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Change-Point Detection in Long-Memory Processes

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We discuss some methods to test for possible changes in the parameters of a long-memory sequence. We obtain the limit distributions of the test statistics under the no-change null hypothesis. The consistency of the tests is also investigated.

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1. INTRODUCTION AND RESULTS

Let $\{\xi_k, -\infty < k < \infty\}$ be a sequence of independent, identically distributed random variables with $E\xi_0 = 0$ and $E\xi_0^2 = \tau^2$. We assume that the distribution of the observation X_k can be written in a parametric form by a finite-dimensional parameter $(\kappa_k, \boldsymbol{\lambda}_k)$, $\boldsymbol{\lambda}_k = (\lambda_{k,1}, \lambda_{k,2}, ..., \lambda_{k,p})$. Namely,

$$X_k = \kappa_k \sum_{-\infty < j < \infty} R(k-j, \boldsymbol{\lambda}_k)\, \xi_j.$$

We assume that $\sum_{-\infty < j < \infty} R^2(j, \boldsymbol{\lambda}_k) < \infty$. This assumption implies that $EX_k^2 < \infty$. We note that $\{X_k, -\infty < k < \infty\}$ is a linear process, which is used very often to model dependence between observations. In this paper we assume that the observations exhibit long-range dependence. Statistical modeling using long-range dependence has received considerable attention during the past 20 years. For surveys we refer the reader to Taqqu (1986) and Beran (1992, 1994).

As Beran and Terrin (1996) pointed out, for some time series the long-term dependence structure seems to change over time. An application to telecommunications engineering is discussed in Beran *et al.* (1995). Beran and Terrin (1996) suggested a procedure to test for the stability of the long–memory parameter. They were testing the null hypothesis $\lambda_{1,1} = \lambda_{2,1} = \cdots = \lambda_{n,1}$. The correct limit distribution of their test statistic was obtained by Horváth and Shao (1999).

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In this paper we wish to test the null hypothesis

$$H_0 : (\kappa_1, \boldsymbol{\lambda}_1) = (\kappa_2, \boldsymbol{\lambda}_2) = \cdots = (\kappa_n, \boldsymbol{\lambda}_n)$$

against the alternative

H_A : there is an integer k^*, $1 \leqslant k^* < n$, such that

$$(\kappa_1, \boldsymbol{\lambda}_1) = \cdots = (\kappa_{k^*}, \boldsymbol{\lambda}_{k^*}) \neq (\kappa_{k^*+1}, \boldsymbol{\lambda}_{k^*+1}) = \cdots = (\kappa_n, \boldsymbol{\lambda}_n).$$

Our procedure is based on the comparison of Whittle's estimates of the parameters. Let

$$D(x; \boldsymbol{\lambda}) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{\mathbf{i}tx} \frac{1}{f(t; \boldsymbol{\lambda})}\, dt,$$

where $f(t; \boldsymbol{\lambda}) = 2\pi\, |\hat{R}(t; \boldsymbol{\lambda})|^2$ and

$$R(t; \boldsymbol{\lambda}) = \int_{-\pi}^{\pi} e^{\mathbf{i}tx} \hat{R}(x; \boldsymbol{\lambda})\, dx.$$

If H_0 holds, then $\kappa_0^2 (2\pi)^{-2} f(t; \boldsymbol{\lambda}_0)$ is the spectral density of X_k, where $(\kappa_0, \boldsymbol{\lambda}_0)$ is the common value of the parameter vector under H_0. Next we define

$$\Lambda_k(\boldsymbol{\lambda}) = \frac{1}{k} \sum_{1 \leqslant i,\, j \leqslant k} D(i-j; \boldsymbol{\lambda})\, X_i X_j$$

and

$$\Lambda_k^*(\boldsymbol{\lambda}) = \frac{1}{n-k} \sum_{k < i,\, j \leqslant n} D(i-j; \boldsymbol{\lambda})\, X_i X_j.$$

We split the data into two subsets after X_k. We compute Whittle's estimates of the parameters from both subsets. Using $X_1, X_2, ..., X_k$, Whittle's estimates $(\hat{\kappa}_k, \hat{\boldsymbol{\lambda}}_k)$, $\hat{\boldsymbol{\lambda}}_k = (\hat{\lambda}_{k,1}, \hat{\lambda}_{k,2}, ..., \hat{\lambda}_{k,p})$ are the solutions of the equations

$$\sum_{1 \leqslant i,\, j \leqslant k} \frac{\partial}{\partial \boldsymbol{\lambda}} D(i-j; \hat{\boldsymbol{\lambda}}_k)\, X_i X_j = \mathbf{0}$$

and

$$\hat{\kappa}_k^2 = \Lambda_k(\hat{\boldsymbol{\lambda}}_k).$$

Similarly, the estimators $(\tilde{\kappa}_k, \tilde{\boldsymbol{\lambda}}_k)$, $\tilde{\boldsymbol{\lambda}}_k = (\tilde{\lambda}_{k,1}, \tilde{\lambda}_{k,2}, ..., \tilde{\lambda}_{k,p})$, are based on $X_{k+1}, X_{k+2}, ..., X_n$ and satisfy

$$\sum_{k<i,\, j\leqslant n} \frac{\partial}{\partial \boldsymbol{\lambda}} D(i-j; \tilde{\boldsymbol{\lambda}}_k)\, X_i X_j = \mathbf{0}$$

and

$$\tilde{\kappa}_k^2 = \Lambda_k^*(\tilde{\boldsymbol{\lambda}}_k).$$

We assume that the parameter set $\mathscr{K} \times \mathscr{L} \subset (0, \infty) \times R^p$ is an open, relatively compact set. We recall that the common value of the parameters under H_0 is denoted by $(\kappa_0, \boldsymbol{\lambda}_0)$. We assume the normalization $R(0; \boldsymbol{\lambda}) = 1$, $\boldsymbol{\lambda} \in \mathscr{L}$ or, equivalently,

$$\int_{-\pi}^{\pi} \log f(t; \boldsymbol{\lambda})\, dt = 0, \qquad \boldsymbol{\lambda} \in \mathscr{L}.$$

