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On the density of log-spot in the Heston volatility model

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Abstract

This paper proves that the log-spot in the Heston model has a $\mathcal{C}^\infty$ density and gives an expression of this density as an infinite convolution of Bessel type densities. Such properties are deduced from a factorization of the characteristic function, mainly obtained through an analysis of the complex moment generating function. As an application a new algorithm to simulate spot is developed.

MSC: 60H10; 60E10; 91B28

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1. Introduction

The Heston volatility model [19] is defined by the system of stochastic differential equations

$$\left.\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + S_t\sqrt{V_t}\,\mathrm{d}Z(t) \\ \mathrm{d}V_t &= a(b - V_t)\mathrm{d}t + c\sqrt{V_t}\,\mathrm{d}W(t) \end{aligned}\right\} \tag{1}$$

with initial conditions $S_0 = s_0 > 0$ and $V_0 = v_0 \geq 0$, where $a,\ b,\ t > 0$, $c \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R}$. The processes W and Z are two standard correlated Brownian motions such that $\langle Z, W\rangle_t = \rho t$

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for some $\rho \in [-1, 1]$. The process V_t is a Feller diffusion [13] or, in the financial literature, a CIR model [7]. It is well known that (1) has a unique strong solution. Write $X_t = \log S_t - \mu t$, which stands for *log-spot*.

Although its great importance in applications, the probability density function of X_t is not well known, and may not be a closed expression for such a density; see [10] for the behaviour of the density for large t, and Gulisashvili and Stein [18] for the asymptotic study (in the variable of the density function for a fixed t) in the case of uncorrelated Brownian motions. Therefore, the computations relative to this model are done mainly with the characteristic function, which, surprisingly, has a nice expression in terms of elementary functions. In this paper we prove that X_t has a $\mathcal{C}^\infty$ density and give an expression of it as an infinite convolution of Bessel types densities. Besides its theoretical interest, this result has some practical consequences; in particular, an infinite convolution of densities allows, by truncation, a quick and simple simulation algorithm of the random variable.

We remark that for certain combinations of the parameters, log-spot X_t is a sum of independent non-centered χ^2 random variables, and we can identify X_t as a member of the non-homogeneous second Wiener chaos (see [20, chapter 6]); this agrees with the fact that for some settings the stochastic volatility, given by a CIR model [7], has the law of the sum of the squares of a finite number of independent Ornstein–Uhlenbeck processes that are in the second Wiener chaos, and such a property is transferred to X_t.

A main tool of our research is that the characteristic function of X_t is analytic, that means there is a function $\Phi(z)$ of the complex variable z, analytic in a neighborhood of 0, such that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}uX_t}] = \Phi(\mathrm{i}u)$$

for u (real) in a neighborhood of 0 (see [25, chapter 7] for an equivalent definition and the main properties of analytic characteristic functions).

The fact that a characteristic function is analytic has important consequences, and we here exploit some of them. However, the standard form of writing the Heston characteristic function hides its analyticity and difficult its utilization. So we first give a new expression of the characteristic function, but more importantly, we use the equivalence between the analytic property and existence of (real) moment generating function in a neighborhood of the origin: there is $\varepsilon > 0$ such that

$$M(u) := \mathbb{E}[\mathrm{e}^{uX_t}] < \infty, \quad -\varepsilon < u < \varepsilon.$$

In that case, the function $M(u)$ is a real analytic function in $(-\epsilon, \epsilon)$ and the main properties of the characteristic function can be studied through $M(u)$, which is much simpler to analyze.

The paper is organized as follows. First we deduce the moment generating function and the characteristic function of X_t; as far as we know, these expressions are new. Both can be obtained manipulating the expressions derived by Dufresne [12] or the standard expressions of the characteristic functions (see for example, [15] or [1]), however we prefer to give a new deduction from scratch since our procedure is quite general and can be applied to other problems. In Section 3 we review the main results of the paper; namely, the domain of the moment generating function; the factorization of the characteristic function; the existence of a $\mathcal{C}^\infty$ density and the expression of the density function as an infinite convolution of Bessel type densities. These results are applied to set up a procedure to simulate spot. Sections 4 and 5 are devoted to prove the main results and the Appendix contains some technical details which may hinder the fluent reading of the paper.

2. The Heston model

We will consider the system

$$\left.\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}Z_t \\ \mathrm{d}V_t &= a(b - V_t)\,\mathrm{d}t + c\sqrt{V_t}\,\mathrm{d}W(t) \end{aligned}\right\} \tag{2}$$

deduced from (1) applying the Itô formula to $X_t = \log S_t - \mu\, t$.

2.1. The moment generating function of X_t

First, we check that X_t (indeed the pair (X_t, V_t)) has a moment generating function, and later we deduce its expression as the solution of a (real) PDE obtained from Itô formula.

For every $u, v \in \mathbb{R}$, the random variable $\mathrm{e}^{uX_t+vV_t}$ is positive, so we can compute its expectation but it can be infinite. Write

$$Z_t = \rho W(t) + \rho' W'(t),$$

where W' is a Brownian motion independent of W, and $\rho' = \sqrt{1-\rho^2}$. Combining both equations of (2) we get

$$\begin{aligned} h(u, v, t) &:= \mathbb{E}\big[\mathrm{e}^{uX_t+vV_t}\big] \\ &= \exp\left\{x_0 u - \frac{u v_0 \rho}{c} - \frac{u\rho a b t}{c}\right\} \\ &\quad \times \mathbb{E}\left[\exp\left\{\Big(v + \frac{\rho u}{c}\Big) V_t + \Big(-\frac{u}{2} + \frac{u a\rho}{c}\Big)\int_0^t V_s\,\mathrm{d}s + u\rho'\int_0^t \sqrt{V_s}\,\mathrm{d}W'_s\right\}\right]. \end{aligned}$$

Now use that $\mathbb{E}[\ldots] = \mathbb{E}[\mathbb{E}[\ldots|\mathcal{F}_t]]$, where $\mathcal{F}_t = \sigma(W_s,\ s \in [0, t])$. Given that $V_s, s \in [0, t]$ is $\mathcal{F}_t$ measurable,

$$\begin{aligned} h(u, v, t) &= \exp\left\{x_0 u - \frac{u v_0 \rho}{c} - \frac{u\rho a b t}{c}\right\} \\ &\quad \times \mathbb{E}\left[\exp\left\{\Big(v + \frac{\rho u}{c}\Big) V_t + \Big(-\frac{u}{2} + \frac{u a\rho}{c}\Big)\int_0^t V_s\,\mathrm{d}s\right\}\right. \\ &\quad \times\ \left.\mathbb{E}\left[\exp\left\{u\rho'\int_0^t \sqrt{V_s}\,\mathrm{d}W'_s\right\}\bigg|\,\mathcal{F}_t\right]\right]. \end{aligned}$$

From the independence between the Brownian motions W and W', for every $y \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left\{y\int_0^t \sqrt{V_s}\,\mathrm{d}W'_s\right\}\bigg|\,\mathcal{F}_t\right] = \exp\left\{\frac{y^2}{2}\int_0^t V_s\,\mathrm{d}s\right\}$$

(see Appendix A.1 in the Appendix). So we have

$$\begin{aligned} h(u, v, t) &= \exp\left\{x_0 u - \frac{u v_0 \rho}{c} - \frac{u\rho a b t}{c}\right\} \\ &\quad \mathbb{E}\left[\exp\left\{\Big(v + \frac{\rho u}{c}\Big) V_t + \Big(\frac{u^2}{2} - \frac{u}{2} - \frac{\rho^2 u^2}{2} + \frac{u a\rho}{c}\Big)\int_0^t V_s\,\mathrm{d}s\right\}\right]. \end{aligned}$$

Note that the coefficient of $\int_0^t V_s \, \mathrm{d}s$ is the equation of a parabola in u through the origin, hence if u is near zero so is the coefficient. Since both V_t and $\int_0^t V_s \, \mathrm{d}s$ have moment generating function (see [11]) it follows that the expectation is finite for (u, v) in a neighborhood of $(0, 0)$. Fix $t > 0$, apply the Itô formula to $\exp\{uX_t + vV_t\}$ and take expectations to end up with

$$\begin{aligned} h(u, v, t) &= \mathrm{e}^{ux_0 + vv_0} + vab \int_0^t h(u, v, s) \, \mathrm{d}s \\ &\quad + \underbrace{\left(-\frac{u}{2} + \frac{u^2}{2} + \frac{v^2 c^2}{2} + uv\rho c - va\right)}_{p(u,v)} \int_0^t \frac{\partial h(u, v, s)}{\partial v} \, \mathrm{d}s. \end{aligned}$$

Finally we differentiate with respect to t the above expression to get the PDE that drives the function $h(u, v, t)$

$$\frac{\partial h(u, v, t)}{\partial t} - p(u, v) \frac{\partial h(u, v, t)}{\partial v} = abvh(u, v, t).$$

The method of characteristics gives the unique solution

$$h(u, v, t) = \left(\frac{p\big(u, \phi(u, v, t)\big)}{p(u, v)} \right)^{ab/c^2} \exp\left\{ ux_0 + \phi(u, v, t) v_0 - \frac{abt\rho u}{c} + \frac{a^2 bt}{c^2} \right\},$$

where

$$\begin{aligned} P(u) &:= \sqrt{(a - \rho c u)^2 + c^2 (u - u^2)}, \\ \gamma(u, v) &:= -2 \operatorname{arctan h}\big((c^2 v + c\rho u - a)/P(u) \big)/P(u) \\ \phi(u, v, t) &:= -\frac{\rho u}{c} + \frac{a}{c^2} - \frac{1}{c^2} P(u) \tanh\Big(P(u)\big(t + \gamma(u, v)\big)/2 \Big). \end{aligned}$$

Let $v = 0$ in the last expression of $h(u, v, t)$ to get the moment generating function of X_t, $M_t(u)$; when there is no confusion we will suppress the subindex t and write $M(u)$. After some tedious manipulations, $M(u)$ can be written as

$$\begin{aligned} M(u) &= \mathbb{E}\Big[\exp\big\{ uX_t \big\} \Big] \\ &= \exp\{x_0 u\} \left(\frac{\mathrm{e}^{(a - c\rho u)t/2}}{\cosh(P(u)t/2) + \big(a - c\rho u\big) \sinh(P(u)t/2)/P(u)} \right)^{2ab/c^2} \\ &\quad \times \exp\left\{ -v_0 \frac{(u - u^2) \sinh(P(u)t/2)/P(u)}{\cosh(P(u)t/2) + \big(a - c\rho u\big) \sinh(P(u)t/2)/P(u)} \right\}. \end{aligned} \tag{3}$$

Remark 2.1. Formula (3) coincides with the one that can be deduced from the joint Laplace–Mellin transformation of S_t, V_t and $\int_0^t V_s \, \mathrm{d}s$ given by Dufresne [12, Theorem 12].

