Available online at www.sciencedirect.com
ScienceDirect
PHYSICS LETTERS A
Physics Letters A 358 (2006) 345–353
www.elsevier.com/locate/pla

Ladder operators and recursion relations for the associated Bessel polynomials

H. Fakhri [a,b,*], A. Chenaghlou [a,c]

[a] Institute for Studies in Theoretical Physics and Mathematics (IPM), PO Box 19395-5531, Tehran, Iran
[b] Department of Theoretical Physics and Astrophysics, Physics Faculty, Tabriz University, PO Box 51666-16471, Tabriz, Iran
[c] Physics Department, Faculty of Science, Sahand University of Technology, PO Box 51335-1996, Tabriz, Iran

Received 25 January 2006; received in revised form 18 March 2006; accepted 19 May 2006

Available online 5 June 2006

Communicated by A.P. Fordy

Abstract

Introducing the associated Bessel polynomials in terms of two non-negative integers, and under an integrability condition we simultaneously factorize their corresponding differential equation into a product of the ladder operators by four different ways as shape invariance symmetry equations. This procedure gives four different pairs of recursion relations on the associated Bessel polynomials. In spite of description of Bessel and Laguerre polynomials in terms of each other, we show that the associated Bessel differential equation is factorized in four different ways whereas for Laguerre one we have three different ways.

PACS: 02.30.Hq; 02.30.Gp; 12.39.St; 03.65.Fd; 11.30.Pb

Keywords: Factorization methods; Shape invariance; Ordinary differential equations; Bessel polynomials; Special functions; Supersymmetry

1. Introduction

The shift operators corresponding to the square integrable functions have an important role in the development of some physical contexts as (para)supersymmetric quantum mechanics, coherent states and representation theory of Lie algebras. The hypergeometric-type polynomials and functions have found many applications in the quantum mechanical description of the physical and chemical systems. The associated (generalized) Bessel functions (polynomials) are known to arise naturally in a number of different contexts [1–7]. In Refs. [8–12], the raising and lowering operators and the second-order differential equation satisfied by the (orthogonal) hypergeometric-type polynomials have been discussed in different ways. Here, we follow the discussion for the associated Bessel functions on the basis of our approach for the associated (confluent) hypergeometric functions [13,14] and obtain the explicit forms of the proposed ladder operators.

Despite the fact that the Bessel and Laguerre polynomials are described in terms of each other [15], however, the associated differential equations corresponding to them lead to the different results when they are factorized. There are three different ways of factorization for the associated Laguerre differential equation (or three different types of ladder operators for the associated Laguerre functions [14]). In this Letter we show that there exist four different ways of factorization for the associated Bessel differential equation (or four different types of ladder operators for the associated Bessel functions). The mathematical reason for the mentioned point is the existence of a remarkable difference between two differential equations. Indeed, the coefficients of the terms involving second order derivatives (master functions [16]) are first and second order from x i.e. $A(x) = x$ and $A(x) = x^2$

* Corresponding author.
E-mail addresses: hfakhri@ipm.ir (H. Fakhri), a.chenaghlou@sut.ac.ir (A. Chenaghlou).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.05.070

for the Laguerre and Bessel, respectively. The associated Bessel functions are necessarily polynomial but the associated Laguerre functions are not such as them. Meanwhile, the realization of square integrability condition for the associated Laguerre functions does not impose any restriction on the non-negative integer n. However, in connection with the associated Bessel functions, the condition gives rise to a limitation on the parameter n. On the other hand, there is a similarity between the differential equations of the associated Bessel and hypergeometric functions, and it is the fact that both of the coefficients of the terms including the second order derivatives are second order from x i.e. $A(x) = x^2$ and $A(x) = 1 - x^2$ [13], respectively. The mentioned fact, in turn, leads to four different ways of factorization for both of the models. There are many physical problems that can be solved by using the associated Laguerre differential equation. This Letter shows that one can also consider those problems by using the associated Bessel differential equation and may obtain much more rich physical results.

Here, we introduce the associated Bessel functions in terms of two non-negative integers n and m such that their corresponding differential equation is factorized into a product of the ladder operators by four different ways as shape invariance symmetry equations for the indices (n, m) and $(n-1, m)$, (n, m) and $(n, m-1)$, $(n, m-1)$ and $(n-1, m)$ as well as (n, m) and $(n-1, m-1)$. It is shown that the shape invariance symmetry is realized by the ladder operators shifting only n, shifting only m, shifting indices n and m simultaneously and inversely, and shifting indices n and m simultaneously and agreeably. Meanwhile for every shape invariance symmetry, a pair of recursion relations on three associated Bessel functions is derived.

2. Realization of shape invariance equations with respect to n and m separately

Let us firstly consider a linear second order differential operator with given real parameters $\alpha < -2$ and $\beta > 0$ as

$$\mathcal{L}^{(\alpha,\beta)}(x) := x^{-\alpha} e^{\frac{\beta}{x}} \frac{d}{dx}\left(x^{\alpha+2} e^{\frac{-\beta}{x}} \frac{d}{dx}\right). \tag{1}$$

Lemma 1. *The operator $\mathcal{L}^{(\alpha,\beta)}(x)$ has the following properties*:

(a) *It is a self-adjoint operator with respect to an inner product with the weight function $x^\alpha e^{-\frac{\beta}{x}}$ in the interval $x \in (0, \infty)$.*

(b) *The action of the operator $\mathcal{L}^{(\alpha,\beta)}(x)$ on any polynomial of arbitrary degree is such that the degree of the polynomial is not increased.*

(c) *If we show the eigenfunctions of the operator $\mathcal{L}^{(\alpha,\beta)}(x)$ with $B_n^{(\alpha,\beta)}(x)$ as a polynomial (the so-called Bessel) exactly of degree n, then we can conclude its eigenvalue equation as follows* [17]

$$x^2 B_n''^{(\alpha,\beta)}(x) + \big[(\alpha+2)x + \beta\big] B_n'^{(\alpha,\beta)}(x) - n[\alpha + n + 1] B_n^{(\alpha,\beta)}(x) = 0, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Proof. The proof is straightforward. □

The differential equation (2) can be considered as a generalization of the differential equation corresponding to the orthogonal Bessel polynomials that has been introduced in Refs. [15,18].

