MINIMAL RECURRENCE FORMULAS FOR ORTHOGONAL POLYNOMIALS ON BERNOULLI'S LEMNISCATE

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INTRODUCTION.

The study of the recurrence formulas as a method of generating orthogonal polynomial sequences associated with a m-distribution function, defined on a curve of the complex plane, began in [5] as an algebraic alternative to the classical asymptotical results (see [9] and [10]) in the case of Jordan curves and analytic arcs. In this paper is presented the election of a family of parameters by which an inner product relative to the Bernouilli's lemniscate can be generated as well as the classification of the "short" recurrence formulas, which verify the associated orthogonal plynomials, keeping in mind the algebraic properties of such parameter sequence. Finally, results related to the asymptotical behavior of the quotient $\hat{P}_n(z)/\hat{P}_{n-2}(z)$ outside of the Bernouilli's lemniscate, are also obtained, where $\{\hat{P}_n(z)\}$ is the sequence of orthonormal polynomials associated with a particular m-distribution function on such a curve.

1. ORTHOGONAL POLYNOMIALS ASSOCIATED WITH A DOUBLE FAMILY OF PARAMETERS.

It is well known (see [1] and [6]) that the elements of the matrix $(c_{kj})_{k,j\in N}$, associated to an m-distribution function on the Bernouilli's lemniscate

$$BL = \{z \in \mathbb{C} : |z^2-1| = 1\} ,$$

satisfy the recurrence relation

$$(1) \qquad c_{k+2,j+2} = c_{k+2,j} + c_{k,j+2} \qquad (k, j \in N)$$

Generalizing the preceding result, a matrix $(c_{kj})_{k,j\in N}$ is said to be relative to BL if it is hermitian positive definite and its elements verify a recurrent relation analogous to (1).

Let $\mathcal{P}$ the vector space $\mathbb{C}[z]$. We define a moment functional

$$\mathcal{L} : \mathcal{P}\times\mathcal{P} \longrightarrow \mathbb{C}$$

through linear extension of $\mathcal{L}[z^k, z^j] = c_{kj}$. Thus, the matrix (c_{kj}), relative to BL, has associated a moment functional $\mathcal{L}$, which is said relative to BL, uniquelly determined by (c_{kj}). $\mathcal{L}$ induces an inner product in $\mathcal{P}$. If $\{\tilde{P}_n(z)\}_{n\in N}$ is the sequence of monic orthogonal polynomials (MOPS) defined by this inner product, our pourpose is to construct such a MOPS and the functional $\mathcal{L}$ by using the parameters $\{\tilde{P}_n(1)\}$ and $\{\tilde{P}_n(-1)\}$ in instead of the moments c_{kj}.

Consider the two families of parameters in $\mathbb{C}$:

(2) $\qquad \{a_n^{(1)}\}_{n\in N}, \quad \{a_n^{(2)}\}_{n\in N}$

verifying

(3) $\qquad a_o^{(i)} = 1, \quad a_1^{(1)} - a_1^{(2)} = 2,$

which implies that $\quad \det\left[\left(a_j^{(i)}\right)_{j=0,1}^{i=1,2}\right] \neq 0.$

Let e_o, e_1 arbitrary positive real numbers.

Having stablished these initial conditions, a polynomial sequence $\{\tilde{P}_n(z)\}_{n\in N}$ is desired, such that:

[SP 1] degree of $\tilde{P}_n(z) = n$;

[SP 2] leading coefficient of $\tilde{P}_n(z) = 1$;

[SP 3] $\tilde{P}_n(1) = a_n^{(1)}$, $\tilde{P}_n(-1) = a_n^{(2)}$.

Define:

<u>1st</u>. The monic polynomials

(4) $\qquad \tilde{P}_o(z) = 1, \; \tilde{P}_1(z) = z - 1 + a_1^{(1)}.$

<u>2nd</u>. The n-kernel

(5) $\qquad K_o(z,y) = 1/e_o,$

which verifies $K_o(\alpha_i, \alpha_j) = 1/e_o$ $(\alpha_1 = 1, \; \alpha_2 = -1)$.