2.2. *The characteristic function of X_t*

Let $z \in \mathbb{C}$ and consider the function

$$\Phi(z) := \exp\{x_0 z\}\left(\frac{\mathrm{e}^{(a-c\rho z)t/2}}{\cosh(P(z)t/2)+(a-c\rho z)\sinh(P(z)t/2)/P(z)}\right)^{2ab/c^2}$$
$$\times\exp\left\{-v_0\,\frac{(z-z^2)\sinh(P(z)t/2)/P(z)}{\cosh(P(z)t/2)+(a-c\rho z)\sinh(P(z)t/2)/P(z)}\right\}. \tag{4}$$

Write

$$p(z) := (a-\rho c z)^2 + c^2(z-z^2) \tag{5}$$

to denote the second degree polynomial within $P(z)$, that is $P(z) = \sqrt{p(z)}$. Since $p(0) = a^2 > 0$ it follows that $\Phi(z)$ is well defined and analytic in a neighborhood of 0. Obviously, $\Phi(u) = M(u)$ on a (real) neighborhood of 0. Therefore the characteristic function of X_t is $\varphi(u) = \Phi(\mathrm{i}u)$. Explicitly, for $u \in \mathbb{R}$,

$$\varphi(u) := \mathbb{E}[\mathrm{e}^{\mathrm{i}uX_t}] = \Phi(\mathrm{i}u) = \exp\{\mathrm{i}x_0 u\}\left(\frac{\mathrm{e}^{\xi t/2}}{\cosh(\mathrm{d}t/2)+\xi\sinh(\mathrm{d}t/2)/\mathrm{d}}\right)^{2ab/c^2}$$
$$\times\exp\left\{-v_0\,\frac{(\mathrm{i}u+u^2)\sinh(\mathrm{d}t/2)/\mathrm{d}}{\cosh(\mathrm{d}t/2)+\xi\sinh(\mathrm{d}t/2)/\mathrm{d}}\right\}, \tag{6}$$

where

$$d := P(\mathrm{i}u) = \sqrt{(a-c\rho\mathrm{i}u)^2 + c^2(\mathrm{i}u+u^2)},$$
$$\xi := a - c\rho u\mathrm{i}.$$

After some computations we arrive at the formula of Albrecher et al. [1]

$$\varphi(u) = \exp\{\mathrm{i}x_0 u\}\exp\left\{\frac{ab}{c^2}\left((\xi-d)t - 2\log\frac{1-g\mathrm{e}^{-\mathrm{d}t}}{1-g}\right)\right\}$$
$$\times\exp\left\{\frac{v_0}{c^2}(\xi-d)\frac{1-\mathrm{e}^{-\mathrm{d}t}}{1-g\mathrm{e}^{-\mathrm{d}t}}\right\}, \tag{7}$$

where

$$g := \frac{\xi-d}{\xi+d}.$$

Of course, formula (6) looks more complex than the compact (7). However, when one recovers from the shock, one realizes that the former is easier to handle than the latter.

2.3. *Infinite divisibility and reciprocal Heston model*

Proposition 2.2. *The characteristic function of X_t is infinitely divisible.*

Proof. Denote by $\varphi(u)$ the characteristic function of X_t in a Heston model with parameters $(a, b, c, \rho, x_0, v_0, \mu)$. For $n \geq 1$ consider a Heston process with parameters $(a, b/n, c, \rho, x_0/n, v_0/n, \mu/n)$ and denote the characteristic function of $X_t = \log S_t - \mu t/n$ by φ_n. It is clear from (6) that

$$\varphi(u) = \left(\varphi_n(u)\right)^n. \quad \square$$

Recent interesting results of del Baño Rollin [8] show that if S is given by a Heston model with parameters $(a, b, c, \rho, s_0, v_0, \mu)$ such that $a > c\rho$, then S^{-1} follows a Heston model with parameters

$$(a - c\rho, ab/(a - c\rho), c, -\rho, s_0^{-1}, v_0, -\mu) \tag{8}$$

with respect to a new probability $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{X_T}.$$

To prove this property, it is needed to work with the whole process S. However, an easy verification can be done using the moment generating function (3). We will use this property in Section 3.1 to compute $\mathbb{E}[\exp\{uX_t\}]$ for negative values of u.

Remark. It can also be proved that the law of Heston models with parameters $(a, b, c, \rho, s_0, v_0, \mu)$ and $(a, b, -c, -\rho, s_0, v_0, \mu)$ is the same. Again, the proof needs to consider the entire process, but a quick check is deduced from the expression of the moment generating function (3). So, without loss of generality we will assume from now on that $c > 0$.

3. Main results

In this section we review the main results of the paper. Some proofs are done in posterior sections.

3.1. The domain of the moment generating function

The domain of the moment generating function (3) is not evident because we did not obtain $M(u)$ by the computation of the expectation $\mathbb{E}[e^{uX_t}]$ but by an indirect way. So, we only know that the moment generating function equals the function given in the right hand side of (3) in a neighborhood of zero. Recall that the domain, D, of a moment generating function coincides with its domain of analyticity and is an interval of $\mathbb{R}$ (finite or infinite, open or closed from one side or the other, that always include the origin, and it may be just $\{0\}$). The left (resp. right) extreme of D is called the left (resp. right) abscissa of convergence. It is clear from (3) that the abscissæ of convergence will be given by the largest negative zero and the smallest positive zero of

$$f(u) := \cosh(P(u)t/2) + (a - c\rho u)\,\frac{\sinh(P(u)t/2)}{P(u)}. \tag{9}$$

In order to locate such zeroes we need to study the behaviour of $p(u)$ defined by (5) in the real line, since the above function is well defined for $p(u) > 0$. When $\rho \neq \pm 1$, $p(u)$ represents a parabola with leading coefficient $c^2(\rho^2 - 1) \leq 0$ and $p(0) = a^2 > 0$; see Fig. 1. Therefore, it is an inverted parabola with real roots, $u_- < 0 < u_+$, given by

$$u_\pm = \frac{c - 2a\rho \pm \sqrt{4a^2 + c^2 - 4ac\rho}}{2c(1 - \rho^2)}.$$

When $\rho = \pm 1$, there are degenerate cases that can be managed easily.

Thus, for every $\rho \in [-1, 1]$, we have that $p(u) \geq 0$ on $[u_-, u_+]$, hence the function $f(u)$ which is the main ingredient in (3), is well defined and analytic in such (possibly infinite) interval. So, at first glance, $M(u)$ is defined in the intersection of $[u_-, u_+]$ and the interval between the

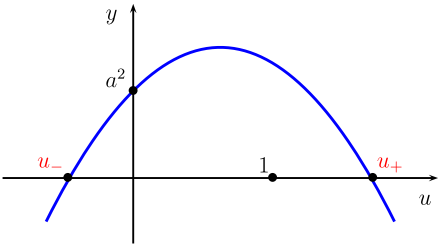


Fig. 1. Parabola $p(u) = (a - \rho c u)^2 + c^2(u - u^2)$.

largest negative zero and the smallest positive zero of $f(u)$. However, a deep insight into the question reveals that if we can find an analytic function defined in a larger interval that coincides with $M(u)$ in a neighborhood of 0, then $M(u)$ is also well defined in the larger interval by analytic continuation. With this purpose, consider the function $\cosh\sqrt{x}$, for $x > 0$. Its Taylor expansion is

$$\cosh\sqrt{x} = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}.$$

The series on the right defines an entire function, say $L_1(x)$, for $x \in \mathbb{R}$. However, when $x < 0$, that series coincides with the Taylor expansion of $\cos\sqrt{-x}$. Hence, $L_1(x)$ is an entire function that, when written as the composition of elementary functions, has different expression according to whether $x > 0$ or $x < 0$, that is

$$L_1(x) := \begin{cases} \cosh\sqrt{x}, & \text{if } x \geq 0, \\ \cos\sqrt{-x}, & \text{if } x \leq 0. \end{cases}$$

In a similar way, the function $(\sinh\sqrt{x})/\sqrt{x},\ x > 0$, can be analytically continued to negative x by $(\sin\sqrt{-x})/\sqrt{-x}$.

Denote by $u_-^* \geq -\infty$ the left abscissa of convergence of $M(u)$ and by $u_+^* \leq \infty$ the right abscissa and define $\widetilde{P}(u) := \sqrt{-p(u)}$. Therefore, by analytic continuation, the moment generating function for $u \in (u_-^*, u_-)$ or $u \in (u_+, u_+^*)$ is

$$M(u) = \exp\{x_0 u\} \left(\frac{\mathrm{e}^{(a - c\rho u)t/2}}{\cos(\widetilde{P}(u)t/2) + (a - c\rho u)\sin(\widetilde{P}(u)t/2)/\widetilde{P}(u)} \right)^{2ab/c^2}$$
$$\times \exp\left\{ -v_0 \frac{(u - u^2)\sin(\widetilde{P}(u)t/2)/\widetilde{P}(u)}{\cos(\widetilde{P}(u)t/2) + \big(a - c\rho u\big)\sin(\widetilde{P}(u)t/2)/\widetilde{P}(u)} \right\}.$$

In that expression, the main part is the function

$$\widetilde{f}(u) := \cos(\widetilde{P}(u)t/2) + (a - c\rho u)\frac{\sin(\widetilde{P}(u)t/2)}{\widetilde{P}(u)}. \tag{10}$$

Both f and $\widetilde{f}$ are defined in disjoint sets, and can be combined in a new function

$$F(u) := \begin{cases} f(u), & \text{if } u \in [u_-, u_+], \\ \widetilde{f}(u), & \text{if } u < u_- \text{ or } u > u_+, \end{cases} \tag{11}$$

that is analytic in $\mathbb{R}$. See Fig. 2 for a plot of that function.

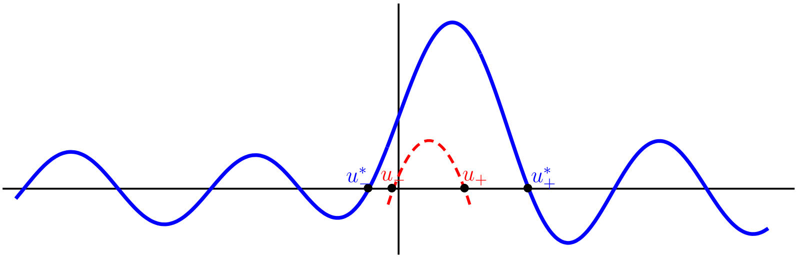


Fig. 2. Solid line: plot of $F(u)$. Dashed line: plot of the parabola $p(u)$. The points u_-^* and u_+^* are the abscissæ of convergence of $M(u)$.