Lemma 2. *The orthogonal Bessel polynomials as particular solutions of the differential equation (2) have a representation the so-called Rodrigues formula*:

$$B_n^{(\alpha,\beta)}(x) = \frac{a_n(\alpha,\beta)}{x^\alpha e^{\frac{-\beta}{x}}} \left(\frac{d}{dx}\right)^n \left(x^{\alpha+2n} e^{\frac{-\beta}{x}}\right), \tag{3}$$

where $a_n(\alpha,\beta)$'s are the normalization coefficients.

Proof. A complete proof can be seen in Ref. [15]. □

Now, we define a non-negative integer N for a given α as

$$N := \begin{cases} \left[\frac{-1}{2}(\alpha+1)\right] + 1 & \text{for } \frac{-1}{2}(\alpha+1) \notin \{1, 2, 3, \ldots\}, \\ \frac{-1}{2}(\alpha+1) & \text{for } \frac{-1}{2}(\alpha+1) \in \{1, 2, 3, \ldots\}, \end{cases} \tag{4}$$

where the symbol [ ] means the integer part. Assuming $n \in \{0, 1, 2, \ldots, N-1\}$ and by using the Rodrigues formula (3), one may calculate the coefficient of the highest power of x, i.e. x^n, for the Bessel functions $B_n^{(\alpha,\beta)}(x)$ as

$$B_n^{(\alpha,\beta)}(x) = a_n(\alpha,\beta)(-1)^n \frac{\Gamma(-\alpha-n)}{\Gamma(-\alpha-2n)} x^n + O\big(x^{n-1}\big). \tag{5}$$

Lemma 3. *The inner product of the orthogonal Bessel polynomials with respect to the weight function $x^\alpha e^{\frac{-\beta}{x}}$ in the interval $x \in (0, \infty)$ is computed as follows*

$$\int_0^\infty B_n^{(\alpha,\beta)}(x) B_{n'}^{(\alpha,\beta)}(x) x^\alpha e^{\frac{-\beta}{x}}\, dx = \delta_{nn'} h_n^2(\alpha, \beta), \quad n, n' \in \{0, 1, 2, \ldots, N-1\}, \tag{6}$$

where

$$h_n^2(\alpha, \beta) = a_n^2(\alpha, \beta) \frac{\Gamma(n+1)\Gamma(-\alpha-n)}{\beta^{-\alpha-2n-1}(-\alpha-2n-1)}. \tag{7}$$

Proof. This follows immediately from integration by parts. □

Lemma 4. *We have the following associated Bessel functions* (*the generalized Bessel polynomials*) *differential equation*

$$x^2 B_{n,m}''^{(\alpha,\beta)}(x) + \left[(\alpha+2)x + \beta\right] B_{n,m}'^{(\alpha,\beta)}(x) - \left[n(\alpha+n+1) + \frac{m\beta}{x}\right] B_{n,m}^{(\alpha,\beta)}(x) = 0, \quad 0 \leqslant m \leqslant n \leqslant N-1 \tag{8}$$

with the solutions as

$$B_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha,\beta)}{x^{\alpha+m} e^{\frac{-\beta}{x}}} \left(\frac{d}{dx}\right)^{n-m} \left(x^{\alpha+2n} e^{\frac{-\beta}{x}}\right), \tag{9}$$

where $a_{n,m}(\alpha, \beta)$'s are the normalization coefficients.

Proof. By differentiating the differential equation (2) m times we obtain a new differential equation similar to (2), but with new parameters $\alpha + 2m$ and $n - m$ instead of α and n, respectively. Thus for the obtained differential equation, we have a polynomial solution of degree $n - m$ as $B_{n-m}^{(\alpha+2m,\beta)}(x)$. Finally, it is easily seen that the associated Bessel functions

$$B_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha,\beta)}{a_{n-m}(\alpha+2m,\beta)} x^m B_{n-m}^{(\alpha+2m,\beta)}(x) \tag{10}$$

satisfy the differential equation (8). □

It is evident that by choosing $m = 0$, the associated Bessel differential Eq. (8) is reduced to the Bessel differential equation (2). Also, Eq. (10) clearly shows that the associated Bessel functions are polynomial, whereas according to Eq. (11) of Ref. [14] the associated Laguerre functions are not always polynomial.

Lemma 5. *We have*

$$\int_0^\infty B_{n,m}^{(\alpha,\beta)}(x) B_{n',m}^{(\alpha,\beta)}(x) x^\alpha e^{\frac{-\beta}{x}}\, dx = \delta_{nn'} h_{n,m}^2(\alpha, \beta), \quad n, n' \leqslant N-1,\ 0 \leqslant m \leqslant \min\{n, n'\}, \tag{11}$$

where

$$h_{n,m}^2(\alpha, \beta) = a_{n,m}^2(\alpha, \beta) \frac{\Gamma(n-m+1)\Gamma(-\alpha-n-m)}{\beta^{-\alpha-2n-1}(-\alpha-2n-1)}. \tag{12}$$

Proof. The proof follows by using the Lemma 3 and the formula (10). □

We shall determine the normalization coefficients by realizing the laddering equations. Before investigating the laddering equations, we obtain the shape invariance symmetry equations with respect to the parameter n as well as m.

