Doubly-Connected Minimal Surfaces

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Let a doubly-connected minimal surface in $\mathbb{R}^n$ be defined as the image under a conformal mapping of an annulus in the plane. Our main object of study is the behavior of the function $L(r)$, defined as the length of the image of the circle of radius r. Our basic result (Theorem 1) is that, for surfaces in $\mathbb{R}^3$,

$$\frac{d^2 L}{d(\log r)^2} \geqq L,$$

with equality in only two cases: when the image is again a plane annulus, or the image is a catenoid bounded by a pair of coaxial circles in parallel planes. This particular inequality seems to be new even for the special case of a complex analytic mapping of the plane. In the case of minimal surfaces, it implies that $L(r)$ has slowest growth for the catenoid among all doubly-connected minimal surfaces (Theorem 2).

In Section 1 of this paper, we give proofs of Theorems 1 and 2, together with a number of related results.

In Section 2, we use Theorem 2 to give a proof of the isoperimetric inequality for doubly-connected minimal surfaces (Theorem 3). If L is the total length of the boundary curves, and A the area of the surface, we show that $L^2 > 4\pi A$. The isoperimetric inequality for doubly-connected minimal surfaces was first considered by NITSCHE [9] and later by KAUL [6], both of whom obtained inequalities of the form $L^2 > cA$ with some constant $c < 4\pi$. More general results of this nature are also known from geometric measure theory. (For a historical discussion, see [13].) Of course, the constant 4π is best possible, as may be seen by taking an annulus in the plane, and letting the inner circle shrink to a point. It is interesting that, for doubly-connected surfaces, the inequality is always strict; in fact, what we obtain is a sharper inequality of the form $L^2 - 4\pi A > cL_1 L_2$, where c is a positive constant, and L_1, L_2 are the lengths of the individual boundary curves.

Section 3 is devoted to the question of finding conditions on a pair of curves which imply that they cannot be spanned by any doubly-connected minimal surface. Again this question was first considered by NITSCHE [7], who has given a variety of such conditions [8, 10, 11], and also by BAILYN [1] and HILDEBRANDT [4]. (In fact, in [6], KAUL uses HILDEBRANDT'S non-existence theorem in order to obtain his isoperimetric inequality, referred to above.) The only case considered previously was that of spanning a catenoid between a pair of coaxial circles lying in parallel planes. In that case, it was known that for circles of fixed radius, if the planes are close together, a spanning surface always exists. As the planes are moved farther apart, there is a largest distance for which the surface exists, and, after that, no catenoid can span the given pair of circles. Thus, there is a function $f(L_1, L_2)$ such that if L_1, L_2 are the circumferences of the two circles, and if they lie in parallel planes a distance d apart, then there is a spanning catenoid if and only if $d \leqq f(L_1, L_2)$. (The function f is more or less explicitly known. See the discussion in BLISS [2], Chapter IV.)

Our first result on non-existence (Theorem 4) states that whenever $d > f(L_1, L_2)$ there does not exist *any* doubly-connected surface in $\mathbb{R}^3$ bounded by curves of length L_1, L_2, separated by parallel planes a distance d apart. Thus the catenoid is precisely the extremal surface for this problem. The proof is based on Theorem 2.

In Theorem 5, we generalize a result of NITSCHE [7], in which he shows that for two curves lying in parallel planes in $\mathbb{R}^3$, not only the distance apart of the planes but also a "horizontal shift" in the curves enters as an obstacle to the existence of a spanning surface. In our generalization, we consider surfaces in $\mathbb{R}^n$ and do not require that the boundary curves lie in parallel hyperplanes.

Finally, Theorem 6 sharpens HILDEBRANDT'S theorem on non-existence [4]. HILDEBRANDT showed that if the boundary curves lie in opposite nappes of a certain cone, then no spanning surface can exist. We show here that the same result holds in a larger cone, and furthermore that the one we obtain is optimal, with the catenoid again providing an extremal. The method of proof is quite different from HILDEBRANDT'S. It is related to an argument used by BAILYN [1], and also by GULLIVER, OSSERMAN, & ROYDEN [3]. (A discussion of the use of this method for non-existence theorems is given at the end of [14].) GULLIVER has informed us that he was also aware of this last result.

Section 4 applies the isoperimetric inequality of Section 2 to show that an area inequality obtained recently by ALEXANDER & OSSERMAN [−1] for simply-connected minimal surfaces also holds for doubly-connected surfaces in $\mathbb{R}^3$.

§ 1. Basic Results

We shall use the following notation:

$$\zeta = \xi + i\eta = \mathrm{re}^{i\Theta}; \text{ complex parameter}$$

$$D: \; 0 \leqq r_1 < |\zeta| < r_2 \leqq \infty; \text{ annular domain}$$

$$x: \; D \to \mathbb{R}^n, \qquad x = (x_1, \ldots, x_n)$$

$$\phi_k = \frac{\partial x_k}{\partial \xi} - i\frac{\partial x_k}{\partial \eta}, \qquad \psi_k = \zeta \phi_k, \qquad k = 1, \ldots, n$$

$$\lambda = \sqrt{\tfrac{1}{2}\sum_{k=1}^{n} |\phi_k|^2}; \qquad \mu = r\lambda = \sqrt{\tfrac{1}{2}\sum_{k=1}^{n} |\psi_k|^2}$$

$$L(r) = \int_0^{2\pi} \mu(\mathrm{re}^{i\Theta})\, d\Theta; \qquad t = \log r.$$

We shall assume throughout that $x(\zeta)$ defines a minimal surface in isothermal parameters. Then, as is well known (see for example [12], § 4), each x_k is harmonic in ξ, η, while the functions ϕ_k are analytic in ζ and satisfy

$$\sum_{k=1}^{n} \phi_k^2 \equiv 0, \tag{1}$$

$$\sum_{k=1}^{n} |\phi_k|^2 \neq 0. \tag{2}$$

Furthermore, the element of arc length ds on the surface is given by

$$ds^2 = \lambda^2 |d\zeta|^2. \tag{3}$$

It follows that $L(r)$ is the length of the curve in $\mathbb{R}^n$ which is the image of the circle $|\zeta| = r$.

Lemma 1. *$L(r)$ is a strictly convex function of* $\log r$.

Proof. We have

$$\frac{d^2 L}{dt^2} = r\frac{d}{dr}\left(r\frac{dL}{dr}\right) = \int_0^{2\pi} r\frac{\partial}{\partial r}\left(r\frac{\partial \mu}{\partial r}\right) d\Theta = \int_0^{2\pi} r^2 \Delta\mu \, d\Theta, \tag{4}$$

since

$$\Delta\mu = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mu}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\mu}{\partial\Theta^2}$$

and

$$\int_0^{2\pi} \frac{\partial^2\mu}{\partial\Theta^2} d\Theta = \left.\frac{\partial\mu}{\partial\Theta}\right|_0^{2\pi} = 0.$$

But a computation, using $\Delta\mu = 4\partial^2\mu/\partial\zeta\,\partial\bar{\zeta}$, yields

$$\Delta\mu = \frac{1}{2\mu}\left[\sum_{k=1}^{n} |\psi_k'|^2 + \frac{\sum_{1 \le j < k \le n} |\psi_j \psi_k' - \psi_k \psi_j'|^2}{\sum_{k=1}^{n} |\psi_k|^2}\right]. \tag{5}$$

It follows that $\Delta\mu > 0$ except at points where all $\psi_k' = 0$. We claim that these points are isolated. Indeed if they were not, then all the ψ_k would be constant. This means that $\phi_k = c_k/\zeta$, and

$$x_k = \mathrm{Re}\{\textstyle\int \phi_k d\zeta\} = \mathrm{Re}\{c_k \log\zeta\}.$$

Since each x_k is single-valued, each c_k is real, and equation (1) implies that all c_k vanish, which contradicts (2).

It follows that the right-hand side of (4) is strictly positive, proving the lemma.