Generally, for $(a_n^{(i)})_{i=1,2}$ verifying

(6) $\qquad e_{n-2} - \sum_{i,j=1}^{2} a_n^{(i)} \overline{a_n^{(j)}} M_{ji}^{(n-1)} = e_n > 0, \; \forall\, n \geq 2,$

where $[M_{ji}^{(k)}]_{j,i=1,2} = \left([K_k(\alpha_i,\alpha_j)]_{i,j=1,2}\right)^{-1}$, we define

(7) $\qquad K_n(\alpha_i,\alpha_j) = \frac{1}{e_n} a_n^{(i)} \overline{a_n^{(j)}} + K_{n-1}(\alpha_i,\alpha_j), \quad \forall\, n \geq 1.$

<u>Proposition 1</u>. If the family of parameters in (2) verifies (3) and (6) for $n = 2,\ldots,p$, then $[K_p(\alpha_i,\alpha_j)]$ is regular. (See [6]).

<u>Corollary</u>. (i) The matrix $[K_p(\alpha_i,\alpha_j)]$ is hermitian positive definite.

(ii) The $\{e_{2n} : n\in N\}$ and $\{e_{2n+1} : n\in N\}$ are decreasing sequences.

(iii) The parameter system (2) verifying (3) and (6) allows us to define $K_n(\alpha_i,\alpha_j)$ by (7), and

(8) $\quad [M_{ji}^{(n)}] = [K_n(\alpha_i,\alpha_j)]^{-1} \quad (n \geq 1),$

which is an hermitian positive definite matrix. (See [6]).

Definition. Let (2) be the parameter system which verifies (3) and (6), the following is defined:

$$\tilde{P}_{n+1}(z) = (z^2-1)\tilde{P}_{n-1}(z) + \sum_{i=1}^{2} a_{n+1}^{(i)} \phi_n^{(i)}(z) \quad (n \geq 1);$$

$$\tilde{P}_o(z) = 1, \quad \tilde{P}_1(z) = z - 1 + a_1^{(1)}.$$

$$K_{n+1}(z,y) = \frac{1}{e_{n+1}} \tilde{P}_{n+1}(z)\ \overline{\tilde{P}_{n+1}(y)} + K_n(z,y) \quad (n \geq 0),$$

$$K_o(z,y) = 1/e_o.$$

$$[\phi_{n+1}^{(i)}(z)]_{i=1,2} = [M_{ji}^{(n+1)}]\,[K_{n+1}(z,\alpha_i)]_{i=1,2} \quad (n \geq 0).$$

Proposition 2. In the conditions of the above definition, the following is proved:

(i) The polynomials $\{\tilde{P}_n(z)\}_{n\in N}$ satisfy [SP 1]-[SP 3].

(ii) The polynomials $\{K_n(z,y)\}_{n\in N}$ satisfy (7) for $z = \alpha_1$, $y = \alpha_2$.

(iii) The polynomials $\{\phi_n^{(i)}(z)\}_{n\in N}$ are such that $\phi_n^{(i)}(\alpha_k) = \delta_{ik}$ $(i,k = 1,2)$, and, in addition, for all $n \geq 1$,

(9) $$\phi_{n+1}^{(i)}(z) = \phi_n^{(i)}(z) + \frac{1}{e_{n+1}} \sum_{j=1}^{2} M_{ji}^{(n+1)}\ \overline{a_{n+1}^{(j)}}\ (z^2-1)\ \tilde{P}_{n-1}(z).$$

Proof. By induction, follows inmediatly. #

It is clear that $\{\tilde{P}_n(z)\}$ is a basis of $\mathcal{P}$. We define a moment functional

$$\mathcal{L} : \mathcal{P}\times\mathcal{P} \longrightarrow \mathbb{C}$$

through the linear extension of

$$\mathcal{L}\,[\tilde{P}_n(z),\ \tilde{P}_m(z)] = e_n\ \delta_{nm} \quad (n,\ m \in N).$$

The functional $\mathcal{L}$ is positive definite (since $e_n > 0$, $\forall\, n \in N$), and induces an inner product $<,>$ in $\mathcal{P}$; $\{\tilde{P}_n(z)\}_{n\in N}$ is the MOPS with such an inner product. Evidently, the following is true:

$<\tilde{P}_n(z),\ p(z)> = 0$ for every polynomial p of degree $m < n$;

$<\tilde{P}_n(z),\ p(z)> \neq 0$ for every polynomial p of degree n.