Theorem 1. *The associated Bessel polynomials differential equation* (8) *is factorized into a product of first order differential operators as*:

(a) *Shape invariance symmetry equations* (*of first type*) *with respect to n, i.e. as equations* (n, m) *and* $(n-1, m)$

$$A_{n,m}^+ A_{n,m}^- B_{n,m}^{(\alpha,\beta)}(x) = E_{n,m} B_{n,m}^{(\alpha,\beta)}(x), \qquad A_{n,m}^- A_{n,m}^+ B_{n-1,m}^{(\alpha,\beta)}(x) = E_{n,m} B_{n-1,m}^{(\alpha,\beta)}(x), \tag{13}$$

with

$$A_{n,m}^+ = x^2 \frac{d}{dx} + (\alpha+n)x + \frac{(\alpha+n+m)\beta}{\alpha+2n}, \qquad A_{n,m}^- = -x^2 \frac{d}{dx} + nx - \frac{(n-m)\beta}{\alpha+2n}, \tag{14}$$

$$E_{n,m}=\frac{(n-m)(-\alpha-n-m)\beta^2}{(\alpha+2n)^2}. \tag{15}$$

(b) *Shape invariance symmetry equations* (*of second type*) *with respect to m, i.e. as equations* (n,m) *and* $(n,m-1)$

$$A_m^+A_m^-B_{n,m}^{(\alpha,\beta)}(x)=\mathcal{E}_{n,m}B_{n,m}^{(\alpha,\beta)}(x), \qquad A_m^-A_m^+B_{n,m-1}^{(\alpha,\beta)}(x)=\mathcal{E}_{n,m}B_{n,m-1}^{(\alpha,\beta)}(x), \tag{16}$$

with

$$A_m^+=x\frac{d}{dx}-m+1, \qquad A_m^-=-x\frac{d}{dx}-\frac{\beta}{x}-\alpha-m, \tag{17}$$

$$\mathcal{E}_{n,m}=(n-m+1)(-\alpha-n-m). \tag{18}$$

Proof. The proof can be made by means of a direct substitution of the explicit forms of $A_{n,m}^{\pm}$, $E_{n,m}$, $A_m^{\pm}$ and $\mathcal{E}_{n,m}$ in Eqs. (13) and (16), and converting them into the differential equation (8). □

Note that the operators A_m^+ and A_m^- ($A_{n,m}^+$ and $A_{n,m}^-$) are (not) adjoint of each other with respect to the inner product (6) with the weight function $x^\alpha e^{-\frac{\beta}{x}}$ in the interval $x\in(0,\infty)$. Now by using the shape invariance symmetry equations (13) and (16), we can obtain the raising and lowering relations of the indices n and m of the associated Bessel polynomials $B_{n,m}^{(\alpha,\beta)}(x)$.

3. Ladder operators for separate shift of *n* and *m* simultaneously

Clearly, the realization of the shape invariance symmetry equations (13) and (16) does not impose any constraint on the normalization coefficients $a_{n,m}(\alpha,\beta)$. However, the realization of the laddering equations with respect to n and m imposes separately recursion relations on the normalization coefficients with respect to n and m, respectively. These recursion relations determine that how function $a_{n,m}(\alpha,\beta)$ is from n and m.

Theorem 2. *For a given m, the raising and lowering relations of the index n,*

$$A_{n,m}^+B_{n-1,m}^{(\alpha,\beta)}(x)=\sqrt{E_{n,m}}\,B_{n,m}^{(\alpha,\beta)}(x), \tag{19a}$$

$$A_{n,m}^-B_{n,m}^{(\alpha,\beta)}(x)=\sqrt{E_{n,m}}\,B_{n-1,m}^{(\alpha,\beta)}(x), \tag{19b}$$

and for a given n, the raising and lowering relations of the index m,

$$A_m^+B_{n,m-1}^{(\alpha,\beta)}(x)=\sqrt{\mathcal{E}_{n,m}}\,B_{n,m}^{(\alpha,\beta)}(x), \tag{20a}$$

$$A_m^-B_{n,m}^{(\alpha,\beta)}(x)=\sqrt{\mathcal{E}_{n,m}}\,B_{n,m-1}^{(\alpha,\beta)}(x), \tag{20b}$$

are simultaneously established if the normalization coefficient $a_{n,m}(\alpha,\beta)$ *is chosen as*

$$a_{n,m}(\alpha,\beta)=\frac{(-1)^m}{\beta^n}\frac{C(\alpha,\beta)}{\sqrt{\Gamma(n-m+1)\Gamma(-\alpha-n-m)}}, \qquad 0\leqslant m\leqslant n\leqslant N-1, \tag{21}$$

where $C(\alpha,\beta)$ *is an arbitrary real constant independent of n and m. Therefore,* $A_{n,m}^{\pm}$ *and* $A_m^{\pm}$ *are the ladder operators on the indices n and m of the associated polynomials* $B_{n,m}^{(\alpha,\beta)}(x)$*, respectively.*

