A Gamma Ornstein-Uhlenbeck model driven by a Hawkes process*

Guillaume Bernis[†], Simone Scotti[‡] and Carlo Sgarra[§]

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Abstract: We propose an extension of the Γ-OU Barndorff-Nielsen and Shephard model taking into account jumps clustering phenomena. We assume that the intensity process of the Hawkes driver coincides, up to a constant, with the variance process. By applying the theory of continuous-state branching processes with immigration, we prove existence and uniqueness of strong solutions of the SDE governing the asset price dynamics. We introduce a measure change of Esscher type in order to describe the relation between the risk-neutral and the historical dynamics. By exploiting the affine features of the model we provide an explicit form for the Laplace transform of the asset log-return, for its quadratic variation and for the ergodic distribution of the variance process. We show that the model proposed exhibits a larger flexibility in comparison with the Γ-OU model, in spite of the same number of parameters required. In particular, we illustrate numerically that the left wing implied volatility could be first fit by using the original Γ-OU model and then the right wing can be arranged by a trigger of the intensity and variance processes. Moreover, implied volatility of variance swap options is upward-sloped due to the self-exciting property of Hawkes processes.

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*The analysis and views expressed in this chapter are those of the authors and do not necessarily reflect those of Natixis Assurances.

[†]Natixis Assurances. eMail: guillaume.bernis@natixis.com

[‡]LPSM, Université Paris Diderot. eMail: simone.scotti@univ-paris-diderot.fr

[§]Politecnico di Milano. eMail: carlo.sgarra@polimi.it

1 Introduction

In recent years, the implied volatility indices, such as VIX in the US and V2X in Europe, proved themselves to be key financial instruments for investment, hedging, see for instance Rhoads [43], Ben Dor and Guan [8], and also indicators of the "stress" on the market. This growing importance has shed light on the peculiar form of the implied volatility dynamics: the occurrence of large variations on very short periods, with a tendency to form clusters of spikes. Moreover, new derivatives products appear, due to market regulation and standardization, like volatility options, with a significant and increasing demand. Implied volatility of these product exhibits an upward slope, i.e. a clear evidence of market volatility risk aversion pushing people to buy options to cover this risk, see Nicolato et al. [42]. The existence of such stylized facts accounts for the emergence of a wide range of financial models taking into account these features. Starting from the Heston model [25], in which volatility follows a CIR diffusion [12], Bates [6] adds jumps in asset dynamics, while Sepp [45] includes jumps in both asset returns and the variance, both papers using Poisson processes. In their seminal paper, Duffie et al. [15] generalize the Heston framework to include a self-exciting structure in the affine jump-diffusion framework for both asset and volatility, however, their examples mainly focus on Poisson drivers. Kallsen et al. [31] consider the case where jumps are added in the stock via a time-changed Lévy process. More recently, a large literature develops rough volatility models, see for example Bayer et al. [7], El Euch and Rosenbaum [17] and El Euch et al. [18]. The generalization of affine Volterra processes is proposed by Abi Jaber et al. [1].

The aim of our paper is to propose a new model for asset pricing able to capture the main stylized features but preserving both mathematical and numerical tractability. Our idea is to build up our model as an extension of the Barndorff-Nielsen and Shephard (BN-S) model [4, 5], in order to include jumps clusters in both the volatility and the stock return dynamics. In their celebrated papers Barndorff-Nielsen and Shephard propose a stochastic volatility model of Ornstein-Uhlenbeck type driven by a subordinator; by considering as a concrete specification for the subordinator a compound Poisson process with exponentially distributed jumps size, they refer to the model as to the Gamma-OU model. In the Gamma-OU model both the variance and the log-returns are driven by the same Poisson process. This type of stochastic volatility model, originally introduced to fit empirical data, was later shown to be suitable for option pricing by Nicolato and Venardos [41].

As pointed out by a large amount of literature, see for instance Veraart and Veraart [46], the BN-S model, and more generally single factor models, are not flexible enough and they are outperformed by multi-factor stochastic volatility models in practice. However, multi-factor models have a main drawback in the huge number of parameters required, giving birth to fitted parameters instability, see for instance the discussion in Bates [6].

Our purpose is to extend the Gamma-OU model in order to include jump clustering features. The clustering effect of jumps in the Hawkes process [24] is well adapted to take into account the periods of turmoil in the implied volatility, that can be observed in implied volatility indices. These clustering features have been studied throughout various financial markets: see [20] for FX rates, [23, 28] for interest rates and [29] for energy prices. In this paper, we will insist on the stylized facts related to the clustering effects of the implied volatility indices, such as VIX index, and the related options.

In the model we are going to propose, the jump components of the diffusion is taken into account through a marked-Hawkes process, with exponentially distributed jump sizes. Thus, the model has three dimensions, the stock, the variance process and the intensity of the jumps. In order to prevent parameters instability, we assume that the intensity process coincides with the variance process up to an additive constant, representing the minimal intensity of jumps arrival. Our model, that will be referred to as Gamma-OU Hawkes volatility model, shares the same number of parameters than Gamma-OU, since the constant intensity is replaced by the shifting constant linking the intensity and the variance processes. In spite of this, the Gamma-OU Hawkes model is intrinsically multi-factors, since both variance and intensity are stochastic processes. It is also more flexible than the Gamma-OU model. We shall show that, first, it is possible to fit the BN-S Gamma-OU model to obtain the skew, that is the slope of implied volatility and the left wing of the implied volatility. Second, the clustering effect can be used to induce a smile, that is the convexity of implied volatility, and then to calibrate the right part of implied volatility by triggering the volatility and the minimal intensity, since both contribute positively to the variance of the log-return. This result is probably the most counterintuitive result of the paper, since jumps are negative in log-return in order to obtain a leverage effect. As a consequence, we could expect that changing in intensity affects mainly the left wing of the implied volatility with no or really small effect on the right side. We highlight numerically that the triggering between volatility and intensity is achieving the precise opposite. The explanation of this counter-intuitive effect is the self-exciting effect of jumps and mainly relative to the intensity of jumps that is no longer constant. The drift of the log-return depends positively and linearly on this drift, in order to preserve risk neutral property. Then a growing intensity induces a strong leptokurtic distribution and produces a more pronounced smile in the right wing of implied volatility even in the absence of positive jumps. Moreover, by focusing on the options written on variance swaps, the same self-exciting effect gives birth to an upward-sloped implied volatility. This upward slope behaviour is coherent with market data, see Nicolato et al. [42], but is exhibited by very few models in the literature. In [42], the authors study the implied volatility of variance options for different stochastic volatility models with jumps and show that only inverse-Gamma Ornstein-Uhlenbeck is able to reproduce an upward slope. However, it is easy to check that the related variance process has only a limited number of finite positive moments due to the power-type decay of inverse-Gamma low. It is easy to deduce that the upward slope is induced by a strong distortion of the distribution paying a limit on the variance moments. Finally, the Gamma-OU Hawkes volatility model behaves in the same way as a hidden Markov chain with one regime with large volatility and many jumps, in clustered periods, and a second regime consisting of long periods of persistency of low volatility with few jumps. This self-exciting effect could capture, in an intrinsically bi-dimensional parsimonious model, the main features of market data, i.e. turmoil periods and jump clusters but also persistency of low volatility periods; the leverage effect and the freedom to calibrate independently the smile convexity; and finally an upward-sloped implied volatility for variance options preserving the existence of positive moments for the variance process.

