VARIANCE OF THE NUMBER OF ZEROES OF SHIFT-INVARIANT GAUSSIAN ANALYTIC FUNCTIONS

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Abstract. Following Wiener, we consider the zeroes of Gaussian analytic functions in a strip in the complex plane, with translation-invariant distribution. We show that the variance of the number of zeroes in a long horizontal rectangle $[0,T]\times[a,b]$ is asymptotically between cT and CT^2, with positive constants c and C. We also supply with conditions (in terms of the spectral measure) under which the variance asymptotically grows linearly with T, as a quadratic function of T, or has intermediate growth.

1. Introduction

Following Wiener, we consider Gaussian Analytic Functions in a strip with shift-invariant distribution. In a previous work [5] we gave a law of large numbers for the zeroes in a long horizontal rectangle $[0,T]\times[a,b]$ (Theorem A below), which extends a result of Wiener [11, chapter X]. Here we go further to study the variance of the number of zeroes in such a rectangle. In Theorems 1 and 2 we show that this number is asymptotically between cT and CT^2 with positive constants c and C, and give conditions (in terms of the spectral measure) for the asymptotics to be exactly linear or quadratic in T. In theorem 3 we give some conditions for intermediate variance. We begin with basic definitions.

1.1. Definitions.

A *Gaussian Analytic Function (GAF)* in the strip $D=D_\Delta=\{z:|\mathrm{Im}z|<\Delta\}$ is a random variable taking values in the space of analytic functions on D, so that for every $n\in\mathbb{N}$ and every $z_1,\dots,z_n\in D$ the vector $(f(z_1),\dots,f(z_n))$ has a mean zero complex Gaussian distribution.

A GAF in D is called *stationary*, if it is distribution-invariant with respect to all horizontal shifts, i.e., for any $t\in\mathbb{R}$, any $n\in\mathbb{N}$, and any $z_1,\dots,z_n\in D$,

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the random n-tuples

$$\big(f(z_1), \dots, f(z_n)\big) \qquad \text{and} \qquad \big(f(z_1+t), \dots, f(z_n+t)\big)$$

have the same distribution.

For a stationary GAF, the covariance kernel

$$K(z,w) = \mathbb{E}\{f(z)\overline{f(w)}\}$$

may be written as

$$K(z,w) = r(z - \overline{w}).$$

For $t \in \mathbb{R}$, the function $r(t)$ is positive-definite and continuous, and so it is the Fourier transform of some positive measure ρ on the real line:

$$r(t) = \mathcal{F}[\rho](x) = \int_{\mathbb{R}} e^{-2\pi i t\lambda} d\rho(\lambda).$$

Moreover, since $r(t)$ has an analytic continuation to the strip $D_{2\Delta}$, ρ must have a finite exponential moment:

$$\text{for each } |\Delta_1| < \Delta, \ \int_{-\infty}^{\infty} e^{2\pi\cdot 2\Delta_1|\lambda|} d\rho(\lambda) < \infty\,. \tag{1}$$

The measure ρ is called the *spectral measure* of f. A stationary GAF is *degenerate* if its spectral measure consists of exactly one atom.

For a holomorphic function f in a domain D, we denote by Z_f the zero-set of f (counted with multiplicities), and by n_f the *zero-counting measure*, i.e.,

$$\forall \varphi \in C_0(D): \qquad \int_D \varphi(z) dn_f(z) = \sum_{z\in Z_f} \varphi(z),$$

where $C_0(D)$ is the set of compactly supported continuous functions on D. We use the abbreviation $n_f(B) = \displaystyle\int_B dn_f(z)$ for the number of zeroes in a Borel subset $B \subset D$.

1.2. **Results.** First, we present a previous result which will serve as our starting point. This result can be viewed as a "law of large numbers" for the zeroes of stationary functions.

Theorem A. [5, Theorem 1] *Let f be a stationary non-degenerate GAF in the strip D_Δ, where $0 < \Delta \le \infty$. Let $\nu_{f,T}$ be the non-negative locally-finite random measure on $(-\Delta, \Delta)$ defined by*

$$\nu_{f,T}(Y) = \frac{1}{2T} n_f([-T,T) \times Y), \ \ Y \subset (-\Delta,\Delta) \textit{ measurable.}$$

Then:

(i) *Almost surely, the measures $\nu_{f,T}$ converge weakly and on every interval to a measure ν_f when $T \to \infty$.*

(ii) *The measure* ν_f *is not random (i.e.* $\operatorname{var} \nu_f = 0$*) if and only if the spectral measure* ρ_f *has no atoms.*

(iii) *If the measure* ν_f *is not random, then* $\nu_f(Y) = \mathbb{E} n_f([0,1] \times Y)$ *and it has density:*

$$L(y) = \frac{d}{dy}\left(\frac{\int_{-\infty}^{\infty} \lambda e^{4\pi y \lambda} d\rho(\lambda)}{\int_{-\infty}^{\infty} e^{4\pi y\lambda} d\rho(\lambda)}\right) = \frac{1}{4\pi}\frac{d^2}{dy^2}\log\left(r(2iy)\right).$$

In the above and in what follows, the term "density" means the Radon-Nikodym derivative w.r.t. the Lebegue measure on $\mathbb{R}$.

A natural question is, how big are the fluctuations of the number of zeroes in a long rectangle? More rigorously, define

$$R_T^{a,b} = [-T,T) \times [a,b], \;\; V_f^{a,b}(T) = \operatorname{var}\left[n_f(R_T^{a,b})\right],$$

where for a random variable X the variance is defined by

$$\operatorname{var}(X) = \mathbb{E}\left(X - \mathbb{E}X\right)^2.$$

We are interested at the asymptotic behavior of $V_f^{a,b}(T)$ as T approaches infinity. The next two theorems show that $V_f^{a,b}(T)$ is asymptotically bounded between cT and CT^2 for some $c, C > 0$, and give conditions under which each of the bounds is achieved. We begin by stating the upper bound result, a relatively easy consequence of Theorem A.

Theorem 1. *Let f be a non-degenerate stationary GAF in a strip D_Δ. Then for all $-\Delta < a < b < \Delta$ the limit*

$$L_2 = L_2(a,b) := \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T^2} \in [0,\infty)$$

exists. This limit is positive if and only if the spectral measure of f has a non-zero discrete component.

The lower bound result, which is our main result, is stated in the following theorem.

Theorem 2. *Let f be a non-degenerate stationary GAF in a strip D_Δ. Then for all $-\Delta < a < b < \Delta$ the limit*

$$L_1 = L_1(a,b) := \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T} \in (0,\infty]$$

exists. Moreover, the limit $L_1(a,b)$ is finite if ρ is absolutely continuous with density $d\rho(\lambda) = p(\lambda)d\lambda$, such that

(2) $$(1+\lambda^2)e^{2\pi\cdot 2y\lambda}p(\lambda) \in L^2(\mathbb{R}), \;\; \textit{for } y \in \{a,b\}.$$

Remark 1.1. Another form of condition (2) is the following: For $y \in \{2a, 2b\}$,

$$\int_{\mathbb{R}} |r(x+iy)|^2 dx, \quad \int_{\mathbb{R}} |r''(x+iy)|^2 dx < \infty.$$

This implies also that $\int_{\mathbb{R}} |r'(x+iy)|^2 dx < \infty$. Moreover, since the set $\{c : \ e^{2\pi \cdot c\lambda} p(\lambda) \in L^2(\mathbb{R})\}$ is convex, it implies the same condition for all $y \in [2a, 2b]$.

The next theorem deals with conditions under which $L_1(a, b)$ is infinite.

Theorem 3. *Let f be a non-degenerate stationary GAF in a strip D_Δ.*

(i) *Suppose $J \subset (-\Delta, \Delta)$ is a closed interval such that for every $y \in J$, the function $\lambda \mapsto (1+\lambda^2)e^{2\pi \cdot 2y\lambda} p(\lambda)$ does not belong to $L^2(\mathbb{R})$. Then for every $\alpha \in J$ the set $\{\beta \in J : \ L_1(\alpha, \beta) < \infty\}$ is at most finite.*

(ii) *The limit $L_1(a, b)$ is infinite for particular a, b if either ρ does not have density, or, if it has density p and for any two points $\lambda_1, \lambda_2 \in \mathbb{R}$ there exists intervals I_1, I_2 such that I_j contains λ_j $(j = 1, 2)$ and*

$$(1+\lambda)e^{2\pi \cdot 2y\lambda} p(\lambda) \notin L^2(\mathbb{R} \setminus (I_1 \cup I_2)), \tag{3}$$

for at least one of the values $y = a$ or $y = b$.

Remark 1.2. There is a gap between the conditions given for linear variance (in Theorem 2) and those for super-linear variance (in Theorem 3). For instance, the theorems do not decide about all the suitable pairs (a, b) in case the spectral measure has density $\frac{1}{\sqrt{|\lambda|}} 1\!\!1_{[-1,1]}(\lambda)$. On the other hand, we are ensured to have super-linear variance in case ρ has a singular part. If ρ has density $p \in L^1(\mathbb{R})$ which is bounded on any compact set, then $(1+\lambda^2)p(\lambda) \in L^2(\mathbb{R})$ implies asymptotically linear variance, and $(1+\lambda)p(\lambda) \notin L^2(\mathbb{R})$ implies asymptotically super-linear variance.

Remark 1.3. Minor changes to the developments in this paper may be made in order to prove analogous results regarding the increment of the argument of a stationary GAF f along a horizontal line. Namely, let $V^{a,a}(T)$ denote the variance of the increment of the argument of f along the line $[0, T] \times \{a\}$ (for some $-\Delta < a < \Delta$). Then:

- the limit $L_2(a) = \lim_{T\to\infty} \frac{V^{a,a}(T)}{T^2}$ exists, belongs to $[0, \infty)$, and is positive if and only if the spectral measure contains an atom.
- the limit $L_1(a) = \lim_{T\to\infty} \frac{V^{a,a}(T)}{T}$ exists, belongs to $(0, \infty]$, and is finite if ρ has density $p(\lambda)$ such that $(1+\lambda^2)e^{2\pi\cdot 2a\lambda} p(\lambda) \in L^2(\mathbb{R})$. Moreover, $L_1(a)$ is infinite if for any $\lambda_0 \in \mathbb{R}$ there is an interval I containing λ_0 such that the measure $(1+\lambda)e^{2\pi\cdot 2a\lambda} d\rho(\lambda)$ restricted to $\mathbb{R} \setminus I$ is not in $L^2(\mathbb{R})$.

