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ON THE NUMBER OF INTERSECTIONS OF A LEVEL BY A GAUSSIAN STOCHASTIC PROCESS, I

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(*Translated by R. H. Rodine*)

Questions conserning the intersections of a level by a random process have attracted attention for a comparatively long time. Among the earliest investigations we single out the works of S. O. Rice [1] and V. I. Bunimovich [2]. Obtaining the exact probability distributions of the number of intersections is, obviously, a very complicated problem. Easier to study is the limiting behavior of the number of intersections, when the height of the level tends to infinity [3], [4]. V. A. Volkonskii and Yu. A Rozanov [3] utilized the property of strong mixing which some Gaussian processes have, for limit theorems [5]. Another approach is possible, however, based on a study of the asymptotic behavior of the higher order moments of the number of intersections of a level by a Gaussian process. In the present work, an explicit expression for the factorial moments of the number of intersections of a level is obtained. Starting from this expression, it is possible to prove a series of limit theorems for some Gaussian processes, not having the strong mixing property. The first part of the work was done during a time of scientific study in Sweden. The author expresses appreciation to Professors H. Cramér and U. Grenander for attention to his work and helpful remarks.

Let ξ_t be a Gaussian random process, $\mathbf{E}\xi_t = 0$, $\mathbf{E}\xi_t\xi_s = r(t, s)$. We assume also that with probability 1 almost all sample functions of the process ξ_t are continuous. We say that the process ξ_t intersects the continuous functions a_t at the moments s, if $\xi_s = a_s$; it intersects a_t from below upwards if $\xi_s = a_s$ and for moments t, sufficiently close to s, $a_t > \xi_t$ for $t < s$, $a_t < \xi_t$ for $t > s$. Intersections from above downwards are defined in the obvious manner. We denote by $\eta_a(\Delta)$ the number of intersections from below upwards happening on the interval of time Δ. We denote the k-th moment of the number of intersections by

$$\alpha_k(\Delta; a) = \mathbf{E}[\eta_a(\Delta)]^k,$$

and the k-th factorial moment by

$$\alpha_{(k)}(\Delta; a) = \mathbf{E}\{\eta_a(\Delta)[\eta_a(\Delta)-1] \cdots [\eta_a(\Delta)-k+1]\}.$$

In the present work, explicit expressions are obtained for $\alpha_k(\Delta; a)$ and $\alpha_{(k)}(\Delta; a)$.

Theorem 1. *If the univariate density of the process ξ_t is bounded, and the derivative $\dot{\xi}_t$ is continuous with probability* 1, *then the number of intersections of the process ξ_t with a continuously differentiable function a_t is finite with probability* 1 *on any finite interval of time t. In addition, the probability that ξ_t is tangent to a_t, is equal to zero.*

The proof of the theorem follows easily from the results of E. V. Bulinskaya [6]. For this it is sufficient to apply the theorem of E. V. Bulinskaya for the number of zeros of the process $\eta_t = \xi_t - a_t$. From Theorem 1 it follows that for Gaussian processes, continuously differentiable with probability 1, the number of intersections from below upwards with a continuously differentiable function a_t is finite and that with probability 1 they occur at a positive angle, i.e., at the moment of intersection s, $\dot{\xi}_s > \dot{a}_s$. We note also that with probability 1 all intersections are intersections from below upwards or from above downwards.

Lemma 1. *If ξ is a random variable, having finite moment $\alpha_k = \mathbf{E}\xi^k$ and $\alpha_{(l)} = \mathbf{E}\xi(\xi-1)\cdots(\xi-l+1)$ as its l-th factorial moment, then*

$$\alpha_k = \sum_{l=1}^{k} c_{(k,l)}\alpha_{(l)}, \qquad \alpha_{(k)} = \sum_{l=1}^{k} c_{k,l}\alpha_l, \tag{1}$$

where the coefficients $c_{(k,l)}$, and $c_{k,l}$ are found from the recursion relations

$$\begin{aligned} c_{(k,l)} &= lc_{(k-1,l)}+c_{(k-1,l-1)}, \qquad c_{k,l} = (1-k)c_{k-1,l}+c_{k-1,l-1}, \\ c_{(k,l)} &= c_{k,l} = 0, \quad l > k, \quad l \leqq 0, \quad c_{(1,1)} = c_{(1,1)} = 1. \end{aligned} \tag{2}$$

PROOF. From the identity

$$\xi(\xi-1)\cdots(\xi-k+1) = \sum_{l=1}^{k} c_{k,l}\xi^l = \Big(\sum_{l=1}^{k-1} c_{k-1,l}\xi^l\Big)(\xi-k+1) \tag{3}$$

we obtain the right-hand recursion relation in formula (2). The left-hand recursion relation in (2) we find from the identity

$$\xi^k = \sum_{l=1}^{k} c_{(k,l)}\xi(\xi-1)\cdots(\xi-l+1) = \xi\Big[\sum_{l=1}^{k-1} c_{(k-1,l)}\xi\cdots(\xi-l+1)\Big]. \tag{4}$$

Equations (1) are obtained as a result of taking expectations in the right-hand and left-hand sides of the identities (3) and (4).

