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BIMEASURES AND SAMPLING THEOREMS FOR WEAKLY HARMONIZABLE PROCESSES

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ABSTRACT

If $\{X(t), t\in\mathbb{R}\}$ is a weakly harmonizable process, conditions on the process are found in order that $X(t) = \sum_{n=-\infty}^{\infty} a_n(t)X(n\pi/\alpha)$ for a suitable $\alpha > 0$ and coefficients $a_n(t)$, the series converging in $L^2(P)$-mean. Consequently the process can be determined by sampling at fixed intervals nh, $n = 0,\pm 1,\ldots$, $h = \pi/\alpha > 0$. A corresponding result is also obtained for a more general Cramér class. To carry out this analysis, it is necessary to use the properties of bimeasures. Some aspects of the bimeasure theory and its distinction from the Lebesgue theory are included. This is used essentially for the analysis of harmonizable processes, and has independent interest.

I. <u>INTRODUCTION</u>. Let (Ω,Σ,P) be a probability space and $L_0^2(P)$ be the subspace of square integrable complex valued random variables on Ω with means zero, i.e., $X\in L_0^2(P)$ iff $E(|X|^2) < \infty$ and $E(X) = 0$ where $E(X) =$

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$\int_\Omega X dP$, the expectation. A second order process $\{X(t), t \in \mathbb{R}\}$ of interest in this paper is a mapping $X:\mathbb{R} \to L_0^2(P)$ and let $r_x(s,t) = E(X(s)\overline{X(t)})$, $r_x(\cdot,\cdot)$ being the covariance function. The types of processes considered here are classified according to the form of the covariance function r_x of X . The process is said to be strongly (or Loève, cf. [8]) harmonizable if r_x admits a representation as:

$$r_x(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda - it\lambda'} F_x(d\lambda, d\lambda') \tag{1}$$

where $F_x:\mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is a positive definite function of bounded variation in the plane, and the integral is defined in the standard Lebesgue sense. Here bounded variation is understood in the sense of Vitali so that one has

$$|F_x|(\mathbb{R}\times\mathbb{R}) = \sup\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |F_x(A_i, B_j)| : A_i \subset \mathbb{R}, B_j \subset \mathbb{R} \text{ are intervals}\} < \infty , \tag{2}$$

where $F_x(A,B)$ denotes the increment, for $A = (a,b]$, $B = (c,d]$, given by

$$F_x(A,B) = F_x(b,d) - F_x(b,c) - F_x(a,d) + F_x(a,c) .$$

This is a generalization of the classical notion of (weak) stationarity since, by definition, the latter is a process whose covariance $r_x(\cdot,\cdot)$ is continuous and depends only on the difference $s - t$ ("invariant" covariance under translations of the time axis $\mathbb{R}$) so that $r_x(s,t) = \tilde{r}_x(s-t)$ and by Bochner's theorem the continuous positive definite function $\tilde{r}_x$ can be uniquely expressed as:

$$\tilde{r}_x(s-t) = \int_{\mathbb{R}} e^{i(s-t)\lambda} G_x(d\lambda) \tag{3}$$

for a positive bounded nondecreasing function G_x. Thus (1) becomes (3) if F_x concentrates on the diagonal $\lambda = \lambda'$ so that $G_x(\lambda) = \delta_{\lambda\lambda'} F_x(\lambda,\lambda')$.

If $T:L_0^2(P) \to L_0^2(P)$ is a bounded linear mapping, and $\{X_t, t\in\mathbb{R}\}$ is a stationary process, then in some applications it is desired to consider the transformed (e.g., filtered) process $Y_t = TX_t$, $t\in\mathbb{R}$. [Here and below $X_t = X(t)$ are interchangeably written, for convenience.] However Y_t is not stationary in most cases. For instance, if T is a projection operator with a finite dimensional range, then the $\{Y_t, t\in\mathbb{R}\}$ is strongly harmonizable but not generally stationary. If T has an infinite dimensional range (but $T \neq$ identity), then $\{Y_t, t\in\mathbb{R}\}$ is generally *not even* strongly harmonizable. It is weakly harmonizable in the sense that its covariance function $r_y:(s,t) \mapsto E(Y_s\bar{Y}_t)$ is representable as:

$$r_y(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda - it\lambda'} F_y(d\lambda, d\lambda'), \quad s,t\in\mathbb{R}, \tag{4}$$

where $F_y:\mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is a positive definite function with finite Fréchet variation. This means:

$$\|F_y\|(\mathbb{R}\times\mathbb{R}) = \sup\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{a}_j F_y(A_i,A_j) : |a_i|\le 1, a_i\in\mathbb{C}, A_i \subset \mathbb{R} \text{ is an interval, } i = 1,\ldots,n\} < \infty. \tag{5}$$

Evidently $\|F_y\|(\mathbb{R}\times\mathbb{R}) \le |F_y|(\mathbb{R}\times\mathbb{R}) \le \infty$, usually with a strict first inequality when the second is infinite. But then what is the meaning of the integral in (4)? This is generally a nonabsolute (hence not a Lebesgue) integral and is taken in the sense of Morse-Transue [9]. Some of its properties are essential for the following work and hence some aspects of this theory of bimeasures determined by such F_y, called the spectral measure of the process, will be given in the next section. Because of its use in other studies on the subject and because of independent interest, the work presented here will be somewhat more than what is needed for our present purposes. It is useful to remark that if $\{Y_t, t\in\mathbb{R}\}$ is any weakly harmonizable process and T is any bounded linear mapping, then $\{TY_t, t\in\mathbb{R}\}$ is also weakly harmonizable. These processes are particularly suited in applications because of such closure properties. Section 3 contains a description of these processes, and their integral representations. Since in applications it is difficult (or expensive) to observe a process $\{Y_t, t\in\mathbb{R}\}$ on all of $\mathbb{R}$, one often wants to sample it preferably at equidistant points nh, $n = 0,\pm1,\pm2,\ldots$, and get a good approximation to the whole process in such a way that there is no "aliasing," so that two different processes shall not have the same realizations at these points. Reasonably good sufficient conditions on the spectral functions are obtained in order that such sampling is possible for weakly harmonizable processes. This requires a different technique than the strongly harmonizable case. These results constitute Section 4. Several related remarks are given in the last section.

It will be seen that, once the representation theory is embarked, one can include processes more general than harmonizable classes. Such a generalization was already introduced by Cramér [3]. Thus a process $Z:\mathbb{R} \to L_0^2(P)$ is said to be of <u>Cramér class</u>, if the covariance function r_Z of Z can be expressed as:

$$r_Z(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_Z(s,\lambda)\overline{g_Z(t,\lambda')}F_Z(d\lambda,d\lambda') , \qquad (6)$$

where $F_Z:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is positive definite, and is of finite Vitali variation on each finite domain of $\mathbb{R}^2$, and $\{g_z(s,\cdot), s\in\mathbb{R}\}$ is a class of Borel functions for which the (Lebesgue) integral satisfies: $\int_{\mathbb{R}}\int_{\mathbb{R}} g_z(s,\lambda)\overline{g_z(s,\lambda')}F_z(d\lambda,d\lambda') < \infty$, $s\in\mathbb{R}$. If $g_z(s,\lambda) = \exp\{is\lambda\}$, and F_z is of finite variation on the whole plane, then the Cramér class reduces to the strongly harmonizable case. A sampling theorem for this class is also obtained, and it extends an earlier result of Piranashvili [11]. It is possible to extend this material if the variations here are replaced by Fréchet variations, in these definitions. The existence of all these processes is also discussed in Section 3 below.

Sampling theorems and their importance have been noted in information and engineering applications. An early general and precise result for stationary processes was obtained by Lloyd [7], and his result (the sufficiency part) was extended in [14] for strongly harmonizable processes, and another extension for the latter can be found in [11]. The type of sampling theorems given below is influenced by the point of view (but not the methods which do not extend) of [11].

For a smooth sailing later on, the measure aspects will now be discussed in some detail. However, a reader interested in seeing the stochastic theory immediately, and is willing to accept (temporarily) the properties of bimeasures, can now go directly to Sections 3 - 5.

II. ASPECTS OF BIMEASURES AND THEIR INTEGRALS.

A starting point for the study of weakly harmonizable processes is to define the integral in (4). This will emerge from the general theory of $\mathbb{C}$-bimeasures developed by Morse and Transue [9] but now their general study has to be slightly restricted. To see what is precisely needed, a brief account of this development will be presented here since it is not available in the existing papers. It will supplement [19].