In fact, the first item is essentially proved in this paper (Claim 9 below).

The rest of the paper is organized as follows: Theorem 1 concerning quadratic growth of variance is proved in Section 2, and is mainly a consequence of Theorem A. For theorems 2 and 3 we develop in Section 3 an asymptotic formula for $V_f^{a,b}(T)/T$ (Proposition 3.1 below). Then we prove Theorem 2 by analyzing this formula and using tools from harmonic analysis. We end by proving Theorem 3 in Section 5.

1.3. **Discussion.** We mention here some related results in the literature (though they do not seem to apply directly to our case). The question for real processes (not necessarily real-analytic) was treated by many authors. An asymptotic formula for the variance was given in Cramer and Leadbetter [3], but the rate of growth is not apparent from it. Cuzick [4] proved a Central Limit Theorem (CLT) for the number of zeroes, whose main condition is linear growth of the variance. Later, Slud [13], using stochastic integration methods he developed earlier with Chambers [2], proved that in case the spectral measure has density which is in $L^2(\mathbb{R})$, this condition is satisfied. It is interesting to note that the condition for linear variance in the present theorem (condition (2)) is the main assumption in the work by Slud for real (non-analytic) processes.

More recently, Granville and Wigman [7] studied the number of zeroes of a Gaussian trigonometric polynomial of large degree N in the interval $[0, 2\pi]$, and showed the variance of this number is linear in N. This work was extended to other level-lines by Azaïs and León [1].

Sodin and Tsirelson [14] and Nazarov and Sodin [10] studied fluctuations of the number of zeroes of a planar GAF (a special model which is invariant to plane isometries), proving linear growth of variance and a CLT for the zeroes in large balls (as the radius approaches infinity).

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2. Theorem 1: Quadratic Variance

Recall the notation $R_T = R_T^{a,b} = [-T, T] \times [a, b]$. From Theorem A we know that

$$\lim_{T\to\infty} \frac{n_f(R_T)}{T} = Z,$$

where Z is some random variable and the limit is in the almost sure sense. Moreover, $\operatorname{var} Z > 0$ if and only if the spectral measure of f contains an atom. Clearly

$$\operatorname{var}\left(\lim_{T\to\infty} \frac{n_f(R_T)}{T}\right) = \operatorname{var} Z$$

Theorem 1 would be proved if we could change the limit with the variance on the left-hand side. By dominant convergence, it is enough to find an integrable majorant for the tails of

$$X_T = \frac{n_f(R_T)}{T} \text{ and } X_T^2 = \frac{n_f(R_T)^2}{T^2}.$$

To this end we refer to an Offord-type estimate (Proposition 5.1 in the paper [5]), which provides exponential bounds on tails of X_T:

Proposition 2.1. *Let f be a stationary GAF in some horizontal strip, then using the notation above we have*

$$\exists C, c > 0: \sup_{T\geq 1} \mathbb{P}(X_T > s) < Ce^{-cs} = h(s).$$

We comment that the statement and proof provided in [5] are for a slightly different family of random functions, so-called symmetric GAFs; nonetheless the result for GAFs requires but mild modifications, and is generally easier.

We may then conclude that

$$\sup_{T\geq 1} \mathbb{P}(X_T^2 > s) < Ce^{-c\sqrt{s}} = h(\sqrt{s}).$$

Since both $h(s)$ and $h(\sqrt{s})$ are integrable on $\mathbb{R}$, we have the desired majorants. Exchanging limit and variance then yields:

$$\lim_{T\to\infty} \frac{\operatorname{var}\,(n_f(R_T))}{T^2} = \lim_{T\to\infty} \operatorname{var}\left(\frac{n_f(R_T)}{T}\right) = \operatorname{var}\left(\lim_{T\to\infty}\frac{n_f(R_T)}{T}\right) = \operatorname{var} Z,$$

and the result is proved.

3. An Asymptotic Formula for the Variance

This section is devoted to the derivation of a formula for the variance $V_f^{a,b}(T) = \operatorname{var} n_f([-T,T]\times[a,b])$ where T is large. We prove the following:

Proposition 3.1. *Let f be a stationary GAF in D_Δ with spectral measure ρ. Suppose ρ has no discrete component. Then for any $-\Delta < a < b < \Delta$, and any $T \in \mathbb{R}$, the series*

$$v^{a,b}(T) = \frac{1}{4\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\operatorname{sinc}^2\left(2\pi T(\lambda-\tau)\right) h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)$$

converges, and

$$\lim_{T\to\infty}\left(\frac{V^{a,b}(T)}{2T}-v^{a,b}(T)\right)=0.$$

Here ρ^{*k} *is the* k*-fold convolution of* ρ*,* $\operatorname{sinc}(x)=\frac{\sin x}{x}$*, and*

$$h_k^{a,b}(\lambda)=\left(l_k^a(\lambda)e^{2\pi a\lambda}-l_k^b(\lambda)e^{2\pi b\lambda}\right)^2,$$

where

$$l_k^y(\lambda)=\frac{\partial}{\partial y}\left(\frac{1}{r^k(2iy)}\right)+\frac{2\pi}{r^k(2iy)}\lambda,\ \textit{for } y\in(-\Delta,\Delta),k\in\mathbb{N}.$$

3.1. **Integrals on significant edges.** The boundary of the rectangle $R_T=[-T,T]\times[a,b]$ is composed of four segments $\partial R_T=\bigcup_{1\le i\le 4}I_j$ with induced orientation from the counter-clockwise orientation of ∂R_T, where $I_1=[-T,T]\times\{a\}$ and $I_3=[T,-T]\times\{b\}$. By the argument principle,

$$n_f(R_T)=\sum_{1\le i\le 4}\frac{1}{2\pi}\triangle_i^T\arg f,$$

where $\triangle_i^T\arg f$ is the increment of the argument of f along the segment I_i (a.s. f has no zeroes on the boundary of the rectangle R_T [1]).

Then, by the argument principle,

$$V_f^{a,b}(T)=\operatorname{var}\left[n_f(R_T)\right]=\frac{1}{4\pi^2}\sum_{1\le i,j\le 4}\operatorname{cov}\left(\triangle_i^T\arg f,\ \triangle_j^T\arg f\right),\tag{4}$$

where

$$\operatorname{cov}(X,Y)=\mathbb{E}[XY]-\mathbb{E}X\cdot\mathbb{E}Y.$$

Our first claim is that asymptotically when T is large, the terms involving the (short) vertical segments are negligible in this sum.

Claim 1. *As* $T\to\infty$*,*

$$V_f^{a,b}(T)=\frac{1}{4\pi^2}\sum_{i,j\in\{1,3\}}\operatorname{cov}\left(\triangle_i^T\arg f,\ \triangle_j^T\arg f\right)$$
$$+O\left(1+\sqrt{\operatorname{var}\left(\triangle_1^T\arg f\right)}+\sqrt{\left(\triangle_3^T\arg f\right)}\right).$$

[1] To see this, first notice that the distribution of $n_f(I_j)$ for $j=2,4$ (the number of zeroes in a "short" vertical segments) does not depend on T. If it were not a.s. zero, then $\mathbb{E}n_f(I_2)>0$. Now for any finite set of points $\{t_j\}_{j=1}^N\subset[0,1]$, we have $\mathbb{E}n_f([0,1]\times[a,b])\ge\sum_{j=1}^N\mathbb{E}n_f(\{t_j\}\times[a,b])=N\mathbb{E}n_f(I_2)$, yielding $\mathbb{E}n_f([0,1]\times[a,b])=\infty$ - which is false. For $j=1,3$, recall that since there are no atoms in the spectral measure, f is ergodic with respect to horizontal shifts (this is Fomin-Grenander-Maruyama Theorem, see explanation and references within [5]). This implies that each horizontal line (such as $L_a=\mathbb{R}\times\{a\}$) either a.s. contains a zero or a.s. contains no zeroes. If the former holds, then also $\mathbb{E}n_f([0,1]\times\{a\})>0$, and the measure ν_f from Theorem A has an atom at a - contradiction to part (iii) of that Theorem.

Proof. We demonstrate how to bound one of the terms in (4) involving a "short" vertical segment (corresponding, say, to $i = 2$). We have by stationarity:

$$\mathrm{var}\,(\triangle_2^T \arg f) = \mathrm{var}\,(\triangle_2^0 \arg f) =: c^2$$

Therefore by Cauchy-Shwartz,

$$\begin{aligned}\mathrm{cov}\ \left(\triangle_1^T \arg f,\ \triangle_2^T \arg f\right) &\leq \sqrt{\mathrm{var}\,(\triangle_1^T \arg f)}\sqrt{\mathrm{var}\,(\triangle_2^T \arg f)} \\ &= c\cdot\sqrt{\mathrm{var}\,(\triangle_1^T \arg f)}.\end{aligned}$$

□

We now give an alternative formulation of Claim 1. Using Cauchy-Riemann equations we write:

$$\triangle_1^T \arg f = \int_{-T}^{T}\left(\frac{\partial}{\partial x}\arg f(x+ia)\right)dx = -\int_{-T}^{T}\frac{\partial}{\partial a}\log|f(x+ia)|\,dx =: -X^a(T)$$

$$\triangle_3^T \arg f = -\int_{-T}^{T}\left(\frac{\partial}{\partial x}\arg f(x+ib)\right)dx = \int_{-T}^{T}\frac{\partial}{\partial b}\log|f(x+ib)|\,dx = X^b(T)$$

Denoting $C^{a,b}(T) = \mathrm{cov}\,(X^a(T), X^b(T))$ we may rewrite Claim 1 as

$$V_f^{a,b}(T) = \frac{1}{4\pi^2}\left(C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)\right) + O\left(1 + \sqrt{C^{a,a}(T)} + \sqrt{C^{b,b}(T)}\right),$$

and so we arrive at:

Claim 1a.

$$\frac{V^{a,b}(T)}{2T} = \frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2\cdot 2T} + O\left(\frac{\sqrt{C^{a,a}(T)} + \sqrt{C^{b,b}(T)}}{T}\right), T\to\infty.$$

Later on we shall prove that $\lim_{T\to\infty}\frac{\sqrt{C^{a,a}(T)}}{T} = 0$ if no atom is present in the spectral measure (Claim 9 below). This may be viewed as a one-dimensional counterpart of Theorem 1 (though the methods of proof are different). In the mean time, we turn to find an expression for $C^{a,b}(T)$, which will be refined through most of the section.

