3-s)}\sqrt{\frac{(4^s-1)\,\Gamma(5-2s)}{\zeta(2s-2)}}\times$$
$$\times\int_0^R \frac{r^{\frac{5}{2}-s}}{\left((4^s-2)\sin(2r)+4^s\right)\left(4^s-(4^s-8)\sin(2r)\right)}\,dr\,.$$

Applying Lemma 4.6 once again, one obtains the desired formula. □

4.8. The case $s = \frac{5}{2}$

The next lemma shows that at this regularity level, there is another transition in the asymptotic behavior of the expected number of critical points of u:

Lemma 4.16. *If $s = \frac{5}{2}$, $\mathbb{E}N(\nabla u, R) \sim \tilde{\kappa}_{\frac{5}{2}} R\sqrt{\log R}$ with*

$$\tilde{\kappa}_{\frac{5}{2}} := \frac{4}{\pi^2}\sqrt{\frac{31}{\zeta(3)}}\int_0^{\pi}\frac{dr}{(16+15\sin 2r)\sqrt{4-3\sin 2r}} \approx 0.497339\,.$$

Proof. Arguing as before, one find that $\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$ with

$$\Sigma^0(r) = \frac{1}{\pi}\begin{pmatrix} \frac{4\log r}{r} & 0 & \frac{\zeta(3)(12\sin 2r+19)}{r(15\sin 2r+16)} \\ 0 & \frac{7\zeta(3)}{4r-3r\sin 2r} & 0 \\ \frac{\zeta(3)(12\sin 2r+19)}{r(15\sin 2r+16)} & 0 & \frac{31\zeta(5)}{64r+60r\sin 2r} \end{pmatrix}$$

and $\mathcal{R}_{ij}(r) = \Sigma^0_{ij}(r)\, o(1)$. This eventually yields the asymptotic formula

$$\mathcal{I}(r) \sim \frac{2}{\pi^2}\sqrt{\frac{31}{\zeta(3)}}\frac{\sqrt{\log r}}{(16+15\sin 2r)\sqrt{4-3\sin 2r}}\,,$$

which implies

$$\mathbb{E}N(\nabla u, R) \sim \frac{4}{\pi}\sqrt{\frac{31}{\zeta(3)}}\int_0^R \frac{\sqrt{\log r}}{(16+15\sin 2r)\sqrt{4-3\sin 2r}}dr$$

by Lemmas 4.3 and 4.5. Lemma 4.6 then yields the desired asymptotic behavior. □

4.9. *The case* $\boldsymbol{s>\frac{5}{2}}$

In this regime, the proof goes as before, showing that the expected number of critical points contained in a large ball grows asymptotically like the radius. However, the explicit formulas one obtains for the proportionality constant are extremely cumbersome.

Lemma 4.17. *For $s > \frac{5}{2}$, there exists an explicit constant $\kappa(s) > 0$ such that*

$$\mathbb{E}N(\nabla u, R) \sim \kappa(s)R\,.$$

Proof. As in the previous cases, let us write $\Sigma(r) = \Sigma^0(r) + \mathcal{R}$ with $\mathcal{R}_{ij} = \Sigma^0(R)\, o(1)$ and

$$\Sigma^0(r) = \frac{1}{\pi r}$$
$$\times \begin{pmatrix} \Sigma_{11}(r) & 0 & \frac{2^{3-2s}\zeta(2s-2)(2^{3-2s}-3\sin 2r-5)}{(2^{1-2s}-1)\sin 2r-1} \\ 0 & -\frac{2^{6-2s}(2^{2-2s}-1)\zeta(2s-2)}{(2^{3-2s}-1)\sin 2r+1} & 0 \\ \frac{2^{3-2s}\zeta(2s-2)(2^{3-2s}-3\sin 2r-5)}{(2^{1-2s}-1)\sin 2r-1} & 0 & \frac{2^{4-2s}(2^{-2s}-1)\zeta(2s)}{(2^{1-2s}-1)\sin 2r-1} \end{pmatrix}.$$

Here

$$\Sigma_{11}(r) := 4\zeta(2s-4)\left(\left(2^{5-2s}-1\right)\sin 2r+1\right) + \frac{4\left(2^{3-2s}-1\right)^2\cos^2(2r)\zeta(2s-2)^2}{\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin 2r-1\right)}\,.$$

Note that all the nonzero matrix components are exactly of order $1/r$. While this fact does not make the problem any harder from a conceptual point of view, it leads to cumbersome expressions for the various quantities appearing in the equations.

Specifically, it is not hard to show that

$$\sigma(r) \sim -\frac{16\zeta(2s-2)\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin 2r-1\right)\left(\left(2^{3-2s}-1\right)\sin 2r+1\right)}{r^2}\,.$$

Plugging this formula in the expression for $I(r,z)$, one finds that

$$\mathcal{I}(r) \sim \int_{\mathbb{R}^3} |Az_1^2 + Bz_2^2 + 2Cz_1z_2| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz\,,$$

where the constants

$$\alpha := \frac{1}{\pi}\Big[\zeta(2s-2)\zeta(2s)[1+(1-2^{1-2s})\sin 2r][1+(2^{3-2s}-1)\sin 2r]\Big]^{-\frac{1}{2}}$$

$$A := \alpha 2^{-2s}\zeta(2s-2)\frac{5-2^{3-2s}+3\sin 2r}{1-(1-2^{1-2s})\sin 2r}\,,$$

$$B := \alpha 2^{3-2s}\zeta(2s-2)\frac{2^{2-2s}-1}{1+(2^{3-2s}-1)\sin 2r}\,,$$

$$\begin{aligned} C :=\ & \frac{\alpha 2^{-s-1}}{1+(1-2^{1-2s})\sin 2r}\times \\ & \times\Big[(1-2^{-2s})\zeta(2s-4)\zeta(2s)[1+(1-2^{1-2s})\sin 2r][1+(-1+2^{5-2s})\sin 2r] \\ & \quad + \zeta(2s-2)^2\big[-(1-2^{-2s})(1-2^{3-2s})^2\cos^2 2r - 2^{-2s}(2^{3-2s}-3\sin 2r-5)^5\big]^2\Big]^{\frac{1}{2}} \end{aligned}$$

are smooth functions of $\sin 2r$.

