EMBEDDING PROBLEMS: GEOMETRIC AND TOPOLOGICAL ASPECTS

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A survey of results obtained after 1976 on questions of the isometric immersions of Riemannian spaces in a Euclidean space, the immersions and embeddings of differential manifolds, and the immersions with minimal absolute curvature is presented.

Problems of the immersions and embeddings of manifolds in Euclidean and other spaces are some of the central problems both in differential geometry as well as in topology and are studied in these disciplines from the most different points of view. An acquaintance with the results established in them, the methods of obtaining them, and the problems that emerge can prove useful for subsequent development. Our survey is devoted mainly to three aspects: isometric immersions of Riemannian spaces, smooth topological immersions and embeddings of differentiable manifolds, and immersions with minimal total absolute curvature. The connection between the geometric and topological aspects shows up particularly strongly in the third aspect.

1. Isometric Immersions

The isometric immersion of a Riemannian space in a Euclidean space or in another Riemannian space is one of the methods for constructing submanifolds of these spaces, possessing new interesting geometric properties. The theory of isometric immersions is connected with difficult questions of the solvability of nonlinear systems of differential equations locally and in particular with difficult questions when the consideration is global, as well as with topological questions; in this theory, together with drawing in a broad arsenal of mathematical tools, intuitive geometric ideas also are used. Many papers have been devoted to this theory; those up to 1976 have been surveyed in [29, 41, 42]. We shall deal with more recent papers.

1. Isometric Immersions of Manifolds of Nonpositive Curvature. By a well-known Hilbert theorem the Lobachevskii plane cannot be immersed regularly and isometrically into the three-dimensional Euclidean space E^3. In [25] Efimov strengthened this theorem, having proved the nonimmersibility of class C^4 of the Lobachevskii halfplane, i.e., of an infinite domain on the Lobachevskii plane bounded by a total geodesic, into E^3. Thus, the property of nonimmersibility can hold not only for complete manifolds but also for manifolds with boundary, even if the curvature does not change sign. The impossibility of a class C^2-isometric immersion of the Lobachevskii halfplane was proved in [18].

Since in [24] Efimov proved a well-known general theorem on the impossibility of a class C^4 isometric immersion of a complete two-dimensional manifold of Gaussian curvature $K \leq -\alpha^2 < 0$, where $\alpha = \text{const}$, into E^3, the assumption emerged that a halfplane, i.e., a domain bounded by a total geodesic, on a manifold with variable curvature $K \leq -\alpha^2 < 0$ also cannot be immersed into E^3. In [19] this question was resolved under an additional condition that the normal curvature along the geodesic edge cannot change sign. Here the condition on the edge is not intrinsic; therefore, we can make a somewhat weaker assumption, now formulatable in intrinsic terms, on the nonimmersibility of a plane with metric curvature $K \leq -\alpha^2 < 0$, from which some domain has been excised, say, a geodesic circle of definite radius. Efimov's proof of the nonimmersibility of the Lobachevskii halfplane was based on the following two statements: 1) the area of any polygon on the immersed domain, whose sides are segments of geodesic or asymptotic lines, is bounded by a number which depends solely on the number of sides; 2) in E^3 a complete asymptotic line cannot have an ε-strip with a metric of constant negative curvature for any $\varepsilon > 0$. The second statement was proved by Efimov in [26]. Analogous statements have not been proved in the case of a variable curvature. In connection with the mentioned

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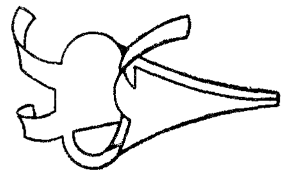

Fig. 1

Efimov's theorem a natural question arises on the description of those domains of the Lobachevskii plane which can be isometrically immersed into E^3. In [40] Poznyak considered isometric immersions of infinite polygons with a finite or an infinite number of sides, not containing the halfplane. The intersection of closed halfplanes whose boundaries are straight lines on L^2 not having common points is called an infinite polygon. We can examine the set of polygons not containing the halfplane and in which each side has a parallel side called an adjacent side. Two classes M_1 and M_2 of such polygons are introduced. The first class M_1 consists of all those polygons for which a horocycle O exists in the plane, such that the lower bound of the lengths of the orthogonal projections of the sides onto this horocycle is positive. The second class M_2 consists of all those infinite polygons for which the lower bound of the lengths of the orthogonal projections of the sides onto one of the sides is positive. The following theorem is proved in [40]:

THEOREM. Any polygon of class M_1 or M_2 can be isometrically immersed into E^3.

This theorem's proof is based on the following geometric idea. We explain it for polygons of the first class. A horocycle O_α, equivalent to the horocycle O, is found, cutting off all the vertices of the polygon. As is well known, a universal covering of the pseudosphere, or rather one half of it, yields an isometric immersion of some horocircle, as which we can take O_α. Therefore, on the part of the polygon, included within O_α, there arises a regular Chebyshev net. It remains to extend it onto the parts located outside O_α. We join the polygon's vertices to the point at infinity Q of the horocircle O_α by geodesic lines a_i and we construct an infinite equidistant strip along a_i of width 2h. For some width 2h the polygon's part left outside the horocircle O_α falls into this strip. In this strip we introduce a semigeodesic coordinate system and in it we write the immersion's system of equations. The initial data for solving this system are prescribed on a_i such that on the part of a_i lying outside O_α they extend the values of the desired functions onto the part of a_i inside O_α corresponding to the pseudosphere (see Fig. 1).

Later we use the existence and uniqueness theorem for the solution of the immersion's system in the strip, proved by Poznyak [36]. Thus, we can obtain the immersions of polygons of class C^∞.

As is well known, the question of the immersion of the Lobachevskii plane L^2 into E^3 reduces to the finding of the solution of the equation for the angle ω between the asymptotic lines

$$\omega_{uv} = \operatorname{Sin} \omega. \tag{1}$$

The inequalities $0 < \omega < \pi$ must be satisfied on a regular surface. If the initial data $\omega(0, v)$ and $\omega(u, 0)$ of class C^k or C^∞ are prescribed on the lines $u = 0$ and $v = 0$, then on the whole plane (u, v) there exists a solution of Eq. (1), reducing to the prescribed functions on the coordinate lines and belonging to the same regularity class.

In [37] Poznyak proved that to any such solution $\omega(u, v)$ there corresponds a vector function $r = r(u, v) \in C^3$, specified on the plane (u, v), such that the graph of this function in the region where $\omega \neq k\pi$ is a surface of constant negative curvature $K = -1$. Here the coordinate lines u and v on the surface indicated form an asymptotic net with grid angle $\omega(u, v)$.

Shikin [59] proved that any geodesic circle with metric negative curvature K, being, in a neighborhood of each point, a function of class C^1 in some semigeodesic coordinate system, can be immersed isometrically as a surface of class C^2. The possibility of an immersion under the condition that function K is of class $C^{2,\alpha}$ was proved earlier by Poznyak [36], while in the case of C^2, by Shikin [61].

Rozendorn [43] defines a helical surface in a four-dimensional Euclidean space as a surface whose metric is the gyration metric $ds^2 = du^2 + B^2(u)dv^2$ and after its reduction to this form the torsion coefficients and the second quadratic forms are independent of the coordinate v. It is proved that the Lobachevskii plane admits of an isometric immersion into E^4 as a regular helical surface. Kadomtsev [27] proved the impossibility of the isometric immersion of L^2 into E^n, under which any two points have congruent neighborhoods matching one another by a motion along a geodesic joining these points. Immersions with a pole were examined as well.

In [150] Moore examines the isometric immersions of space forms into space forms. A proof is presented of Cartan's theorem that n principal directions exist under a local immersion of an n-dimensional Lobachevskii space into E^{2n-1}. A direction on a submanifold is called principal if it is principal for the second form defined by any normal vector. The existence was proved in [150] of 2^n asymptotic directions, i.e., of directions for which all the second quadratic forms are simultaneously zero. It was proved that if an M^n-complete simply connected Riemannian manifold of constant curvature k is isometrically immersed into a (2n — 1)-dimensional Riemannian manifold of constant curvature $K > k$, then any n linearly independent asymptotic vector fields $Z_1, \ldots, Z_n$ determine a global coordinate system with coordinate vectors Z_i.

Isometric immersions of domains of the n-dimensional Lobachevskii space L^n into E^{2n-1} were studied in greater detail by Aminov [2, 3]. Submanifolds with a variable curvature were examined in parallel. The following general statement was proved: if n principal directions exist on a submanifold M^n of negative curvature in a Euclidean space E^N, then they are holonomic, i.e., in a neighborhood of each point of M^n there exist coordinates such that the coordinate lines are tangent to the principal directions. In these coordinates the metric in L^n can be written as $ds^2 = \sum_{i=1}^{n} \operatorname{Sin}^2\sigma_i du_i^2$, where $\sum_{i=1}^{n} \operatorname{Sin}^2\sigma_i = 1$. We remark that systems of orthogonal coordinates in an n-dimensional Euclidean space with the condition of constancy of the sum of the metric coefficients were examined initially by Guichard and then by Darboux and Bianchi. In [2] it was proved that the Codazzi—Ricci equations have the Gauss equations as their integrability condition. Therefore, the whole system of equations for the immersion of L^n into E^{2n-1} reduces to the system

$$R_{ijij} = \operatorname{Sin}^2\sigma_i \operatorname{Sin}^2\sigma_j, \quad R_{ijkj} = 0, \quad i \neq k, \quad \sum_{i=1}^{n} \operatorname{Sin}^2\sigma_i = 1. \tag{2}$$

It was proved that to construct an arbitrary local immersion of L^n into E^{2n-1} it is necessary and sufficient to indicate a method for constructing in L^n an orthogonal coordinate system with the condition $\sum_{i=1}^{n} \operatorname{Sin}^2\sigma_i = 1$, which leads to system (2). It was established that the arbitrariness in the prescribing of the initial data for solving this system consists of n(n — 1) functions of one argument. System (2) is the multidimensional analog of the "sine-Gordon" equation $\omega_{xx} - \omega_{yy} = \operatorname{Sin}\omega$ or is another way of writing (1). The "sine-Gordon" equation is studied as well in theoretical physics when describing various physical phenomena. The three-dimensional "sine-Gordon" equation

$$\omega_{zz} - \omega_{xx} - \omega_{yy} = \operatorname{Sin}\omega \tag{3}$$

is derived in [28]. A certain combination of Eqs. (2) leads to an equation differing from (3) only by a certain supplementary summand in the right hand side (see (35) in [3]). It can be expected that the whole system (2) is of interest from the physical point of view.