Our main mathematical contributions can be resumed as follows. First, we propose an original approach, based on the theory of continuous branching processes with immigration (CBI), in order to deal with the questions of existence of solutions, their regularity properties and the construction of equivalent martingale measures. In particular, we harness the Dawson and Li formulation of CBI, see [13], in order to provide a proof of these results. We obtain moreover the Laplace transform in closed form. Then the law of variance swaps is computed in almost closed form (that is in closed form up to an integration step) by applying the general result by Kallsen et al. [31]. Second, by introducing an Esscher type transform for continuous branching processes with immigration, similar to that proposed by Jiao et al [28, 29], we derive a specific class of equivalent martingale measures preserving the structure of the Gamma-OU Hawkes volatility model, i.e. under the equivalent martingale measure, the couple of log-return and volatility is described by the same Gamma-OU Hawkes volatility model with different parameters. In particular, we show that the mean reverting speed is not preserved under change of probability in contrast with previous models as for instance standard Γ-OU, see Nicolato and Venardos [41]. Finally, we detail an efficient strategy to price asset and variance options based only on the simulation of the Hawkes process.

The paper is organized as follows. Section 2 details a statistical analysis of VIX prices pointing out the existence of jump clusters and giving a first fit to data. In section 3, we introduce the Hawkes volatility model, by providing theoretical results about existence and characterization. Section 4 deals with the Esscher transform and the equivalent martingale measures, and the option pricing. Finally, Section 5 is devoted to numerical applications on pricing.

2 Clusters in VIX: stylized facts

In this section, we propose to shed light on the clustering effects of the VIX. A general analysis of this index is performed by Avellaneda and Papanicolaou [2]. This point provides us with an empirical justification of our approach involving clustering effects both on the jumps of the asset price and on the volatility. Moreover, we also see the particular importance of downward jumps.

The VIX index represents the square-root of the implied variance extracted from short dated options on the S&P 500 index. Hence, we consider the square of VIX index since it coincides with the variance swap price, that is a linear functional with respect to the time, converging to the quadratic variation of the logarithm of the equity process. See for instance Kallsen et al. [31], for details on this issue.

Our sample consists of daily observations from 25/09/2006 to 23/10/2018. Thus, it involves periods of important turmoil (post Lehman Brothers credit crisis after September 2008, European sovereign debt crisis in summer 2011, etc.). The sample starts in a period of mild volatility. Thus, we can reasonably assume that the Hawkes intensity, at this time, is close to its minimal level, which we denote by $\underline{\lambda}$.

The jumps are detected as the larger positive fluctuations using the algorithm detailed in

Callegaro et al. [10]. The first analysis concerns the distribution of gaps between two jumps in order to confirm that the jumps are clustered. Given the total number of jumps in a given period and assuming a constant intensity, the conditional distribution of gap lengths is uniform. It is now easy to test this hypothesis using a Kolmogorov-Smirnov (K-S) test. The renormalized K-S statistics gives a value of 4.45 that is really large compared to all critical valued usually considered: for instance, for a rejection of the null hypothesis at level of 0.01, the critical value is 1.62, for 0.001 it is 1.95. We then reject the Poisson arrival rate of jumps. In contrast, we can test the hypothesis of an intensity proportional to the index VIX^2 itself and the related renormalised K-S statistics gives a value of 1.04 that is small compared to the critical valued usually considered, for instance 1.22 for a significance threshold 0.1. As a consequence, we cannot reject the pure self-excited framework. Figure 1 illustrates the result of the goodness of fit. It can be remarked that the values obtained assuming a constant intensity are really far from the ideal diagonal whereas the ones assuming an intensity proportional to the VIX^2 itself are really close.

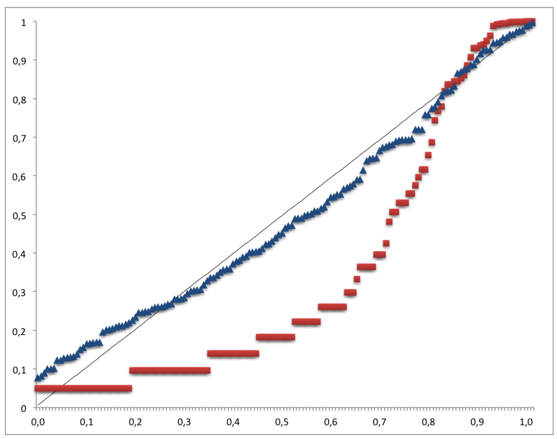

Figure 1: Kolmogorov-Smirnov test for constant (red squares) and proportional (blue triangles) intensity for the jump arrivals.

We now turn on the joint analysis of S&P and VIX^2. We identify, in an independent way, the jump times and the size of the relative increments of S&P and the absolute increments of VIX^2. Our study shows that almost all large negative jumps of S&P coincide with the positive jumps in VIX^2, whereas only one half of positive jumps of S&P coincides with the negative ones in VIX^2. By studying the arrival times of the negative jumps of VIX^2, we see that they coincide with the very large values of the VIX^2 itself and follow large positive jumps. We can reproduce this effect with only positive jumps in VIX and an exponential mean-reversion speed. Moreover, the method adopted in order to identify jumps, see Callegaro et al. [10], is unable to detect jumps relatively small. In particular, jumps smaller than three standard deviations of the other increments are classified as usual Brownian noise. The fact that we identify almost 2.5 more positive jumps in VIX^2 than negative jumps in S&P could be explained by the presence of a larger Brownian contribution, covering a part of relatively small jumps. Indeed, we stop our iteration at the fifth loop since the ratio of the variance at the fourth and fifth loop in S&P is 0.975, i.e. really near 1, showing that the split between the jumps and the Brownian component is done. However, the same ratio for VIX^2 increments is only 0.83 showing that the Brownian part is certainty overestimated. In order to reach a variance ratio for VIX^2 similar to the one of S&P more than eleven loops are required. The total number of large fluctuations reaches a frequency of one every ten days that pushes us to cut the Brownian component in the variance process in ageement with the Barndorff-Nielsen and Shephard model.

We now focus on the law of jumps sizes in VIX^2 and S&P. According to the Kou model, see [35, 36], we obtain a quite acceptable fit for an exponential law truncated at the threshold used to split increments between jumps and Brownian oscillations. The Kolmogorov-Smirnov test gives values of 1.06 and 1.17 respectively for the positive jumps of VIX^2 and negative ones of S&P, that could not authorise to reject the exponential law hypothesis, see also Figure 2.

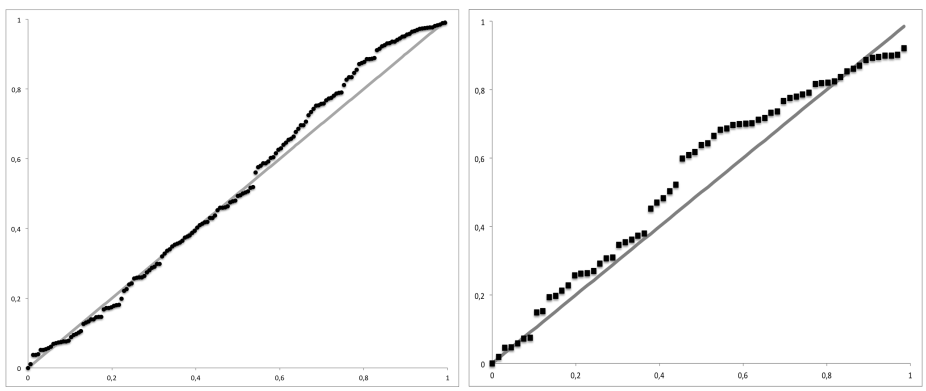

Figure 2: Kolmogorov-Smirnov test for exponential law for positive (resp. negative) fluctuation in VIX^2 (resp. S&P).

