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Fluctuations for zeros of Gaussian Taylor series

Avner Kiro and Alon Nishry

Abstract

We study fluctuations in the number of zeros of random analytic functions given by a Taylor series whose coefficients are independent complex Gaussians. When the functions are entire, we find sharp bounds for the asymptotic growth rate of the variance of the number of zeros in large disks centered at the origin. To obtain a result that holds under no assumptions on the variance of the Taylor coefficients, we employ the Wiman–Valiron theory. We demonstrate the sharpness of our bounds by studying well-behaved covariance kernels, which we call admissible (after Hayman).

1. *Introduction*

Some of the earliest works concerning random analytic functions are the ones of Littlewood and Offord [**19, 20**], who showed that the structure of the zero set of these functions is very regular. More recently, these classical results were sharpened in the papers [**15, 22**]. In this paper, we consider the typical size of fluctuations in the number of zeros of random analytic functions whose coefficients are independent complex Gaussians. This is the most well-studied and best understood model (see the book [**14**] and ICM notes [**23**]).

Given a sequence $\{a_n\}_{n\geqslant 0}$ of non-negative numbers, we consider random Taylor series

$$f(z) = \sum_{n\geqslant 0} \xi_n a_n z^n, \tag{1.1}$$

where ξ_n are independent and identically distributed standard complex Gaussians. We only consider *transcendental* analytic functions, that is, sequences $\{a_n\}$ which contain infinitely many non-zero terms. Denote by $\mathcal{Z}_f = f^{-1}\{0\}$ the zero set of f; its properties are determined by the covariance kernel

$$K(z,w) = \mathbb{E}\Big[f(z)\overline{f(w)}\Big] =: G(z\bar{w}), \quad \text{where} \quad G(z) := \sum_{n\geqslant 0} a_n^2 z^n.$$

We will call G the *covariance function* of f; denote by R_G the radius of convergence of G around the origin. We consider both $R_G < \infty$ and $R_G = \infty$, and in the former case, without loss of generality, we may assume $R_G = 1$. Then, it is not difficult to check that the radius of convergence of f is almost surely R_G in both cases (see [**14**, Lemma 2.2.3]). When $R_G = \infty$, we call f a *Gaussian entire function*.

Let $n_f(r)$ be the number of zeros of the function f inside the disk $\{|z| \leqslant r\}$, where $r < R_G$. We are interested in the asymptotic statistical properties of the random variable $n_f(r)$,

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as $r \to R_G$. In order to study this asymptotics, it will be convenient to define the following functions

$$a(z) = a_G(z) := z(\log G(z))' = \frac{zG'(z)}{G(z)}, \quad b(z) = b_G(z) := za'(z),$$

borrowing the notation used in [12]. Since the Taylor coefficients of G are non-negative, the function $r \mapsto \log G(e^r)$ is *convex*, hence $a(r)$ is increasing, and $b(r)$ is non-negative for all $r < R_G$.

The Edelman–Kostlan formula [14, p. 25] (see also the Appendix) states that for *any* Gaussian analytic function f of the form (1.1), we have

$$\mathbb{E}[n_f(r)] = a\big(r^2\big), \quad \text{for all } r < R_G.$$

However, the expected value provides little information about the distribution of the random variable. Here we will be interested in the asymptotic growth rate of the variance $\mathrm{Var}(n_f(r))$ in terms of the functions a and b, in general under no additional assumptions on the nature of the coefficients a_n.

In order to present the results, we will need the following notation. We say that a set $E \subset \mathbb{R}^+$ is of *finite logarithmic measure* if

$$\int_E \frac{\mathrm{d}t}{t} < \infty.$$

If $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are non-negative functions, we write $g_1 \lesssim_L g_2$ if there is a constant $C > 0$, and a set $E \subset \mathbb{R}^+$ of finite logarithmic measure, such that $g_1 \leqslant Cg_2$ in $\mathbb{R}^+ \backslash E$. Finally, we write $g_1 \asymp_L g_2$ if $g_1 \lesssim_L g_2$ and $g_1 \gtrsim_L g_2$ both hold.

THEOREM 1.1. *For* any *Gaussian entire function f with a transcendental covariance function G, and any $\varepsilon > 0$*

$$\mathrm{Var}(n_f(r)) \gtrsim_L \frac{b^2\big(r^2\big)}{a(r^2)^{\frac{3}{2}+\varepsilon}}.$$

In addition, if b is a non-decreasing *function, then*

$$\mathrm{Var}(n_f(r)) \gtrsim_L \sqrt{b(r^2)}.$$

REMARK 1.1. With some more work the factor $a(r^2)^\varepsilon$ in Theorem 1.1 can be replaced by a power of $\log a(r^2)$, we will not pursue this here.

REMARK 1.2. By [29, Lemma 1], it follows that for every $\varepsilon > 0$ we have that $b(r) \lesssim_L a(r)^{1+\varepsilon}$.

It turns out that the upper bound for the variance may be considerably larger asymptotically. The following result holds without *any* restrictions on G.

THEOREM 1.2. *Let f be a Gaussian analytic function with covariance function G, then for every $r < R_G$*

$$\mathrm{Var}(n_f(r)) \leqslant b\big(r^2\big).$$

REMARK 1.3. If $G(z)$ is of the form $a_n^2 z^n + a_m^2 z^m$, then one can check that $\mathrm{Var}(n_f(r)) = b_G(r^2)$. This implies that for any non-decreasing and unbounded function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$, there is a covariance function G_β, so that if f is the Gaussian entire function whose covariance function is G_β, there is a sequence $r_n \to \infty$ so that

$$\operatorname{Var}(n_f(r_n)) = (1+o(1))\beta(r_n) = (1+o(1))b_{G_\beta}(r_n^2), \quad n \to \infty,$$

and in particular

$$\limsup_{r\to\infty} \frac{\operatorname{Var}(n_f(r))}{b_{G_\beta}(r^2)} = 1.$$

By Theorem 1.2 and Remark 1.2, we get the following conclusion.

COROLLARY 1.3. *For any Gaussian entire function f with a transcendental covariance function G, and every $\varepsilon > 0$,*

$$\operatorname{Var}(n_f(r)) \lesssim_L \mathbb{E}[n_f(r)]^{1+\varepsilon}.$$

1.1. *Well-behaved covariance functions*

If the covariance function G of f is sufficiently well behaved, such as e^z, e^{e^z}, and the Mitag–Leffler functions

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

then it is possible to find the asymptotics of the variance. A notable example is the Gaussian Entire Function (GEF), with $G(z) = e^z$, whose zero set is *invariant* with respect to the isometries of the complex plane (see the book [**14**, Chapter 2.3]). Forrester and Honner [**8**] found the precise asymptotic growth of the variance for the GEF (see also [**24**]). In order to extend this result, we define two classes of *admissible* covariance functions, which in particular include all the previous examples. For the precise definitions see Sections 5.1 and 7.2. More examples, including Gaussian analytic functions with an admissible covariance function in the unit disk are described in Section 2.

THEOREM 1.4. *Let f be a Gaussian analytic function with a type I admissible covariance function G. Then*

$$\operatorname{Var}(n_f(r)) = (1+o(1))\frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}}\sqrt{b(r^2)}, \quad r \to R_G^-,$$

where $\zeta(u)$ is the Riemann zeta function.

REMARK 1.4. For the GEF, $a(r) = b(r) = r$, recovering the results of [**8, 24**].

REMARK 1.5. This indicates that the lower bound in Theorem 1.1 is sharp.

Suppose G is a sufficiently regular covariance function (see Section (7.2) for the precise requirements). In the next theorem, we construct a Gaussian entire function $\widetilde{f}$ with covariance function $\widetilde{G}$, so that the variance of the number of zeros of $\widetilde{f}$ is large outside a small exceptional set of values of r. The statement of the theorem requires the following definitions.