Also considered in [3] were the properties of the Grassmann image of an arbitrary submanifold in Euclidean space, including $L^n \subset E^{2n-1}$. It turns out that the Grassmann image of L^n in E^{2n-1} is a regular submanifold Γ^n in the Grassmann manifold $G_{n-1,2n-1}$ and, if $G_{n-1,2n-1}$ has been endowed with the standard metric, then the metric in Γ^n has the form $ds^2 = \sum_{i=1}^{n} \operatorname{Cos}^2\sigma_i \cdot (du^i)^2$. The intrinsic curvature of the Grassmann image can be of any sign, but it turns out that the curvature $\bar{K}$ of the manifold $G_{n-1,2n-1}$ for area elements tangent to Γ^n is not arbitrary. Wong [200] proved that the curvature of the Grassmann manifold $G_{n,m}$ with min(n, m) >1 lies in the interval [0, 2]. It was proved in [3] that the curvature $\bar{K}$ for area elements tangent to Γ^n lies in the open interval (0, 1). Here, however, it has been established that for an area element tangent to the Grassmann image of an arbitrary submanifold V^n in the Euclidean space E^N, the curvature $\bar{K}$ can be expressed in terms of the second quadratic forms of submanifold V^n. If V^n has n principal directions, then the curvature $\bar{K} \in [0, 1]$. Submanifolds of a Euclidean space can be classified by the type of the Grassmann image: if the curvature $\bar{K} < 1$ for any area element tangent to Γ^n, then the type is hyperbolic, if $\bar{K} = 1$, then the type is parabolic, and if $\bar{K} > 1$, then the type is elliptic. However, if there are area elements with curvature greater than or less than unity, then the type can be mixed.

When $n = 3$ two classes of solutions of system (2) have been studied: 1) the curvature lines of one family are geodesic and in this case system (2) reduces to two equations: an ordinary differential equation and the "sine-Gordon" equation (both these equations are interconnected by constants); 2) functionally degenerate immersions, i.e., those for which $\sin \sigma_i$ depend on one argument $t(u_1, u_2, u_3)$ (in mechanics solutions of such kind are called selfsimilar). We remark that functionally degenerate immersions can be examined also for other manifolds by requiring that the second fundamental forms and the torsion coefficients depend on a smaller number of arguments than the manifold's dimension.

In [9] it is proved that the multidimensional analog of the Bianchi transformation of an n-dimensional submanifold of constant negative curvature in E^{2n-1} leads to a submanifold of the same constant negative curvature.

System (2) is examined as well by Tenenblat and Terng [184, 185], but without proving the fact that the whole system of equations of the immersion of L^n into E^{2n-1} reduces to this system. A multidimensional Backlund transformation is constructed, taking one immersion of a domain in L^n into another. The Backlund transformation for submanifolds in space forms is constructed by Tenenblat [182]. Asymptotic submanifolds in an immersed manifold $M^n \subset E^N$ are examined in [182]. They are defined as follows. Let s be the second fundamental form of manifold M^n. Then the q-dimensional linear subspace L, $0 < q < n$, from the tangent space T_xM is called asymptotic if a vector ξ orthogonal to T_xM exists such that $\langle s(X, Y), \xi \rangle = 0$ for all vectors X and $Y \in L$. A submanifold $V^q \subset M$ is called asymptotic if at each point of it the tangent space is an asymptotic subspace in T_xM. It is proved that if M^n has been nonsingularly isometrically immersed in E^N, then an $(n - 1)$-dimensional submanifold in M^n is characteristic for the immersion's system (in Cartan's sense) if and only if this submanifold is asymptotic.

Shefel' gave several generalizations of the concept of a saddle surface on multidimensional submanifolds of a Eulidean space. One of them has the following form: a complete m-dimensional $(m \geqslant 2)$ surface S in E^{m+p} is called a saddle surface if on any closed contour L belonging to the intersection of S with an arbitrary plane E^r $(2 \leqslant r < p + m)$ in E^{m+p} and deformable to a point on surface S we can span a two-dimensional simply connected surface contained in $S \cap E^r$. Glazyrin [20] proved that the Riemannian curvature of such a surface in some two-dimensional direction equals zero. A second generalization of a saddle surface by Shefel' is in terms of homology groups. Glazyrin presents a formulation of this generalization in [21]: a complete n-dimensional surface $F^n \subset E^{n+m}$ is called a k-saddle surface if for any r-dimensional plane E^r $(2 < r < n + m)$ the $(k + 1)$-dimensional Vietoris relative homology group $H_{k+1}(F, F \cap E^r) = 0$. In the paper there is proved the equivalence of this definition to the condition: for each normal the second quadratic form has no more than k eigenvalues of one sign. A generalization of the Chern—Kuiper theorem is proved as well.

Borovskii and Shefel' [15] have a geometric interpretation of Otsuki's lemma, basing the proof on the Chern—Kuiper theorem on the nonimmersibility into E^{2n-1} of an n-dimensional compact Riemannian manifold of nonpositive curvature. This interpretation is the following: n n-dimensional ellipsoids in E^{n+1} with a common center O can intersect a two-dimensional plane E^2 passing through point O in such a way that the ellipses obtained in the sections will be similar. Otsuki's lemma was generalized by Borisenko [12] to k-dimensional sectional curvatures; using this generalization he proved that an l-dimensional Riemannian manifold of strictly negative k-dimensional curvature cannot be locally immersed in E^{l+p} when $p \leqslant [(l-1)/(k-1)]-1$. The following generalization of the Chern—Kuiper theorem was proved: a compact l-dimensional Riemannian manifold M^l of nonpositive k-dimensional curvature cannot be isometrically immersed in an $(l + p)$-dimensional Euclidean space when $p \leqslant (l - 1)/(k - 1)$. The order of growth of the volume of complete noncompact surfaces of nonpositive k-dimensional curvature in a Euclidean space was estimated and it was shown, for example, that a manifold of nonpositive two-dimensional sectional curvature with a finite volume cannot be immersed isometrically into a Euclidean space of dimension $(3/2)l - 1$. Almgren's theorem on minimal surfaces homeomorphic to a sphere in the spherical space S^3 was generalized in this same paper. Borisenko proved that an analytic surface F^2, homeomorphic to a sphere in the spherical space S^3 of curvature 1, with Gaussian curvature $K \leqslant 1$, is a great sphere. Certain theorems were obtained in [10] on the nonimmersibility of compact manifolds into Riemannian manifolds with nonpositive exterior curvature. Immersions of complete connected pseudo-Riemannian manifolds of constant negative curvature into pseudo-Euclidean spaces were studied in [11]. If $R^k{}_{p,q}$ is a pseudo-Riemannian space the matrix of whose metric tensor has p positive and q negative eigenvalues, then in [11] it was proved that a complete connected pseudo-Riemannian manifold $V^l{}_{p,q}$ of constant curvature with $q \neq 0, 1, 3, 7$ cannot be immersed isometrically into a pseudo-Euclidean space $E_{p',q}{}^{2l-1}$, where $p' = 2l - 1 - q$.

We note as well that isometric immersions of pseudo-Riemannian spaces were examined also by Graves [102] and by Graves and Nomizu [103].

An interesting question is that of estimating the external diameter of an immersed manifold, which has been examined in many papers. A recent monograph [16] by Burago and Zalgaller presents numerous results of various authors on geometric inequalities, including the estimates of a submanifold's exterior diameter. Here a simple proof is given of a lower bound obtained earlier by Burago for the radius R of a ball containing a surface in E^3, in terms of the length L of the boundary, the area S, and the positive part of the integral curvature ω^+: $S \leqslant C(\omega^+ R^2 + LR)$, if the Euler characteristic $\chi(M) = 1$ and $S \leqslant C[\omega^+ R^2 - 2\pi\chi R^2 + LR]$ for $\chi \leqslant 0$, C is some constant. In [5, 6] Aminov gave a very short integral method for obtaining estimates of the exterior diameter of a hypersurface in a Euclidean space, based on the Darboux equation for the square of the length of the radius-vector. This method enables him to obtain estimates either for closed hypersurfaces or for domains in which a regular family of geodesic spheres exists. For a geodesic circle of radius r on a surface $F^2 \subset E^3$ of nonpositive Gaussian curvature bounded from below by the number $-a^2$ the estimate has the form $R \geqslant r/\sqrt{b^2 + \frac{3}{2}a^2r^2}$. The surface $F^2 \subset E^3$, unbounded in the intrinsic sense, whose Gaussian curvature K(P) tends to zero as point P goes to infinity in the intrinsic sense, is not bounded in the space. In this theorem the Gaussian curvature can change sign, but F^2 has a boundary. Rozendorn [44] constructed an example of a surface complete in the intrinsic sense with $K \leqslant 0$, which lies in a ball of finite radius. In this example a countable number of singular points exist at which the surface is of class C^1, while at the other points, of class C^2. Apparently, the presence of these singular points is not vital. Since in this example $\inf K = -\infty$, a question is posed in [5] on the existence of a complete saddle surface, bounded in space, with a curvature K bounded from below. An answer to this question, based on an interesting paper [161] by Omori, was recently given by Baikousis and Koufogiogos [66]. It was proved that a complete two-dimensional Riemannian space immersed isometrically in a three-dimensional Euclidean space with a Gaussian curvature K satisfying the inequality $-\infty < -a^2 \leqslant K \leqslant 0$ is not externally bounded. Aminov [4] obtained an estimate for the outer diameter of a domain of a submanifold F^n in the Euclidean space E^N. Passing to the limit in this estimate, for a complete submanifold with Ricci curvature $\mathrm{Ric} \geqslant -a^2$ and bounded in modulus by the mean curvature vector $|H| \leqslant H_0$, yields an estimate for the radius of the ball containing the surface: $R \geq 1/(H_0 + a^2(n-2))$. Hence, $R \geqslant 1/H_0$ for a two-dimensional surface, and a complete minimal surface with a curvature bounded from below is unbounded in the space. This partially answers one of Chern's questions. The estimate $R \geqslant 1/H_0$ mentioned was next proved also for $n > 2$ for submanifolds with Ricci curvature bounded from below, by Hasanis and Koutroufiotis [112] on the basis of Omori's theorem. The possibility of applying this theorem to obtaining estimates of the diameter was first noted by Hasanis [111]. In [161] Omori proved that for a smooth function f, bounded from above, on a complete Riemannian manifold M with curvature K bounded from below by any $\varepsilon > 0$, we can find a point $p \in M$ such that $|\mathrm{grad}\, f| < \varepsilon$ and $\max_X \{X^i X^j \nabla_i \nabla_j f(p)\} < \varepsilon$, $|X| = 1$. As shown by examples, the boundedness from below of curvature K is an essential condition. Using this theorem Omori shows that if $\varphi: M \to E^N$ is an isometric immersion of a complete Riemannian manifold with a sectional curvature bounded from below into E^N and if a unit vector n exists such that $\frac{\langle \varphi(p) \;\; \rangle}{|\varphi(p)|} \geqslant \delta > 0$ for all $p \in M$, then a point p_0 and a unit vector ξ normal to M at p_0 exist for which the second fundamental form is positive definite. The method of proving these theorems makes essesntial use of the manifold's completeness. For a closed Riemannian manifold $M^n \subset E^{2n-1}$ with curvature $K \leqslant K_0$ Jacobowitz established an estimate $R \geqslant 1/\sqrt{K_0}$ for the radius of the ball containing M^n. Using this result Aminov [6] established an estimate of R for a Riemannian manifold $M^n \subset E^{2n-3}$ with boundary, expressed in terms of intrinsic quantities. We remark that if the dimension of the enveloping space equals 2n, then one cannot obtain a similar estimate.