The following regularity conditions are taken from Giraitis and Surgailis (1990) (cf. also Fox and Taqqu (1986)): There exist $0 < \gamma = \gamma(\boldsymbol{\lambda}) < 1$ and $0 < C = C(\boldsymbol{\lambda}) < \infty$ such that

$$\int_{-\pi}^{\pi} \log f(t; \lambda)\, dt (\equiv 0) \text{ is twice differentiable in } \boldsymbol{\lambda}$$
$$\text{under the integral sign} \tag{1.1}$$

$$f(t; \boldsymbol{\lambda}) \text{ is continuous at all } (t; \boldsymbol{\lambda}),\ t \neq 0, \text{ and } |f(t; \boldsymbol{\lambda})| \leqslant C\, |t|^{-\gamma}, \tag{1.2}$$

$$1/f(t; \boldsymbol{\lambda}) \text{ is continuous at all } (t; \boldsymbol{\lambda}), \tag{1.3}$$

$$\frac{\partial}{\partial \lambda_i} \frac{1}{f(t; \boldsymbol{\lambda})} \text{ is continuous at all } (t; \boldsymbol{\lambda}), \qquad 1 \leqslant i \leqslant p, \text{ and}$$
$$\left| \frac{\partial}{\partial \lambda_i} \frac{1}{f(t; \boldsymbol{\lambda})} \right| \leqslant C\, |t|^{\gamma}, \qquad 1 \leqslant i \leqslant p, \tag{1.4}$$

$$\frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \frac{1}{f(t; \boldsymbol{\lambda})} \text{ is continuous at all } (t; \boldsymbol{\lambda}), \qquad 1 \leqslant i, j \leqslant p, \tag{1.5}$$

and

$$\left| \frac{\partial^2}{\partial \lambda_i \partial t} \frac{1}{f(t; \boldsymbol{\lambda})} \right| \leqslant C\, |t|^{\gamma-1}, \qquad 1 \leqslant i \leqslant p. \tag{1.6}$$

Let $\mathscr{W}(\boldsymbol{\lambda})$ be a $p \times p$ matrix with entries

$$w_{ij}(\boldsymbol{\lambda}) = \int_{-\pi}^{\pi} f(t; \boldsymbol{\lambda}) \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \frac{1}{f(t; \boldsymbol{\lambda})} \, dt.$$

We assume that

$$\mathscr{W}^{-1}(\boldsymbol{\lambda}_0) \text{ exists.} \tag{1.7}$$

Let

$$c_i(k) = \kappa_0 R(k; \boldsymbol{\lambda}_0) \frac{\partial}{\partial \lambda_i} D(0; \boldsymbol{\lambda}_0)$$
$$+ 2\kappa_0 \sum_{1 \leqslant j < \infty} R(k-j; \boldsymbol{\lambda}_0) \frac{\partial}{\partial \lambda_i} D(j; \boldsymbol{\lambda}_0), \qquad 1 \leqslant i \leqslant p. \tag{1.8}$$

We note that $c_i(k) = \kappa_0 (R * (\partial/\partial\lambda_i) D)(k)$, where $*$ denotes the convolution. We also assume that

$$R(k; \boldsymbol{\lambda}_0) = O(|k|^{-\alpha - 1/2}), \qquad \max_{1 \leqslant i \leqslant p} \left| \frac{\partial}{\partial \lambda_i} D(k; \boldsymbol{\lambda}_0) \right| = O(|k|^{-\theta - 1/2})$$
$$\max_{1 \leqslant i \leqslant p} |c_i(k)| = O(|k|^{-\beta - 1/2}) \qquad \text{with some} \quad \alpha > 0, \quad \beta > 0 \tag{1.9}$$
$$\text{and} \quad \theta > 0 \quad \text{satisfying} \quad \alpha + \beta > 1/2 \quad \text{and} \quad \theta + 2\alpha > 1,$$

and

$$E |\xi_0|^{4+\rho} < \infty \qquad \text{with some} \quad \rho > 0. \tag{1.10}$$

We note that condition (1.9) may be replaced by some smoothness condition on f.

First we show that $n^{1/2}(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]})$, $0 < t < 1$ converges in the q-metric. Let

$$FC_{0,1} = \{q : \inf_{\delta \leqslant t \leqslant 1-\delta} q(t) > 0 \text{ for all } 0 < \delta < 1/2, \ q \text{ is nondecreasing in a}$$
$$\text{neighborhood of 0 and nonincreasing in a neighborhood of } 1\}$$

and

$$I(q, c) = \int_0^1 \frac{1}{t(1-t)} \exp\left(-\frac{c q^2(t)}{t(1-t)} \right) dt.$$

We say that $\{\mathbf{\Gamma}(t), 0 \leqslant t \leqslant 1\}$ is a p-dimensional Brownian bridge with covariance matrix $4\pi\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$, if $\mathbf{\Gamma}(t)$ is Gaussian, $E\mathbf{\Gamma}(t) = \mathbf{0}$, and $E\mathbf{\Gamma}(t)\,\mathbf{\Gamma}^T(s) = (\min(t, s) - ts)\, 4\pi\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$, where $\mathbf{x}^{\mathbf{T}}$ denotes the transpose of $\mathbf{x}$. The maximum norm of vectors will be denoted by $\|\cdot\|$.

THEOREM 1.1. *We assume that* H_0, (1.1)–(1.7), (1.9), *and* (1.10) *are satisfied. If* $q \in FC_{0,1}$ *and* $I(q, c) < \infty$ *for all* $c > 0$, *then there is a sequence of p-dimensional Brownian bridges* $\{\mathbf{\Gamma}_n(t), 0 \leqslant t \leqslant 1\}$ *with covariance matrix* $4\pi\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$ *such that*

$$\sup_{0 \leqslant t \leqslant 1} \|n^{1/2} t(1-t)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]}) - \mathbf{\Gamma}_n(t)\| / q(t) = o_P(1).$$

Next we discuss two immediate consequences of the weighted approximation in Theorem 1.1. Since the function $q(t) = 1$ satisfies the conditions of Theorem 1.1, we get the weak convergence of $n^{1/2} t(1-t)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]})$ in $\mathscr{D}^p[0, 1]$, in the space of R^p-valued right-continuous functions on $[0, 1]$ with left-hand limits.