DEFINITION 1.5. We will say that two covariance kernels G and $\widetilde{G}$ are *similar* if

$$a_{\widetilde{G}}(r) \asymp_L a_G(r) \text{ and } b_{\widetilde{G}}(r) \asymp_L b_G(r).$$

Definition 1.6. Let $G(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function. A function $\widetilde{G}$ is a *Taylor series restriction* of G, if $\widetilde{G}(z) = \sum_{n=0}^{\infty} \delta_n c_n z^n$ with $\delta_n \in \{0,1\}$ for all $n \in \mathbb{N}$.

Theorem 1.7. *Let G be a type II admissible function. There exists $\widetilde{G}$ which is a Taylor series restriction of and similar to G, so that*

$$\mathrm{Var}\Big(n_{\widetilde{f}}(r)\Big) \asymp_L b_{\widetilde{G}}(r^2),$$

where $\widetilde{f}$ is a Gaussian entire function with covariance kernel $\widetilde{G}$.

Remark 1.6. The theorem shows that the upper bound in Theorem 1.2 is sharp (up to a constant) for certain *transcendental* entire functions *outside* a set of finite logarithmic measure (cf. Remark 1.3).

Remark 1.7. By the example in Section 2.2 (which is type II admissible), it follows that in general ε *cannot* be removed in Corollary 1.3.

1.2. *Background and related results*

Following earlier work by Edelman and Kostlan [**6**], and Offord [**27**], some fundamental properties of zeros of Gaussian analytic functions (GAFs) were developed by Sodin [**33**] (see also [**16**, Chapter 13]). Sodin and Tsirelson [**34**] found the asymptotics of the variance and proved a central limit theorem for smooth linear statistics for planar, spherical, and hyperbolic GAFs. More general results about linear statistics were obtained by Nazarov and Sodin [**24, 25**].

For the family of hyperbolic GAFs, whose zero sets are invariant with respect to the isometries of the unit disk, Buckley [**4**] found the asymptotics of the variance of the number of zeros (see also Section 2.4). Buckley and Sodin [**5**] studied fluctuations in the increment of the argument along curves for the planar GAF (GEF). Feldheim [**7**] derived bounds for the growth of the variance of zeros for GAFs which are invariant with respect to shifts. Ghosh and Peres [**9**] showed that the fast decay rate of the variance of smooth linear statistics of the GEF implies the rigidity of the zero set. Their technique was recently used by the authors in [**17**] to construct examples of 'completely rigid' Gaussian entire functions.

Considerable amount of research is devoted to the study of zero sets of random algebraic and trigonometric polynomials. Maslova [**21**], Granville and Wigman [**11**], Azaïs, Dalmao, and León [**2**], and Nguyen and Vu [**26**] (by no means an exhaustive list) proved central limit theorems for real zeros of random polynomials. Bally, Caramellino, and Poly [**3**] studied the dependence of the variance of the number of zeros on the distribution of the coefficients. Very recently, following the earlier work [**32**], Shiffman [**30**] found an asymptotic expansion for the variance of smooth statistics of random zeros on complex manifolds.

2. *Examples of admissible covariance functions*

Here are some explicit examples for covariance functions which are type I and type II admissible (see Sections 5.1 and 7.2, respectively, for the precise definitions).

2.1. *The Mittag–Leffler function*

We consider the Gaussian entire function f whose covariance function G is given by the Mittag–Leffler function

$$G(z) = G_\alpha(z) = \sum_{n\geqslant 0} \frac{z^n}{\Gamma(1+\alpha^{-1}n)},$$

where $\alpha > 0$ is a parameter. Note that $G_1(z) = e^z$ and $G_{\frac{1}{2}}(z) = \cosh\sqrt{z}$. The asymptotic behavior of G and G' is well known, see, for example, [10, Section 3.5.3]. In particular, as $|z| \to \infty$, and uniformly in $\arg z$,

$$G(z) = \begin{cases} \alpha e^{z^\alpha} + O(1), & |\arg z| \leqslant \frac{\pi}{2\alpha}; \\ O(1), & \text{otherwise,} \end{cases}$$

and

$$zG'(z) = \begin{cases} \alpha^2 z^\alpha e^{z^\alpha} + O(1), & |\arg z| \leqslant \frac{\pi}{2\alpha}; \\ O(1), & \text{otherwise.} \end{cases}$$

REMARK 2.1. Note that for $\alpha \in (0, \frac{1}{2}]$ the complement of $\{|\arg z| \leqslant \frac{\pi}{2\alpha}\}$ is empty.

From this asymptotic description, one easily verifies that G is type I and type II admissible. Since

$$a(r) = \frac{rG'(r)}{G(r)} \sim \alpha r^\alpha, \quad b(r) = ra'(r) \sim \alpha^2 r^\alpha, \quad r \to \infty,$$

we have

$$\mathbb{E}[n_f(r)] = a(r^2) \sim \alpha r^{2\alpha}, \quad r \to \infty.$$

By Theorem 1.4,

$$\mathrm{Var}(n_f(r)) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} \sqrt{b(r^2)} \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} \cdot \alpha r^\alpha, \quad r \to \infty.$$

2.2. *The double exponent*

Here we consider the Gaussian entire function f with covariance function

$$G(z) = e^{e^z}.$$

The function G is type I and type II admissible, and has an infinite order of growth, with

$$a(r) = re^r, \quad b(r) = r(r+1)e^r.$$

Thus,

$$\mathbb{E}[n_f(r)] = r^2 e^{r^2},$$

and by Theorem 1.4

$$\mathrm{Var}(n_f(r)) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} r^2 e^{\frac{1}{2}r^2}, \quad r \to \infty.$$

2.3. *The Lindelöf functions*

For $\alpha > 0$, we consider the Gaussian entire function f with covariance function

$$G(z) = G_\alpha(z) = \sum_{n \geqslant 0} \frac{z^n}{\log^{\alpha n}(n+e)}.$$

The function G has infinite order of growth, and it follows from [18, Example 1.4.1] that it is type I and type II admissible with

$$\log G(r) \sim \frac{\alpha}{e} r^{-\frac{1}{\alpha}} \exp\left(r^{\frac{1}{\alpha}}\right), \quad r \to \infty,$$

and

$$a(r) \sim \frac{1}{e}\exp\left(r^{1/\alpha}\right), \quad b(r) \sim \frac{r^{1/\alpha}}{\alpha e}\exp\left(r^{1/\alpha}\right), \quad r \to \infty.$$

In this case,

$$\mathbb{E}[n_f(r)] \sim e^{-1}e^{r^{2/\alpha}}, \quad r \to \infty,$$

$$\operatorname{Var}(n_f(r)) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi e\alpha}} \cdot r^{1/\alpha} e^{\frac{1}{2}r^{2/\alpha}}, \quad r \to \infty.$$

2.4. *An example with radius of convergence* 1

For $\alpha > 0$, we consider the Gaussian analytic function f, where now the covariance function is given by

$$G(z) = \exp\left(\frac{1}{(1-z)^\alpha}\right).$$

One can check that this function is type I admissible with $R_G = 1$ and $C_G > 2$ sufficiently large depending on α. Since

$$a(r) = \frac{\alpha r}{(1-r)^{\alpha+1}}, \quad b(r) = \frac{\alpha r}{(1-r)^{\alpha+1}} + \frac{\alpha(\alpha+1)r^2}{(1-r)^{\alpha+2}},$$

we have

$$\mathbb{E}[n_f(r)] = \frac{\alpha r^2}{(1-r^2)^{\alpha+1}}, \quad r < 1,$$

and Theorem 1.4 yields

$$\operatorname{Var}(n_f(r)) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} \frac{\sqrt{\alpha(\alpha+1)}}{(1-r^2)^{\frac{1}{2}\alpha+1}}, \quad r \to 1^-.$$

REMARK 2.2. For functions G of *slower* growth, the above asymptotics *no longer* holds. Buckley [4] found the asymptotic of the variance for the following special choice

$$G(z) = \frac{1}{(1-z)^L}, \quad \text{with } L > 0,$$

which corresponds to Gaussian analytic functions whose zero sets are *invariant* with respect to the isometries of the hyperbolic disk (see [14, Chapter 2.3]). Earlier Peres and Virág [28] computed the variance in the case $L = 1$, where the zero set forms a *determinantal* point process.