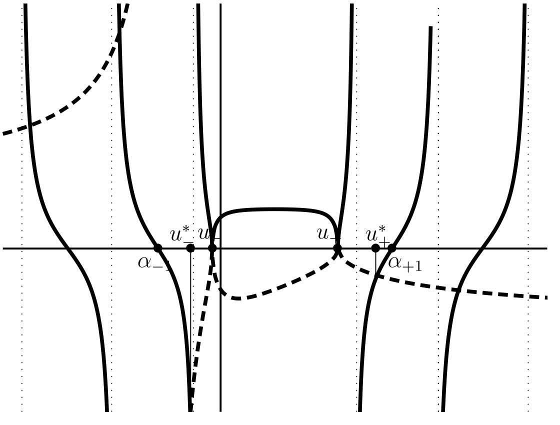


Fig. 3. Solid line: plot of $\tanh\Big(P(u)t/2\Big)$ for $u \in [u_-, u_+]$ or $\tan\Big(\widetilde{P}(u)t/2\Big)$ in the complementary. Dashed line: plot of $-P(u)/(a - \rho c u)$ for $u \in [u_-, u_+]$ or $-\widetilde{P}(u)/(a - \rho c u)$ in the complementary.

Then, to find the right abscissa of convergence, we need to find the zero of the function $f(u)$ in $[1, u_+]$, and if there is no zero, then to find the smallest zero $u > u_+$ of $\widetilde{f}(u)$. Note also that the real zeroes of $\widetilde{f}(u)$ are the real solutions of the equation (see Fig. 3)

$$\tan(\widetilde{P}(u)t/2) = -\frac{\widetilde{P}(u)}{a - c\rho u}. \tag{12}$$

(Except when $a/c\rho = k\pi/2$ for some natural number k). For the left abscissa of convergence we need to look for the largest solution $u < u_-$ of $\widetilde{f}(u) = 0$, or, equivalently, to work with Eq. (12).

In order to give a more precise location for the abscissæ of convergence, denote by $\alpha_{\pm 1}$ the solutions of the equation $p(u) = -4\pi^2/t^2$, which are real for all $t > 0$ and such that $\alpha_{-1} < u_- < u_+ < \alpha_{+1}$. Also write $\beta_{\pm 1}$ for the solutions of $p(u) = -\pi^2/t^2$. Finally, we prove (see Section 4):

Theorem 3.1. *With the above notations,*

1. *If $a \geq \rho c$, the right abscissa of convergence u_+^* is the smallest zero $u > u_+$ of $\widetilde{f}(u) = 0$, and $u_+^* \in (u_+, \alpha_{+1})$.*
2. *If $\rho = 1$ and $2a = c$, the right abscissa of convergence is $u_+^* = (1 - \mathrm{e}^{-at})^{-1}$.*

3. *If* $a < \rho c$ *(except for* $\rho = 1$ *and* $2a = c$*), let* $t_0 = 2/(c\rho u_+ - a) > 0$.
   (i) *If* $t < t_0$*, then* u_+^* *is smallest zero* $u > u_+$ *of* $\widetilde{f}(u) = 0$*, and* $u_+^* \in (u_+, \beta_{+1})$.
   (ii) *If* $t \geq t_0$*, then* u_+^* *is the zero of* $f(u)$ *in* $(1, u_+]$.
4. *In any case, the left abscissa of convergence,* u_-^* *is the largest zero* $u < u_-$ *of* $\widetilde{f}(u) = 0$ *and* $u_-^* \in (\alpha_{-1}, u_-)$.

Remark 3.2. 1. When $a > \rho c$ the inversion formula (8) can be used to compute u_-^* in terms of u_+^* of the inverted model. Specifically,

$$u_-^*(a, b, c, \rho, x_0, v_0, t) = -u_+^*(a - c\rho, ab/(a - c\rho), c, -\rho, -x_0, v_0, t) + 1.$$

2. This theorem gives a direct procedure to invert the formulæ of Andersen and Piterbarg [2, Proposition 3.1].

3.2. Factorization of the characteristic function and existence of a $\mathcal{C}^\infty$ density

The factorization of the characteristic function is obtained using classical theorems of complex analysis applied to the complex moment generating function (4). We prove (Section 5):

Theorem 3.3. *The characteristic function of* X_t *can be written as*

$$\varphi(u) = \exp\{\mathrm{i}ud\} \prod_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}c_n u} \left(1 - \frac{\mathrm{i}u}{a_n}\right)^{-2ab/c^2} \exp\left\{\frac{v_0 b_n \mathrm{i}u/a_n}{1 - \mathrm{i}u/a_n}\right\} \tag{13}$$

where

1. a_n *are the roots of the function* $F(u)$ *given in* (11).
2. $b_n = 4p(a_n)(1 - a_n)\left[tp'(a_n)p(a_n) - 4c\rho p(a_n) - p'(a_n)(a - c\rho a_n)((a - c\rho a_n)t + 2)\right]^{-1}$, *where* $p(u)$ *is the second degree polynomial* (5)*. By Theorem 5.9 it follows that* $b_n > 0$.
3. $c_n = -(v_0 b_n + 2abc^{-2})a_n^{-1}$.
4. $d = x_0 - \rho abtc^{-1} - 2ab\nu c^{-2} - v_0(1 - \mathrm{e}^{-at})(2a)^{-1}$*, where* ν *is determined by*

$$\cosh\bigl(|a - \rho c|t/2\bigr) + \mathit{sign}\,\{a - \rho c\}\, \sinh\bigl(|a - \rho c|t/2\bigr) = \mathrm{e}^{at/2}\mathrm{e}^{\nu} \prod_{n=1}^{\infty} \left(1 - \frac{1}{a_n}\right) \exp\left\{\frac{1}{a_n}\right\}. \tag{14}$$

We also have $d = \mathbb{E}[X_t]$. The above formula will prove useful in the next section.

Corollary 3.4. *The random variable* X_t *has a* $\mathcal{C}^\infty$ *density.*

Proof. We use a classical result from Fourier Analysis that is in the basis of the Malliavin Calculus: if the characteristic function φ of a random variable satisfies that for all $k \in \mathbb{N}$

$$\int_{-\infty}^{\infty} |u|^k\, |\varphi(u)|\, \mathrm{d}u < \infty,$$

then the random variable possesses a $\mathcal{C}^\infty$ density; see, for example, the proof of Lemma 2.1.5 in [27].

Take a generic term of the infinite product (13)

$$\mathrm{e}^{\mathrm{i}c_n u} \left(1 - \frac{\mathrm{i}u}{a_n}\right)^{-2ab/c^2} \exp\left\{\frac{v_0 b_n \mathrm{i}u/a_n}{1 - \mathrm{i}u/a_n}\right\},$$

which is the product of three characteristic functions; namely, a constant random variable, a positive or negative gamma law and a compound Poisson random variable with a gamma jump size distribution. Hence,

$$\left| e^{\mathrm{i}c_n u}\left(1-\frac{\mathrm{i}u}{a_n}\right)^{-2ab/c^2}\exp\left\{\frac{v_0 b_n \mathrm{i}u/a_n}{1-\mathrm{i}u/a_n}\right\}\right| \le \left|\left(1-\frac{\mathrm{i}u}{a_n}\right)^{-2ab/c^2}\right| \le \frac{1}{(1+u^2/a_n^2)^{ab/c^2}} \le 1.$$

Given $k \ge 0$, choose an integer K such that

$$2K\frac{ab}{c^2} > k+1,$$

and let $\alpha = \max\{|a_1|, \dots, |a_K|\}$. We have

$$\int_{-\infty}^{\infty} |u|^k\, |\varphi(u)|\, \mathrm{d}u \le 2\int_0^{\infty} \frac{u^k}{(1+u^2/\alpha^2)^{Kab/c^2}}\, \mathrm{d}u < \infty. \quad \square$$

3.3. *Expression of the density function*

Each factor of (13) can be identified as characteristic functions of Bessel type laws or *negative* Bessel laws. So the infinite product in (13) corresponds to the convergent infinite convolution of such laws. Using a classical result of Wintner [34,33], we can deduce an expression of the density as an infinite convolution of densities. Let a generic factor of the infinite product be decomposed in the following way:

$$\underbrace{e^{\mathrm{i}c_n u}}_{(*)}\underbrace{\left(1-\frac{\mathrm{i}u}{a_n}\right)^{-2ab/c^2}\exp\left\{\frac{v_0 b_n \mathrm{i}u/a_n}{1-\mathrm{i}u/a_n}\right\}}_{(**)}. \tag{15}$$

The term $(*)$ is a translation of the random variable. When $a_n > 0$ the factor $(**)$ is the characteristic function of a Bessel law that is reported, for example, in [26, Page 46], or in [9, Page 469], where the name Polya–Aeppli is also used. In Lukacs and Laha notation, the parameters are

$$\lambda = \frac{2ab}{c^2}, \quad \theta = a_n \quad \text{and} \quad \beta = 2\sqrt{v_0 b_n a_n}. \tag{16}$$

The density function (with Lukacs and Laha notation) is

$$B(x) = C x^{(\lambda-1)/2}\exp\{-\theta x\} I_{\lambda-1}(\beta\sqrt{x})\mathbf{1}_{(0,\infty)}(x), \tag{17}$$

where the constant C is

$$C = \left(\frac{1}{2\beta}\right)^{\lambda-1}\theta^{\lambda}\exp\{-\beta^2/4\theta\},$$

and $I_\tau(x)$ is the modified Bessel function of the first kind of index τ (see, for example, Revuz and Yor, page 549). When $a_n < 0$, such a characteristic function corresponds to *negative* Bessel random variables with density

$$B(x) = C(-x)^{(\lambda-1)/2}\exp\{\theta x\} I_{\lambda-1}(\beta\sqrt{-x})\mathbf{1}_{(-\infty,0)}(x), \tag{18}$$

where $\theta = -a_n$, $\beta = 2\sqrt{-v_0 b_n a_n}$ and λ is as in (16).

Remark 3.5. When $4ab/c^2 = k$ is a natural number, we can identify the characteristic function (15) as the one given by Janson [20, Theorem 6.2], where each eigenvalue has multiplicity k. That means, for such parameters, X_t is in the (non-homogeneous) second Wiener chaos. It is well known that for such a combination of parameters the CIR model has the law of a sum of the squares of k independent Ornstein–Uhlenbeck processes, and this fact has been used for many applications. See, for example, Grasselli and Hurd [17] and the references therein.