Let ξ_t be a Gaussian process, continuously differentiable with probability 1, and [1]

$$\frac{\partial^2 r(t,s)}{\partial t\,\partial s} \in H_\varepsilon, \qquad \varepsilon < 0.$$

It is possible to weaken the requirement that of $\delta^2 r(t,s)/\partial t\partial s$ belong to the class H_ε. In the stationary case, it is sufficient that

$$\left|\frac{\partial^2 r(t-s)}{\partial(t-s)^2} - r''(0)\right| < C\frac{|t-s|}{|\ln|t-s||^a}, \qquad a > 1.$$

Let $t_1, \cdots, t_k$ be fixed instants of time. We introduce the following notation for row vectors:

$$\boldsymbol{\xi}' = (\xi_{t_1}, \cdots, \xi_{t_k}), \quad \dot{\boldsymbol{\xi}}' = (\dot{\xi}_{t_1}, \cdots, \dot{\xi}_{t_k}), \quad \mathbf{x}' = (x_1, \cdots, x_k), \quad \mathbf{y}' = (y_1, \cdots, y_k);$$

correspondingly, $\boldsymbol{\xi}$, $\dot{\boldsymbol{\xi}}$, $\mathbf{x}$, $\mathbf{y}$ will be k-dimensional column vectors. In this notation, for example, $\boldsymbol{\xi}\boldsymbol{\xi}'$ is a square matrix with elements $a_{ij} = \xi_{t_i}\xi_{t_j}$. We introduce the matrices $\mathbf{R}_{11} = \mathbf{E}\boldsymbol{\xi}\boldsymbol{\xi}'$, $\mathbf{R}_{22} = \mathbf{E}\dot{\boldsymbol{\xi}}\dot{\boldsymbol{\xi}}'$, $\mathbf{R}_{21} = \mathbf{E}\dot{\boldsymbol{\xi}}\boldsymbol{\xi}'$; $\mathbf{R}$ and $\mathbf{R}^{-1}$ are mutually inverse matrices of the form

$$\mathbf{R} = \left\{\begin{array}{c:c} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \hdashline \mathbf{R}_{21} & \mathbf{R}_{22} \end{array}\right\}, \qquad \mathbf{R}^{-1} = \left\{\begin{array}{c:c} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \hdashline \mathbf{S}_{21} & \mathbf{S}_{22} \end{array}\right\}.$$

Everywhere below we shall assume that $\det \mathbf{R} = |\mathbf{R}| > 0$, $|\mathbf{R}_{11}| > 0$, $|\mathbf{R}_{22}| > 0$ for all sets $t_1, \cdots, t_k$, $t_i \neq t_j$. We note that this assumption is fulfilled for stationary Gaussian processes whose spectral function contains a continuous component. We denote by $p_{t_1}, \cdots, t_k(\mathbf{x}', \mathbf{y}')$ the joint density distribution of the values $\xi_{t_i} = x_i$, $\dot{\xi}_{t_i} = y_i$, $i = 1, 2, \cdots, k$. In this notation the following lemma, whose existence was indicated to the author by H. Cramér, holds.

Lemma 2. *The joint density distribution of $\boldsymbol{\xi}$ and $\dot{\boldsymbol{\xi}}$ can be written in the following form*

$$\begin{aligned} p_{t_1,\cdots,t_k}(\mathbf{x}', \mathbf{y}') &= (2\pi)^{-k/2}|\mathbf{R}_{11}|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\mathbf{x}'\mathbf{R}_{11}^{-1}\mathbf{x}\}(2\pi)^{-k/2}|\mathbf{B}_{22}|^{-\frac{1}{2}} \\ &\times \exp\{-\tfrac{1}{2}(\mathbf{y}-\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{x})'\mathbf{B}_{22}^{-1}(\mathbf{y}-\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{x})\}. \end{aligned} \tag{5}$$

[1] $f \in H_\varepsilon$ means that in each of its arguments, the function satisfies Hölder's condition with exponent $\varepsilon > 0$.

Thus, $\mathbf{R}_{21}\mathbf{R}_{11}^{-1}x$ is a vector of conditional expectations, and $\mathbf{B}_{22}$ is a conditional covariance matrix for the values of $\dot{\boldsymbol{\xi}}$ under the condition that $\xi_{t_i} = x_i$, $i = 1, \cdots, k$.