3.2. **Passing to covariance of logarithms.** Our first step is a technical change of order of operations.

Claim 2.

$$C^{a,b}(T) = \frac{\partial^2}{\partial a\ \partial b}\int_{-T}^{T}\int_{-T}^{T}\ \mathrm{cov}\ (\log|f(t+ia)|,\ \log|f(s+ib)|)\,dt\ ds.$$

We comment that the right-hand-side (RHS) of the equation contains a mixed partial derivative, so for $C^{a,a}(T)$ the computation is as follows: take the prescribed mixed derivative (as if $a\neq b$) and then substitute $b = a$.

Proof. Following the definition of $C^{a,b}(T)$, we should check that

$$\mathbb{E}\left\{\int_{-T}^{T} dt \int_{-T}^{T} ds \left(\frac{\partial}{\partial a}\log|f(t+ia)| \ \frac{\partial}{\partial b}\log|f(s+ib)|\right)\right\}$$
$$- \mathbb{E}\left\{\int_{-T}^{T} \frac{\partial}{\partial a}\log|f(t+ia)|\,dt\right\}\mathbb{E}\left\{\int_{-T}^{T} \frac{\partial}{\partial b}\log|f(s+ib)|\,ds\right\}$$

coincides with

$$\frac{\partial^2}{\partial a\,\partial b}\int_{-T}^{T}\int_{-T}^{T} \big[\mathbb{E}\{\log|f(t+ia)|\ \log|f(s+ib)|\}$$
$$- \mathbb{E}\log|f(t+ia)|\,\mathbb{E}\log|f(s+ib)|\ \big]dtds.$$

Notice that

$$\left|\frac{\partial}{\partial a}\log|f(x+ia)|\right| \le \left|\frac{f'(x+ia)}{f(x+ia)}\right|.$$

By the dominated convergence theorem and Fubini's theorem, we will be done if we show the following two statements:

(I)
$$\int_{-T}^{T} \mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\right| dt < \infty$$

(II)
$$\int_{-T}^{T}\int_{-T}^{T} \mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\frac{f'(s+ib)}{f(s+ib)}\right| dt\ ds < \infty$$

Let us first explain item (I): Let $z = t+ia$ be fixed. The vector $(f(z), f'(z))$ is jointly Gaussian, in fact we may write

$$f(z) = \rho f'(z) + Y(z)$$

where ρ is a number and $Y(z)$ is a Gaussian random variable independent of $f'(z)$. Therefore,

$$\big(f(z)|f'(z)\big) \sim \mathcal{N}_{\mathbb{C}}(0,\sigma^2) + \mu(f'(z));$$

that is, $f(z)$ conditioned on the value of $f'(z)$ is Gaussian, with mean depending on $f'(z)$ and variance not depending on it (equal to $\sigma^2 = \mathrm{var}\,(Y(z))$). The following is a straightforward computation.

Lemma 3.1. *Let $\sigma > 0$ and $\zeta \sim \mathcal{N}_{\mathbb{C}}(0,\sigma^2)$. Then there is a constant $C_0 > 0$ such that for any $\mu \in \mathbb{C}$, $\mathbb{E}\frac{1}{|\zeta+\mu|} < C_0$.*

Using this lemma, we have

$$\mathbb{E}\left|\frac{f'(z)}{f(z)}\right| = \mathbb{E}\,\mathbb{E}\left(\left|\frac{f'(z)}{f(z)}\right| \ \Big|\ f'(z)\right) \le \mathbb{E}\left(|f'(z)|\cdot C_0\right),$$

where the notation $\mathbb{E}\mathbb{E}(X|Y)$ for random variables X, Y means first taking the conditional expectation of X given Y (which results in a function of Y), then taking expectation of this function. Now (I) follows easily.

Similarly, for (II), notice that for any points $z, w \in D_\Delta$,

$$\big(f(z), f(w) | f'(z), f'(w)\big) \sim \mathcal{N}_{\mathbb{C}}^2(0, \Sigma) + \mu(f'(z), f'(w))$$

where Σ is a given covariance matrix, not depending on the values of $f'(z), f'(w)$. If $z \neq w$, then $\mathrm{rank}(\Sigma) = 2$.

Lemma 3.2. *Let Σ be a 2×2 complex covariance matrix,* $\mathrm{rank}(\Sigma) = 2$*, and let $(\zeta, \eta) \sim \mathcal{N}_{\mathbb{C}}(0, \Sigma)$. Then there exists $C > 0$ (depending continuously on the entries of Σ) such that for any $(\mu_1, \mu_2) \in \mathbb{C}^2$,*

$$\mathbb{E}\frac{1}{|\zeta + \mu_1||\eta + \mu_2|} < C.$$

The lemma concludes the proof of (II), since whenever $z \neq w$ (which is of full measure in the integration domain), we have:

$$\mathbb{E}\left|\frac{f'(z)f'(w)}{f(z)f(w)}\right| = \mathbb{E}\,\mathbb{E}\left(\left|\frac{f'(z)f'(w)}{f(z)f(w)}\right| \;\Big|\; f'(z), f'(w)\right) \le \mathbb{E}\left|f'(z)f'(w)\right| \cdot C(z,w),$$

where $C(z, w)$ is the constant derived from the lemma. It is also asserted that $C(z, w)$ is continuous in z, w, therefore the integral in (II) is finite.

We include a proof of the last lemma for completion. Denote by $g(u, v)$ $(u, v \in \mathbb{C})$ the density function of $\mathcal{N}(0, \Sigma)$. We have

$$\begin{aligned}
\mathbb{E}\frac{1}{|\zeta + \mu_1||\eta + \mu_2|} &= \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{1}{|u + \mu_1||v + \mu_2|}\; g(u, v) du\; dv \\
&= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^{\infty} dr_1 \int_0^{\infty} dr_2\; \left\{\frac{r_1 r_2}{r_1 r_2}\; g(-\mu_1 + r_1 e^{i\theta_1}, -\mu_2 + r_2 e^{i\theta_2})\right\} \\
&\le \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_1^{\infty} dr_1 \int_1^{\infty} dr_2\; \left\{r_1 r_2\; g(-\mu_1 + r_1 e^{i\theta_1}, -\mu_2 + r_2 e^{i\theta_2})\right\} \\
&\quad + \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^{1} dr_1 \int_0^{1} dr_2\; g(-\mu_1 + r_1 e^{i\theta_1}, -\mu_2 + r_2 e^{i\theta_2}) \\
&\le 1 + 4\pi^2 \max_{\mathbb{C}\times\mathbb{C}} g.
\end{aligned}$$

□

3.3. **Expansion in terms of the original covariance function.**

The covariance between logarithms of two Gaussians can be expressed as a power series, using the following claim.

Claim 3. *Let $\xi^*, \eta^* \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ be standard complex Gaussian random variables. Then*

$$\mathrm{cov}\left(\log|\xi^*|, \log|\eta^*|\right) = \frac{1}{4}\sum_{k=1}^{\infty} \frac{|\mathbb{E}\xi^*\overline{\eta^*}|^{2k}}{k^2}.$$

A proof is included in the book [8, Lemma 3.5.2], or in a slightly different language in the paper by Nazarov and Sodin [10, Lemma 2.2].

For any centered complex Gaussian random variable $\xi \sim \mathcal{N}_{\mathbb{C}}(0,\sigma^2)$ we may write $\xi = \sigma\xi^*$ where $\xi^* \sim N_{\mathbb{C}}(0,1)$, and thus get

$$\log|\xi| - \mathbb{E}\log|\xi| = \log|\xi^*| - \mathbb{E}\log|\xi^*|.$$

Therefore Claim 3 implies that for any centered complex Gaussians ξ and η we have:

$$\operatorname{cov}(\log|\xi|, \log|\eta|) = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{|\mathbb{E}(\xi\overline{\eta})|^2}{\mathbb{E}|\xi|^2\mathbb{E}|\eta|^2}\right)^k.$$

We now apply this formula for $\xi = f(t+ia)$ and $\eta = f(s+ib)$: By stationarity and our notation, we have

$$\frac{|\mathbb{E}(f(t+ia)\overline{f(s+ib)})|^2}{\mathbb{E}|f(t+ia)|^2\,\mathbb{E}|f(s+ib)|^2} = \frac{|r(t-s+ia+ib)|^2}{r(2ia)\,r(2ib)} =: q(t-s,a,b),$$

so that Claim 2 gives:

$$\begin{aligned} C^{a,b}(T) &= \frac{1}{4}\int_{-T}^{T}dt\int_{-T}^{T}ds\,\sum_{k=1}^{\infty}\frac{\partial^2}{\partial a\,\partial b}\frac{1}{k^2}q(t-s,a,b)^k \\ &= \frac{1}{4}\frac{\partial^2}{\partial a\,\partial b}\int_{-T}^{T}dt\int_{-T}^{T}ds\,\sum_{k=1}^{\infty}\frac{1}{k^2}q(t-s,a,b)^k. \qquad (5)\end{aligned}$$

3.4. **Some properties of q.** We digress shortly to summarize some properties of q, which we will use later in our proofs. In the following, when we do not specify the variables we mean the statements holds on all the domain of definition. We use the subscript notation for partial derivatives (such as q_a for $\frac{\partial}{\partial a}q$).