COROLLARY 1.1. *If* H_0, (1.1)–(1.7), (1.9), *and* (1.10) *are satisfied, then*

$$n^{1/2} t(1-t)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]}) \xrightarrow{\mathscr{D}^p[0,1]} \mathbf{\Gamma}(t),$$

where $\{\mathbf{\Gamma}(t), 0 \leqslant t \leqslant 1\}$ *is a p–dimensional Brownian bridge with covariance matrix* $4\pi\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$.

The next test is based on quadratic forms of Whittle's estimates. The proposed tests are analogues of the union-intersection and Wald's tests proposed by Hawkins (1989) to detect changes in the parameters of a linear model. For the asymptotic properties of the union-intersection (U–I) test in linear models we refer the reader to Horváth and Shao (1995). Let

$$Z_n(t) = \frac{1}{4\pi} n^{1/2} t(1-t) \{ (\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]})\, \mathscr{W}(\boldsymbol{\lambda}_0) (\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]})^T \}^{1/2}, \qquad 0 \leqslant t \leqslant 1.$$

Functionals of $Z_n(t)$ can be used for hypothesis testing. The supremum functional of $Z_n(t)$ gives a version of the union-intersection test. Theorem 1.1 immediately implies that $Z_n(t)$ converges weakly and the limit process is

$$M(t) = \left(\sum_{1 \leqslant j \leqslant p} B^2_{(j)}(t) \right)^{1/2}, \qquad 0 \leqslant t \leqslant 1,$$

where $\{B_{(1)}(t), 0 \leqslant t \leqslant 1\}, ..., \{B_{(p)}(t), 0 \leqslant t \leqslant 1\}$ are independent Brownian bridges.

Corollary 1.2. *We assume that* H_0, (1.1)–(1.7), (1.9), *and* (1.10) *are satisfied. If* $q \in FC_{0,1}$ *and* $I(q, c) < \infty$ *for all* $c > 0$, *then there is a sequence of stochastic processes* $\{M_n(t), 0 \leqslant t \leqslant 1\}$ *satisfying*

$$\{M_n(t), 0 \leqslant t \leqslant 1\} \overset{\mathscr{D}}{=} \{M(t), 0 \leqslant t \leqslant 1\} \qquad \textit{for each } n$$

and

$$\sup_{0 \leqslant t \leqslant 1} |Z_n(t) - M_n(t)|/q(t) = o_P(1).$$

We note that we can choose $q(t) = 1$ in Corollary 1.2 so the weighted approximation includes the weak convergence in $\mathscr{D}[0, 1]$.

Since $\mathscr{W}(\boldsymbol{\lambda}_0)$ is unknown, we must estimate it from the random sample if we wish to use our limit theorems for hypothesis testing. For any k, $1 \leqslant k < n$, we can use

$$\hat{\mathscr{W}}(k) = \frac{k}{n} \mathscr{W}(\hat{\boldsymbol{\lambda}}_k) + \frac{n-k}{n} \mathscr{W}(\tilde{\boldsymbol{\lambda}}_k)$$

to estimate $\mathscr{W}(\boldsymbol{\lambda}_0)$. We shall see in the proofs that $\hat{\mathscr{W}}(k)$ is weakly asymptotically consistent uniformly in k. In addition to the conditions of Theorem 1.1 we assume that $\mathscr{W}(\boldsymbol{\lambda})$ is continuous in a neighborhood of $\boldsymbol{\lambda}_0$. Corollary 1.1 yields that

$$\frac{1}{(4\pi)^{1/2}} n^{1/2} t(1-t)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]}) \hat{\mathscr{W}}^{1/2}(nt) \xrightarrow{\mathscr{D}[0,1]} \mathbf{B}(t), \tag{1.11}$$

where $\mathbf{B}(t) = (B_{(1)}(t), ..., B_{(p)}(t))$ and $B_{(1)}, ..., B_{(p)}$ are independent Brownian bridges. Similarly,

$$\hat{Z}_n(t) \xrightarrow{\mathscr{D}[0,1]} M(t), \tag{1.12}$$

where

$$\hat{Z}_n(t) = \frac{1}{4\pi} n^{1/2} t(1-t)\{(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]}) \hat{\mathscr{W}}(nt)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]})^T\}^{1/2}.$$

Next we discuss briefly the behavior of the processes in (1.11) and (1.12) under the alternative. We assume that $k^* = [n\tau^*]$ with some $0 < \tau^* < 1$, and the parameters before and after the change will be denoted by

$(\kappa^{(1)}, \boldsymbol{\lambda}^{(1)})$ and $(\kappa^{(2)}, \boldsymbol{\lambda}^{(2)})$, respectively. By Giraitis and Surgailis (1990) we have

$$\mathscr{W}(k^*) \xrightarrow{P} \tau^* \mathscr{W}(\boldsymbol{\lambda}^{(1)}) + (1-\tau^*)\, \mathscr{W}(\boldsymbol{\lambda}^{(2)}),$$

$$\hat{\boldsymbol{\lambda}}_{k^*} \xrightarrow{P} \boldsymbol{\lambda}^{(1)},$$

and

$$\tilde{\boldsymbol{\lambda}}_{k^*} \xrightarrow{P} \boldsymbol{\lambda}^{(2)}.$$

Hence

$$\begin{aligned} &k^*(n-k^*)\, n^{-2}(\hat{\boldsymbol{\lambda}}_{k^*} - \tilde{\boldsymbol{\lambda}}_{k^*})\, \hat{\mathscr{W}}^{1/2}(k^*) \\ &\quad \xrightarrow{P} \tau^*(1-\tau^*)(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})(\tau^* \mathscr{W}(\boldsymbol{\lambda}^{(1)}) + (1-\tau^*)\, \mathscr{W}(\boldsymbol{\lambda}^{(2)}))^{1/2}. \qquad (1.13) \end{aligned}$$