The proof follows from the identities

$$\mathbf{y}'\mathbf{S}_{22}\mathbf{y}+\mathbf{y}'\mathbf{S}_{21}\mathbf{x}+\mathbf{x}'\mathbf{S}_{21}\mathbf{y}+\mathbf{x}'\mathbf{S}_{11}\mathbf{x} = (\mathbf{y}-\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{x})'\mathbf{B}_{22}^{-1}(\mathbf{y}-\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{x}),$$
$$\mathbf{B}_{22} = \mathbf{R}_{22}-\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{R}_{12}.$$

These identities may be obtained by generalizing to the multidimensional case the well-known formulas for conditional distributions in the book [7].

Lemma 3. *If ξ_t is a continuously differentiable Gaussian process, $r''(t, s) \in H_\varepsilon$, and a_t is a continuously differentiable function, then for any moments of time $t_1, \cdots, t_k$, $t_i \neq t_j$,*

$$(6)\quad \lim_{h\to 0} h^{-k}\mathbf{P}\{\Gamma_{ht_1\cdots t_k}\} = \int_{\Gamma_{t_1\cdots t_k}} \prod_{i=1}^{k}(y_i-\dot{a}_{t_i})p_{t_1\cdots t_k}(a_{t_1},\cdots, a_{t_k}, y_1,\cdots, y_k)\,dy_1\cdots dy_k,$$

where the events

$$\Gamma_{ht_1\cdots t_k} = \{\xi_{t_i} < a_{t_i}, \xi_{t_i+h} > a_{t_i+h}, \quad i = 1, \cdots, k\},$$
$$\Gamma_{t_1\cdots t_k} = \{y_i > \dot{a}_{t_i}, \quad i = 1, \cdots, k\}.$$

Proof. If we observe that

$$\Gamma_{ht_1\cdots t_k} = \left\{a_{t_i} - \frac{\Delta_h(\xi_{t_i}-a_{t_i})}{h}h < \xi_{t_i} < a_{t_i};\ \frac{\Delta_h(\xi_{t_i}-a_{t_i})}{h} > 0, \quad i = 1, \cdots, k\right\},$$

and if we use the formula on total probability, we may write

$$I = \mathbf{P}(\Gamma_{ht_1\cdots t_k}) = \int_{A_h} \mathbf{P}\left\{a_{t_i} - \left(z_i - \frac{\Delta_h a_{t_i}}{h}\right)h < \xi_{t_i} < a_{t_i} \,\middle|\, \frac{\Delta_h \xi_{t_i}}{h} = z_i,\right.$$
$$(7)\qquad \left. i = 1, \cdots, k\right\} p_{ht_1\cdots t_k}(z_1, \cdots, z_k)\,dz_1, \cdots, dz_k,$$

where $A_h = \bigcap_{i=1}^{k}[z_i > \Delta_h a_{t_i}/h]$, and $p_{ht_1}, \cdots, {}_{t_k}(z_1, \cdots, z_k)$ is the density of the probability distribution of the random variables $\Delta_h \xi_{t_i}/h$. We divide the integral I, defined by equation (7), into two parts: $I = I_1+I_2$, where I_1 and I_2 are the integrals over the regions

$$B_h = \bigcap_{i=1}^{k}\left[\frac{\Delta_h a_{t_i}}{h} < z_i < h^{-\delta}\right], \qquad 0 < \delta < \min\left(\tfrac{1}{2}, \frac{\varepsilon}{2}\right),$$
$$A_h\bar{B}_h \subset \bigcup_{i=1}^{k}[z_i > h^{-\delta}],$$

respectively. By replacing the conditional probability under the integral sign in I_2 by unity, it can easily be shown that $I_2 = o(h^k)$ for any $k > 0$ and $h \to 0$. Uniformly for all values of $z_i < h^{-\delta}$, $i = 1, \cdots, k$, we have

$$(8)\quad \lim_{h\to 0} \frac{\mathbf{P}\left\{a_{t_i} - \left(z_i - \frac{\Delta_h a_{t_i}}{h}\right)h < \xi_{t_i} > a_{t_i} \,\middle|\, \frac{\Delta_h \xi_{t_i}}{h} = z_i, \quad i = 1, \cdots, k\right\}}{h^k \prod_{i=1}^{k}\left(z_i - \frac{\Delta_h a_{t_i}}{h}\right) p_{t_1\cdots t_k}(a_{t_1} \cdots a_{t_k}|z_1, \cdots, z_k)} = 1,$$

$$(9)\quad \lim_{h\to 0} p_{ht_1\cdots t_k}(z_1, \cdots, z_k) = \tilde{p}_{t_1\cdots t_k}(z_1, \cdots, z_k).$$

Here $\tilde{p}_{t_1}, \cdots, t_k\ (z_1, \cdots, z_k)$ is the probability density of the vector $\dot{\boldsymbol{\xi}}$. Using (8), (9) for an estimate of the limit $h^{-k}\mathbf{P}(\Gamma_{ht_1} \cdots t_k)$, we obtain the assertion of the lemma.