Let S_1, S_2 be a pair of locally compact spaces and $\mathcal{K}(S_i)$ be the space of continuous complex functions on S_i with compact supports, $i = 1,2$. The concept of a $\mathbb{C}$-bimeasure of [9] is as follows:

<u>Definition 2.1</u>. A complex valued bilinear mapping Λ on the product space $\mathcal{K}(S_1)\times\mathcal{K}(S_2)$ is a $\mathbb{C}$-<u>bimeasure</u> on $S_1\times S_2$ if $\Lambda(\cdot,v):\mathcal{K}(S_1)\to\mathbb{C}$ and $\Lambda(u,\cdot):\mathcal{K}(S_2)\to\mathbb{C}$ are relatively bounded linear functionals for each $u\in\mathcal{K}(S_1)$, $v\in\mathcal{K}(S_2)$. [Relatively bounded $\Lambda(\cdot,v)$ means that if $K\subset S_1$ is compact and $\mathcal{K}(K)\subset\mathcal{K}(S_1)$ is considered, then $\Lambda(\cdot,v):\mathcal{K}(K)\to\mathbb{C}$ is bounded.]

Since each such relatively bounded functional on $\mathcal{K}(S_1)$ is uniquely representable by a signed Baire measure on S_1, by the classical Riesz representation theorem and then such a measure has a unique extension of being a Radon measure by the standard theory of Bourbaki [1], one refers to each such functional itself as a Radon measure, as is done in [9]. Given such a $\mathbb{C}$-bimeasure Λ, one defines the integral of [9] as:

<u>Definition 2.2</u>. Let $f:S_1\to\mathbb{C}$, $g:S_2\to\mathbb{C}$ be Baire functions and Λ be a $\mathbb{C}$-bimeasure on $S_1\times S_2$. Then (f,g) is said to be <u>Morse-Transue</u> (or <u>MT</u>-, cf. [9], p. 482) integrable if (a) f is $\Lambda(\cdot,v)$- and g is $\Lambda(u,\cdot)$-integrable for each $u\in\mathcal{K}(S_1), v\in\mathcal{K}(S_2)$ (the integrals are denoted $\Lambda(\cdot,v)(f)$ and $\Lambda(u,\cdot)(g)$), (b) the linear functionals $\Lambda(\cdot,g):u\mapsto\Lambda(u,\cdot)(g)$, $\Lambda(f,\cdot):v\mapsto\Lambda(\cdot,v)(f)$ are Radon measures, and (c) the integrals $\Lambda(f,\cdot)(g)$ and $\Lambda(\cdot,g)(f)$ exist and are equal. This common value is denoted by

$$\Lambda(f,g) = (MT)\int_{S_1}\int_{S_2}(f,g)d\Lambda = \Lambda(f,\cdot)(g) = \Lambda(\cdot,g)(f) . \quad (7)$$

The reason for condition (b) in this definition is that $\Lambda(\cdot,\cdot)$ should be definable for measurable (f,g), not merely continuous ones. However, as a consequence of his work on vector measures, Thomas ([17], p. 144) has shown that this is actually redundant, and hence (a) and (c) suffice in the definition. Although the integrability concept of (7) is needed below, the Bourbaki point of view of integration employed is somewhat inconvenient for our work. So an ensemble point of view, due to Ylinen [19], will now be presented and the results will be compared with the above.

Let (Ω_i,Σ_i), $i=1,2$, be a pair of measurable spaces and $\mathbb{C}$ the usual complex plane. Then a mapping β: $\Sigma_1\times\Sigma_2\to\mathbb{C}$ is termed a <u>bimeasure</u> if $\beta(\cdot,F)$ and $\beta(E,\cdot)$ are σ-additive for each $E\in\Sigma_1$, and $F\in\Sigma_2$, respectively. The Vitali and Fréchet variations of β are defined exactly as in (2) and (5) where in (5) the sum now is replaced by $|\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{b}_j\beta(A_i,B_j)|$ for disjoint collections $\{A_i\}_1^n,\{B_j\}_1^n$ of Σ_1,Σ_2. It follows from classical theory (cf. [4], IV.10.2) that $\|\beta\|(\Omega_1,\Omega_2)<\infty$ always, though $|\beta|(\Omega_1,\Omega_2)$ is not necessarily finite. Clearly $\|\beta\|(A,B)\le|\beta|(A,B)$. An integral relative to β is introduced as follows:

<u>Definition 2.3</u>. Let $f_i:\Omega_i\to\mathbb{C}$, $i=1,2$, be measurable functions and $\beta:\Sigma_1\times\Sigma_2\to\mathbb{C}$ be a bimeasure. Then (f_1,f_2) is β-<u>integrable</u> if (a) for each $E\in\Sigma_1,F\in\Sigma_2$, f_1 is $\beta(\cdot,F)$- and f_2 is $\beta(E,\cdot)$-integrable in Lebesgue's sense, so that $_{f_1}\beta(\Omega_1,\cdot):F\mapsto\int_{\Omega_1} f_1(\omega_1)\beta(d\omega_1,F)$ is a complex measure on Σ_2 and similarly $\beta_{f_2}(\cdot,\Omega_2)$ is a complex measure on Σ_1, and (b) f_1 is $\beta_{f_2}(\cdot,\Omega_2)$-inte-

grable and f_2 is ${}_{f_1}\beta(\Omega_1,\cdot)$-integrable (in the Lebesgue sense) and

$$\int_{\Omega_1} f_1(\omega_1)\beta_{f_2}(d\omega_1,\Omega_2) = \int_{\Omega_2} f_2(\omega_2)\,{}_{f_1}\beta(\Omega_1,d\omega_2) = \int_{\Omega_1}\int_{\Omega_2}(f_1,f_2)d\beta \ , \tag{8}$$

where the last symbol is, by definition, the common value of the other two.

The two integrability concepts introduced above are related as follows. For simplicity, $\Omega_i = S_i = \mathbb{R}$ is taken, but the argument is seen to be valid if $\mathbb{R}$ is replaced by a σ-compact Hausdorff space.

Theorem 2.4. Let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$, and $\beta:\mathcal{B}\times\mathcal{B}\to\mathbb{C}$ be a bimeasure. Then each pair $f_i\in\mathcal{K}(\mathbb{R})$, $i=1,2$, is β-integrable in the sense of Definition 2.3, and $\Lambda:(f_1,f_2)\mapsto\int_{\mathbb{R}}\int_{\mathbb{R}}(f_1,f_2)d\beta$ given by (8) defines a bounded $\mathbb{C}$-bimeasure of Definition 2.1 on $\mathcal{K}(\mathbb{R})\times\mathcal{K}(\mathbb{R})$. Moreover, if (f_1,f_2) is any β-integrable pair, it is also MT-integrable (relative to Λ) and the integrals agree:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(f_1,f_2)d\beta = (MT)\int_{\mathbb{R}}\int_{\mathbb{R}}(f_1,f_2)d\Lambda \ . \tag{9}$$

On the other hand, if $\Lambda:\mathcal{K}(\mathbb{R})\times\mathcal{K}(\mathbb{R})\to\mathbb{C}$ is a bounded (not merely relatively bounded) $\mathbb{C}$-bimeasure, then there exists a bimeasure $\beta:\mathcal{B}\times\mathcal{B}\to\mathbb{C}$ such that each pair (f_1,f_2) (from $\mathcal{K}(\mathbb{R})\times\mathcal{K}(\mathbb{R})$, or more generally) of β-integrable functions, is MT-integrable relative to Λ and the relation (9) holds.

Proof. In proving this result several properties of β-integrals established in [19] and those of MT-integrals from [9] and [17] will be utilized. Indeed it is an elaboration of ([19], Thm. 7.2) and complements some of its statements.

Let $\beta:\mathcal{B}\times\mathcal{B}\to\mathbb{C}$ be a bimeasure so that $\|\beta\|(\mathbb{R}\times\mathbb{R})<\infty$, and let (f,g) be β-integrable. The definition of β-integrability implies that each pair of bounded Baire functions is β-integrable (cf. [19], p. 126). So, if $u,v\in\mathcal{K}(\mathbb{R})$, then (u,v) is β-integrable and the mapping $\Lambda:(u,v)\mapsto\int_{\mathbb{R}}\int_{\mathbb{R}}(u,v)d\beta$ is well defined and Λ is a bounded bilinear functional. In fact, $|\Lambda(u,v)|\le \|\beta\|(\mathbb{R},\mathbb{R})\|u\|_\infty\|v\|_\infty$ where $\|\cdot\|_\infty$ is the usual supremum norm on $\mathcal{K}(\mathbb{R})$. Thus Λ is a bounded $\mathbb{C}$-bimeasure, by Definition 2.1. Consequently $\Lambda(\cdot,v)$ and $\Lambda(u,\cdot)$ define bounded (complex) Radon measures on $\mathcal{B}$, because every σ-additive function on the Baire σ-algebra of a σ-compact space into a Banach space has a (bounded) Radon extension onto its Borel σ-algebra, by the standard measure theory. Using the same symbol $\Lambda(\cdot,v)$ for this extended measure, one has

$$\begin{aligned}\Lambda(f,\cdot)(v) = \Lambda(\cdot,v)(f) &= \int_{\mathbb{R}} f(x)\beta_v(dx,\mathbb{R}), \quad v\in\mathcal{K}(\mathbb{R}), \\ &= \int_{\mathbb{R}} v(y)\,{}_f\beta(\mathbb{R},dy), \text{ by (8) since} \\ &\qquad (f,v) \text{ is } \beta\text{-integrable}, \qquad (10)\end{aligned}$$

where the first equality obtains by definition of this symbol (cf. [9]). Since ${}_f\beta(\mathbb{R},\cdot)$ is also a regular measure on $\mathcal{B}$, (10) shows that $\Lambda(f,\cdot)$ on $\mathcal{K}(\mathbb{R})$ is a bounded linear functional and hence defines a (complex) Radon measure on $\mathcal{B}$. So (10) holds if v is replaced by any Baire function g, provided the integral on the right side exists. Since (f,g) is β-integrable, it does exist by (8), and one has:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(f,g)d\beta = \int_{\mathbb{R}} g(y)_f\beta(\mathbb{R},dy) = \int_{\mathbb{R}} g(y)\Lambda(f,dy) = \Lambda(f,\cdot)(g) \ . \tag{11}$$