Lemma 2. *Define $\bar{x}_k(r)$ by*

$$\bar{x}_k(r) = \frac{1}{2\pi}\int_0^{2\pi} x_k(re^{i\Theta})\, d\Theta. \tag{6}$$

Then $\bar{x}_k$ is a linear function of $\log r$.

Proof. As in the proof of Lemma 1,

$$\frac{d^2 \bar{x}_k}{dt^2} = r^2 \int_0^{2\pi} \Delta x_k \, d\Theta = 0.$$

Corollary. *By making a rotation of coordinates, if necessary, we may assume that*

$$\int_0^{2\pi} \psi_k(\mathrm{re}^{i\Theta}) \, d\Theta = 0, \qquad k = 1, \ldots, n-1. \tag{7}$$

Proof. By the lemma, $\bar{x}_k = c_k \log r + d_k$, where c_k and d_k are constants. Thus the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ moves along a straight line, which, after a rotation, we may assume to be parallel to the x_n-axis. Then $\bar{x}_1, \ldots, \bar{x}_{n-1}$ are constants, so that

$$0 = r \frac{d\bar{x}_k}{dr} = \int_0^{2\pi} \frac{\partial x_k}{\partial r} r \, d\Theta, \qquad k = 1, \ldots, n-1. \tag{8}$$

This implies that

$$x_k = \mathrm{Re}\{f_k\}, \qquad k = 1, \ldots, n-1, \tag{9}$$

where $f_k(\zeta)$ is a single-valued analytic function in D. But then

$$\phi_k = f_k', \qquad k = 1, \ldots, n-1, \tag{10}$$

and

$$\int_0^{2\pi} \psi_k(\mathrm{re}^{i\Theta}) \, d\Theta = \int_0^{2\pi} \zeta \phi_k(\mathrm{re}^{i\Theta}) \, d\Theta = -i \int_{|\zeta| = r} f_k'(\zeta) \, d\zeta = 0, \qquad k = 1, \ldots, n-1. \tag{11}$$

Example. The *catenoid*:

$$x_1 = \tfrac{1}{2}\left(r + \frac{1}{r}\right) \cos\Theta, \qquad x_2 = \tfrac{1}{2}\left(r + \frac{1}{r}\right) \sin\Theta, \qquad x_3 = \log r.$$

$$\bar{x}(r) = (0, 0, \log r),$$

$$\phi_1 = \tfrac{1}{2}\left(1 - \frac{1}{\zeta^2}\right), \qquad \phi_2 = -\frac{i}{2}\left(1 + \frac{1}{\zeta^2}\right), \qquad \phi_3 = \frac{1}{\zeta},$$

$$\psi_1 = \tfrac{1}{2}\left(\zeta - \frac{1}{\zeta}\right), \qquad \psi_2 = -\frac{i}{2}\left(\zeta + \frac{1}{\zeta}\right), \qquad \psi_3 = 1,$$

$$\lambda = \tfrac{1}{2}\left(1 + \frac{1}{r^2}\right), \qquad \mu = \tfrac{1}{2}\left(r + \frac{1}{r}\right),$$

$$L(r) = \pi \left(r + \frac{1}{r}\right) = 2\pi \cosh t,$$

$$\frac{d^2 L}{dt^2} = L.$$

Theorem 1. *For an arbitrary doubly-connected minimal surface in* $\mathbb{R}^3$,

$$\frac{d^2 L}{dt^2} \geqq L. \tag{12}$$

Equality holds if and only if the surface is the portion of a catenoid bounded by parallel coaxial circles, or an annulus in the plane.

Proof. In $\mathbb{R}^3$, one has the representation theorem

$$\phi_1 = \tfrac{1}{2} f(1-g^2), \quad \phi_2 = \frac{i}{2} f(1+g^2), \quad \phi_3 = fg, \tag{13}$$

where f is analytic in D, g is meromorphic in D, g has poles precisely at the zeros of f, and each zero of f has order equal to twice the order of the pole of g at that point (see [12], § 8). Then

$$\lambda = \tfrac{1}{2} |f| (1+|g|^2) = \tfrac{1}{2} |\phi_3| \left(\frac{1}{|g|} + |g| \right)$$

and

$$\mu = \tfrac{1}{2} |\psi_3| \left(\frac{1}{|g|} + |g| \right). \tag{14}$$

Further, from (13),

$$\psi_1 = \tfrac{1}{2} \psi_3 \left(\frac{1}{g} - g \right), \quad \psi_2 = \frac{i}{2} \psi_3 \left(\frac{1}{g} + g \right),$$

so that

$$\frac{\psi_3}{g} = \psi_1 - i\psi_2, \quad \psi_3 g = -\psi_1 - i\psi_2. \tag{15}$$

By the corollary to Lemma 2, we may assume (after making a rotation of coordinates) that equation (7) holds, and therefore from (15),

$$\int_0^{2\pi} \frac{\psi_3}{g} (re^{i\Theta})\, d\Theta = 0, \quad \int_0^{2\pi} \psi_3\, g(re^{i\Theta})\, d\Theta = 0. \tag{16}$$

From (4) and (14), we have

$$\frac{d^2 L}{dt^2} = \frac{1}{2} \int_0^{2\pi} \left(r^2 \Delta \left| \frac{\psi_3}{g} \right| + r^2 \Delta\, |\psi_3 g| \right) d\Theta, \tag{17}$$

while

$$L = \tfrac{1}{2} \int_0^{2\pi} \left(\left| \frac{\psi_3}{g} \right| + |\psi_3 g| \right) d\Theta. \tag{18}$$

From (4) and (5), we see that $d^2 L/dt^2$ (as well as L) is continuous. It follows from (16), (17), (18) that in order to prove (12) it suffices to prove the following lemma.

Lemma 3. *Let* $F(\zeta)$ *be analytic in* D, *and satisfy*

$$\int_0^{2\pi} F(re^{i\Theta})\, d\Theta = 0. \tag{19}$$

Then on every circle $|\zeta|=r$ where F has no zeros, the inequality

$$\int_0^{2\pi} r^2 \Delta |F|\, d\Theta \geqq \int_0^{2\pi} |F|\, d\Theta \tag{20}$$

is valid. Equality holds if and only if F is a constant multiple of ζ or $1/\zeta$.

Proof. Let $G(\zeta)$ be analytic in an annulus, and have there the Laurent expansion

$$G(\zeta)=\sum_{n=-\infty}^{\infty} a_n \zeta^n. \tag{21}$$

Then

$$\int_0^{2\pi} |G(re^{i\Theta})|^2\, d\Theta = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}$$

and

$$\int_0^{2\pi} |G'(re^{i\Theta})|^2\, d\Theta = 2\pi \sum_{n=-\infty}^{\infty} n^2 |a_n|^2 r^{2n-2}.$$

Further,

$$\Delta |G(\zeta)|^2 = 4 \frac{\partial^2}{\partial\zeta\, \partial\bar{\zeta}} G(\zeta)\, \overline{G(\zeta)} = 4\, |G'(\zeta)|^2.$$

Thus

$$\int_0^{2\pi} r^2 |G'|^2\, d\Theta \geqq \int_0^{2\pi} |G|^2\, d\Theta - 2\pi |a_0|^2 \tag{22}$$

and

$$\int_0^{2\pi} r^2 \Delta |G|^2\, d\Theta \geqq 4 \int_0^{2\pi} |G|^2\, d\Theta - 8\pi |a_0|^2. \tag{23}$$

Since $F \neq 0$ on the circle $|\zeta|=r$, we may choose an annulus (by "thickening" this circle) in which F is non-zero. In this annulus, there are two possibilities; either

$$\text{Case 1. } F=G^2 \quad \text{or} \quad \text{Case 2. } F=\zeta G^2,$$

where G is analytic in the annulus. Namely,

$$\int_0^{2\pi} \frac{\partial}{\partial\Theta} \arg F(re^{i\Theta})\, d\Theta = 2\pi k$$

for some integer k. Cases 1 and 2 correspond to k even and odd respectively. We have to treat these cases separately.