Proof. Using Eq. (10) in (19a) and applying Eq. (5) as well, one may compare the coefficients of the highest power of x, i.e. x^n, on the both sides then the following recursion relation with respect to the index n is obtained:

$$a_{n,m}(\alpha,\beta)=\frac{1}{\beta}\sqrt{\frac{-\alpha-n-m}{n-m}}\,a_{n-1,m}(\alpha,\beta), \qquad N>n>m\geqslant 0. \tag{22}$$

If we follow similar procedure in connection with Eq. (19b), then we will find out that the coefficient of the highest power of x, that is x^{n+1}, is identically zero on the both sides. Repeated application of the recursion relation (22) results in

$$a_{n,m}(\alpha,\beta)=\frac{1}{\beta^{n-m}}\sqrt{\frac{\Gamma(-\alpha-2m)}{\Gamma(n-m+1)\Gamma(-\alpha-n-m)}}\,a_{m,m}(\alpha,\beta), \qquad N>n\geqslant m\geqslant 0. \tag{23}$$

Moreover, using Eq. (10) in each of Eqs. (20a) and (20b) then applying (5), one may obtain the following recursion relation on the index m by means of comparing the coefficients of the highest power of x, i.e. x^n, on the both sides of them:

$$a_{n,m}(\alpha,\beta)=\frac{-1}{\sqrt{(n-m)(-\alpha-n-m-1)}}a_{n,m+1}(\alpha,\beta), \qquad N>n>m\geqslant 0. \tag{24}$$

The recursion relation (24) immediately gives

$$a_{n,m}(\alpha,\beta)=(-1)^{n-m}\sqrt{\frac{\Gamma(-\alpha-2n)}{\Gamma(n-m+1)\Gamma(-\alpha-n-m)}}\,a_{n,n}(\alpha,\beta),\qquad N>n\geqslant m\geqslant 0. \tag{25}$$

Comparing the results (23) and (25), it appears that

$$a_{n,n}(\alpha,\beta)=\frac{C(\alpha,\beta)}{(-\beta)^n\sqrt{\Gamma(-\alpha-2n)}},\qquad n=0,1,2,\dots,N-1. \tag{26}$$

Certainly, a relation similar to (26) is satisfied when $n=m$. Therefore by using each of Eqs. (23) and (25), the relation (21) for the normalization coefficients is obtained. Although in deriving the relation (21) we have not used (19b) however, one may notice the realization of the relation (19b) by means of using (21). This completes the proof. □

It is important to note that in (21) we have essentially derived the normalization coefficients in terms of n and m in such a way that the associated Bessel polynomials satisfy simultaneously the laddering relations with respect to the both indices n and m. Clearly, the laddering relations (19) and (20) are both finite since $m\leqslant n\leqslant N-1$. Such limitation was not imposed by an integer for the associated Laguerre functions (see [14]). The norm of the associated Bessel polynomials is determined by fixing the normalization coefficients as (21).

Corollary 1. *For given n and m, the norm of the associated Bessel polynomials $B_{n,m}^{(\alpha,\beta)}(x)$ is independent of m and is given by*

$$h_{n,m}(\alpha,\beta)=\frac{\beta^{\frac{\alpha+1}{2}}C(\alpha,\beta)}{\sqrt{-\alpha-2n-1}}. \tag{27}$$

Proof. It follows immediately by substituting (21) in (12). □

Corollary 2. *There are the following two algebraic solutions for the associated Bessel polynomials differential equation*:

$$B_{n,m}^{(\alpha,\beta)}(x)=\frac{A_{n,m}^{+}A_{n-1,m}^{+}\cdots A_{m+1,m}^{+}B_{m,m}^{(\alpha,\beta)}(x)}{\sqrt{E_{n,m}E_{n-1,m}\cdots E_{m+1,m}}},\qquad n\geqslant m+1, \tag{28}$$

$$B_{n,m}^{(\alpha,\beta)}(x)=\frac{A_{m+1}^{-}A_{m+2}^{-}\cdots A_{n}^{-}B_{n,n}^{(\alpha,\beta)}(x)}{\sqrt{\mathcal{E}_{n,m+1}\mathcal{E}_{n,m+2}\cdots\mathcal{E}_{n,n}}},\qquad m\leqslant n-1, \tag{29}$$

where

$$B_{m,m}^{(\alpha,\beta)}(x)=a_{m,m}(\alpha,\beta)x^m. \tag{30}$$

Proof. Considering $E_{m,m}=\mathcal{E}_{n,n+1}=0$, the following first order differential equations are obtained from Eqs. (19b) and (20a)

$$A_{m,m}^{-}B_{m,m}^{(\alpha,\beta)}(x)=0, \tag{31}$$

$$A_{n+1}^{+}B_{n,n}^{(\alpha,\beta)}(x)=0. \tag{32}$$

The solution of the differential equation (31) is (30) which is also the solution of the differential equation (32) if m is replaced by n. For given m and n, using the laddering relations (19a) and (20b) one may obtain the algebraic solutions (28) and (29) for the associated Bessel polynomials, respectively. □

Note that the algebraic solution (30) is consistent with the analytic solution (9).