3 The Gamma-OU model driven by a Hawkes process

As the BN-S model, the Gamma-OU Hawkes volatility model is a stochastic volatility model with the same jumps driving both the volatility and the log-return processes, including a leverage effect. Here, we chose the variance process equal to the Hawkes intensity, exhibiting the suited mean-reverting form. However, the intensity of the Hawkes process is bounded from below by a strictly positive constant, denoted by $\underline{\lambda}$. We subtract this constant to the variance, in order to ensure that the variance process could visit all the positive values.

Since the model is constructed in order to reproduce the VIX^2 behavior, which is naturally defined with respect to the risk neutral probability, we first consider the risk neutral framework.

3.1 Definitions

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space supporting a standard Brownian motion $W := \{W_t\}_{t\geq 0}$ and a marked Hawkes process independent on W. The Hawkes process can be characterized by its Poisson measure $\mu(dt, dz)$ defined on $(\mathbb{R}^+)^2$ and by the sequence of jump times $\{\tau_i\}_{i\in\mathbb{N}}$ and marks $\{Z_i\}_{i\in\mathbb{N}}$, representing the jump times and jump sizes respectively. In order to describe the leverage effect mentioned before, the jump process driving the log-return dynamics is multiplied by $-\rho$, with $\rho > 0$. The compensator of μ can be written as $\lambda_t\theta(dz)dt$, where $\theta(dz)$ is a probability distribution on the measurable space R^+, and the compensated measure is denoted by $\tilde{\mu}(dz, dt) = \mu(dz, dt) - \lambda_t\theta(dz)dt$, In this paper we assume that θ is of exponential type $\theta(dz) := \nu\alpha e^{-\alpha z}dz$. The intensity process reads:

$$\lambda_t := \lambda_0 + \beta \int_0^t (\underline{\lambda} - \lambda_s)\, ds + \int_0^t \int_{\mathbb{R}^+} z\, \mu(ds, dz), \tag{1}$$

where $\underline{\lambda} < \lambda_0$ and β are positive constants. From now on, we require that the following integrability assumption holds:

Assumption 1 *The following condition holds:*

$$\widetilde{\beta} := \beta - \int_{\mathbb{R}^+} z\, \theta(dz) > 0, \tag{2}$$

The last condition, with our assumption on the jump size distribution can be written as $\widetilde{\beta} := \beta - \frac{\nu}{\alpha} > 0$.

This condition could be considered as the usual condition of non-explosion for Hawkes process in a marked Hawkes framework, see Bernis et al. [9].

The stock price process $\{S_t\}_{t\in\mathbb{R}^+}$ is defined via its log-return process: $\{X_t\}_{t\in\mathbb{R}^+}$: $\{X_t\} = \ln\{S_t\}$. By assuming that both interest rate and dividends vanish, we can simply write $S_t = \exp(X_t)$, and $S_0 = \exp(X_0)$ denotes the value of S at time $t = 0$. We assume that X satisfies the following SDE:

$$dX_t = \left[\frac{\underline{\lambda}}{2} - \left(\frac{1}{2} + \int_{\mathbb{R}^+} \left(e^{-\rho z} - 1\right)\theta(dz)\right)\lambda_t\right] dt + \sqrt{\lambda_t - \underline{\lambda}}\, dW_t - \rho \int_{\mathbb{R}^+} z\, \mu(dt, dz),$$

which under our assumption on the jump size distribution becomes:

$$dX_t = -\left(\frac{1}{2}\sigma_t^2 - \gamma\lambda_t\right) dt + \sigma_t\, dW_t - \rho \int_{\mathbb{R}^+} z\, \mu(dt, dz), \tag{3}$$

where $\gamma = \frac{\nu\rho}{\rho+\alpha}$ and we have introduced the variance process $\sigma_t^2 = \lambda_t - \underline{\lambda}$

$$d\sigma_t^2 = -\beta\sigma_t^2 dt + \int_{\mathbb{R}^+} z\, \mu(dt, dz)\,. \tag{4}$$

Note that, thanks to the negative sign in front of the jump term and the domain of Z_i, jumps in the log-return process are negative and the compensator is positive. The natural filtration of the point process will be denoted by $\mathbb{F}^\mu := \{\mathcal{F}_t^\mu\}_{t\geq 0}$, while the natural filtration of (λ, S) will be denoted by $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$.

3.2 Existence and basic properties

In this subsection we prove existence and regularity of the solution of SDE (1, 3, 4), by using the tools of continuous state branching processes with immigration (CBI). We first write the evolution of the compensated version of the SDE satisfied by the triplet (λ, σ, X). We then characterize the evolution as a multi-dimensional exponential affine process where λ is an autonomous CBI. A by product of this characterization is that our model could be rewritten in the so-called Dawson and Li representation of CBI, see [13, 14, 39]. This last characterization provides access to the ergodic distribution of the process λ and σ. Under hypothesis 1, the process λ is exponential ergodic with parameter $\widetilde{\beta}$, see for instance Jiao et al. [28].

Lemma 1 (Compensated representation) *The triplet (λ, σ, X) satisfies the following SDE where the jump component is compensated.*

$$\begin{cases} d\lambda_t &= \widetilde{\beta}\left(\frac{\beta}{\widetilde{\beta}}\underline{\lambda} - \lambda_t\right)dt + \int_{\mathbb{R}^+} z\,\widetilde{\mu}(dt, dz) \\ d\sigma_t^2 &= \widetilde{\beta}\left(\frac{\nu}{\alpha\widetilde{\beta}}\underline{\lambda} - \sigma_t^2\right)dt + \int_{\mathbb{R}^+} z\,\widetilde{\mu}(dt, dz) \\ dX_t &= -\left(\frac{1}{2}\sigma_t^2 - \frac{\rho}{\alpha}\gamma\lambda_t\right)dt + \sigma_t\,dW_t - \rho\int_{\mathbb{R}^+} z\,\widetilde{\mu}(dt, dz) \end{cases} \tag{5}$$

The proof is a straightforward computation, the crucial point is that the compensator is proportional to the process λ. In order to exhibit explicitly this dependency on λ, we resort to the general theory of continuous state branching processes with immigration and in particular the Dawson and Li representation of the SDE satisfied by (λ, σ, X). The next result shows that the the couple (λ, X) satisfies the Dawson-Li representation of CBI in an extended probability space.