3. *Definitions and preliminaries*

Given an analytic function $G(z) = \sum_{n \geqslant 0} a_n^2 z^n$, we denote its radius of convergence by R_G, and assume from here on that $R_G \in \{1, \infty\}$. We always assume a_n are non-negative, and contain infinitely many non-zero terms (that is, G is *transcendental*). We recall the following notation

$$a(z) = a_G(z) = z\frac{G'(z)}{G(z)}, \quad b(z) = b_G(z) = z a_G'(z).$$

We use little-o and big-O notation in the standard way. Given two functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}^+$, we write $g_1 \lesssim g_2$ if $g_1 = O(g_2)$, possibly on a subset of $\mathbb{R}$ (depending on the context). We also write $g_1 \sim g_2$ when $g_1(x) = (1+o(1))g_2(x)$ as $x \to \infty$. Recall that $g_1 \lesssim_L g_2$ when there exists a set $\mathcal{N} \subset \mathbb{R}^+$ and a constant $C > 0$ so that $g_1(x) \leqslant C g_2(x)$ for all $x \in \mathcal{N}$, and $\mathbb{R}^+ \backslash \mathcal{N}$ is a set of *finite* logarithmic measure. Let $I \subset \mathbb{R}^+$ be a open interval, we denote the fact that $h : I \to \mathbb{R}^+$ is a *non-decreasing* and *unbounded* function on I by writing $h \uparrow \infty$.

3.1. *A formula for the variance*

Let $G(z) = \sum_{n=0} a_n^2 z^n$ be the covariance function of a Gaussian analytic function f, with radius of convergence $R_G \in \{1, \infty\}$. For the rest of the paper, it will be convenient to put $e^t = r^2$, and use the exponential change of variables

$$H(t) = G(e^t) = \sum_{n \geqslant 0} a_n^2 e^{nt},$$

and also define

$$t_G \coloneqq \log R_G, \quad A(z) \coloneqq (\log H(z))' = a(e^z), \quad B(z) \coloneqq A'(z) = b(e^z).$$

Note that $A(z), B(z)$ are meromorphic functions which are given by

$$A(z) = \frac{H'(z)}{H(z)}, \quad B(z) = \frac{H''(z)}{H(z)} - \left(\frac{H'(z)}{H(z)}\right)^2 = H^{-2}(z)\left(\sum_{n<m} (n-m)^2 a_n^2 a_m^2 e^{(m+n)z}\right).$$

We will repeatedly use the following formula for the variance of $n_f(r)$:

$$\operatorname{Var}(n_f(r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|A(t+i\theta) - A(t)|^2}{\exp\left(2\int_0^\theta \operatorname{Im}[A(t+i\varphi)]\mathrm{d}\varphi\right) - 1} \mathrm{d}\theta, \tag{3.1}$$

for its proof see Claim A.2 in the Appendix (cf. [**16**, p. 195]). We will also use the following equivalent form

$$\operatorname{Var}(n_f(r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H(t)H'(t+i\theta) - H(t+i\theta)H'(t)|^2}{H^2(t)(H^2(t) - |H^2(t+i\theta)|)} \mathrm{d}\theta. \tag{3.2}$$

3.2. *Local admissibility*

In order to bound the integral in (3.1) from below, we will introduce the following definition, which is motivated by a result from the Wiman–Valiron theory about the value distribution of entire functions, more precisely the asymptotics of such functions near their points of maximum modulus (see [**13**, Theorem 10]).

Definition 3.1. An analytic function H is called *local δ-admissible* on a set $T \subset (-\infty, t_G)$ if there is a function $\delta(t) : [-\infty, t_G) \to (0, \pi)$ so that for any $\varepsilon > 0$ there exists an $\eta > 0$, such that for $t \in T \cap (t_0(\varepsilon), t_G)$ and $|\tau| \leqslant \eta\delta(t)$

$$\log \frac{H(t+\tau)}{H(t)} = \tau A(t) + \frac{1}{2}\tau^2 B(t) + h_t(\tau), \quad \text{where } |h_t(\tau)| \leqslant \varepsilon|\tau|^2 B(t). \tag{3.3}$$

Remark 3.1. It is implicitly assumed that $t + \delta(t) < t_G$ for all $t < t_G$.

We will show in Section 4.2 that if $B \uparrow \infty$, then H is local δ-admissible outside a set of finite logarithmic measure, with $\delta(t) = \frac{1}{\sqrt{B(t)}}$. With a (smaller) appropriate choice of δ, this statement is also true without making *any* assumptions on B, for the details see Section 4.2.3.

3.3. *Lower bound for the variance assuming local admissibility*

Let f be a Gaussian analytic function with covariance function $G(z) = \sum_{n=0} a_n^2 z^n$, and radius of convergence $R_G \in \{1, \infty\}$.

In the next lemma, we use Cauchy's integral formula to obtain estimates for A, B when H is a local δ-admissible function.

LEMMA 3.2. *Let H be a local $\delta-$admissible function on T with $t_G \in \{0, \infty\}$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $t \in T \cap (t_0(\varepsilon), t_G)$, and $|\theta| \leqslant \frac{\eta}{2}\delta(t)$, we have*

$$(1-\varepsilon)B(t)|\theta| \leqslant |A(t+i\theta) - A(t)| \leqslant (1+\varepsilon)B(t)|\theta|,$$

and

$$\frac{1-\varepsilon}{2}\theta^2 B(t) \leqslant \int_0^\theta \mathrm{Im}[A(t+i\varphi)]d\varphi \leqslant \frac{1+\varepsilon}{2}\theta^2 B(t).$$

Proof. Given $\varepsilon > 0$, choose $\eta > 0$ as in the definition of a local $\delta-$admissible function, and let $0 < |\tau| \leqslant \frac{\eta}{2}\delta(t)$. Differentiating (3.3) with respect to τ, we obtain

$$A(t+\tau) = \frac{H'(t+\tau)}{H(t+\tau)} = A(t) + \tau B(t) + h_t'(\tau), \tag{3.4}$$

with

$$h_t'(\tau) = \frac{1}{2\pi i}\int_\Gamma \frac{h_t(z)}{(z-\tau)^2}\mathrm{d}z, \quad \text{where } \Gamma = \{z : |z-\tau| = |\tau|\}.$$

By local-admissibility, we have

$$|h_t'(\tau)| \leqslant \frac{1}{|\tau|}\cdot \varepsilon|\tau|^2 B(t) = \varepsilon|\tau|B(t).$$

It follows from (3.4) that for $t \in T \cap (t_0(\varepsilon), t_G)$ and $|\theta| \leqslant \frac{\eta}{2}\delta(t)$, we have

$$(1-\varepsilon)B(t)|\theta| \leqslant |A(t+i\theta) - A(t)| \leqslant (1+\varepsilon)B(t)|\theta|.$$