A similar reasoning shows that $\Lambda(\cdot,g)(f)$ exists and equals $\int_{\mathbb{R}}\int_{\mathbb{R}}(f,g)d\beta$, so that $\Lambda(f,\cdot)(g) = \Lambda(\cdot,g)(f)$, and (f,g) is MT-integrable.

For the last part, let Λ be a bounded $\mathbb{C}$-bimeasure on $\mathcal{K}(\mathbb{R})\times\mathcal{K}(\mathbb{R})$. Then a pair of bounded Baire functions (f,g) is MT-integrable as a consequence of ([9], Thm. 11.1), since Λ is a bounded $\mathbb{C}$-bimeasure. By the theory of [9], for each bounded Baire function f or g , $\Lambda(\cdot,g)$ and $\Lambda(f,\cdot)$ determine (complex) bounded Radon measures on $\mathcal{B}$. If we define a function $\beta:\mathcal{B}\times\mathcal{B}\to\mathbb{C}$ by $\beta(E,F) = \Lambda(\chi_E,\chi_F)$, then β is seen, after a standard computation, to be a complex bimeasure (hence bounded). If now $\Lambda':(h,k)\mapsto\int_{\mathbb{R}}\int_{\mathbb{R}}(h,k)d\beta$ is defined for each pair of bounded Baire functions (h,k) , then by the first part Λ' is a $\mathbb{C}$-bimeasure and $\Lambda'(\chi_E,\chi_F) = \Lambda(\chi_E,\chi_F)$ so that $\Lambda'(u,v) = \Lambda(u,v)$ for u,v in $\mathcal{K}(\mathbb{R})$. The extension procedure then shows that $\Lambda' = \Lambda$ and (9) holds. This completes the proof.

<u>Remark</u>. There exist (unbounded) $\mathbb{C}$-bimeasures Λ on $\mathcal{K}(\mathbb{R})\times\mathcal{K}(\mathbb{R})$ (or any noncompact σ-compact space S in place of $\mathbb{R}$) such that it induces a complex bimeasure β , which is thus bounded, and for which (9) holds for all β-integrable pairs (f,g) ; but there exists a Λ-integrable (not bounded) pair (h,k) which is not β-integrable, i.e., (8) does not hold. An example of this phenomenon is given in ([19], Remark 7.3). Thus in the above theorem, it is essential that Λ remains bounded after it is extended from $\mathcal{K}(S_1)\times\mathcal{K}(S_2)$ to more functions.

The integrability of a pair (f,g) relative to a bimeasure β as in Definition 2.3, and hence for the

MT-integral (by the above theorem), is not absolute. Indeed, the integrability of (f,g) for β does not necessarily imply the same for $(f\chi_A, g\chi_B)$, $A\in\Sigma_1, B\in\Sigma_2$, in contrast to the Lebesgue theory. This is seen from the following example. We use the fact that the theorem is true if $\mathbb{R}$ is replaced by a σ-compact set.

<u>Example 2.5</u>. Let the underlying spaces $S_1 = S_2 = \mathbb{Z}$ (the integers), $\Sigma = \mathcal{P}(\mathbb{Z})$ the power set. Let $\beta(\{m\},\{n\}) = (|m|+|n|)^{-4}$ for $|n|+|m| \neq 0$, $= 0$ otherwise, $m,n\in\mathbb{Z}$. Now $\sum\limits_m\sum\limits_n (|m|+|n|)^{-4} < \infty$ so that for any $E,F\in\mathcal{P}(\mathbb{Z})$ we can define $\beta(E,F)$, and extend it to $\mathcal{P}(\mathbb{Z}^2)$ by additivity (with values in $\mathbb{R}^+$). Then one can verify that $|\beta|(\mathbb{Z},\mathbb{Z}) < \infty$, i.e., (2) holds. Let $f(x) = g(x) = x$, $x\in\mathbb{Z}$. Then for each $E \subset \mathbb{Z}$, we have

$$\int_E f(x)\beta(dx,\{m\}) = \sum_{n\in E-\{0\}} n(|n|+|m|)^{-4}, \quad m\in\mathbb{Z}.$$

Since the series is convergent, f is $\beta(\cdot,F)$-integrable for each $F\in\mathcal{P}(\mathbb{Z})$, and

$$\int_E f(x)\beta(dx,F) = \sum_{n\in E-\{0\}}\sum_{m\in F} n(|m|+|n|)^{-4}.$$

Similarly $\int_F g(y)\beta(E,dy)$ exists. Also for each $F \subset S_2 = \mathbb{Z}$,

$$f^{\beta}(S_1,F) = \int_{S_1} f(x)\beta(dx,F) = \sum_{m\in F}\sum_{n\in\mathbb{Z}-\{0\}} n(|m|+|n|)^{-4} = 0$$

and likewise $\beta_g(E,S_2) = 0$ for each $E \subset S_1 = \mathbb{Z}$. Hence (f,g) is β-integrable and $\int_{S_1}\int_{S_2}(f,g)d\beta = 0$.

Now let $A = B = \mathbb{Z}^+$. Since $f^{\beta}(A,F) = \int_A f(x)\beta(dx,F)$ $= \sum\limits_{m\in F}\sum\limits_{n\geq 1} n(n+|m|)^{-4}$, and $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\frac{mn}{(m+n)^4} = \infty$, so

$\int_B g(y)_f\beta(A,dy)$ does not exist. But $_f\beta(A,\cdot) = \chi_A f^\beta(S_1,\cdot)$. Hence $\int_{S_2}(\chi_B g)(y)_{\chi_A f}\beta(S_1,dy)$ does not exist. Thus $(\chi_A f,\chi_B g)$ is not integrable in the sense of Definition 2.3 even though (f,g) is.

Now by Theorem 2.4, if Λ is the induced $\mathbb{C}$-bimeasure by β, then (f,g) is also MT-integrable and its integral vanishes by (9). But again $(\chi_A f,\chi_B g)$ is not MT-integrable as follows from a similar computation. Thus the pathology is present also for bounded $\mathbb{C}$-bimeasures.

To eliminate this unpleasant behavior, we shall now strengthen the definition of integrability relative to a bimeasure in such a way that all bounded functions are again integrable. Thus we shall restrict the (unbounded) set of integrable pairs of functions, <u>but the bimeasure itself is left relatively unrestricted</u>. However, it may be of interest to note that $\|\beta\|(\mathbb{R}\times\mathbb{R}) < |\beta|(\mathbb{R}\times\mathbb{R}) = \infty$ can happen even if β is a positive definite bimeasure. Counterexamples illustrating these points are not obvious. Using a modification of an example due to H. Helson and D. Lowdenslager, an example was discussed in [13]. It uses a deep result, due to S. Bochner, on the planar extension of the F. and M. Riesz theorem on absolute continuity of measures. A general counterexample can also be obtained by a nontrivial modification of that of ([2], p. 840) to show that $\|\beta\|(\mathbb{R}\times\mathbb{R}) < |\beta|(\mathbb{R}\times\mathbb{R}) = \infty$ holds when β is a bimeasure which is not necessarily positive definite. For brevity, the details of this will not be included here.