Claim 4. *The function*

$$q(x,a,b) = \frac{|r(x+ia+ib)|^2}{r(2ia)\,r(2ib)} \qquad (6)$$

is well-defined, infinitely differentiable on $\mathbb{R}\times(-\Delta,\Delta)^2$, and satisfies the following properties:

1. $q(x,y_1,y_2) \in [0,1]$.
   $q(x,y_1,y_2) = 1$ *if and only if* $(x=0$ *and* $y_1 = y_2)$.
2. $\sup_{|x|\geq 1} q(x,y_1,y_2) < 1$, *and if $y_1 \neq y_2$ also* $\sup_{x\in\mathbb{R}} q(x,y_1,y_2) < 1$.
3. *For fixed y_1 and y_2 let $g_{y_1,y_2}(x)$ be one of the functions q, q_a, q_b, q_{ab} evaluated on the line $\{(x,y_1,y_2) : x\in\mathbb{R}\}$. Then $g_{y_1,y_2} \in L^\infty \cap C_0(\mathbb{R})$ (i.e., is bounded and tends to zero as $x\to\pm\infty$). If condition* (2) *holds, then for any $y_1, y_2 \in [a,b]$ we have also $g_{y_1,y_2} \in L^1(\mathbb{R})$.*
4. $q_a(0,t,t) = 0$, *for any $t\in(-\Delta,\Delta)$.*

Proof. Since $r(2iy) > 0$ for all $y \in \mathbb{R}$, the function q is indeed well-defined; differentiability follows from that of $r(z)$.

For item 1, notice that

$$q(x,a,b) = \frac{\left(\int e^{2\pi(a+b)\lambda} e^{-2\pi i x\lambda} d\rho(\lambda)\right)^2}{\int e^{2\pi\cdot 2a\lambda} d\rho(\lambda) \int e^{2\pi\cdot 2a\lambda} d\rho(\lambda)}$$

and so, by Cauchy-Schwartz, is in $[0,1]$. Equality $q(x,a,b) = 1$ holds only if the function $\lambda \mapsto e^{2\pi\cdot a\lambda} e^{-2\pi i x\lambda}$ is a constant times the function $\lambda \mapsto e^{2\pi\cdot b\lambda}$, ρ-a.e., but, if ρ is non-atomic, this is impossible unless $x = 0$ and $a = b$.

Item 2 will be clear once we prove item 3 and recall the continuity of q.

For item 3, notice any one of the functions q, q_a, q_b, q_{ab} is the sum of summands of the form

$$C(a,b)\ r^{(j)}(x + ia + ib)\ r^{(m)}(-x + ia + ib), \tag{7}$$

where $0 \le j, m \le 2$ are integers. It is enough therefore to explain why $r^{(j)}(x+ia+ib)$ is bounded and approaches zero as $x \to \pm\infty$, for any integer $0 \le j \le 2$. Recall that

$$r^{(j)}(x + iy) = c_j \mathcal{F}_\lambda[\lambda^j e^{2\pi y\lambda}\, d\rho(\lambda)](x),$$

where c_j is some constant. As a function of x, this is a Fourier transform of a non-atomic measure, therefore has the desired properties.

If condition (2) holds, then $d\rho(\lambda) = p(\lambda)d\lambda$, and the function $\lambda \mapsto \lambda^j e^{2\pi(y_1+y_2)\lambda} p(\lambda)$ is in $L^2(\mathbb{R})$. Then, its Fourier transform $r^{(j)}(x+iy_1+iy_2)$ is also in $L^2(\mathbb{R})$, and each summand of the form (7) is in $L^1(\mathbb{R})$, as anticipated.

For item 4, notice that for all $x \in \mathbb{R}$ and all $a, b \in (-\Delta, \Delta)$ we have the symmetry $q(x,a,b) = q(x,b,a)$, and therefore for all $t \in \mathbb{R}$: $q_a(x,t,t) = q_b(x,t,t)$. On the other hand, for all $t \in (-\Delta, \Delta)$ it holds that $q(0,t,t) = 1$, so taking derivative by t we get $q_a(0,t,t) \cdot 1 + q_b(0,t,t) \cdot 1 = 0$. This proves the result. $\square$

3.5. **From double to single integral.** Next, we pass to a one-dimensional integral using a simple change of variables:

Claim 5. *For any function $Q \in L^1([-2T, 2T])$, the following equality holds:*

$$\int_{-T}^{T} \int_{-T}^{T} Q(t-s)\,dt\,ds = 2\int_{-2T}^{2T} (2T - |x|) Q(x)\,dx$$

Then, since $x \mapsto q(x,a,b)$ is continuous and takes values in $[0,1]$, the sum $Q(x) := \sum_{k\ge1} \frac{1}{k^2} q^k(x,a,b)$ converges uniformly to a continuous function. We may apply Claim 5 to equation (5) and get:

$$\frac{C^{a,b}(T)}{2T} = \frac{1}{2}\frac{\partial^2}{\partial a\ \partial b}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\sum_{k\geq 1}\frac{q^k(x,a,b)}{k^2}\ dx$$

Once again by uniform convergence of the series:

$$\frac{C^{a,b}(T)}{2T} = \frac{1}{2}\frac{\partial^2}{\partial a\ \partial b}\sum_{k\geq 1}\frac{1}{k^2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\ q^k(x,a,b)\ dx. \tag{8}$$

3.6. **Parseval's identity.** The next claim is a special case of Parseval's identity for measures (see Katznelson [9, VI.2.2]):

Claim 6. *For any finite measure γ on $\mathbb{R}$,*

$$\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\mathcal{F}[\gamma](x)dx = \int_{\mathbb{R}} 2T\text{sinc}^2(2\pi T\xi)d\gamma(\xi).$$

where $\text{sinc}(\xi) = \frac{\sin\xi}{\xi}$ *and* $\mathcal{F}[\gamma]$ *is the Fourier transform of* γ*.*

In order to apply this claim to simplify equation (8), we must find a finite measure $\gamma_k^{a,b}$ such that $\mathcal{F}[\gamma_k^{a,b}](x) = q(x,a,b)^k$. This is done in the next step.

3.7. **The search for an inverse Fourier transform.** For now, we keep a, b and k fixed. Our goal is to find a measure whose Fourier transform results in $q^k(x,a,b)$ (or, instead, in $|r(x+ia+ib)|^{2k}$). This measure is given in Claim 7 in the end of this subsection. In order to present it we must first discuss some definitions and relations between operations on measures.

Denote by $\mathcal{M}(\mathbb{R})$ the space of all finite measures on $\mathbb{R}$, similarly $\mathcal{M}^+(\mathbb{R})$ denotes all finite non-negative measures on $\mathbb{R}$. For two measure $\mu,\nu\in\mathcal{M}(\mathbb{R})$ the *convolution* $\mu*\nu\in\mathcal{M}(\mathbb{R})$ is a measure defined by:

$$\forall\varphi\in C_0(\mathbb{R}):\ (\mu*\nu)(\varphi) = \iint\varphi(\lambda+\tau)d\mu(\lambda)d\nu(\tau).$$

When both measures have density, this definition agrees with the standard convolution of functions. We write μ^{*k} for the iterated convolution of μ with itself k times.

Next recall that

$$r(z) = \int_{\mathbb{R}} e^{-2\pi iz\lambda}d\rho(\lambda) =: \mathcal{F}[\rho](z).$$

By properties of Fourier transform,

$$r^k(z) = \mathcal{F}[\rho^{*k}](z),$$

or, writing $z = x+iy$ we have

$$r^k(x+iy) = \int_{\mathbb{R}} e^{-2\pi ix\lambda}e^{2\pi y\lambda}d\rho^{*k}(\lambda). \tag{9}$$

This gives rise to the following notation: for a measure $\mu \in \mathcal{M}^+(\mathbb{R})$ having exponential moments up to 2Δ (i.e., obeying condition (1)), and a number $y \in (-2\Delta, 2\Delta)$, we define the *exponentially rescaled measure* $\mu_y \in \mathcal{M}^+(\mathbb{R})$ by

$$\forall \varphi \in C_0(\mathbb{R}):\ \mu_y(\varphi) = \mu(e^{2\pi y\lambda}\varphi(\lambda)) = \int_{\mathbb{R}} e^{2\pi y\lambda}\varphi(\lambda)d\mu(\lambda)$$

Observation. *For any $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and any $|y| < 2\Delta$,*

$$(\mu * \nu)_y = \mu_y * \nu_y.$$

Proof. for any test function $\varphi \in C_0(\mathbb{R})$ we have:

$$\begin{aligned}\int \varphi\ d(\mu_y * \nu_y) &= \iint \varphi(\lambda+\tau)\ d\mu_y(\lambda)d\nu_y(\tau)\\ &= \iint \varphi(\lambda+\tau)e^{2\pi y(\lambda+\tau)}\ d\mu(\lambda)d\nu(\tau) = \int \varphi\ d(\mu * \nu)_y\end{aligned}$$

□

As a corollary, we get that for any $|y| < 2\Delta$ and any $k \in \mathbb{N}$,

$$(\rho_y)^{*k} = (\rho^{*k})_y.$$

Therefore there will be no ambiguity in the notation ρ_y^{*k}.

Next, we define for $\mu \in \mathcal{M}(\mathbb{R})$ the *flipped measure* $\mathrm{flip}\{\mu\} \in \mathcal{M}(\mathbb{R})$ by:

$$\mathrm{flip}\{\mu\}(I) = \mu(-I) \text{ for any interval } I \subset \mathbb{R},$$

and the *cross-correlation* of measures $\mu, \nu \in \mathcal{M}(\mathbb{R})$ by:

$$\mu \star \nu := \mu * \mathrm{flip}\{\nu\}.$$

An alternative definition, via actions on test-functions, would be:

$$\forall \varphi \in C_0(\mathbb{R}):\ (\mu \star \nu)(\varphi) = \iint \varphi(\lambda - \tau)d\mu(\lambda)d\nu(\tau).$$

Notice that the cross-correlation operator is bi-linear, but not commutative.

