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Universality of the correlations between eigenvalues of large random matrices

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The distribution of eigenvalues of random matrices appears in a number of physical situations, and it has been noticed that the resulting properties are universal, i.e. independent of specific details. Standard examples are provided by the universality of the conductance fluctuations from sample to sample in mesoscopic electronic systems, and by the spectrum of energy levels of a non-integrable classical hamiltonian (the so-called quantum chaos). The correlations between eigenvalues, measured on the appropriate scale, are described in all those cases by simple gaussian statistics. Similarly numerical experiments have revealed the universality of these correlations with respect to the probability measure of the random matrices. A simple renormalization group argument leads to a direct understanding of this universality; it is a consequence of the attractive nature of a gaussian fixed point. Detailed calculations of these correlations are given for a general probability distribution (in which the logarithm of the probability is the trace of a polynomial of the matrix); the universality is shown to follow from an explicit asymptotic form of the orthogonal polynomials with respect to a non-gaussian measure. In addition it is found that the connected correlations, when suitably smoothed, exhibit, even when the eigenvalues are not in the scaling region, a higher level of universality than the density of states.

1. Introduction

Some forty years ago, Wigner proposed that the energy levels of complex quantum systems, such as large nuclei, may be characterized statistically [1]. Since then it has been progressively understood that a number of physical systems can be accurately described by the statistics of the eigenvalues of ensembles of random matrices. For instance if one considers a classically non-integrable hamiltonian system, the corresponding quantum system seems to exhibit a statistics of energy levels which is identical to that of a gaussian ensemble of random matrices. A typical example is the hydrogen atom in a constant magnetic field [2]. This seems

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to be a general property [3], if not yet a theorem. Similarly it is well-known that the sample to sample fluctuations in small electronic systems, the so-called mesoscopic systems, exhibit similar behaviour. An attempt to understand the universality of these fluctuations has been given recently [4].

Within random matrices themselves it has been known "experimentally" for quite some time that the correlation properties of the eigenvalues (on a scale which will be defined below) are independent of the probability distribution [5], within a class of given symmetry (it does matter whether the measure is invariant under the unitary or the orthogonal group for instance).

This paper is written to be as self-contained as possible. The basic formulae and identities are reviewed in two appendices. We hope that readers with no prior knowledge of the subject will find this paper accessible.

There has been a recent burst of interest in random matrices when it was understood that they can be used to generate triangulated random surfaces of arbitrary genera [6], and a number of powerful techniques were added to our understanding of random matrices. It was understood, for instance, that the well-known "double scaling limit" of these models [7], governs the behaviour of the density of eigenvalues near the edge of the support of the spectrum [8]. It turns out to be universal in a small range related to the order of multicriticality. In this framework one is led to consider probability distribution of N by N matrices φ, with a weight

$$P(\varphi) = \frac{1}{Z} \exp(-N \operatorname{Tr}[V(\varphi)]), \tag{1.1}$$

in which V is a polynomial of degree $2p$. (For simplicity we shall limit ourselves to even potentials, but the qualitative conclusions should not change for an arbitrary V.) The density of eigenvalues for a gaussian model, a quadratic V,

$$V = \tfrac{1}{2}m^2\varphi^2, \tag{1.2}$$

in the large-N limit, obeys the celebrated Wigner semi-circle law [9]:

$$\rho_0(\lambda) = \frac{m^2}{2\pi}\sqrt{\frac{4}{m^2} - \lambda^2} \tag{1.3}$$

(We define the density of eigenvalues with the normalization

$$\rho(\lambda) = \langle \frac{1}{N} \operatorname{Tr}\, \delta(\lambda - \varphi)\rangle,$$

so that the integral of ρ over the whole spectrum is one.) It is easy to extend this calculation to an arbitrary potential [10],

$$V = \sum_{k=1}^{p} \frac{g_k}{2k} \varphi^{2k}. \tag{1.4}$$

The result depends in a detailed way on the potential; we quote here the result for later use:

$$\rho(\lambda) = \frac{1}{\pi} P(\lambda) \sqrt{a^2 - \lambda^2}, \tag{1.5}$$

in which P is the even polynomial of degree $2p-2$ given by

$$P(\lambda) = \tfrac{1}{2} \sum_{k=1}^{p} g_k \sum_{n=0}^{k-1} \binom{2n}{n} \left(\frac{a^2}{4}\right)^n \lambda^{2k-2n-2} \tag{1.6}$$

and a, the endpoint of the spectrum, is given by the algebraic equation

$$\tfrac{1}{2} \sum_{k=1}^{p} g_k \binom{2k}{k} \left(\frac{a^2}{4}\right)^k = 1. \tag{1.7}$$