We note here a few additional properties:

1st. Reproductive property of n-kernel $K_n(z,y)$: Given $p \in \mathcal{P}_n$ (subspace of $\mathcal{P}$ of the polynomials of degree less than or equal to n),

$$< K_n(z,y),\ p(z)> = \overline{p(y)}.$$

2nd. $\{K_n(z,1),\ K_n(z,-1)\}$ constitutes a linearly independent system (which is inmediate because $\det[K_n(\alpha_i,\alpha_j)] \neq 0$), and the n-kernel being orthogonal to the vector subspace $(z^2-1)\mathcal{P}_{n-2}$ of $\mathcal{P}_n$. Then, $\{K_n(z,\alpha_i)\}_{i=1,2}$ constitutes a basis of the orthogonal subspace of

$(z^2-1)\mathcal{P}_{n-2}$ in $\mathcal{P}_n$, $[(z^2-1)\mathcal{P}_{n-2}]^{\perp n}$.

In the same way, since $[M_{ji}^{(n)}]$ is regular, $\{\phi_n^{(i)}(z)\}_{i=1,2}$ constitutes a basis of $[(z^2-1)\mathcal{P}_{n-2}]^{\perp n}$, with $<\phi_n^{(i)}(z),\phi_n^{(j)}(z)> = M_{ji}^{(n)}$.

3rd. $\{(z^2-1)\tilde{P}_n(z)\}_{n\in N}$ is an orthogonal system in $\mathcal{P}$, and a basis of the ideal $(z^2-1)\mathcal{P}$.

Through linear extension of the third property, we have:

Proposition 3. Let $A:\mathcal{P}\longrightarrow\mathcal{P}$ be the operator defined by

$$A[p(z)] = (z^2-1)p(z).$$

Then, A is isometric related to $\mathcal{L}$, $<Ap(z),Aq(z)> = <p(z),q(z)>$.

2. RECURRENCE.

Having obtained the MOPS $\{\tilde{P}_n(z)\}$ in the above paragraph, if $n \geqslant 1$ is verified:

$$(10)\quad \tilde{P}_{n+1}(z) = (z^2-1)\,\tilde{P}_{n-1}(z) + \sum_{i=1}^{2} a_{n+1}^{(i)}\,\phi_n^{(i)}(z)$$

$$(11)\quad \phi_{n+1}^{(i)}(z) = \phi_n^{(i)}(z) + \frac{1}{e_{n+1}}\,A_{n+1}^{(i)}\,(z^2-1)\tilde{P}_{n-1}(z) \qquad (i=1,2),$$

where

$$A_n^{(i)} = \sum_{j=1}^{2} M_{ji}^{(n)}\,\overline{a_n^{(j)}} \qquad (i = 1,2).$$

Proposition 4. For $n \geqslant 1$, it is shown that

$$\sum_{i=1}^{2} A_n^{(i)}\,a_n^{(i)} = \frac{e_n}{e_{n-2}}\,(e_{n-2} - e_n).$$

Therefore,

$$0 \leqslant \sum_{i=1}^{2} A_n^{(i)}\,a_n^{(i)} \leqslant e_{n-2} - e_n\,,$$

and $A_n^{(i)} = 0$, $\sum A_n^{(i)}a_n^{(i)} = 0$ iff $a_n^{(1)} = a_n^{(2)} = 0$.

Proof. Since (10) and (11):

$$\tilde{P}_{n+1}(z) = \left[1 - \frac{1}{e_{n+1}}\sum A_{n+1}^{(i)}\,a_{n+1}^{(i)}\right](z^2-1)\,\tilde{P}_{n-1}(z) + $$

$$+ \sum a_{n+1}^{(i)}\ {}_{n+1}^{(i)}(z)\ .$$

Thus,

$$<\tilde{P}_{n+1}(z),(z^2-1)\tilde{P}_{n-1}(z)> = e_{n-1}\left[1 - \frac{1}{e_{n+1}}\quad A_{n+1}^{(i)}\,a_{n+1}^{(i)}\right].\ (*)$$