Now relation (9) easily implies:

- $r^k(x+iy) = \mathcal{F}[\rho_y^{*k}](x)$
- $\overline{r^k(x+iy)} = \mathcal{F}[\rho_y^{*k}](-x) = \mathcal{F}[\mathrm{flip}\{\rho_y^{*k}\}](x)$,

which leads at last to the end of our investigation:

Claim 7. *For any $x \in \mathbb{R}$, $|y| < 2\Delta$ and $k \in \mathbb{N}$, we have:*

$$|r^k(x+iy)|^2 = \mathcal{F}\left[(\rho_y^{*k}) \star (\rho_y^{*k})\right](x).$$

This measure acts on a test-function $\varphi \in C_0(\mathbb{R})$ in the following way:

$$(\rho_y^{*k} \star \rho_y^{*k})(\varphi) = \iint \varphi(\lambda - \tau)e^{2\pi y(\lambda+\tau)}d\rho^{*k}(\lambda)d\rho^{*k}(\tau).$$

3.8. **Taking the double derivative.** Using Claims 6 and 7, we rewrite equation (8):

$$\begin{aligned}\frac{C^{a,b}(T)}{2T} &= \frac{\partial^2}{\partial a\ \partial b}\sum_{k\geq 1}\frac{1}{2k^2\ r^k(2ia)r^k(2ib)}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\mathcal{F}[\rho_{a+b}^{*k}\star\rho_{a+b}^{*k}](x)\ dx\\ &= \frac{\partial^2}{\partial a\ \partial b}\sum_{k\geq 1}\frac{1}{k^2\ r^k(2ia)r^k(2ib)}\int_{\mathbb{R}} T\mathrm{sinc}^2(2\pi T\xi)\ d\left(\rho_{a+b}^{*k}\star\rho_{a+b}^{*k}\right)(\xi)\\ &= \frac{\partial^2}{\partial a\ \partial b}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\mathrm{sinc}^2(2\pi T(\lambda-\tau))\ \frac{e^{2\pi(a+b)(\lambda+\tau)}}{r^k(2ia)r^k(2ib)}d\rho^{*k}(\lambda)d\rho^{*k}(\tau).\end{aligned}$$

Let us now look at the same expression, with the only change being that the derivative $\frac{\partial^2}{\partial a\ \partial b}$ has passed through the sum and through the integrals. The result is:

$$\begin{aligned}\frac{E^{a,b}(T)}{2T} = \sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} & T\mathrm{sinc}^2(2\pi T(\lambda-\tau))\\ &\cdot l_k^a(\lambda+\tau)l_k^b(\lambda+\tau)e^{2\pi(a+b)(\lambda+\tau)}d\rho^{*k}(\lambda)d\rho^{*k}(\tau),\end{aligned}$$

where $l_k^a(\lambda), l_k^b(\lambda)$ are linear functions in λ, given by

$$l_k^a(\lambda) = \frac{\partial}{\partial a}\left(\frac{1}{r^k(2ia)}\right) + \frac{2\pi}{r^k(2ia)}\lambda = \frac{2}{r^k(2ia)}\left(-ik\frac{r'(2ia)}{r(2ia)}+\pi\lambda\right).$$

Recalling Claim 1a, we calculate the expression which, we hope, is asymptotically equivalent to $\frac{V_f^{a,b}(T)}{2T}$:

$$\begin{aligned}&\frac{1}{4\pi^2}\cdot\frac{1}{2T}\left(E^{a,a}(T)-2E^{a,b}(T)+E^{a,b}(T)\right)\\ &\qquad = \sum_{k\geq 1}\frac{1}{4\pi^2k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\mathrm{sinc}^2(T(\lambda-\tau))h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)\\ &\qquad =: \sum_{k\geq 1}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi_k(\lambda,\tau)d\lambda d\tau, \qquad (10)\end{aligned}$$

where

$$h_k^{a,b}(\lambda) = \left(l_k^a(\lambda)e^{2\pi a\lambda} - l_k^b(\lambda)e^{2\pi b\lambda}\right)^2. \qquad (11)$$

This is the expression which appears in Proposition 3.1.

Now we justify the exchange of operations in two steps. In both, we regard $\frac{\partial^2}{\partial a\ \partial b}$ as a limit, and apply classical theorems in order to exchange it with the sum and the integral. We begin from the RHS of equation (10).

1. exchange of $\int_{\mathbb{R}}\int_{\mathbb{R}}$ and $\frac{\partial^2}{\partial a\ \partial b}$ (for a fixed $k\in\mathbb{N}$): we use the dominated convergence theorem. For given $k\in\mathbb{N}$, there is a $\Delta_1\in(0,\Delta)$ such that:

$$h_k^{a,b}(\lambda)\leq C_k e^{2\pi\cdot 2\Delta_1|\lambda|},$$

so that $|\psi_k(\lambda,\tau)| \le \widetilde{C}_k e^{2\pi\cdot 2\Delta_1|\lambda+\tau|}$. By condition (1) this is an integrable majorant with respect to $d\rho^{*k}(\lambda)d\rho^{*k}(\tau)$.

2. exchange of $\sum_{k\geq 1}$ and $\frac{\partial^2}{\partial a\ \partial b}$: by the monotone convergence theorem. After passing in the derivative, each term in the RHS of (10) is non-negative, therefore the exchange is justified.

We summarize the result in the following claim.

Claim 8.

$$\frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2\cdot 2T} = \sum_{k\geq 1}\frac{1}{4\pi^2 k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\mathrm{sinc}^2(2\pi T(\lambda-\tau))h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau),$$

where $h_k^{a,b}$ is given by (11)*.*

One more step is required in order to establish Proposition 3.1.

3.9. **The error term.** At last, we show that the error term in Claim 1a approaches zero as T tends to infinity.

Claim 9. *If ρ contains no atoms, then for any $a\in(-\Delta,\Delta)$:*

$$\lim_{T\to\infty}\frac{C^{a,a}(T)}{T^2} = 0.$$

Proof. Starting from equation (5) and applying Claim 5, we have:

$$\begin{aligned}\frac{C^{a,a}(T)}{4T^2} &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} Q(t-s,a,a)dt\,ds\\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)Q(x,a,a)\,dx,\end{aligned}$$

where

$$\begin{aligned}Q(x,a,b) &= \frac{\partial^2}{\partial a\ \partial b}\mathrm{cov}\left(\log|f(x+ia)|,\log|f(ib)|\right)\\ &= \frac{\partial^2}{\partial a\ \partial b}\sum_{k=1}^{\infty}\frac{q(x,a,b)^k}{k^2},\end{aligned}\tag{12}$$

and $q(x,a,b)$ is given by (6). (Note that by $Q(x,a,a)$ we mean the evaluation of the same mixed partial derivative at the point (x,a,a).)

It is thus enough to show that

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}|Q(x,a,a)|dx = 0,$$

or the stronger claim

$$\lim_{x\to\pm\infty} Q(x,a,a) = 0.\tag{13}$$

We would like to take derivative term-by-term in (12), at least in the region $|x| \geq 1$. For shortness, we do not write the variables (x, a, b), and use again the subscript notation for partial derivatives. We compute:

$$S_k^{a,b} := \frac{\partial^2}{\partial a \, \partial b} \left\{ q^k \right\} = \begin{cases} q_{ab}, & k = 1 \\ k(k-1)q^{k-2} q_a q_b, + k q^{k-1} q_{ab} & k > 1. \end{cases}$$

We see that the derivative of the k-th term in the sum (12), i.e., $\frac{S_k}{k^2}$, is bounded in absolute value by $q^{k-2}|q_a q_b| + \frac{1}{k} q^{k-1}|q_{ab}|$. In the region $|x| \geq 1$ we have $q(x, a, a) \leq \alpha < 1$ (part 2 of Claim 4), and since q_a, q_b and q_{ab} are bounded on the line $\{(x, a, a) : x \in \mathbb{R}\}$ (part 3 of that claim), we see that the change of derivative $\frac{\partial^2}{\partial a \, \partial b}$ with the sum in k in (12) is legal (by dominant convergence). Now,

$$\lim_{x \to \infty} |Q(x, a, a)| \leq \lim_{x \to \infty} \left(|q_{ab}| + (|q_a q_b| + |q_{ab}|) \frac{\alpha}{1 - \alpha} \right).$$

By part 3 of Claim 4 the limit on the RHS is zero, so (13) holds as anticipated. □

4. Theorem 2: Linear and Intermediate Variance

The proof is divided into two parts. First we prove the existence of the limit L_1 and its positivity, and later we prove that it is finite under condition (2).

4.1. **Existence and Positivity.** Using the formula for the variance obtained in Proposition 3.1, and recalling the functions $h_k^{a,b}$ are non-negative, we see that the limit L_1 exists and is in $[0, \infty]$. More effort is needed in order to establish that $L_1 > 0$. We begin with a simple bound arising from Proposition 3.1:

$$\begin{aligned} \liminf_{T \to \infty} \frac{V_f^{a,b}(T)}{2T} &= \frac{1}{4\pi^2} \liminf_{T \to \infty} \sum_{k \geq 1} \frac{1}{k^2} \int_{\mathbb{R}} \int_{\mathbb{R}} T \operatorname{sinc}^2(2\pi T(\lambda - \tau)) h_k^{a,b}(\lambda + \tau) d\rho^{*k}(\lambda) d\rho^{*k}(\tau) \\ &\geq \frac{1}{4\pi^2} \liminf_{T \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} T \operatorname{sinc}^2(2\pi T(\lambda - \tau)) \; h_1^{a,b}(\lambda + \tau) \; d\rho(\lambda) d\rho(\tau) \\ &\geq C_0 \liminf_{\varepsilon \to 0+} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\varepsilon} \mathbb{1}\{(\lambda, \tau) : |\lambda - \tau| < \varepsilon\} h_1^{a,b}(\lambda + \tau) \; d\rho(\lambda) d\rho(\tau), \end{aligned} \tag{14}$$

where $C_0 > 0$ is an absolute constant. The last step follows from ignoring the integration outside

$$\mathrm{Diag}_\varepsilon = \mathbb{1}\{(\lambda, \tau) : |\lambda - \tau| < \varepsilon\},$$

for $\varepsilon < \frac{1}{4T}$. Next we turn to investigate $h_1^{a,b}$. Recall its form is given in Proposition 3.1 or more recently in (11).