If $(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})(\mathscr{W}(\boldsymbol{\lambda}^{(1)}) + \mathscr{W}(\boldsymbol{\lambda}^{(2)}))(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})^T > 0$, then by (1.13) we have that

$$\sup_{0 \leqslant t \leqslant 1} \| n^{1/2} t(1-t)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]})\, \hat{\mathscr{W}}^{1/2}(nt) \| \xrightarrow{P} \infty \qquad (1.14)$$

and

$$\sup_{0 \leqslant t \leqslant 1} |\hat{Z}_n(t)| \xrightarrow{P} \infty, \qquad (1.15)$$

and the rate of convergence to ∞ in (1.14) and (1.15) is a least $n^{1/2}$ in probability. So we have the consistency of procedures based on (1.11) and (1.12) if $\boldsymbol{\lambda}^{(1)} \neq \boldsymbol{\lambda}^{(2)}$.

2. PROOFS

Let

$$\mathbf{U}_n(t) = \frac{2\pi}{\kappa_0^2} \left\{ \frac{1}{nt}\, \mathbf{Q}(nt) - \frac{1}{n(1-t)}\, \mathbf{Q}^*(nt, n) \right\} \mathscr{W}^{-1}(\boldsymbol{\lambda}_0),$$

where

$$\frac{1}{k}\, \mathbf{Q}(k) = \frac{\partial}{\partial \boldsymbol{\lambda}}\, \Lambda_k(\boldsymbol{\lambda}_0) - E\, \frac{\partial}{\partial \boldsymbol{\lambda}}\, \Lambda_k(\boldsymbol{\lambda}_0)$$

and

$$\frac{1}{n-k}\,\mathbf{Q}^*(k,n)=\frac{\partial}{\partial\boldsymbol{\lambda}}\,\Lambda_k^*(\boldsymbol{\lambda}_0)-E\,\frac{\partial}{\partial\boldsymbol{\lambda}}\,\Lambda_k^*(\boldsymbol{\lambda}_0).$$

LEMMA 2.1. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$\sup_{0\leqslant t\leqslant 1}\frac{n^{1/2}t(1-t)}{q(t)}\,\|(\hat{\boldsymbol{\lambda}}_{[nt]}-\tilde{\boldsymbol{\lambda}}_{[nt]})-\mathbf{U}_n(t)\|=o_P(1).$$

Proof. This is the first step in the proof of Theorem 2.2 in Horváth and Shao (1999).

Introducing

$$v_m(t)=\frac{\partial}{\partial\lambda_m}\,D(t;\boldsymbol{\lambda}_0),\qquad 1\leqslant m\leqslant p$$

we can write

$$Q_m(k)=\sum_{1\leqslant i,\,j\leqslant k} v_m(i-j)\{X_iX_j-EX_iX_j\}$$

and

$$Q_m^*(k,n)=\sum_{k<i,\,j\leqslant n} v_m(i-j)\{X_iX_j-EX_iX_j\},$$

where $\mathbf{Q}(k)=(Q_1(k),\,...,\,Q_p(k))$ and $\mathbf{Q}^*(k,n)=(Q_1^*(k,n),\,...,\,Q_p^*(k,n))$. According to Lemma 2.1 it is enough to study the asymptotic properties of the quadratic forms $\mathbf{Q}(k)$ and $\mathbf{Q}^*(k,n)$, $1\leqslant k<n$. The next lemma shows that these quadratic forms can be approximated with martingales. Let

$$d_m(k)=\sum_{1\leqslant j<\infty} R(j-k)\,c_m(j-k)$$

and

$$d_m(k,\ell)=\sum_{1\leqslant j<\infty}\{R(j-k)\,c_m(j-\ell)+R(j-\ell)\,c_m(j-k)\},$$

where $c_m(k)$ is defined in (1.8). Next we define $\mathbf{Y}(k)=(Y_1(k),\,...,\,Y_p(k))$ and $\mathbf{z}(k,n)=(z_1(k,n),\,...,\,z_p(k,n))$, where

$$Y_m(k)=d_m(k)(\xi_k^2-\tau^2)+\xi_k\sum_{1\leqslant\ell\leqslant k-1} d_m(k,\ell)\,\xi_\ell$$

and

$$z_m(k, n) = z_m(n+1-k)$$

with

$$z_m(k) = d_m(k)(\xi_k^2 - \tau^2) + \xi_k \sum_{k<\ell\leqslant n} d_m(k, \ell)\, \xi_\ell.$$

LEMMA 2.2. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$\max_{1\leqslant k\leqslant n} \|\mathbf{Q}(k) - \sum_{1\leqslant i\leqslant k} \mathbf{Y}(i)\|/k^{-\delta+1/2} = O_P(1), \tag{2.1}$$

$$\max_{1\leqslant k<n} \left\| \mathbf{Q}^*(k, n) - \sum_{1\leqslant i\leqslant n-k} \mathbf{z}(i, n) \right\| \Big/ (n-k)^{-\delta+1/2} = O_P(1) \tag{2.2}$$

and

$$\begin{aligned}\max_{1\leqslant k\leqslant n} \Bigg\| & \sum_{1\leqslant i\leqslant k} \mathbf{Y}(i) + \sum_{1\leqslant i\leqslant n-k} \mathbf{z}(i, n) \\ & - \Bigg(\sum_{1\leqslant i\leqslant n/2} \mathbf{Y}(i) + \sum_{1\leqslant i\leqslant n/2} \mathbf{z}(i, n) \Bigg) \Bigg\| \\ & = O_P(n^{-\delta+1/2})\end{aligned} \tag{2.3}$$

with some $\delta > 0$

Proof. Lemmas 3.2 and 3.4 of Horváth and Shao (1999) contain the proofs of (2.1) and (2.2). The relation in (2.3) is established in the proof of Theorem 1.2 in Horváth and Shao (1999, p. 156).