The density ρ depends explicitly upon V, and also the correlations function; however, if the correlation functions are measured on the right scale set by the density, they are indeed universal, and it is the aim of this paper to prove it. More specifically let us consider the (connected) correlation function ($\mu \neq \nu$):

$$\rho_c(\mu, \nu) = \left\langle \frac{1}{N} \mathrm{Tr}\ \delta(\mu - \varphi) \frac{1}{N} \mathrm{Tr}\ \delta(\nu - \varphi) \right\rangle - \rho(\mu)\rho(\nu). \tag{1.8}$$

We do not know of any explicit form, analogous to (1.5), of this function in the large-N limit. However, if we take μ and ν at a distance of order $1/N$, but both at a finite distance from the endpoints of the spectrum, this function has indeed a universal form; (note that there are N eigenvalues in the interval $[-a, +a]$, and it is thus natural to consider a scaling limit in which $N(\mu - \nu)$ is finite). Similarly it will be shown that higher correlation functions do exhibit universal properties in the same scaling range. In sect. 2 we show how to calculate $\rho(\mu, \nu)$ and the higher connected correlation functions. The key to the solution is provided by an explicit asymptotic representation of the orthogonal polynomials with respect to the measure $\exp[-NV(\lambda)]$. In sect. 3 we outline another argument in favour of universality based upon renormalization group ideas: the existence of a stable gaussian fixed point provides a simple and intuitive understanding of the ubiquitous gaussian statistics.

2. Correlation functions of eigenlevels

In appendix A it is shown that the density of eigenvalues, as well as the higher connected correlation functions, are determined by the kernel

$$K(\mu, \nu) = \frac{1}{N} \sum_{0}^{N-1} \psi_n(\mu)\psi_n(\nu), \tag{2.1}$$

in which

$$\psi_n(\lambda) \equiv p_n(\lambda) \exp(-NV(\lambda)/2), \tag{2.2}$$

and the p_n's are the orthogonal polynomials defined by

$$\int_{-\infty}^{+\infty} d\lambda \exp[-NV(\lambda)]\, p_n(\lambda) p_m(\lambda) = \delta_{n,m}. \tag{2.3}$$

One finds

$$\rho(\lambda) = K(\lambda, \lambda), \tag{2.4}$$

$$\rho_c(\mu, \nu) = -[K(\mu, \nu)]^2, \tag{2.5}$$

and similar formulae for higher correlation functions *; (the m-point function is simply the determinant of the m by m matrix $K(\lambda_i, \lambda_j)$). A standard identity (Christoffel–Darboux) allows one to express the kernel K in terms of the two polynomials p_N and p_{N-1} only (A.12).

Our approach consists of guessing (or to put it more academically, of proposing an ansatz for) the form of the polynomials $p_n(\lambda)$ for $n = N + O(N^0)$ (which, incidentally, is not the same as $n = O(N)$). We guess that ψ_n is given in that range by

$$\psi_n(\lambda) = \frac{1}{\sqrt{f(\lambda)}} \cos[N\zeta(\lambda) - (N-n)\varphi(\lambda) + \chi(\lambda)], \tag{2.6}$$

with $f(\lambda)$, $\zeta(\lambda)$, $\varphi(\lambda)$ and $\chi(\lambda)$ to be determined; $(N-n)$ is an integer much smaller than N. This guess is motivated by the known asymptotic form of $\psi_n(\lambda)$ for the gaussian case.