We need now to fix some notation for next results. The product of two characteristic functions of absolutely continuous laws with densities f_1 and f_2 is the characteristic function of the convolution of the laws, that has a density given by the convolution of the densities

$$f_1 \star f_2(x) := \int_{-\infty}^{+\infty} f_1(y) f_2(x-y)\,\mathrm{d}y.$$

The product $f_1 \star \cdots \star f_n$ is denoted by $\bigstar_{j=1}^{n} f_j$. For a sequence of densities, $\{f_n, n \geq 1\}$, we will say that $\bigstar_{n=1}^{\infty} f_n$ is a convergent infinite convolution if the sequence $\{\bigstar_{j=1}^{n} f_j, n \geq 1\}$ converges (pointwise) to a density, that it is written $\bigstar_{n=1}^{\infty} f_n$.

Theorem 3.6. *The density of X_t is $g(x) = \bigstar_{j=1}^{\infty} h_j(x - d(t))$, where $h_n(x) = B_n(x - c_n(t))$ and B_n is the probability density function given in (17) or (18).*

Proof. The main ingredient of the proof is the following classical result of Wintner ([34], see also [33]): Consider a convergent infinite convolution of probability measures $\bigstar_{j=1}^{\infty} P_j$. If at least one of the probability measures has a continuous density of bounded variation on $\mathbb{R}$, then so does $\bigstar_{j=1}^{\infty} P_j$, and the continuous density of $\bigstar_{j=1}^{n} P_j$ tends, as $n \to \infty$, to that of $\bigstar_{j=1}^{\infty} P_j$ uniformly on every bounded interval.

So we need only to check that the convergent infinite convolution of the laws corresponding to the factorization (13) has an element with a density that is continuous and of bounded variation on $\mathbb{R}$. Consider a finite product of K terms in the factorization (13), and a decomposition of a generic term as in the proof of Corollary 3.4. Reorder the convolution so the characteristic functions of gamma laws are first. The convolution of positive and negative gamma laws belongs to the class of extended generalized gamma convolutions (see [3], chapter 7), and hence is unimodal [3, Page 107]. Therefore, as in the proof of Corollary 3.4 we can take K large enough to get a law with continuous density, and the proof is finished. □

3.4. Numerical applications

3.4.1. Moment explosion

From Theorem 3.1 and Section 4.2 it is deduced that we are able to localize the abscissæ of convergence in certain intervals depending on the parameters. In each interval we specify which function needs to be solved, and from the proof of such a result and Lucic [24] it follows that this function has only one root (except in the case 1 that may have two) in the given interval. Therefore the numerical computation of the domain of convergence of the moment generating function is extremely efficient and fast.

The knowledge of the abscissæ of convergence implies the knowledge of the edge from which the moment of spot is no longer finite. These edges play a major role in many circumstances. For instance, an outstanding result of Roger Lee [22] gives a relationship between the asymptotic behaviour of the volatility smile and the abscissæ of convergence. This can be of interest in designing sensible smile interpolation and extrapolation schemes as has been shown by Gatheral

[14]. Furthermore, when using the Fast Fourier Transform proposed by Carr and Madan in [5] to price vanilla options, we need to choose a parameter α such that $\mathbb{E}[S_t^{\alpha+1}]$ exists.

3.4.2. Simulation of X_t

The factorization of the characteristic function, Theorem 3.3, and the identification of the factors in the previous section suggest to approximate X_t by a finite sum of independent random variables. We can reorder freely the different terms in a finite sum, thus we consider the approximation

$$X_t^N := d_N + \sum_{n=1}^{N} (X_n + c_n),$$

where $X_1, \ldots, X_N$ are independent Bessel random variables given by (17) or (18), c_n is defined in Theorem 3.3 and d_N has the same definition as d has in Theorem 3.3, but we exchange ν for ν_N which is the solution of

$$\begin{aligned}&\cosh\big(|a-\rho c|t/2\big) + \operatorname{sign}\{a-\rho c\}\,\sinh\big(|a-\rho c|t/2\big)\\&\quad = e^{at/2}e^{\nu_N}\prod_{j=1}^{N}\left(1-\frac{1}{a_j}\right)\exp\left\{\frac{1}{a_j}\right\}.\end{aligned}$$

As Devroye [9, Page 469] explains, a Bessel random variable can be simulated using a Poisson, a gamma random variable plus a variable number of exponentials.

Mean square rate of convergence

Using a procedure similar to Rydén and Wiktorsson [29] it is deduced that $\mathbb{E}\big[\big(X_t - X_t^N\big)^2\big] = O(N^{-1})$. Let $\tilde{M}_N(u)$ denote the moment generating function of the random variable $Y_N := X_t - X_t^N$. In virtue of Theorem 3.3 we can write

$$\tilde{M}_N(u) = \exp\{(d-d_N)u\}\prod_{n=N+1}^{\infty}\left(1-\frac{u}{a_n}\right)^{-2ab/c^2}\exp\left\{-\frac{2ab}{c^2}\frac{u}{a_n} + v_0\frac{u^2 b_n}{a_n^2(1-u/a_n)}\right\},$$

and thus

$$\mathbb{E}\big[\big(X_t - X_t^N\big)^2\big] = \tilde{M}_N''(0) = (d-d_N)^2 + \sum_{n=N+1}^{\infty}\left(\frac{2ab}{c^2} + v_0 b_n\right)\frac{1}{a_n^2} = \hat{\sigma}_N^2 + \tilde{\sigma}_N^2,$$

where

$$\hat{\sigma}_N^2 := (d-d_N)^2 \quad \text{and} \quad \tilde{\sigma}_N^2 := \sum_{n=N+1}^{\infty}\left(\frac{2ab}{c^2} + v_0 b_n\right)\frac{1}{a_n^2}.$$

We use now two results which will be shown in Section 5. The first result is the fact that $\lim_{n\to\infty} b_n = C$, where C is a positive constant (see the proof of Theorem 5.9). The second result states that $a_N = O(N)$ (see the proof of Proposition 5.4).

Since the limit of $\{b_n\}_n$ exists, there also exists $\varepsilon > 0$ and N_ε such that for $N \ge N_\varepsilon$ then

$$\begin{aligned}\left(\frac{2ab}{c^2} + v_0(C-\varepsilon)\right)\sum_{n=N+1}^{\infty}\frac{1}{a_n^2} &\le \sum_{n=N+1}^{\infty}\left(\frac{2ab}{c^2} + v_0 b_n\right)\frac{1}{a_n^2}\\ &\le \left(\frac{2ab}{c^2} + v_0(C+\varepsilon)\right)\sum_{n=N+1}^{\infty}\frac{1}{a_n^2}.\end{aligned}$$

Finally, $\sum_{n=N+1}^{\infty} \frac{1}{a_n^2} \sim \frac{1}{N}$ and thus $\tilde{\sigma}_N^2 = \mathrm{O}(N^{-1})$. On the other hand

$$|d - d_N|^2 = \left(\frac{2ab}{c^2}\right)^2 |\nu - \nu_N|^2,$$

and from the definition of ν and ν_N it turns out that

$$\begin{aligned}
|\nu - \nu_N|^2 &= \left[\ln\left(\prod_{n=N+1}^{\infty}\left(1 - \frac{1}{a_n}\right)\exp\left\{\frac{1}{a_n}\right\}\right)\right]^2 \\
&= \left[\sum_{n=N+1}^{\infty} \ln\left(1 - \frac{1}{a_n}\right) + \frac{1}{a_n}\right]^2 \sim \left[\frac{-1}{2}\sum_{n=N+1}^{\infty}\frac{1}{a_n^2}\right]^2,
\end{aligned}$$

which implies $\hat{\sigma}_N^2 = \mathrm{O}(N^{-2})$. Therefore $\mathbb{E}\big[\big(X_t - X_t^N\big)^2\big] \sim \tilde{\sigma}_N^2$.

Tail approximation

Following again the scheme of Rydén and Wiktorsson [29], we will now prove that Y_N/σ_N converges in distribution to a Gaussian random variable, where

$$\sigma_N^2 := \sum_{n=N+1}^{\infty} \frac{1}{a_n^2}.$$

The proof is very easy because Polya–Aeppli random variables have a very simple structure of the cumulants: for a random variable with moment generating function

$$g(u) = (1 - u/\gamma)^{-\alpha} \exp\left\{\frac{\beta u/\gamma}{1 - u/\gamma}\right\},$$

the cumulants are

$$\kappa_j = \frac{(j-1)!\alpha}{\gamma^j} + \frac{j!\beta}{\gamma^j}, \quad \text{for } j \geq 1.$$

It is deduced that the cumulants of the tail Y_N are $\kappa_1(Y_N) = d - d_N$ and

$$\kappa_j(Y_N) = (j-1)!\frac{2ab}{c^2}\sum_{n=N+1}^{\infty}\frac{1}{a_n^j} + j!v_0\sum_{n=N+1}^{\infty}\frac{b_n}{a_n^j} \quad \text{for } j \geq 2.$$

By the results obtained in the computation of the mean square rate of convergence it is straightforward to calculate the cumulants of the normalized tail Y_N/σ_N. For instance, $\kappa_1(Y_N/\sigma_N) = (d - d_N)/\sigma_N = \mathrm{O}(N^{-1/2})$ and thus $\lim_{N\to\infty}\kappa_1(Y_N/\sigma_N) = 0$. It is also easy to check that

$$\lim_{N\to\infty}\kappa_2(Y_N/\sigma_N) = \lim_{N\to\infty}\kappa_2(Y_N)/\sigma_N^2 = \frac{2ab}{c^2} + 2v_0\lim_{n\to\infty} b_n.$$

For the rest of the cumulants $j \geq 3$, note that

$$\frac{1}{\sigma_N^j}\sum_{n=N+1}^{\infty}\frac{1}{a_n^j} = \mathrm{O}(N^{1-\frac{j}{2}}),$$

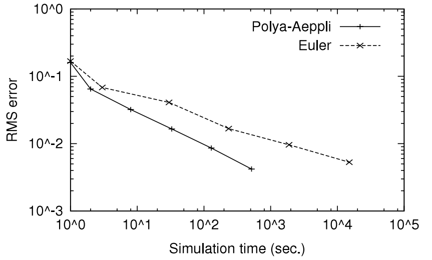
and hence $\lim_{N\to\infty}\kappa_j(Y_N/\sigma_N) = \lim_{N\to\infty}\kappa_j(Y_N)/\sigma_N^j = 0$ for $j \geq 3$. So by the moments method, the limit in law of Y_N/σ_N is a centered Gaussian random variable.