To avoid the troubles of Example 2.5, let us introduce the following:

<u>Definition 2.6</u>. Let (Ω_i,Σ_i) be measurable spaces, $f_i:\Omega_i \to \mathbb{C}$ be measurable functions, $i = 1,2$. If β:

$\Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a bimeasure, then the pair (f_1,f_2) will be termed <u>strictly β-integrable</u> if:

(a) f_1 is $\beta(\cdot,F)$- and f_2 is $\beta(E,\cdot)$-integrable for each $E\in\Sigma_1, F\in\Sigma_2$, so that ${}_{f_1}\beta(E,\cdot):\Sigma_2 \to \mathbb{C}$ and $\beta_{f_2}(\cdot,F):\Sigma_1 \to \mathbb{C}$ given by ${}_{f_1}\beta(E,B) = \int_E f_1(\omega_1)\beta(d\omega_1,B)$, $B\in\Sigma_2$, and $\beta_{f_2}(A,F) = \int_F f_2(\omega_2)\beta(A,d\omega_2)$, $A\in\Sigma_1$, are complex measures for each $E\in\Sigma_1, F\in\Sigma_2$, and

(b) f_1 is $\beta_{f_2}(\cdot,F)$-integrable for each $F\in\Sigma_2$, and f_2 is ${}_{f_1}\beta(E,\cdot)$-integrable for each $E\in\Sigma_1$, and $\int_E f_1(\omega_1)\beta_{f_2}(d\omega_1,F) = \int_F f_2(\omega_2)\,{}_{f_1}\beta(E,d\omega_2)$ holds for $E\in\Sigma_1, F\in\Sigma_2$. When these conditions obtain, the integral is denoted by

$$\int_E \int_F^* (f_1,f_2)d\beta = \int_{S_1}\int_{S_2}(\chi_E f_1,\chi_F f_2)d\beta = \int_E f_1(\omega_1)\beta_{f_2}(d\omega_1,F) . \tag{12}$$

It is not difficult to verify that each strictly β-integrable pair is β-integrable in the sense of Definition 2.3 with the same value. If $\mu:(A,B) \mapsto \int_{S_1}\int_{S_2}(\chi_A f_1,\chi_B f_2)d\beta$, $A\in\Sigma_1, B\in\Sigma_2$ and (f_1,f_2) is strictly β-integrable, then μ is a bimeasure on $\Sigma_1\times\Sigma_2$ since $(\mu = \mu_{f_1,f_2})$:

$$\mu(A,B) = \int_A f_1(\omega_1)\beta_{f_2}(d\omega_1,B) , \qquad A\in\Sigma_1, B\in\Sigma_2 .$$

It may be easily seen that the results on β-integrability given in ([19], Thms. 5.4 and 5.6, Corol. 5.7) are also valid for strict β-integrals. In particular, each bounded measurable pair (f,g) is always strictly β-integrable. The following "change of variables" formula holds:

Theorem 2.7. Let (f,g) be strictly β-integrable on $\{(\Omega_i,\Sigma_i), i=1,2\}$ for a bimeasure β on $\Sigma_1\times\Sigma_2$. Let $\mu(A,B) = \int_{\Omega_1}\int_{\Omega_2}(\chi_A f,\chi_B g)d\beta$, $A\in\Sigma_1, B\in\Sigma_2$. If $h:\Omega_1\to\mathbb{C}$, $k:\Omega_2\to\mathbb{C}$ are bounded measurable functions, then the following formula holds: (again $\mu = \mu_{f,g}$)

$$\int_A\int_B^* (fh,gk)d\beta = \int_A\int_B^*(h,k)d\mu \,, \qquad A\in\Sigma_1, B\in\Sigma_2 \,. \tag{13}$$

Proof. As noted above, μ is a bimeasure on $\Sigma_1\times\Sigma_2$. Define a linear functional k_F' on the space of scalar measures $ca(\Omega_2,\Sigma_2)$ as:

$$k_F'(\lambda) = \int_F k(\omega_2)\lambda(d\omega_2) \,, \qquad F\in\Sigma_2 \,, \quad \lambda\in ca(\Omega_2,\Sigma_2) \,,$$

where k is given in the statement. By the structure of measurable functions there is a sequence of step functions $k_n\to k$ pointwise, $|k_n|\le|k|$, and so let $k_n = \sum_{j=1}^{\ell_n} b_j^n \chi_{B_j^n}$. Then for each $E\in\Sigma_1$, one has

$$\begin{aligned}
\beta_{gk}(E,F) &= \int_F (gk)(\omega_2)\beta(E,d\omega_2) \,, \text{ by definition,}\\
&= \lim_{n\to\infty}\int_F (gk_n)(\omega_2)\beta(E,d\omega_2) \,, \text{ by the dominated convergence,}\\
&= \lim_{n\to\infty}\sum_{j=1}^{\ell_n} b_j^n\beta_g(E,F\cap B_j^n)\\
&= \lim_{n\to\infty}\int_F k_n(\omega_2)\beta_g(E,d\omega_2)\\
&= \int_F k(\omega_2)\beta_g(E,d\omega_2) = k_F'(\beta_g(E,\cdot)) \,.
\end{aligned}$$

Hence using the standard theory ([4], p. 180),

$$\int_F (gk)(\omega_2)_f\beta(E,d\omega_2) = k_F'(\int_{(\cdot)} g(\omega_2)_f\beta(E,d\omega_2))$$
$$= k_F'(\int_E f(\omega_1)\beta_g(d\omega_1,\cdot))$$
$$= \int_E f(\omega_1)k_F'(\beta_g(d\omega_1,\cdot)) \text{ , by ([4], p. 324),}$$
$$= \int_E f(\omega_1)\beta_{gk}(d\omega_1,F) \ .$$

This shows that (f,gk) is strictly β-integrable and further

$$\int_E\int_F^* (f,gk)d\beta = k_F'(\mu(E,\cdot)) = \int_F k(\omega_2)\mu(E,d\omega_2) = \tilde{\mu}_k(E,F) \ . \quad (14)$$

By a similar argument, one shows that (fh,gk) is strictly β-integrable and

$$\int_E\int_F^* (fh,gk)d\beta = \int_E h(\omega_1)\tilde{\mu}_k(d\omega_1,F) = \int_E\int_F^*(h,k)d\mu \ .$$

This is (13) and the proof is complete.

The corresponding statement for the bounded $\mathbb{C}$-bimeasures and MT-integration may be stated as follows:

<u>Theorem 2.8</u>. <u>Let</u> $(S_i,\mathcal{B}_i)$, $i=1,2$ <u>be σ-compact Borelian spaces, and</u> f,g <u>be $\mathcal{B}_1,\mathcal{B}_2$-measurable scalar functions</u>. <u>Let</u> Λ <u>be a bounded $\mathbb{C}$-bimeasure on</u> $S_1\times S_2$. <u>Then the pairs</u> (fh,gk) <u>are MT-integrable relative to</u> Λ <u>for all bounded scalar Baire functions</u> h,k <u>on</u> S_1,S_2 , <u>iff</u> $(\chi_A f,\chi_B g)$ <u>are MT-integrable for all</u> $A\in\mathcal{B}_1$ <u>and</u> $B\in\mathcal{B}_2$.

The proof of one direction is immediate, and the converse is similar to the preceding result using the interplay (of Bourbaki's) between the bounded linear functionals and the (bounded) Radon measures. The details are omitted.

The following consequence of Theorem 2.7 will be noted for applications.

Corollary 2.9. *Let $(S_i, \mathcal{B}_i)$, $i=1,2$ be measurable spaces and β be a bimeasure on $\mathcal{B}_1 \times \mathcal{B}_2$ into $\mathbb{C}$. Then one has:*

(i) *A pair (f,g) is strictly β-integrable iff the pair $(|f|,|g|)$ is.*

(ii) *If the $f_i : S_i \to \mathbb{C}$ are measurable, $i = 1,2$, and $|f_1| \le |f|$, $|f_2| \le |g|$ and (f,g) is strictly β-integrable, then so is (f_1, f_2).*

(iii) *If (f,g) is strictly β-integrable, f_n, g_n are sequences of measurable functions, $|f_n| \le |f|$, $|g_n| \le |g|$ and $f_n \to \tilde{f}$, $g_n \to \tilde{g}$ pointwise, then $(\tilde{f}, \tilde{g})$ is strictly β-integrable and*

$$\int_E \int_F^* (\tilde{f}, \tilde{g}) d\beta = \lim_{m\to\infty} \lim_{n\to\infty} \int_E \int_F^* (f_m, g_n) d\beta \ , \quad E \in \mathcal{B}_1, F \in \mathcal{B}_2 \ , \tag{15}$$

and similarly if m,n are interchanged in the limit.

Proof. (i) and (ii) are immediate. Regarding (iii), by (ii) $(\tilde{f}, \tilde{g})$ is strictly β-integrable. Further,

$$\begin{aligned} \int_E \int_F^* (\tilde{f}, \tilde{g}) d\beta &= \int_E f(x) \beta_{\tilde{g}}(dx, F) \\ &= \lim_{n\to\infty} \int_E f_n(x) \beta_{\tilde{g}}(dx, F) \\ &= \lim_{n\to\infty} \int_F \tilde{g}(y)\, {}_{f_n}\beta(E, dy) \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \int_F g_m(y)\, {}_{f_n}\beta(E, dy) \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \int_E \int_F^* (f_n, g_m) d\beta \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \int_E \int_F^* (f_n, g_m) d\beta \ , \text{ by a similar argument.} \end{aligned}$$

This is (15), and the result follows.