Blecker [78] has proved that if a C^4-regular complete surface in E^3 does not have umbilical points and its Gaussian curvature K outside some compact set is constant and the sum of the squares of the principal curvatures is nonzero, then its integral curvature equals zero. This assertion is connected to a conjecture by Milnor.

2. Isometric Immersions of Spaces of Positive Curvature. In [35] Pogorelov investigates the question of the extent to which the regularity of the metric of a convex hypersurface in a Euclidean space implies the regularity of the hypersurface. It is proved that a smooth convex hypersurface with a regular metric and a positive sectional curvature is regular. To be precise, if the metric is k times differentiable, $k \geqslant 2$, then the hypersurface too is k times differentiable. If the hypersurface's metric is analytic, then the hypersurface is analytic. Milka [31], using a newly found property of shortest lines on convex manifolds, proves this statement without requiring the hypersurface's smoothness. More precisely, a convex hypersurface in a Euclidean or spherical space or in a Lobachevskii space R with a C^2-regular metric and with a two-sectional curvature greater than the space's curvature is C^2-regular. A generalization of a theorem of Aleksandrov is proved as well: a convex hypersurface in R with a two-sidedly bounded curvature greater than the space's curvature is strictly convex and smooth.

By Pogorelov's well-known theorem [34] on the regularity of a convex surface with a regular metric and Aleksandrov's theorem [1] on the realizability of convex metrics in a three-dimensional Euclidean space E^3, a metric of class C^k and of positive curvature can be immersed into E^3 as a convex surface regular of class $C^{k-1,\beta}$. In [48] Sabitov refined the theorem on the regularity of a surface. It was proved: if a convex surface S is isometric with some two-dimensional Riemannian manifold M of class $C^{k,\alpha}$, $k \geqslant 2$, $0 < \alpha < 1$, whose metric has a Gaussian curvature $K > 0$, then the isometry between M^2 and S is of class $C^{k,\alpha}$. Surface S is regular of class $C^{k,\alpha}$. Sabitov and Shefel' [49] examine the converse statement of the question: on the regularity of the metric if the surface's regularity is known. The smoothness of a surface's metric is, in general, less than the smoothness of the surface. Nevertheless, it is proved in [49] that if the metric ds^2 is the metric of some surface, then a certain special coordinate system can be selected, in which ds^2 has the same or "almost" the same smoothness as the surface itself. More precisely, every $C^{m,\alpha}$-smooth ($m \geqslant 2$, $0 < \alpha < 1$) k-dimensional surface S^k ($2 \leqslant k \leqslant n-1$) in an n-dimensional Riemannian space $R^n(m, \alpha)$ is a $C^{m,\alpha}$-smooth isometric immersion of some $C^{m,\alpha}$-smooth Riemannian manifold. Such special coordinates in which a metric acquires the smoothest form are harmonic coordinates. Sabitov [47] studies the immersion of two-dimensional metrics of class $C^{n,\alpha}$ ($n \geqslant 2$, $0 < \alpha < 1$), specified in a domain G homeomorphic to a disk in E^3. It is proved, for instance, that each point of a $C^{n,\alpha}$-smooth manifold of positive curvature has a neighborhood immersible into E^3 as a $C^{n,\alpha}$-smooth convex surface with boundary on a sphere of a radius specified in advance. The possibility of an immersion with a circular boundary is proved as well.

Shefel' [56] studies the connection of the class of regularity of a surface with the transformation of inversion. It is proved that a surface F in E^n with a $C^{l,\alpha}$-smooth metric ($l \geqslant 2$, $0 < \alpha < 1$), retaining this smoothness class under $(n+1)$ inversions in E^n with centers at the vertices of a nondegenerate simplex, is $C^{l,\alpha}$-smooth.

In [14] Borisov introduces classes of immersions connected with the Holder indices. The set of all $C^{l,\alpha}$-isometric immersions such that they are not isometric immersions with $\alpha' > \alpha$ even at one point is called the class $C^{l,\alpha}$ of the immersions. An immersion class is called universal for n-dimensional spaces if every space V^n admits of a local immersion of the given class into E^{n+1}. There holds the

THEOREM. For any $\alpha < 1/(n^2+n+1)$ the immersion class $C^{l,\alpha}$ is universal for spaces V^n.

Developing an idea of Nash and Kuiper, Jacobowitz [121] continued the study of immersions of submanifolds with small regularity.

In [125] Kallen established a theorem on the immersions of metrics with small regularity but belonging precisely to the Holder classes H^β.

THEOREM. If a metric G of a compact n-dimensional manifold is of class H^β, $0 < \beta \leqslant 2$, then for a sufficiently large N the equation

$$(dU, dU) = G$$

has the solution $U \in H^{\alpha}$, $\alpha < 1 + \frac{\beta}{2}$, where U is an N-dimensional vector. On the other hand, if $0 \leq \beta < 2$, then the set of all metrics $G \subset H^{\beta}$ for which this equation has the solution $U \in H^{\alpha}$ with $\alpha > 1 + \beta/2$ is of first category.

In [162] O'Neill studies the existence of umbilical points on submanifolds of constant curvature of codimension 2. A point $m \in M$ at which all vectors of the normal curvature are equal is called an umbilical point of the isometric immersion $\psi: M^d \to \bar{M}^{d+k}$. It is proved that if M^d is a complete Riemannian manifold of constant curvature $c > 0$ and of dimension $d \geq 4$, then each isometric immersion $\psi: M^d \to \bar{M}^{d+k}$ into a manifold of constant curvature $\bar{c} < c$ has an umbilical point. Examples show that the theorem's assertion does not holds when $c \leq 0$ or when $k > 2$. For an isometric immersion $\psi: M^d \to \bar{M}^{d+k}$ with $k \leq d - 2$ and $\bar{c} < c$ it is proved that each tangent space M_m contains an umbilical subspace U of dimension $r \geq d - k - 1$ (i.e., all directions in U have one and the same length of the normal curvature vector). At the points of M "regular" relative to ψ (i.e., at those points where the normal vector field Z and the field U of umbilical subspaces ψ are differentiable) these umbilical subspaces can be integrated precisely to obtain an umbilical submanifold in M and in $\bar{M}$. Investigations in this direction were continued by Otsuki and by Reckziegel [170, 171].

It is proved in [163] that if an M^d-complete Riemannian manifold has a constant curvature $c > 0$, then M^d can be an immersion into a sphere $S^{d+1}(c)$ of curvature c if and only if M^d is isometric with $S^d(c)$. Any such immersion is an embedding into a great d-sphere. If, however, the M^d with a constant negative curvature c is isometrically immersed into a hyperbolic space of the same curvature c, then the cohomology groups $H^i(M^d) = 0$ for $i \geq 2$.