By (10):

$$<\tilde{P}_{n+1}(z),(z^2-1)\tilde{P}_{n-1}(z)> = e_{n+1}\ .\ (**)$$

Since (*) and (**), the proposition follows. #

From the formulas (10) and (11), the equation system follows

$$(12)\begin{cases} \tilde{P}_{n+1}(z) = (z^2-1)\tilde{P}_{n-1}(z) + \frac{1}{e_n}\sum A_n^{(i)}\, a_{n+1}^{(i)}\,(z^2-1)\tilde{P}_{n-2}(z) + \\ \qquad + \frac{1}{e_{n-1}}\sum A_{n-1}^{(i)}\, a_{n+1}^{(i)}\,(z^2-1)\tilde{P}_{n-3}(z) + \sum a_{n+1}^{(i)}\,\phi_{n-2}^{(i)}(z). \\ \tilde{P}_n(z) = (z^2-1)\tilde{P}_{n-2}(z) + \frac{1}{e_{n-1}}\sum A_{n-1}^{(i)}\, a_n^{(i)}\,(z^2-1)\tilde{P}_{n-3}(z) + \\ \qquad + \sum a_n^{(i)}\,\phi_{n-2}^{(i)}(z). \\ \tilde{P}_{n-1}(z) = (z^2-1)\tilde{P}_{n-3}(z) + \sum a_{n-1}^{(i)}\,\phi_{n-2}^{(i)}(z) \end{cases}$$

(valid if $n \geqslant 3$), which represents a system of equations in the unknown quantities $\phi_{n-2}^{(i)}(z)$ $(i=1,2)$, and must be compatible. We note that, in setting the determinant in (12) equal to zero, an expression in $\tilde{P}_k(z)$ appears, with $n-3 \leqslant k \leqslant n+1$, being minimal respect to the number of polynomials $\tilde{P}_k(z)$, and thereby is called "short recurrence" (SR):

$$\begin{vmatrix} a_{n+1}^{(1)} & a_{n+1}^{(2)} & -\tilde{P}_{n+1} + (z^2-1)\tilde{P}_{n-1} + \frac{1}{e_n}\sum A_n^{(i)} a_{n+1}^{(i)}(z^2-1)\tilde{P}_{n-2} + \frac{1}{e_{n-1}}\sum A_{n-1}^{(i)} a_{n+1}^{(i)}(z^2-1)\tilde{P}_{n-3} \\ a_n^{(1)} & a_n^{(2)} & -\tilde{P}_n + (z^2-1)\tilde{P}_{n-2} + \frac{1}{e_{n-1}}\sum A_{n-1}^{(i)} a_n^{(i)}(z^2-1)\tilde{P}_{n-3} \\ a_{n-1}^{(1)} & a_{n-1}^{(2)} & -\tilde{P}_{n-1} + (z^2-1)\tilde{P}_{n-3} \end{vmatrix}$$

are equal to 0.

The coefficients of the polynomials $\tilde{P}_k(z)$ in the SR can be obtained adding a column to the matrix

$$(13) \qquad \begin{pmatrix} a_{n+1}^{(1)} & a_{n+1}^{(2)} \\ a_n^{(1)} & a_n^{(2)} \\ a_{n-1}^{(1)} & a_{n-1}^{(2)} \end{pmatrix},$$

with the following columns:

coeff. of	$\tilde{P}_{n+1}$	$\tilde{P}_n$	$\tilde{P}_{n-1}$	$(z^2-1)\tilde{P}_{n-2}$	$(z^2-1)\tilde{P}_{n-3}$
Column	-1	0	z^2-1	$\frac{1}{e_n}\sum A_n^{(i)} a_{n+1}^{(i)}$	$\frac{1}{e_{n-1}}\sum A_{n-1}^{(i)} a_{n+1}^{(i)}$
	0	-1	0	1	$\frac{1}{e_{n-1}}\sum A_{n-1}^{(i)} a_n^{(i)}$
	0	0	-1	0	1