Case 1. If G has the expansion (21), then the constant term in the expansion of $F=G^2$ is

$$a_0^2 + 2\sum_{n=1}^{\infty} a_n a_{-n}.$$

But condition (19) is equivalent to the vanishing of the constant term in the Laurent expansion of F. Thus

$$a_0^2 = -2\sum_{n=1}^{\infty} a_n a_{-n}$$

and

$$|a_0^2| = 2\left|\sum_{n=1}^{\infty} a_n r^n a_{-n} r^{-n}\right| \leqq \sum_{n=1}^{\infty} (|a_n|^2 r^{2n} + |a_{-n}|^2 r^{-2n})$$
$$= \frac{1}{2\pi} \int_0^{2\pi} |G(\mathrm{re}^{i\Theta})|^2 \, d\Theta - |a_0|^2,$$

or

$$4\pi |a_0|^2 \leqq \int_0^{2\pi} |G|^2 \, d\Theta.$$

Substituting this in (23), and using $|F| = |G|^2$, yields

$$\int_0^{2\pi} r^2 \Delta |F| \, d\Theta \geqq 2 \int_0^{2\pi} |F| \, d\Theta. \tag{24}$$

Thus in Case 1, not only does (20) hold but in fact a stronger form is valid, implying in particular that the inequality in (20) is strict.

Case 2. Here $F = \zeta G^2$; introducing the function $H = \zeta G$, we have $\zeta F = H^2$. Then

$$2HH' = \zeta F' + F, \qquad |H'|^2 = \frac{|\zeta F' + F|^2}{4|\zeta F|},$$

$$2GG' = \frac{\zeta F' - F}{\zeta^2}, \qquad |G'|^2 = \frac{|\zeta F' - F|^2}{4|\zeta^3 F|}.$$

Since the constant term in the Laurent series of H is just a_{-1}, if we apply (22) to both G and H we obtain

$$\int_0^{2\pi} |\zeta F| \, d\Theta = \int_0^{2\pi} |H|^2 \, d\Theta \leqq \frac{r^2}{4} \int_0^{2\pi} \frac{|\zeta F' + F|^2}{|\zeta F|} \, d\Theta + 2\pi |a_{-1}|^2,$$

$$\int_0^{2\pi} \left|\frac{F}{\zeta}\right| d\Theta = \int_0^{2\pi} |G|^2 \, d\Theta \leqq \frac{r^2}{4} \int_0^{2\pi} \frac{|\zeta F' - F|^2}{|\zeta^3 F|} \, d\Theta + 2\pi |a_0|^2.$$

Since $|\zeta F' + F|^2 + |\zeta F' - F|^2 = 2|\zeta F'|^2 + 2|F|^2$, if we divide the first of the above equations by $r = |\zeta|$, multiply the second one through by r, and add, we obtain

$$2\int_0^{2\pi} |F| \, d\Theta \leqq \tfrac{1}{2} \int_0^{2\pi} \left[\frac{|\zeta F'|^2}{|F|} + |F|\right] d\Theta + 2\pi \frac{|a_{-1}|^2}{r} + |a_0|^2 r,$$

or

$$3\int_0^{2\pi} |F| \, d\Theta \leqq \int_0^{2\pi} r^2 \frac{|F'|^2}{|F|} \, d\Theta + 4\pi r \left[\frac{|a_{-1}|^2}{r^2} + |a_0|^2\right]. \tag{25}$$

But

$$\int_0^{2\pi} |F| \, d\Theta = r \int_0^{2\pi} |G|^2 \, d\Theta = 2\pi r \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n} \geqq 2\pi r \left[\frac{|a_{-1}|^2}{r^2} + |a_0|^2\right],$$

with equality if and only if $a_n = 0$ for all $n \neq 0, -1$. Substituting this in (25) yields

$$\int_0^{2\pi} |F| \, d\Theta \leqq \int_0^{2\pi} r^2 \frac{|F'|^2}{|F|} \, d\Theta. \tag{26}$$

However, for an analytic function F, one has

$$\Delta |F| = \frac{|F'|^2}{|F|}, \tag{27}$$

so that (26) is equivalent to (20).

As for equality, it can only hold if $a_n = 0$ for $n \neq 0, -1$; that is, if

$$G(\zeta) = a_0 + \frac{a_{-1}}{\zeta},$$

$$F = \zeta G^2 = a_0{}^2 \zeta + 2a_0 a_{-1} + \frac{a_{-1}{}^2}{\zeta}.$$

Condition (19) then implies that $a_0 a_{-1} = 0$, so that F is of the form stated. This proves Lemma 3 and hence inequality (12). Before completing the proof of Theorem 1, we make several comments.

Remarks. 1. Note that in Case 2 the assumption (19) is not needed to deduce the inequality (20). Only in Case 1 did we use it, and there it is clearly necessary since, for example, (20) is false if F is constant.

2. If we put

$$I(r) = \int_0^{2\pi} |F(\mathrm{re}^{i\Theta})| \, d\Theta, \qquad t = \log r,$$

then (20) is equivalent to

$$\frac{d^2 I}{dt^2} \geqq I.$$

3. Lemma 3 is equivalent to the fact that, if $f(\zeta)$ is analytic in an annulus, and if

$$L(r) = \int_0^{2\pi} |f'(\zeta)| \, |d\zeta|$$

is the length of the image of $|\zeta| = r$, then (with $t = \log r$)

$$\frac{d^2 L}{dt^2} \geqq L,$$

with equality if and only if $f(\zeta) = c\zeta$ or c/ζ.

Namely, setting $F(\zeta) = \zeta f'(\zeta)$, we have

$$L(r) = \int_0^{2\pi} |f'(\zeta)| \, r \, d\Theta = \int_0^{2\pi} |F| \, d\Theta$$

and

$$\int_0^{2\pi} F(\mathrm{re}^{i\Theta}) \, d\Theta = -i \int_{|\zeta|=r} f'(\zeta) \, d\zeta = 0.$$

Conversely, (19) implies that $F(\zeta)/\zeta$ is the derivative of a single-valued analytic function in the annulus.

We now complete the proof of Theorem 1 by analyzing when equality can hold in (12). Returning to (18), we see that for equality to hold in (12) we must have

$$\frac{\psi_3}{g}=c_1\zeta \quad \text{or} \quad \frac{c_2}{\zeta}, \qquad \psi_3 g=b_1\zeta \quad \text{or} \quad \frac{b_2}{\zeta}.$$

We therefore have four cases.

Case 1.

$$\frac{\psi_3}{g}=c_1\zeta, \qquad \psi_3 g=b_1\zeta.$$

Then g is a constant, and so is ϕ_3. It follows from (13) that

$$\phi_1=\tfrac{1}{2}c\left(\frac{1}{d}-d\right), \qquad \phi_2=\frac{i}{2}c\left(\frac{1}{d}+d\right), \qquad \phi_3=c.$$

It follows that the image surface is a plane, and the map $x:D\to\mathbb{R}^3$ is a complex linear map into the plane.

Case 2.

$$\frac{\psi_3}{g}=\frac{c_2}{\zeta}, \qquad \psi_3 g=\frac{b_2}{\zeta}.$$

Again g is constant, but this time

$$\phi_1=\frac{1}{2}\frac{c}{\zeta^2}\left(\frac{1}{d}-d\right), \qquad \phi_2=\frac{i}{2}\frac{c}{\zeta^2}\left(\frac{1}{d}+d\right), \qquad \phi_3=\frac{c}{\zeta^2}.$$

The image is again a plane, but the map is this time the composition of $1/\zeta$ with a complex linear map.