Claim 10. *The function $h_1^{a,b}$ has exactly two real zeroes.*

Proof. By the form of $h_1^{a,b}$, $h_1^{a,b}(\lambda) = 0$ if and only if

$$e^{2\pi(b-a)\lambda} = \frac{l_1^a(\lambda)}{l_1^b(\lambda)} = \frac{\frac{1}{r(2ia)}\left(\pi\lambda - i\frac{r'}{r}(2ia)\right)}{\frac{1}{r(2ib)}\left(\pi\lambda - i\frac{r'}{r}(2ib)\right)} = C \cdot \frac{\lambda - \psi(a)}{\lambda - \psi(b)},$$

where $C > 0$ is a positive constant and $\psi(y) = \frac{1}{2\pi}\frac{d}{dy}\left[\log r(2iy)\right]$. Since $y \mapsto \log r(2iy)$ is a convex function, for $a < b$ we have $\psi(a) < \psi(b)$. Therefore, $\lambda \mapsto C\,\frac{\lambda-\psi(a)}{\lambda-\psi(b)}$ is a strictly decreasing function, with a pole at $\psi(b)$ and with the same positive limit at $\pm\infty$. Thus, it crosses exactly twice the increasing exponential function $e^{2\pi(b-a)\lambda}$. $\square$

The next claim will enable us to bound $h_1^{a,b}$ from below, on most of the real line. Denote by $z_1, z_2 \in \mathbb{R}$ ($z_1 < z_2$) the two real zeroes of $h_1^{a,b}$ whose existence is guaranteed by Claim 10. We also use the notation $B(x,\delta)$ for the interval of radius $\delta > 0$ around $x \in \mathbb{R}$.

Claim 11. *For all $\delta_0 > 0$, there exists $c_\delta > 0$ such that for all $\lambda \in \mathbb{R} \setminus (B(z_1,\delta_0) \cup B(z_2,\delta_0))$:*

$$h_1^{a,b}(\lambda) > c_\delta(1+\lambda^2)\max(e^{2a\cdot 2\pi\lambda}, e^{2b\cdot 2\pi\lambda}).$$

Proof. Since the function $\frac{h_1^{a,b}(\lambda)}{(1+\lambda^2)e^{2a\cdot 2\pi\lambda}} = \left(\frac{l_1^a(\lambda) - l_1^b(\lambda)e^{2\pi(b-a)\lambda}}{\sqrt{1+\lambda^2}}\right)^2$ approaches strictly positive limits as $|\lambda| \to \infty$, there exist $M_a, c_a > 0$ such that

$$\forall |\lambda| \geq M_a : h_1^{a,b}(\lambda) \geq c_a(1+\lambda^2)e^{2a\cdot 2\pi\lambda}.$$

Similarly, there exist some $M_b, c_b > 0$ such that $\forall |\lambda| \geq M_b : h_1^{a,b}(\lambda) \geq c_b(1+\lambda^2)e^{2b\cdot 2\pi\lambda}$. Take $M = \max(M_a, M_b)$. Since $h(\lambda)$ attains a positive minimum on $[-M, M] \setminus (B(z_1,\delta_0) \cup B(z_2,\delta_0))$, there exists some $c > 0$ such that for all λ in this set, $h(\lambda) \geq c(1+\lambda^2)\max(e^{2a\cdot 2\pi\lambda}, e^{2b\cdot 2\pi\lambda})$. Choosing now $c_\delta = \min(c, c_a, c_b)$ will yield the result. $\square$

The next claim is a slight modification of the previous one, in order to fit our specific need.

Claim 12. *For every $\delta > 0$ there exist a set $F = F_\delta = \mathbb{R} \setminus (I_1 \cup I_2)$ such that I_j is an interval containing z_j and of length at most δ ($j = 1, 2$), $\rho(F) > 0$, and there exists $c_\delta > 0$ such that for all small enough ε,*

$$h(\lambda+\tau) \geq c_\delta(1+(\lambda+\tau)^2)\max\left(e^{2a\cdot 2\pi(\lambda+\tau)}, e^{2b\cdot 2\pi(\lambda+\tau)}\right),$$

for all $\lambda, \tau \in (F \times F) \cap Diag_\varepsilon$.

Proof. Choose $F = \mathbb{R} \setminus \left(B\left(\frac{z_1}{2}, \delta_0\right) \cup B\left(\frac{z_2}{2}, \delta_0\right)\right)$, where $\delta_0 \leq \delta$ is small enough so that $\rho(F) > 0$. Then, for $\varepsilon \leq \delta_0$ and $(\lambda, \tau) \in (F \times F) \cap \mathrm{Diag}_\varepsilon$, we have

$$|\lambda + \tau - z_j| \geq |2\tau - z_j| - |\lambda - \tau| \geq 2\delta_0 - \varepsilon \geq \delta_0.$$

Choosing the constant $c_\delta > 0$ which is the consequence of applying Claim 11 will end our proof. $\square$

Fix a parameter $\delta > 0$, and fix $F = F_\delta$ to be the set provided by Claim 12. Continuing from equation (14), we have

$$\begin{aligned}
\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} &\geq c_\delta \liminf_{\varepsilon\to 0} \iint_{F\times F} \frac{1}{2\varepsilon} 1\!\!1_{\mathrm{Diag}_\varepsilon}(\lambda, \tau)\ e^{2\pi\cdot 2a(\lambda+\tau)} d\rho(\lambda) d\rho(\tau) \\
&= c_\delta \liminf_{\varepsilon\to 0} \int_F \frac{1}{2\varepsilon} \rho_{2a}\left((\tau - \varepsilon, \tau + \varepsilon) \cap F\right) d\rho_{2a}(\tau) \\
&= c_\delta \liminf_{\varepsilon\to 0} \int_{\mathbb{R}} \frac{1}{2\varepsilon} \mu\left(\tau - \varepsilon, \tau + \varepsilon\right) d\mu(\tau),
\end{aligned}$$

where μ is the restriction of ρ_{2a} to F, i.e. $\mu(\varphi) = \rho_{2a}(1\!\!1_F \cdot \varphi)$ for any test-function φ. Notice that by the choice of F, $\mu(\mathbb{R}) = \rho_{2a}(F) > 0$. The next lemma characterizes the limit we are investigating.

Lemma 4.1. *Let $\mu \in \mathcal{M}^+(\mathbb{R})$ ($\mu \not\equiv 0$). Then the following limit exists (finite or infinite):*

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}} \frac{1}{2\varepsilon} \mu\left(\tau - \varepsilon, \tau + \varepsilon\right) d\mu(\tau) = \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x) dx.$$

Positivity of the lower bound which we gave for the limit L_1 is now clear.

Proof of Lemma 4.1. Denote $\varphi_\varepsilon = \frac{1}{2\varepsilon} 1\!\!1_{(-\varepsilon,\varepsilon)}$ for $\varepsilon > 0$. Rewriting the integral and using Parseval's identity, we get:

$$\begin{aligned}
I_\mu(\varepsilon) &:= \frac{1}{2\varepsilon} \int_{\mathbb{R}} \mu\left(\tau - \varepsilon, \tau + \varepsilon\right) d\mu(\tau) \\
&= \int_{\mathbb{R}} (\mu * \varphi_\varepsilon)(\tau) d\mu(\tau) \\
&= \int_{\mathbb{R}} (\mathcal{F}[\mu] \cdot \mathcal{F}[\varphi_\varepsilon])(x) \mathcal{F}[\mu](-x) dx \\
&= \int_{\mathbb{R}} \mathrm{sinc}(2\pi\varepsilon x) |\mathcal{F}[\mu]|^2(x) dx
\end{aligned}$$

Since $|\mathrm{sinc}(2\pi\varepsilon x)| \leq 1$, we have the upper bound $I_\mu(\varepsilon) \leq \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x) dx$.

For a lower bound we shall use the following general fact:

Observation. *For any $\psi_1, \psi_2 \in C_0(\mathbb{R})$ and $\mu \in \mathcal{M}^+(\mathbb{R})$,*

$$\int \psi_1\, d(\mu * \psi_2) = \int\ (\psi_1 * \mathrm{flip}\{\psi_2\})\, d\mu.$$

Proof.

$$\begin{aligned}\int \psi_1 \, d(\mu * \psi_2) &= \int \psi_1(x+y)\, d\mu(x)\psi_2(y)dy \\ &= \int \left(\int \psi_1(x+y)\, \mathrm{flip}\{\psi_2\}(-y)\, dy \right) d\mu(x) = \int (\psi_1 * \mathrm{flip}\{\psi_2\})(x) d\mu(x).\end{aligned}$$

□

Using the last observation and the fact that $\varphi_\varepsilon * \varphi_\varepsilon \le 2\varphi_{2\varepsilon}$ we get:

$$\int_{\mathbb{R}} (\mu * \varphi_\varepsilon) \; d(\mu * \varphi_\varepsilon) = \int_{\mathbb{R}} \mu * (\varphi_\varepsilon * \varphi_\varepsilon) d\mu \le 2 \int_{\mathbb{R}} \mu * \varphi_{2\varepsilon} \; d\mu = 2I_\mu(2\varepsilon)$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} (\mu * \varphi_\varepsilon) \; d(\mu * \varphi_\varepsilon) &= \int |\mathcal{F}[\mu * \varphi_\varepsilon]|^2 = \int |\mathcal{F}[\mu]|^2 \cdot \mathrm{sinc}^2(2\pi\varepsilon x) \; dx \\ &\ge \int_K |\mathcal{F}[\mu]|^2 \cdot \mathrm{sinc}^2(2\pi\varepsilon x) \; dx\end{aligned}$$

for any compact set $K \subset \mathbb{R}$. Since the limit $\lim\limits_{\varepsilon \to 0+} \mathrm{sinc}(2\pi\varepsilon x) = 1$ is uniform in $x \in K$, the last expression approaches $\int_K |\mathcal{F}[\mu]|^2$ as $\varepsilon \to 0+$. Thus, by choosing K and then $\varepsilon > 0$ properly, the lower bound may be made arbitrarily close to $\int_{\mathbb{R}} |\mathcal{F}[\mu]|^2$. This concludes the proof. □