We note that formula (6) with the help of conditional expectations may be written in the following form:

$$(10)\quad \lim_{h\to 0} h^{-k}\mathbf{P}\{\Gamma_{ht_1\cdots t_k}\} = \mathbf{E}\left\{\prod_{i=1}^{k}(\xi_{t_i}-\dot a_{t_i})^{+}\,|\,\xi_{t_i}=a_{t_i},\qquad i=1,\cdots,k\right\}p_{t_1\cdots t_k}(a_{t_1},\cdots,a_{t_k}),$$

where $p_{t_1}\cdots_{t_k}(a_{t_1},\cdots,a_{t_k})$ is the joint density of the distribution of the variables $\xi_{t_1},\cdots,\xi_{t_k}$ at the points $\xi_{t_i}=a_{t_i}$, $i=1,\cdots,k$, and $x^+=x$, if $x>0$, and $x^+=0$ if $x\leqq 0$.

For brevity in writing, we introduce the following notation:

$$(11)\quad J_{(l)}(\Delta)=\int_{\{t_i\in\Delta,\,t_i\neq t_j,\,j=1,\cdots,l\}}\mathbf{E}\{\prod_{i=1}^{l}(\xi_{t_i}-\dot a_{t_i})^{+}\,|\,\xi_{t_j} = a_{t_j}, j=1,\cdots,l\}p_{t_1\cdots t_l}(a_{t_1}\cdots a_{t_l})dt_1\cdots dt_l.$$

Theorem 2. *The k-th factorial moment for the number of intersections $\eta_a(0,T)$ of a continuously differentiable function a_t with a random Gaussian process ξ_t is given by the equation*

$$(12)\qquad \alpha_{(k)}(0,T;a)=J_{(k)}(0,T),$$

where $J_{(k)}(0,T)$ is determined by formula (11).

Proof. As a corollary to Theorem 1, we get that the intersections from below upwards of the function a_t occur at the random moments $0\leqq t_1<\cdots<t_\eta\leqq T$, where $\eta=\eta_a(0,T)$ is a random variable, taking finite values with probability 1. Hence it follows that $\eta_a(0,T)$ may be considered as the number of points in a k-dimensional cube $L_k(T)=\{(x_1,\cdots,x_k)\}$, $0\leqq x_i\leqq T$, $i=1,\cdots,k$, with coordinates $(t_{\alpha_1},\cdots,t_{\alpha_k})$. The indices $\alpha_1,\cdots,\alpha_k$ take arbitrary values of combinations of integers from 1 to $\eta=\eta_a(0,T)$. We must particularly single out those points $(t_{\alpha_1},\cdots,t_{\alpha_k})\in L_k(T)$, for which some of the indices coincide. These coincidences correspond to the cases when the points are situated on the intersections of the cube $L_k(T)$ with the hyperplanes which are defined by equations of the form $x_{i_1}=\cdots=x_{i_l}$. The calculation of $[\eta_a(0,T)]^k$ may be carried out successively: first, count all points of the form $(t_{a_1},\cdots,t_{a_k})$ for which all indices α are distinct, next, those points for which two indices coincide, then three, and so on. Thus, we may write

$$(13)\qquad [\eta_a(0,T)]^k=\sum_{l=1}^{k}\sum_{i=1}^{c(k,l)}\eta_{i,l,k},$$

where $\eta_{i,l,k}$ is the number of points $(t_{\alpha_1},\cdots,t_{\alpha_\eta})$ for which exactly $k-(l-1)$ values of the indices $\alpha_{i_1}=\cdots=\alpha_{i_{k-l+1}}$ coincide; all these points lie on one and the same hyperplane, intersecting the cube $L_k(T)$. Successively increasing k, it may be shown that the numbers $c_{(k,i)}$ satisfy the recursion relation (2), which justifies the notation introduced for them. If we say that the $k-(l-1)$ coincident coordinates of the points, entering into $\eta_{i,l,k}$, represent one common coordinate, then we can consider them to be points with different coordinates, but now from the cube $L_{i,l,k}(T)$, having l dimensions. The probability characteristics of the distribution of points $(t_{\alpha_1},\cdots,t_{\alpha_l})$, $\alpha_r\neq\alpha_s$, in the cubes $L_{i,l,k}(T)$ are the same for all $i=1,\cdots,c_{(k,l)}$. We consider in the cube $L_{i,l,k}(T)$ a net $S_{n,\varepsilon}$ of points of the form $(Tm_1/2^n,\cdots,Tm_l/2^n)$, $0\leqq m_i\leqq 2^n$, $(m_i/2^n-m_j/2^n)T>\varepsilon$, $i,j=1,\cdots,l$. We introduce the random quantity $\eta_{i,l,k}(n,\varepsilon)$ equal to the number of points in the net $S_{n,\varepsilon}$ for which the inequalities