Since $A(t) \in \mathbb{R}$, we also have there

$$\begin{aligned}\int_0^\theta \mathrm{Im}[A(t+i\varphi)]\mathrm{d}\varphi &\leqslant \int_0^\theta \mathrm{Im}[A(t) + i\varphi B(t)(1+\varepsilon)]\mathrm{d}\varphi \\ &= \frac{(1+\varepsilon)}{2}\theta^2 B(t),\end{aligned}$$

and similarly for the lower bound. □

For $J \subset \mathbb{T}$, we define the following integral

$$\mathcal{I}(H,t,J) := \frac{1}{2\pi}\int_J \frac{|A(t+i\theta) - A(t)|^2}{\exp\left(2\int_0^\theta \mathrm{Im}[A(t+i\varphi)]\mathrm{d}\varphi\right) - 1}\mathrm{d}\theta. \tag{3.5}$$

COROLLARY 3.3. *Let H be a local $\delta-$admissible function on T with $t_G \in \{0, \infty\}$, then for $t \in T$ we have*

$$\mathrm{Var}(n_f(r)) = \mathcal{I}(H,t,\mathbb{T}) \gtrsim \min\left\{\delta(t)B(t), \sqrt{B(t)}\right\}.$$

Proof. Applying Lemma 3.2 with $\varepsilon = \frac{1}{2}$, there exists an $\eta > 0$ so that for t sufficiently large and $|\theta| \leqslant \frac{\eta}{2}\delta(t)$,

$$\frac{|A(t+i\theta) - A(t)|^2}{\exp\left(2\int_0^\theta \operatorname{Im}[A(t+i\varphi)]\mathrm{d}\varphi\right) - 1} \geqslant \frac{\frac{1}{4}B^2(t)\theta^2}{\exp\left(\frac{3}{2}\theta^2 B(t)\right) - 1}.$$

Put $\Delta(t) := \min\{\frac{\eta}{2}\delta(t), \frac{1}{\sqrt{B(t)}}\}$, by the inequality $e^x - 1 \leqslant 4x$ which is valid for $x \in [0,2]$, we find

$$\mathcal{I}(H,t,\mathbb{T}) \geqslant \mathcal{I}(H,t,[-\Delta(t),\Delta(t)]) \geqslant \frac{1}{4}B^2(t)\int_{-\Delta(t)}^{\Delta(t)} \frac{\theta^2}{6B(t)\theta^2}\,\mathrm{d}\theta \geqslant \frac{1}{24}B(t)\Delta(t).$$

□

4. *Lower bound for the variance*

In this section, we prove Theorem 1.1. First we assume that b is non-decreasing. Below the letters λ, t, θ, y denote real quantities, and τ is a complex number. It will be convenient to put $e^t = r^2$.

4.1. *Normal values of t and the set $\mathcal{X}$*

We will now define a set $\mathcal{X} \subset \mathbb{R}^+$ whose *complement* is of *finite* Lebesgue measure, where the function B increases slowly. Since B is unbounded, we may choose a sequence $t_\ell \uparrow \infty$ so that

$$B(t_\ell) = \ell^6, \quad \ell \geqslant 1.$$

We then define a sequence of intervals $\{T_\ell\}_{\ell=1}^\infty$ by

$$T_\ell = [t_\ell, t_{\ell+1}], \quad |T_\ell| = t_{\ell+1} - t_\ell.$$

DEFINITION 4.1. The interval T_ℓ is *long* if

$$|T_\ell| \geqslant \frac{8}{\ell^2}, \tag{4.1}$$

otherwise it is *short*. For a long interval T_ℓ, we define

$$\mathring{T}_\ell := \left[t_\ell + \frac{2}{l^2}, t_{\ell+1} - \frac{2}{\ell^2}\right].$$

REMARK 4.1. Note that for long intervals, $\mathring{T}_\ell$ is non-empty.

DEFINITION 4.2. The set $\mathcal{X}$ of *normal* values of t is given by

$$\mathcal{X} := \bigcup_{T_\ell \text{ long}} \mathring{T}_\ell.$$

REMARK 4.2. Note that $\sum_{T_\ell \text{ short}} |T_\ell| < \infty$ and also $\sum_{T_\ell \text{ long}} |T_\ell \setminus \mathring{T}_\ell| < \infty$. Therefore, the Lebesgue measure of the set $\mathbb{R}^+ \setminus \mathcal{X}$ is finite.

REMARK 4.3. Throughout the proof we may need to take the value of ℓ to be sufficiently large, thus we may drop finitely many intervals $\mathring{T}_\ell$ from $\mathcal{X}$ without explicitly stating it.

4.2. *Proof of local admissibility for non-decreasing B*

Let T_ℓ be a long interval with $\ell \geqslant 4$. Since B is non-decreasing, we have for all $t \in T_\ell$

$$B(t_\ell) \leqslant B(t) \leqslant B(t_{\ell+1}) = (\ell+1)^6 \leqslant 4\ell^6 = 4B(t_\ell). \tag{4.2}$$

By the Lagrange formula for the remainder in the Taylor approximation for $\log \frac{H(y+\lambda)}{H(y)}$ near $\lambda = 0$, we have

$$\log \frac{H(y+\lambda)}{H(y)} = \lambda A(y) + \frac{\lambda^2}{2} B(c),$$

where $|c-y| \leqslant |\lambda|$. If $y+\lambda, y \in T_\ell$, then $c \in T_\ell$, and we deduce that

$$\left|\log \frac{H(y+\lambda)}{H(y)} - \lambda A(y)\right| \leqslant \frac{\lambda^2}{2} B(c) \leqslant 2\lambda^2 B(t_\ell). \tag{4.3}$$

4.2.1. *An adaptation of Rosenbloom's method.* Recall that

$$H(z) = \sum_{n=0}^{\infty} a_n^2 e^{nz},$$

where a_n are non-negative. Following Rosenbloom [29], we define for $t \in \mathbb{R}$, the random variable $X_t \in \mathbb{N}$ as follows:

$$\mathbb{P}[X_t = k] = \frac{a_k^2 e^{kt}}{H(t)}, \quad k \in \mathbb{N}.$$

Then

$$\mathbb{E}[X_t] = \frac{1}{H(t)} \sum_{k=0}^{\infty} k a_k^2 e^{kt} = \frac{H'(t)}{H(t)} = A(t),$$

and moreover

$$\begin{aligned}\operatorname{Var}(X_t) = \mathbb{E}\left[X_t^2\right] - (\mathbb{E}[X_t])^2 &= \frac{1}{H(t)} \sum_{k=0}^{\infty} k^2 a_k^2 e^{kt} - A^2(t) \\ &= \frac{H''(t)}{H(t)} - \left(\frac{H'(t)}{H(t)}\right)^2 = B(t).\end{aligned}$$

In order to approximate H by an appropriate (exponential) polynomial, we first prove the following lemma.

LEMMA 4.3. *For $t \in \mathring{T}_\ell$, and $|\tau - t| < \frac{1}{\sqrt{B(t)}}$, we have*

$$|E(\tau)| := \left|\sum_{|k-A(t)|>s\sqrt{B(t)}} a_k^2 e^{k\tau}\right| \leqslant 2H(\operatorname{Re}[\tau]) \exp\left(-\frac{1}{8}(s-4)^2\right),$$

for all $4 < s < B^{1/6}(t)$.