Henke [114, 115] and Erbacher [93] have studied isometric immersions of an m-dimensional sphere into a Euclidean space E^{m+2}. It is easy to construct such immersions if we consider the immersion of E^{m+1} (or of a spatial neighborhood S^m) into E^{m+2} (for example, by winding onto a cylinder or onto a cone) and if this mapping is restricted to S^m. The question is posed: to what extent, conversely, can an arbitrary prescribed isometric C^{∞}-immersion $f: S^m \to E^{m+2}$ be treated as the restriction of an isometric C^{∞}-immersion of E^{m+1} into E^{m+2}. If mapping f is a restriction to S^m of some isometric mapping $F: E^{m+1}$ into E^{m+2} of neighborhood S^m, then f is said to be extendable. It turns out that the existence of such an extension in some neighborhood of a point $x \in S^m$ depends on whether or not point x is a boundary point of the set of umbilical points of immersion f. (If $f: M \to \bar{M}$ is an isometric C^{∞}-immersion, then a point $p \in M$ is called umbilical if for each normal vector $\eta \in (f_* T_p M)^{\perp}$ there exists a real number λ such that the second fundamental tensor A_{η} at this point has the form $A_{\eta} = \lambda \mathrm{id}_{T_pM}$.) Erbacher [93] had proved the existence of an isometric extension F of the mapping $f: S^m \to E^{m+2}$, $m \geq 4$, in a neighborhood of a nonumbilical point $x \in S^m$. In [114] Henke proves the uniqueness of such an extension if x is a nonumbilical point, i.e., a certain form of "rigidity" holds for these points. If x is an interior point of the set of umbilical points, then a local isometric extension of f in a neighborhood of x exists as well, but is not unique; conversely, extensions of f exist which do not coincide in even one neighborhood. From these statements it follows that in case $m \geq 4$ a local isometric immersion $f: S^m \to E^{m+2}$ has an extension at all points of S^m, excepting the boundary set of all umbilical points. An example is constructed in [114] of an isometric C^{∞}-immersion $f: S^m \to E^{m+2}$ such that a point $p \in S^m$ exists in whose neighborhood f is not locally isometrically extendable. The geometric structure of the set Z of nonumbilical points is studied as well. There exist two affine hyperplanes H_1 and $H_2 \subset E^{m+1}$ with $H_1 \cap S^m \neq \varnothing$, $H_2 \cap S^m \neq \varnothing$ and $H_1 \cap H_2 \cap S^m = \varnothing$ such that the open halfspaces H_1^+ and H_2^+ in intersection with S^m yield Z, i.e., $Z = H_1^+ \cap H_2^+ \cap S^m$ (see Fig. 2).

In [113] Henke has investigated questions on the isometric immersion of m-dimensional spherical space forms of codimension 2 in the Euclidean space E^{m+2}, the spherical space S^{m+2}, a real projective space P_D^{m+2} of curvature D, and a hyperbolic space H_D^{m+2} of curvature $D < 0$.

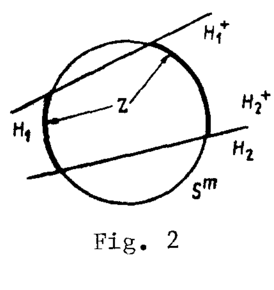

Fig. 2

Fundamental Result. Let M be an m-dimensional space form of curvature $c > 0$ ($m \geqslant 4$) and $\bar{M}$ be an $(m+2)$-dimensional standard space of curvature $\bar{c} < c$. If an isometric real-analytic immersion $M \to \bar{M}$ exists, then M is simply connected, i.e., is isometric with S_c^m.

COROLLARY. There does not exist even one isometric real-analytic immersion of an m-dimensional multiply connected spherical space form ($m \geqslant 4$) into some $(m+2)$-dimensional simply connected space form.

The following proposition has been proved for immersions of class C^∞.

If an isometric C^∞-immersion $g: M \to \bar{M}$ exists such that the open set of all nonumbilical points of g consists of the largest of the finite number of simply connected components, then M is simply connected.

In their papers O'Neill and Moore further studied isometric immersions of spaces of constant curvature with a large codimension.

If M^n is a Riemannian space of constant curvature k that has been immersed into a Euclidean space E^N and $\alpha(X, Y)$ is the immersion's second fundamental form, which can be treated as a symmetric bilinear mapping $\alpha: T_pM \times T_pM \to N_pM$ defined by the formula $\alpha(X, Y) =$ normal component of $\nabla_X Y$, where ∇ is the covariant derivative in E^N, then the following theorem is proved in [152].

THEOREM. Assume that $K > 0$ and $N \leqslant 2n - 1$. If a unit vector $u \in T_pM$ exists such that $\langle \alpha(u, u), \alpha(u, u) \rangle < K$, then $N = 2n - 1$ and α is a diagonalizable form, i.e., a basis $e_1, \ldots, e_n$ exists in T_pM such that $\alpha(e_i, e_j) = 0$ for $i \neq j$. Furthermore, the principal curvature vectors $\alpha(e_i, e_i)$ generate N_pM, and only one vector, say, e_1, satisfies the inequality $\langle \alpha(e_1, e_1) \alpha(e_1, e_1) \rangle < K$.

The following proposition was proved in [163] for $N \leqslant 2n - 2$ and in [152] for $N \leqslant 2n - 1$.

If $\langle \alpha(u, u), \alpha(u, u) \rangle \geqslant K$ for all unit tangent vectors, then a vector $e \in N_pM$ of unit length exists such that

$$\langle \alpha(x, y) e \rangle = \sqrt{K} \langle x, y \rangle$$

for all $x, y \in T_pM$. Furthermore, the vector e is uniquely defined by the condition that it satisfy this relation.

This proposition leads to a new concept of umbilicity. A point $p \in M^n$ is said to be strictly umbilical if a vector $e \in N_pM$ of unit length exists such that the second fundamental form can be written as

$$\alpha(x, y) = \sqrt{K} \langle x, y \rangle e, \text{ for } x, y \in T_pM.$$

The surface M^n of strictly umbilical points is a part of the standard sphere of radius $1/\sqrt{K}$ in E^{n+1}. A point $p \in M^n$ is said to be weakly umbilical if a unit vector $e \in N_pM$ exists such that

$$\langle \alpha(x, y), e \rangle = \sqrt{K} \langle x, y \rangle.$$

The question arises on the form of a manifold M^n each point of which is weakly umbilical.

Moore proved the following theorem in [152].

THEOREM. Let M^n be a compact Riemannian manifold of constant curvature 1. If M^n has an isometric immersion into E^N and $N \leqslant 3n/2$, then M^n is simply connected and, consequently, is isometric with S^n.

However, if M^n has a variable but positive curvature and is isometrically immersed into E^{n+2}, then manifold M^n is homotopic to a sphere [153].

Calabi, and Wallach and do Carmo, have studied isometric minimal immersions of an n-dimensional sphere $S_K{}^n$ of constant sectional curvature K into the unit sphere $S_1{}^l$ in the Euclidean space E^{l+1}. Calabi proved that for each positive integer s there exists an isometric minimal immersion $\psi_{2,s}: S^2{}_{K(s)} \to S_1^{2s}$, where $K(s) = 2/s(s+1)$. If $\varphi: S_K^2 \to S_1{}^l$ is an arbitrary isometric minimal immersion such that $\varphi(S_K^2)$ is not contained in a hyperplane of space E^{l+1}, then, to within the motion $\varphi = \psi_{2,s}$ for some s, in particular $K = K(s)$, $l = 2s$. In [191] there is constructed an isometric minimal immersion $\psi_{n,s}$ of multidimensional spheres $S^n{}_{K(s)}$ into $S_1{}^{m(s)}$, where $K(s) = n/s(s+n-1)$ and

$$m(s) = (2s + n - 1)\frac{(s+n-2)!}{s!\,(n-1)!} - 1.$$

These immersions are constructed with the aid of s-th-order spherical harmonics, i.e., restrictions to S_1^n of s-th-degree homogeneous polynomials $P(x_0, x_1, \ldots, x_n)$ in E^{n+1}, satisfying the equation $\sum_i \partial^2 P/\partial x_i^2 = 0,\ i = 0, \ldots, n$. The dimension of the space of the s-th-order spherical harmonics equals m(s)+1 and if $f_0, \ldots, f_m$ is an orthonormal basis of this space, then the desired isometric immersion $M \to E^{m+1}$ has the form

$$\psi(p) = (f_0(p), \ldots, f_m(p)).$$

Analogously there is constructed the isometric immersion of complex projective spaces $P^d(C)$ of constant holomorphic curvature 1 into the unit sphere S_1^m. In this case we use the k-th-degree homogeneous polynomials $P(z_0, \ldots, z_d, \bar{z}_0, \ldots, \bar{z}_d)$ satisfying the equation $\sum_i \partial^2 P/\partial z_i \partial \bar{z}_i = 0$. These immersions are minimal, are embeddings when d = 1, and they include the Segre manifold (Cartan).

O'Neill [164] defined isotropic isometric immersions as immersions for which the length of each normal curvature vector is one and the same at each point, i.e., the length can depend only on the point. These properties are possessed by the isometric minimal embeddings constructed by him of projective spaces $P^n(\sigma_n)$ of curvature $\sigma_n = n/2(n+1)$ into the unit sphere of dimension $(n(n+3)/2) - 1$. These embeddings are generalizations of the Veronese surface.

Isotropic immersions were studied in detail in [118, 119]. It was proved that under specific conditions an n-dimensional compact orientable space of form $M^n(c)$, isotropically immersed into another space of form $M^{n+p}(c)$, is a Veronese manifold in a certain umbilical hypersurface $M^{n+p}(\bar{c})$.

In [148] Millson has obtained formulas for the conformal Chern—Simons invariants and with their aid has proved, for example, that a 15-dimensional lens cannot be conformally immersed into E^{22}, although it can be smoothly immersed in E^{16}. Moore and Morvan [154] study conformally flat submanifolds of codimension 4. It is shown that if $M^n \subset E^{n+p}$ and $n \geqslant 4$, $p \leqslant 4$ and $p \leqslant n - 3$, then M^n is quasiumbilical if and only if M^n is conformally flat. Characteristic classes and conformal invariants were examined by Moore and White [155].

Borisenko [13] proved the vanishing of certain Pontryagin classes. In the same place is proved that the Euler characteristic of a compact strictly convex surface $F^{2l} \subset E^{2l+p}$ is positive. Here the surface is called strictly convex if at each point there exists an orthogonal basis of normals $n_1, \ldots, n_p$ with respect to which the second quadratic forms are sign-definite and one of the forms is positive definite.

2. Immersions and Embeddings of Differentiable Manifolds

The immersion of a differentiable manifold M^k into a second one V^n, $n > k$, is a mapping f of class C^1 with Jacobian rank k of manifold M^k into V^n. If for different points p and $q \in M^k$ their images are different, then we speak of an embedding of M^k into V^n. In a well-known paper [195] Whitney proved that any k-dimensional manifold can be immersed into E^{2k-1} and embedded into E^{2k}.