$$\xi_{Tm_i/2^n}<a_{Tm_i/2^n},\quad \xi_{T(m_i+1)/2^n}>a_{T(m_i+1)/2^n},\qquad i=1,\cdots,l,$$

hold. Since the intersections of the curve a_t occur with probability 1 at positive angles, we have as $n\to\infty$ and then $\varepsilon\to 0$, $\eta_{i,l,k}(n,\varepsilon)\uparrow\eta_{i,l,k}$. From formula (10) of Lemma 3 and the Lebesgue-Fatou theorem on monotone passage to the limit under the integral sign, we obtain

$$\mathbf{E}\eta_{i,l,k} = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\mathbf{E}\eta_{i,l,k}(n,\varepsilon) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{(t_1,\cdots,t_l)\in S_{n,\varepsilon}} \mathbf{P}(\Gamma_{ht_1\cdots t_l})$$

$$(14)\qquad = \int_{t_i\neq t_j;\,0\leqq t_i\leqq T,\,i,\,j=1,\cdots,l]} \mathbf{E}\{\prod_{i=1}^{l}(\xi_{t_i}-\dot{a}_{t_i})^+|\xi_{t_j} = a_{t_j},$$

$$j = 1,\cdots,l\}p_{t_1\cdots t_l}(a_{t_1},\ldots,a_{t_l})dt_1,\cdots,dt_l.$$

Taking the expectation of both parts of equation (13), we obtain, taking (14) into account,

$$(15)\qquad \alpha_k(0,T;a) = \sum_{l=1}^{k} c_{(k,l)}J_{(l)}(0,T).$$

Comparing (15) with formula (1) of Lemma 1, we get (12). The theorem is proved.

REMARK 1. As it is easy to see from the proofs of Lemma 3 and Theorem 2, formula (12) may be obtained for a class of processes broader than the Gaussian processes. However, the Gaussian processes are attractive from the point of view of the possibility of formulating theorems in terms of correlation or spectral functions.

REMARK 2. Formula (12) can be simplified somewhat in case intersections with a horizontal line $a_t \equiv a$ are considered. Here we have

$$(16)\qquad \alpha_{(k)}(0,T;a) = \int_{[t_i\neq t_j;\,0\leqq t_i\leqq T;\,I,\,j=k]} \mathbf{E}\{\prod_{i=1}^{k}\dot{\xi}_{t_i}^+|\xi_t = a,$$

$$j = 1,\cdots,k\}p_{t_1\cdots t_k}(a,\cdots,a)dt_1\cdots dt_k.$$

We consider now the problem of existence of moments of higher order for the number of intersections of a level. For simplicity of writing, we restrict ourselves to the case $a_t \equiv a$. In the following we shall need.

Lemma 4. *If $\xi_1,\cdots,\xi_k$ are Gaussian random variables with means $\mathbf{E}\xi_i = \mu_i$ and variances $D\xi_i = \sigma_i^2$, $i = 1,\cdots,k$, then the following inequality holds*:

$$(17)\qquad \mathbf{E}\xi_1^+\cdots\xi_k^+ \leqq \sum C_l|\mu_{i_1}|\cdots|\mu_{i_l}|\sigma_{j_1}\cdots\sigma_{j_{k-l}},$$

where C_l is some constant, and the summation is taken over all collections of numbers $1\leqq i_1 < i_2 < \cdots < i_l \leqq k$, $l = 1,\cdots,k$. The complete set of numbers $i_1,\cdots,i_l$, $j_1,\cdots,j_{k-l}$ coincides with the set $1, 2,\cdots,k$.

PROOF. Making use of the Bunyakovski-Schwarz inequality, we may write

$$(18)\qquad \mathbf{E}(\prod_{i=1}^{k}|\xi_i-\mu_i|) \leqq (\mathbf{E}|\xi_1-\mu_1|^2)^{\frac{1}{2}}(\mathbf{E}|\xi_2-\mu_2|^4)^{\frac{1}{4}}\mathbf{E}|\xi_k-\mu_k|^{2^k})^{2^{-k}}.$$

For any Gaussian random variable η, $\mathbf{E}\eta = 0$, $\mathbf{E}\eta^2 = \sigma^2$, the equation

$$(19)\qquad \mathbf{E}|\eta|^k = \frac{2^{k/2+1}}{\sqrt{2\pi}}\Gamma\left(\frac{k+1}{2}\right)\sigma^k$$

holds. Taking (18) and (19) into account, we obtain

$$\mathbf{E}\xi_1^+\cdots\xi_k^+ \leqq \mathbf{E}|\xi_1|\cdots|\xi_k| \leqq \mathbf{E}\prod_{i=1}^{k}(|\xi_i-\mu_i|+\mu_i)$$

$$= \sum\mu_{i_1}\cdots\mu_{i_l}\mathbf{E}\prod_{s=1}^{k-1}|\xi_{j_s}-\mu_{j_s}| \leqq \sum C_l\mu_{i_1}\cdots\mu_{i_l}\sigma_{j_1}\cdots\sigma_{j_{k-l}}.$$