Lemma 4.7 then shows that

$$\mathcal{I}(r) \sim F(s, \sin 2r)$$

for some explicit smooth function of the form

$$F(s,\sin 2r) = \frac{2}{\pi}\int_0^\infty \frac{1-a(t,s,\sin 2r)\cos\frac{1}{2}\Phi(t,s,\sin 2r)}{t^2}\,dt\,.$$

Since

$$a(t,s,\sin 2r) = \Big[(1+4B^2t^2)\big[(1+4C^2t^2)^2+4A^2t^2\big]\Big]^{-\frac{1}{4}} < 1$$

for all r and all $t>0$, it stems that

$$F(s,\sin 2r) > 0\,.$$

Lemmas 4.3, 4.5 and 4.6 then ensure that

$$\mathbb{E}N(\nabla u, R) \sim \kappa(s)R$$

with

$$\kappa(s) := 2\int_0^{\pi} F(s, \sin 2r)\, dr\,. \quad \square$$

One can now read the asymptotic behavior of $\mathbb{E}N(\nabla u, R)$ in any regularity regime from the lemmas that we have established in this section. Theorem 1.1 is therefore proven.

5. Asymptotics for the number of critical points in the high regularity case

This section is devoted to the proof of Theorem 1.3. As all along this paper, we shall take the definition (4.1) for the Gaussian random function u.

5.1. Some non-probabilistic lemmas

Before presenting the proof of this theorem, we need to prove a few auxiliary results that do not use the fact that u and f are random functions. Specifically, these lemmas concern solutions to the Helmholtz equation on $\mathbb{R}^2$ of the form

$$v(x) := \int_{\mathbb{T}} e^{-ix\cdot E(\phi)}\, g(\phi)\, d\phi$$

where $g \in H^m(\mathbb{T})$ for a certain real m and the standard embedding $E : \mathbb{T} \to \mathbb{R}^2$ is given by (1.2).

We start by recalling the following result on the asymptotic behavior of v, which we proved in [10, Proposition 2.2 and Remark 3.2]. In what follows, we will denote the real and imaginary parts of a function g by g_{R} and g_{I}, respectively.

Lemma 5.1. *If $m > 9/2$, for $r \gg 1$ one has*

$$v = \left(\frac{8\pi}{r}\right)^{\frac{1}{2}} \left[g_{\mathrm{I}}(\theta)\, \sin(r - \tfrac{\pi}{4}) + g_{\mathrm{R}}(\theta)\, \cos(r - \tfrac{\pi}{4}) + \mathcal{R}_1\right],$$

$$\partial_r v = \left(\frac{8\pi}{r}\right)^{\frac{1}{2}} \left[g_{\mathrm{I}}(\theta)\, \cos(r - \tfrac{\pi}{4}) - g_{\mathrm{R}}(\theta)\, \sin(r - \tfrac{\pi}{4}) + \mathcal{R}_2\right],$$

$$\partial_\theta v = \left(\frac{8\pi}{r}\right)^{\frac{1}{2}} \left[g_{\mathrm{I}}'(\theta)\, \sin(r - \tfrac{\pi}{4}) + g_{\mathrm{R}}'(\theta)\, \cos(r - \tfrac{\pi}{4}) + \mathcal{R}_3\right],$$

where the errors are bounded as

$$|\mathcal{R}_1| + |\nabla\mathcal{R}_1| + |\nabla^2\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_3| \lesssim \frac{1}{r}\,.$$

The following theorem provides very precise asymptotic information about the critical points of v:

Lemma 5.2. *Assume that $m > 9/2$, that g does not vanish on $\mathbb{T}$, and that all the critical points of $|g|$ are non-degenerate. If ϕ^* is a critical point of $|g|$, then for each large enough positive integer n there exists a critical point (r_n^*, θ_n^*) of v such that*

$$|\phi^* - \theta_n^*| + \left|\pi n + \tfrac{\pi}{4} + \arg g(\phi^*) - r_n^*\right| \lesssim \frac{1}{n}\,.$$

Conversely, if (r^*, θ^*) is a critical point of v, there is some critical point ϕ^* of $|g|$ such that

$$|\phi^* - \theta^*| \lesssim \frac{1}{r^*}\,.$$

Proof. Let us consider the function

$$V := \operatorname{Re}\left[g(\theta)e^{-i(r-\frac{\pi}{4})}\right] = g_{\mathrm{I}}(\theta)\,\sin(r - \tfrac{\pi}{4}) + g_{\mathrm{R}}(\theta)\,\cos(r - \tfrac{\pi}{4})\,,$$

whose critical points (r^*, θ^*) are the solutions to the equations

$$\operatorname{Im}\left[g(\theta^*)e^{-i(r^*-\frac{\pi}{4})}\right] = 0\,, \qquad \operatorname{Re}\left[g'(\theta^*)e^{-i(r^*-\frac{\pi}{4})}\right] = 0\,.$$

Writing $g = |g|e^{i\arg g}$, an elementary calculation shows that (r^*, θ^*) is a critical point of V if and only if $r^* = \arg g(\theta^*) + \frac{\pi}{4} + \pi n$ for some integer n and $\operatorname{Re}[\overline{g(\theta^*)}g'(\theta^*)] = 0$. As g does not vanish on $\mathbb{T}$, the latter condition simply means that θ^* is a critical point of $|g|$. Furthermore, the Hessian of V at the critical points is

$$D^2V(r^*, \theta^*) = (-1)^n \begin{pmatrix} -|g(\theta^*)| & |g(\theta^*)|(\arg g)'(\theta^*) \\ |g(\theta^*)|(\arg g)'(\theta^*) & |g|''(\theta^*) - |g(\theta^*)|[(\arg g)'(\theta^*)]^2 \end{pmatrix}.$$

Therefore,

$$\det D^2V(r^*, \theta^*) = -|g(\theta^*)|\,|g|''(\theta^*) \neq 0 \tag{5.1}$$

because the critical points of $|g|$ are, by hypothesis, nondegenerate.