In their separate papers Smale and Hirsch continued the study of immersions and of their classification with respect to regular homotopy. A homotopy $f_t: M^k \to V^n$ is called regular if at each stage it is a regular immersion and induces a continuous homotopy of tangent bundles. Smale [176] considered the immersion of sphere S^k into the n-dimensional Euclidean space E^n. His result can be stated thus. On sphere S^k we select a point and a k-frame at this point, and in E^n we select a point and a k-frame at this point. An immersion $f: S^k \to E^n$ is called a base immersion if its leads the point and k-frame selected on sphere S^k into the point and k-frame selected on E^n. Base regular homotopies are examined, which for each t are base immersions. Let $V_{n,k}$ be the Stiefel manifold of all k-frames in E^n (not necessarily orthogonal). It is shown that a one-to-one correspondence exists between the elements $\pi_k(V_{n,k})$ and the base regular homotopic classes of immersions of S^k into E^n. For any two base immersions f and g an invariant $\Omega(f, g) \in \pi_k(V_{n,k})$ exists such that f and g will be base regular homotopic if and only if $\Omega(f, g) = 0$.

The invariant $\Omega(f, g)$ is determined thus. If the base immersions f and g have been specified, then by a small regular homotopy of g they can be made to coincide in some neighborhood U of the point selected on sphere S^k. We can take it that the complement of this neighborhood is homeomorphic to disk D^k. On it we select a fixed field of k-frames. On $f(D^k)$ and $g(D^k)$ arises a field of k-frames, i.e., f and g induce mappings f_* and g_* of disk D^k into the Stiefel manifold $V_{n,k}$; moreover, these mappings coincide on the boundary of D^k. Treating S^k as two copies of D^k glued along the boundary with mappings f_* and g_* on each copy, we obtain a mapping of S^k into $V_{n,k}$. The homotopy class of this mapping is precisely $\Omega(f, g) \in \pi_k(V_{n,k})$. Many of the groups $\pi_k(V_{n,k})$ have been computed. If $n > k+1$, then the invariant $\Omega(f, g)$ can be determined also for nonbase immersions. The statements will be valid in this case for nonbase homotopy classes. Since $\pi_k(V_{n,k}) = 0$ for $n \geqslant 2k+1$, then two C^∞-immersions of S^k into E^n with $n \geqslant 2k+1$ are regularly homotop. In case $n = 2k+1$, then two homotopy classes are characterized by the number I_f of self-intersections.

Whitney defined I_f as the algebraic number of self-intersections of $f: M^k \to E^{2k}$ under the assumption that $f(M^k)$ intersects itself only at isolated points. Smale has proved that two C^∞-immersions of S^k into E^{2k}, $k > 1$, are regularly homotop if and only if they have one and the same number I_f. For an even k Lashof and Smale showed that the k-th normal Stiefel—Whitney class $\bar{w}_k(f) = 2I_f$, i.e., in this case $\bar{w}_k(f)$ characterizes the regular homotopy class of f. Stiefel introduced his own classes by studying the question of the existence, on an n-dimensional manifold, of a system of m continuous vector fields (an m-field) which are linearly independent at each point. He proved that on each manifold M^n and for each prescribed number m ($1 \leqslant m \leqslant n$) we can construct an m-field whose singularities coincide with at most some $(m - 1)$-dimensional complex. The singularities form a certain cycle and its homology class, belonging to an $(m - 1)$-dimensional Betti group M^n, does not depend on a special choice of the m-field. Stiefel called this class the m-th characteristic class. He proved that for the existence of a regular m-field on M^n it is necessary that all the characteristic classes from the first to the m-th vanish. Stiefel showed as well that each three-dimensional closed orientable manifold is parallelizable. Whitney introduced more general classes by examining the space of spheres S(K), which is obtained if each point p of the complex K consists of some ν-dimensional sphere S(p). The Whitney classes arise when considering the possibility of constructing k orthogonal points in each S(p), varying continuously under a continuous variation of p on the whole K. Hirzebruch proposed an axiomatic definition of the Stiefel—Whitney classes (see [32]).

Smale considered as well the question on the possibility of the extension of the immersion $f: S^{k-1} \to E^n$ up to the immersion of disk D^k and showed that this is possible if and only if $\Omega(f, e) = 0$, where e is the standard immersion of S^{k-1} into E^k.

Hirsch obtained theorems on the immersions of arbitrary differentiable manifolds into Euclidean space. In [118] it was shown that a parallelizable manifold can be immersed into a Euclidean space as a hypersurface with normal degree 0. Each closed three-dimensional manifold can be immersed in E^4. If M^k is of dimension $k \equiv 1 \pmod 4$, then M^k can be immersed in E^{2k-2} if and only if $\bar{w}_{k-1}(M) = 0$. Hence it follows that each compact five-dimensional manifold can be immersed in E^8.

If M^k can be immersed into E^{n+r} with a transversal r-field of frames, then M^k can be immersed in E^n. If M^k can be immersed into E^n with a trivial normal bundle, then M^k can be immersed into E^{k+1}. The best possible result has been obtained for projective spaces of dimension $\leqslant 8$. The following immersions of real projective spaces are possible: P^2 and P^3 into E^4 (Milnor), P^5 into E^7, P^6 into E^7, P^7 into E^8. However, if $n > 7$, then P^n cannot be differentially immersed into E^{n+2} (Levine [136]). In this same paper it was proved that P^9 can be immersed into E^{15} but not in E^{14}. If a projective space P_{2^r} can be immersed into E_{2^r+k}, then $k \geqslant 2^r - 1$ (see [32]). Therefore, P^8 can be immersed into E^{15} and not in E^{14}.

Mahowald [142] has shown that if M^n is a compact orientable manifold, $n > 4$, and n is odd, then M^n can be immersed into E^{2n-2}. However, if n is even, then M^n can be immersed into E^{2n-2} if and only if $\bar{w}_2\bar{w}_{n-2} = 0$. The sufficiency of this condition was proved by Massey in [144]; the necessity, in [142]. In the paper cited Hirsch obtained as well certain sufficient conditions that two immersions be regularly homotop. Two invariants Ω and τ were introduced. For a specified immersion $f: S^{k-1} \to E^n$, $k < n$, and for a field f' of vectors transversal to $f(S^{k-1})$ the invariant $\tau(f, f')$ is defined as an element of some homotopy group possessing the following properties: 1) $\tau(f, f')=0$ if and only if f can be extended up to an immersion g of a k-dimensional disk D^k such that its normal derivative on S^{k-1} is f'; 2) $\tau(f, f')=\tau(g, g')$. The invariant $\Omega(f, g)$ is an element of a certain homotopy group such that $\Omega(f, g)=0$ if and only if f and g are regularly homotop relative to a homotopy coinciding with f and g on S^{k-1}. Here f and g: $D^k \to E^n$.

The following conjecture on the immersions of projective spaces is well known. If $\alpha(n)$ is the number of ones in the binary expansion of n, then it was conjectured that RP^n can be immersed into E^{2n-k+1} and not in E^{2n-k}, where $k=2\alpha(n)$ if $\alpha(n)\equiv 1, 2 \pmod 4$, $k=2\alpha(n)+1$ if $\alpha(n)\equiv 0 \pmod 4$, and $k=2\alpha(n)+2$ if $\alpha(n)\equiv 3 \pmod 4$. This conjecture was refuted by Davis and Mahowald [85] who proved that if $n\equiv 7 \pmod 8$ and $\alpha(n)=6$, $n\neq 63$, then RP^n can be immersed in E^{2n-14} and not in E^{2n-13}. Immersions of such manifolds were examined in [76].

Feder and Segal [95] proved that for all n the complex projective space CP^n cannot be immersed in $E^{4n-2\alpha(n)-1}$ and the quaternion projective space QP^n cannot be immersed in $E^{8n-2\alpha(n)-3}$, where $\alpha(n)$ is the number of ones in the binary expansion of n. Here, however, the results obtained were on nonembeddability: CP^n cannot be embedded in $E^{4n-2\alpha(n)}$ and QP^n cannot be embedded in $E^{8n-2\alpha(n)-2}$.

The question of the immersion of CP^n into $E^{4n-2\alpha(n)}$ was considered by Sigrist and Suter [175]. A numerical test is obtained, necessary for the existence of such an immersion. Immersions of complex projective spaces were examined by Davis and Mahowald [86].

Albert [62] defined and studied the characteristic classes of the immersion of a smooth manifold into another manifold with a Riemannian metric of constant curvature. Characteristic classes in connection with immersions and embeddings were studied by Papastavridis [166, 167]. Golbus [101] studied obstructions to immersions of one smooth manifold M^m into another N^n. Sufficient conditions were indicated for the immersibility of M^m into CP^{2k}, expressed in terms of the Stiefel classes $\bar{w}(M)$.

Wintgen [199] studies the immersions of smooth manifolds $N^n \to M^{n+k}$ with higher-order nondegenerating differentials. Morin [156] and Morin and Petit [157, 158] describe the eversion of a sphere in E^3. According to Smale's general theorem there exists a continuous family of regular immersions $S^2 \to E^3$, with the aid of which the interior side of a sphere can be made external, but Smale gave no clue as to how this could be done. Shapiro suggested one eversion process using a two-fold covering of a Boy surface. Kuiper, Philips, and others have published papers on this same theme, which also made use of the Boy surface. In [156-158] the process of everting a sphere is given with the aid of pictures and of analytic expressions.

Characteristic classes of immersed manifolds were studied by Bendersky [74] and Lai [131].

Let us now consider the question of smooth embeddings of smooth manifolds. By Whitney's theorem, each n-dimensional manifold can be embedded into E^{2n}. However, he showed that a real projective space RP^n with $n=2^i$ cannot be embedded into E^{2n-1}. But the case when the dimension n is a power of 2 is an exception. As a rule, a manifold of dimension n can be embedded into E^{2n-1}. To be precise, Haefliger and Hirsch [108] and Wu [201] proved this for every orientable closed manifold of dimension $n\geqslant 5$; the possibility of an embedding was proved in [108] if the manifold is nonorientable, $n\geqslant 5$ and is not a power of two. Every three-dimensional manifold can be embedded into E^5. Hirsch [116] proved this for orientable closed manifolds and Rokhlin [45] for nonorientable closed manifolds. Every connected nonclosed manifold M^n can be embedded in E^{2n-1}.