According to Lemma 2 the following formulae may be written for conditional expectation and conditional variance:

$$(20)\quad \mu_i = \mathbf{E}\{\dot{\xi}_{t_i}|\xi_{t_j} = a; j = 1, \cdots, k\} = -\frac{\begin{vmatrix} \mathbf{R}_{11} & \begin{matrix} \cdots \\ \dfrac{\partial r(t_i, t_j)}{\partial t_i} \\ \cdots \end{matrix} \\ a \cdots a & 0 \end{vmatrix}}{|\mathbf{R}_{11}(t_1, \cdots, t_k)|} = -\frac{|\mathbf{R}_{\mu i}|}{|\mathbf{R}_{11}|},$$

$$(21)\quad \sigma_i^2 = \mathbf{D}\{\dot{\xi}_{t_i}|\xi_{t_j} = a; j = 1, \cdots, k\} = \frac{\begin{vmatrix} \mathbf{R}_{11} & \begin{matrix} \cdots \\ \dfrac{\partial r(t_i, t_j)}{\partial t_i} \\ \cdots \end{matrix} \\ \cdots \dfrac{\partial r(t_i, t_j)}{\partial t_i} \cdots & \dfrac{\partial^2 r(t_i, t_i)}{\partial t_i^2} \end{vmatrix}}{|\mathbf{R}_{11}|} = \frac{|\mathbf{R}_{\sigma i}|}{|\mathbf{R}_{11}|}.$$

We suppose further that there exist k derivatives $\dot{\xi}_t, \xi_t^{(2)}, \cdots, \xi_t^{(k)}$, which are mean square continuous. This corresponds to the assumption that the continuous derivatives

$$\frac{\partial^{2k} r(t, s)}{\partial t^l \partial s^m}, \qquad l+m = 2k, k > 1,$$

exist.

Theorem 3. *If a Gaussian process ξ_t is k times differentiable, then the k-th moment of the number of intersections from below upwards of the level a by the process ξ_t exists.*

Proof. We must prove that the integral $J_{(k)}(0, T)$, given by formula (11), is finite. The integration is carried out over the finite domain $0 \leqq t_i \leqq T, t_i \neq t_j, i, j = 1, \cdots, k$. Singularities may arise in connection with the convergence to zero of the determinant of the matrix $\mathbf{R}_{11}(t_1, \cdots, t_k) = \mathbf{E}\boldsymbol{\xi}\boldsymbol{\xi}'$ when $\min |t_i - t_j| \to 0$, where, as we recall, $\boldsymbol{\xi}' = (\xi_{t_1}, \cdots, \xi_{t_k})$. Since in formula (16) the probability density is equal to

$$p_{t_1, \cdots, t_k}(a, \cdots, a) = (2\pi)^{-k/2}(|\mathbf{R}_{11}(t_1, \cdots, t_k)|)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\mathbf{a}'\mathbf{R}_{11}^{-1}\mathbf{a}\},$$

it is sufficient to prove the boundedness of the ratios

$$(22)\qquad \frac{\mu_{i_1} \cdots \mu_{i_l} \sigma_{j_1} \cdots \sigma_{j_{k-l}}}{\sqrt{|\mathbf{R}_{11}(t_1, \cdots, t_k)|}},$$

when $\min |t_i - t_j| \to 0$, in view of inequality (17).

We introduce two systems of functions $f_l(t_i, \cdots, t_{i-l})$ and $f_l^*(t_i, \cdots, t_{i-l})$, $t_1 < t_2 < \cdots < t_k$. To do this we take an infinitely differentiable function $g(t)$ and we define f_l and f_l^* as the normalizing factors, ensuring the following convergence, for $t_i \to t_1$:

$$L_{t_i t_{i-1}}[g] = \frac{g(t_i) - g(t_{i-1})}{f_1(t_i, t_{i-1})} \to g'(t_0),$$

$$L^*_{t_i t_{i-1}}[g] = \frac{g'(t_i) - L_{t_i t_{i-1}}[g]}{f_1^*(t_i, t_{i-1})} \to g''(t_0),$$

$$L_{t_i t_{i-1} t_{i-2}}[g] = \frac{L_{t_i t_{i-1}}[g] - L_{t_{i-1} t_{i-2}}[g]}{f_2(t_i, t_{i-1}, t_{i-2})} \to g''(t_0),$$

$$L^*_{t_i t_{i-1} t_{i-2}}[g] = \frac{L^*_{t_i t_{i-1}}[g] - L_{t_i t_{i-1} t_{i-2}}[g]}{f_2^*(t_i, t_{i-1}, t_{i-2})} \to g'''(t_0),$$

and so on. All the functions f_i, f_i^* tend to zero when $t_s \to t_0$, $s = i, i-1, \ldots, 1$. In addition, it may be shown that $|f_i^*/f_i| < K_i$. We note that the form of the function $g(t)$ does not affect the choice of the system of functions f_i, f_i^*; this can easily be established by starting from the Taylor series expansion of $g(t)$.