Let us now consider the function

$$F(r, \theta) := DV(r, \theta) - \left(\frac{r}{8\pi}\right)^{\frac{1}{2}} Dv(r, \theta)\,,$$

where $DV := (\partial_r V, \partial_\theta V)$. Lemma 5.1 ensures that

$$|F(r, \theta)| + |DF(r, \theta)| \lesssim \frac{1}{r}\,.$$

As the critical points of V are uniformly non-degenerate by (5.1), Thom's isotopy theorem (as stated, e.g., in [8]) ensures that v has a critical point at a distance at most C/n to each of the critical points (r^*, θ^*) of V as described above, provided that n is large enough. Furthermore, the asymptotic formulas for Dv presented in Lemma 5.1 guarantee that all critical points of v that are far enough from the origin must be of this form. The lemma is then proven. $\square$

5.2. Proof of Theorem 1.3

As $s > 5$, Proposition 2.2 ensures that $f \in H^{s'}(\mathbb{T})$ almost surely for some $s' > \frac{9}{2}$. Therefore, if one can prove that, with probability 1, f does not vanish on $\mathbb{T}$ and all the critical points of $|f|$ are nondegenerate, Theorem 1.3 will follow as an easy consequence of Lemma 5.2.

Proving the first part of this assertion is completely standard, but the second part is harder. In both cases, the proof relies on Bulinskaya's lemma, which one can state as follows [3, Proposition 6.11]:

Lemma 5.3 *(Bulinskaya). Let $Y : \mathbb{T} \to \mathbb{R}^2$ be a random function that is of class $C^1(\mathbb{T})$ almost surely. For each $\phi \in \mathbb{T}$, assume that the random variable $Y(\phi)$ has a probability density $\rho_{Y(\phi)} : \mathbb{R}^2 \to [0, \infty)$ that is bounded in some fixed neighborhood of the origin. Then*

$$\mathbb{P}\{Y(\phi) = 0 \ \text{ for some } \phi \in \mathbb{T}\} = 0\,.$$

Armed with Bulinskaya's lemma, it is easy to show that, almost surely, f does not vanish:

Lemma 5.4. *With probability 1, f does not vanish on $\mathbb{T}$.*

Proof. By the definition of u, cf. Equations (4.1) and (1.5), $\tilde{Y}(\phi) := (f_{\mathrm{R}}(\phi), f_{\mathrm{I}}(\phi))$ is a Gaussian random field $\tilde{Y} : \mathbb{T} \to \mathbb{R}^2$ with zero mean. The covariance of $\tilde{Y}(\phi)$ can be computed just as in Lemma 4.1, obtaining the nondegenerate matrix

$$\operatorname{Var} \tilde{Y}(\phi) = \mathbb{E}[\tilde{Y}(\phi) \otimes \tilde{Y}(\phi)] = \begin{pmatrix} \pi^{-2} \sum_{l>0,\text{even}} l^{-2s} & 0 \\ 0 & \pi^{-2} \sum_{l>0,\text{odd}} l^{-2s} \end{pmatrix} =: \Sigma\,.$$

Therefore, $\tilde{Y}(\phi)$ has a bounded probability density function

$$\rho_{\tilde{Y}(\phi)}(y) := \frac{\exp\left(-\frac{1}{2} y \cdot \Sigma^{-1} y\right)}{2\pi (\det \Sigma)^{1/2}}$$

on $\mathbb{R}^2$ because Σ is a nondegenerate matrix. Lemma 5.3 then ensures that $\tilde{Y}$ does not vanish with probability 1. As the zeros of $\tilde{Y}$ and f obviously coincide, the lemma follows. $\square$

The crux of the proof of Theorem 1.3 is to show that the critical points of $|f|$ are nondegenerate. This is not direct because $|f|$ is not a Gaussian variable, and showing that it has a bounded probability density requires some work. The main ingredient of the proof is the estimate we present in the following lemma. The proof is somewhat involved, so we have relegated it to the next subsection in order to streamline the presentation of the proof of Theorem 1.3. To state the auxiliary result, we will write points in $\mathbb{R}^6$ as

$$z = (z', z'') \in \mathbb{R}^4 \times \mathbb{R}^2$$

with $z' := (z_1, z_2, z_3, z_4)$ and $z'' := (z_5, z_6)$.

Lemma 5.5. *Consider the nonnegative rational function on $\mathbb{R}^6$ given by*

$$Q(z) := |z'|^2 + \frac{(z_5 - z_1 z_3)^2}{z_2^2} + \frac{[(z_5 - z_1 z_3)^2 + z_2^2(z_1 z_4 + z_3^2 - z_6)]^2}{z_2^6} \,. \tag{5.2}$$

For any constant $c > 0$,

$$\sup_{|z''|<\frac{1}{2}} \int_{\mathbb{R}^4} \frac{e^{-c\,Q(z)}}{z_2^2}\, dz' < \infty \,.$$

Assuming for the moment that this technical lemma holds, proving that the critical points of $|f|$ are nondegenerate almost surely is straightforward:

Lemma 5.6. *With probability 1, all the critical points of $|f|$ are nondegenerate.*

Proof. Let us start by noting that

$$|f|\,|f|' = \tfrac{1}{2}(|f|^2)' = \operatorname{Re} \overline{f}\, f' = f_{\mathrm{R}} f_{\mathrm{R}}' + f_{\mathrm{I}} f_{\mathrm{I}}' \,.$$

Differentiating this identity, we obtain

$$|f|\,|f|'' + (|f|')^2 = \operatorname{Re} \overline{f}\, f'' + |f'|^2 = f_{\mathrm{R}} f_{\mathrm{R}}'' + f_{\mathrm{I}} f_{\mathrm{I}}'' + (f_{\mathrm{R}}')^2 + (f_{\mathrm{I}}')^2 \,.$$

Therefore, all the critical points of $|f|$ are nondegenerate if and only if

$$Y := (f_{\mathrm{R}} f_{\mathrm{R}}' + f_{\mathrm{I}} f_{\mathrm{I}}', f_{\mathrm{R}} f_{\mathrm{R}}'' + f_{\mathrm{I}} f_{\mathrm{I}}'' + (f_{\mathrm{R}}')^2 + (f_{\mathrm{I}}')^2) : \mathbb{T} \to \mathbb{R}^2$$

does not vanish.

As $Y \in C^2(\mathbb{T})$ almost surely because $s > 5$, in order to apply Bulinskaya's lemma we only need to show that $Y(\phi)$ has a probability density that is bounded in a neighborhood of the origin. The random variable $Y(\phi)$ is obviously not Gaussian, so in order to compute its density we need to argue in an indirect way.