If we denominate

$$U^{(n)} = \begin{vmatrix} a_{n+1}^{(1)} & a_{n+1}^{(2)} \\ a_n^{(1)} & a_n^{(2)} \end{vmatrix}, \qquad V^{(n)} = \begin{vmatrix} a_{n+1}^{(1)} & a_{n+1}^{(2)} \\ a_{n-1}^{(1)} & a_{n-1}^{(2)} \end{vmatrix},$$

the coefficients of $(z^2-1)\tilde{P}_{n-3}(z)$ and $(z^2-1)\tilde{P}_{n-2}(z)$ are, respectively:

$$\frac{e_{n-1}}{e_{n-3}}\,U^{(n)};\ \frac{U^{(n-1)}}{e_n}\sum A_n^{(i)} a_{n+1}^{(i)} - V^{(n)} = -\frac{e_n}{e_{n-2}}\,V^{(n)} - U^{(n)}\sum A_n^{(i)} a_{n+1}^{(i)}.$$

Here ist must be noted that (12) represents a short recurrence when the matrix (13) has the characteristic 2. In this case, the coefficients of the $\tilde{P}_k(z)$ are:

$\tilde{P}_{n+1}$	$\tilde{P}_n$	$\tilde{P}_{n-1}$	$(z^2-1)\tilde{P}_{n-2}$	$(z^2-1)\tilde{P}_{n-3}$
$-U^{(n-1)}$	$V^{(n)}$	$(z^2-1)U^{(n-1)}-U^{(n)}$	$\frac{U^{(n-1)}}{e_n}\sum A^{(i)}a^{(i)}-V^{(n)}=\frac{e_n}{e_{n-2}}V^{(n)}-U^{(n)}\sum A_n^{(i)}a_{n+1}^{(i)}$	$\frac{e_{n-1}}{e_{n+1}}U^{(n)}$

Related to the number of terms which appear in the SR, the following situations must be considered:

Char. of (13)	Other conditions	Type of SR	
2	$U^{(n)}, U^{(n-1)}, V^{(n)} \neq 0$	5 terms	(SR 1)
2	$V^{(n)}=0;\ U^{(n)}, U^{(n-1)} \neq 0$	4 terms non-consec.	(SR 2)
2	$U^{(n)}=0;\ V^{(n)}, U^{(n-1)} \neq 0$		
2	$U^{(n-1)}=0;\ U^{(n)}, V^{(n)} \neq 0$	4 consec. terms.	(SR 3)
1	$a_n^{(i)}, a_{n-1}^{(i)} \neq 0$ for some $\underline{i}$		
0,1,2	$a_n^{(i)}=0$, for each $i=1,2$	2 terms	(SR 4)

Must be noted that, in (SR 1) and (SR 2), the coefficient of $(z^2-1)\tilde{P}_{n-2}(z)$ can be equal to zero. In this case, (SR 1) and (SR 2) as recurrence relationship of 4 and 3 non-consecutive terms remains.

3. ORTHOGONAL POLYNOMIALS OVER BERNOUILLI'S LEMNISCATE.

Let $\mu(z)$ an m-distribution function, defined over BL. Note the inner product in

$$(14)\qquad \langle p,q\rangle_\mu = \int_{BL} p(z)\,\overline{q(z)}\,d\mu(z)\ ;\ p,\ q \in \mathcal{P}.$$

It can be shown that both the inner product (14) as well as the MOPS $\{\hat{P}_n(z)\}$ induced and univocally determinated by such inner product, satisfy the properties indicated in §1 and §2. (See [1], [6] and [7]).

It is necessary here to sumarize two results obtained by G. Szegő and P. Duren (see [10] and [3]).

Given a Jordan analytic curve C in the complex plane, a continuous and positive function w(z) defined on C, and $\{\hat{P}_n(z)\}$ the orthonormalized polynomial sequence induced by the inner product

$$\langle p,q\rangle = \int_C p(z)\,\overline{q(z)}\,w(z)\,|dz|\ ,$$

the following statements are true:

1) $\lim \hat{P}_{n+1}(z)/\hat{P}_n(z) = \psi(z) = cz + c_o + c_1 z^{-1} + \ldots$, uniformly outside C, where $\zeta = \psi(z)$ is a function which gives the conformal mapping of the exterior of C onto $|\zeta| > 1$.

