

The Wigner-Moyal equation, which governs the time evolution of the Wigner function $W(x, p, t)$,

$$\frac{\partial W}{\partial t} + \frac{p}{m} \frac{\partial W}{\partial x} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\hbar}{i} \right)^n \frac{\partial^{2n} V(x)}{\partial x^{2n}} W = 0$$

In this equation:

- W is the Wigner function, a quasi-probability distribution function in phase space.
- t is time.
- x and p are the position and momentum in phase space, respectively.
- m is the mass of the particle.
- \hbar is the reduced Planck's constant.
- $V(x)$ is the potential energy as a function of position x .
- The series expansion accounts for the quantum corrections to the classical motion, with the first term being the classical motion.

For the one-dimensional quantum harmonic oscillator, a particularly insightful solution to the Wigner-Moyal equation is the coherent state, which is a minimum uncertainty state.

$$W(x, p) = \frac{1}{\pi \hbar} \exp \left(-\frac{2H}{\hbar \omega} \right)$$

where H is the Hamiltonian of the harmonic oscillator, expressed in terms of position x and momentum p :

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

Substituting the expression for H into the Wigner function, we obtain:

$$W(x, p) = \frac{1}{\pi \hbar} \exp \left(-\frac{p^2}{m \hbar \omega} - \frac{m \omega}{\hbar} x^2 \right)$$

Yes, the Wigner function for a state centered at an arbitrary complex number in phase space is given by:

A coherent state $|\alpha\rangle$ is often described by a complex number α , where α is the eigenvalue of the annihilation operator a :

$$W_{\alpha}(x, p) = \frac{1}{\pi \hbar} \exp \left(-\frac{m \omega}{\hbar} (x - x_0)^2 - \frac{p_0^2}{m \hbar \omega} \right)$$

In this expression, $x_0 = \sqrt{\frac{2\hbar}{m\omega}} \text{Re}(\alpha)$ and $p_0 = \sqrt{2\hbar m \omega} \text{Im}(\alpha)$.