Proof. Since $|a_k^2 e^{k\tau}| = a_k^2 e^{k\operatorname{Re}[\tau]}$, by the triangle inequality, we may assume τ is real. Note that

$$\sum_{|k-A(t)|>s\sqrt{B(t)}} a_k^2 e^{k\tau} = H(\tau)\mathbb{P}\left[|X_\tau - A(t)| > s\sqrt{B(t)}\right],$$

and also

$$\mathbb{E}\left[e^{\lambda X_\tau}\right] = \frac{1}{H(\tau)}\sum_{k=0}^{\infty} e^{\lambda k} a_k^2 e^{k\tau} = \frac{H(\tau+\lambda)}{H(\tau)}.$$

Fix $\lambda > 0$ so that $\tau + \lambda \in T_\ell$. By Markov's inequality,

$$\begin{aligned}\mathbb{P}\left[X_\tau > A(t) + s\sqrt{B(t)}\right] &\leqslant \frac{\mathbb{E}\left[e^{\lambda X_\tau}\right]}{\exp\left(\lambda\left(A(t) + s\sqrt{B(t)}\right)\right)} \\ &= \exp\left(\log\frac{H(\tau+\lambda)}{H(\tau)} - \lambda\left(A(t) + s\sqrt{B(t)}\right)\right).\end{aligned}$$

By (4.3) and (4.2), we have

$$\begin{aligned}\log\frac{H(\tau+\lambda)}{H(\tau)} - \lambda A(t) &= \log\frac{H(\tau+\lambda)}{H(\tau)} - \lambda A(\tau) + \lambda[A(\tau) - A(t)] \\ &\leqslant \lambda[A(\tau) - A(t)] + 2\lambda^2 B(t_\ell) \\ &\leqslant 4\lambda|\tau - t|B(t_\ell) + 2\lambda^2 B(t_\ell) \leqslant 4\lambda\sqrt{B(t)} + 2\lambda^2 B(t).\end{aligned}$$

Therefore, by taking $\lambda = \frac{s-4}{4\sqrt{B(t)}}$, we get

$$\begin{aligned}\log\mathbb{P}\left[X_\tau > A(t) + s\sqrt{B(t)}\right] &\leqslant 4\lambda\sqrt{B(t)} + 2\lambda^2 B(t) - \lambda s\sqrt{B(t)} \\ &\leqslant -\frac{1}{8}(s-4)^2.\end{aligned}$$

Since $s < B^{1/6}(t)$, we obtain by (4.2)

$$\begin{aligned}\tau + \lambda &< t_{\ell+1} - \frac{2}{\ell^2} + \frac{1}{\sqrt{B(t)}} + \frac{1}{4B(t)^{1/3}} \leqslant t_{\ell+1} - \frac{2}{\ell^2} + \frac{1}{\sqrt{B(t_\ell)}} + \frac{1}{4B(t_\ell)^{1/3}} \\ &= t_{\ell+1} - \frac{2}{\ell^2} + \frac{1}{\ell^3} + \frac{1}{4\ell^2} < t_{\ell+1},\end{aligned}$$

in the same way we see that $\tau + \lambda > t_\ell$, so that $\tau + \lambda \in T_\ell$ as required. The bound for $\mathbb{P}[X_t < A(t) + s\sqrt{B(t)}]$ is obtained similarly. □

DEFINITION 4.4. We say that an exponential polynomial with *real* exponents is of *width* at most ω, if it is of the form

$$\sum_{k=0}^{m} c_k e^{\alpha_k z}, \quad \text{with } \max_k |\alpha_k| \leqslant \omega.$$

The following lemma adapts [**13**, Lemma 8] to exponential polynomials.

LEMMA 4.5. *Let $P(z)$ be an exponential polynomial of width at most ω, and non-negative coefficients. We have for any $t \in \mathbb{R}$, and $|\tau - t| \leqslant \frac{1}{5\omega}$ that*

$$\frac{3}{4}P(t) \leqslant |P(\tau)| \leqslant \frac{5}{4}P(t).$$

Proof. Suppose $P(z) = \sum_{k=0}^{m} c_k e^{\alpha_k z}$ with $\max_k |\alpha_k| \leqslant \omega$, and $c_k \geqslant 0$. Then

$$|P'(\tau)| = \left|\sum_{k=0}^{m} c_k \alpha_k e^{\alpha_k \tau}\right| = \left|\sum_{k=0}^{m} c_k e^{\alpha_k t} \alpha_k e^{\alpha_k(\tau - t)}\right| \leqslant \max_k \left|\alpha_k e^{\alpha_k(\mathrm{Re}[\tau]-t)}\right| \cdot P(t)$$
$$\leqslant \max_k |\alpha_k| e^{|\alpha_k||\tau - t|} \cdot P(t) \leqslant \omega e^{\omega|\tau - t|} P(t).$$

Now,

$$|P(\tau) - P(t)| = \left|\int_t^\tau P'(s)\,\mathrm{d}s\right| \leqslant \omega e^{\omega|\tau-t|} P(t) \cdot |\tau - t| \leqslant \frac{1}{5} e^{1/5} P(t) \leqslant \frac{1}{4} P(t). \qquad \square$$

4.2.2. *Proof of second part of Theorem* 1.1. The proof of Theorem 1.1 in the case $b \uparrow \infty$ follows by combining the next proposition with Corollary 3.3. We will use the Schwarz integral formula (see [**1**, Chapter 4, Section 6.3]), *if g is an analytic function in the closed unit disk, then*

$$g(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \mathrm{Re}[g(\zeta)] \frac{\mathrm{d}\zeta}{\zeta} + i\mathrm{Im}[g(0)], \tag{4.4}$$

for all $|z| < 1$, where $\mathbb{T}$ is the unit circle.

Proposition 4.6 (cf. [**13**, Theorem 10]). *If $B(t) \uparrow \infty$, then H is* local δ-admissible *on $\mathcal{X}$, with $\delta(t) = \frac{1}{\sqrt{B(t)}}$, that is, for any $\varepsilon > 0$ there exists $\eta > 0$, such that for $t \in \mathcal{X} \cap (t_0(\varepsilon), t_G)$ and $|w| \leqslant \frac{\eta}{\sqrt{B(t)}}$, we have*

$$\log \frac{H(t+w)}{H(t)} = wA(t) + \frac{1}{2} w^2 B(t) + h_t(w), \quad \textit{where } |h_t(w)| \leqslant \varepsilon |w|^2 B(t).$$

Proof. Let $\varepsilon > 0$ be given, and $\eta > 0$ depending on ε to be chosen later, also let $t \in \mathcal{X}$. Put

$$Q(\tau) := \sum_{|k - A(t)| \leqslant s\sqrt{B(t)}} a_k^2 e^{k\tau} = H(\tau) - E(\tau),$$

then by Lemma 4.3 for $|\tau - t| < \frac{1}{\sqrt{B(t)}}$, we have $|E(\tau)| \leqslant 2H(\mathrm{Re}[\tau]) \exp(-\frac{1}{8}(s-4)^2)$ for all $4 < |s| < B^{1/6}(t)$. We also write $P(\tau) = e^{A(t)\tau} Q(\tau)$ so that P is an exponential polynomial with non-negative coefficients of width at most $s\sqrt{B(t)}$. Choosing $s = 9$ so that $2\exp(-\frac{1}{8}(s-4)^2) < \frac{1}{4}$, we have

$$|E(\tau)| < \frac{1}{4} H(\mathrm{Re}[\tau]), \tag{4.5}$$

and in particular

$$\frac{3}{4} H(\mathrm{Re}[\tau]) < Q(\mathrm{Re}[\tau]) = H(\mathrm{Re}[\tau]) - E(\mathrm{Re}[\tau]) < \frac{5}{4} H(\mathrm{Re}[\tau]). \tag{4.6}$$

By Lemma 4.5 and the previous inequality for $\mathrm{Re}[\tau] = t$, we have for $|\tau - t| \leqslant \frac{1}{45\sqrt{B(t)}}$,

$$\left(\frac{3}{4}\right)^2 H(t) \leqslant \frac{3}{4} Q(t) \leqslant |Q(\tau)| \leqslant \frac{5}{4} Q(t) \leqslant \frac{5}{4} H(t). \tag{4.7}$$