As an example of the proof of the boundedness of the ratios (22) we consider the case when $\max_{1<i\leqq k}|t_i-t_1| \to 0$, i.e., when all the time coordinates approach each other. In the remaining cases, the proof of the boundedness can be carried out analogously.

Lemma 5. *If $t_i \to t_0$, $i = 1, \cdots, k$, then the relation*

$$\lim_{t_i\to t_0} \frac{|\mathbf{R}_{11}(t_1, \cdots, t_k)|}{\prod\limits_{l=1,\cdots,k-1;\, i=l+1,\cdots,k} f_l^2(t_i, \cdots, t_{i-l})} = |\mathbf{E}\eta\eta'| \tag{23}$$

holds, for the determinant of the correlation matrix $\mathbf{R}_{11}(t_1, \cdots, t_k) = \mathbf{E}\xi\xi'$, *where the vector* $\eta' = (\xi_t, \dot{\xi}_t, \cdots, \xi_t^{(k-1)})$.

Proof. We perform successively the following transformations on the determinant of the correlation matrix. As the first step, we subtract from each i-th column the $(i-1)$-st column and from each i-th row the $(i-1)$-st, $i = k, k-1, \cdots, 2$. In the resulting matrix, we divide the i-th row and i-th column by $f_1(t_i, t_{i-1})$ and denote the transformed matrix by $\mathbf{R}_{11}^{(1)}(t_1, \cdots, t_k) = ||a_{ij}^{(1)}||$,

$$a_{ij}^{(1)} = \mathbf{E}\left\{\frac{\xi_{t_i}-\xi_{t_{i-1}}}{f_1(t_i, t_{i-1})} \cdot \frac{\xi_{t_j}-\xi_{t_{j-1}}}{f_1(t_j, t_{j-1})}\right\}, \qquad i > 1, j > 1,$$

$$a_{ij}^{(1)} = a_{j1}^{(1)} = \mathbf{E}\left\{\xi_{t_1} \cdot \frac{\xi_{t_j}-\xi_{t_{j-1}}}{f_1(t_j, t_{j-1})}\right\}, \qquad j = 1, \cdots, k.$$

As a result of the first step, we obtain the relation

$$|\mathbf{R}_{11}(t_1, \cdots, t_k)| = |\mathbf{R}_{11}^{(1)}(t_1, \cdots, t_k)| \prod_{i=2}^{k} f_1^2(t_i, t_{i-1}).$$

As the second step, we again subtract, in the matrix $\mathbf{R}_{11}^{(1)}$, from the i-th column the $(i-1)$-st, $2 < i \leqq k$, and divide the resulting difference by $f_2(t_i, t_{i-1}, t_{i-2})$. We treat the rows in an analogous way, and so on. As a result of the $(k-1)$-st step, we arrive at the identity

$$|\mathbf{R}_{11}(t_1, \cdots, t_k)| = |\mathbf{R}_{11}^{(k-1)}(t_1, \cdots, t_k)| \prod_{l=1,\cdots,k-1;\, i=l+1,\cdots,k} f_l^2(t_i, \cdots, t_{i-l}), \tag{24}$$

where the elements of the matrix $\mathbf{R}_{11}^{(k-1)} = ||a_{rs}^{(k-1)}||$ have the form

$$a_{rs}^{(k-1)} = \mathbf{E}(L_{t_s \cdots t_1}[\xi_t])(L_{t_r \cdots t_1}[\xi_t]).$$

If $t_i \to t_0$, then $a_{rs}^{(k-1)} \to \mathbf{E}\xi_{t_0}^{(r-1)}\xi_{t_0}^{(s-1)}$; correspondingly,

$$|\mathbf{R}_{11}^{(k-1)}(t_1, \cdots, t_k)| \to |\mathbf{E}\eta\eta'|. \tag{25}$$

The assertion (23) of the lemma is a consequence of the relations (14) and (25).