4.2. **Linear Variance.** We recall that by the development in Section 3, we had (Claims 1a and 9):

$$\frac{V^{a,b}(T)}{2T} = \frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2 \cdot 2T} + o(1), \text{ as } T \to \infty,$$

where (relation (8))

$$\frac{C^{a,b}(T)}{2T} = \frac{1}{2} \frac{\partial^2}{\partial a \; \partial b} \sum_{k \ge 1} \frac{1}{k^2} \int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \; q^k(x,a,b) \; dx.$$

The function $q(x,a,b)$ was a normalized covariance (recall (6)), i.e.:

$$q(x,a,b) = \frac{\left| \mathbb{E}\left[f(x+ia)\overline{f(ib)} \right] \right|^2}{\mathbb{E}|f(ia)|^2 \, \mathbb{E}|f(ib)|^2} = \frac{|r(x+ia+ib)|^2}{r(2ia)\, r(2ib)}.$$

By a change of operations, we anticipate that:

(15)

$$\frac{V^{a,b}(T)}{2T} = \frac{1}{8\pi^2} \sum_{k \ge 1} \frac{1}{k^2} \int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \; \left(S_k^{a,a}(x) - 2S_k^{a,b}(x) + S_k^{b,b}(x) \right) \; dx + o(1),$$

where

$$S_k^{a,b}(x) := \frac{\partial^2}{\partial a \; \partial b} \left\{ q^k(x,a,b) \right\},$$

and $S_k^{a,a}(x)$ denotes the evaluation of the same mixed partial derivative at the point (x,a,a).

Indeed, by the justifications in Section 3.8, we may exchange the derivative $\frac{\partial^2}{\partial a\ \partial b}$ with the sum $\sum_{k\geq 1}$ when writing the total expression for the variance (by positivity of each term, which is apparent from there). Then we may exchange $\frac{\partial^2}{\partial a\ \partial b}$ with the integral $\int_{-2T}^{2T}$ as the resulting function $S_k^{a,b}(x)$ is continuous, hence in $L^1([-2T,2T])$ with respect to the variable x.

In fact, the following claim is much stronger then the last argument, and is the main tool to what follows.

Claim 13. *If condition* (2) *is satisfied, then for every $k\in\mathbb{N}$ the functions $S_k^{a,a}(x)$, $S_k^{a,b}(x)$ and $S_k^{b,b}(x)$ are in $L^1(\mathbb{R})$ with respect to the variable x. Moreover,*

$$\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}} S_k(x)dx \ \textit{converges},$$

with any of the three possible superscripts on the letter S.

Let us first see how to finish the proof of linear variance using this claim. Again, as we saw in section 3.8, each term of the series in the RHS of (15) is non-negative. Therefore, by the monotone convergence theorem:

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}=\frac{1}{8\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\lim_{T\to\infty}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\ (S_k^{a,a}(x)-2S_k^{a,b}(x)+S_k^{b,b}(x))\ dx.$$

The limit in each term can be computed using the following:

Claim 14. *If $Q\in L^1(\mathbb{R})$, then*

$$\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)Q(x)dx=\int_{\mathbb{R}}Q.$$

Proof. First, notice that the claim holds for $|Q|$, since

$$\int_{-\sqrt{T}}^{\sqrt{T}}\left(1-\frac{1}{\sqrt{T}}\right)|Q(x)|dx\leq\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)|Q(x)|dx\leq\int_{-T}^{T}|Q(x)|dx,$$

and both ends of the inequality approach the limit $\int_{\mathbb{R}}|Q|$. Now,

$$\left|\int_{-T}^{T}Q(x)dx-\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)Q(x)dx\right|\leq\int_{-T}^{T}\frac{|x|}{T}|Q(x)|dx\to 0,\ \text{as } T\to\infty.$$

□

We conclude that

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}=\frac{1}{8\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}(S_k^{a,a}(x)-2S_k^{a,b}(x)+S_k^{b,b}(x))\ dx,$$

which is finite by Claim 13.

Lastly, once we know the limit is finite we may obtain another formula of it using Proposition 3.1. We may take term-by-term limit of $T \to \infty$, again by monotone convergence, and get the form presented in Remark **??**:

$$\lim_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} = \frac{1}{8\pi^3} \sum_{k\geq 1} \frac{1}{k^2} \int_{\mathbb{R}} \left(p^{*k}(\lambda)\right)^2 h_k^{a,b}(2\lambda) d\lambda \in (0,\infty).$$

All that remains now is to prove Claim 13.

Proof of Claim 13. We recall that $S_k^{a,b}$ was computed in the proof of Claim 9, to be:

$$S_k = \frac{\partial^2}{\partial a\ \partial b}\left\{q^k\right\} = \begin{cases} q_{ab}, & k=1 \\ k(k-1)q^{k-2}q_a q_b, +kq^{k-1}q_{ab} & k>1. \end{cases} \tag{16}$$

Step 1: Let g be one of the functions q, q_a, q_b or q_{ab}. Then $g(x,a,a)$, $g(x,a,b)$ and $g(x,b,b)$ are all in $\left(L^1 \cap L^\infty\right)(\mathbb{R})$ with respect to the variable x.

This is, in fact, part 3 of Claim 4. This step ensures that $S_k^{a,a}$, $S_k^{a,b}$ and $S_k^{b,b}$ are in $L^1(\mathbb{R})$ with respect to x.

We turn now to prove the "moreover" part of the claim. We use (16) in order to rewrite the desired series:

$$\begin{aligned} &\sum_{k\geq 1} \frac{1}{k^2} \int_{\mathbb{R}} S_k(x) dx \\ &= \int_{\mathbb{R}} q_{ab}\ dx + \sum_{k\geq 2} \int_{\mathbb{R}} q^{k-2} q_a q_b\ dx + \sum_{k\geq 2} \frac{1}{k} \int_{\mathbb{R}} q^{k-2}\left(q q_{ab} - q_a q_b\right)\ dx. \end{aligned} \tag{17}$$

Once again, all functions are evaluated at (x,a,a), (x,a,b) or (x,b,b) and what follows holds for each of the three options. By step 1,

$$\int_{\mathbb{R}} |q_{ab}|\ dx < \infty, \quad \int_{\mathbb{R}} |q_a q_b|\ dx < \infty. \tag{18}$$

For the middle sum in (17), it is therefore enough to show that:

Step 2: The sum $\sum_{m\geq 1} \int_{\mathbb{R}} q^m q_a q_b\ dx$ converges.

Proof. We will show, in fact, that the positive series $\sum_{m\geq 1} \int_{\mathbb{R}} q^m |q_a q_b|\ dx$ converges.

First, in case we are evaluating at (x,a,b) $(a<b)$, our series converges due to (18) and the bound in part 2 of Claim 4. Now assume we are evaluating at (x,t,t) (where $t \in \{a,b\}$). As we deal with a positive series, it is enough to show that both

(I) $\sum_{m\geq 1} \int_{-1}^{1} q^m |q_a q_b|\ dx < \infty$, and
(II) $\sum_{m\geq 1} \int_{|x|\geq 1} q^m |q_a q_b|\ dx < \infty$.

Denote by $C = \sup_{x\in\mathbb{R}} |q_a q_b(x,t,t)| \in (0,\infty)$. The sum in (II) is bounded by

$$C\sum_{m\geq 1}\int_{|x|\geq 1} q^m(x,t,t)\ dx = C\int_{|x|\geq 1}\frac{q}{1-q}(x,t,t)\ dx \leq C'\int_{\mathbb{R}} q(x,t,t)\ dx,$$

where $C' \in (0,\infty)$ is another constant. C, C' and $\int_{\mathbb{R}} q(x,t,t)dx$ are all finite by Claim 4.

We turn to show (I). By parts 1 and 4 of Claim 4, the sum $\sum_{m\geq 1} q^m |q_a q_b|\ dx = \frac{|q_a q_b|}{1-q}$ is well-defined for all x (including $x=0$). By the monotone convergence theorem, item (I) is then equivalent to

$$\int_{-1}^{1}\frac{|q_a q_b|}{1-q}(x,t,t)\ dx < \infty,$$

which is indeed finite as an integral of a continuous function on $[-1,1]$. □

At last, only the right-most sum in (17) remains. Using the boundedness and integrability guaranteed in Step 1, it is enough to show:

Step 3: The sum $\sum_{m\geq 1}\frac{1}{m+2}\int_{\mathbb{R}} q^m\ dx$ converges.

Proof. We use a fact which is the basis for on of the standard proofs of the Central Limit Theorem (CLT). For completeness, we include a proof in the end of this subsection.

Lemma 4.2. *Let $g \in L^1(\mathbb{R})$ be a probability density, i.e., $g \geq 0$ and $\int_{\mathbb{R}} g = 1$. Suppose further that*

(a) $\int_{\mathbb{R}} |\lambda|^k g(\lambda) d\lambda < \infty$ for $k = 1,2,3$ and
(b) $\int_{\mathbb{R}} |\mathcal{F}[g](x)|^\nu\ dx < \infty$ for some $\nu \geq 1$.