We may note in passing that even in the recent mathematical literature [12] the form of $\psi_n(\lambda)$ in the same asymptotic regime is known for only a few specific

* One possible approach to explore universality may be to study whether or not $K(\mu, \nu)$ changes when we vary $V(\lambda)$ in such a way to keep, say, $\rho(\mu)$ and $\rho(\nu)$ unchanged; see ref. [11]. From our explicit result (2.15) and (2.17), we see that it is not true in general. For $(\mu-\nu)$ of order $1/N$, we may perhaps consider the variation of $K(\mu, \nu)$ when we vary $V(\lambda)$ keeping $\rho(\mu+\nu)/2)$ fixed. The difficulty is that in this regime one cannot naively differentiate the orthogonal polynomials.

choices of $V(\lambda)$. Our ansatz (2.6), as will be proved below, thus represents a significant advance in the study of orthogonal polynomials.

Orthogonal polynomials are completely determined by their defining relations (2.3) (see appendix B). Orthonormality implies that

$$\delta_{n,m} = \int_{-\infty}^{+\infty} d\lambda\ \psi_n(\lambda)\psi_m(\lambda)$$

$$= \tfrac{1}{2}\int_{-\infty}^{+\infty} d\lambda\ [f(\lambda)]^{-1}[\cos\{(n-m)\varphi(\lambda)\}$$

$$+\cos\{2N\zeta(\lambda) - (2N-n-m)\varphi(\lambda) + 2\chi(\lambda)\}]. \tag{2.7}$$

In the large-N limit, with n and m close to N, the second term oscillates wildly as $N \to \infty$ as long as $\zeta(\lambda)$ varies over some finite range as λ runs over the whole real axis, and so does not contribute. For $n = m$, we then have the condition

$$\tfrac{1}{2}\int d\lambda\ [f(\lambda)]^{-1} = 1. \tag{2.8}$$

(A priori the integral runs over the whole real axis, but it will become clear below that it is in fact limited to a finite symmetric interval $[-a, +a]$.)

Orthogonality is satisfied for $(n-m)$ odd if we suppose that

$$\varphi(-\lambda) = \pi - \varphi(\lambda) \tag{2.9}$$

and for $(n-m)$ even if we set $d\lambda/d\varphi$ to be proportional to $f(\lambda)$. The constant of proportionality is fixed by the fact that $\varphi(0) = \pi/2$, as required from (2.9) by continuity.

We thus determine

$$\varphi(\lambda) = \frac{\pi}{2}\int_{\lambda} d\lambda'\ [f(\lambda')]^{-1}. \tag{2.10}$$

We next check the identity (B.3) relating $\lambda\psi_n(\lambda)$ to $\psi_{n\pm1}$ (and assuring us that $p_n(\lambda)$ as defined in (2.2) is in fact a polynomial). The identity implies that

$$\lambda = 2Q_{N,N-1}\cos\varphi(\lambda). \tag{2.11}$$

If we define

$$a \equiv 2Q_{N,N-1}, \tag{2.12}$$

we have

$$\lambda = a\cos\varphi(\lambda) \tag{2.11'}$$

and

$$f(\lambda) = \frac{\pi}{2}\sqrt{a^2 - \lambda^2} = \frac{\pi}{2} a \sin[\varphi(\lambda)]. \tag{2.13}$$

We thus learn that the upper limit of integration in (2.10) should be a.

At this stage we still have to determine $\zeta(\lambda)$. Requiring $K(\lambda, \lambda) = \rho(\lambda)$ we obtain

$$\frac{\mathrm{d}\zeta(\lambda)}{\mathrm{d}\lambda} = -\pi\rho(\lambda). \tag{2.14}$$

We have thus reached our goal of determining the kernel K directly in terms of the density $\rho(\lambda)$:

$$K(\mu, \nu) = \frac{a}{2N}(\mu - \nu)^{-1}[f(\mu)f(\upsilon)]^{-1/2} \cos[Nh(\mu)] \cos[Nh(\nu)]$$
$$\times\{\cos[\varphi(\nu)] + \tan[Nh(\nu)] \sin[\varphi(\nu)] - (\nu \leftrightarrow \mu)\}, \tag{2.15}$$

where we have defined

$$h(\lambda) = \zeta(\lambda) + \frac{1}{N}\chi(\lambda). \tag{2.16}$$

An alternative form reads

$$K(\mu, \nu) = -\frac{a}{4N}(\mu - \nu)^{-1}[f(\mu)f(\upsilon)]^{-1/2}$$
$$\times\{[\cos[\varphi(\mu)] - \cos[\varphi(\nu)]][\cos[N(h(\mu) + h(\nu))]$$
$$+ \cos[N(h(\mu) - h(\nu))]]$$
$$+ [\sin[\varphi(\mu)] - \sin[\varphi(\nu)]] \sin[N(h(\mu) + h(\nu))]$$
$$+ [\sin[\varphi(\mu)] + \sin[\varphi(\nu)]] \sin[N(h(\mu) - h(\nu))]\}. \tag{2.17}$$