Corollary 3. *There exist the following two independent recursion relations* (*of first type*) *on the index n for the associated Bessel polynomials*

$$\left[(\alpha+2n+1)x+\frac{(\alpha+n+m+1)\beta}{\alpha+2n+2}-\frac{(n-m)\beta}{\alpha+2n}\right]B_{n,m}^{(\alpha,\beta)}(x)=\sqrt{E_{n+1,m}}\,B_{n+1,m}^{(\alpha,\beta)}(x)+\sqrt{E_{n,m}}\,B_{n-1,m}^{(\alpha,\beta)}(x),$$

$$\left[2x^2\frac{d}{dx}+(\alpha+1)x+\frac{(\alpha+n+m+1)\beta}{\alpha+2n+2}+\frac{(n-m)\beta}{\alpha+2n}\right]B_{n,m}^{(\alpha,\beta)}(x)=\sqrt{E_{n+1,m}}\,B_{n+1,m}^{(\alpha,\beta)}(x)-\sqrt{E_{n,m}}\,B_{n-1,m}^{(\alpha,\beta)}(x). \tag{33}$$

Proof. In order to derive these recursion relations it is sufficient to change n to $n+1$ in Eq. (19a) then the obtained result must be added to and subtracted from (19b). □

Corollary 4. *There are the following two independent recursion relations (of second type) on the index m for the associated Bessel polynomials*

$$\left[\frac{-\beta}{x}-\alpha-2m\right]B_{n,m}^{(\alpha,\beta)}(x)=\sqrt{\mathcal{E}_{n,m+1}}\,B_{n,m+1}^{(\alpha,\beta)}(x)+\sqrt{\mathcal{E}_{n,m}}\,B_{n,m-1}^{(\alpha,\beta)}(x),$$
$$\left[2x\frac{d}{dx}+\frac{\beta}{x}+\alpha\right]B_{n,m}^{(\alpha,\beta)}(x)=\sqrt{\mathcal{E}_{n,m+1}}\,B_{n,m+1}^{(\alpha,\beta)}(x)-\sqrt{\mathcal{E}_{n,m}}\,B_{n,m-1}^{(\alpha,\beta)}(x). \tag{34}$$

Proof. The proof is quite similar to the proof of the Corollary 3. □

4. Ladder operators for simultaneous shift of *n* and *m* inversely and agreeably

The laddering equations (19) and (20), which shift n and m respectively, lead to the derivation of two new types of the factorization for the differential equation (8) as the shape invariance symmetry equations with the indices $(n,m-1)$ and $(n-1,m)$ as well as (n,m) and $(n-1,m-1)$. Each of these factorizations is realized by a pair of the ladder operators whose corresponding laddering equations shift both of the indices n and m simultaneously and inversely, and simultaneously and agreeably, respectively.

Theorem 3. *Let us define the following two new ladder operators*

$$A_{n,m}^{+,-}:=A_{n,m-1}^{+}A_{m}^{-}-A_{m}^{-}A_{n,m}^{+},\qquad A_{n,m}^{-,+}:=A_{m}^{+}A_{n,m-1}^{-}-A_{n,m}^{-}A_{m}^{+}. \tag{35}$$

(a) *They satisfy the raising and lowering relations with respect to n and m, simultaneously as*

$$A_{n,m}^{+,-}B_{n-1,m}^{(\alpha,\beta)}(x)=\frac{\beta\sqrt{(n-m)(n-m+1)}}{\alpha+2n}B_{n,m-1}^{(\alpha,\beta)}(x), \tag{36a}$$
$$A_{n,m}^{-,+}B_{n,m-1}^{(\alpha,\beta)}(x)=\frac{\beta\sqrt{(n-m)(n-m+1)}}{\alpha+2n}B_{n-1,m}^{(\alpha,\beta)}(x). \tag{36b}$$

Hence, the operator $A_{n,m}^{+,-}$ increases n and decreases m however, the operator $A_{n,m}^{-,+}$ decreases n and increases m.

(b) *They satisfy shape invariance symmetry equations (of third type) with respect to the indices n and m as equations $(n,m-1)$ and $(n-1,m)$:*

$$A_{n,m}^{+,-}A_{n,m}^{-,+}B_{n,m-1}^{(\alpha,\beta)}(x)=\frac{(n-m)(n-m+1)\beta^2}{(\alpha+2n)^2}B_{n,m-1}^{(\alpha,\beta)}(x),$$
$$A_{n,m}^{-,+}A_{n,m}^{+,-}B_{n-1,m}^{(\alpha,\beta)}(x)=\frac{(n-m)(n-m+1)\beta^2}{(\alpha+2n)^2}B_{n-1,m}^{(\alpha,\beta)}(x). \tag{37}$$

(c) *They are first order differential operators with the following explicit forms*

$$A_{n,m}^{+,-}=x\left[\frac{\beta}{\alpha+2n}+x\right]\frac{d}{dx}+(\alpha+n)x+\frac{\beta^2}{(\alpha+2n)x}+\frac{(2\alpha+2n+m)\beta}{\alpha+2n},$$
$$A_{n,m}^{-,+}=-x\left[\frac{\beta}{\alpha+2n}+x\right]\frac{d}{dx}+nx+\frac{(m-1)\beta}{\alpha+2n}. \tag{38}$$

Proof. The laddering relations (36) are proved by applying the laddering relations (19) and (20) in the definitions (35). The shape invariance symmetry equations (37) are established by the laddering relations (36). The explicit forms of the differential operators $A_{n,m}^{\pm,\mp}$ are obtained by the explicit forms of the differential operators $A_{n,m}^{\pm}$ and $A_{m}^{\pm}$ which are given by Eqs. (14) and (17), respectively. □