Then there exists $C > 0$ such that for all $m \geq \nu$,

$$\int_{\mathbb{R}} |\mathcal{F}[g](x)|^m dx < \frac{C}{\sqrt{m}}.$$

We would like to apply the lemma to

$$g^{a,b}(\lambda) = \frac{e^{2\pi(a+b)\lambda}p(\lambda)}{r(ia+ib)}.$$

Notice that indeed this is probability measure, as by equation (9) with $k=1$:

$$\mathcal{F}[g^{a,b}](x) = \frac{r(x+ia+ib)}{r(ia+ib)},$$

and in particular $\mathcal{F}[g^{a,b}](0) = \int_{\mathbb{R}} g^{a,b} = 1$. This choice also obeys the extra integrability conditions in the lemma (as condition (1) implies (a) and (2) implies (b) with $\nu = 2$). We see now that

$$q(x,a,b) = \frac{r(ia+ib)^2}{r(2ia)r(2ib)}\cdot|\mathcal{F}[g^{a,b}](x)|^2 \leq |\mathcal{F}[g^{a,b}](x)|^2,$$

the last inequality following from the log-convexity of $y \mapsto r(iy)$. Similarly we define $g^{a,a}$ and have $q(x,a,a) = |\mathcal{F}[g^{a,a}](x)|^2$. Thus in all three cases of evaluation, using the lemma with the appropriate function g yields:

$$\sum_{m\geq 1} \frac{1}{m+2} \int_{\mathbb{R}} q^m \, dx \leq \sum_{m\geq 1} \frac{1}{m+2} \int_{\mathbb{R}} |\mathcal{F}[g](x)|^{2m} dx < C \sum_{m\geq} \frac{1}{(m+2)\sqrt{2m}} < \infty,$$

as required. $\square$

Combining all three steps with (17), we end the proof of Claim 13. $\square$

Our last debt now is to prove Lemma 4.2. The proof is a minor variation of the proof for CLT appearing in Feller [6, Chapter XV.5].

Proof of Lemma 4.2. Write $G(x) = \mathcal{F}[g](x)$. We may assume that $\int_{\mathbb{R}} \lambda g(\lambda) = 0$ (otherwise we shall consider, instead of g, the function $g_\mu(\lambda) = g(\lambda + \mu)$ where $\mu := \int_{\mathbb{R}} \lambda g(\lambda) d\lambda$. There is no penalty since $|\mathcal{F}[g_\mu](x)| = |\mathcal{F}[g](x)|$ for all $x \in \mathbb{R}$). By assumption (a), $G(x)$ is thrice differentiable, and by the above assumptions $G(0) = 1$ and $G'(0) = 0$.

To prove the lemma, it is enough to show that

$$\lim_{m\to\infty} \sqrt{m} \int_{\mathbb{R}} |G(x)|^m dx \text{ exists and is finite.}$$

Notice that $\sqrt{m} \int_{\mathbb{R}} |G(x)|^m dx = \int_{\mathbb{R}} |G(x/\sqrt{m})|^m dx$, and so it is enough to show that

$$\lim_{m\to\infty} \int_{\mathbb{R}} \left| \left| G\left(\frac{x}{\sqrt{m}}\right)\right|^m - e^{-\frac{\alpha x^2}{2}} \right| dx = 0, \tag{19}$$

for some value of $\alpha > 0$, which in fact is $\alpha := G''(0)$.

We shall achieve (19) by splitting the integral into three parts, and showing each could be made less than a given $\varepsilon > 0$ if $m \geq \nu$ is chosen large enough.

Fix $R > 0$ (to be determined later). By Taylor expansion,

$$G(x) = G(0) + xG'(0) + \frac{x^2}{2} G''(0) + o(x^2) = 1 + \frac{\alpha x^2}{2} + o(x^2),\ x \to 0 \tag{20}$$

and so $|G(x/\sqrt{m})|^m \to e^{-\alpha x^2/2}$ as $m \to \infty$, uniformly in $x \in [-R, R]$. Thus the integral in (19) computed on $[-R, R]$ converges to zero as $m \to \infty$.

From the expansion (20) we get

$$\exists \delta > 0\ \forall |x| < \delta :\ |G(x)| \leq e^{-\frac{\alpha x^2}{4}}.$$

Consider the integration in (19) for $R \leq |x| \leq \delta\sqrt{m}$. For such x we have $|G(x/\sqrt{m})|^m \leq e^{-\frac{\alpha x^2}{4}}$, and so the integrand is less than $2e^{-\frac{\alpha x^2}{4}}$. Choosing R so that $4 \int_R^\infty e^{-\frac{\alpha x^2}{4}} < \varepsilon$ will satisfy our needs.

Lastly, consider the integration on $\delta\sqrt{m} \le |x| < \infty$. By properties of Fourier transform, $\eta := \sup_{|x|\ge\delta} |G(x)| \in (0,1)$. Thus

$$\int_{|x|\ge\delta\sqrt{m}} \left| \left|G\left(\frac{x}{\sqrt{m}}\right)\right|^m - e^{-\frac{\alpha x^2}{2}} \right| dx \le \eta^{m-\nu} \sqrt{m} \int_{\mathbb{R}} |G|^\nu + \int_{|x|\ge\delta\sqrt{m}} e^{-\frac{\alpha x^2}{2}} < \varepsilon,$$

for m large enough. Here we have used condition (b).

□

5. Theorem 3: Super-linear variance

In this section we prove the two items of Theorem 3, in reverse order.

5.1. **Item** (ii)**: Super-linear variance for particular** a, b**.** Assume condition (3) holds for the particular a and b at hand. Fix a parameter $\delta > 0$, and let $F = F_\delta$ be the set provided by Claim 12. The premise ensures that, if δ is small enough, at least one of the measures $(1+\lambda)\rho_{2a}|_{F_\delta}$ and $(1+\lambda)\rho_{2b}|_{F_\delta}$ does not have L^2-density. WLOG assume it is the former. At first, assume also $\rho_{2a}|_F$ is not in L^2. Repeating the arguments of the Subsection 4.1 we get the lower bound

$$\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} \ge c_\delta \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx,$$

where $\mu = \rho_{2a}|_F$ and $c_\delta > 0$. The LHS is therefore infinite, and so $L_1 = \infty$.

We are left with the case that $\lambda\rho_{2a}|_{F_\delta}$ does not have L^2-density, but $\rho_{2a}|_{F_\delta}$ does (denote it by p_{2a}). The argument is similar. Continuing from (14) and employing Claim 12, we get

$$\begin{aligned}\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} &\ge c_\delta \liminf_{\varepsilon\to 0} \int_F \int_F \frac{1}{2\varepsilon} 1\!\!1_{(\tau-\varepsilon,\tau+\varepsilon)}(\lambda)(\lambda+\tau)^2 p_{2a}(\lambda)\ p_{2a}(\tau)d\lambda d\tau \\ &\ge c_\delta \cdot 4 \int_K \lambda^2 p_{2a}(\lambda)^2 d\lambda,\end{aligned}$$

where $K \subset F$ is compact. But, by our assumption, by choosing K properly the last bound can be made arbitrarily large, so that $\lim_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} = \infty$.

5.2. **Item** (i)**: Super-linear variance for almost all** a, b**.** Let ρ be such that the condition in item (i) holds. If ρ has a singular component, then the condition in item (ii) holds for all possible a, b and so $L_1(a,b) = \infty$ with no exceptions. Otherwise, ρ has density $p(\lambda)$. Define the set

$$E = \{(a,b) :\ a, b \in J,\ a < b,\ \text{the condition in item (ii) fails for } a, b\}.$$

If $E = \emptyset$, once again $L_1(a,b) = \infty$ for all $a, b \in J$ with no exceptions.

Assume then there is some $(a_0, b_0) \in E$. This means there exists λ_1, λ_2 such that for any pair of intervals I_1, I_2 such that $\lambda_j \in I_j$ $(j = 1, 2)$, both the functions $(1+\lambda^2)e^{2\pi\cdot 2a_0\lambda}p(\lambda)$ and $(1+\lambda^2)e^{2\pi\cdot 2b_0\lambda}p(\lambda)$ are in $L^2(\mathbb{R}\setminus(I_1\cup I_2))$, but at least one of them (WLOG, the former) is not in $L^2(\mathbb{R})$. Observe that the existence of such λ_1, λ_2 depends solely on $p(\lambda)$, and may therefore be regarded as independent of the point $(a_0, b_0) \in E$. Moreover, at least one among λ_1 and λ_2 (say, λ_1) is such that for any neighborhood I containing it, $p \notin L^2(I)$.

Suppose now $a, b \in E$ are such that

$$h_1^{a,b}(\lambda_1) > 0, \tag{21}$$

where $h_1^{a,b}(\lambda) = \left(l_1^a(\lambda)e^{2\pi a\lambda} - l_1^b(\lambda)e^{2\pi b\lambda}\right)^2$ is the function appearing in the the first term of our asymptotic formula, and in the lower bound in inequality (14). Recall $h_1^{a,b}$ is non-negative and has only two zeroes by Claim 10.

We may choose $\delta > 0$ smaller than the minimal distance between λ_1 and a zero of $h_1^{a,b}$, and then construct $F = F_\delta$ as in Claim 12. Certainly $\lambda_1 \in F_\delta$, and so the measure $\mu = \rho_{2a}|_{F_\delta}$ is not in $L^2(\mathbb{R})$ (it is even not in $L^2(I)$ for any neighborhood I of λ_1). Just as in subsection 4.1 we shall get

$$\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} \geq c_\delta \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx = \infty.$$

We end by showing that for a given point $\lambda_1 \in \mathbb{R}$ and a given $a \in J$, the set of $b \in J$ which do not obey (21) is finite. Indeed, this is the set

$$\{b \in J : \ h^{a,b}(\lambda_1) = 0\} = \{b \in J : \ \varphi(a) = \varphi(b)\}$$

where

$$\varphi(y) = e^{2\pi y\lambda_1} l_1^y(\lambda_1) = \frac{\partial}{\partial y}\left(\frac{e^{2\pi\lambda_1 y}}{r(2iy)}\right).$$

Suppose the desired set is not finite. Since φ is real-analytic, it must be constant on J. But then $r(2iy) = \frac{e^{2\pi\lambda_1 y}}{cy+d}$ for some $c, d \in \mathbb{R}$, and the corresponding spectral density would satisfy condition (2) for all relevant a, b. This contradiction ends the proof.

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