In recent years the embedding problem for projective spaces was studied in many papers. In [136] Levine proved that if (n — 1) is a power of 2, then P^n cannot be embedded in E^{2n-2}. According to Mahowald [140] and Handel [110], RP^n can be embedded into E^{2n-2} if and only if $n=2^r+s$, where $2\leqslant s<2^r$. In [121]* Hopf and James constructed the imbedding of P^{2^k+1} into a $(2^{k+1}+1)$-dimensional Euclidean space. In [141] Mahowald has shown that this embedding of RP^n is the best possible. Precisely, it was proved that if $2^{k-1}<n<2^k$, then P^n cannot be differentiably embedded into E^{2^k}. For example, P^3 can be embedded in E^5, P^5 in E^9, P^9 in E^{17}, but these spaces cannot be embedded into Euclidean spaces of lower dimension.

*Translator's Note: Reference [121] is not by Hopf and James. There is no citation of a joint paper by Hopf and James.

Using Pontryagin classes, Chern proved that a complex projective space CP^m of complex dimension m cannot be differentiably embedded into a Euclidean space of dimension $3m - 1$ or $3m - 2$ if m is even or odd, respectively. We note that CP^2 can be embedded in E^7, the quaternion projective plane QP^2 can be embedded in E^{13}, and the Cayley plane can be embedded in E^{25}. The dimensions of these Euclidean spaces are the best possible. A number of results on the nonembeddability of the Grassmann manifolds $G_{2,n}$ and $G_{3,n}$ into a Euclidean space were obtained by Oproiu [165]. Here it is shown, for example, that if a number $s = 2^r$ is such that $2^{r-1} \leqslant n \leqslant 2^r$ and either 1) $n \neq s - 1$, then $G_{2,n}$ cannot be embedded in E^{2s-2} and cannot be immersed in E^{2s-3}, or 2) $n = s - 1$, then $G_{2,s-1}$ cannot be embedded in E^{3s-2}. Analogous results have been obtained for $G_{3,n}$.

The proof of embedding impossibility is connected with the study of Stiefel—Whitney classes which are not arbitrary for embedded manifolds. If a manifold M^n has been smoothly embedded as a closed subset into a Euclidean space E^{n+k}, then the k-th Stiefel—Whitney class of the normal bundle $\bar{w}_k = 0$ (see Milnor and Stasheff [32]). Lashof, Smale, Massey, and others have studied conditions on the Stiefel—Whitney classes. Massey [145] proved that if the class $\bar{w}_{n-q} = 0$ for a compact manifold M^n, $0 < q < n$, then constant numbers $h_1, h_2, \ldots, h_q$ exist such that $h_1 \geqslant \ldots \geqslant h_q$ and $n = 2^{h_1} + \ldots + 2^{h_q}$. If n is even and M^n is orientable, then $\bar{w}_{n-1} = 0$. This generalizes a result of Whitney on the fact that $\bar{w}_3$ of an orientable M^4 equals zero. The triviality of the Stiefel—Whitney—Euler classes for a manifold embedded in a Euclidean space whose second quadratic form satisfies certain inequalities has been proved by Morvan [159]. Banchoff and McCrory [71] give an interpretation of the Stiefel—Whitney classes of smooth manifolds with the aid of the singularities of the projections onto coordinate spaces. Bausum [73] proves that if $\bar{w}_{n-i} = 0$ for $i \leqslant 4$ and either M^n is orientable and $m \equiv 0, 1 \pmod 4$ or M^n is not orientable and $m \equiv 2, 3 \pmod 4$, then M^n can be embedded into E^{2n-2}.

The question on the classification of embeddings, which is considerably more difficult that the classification of immersions, has been intensively studied. In the case of multidimensional knots S^n and S^{n+q} with codimension $q > 2$ a significant result was obtained by Haefliger [105] who classified these knots with the aid of homotopy groups. It turns out that when $q > 2$ the isotopy problem can be reduced to a concordance problem. Two differentiable embeddings f_0 and f_1 of manifolds $M \to X$ are concordant if an embedding $F: M \times I \to X \times I$ exists such that $F(x, t) = (f_i(x), i)$; $i = 0, 1$. Two concordant embeddings are said to be equivalent, while the set of classes of equivalent embeddings $S^n \to S^{n+q}$ is denoted by C_n^q. When $q > 2$ two concordant embeddings will be isotop. An operation is introduced of the sum of classes from C_n^q, which turns set C_n^q into an Abelian group. The main result is that the group of isotopic classes C_n^q is isomorphic with the triad homotopy group $\pi_{n+1}(G, SO, G_q)$, where G_q is the space of mappings of the power of one sphere S^{q-1} onto itself, G is the inductive limit of G_k as $k \to \infty$, and SO is the inductive limit of the rotation spaces SO_k. An element of this homotopy group is represented by a continuous mapping $f: D^{n+1} \times S^{N-1} \to S^{N-1}$ for some large N, possessing certain properties. In a number of cases the Haefliger homotopy groups can be computed to completion. For example, Milgram [147] has indicated methods enabling the computation of group C^k_{2k-3+j} for $k \equiv 2$ or $3 \pmod 8$, $j = 0, \ldots, 6$. In [107] Haefliger proves that any differentiable embedding of a $(4k - 1)$-dimensional sphere into an m-dimensional sphere S^m with $m > 6k$ is unknotted, i.e., restricts the 4k-dimensional disk to S^m. Under the condition $m = 6k$ an infinite number of isotopic classes of the embeddings of a $(4k - 1)$-dimensional sphere into the 6k-dimensional sphere S^{6k} exist. The group $\Sigma^{m,n}$ of h-cobordism classes of embedded spheres S^n in S^m is determined. An embedded sphere S^n in S^m is called h-cobordant to zero if it restricts a differentiable $(n+1)$-dimensional disk D^{n+1} to the unit disk D^{m+1}. As Smale showed, the elements of $\Sigma^{m,n}$ correspond, in a majority of cases, to the isotopic classes of the embeddings of S^n in S^m. Haefliger proves that the group $\Sigma^{6k,4k-1}$ is isomorphic to the integer group Z if $k > 1$. A certain standard embedding of S^{4k-1} into S^{6k}, which is a generator of group $\Sigma^{6k,4k-1}$, is indicated. Its construction is totally geometric. In a Euclidean space E^{3d} with coordinates $(x, y, z) = (x_1, \ldots, x_d, y_1, \ldots, y_d, z_1, \ldots, z_d)$ three embeddings are examined of $(2d - 1)$-dimensional spheres lying in the "coordinate" spaces $S_1: x=0, \frac{y^2}{\alpha^2}+\frac{z^2}{\beta^2}=1$, $S_2: y=0, \frac{z^2}{\alpha^2}+\frac{x^2}{\beta^2}=1$, $S_3: z=0, \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, where the numbers $\alpha > \beta > 0$. Next, S_1 is joined with S_2 and S_2 with S_3 by two thin cylinders (Fig. 3).

When $d = 2k$ we obtain a standard embedding of S^{4k-1} into S^{6k}, whose h-cobordism class is independent of a special choice of the joining cylinders.

A number of papers have drawn in the homotopy groups of the knot's complement. Stallings and Levine [137] proved that for $n \geqslant 5$ the knot $\sigma: S^{n-2} \subset S^n$ will be unknotted in a

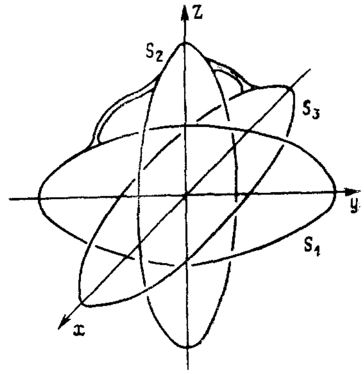

Fig. 3

piecewise-linear and topological category if and only if $\pi_q(S^n \setminus S^{n-2}) \cong \pi_q(S^1)$ for $q \leqslant n$. For $n \geqslant 4$ this question is examined also in [91]. Lashof and Shaneson [135] showed that when $n \geqslant 5$ there exist no more than two nonequivalent knots $S^{n-2} \subset S^n$ with diffeomorphic complements. Lashof [134] examines the embedding of a smooth manifold M^m into a smooth manifold N^n under the condition $n \geqslant m+3$. Daverman [84] studies the embeddings of an $(n-1)$-dimensional sphere into E^n.

For a regular mapping $f: M^n \to E^{2n-1}$ of a simply connected manifold Novikov [33] defined an invariant from Z_2, whose equality to zero is necessary and sufficient for f to be regularly homotop to an embedding. It is proved that there exists a mapping $g: M^n \to E^{2n-1}$, C^1-close to f, such that the set of points in M^n, having one and the same image under mapping g, can be separated into a pair of circles called singular. Each pair of singular circles S_1 and S_2 are glued together by disks $\sigma_i \subset M^n$ such that $\sigma_1 \cap \sigma_2 = \varnothing$. At each point of disk σ_i there are prescribed a system of fields transversal to it and a transversal to S_i lying in σ_i. Then on the image $g(S_1) = g(S_2)$ of the circles there are defined $(2n-2)$ vector fields transversal to $g(S_i)$ and independent. They determine an element $\alpha \in \pi_1(GL(2n-2))$. Under specific conditions α is independent of the choice of σ_i. The sum $\Sigma\alpha_k$ of the invariants over all singular pairs is an invariant of mapping $g: M^n \subset E^{2n-1}$, whose vanishing permits the removal of self-intersections of $g(M^n)$.