Next we collect some properties of $Y_m(k)$ and $z_m(k, n)$. Let

$$g_m(k) = \sum_{-\infty<i<\infty} \{R(i)\, c_m(i+k) + R(i+k)\, c_m(i)\}$$

and

$$\begin{aligned}\mathscr{T}(i, j) = \chi_4 & \Bigg\{ \sum_{-\infty<\ell<\infty} R(\ell)\, c_i(\ell) \Bigg\} \Bigg\{ \sum_{-\infty<\ell<\infty} R(\ell)\, c_j(\ell) \Bigg\} \\ & + \tau^4 \sum_{1\leqslant \ell<\infty} g_i(\ell)\, g_j(\ell),\end{aligned}$$

where $\chi_4 = E(\xi_0^2 - \tau^2)^2$. The σ-algebra generated by $\xi_1, \xi_2, ..., \xi_m$ will be denoted by $\mathscr{F}_m$. Similarly, $\mathscr{F}_k(n) = \sigma(\xi_n, \xi_{n-1}, ..., \xi_{n-k+1})$.

LEMMA 2.3. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$E\,|Y_i(k)|^{(4+\rho)/2} \leqslant C, \tag{2.4}$$

$$E\,|z_i(k,n)|^{(4+\rho)/2} \leqslant C, \tag{2.5}$$

$$\max_{1\leqslant k\leqslant n} k^{\delta}\left|\frac{1}{k}\sum_{1\leqslant m\leqslant k} EY_i(m)\,Y_j(m)-\mathscr{T}(i,j)\right| = O(1), \tag{2.6}$$

$$\max_{1\leqslant k< n} k^{\delta}\left|\frac{1}{k}\sum_{1\leqslant m\leqslant k} Ez_i(m,n)\,z_j(m,n)-\mathscr{T}(i,j)\right| = O(1), \tag{2.7}$$

$$\max_{1\leqslant k\leqslant n} k^{\delta-1}\left|\sum_{1\leqslant m\leqslant k}\{E(Y_i(m)\,Y_j(m)\mid \mathscr{F}_{m-1})-EY_i(m)\,Y_j(m)\}\right|$$
$$= O_P(1) \tag{2.8}$$

and

$$\max_{1\leqslant k< n} k^{\delta-1}\left|\sum_{1\leqslant m\leqslant k}\{E(z_i(m,n)\,z_j(m,n)\mid \mathscr{F}_{m-1}(n))\right.$$
$$\left. -Ez_i(m,n)\,z_j(m,n)\}\right| = O_P(1) \tag{2.9}$$

for all $1\leqslant i, j\leqslant p$ *with some* $C>0$ *and* $\delta>0$.

Proof. Observing that

$$|d_i(k,m)| \leqslant C(1+|k-m|)^{-\mu} \qquad \text{with some } C \text{ and } \quad 1/2<\mu<1 \tag{2.10}$$

(cf. Lemma 4.5 in Horváth and Shao, 1999), Rosenthal's inequality (cf. Theorem 2.12 in Hall and Heyde, 1980) gives (2.4) and (2.5).

Elementary arguments give

$$\sum_{1\leqslant m\leqslant k} EY_i(m)\,Y_j(m) = \chi_4 \sum_{1\leqslant m\leqslant k} d_i(m)\,d_j(m)$$
$$+\tau^4 \sum_{1\leqslant m\leqslant k}\;\sum_{1\leqslant \ell\leqslant m-1} d_i(m,\ell)\,d_j(m,\ell).$$

Next we write

$$d_i(m) = \sum_{-\infty<\ell<\infty} R(\ell)\,c_i(\ell) - \sum_{\ell\leqslant -k} R(\ell)\,c_i(\ell).$$

Using Lemma 4.5 of Horváth and Shao (1999), one can derive from condition (1.9) that

$$\sum_{k \leqslant |\ell|} |R(\ell)\, c_i(\ell)| = O(k^{-\mu}) \qquad \text{with some} \quad 1/2 < \mu < 1 \tag{2.11}$$

and therefore

$$\left| \sum_{1 \leqslant m \leqslant k} d_i(m)\, d_j(m) - k \left\{ \sum_{-\infty < \ell < \infty} R(\ell)\, c_i(\ell) \right\} \left\{ \sum_{-\infty < \ell < \infty} R(\ell)\, c_j(\ell) \right\} \right| = O(k^{1-\mu}), \tag{2.12}$$

as $k \to \infty$.

Similarly to (2.12) we have that

$$\sum_{-\infty < \ell < \infty} \{ |R(\ell)\, c_i(\ell + m)| + |R(\ell + m)\, c_i(\ell)| \} = O(m^{-\mu}) \qquad \text{with some} \quad 1/2 < \mu < 1 \tag{2.13}$$

(cf. Lemma 4.5 in Horváth and Shao, 1999). Using (2.13) we obtain that

$$\begin{aligned}
&\sum_{1 \leqslant m \leqslant k} \sum_{1 \leqslant \ell \leqslant m-1} d_i(m, \ell)\, d_j(m, \ell) \\
&\quad = \sum_{1 \leqslant m \leqslant k} \sum_{1 \leqslant \ell \leqslant m-1} g_i(m - \ell)\, g_j(m - \ell) \\
&\qquad + \sum_{1 \leqslant m \leqslant k} \sum_{1 \leqslant \ell \leqslant m-1} \{ d_i(m, \ell)\, d_j(m, \ell) - g_i(m - \ell)\, g_j(m - \ell) \} \\
&\qquad + \sum_{1 \leqslant m \leqslant k} \sum_{1 \leqslant \ell \leqslant m-1} g_i(m - \ell) \{ d_j(m, \ell) - g_j(m - \ell) \} \\
&\qquad + \sum_{1 \leqslant m \leqslant k} \sum_{1 \leqslant \ell \leqslant m-1} g_j(m - \ell) \{ d_i(m, \ell) - g_i(m - \ell) \} \\
&\quad = k \sum_{1 \leqslant \ell < \infty} g_i(\ell)\, g_j(\ell) + O(k^{2-2\mu})
\end{aligned}$$

with some $1/2 < \mu < 1$, which also completes the proof of (2.6). Similar arguments give (2.7).