2) If $\{\hat{P}_n(z)\}$ satisfies a three terms recurrence relation, as

$$a_n \hat{P}_n(z) + (b_n - z)\hat{P}_{n+1}(z) + c_n \hat{P}_{n+2}(z) = 0,$$

then C is an ellipse.

This last result is obtained through the first one. Duren presents the validity of both as an open problem when C is a rectifiable Jordan curve.

In this paper, it shall be demostrated the convergence of the quotient $\hat{P}_n(z)/\hat{P}_{n-2}(z)$ towards a function $\psi(z)$ uniformly outside BL (union of Jordan curves), being $\mu(z) = \mathrm{Arg}(z)$.

Consider $\phi = \mathrm{Arg}(z)$. We have:

1. $$<(z^2-1)^k,(z^2-1)^j> = \int_{BL} (z^2-1)^k(\bar{z}^2-1)^j\, d\phi = \int_{-\pi/4}^{\pi/4} e^{4i(k-j)\phi}\, d\phi + \int_{3\pi/4}^{5\pi/4} e^{4i(k-j)\phi}\, d\phi = 4\int_0^{\pi/4} \cos 4(k-j)\phi\, d\phi = \pi\,\delta_{kj}.$$

2. $$<z(z^2-1)^k,(z^2-1)^j> = \int_{BL} z(z^2-1)^k(\bar{z}^2-1)^j\, d\phi =$$
$$= \int_{-\pi/4}^{\pi/4} \sqrt{2\cos 2\phi}\; e^{i\phi}e^{4i(k-j)\phi}\, d\phi + \int_{3\pi/4}^{5\pi/4} \sqrt{2\cos 2\phi}\; e^{i\phi}e^{4(k-j)\phi}\, d\phi =$$
$$= 0.$$

3. $$<z(z^2-1),z(z^2-1)> = \int_{BL} (z^2-1)^k(\bar{z}^2-1)^j\, |z|^2\, d\phi =$$
$$= \int_{-\pi/4}^{\pi/4} \cos 2\phi\; e^{4i(k-j)\phi}\, d\phi + \int_{3\pi/4}^{5\pi/4} \cos 2\phi\; e^{4i(k-j)\phi}\, d\phi =$$
$$= 2\int_{-\pi/4}^{\pi/4} (e^{2i\phi} + e^{-2i\phi})\; e^{4i(k-j)\phi}\, d\phi = \frac{(-1)^{k-j}\,4}{1-4(k-j)} = d_{kj}.$$

In particular:

Proposition 5. The MOPS $\{\tilde{P}_n(z)\}$ associated to the m-distribution function over BL $\phi(z) = \mathrm{Arg}(z)$, is given by:

$\tilde{P}_o(z) = 1;\ \tilde{P}_1(z) = z;\ \tilde{P}_{2n}(z) = (z^2-1)^n \quad (n \geq 1);$

$$\tilde{P}_{2n+1}(z) = \frac{z}{D_{n-1}} \begin{vmatrix} d_{oo} & \cdots & d_{no} \\ \cdots & \cdots & \cdots \\ d_{o,n-1} & \cdots & d_{n,n-1} \\ 1 & \cdots & (z^2-1)^n \end{vmatrix} \quad (n \geq 1),$$

where $D_n = \det\left[(d_{kj})_{k,j=0}^{n}\right]$.

Furthermore, the sequence $\{\tilde{P}_{2n+1}(z)\}_{n\in N}$ verifies:

(15) $\quad \tilde{P}_{2n+1}(z) = (z^2-1)\,\tilde{P}_{2n-1}(z) + \tilde{P}_{2n+1}(1)\,\tilde{P}^*_{2n-1}(z),$

where

$$\tilde{P}^*_{2n+1}(z) = \frac{z}{D_{n-1}} \begin{vmatrix} d_{oo} & \cdots & d_{on} \\ \cdots & \cdots & \cdots \\ d_{n-1,0} & \cdots & d_{n-1,n} \\ (z^2-1)^n & \cdots & 1 \end{vmatrix} .$$

Note that (15) is a recurrence relation of two terms.