Proposition 1 (Dawson-Li representation) *There exist an extended probability space where there exist a white noise $W(dt, du)$, defined on $(\mathbb{R}^+)^2$, and a compensated Poisson measure $\widetilde{N}(dt, du, dz)$, defined on $(\mathbb{R}^+)^3$, with compensator $dt\,du\,\theta(dz)$, such that the triplet (λ, σ, X) satisfies the SDE*

$$\begin{aligned} \lambda_t &= \lambda_0 + \widetilde{\beta}\int_0^t \left(\frac{\beta}{\widetilde{\beta}}\underline{\lambda} - \lambda_s\right)ds + \int_0^t\int_0^{\lambda_{s-}}\int_{\mathbb{R}^+} z\,\widetilde{N}(ds, du, dz) \\ \sigma_t^2 &= \sigma_0^2 + \widetilde{\beta}\int_0^t \left(\frac{\nu}{\alpha\widetilde{\beta}}\underline{\lambda} - \sigma_s^2\right)ds + \int_0^t\int_0^{\lambda_{s-}}\int_{\mathbb{R}^+} z\,\widetilde{N}(ds, du, dz) \\ X_t &= X_0 - \int_0^t \left(\frac{1}{2}\sigma_s^2 - \frac{\gamma\rho}{\alpha}\lambda_s\right)ds + \int_0^t\int_0^{\sigma_s^2} W(ds, du) - \rho\int_0^t\int_0^{\lambda_{s-}}\int_{\mathbb{R}^+} z\,\widetilde{N}(ds, du, dz)\,. \end{aligned}$$

Moreover, this SDE admits a unique solution, which coincides almost surely with the solution of the SDE (5).

The main advantage of this representation is to highlight that the mean reverting speed and level of the intensity between two jumps change due to the self-exciting property. Moreover one of the consequences of the previous result is the affine structure of the couple $(\lambda, \log S)$. Next result characterizes its Laplace transform. For a comprehensive introduction to affine processes see Duffie et al. [16], Filipovic [22] and for the one-factor case [21].

Proposition 2 (Laplace transform) *The couple (λ, X) is an exponential affine process. That is the Laplace transform of (λ, X) satisfies:*

$$\log \mathbb{E}\left[e^{uX_t+w\lambda_t}\right] = uX_0 + \lambda_t \psi_{u,w}(t) + \phi_{u,w}(t),$$

where the set of definition of (u,w) will be defined after; ψ and ϕ satisfy:

$$\begin{aligned}\psi'_{u,w}(t) &= R(u, \psi_{u,w}(t))\\ \phi'_{u,w}(t) &= F(u, \psi_{u,w}(t)),\end{aligned}$$

where

$$\begin{aligned} R(u,w) &:= \frac{1}{2}\left(u^2-u\right) + \gamma u - \beta w + \nu \frac{w-\rho u}{\alpha+\rho u - w} \\ F(u,w) &:= \underline{\lambda}\left[\beta w - \frac{1}{2}\left(u^2-u\right)\right] \end{aligned} \tag{6}$$

with the starting conditions $\psi_{u,w}(0) = w$ and $\phi_{u,w}(0) = 0$.

Proof. The main statement is a direct application of Proposition 1 in Duffie et al. [15]. The explicit form for the ODEs satisfied by ψ and ϕ is the result of a standard but tedious computation. □

For sake of readability, we will write again the generalized Riccati operator in the following form

$$R(u,w) = -\frac{\beta w^2 + p(u)w + uq(u)}{w - (\alpha+\rho u)} \tag{7}$$

where $p(u)$ and $q(u)$ are the two following quadratic polynomials:

$$\begin{aligned} p(u) &:= \nu + \frac{1}{2}u\,(1-u-2\gamma) - \beta(\alpha+\rho u) \\ q(u) &:= -\frac{1}{2}\left[(\alpha+\rho u)\,(1-u-2\gamma) + 2\rho\nu\right] \end{aligned}$$

The effective domain of R is $\{(u,w)\,|\,w > \alpha+\rho u\}$. It is easy to remark that $\mathbb{E}\left[e^{u\,X_t}\right] = \infty$ for all $u \leq -\alpha/\rho$ due to the exponential law of the jumps. In a similar way $\mathbb{E}\left[e^{u\,X_t + w\,\lambda_t}\right] = \infty$ if $w - \rho u < \alpha$ since the negative jumps on the asset are partially compensated by positive jumps on the volatility. We then extend the generalized Riccati operator R to $\mathbb{R}^2$ by convexity in agreement with Keller-Ressel [34], then $R(u,w) = \infty$ if $w < \alpha + \rho u$.

The next result is a direct consequence of the affine structure of the model. The proof of more general results and an explicit computation can be found in Keller-Ressel and Steiner [33, theorem 3.16], Li [40, theorem 3.20] and Jiao et al. [28, proposition 3.7].

Corollary 1 (ergodic distribution) *Under assumption 1, the intensity process λ is exponential ergodic and the moment generating function of the invariant distribution is given by*

$$\mathbb{E}\left[e^{w\lambda_\infty}\right] = e^{w\underline{\lambda}}\left(1-\widetilde{\beta}w\right)^{-\frac{\nu}{\beta}\underline{\lambda}}$$

for $w \in \left(-\infty, \widetilde{\beta}^{-1}\right)$. The process σ^2 is also exponential ergodic with a Gamma invariant law with scale parameter $\widetilde{\beta}$ and shape parameter $\frac{\nu}{\beta}\underline{\lambda}$.

3.3 Moments explosion

In order to perform our analysis on moments explosions we need to compute the function $w(u)$ such that $R(u, w(u)) = 0$. According to Lemma 3.2 of [34] this function is uniquely defined on a maximal interval I and verifies $w(0) = w(1) = 0$ and it is convex. Looking at equation (7) and introducing

$$\Delta(u) := p^2(u) - 4\beta u q(u)\,,$$

we deduce that the interval $I = \{u|\Delta(u) \geq 0\}$, we have easily that

$$w_\pm(u) := \frac{-p(u) \pm \sqrt{\Delta(u)}}{2\beta}$$

on I and we could write $R|_I$ as

$$R|_I(u,w) = -\beta\frac{\left(w - w_+(u)\right)\left(w - w_-(u)\right)}{w - (\alpha + \rho u)}$$

Moreover, in agreement with [34], by introducing:

$$\begin{aligned} f_+(u) &:= \sup\{w \geq 0 : F(u,w) < \infty\} = \infty \\ r_+(u) &:= \sup\{w \geq 0 : R(u,w) < \infty\} = \begin{cases} \infty & u > -\alpha/\rho \\ \alpha + \rho u & u \leq -\alpha/\rho \end{cases}, \end{aligned}$$

we have the following moment explosion time:

$$T_\star(u) = \begin{cases} \infty & if \quad u > -\alpha/\rho \\ 0 & if \quad u \leq -\alpha/\rho \end{cases}$$

Looking at ODE satisfied by ψ, we easily remark that it is non-linear but of first order and separable. Then, we can formally solve it as indicated in the following corollary, which proof is a straightforward but really tedious computation of an integral of a ratio of polynomials.