Case 3.

$$\frac{\psi_3}{g}=c_1\zeta, \qquad \psi_3 g=\frac{b_2}{\zeta}.$$

Then we have $\psi_3=c$, $g=d/\zeta$, and, by (13),

$$\phi_1=\frac{c}{2}\left(\frac{1}{d}-\frac{d}{\zeta^2}\right), \qquad \phi_2=\frac{ic}{2}\left(\frac{1}{d}+\frac{d}{\zeta^2}\right), \qquad \phi_3=\frac{c}{\zeta},$$

$$x_1=\operatorname{Re}\frac{c\zeta}{2d}+\frac{cd}{2\zeta}, \qquad x_2=\operatorname{Re}\frac{ic\zeta}{2d}-\frac{icd}{2\zeta}, \qquad x_3=\operatorname{Re}\{c\log\zeta\},$$

$$x_3 \text{ single-valued} \Rightarrow c \text{ real} \Rightarrow x_3=c\log|\zeta|.$$

If we set $d=\alpha e^{i\beta}$, $\zeta=\mathrm{r}e^{i}\theta$, then

$$x_1=\frac{c}{2}\left[\frac{r}{\alpha}+\frac{\alpha}{r}\right]\cos(\Theta-\beta), \qquad x_2=-\frac{c}{2}\left[\frac{r}{\alpha}+\frac{\alpha}{r}\right]\sin(\Theta-\beta),$$

$$x_3=c\left[\log\frac{r}{\alpha}+\log\alpha\right].$$

But this is a catenoid; setting

$$y_1=\frac{x_1}{c}, \qquad y_2=\frac{x_2}{c}, \qquad y_3=\frac{x_3}{c}-\log\alpha,$$

we obtain
$$y_1 - i y_2 = e^{i(\Theta-\beta)} \cosh y_3, \qquad {y_1}^2 + {y_2}^2 = (\cosh y_3)^2.$$
Note also that circles $|\zeta| = r$ map onto horizontal circles.

Case 4.
$$\frac{\psi_3}{g} = \frac{c_2}{\zeta}, \qquad \psi_3 g = b_1 \zeta.$$
This case is exactly analogous to Case 3, and again yields a catenoid.

This completes the proof of Theorem 1.

In order to state our next result, it is necessary to observe that doubly-connected minimal surfaces in $\mathbb{R}^n$ fall into two distinct classes having quite different properties. In fact, it is remarkable that Theorem 1 holds equally for both types.

We recall from Lemma 2 that the point

(28)
$$\bar{x}(r) = (\bar{x}_1(r), \ldots, \bar{x}_n(r)) = \frac{1}{2\pi} \int_0^{2\pi} x(re^{i\Theta})\, d\Theta$$

satisfies

(29)
$$\bar{x} = tc + d, \qquad t = \log r,$$

where c and d are fixed vectors.

Definition. A doubly-connected minimal surface is of *Type 1* if $c = 0$ in (29), and of *Type 2* if $c \neq 0$. A surface of Type 2 is *normalized* if

(30)
$$\bar{x}(r) = (0, \ldots, 0, \log r).$$

Note that every surface of Type 2 may be normalized by using a translation, rotation, and similarity transformation.

Lemma 4. *For a normalized surface of Type 2,*

(31)
$$L(r) \geqq 2\pi \qquad \textit{for all } r.$$

Equality can hold for at most one value of r. Moreover $L(r_0) = 2\pi$ *if and only if the circle* $|\zeta| = r_0$ *maps onto a horizontal plane* $x_n = c$, *and each radial direction along the circle maps into a vertical direction (parallel to the* x_n*-axis).*

Proof. Condition (30) implies that $x_n - \log r$ has a single-valued conjugate function in the annulus, or that
$$\phi_n(\zeta) = \frac{1}{\zeta} + f_n'(\zeta),$$
where $f_n(\zeta)$ is a single-valued analytic function in the annulus. But equation (1) yields
$$\sum_{k=1}^{n-1} {\phi_k}^2 = -{\phi_n}^2,$$
and hence
$$\lambda^2 = \tfrac{1}{2} \sum_1^n |\phi_k|^2 = \tfrac{1}{2} \left[\sum_1^{n-1} |\phi_k|^2 + |\phi_n|^2 \right]$$
$$\geqq \tfrac{1}{2} \left[\left| \sum_1^{n-1} {\phi_k}^2 \right| + |\phi_n|^2 \right] = |\phi_n|^2.$$

Thus

$$
(32)\qquad
\begin{aligned}
L(r) &= \int_0^{2\pi} \lambda(re^{i\Theta})\, r\, d\Theta \geqq \int_0^{2\pi} |\phi_n|\,|\zeta|\, d\Theta \\
&\geqq \left| \int_0^{2\pi} \zeta\, \phi_n(\zeta)\, d\Theta \right| = \left| \int_0^{2\pi} (1+\zeta f_n'(\zeta))\, d\Theta \right| \\
&= \left| 2\pi - i \int_{|\zeta|=r} f_n'(\zeta)\, d\zeta \right| = 2\pi .
\end{aligned}
$$

The fact that $L(r)$ can attain the minimum value 2π for at most one value of r is an immediate consequence of Lemma 1, that L is a strictly convex function of $\log r$.

$L(r_0)=2\pi$ if and only if both of the inequalities in (32) become equalities. This means

$$
(33)\qquad \lambda(r_0 e^{i\Theta}) = |\phi_n(r_0 e^{i\Theta})|, \qquad 0 \leqq \Theta \leqq 2\pi,
$$

$$
(34)\qquad \int_0^{2\pi} |\psi_n(r_0 e^{i\Theta})|\, d\Theta = \left| \int_0^{2\pi} \psi_n(r_0 e^{i\Theta})\, d\Theta \right| .
$$

Using the relation $\psi_n(\zeta) = 1 + \zeta f_n'(\zeta)$ gives

$$
\begin{aligned}
2\pi &= \int_0^{2\pi} \psi_n(r_0 e^{i\Theta})\, d\Theta \\
&= \int_0^{2\pi} \operatorname{Re}\{\psi_n(r_0 e^{i\Theta})\}\, d\Theta + i \int_0^{2\pi} \operatorname{Im}\{\psi_n(r_0 e^{i\Theta})\}\, d\Theta .
\end{aligned}
$$

Hence

$$
\int_0^{2\pi} \operatorname{Im}\{\psi_n(r_0 e^{i\Theta})\}\, d\Theta = 0
$$

and

$$
(35)\qquad
\begin{aligned}
\left| \int_0^{2\pi} \psi_n(r_0 e^{i\Theta})\, d\Theta \right| &= \left| \int_0^{2\pi} \operatorname{Re}\{\psi_n(r_0 e^{i\Theta})\}\, d\Theta \right| \\
&\leqq \int_0^{2\pi} |\operatorname{Re}\{\psi_n(r_0 e^{i\Theta})\}|\, d\Theta \leqq \int_0^{2\pi} |\psi_n(r_0 e^{i\Theta})|\, d\Theta .
\end{aligned}
$$

For (34) to hold, we must have

$$
(36)\qquad \operatorname{Im}\{\psi_n(r_0 e^{i\Theta})\} = 0, \qquad 0 \leqq \Theta \leqq 2\pi .
$$

But

$$
(37)\qquad
\begin{aligned}
\psi_n &= (\xi + i\eta)\left(\frac{\partial x_n}{\partial \xi} - i \frac{\partial x_n}{\partial \eta} \right) \\
&= \xi \frac{\partial x_n}{\partial \xi} + \eta \frac{\partial x_n}{\partial \eta} + i \left(\eta \frac{\partial x_n}{\partial \xi} - \xi \frac{\partial x_n}{\partial \eta} \right) \\
&= r \frac{\partial x_n}{\partial r} - i \frac{\partial x_n}{\partial \Theta} .
\end{aligned}
$$

Thus (36) holds if and only if $x_n(r_0 e^{i\Theta})$ is constant, and grad x_n is orthogonal to the circle $|\zeta|=r_0$. From (33), at each point of $|\zeta|=r_0$,

$$\lambda=|\text{grad } x_n|=\left|\frac{\partial x_n}{\partial r}\right|. \tag{38}$$

But $\lambda=|\partial x/\partial r|$, and so (38) holds if and only if $\partial x_k/\partial r=0$ for $k=1, \ldots, n-1$.