The starting point is the fact that the 2-jet of f,

$$Z := (f_{\mathrm{R}}, f_{\mathrm{I}}, f'_{\mathrm{R}}, f'_{\mathrm{I}}, f''_{\mathrm{R}}, f''_{\mathrm{I}}),$$

defines a Gaussian random variable $Z : \mathbb{T} \to \mathbb{R}^6$ with zero mean. Its variance

$$\operatorname{Var} Z(\phi) := \mathbb{E}[Z(\phi) \otimes Z(\phi)],$$

which does not depend on ϕ, can be computed from the definition

$$f(\phi) := \frac{1}{2\pi} \sum_{l \neq 0} i^l a_l |l|^{-s} e^{il\phi}$$

by arguing just as in the proof of Lemma 4.1. It turns out that $\operatorname{Var} Z(\phi) = \Sigma$, where Σ is the 6×6 matrix

$$\Sigma := \begin{pmatrix} a_0 & 0 & 0 & 0 & -b_0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & -b_1 \\ 0 & 0 & b_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & 0 & 0 \\ -b_0 & 0 & 0 & 0 & c_0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 & c_1 \end{pmatrix},$$

where

$$a_i := \pi^{-2} \sum_{m=0}^{\infty} \sigma_{i+2m}^2, \qquad b_i := \pi^{-2} \sum_{m=0}^{\infty} \sigma_{i+2m}^2 (i+2m)^2,$$
$$c_i := \pi^{-2} \sum_{m=0}^{\infty} \sigma_{i+2m}^2 (i+2m)^4$$

and we have set $\sigma_l := |l|^{-s}$ for $l \neq 0$ and $\sigma_0 := 0$. We have chosen to write this formula in terms of σ_l so that it is apparent that the result only uses the asymptotic properties of the sequence σ_l. Note that these sums are all convergent because $s > 5$.

The determinant of Σ is

$$\det \Sigma = b_0 b_1 \left(b_0^2 - a_0 c_0\right) \left(b_1^2 - a_1 c_1\right).$$

As $a_i c_i > b_i^2$ strictly by the Cauchy–Schwartz inequality, the matrix Σ is invertible. Therefore, for each $\phi \in \mathbb{T}$, the probability density distribution of $Z(\phi)$ is given by the Gaussian function

$$g(z) := (2\pi)^{-3} (\det \Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2} z \cdot \Sigma^{-1} z} \in C^\infty(\mathbb{R}^6).$$

Consider now the map $H : \mathbb{R}^6 \to \mathbb{R}^6$ given by

$$H(z) := \left(z_1, z_2, z_3, z_5, z_1 z_3 + z_2 z_4, z_1 z_5 + z_2 z_6 + z_3^2 + z_4^2\right) . \tag{5.3}$$

This map is invertible outside the hyperplane $\{z_2 = 0\}$, with inverse

$$H^{-1}(z) := \left(z_1, z_2, z_3, \frac{z_5 - z_1 z_3}{z_2}, z_4, -\frac{(z_5 - z_1 z_3)^2}{z_2^3} - \frac{z_1 z_4 + z_3^2 - z_6}{z_2}\right) ,$$

and its corresponding Jacobian determinant is $\det \nabla H^{-1}(z) = -z_2^{-2}$. Therefore, the probability density distribution of the random variable $H[Z(\phi)]$ is obtained by pulling back with the map H the probability distribution of $Z(\phi)$:

$$\rho_{H[Z(\phi)]}(z) = |\det \nabla H^{-1}(z)|\, g[H^{-1}(z)] = (2\pi)^{-3}(\det \Sigma)^{-\frac{1}{2}} z_2^{-2} e^{-Q_H(z)} . \tag{5.4}$$

with $Q_H(z) := \frac{1}{2} H^{-1}(z) \cdot \Sigma^{-1} H^{-1}(z)$.

Now let $\widetilde{H} : \mathbb{R}^6 \to \mathbb{R}^2$ denote the last two components of the map (5.3), that is,

$$\widetilde{H}(z) := \left(z_1 z_3 + z_2 z_4, z_1 z_5 + z_2 z_6 + z_3^2 + z_4^2\right) .$$

As the random variables $Y(\phi)$ and $Z(\phi)$ are related by

$$Y(\phi) = \widetilde{H}[Z(\phi)] ,$$

it then follows from (5.4) that the density of $Y(\phi)$ is given by the marginal distribution

$$\rho_{Y(\phi)}(z'') = \int_{\mathbb{R}^4} \rho_{H[Z(\phi)]}(z)\, dz' .$$

Now notice that the function $Q(z)$ defined in (5.2) is simply

$$Q(z) = |H^{-1}(z)|^2 .$$

As the matrix Σ is positive definite, therefore there is a positive constant $c > 0$ such that

$$\rho_{Y(\phi)}(z'') \lesssim \int_{\mathbb{R}^4} \frac{e^{-cQ(z)}}{z_2^2}\, dz' .$$

Lemma 5.5 then ensures that $\sup_{|z''|<\frac{1}{2}} \rho_{Y(\phi)}(z'') \lesssim 1$. Lemma 5.3 then guarantees that the random function Y does not vanish on $\mathbb{T}$ almost surely, and the theorem follows. □

Theorem 1.3 is then proven, modulo the proof of Lemma 5.5, which we will address next.

5.3. Proof of the main technical lemma

Let us now present the proof of Lemma 5.5. To make the exposition clearer, we will divide the proof in three steps.

5.3.1. The integral $\widetilde{I}$

The first step is to rewrite the integral

$$I := \int_{\mathbb{R}^4} \frac{e^{-cQ(z)}}{z_2^2} dz'$$

in a more convenient way. For this, let us set

$$\varrho := z_1 z_3 - z_5 \,, \qquad \tau := \frac{z_1 z_3 - z_5}{z_2} \,.$$

The map $z' \mapsto (\varrho, \tau, z_3, z_4)$ is invertible outside the hyperplane $z_3 = 0$ and the set $\tau = 0$. In terms of these variables, the integral reads as

$$I = \int_{\mathbb{R}^4} \frac{e^{-c\,Q_1}}{|\varrho z_3|} \, d\varrho \, d\tau \, dz_3 \, dz_4$$

with

$$\begin{aligned} Q_1 &:= Q_2 + z_4^2 \left[1 + \left(\frac{\tau(\varrho + z_5)}{\varrho z_3} \right)^2 \right] + 2 z_4 (\tau^2 + z_3^2 - z_6) \frac{\tau^2(\varrho + z_5)}{\varrho^2 z_3} \,, \\ Q_2 &:= z_3^2 + \tau^2 + \frac{\varrho^2}{\tau^2} + \left(\frac{\tau(\tau^2 + z_3^2 - z_6)}{\varrho} \right)^2 + \left(\frac{\varrho + z_5}{z_3} \right)^2 . \end{aligned} \tag{5.5}$$