To summarize, given $\rho(\lambda)$ we determine $\zeta(\lambda)$ by (2.14) and use $f(\lambda)$ and $\varphi(\lambda)$ as given in (2.11) and (2.13). The kernel K and hence all the relevant correlation functions are determined as in (2.16) and (2.17). Notice that the function $\chi(\lambda)$ cannot be determined without dragging in less leading terms in $1/N$. In the gaussian case $\chi(\lambda) = \frac{1}{2}\varphi(\lambda) - \frac{1}{4}\pi$ and this formula is known to hold also for a pure quartic potential [12]; we can conjecture that it holds in general. However, it will turn out that $\chi(\lambda)$ drops from the scaling form of the correlation functions. Since

it appears inside trigonometric functions it would be erroneous to drop it relative to $N\zeta(\lambda)$ (cf. (2.16)) as one would be tempted to do naively.

At this point it is worth pointing out that the kernel K is indeed a function of μ, ν and a functional of the density $\rho(\lambda)$ which may be related to the potential V (see (1.5)–(1.7)); indeed the potential does not enter directly, except from the dependence through ρ, but K is not a *function* of μ, ν, $\rho(\mu)$ and $\rho(\nu)$, but a functional of the whole ρ. The form (2.17) of the kernel K should be tested numerically for arbitrary μ and ν and fitted to the unknown function. For $\mu - \nu$ of order $1/N$ (so that there can still be a finite number of eigenvalues between μ and ν), and both at a finite distance from the end points $\pm a$ of the spectrum, we do obtain a universal result. In this limit only the third term in (2.17) survives and we obtain

$$K(\mu, \nu) \xrightarrow[\mu-\nu\sim O(1/N)]{} \frac{\sin[2\pi N\delta\mu\rho(\bar{\mu})]}{2\pi N\delta\mu}, \tag{2.18}$$

where $\bar{\mu} \equiv \frac{1}{2}(\mu + \nu)$ and $\delta\mu \equiv \frac{1}{2}(\mu - \nu)$. This means that the correlation function takes the form

$$\rho(\mu, \nu) = \rho(\mu)\rho(\nu)\left[1 - \left(\frac{\sin x}{x}\right)^2\right] \tag{2.19}$$

with

$$x = 2\pi N\delta\mu\rho(\bar{\mu}). \tag{2.20}$$

For $(\mu - \nu)$ small this implies that $\rho(\mu, \nu)$ vanishes as $(\mu - \nu)^2$, as expected from the Van der Monde determinant in the measure:

$$\rho(\mu, \nu) \sim \tfrac{1}{3}\pi^2\rho(\bar{\mu})^4[N(\mu - \nu)]^2. \tag{2.21}$$

Note that the correct limit is obtained if we do keep the oscillating sine function in (2.19).

For $(\mu - \nu)$ far apart, that is of order 1, $K(\mu, \nu)$ and hence $\rho_c(\mu, \nu)$ oscillates wildly as is expected physically, since there are $O(N)$ eigenlevels between μ and ν. One possibility is to define a smoothed $\rho_c(\mu, \nu)$ by averaging over intervals $\Delta\mu$ and $\Delta\nu$ much less than $O(1)$ but larger than $O(N^{-1})$, so that in each interval, $\Delta\mu$ or $\Delta\nu$, there are many eigenlevels. Upon smoothing, i.e. replacing $\cos^2(Nh)$ by $\frac{1}{2}$, we obtain the remarkably simple result

$$\rho_c^{\text{smooth}}(\mu, \nu) = -\langle K(\mu, \nu)]^2\rangle = \frac{-1}{2N^2\pi^2}\frac{1}{(\mu - \nu)^2}\frac{(a^2 - \mu\nu)}{[(a^2 - \mu^2)(a^2 - \nu^2)]^{1/2}}. \tag{2.22}$$

Perhaps surprisingly, $\rho_c(\mu, \nu)$ in the smoothed sense described here, exhibits universality in a way that $\rho(\mu)$ does not: the only dependence of $\rho_c^{\text{smooth}}(\mu, \nu)$ on the potential V is through the endpoint of the spectrum a. The result in (2.22) is consistent with that in (2.18).