Conversely, $\partial x/\partial r$ vertical means that (38) holds, and this implies $\partial x_n/\partial r \neq 0$. Thus by (37), $\operatorname{Re} \psi_n$ cannot change sign on the circle $|\zeta|=r_0$. The condition that x_n is constant on this circle implies (36), again using (37). These two facts yield equality in (35), hence in (34), while (38) gives equality in (33). This completes the proof of the lemma.

Lemma 5. *Let $f(t)$ satisfy $f''(t) \geqq f(t)$ in some interval. Then for all t_0, t in the interval,*

$$f(t) \geqq f(t_0) \cosh(t-t_0)+f'(t_0) \sinh(t-t_0). \tag{39}$$

Equality holds for some $t_1 \neq t_0 \Leftrightarrow$ it holds for all t between t_0 and $t_1 \Leftrightarrow f''(t)=f(t)$ for t between t_0 and t_1.

Proof. We have

$$\frac{d}{dt}(f'(t) \cosh t-f(t) \sinh t)=(f''-f) \cosh t \geqq 0.$$

Hence $t>t_0$ implies

$$f'(t) \cosh t-f(t) \sinh t \geqq f'(t_0) \cosh t_0-f(t_0) \sinh t_0.$$

Thus

$$\frac{d}{dt} \frac{f(t)}{\cosh t} \geqq [f'(t_0) \cosh t_0-f(t_0) \sinh t_0] \operatorname{sech}^2 t$$

and

$$\frac{f(t)}{\cosh t}-\frac{f(t_0)}{\cosh t_0} \geqq [f'(t_0) \cosh t_0-f(t_0) \sinh t_0][\tanh t-\tanh t_0].$$

Multiplying out and simplifying, we obtain (39).

An analogous argument holds for $t<t_0$.

For equality to hold, it must hold in the first step, so that $f'' \equiv f$.

Theorem 2. *Let $x(\zeta)$ be a doubly-connected minimal surface in $\mathbb{R}^3$. Further assume (by a reparametrization of the form $z=\zeta/c$ if necessary) that $L(r)$ attains a minimum L_0 for $r=1$. Then the lengths of the boundary curves are greater than or equal to $L_0/2\pi$ times the lengths of the corresponding boundary curves of the standard catenoid based on the same annulus. Equality can hold only if $x(\zeta)$ is itself the standard catenoid.*

Proof. There are three cases, depending on whether $L(r)$ is increasing throughout, decreasing throughout, or has an interior minimum. Again using the notation $t=\log r$, the minimum occurs at $r=1$ or $t=0$. If we use primes to denote derivatives with respect to t, then by Theorem 1, $L''(t) \geqq L(t)$, and, by Lemma 5,

$$L(t) \geqq L(0) \cosh t + L'(0) \sinh t.$$

In the case of an interior minimum, $L'(0)=0$ and

$$L(t_1) \geqq L_0 \operatorname{cdsh} t_1, \qquad L(t_2) \geqq L_0 \cosh t_2$$

for the values t_1, t_2 corresponding to the boundary curves. But we have seen that, for the catenoid, the length function is $L(t)=2\pi \cosh t$.

If L is increasing, then the boundary values are $t=0$ and $t=t_1>0$. Using $L'(0) \geqq 0$ we obtain $L(t_1) \geqq L_0 \cosh t_1$, and the result is again true. A similar procedure applies if L is decreasing.

For equality to hold in any of these cases, it follows from Lemma 5 that $L''(t)=L(t)$. According to Theorem 1, $x(\zeta)$ must be a standard catenoid or else a plane annulus. However, a direct computation shows that, for the annulus, one has either $L(t)=L_0 e^t$, $t \geqq 0$, or $L(t)=L_0 e^{-t}$, $t \leqq 0$, and this is strictly greater than $L_0 \cosh t$.

§ 2. The Isoperimetric Inequality

Lemma 6. *Let S be a doubly-connected minimal surface in $\mathbb{R}^n$. If S lies on a simply-connected minimal surface, then it is of Type 1. If the boundary curves of S can be separated by a hyperplane, then it is of Type 2.*

Proof. Let C_r be the image of $|\zeta|=r$. Then along C_r,

$$\frac{\partial x}{\partial r} = \left|\frac{\partial x}{\partial r}\right| \frac{\partial x/\partial r}{|\partial x/\partial r|} = \lambda v,$$

where v is a unit vector tangent to the surface and normal to C_r. Since C_r is an analytic curve, it can have at most a finite number of self-intersections. Let C_j be a simple loop of C_r. Then since S lies on a simply-connected minimal surface, C_j bounds a simply-connected domain D_j on this surface. Along C_j, v is a unit normal vector field, tangent to the surface. It follows from standard formulas that $\int_{C_j} v\, ds = 0$. (See, for example, the discussion preceding equation (15) in [13].) Hence

$$0 = \int_{C_r} v\, ds = \int_0^{2\pi} \left(\frac{\partial x}{\partial r} \bigg/ \left|\frac{\partial x}{\partial r}\right|\right) \lambda r\, d\Theta = r \int_0^{2\pi} \frac{\partial x}{\partial r}\, d\Theta = 2\pi r \frac{\partial \bar{x}}{\partial r}.$$

Thus $\bar{x}$ is constant, and S is of Type 1.

For the second statement in the lemma, suppose that S is of Type 1. Then

$$\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(\mathrm{r}e^{i\Theta})\, d\Theta$$

is independent of r, hence a fixed point. If a curve C_r lies on one side of a hyperplane, then so does the point $\bar{x}$. Thus no two such curves can lie on opposite sides of any hyperplane.

Theorem 3. *Let A be the area of a doubly-connected minimal surface in $\mathbb{R}^n$, let L_1, L_2 be the lengths of its boundary curves, and let $L=L_1+L_2$. If the boundary* *curves cannot be separated by a hyperplane, then*

(40) $$L_1{}^2 + L_2{}^2 \geqq 4\pi A$$

or, equivalently,

(41) $$L^2 - 4\pi A \geqq 2L_1 L_2 .$$

For an arbitrary doubly-connected minimal surface in $\mathbb{R}^3$,

(42) $$L^2 - 4\pi A > 2L_1 L_2 (1 - \log 2).$$

Proof. We use the following identity:

(43) $$L_1{}^2 + L_2{}^2 - 4\pi A = M_1 + M_2 - 2\pi[p_1 \cdot \int_{C_1} v\,ds + p_2 \cdot \int_{C_2} v\,ds],$$

where C_1, C_2 are the boundary curves, ds the element of arc length, v the outward-directed unit vector tangent to the surface and normal to the boundary curve, p_j an arbitrary point of C_j, and M_j a non-negative quantity depending on p_j. (See OSSERMAN [13], equation (12); also KAUL [5], § 5 and § 8.)

If the surface is given by a map

$$x\colon\ r_1 \leqq |\zeta| \leqq r_2 \to \mathbb{R}^n,$$

then along the circle $|\zeta| = r_2$, mapping into C_2:

$$ds = \lambda r_2\, d\Theta = \left|\frac{\partial x}{\partial r}\right| r_2\, d\Theta, \qquad v = \frac{\partial x}{\partial r} \Big/ \left|\frac{\partial x}{\partial r}\right|,$$

and

(44) $$\int_{C_2} v\,ds = r_2 \int_0^{2\pi} \frac{\partial x}{\partial r}\, d\Theta = \left[2\pi r \frac{\partial \bar{x}}{\partial r}\right]_{r=r_2}.$$

Similarly,

(45) $$\int_{C_1} v\,ds = \left[-2\pi r \frac{\partial \bar{x}}{\partial r}\right]_{r=r_1}.$$

By (29), it follows that

$$p_1 \cdot \int_{C_1} v\,ds + p_2 \cdot \int_{C_2} v\,ds = 2\pi c \cdot (p_2 - p_1).$$

If the surface is of Type 1, then $c = 0$ and the right-hand side of (43) equals $M_1 + M_2$ for any choice of p_1, p_2. If the surface is of Type 2, then $c \neq 0$, and the set of real numbers d, for which the hyperplane $c \cdot x = d$ intersects each curve C_1, C_2, forms a non-empty closed interval. The assumption that no hyperplane separates C_1 from C_2 implies that these two intervals cannot be disjoint. Choosing a value d in the intersection, we can find points $p_1 \in C_1$, $p_2 \in C_2$, with $c \cdot p_1 = c \cdot p_2 = d$. For this choice of p_1, p_2, the right-hand side of (43) equals $M_1 + M_2$. Thus (40) follows from (43) for arbitrary surfaces of Type 1, and for surfaces of Type 2 whose boundary curves cannot be separated by hyperplanes.