As Q_1 is a second order polynomial in z_4, one can explicitly integrate in this variable, obtaining

$$I(z'') = \sqrt{\frac{\pi}{c}} \int_{\mathbb{R}^3} \frac{e^{-cQ_3}}{\sqrt{\varrho^2 z_3^2 + \tau^2(\varrho + z_5)^2}} \, d\varrho \, d\tau \, dz_3 \,,$$

with

$$Q_3 := z_3^2 + \tau^2 + \frac{\varrho^2}{\tau^2} + \left(\frac{\tau z_3 (\tau^2 + z_3^2 - z_6)}{(z_3^2 \varrho^2 + \tau^2(\varrho + z_5)^2)^{1/2}} \right)^2 + \left(\frac{\varrho + z_5}{z_3} \right)^2 .$$

Let us now consider polar coordinates $(\sigma, \alpha) \in \mathbb{R}^+ \times \mathbb{T}$, defined as

$$z_3 =: \sigma \cos\alpha \,, \qquad \tau =: \sigma \sin\alpha \,.$$

Still denoting by Q_2 the expression of (5.5) in these variables, and similarly with the other functions Q_j, we get

$$Q_2 = \frac{\varrho^2}{\sigma^2}\csc^2\alpha + \left(\frac{\varrho + z_5}{\sigma}\right)^2 \sec^2\alpha + \sigma^2 + \left(\frac{\sigma(\sigma^2 - z_6)\sin\alpha}{\varrho}\right)^2 .$$

This enables us to write

$$I = \sqrt{\frac{\pi}{c}} \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \frac{e^{-cQ_3}}{\sqrt{\varrho^2\cos^2\alpha + (\varrho + z_5)^2\sin^2\alpha}}\, d\sigma\, d\alpha\, d\varrho\,.$$

As $|z''| < \frac{1}{2}$, the denominator is nonzero for $|\varrho| > 1$, so one obviously has

$$\int_{\mathbb{R}\backslash[-1,1]}\int_0^{2\pi}\int_0^{\infty} \frac{e^{-cQ_3}}{\sqrt{\varrho^2\cos^2\alpha + (\varrho + z_5)^2\sin^2\alpha}}\, d\sigma\, d\alpha\, d\varrho \lesssim \int_{\mathbb{R}}\int_0^{\infty} e^{-c(\sigma^2 + \frac{\varrho^2}{\sigma^2})}\, d\sigma\, d\varrho \lesssim 1\,.$$

We can then write

$$I \lesssim 1 + \int_{-1}^{1}\int_0^{2\pi}\int_0^{\infty} \frac{e^{-cQ_3}}{\sqrt{\varrho^2\cos^2\alpha + (\varrho + z_5)^2\sin^2\alpha}}\, d\sigma\, d\alpha\, d\varrho =: 1 + \widetilde{I}\,. \tag{5.6}$$

5.3.2. The case $z_5 = 0$

Let us start by assuming that $z_5 = 0$, so that

$$\widetilde{I} = \int_{-1}^{1}\int_0^{2\pi}\int_0^{\infty} \frac{e^{-cQ_3}}{|\varrho|}\, d\sigma\, d\alpha\, d\varrho \leqslant 2\int_0^{1}\int_0^{2\pi}\int_{-\infty}^{\infty} \frac{e^{-c\sigma^2 - c\varrho^{-2}\sigma^2(\sigma^2 - z_6)^2\sin^2\alpha\cos^2\alpha}}{\varrho}\, d\sigma\, d\alpha\, d\varrho\,.$$

The integral in ϱ can be computed in terms of the incomplete Gamma function

$$\Gamma(\lambda, x) := \int_x^{\infty} t^{\lambda-1}e^{-t}\, dt\,,$$

obtaining

$$\widetilde{I} \leqslant \int_0^{2\pi}\int_{-\infty}^{\infty} e^{-c\sigma^2}\Gamma[0, c\sigma^2(\sigma^2 - z_6)^2\sin^2\alpha\cos^2\alpha]\, d\sigma\, d\alpha\,.$$

Then the bound

$$\Gamma(0, x) \lesssim \log\left(2 + \frac{1}{x}\right),$$

valid for all $x > 0$, immediately implies that

$$\sup_{|z_6|<\frac{1}{2}} \widetilde{I} \lesssim \int_0^{2\pi} \int_{-\infty}^{\infty} e^{-c\sigma^2} \log\left(2 + \frac{1}{c\sigma^2(\sigma^2-1/2)^2 \sin^2\alpha\cos^2\alpha}\right) d\sigma\, d\alpha \lesssim 1 \tag{5.7}$$

when $z_5 = 0$.

5.3.3. The case $z_5 \neq 0$

In view of the estimate (5.7), from now on, we shall assume that $z_5 \neq 0$. Let us now define the new variable $\tilde{\varrho} := -\varrho/z_5$, in terms of which the integral $\widetilde{I}$ reads as

$$\widetilde{I} \leqslant \int_{-1/|z_5|}^{1/|z_5|} \int_0^{2\pi} \int_0^{\infty} \frac{e^{-cQ_4}}{S(\tilde{\varrho},\alpha)}\, d\sigma\, d\alpha\, d\tilde{\varrho}\,.$$

Here we have used that

$$\sqrt{\varrho^2\cos^2\alpha + (\varrho+z_5)^2\sin^2\alpha} = |z_5|\, S(\tilde{\varrho},\alpha)$$

with

$$S(\tilde{\varrho},\alpha) := \sqrt{\tilde{\varrho}^2\cos^2\alpha + (\tilde{\varrho}-1)^2\sin^2\alpha}$$

and Q_4 is defined as

$$Q_4 := \sigma^2 + \frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2}\sin^2\alpha\cos^2\alpha.$$