In exactly the same manner, if the $(\chi_A f, \chi_B g)$ are MT-integrable relative to a $\mathbb{C}$-bimeasure $\Lambda: \mathcal{K}(S_1)\times\mathcal{K}(S_2) \to \mathbb{C}$, then the above three statements hold for the MT-integrals, using Theorem 2.8.

These results will be sufficient to present the work on harmonizable processes and their extensions to Cramér classes.

III. INTEGRAL REPRESENTATIONS OF PROCESSES. The preceding analysis justifies the integration in (4) and one uses the strict β- (or F_y- in (4)) integrability. In what follows only this strict integral will be used and the word "strict" will be dropped hereafter since no other concepts will be employed. In (4), the integrand is a bounded continuous function. If β is restricted to functions with Vitali variation finite, then the "strict" and the "ordinary" (those in the sense of Definitions 2.2 and 2.3) integrals coincide.

The purpose of this section is to show that every weakly harmonizable process admits an integral representation; in fact to show that it is the Fourier transform of an $L_0^2(P)$-valued (or a stochastic) measure. Let us present the result in a form which applies to the Cramér class also, extending the work of ([5], Sec. 4.4).

Let $r: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ be a positive definite continuous function. It is then a covariance function in the sense that there exists a probability space and a stochastic process $\{X_t, t\in\mathbb{R}\}$ on it with the given r as its covariance function. To see this let $t_1,\ldots,t_n$ be n points from $\mathbb{R}$, and note that $(r(t_i,t_j), 1\le i,j\le n)$ is a positive definite matrix for each n. Let $F_{t_1,\ldots,t_n}$ be a Gaussian d.f. with mean zero, and this matrix as its covariance matrix. Such d.f.'s clearly exist on $\mathbb{R}^n$ for each $n \ge 1$. The family of all these d.f.'s has the consistency property, i.e., (i) $\lim_{x_n \to \infty} F_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = F_{t_1,\ldots,t_{n-1}}(x_1,\ldots,x_{n-1})$, and (ii) $F_{t_{i_1},\ldots,t_{i_n}}(x_{i_1},\ldots,x_{i_n}) = F_{t_1,\ldots,t_n}(x_1,\ldots,x_n)$ for any permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$. It then follows from a fundamental result of Kolmogorov (and Bochner) that there exists a probability space (Ω,Σ,P), and a real process $\{X_t, t\in\mathbb{R}\}$ on it such that $P[X_{t_1}<x_1,\ldots,X_{t_n}<x_n] = F_{t_1,\ldots,t_n}(x_1,\ldots,x_n)$. Here the finite dimensional d.f.'s are the given F's so that it is Gaussian and hence the covariance function is the given r and mean function is zero. (For a proof of this statement and related material on the existence of such processes, see [12], Sec. I.3.) This result also implies that the covariance functions given by (1), (4) or (6) are in fact such functions of concrete stochastic processes and not some fictitious objects, and the theory thereby gains importance for applications. Consequently, if $F_z:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is a positive definite function which is only of locally finite Fréchet variation in the sense that on each compact rectangle $I\times I\subset\mathbb{R}\times\mathbb{R}$, F_z satisfies $\|F_z\|(I\times I)<\infty$ so that (5) holds on $I\times I$ (but $\|F_z\|(\mathbb{R}\times\mathbb{R})=\infty$ is possible), let $\{g_z(s,\cdot), s\in\mathbb{R}\}$ be a family of Baire functions on $\mathbb{R}\to\mathbb{C}$. Suppose that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} g_z(s,\lambda)\overline{g_z(s,\lambda')}F_z(d\lambda,d\lambda')<\infty, \qquad s\in\mathbb{R}, \tag{16}$$

where the symbol denotes the strict integral relative to the (local) bimeasure F_z. For instance each $g_z(s,\cdot)$ can have compact supports. It is clear how the integration theory of the preceding section can be adapted to this situation with a modification of the classical methods (cf. e.g., [1]). If r_z is defined, for a family of F_z-integrable Baire functions $\{g(s,\cdot), s\in\mathbb{R}\}$, as

$$r_z(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_z(s,\lambda)\overline{g_z(t,\lambda')}F_z(d\lambda,d\lambda') < \infty \ , \qquad (17)$$

then it is a covariance function. The corresponding process $\{Z_t, t\in\mathbb{R}\}$ with this r_z as its covariance will be called <u>weakly of Cramér class</u>. This is the most general nonstationary family that can be studied with these methods. If $g_z(s,\lambda) = e^{is\lambda}$, then this reduces to the weakly harmonizable case, provided the F_z is a bimeasure on $\mathbb{R}\times\mathbb{R}$ (i.e., not merely locally).

To present the general representation, it is also necessary to recall the integral of Dunford and Schwartz ([4], Sec. IV.10) of a scalar function relative to a vector measure in the form needed here.

<u>Definition 3.1</u>. Let $(\mathbb{R},\mathcal{B})$ be the Borelian line and (Ω,Σ,P) be a probability space. If $L^2(P)$ is the usual Hilbert space on (Ω,Σ,P), then a mapping $Z:\mathcal{B}\to L^2(P)$ which is σ-additive in the norm topology of $L^2(P)$ is called a vector measure, and a stochastic measure if also $E(Z(A)) = 0$ for each $A\in\mathcal{B}$. If $f_n = \sum_{i=1}^{k_n} a_i^n\chi_{A_i^n}$, $A_i^n\in\mathcal{B}$ is a step function, then, as usual, we let $g_n^B = \int_B f_n dZ = \sum_{i=1}^{k_n} a_i^n Z(A_i^n\cap B)$, $B\in\mathcal{B}$ (which is seen to be uniquely defined) and if $f_n\to f$ pointwise, and $\{g_n^B, n\geq 1\}\subset L^2(P)$ is a Cauchy sequence for each $B\in\mathcal{B}$, then the unique limit $g^B = \lim_{n\to\infty}\int_B f_n dZ$ is denoted $\int_B f dZ = \lim_{n\to\infty}\int_B f_n dZ$, $B\in\mathcal{B}$, called the <u>Dunford-Schwartz (or D-S)</u> integral.

This concept is a specialized form (to $L^2(P)$) of that given in ([4], IV.10.7) where it is shown that the integral is uniquely defined, does not depend on the sequence used, and is linear. Taking $Z(\cdot)$ as the stochastic measure, the integral was defined differently in [3], [8] and others, but it can be seen to be a specialized version of the above definition. If $Z:\mathcal{B}(K) \to L^2(P)$ is a stochastic measure for each compact $K \subset \mathbb{R}$ ($\mathcal{B}(K)$ is the trace σ-algebra of $\mathcal{B}$ on K), so that Z will be a stochastic measure on the <u>δ-ring</u> $\mathcal{B}_0 \subset \mathcal{B}$ of bounded Borel sets, then again the D-S integration extends to this "local" situation with only simple modifications. (A special development to this case was given by Cramér in [2] to use it with (6); and a similar result holds with (17).)

We can now present a general integral representation for processes whose covariance is of the type (1), (4), (6) or (17). This is essential for the sampling theory of the next section and the interplay between the D-S integral and the strict integral of Definition 2.6 plays an important role in the present work.

<u>Theorem 3.2</u>. <u>Let</u> $\{X_t, t\in\mathbb{R}\}$ <u>be a second order process on</u> (Ω,Σ,P), <u>weakly of Cramér class, in the sense that</u> $E(X_t) = 0$ <u>and its covariance function</u> r_x <u>is representable by (17) relative to a positive definite local bimeasure</u> F_x <u>and a family of functions</u> $\{g_x(s,\cdot), s\in\mathbb{R}\}$ <u>which are strictly</u> F_x<u>-integrable</u>. <u>Then there exists a stochastic measure</u> $Z_x:\mathcal{B}_0 \to L^2(P)$ <u>such that</u>

$$\text{(i)} \quad E(Z_x(A)\overline{Z_x}(B)) = F_x(A,B), \quad A,B\in\mathcal{B}_0$$

$$\text{(ii)} \quad X(t) = \int_{\mathbb{R}} g_x(t,\lambda)Z_x(d\lambda), \quad t\in\mathbb{R}, \tag{18}$$

<u>where</u> $\mathcal{B}_0$ <u>is the δ-ring of bounded Borel sets of</u> $\mathbb{R}$, <u>and the integral on</u> $\mathcal{B}_0$ <u>in (18) is in the D-S sense</u>.

<u>If</u> F_x <u>is a bimeasure on</u> $\mathbb{R}^2$, <u>then</u> $\mathcal{B}_0$ <u>can be replaced by the Borel</u> σ<u>-algebra</u> $\mathcal{B}$ <u>and</u> $\int_{\mathbb{R}}\int_{\mathbb{R}}^{*} g_x(s,\lambda)\overline{g_x(s,\lambda')}F_x(d\lambda,d\lambda') < \infty$, $s\in\mathbb{R}$, <u>for any bounded Baire family</u> $\{g_x(s,\cdot), s\in\mathbb{R}\}$.