The results in (2.8) and (2.9) are taken from Horváth and Shao (1999).

LEMMA 2.4. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P \left\{ \sup_{0 \leqslant t \leqslant \delta} n^{-1/2} \left\| \sum_{1 \leqslant k \leqslant nt} \mathbf{Y}(k) \right\| \Big/ q(t) > x \right\} = 0 \tag{2.14}$$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\left\{ \sup_{0 \leqslant t \leqslant \delta} n^{-1/2} \left\| \sum_{1 \leqslant k \leqslant nt} \mathbf{z}(k, n) \right\| \Big/ q(t) > x \right\} = 0 \quad (2.15)$$

for all $x > 0$.

Proof. Horváth and Shao (1999, Lemma 3.1) showed that there are constants $\tilde{\sigma} \geqslant 0$ and Wiener processes $\{\tilde{W}_i(t), 0 \leqslant t < \infty\}$ such that

$$\left| \sum_{1 \leqslant k \leqslant m} Y_i(k) - \tilde{\sigma}_i \tilde{W}_i(m) \right| = O(m^{-\nu + 1/2}) \qquad \text{a.s.} \qquad (2.16)$$

with some $\nu > 0$. By the scale transformation of Wiener processes we get that $\tilde{W}_{i,n}(t) = n^{-1/2} \tilde{W}_i(nt)$ are also Wiener processes and (2.16) yields

$$n^{\nu} \sup_{1/n \leqslant t \leqslant 1} \left| n^{-1/2} \sum_{1 \leqslant k \leqslant nt} Y_i(k) - \tilde{\sigma}_i \tilde{W}_{i,n}(t) \right| \Big/ t^{-\nu + 1/2} = O_P(1) \quad (2.17)$$

with some $\nu > 0$. It is well known (cf. Csörgő and Horváth, 1993, p. 181) that

$$\lim_{t \downarrow 0} t^{1/2}/q(t) = 0 \qquad (2.18)$$

and

$$\lim_{\delta \to 0} P\{ \sup_{0 \leqslant t \leqslant \delta} |\tilde{W}_{i,n}(t)|/q(t) > x\} = 0 \qquad (2.19)$$

for all $x > 0$, $1 \leqslant i \leqslant p$, and $1 \leqslant n < \infty$. By (2.17) have that

$$\begin{aligned}
&\sup_{0 \leqslant t \leqslant \delta} \left| n^{-1/2} \sum_{1 \leqslant k \leqslant nt} Y_i(k) - \tilde{\sigma}_i \tilde{W}_{i,n}(t) \right| \Big/ q(t) \\
&\quad \leqslant \tilde{\sigma}_i \sup_{0 \leqslant t \leqslant 1/n} |\tilde{W}_{i,n}(t)|/q(t) \\
&\qquad + \left\{ \sup_{1/n \leqslant t \leqslant \delta} \left| n^{-1/2} \sum_{1 \leqslant k \leqslant nt} Y_i(k) - \tilde{\sigma}_i \tilde{W}_{i,n}(t) \right| \Big/ t^{-\nu + 1/2} \right\} \\
&\qquad \times \left\{ n^{\nu} \sup_{1/n \leqslant t \leqslant \delta} t^{1/2}/q(t) \right\} \\
&\quad = o_P(1) + O_P(1) \sup_{0 \leqslant t \leqslant \delta} t^{1/2}/q(t),
\end{aligned}$$

and therefore (2.14) follows from (2.18).

The proof of (2.15) is similar to that of (2.14) and therefore it is omitted.

We say that $\mathbf{\Gamma}^*(t)=(\Gamma_1^*(t), ..., \Gamma_p^*(t))$ is a p-dimensional Wener process with covariance matrix $\mathscr{T}$, if $\mathbf{\Gamma}^*(t)$ is Gaussian with $E\Gamma_i^*(t)=0$ and $E\Gamma_i^*(t)\,\Gamma_j^*(s)=\mathscr{T}(i, j)\min(t, s)$.

LEMMA 2.5. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$n^{-1/2}\sum_{1\leqslant m\leqslant nt}\mathbf{Y}(m)\xrightarrow{\mathscr{D}^p[0,1]}\mathbf{\Gamma}^*(t) \tag{2.20}$$

and

$$n^{-1/2}\sum_{1\leqslant m\leqslant nt}\mathbf{z}(m, n)\xrightarrow{\mathscr{D}^p[0,1]}\mathbf{\Gamma}^*(t), \tag{2.21}$$

where $\{\mathbf{\Gamma}^*(t), 0\leqslant t<\infty\}$ *is a p-dimensional Wiener process with covariance matrix* $\mathscr{T}$.

Proof. We prove only (2.20) because the proof of (2.21) is essentially the same.

The tightness of $n^{-1/2}\sum_{1\leqslant m\leqslant nt}\mathbf{Y}(m)$ follows from (2.16).

Let $0\leqslant t_1, t_2, ..., t_N\leqslant 1$ and $v(i, j)$, $1\leqslant i\leqslant p$, $1\leqslant j\leqslant N$, be constants. We need the convergence of the final dimensional distributions, so it is enough to show that

$$n^{-1/2}\sum_{1\leqslant i\leqslant p}\;\sum_{1\leqslant j\leqslant N}v(i, j)\sum_{1\leqslant k\leqslant[nt_j]}Y_i(k)\xrightarrow{\mathscr{D}}N(0, A^2), \tag{2.22}$$

where $N(0, A^2)$ is a normal random variable with zero mean and variance,

$$A^2=\sum_{1\leqslant i, i'\leqslant p}\;\sum_{1\leqslant j, j'\leqslant N}\min(t_j, t_{j'})\,v(i, j)\,\mathscr{T}(i, i')\,v(i', j').$$

Let

$$e(m)=\sum_{1\leqslant k\leqslant m}\eta_m,$$

where

$$\eta_m=\sum_{1\leqslant i\leqslant p}\;\sum_{1\leqslant j\leqslant N}v(i, j)\;Y_i(k)\;I\{k\leqslant[nt_j]\}.$$