Proof. Since

$$\mathcal{P}_{2n}(z) = \mathcal{P}_n(z^2-1) \oplus z\,\mathcal{P}_{n-1}(z^2-1)$$
$$\mathcal{P}_{2n+1}(z) = \mathcal{P}_n(z^2-1) \oplus z\,\mathcal{P}_n(z^2-1) ,$$

we have:

(a) For the polynomials $\tilde{P}_{2n}(z)$ $(n \in N)$:

$$\tilde{P}_{2n}(z) = \sum_{k=0}^{n} a_{kn}\,(z^2-1)^k + z \sum_{k=0}^{n-1} b_{kn}\,(z^2-1)^k$$

and $a_{kn} = \frac{1}{\pi} < \tilde{P}_{2n}(z),(z^2-1)^k > = 0$ if $k < n$. But, $\tilde{P}_{2n}$ is a monic polynomial, hence $a_{nn} = 1$.

On the other hand, $< \tilde{P}_{2n}(z), z(z^2-1) > = 0$ $(j = 0,1,\dots,n-1)$. Thus, b_{kn} are given by the system

$$\sum_{k=0}^{n-1} < z(z^2-1)^k, z(z^2-1)^j > b_{kn} = 0 \quad (j = 0,1,\dots,n-1),$$

being the coefficients matrix Gramm's type. Hence, $b_{kn} = 0$, and

$$\tilde{P}_{2n} = (z^2-1)^n , \quad \hat{P}_{2n} = \frac{1}{\sqrt{\pi}}\,(z^2-1)^n \quad (n \in N).$$

(b) For the polynomials $\tilde{P}_{2n+1}(z)$ $(n \in N)$:

$$\tilde{P}_{2n+1}(z) = \sum_{k=0}^{n} a_{kn}\,(z^2-1)^k + z \sum_{k=0}^{n} b_{kn}(z^2-1)^k,$$

where $a_{kn} = \frac{1}{e_{2n+1}} < \tilde{P}_{2n+1}(z),(z^2-1)^k > = 0$, and $< \tilde{P}_{2n+1}(z), z(z^2-1)^j > =$ $= e_{2n+1}\,\delta_{jn}$ $(j=0,1,\dots,n)$. Thus, b_{kn} are given by

$$\sum_{k=0}^{n} < z(z^2-1)^k, z(z^2-1)^j > b_{kn} = e_{2n+1}\,\delta_{jn} \quad (j = 0,1,\dots,n),$$

with $b_{nn} = 1$. Hence the above system remains

$$\begin{cases} \sum_{k=0}^{n} d_{kj}\,b_{kn} = 0 & (j = 0,1,\dots,n-1) \\ b_{nn} = 1 \end{cases}$$

i.e.,

$$\begin{cases} d_{00}\,b_{0n} + d_{10}\,b_{1n} + \dots + d_{n-1,0}\,b_{n-1,n} = \\ d_{0,n-1}b_{0n} + d_{1,n-1}b_{1n} + \dots + d_{n-1,n-1}\,b_{n-1,n} = -\,d_{n,n-1} \end{cases}$$

But, $z\,b_{0n} + \dots + z(z^2-1)^{n-1}\,b_{n-1,n} = \tilde{P}_{2n+1}(z) - z(z^2-1)^n$, hence

$$\tilde{P}_{2n+1}(z) = \frac{z}{D_{n-1}} \begin{vmatrix} d_{oo} & \dots & d_{no} \\ \dots & \dots & \dots \\ d_{o,n-1} & \dots & d_{n,n-1} \\ 1 & \dots & (z^2-1)^n \end{vmatrix} = \frac{z}{D_{n-1}} Q_n(z^2-1) = z\, \tilde{Q}_n(z^2-1)$$

follows, where

$$Q_n(w) = \begin{vmatrix} d_{oo} & \dots & d_{no} \\ \dots & \dots & \dots \\ d_{o,n-1} & \dots & d_{n,n-1} \\ 1 & \dots & w^n \end{vmatrix}, \quad \tilde{Q}_n(w) = \frac{1}{D_{n-1}} Q_n(w) \quad \text{monic polynomial.}$$

Also, $\hat{P}_{2n+1}(z) = z\, \hat{Q}_n(z^2-1)$.