Sosinskii [53] considered the question of the finite decomposability of a knot $\sigma: S^{n-2} \subset S^n$ into a sum of simple knots, i.e., indecomposable knots σ_1 and σ_2. A knot is said to be decomposable into a sum $\sigma = \sigma_1 \# \sigma_2$ if nontrivial knots $\sigma_i: S^{n-2} \subset S^n$ exist whose intersection with each other and with some equator S^{n-1} is a disk D^{n-2}, while their union less $\mathrm{Int} D^{n-2}$ yields a knot decomposed transversally to S^{n-1}. It was proved that if $n \geqslant 5$, then every knot σ can be represented as a sum of a finite number of simple knots $\sigma = \sigma_1 \# \ldots \# \sigma_m$.

In [51] Smirnov classified knots S^m on a manifold M^n without boundary with a contractible universal covering and $1 < m < n-2$. Using the numbers m and n and the group $\pi_1(M)$ he defined a certain Abelian group $\Gamma(m, n, \pi_1(M))$, related to Haefliger groups, such that the set of smooth isotopy classes is isomorphic with this group if M is orientable or with some factorization of it if M is unorientable. Embeddings and immersions were studied by Smirnov in [50] and by Farber in [54, 55]. A construction of the knotting of an $(n-2)$-dimensional submanifold with the aid of some knot $\sigma: S^{n-2} \subset S^n$ is used by Viro [17] to construct smooth embeddings of closed unorientable surfaces in E^4 with a fundamental complement group different from Z_2. In the well-known standard embeddings $P^2 \subset E^4$ this is the group Z_2. The homotopy groups of the space of proper embeddings of one open manifold into another are studied in [79] by Bourgeois and Spring. Dymov [23] proves embedding theorems for n-dimensional open manifolds into compact manifolds of dimensions n, n+1, and n+2q+2.

A number of authors have examined the interesting question on the representability of homology group elements of a manifold M by regular submanifolds, a topic initiated by Milnor, Kervaire, Rokhlin, and others. Tristram in [187] proved for $M = S^2 \times S^2$ that an element $\xi \in H_2(M)$ of form $\xi = n_1\xi_1 + n_2\xi_2$, where ξ_1 and ξ_2 are the natural generators of this group and n_1 and n_2 are relative primes, cannot be realized by a sphere. In [46] Rokhlin gave an upper bound for the genus of a two-dimensional surface realizing an element ξ of group $H_2(M)$ of a four-dimensional manifold M, in terms of the selfintersection index, the signature of M, and the Betti number b_2. When in the above-mentioned example n_1 and n_2 are even, this bound yields $g \geqslant (|n_1 n_2|/2) - 1$. The question on the representability of homology classes by embedded submanifolds is studied by Meeks and Patrusky [146], Bennequin [75], and Papastavridis [168]. For instance, it is shown in [168] that if M^{2n+1} is a smooth closed manifold and $2n+1 = 2^k - 1$, then every element $\alpha \in H_{n+1}(M, Z_2)$ can be realized by a submanifold; however, if $2n \neq 2^k - 2$, then a $(n-1)$-connected manifold M^{2n+1} exists, some cycle of which is not realized by a submanifold. If M^{2n+1} is orientable, all elements of $H_{n+1}(M, Z_2)$ are realized by submanifolds.

Yasui has devoted his papers [202-204] to isotopy classes of the embeddings of projective spaces RP^n and CP^n. With the aid of the results of Postnikov and Haefliger it is proved that if n is even, $n \neq 2^r$, and $n \geqslant 10$, then only one isotopy class exists of the embeddings of RP^n into E^{2n-2}. If $n \geqslant 5$ and $n \neq 2^r + 2^s$ $(r \geqslant s \geqslant 0)$, then an n-dimensional complex projective space CP^n can be embedded into E^{4n-3} and a countable number of different isotopy classes of such embeddings exist. The isotopy classes of the embeddings of RP^n into E^N with $N = 2n$, $2n-1$, and $2n-2$ and the normal bundles are examined by Larmore and Rigdon [133].

Intensive efforts were expended on the problem of immersions and embeddings of lens spaces. Recently this question was examined by Dediu [87, 88], Kobayashi [126], and Berrick [77]. A lens space $L^{2n+1}[p; p_0, p_1, \ldots, p_n]$ is a factorization of the sphere $S^{n+1} = \{(z_0, \ldots, z_n) \in C^{n+1} \mid \sum_{k=0}^{n} |z_k|^2 = 1\}$ by the action of the group generated by rotation γ:

$$(z_0, \ldots, z_n) \xrightarrow{\gamma} \left(l^{\frac{2\pi i p_0}{p}} z_0, l^{\frac{2\pi i p_1}{p}} z_1, \ldots, l^{\frac{2\pi i p_n}{p}} z_n \right),$$

where $p, p_0, \ldots, p_n$ are positive integers relatively prime with p. If $p_0 = \ldots = p_n = 1$, then the lens space is denoted $L^n(p)$. Parallelizable lens spaces exist, for example $L^1(p)$, $L^3(p)$; by Hirsch's theorem they are immersed into a Euclidean space as a hypersurface [88]. In [126] it is proved that $L^n(5)$ cannot be embedded into E^{3n+2} for $n = 3 \cdot 5^{t+1} + 5^t$, but can be embedded in E^{3n+3}. It has been shown in [76] that $L^n(p)$ with p a multiple of the number $2^{2n-1-\alpha(n-1)}$ cannot be immersed in a Euclidean space of dimension $4n - 2\alpha(n) - 2$.

3. Immersions with Minimal Absolute Curvature

The well-known Fenchel inequality says that the integral of the absolute curvature k of a closed curve in E^3 is greater than or equal to 2π; moreover, its equality to 2π obtains if and only if the curve is planar and convex.

Fary [94] and Milnor [149] proved Borsuk's conjecture that for a knotted curve in E^3 this integral is greater than or equal to 4π. Various generalizations of these inequalities have been examined. For a knotted curve C which is the boundary of a surface M of genus g, Langevin and Rosenberg [132] proved the following inequality:

$$2\int_C k\,ds + \int_M |K|\,dS \geqslant 2\pi(2g+3),$$

where K is the Gaussian curvature of surface M. In the same place it was proved that for a knotted torus T in E^3

$$\int_T |K|\,dS \geqslant 16\pi.$$

The Fenchel inequality on closed curves in Riemannian spaces has been generalized in [179, 188].

We remark that although the integral of the modulus of torsion $\varkappa$ of a closed curve C can be arbitrarily small, for it we can indicate the following estimate by Segre [174]. By δ we denote the spherical diameter of a disk circumscribed around the indicatrix of the binormals if this indicatrix is contained on the halfsphere, and we set $\delta = \pi$ if this indicatrix does not lie on the halfsphere. Then

$$\int |\varkappa|\,ds > 2\delta.$$

If the closed curve C is located on a locally convex cylinder whose directrix is a curve of index m and if curve C is not tangent to the curvilinear generators, then for the integral of the torsion we can indicate an upper bound established by Aminov [8]:

$$\left|\int_C \varkappa ds\right| < 2\pi m.$$

In [193] Weiner proves the following inequality for a nonplanar closed curve in E^3 of length L, lying in a ball of radius R:

$$L\int_C \varkappa^2 ds \leqslant R^2\left[\int_C k^2 ds \int_C \varkappa^2 ds - \left(\int_C k\varkappa ds\right)^2\right].$$

As a corollary, for a closed curve in E^3 we obtain the inequality

$$16\int_C \varkappa^2 ds \leqslant L\left[\int_C k^2 ds \int_C \varkappa^2 ds - \left(\int_C k\varkappa ds\right)^2\right].$$

Corresponding inequalities have been derived as well for curves in a multidimensional space.

Let us now proceed to consider a generalization of the Fenchel inequality to multidimensional submanifolds. We mention the surveys [130, 89] on this subject.

Let M^n be a compact differentiable manifold of dimension n and let $\varphi: M^n \to E^{n+k}$ be some immersion of M^n into the Euclidean space E^{n+k}. The set of unit normal vectors to M forms a bundle of $(k-1)$-dimensional spheres over M, i.e., a manifold B of dimension $(n+k-1)$. We consider a mapping $\nu: B \to S^{n+k-1}$ associating with each unit normal vector to B a unit vector passing through the origin and parallel to the normal vector. Let ν^* be the induced mapping of the tangent spaces. Then the total absolute curvature is

$$\tau(M, \varphi, E^{n+k}) = \frac{1}{C_{n+k-1}} \int_B |\nu^* d\sigma|,$$

where C_{n+k-1} is the volume of the unit sphere S^{n+k-1} and $d\sigma$ is a volume element of S^{n+k-1}. The integrand on the right-hand side can be written with the aid of the Lipschitz—Killing curvature defined for each normal vector field. Immersion φ is said to be minimal or tight if for it $\tau(M, \varphi, E^{n+k})$ takes the smallest value under a change of φ and k. Not every compact differentiable manifold M can be immersed minimally into a Euclidean space. The total absolute curvature can as well be determined with the aid of the height function $h_\alpha(x)$, i.e., the distance from point $\varphi'(x)$ up to the plane passing through the origin 0, orthogonal to vector a. For almost all a this function has only nondegenerate critical points. Let $\beta(M, f)$ be the number of critical points of the differentiable function f. Then

$$\tau(M, \varphi, E^{n+k}) = \int_{a\in S^{n+k-1}} \beta(M, h_a)\, d\sigma.$$

Chern and Lashof showed that if the total absolute curvature equals 2, then φ is an embedding of M as a convex hypersurface into E^{n+1}, while Kuiper showed that

$$\inf_{\varphi, k} \tau(M, \varphi, E^{n+k}) = \inf_f \beta(M, f).$$

Therefore, a tight immersion is defined also as the immersion for which almost all linear height functions have the least admissible number of critical points.

Kuiper pointed out that an exotic sphere cannot be minimally embedded in a Euclidean space, while Ferus [97] proved that each embedded exotic n-sphere, $n \geqslant 5$, in E^{n+2} has a total curvature $\geqslant 4$.

Analogously, there is introduced the concept of a taut immersion as one for which, for almost all points $p \in E^m$, the functions of distance from these points have the least admissible number of critical points. If $\varphi(M)$ is not contained in even one hyperplane of E^{n+k}, then immersion φ is called substantial or proper. Banchoff proved that a taut compact surface in E^3 must be a sphere or a Dupin cyclide.