(a) Euler discretisation

Simulation	Steps	Bias	SE	RMS	%Abs.	sec.
10000	100	0.1001	0.1350	0.1681	4.67	1
40000	200	0.0136	0.0668	0.0682	3.02	3
160000	400	0.0243	0.0329	0.0409	1.94	30
640000	800	0.0030	0.0164	0.0167	1.24	235
2560000	1600	0.0050	0.0082	0.0096	0.80	1883
10240000	3200	0.0034	0.0041	0.0053	0.51	15042

(b) Polya–Aeppli approximation

Simulation	Bias	SE	RMS	sec.
10000	0.0971	0.1277	0.1604	1
40000	0.0055	0.0646	0.0648	2
160000	0.0105	0.0325	0.0321	8
640000	0.0038	0.0161	0.0165	33
2560000	0.0030	0.0080	0.0086	129
10240000	0.0012	0.0040	0.0042	516



Parameters	
S_0	100
V_0	0.010201
a	6.21
b	0.019
c	0.61
ρ	-0.7
t	1
μ	0.0319

Fig. 4. Simulation where the Feller condition is violated.

In the light of the results of Rydén and Wiktorsson [29] for the Lévy area, one may expect that the simulation can be improved adding to X_N a centered Gaussian random variable of suitable variance. We do not continue here such a theoretical analysis that likely requires to find a coupling for the tail, as it is beyond the purpose of the present paper.

Numerical results

The aim of this section is to show that our method produces slightly better results than the Euler discretisation of the Heston model, but is much less time consuming. Furthermore it shares its programming simplicity and thus it is a good candidate to be used as a benchmark for other simulating methods. For an overview of more sophisticated discretisation algorithms we refer to the reader to Lord et al. [23] and the references therein.

The main weakness of the Euler discretisation is the need for an external condition from the model to guarantee the non-negativeness of the volatility term. Typically this external condition is an absorbing barrier at 0. This absorbing barrier is more often reached when the Feller condition is not satisfied ($2ab \leq c^2$), making the approximation biased. Moreover, the coefficients of the stochastic differential equation of the Heston model do not satisfy the classical hypotheses for the rate of convergence of the Euler discretisation. This also makes that the rate of steps increasing with respect to the simulation trials is to be computed by direct inspection, see for instance Broadie and Kaya [4].

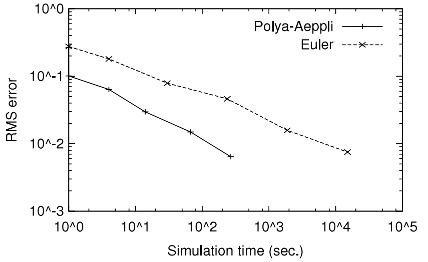
Interestingly enough, the Feller condition has also influence in our method: to simulate a Polya–Aeppli random variable with Devroye's algorithm, we have to be able to simulate a Poisson and a gamma random variable. One can easily check that when the Feller condition is not satisfied we often need to simulate a gamma random variable with shape parameter less than one. It turns out that the algorithms for simulating such random variables are not very efficient (see [21]). For our algorithm we have used algorithms [9, Page 505] and [9, Page 410] to simulate the Poisson and the gamma random variable of shape parameter greater or equal than 1. Algorithm 3 from [21] was chosen to simulate a gamma random variable with shape parameter less than 1. The approximation was made with the first 15 positive and negative Polya–Aeppli random variables and enhanced with an approximation of a normal distribution for the tail. We

(a) Euler discretisation

Simulation	Steps	Bias	SE	RMS	%Abs.	sec.
10000	100	0.1895	0.2009	0.2762	0.19	1
40000	200	0.1489	0.1015	0.1802	0.09	4
160000	400	0.0603	0.0507	0.0788	0.04	30
640000	800	0.0388	0.0253	0.0463	0.02	237
2560000	1600	0.0095	0.0126	0.0158	0.01	1885
10240000	3200	0.0039	0.0063	0.0075	0.00	15055

(b) Polya–Aeppli approximation

Simulation	Bias	SE	RMS	sec.
10000	0.1780	0.1982	0.2665	1
40000	0.0114	0.1002	0.1008	1
160000	0.0394	0.0501	0.0637	4
640000	0.0157	0.0250	0.0296	17
2560000	0.0080	0.0125	0.0149	70
10240000	0.0012	0.0063	0.0064	270



Parameters	
S_0	100
V_0	0.04
a	1.2
b	0.04
c	0.3
ρ	-0.5
t	1
μ	0.05

Fig. 5. Simulation where the Feller condition is fulfilled.

must say that the adjunction of the tail did not produce a significant improvement of the root mean squared error of the simulations.

In the following tables (Figs. 4 and 5) we have compared our method against the Euler discretisation to estimate the expectation of the spot, which is a known parameter of the model since $\mathbb{E}[S_t] = S_0 e^{\mu t}$. The first table is based on an example from Broadie and Kaya [4] where the Feller condition is not satisfied. The second example is from Glasserman [16], where the Feller condition is fulfilled ($2ab > c^2$). For the Euler discretisation we use the same number of time steps as Broadie and Kaya [4]. We compute the bias (Bias), standard error (SE), root mean squared error (RMS), the percentage of simulations absorbed by the barrier (%Abs.) and the computing time (s). What this two examples show is that we are able to obtain the same RMS accuracy with much less computing time (specially when the Feller condition is fulfilled and we do not need to use the rejection method proposed by Kundu and Gupta [21]).

We should also mention that as an alternative to the discretisation procedures, Broadie and Kaya [4] propose a method of direct simulation of the random variables; in agreement with the results that they report, the computing times of that method are worse than the Euler scheme except for very large simulations, and even in that case, much slower than the method that we propose. Moreover, the Broadie and Kaya algorithm has a code much more complex than ours.

4. Proof of the determination of the domain of the moment generating function

4.1. Preliminary study of the domain

As we commented, the main ingredient of the moment generating function is $f(u)$ defined in (9). We saw that $p(u) \geq 0$ for $u \in [u_-, u_+]$, where u_- and u_+ are the roots of $p(u)$, and then $f(u)$ is well defined and analytic in such an interval. The first step in the assessment of the domain of $M(u)$ is to study such an interval. Denote by $D(X_t)$ such a domain. The following proposition summarizes the results of this study.

Proposition 4.1. *For every $\rho \in [-1, 1]$ there is the inclusion $[u_-, 1] \subset D(X_t)$. Moreover,*

1. *When $a \geq c\rho$ (in particular, for every $\rho < 0$), the function $f(u)$ has no zeroes in $[u_-, u_+]$. Consequently $[u_-, u_+] \subset D(X_t)$.*
2. *When $a < c\rho$ (except for $\rho = 1$ and $2a = c$), write $t_0 = 2/(c\rho u_+ - a) \geq 0$*
   (i) *If $t < t_0$, then $[u_-, u_+] \subset D(X_t)$, and $f(u)$ has no zeroes in $[u_-, u_+]$.*
   (ii) *If $t \geq t_0$, then $f(u)$ has no zeroes in $[u_-, 1]$, and has one and only one zero in $(1, u_+]$.*
3. *When $\rho = 1$ and $2a = c$, then $D(X_t) = \big(-\infty, 1/(1 - e^{-at})\big)$.*

Proof. The proof is straightforward and translated to the Appendix. □

Remark 4.2. 1. The cases $\rho = \pm 1$ are specially important. We stress that
   (i) For $\rho = -1$, we are always in case 1 and $u_+ = \infty$.
   (ii) For $\rho = 1$ and $a \geq c$, we are in case 1. For $\rho = 1$ and $a < c \neq 2a$ we are in case 2; however, if $2a < c$ then $u_+ = \infty$ and $t_0 = 0$, thus we are in case (2ii) for all $t > 0$.
2. From the preceding proposition it follows that $\mathbb{E}[S_t] < \infty$ for every $a, b, c > 0$ and $\rho \in [-1, 1]$. Moreover, when $\mu = 0$, by construction, $\{S_t,\ t \in [0, T]\}$ is an exponential local martingale; that is a positive supermartingale, see Revuz and Yor [28, pages 148 and 149], and
$$\mathbb{E}[S_t] = M(1) = e^{x_0},$$
so $\{S_t, t \in [0, T]\}$ is a true martingale. This observation was proved by Andersen and Piterbarg [2, Proposition 2.5] using the Feller explosion criteria and Girsanov Theorem.
3. As a continuation of the preceding point, we should remark that $\{S_t,\ t \in [0, T]\}$ is not always a square integrable martingale.

4.2. Computation of the abscissæ of convergence of the moment generating function

When there is no root of $f(u)$ in $[1, u_+]$ the domain $D(X_t)$ is larger than $[u_-, u_+]$ (in particular, when $a \geq \rho c$). To carry out this study, we need more properties of the moment generating function. Consider an arbitrary random variable X, with moment generating function $M_X(u)$, and domain $D_X = \{u \in \mathbb{R} :\ M_X(u) < \infty\}$. The following property is well known, but since it plays a major role in this paper, we stress it. We give the property for the right abscissa of convergence but a similar statement is true for the left abscissa.

Lemma 4.3. *Let X a random variable such that there is a neighborhood of zero included in D_X. Assume that $(r, s] \subset D_X$, and the existence of an analytic function $h : (p, q) \to \mathbb{R}$ such that*

1. *$(r, s] \subset (p, q)$.*
2. *$M_X(u) = h(u)$, for $u \in (r, s]$.*

Then $M_X = h$ on (p, q). Moreover, if $\lim_{u \nearrow q} h(u) = \infty$, then the right abscissa of convergence of M_X is the point q.

Proof. Denote the interior of D_X by (α, β). If $\beta = \infty$, by analytic continuation, $M_X = h$ on (r, q). Consider $\beta < \infty$, so β is the finite right abscissa of convergence. Then it is well known that the function M_X has a singularity at β (see, for example, Widder [32, Theorem II.5b]; the proof for complex Laplace transforms can be translated to this context). Thus, $s \neq \beta$ because $M_X(s) = h(s)$ and h is analytic in s. Hence, $\beta > s$. In the same way, $\beta < q$ is contradictory, then $\beta \geq q$ and by analytic continuation $M_X = h$ on (r, q). The second part of the lemma is obvious. □

Now we apply the preceding lemma to the moment generating function of the log-spot, $M(u)$. When the function $f(u)$ given in (9) has no zeroes in $[1, u_+]$, it follows that

$$\lim_{u \searrow u_-} M(u) < \infty \quad \text{and} \quad \lim_{u \nearrow u_+} M(u) < \infty,$$

then by Lemma 4.3 the domain of $M(u)$, $D(X_t)$, is larger than $[u_-, u_+]$. We can also easily deduce the results in Section 3.1 by the preceding lemma. In particular, the following cases are the tricky ones.