Let us fix some small $\varepsilon > 0$ and define the sets

$$\mathcal{M}_0 := \{(\tilde{\varrho},\alpha) : |\tilde{\varrho}| < \varepsilon\,,\ |\sin\alpha| < \varepsilon\}\,, \qquad \mathcal{M}_1 := \{(\tilde{\varrho},\alpha) : |\tilde{\varrho}-1| < \varepsilon\,,\ |\cos\alpha| < \varepsilon\}\,.$$

Since $S(\tilde{\varrho},\alpha) \gtrsim 1$ for $(\tilde{\varrho},\alpha) \notin \mathcal{M}_0 \cup \mathcal{M}_1$ (not uniformly in ε), let us consider the set

$$\mathcal{M}_2 := \left(\left(-\frac{1}{|x_5|}, \frac{1}{|x_5|}\right) \times \mathbb{T}\right) \backslash (\mathcal{M}_0 \cup \mathcal{M}_1)$$

and split the above integral as

$$\widetilde{I} = \int_{\mathcal{M}_0}\int_0^{\infty} + \int_{\mathcal{M}_1}\int_0^{\infty} + \int_{\mathcal{M}_2}\int_0^{\infty} =: \widetilde{I}_0 + \widetilde{I}_1 + \widetilde{I}_2\,.$$

To estimate $\widetilde{I}_0$, observe that $\mathcal{M}_0$ consists of two connected components, which are contained in $|\tilde{\varrho}| < \varepsilon$ and either $|\alpha| < C\varepsilon$ or $|\alpha - \pi| < C\varepsilon$, respectively. It is easy to see that both contributions to the integral are of the same size, so we will just consider the first. To analyze it, let us use the bound

$$S(\tilde{\varrho}, \alpha) \gtrsim \sqrt{\tilde{\varrho}^2 + \alpha^2}\,,$$

which clearly holds for $(\tilde{\varrho}, \alpha) \in \mathcal{M}_0^+$, to write

$$\begin{aligned}\widetilde{I}_0 &\lesssim \int_{-\varepsilon}^{\varepsilon} \int_{-C\varepsilon}^{C\varepsilon} \int_0^{\infty} \frac{e^{-cQ_4}}{S(\tilde{\varrho}, \alpha)}\, d\sigma\, d\alpha\, d\tilde{\varrho} \\ &\lesssim \int_{-\varepsilon}^{\varepsilon} \int_{-C\varepsilon}^{C\varepsilon} \int_0^{\infty} \frac{e^{-c\sigma^2}}{\sqrt{\tilde{\varrho}^2 + \alpha^2}}\, d\sigma\, d\alpha\, d\tilde{\varrho}\,.\end{aligned}$$

Once can now introduce a new set of polar coordinates

$$\tilde{\varrho} =: r \cos\beta\,, \qquad \alpha =: r \sin\beta\,,$$

which yields

$$\widetilde{I}_0 \lesssim \int_0^{C\varepsilon} \int_0^{2\pi} \int_0^{\infty} e^{-c\sigma^2}\, d\sigma\, d\beta\, dr \lesssim 1\,.$$

An analogous argument for $\mathcal{M}_1$, where $|\tilde{\varrho} - 1| < \varepsilon$ and either $|\alpha - \frac{\pi}{2}| < C\varepsilon$ or $|\alpha - \frac{3\pi}{2}| < C\varepsilon$, shows that

$$\widetilde{I}_1 \lesssim 1\,.$$

It only remains to bound $\widetilde{I}_2$. As $S(\tilde{\varrho}, \alpha) \gtrsim \langle \tilde{\varrho} \rangle$ on $\mathcal{M}_2$, where $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$ is the Japanese bracket, we can write

$$\begin{aligned}\widetilde{I}_2 &\lesssim \int_{-1/|z_5|}^{1/|z_5|} \int_0^{2\pi} \int_0^{\infty} \frac{1}{\tilde{\varrho}} e^{-c\sigma^2 - c\frac{\sigma^2(\sigma^2 - z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2} \sin^2\alpha \cos^2\alpha}\, d\sigma\, d\alpha\, d\tilde{\varrho} \\ &= 4 \int_{-1/|z_5|}^{1/|z_5|} \int_0^{\pi/2} \int_0^{\infty} \frac{1}{\tilde{\varrho}} e^{-c\sigma^2 - c\frac{\sigma^2(\sigma^2 - z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2} \sin^2\alpha \cos^2\alpha}\, d\sigma\, d\alpha\, d\tilde{\varrho}\,.\end{aligned}$$

As $\cos^2\alpha \sin^2\alpha = \frac{1}{4}\sin^2(2\alpha)$ and $\sin\alpha \gtrsim \alpha$ for $|\alpha| < \frac{\pi}{2}$, the integral in α can be estimated as

$$\int_0^{\pi/2} e^{-c\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 S(\tilde\varrho,\alpha)^2}\sin^2\alpha\cos^2\alpha}\,d\alpha \leqslant \int_0^{\pi/2} e^{-C\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 \tilde S(\tilde\varrho)^2}\sin^2(2\alpha)}\,d\alpha$$

$$= 2\int_0^{\pi/4} e^{-C\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 \tilde S(\tilde\varrho)^2}\sin^2(2\alpha)}\,d\alpha \lesssim \left\langle \frac{\sigma(\sigma^2-z_6)}{z_5\tilde S(\tilde\varrho)}\right\rangle^{-1},$$

where $\tilde S(\tilde\varrho) := \tilde\varrho^2 + (1-\tilde\varrho)^2$. Here we have used that for $c>0$

$$\int_0^{\pi/4} e^{-c^2x^2}\,dx = \frac{\sqrt{\pi}\,\mathrm{Erf}\left(\frac{\pi c}{4}\right)}{2c} \lesssim \langle c\rangle^{-1},$$

where Erf is the error function. Since $|z_6| \leqslant \frac{1}{2}$, this yields

$$\widetilde{I}_2 \lesssim \int_{-1/|z_5|}^{1/|z_5|}\int_0^\infty \frac{e^{-c\sigma^2}}{\langle\tilde\varrho\rangle}\left\langle \frac{\sigma(\sigma^2-z_6)}{z_5\tilde S(\tilde\varrho)}\right\rangle^{-1} d\sigma\, d\tilde\varrho$$

$$= \int_{-1}^{1}\int_0^\infty \frac{e^{-c\sigma^2}}{(z_5^2+\varrho^2)^{\frac12}}\frac{|z_5\tilde S(\tilde\varrho)|}{(\varrho^2+(\varrho+z_5)^2+\sigma^2(\sigma^2-z_6)^2)^{1/2}}\,d\sigma\,d\varrho$$