<u>Conversely, if</u> $\{X(t),t\in\mathbb{R}\}$ <u>is a process defined by (18) for a class</u> $\{g_x(s,\cdot),s\in\mathbb{R}\}$ <u>of D-S integrable functions, then</u> $\{X_t,t\in\mathbb{R}\}$ <u>is weakly of Cramér class, and the</u> $g_x(s,\cdot)$<u>'s are</u> F_x<u>-integrable strictly, where</u> $F_x(\cdot,\cdot)$, <u>given by (i), is (locally) a bimeasure on</u> $\mathcal{B}_0$, <u>the</u> δ<u>-ring of bounded Borel sets of</u> $\mathbb{R}$.

The special case of importance here is when the process is weakly harmonizable, or of Cramér class (6). In the latter case F_x is of finite Vitali variation on each compact rectangle and the integral with F_x is a planar Lebesgue-Stieltjes integral. In this event, the result reduces to the representation established by Cramér himself in [3]. The proof of the above theorem is an extension of [3] using the theory of bimeasures given in the preceding section. The details have been spelled out in ([13], Sec. 3), and will not be reproduced here. Let us state the harmonizable case separately for ready reference and to use it in Section IV.

<u>Theorem 3.3</u>. <u>Let</u> $\{X_t,t\in\mathbb{R}\}$ <u>be a process with</u> $E(X_t) = 0$ <u>and</u> $E(|X_t|^2) \le K_0 < \infty$, $t\in\mathbb{R}$. <u>Then it is weakly harmonizable relative to a positive definite bimeasure</u> F_x <u>on</u> $\mathcal{B}\times\mathcal{B}$, <u>iff there exists a stochastic measure</u> $Z_x:\mathcal{B}\to L^2(P)$ <u>such that</u> $E(Z_x(A)\overline{Z_x}(B)) = F_x(A,B)$, $A,B\in\mathcal{B}$, <u>and</u>

$$X_t = \int_{\mathbb{R}} e^{it\lambda} Z_x(d\lambda), \qquad t\in\mathbb{R}, \tag{19}$$

<u>where the integral is in the D-S sense</u>. <u>The process is strongly harmonizable if</u> F_x, <u>related to</u> Z_x <u>of (19), is of finite Vitali variation</u>, $|F_x|(\mathbb{R}\times\mathbb{R}) < \infty$. <u>In either</u>

<u>case</u> X_t <u>is uniformly continuous in</u> t <u>with the norm topology of the range space</u> $L^2(P)$.

The weakly harmonizable processes were also called "V-bounded" by S. Bochner who was the first to introduce them into the stochastic theory in 1954, and later (independently) were again defined by Yu. A. Rozanov in [16] with an indication of the need for integration akin to Definition 2.6. A comparison (and equivalence) of these concepts and other characterizations are given in [13]. The representation in this case has also been obtained by Niemi ([10], p. 35) by a slightly different method. But he was the first to recognize the use of MT integration in this study with V-boundedness.

If in (6) or (17) the function F_X concentrates on the diagonal of $\mathbb{R}^2$, then one has $F_X(\lambda,\lambda') = \delta_{\lambda\lambda'} G_X(\lambda)$ [$\delta_{\lambda\lambda'}$ is the Kronecker delta] and so

$$r_X(s,t) = \int_{\mathbb{R}} g_X(s,\lambda)\overline{g_X(t,\lambda)}G_X(d\lambda) \ . \tag{20}$$

Processes for which r_X has this special property are said to be of <u>Karhunen class</u>. If $g_X(s,\lambda) = e^{is\lambda}$ then this reduces to the (weakly) stationary case. Thus a harmonizable Karhunen class is simply stationary. If this happens in (18) and (19), the stochastic measure has the additional property of orthogonal increments, i.e., $E(Z_X(A)\overline{Z_X}(B)) = G_X(A \cap B)$. In all these cases F_X (or G_X) the positive definite bimeasure representing r_X is called the <u>spectral measure</u> of the process. This function plays an important role in the sampling results of the next section. Its significance for this problem is further pointed out in the last section. Theorem 3.2 for the Karhunen case becomes simpler, and this is given in ([5], p. 201).

IV. SAMPLING THEOREMS. Since it may be costly or difficult to observe the whole process, it is desirable to sample the observations at fixed intervals. However, by regarding our process as a curve in the Hilbert space $L^2(P)$, it is clear that two essentially different curves (or processes) can pass through a fixed set of equidistant points. This is usually called the "aliasing" problem and it is desirable to avoid this by choosing the spacing unit carefully. Thus the *sampling problem* is to find conditions on the characteristics (or the spectral function) governing the process such that it can be determined from a countable set of observations. In other words, if $L(X) = \overline{sp}\{X_t, t \in \mathbb{R}\} \subset L^2(P)$, and $\mathfrak{m}(X) = \overline{sp}\{X_{t_i}, t_i$ are points at which the process is to be observed$\}$, then $L(X) = \mathfrak{m}(X)$. If $t_n = \pm hn$ where $h > 0$ is the unit to be chosen, then it is called *periodic sampling*, and if $\{t_n, n \geq 1\}$ is a bounded infinite set of distinct values, then it is *nonperiodic sampling*.

For the classes considered in the preceding section, the process characteristic is the spectral function by which one classifies the process. So the conditions should be on such a function. For the periodic sampling of weakly harmonizable processes, the following general result, called a sampling theorem, holds:

Theorem 4.1. Let $\{X_t, t \in \mathbb{R}\}$ be a weakly harmonizable process with zero means and a spectral function F_X related by (4). Given $\varepsilon > 0$, there exists a bounded Borel set A $(= A_\varepsilon) \subset \mathbb{R}$ such that

$$\int_{A^c}\int_{A^c} F_X(d\lambda, d\lambda') < \varepsilon/4 , \qquad A^c = \mathbb{R} - A ,$$

and if σ_0 = diameter of A, then for any $\alpha > \sigma_0$ one has (with $\|X_t\|^2 = E(|X_t|^2)$) an $n(=n_{\varepsilon,t})$ such that

$$\|X(t) - X_n(t)\| \le C(t)\alpha[(\alpha - \sigma_0)n]^{-1} + \varepsilon \ , \tag{21}$$

where $X_n(t) = \sum_{k=-n}^{n} a_k(t;\alpha)X(k\pi/\alpha)$, $t \in \mathbb{R}$, and $0 < C(t) < \infty$ is bounded for t in bounded sets. The coefficients a_k's may be taken to be:

$$a_k(t;\alpha) = \frac{\sin(t\alpha - k\pi)}{(t\alpha - k\pi)} \ . \tag{22}$$

If the spectral function F_x has a bounded support, then we can set $\varepsilon = 0$ in (21).

Proof. Since $\{X_t, t \in \mathbb{R}\}$ is weakly harmonizable, there is a stochastic measure $Z_x : \mathcal{B} \to L^2_{\mathbb{C}}(P)$ satisfying (19), and such that $F_x(A,B) = E(Z_x(A)\overline{Z_x}(B))$. Also

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[-n,n)}(\lambda)\chi_{[-n,n)}(\lambda')F_x(d\lambda,d\lambda') = \left\|\int_{\mathbb{R}} \chi_{[-n,n)}(\lambda)Z_x(d\lambda)\right\|^2$$

$$\to \left\|\int_{\mathbb{R}} 1 \cdot Z_x(d\lambda)\right\|^2 \ , \quad \text{using}$$

([4],IV.10.10)

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} F_x(d\lambda,d\lambda') \ . \tag{23}$$

Hence given $\varepsilon > 0$, there exists n_0 ($= n_0(\varepsilon)$) such that $n \ge n_0 \Rightarrow$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} F_x(d\lambda,d\lambda') - \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[-n,n)}(\lambda)\chi_{[-n,n)}(\lambda')F_x(d\lambda,d\lambda') < \varepsilon^2/256 \ . \tag{24}$$

This may be written alternately in the following form which also obtains from the fact that $1 - \chi_{[-n,n)} \downarrow 0$

boundedly and the same theorem of [4] as in (23) applies:

$$\int_{A_n^c}\int_{A_n^c} F_x(d\lambda, d\lambda') = \left\|\int_{\mathbb{R}} (1 - \chi_{A_n}(\lambda)) Z_x(d\lambda)\right\|^2 = \|Z_x(A_n^c)\|^2 \to 0 \tag{24'}$$

where $A_n = [-n,n)$ so that for A_{n_0} one has $\|Z_x(A_{n_0}^c)\| < \varepsilon/16$ by (24'). (We can actually choose $\|Z_x\|(A_{n_0}^c) < \varepsilon/4$ by [6], Thm. 3.5 here.) That this fact holds for vector measure more generally is proved in ([6], Thm. 3.5).