1. If $a \geq \rho c$, then $u_+^* \in (u_+, \alpha_{+1})$. This can be deduced from the consideration that the image of the function $\tan(\widetilde{P}(u)t/2)$ on that interval is $\mathbb{R}$, and the properties of the function in the right hand side of (10). The only case not clear enough is when $a/(c\rho) = \pi/2$, due to the fact that $-\widetilde{P}(u)/(a - c\rho u)$ has a vertical asymptote at $u = \beta_1$; this case is studied by direct inspection.
2. If $a < \rho c$, then $f(1) > 1$ and
$$\widetilde{f}(\beta_{+1}) = (a - c\rho\beta_{+1})2/\pi < 0,$$
because $\beta_{+1} > 1$ and $\rho > 0$. So $F(u)$ has at least one root in $(1, \beta_{+1})$.

Theorem 3.1 is deduced from such considerations.

5. Factorization of the characteristic function

In this section we will work exclusively with the random variable X_t for $t > 0$ fixed, and the time t will be considered as a parameter. Denote by $\mathcal{HL}(a, b, c, \rho, x_0, v_0, t)$ the law of X_t, that is, a probability on $\mathbb{R}$ that has the moment generating function given by (3). Observe that for all $\lambda > 0$,

$$\mathcal{HL}(\lambda a, \lambda b, \lambda c, \rho, x_0, \lambda v_0, t/\lambda) = \mathcal{HL}(a, b, c, \rho, x_0, v_0, t).$$

Since we are interested in a property true for all parameters, it suffices to prove that property for arbitrary a, b, c, ρ, x_0, v_0, and $t = 2$. So, in all proofs, we will take this value of t.

We will use two powerful theorems of complex analysis that we will apply to the complex moment generating function $\Phi(z)$ given by (4) and introduced in Section 2.2. For easy reference, we here recall the two main theorems expressed in the form that best suits us. We will factorize separately the different factors of $\Phi(z)$, and at a later stage, we will combine them. The first of such theorems is Hadamard's factorization Theorem.

Theorem 5.1. *Let $F(z)$ be an entire function of finite order, $F(0) \neq 0$, with roots $a_1, a_2, \ldots,$ such that*

$$\sum_n 1/|a_n|^2 < \infty.$$

Then F can be represented as

$$F(z) = F(0)e^{Cz} \prod_n \left(1 - \frac{z}{a_n}\right) e^{z/a_n},$$

where C is a constant.

The second theorem is due to Mittag-Leffler (Titchmarsh [31, page 110]).

Theorem 5.2. *Let $G(z)$ be a meromorphic function such that all its poles are simple. Let them be $a_1, a_2, \ldots$, where $0 < |a_1| \leq |a_2| \leq \cdots$, and let the residues at the poles be $b_1, b_2 \ldots$ respectively. Suppose that there is a sequence of closed contours C_n, such that C_n includes $a_1, \ldots, a_n$, but no other poles; and the minimum distance R_n of C_n to the origin tends to infinite with n, while the length of C_n is $O(R_n)$; and $f(z) = o(R_n)$ on C_n. Then*

$$G(z) = G(0) - z \sum_n \frac{b_n/a_n^2}{1 - z/a_n}.$$

5.1. The entire component

Write

$$F(z) = \cosh(P(z)) + (a - c\rho z) \frac{\sinh(P(z))}{P(z)},$$

for the complex version of $f(u)$ defined in (9). Define also the complex analogue of L_1 and L_2 introduced in Section 3.1 by the power series

$$L_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2n)!}, \quad \text{and} \quad L_2(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2n+1)!}.$$

Note that at each z,

$$L_1(z) = \cosh \sqrt{z} \quad \text{and} \quad L_2(z) = \frac{1}{\sqrt{z}} \sinh \sqrt{z},$$

independently of the branch of the square root. Indeed, in every neighborhood that does not include zero, the previous relations are true fixing an arbitrary branch of the square root. However, in the whole $\mathbb{C}$, the functions L_1 and L_2 are defined by the power series and not as a composition of $\cosh z$ or $\sinh z$ with a particular branch of $\sqrt{z}$. We consider the extension of $F(z)$ to an entire function using the functions $L_1(z)$ and $L_2(z)$. Note that both $L_1(z)$ and $L_2(z)$ take real values on $\mathbb{R}$, and that $F(z)$ restricted to $\mathbb{R}$ coincides with the function $F(u)$ defined in (11).

The first interesting property of $F(z)$ is that all its the zeroes are real and simple. This can be deduced from a deep theorem of Lucic [24, Theorem 1] who proves that all the singularities of the (complex) characteristic function of X_t are purely imaginary. We give here an alternative proof in line with the philosophy of Lucic that provides technical tools which will be used in subsequent derivations. We may start by a lemma that will ease the argument.

Lemma 5.3. *For every $\varepsilon > 0$ there is $n_0 \geq 1$ such that for $n \geq n_0$,*

$$\sup_{|z|=(n+\frac{1}{2})\pi} |\coth z| < 1 + \varepsilon \quad \textit{and} \quad \sup_{|z|=n\pi} |\tanh z| < 1 + \varepsilon.$$

Proof. See Appendix A.3. □

Proposition 5.4. *The zeroes of $F(z)$ are all real and simple.*

Proof. Remember that we consider $t = 2$. First, note that for $\rho \neq \pm 1$

$$\lim_{|z|\to\infty} \left| \frac{a - c\rho z}{P(z)} \right| = \frac{|\rho|}{\sqrt{1-\rho^2}}. \tag{19}$$

Consider the following three cases:
Case 1. $|\rho|/\sqrt{1-\rho^2} > 1$. The first objective is to prove that for $n \gg 1$

$$|\cosh P(z)| < \left|(a - c\rho z)\frac{\sinh P(z)}{P(z)}\right| \tag{20}$$

on the contour $C_n = \{z \in \mathbb{C} : |p(z)| = (n+0.5)^2\pi^2\}$. Then, by Rouche's Theorem, both functions $F(z)$ and $(a - c\rho z)\sinh(P(z))/P(z)$ have the same number of zeroes inside C_n. To prove the inequality (20), take $\varepsilon > 0$ such that $1 + \varepsilon < |\rho|/\sqrt{1-\rho^2}$. By Lemma 5.3 we have $|\coth P(z)| < 1 + \varepsilon$, then apply the limit (19) for n large enough to get the desired inequality.

To study of the zeroes of $F(z)$ assume that $\rho \neq 0$. The zeroes of $(a - c\rho z)\sinh(P(z))/P(z)$ are $z = a/(c\rho)$ and $\sinh(P(z))/P(z)$ which can be factorized as

$$\frac{\sinh P(z)}{P(z)} = \prod_{k=1}^{\infty}\left(1 + \frac{p(z)}{k^2\pi^2}\right).$$

Hence, the zeroes of the above function are the roots of the second order polynomial

$$p(z) = -k^2\pi^2, \quad k = 1, 2, \ldots,$$

that we denote by $\alpha_{\pm k}$; write also $u_- = \alpha_{-0}$ and $u_+ = \alpha_{+0}$, in agreement with the notations of Section 3.1. They are all real and $\alpha_{-(k+1)} < \alpha_{-k} < \alpha_{-0} < 0 < \alpha_{+0} < \alpha_k < \alpha_{k+1}$ for $k \geq 1$. Therefore $F(z)$ has $2n+1$ zeroes inside C_n.

Now we count the number of real roots of $F(z)$ inside C_n. Assume that $a > c\rho$ and $\rho > 0$. We saw in the proof of Proposition 4.1 that $a/(c\rho) > u_+$, and assume that $a/(c\rho) \in (\alpha_m, \alpha_{m+1})$ for some $m > 0$ (the case $a/(c\rho) = \alpha_m$ needs to be studied as a particular case). Then,

(i) In each interval (α_k, α_{k+1}), for $k \neq m$, $F(u) = \widetilde{f}(u)$ and $\widetilde{f}(u)$ has one root. This is due to the fact that the roots of $\widetilde{f}(u)$ in such intervals are the solutions of

$$\tan \widetilde{P}(u) = -\frac{\widetilde{P}(u)}{a - c\rho u}.$$

(ii) The function $\widetilde{f}(u)$ has 2 roots in (α_m, α_{m+1}). This claim is proved, observing that $a - c\rho\,\alpha_m > 0$ and $a - c\rho\,\alpha_{m+1} < 0$, and the curve $-\widetilde{P}(u)/(a - c\rho u)$ cuts twice the curve $\tan \widetilde{P}(u)$ in that interval.

(iii) In each $(\alpha_{-(k+1)}, \alpha_{-k})$, $k \geq 0$, the function $\widetilde{f}(u)$ has one root. This is proved as in point (i).

All the other possibilities for a, c and ρ are discussed in a similar way, obtaining that $F(z)$ has at least $2n+1$ real roots in C_n. So the Theorem follows.
Case 2. $|\rho|/\sqrt{1-\rho^2} < 1$. Here use the contours $C'_n = \{z \in C : |p(z)| = n^2\pi^2\}$. Prove that for $z \in C'_n$ and n large enough,

$$\left|(a - c\rho z)\frac{\sinh P(z)}{P(z)}\right| < |\cosh P(z)|.$$

Finish the proof as in Case 1.
Case 3. $\rho = \pm\sqrt{2}/2$. Write $\rho_n = \rho + 1/n$, for $n \geq 4$, and let F_n be the function F with ρ changed by ρ_n. We have $F_n(z) \to F(z)$ as $n \to \infty$, uniformly in every disc. By the Hurwitz theorem (see [31]), the roots of $F(z)$ in such a disc are the limit points of the roots of $F_n(z)$ in the disc. So the roots of $F(z)$ are also real and simple.

The cases $\rho = \pm 1$ are studied in a similar way. □.

Using Theorem 5.1, we deduce the following representation of $F(z)$.

Theorem 5.5. *Let $\{a_n,\ n \geq 1\}$ be the zeroes of $F(z)$. Then*

$$F(z) = \mathrm{e}^{a}\mathrm{e}^{\nu z}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left\{\frac{z}{a_n}\right\}, \tag{21}$$

where $\nu \in \mathbb{R}$ is given by (14).

