attempting to preserve mathematical rigor sometimes prefer to write (2.191a) in a more complicated, but more rigorous form (e.g. as an equation including differentials or an integral equation).[51]

*An equation of the form (2.191a) was first proposed by the physicist Langevin in 1908 for describing the Brownian motion of a free particle. Therefore this equation is often called the *Langevin equation*. If a small particle is immersed in a fluid, then the surrounding medium acts on it in two ways. Firstly, the medium offers resistance to the motion of the particle, which takes the form of a frictional force equal to $-wY(t)$, where $Y(t)$ is the velocity of the particle, and the coefficient w equals $6\pi r\mu$ for a spherical particle of radius r immersed in a fluid of viscosity μ. Secondly, the fluctuations in the number of collisions of molecules of the fluid with the particle appear as an additional "purely random" force $X_1(t) = c_1E(t)$ having a constant spectral density $f(\omega) = f_0 = c_1^2/2\pi$ (cf. p. 117–120). Thus, the one-dimensional equation of motion along any fixed direction for a particle of mass m has the form (2.191a), where $Y(t)$ is the particle velocity, $a = w/m > 0$, and $cE(t) = X_1(t)/m = (c_1/m)E(t)$ is a new white noise with a spectral density $c^2/2\pi = f_0/m^2$.

The solution of the Langevin equation for a Brownian particle velocity $Y(t)$ was investigated by Uhlenbeck and Ornstein in 1930.[52] These authors showed in particular that according to this equation the velocity $Y(t)$ represents a stationary random process with the correlation function $B_{YY}(\tau) = \langle Y(t+\tau)Y(t)\rangle = (\pi f_0/wm)\exp[-(w/m)|\tau|]$ and the spectral density $f(\omega) = f_0/(m^2\omega^2 + w^2)$. These results clearly agree with (2.196). Moreover, the general laws of statistical physics imply that $m\langle Y^2\rangle/2 = kT/2$, where k is the Boltzmann constant and T is the absolute temperature. Hence $\pi f_0/w = kT$ and $f_0 = wkT/\pi$ – this result has already been mentioned on p. 120.

Note that according to the Ornstein–Uhlenbeck theory of Brownian motion the velocity $Y(t)$ of the Brownian particle exists (and has a definite probability distribution), but $Y(t)$ is nondifferentiable and hence the acceleration of the particle has no meaning. Therefore, to study the acceleration we have to use a more exact theory.

In the original Einstein–Smoluchowski theory (see, e.g. Note 10 to the Introduction) the inertia of the particle wa neglected, i.e. a limiting case of the Langevin equation, a $m \to 0$, was used. With this approximation, $wY(t) = X_1(t)$ = $c_1E(t)$ and $Y(t) = (c_1/w)E(t)$, i.e. $Y(t)$ is not an ordinar process but a white noise. Hence the Brownian particl does not even have a velocity according to this theor (Recall that the particle has a zero mass.) However, th integral of the velocity, i.e. the path $W(t)$ traversed by th particle, has real meaning, and is given by the secon equation (2.194a), where $X(s')$ is replaced by $Y(s)$ and c c_1/w. Bearing in mind that $c_1^2 = 2\pi f_0 = 2wkT$, we find, usin the first equation (2.195a), that for this approximation th mean square of the path length traversed by the particle i time τ is given by the formula

$$\langle [W(t+\tau) - W(t)]^2\rangle = (2kT/w)\tau.$$

This is the famous Einstein formula of the theory o Brownian motion. In particular, this formula immediatel implies that the particle cannot have a finite velocity, fo otherwise the mean value of the square of the pat traversed by the particle in a short time $\tau = \Delta t$ would hav to be proportional to $(\Delta t)^2$ and not to Δt. (cf. the relate remark concerning $Z(\omega)$ in Sec. 9). The refined form o the Einstein formula, which is valid for a Brownia particle of nonzero mass, will be given in Sec. 23.

The normality of Maxwell's distribution of molecula velocities implies that the "purely random" force $X_1($ acting on a Brownian particle is a normal (i.e. Gaussian white noise process. (This means that $X_1(t)$ must b regarded as a limit of the sequence of ordinary Gaussia processes. Another, more direct, definition of the Gaussia white noise process will be given in Sec. 24.) Thus, th process $W(t) = \int_0^t X_1(t')dt'$ is also Gaussian in this case. I the Einstein–Smoluchowski theory of Brownian motio this process represents, to within a constant factor, a pat traversed by a Brownian particle; in this relation it wa studied in detail by Wiener (see again Note 10 to th Introduction). Therefore, the Gaussian random proce $W(t)$, which has a mean value zero and a correlation functio