The question on which manifolds can be immersed minimally is a difficult problem.

Kuiper proved that each orientable closed surface and each unorientable closed surface with an Euler characteristic $\leqslant -2$ can be embedded minimally in E^3 and that a real projective plane and Klein bottle cannot be minimally embedded in E^3. A projective plane can be immersed substantially and minimally into E^5 as a Veronese surface. Kuiper showed that if the immersion $\varphi: M \to E^{n+k}$ is minimal and substantial, then $k \leqslant n(n+1)/2$. He also gave examples of minimal and substantial embeddings with codimensions k from 1 to $n(n+1)/2$.

Kobayashi [127] proves that each compact homogeneous Kahlerian manifold can be minimally embedded into a Euclidean space. The Mannoury immersion $P^m(C)$ in $E^{(m+1)^2-1}$ is presented. In [128] he constructs in a unified manner the isometric immersions of compact symmetric spaces into Euclidean spaces, which are minimal and substantial. Tai [181] constructs such immersions for real and quaternion projective spaces and for the projective Cayley plane. Let us describe the embeddings of $P^n(F)$, where F is the field of real, complex, or quaternion numbers. Let F^{n+1} be a vector space over this field, having the real dimension $(n+1)\cdot d$, where $d=1$ for R, $d=2$ for C, $d=4$ for Q. A conjugate element is defined for each element $x\in F$. For the matrix A of an operator on F^{n+1} there is defined the transposed matrix ${}^tA$ and the conjugate matrix $\bar{A}$, as well as $A^* = {}^t\bar{A}$. By $H(n+1, F)$ we denote the space of all $(n+1)$-dimensional square matrices over field F, such that $A^*=A$ (Hermitian matrices over F). Let $P^n(F)$ be a projective space over field F.

For an element $x\in P^n(F)$ we can use the homogeneous coordinates

$$x=\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}\in F^{n+1} \text{ with the condition } x^*x=1.$$

The mapping $\varphi: P^n(F)\to H(n+1, F)$ is defined as follows:

$$\varphi(x)=xx^*=\begin{pmatrix} |x_0|^2, & x_0\bar{x}_1, & \ldots, & x_0\bar{x}_n \\ x_1\bar{x}_0, & |x_1|^2, & \ldots, & x_1\bar{x}_n \\ \cdot & \cdot & \cdot & \cdot \\ x_n\bar{x}_0, & \ldots & & , |x_n|^2 \end{pmatrix}.$$

The image of this mapping lies in some $(\binom{n+1}{2}d+n)$-dimensional Euclidean space. The scalar product $(xy)=\mathrm{Re}(x^*y)$ is defined for two elements x and y of F^{n+1}. For two matrices A and $B\in H(n+1, F)$ the scalar product has the form $(AB)=\mathrm{Tr}(AB)$. It turns out that mapping φ is isometric relative to the metric thus introduced, is minimal, and is substantial.

In [197] Wilson proves that the embedding of the orthogonal group SO(n) into E^{n^2}, obtained by considering all elements of the matrices from SO(n) as coordinates in E^{n^2}, is minimal. It is shown as well that the Grassmann coordinates of the Grassmann manifold $G_{2,n}$ of oriented 2-planes in an n-dimensional plane yield a minimal embedding. Minimal embeddings of U(n) into Sp(n) are given as well.

The total absolute curvature for immersions of M^n into a Riemannian space is determined as the normed integral of the absolute value of the Lipschitz—Killing curvature taken over the manifold of all unit normal vectors. Here the Lipschitz—Killing curvature is defined as the determinant of the second fundamental form for each normal vector field.

In [69] Banchoff defined the central curvature of curves, which is a generalization of the concept of total absolute curvature. In the planar case this concept is defined as follows. Let $f: S^1\to E^2$ be a continuous mapping of the unit disk S^1 into a plane. The support line to f at point x is a straight line containing x and being a support line to some neighborhood of point x in S^1. Let $\tau_p(f)$ be the number of support lines passing through the point $p\in E^2$. The curve's central curvature $\tau_C(f)$ with respect to circle C is the average value of $\tau_p(f)$ for points $p\in C$, i.e.,

$$\tau_c(f)=\frac{1}{l(C)}\int_{p\in C}\tau_p(f)\,ds_c,$$

where $l(C)$ is the length of circle C. It is proved that if a curve $f(S^1)$ is contained in C, then $\tau_C(f)$ is independent of the position of C. The classical total absolute curvature $\tau(f)$ can be treated as $\tau_C(f)$ with respect to an infinitely great circle. For space curves the total central curvature $\tau_S(f)$ with respect to a sphere is determined with the aid of stereographic projections onto the plane passing through the center of S.

Banchoff [68] introduced the concept of the two-piece property (TPP) for a topological mapping f of a manifold M into the Euclidean space E^N. A mapping f is TPP if for any hyperplane $H\subset E^N$ the preimage $f^{-1}[f(M)\setminus H]$ has no more than two connected components. He showed that for compact closed manifolds of dimension $\leqslant 2$ tightness is equivalent to this property. Kuiper and Pohl [129] proved that if a topological substantial embedding of a real projective plane into E^N, $N\geqslant 5$, is TPP, then $N=5$ and the image of the embedding is either the Veronese surface or Banchoff's piecewise-linear embedding with six vertices. For a regular mapping the only solution will be the Veronese surface.

A new definition of total absolute curvature for submanifolds immersed in Euclidean spheres was given by Weiner [192]. Having selected a point p on a sphere S^{n+k} such that $-p$ does not lie on the submanifold M^n, he parallelly displaced into the tangent space to the sphere at the point p any normal vector to M^n, taken at a point $q \in M^n$, along any geodesic joining p and q. The thus-obtained mapping of the normal bundle $\nu(M)$ into the unit sphere S_pS^{n+k} in the tangent space to sphere S^{n+k} at point p is denoted e_p. It is proved that an analog of total curvature, viz., the integral of the Jacobian of mapping e_p, taken with respect to $\nu(M)$, equals the Euler—Poincare χ-characteristic if $-p \notin M$ and equals χ minus twice the number of passages of M through $-p$ if $-p \in M$. The total absolute curvature $\tau_p(M)$ with respect to p is defined as the integral of $|e_p^*|$ with respect to the normal bundle $\nu(M)$. For a compact orientable submanifold $M^n \subset S^{n+k}$, under the condition $-p \notin M^n$, there is proved: 1) $\tau_p(M) \geqslant$ sums of the Betti numbers of M; 2) if $\tau_p(M) < 3$, then M is homeomorphic to a sphere; 3) if $\tau_p(M) = 2$, then M is the hypersurface of a small (n + 1)-dimensional sphere passing through $-p$.

Cecil and Ryan [82] generalized the concepts of tightness and tautness to immersions into a hyperbolic space H^m, by examining the distance function from a totally geodesic plane and from a point. It was proved that the immersion $f:M \to H^m$ is tight if and only if the composition of f with the stereographic projection P of space H^m into E^m yields a tight immersion of M into E^m in the Euclidean sense for any choice of P. It is proved that there are differences between tight immersions into H^m and into E^m. To be precise, if $f:M \to H^m$ is any tight immersion and $i:H^m \to H^{m+k}$ is a totally geodesic embedding, then the composition $if:M \to H^{m+k}$ is not necessarily a tight immersion, in contradiction to the Euclidean case. Chen [83] proves that a substantial tight embedding $f:S^m \times S^n$ into E^{m+n+2} (where $m/n \neq 2$ or 1/2) with an image lying on an ovaloid is projectively equivalent (in Kuiper's sense) to the product of two ovaloids of dimensions m and n, respectively. This theorem generalizes a theorem of Chern and Lashof.

Sunday [178] examines knotted embeddings $x(S^n)$ of an n-dimensional sphere S^n into E^{n+2}. If g is the minimal number of generators of the fundamental group $E^{n+2} \setminus S^n$, then the inequality

$$K^* \geqslant 2g(x)C_{n+1}$$

is proved for the total absolute integral curvature K^*. Hence it follows that if $K^* < 4C_{n+1}$, then the fundamental group $E^{n+2} \setminus S^n$ is Z. Wintgen [199] gives an estimate for the total integral curvature of an n-dimensional closed manifold $M \subset E^{n+2}: K^* \geqslant [\mu(M) + 4(g-1)]C_{n+1}$, where $\mu(M)$ is the Morse number of manifold M. A method is indicated for obtaining a lower bound for the total absolute curvature in a given isotopy class of knots.

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CERTAIN QUESTIONS OF LOBACHEVSKII GEOMETRY, CONNECTED WITH PHYSICS

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An historical survey of the development of Lobachevskii geometry as a typical representative of a geometry of negative curvature, Friedman cosmology, Lobachevskii geometry, and an interpretation of the velocity space in the special theory of relativity as a Lobachevskii space are presented.

1. Lobachevskii Geometry — A Typical Representative of a Geometry of Negative Curvature

Historically, Lobachevskii geometry appeared as the first non-Euclidean geometry realized to be such. This was precisely the turning point in the development of geometry from the still intuitively operating geometry of Euclid to modern geometry absorbing within itself both the concept of a curved Riemannian space as well as the algebraic-group ideas of Klein. It happens that this all-encompassing aspect was the principal one in the development of Lobachevskii geometry in the last century. Just as he apparently was recognized as the foremost among the three scholars tending the cradle of this science (Lobachevskii, Bolyai, Gauss), it is his name that the geometry validly bears. In particular, this is manifest in the complex evolution of Lobachevskii's views on the question of a geometry of the real world, which resulted in the statement on the possibility or impossibility of geometric constructions not having a direct relation to real space. We stress the profound nontriviality of recognizing the geometry constructed as a non-Euclidean one. Strictly speaking, the fact is that this geometry was not the first, but the second non-Euclidean geometry. Since the time of Greek (or, anyway, Hellenistic) antiquity another non-Euclidean geometry — spherical geometry — has been very well known. Even then the degree of its development was not principally different from the degree of development of Euclidean geometry. Apparently, the

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