$$\leqslant \int_{-1}^{1}\int_0^\infty \frac{e^{-c\sigma^2}}{(\varrho^2+(\varrho+z_5)^2+\sigma^2(\sigma^2-1/2)^2)^{1/2}}\,d\sigma\,d\varrho\,.$$

where we have used that if $z_5 = a\rho$

$$\frac{\left(z_5\tilde S(\tilde\varrho)\right)^2}{\rho^2+z_5^2} = \frac{\rho^2+(\rho+z_5)^2}{\rho^2+z_5^2} = \frac{a^2+2a+2}{a^2+1} < C$$

for some $C>0$ and for all $a\in\mathbb{R}$. To integrate in ϱ, we need that

$$\int_{-1}^{1}\frac{1}{\sqrt{((a+\rho)^2+\rho^2)+b}}\,d\rho$$
$$= \frac{1}{\sqrt2}\log\left(\frac{\left(\sqrt2\sqrt{(a-2)a+b+2}-a+2\right)\left(\sqrt2\sqrt{a(a+2)+b+2}+a+2\right)}{a^2+2b}\right).$$

Using that $|z_5| < \frac12$ we conclude

$$\widetilde{I}_2 \lesssim \int_0^\infty e^{-c\sigma^2}\log\left(\frac{\left(\sqrt2\sqrt{4\sigma^6-4\sigma^4+\sigma^2+13}+5\right)^2}{2\sigma^2\left(1-2\sigma^2\right)^2}\right)d\sigma$$

Thus, we obtain the bound

$$\widetilde{I}_2 \lesssim 1\,,$$

from the fact that the logarithmic singularities at $\sigma = 0$ and $\sigma = 1/\sqrt{2}$ are integrable. Lemma 5.5 in then proven.

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Appendix A. Monochromatic waves with many nondegenerate critical points

In this Appendix we aim to prove that there exist solutions to the Helmholtz equation

$$\Delta v + v = 0$$

on the plane with many isolated critical points. Specifically, let

$$N^*(\nabla v, R) := \big\{x \in B_R : \nabla v(x) = 0\,,\ \det \nabla^2 v(x) \neq 0\big\}$$

be the number of nondegenerate critical points of v contained in the ball of radius R. One can then prove the following:

Proposition A.1. *Given any continuous function $\rho : \mathbb{R}^2 :\to \mathbb{R}^+$, there exists a solution to the Helmholtz equation on $\mathbb{R}^2$ such that*

$$N^*(\nabla v, R) > \rho(R)$$

for all $R > 1$.

Proof. Without any loss of generality, let us assume that the function ρ is increasing. Take a set of distinct points $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$ without any accumulation points such that

$$\#\{k \in \mathbb{N} : x_k \in B_R\} > \rho(R + \tfrac{1}{2}) \tag{A.1}$$

for all $R > \frac{1}{8}$. At each point x_k, consider the number

$$r_k := \frac{1}{8} \min \left\{ 1, \inf_{j \in \mathbb{N} \setminus \{k\}} |x_k - x_j| \right\} ,$$

which is positive because the set $\{x_k\}_{k\in\mathbb{N}}$ does not have any accumulation points.

The function $v_k(x) := J_0(|x - x_k|)$ satisfies the Helmholtz equation on the plane and x_k is a nondegenerate maximum of v_k (in fact, $D^2 v_k(x_k) = -\frac{1}{2}I$). Therefore, the implicit function theorem ensures that there exists some $\varepsilon_k > 0$ such that any function v with $\|v_k - v\|_{C^2(B(x_k,2r_k))} < \varepsilon_k$ has a nondegenerate local maximum inside the ball $B(x_k, r_k)$. Notice that $B(x_k, 2r_k) \cap B(x_j, 2r_j) = \emptyset$ if $k \neq j$.

The better-than-uniform global approximation theorem for the Helmholtz equation [8, Lemma 7.2] ensures that there exists a solution v to the Helmholtz equation on $\mathbb{R}^2$ such that

$$\sup_{k\in\mathbb{N}} \frac{\|v_k - v\|_{C^2(B(x_k,2r_k))}}{\varepsilon_k} < 1 .$$

One then infers that v has a nondegenerate critical point in each disk $B(x_k, r_k)$. The property (A.1) then ensures that $N^*(\nabla v, R) > \rho(R)$ for all $R > 1$, as claimed. □

Remark A.2. The result and the proof remain valid in higher dimensions. The only modification is that, on $\mathbb{R}^n$, one must define $v_k(x) := |x - x_k|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x - x_k|)$.

Remark A.3. The function v may not be polynomially bounded at infinity, so v does not need to have a Fourier transform. In particular, it does not need to be the Fourier transform of a distribution supported on the unit sphere.

Appendix B. The translation-invariant case

In this Appendix we shall see why the evaluation of the Kac–Rice integral that gives the asymptotic behavior of $\mathbb{E}N(\nabla u, R)$ (cf. Lemma 4.3) is so much easier in the translation-invariant case (that is, when $s = 0$ following Remark 4.2).

In the translation-invariant case, it is easy to work directly in Cartesian coordinates, instead of using polar coordinates. This is because all one needs to know about u in order to apply the Kac–Rice formula are expectation values of the form $\mathbb{E}[\partial^\alpha u(x)\, \partial^\beta u(x)]$, where α, β are multiindices of order at most 2. These quantities can be computed exactly using that, as discussed in Remark 4.2, for $s = 0$ the covariance kernel is (up to a normalizing constant)

$$K(x, x') = J_0(|x - x'|) = \int_{\mathbb{T}} e^{i\xi\cdot(x-x')}\, d\sigma(\xi) . \tag{B.1}$$

Indeed, taking derivatives in this expression one finds that

$$\mathbb{E}[\partial^\alpha u(x)\,\partial^\beta u(x)] = i^{|\alpha|-|\beta|}\int_{\mathbb{T}} \xi^\alpha\,\xi^\beta\,d\sigma(\xi)\,.$$

The last integral can be computed in closed form because [12]

$$\int_{\mathbb{T}} \xi^\alpha\,d\sigma(\xi) = \begin{cases} \pi^{-1}\big[\prod_{j=1}^2 \Gamma(\frac{\alpha_j+1}{2})\big]/\Gamma(\frac{|\alpha|+2}{2}) & \text{if } \alpha_1,\alpha_2 \text{ are even,}\\ 0 & \text{otherwise.}\end{cases}$$