Proof. Since $p(u)$ is a second degree polynomial, there are constants $K_1,\ K_2 > 0$ such that for $|u|$ sufficiently large,

$$K_1\, u^2 \leq |p(u)| \leq K_2\, u^2,$$

and then

$$\frac{1}{\sqrt{K_2}} n\pi \leq |\alpha_{\pm n}| \leq \frac{1}{\sqrt{K_1}} n\pi.$$

By Proposition 5.4 there is one and only one root (except possibly in a particular interval which may have two zeroes) of $F(z)$ in the intervals of the form $(\alpha_{-(n+1)}, \alpha_{-n})$ or (α_n, α_{n+1}). Hence,

$$\sum_n \frac{1}{a_n^2} < \infty \quad \text{and} \quad \sum_n \frac{1}{|a_n|} = \infty.$$

By Hadamard's factorization Theorem 5.1 we obtain the representation (21). □

Remark 5.6. An alternative way to prove the previous result is using that the order (as entire function) of $F(z)$ is less or equal to 1. This can be deduced from the fact that the order of the function $\cosh(\sqrt{z})$ is $1/2$ and

$$|L_j(z)| \leq \cosh\left(\sqrt{|z|}\right), \quad j = 1, 2.$$

Remark 5.7. Along this section and the next one, we exclude the case $\rho = 1$ and $2a = c$ because the moment generating function simplifies and has only one pole. Then the factorization is trivial.

5.2. The meromorphic component

In order to deal with the other factor of the function $\Phi(z)$ given in (4), write

$$G(z) := \frac{(1-z)\sinh(P(z))/P(z)}{\cosh(P(z)) + (a - c\rho z)\sinh(P(z))/P(z)},$$

where, as above, the function is extended to $\mathbb{C}$ using the entire functions $L_1(z)$ and $L_2(z)$. The function $G(z)$ is meromorphic with simple poles at the roots of $F(z)$.

In order to apply Mittag-Leffler's Theorem, we also need the properties of the contour appointed by the polynomial $p(z)$. For $K > 0$, let $C_K = \{z \in \mathbb{C} : |p(z)| = K\}$, and denote by L_K its length and by R_K the minimum distance from L_K to the origin.

Lemma 5.8. *With the above notations, for K large enough, C_K turns out to be a Jordan curve, $R_K = O(\sqrt{K})$ and $L_K = O(R_K)$.*

Proof. The polynomial $p(z)$ has real coefficients and has a positive and a negative root ($\rho \neq \pm 1$), and since the length of a contour does not change by translation, we can consider that $p(z) = z^2 - \alpha$, with $\alpha > 0$ (redefine K to make the coefficient of z^2 equal to 1). Write $z = x + \mathrm{i}y$; then the equation that determines C_K is

$$(x^2 - y^2 - \alpha)^2 + 4y^2x^2 = K^2.$$

In polar coordinates the contour is given by

$$r^4 - 2\alpha \cos(2\theta) r^2 + \alpha^2 - K^2, \ \theta \in [0, 2\pi].$$

For $K \gg \alpha$ we need only to consider the solution

$$r(\theta) = \sqrt{\alpha \cos(2\theta) + \sqrt{\alpha^2 \sin^2(2\theta) + K^2}},$$

that determines a spiric section which is a Jordan curve, in fact it looks like a slightly perturbed circumference. Obviously, the distance of the curve to the origin goes to infinity as $K \to \infty$. To compute the length L_K we need to evaluate the rate of growth of

$$L_K = \int_0^{2\pi} \sqrt{r^2(\theta) + \left(r'(\theta)\right)^2}\, \mathrm{d}\theta.$$

Since $\lim_{K\to\infty} \frac{r^2(\theta)}{K} = 1$ and $\lim_{K\to\infty} \frac{(r'(\theta))^2}{K} = 0$, uniformly in $\theta \in [0, 2\pi]$, it follows that $L_K = \mathrm{O}(R_K)$ and the lemma. $\square$

Now we are ready to use Theorem 5.2 to find a decomposition of $G(z)$:

Theorem 5.9. *Let $0 < |a_1| \leq |a_2| \leq \cdots$ be the zeroes of $F(z)$, let b_n be the pole of $F(z)$ at a_n. Then, for $z \notin \{a_n,\ n \geq 1\}$,*

$$G(z) = \frac{1}{2a}(1 - \mathrm{e}^{-at}) - z \sum_{n=1}^{\infty} \frac{b_n}{a_n^2(1 - z/a_n)}, \tag{22}$$

where $b_n > 0$. Moreover $\sum_n b_n a_n^{-2} < \infty$ and the series converges uniformly in every compact set included in the disc $|z| < |a_1|$.

Proof. In a similar way as in the proof of Proposition 5.4, we are going to show that there is a constant $C > 0$ and n large enough such that for $z \in C_n$ or $z \in C_n'$ we have $|G(z)| < C$, where C_n and C_n' were defined in Proposition 5.4. Then, thanks to Lemma 5.8, we can apply the Theorem of Mittag-Leffler 5.2 that gives the expression (22). Consider three cases:
Case 1. $|\rho|/\sqrt{1-\rho^2} > 1$. Take $\varepsilon > 0$ such that $1 < 1 + \varepsilon < |\rho|/\sqrt{1-\rho^2}$. By Lemma 5.3 $|\coth P(z)| < 1 + \varepsilon$ if $z \in C_n$. Therefore

$$|G(z)| \leq \frac{|(1-z)/P(z)|}{||\coth P(z)| - |(a - c\rho z)/P(z)||} = \frac{|(1-z)/P(z)|}{|(a - c\rho z)/P(z)| - |\coth P(z)|} < C.$$

Case 2. $|\rho|/\sqrt{1-\rho^2} < 1$. Let $\delta > 0$ be such that $|\rho|/\sqrt{1-\rho^2} < \delta < 1$, and $\varepsilon > 0$ such that $(1+\varepsilon)\delta < 1$. Then, for $z \in C_n'$,

$$|G(z)| = \left| \frac{(1-z)\tanh P(z)/P(z)}{1 + (a - c\rho z)\tanh P(z)/P(z)} \right| \leq \frac{|\tanh P(z)|\,|(1-z)/P(z)|}{1 - |\tanh P(z)|\,|(a - c\rho z)/P(z)|} < C.$$

In both cases 1 and 2, Theorem 5.2 gives (22). Since $G(z)$ is the quotient of two entire functions, and the pole is simple,

$$b_n = \frac{(1-a_n)\sinh(P(a_n))/P(a_n)}{F'(a_n)}.$$

Use that a_n is a root of $F(z)$ to obtain the equivalent formulation

$$b_n = \frac{2p(a_n)(1-a_n)}{p'(a_n)p(a_n) - 2c\rho p(a_n) - p'(a_n)(a-c\rho a_n)(a-c\rho a_n+1)}.$$

In order to determine the behaviour of b_n, convert the above expression into a function and check the limits at infinity, that is

$$\lim_{u\to\pm\infty} \frac{2p(u)(1-u)}{p'(u)p(u) - 2c\rho p(u) - p'(u)(a-c\rho u)(a-c\rho u+1)} = \begin{cases} \dfrac{1}{c^2}, & \text{if } \rho^2 \neq 1 \\ \dfrac{2}{c^2}, & \text{if } \rho^2 = 1 \text{ and } c \neq 2a\rho. \end{cases}$$

(For case $\rho = 1$ and $2a = c$, see Remark 5.7). So, for large n, it is clear that $b_n > 0$. For small n the positivity of b_n is proved through an analysis of the sign of $F(u)$ and $F'(u)$ in the different intervals where the roots of F are located. These roots and its location are clearly studied in Lucic [24]. On the other hand it is clear that the sequence $\{b_n,\ n \geq 1\}$ is bounded, and hence $\sum_n b_n a_n^{-2} < \infty$.

Case 3. $\rho = \pm\sqrt{2}/2$. As in case 3 of Proposition 5.4, the result is obtained by continuity, using monotone convergence Theorem. □

Combining Theorems 5.1 and 5.9, the factorization of the characteristic function of Theorem 3.3 follows.

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Appendix

A.1. The conditional expectation of a stochastic integral

We use the notations introduced in Section 2.1. It is well known that the law of $\int_0^t \sqrt{V_s}\,\mathrm{d}W_s'$ conditional on $\mathcal{F}_t$ is centered Gaussian with variance $\int_0^t V_s\,\mathrm{d}s$; see Scott [30, Page 416]. For the sake of completeness, we present here a proof of the property that we use, in a more general formulation:

Proposition A.1. *With the above notations, let $H = \{H_s,\ s \in [0,t]\}$ be a continuous process, adapted to the filtration generated by W and such that $\mathbb{E}\int_0^t H_s^2\,\mathrm{d}s < \infty$. Then for all $u \in \mathbb{R}$,*

$$\mathbb{E}\left[\exp\left\{u\int_0^t H_s \mathrm{d}W_s'\right\} |\mathcal{F}_t\right] = \mathrm{e}^{\frac{u^2}{2}\int_0^t H_s^2\,\mathrm{d}s}.$$

Proof. To simplify the notations write

$$X = \int_0^t H_s \mathrm{d}W_s' \quad \text{and} \quad Y = \int_0^t H_s^2\,\mathrm{d}s.$$

Fix $u \in \mathbb{R}$. Approximating H by elementary processes, thanks to the independence between W and W', and using the dominated convergence Theorem it is deduced that

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}uX}|\mathcal{F}_t\big] = \mathrm{e}^{-\frac{u^2}{2}Y}.$$

Denote by $p : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$ a regular conditional probability of X given $\mathcal{F}_t$ (see, for example, Chow and Teicher, [6, Chapter 7]), that is,

(i) Fixing $\omega \in \Omega$, the mapping $p(\omega,\cdot) : \mathcal{B}(\mathbb{R}) \to [0,1]$ is a probability measure.
(ii) For each $B \in \mathcal{B}(\mathbb{R})$, the mapping $p(\cdot, B) : \Omega \to [0,1]$ is $\mathcal{F}$ measurable and

$$\mathbb{P}(X \in B|\mathcal{F}_t)(\omega) = p(\omega, B), \text{ a.s.}$$

Moreover, for a measurable function $h : \mathbb{R} \to \mathbb{R}$ positive or such that $\mathbb{E}[|h(X)|] < \infty$,

$$\mathbb{E}\big[h(X)|\mathcal{F}_t\big](\omega) = \int_{\mathbb{R}} h(x)p(\omega, \mathrm{d}x), \text{ a.s.}$$

So we have that for every $u \in \mathbb{R}$, there is an event N_u of zero probability such that for $\omega \in N_u^c$,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} p(\omega, \mathrm{d}x) = \mathrm{e}^{-\frac{u^2}{2}Y(\omega)}.$$

Let $N = \bigcup_{u\in\mathbb{Q}} N_u$. By continuity of both sides of the above expression, for $\omega \in N^c$,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} p(\omega, \mathrm{d}x) = \mathrm{e}^{-\frac{u^2}{2}Y(\omega)}, \quad \forall u \in \mathbb{R}.$$

Hence, $p(\cdot, \omega)$ is a centered Gaussian law with variance $Y(\omega)$. It follows that

$$\int_{\mathbb{R}} \mathrm{e}^{ux} p(\omega, \mathrm{d}x) = \mathrm{e}^{\frac{u^2}{2}Y(\omega)}, \quad \forall u \in \mathbb{R}.$$

Since e^{ux} is positive,

$$\mathbb{E}\big[\mathrm{e}^{uX}|\mathcal{F}_t\big] = \mathrm{e}^{\frac{u^2}{2}Y}. \quad \square$$

A.2. Proof of Proposition 4.1

Notice that $p(0) = a^2$ and $p(1) = (a - \rho c)^2 \geq 0$, then $1 \leq u_+$. Hence, the inclusion $[u_-, 1] \subset D(X_t)$ is obtained from the other assertions of the proposition.