Let $w = z^2-1$ be, or, equivalently, $\theta = 4\phi$ (where $\theta = \operatorname{Arg}(w)$ and $\phi = \operatorname{Arg}(z)$). We have:

$$d_{kj} = \int_{BL} (z^2-1)^k (\bar{z}^2-1)^j \, |z|^2 \, d\phi = 4 \int_{-\pi/4}^{\pi/4} \cos 2\phi \; e^{4i(k-j)\phi} \, d\phi =$$

$$= 4 \int_{-\pi}^{\pi} \cos \frac{\theta}{2} \, e^{i(k-j)\theta} \, d\theta = \int_{-\pi}^{\pi} \cos \frac{\theta}{2} \, e^{i(k-j)\theta} \, d\theta \, .$$

Hence, $(d_{kj})_{k,j\in N}$ is the moment matrix with respect to the m-distribution function $\sigma(\theta)$ over the unit circle U, defined as

$$d\sigma(\theta) = \cos \frac{\theta}{2} \, d\theta .$$

Thus, $\{\tilde{Q}_n(w)\}$ is a MOPS over U, satisfying a recurrence relationship

(16) $\quad \tilde{Q}_n(w) = w\, \tilde{Q}_{n-1}(w) + \tilde{Q}_n(0)\, \tilde{Q}^*_{n-1}(w),$

being

$$w^{n-1}\, \overline{\tilde{Q}_{n-1}(\tfrac{1}{\bar{w}})} = \frac{1}{D_{n-2}} \begin{vmatrix} d_{oo} & \dots & d_{on} \\ \dots & \dots & \dots \\ d_{n-1,0} & \dots & d_{n-1,n} \\ w^n & \dots & 1 \end{vmatrix} = \tilde{Q}^*_{n-1}(w) .$$

If $w = z^2-1$, $z\, \tilde{Q}^*_{n-1}(z^2-1) = \tilde{P}^*_{2n-1}(z)$, and (16) remains

$$\frac{1}{z}\, \tilde{P}_{2n+1}(z) = \frac{z^2-1}{z}\, \tilde{P}_{2n-1}(z) + \tilde{P}_{2n+1}(1)\, \frac{1}{z}\, \tilde{P}^*_{2n-1}(z),$$

being $z \neq 0$. Thus, we obtained

$$\tilde{P}_{2n+1}(z) = (z^2-1)\, \tilde{P}_{2n-1}(z) + \tilde{P}_{2n+1}(1)\, \tilde{P}^*_{2n-1}(z),$$

also holds for $z = 0$, because $\tilde{P}_{2n+1}(0) = \tilde{P}_{2n-1}(0) = \tilde{P}^*_{2n-1}(0) = 0$. #

It is well known that $\{\tilde{Q}_n(w)\}$ satisfies a recurrence relationship of three terms

$$\tilde{Q}_{n+1}(w) = (w - a_n)\, \tilde{Q}_n(w) + b_n w\, \tilde{Q}_{n-1}(w) \; ,$$

hence, the $\{\tilde{P}_{2n+1}\}$ sequence verifies a three terms relation, (SR 2) type.

Proposition 6. The quotient $\hat{P}_n(z)/\hat{P}_{n-2}(z)$ converges to z^2-1 pointwise in Ext(BL), and uniformly for each compact subset of Ext(BL).

Proof. For $\hat{P}_{2n}(z)$, we have that $\hat{P}_{2n}(z)/\hat{P}_{2n-2}(z) = z^2-1$ $(z \neq \pm 1)$.

For $\hat{P}_{2n+1}(z)$,

$$\frac{\hat{P}_{2n+1}(z)}{\hat{P}_{2n-1}(z)} = \frac{z\,\hat{Q}_n(z^2-1)}{z\,\hat{Q}_{n-1}(z^2-1)} = \frac{\hat{Q}_n(z^2-1)}{\hat{Q}_{n-1}(z^2-1)} .$$

Making $w = z^2-1$, the above quotient remains as $\hat{Q}_n(w)/\hat{Q}_{n-1}(w)$, which converges to w, pointwise in $|w| > 1$, and uniformly for each compact subset of $|w| > 1$ (see [10]). #

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