Similarly one can define smoothed higher point correlations [13]. One advantage of our formalism is that all these functions may be calculated explicitly. We find by direct calculation that, remarkably enough, the connected three-point correlation

$$\rho_c(\mu, \nu, \omega) = 2K(\mu, \nu)K(\nu, \omega)K(\omega, \mu) \tag{2.23}$$

and the connected four-point correlation

$$\rho_c(\mu, \nu, \omega, \sigma) = -K(\mu, \nu)K(\nu, \omega)K(\omega, \sigma)K(\sigma, \mu) + \text{five other similar terms} \tag{2.23'}$$

vanish identically under smoothing. This results suggests strongly that all the higher connected correlations vanish upon smoothing. This property that the smoothed $\rho_c(\mu_1, \mu_2, \ldots, \mu_P)$ vanishes for $P > 2$ is easy to prove in the regime in which $N(\mu_i - \mu_j)$ is of order one for all pairs i and j. The statistical distribution of the eigenvalues, when smoothed in the sense defined here, is thus universal and of Gaussian nature, but with non-trivial one-point and two-point functions. The clue to understanding this surprising result may lie in the renormalization group of the next section.

It is perhaps worth noting that we have obtained results in several regimes: (i) for $(\mu - \nu) \sim O(1)$ the explicit form (2.17) and the related universal smoothed forms (2.22) and (2.23) (ii) for $N(\mu - \nu) \sim O(1)$ the universal kernel (2.18) and (iii) the regime $N(\mu - \nu)$ much smaller than one (2.21).

We emphasize here that we have determined K, and hence all the relevant correlations, without referring to the potential $V(\lambda)$. The reader may wonder how we can determine ρ and hence K for a specific given $V(\lambda)$ within this approach. The key is to use (B.8) and (B.9). Differentiating (2.6) directly and using (2.14) we obtain for large N,

$$\frac{d\psi_n}{d\lambda} = \frac{N}{\sqrt{f(\lambda)}} \pi\rho(\lambda) \sin[N\zeta(\lambda) - (N-n)\varphi(\lambda) + \chi(\lambda)]. \tag{2.24}$$

According to (B.9) this should be equal to

$$\frac{d\psi_n}{d\lambda} = \frac{1}{\sqrt{f(\lambda)}} \sum_m \tilde{P}_{nm} \cos[N\zeta(\lambda) - (N-n)\varphi(\lambda) + \chi(\lambda)]. \tag{2.25}$$

This is possible only if the density of eigenvalues $\rho(\lambda)$ has the form

$$\rho(\lambda) = \frac{a}{\pi} \sin \varphi \sum_{j=0}^{p-1} c_j a^{2j} \cos^{2j}\varphi \tag{2.26}$$

for a potential V of degree $2p$ (we have taken out some factors of a for convenience). By equating (2.24) and (2.25) and using (B.8) we determine easily the coefficients c_j in terms of V:

$$c_j = \tfrac{1}{2} \sum_{k=0}^{p-j-2} g_{j+k-1} \binom{2k}{k} \left(\frac{a^2}{4}\right)^k . \tag{2.27}$$

Indeed (2.26), after relating φ to λ by (2.11'), is in complete agreement with the form (1.5) and (1.6) for ρ which has been derived easily earlier [10] from the saddle-point equations (which lead to an integral equation solved by the Riemann–Hilbert method).

An interesting point is that the quantity $a \equiv 2Q_{N,N-1}$ which appears in the course of our calculation (in (2.12)) has acquired physical meaning as the endpoint of the spectrum of eigenvalues. It may be shown that the endpoint a as determined here (cf. (B.12)) agrees with that derived earlier and stated in (1.7).