We note that

$$e(n)=\sum_{1\leqslant i\leqslant p}\ \sum_{1\leqslant j\leqslant N} v(i,\,j)\ Y_i(k).$$

Elementary arguments show that $\{e(m), \mathscr{F}_m, 1\leqslant m\leqslant n\}$ is a martingale. By Skorokhod's representation theorem (cf. Hall and Heyde, 1980, pp. 269) there is a Wiener process $\{W(t), 0\leqslant t<\infty\}$ and stopping times $s_1, s_2, ..., s_n$ such that

$$e(m)=W(s_m), \qquad 1\leqslant m\leqslant n, \tag{2.23}$$

$$E(s_m\mid \mathscr{F}_{m-1})=E(\eta_m^2\mid \mathscr{F}_{m-1}), \qquad 1\leqslant m\leqslant n \tag{2.24}$$

and

$$E\,|s_m|^{1+\rho/4}\leqslant c(1+\rho/4)\,E\,|\eta_m|^{2+\rho/2}, \qquad 1\leqslant m\leqslant n. \tag{2.25}$$

Next we show that

$$\sum_{1\leqslant m\leqslant n} s_m-nA^2=O_P(n^{1-\delta}) \tag{2.26}$$

with some $\delta>0$. By (2.24) we can write

$$\begin{aligned}\sum_{1\leqslant m\leqslant n} s_m-nA^2=&\sum_{1\leqslant m\leqslant n}\{s_m-E(s_m\mid \mathscr{F}_{n-1})\}\\ &+\sum_{1\leqslant m\leqslant n}\{E(\eta_m^2\mid \mathscr{F}_{m-1})-E\eta_m^2\}\\ &+\sum_{1\leqslant m\leqslant n} E\eta_m^2-nA^2.\end{aligned}$$

By (2.4) and (2.25) we obtain that

$$E\,|s_m|^{1+\rho/4}\leqslant C, \qquad 1\leqslant m\leqslant n \tag{2.27}$$

with some constant C. Hence Theorem 2.18 in Hall and Heyde (1980) yields

$$\sum_{1\leqslant m\leqslant n}\{s_m-E(s_m\mid \mathscr{F}_{m-1})\}=O(n^{(8+\rho)/(8+2\rho)}) \qquad \text{a.s.}$$

Using (2.8) we get

$$\sum_{1\leqslant m\leqslant n}\{E(\eta_m^2\mid \mathscr{F}_{m-1})-E\eta_m^2\}=O_P(n^{1-\delta}),$$

while (2.6) implies that

$$\sum_{1\leqslant m\leqslant n} E\eta_m^2 - nA^2 = O(n^{1-\delta})$$

with some $\delta>0$. The proof of (2.26) is complete

Combining (2.23) and (2.26) with the modulus of continuity of W we conclude that

$$e(n) - W(nA^2) = o_P(n^{1/2}),$$

which implies (2.22).

Let

$$\boldsymbol{\omega}_n(t) = n^{1/2}t(1-t)\left\{\frac{1}{nt}\sum_{1\leqslant m\leqslant nt}\mathbf{Y}(m) - \frac{1}{n-nt}\sum_{1\leqslant m\leqslant n-nt}\mathbf{z}(m,n)\right\},\quad 0\leqslant t\leqslant 1.$$

Lemma 2.6. *If the conditions of Theorem* 1.1 *are satisfied, then there is a sequence of p-dimensional Brownian bridges* $\{\hat{\boldsymbol{\Gamma}}_n(t), 0\leqslant t\leqslant 1\}$ *with covariance matrix* $\mathscr{T}$ *such that*

$$\sup_{0\leqslant t\leqslant 1}\|\boldsymbol{\omega}_n(t) - \hat{\boldsymbol{\Gamma}}_n(t)\|/q(t) = o_P(1).$$

Proof. First we note that $\{\mathbf{Y}(m), 1\leqslant m\leqslant n/2\}$ and $\{\mathbf{z}(m,n), 1\leqslant m< n/2\}$ are independent for each n. So by Lemma 2.5,

$$n^{-1/2}\left(\sum_{1\leqslant m\leqslant nt}\mathbf{Y}(m), \sum_{1\leqslant m\leqslant nt}\mathbf{z}(m,n)\right)\xrightarrow{\mathscr{D}^{2p}[0,1/2]}(\boldsymbol{\Gamma}^{(1)}(t), \boldsymbol{\Gamma}^{(2)}(t)), \tag{2.28}$$

where $\boldsymbol{\Gamma}^{(1)}$ and $\boldsymbol{\Gamma}^{(2)}$ are independent p-dimensional Wiener processes with covariance matrix $\mathscr{T}$. Using (2.3) we get that

$$\begin{aligned}\sup_{0\leqslant t\leqslant 1/2}\Bigg\|\boldsymbol{\omega}_n(t) - n^{-1/2}\Bigg\{&\sum_{1\leqslant m\leqslant nt}\mathbf{Y}(m)\\ &-t\left(\sum_{1\leqslant m\leqslant n/2}\mathbf{Y}(m)+\sum_{1\leqslant m\leqslant n/2}\mathbf{z}(m,n)\right)\Bigg\}\Bigg\|\Big/ t = O_P(n^{-\delta})\end{aligned} \tag{2.29}$$

and

$$\sup_{1/2\leqslant t\leqslant 1}\left\|\boldsymbol{\omega}_n(t)-n^{-1/2}\left\{-\sum_{1\leqslant m\leqslant n-nt}\mathbf{z}(m,n)\right.\right.$$
$$\left.\left.+(1-t)\left(\sum_{1\leqslant m\leqslant n/2}\mathbf{Y}(m)+\sum_{1\leqslant m\leqslant n/2}\mathbf{z}(m,n)\right)\right\}\right\|\Big/(1-t)=O_P(n^{-\delta})$$
(2.30)

with some $\delta>0$. Thus (2.28) implies that

$$\boldsymbol{\omega}_n(t)\xrightarrow{D^p[0,1]}\hat{\boldsymbol{\Gamma}}(t), \tag{2.31}$$

where

$$\hat{\boldsymbol{\Gamma}}(t)=\begin{cases}\boldsymbol{\Gamma}^{(1)}(t)-t(\boldsymbol{\Gamma}^{(1)}(1/2)+\boldsymbol{\Gamma}^{(2)}(1/2)), & 0\leqslant t\leqslant 1/2\\ -\boldsymbol{\Gamma}^{(2)}(1-t)+(1-t)(\boldsymbol{\Gamma}^{(1)}(1/2)+\boldsymbol{\Gamma}^{(2)}(1/2)), & 1/2\leqslant t\leqslant 1.\end{cases}$$

Computing the covariance structure of $\hat{\boldsymbol{\Gamma}}(t)$, one can easily verify that $\hat{\boldsymbol{\Gamma}}(t)$ is a Brownian bridge with covariance matrix $\mathscr{T}$.