Corollary 2 (Explicit form of Laplace transform) *We have*

$$t = \int_w^{\psi_{u,w}(t)} \frac{dy}{R(u,y)} = L(\psi_{u,w}(t), u, w) + H(\psi_{u,w}(t), u, w)$$

where

$$
\begin{aligned}
L(\psi_{u,w}(t),u,w) &:= \frac{1}{2\beta}\log\left|\frac{\beta\,\psi^2_{u,w}(t)+p(u)\,\psi_{u,w}(t)+uq(u)}{\beta w^2+p(u)w+uq(u)}\right| \\
H(\psi_{u,w}(t),u,w) &:= \begin{cases} \frac{[p(u)+2\beta(\alpha+\rho u)]\sqrt{\Delta(u)}}{2\beta^3}\log\left|\frac{\left(\psi_{u,w}(t)-w_+(u)\right)\left(w-w_-(u)\right)}{\left(w-w_+(u)\right)\left(\psi_{u,w}(t)-w_-(u)\right)}\right| & \Delta(u)\geq 0 \\ \frac{p(u)+2\beta(\alpha+\rho u)}{\beta\sqrt{-\Delta(u)}}\left\{\arctan\left[\frac{2\beta}{\sqrt{-\Delta(u)}}\left[\psi_{u,w}(t)+\frac{p(u)}{2\beta}\right]\right]+\right. & \textit{elsewhere} \\ \left.\qquad -\arctan\left[\frac{2\beta}{\sqrt{-\Delta(u)}}\left[w+\frac{p(u)}{2\beta}\right]\right]\right\} & . \end{cases}
\end{aligned}
$$

We remark in particular that the function H coincides, up to some constants, to the corresponding function of the Heston model. The function L, that does not appear in Heston case, is the logarithm of a quadratic function of ψ.

We resume, in the following remark, the main similarities and differences between Heston, Gamma-OU BN-S and our model in the point of view of modeling. Other differences will be founded in the next section devoted to vanilla options.

Remark 1 *By comparing Gamma-OU Hawkes, BN-S Gamma-OU and Heston volatility models, we could remark that*

Ergodic distribution of variance process: *The three models share the same ergodic distribution, that is of Gamma type. However, a main difference between Heston and Gamma-OU Hawkes is that the volatility process can not reach* 0*. This property could be easily deduced by the evolution* (1) *assuming* $\lambda_0 > \underline{\lambda}$*.*

Laplace transform: *The three models are exponential affine and the associated Riccati equation could be solved explicitly. In particular, the inverse of the joint Laplace transform of* (X_t, V_t) *is an explicit function containing only functions of* $\ln$ *and* $\arctan$ *type as in Heston model. Given the huge amount of literature devoted to numerical methods for option pricing based on Fourier transform inversion methods, see [11], [38], [19] we expect that at least a significant part of these methods can be adapted to the present framework of the Gamma-OU Hawkes. However, in section 4.2 we shall show that vanilla option pricing could be performed in a very fast way by simulating the Hawkes process and by applying a conditional Black Scholes formula in the same spirit of Hull and White [27] and Romano and Touzi [44]. Moreover, it will be not surprising that the term structure and the evolution in time of the implied volatility for intermediate and longer maturities will be similar to the one described by Heston model. However, for very short maturities, the implied volatility behavior is definitively more similar to that exhibited by the BN-S Gamma-OU.*

Moments explosion: *Gamma-OU Hawkes has not the undesirable property that moments of order higher than* 1 *can explode in finite time, see Andersen and Piterbarg [3]. This is a main difference from the Heston case and many other stochastic volatility models, see for instance inverse-Gamma OU in Nicolato et al. [42]. As long as the moment* *generating function of the jump size distribution exists, the moments of any order exist for all finite maturities. From this point of view, Gamma-OU Hawkes is more similar to BN-S Gamma-OU.*

Leverage effect: *It is guaranteed by the negative sign of the parameter ρ multiplying the jump driver as in the usual BN-S Gamma-OU.*

Volatility and jumps clusters: *This is the main difference between our model and the two other models. As a matter of fact, the Hawkes driver implies clusters by construction. That is, asset can experience turmoil periods with spikes and high volatility levels followed by very long periods with a persistence of flat volatility.*

3.4 Variance swap analysis

The next proposition, giving the Laplace transform and the expectation of the quadratic variation, is a direct application of Lemma 4.2 and 4.4 in Kallsen et al. [31].

Proposition 3 (Laplace transform of Quadratic variation) *Assuming $u \in \mathbb{C}$ and $\mathcal{RE}(u) \geq 0$, we have*

$$\mathbb{E}\left[e^{-u[X]_T}\middle|\mathcal{F}_t\right] = \exp\left\{\Psi_u(T-t)\lambda_t - \beta\underline{\lambda}\int_t^T \Psi_u(y-t)dy - u[X]_t\right\}$$

where Ψ is the unique solution of the following non-linear ordinary differential equation of the first order

$$\Psi'_u(t) = -\beta\Psi_u(t) - u + \nu\left[1 - \alpha\sqrt{\frac{\pi}{\rho u}}e^{\frac{(\Psi_u(t)-\alpha)^2}{4\rho u}}\mathcal{N}\left(\sqrt{\frac{1}{2\rho u}}\left(\Psi_u(t)-\alpha\right)\right)\right],$$

with $\Psi_u(0) = 0$ and where $\mathcal{N}$ denotes the cumulative distribution function of the standard Gaussian random variable. Moreover, under Assumption 1, we have

$$\mathbb{E}\Big[[X]_T\Big|\mathcal{F}_t\Big] = \Phi(T-t)\lambda_t + \beta\underline{\lambda}\int_t^T \Phi(y-t)dy + [X]_t$$

where $\Phi(s) := -\left.\frac{\partial\Psi_u(s)}{\partial u}\right|_{u=0}$

With this result at hand, we can see the influence of the clustering effect on the bracket of $\log(S)$, and, therefore, on the price of the variance swap. As this quantity is directly related to the VIX index, we see that the Hawkes volatility model can explain the clustering effects observed in Section 2.

By direct integration we obtain the following result.

Corollary 3 *The function* $\Psi_u(t)$ *admits an almost-closed form, that is* $\Psi_u(t) = g_u^{-1}(t)$ *where* g *reads*

$$g_u(x) \quad := \quad \int_0^x \left\{ \beta y + u + \nu \left[\alpha \sqrt{\frac{\pi}{\rho u}} e^{\frac{(y-\alpha)^2}{4\rho u}} \mathcal{N}\left(\frac{\alpha - y}{\sqrt{2\rho u}} \right) - 1 \right] \right\}^{-1} dy \, .$$

4 Change of probability and vanilla options

In this section, we investigate both the risk-neutral change of probability and the pricing of vanilla options, in the framework defined above. First, we present the change of probability from risk-neutral to historical probability through an Esscher type transform for self-exciting processes. This issue is not plain since we are not directly into the setting of Lévy processes. The main result is that the class of stochastic volatility model driven by Hawkes is stable under the Esscher type transform. Secondly, we investigate the form of the smile induced by the Hawkes volatility model, with a comparison to the BN-S model.

4.1 The Esscher type transform

One of the main advantages of the integral representation detailed in proposition 1 is to give rise to a natural extension of the Esscher transform in the jump clustering framework. The main result is detailed in the following proposition showing that the class of Gamma-OU models is stable under a self-exciting Esscher type change of probability, see [28] and [29] for similar results. We point out that the Esscher transform we are going to introduce it is not the the Esscher transform for the semimartingale S_t, as defined by Kallsen and Shiryaev [32], but the Esscher transform of the driving Hawkes process, strictly analogous to the transform defined by Nicolato and Venardos [41, eq. 3.14, pag.455]. For a critical presentation of the Esscher transform for BN-S models, see [26]. The big advantage of the class of measure change proposed is that it is structure preserving, because it preserves the model dynamics, i.e. the time evolution under the new measure is described by the same equations with (eventually) different parameters.