Consider

$$X(t) = \int_{\mathbb{R}} e^{it\lambda} Z(d\lambda) = \int_{A_{n_0}} e^{it\lambda} Z_x(d\lambda) + \int_{A_{n_0}^c} e^{it\lambda} Z_x(d\lambda)$$
$$= X_1(t) + X_2(t) \quad \text{(say)}. \tag{25}$$

Writing $\tilde{Z}_1 = Z_x(A_{n_0} \cap \cdot)$ and $\tilde{Z}_2 = Z_x(A_{n_0}^c \cap \cdot)$, and noting that these are again stochastic measures, it follows that X_1 and X_2 are also weakly harmonizable. Moreover,

$$\|X(t) - X_1(t)\| = \|X_2(t)\| = \left\|\int_{A_{n_0}^c} e^{it\lambda} Z_x(d\lambda)\right\| \le 1 \cdot \|Z_x\|(A_{n_0}^c)$$
$$\le 4 \sup_{E \subset A_{n_0}^c} \|Z_x(E)\| \le \varepsilon/4, \text{ by ([4], IV.10.4(b)) (cf. also [6], Thm. 3.5).} \tag{26}$$

Since, if $f_t(\lambda) = e^{it\lambda}$, $f_t(\cdot)$ is an entire function of exponential type (with finite exponent [= 1 here]), the classical results on approximation imply (cf. [18], Sec. 4.3; these are given in the form needed by Piranashvili [10], but the present one is simply the classical Kotel'nikov-Shannon formula, cf. [5], p. 204) with $z = \lambda_1 + i\mu_1$ and $\sigma_0 \geq n_0$, for any $\alpha > \sigma_0$:

$$\left| e^{iz\lambda} - \sum_{k=-n}^{n} \exp\{ i\lambda \frac{k\pi}{\alpha}\} \cdot \frac{\sin\alpha(z-\frac{k\pi}{\alpha})}{\alpha(z-\frac{k\pi}{\alpha})} \right| < \frac{L_0(z)\alpha}{(\alpha-\sigma_0)n} \tag{27}$$

where $L_0(z)$ is finite for z in bounded domains in the complex plane. (A more general form of this estimate for any entire function of finite exponential type appears in the proof of the next result.) Define $X_{n_0}(t)$ by:

$$X_{n_0}(t) = \sum_{k=-n_0}^{n_0} X(\frac{k\pi}{\alpha}) \frac{\sin\alpha(t-\frac{k\pi}{\alpha})}{\alpha(t-\frac{k\pi}{\alpha})} . \tag{28}$$

Taking $a_k(t;\alpha)$ as in (22), it is asserted that this $X_{n_0}(t)$-process satisfies (21).

For, let $\mathcal{S}_{n_0}(z)$ be the nonnegative left side quantity of (27). Then

$$\begin{aligned} \|X(t)-X_{n_0}(t)\| &\leq \Big\| \int_{A_{n_0}} (e^{it\lambda} - \sum_{k=-n_0}^{n_0} e^{i\lambda\frac{k\pi}{\alpha}} a_k(t;\alpha))\tilde{Z}_1(d\lambda) \Big\| + \\ &\quad \|X_2(t)-X_{2,n_0}(t)\| , \text{ using (25) and (28)} \\ &\quad \text{with } X_{2,n_0} \text{ in place of } X_{n_0} \text{ if } X \\ &\quad \text{is replaced by } X_2 \ ; \\ &\leq \mathcal{S}_{n_0}(t)\|Z_x\|(A_{n_0}) + \|X_2(t)\| + \|X_{2,n_0}(t)\| , \text{ by} \\ &\quad \text{the triangle inequality and ([4], IV.10.7)} \\ &\leq \frac{L_0(t)\alpha}{(\alpha-\sigma_0)n_0}\|Z_x\|(\mathbb{R}) + \frac{\varepsilon}{4} + \|X_{2,n_0}(t)\| , \text{ by (26)} \\ &\quad \text{and (27).} \end{aligned} \tag{29}$$

Now consider,

$$\|X_{2,n_0}(t)\| = \|\sum_{k=-n_0}^{n_0} a_k(t;\alpha)X_2(\tfrac{k\pi}{\alpha})\|$$

$$= \|\int_{\mathbb{R}} (\sum_{k=-n_0}^{n_0} e^{i\lambda\frac{k\pi}{\alpha}} a_k(t;\alpha))\tilde{Z}_2(d\lambda)\|$$

$$\le \sup_{\lambda\in\mathbb{R}} |\sum_{k=-n_0}^{n_0} e^{i\lambda\frac{k\pi}{\alpha}} a_k(t;\alpha)| \, \|\tilde{Z}_2\|(\mathbb{R})$$

$$\le [1 + \frac{L_0(t)\alpha}{(\alpha-\sigma_0)n_0}]\frac{\varepsilon}{4} \text{ , by (27) and (26)}$$

$$\le [1 + \frac{1}{2}]\frac{\varepsilon}{4} = \frac{3\varepsilon}{8} \text{ , if } n_0 \ge (\frac{2L_0(t)\alpha}{\alpha-\sigma_0}).$$

Substituting this in (29) one gets (21) with $C(t) = L_0(t)\|Z_x\|(\mathbb{R})$.

If the support of F_x is enclosable in a compact rectangle, then for a suitable n_0 , it will be in $A_{n_0} \times A_{n_0}$ and hence $X_2(t) = 0$. Thus $\varepsilon = 0$ is possible in the above estimates. This completes the proof.

<u>Remark</u>. If the process is strongly harmonizable, then F_x has finite Vitali variation and the analysis proceeds with Lebesgue integration, and the result can be deduced from [11]. However, in the weakly harmonizable case, this is not possible and the vector (or D-S) integration in $L^2(P)$ and its relation with the MT-integration of bimeasures must play a central role.

To illuminate these ideas further, we present another (periodic) sampling theorem for some Cramér class processes. Recall that a process $\{X_t, t\in\mathbb{R}\}$ is of weakly Cramér class if it has means zero and its covariance function r_x admits a representation as (17) relative to a spectral measure function F_x, of local finite Fréchet variation and a family $\{g_x(s,\cdot), s\in\mathbb{R}\}$ of F_x-integrable complex functions. Then by Theorem 3.2, the X_t process admits an integral representation (18) relative to a σ-additive $Z:\mathcal{B}_0\to L^2(P)$, the integral being (an extended) one of D-S type. In a related terminology $(\mathbb{R},\mathcal{B},\mathcal{B}_0,Z)$ becomes a "semi-standard quasi-measure space" (cf. [6], p. 210) and the integral becomes the same vector integral of [6]. For the following, the $g(s,\lambda)$'s will be assumed to satisfy two growth and smoothness conditions:

(i) each $g(\cdot,\lambda)$ can be extended to be an entire function, $\lambda\in\mathbb{R}$;

(ii) if $c_n(\lambda) = \frac{\partial^n g}{\partial z^n}(z,\lambda)\big|_{z=0}$, then $c^*(\lambda) = \limsup_{n\to\infty} [|c_n(\lambda)|]^{1/n} \le \sigma_0 < \infty$, and there is an integer $m \ge 0$, such that
$|g(z,\lambda)| \le L(\lambda)(1+|z|^m)\exp\{c^*(\lambda)|y|\}$, $z=x+iy$.

In the preceding case $g(t,\lambda) = e^{it\lambda}$, and this satisfies (i) automatically. Regarding (ii), $c^*(\lambda) = |\lambda|$ in that case. So there is such a $\sigma_0<\infty$ only if the λ's vary in a compact set and this will obtain if F_x has a bounded support. In the present case, if the F_x is not restricted, then the $g(s,\cdot)$ should have compact supports. This is reasonable in the Cramér class, and is one of the reasons for this generalization. Within such a framework, the following (periodic) sampling theorem holds:

<u>Theorem 4.2. Let</u> $\{X(t),t\in\mathbb{R}\}$ <u>be of weakly Cramér class with its g-family satisfying the growth conditions (i) and (ii) above. Suppose that</u> $L(\cdot)$ <u>in (ii) is strictly</u>

<u>integrable relative to the spectral measure</u> F_X . <u>Then for each</u> $\alpha > \sigma_0$, <u>if</u> $X_n(t)$ <u>is given by</u>:

$$X_n(t) = \sum_{k=-n}^{n} X(\frac{k\pi}{\alpha}) \frac{\sin(\alpha t-k\pi)\sin^q\beta(t-k\pi/\alpha)}{(\alpha t-k\pi)\beta^q(t-k\pi/\alpha)} , \tag{30}$$

<u>where</u> $q \geq m$, <u>and</u> $\beta < (\alpha-\sigma_0)/q$, <u>we have for a constant</u> $C_0(t,\alpha,q) < \infty$:

$$\|X(t) - X_n(t)\| \leq C_0(t,\alpha,q)/n . \tag{31}$$

<u>In other words</u> $\{X(k\pi/\alpha), k=0,\pm 1,\pm 2,\ldots\}$ <u>spans the same space as</u> $\{X(t), t\in\mathbb{R}\}$.