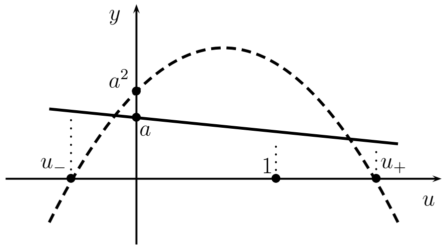

Fig. 6. Solid line: straight line $y = a - \rho c u$. Dashed line: parabola $p(u)$.

1. Consider the case $a \geq \rho c$ for $\rho \neq \pm 1$. We have that $a - \rho c u_- > 0$ and $a - c\rho u_+ > 0$; this is proved in the following way: for $\rho \in [0, 1)$ and $a \geq \rho c$ then

$$
\begin{aligned}
&u_+ = \frac{2a\rho - c - \sqrt{4a^2 + c^2 - 4ac\rho}}{2c(\rho^2 - 1)} \leq \frac{a}{c\rho} \\
&-\sqrt{4a^2 + c^2 - 4ac\rho} \geq \frac{a2(\rho^2 - 1)}{\rho} + c - 2a\rho \\
&\rho^2(4a^2 - 4ac\rho) \leq 4a^2 - 4a\rho c \\
&(\rho^2 - 1)4a(a - c\rho) \leq 0,
\end{aligned}
$$

thus $0 < a - c\rho u_+$. The same sort of arguments apply for $\rho \in (-1, 0]$ to derive $0 < a - c\rho u_-$. Any other casuistry is trivial.

So the straight line $y = a - \rho c u$ is positive for $u \in [u_-, u_+]$ (see Fig. 6). Since $\sinh x > 0$ for $x > 0$ and $\cosh x \geq 1$, it follows that $f(u) > 0$ in $[u_-, u_+]$. Note that if $a = \rho c$, then $u_+ = 1$.

When $\rho = -1$, then $u_- = -a^2/(c(c + 2a)) < 0$, and it is trivial to check $a + c u_- > 0$. So the straight line $y = a + cu$ is positive for $u \geq u_-$, and hence $f(u) > 0$ in $[u_-, \infty)$.

When $\rho = 1$, then $u_+ = a^2/(c(2a - c))$. Standard manipulations show that

$$
\frac{a}{c} \geq \frac{a^2}{c(2a - c)},
$$

and hence $a - c u_+ \geq 0$. Then $a - cu \geq 0$ for all $u \leq u_+$, and it follows $f(u) > 0$ in $(-\infty, u_+]$.

2. Consider the case $a < \rho c$ (note that this implies $\rho > 0$). Then $a - c\rho u_- > 0$. To study the behaviour of $f(u)$ in $[u_-, u_+]$, we will divide the interval into three parts and separately inspect the roots of $f(u)$. Be the next division $[u_-, u_+] = [u_-, a/(c\rho)] \cup [a/(c\rho), 1) \cup [1, u_+]$.

On the first subinterval consider the straight line $y = a - c\rho u$ which is non-negative for $u \in [u_-, a/(c\rho)]$. Thus $f(u)$ has no zeroes in such an interval.

Notice that the roots of $f(u)$ in the second subinterval are the same as the ones corresponding to the function $g(u)$ given by

$$
g(u) := 1 + \frac{a - c\rho u}{P(u)} \tanh(P(u)t/2) > 1 + \frac{a - c\rho u}{P(u)} > 0,
$$

where that last inequality holds due to the fact that $c^2(u - u^2) > 0$ for $u \in [a/(c\rho), 1)$. Thus $f(u) > 0$ in $[a/(\rho c), 1)$.

Now, we analyze the behaviour of $f(u)$ for $u \in [1, u_+]$. Recall again the straight line $y = a - c\rho u$ and check that $y(0) > 0$ and $y(1) < 0$. Therefore $a - c\rho u_+ < 0$ since $u_+ > 1$. Let $t_0 = 2/(c\rho u_+ - a) > 0$.

(i) If $t < t_0$, then $0 < (c\rho u - a)t/2 < 1$ for all $u \in [1, u_+]$. Hence,

$$0 < \frac{(c\rho u - a)t}{2} \frac{\sinh\big(P(u)t/2\big)}{P(u)t/2} < \frac{\sinh\big(P(u)t/2\big)}{P(u)t/2} \le \cosh\big(P(u)t/2\big)$$

in the third subinterval, where the last inequality holds due to $\sinh x/x \le \cosh x$. Finally $f(u) > 0$ in such an interval.

(ii) Let $t \ge t_0$. We claim that $f(1) > 0$ and $f(u_+) < 0$. The first inequality is straightforward, and the second follows from the fact that

$$\lim_{u \nearrow u_+} \frac{\sinh\big(P(u)t/2\big)}{P(u)t/2} = 1$$

and thus

$$\lim_{u \nearrow u_+} f(u) = 1 + (a - c\rho u_+)t/2.$$

Therefore there is at least one zero of $f(u)$ in $[1, u_+]$ by Bolzano's theorem. The fact that there is only one zero is commented in Section 3.4.1.

3. Finally, the case $\rho = 1$ and $2a = c$ is studied in a similar way.

A.3. *Proof of Lemma 5.3*

From the formula

$$|\coth z|^2 = \frac{\sinh^2 x + \cos^2 y}{\sinh^2 x + \sin^2 y} \tag{23}$$

it is clear that $|\coth z|$ is the same for the points $z = x + \mathrm{i}y,\ -x + \mathrm{i}y,\ x - \mathrm{i}y,\ -x - \mathrm{i}y$. Hence, to bound $\coth z$ in the circle $\{z \in \mathbb{C} :\ |z| = (n + \frac{1}{2})\pi\}$ we can restrict ourselves to study the arc with $\Re z \ge 0$ and $\Im z \ge 0$. (see Fig. 7 for $n = 2$). Given the periodicity $\coth(z + k\pi \mathrm{i}) = \coth z$, for all $k \in \mathbb{Z}$, it suffices to bound the translation to the strip $0 \le \Im z \le \pi$ (see Fig. 8). For K large enough and $n \ge n_0$ all translations are included in the shaded region $A_K \cup B_K$ of Fig. 9. Thus, it suffices to prove that for K large enough, $\sup_{z \in A_K \cup B_K} |\coth z| < 1 + \varepsilon$. For $z = x + \mathrm{i}y \in A_K$, (that is, $x > K$), we have

$$|\coth z|^2 = \frac{\sinh^2 x + \cos^2 y}{\sinh^2 x + \sin^2 y} \le \frac{\sinh^2 x + 1}{\sinh^2 x} = 1 + \frac{1}{\sinh^2 x} \le 1 + \frac{1}{\sinh^2 K}, \tag{24}$$

that goes to 1 when $K \to \infty$.

Now we study the bound on the triangle B_K of Fig. 9. By the maximum principle, we only need to study the function on the border of B_K. On the vertical side, $z = K + \mathrm{i}y$ for $y \in [0, \pi/2]$, works the bound (24). On the horizontal side, $z = x + \mathrm{i}\frac{\pi}{2}$ for $x \in [0, K]$, we can use formula (23) and end up with

$$|\coth z|^2 = \frac{\sinh^2 x}{\sinh^2 x + 1} \le 1.$$

Finally, the hypotenuse is $z = x + \mathrm{i}y$ with $y \in [0, \pi/2]$ and $x = -2Ky/\pi + K$. For $y \in [\pi/4,\ \pi/2]$, we have that $\cos 2y \le 0$, and by

$$|\coth z|^2 = \frac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y},$$

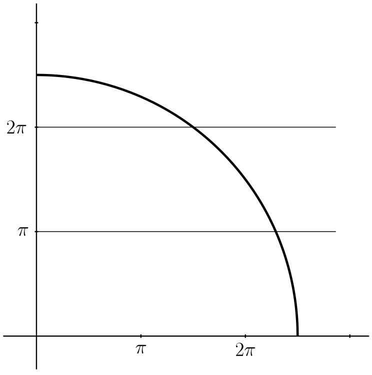

Fig. 7. Arc $|z| = (n + \frac{1}{2})\pi$, $\Re z \geq 0$, $\Im z \geq 0$ for $n = 2$.

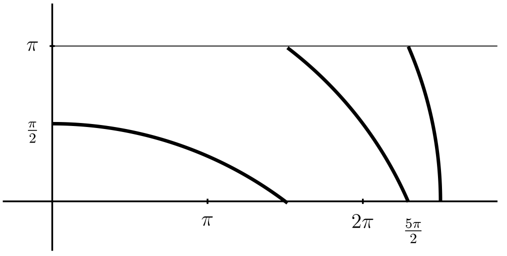

Fig. 8. Translation to the arc of Fig. 7 to the strip $0 \leq |\Im z| \leq \pi$.

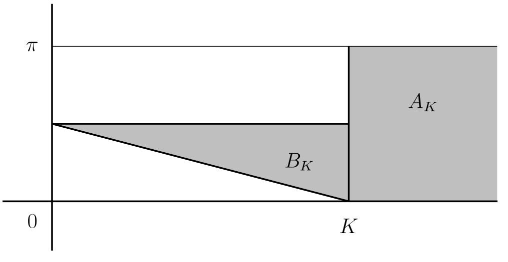

Fig. 9. Translations for all $n \geq n_0$ are included in the shadow region.

we deduce that

$$|\coth z| \leq 1.$$

For $y \in [0, \pi/4]$, we have $x \in [K/2, K]$, and again by (23) we obtain the bound.

To obtain the inequality for $|\tanh z|$ and $|z| = n\pi$ proceed as before and use the relationship

$$\tanh z = \coth\left(z - \mathrm{i}\frac{\pi}{2}\right).$$

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