In this section, in order to avoid ambiguity, we add a superscript $\mathbb{P}$ or $\mathbb{Q}$ on the various quantities of (5) depending on the reference probability.

Proposition 4 (Self-exciting Esscher transform)
Let (λ, σ, X) *be as in Proposition 1 under the risk neutral probability* $\mathbb{Q}$*. Fix* $(\eta, \xi) \in \mathbb{R} \times (-\alpha^{\mathbb{Q}}, \infty)$ *and define*

$$U_t := \eta \int_0^t \int_0^{\sigma_s^2} W(ds, du) + \int_0^t \int_0^{\lambda_{s-}} \int_{\mathbb{R}^+} \left(e^{-\xi z} - 1 \right) \widetilde{N}(ds, du, dz)$$

Then the Doléans-Dade exponential $\mathcal{E}(U)$ *is a martingale and the probability measure* $\mathbb{P}$ *defined by* $d\mathbb{P}/d\mathbb{Q}|_{\mathcal{F}_t} := \mathcal{E}(U)_t$ *is equivalent to* $\mathbb{Q}$*. Moreover, under* $\mathbb{P}$*, the couple* (λ, X)

satisfies the evolution of exponential affine class, see equation (5) *with parameters*

$$\alpha^{\mathbb{P}} \;:=\; \alpha^{\mathbb{Q}} + \xi, \qquad \nu^{\mathbb{P}} := \nu^{\mathbb{Q}} \frac{\alpha^{\mathbb{Q}}}{\alpha^{\mathbb{Q}} + \xi} \qquad \widetilde{\beta}^{\mathbb{P}} := \widetilde{\beta}^{\mathbb{Q}} + \frac{\nu^{\mathbb{Q}}}{\alpha^{\mathbb{Q}}} \left[1 - \frac{\left(\alpha^{\mathbb{Q}}\right)^2}{\left(\alpha^{\mathbb{Q}} + \xi\right)^2} \right].$$

and the dynamics with respect to $\mathbb{P}$ *takes the following form:*

$$\begin{aligned} dX_t &= -\left[\left(\frac{1}{2} + \eta \right) \sigma_t^2 - \rho \nu^{\mathbb{Q}} \left(\frac{1}{\rho + \alpha^{\mathbb{Q}}} - \frac{\alpha^{\mathbb{Q}}}{(\alpha^{\mathbb{Q}} + \xi)^2} \right) \lambda_t \right] dt + \sigma_t dW_t^{\mathbb{P}} - \rho \int_{\mathbb{R}^+} z \, \widetilde{\mu}^{\mathbb{P}}(dt, dz) \\ d\lambda_t &= \widetilde{\beta}^{\mathbb{P}} \left(\frac{\beta}{\widetilde{\beta}^{\mathbb{P}}} \underline{\lambda} - \lambda_t \right) dt + \int_{\mathbb{R}^+} z \, \widetilde{\mu}^{\mathbb{P}}(dt, dz) \end{aligned}$$

Proof. First, we remark that $\int_{\mathbb{R}^+} e^{-\xi z} \, \theta^{\mathbb{P}}(dz) < \infty$, since $\xi \in (-\alpha^{\mathbb{Q}}, \infty)$. It is easy to show that the triplet (λ, X, Y), where $Y := \mathcal{E}(U)$, is Markovian and exponential affine by applying the same argument of Proposition 1. Thanks to the integrability property we can then apply Corollary 3.9 in Kallsen and Muhle-Karbe [30] and this will imply that Y is a martingale, that $\mathbb{P}$ exists and it is equivalent to $\mathbb{Q}$. We have easily that $dY_t = Y_{t-} dU_t$.

For the second statement, let $f \in C_b^2(\mathbb{R}^+ \times \mathbb{R})$, we apply Itō formula to $H_t := f(\lambda_t, X_t)\, Y_t$. Let's denote by f_λ (resp. f_X) the first derivative of f with respect to λ (resp. X). A standard but tedious computation gives

$$\begin{aligned} dH_t \;=\; & \text{Local Martingale } + Y_{t-} \Bigg\{ \frac{1}{2} f_{XX}(\lambda_t, X_t)\, \lambda(t) \\ & + f_\lambda(\lambda_{t-}, X_{t-}) \left[\left(\beta \underline{\lambda} - \widetilde{\beta}^{\mathbb{Q}} \lambda_{t-} \right) + \lambda_{t-} \int_{\mathbb{R}^+} z \left(e^{-\xi z} - 1 \right) \theta^{\mathbb{Q}}(dz) \right] \\ & + f_X(\lambda_{t-}, X_{t-}) \left[- \left(\frac{1}{2} + \eta \right) \sigma_t^2 + \rho \lambda_{t-} \int_{\mathbb{R}^+} z \left(e^{-\xi z} - 1 \right) \theta^{\mathbb{Q}}(dz) \right] \\ & + \lambda_{t-} \int_{\mathbb{R}^+} \Big[f(\lambda_{t-} + z, X_{t-} - \rho z) - f(\lambda_{t-}, X_{t-}) - z f_\lambda(\lambda_{t-}, X_{t-}) \\ & \qquad + \rho z f_X(\lambda_{t-}, X_{t-}) \Big] e^{-\xi z} \, \theta^{\mathbb{Q}}(dz) \Bigg\} dt \end{aligned}$$

By identifying the terms, we obtain the evolution of (λ, X) under $\mathbb{P}$. □

4.2 The pricing of vanilla options

From now, we consider the risk neutral probability. We assume moreover that the risk free rate is vanishing and no dividends are distributed to shareholders during the option lifetime. Since no ambiguity can arise, we shall not indicate the probability $\mathbb{Q}$ in the rest of the section. Accordingly, all the expected values are computed under $\mathbb{Q}$. The asset price can be written as follows:

$$S_t = S_0 \exp\left\{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds + \gamma\Lambda_t - \rho\int_0^t \int_{\mathbb{R}^+} z\,\mu(ds,dz)\right\}, \tag{8}$$

where $\Lambda_t = \int_0^t \lambda_s ds$ and S_0 denotes the initial value of the asset.

Let us consider a put option with strike K and maturity $T > 0$. We recall that the price of such an option, within the Black and Scholes framework reads:

$$\mathcal{P}(S_0, \sigma, K, T) = K\Phi(-d_2) - F\Phi(-d_1) \tag{9}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \frac{\sigma^2}{2}T\right],$$

and $d_2 = d_1 - \sigma\sqrt{T}$.

In the Gamma-OU Hawkes volatility model, we obtain the following results for the price of a vanilla put option. For the sake of simplicity, we give only the put option price at initial time. A similar result could be easily found for intermediate time.

Proposition 5 *Assume that S follows Equation (8). Then, the price of a put option, of strike K and maturity T, writes*

$$\pi^*(K,T) := \mathbb{E}\left[\mathcal{P}\left(S_0 e^{N_T}, \sqrt{T^{-1}\Lambda_T - \underline{\lambda}}, K, T\right)\right]$$

where $\mathcal{P}$ is the Black-Scholes price of a put given by Equation (9) and $\{N_t\}_{t\geq 0} = \gamma\Lambda_t - \rho\int_0^t \int_{\mathbb{R}^+} z\mu(ds,dz)$ is the compensated jump process.