Our result for the orthogonal polynomials stated in (2.6) with $f(\lambda)$, $\zeta(\lambda)$ and $\varphi(\lambda)$ given later, may have applications elsewhere. The form in (2.6) looks like a WKB approximation to some second-order differential equation. It is worth remarking, however, that in fact orthogonal polynomials $p_n(\lambda)$, and by extension $\psi_n(\lambda)$, satisfy a second-order differential equation only in certain simple cases, such as $V(\lambda)$ quadratic. In this connection, note that the overall factor $[f(\lambda)]^{-1/2} = [\frac{1}{2}\pi\sqrt{a^2-\lambda^2}\,]^{-1/2}$ is not $[\rho(\lambda)]^{-1/2}$ as one might have guessed; $f(\lambda)$ has indeed the same form as $\rho(\lambda)$ in the gaussian case but with the important conceptual difference that the true endpoint a, that is the endpoint relevant for the potential $V(\lambda)$, enters here.

It is tempting to conjecture that our conclusions would hold also for non-polynomial V. It would be interesting to see if our approach would also hold for multi-matrix models.

3. Renormalization group approach

The idea of understanding ensembles of large random matrices by a renormalization group technique was introduced in ref. [14] and we repeat here the main points for completeness. The basic and simple minded idea is that if we consider correlations involving only a finite number of eigenvalues of N by N matrices, in

the large-N limit it should not matter whether the matrices are N by N or $(N-1)$ by $(N-1)$. Therefore we can try to integrate out one column and one row and find the corresponding flow equations; we thus represent the N by N matrix φ_N as

$$\varphi_N = \begin{bmatrix} \varphi_{N-1} & u \\ u^* & \alpha \end{bmatrix}, \tag{3.1}$$

in which φ_{N-1} is an $N-1$ by $N-1$ matrix, u an (N − 1)-component complex vector and α a c-number. The renormalization group flow is defined by

$$\frac{1}{Z_{N-1}} \exp\left(-\left[(N-1)\ \mathrm{Tr}\{V'(\varphi_{N-1})\}\right]\right)$$

$$= \frac{1}{Z_N} \int \mathrm{d}\alpha\ \mathrm{d}^{N-1}u\ \mathrm{d}^{N-1}u^* \exp\left(-\left[N\ \mathrm{Tr}\{V(\varphi_N)\}\right]\right). \tag{3.2}$$

Of course, except for the trivial gaussian case, one is unable to write explicitly the corresponding flow equations; in fact our way of writing the left-hand side of (3.2) is slightly misleading: in general V' is not a polynomial any more and products of traces of powers of φ_{N-1} are also generated [15]. However, it is easy to do a perturbation expansion near a gaussian. Starting for simplicity from a φ^4 potential

$$V(\varphi) = \tfrac{1}{2}\varphi^2 + \tfrac{1}{4}g\varphi^4 \tag{3.3}$$

one finds that the flow (3.2) does not generate new terms at one-loop and the result (see ref. [13] for the explicit calculation) is simply

$$g' = g - \frac{1}{N}\beta(g) + \mathrm{O}(N^{-2}) \tag{3.4}$$

with

$$\beta(g) = -g - 6g^2 + \mathrm{O}(g^3). \tag{3.5}$$

The (unstable) fixed point $g^* = -\frac{1}{6}$ governs the "double scaling limit" and is of no interest to us here. However, there is a stable fixed point at the origin and the result can be easily extended to more general potentials than (3.3). The universal properties of large matrices, the fact that their correlations when expressed in terms of the density, are identical to that of a gaussian ensemble, is clearly related to the triviality of the fixed point. However, this argument does not provide an explicit form for these correlations and is thus much less powerful than the detailed calculations of the previous section.