From (4.5), (4.6), and (4.7), we get

$$|E(\tau)| \leqslant \frac{1}{4}\cdot\frac{4}{3}Q(\mathrm{Re}[\tau]) \leqslant \frac{1}{3}\cdot\frac{5}{4}H(t).$$

Thus, we have

$$|H(\tau)| \leqslant |Q(\tau)| + |E(\tau)| \leqslant \left[\frac{5}{4}+\frac{1}{3}\cdot\frac{5}{4}\right]H(t) \leqslant 2H(t),$$

and

$$|H(\tau)| \geqslant |Q(\tau)| - |E(\tau)| \geqslant \left[\left(\frac{3}{4}\right)^2 - \frac{1}{3}\cdot\frac{5}{4}\right]H(t) \geqslant \frac{1}{8}H(t).$$

We found that

$$-\log 8 \leqslant \log\frac{|H(\tau)|}{H(t)} \leqslant \log 2. \tag{4.8}$$

Now let us define the analytic function

$$\phi(w) = \log H(t+w) - \log H(t), \quad \phi(w) = \sum_{n=1}^{\infty}\phi_n w^n, \quad |w| \leqslant \frac{1}{45\sqrt{B(t)}} =: \lambda_0.$$

By (4.8), we have that $|\mathrm{Re}[\phi(w)]| \leqslant \log 8$, and therefore by (4.4) for $|w| \leqslant \frac{1}{2}\lambda_0$, we get (since $\phi(0) = 0$)

$$|\phi(w)| \leqslant \frac{1+\frac{1}{2}}{1-\frac{1}{2}}\cdot\log 8 < 7.$$

By Cauchy's estimates $|\phi_n| \leqslant 7(\frac{2}{\lambda_0})^n$, and therefore, for $|w| \leqslant \frac{1}{4}\lambda_0$

$$\begin{aligned}\left|\phi(w) - wA(t) - \frac{1}{2}w^2B(t)\right| &= \left|\sum_{n=3}^{\infty}\phi_n w^n\right| \\ &\leqslant 7\sum_{n=3}^{\infty}\left(\frac{2w}{\lambda_0}\right)^n \leqslant 7\cdot 8\left(\frac{w}{\lambda_0}\right)^3\sum_{n=0}^{\infty}\frac{1}{2^n} = 112\left(\frac{w}{\lambda_0}\right)^3.\end{aligned}$$

Thus in order to obtain the result it remains to choose $\eta = \frac{\varepsilon}{112\cdot 45^3}$. □

4.2.3. *Proof of first part of Theorem 1.1.* Fix an entire function $G(z) = \sum_{n\geqslant 0} a_n^2 z^n$, with non-negative coefficients a_n, and recall that

$$a(r) = r\frac{G'(r)}{G(r)}, \quad b(r) = ra'(r).$$

In order to get a lower bound for the variance without any assumptions on the function $b(r)$, we will use some results about G obtained by the Wiman–Valiron method (see [**13**]). We recall some of the terminology regarding entire functions: $\mu(r) = \max_n\{a_n^2 r^n\}$ is called the *maximal term* of G, and $N(r) = \max\{n : a_n^2 r^n = \mu(r)\}$ the *central index*.

One of the main results of the Wiman–Valiron method is that there is a set $\mathcal{N} \subset \mathbb{R}^+$, such that $G(z)$ has desirable properties if $|z| \in \mathcal{N}$, and that $\mathbb{R}^+\backslash\mathcal{N}$ has finite logarithmic measure (see [**13**, Sections 2 and 3]). We fix a parameter $\gamma \in (0, \frac{1}{2})$; the set $\mathcal{N}$ will depend on G and γ. By [**13**, Theorem 2] for $r \in \mathcal{N}$ we have for all $n \in \mathbb{N}$

$$a_n^2 r^n \leqslant \mu(r)\exp\left(-\frac{k^2}{(|k|+N(r))^{1+\gamma}}\right), \quad \text{where } k = n - N(r),$$

hence the summands $a_n^2|z|^n$ of the series $G(z)$ with $|z| = r$, corresponding to the 'window' of indices

$$\{n \in \mathbb{N} : |n - N(r)| \leqslant K(r)\} \quad \text{with } K(r) := N(r)^{\frac{1+\gamma}{2}}, \tag{4.9}$$

are the largest ones. In particular, this implies that $N(r)$ and $a(r)$ are asymptotically comparable, see Claim 4.7. Note that by applying the change of variables $x \mapsto \sqrt{x}$, the set $\{r > 0 : r^2 \notin \mathcal{N}\}$ is of finite logarithmic measure as well.

It is known that for any $\gamma > 0$ we have

$$b(r) \leqslant a(r)^{1+\gamma} \tag{4.10}$$

outside a set of finite logarithmic measure (see [**29**, Lemma 1]). Thus, we may and will assume that (4.10) is satisfied for all $r \in \mathcal{N}$.

Claim 4.7. For $r \in \mathcal{N}$, we have

$$a(r) = N(r) + O(K(r)), \quad \text{as } r \to \infty.$$

Proof. Since $a_n^2 \geqslant 0$, we have that $\max_{|z|=r} |G(z)| = G(r)$ and hence, by [**13**, Theorem 12] with $q = 1, f = G, k = K(r)$, we have

$$\frac{r}{N(r)} G'(r) = G(r) + O\left(\frac{K(r)}{N(r)}\right) G(r),$$

and therefore

$$a(r) = N(r) + O(K(r)).$$

□

Fixing $r \in \mathcal{N}$ we can approximate the function G near $|z| = r$ by a polynomial of degree about $K(r)$ (cf. Lemma 4.3). This allows us to obtain rather precise Taylor expansion asymptotics for $\log G(e^\tau)$ near $\tau = 2\log r$.

The following lemma is a special case of [**13**, Lemma 2].

LEMMA 4.8. *Suppose $e^t = r^2 \in \mathcal{N}$ and K, N are as above, then for $|\tau - t| \leqslant \frac{2}{K(e^t)}$*

$$\left| \sum_{|k-N(e^t)|>K(e^t)} a_k^2 e^{k\tau} \right| \leqslant \frac{1}{4} H(\mathrm{Re}[\tau]).$$

Combining the lemma above with Claim 4.7, we find that we can replace N by A.

COROLLARY 4.9. *There exists a constant $s > 0$, so that for $e^t = r^2 \in \mathcal{N}$ and $|\tau - t| \leqslant \frac{2}{K(e^t)}$,*

$$\left| \sum_{|k-A(t)|>sK(e^t)} a_k^2 e^{k\tau} \right| \leqslant \frac{1}{4} H(\mathrm{Re}[\tau]).$$

Repeating the proof of Proposition 4.6 using the previous corollary instead of Lemma 4.3, we obtain the following result.

PROPOSITION 4.10. *Any entire function G with non-negative coefficients is local $\delta-$admissible on a set $\mathcal{N} = \mathcal{N}_G$ with $\delta(t) = \frac{B(t)}{K^3(e^t)}$. Here $\mathbb{R}^+ \backslash \mathcal{N}$ is a set of finite logarithmic measure.*

We are now ready to prove the first part of Theorem 1.1. We recall that f is a Gaussian entire function with covariance kernel G.