<u>Proof</u>. The idea of proof is similar to that of the above theorem, and we can quickly sketch the argument. The key again is the approximation result from function theory. This is based on a theorem of M. L. Cartwright (cf. [18], p. 186) as modified in Piranashvili ([11], p. 648). By this work, one has

$$\begin{aligned} &\left|g(z,\lambda) - \sum_{k=-n}^{n} g(\frac{\pi k}{\alpha},\lambda) \frac{\sin(\alpha z-k\pi)\sin^q\beta(z-k\pi/\alpha)}{(\alpha z-k\pi)\beta^q(z-k\pi/\alpha)}\right| \\ &\qquad < \frac{L(\lambda)\tilde{L}_q(z)}{\beta^q(\alpha-\sigma-\beta q)} \cdot \frac{\alpha}{n}[(\frac{\alpha}{n})^q + (\frac{\alpha}{n})^{q-m}] , \end{aligned} \tag{32}$$

where $L(\cdot)$ is as in (ii) of the growth condition and $\tilde{L}_q(z)$ is a positive finite number for z in bounded sets of the complex plane. So if $X_n(t)$ is defined as in (30), and $\zeta_n(t) = X(t) - X_n(t)$, then, if $|v_n(z,\lambda)|$ is the left side quantity , $L(\lambda)h_n(\alpha,z,q)$ the right side, of (32)

$$\zeta_n(t) = \int_{\mathbb{R}} v_n(t,\lambda)Z(d\lambda)$$

and $|v_n(t,\lambda)| \leq L(\lambda)\cdot h_n(\alpha,t,q)$ which by hypothesis is

F_x-integrable and which in turn implies that $L(\cdot)$ is integrable for $Z(\cdot)$. Hence by ([6], Thm. 6.11(e)), one has

$$\begin{aligned}\|\zeta_n(t)\| &\le 4\sup\{\|\int_A L(\lambda)\cdot h_n(\alpha,t,q)Z(d\lambda)\|:A\in\mathcal{B}\}\\ &\le \frac{4\tilde{L}_q(t)}{\beta^q(\alpha-\sigma-\beta q)}\cdot\frac{\alpha}{n}[(\frac{\alpha}{n})^q+(\frac{\alpha}{n})^{q-m}]\cdot\sup\{\|\int_A L(\lambda)Z(d\lambda)\|:A\in\mathcal{B}\}\\ &= M_0\frac{\tilde{L}_q(t)}{\beta^q(\alpha-\sigma-\beta q)}\cdot\frac{\alpha}{n}[(\frac{\alpha}{n})^q+(\frac{\alpha}{n})^{q-m}] \quad \text{(say)}. \qquad (33)\end{aligned}$$

The right side $\to 0$ as $n\to\infty$ for t in bounded sets, $\alpha>\sigma_0$. Setting $C_0(t,\alpha,q)$ as the coefficient of n^{-1} in (33), (31) results and completes the proof.

Without doubt, this result can be extended to get an ε-approximation by relaxing some conditions, to match with the above theorem. We shall not go into such a formulation here. Instead some related remarks will be given in the final section.

V. FINAL REMARKS. 1. For stationary processes, the spectral measure F is real (and positive) and a more precise result can be obtained. A characterization for a periodic sampling theorem is obtained by S. P. Lloyd [7] involving only the support of F_x, namely its translates should be disjoint when the translation is suitably related to the sampling interval $h(=\pi/\alpha)$. An extension of that result for strongly harmonizable processes can be formulated. The sufficiency of it has been given in [14]. It was stated there without proof that the converse also holds. This was clearly an oversight and the problem is still open, i.e., if a periodic sampling theorem holds for a (weakly or strongly) harmonizable process, describe the precise property of the support of the spectral measure.

For this reason in the preceding section only sufficient conditions are considered which however are the most useful ones for applications. In [11], independently of [14], the method based on approximation theory, of functions of a real variable, was presented. The underlying ideas extend, as shown here, though the method of proof of [11] does not generalize. The vector integration, and the bimeasure theory are useful in this extension.

2. As the above discussion indicates, historically the philosophy of sampling theorms for processes is to analyze the behavior of the spectral function where it is available. However, one cannot always consider processes having spectral functions as in Section 1. Then this type of sampling theory becomes meaningless. Instead one can consider some approximations from a different point of view. For instance most second order processes admit their covariance representation, called "generalized triangular covariances," extending (20) in which $\mathbb{R}$ is replaced by a more complicated set. This may be stated as follows. Let T be a subset of $\mathbb{R}$ ($= \mathbb{R}$ is possible) and $r(\cdot,\cdot) : T{\times}T \to \mathbb{C}$ be a positive definite mapping. Consider a space $H_0 \subset T^{\mathbb{C}}$, called the reproducing kernel inner product space: $f \in H_0$ iff $f = \sum_{i=1}^{n} c_i r(s_i,\cdot)$, $c_i \in \mathbb{C}$. If also $g = \sum_{j=1}^{m} d_j r(\cdot,t_j) \in H_0$, then introduce the inner product $\langle f,g\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \bar{d}_j\, r(s_i,t_j)$. This is well-defined and is an inner product. If $\mathcal{H}_r$ is the closure of H_0 in this inner product, then $\mathcal{H}_r$ is called the <u>Aronszajn space</u>. This is separable if for instance r satisfies a smoothness condition. Let $(\mathbb{R}, \mathcal{B})$ be the usual Borelian line, $\Omega = \mathbb{R} \times \mathbb{Z}$, $\Sigma = \mathcal{B} \otimes \mathcal{P}(\mathbb{Z})$. Then the following result holds:

Proposition 5.1. *Let* $r : T\times T \to \mathbb{C}$ *be a covariance function such that the associated Aronszajn space* $\mathcal{H}_r$ *is separable. If* (Ω,Σ) *is as above, then there exists a Lebesgue-Stieltjes* σ*-finite measure* ν *on* Σ *and a measurable family of complex functions* $\{\psi(t,\cdot),\ t\in T\}$ *such that*

$$r(s,t) = \int_\Omega \psi(s,\omega)\overline{\psi(t,\omega)}\nu(d\omega)\ ,\ s,t\in T\ . \qquad (34)$$

The actual structure of ψ's and ν as well as the proof of this result are given in ([15], Sec. 6.2) . But the ψ-functions do not generally have any of the properties of the g-functions of the last section. While ν can be regarded as a "generalized spectral measure" of r , it is hard to relate these to those of the process X , as in the preceding theory. This is why the considerations there were given to a subclass.

3. One can consider nonperiodic sampling theorems also. Let $X(t)$ be of weakly Cramér class relative to a family $\{g(s,\cdot),s\in\mathbb{R}\}$ so that (17) holds. Suppose that $g(\cdot,\lambda)$, $\lambda\in\mathbb{R}$, is *analytic* and $g^{(n)}(s,\cdot)$ is strictly F_x - integrable for each $s\in\mathbb{R}$ and $n\geq 1$ where F_x is the spectral measure of $X(t)$'s. This implies through (15) and ([4], Thm. IV.10.8) that $\{X(t),t\in\mathbb{R}\}$ is a second order analytic random function since the covariance $r_x(\cdot,\cdot)$ is infinitely differentiable. Using the fact that an analytic function is uniquely determined if it is known at a countable sequence of points which tend to a limit point, one can deduce the sampling theorem of the following type as in ([14], p. 68): *If* $\{t_n,n\geq 1\}\subset\mathbb{R}$ *is an infinite bounded set of distinct points, and* $X(t_n)$ *is known at each "time"* t_n , *where the* $g(s,\cdot)$*-set satisfies the smoothness conditions noted above, then the samples* $\{X(t_n),n\geq 1\}$ *also determine the*

process. A proof of this statement will be omitted since it can be constructed from the above remarks easily.

4. It is possible, in many cases, to put conditions on the covariance function r_X of a process X such that it admits a Mercer type series expansion. This in turn easily implies an orthogonal series expansion of $X(t) = \sum_{n=1}^{\infty} a_n(t)f_n$, where $\{f_n, n\geq 1\}$ is an uncorrelated sequence of random variables each with mean zero and unit variance. The partial sums of the series can be used as the "sampling sequences." Here the problem of computing the "coefficient functions" $a_n(\cdot)$ is nontrivial, and this approach has other drawbacks. We therefore do not consider this set in the general format of the sampling theory of random processes, which is based on the Kotel'nikov-Shannon type series. Such expansions, however, have utility for other problems.

5. Finally, formulas (1) and (3) show, in case F_X, G_X are absolutely continuous, that F_X' and G_X' are Fourier transforms $\hat{r}_X$ of r_X and so one may seek conditions on these to obtain some "sampling theorems." These are very special assumptions and for (4) or (7) no such assumption is meaningful since the respective integrals are not in Lebesgue's sense. From all these considerations, it appears that the theory of bimeasures plays a vital part for processes of the type having spectral functions in $\mathbb{R}^2$, i.e., for the classes of processes treated in this paper.

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