4. Conclusion

We have found a remarkable universality in the correlation between eigenvalues in two regimes. In the first regime we have smoothed the oscillations on a small interval large compared to N^{-1}. In the second one we have shown that the correlation functions for eigenvalues at a distance of order $1/N$, on a scale in which the N eigenvalues spread out over a finite interval, are universal and thus equal to that of a simple gaussian ensemble. This is true provided the eigenvalues stay away from the endpoints of the spectrum. We have established this result for a probability measure of the form

$$P(\varphi) = \frac{1}{Z}\exp(-N \operatorname{Tr}[V(\varphi)])$$

with any polynomial V. Are there unexpected limitations and other correlations of a different type that the present numerical experiments have failed to see? How much more general is this universality? These are all questions worth investigating. It is clear that by taking appropriate limits on the polynomials one can generate many distributions. We are under the impression that within random matrices there is little room left for other possibilities. This does not say, of course, that for the so-called "quantum chaos", there is no possibility for spectra "in-between" the integrable case and the gaussian statistics of non-integrable systems.

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Appendix A

We collect here, for the sake of completeness, and also to fix our normalization, some of the basic facts about random matrices. The weight in (1.1) may be written in terms of the probability distribution of the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the matrix φ (once the unitary transformation which diagonalizes φ is integrated over):

$$\rho(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{N}\Delta^2\{\lambda\} \exp\left(-N\sum_{i=1}^{N} V(\lambda_i)\right), \tag{A.1}$$

where

$$\Delta\{\lambda\} = \prod_{i>j}(\lambda_i - \lambda_j) = \det\{(\lambda_i)^{j-1}\}. \tag{A.2}$$

(For the reader who has never seen this well-known jacobian let us simply note that the parameterization to eigenvalues and eigenvectors is singular only if two eigenvalues coincide; therefore the jacobian is a power of Δ and the power two is easily fixed by counting dimensions.) Long ago Mehta [1] noticed that by taking linear combinations of the columns of the matrix $(\lambda_i)^{j-1}$ one can, up to a normalization factor, replace the monomials $(\lambda)^{j-1}$ by arbitrary polynomials $p_n(\lambda)$ with degrees $n = 0, 1, \ldots, N-1$. One chooses those polynomials to be the orthogonal polynomials defined by

$$\int_{-\infty}^{+\infty} d\lambda \exp[-NV(\lambda)]\, p_n(\lambda) p_m(\lambda) = \delta_{n,m}, \tag{A.3}$$

with $p_n(\lambda) = a_n\lambda^n +$ terms of lower degree, and a_n positive. We can then write

$$\Delta\{\lambda\} = \left(\prod_0^{N-1} a_i^{-1}\right) \det\{p_i(\lambda_j)\}, \tag{A.4}$$

namely in terms of a determinant of a matrix whose ijth element is $p_i(\lambda_j)$, with $i = 0, 1, \ldots, N-1$ and $j = 1, 2, \ldots, N$.

It is convenient to introduce the functions

$$\psi_n(\lambda) = p_n(\lambda) \exp(-NV(\lambda)/2), \tag{A.5}$$

which are orthogonal with a flat measure. Putting all this into (A.1) we find

$$\rho(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{N^N}{N!} \det\{K(\lambda_i, \lambda_j)\}, \tag{A.6}$$

where we have defined the kernel

$$K(\mu, \nu) = \frac{1}{N} \sum_{i=0}^{N-1} \psi_i(\mu)\psi_i(\nu). \tag{A.7}$$

More precisely, in (A.6) we have the determinant of a matrix whose (ij)th element is the function $K(\mu, \nu)$ evaluated at $\mu = \lambda_i$ and $\nu = \lambda_j$.

The normalization

$$\int d\lambda_1\, d\lambda_2 \ldots\, d\lambda_N\, \rho(\lambda_1, \lambda_2, \ldots, \lambda_N) = 1 \tag{A.8}$$

fixes Z in (A.1) to be precisely such that it cancels the $\prod_i a_i$ factor in (A.4).