Corollary 5. *There are two independent recursion relations (of third type) among three associated Bessel polynomials as*

$$\left[\frac{-2\beta x}{(\alpha+2n)(\alpha+2n+2)}\frac{d}{dx}+(\alpha+2n+1)x+\frac{\beta^2}{(\alpha+2n+2)x}+\frac{(2\alpha+2n+m+2)\beta}{\alpha+2n+2}+\frac{(m-1)\beta}{\alpha+2n}\right]B_{n,m}^{(\alpha,\beta)}(x)$$
$$=\frac{\beta\sqrt{(n-m+1)(n-m+2)}}{\alpha+2n+2}B_{n+1,m-1}^{(\alpha,\beta)}(x)+\frac{\beta\sqrt{(n-m-1)(n-m)}}{\alpha+2n}B_{n-1,m+1}^{(\alpha,\beta)}(x),$$
$$\left[2x\left(x+\frac{(\alpha+2n+1)\beta}{(\alpha+2n)(\alpha+2n+2)}\right)\frac{d}{dx}+(\alpha+1)x+\frac{\beta^2}{(\alpha+2n+2)x}+\frac{(2\alpha+2n+m+2)\beta}{\alpha+2n+2}-\frac{(m-1)\beta}{\alpha+2n}\right]B_{n,m}^{(\alpha,\beta)}(x)$$
$$=\frac{\beta\sqrt{(n-m+1)(n-m+2)}}{\alpha+2n+2}B_{n+1,m-1}^{(\alpha,\beta)}(x)-\frac{\beta\sqrt{(n-m-1)(n-m)}}{\alpha+2n}B_{n-1,m+1}^{(\alpha,\beta)}(x). \tag{39}$$

Proof. To prove this Corollary one may change n and m to $n+1$ and $m+1$ in Eqs. (36a) and (36b), respectively, and then the obtained results should be added to and subtracted from each other. □

Theorem 4. *Let us define two new ladder operators as*

$$A^{+,+}_{n,m} := A^{+}_{m}A^{+}_{n,m-1} - A^{+}_{n,m}A^{+}_{m}, \qquad A^{-,-}_{n,m} := A^{-}_{n,m-1}A^{-}_{m} - A^{-}_{m}A^{-}_{n,m}. \tag{40}$$

(a) *They satisfy the raising and lowering relations with respect to n and m, simultaneously as*

$$A^{+,+}_{n,m}B^{(\alpha,\beta)}_{n-1,m-1}(x) = \frac{\beta\sqrt{(-\alpha-n-m)(-\alpha-n-m+1)}}{\alpha+2n}B^{(\alpha,\beta)}_{n,m}(x), \tag{41a}$$

$$A^{-,-}_{n,m}B^{(\alpha,\beta)}_{n,m}(x) = \frac{\beta\sqrt{(-\alpha-n-m)(-\alpha-n-m+1)}}{\alpha+2n}B^{(\alpha,\beta)}_{n-1,m-1}(x). \tag{41b}$$

So, the operator $A^{+,+}_{n,m}$ increases both of the indices n and m but the operator $A^{-,-}_{n,m}$ decreases both of them.

(b) *They satisfy shape invariance symmetry equations* (*of fourth type*) *with respect to the indices n and m as equations* (n,m) *and* $(n-1,m-1)$*:*

$$A^{+,+}_{n,m}A^{-,-}_{n,m}B^{(\alpha,\beta)}_{n,m}(x) = \frac{(-\alpha-n-m)(-\alpha-n-m+1)\beta^2}{(\alpha+2n)^2}B^{(\alpha,\beta)}_{n,m}(x),$$
$$A^{-,-}_{n,m}A^{+,+}_{n,m}B^{(\alpha,\beta)}_{n-1,m-1}(x) = \frac{(-\alpha-n-m)(-\alpha-n-m+1)\beta^2}{(\alpha+2n)^2}B^{(\alpha,\beta)}_{n-1,m-1}(x). \tag{42}$$

(c) *They are first order differential operators with the following explicit forms*

$$A^{+,+}_{n,m} = x\left[x - \frac{\beta}{\alpha+2n}\right]\frac{d}{dx} + (\alpha+2n)x + \frac{(m-1)\beta}{\alpha+2n},$$
$$A^{-,-}_{n,m} = -x\left[x - \frac{\beta}{\alpha+2n}\right]\frac{d}{dx} + nx + \frac{\beta^2}{\alpha+2n}\frac{1}{x} - \frac{(2n-m)\beta}{\alpha+2n}. \tag{43}$$

Proof. The proof is similar to the proof of the Theorem 3. □

Corollary 6. *There are two independent recursion relations* (*of third type*) *among three associated Bessel polynomials as*

$$\left[\frac{2\beta x}{(\alpha+2n)(\alpha+2n+2)}\frac{d}{dx} + (\alpha+3n+2)x + \frac{\beta^2}{(\alpha+2n)x} + \frac{m\beta}{\alpha+2n+2} - \frac{(2n-m)\beta}{\alpha+2n}\right]B^{(\alpha,\beta)}_{n,m}(x)$$
$$= \frac{\beta\sqrt{(-\alpha-n-m-2)(-\alpha-n-m-1)}}{\alpha+2n+2}B^{(\alpha,\beta)}_{n+1,m+1}(x) + \frac{\beta\sqrt{(-\alpha-n-m)(-\alpha-n-m+1)}}{\alpha+2n}B^{(\alpha,\beta)}_{n-1,m-1}(x),$$
$$\left[2x\left(x - \frac{(\alpha+2n+1)\beta}{(\alpha+2n)(\alpha+2n+2)}\right)\frac{d}{dx} + (\alpha+n+2)x - \frac{\beta^2}{(\alpha+2n)x} + \frac{m\beta}{\alpha+2n+2} + \frac{(2n-m)\beta}{\alpha+2n}\right]B^{(\alpha,\beta)}_{n,m}(x)$$
$$= \frac{\beta\sqrt{(-\alpha-n-m-2)(-\alpha-n-m-1)}}{\alpha+2n+2}B^{(\alpha,\beta)}_{n+1,m+1}(x) - \frac{\beta\sqrt{(-\alpha-n-m)(-\alpha-n-m+1)}}{\alpha+2n}B^{(\alpha,\beta)}_{n-1,m-1}(x). \tag{44}$$

Proof. In order to prove this Corollary it is sufficient to increase each of the indices n and m by one unit in Eq. (41a) then, the obtained result must be added to and subtracted from Eq. (41b). □

Note that each of Eqs. (13), (16), (37) and (42) is converted to the differential equation (8) after some manipulation. In fact, they are different types of the factorizations for (8) as the shape invariance symmetry equations.