To complete the theorem, it suffices to prove (42) for surfaces of Type 2 in $\mathbb{R}^3$. Since a similarity transformation will multiply both sides of (42) by a positive constant, we may assume the surface to be normalized. Then from (30),

$$2\pi r \frac{\partial \bar{x}}{\partial r} = (0, 0, 2\pi).$$

Further,

$$\frac{1}{2\pi} \int_0^{2\pi} x_3(\mathrm{re}^{i\Theta})\, d\Theta = \bar{x}_3(r) = \log r$$

This implies that the plane $x_3 = \log r_1$ intersects C_1, and the plane $x_3 = \log r_2$ intersects C_2. Let p_1, p_2 be points in these intersections, respectively. Then by (44) and (45),

$$p_2 \cdot \int_{C_2} v\, ds = 2\pi \log r_2, \qquad p_1 \cdot \int_{C_1} v\, ds = -2\pi \log r_1.$$

Hence (43) becomes

$$L_1{}^2 + L_2{}^2 - 4\pi A = M_1 + M_2 - 4\pi^2 \log \frac{r_2}{r_1}.$$

Since M_1 and M_2 are non-negative,

$$L^2 - 4\pi A \geqq 2 L_1 L_2 - 4\pi^2 \log \frac{r_2}{r_1}. \tag{46}$$

We now apply Theorem 2. We may assume

$$r_1 \leqq 1 \leqq r_2$$

and that $L(r)$ is minimum for $r = 1$. We let

$$K_j = \pi \left(r_j + \frac{1}{r_j} \right), \qquad j = 1, 2,$$

be the lengths of the corresponding boundary curves on the catenoid. Then

$$\pi^2 \frac{r_2}{r_1} < K_1 K_2 < 4\pi^2 \frac{r_2}{r_1}.$$

By Theorem 2 and Lemma 4, $K_1 K_2 \leqq L_1 L_2$. Finally, if we let $k_j = L_j/\pi$, we have

$$\begin{aligned} 2L_1 L_2 - 4\pi^2 \log \frac{r_2}{r_1} &= 2\pi^2 \left(k_1 k_2 - 2\log \frac{r_2}{r_1} \right) \\ &> 2\pi^2 \left(k_1 k_2 - 2\log \frac{K_1 K_2}{\pi^2} \right) \\ &\geqq 2\pi^2 (k_1 k_2 - 2\log k_1 k_2) \\ &> 2\pi^2 k_1 k_2 (1 - \log 2). \end{aligned}$$

The last inequality follows from the elementary fact that

$$2\log x < x \log 2 \qquad \text{for } x > 4,$$

combined with $K_j \geqq 2\pi$, $k_1 k_2 = L_1 L_2/\pi^2 \geqq K_1 K_2/\pi^2 \geqq 4$. Substituting in (46) gives (42), and the theorem is proved.

Remark. Clearly, the limitation in the second half of the theorem to $\mathbb{R}^3$ depends on the fact that Theorem 2 was only proved in that case. We conjecture that Theorem 2 is valid in $\mathbb{R}^n$, and this would give the isoperimetric inequality.

§ 3. Non-Existence Theorems

Lemma 7. *Let*

$$x\colon \{r_1<|\zeta|<r_2\}\to\mathbb{R}^n$$

define a doubly-connected minimal surface S. If the boundary curves can be separated by a pair of parallel hyperplanes a distance d apart, then

$$\log\frac{r_2}{r_1}\geqq\frac{2\pi}{L_0}\,d,$$

where $L_0=\inf_{r_1<r<r_2}L(r)$.

Proof. It follows from Lemma 6 that S must be of Type 2; by (29),

$$\bar{x}(r_2)-\bar{x}(r_1)=\left(\log\frac{r_2}{r_1}\right)c,$$

where

$$c=\frac{d\bar{x}}{dt}=r\frac{d\bar{x}}{dr}=\frac{1}{2\pi}\int_0^{2\pi}r\frac{\partial x}{\partial r}\,d\Theta.$$

Hence

$$|c|\leqq\frac{1}{2\pi}\int_0^{2\pi}r\left|\frac{\partial x}{\partial r}\right|d\Theta=\frac{1}{2\pi}\int_0^{2\pi}r\lambda\,d\Theta\equiv\frac{L(r)}{2\pi}.$$

However, any hyperplanes separating the boundary curves must also separate $\bar{x}(r_1)$ from $\bar{x}(r_2)$. Thus $|\bar{x}(r_2)-\bar{x}(r_1)|\geqq d$, and hence

$$d\leqq\left(\log\frac{r_2}{r_1}\right)\frac{L(r)}{2\pi}\qquad\text{for all } r.$$

Taking the infimum over r gives the desired inequality.

Theorem 4. *Let L_1, L_2, d be any three positive numbers. Necessary and sufficient that there exists a doubly-connected minimal surface in $\mathbb{R}^3$ whose boundary curves C_1, C_2 have lengths L_1, L_2 and are separated by a pair of parallel planes a distance d apart, is that there exists a catenoid for which this is true.*

Proof. Let S be some doubly-connected minimal surface whose boundary curves have lengths L_1, L_2 and are separated by a slab of width d. What we must show is that there is also a catenoid with this property.

Note first of all that, by Lemma 6, S is of Type 2, and by Lemma 7,

$$d\leqq\frac{L_0}{2\pi}(t_2-t_1),$$

where $t_j=\log r_j$. But by Theorem 2,

$$L_1\geqq L_0\cosh t_1,\qquad L_2\geqq L_0\cosh t_2,$$

where the parameter domain has been chosen so that

$$t_1 \leqq 0 \leqq t_2.$$

Letting $\lambda = L_0/2\pi$, $\ell_1 = L_1/2\pi$, $\ell_2 = L_2/2\pi$, we have

$$d \leqq \lambda \left[\operatorname{Cosh}^{-1}\frac{\ell_2}{\lambda} + \operatorname{Cosh}^{-1}\frac{\ell_1}{\lambda}\right],$$

where $\operatorname{Cosh}^{-1} u$, for $u \geqq 1$, denotes the non-negative value of $\cosh^{-1} u$. Since we do not know the value of L_0, but simply that it is less than or equal to L_1 and L_2, all we can say is that there exists some value of λ, $0 < \lambda \leqq \min\{\ell_1, \ell_2\}$, such that the above inequality holds. Using such a value of λ, we consider the catenoid

$$r = \lambda \cosh\frac{x_3}{\lambda}, \qquad -\lambda \operatorname{Cosh}^{-1}\frac{\ell_1}{\lambda} \leqq x_3 \leqq \lambda \operatorname{Cosh}^{-1}\frac{\ell_2}{\lambda},$$

where $r^2 = x_1{}^2 + x_2{}^2$. It has boundary circles of length $L_1 = 2\pi\ell_1$ and $L_2 = 2\pi\ell_2$, lying in parallel planes whose distance apart is

$$\lambda \operatorname{Cosh}^{-1}\frac{\ell_2}{\lambda} + \lambda \operatorname{Cosh}^{-1}\frac{\ell_1}{\lambda} \geqq d.$$

This proves the theorem.