Proof of Theorem 1.1. Choose γ sufficiently small so that $\frac{3}{2}(1+\gamma)+\gamma \leqslant \frac{3}{2}+\varepsilon$. By (4.10), (4.9), and Claim 4.7, we have

$$\frac{B^2(t)}{K^3(e^t)} \leqslant \frac{2B^2(t)}{A(t)^{\frac{3}{2}(1+\gamma)}} \leqslant 2\sqrt{B(t)} \quad \text{as } t\to\infty, \quad e^t \in \mathcal{N},$$

and therefore by Corollary 3.3, with $e^t = r^2$, $H(z) = G(e^z)$, and the previous proposition

$$\mathrm{Var}(n_f(r)) = \mathcal{I}(H,t,\mathbb{T}) \geqslant \min\left\{\delta(t)B(t), \sqrt{B(t)}\right\} \gtrsim \frac{B^2(t)}{K^{3+\gamma}(e^t)} \gtrsim \frac{b^2(r^2)}{a^{\frac{3}{2}(1+\gamma)+\gamma}(r^2)}. \qquad \square$$

5. *Asymptotics of the variance — proof of Theorem* 1.4

Let f be a Gaussian analytic function with covariance function G, recall $H(t) = G(e^t)$, and that

$$t_G = \log R_G \in \{0,\infty\}, \quad A(z) = (\log H(z))' = a(e^z), \quad B(z) = A'(z) = b(e^z).$$

In this section, we find the asymptotic growth of $\mathrm{Var}(n_f(r))$ as $r \to R_G^-$ when the covariance kernel G is type I admissible.

5.1. *Type I admissible covariance functions*

To find precise asymptotics for the integral (3.1), we make certain assumptions on the function H, motivated by the Hayman [**12**] admissibility condition.

Definition 5.1. We call G *type I admissible* if the function H has the following properties.

(1) $B(t) \to \infty$ as $t \to t_G^-$.
(2) $A(t) = O(B^2(t))$ as $t \to t_G^-$.
(3) There is a constant $C_G > 2$ such that

$$\log\frac{H(t+i\theta)}{H(t)} = i\theta A(t) - \frac{1}{2}\theta^2 B(t)(1+o(1)), \quad \text{as } t \to t_G^-,$$

uniformly for all $|\theta| \leqslant \delta(t) := \sqrt{C_G \frac{\log B(t)}{B(t)}}$.

(4) $|H(t+i\theta)| = O(\frac{H(t)}{B^2(t)})$ and $|H'(t+i\theta)| = O(\frac{H'(t)}{B^2(t)})$ as $t \to t_G^-$, uniformly in $|\theta| \in [\delta(t),\pi]$.

Remark 5.1. By the proof of [**12**, Lemma 4], it follows that an admissible function is local δ-admissible on $\mathbb{R}^+$ with $\delta(t) = \frac{c}{\sqrt{B(t)}}$ with some constant $c > 0$.

Remark 5.2. Since $B = A' = (\log H)''$, it follows that Assumption 2 puts a restriction on the minimal growth rate of the function H.

Remark 5.3. The choice of the constant C_G is not important, we choose it in this way so that Assumptions 3 and 4 will agree for $\theta = \delta(t)$.

5.2. *Asymptotics of the variance*

We put $r^2 = e^t$, and split the integral in (3.1) into two parts:

$$\begin{aligned}\operatorname{Var}(n_f(r)) = \mathcal{I}(H,t,\mathbb{T}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|A(t+i\theta)-A(t)|^2}{\exp\left(2\int_0^\theta \operatorname{Im}[A(t+i\varphi)]\mathrm{d}\varphi\right)-1}\,\mathrm{d}\theta \\ &= \mathcal{I}(H,t,[-\delta(t),\delta(t)]) + \mathcal{I}(H,t,\mathbb{T}\setminus[-\delta(t),\delta(t)]) \\ &=: J_1(r)+J_2(r),\end{aligned}$$

where we used the definition of $\mathcal{I}$ in (3.5).

5.2.1. *Evaluating J_1.* For $|\theta| \leqslant \delta(t)$, Assumption 3 implies

$$|A(t+i\theta)-A(t)| = B(t)|\theta|(1+o(1)), \quad t\to t_G^-, \tag{5.1}$$

uniformly in $|\theta|\leqslant\delta(t)$. Since $A(t)\in\mathbb{R}$,

$$\begin{aligned}\int_0^\theta \operatorname{Im}[A(t+i\varphi)]\mathrm{d}\varphi &= \int_0^\theta \operatorname{Im}[A(t)+i\varphi B(t)(1+o(1))]\mathrm{d}\varphi \\ &\sim -\frac{1}{2}\theta^2 B(t), \quad t\to t_G^-,\end{aligned} \tag{5.2}$$

also uniformly in $|\theta|\leqslant\delta(t)$. Making the change of variables

$$u = \sqrt{-2\int_0^\theta \operatorname{Im}[A(t+i\varphi)]\mathrm{d}\varphi} \sim \theta\sqrt{B(t)}$$

yields by (5.1) and (5.2)

$$J_1(r) \sim \frac{\sqrt{B(t)}}{2\pi}\int_{|u|\leqslant\delta(t)\sqrt{B(t)(1+o(1))}} \frac{u^2}{\exp(u^2)-1}\,\mathrm{d}u.$$

By Assumption 1 and the definition of δ, we have that $\delta(t)\sqrt{B(t)}\to\infty$, as $t\to t_G^-$. Thus

$$\int_{|u|\leqslant\delta(t)\sqrt{B(t)(1+o(1))}} \frac{u^2}{\exp(u^2)-1}\,\mathrm{d}u \sim \int_{\mathbb{R}} \frac{u^2}{\exp(u^2)-1}\,\mathrm{d}u \overset{(x=u^2)}{=} \int_0^\infty \frac{\sqrt{x}}{e^x-1}\mathrm{d}x, \quad \text{as } t\to t_G^-.$$

Using the formula for the Riemann zeta function:

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}\mathrm{d}x,$$

we find that the integral on the right-hand side is equal to $\frac{\sqrt{\pi}}{2}\zeta(\frac{3}{2})$. This gives the main term in the asymptotic behavior of the variance

$$J_1(r) \sim \frac{\zeta(\frac{3}{2})}{4\sqrt{\pi}}\sqrt{B(t)}, \quad t\to t_G^-. \tag{5.3}$$

5.2.2. *Bounding J_2.* Again we write $r^2=e^t$. The admissibility assumptions allow us to control the size of G also in the range $\delta(t)\leqslant|\theta|\leqslant\pi$. Note the identity

$$\frac{|A(t+i\theta)-A(t)|^2}{\exp\left(2\int_0^\theta \operatorname{Im}[A(t+i\varphi)]\mathrm{d}\varphi\right)-1} = \frac{|H(t)H'(t+i\theta)-H(t+i\theta)H'(t)|^2}{H^2(t)(H^2(t)-|H^2(t+i\theta)|)}.$$

By Assumptions 1 and 4, we have that $H^2(t) - |H^2(t+i\theta)| \sim H^2(t)$. Therefore, again by Assumption 4, we get

$$\begin{aligned}\frac{|A(t+i\theta) - A(t)|^2}{\exp\left(2\int_0^\theta \mathrm{Im}[A(t+i\varphi)]\mathrm{d}\varphi\right) - 1} &\leqslant (2+o(1))\left[\frac{|H(t)H'(t+i\theta)|^2}{H^4(t)} + \frac{|H(t+i\theta)H'(t)|^2}{H^4(t)}\right]\\ &= (2+o(1))\left[\frac{|H'(t+i\theta)|^2}{H^2(t)} + \frac{|H(t+i\theta)|^2}{H^2(t)}A^2(t)\right]\\ &= O\left(\frac{A^2(t)}{B^4(t)}\right).\end{aligned}$$

Finally, by Assumption 2

$$J_2(r) = O\left(\frac{A^2(t)}{B^4(t)}\right) = O(1), \quad t \to t_G^-,$$

and combining this with (5.3) we conclude that

$$\mathrm{Var}(n_f(r)) = J_1(r) + J_2(r) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}}\sqrt{b(r^2)}, \quad r \to R_G^-,$$

thus proving Theorem 1.4.