5. Concluding remarks

It must be emphasized that the functional forms of the Bessel polynomials $B^{(\alpha,\beta)}_{n}(x)$ from the indices n and α are completely different and for this reason, the shifts of the indices are two different subjects. So, the relation (10) shows that the shifts of the indices n and m in the associated Bessel polynomials $B^{(\alpha,\beta)}_{n,m}(x)$ by four methods (19), (20), (36) and (41) are independent from each other. Consequently, four recursion relations (33), (34), (39) and (44) are also independent from each other. In Fig. 1 we have schematically shown four different types of the shifts of the indices n and m (or α) corresponding to the associated Bessel polynomials. In Ref. [13], for the associated hypergeometric differential equation with the parameters $a,b>-1$,

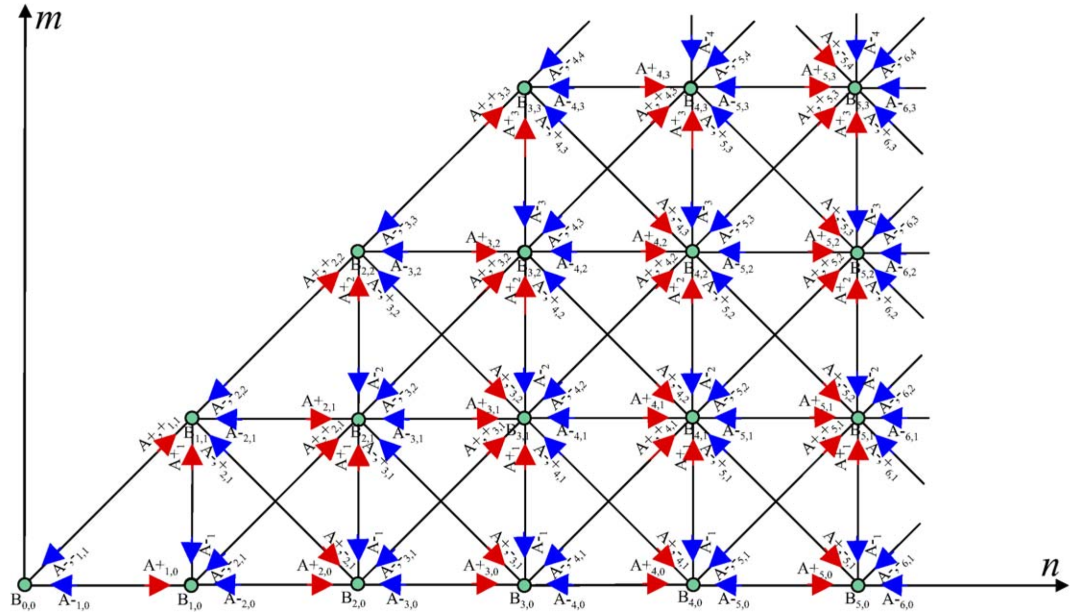

Fig. 1. The diagram of shift corresponding to the indices of the associated Bessel polynomials in four different ways by the ladder operators.

$$x(1-\omega x)F''^{(a,b)}_{n,m}(x)+\big[a+1-(a+b+2)\omega x\big]F'^{(a,b)}_{n,m}(x)$$
$$+\left[n\omega(a+b+n+1)+\frac{m[2(a-b)\omega x-(2a+m)]}{4x(1-\omega x)}\right]F^{(a,b)}_{n,m}(x)=0,\quad 0\leqslant m\leqslant n \tag{45}$$

in the interval $x\in(0,\frac{1}{\omega})$, two bunches of shape invariance equations with respect to the parameters n and m have been obtained as follow

$$A_+(n,m;x)A_-(n,m;x)F^{(a,b)}_{n,m}(x)=E(n,m)F^{(a,b)}_{n,m}(x),$$
$$A_-(n,m;x)A_+(n,m;x)F^{(a,b)}_{n-1,m}(x)=E(n,m)F^{(a,b)}_{n-1,m}(x), \tag{46}$$

$$A_+(m;x)A_-(m;x)F^{(a,b)}_{n,m}(x)=\mathcal{E}(n,m)F^{(a,b)}_{n,m}(x),$$
$$A_-(m;x)A_+(m;x)F^{(a,b)}_{n,m-1}(x)=\mathcal{E}(n,m)F^{(a,b)}_{n,m-1}(x), \tag{47}$$

where the explicit forms of the shift operators, $E(n,m)$ and $\mathcal{E}(n,m)$ are given by

$$A_+(n,m;x)=x(1-\omega x)\frac{d}{dx}-(a+b+n)\omega x+\frac{1}{2}(2a+n)-\frac{(n-m)(a-b)}{2(a+b+2n)},$$
$$A_-(n,m;x)=-x(1-\omega x)\frac{d}{dx}-n\omega x+\frac{n}{2}-\frac{(n-m)(a-b)}{2(a+b+2n)}, \tag{48}$$