Proof: The proof follows the main idea of Proposition 4.1 in Romano and Touzi [44]. As both jumps and marks are independent from the Brownian motion, we can take the conditional expectation of the pay-off with respect to the marked process. Equation (8) immediately yields that, given the path of the marked Hawkes process, S_T follows a log-normal law:

$$S_T = S_0 e^{N_T} \exp\left\{\int_0^T \sigma_t\, dW_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right\}.$$

Then, using the form of Equation (9), we have

$$\mathbb{E}\left[\left(K - S_T\right)_+ \mid \mathcal{F}_T^{\mu}\right] = \mathcal{P}\left(S_0 e^{N_T}, \sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt}, K, T\right).$$

And the result follows by computing the expectation. $\square$

4.3 Option on realized variance

In this subsection, we focus on the realized variance of the asset. In order to simplify the expressions, we assume that *both rates and dividends are equal to* 0. Fix an interval $[0,T]$ and introducing a partition $\Pi_N := \{t_i\}_{i=0,\dots,N}$ of $[0,T]$ and let

$$V_{\Pi_N} := \sum_{i=1}^{N}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 = \sum_{i=1}^{N}\left(X_{t_i}-X_{t_{i-1}}\right)^2$$

then the realized variance could be defined as the limit of V_{Π_N} when the mesh step of Π_N goes to zero. We recall that, up to a normalization constant, the expression of V_{Π_n} with a daily time step, is the variable part of the pay-off of a variance swap.

For a continuous paths SDE, the realized variance coincides with the integral of the square of volatility. In our framework we have the following result

Proposition 6 *The realized variance of Gamma-OU Hawkes volatility model reads*

$$V_{[t,T]} = \Lambda_T - \Lambda_t - (T-t)\underline{\lambda} + \rho^2\int_t^T\int_{\mathbb{R}^+} z^2\,\mu(ds,dz).$$

Moreover, we have the following variance swap forward rate on the interval $[T_1,T_2]$ *at time* $t<T_1$.

$$VS(\lambda_t,[T_1,T_2]) = \mathbb{E}\left[V_{[T_1,T_2]}|\mathcal{F}_t\right] = \left(1+\frac{2\rho^2\nu}{\alpha^2}\right)M(\lambda_t,T_1,T_2) - \underline{\lambda}(T_2-T_1)$$

where

$$M(\lambda_t,T_1,T_2) = \lambda_t\left[\frac{e^{-\widetilde{\beta}T_1}-e^{-\widetilde{\beta}T_2}}{\widetilde{\beta}}\right] + \frac{\beta\underline{\lambda}}{\widetilde{\beta}}(T_2-T_1) - \frac{\beta\underline{\lambda}}{\widetilde{\beta}}\left[\frac{e^{-\widetilde{\beta}T_1}-e^{-\widetilde{\beta}T_2}}{\widetilde{\beta}}\right]$$

Proof. The expression of $V_{[t,T]}$ denotes the quadratic variation of X. It stems from two parts: the integral with respect to the Brownian motion and the sum of the squares of the jumps. It yields

$$V_{[t,T]} = \int_t^T(\lambda_s-\underline{\lambda})ds + \rho^2\int_t^T\int_{\mathbb{R}^+} z^2\,\mu(ds,dz).$$

The derivation of the variance swap forward rate is based on a direct computation performed by using the property of the compensator:

$$\begin{aligned}
\mathbb{E}\left[V_{[T_1,T_2]}|\mathcal{F}_t\right] &= \mathbb{E}\left[\Lambda_{T_2}-\Lambda_{T_1}|\mathcal{F}_t\right] - \underline{\lambda}(T_2-T_1) + \mathbb{E}\left[\rho^2\int_{T_1}^{T_2}\int_{\mathbb{R}^+} z^2\,\mu(ds,dz)\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[\Lambda_{T_2}-\Lambda_{T_1}|\mathcal{F}_t\right] - \underline{\lambda}(T_2-T_1) + \rho^2\mathbb{E}\left[\int_{T_1}^{T_2}\int_{\mathbb{R}^+} z^2\theta(dz)\,\lambda_s ds\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[\Lambda_{T_2}-\Lambda_{T_1}|\mathcal{F}_t\right] - \underline{\lambda}(T_2-T_1) + \frac{2\nu\rho^2}{\alpha^2}\int_{T_1}^{T_2}\mathbb{E}\left[\lambda_s|\mathcal{F}_t\right]ds
\end{aligned}$$

The next step consists in calculating the conditional expectation of $\Lambda_{T_2} - \Lambda_{T_1}$. It can easily performed by noticing that $\mathbb{E}\left[\Lambda_{T_2} - \Lambda_{T_1} | \mathcal{F}_t\right] = \int_{T_1}^{T_2} \mathbb{E}\left[\lambda_s | \mathcal{F}_t\right] ds$. The derivation of the first order moment of λ is well-known: one needs to solve a first order linear differential equation. It yields $\mathbb{E}\left[\lambda_s | \mathcal{F}_t\right] = \lambda_t \, e^{-\widetilde{\beta}(s-t)} + \frac{\beta \underline{\lambda}}{\widetilde{\beta}} \left[1 - e^{-\widetilde{\beta}(s-t)}\right]$. Replacing into the previous formulas, we obtain the result. $\square$

From this result, we can see that the variance swap forward rate is equal to a deterministic part plus a term driven by the current level of the intensity λ. This term involves the clustering effect of the intensity. It is multiplied by the term $\left(1 + \frac{2\nu\rho^2}{\alpha^2}\right)$ which incorporates a part stemming from the second order moment of the jumps. This term is damped with a rate of decay $\widetilde{\beta}$. Therefore, this term exhibits a behaviour which is able to capture the occurrence of the numerous spikes observed on VIX and V2X indices, with clustering effects described in Section 2. A numerical illustration is provided in Section 5.

We focus now on variance swap options and in particular on the put option. That is an option to enter, at time T_1, into a variance swap with maturity T_2 and rate K. The pay-off of such product reads $\left(K - V_{[T_1,T_2]}\right)_+$. We have the following immediate result:

Proposition 7 *Let $t < T_1 < T_2$. The price, at time t, of a put option on a variance swap starting in T_1 and ending in T_2 is given by*

$$P_t = \mathbb{E}\left[\left(K - \Lambda_{T_2} + \Lambda_{T_1} + (T_2 - T_1)\underline{\lambda} - \rho^2 \int_{T_1}^{T_2} \int_{\mathbb{R}^+} z^2 \, \mu(ds, dz)\right)^+ \middle| \mathcal{F}_t\right].$$

5 Numerical applications

In this section we test numerically our model in order to illustrate the behaviour of contingent claims implied volatility. A first analysis deals with the calibration of a set of parameters able to reproduce the implied volatility observed in the market. In agreement with the market volatility observed in September 2018, we fix the following set of parameters:

$\underline{\lambda}$	α	ν	β	ρ	σ_0^2
3.9	90	1	0.02	6	0.05

These parameter values are very close to those proposed by Nicolato and Venardos [41]. Moreover, our numerical tests show that a calibration could be split into two steps, first fitting the Gamma-OU model as in [41] focusing mainly on the ATM skew and on the strikes lower than the ATM. The second step is to fit the right wing of implied volatility by trigging between the initial volatility σ_0 and the minimal intensity of the jumps arrival $\underline{\lambda}$. For instance, figure 3-(a) shows the implied volatility for three models, i.e. the Gamma-OU BN-S model with intensity $\lambda = 4$ and two different specifications of our model, the first one with parameters in agreement with the previous table, and the second one with

$\sigma_0^2 = 0.04$ and $\underline{\lambda} = 4.10$. We could remark that the three models exhibit similar shapes for strikes smaller or equal to the strike of the option at the money. Our model exhibits a more pronounced smile (convexity) compared to Gamma-OU. This convexity increases when the source of volatility moves from the Brownian component to the jump component like in the modified set of parameters. This result at first sight can look surprising since we

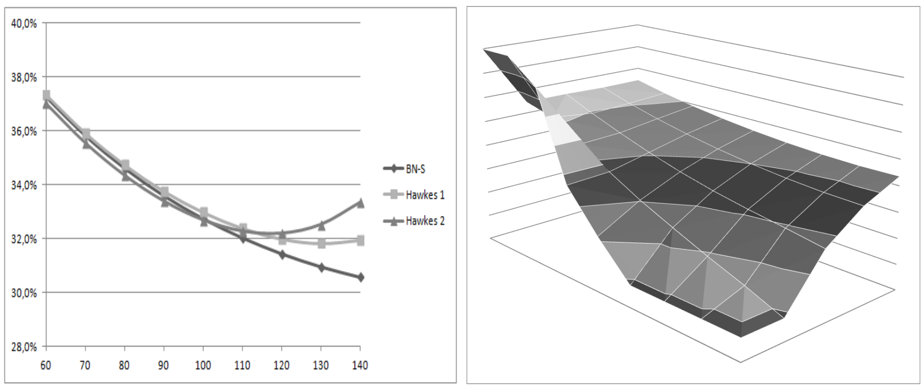

Figure 3: (a) Implied volatilities in the BN-S model and in two settings of the Γ-OU Hawkes volatility model. (b) Surface of implied volatility

have assumed only negative jumps in the log-return dynamics, but the main consequence of this choice is to push upward the right wing of the implied volatility, whereas the left one is fundamentally unchanged. The main effect of introducing self-exciting jumps can then be detected where the jumps are not acting. We explain this uncanny effect by looking at the process N introduced in Proposition (5). The process N is a martingale, so when the jump intensity λ increases due to a jump arrival, the compensator increases too. The final consequence is to reinforce the probability of extremal trajectories and then the pushing up of both wings, not only the left one.

We also point out a switch between the initial volatility σ_0^2 and the minimal intensity $\underline{\lambda}$. As a matter of fact, the implied volatility behavior is driven by a combination of the two previous processes. When the intensity of jumps increases, growing $\underline{\lambda}$ at the expense of σ_0^2, the probability of extremal events increases and so does its main impact on the right wing. Figure 3-(b) shows the implied surface of volatility for different strikes and maturities.

Therefore, the Hawkes volatility model proposed overcomes one of the main drawbacks of the Barndorff-Nielsen and Shephard model and more generally of all the single factor stochastic volatility models. Veraart and Veraart [46] pointed out how multi-factor stochastic volatility models outperform single-factor models, but they introduce a stochastic leverage in order to fit the implied volatility surface. So their model requires a third process characterised by its own parameters. The big advantage of the Hawkes volatility model is that smile and skew can be adjusted separately by fine tuning the self-exciting effect, by keeping the same numbers of parameters required by the BN-S model.

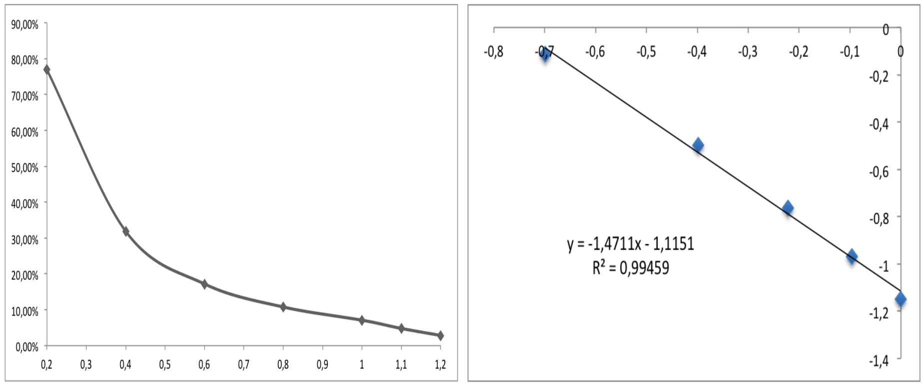

Figure 4: Time decay of ATM skew of implied volatility in the Γ-OU Hawkes volatility model in linear (a) and logarithmic (b) scales.

Another interesting behaviour to be investigated in modeling implied volatility is the time decay of the skew around the ATM values. This result is illustrated in Figure 4. The form can be perfectly fit by a power function. That is coherent with the behaviour observed in the market for maturities shorter than one year. We obtain the same power decay for the ATM slope also by changing the parameters of the model, and we could deduce that the power decay is intrinsic in our model.

We conclude by focusing on variance swaps and their options. We consider a put option written on one-month variance swap which starts in three months. Figure 5 shows the related implied variance swap. We remark that the implied volatility of variance swap is increasing with respect to the strike. This is coherent with the implied volatility observed in variance market, see Nicolato et al. [42]. Very few models in literature exhibit this property, for instance [42] focusing on stochastic volatility models with jumps, obtains that only Ornstein-Uhlenbeck with inverse gamma jumps (OU-IΓ) exhibits this upward-sloped implied volatility for variance options. However, it is easy to check that OU-IΓ model has positive moments for the variance process up to a threshold related the power parameter of the inverse gamma law. It could be then seen as a consequence of Lee [37] result for large strike.

Our model exhibits this property of upward-sloped implied volatility for variance options but all the positive moments of the variance processes are well-defined. We conclude that this behaviour is not related to the Lee result for large strike but requires another explanation. A possible mechanism is related to the self-exciting jump structure. Considering call options on variance, the probability that the variance process reaches a very high threshold, and then the related call option has an intrinsic value, increases moving from a Poisson to a Hawkes framework, since clusters will modify the probability distribution in a more significant way in comparison with the usual Poisson setting. As a consequence, the usual downward implied volatility could move upward, in agreement with market data, see for instance [42].

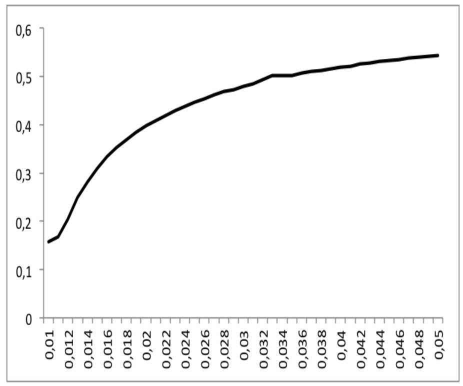

Figure 5: Implied volatility on variance swap options.

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