Indeed, using (A.6) repeatedly, we can calculate all correlation functions. In particular the density of eigenvalues

$$\rho(\lambda) \equiv \int d\lambda_2 \dots d\lambda_N \, \rho(\lambda, \lambda_2, \dots, \lambda_N) = K(\lambda, \lambda) \tag{A.9}$$

and the two-point correlation function

$$\rho(\mu, \nu) \equiv \int d\lambda_3 \dots d\lambda_N \, \rho(\mu, \nu, \lambda_3, \dots, \lambda_N)$$

$$= \frac{1}{N}\delta(\mu - \nu)\rho(\mu) + K(\mu, \mu)K(\nu, \nu) - [K(\mu, \nu)]^2. \tag{A.10}$$

The first term on the right-hand side of (A.10) can be ignored if we always consider $\rho(\mu, \nu)$ for $\mu \neq \nu$. The connected correlation function is given by

$$\rho^c(\mu, \nu) \equiv \rho(\mu, \nu) - \rho(\mu)\rho(\nu) = -[K(\mu, \nu)]^2. \tag{A.11}$$

If we can determine the kernel $K(\mu, \nu)$ then we know all the correlation functions. A useful form for K is obtained by multiplying and dividing (A.7) by $(\mu - \nu)$ and using (B.3) (the Christoffel–Darboux identity):

$$K(\mu, \nu) = \frac{1}{N} Q_{N-1,N} \frac{[\psi_N(\mu)\psi_{N-1}(\nu) - \psi_{N-1}(\mu)\psi_N(\nu)]}{\mu - \nu}. \tag{A.12}$$

Appendix B

To make the paper self-contained, we review here some well-known facts about orthogonal polynomials that we will need in the text.

Since $\lambda p_n(\lambda)$ is a polynomial of degree $(n + 1)$ we may write

$$\lambda p_n(\lambda) = \sum_m^{n+1} Q_{nm} p_m(\lambda) \tag{B.1}$$

and (A.3) tells us that the expansion coefficients are given by

$$Q_{nm} = \int d\lambda \, \exp(-NV(\lambda)) \lambda p_n(\lambda) p_m(\lambda). \tag{B.2}$$

Thus Q_{nm} considered as a matrix is symmetric and of type [1, 1] (a matrix is called type $[a, b]$ if all the matrix elements with $n - m > a$ and $m - n > b$ vanish). Furthermore if $V(\lambda)$ is even in λ, then $p_n(\lambda)$ is even for n even, and odd for n odd, and so $Q_{n,n} = 0$. It follows that (B.1) reads

$$\lambda p_n(\lambda) = Q_{n,n+1} p_{n+1}(\lambda) + Q_{n,n-1} p_{n-1}(\lambda). \tag{B.3}$$

Similarly to (B.1) we can write

$$\frac{\mathrm{d}p_n(\lambda)}{\mathrm{d}\lambda} = \sum_m^{n-1} P_{nm} p_m(\lambda). \tag{B.4}$$

Evaluating the right-hand side of the identity

$$0 = \int \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}(p_n p_m \exp(-NV(\lambda))) \tag{B.5}$$

and using (A.3) and (B.1) we learn that

$$P_{nm} + P_{mn} - N[V'(Q)]_{mn} = 0. \tag{B.6}$$

In other words the matrix $\tilde{P}$ defined by

$$\tilde{P} = P - \tfrac{1}{2}NV'(Q) \tag{B.7}$$

is antisymmetric and hence of type $[2p-1, 2p-1]$ for $V(\lambda)$ a polynomial of degree $2p$. These facts suffice to determine $\tilde{P}$ completely

$$\tilde{P}_{mn} = -\tfrac{1}{2}N[V'(Q)]_{mn} \tag{B.8}$$

for $m > n$, 0 for $m = n$, and for $m < n$ determined by antisymmetry. Note that the matrix $\tilde{P}$ appears naturally in

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}\lambda} = \sum_m \tilde{P}_{nm}\psi_m(\lambda). \tag{B.9}$$

From their definitions we see that the matrices P and Q satisfy the commutation relation $[P, Q] = -1$, which may be solved to give

$$P_{n+1,n}Q_{n+1,n} = n + 1. \tag{B.10}$$

P is explicitly determined in terms of $V'(Q)$ according to (B.7) and (B.8). For n and $m = N + O(N^0)$ the matrix elements Q_{nm} are equal to some number Q up to corrections of order $1/N$ and so to leading order

$$(Q^{2k-1})_{n+2j+1,n} = \binom{2k-1}{k+j} Q^{2k-1}. \tag{B.11}$$

Thus, (B.10) reduces to

$$1 = \tfrac{1}{2}\sum_{k=1}^{p} g_k \binom{2k}{k} Q^{2k}. \tag{B.12}$$

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