By the weak convergence in (2.31) we can find p-dimensional Brownian bridges $\{\hat{\boldsymbol{\Gamma}}_n(t), 0\leqslant t\leqslant 1\}$ such that

$$\{\hat{\boldsymbol{\Gamma}}_n(t), 0\leqslant t\leqslant 1\}\overset{\mathscr{D}}{=}\{\hat{\boldsymbol{\Gamma}}(t), 0\leqslant t\leqslant 1\}$$

for each n and

$$\sup_{0\leqslant t\leqslant 1}\|\boldsymbol{\omega}_n(t)-\hat{\boldsymbol{\Gamma}}_n(t)\|=o_P(1).$$

Using Lemma 2.14, (2.29), and (2.30) we obtain that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}P\{\sup_{0\leqslant t\leqslant\delta}\|\boldsymbol{\omega}_n(t)\|/q(t)>x\}=0,$$
$$\lim_{\delta\to 0}\limsup_{n\to\infty}P\{\sup_{1-\delta\leqslant t\leqslant 1}\|\boldsymbol{\omega}_n(t)\|/q(t)>x\}=0$$

and by Csörgő and Horváth (1993, p. 189)

$$\lim_{\delta\to 0}P\{\sup_{0\leqslant t\leqslant\delta}\|\hat{\boldsymbol{\Gamma}}(t)\|/q(t)>x\}=0,$$
$$\lim_{\delta\to 0}P\{\sup_{1-\delta\leqslant t\leqslant 1}\|\hat{\boldsymbol{\Gamma}}(t)\|/q(t)>x\}=0$$

for all x. The proof of Lemma 2.6 is complete.

Proof of Theorem 1.1. Combining Lemmas 2.1 and 2.1 with (2.6) we get that

$$\sup_{0\leqslant t\leqslant 1} \left\| n^{1/2} t(1-t)(\hat{\boldsymbol{\lambda}}_{[nt]} - \tilde{\boldsymbol{\lambda}}_{[nt]}) - \frac{2\pi}{\kappa_0^2} \hat{\boldsymbol{\Gamma}}_n(t)\, \mathscr{W}^{-1}(\boldsymbol{\lambda}_0) \right\| \Big/ g(t) = o_P(1).$$

Next we note that $(2\pi/\kappa_0^2)\, \hat{\boldsymbol{\Gamma}}_n(t)\, \mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$ is a p-dimensional Brownian bridge with covariance matrix $(2\pi/\kappa_0^2)^2\, \mathscr{W}^{-1}(\boldsymbol{\lambda}_0)\, \mathscr{T}(\boldsymbol{\lambda}_0)\, \mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$. Giraitis and Surgailis (1990) showed that $n^{1/2}(\boldsymbol{\lambda}_n - \boldsymbol{\lambda}_0)$ is asymptotically normal with mean $\mathbf{0}$ and covariance matrix $4\pi \mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$. Hence $(2\pi/\kappa_0^2)^2\, \mathscr{W}^{-1}(\boldsymbol{\lambda}_0)\, \mathscr{T}(\boldsymbol{\lambda}_0)\, \mathscr{W}^{-1}(\boldsymbol{\lambda}_0) = 4\pi \mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$, which completes the proof of Theorem 1.11.

REFERENCES

1. J. Beran, Statistical methods for data with long-range dependence (with discussion), *Statist. Sci.* **7** (1992), 404–416.
2. J. Beran, "Statistics for Long-Memory Processes," Chapman & Hall, New York, 1994.
3. J. Beran, R. Sherman, M. N. Taqqu, and W. Willinger, The variable-bit-rate video traffic and long-range dependence, *IEEE Trans. Commun.* **42** (1995), 1566–1579.
4. J. Beran, and N. Terrin, Testing for a change of the long-memory parameter, *Biometrika* **83** (1996), 627–638.
5. M. Csörgő and L. Horváth, "Weighted Approximations in Probability and Statistics," Wiley, Chichester, 1993.
6. M. Csörgő and L. Horváth, "Limit Theorems in Change-Point Analysis," Wiley, Chichester, 1997.
7. R. Fox and M. S. Taqqu, Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series, *Ann. Stat.* **14** (1986), 517–532.
8. L. Giraitis and D. Surgailis, A central limit theorem for quadratic forms of strongly dependent linear variables and its application to asymptotic normality of Whittle's estimate, *Probab. Theory Related Fields* **86** (1990), 87–104.
9. P. Hall and C. C. Heyde, "Martingale Limit Theory and Its Application," Academic Press, New York, 1980.
10. D. L. Hawkins, A U-I approach to retrospective testing for shift parameters in a linear model, *Comm. Statist. - Theory Method* **18** (1989), 3117–3134.
11. L. Horváth and Q.-M. Shao, Limit theorems for the union-intersection test, *J. Statist. Planning Inference* **44** (1995), 133–148.
12. L. Horváth and Q.-M. Shao, Limit theorems for quadratic forms with applications to Whittle's estimate, *Ann Appl. Probab.* **9** (1999), 146–187.
13. M. S. Taqqu, A bibliographical guide to self-similar processes and long-range dependence, *in* "Dependence in Probability and Statistics," pp. 137–162, Birkhäuser, Boston, 1986.