These formulas readily show that $\mathbb{E}[\partial_j u\,\partial_{kl}u] = 0$, so ∇u and $\nabla^2 u$ are independent Gaussian random functions, and that the covariance matrices of the first and second derivatives of u are

$$\operatorname{Var}\nabla u(x) = \frac{1}{2}I\,,\qquad \operatorname{Var}\nabla^2 u(x) = \frac{1}{8}\begin{pmatrix} 3 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 3\end{pmatrix}.$$

Again, we have regarded $\nabla^2 u$ as a 3-component vector. By the Kac–Rice formula, these expressions are enough to show

$$\mathbb{E}N(\nabla u, R) = \pi R^2 \int_{\mathbb{R}^3} \frac{\big|z_1^2 + 2\sqrt{2}z_1z_2 - z_2^2\big|}{8\pi}\,\frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}dz = \kappa(0)R^2 \tag{B.2}$$

as in Remark 4.10.

In polar coordinates, one sees essentially the same simplifications. The point is that it suffices to differentiate the addition formula

$$g(r,r',\theta) := J_0\big(\sqrt{r^2 + r'^2 - 2rr'\cos\theta}\big) = \sum_{l=0}^\infty \epsilon_l J_l(r)J_l(r')\cos l\theta\,,$$

where $\epsilon_l := 2-\delta_{l,0}$ is Neumann's factor, to compute in closed form all the sums appearing in the Kac–Rice formula (Lemma 4.3). Incidentally, the addition formula is equivalent to the assertion that the covariance matrix of u is (B.1), written in polar coordinates. For example,

$$\sum_{l=0}^\infty \epsilon_l J_l(r)^2 = g(r,r,0) = 1\,,$$

$$\sum_{l=0}^\infty \epsilon_l J_l'(r)^2 = \partial_r\partial_{r'}g(r,r,0) = \frac{1}{2}\,,$$

$$\sum_{l=0}^\infty \epsilon_l l^2 J_l(r)J_l'(r) = -\frac{1}{2}\partial_r\partial_\theta^2 g(r,r,0) = \frac{r}{4}\,,$$

$$\sum_{l=0}^{\infty} \epsilon_l l^4 J_l(r)^2 = \partial_\theta^4 g(r,r,0) = \frac{r^2(4+3r^2)}{8}\,.$$

These formulas are exact and easy to obtain, as one does not need to carry out the hard frequency analysis that constitutes the core of this paper. Of course, one can plug the values of these sums in Lemma 4.3 to readily recover the formula (B.2) for the expected number of critical points.

References

[1] G. Alessandrini, Critical points of solutions of elliptic equations in two variables, Ann. Sc. Norm. Super. Pisa 14 (1987) 229–256.

[2] G. Alessandrini, R. Magnanini, The index of isolated critical points and solutions of elliptic equations in the plane, Ann. Sc. Norm. Super. Pisa 19 (1992) 567–589.

[3] J.M. Azais, M. Wschebor, Level Sets and Extrema of Random Processes and Fields, Wiley, New York, 2009.

[4] D. Beliaev, V. Cammarota, I. Wigman, Two point function for critical points of a random plane wave, Int. Math. Res. Not. 9 (2019) 2661–2689.

[5] D. Beliaev, S. Muirhead, I. Wigman, No repulsion between critical points for planar Gaussian random fields, Electron. Commun. Probab. 25 (2020) 1–13.

[6] L. Buhovsky, A. Logunov, M. Sodin, Eigenfunctions with infinitely many isolated critical points, Int. Math. Res. Not. 24 (2020) 10100–10113.

[7] Y. Canzani, P. Sarnak, Topology and nesting of the zero set components of monochromatic random waves, Commun. Pure Appl. Math. 72 (2019) 343–374.

[8] A. Enciso, D. Peralta-Salas, Submanifolds that are level sets of solutions to a second-order elliptic PDE, Adv. Math. 249 (2013) 204–249.

[9] A. Enciso, D. Peralta-Salas, Topological aspects of critical points and level sets in elliptic PDEs, Lect. Notes Math. 2220 (2018) 89–119.

[10] A. Enciso, D. Peralta-Salas, A. Romaniega, Asymptotics for the nodal components of non-identically distributed monochromatic random waves, Int. Math. Res. Not. (2021), https://doi.org/10.1093/imrn/rnaa178, in press.

[11] A. Estrade, J. Fournier, Number of critical points of a Gaussian random field: condition for a finite variance, Stat. Probab. Lett. 118 (2016) 94–99.

[12] G.B. Folland, How to integrate a polynomial over a sphere, Am. Math. Mon. 108 (2001) 446–448.

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Reprint of the second edition, Springer, New York, 2015.

[14] D. Jakobson, N. Nadirashvili, Eigenfunctions with few critical points, J. Differ. Geom. 53 (1999) 177–182.

[15] W.V. Li, A. Wei, et al., Gaussian integrals involving absolute value functions, in: High Dimensional Probability V: the Luminy Volume, Institute of Mathematical Statistics, 2009, pp. 43–59.

[16] F. Nazarov, M. Sodin, On the number of nodal domains of random spherical harmonics, Am. J. Math. 131 (2009) 1337–1357.

[17] F. Nazarov, M. Sodin, Asymptotic laws for the spatial distribution and the number of connected components of zero sets of gaussian random functions, J. Math. Phys. Anal. Geom. 12 (2016) 205–278.

[18] F.W. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, NIST Handbook of Mathematical Functions, Cambridge Univ. Press, Cambridge, 2010.

[19] A. Rivera, Expected number of nodal components for cut-off fractional Gaussian fields, J. Lond. Math. Soc. 99 (2019) 629–652.

[20] P. Sarnak, I. Wigman, Topologies of nodal sets of random band-limited functions, Commun. Pure Appl. Math. 72 (2019) 275–342.

[21] J.L. Walsh, The Location of Critical Points of Analytic and Harmonic Functions, AMS, New York, 1950.

[22] S.T. Yau, Problem section, seminar on differential geometry, Ann. Math. Stud. 102 (1982) 669–706.

[23] S.T. Yau, Open Problems in Geometry, Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, 1993, pp. 1–28.