$$A_+(m;x)=\sqrt{x(1-\omega x)}\frac{d}{dx}+\frac{(m-1)(2\omega x-1)}{2\sqrt{x(1-\omega x)}},$$
$$A_-(m;x)=-\sqrt{x(1-\omega x)}\frac{d}{dx}+\frac{2(a+b+m)\omega x-2a-m}{2\sqrt{x(1-\omega x)}}, \tag{49}$$

$$E(n,m)=\frac{(n-m)(a+n)(b+n)(a+b+n+m)}{(a+b+2n)^2}, \tag{50}$$
$$\mathcal{E}(n,m)=(n-m+1)(a+b+n+m)\omega. \tag{51}$$

Now, one may easily show that by setting $a=-\omega\beta-1$ and $b=\alpha+\omega\beta+1$, and then with taking the limit $\omega\to\infty$, the differential equation (45) is converted to (8). Moreover by considering the shape invariance equations (46) and (47), it appears that

$$\lim_{\omega\to\infty}\frac{-1}{\omega}A_\pm(n,m;x)=A^\pm_{n,m},\qquad \lim_{\omega\to\infty}\frac{1}{\omega^2}E(n,m)=E_{n,m},$$
$$\lim_{\omega\to\infty}\frac{-i}{\sqrt{\omega}}A_\pm(m;x)=A^\pm_m,\qquad \lim_{\omega\to\infty}\frac{-1}{\omega}\mathcal{E}(n,m)=\mathcal{E}_{n,m}. \tag{52}$$

Apart from the constant coefficients, the hypergeometric function $F_{n,m}^{(a,b)}(x)$ and the weight function $x^{\alpha}(1-\omega x)^{\beta}$ are reduced to the associated Bessel polynomials $B_{n,m}^{(\alpha,\beta)}(x)$ and the weight function $x^{\alpha}e^{-\frac{\beta}{x}}$, respectively. Eqs. (52) not only verify the calculations of this Letter but also show the necessity for doing the computations. Since, one has to calculate directly the normalization coefficients of the associated Bessel polynomials in such a way that Eqs. (19) and (20) are simultaneously realized.

Although, the Laguerre and Bessel polynomials are described in a well-known manner in terms of each other [15], however, a comparison between the results of this Letter and Ref. [14] shows that there still exist differences between the associated Laguerre functions and the associated Bessel functions. In fact, three different types of the ladder operators are extracted for the associated Laguerre functions [14], whereas for the associated Bessel functions (polynomials) there exist four different types of them. This fact leads to the extraction of an additional property of supersymmetry for the radial bound states of the hydrogen-like atoms [19], when we want to use the associated Bessel differential equation. Note that in solving the radial part of hydrogen-like atoms by the associated Bessel differential equation, $N-m$ plays the role of principal quantum number whereas in solving the problem via the associated Laguerre differential equation, n plays the same role. Useful discussions followed in Refs. [20,21] on the coherent states corresponding to one-dimensional Morse and two-dimensional Landau models can also be considered through the ladder operators introduced in this Letter.

References

[1] H.L. Krall, O. Frink, Trans. Am. Math. Soc. 65 (1949) 100.
[2] L. Infeld, T.E. Hull, Rev. Mod. Phys. 23 (1951) 21.
[3] G. Szegö, Orthogonal Polynomials, fourth ed., American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, Providence, RI, 1975.
[4] E. Grosswald, Bessel Polynomials, Lecture Notes in Mathematics, vol. 698, Springer-Verlag, Berlin, 1978.
[5] F. Galvez, J.S. Dehesa, J. Phys. A: Math. Gen. 17 (1984) 2759.
[6] H.M. Srivastava, Z. Angew. Math. Mech. 64 (1984) 255.
[7] S.D. Lin, I.C. Chen, H.M. Srivastava, Appl. Math. Comput. 137 (2003) 261.
[8] Y. Chen, M.E.H. Ismail, J. Phys. A: Math. Gen. 30 (1997) 7817.
[9] N. Cotfas, J. Phys. A: Math. Gen. 35 (2002) 9355.
[10] N. Cotfas, Cent. Eur. J. Phys. 2 (2004) 456.
[11] Y. Chen, M. Ismail, Proc. Amer. Math. Soc. 133 (2004) 465.
[12] E.L. Basor, Y. Chen, math.CA/0402362, Ramanujan J., in press.
[13] H. Fakhri, A. Chenaghlou, J. Phys. A: Math. Gen. 37 (2004) 3429.
[14] H. Fakhri, A. Chenaghlou, J. Phys. A: Math. Gen. 37 (2004) 7499.
[15] A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics: A Unified Introduction with Applications, Birkhäuser, Basel, 1988.
[16] M.A. Jafarizadeh, H. Fakhri, Ann. Phys. (N.Y.) 262 (1998) 260.
[17] W.D. Evans, W.N. Everitt, K.H. Kwon, L.L. Littlejohn, J. Comput. Appl. Math. 49 (1993) 51.
[18] S.S. Han, K.H. Kwon, Quaest. Math. 14 (1991) 327.
[19] A. Chenaghlou, H. Fakhri, Int. J. Quantum Chem. 101 (2005) 291.
[20] H. Fakhri, A. Chenaghlou, Phys. Lett. A 310 (2003) 1.
[21] H. Fakhri, Phys. Lett. A 313 (2003) 243.