Lemma 8. *Let u be harmonic in an annulus $r_1 < |\zeta| < r_2$. Suppose $b \geqq a$, and*

$$\varliminf_{r \to r_1} u(re^{i\Theta}) \leqq a, \qquad \varlimsup_{r \to r_2} u(re^{i\Theta}) \geqq b.$$

Then for $r_1 < r < r_2$,

$$\int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\Theta})\, r\, d\Theta \geqq 2\pi \frac{b-a}{\log(r_2/r_1)}.$$

Proof. Given $\varepsilon > 0$, let

$$v = u - a - \frac{b-a-\varepsilon}{\log(r_2/r_1)} \log\frac{r}{r_1}.$$

Then v is harmonic in the annulus, and

$$\varliminf_{r \to r_1} v(re^{i\Theta}) \leqq 0, \qquad \varlimsup_{r \to r_2} v(re^{i\Theta}) \geqq \varepsilon. \tag{47}$$

Choose ε', $0 < \varepsilon' < \varepsilon$, such that $\operatorname{grad} v \neq 0$ on the level curve C: $v = \varepsilon'$. Then C must consist of one or more analytic Jordan curves. But if any subset C' of C were homologous to zero in the annulus, the function v would be constant in the domain bounded by C', hence in the whole annulus, which contradicts (47). Thus C consists of a single curve not homologous to zero. Choose δ such that

$$r_1 < \delta < \min_{\zeta \in C} |\zeta|.$$

Then C is homologous to the circle $|\zeta| = \delta$, and hence

$$\int_C \frac{\partial v}{\partial n}\, ds = \int_{|\zeta| = \delta} \frac{\partial v}{\partial n}\, ds.$$

But $v \geqq \varepsilon'$ outside C, and $v = \varepsilon'$ on C. Therefore $\partial v/\partial n \geqq 0$ on C, where $\partial/\partial n$ is the exterior normal derivative. Thus

$$\int_0^{2\pi} \frac{\partial v}{\partial r}(\delta e^{i\Theta})\,\delta\, d\Theta = \int_C \frac{\partial v}{\partial n}\, ds \geqq 0.$$

Using the explicit expression for v, we obtain

$$\int_0^{2\pi} \frac{\partial u}{\partial r}(\delta e^{i\Theta})\,\delta\, d\Theta \geqq 2\pi \frac{b-a-\varepsilon}{\log(r_2/r_1)}.$$

But the expression on the left is independent of δ, hence this inequality holds on every circle $|\zeta| = r$. Since ε was arbitrary, the lemma is proved.

Theorem 5. *Let δ_1, δ_2, c, d be arbitrary positive numbers satisfying*

$$\left(\frac{c^2}{2} + d^2\right)^{1/2} \geqq \delta_1 + \delta_2. \tag{48}$$

Let C_1, C_2 be closed curves in $\mathbb{R}^n$. Let

$$D_1 = \{x \in \mathbb{R}^n \mid x_1{}^2 + \cdots + x_{n-1}{}^2 < \delta_1{}^2,\ x_n = 0\},$$

$$D_2 = \{x \in \mathbb{R}^n \mid (x_1 - c)^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 < \delta_2{}^2,\ x_n = d\}.$$

Then if $C_1 \subset D_1$, $C_2 \subset D_2$, there does not exist any doubly-connected minimal surface spanning C_1 and C_2. More generally, the same conclusion holds if we replace D_k by D'_k, $k = 1, 2$, where

$$D'_1 = \left\{x \in \mathbb{R}^n \,\middle|\, \left(x_1 - \frac{c}{d}x_n\right)^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 \leqq \delta_1{}^2,\ x_n \leqq 0\right\},$$

$$D'_2 = \left\{x \in \mathbb{R}^n \,\middle|\, \left(x_1 - \frac{c}{d}x_n\right)^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 \leqq \delta_2{}^2,\ x_n \geqq d\right\}.$$

Proof. Suppose there does exist a surface S spanning the given curves. We shall show that (48) cannot hold.

We use the notation at the beginning of § 1, where the surface S is given by a map $x(\zeta)$ of an annulus into $\mathbb{R}^n$. Let the boundary curve C_k be the image of the circle $|\zeta| = r_k$, $k = 1, 2$. We define a function $u(\zeta)$ in the annulus by

$$u = \left(x_1 - \frac{c}{d}x_n\right)^2 + x_2{}^2 + \cdots + x_{n-1}{}^2. \tag{49}$$

Then one may compute that

$$\begin{aligned}
\Delta u &= 2\left(\left|\phi_1 - \frac{c}{d}\phi_n\right|^2 + |\phi_2|^2 + \cdots + |\phi_{n-1}|^2\right)\\
&\geqq 2\left(\left|\phi_1 - \frac{c}{d}\phi_n\right|^2 + |\phi_2{}^2 + \cdots + \phi_{n-1}{}^2|\right)\\
&= 2\left(\left|\phi_1 - \frac{c}{d}\phi_n\right|^2 + |\phi_1{}^2 + \phi_n{}^2|\right),
\end{aligned}$$

by virtue of equation (1).

We assert next that, if b is an arbitrary real number, then

$$\min\{|w-b|^2+|w^2+1|\}=\frac{b^2}{2}+1, \tag{51}$$

where the minimum is taken over all complex numbers w. Namely, setting $w=b+\mathrm{r}e^{i\Theta}$ gives

$$\begin{aligned} |w-b|^2+|w^2+1| &= r^2+|b^2+2b\,\mathrm{r}e^{i\Theta}+r^2 e^{2i\Theta}+1| \\ &\geqq r^2+b^2+1+2br\cos\Theta+r^2\cos 2\Theta \\ &= b^2+1+2br\cos\Theta+2r^2\cos^2\Theta \\ &= b^2+1+2r^2\left(\cos\Theta+\frac{b}{2r}\right)^2-\frac{b^2}{2}\geqq\frac{b^2}{2}+1. \end{aligned} \tag{52}$$

This gives a lower bound which is actually attained when $w=b/2$. This proves (51). Returning to (50), we therefore have

$$\Delta u\geqq 2\,|\phi_n|^2\left(\left|\frac{\phi_1}{\phi_n}-\frac{c}{d}\right|^2+\left|\left(\frac{\phi_1}{\phi_n}\right)^2+1\right|\right)\geqq\left(\left(\frac{c}{d}\right)^2+2\right)|\phi_n|^2.$$

Using the notation

$$t=\log r, \qquad U(t)=\frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{r}e^{i\Theta})\,d\Theta, \tag{53}$$

we find, as in the proof of Lemma 1, that

$$\frac{d^2U}{dt^2}=\frac{1}{2\pi}\int_0^{2\pi} r^2\,\Delta u\,d\Theta\geqq\frac{c^2+2d^2}{2\pi d^2}\int_0^{2\pi}|\psi_n|^2\,d\Theta. \tag{54}$$

But

$$2\pi\int|\psi_n|^2\,d\Theta\geqq\left(\int|\psi_n|\,d\Theta\right)^2\geqq\left(\int r\frac{\partial x_n}{\partial r}\,d\Theta\right)^2 \tag{55}$$

by virtue of (37). Now the assumption that $C_1\subset D_1'$, $C_2\subset D_2'$ implies that $x_n(r_1 e^{i\Theta})\leqq 0$ and $x_n(r_2 e^{i\Theta})\geqq d$. By Lemma 8, we have

$$\int_0^{2\pi} r\frac{\partial x_n}{\partial r}\,d\Theta\geqq 2\pi\frac{d}{T}, \tag{56}$$

where

$$T=\log\frac{r_2}{r_1}. \tag{57}$$

Combining (54), (55), (56) gives

$$\frac{d^2U}{dt^2}\geqq\frac{c^2+2d^2}{T^2}. \tag{58}$$

By the definition of D_k', the statement $C_k\subset D_k'$ implies $u(r_k e^{i\Theta})\leqq\delta_k^{\,2}$, and hence

$$U(t_k)\leqq\delta_k^{\,2}, \qquad k=1,2. \tag{59}$$

We may assume that

$$t_1 = \log r_1 = 0, \qquad t_2 = \log r_2 = T.$$

Set

$$A = \frac{c^2}{2} + d^2 \tag{60}$$

so that (58) becomes

$$\frac{d^2 U}{dt^2} \geqq \frac{2A}{T^2}, \qquad 0 < t < T. \tag{61}$$

Define $V(t)$ to be the parabola

$$V(t) = a t^2 + b t + \delta_1{}^2,$$

satisfying

$$\frac{d^2 V}{dt^2} = \frac{2A}{T^2}, \qquad V(0) = \delta_1{}^2, \qquad V(T) = \delta_2{}^2. \tag{62}$$