We evaluate the asymptotic behavior of the determinants of the matrices $\mathbf{R}_{\mu i}(t_1, \cdots, t_k)$ and $\mathbf{R}_{\sigma i}(t_1, \cdots, t_k)$ in an analogous way, beginning with formulae (20) and (21). Here we have

Lemma 6. *If $t_i \to t_0$, $i = 1, \cdots, k$, then the relations*

$$\lim_{t_i\to t_0} \frac{|\mathbf{R}_{\sigma i}(t_1, \cdots, t_k)|}{|\mathbf{R}_{11}(t_1, \cdots, t_k)| \prod\limits_{l=1}^{i-1} f_l^{*2}(t_i, \cdots, t_{i-l})} = \frac{|\mathbf{E}\zeta\zeta'|}{|\mathbf{E}\eta\eta'|}, \tag{26}$$

$$\lim_{t_1\to t_0} \frac{|\mathbf{R}_{\mu i}(t_1, \cdots, t_k)|}{|\mathbf{R}_{11}(t_1, \cdots, t_k)| \prod\limits_{l=1}^{i-1} f_l^{*}(t_i, \cdots, t_{i-l})} = \frac{|\mathbf{E}\zeta_i\zeta_i'|}{|\mathbf{E}\eta\eta'|} \tag{27}$$

hold for the determinants $\mathbf{R}_{\sigma i}$ *and* $\mathbf{R}_{\mu i}$, *where*

$$\eta' = (\xi_{t_0}, \dot{\xi}_{t_0}, \cdots, \xi_{t_0}^{(k-1)}), \quad \zeta' = (\xi_{t_0}, \dot{\xi}_{t_0}, \cdots, \xi_{t_0}^{(k-1)}, \xi_{t_0}^{(k)}),$$

$$\zeta_i' = (\xi_{t_0}, \dot{\xi}_{t_0}, \cdots, \xi_{t_0}^{(k-1)}, \xi_{t_0}^{(i)}).$$

PROOF. The proofs of both of the formulae (26) and (27) are carried out in a way similar to the one used in deducing relation (23). First, we subtract from the last row and the last column of the matrix $\mathbf{R}_{\sigma i}$ the difference between the i-th and $(i-1)$-st columns and rows, divided by $f_1(t_i, t_{i-1})$. We divide by $f_1^*(t_i, t_{i-1})$ the expressions obtained in the last $(k+1)$-st row and column. The second step cousists in subtracting from the $(k+1)$-st column and the $(k+1)$-st row the linear combination of the i-th, $(i-1)$-st, and $(i-2)$-nd columns formed in correspondence with the operator $L_{t_i t_{i-1} t_{i-2}}$. We divide the resulting differences in the $(k+1)$-st column by $f_2(t_i, t_{i-1}, t_{i-2})$, and so on, until the linear combination corresponding to the operator $L_{t_1} \cdots {}_{t_i}$ is used up. The columns and rows with numbers $i = 1, \cdots, k$, generating the minor of $\mathbf{R}_{11}(t_i, \cdots, t_k)$, which remain unchanged will successively undergo the same operations as in the proof of Lemma 5. First we subtract from the k-th column and k-th row the $(k-1)$-st columb and $(k-1)$-st row and next we divide the resulting differences by $f_1(t_k, t_{k-1})$, and so on. Altogether, we arrive at the relation

$$(28) \quad |\mathbf{R}_{\sigma i}(t_1, \cdots, t_k)| = |\mathbf{R}_{\sigma i}'(t_1, \cdots, t_k)| \prod_{r=1,\cdots,k-1;\, s=r+1,\cdots,k} f_r(t_s, \cdots, t_{s-}\ ) \prod_{l=1}^{i-1} f_l^*(t_i, \cdots, t_{i-l}).$$

The matrix $|\mathbf{R}_{\sigma i}'(t_1, \ldots, t_k)| = ||a_{rs}'||$has elements of the form

$$a_{rs}' = \mathbf{E}(L_{t_r t_{r-1} \cdots t_1}[\xi_t])(L_{t_s \cdots t_1}[\xi_t]), \qquad r, s = 1, \cdots, k,$$

$$a_{k+1,r}' = a_{r,k+1}' = \mathbf{E}(L_{t_r, \cdots t_1}[\xi_t])(L_{t_i \cdots t_1}^*[\xi_t]), \qquad r = 1, \cdots, k,$$

$$a_{k+1,k+1}' = \mathbf{E}(L_{t_i \cdots t_1}^*[\xi_t])(L_{t_i \cdots t_1}^*[\xi_t]).$$

From equation (28) and the differentiability of the process ξ_t we obtain the relation (26). The relation (27) for the matrices $\mathbf{R}_{\mu i}$ may be derived by analogous means.

From Lemmas 5 and 6 we obtain the boundedness of the ratios (22) and, correspondingly, in view of Lemma 4, the boundedness of the integrand expressions in formula (16). Theorem 3 is proved.

We note that the assertion of Theorem 3 may be sharpened somewhat if we admit that the expression under the integral sign goes to infinite provided that the resulting singularities are integrable.

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