It follows from (59), (61), (62) that

$$U(t) \leqq V(t), \qquad 0 < t < T. \tag{63}$$

The conditions (62) determine the coefficients a, b of V:

$$a = \frac{A}{T^2}, \qquad b = \frac{1}{T}[\delta_2{}^2 - \delta_1{}^2 - A]. \tag{64}$$

Since $a > 0$, $V(t)$ has a minimum at $t = t_0$, where

$$t_0 = -\frac{b}{2a} = T\left[\frac{1}{2} - \frac{\delta_2{}^2 - \delta_1{}^2}{2A}\right]. \tag{65}$$

It follows that

$$t_0 > 0 \Leftrightarrow \delta_2{}^2 - \delta_1{}^2 < A,$$

$$t_0 < T \Leftrightarrow \delta_2{}^2 - \delta_1{}^2 > -A.$$

Thus

$$0 < t_0 < T \Leftrightarrow |\delta_2{}^2 - \delta_1{}^2| < A. \tag{66}$$

We consider two cases, according to whether (66) does or does not hold. If it does not hold, then

$$A \leqq |\delta_2{}^2 - \delta_1{}^2| = |\delta_2 - \delta_1|\,|\delta_2 + \delta_1| < |\delta_2 + \delta_1|^2. \tag{67}$$

On the other hand, if (66) does hold, then, by virtue of (63) and the fact that $U(t) > 0$ for all t,

$$V(t_0) \geqq U(t_0) > 0.$$

But by (64) and (65),

$$V(t_0) = -\frac{b^2}{4a^2} + \delta_1{}^2 > 0$$

$$\Leftrightarrow b^2 < 4a\delta_1{}^2 \Leftrightarrow (\delta_2{}^2 - \delta_1{}^2)^2 - 2A(\delta_2{}^2 + \delta_1{}^2) + A^2 < 0$$

$$\Rightarrow A < (\delta_2{}^2 + \delta_1{}^2) + \sqrt{(\delta_2{}^2 + \delta_1{}^2)^2 - (\delta_2{}^2 - \delta_1{}^2)^2} = (\delta_2 + \delta_1)^2.$$

Comparing this with (67), we see that in both cases we must have $A<(\delta_1+\delta_2)^2$. But going back to the definition (60) of A, we see that under the assumption that a spanning surface exists, inequality (48) must be violated. This proves the theorem.

Theorem 6. *Let τ be the unique positive number satisfying*

$$\cosh\tau-\tau\sinh\tau=0. \tag{68}$$

Let V denote the cone in $\mathbb{R}^3$ defined by

$$V=\{x\in\mathbb{R}^3 \mid {x_1}^2+{x_2}^2<{x_3}^2\sinh^2\tau\}. \tag{69}$$

Let C_1, C_2 be closed curves, each lying in one of the two components of V. Then there does not exist any connected minimal surface spanning C_1 and C_2.

Proof. Note first that the catenary $y=a\cosh(x/a)$ is tangent to the lines $y=\pm(\sinh\tau)x$ at the points $x=\pm a\tau$, and lies above them elsewhere. In other words,

$$a\cosh\frac{x}{a}\geqq|x|\sinh\tau \qquad \text{for all } x. \tag{70}$$

This means that each catenoid

$$r=a\cosh\frac{x_3}{a}, \qquad r^2={x_1}^2+{x_2}^2,$$

lies outside the cone V. Thus if we let

$$C_a=\left\{x\in\mathbb{R}^3 \,\middle|\, r\leqq a\cosh\frac{x_3}{a}\right\},$$

we have

$$\bar{V}\subset\bigcap_{a>0}C_a.$$

In fact, one has

$$\bar{V}=\bigcap_{a>0}C_a. \tag{71}$$

Namely, by (68) and (69) the coordinates of any point not in $\bar{V}$ satisfy $|x_3|\cosh\tau<r\tau$. Let $a=|x_3|/\tau$. Then $x_3=\pm\tau a$, and $r>a\cosh\tau=a\cosh(x_3/a)$, so that the point is not in C_a. This proves (71).

Suppose now that S is a connected minimal surface whose boundary lies in V. We assert that $S\subset\bar{V}$. Indeed by (71) it is sufficient to show that $S\subset C_a$ for all $a>0$. Suppose then that $S\not\subset C_a$ for some $a>0$. There will then be a largest value of $\lambda<1$ such that the surface λS obtained from S by a similarity transformation (contracting toward the origin by a factor of λ) lies in C_a. Some point p of λS must then lie on the catenoid ∂C_a. From $\partial S\subset V$ follows $\partial(\lambda S)\subset V$, so that p must be an inner point of λS. Then λS and ∂C_a are minimal surfaces tangent at p, while λS lies on one side of ∂C_a since $\lambda S\subset C_a$. This implies that λS coincides with ∂C_a. But this is impossible since $\partial C_a\cap V=\emptyset$, while $\partial(\lambda S)\subset V$.

Thus for any minimal surface S, $\partial S\subset V$ implies $S\subset\bar{V}$. However, S cannot pass through the vertex of $\bar{V}$. It follows that, if S is connected, it must lie in one of the two components of $\bar{V} \setminus \{0\}$. Thus the boundary of S cannot lie in both components of V. This proves the theorem.

In conclusion, we remark that, unlike the previous theorems in this paper, Theorem 6 does not require the surface to be doubly-connected. In fact, the boundary may have an arbitrary number of components, and the proof goes through unchanged.

We note also that the constant $\sinh\tau$ defined by (68) is approximately 3/2. Thus the cone (69) is considerably wider than the cone $x_1{}^2+x_2{}^2<x_3{}^2$ obtained by HILDEBRANDT [4]. Further, no larger constant can be used, for since the catenoids $r=a\cosh(x_3/a)$ are tangent to ∂V, the circles along which they are tangent would lie in opposite nappes of any larger cone, and there would be a spanning catenoid.

§ 4. An Area Inequality

Lemma 9. *Let D be a finitely-connected plane domain, and $x\colon D \to \mathbb{R}^n$ a smooth map satisfying the convex hull property and mapping the boundary of D into the sphere $S^{n-1}(R)$ of radius R. Let D_r be the inverse image of the ball $|x|<r$, for $r<R$. Then for almost all r for which $D_r \neq \emptyset$, D_r has connectivity no greater than that of D.*

Proof. Let d be the distance of $x(D)$ to the origin. Then for almost all r, $d<r<R$, D_r is bounded by a finite number of smooth curves on which $|x|=r$. If there is more than one curve, each of the inner ones must enclose a curve of the boundary of D, since, otherwise, the interior would consist of a subdomain of D on which $|x|>r$, contradicting the convex hull property. Thus D must have at least as many boundary components as D_r, and this proves the lemma.

Theorem 7. *Let D be a plane annulus, and let $x\colon D \to \mathbb{R}^3$ define a minimal surface whose boundary lies on the sphere $S^2(R)$ of radius R. Let d be the distance of $x(D)$ to the origin. Then the area A of $x(D)$ satisfies*

$$A > \pi(R^2 - d^2). \tag{72}$$

Proof. It was shown by ALEXANDER, HOFFMAN & OSSERMAN ([0], Theorem 3.2) that (72) would follow from the isoperimetric inequality $L^2 > 4\pi A$ for each domain $D_r = \{\zeta \in D \,|\, |x(\zeta)| < r\}$, $d<r<R$, having a smooth boundary. But by Lemma 9, each such D_r is either simply- or doubly-connected. The hypotheses imply that $x(D)$ cannot lie on a plane. Thus $L^2 > 4\pi A$ for each simply-connected D_r, as is well-known, and the same is true for doubly-connected D_r by Theorem 3.

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