6. *Upper bound for the variance — proof of Theorem* 1.2

The upper bound for the variance is derived from the following algebraic identity.

Claim 6.1. For $a_1, a_2, a_3 \in \mathbb{R}$ and $b_1, b_2, b_3 \in \mathbb{C}$, the following holds

$$\left|a_1 b_3 - \overline{b_1} b_2\right|^2 = \left(a_1 a_3 - |b_2|^2\right)\left(a_1 a_2 - |b_1|^2\right) - a_1 \cdot \det\begin{pmatrix} a_1 & \overline{b_1} & \overline{b_2} \\ b_1 & a_2 & \overline{b_3} \\ b_2 & b_3 & a_3\end{pmatrix}.$$

Proof. One can directly show this equality holds, but here we will give a geometric reasoning which holds when the matrix in the expression above is positive definite (which is the case we use). In that case, we may take vectors (v_1, v_2, v_3) so that their Gram matrix satisfies

$$\begin{pmatrix} a_1 & \overline{b_1} & \overline{b_2} \\ b_1 & a_2 & \overline{b_3} \\ b_2 & b_3 & a_3\end{pmatrix} = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_2, v_1\rangle & \langle v_3, v_1\rangle \\ \langle v_1, v_2\rangle & \langle v_2, v_2\rangle & \langle v_3, v_2\rangle \\ \langle v_1, v_3\rangle & \langle v_2, v_3\rangle & \langle v_3, v_3\rangle\end{pmatrix} \tag{6.1}$$

and we denote the corresponding Gram determinant by $\mathrm{Gram}(v_1, v_2, v_3)$. On one side, we have

$$\mathrm{dist}^2(v_3, \mathrm{span}\{v_1, v_2\}) = \frac{\mathrm{Gram}(v_1, v_2, v_3)}{\mathrm{Gram}(v_1, v_2)} = \frac{\mathrm{Gram}(v_1, v_2, v_3)}{a_1 a_2 - |b_1|^2}.$$

On the other hand, using the Gram–Schmidt process to find orthogonal vectors (w_1, w_2, w_3), we get

$$\begin{aligned} w_1 &= v_1,\\ w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1,\\ w_3 &= v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2.\end{aligned}$$

Taking into account (6.1), we find that

$$w_1 = v_1,\ w_2 = v_2 - \frac{\overline{b_1}}{a_1} v_1,\ \langle w_2, w_2 \rangle = a_2 - \frac{|b_1|^2}{a_1} = \frac{a_1 a_2 - |b_1|^2}{a_1}$$

and

$$\langle v_3, w_2 \rangle = \left\langle v_3, v_2 - \frac{\overline{b_1}}{a_1} v_1 \right\rangle = \langle v_3, v_2 \rangle - \frac{b_1}{a_1} \langle v_3, v_1 \rangle = \overline{b_3} - \frac{b_1 \overline{b_2}}{a_1} = \frac{\overline{a_1 b_3 - \overline{b_1} b_2}}{a_1}.$$

Finally

$$\begin{aligned} \operatorname{dist}^2(v_3, \operatorname{span}\{v_1, v_2\}) &= \langle v_3, v_3 \rangle - \frac{|\langle v_3, w_1 \rangle|^2}{\langle w_1, w_1 \rangle} - \frac{|\langle v_3, w_2 \rangle|^2}{\langle w_2, w_2 \rangle} \\ &= a_3 - \frac{|b_2|^2}{a_1} - \frac{\left| a_1 b_3 - \overline{b_1} b_2 \right|^2}{a_1 \left(a_1 a_2 - |b_1|^2 \right)}, \end{aligned}$$

and we get the required identity by multiplying the two expressions for $\operatorname{dist}^2(v_3, \operatorname{span}\{v_1, v_2\})$ by $a_1(a_1 a_2 - |b_1|^2)$. By continuity the identity extends to positive semi-definite matrices. □

Put

$$g_j(\theta) = \sum_{n \in \mathbb{Z}} n^j c_n^2 e^{in\theta}, \quad j \in \{0, 1, 2\},$$

and write g for g_0.

Claim 6.2. Assume that $c_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$ and that $\sum_{n \in \mathbb{Z}} n^2 c_n^2 < \infty$, then

$$|g(0) g_1(\theta) - g_1(0) g(\theta)|^2 \leqslant \left(g(0) g_2(0) - g_1^2(0) \right) \left(g^2(0) - |g(\theta)|^2 \right).$$

Proof. By Claim 6.1, and using $\overline{g_j(\theta)} = g_j(-\theta)$, we have

$$\begin{aligned} |g(0) g_1(\theta) - g_1(0) g(\theta)|^2 &= \left(g(0) g_2(0) - g_1^2(0) \right) \left(g^2(0) - |g(\theta)|^2 \right) \\ &\quad - g(0) \cdot \det \begin{pmatrix} g(0) & g(\theta) & g_1(0) \\ g(-\theta) & g(0) & g_1(-\theta) \\ g_1(0) & g_1(\theta) & g_2(0) \end{pmatrix}. \end{aligned}$$

Let $h_j(\theta) = \sum_{n \in \mathbb{Z}} \xi_n n^j c_n e^{in\theta}$ for $j \in \{0, 1\}$, put $V = (h(0), h(-\theta), h_1(0))$ (again with $h = h_0$), then we have

$$\left(\mathbb{E}\left[V_j \overline{V_k} \right] \right)_{j,k=1}^3 = \begin{pmatrix} g(0) & g(\theta) & g_1(0) \\ g(-\theta) & g(0) & g_1(-\theta) \\ g_1(0) & g_1(\theta) & g_2(0) \end{pmatrix}.$$

Thus the matrix above is a covariance matrix, hence positive semi-definite and its determinant is non-negative. □

6.1. *Proof of Theorem* 1.2

Now let f be a Gaussian analytic function with covariance kernel G, whose radius of convergence is R_G. For the proof again it will be more convenient to use the exponential parameterization

$$H(z) \coloneqq G(e^z) = \sum_{n=0}^{\infty} a_n^2 e^{nz},$$

so that

$$t_G = \log R_G, \quad A(z) = \frac{H'(z)}{H(z)} = (\log H(z))', \quad B(z) = A'(z) = (\log H(z))''.$$

We have (see Claim A.2 in the Appendix)

$$\mathrm{Var}(n_f(r)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|H(t)H'(t+i\theta) - H(t+i\theta)H'(t)|^2}{H^2(t)(H^2(t) - |H^2(t+i\theta)|)}\mathrm{d}\theta, \quad \text{where } e^t = r^2.$$

The following claim will finish the proof of Theorem 1.2.

COROLLARY 6.3. *For all $t < t_G$ and $\theta \in [-\pi, \pi]$*

$$\frac{|H(t)H'(t+i\theta) - H(t+i\theta)H'(t)|^2}{H^2(t)(H^2(t) - |H^2(t+i\theta)|)} \leqslant B(t).$$

Proof. The result follows immediately from Claim 6.2, by taking $c_n^2 = a_n^2 e^{nt}$ for $n \in \mathbb{N}$, and $c_n = 0$ otherwise, so that

$$g(\theta) = \sum_{n=0}^{\infty} a_n^2 e^{nt} e^{in\theta} = H(t+i\theta), \quad g_1(\theta) = H'(t+i\theta), \quad g_2(\theta) = H''(t+i\theta),$$

and

$$\frac{g(0)g_2(0) - g_1^2(0)}{g^2(0)} = \frac{H(t)H''(t) - (H'(t))^2}{H^2(t)} = (\log H(t))'' = B(t).$$

Note that for $t < t_G$ we have that $\sum_{n\in\mathbb{Z}} n^2 c_n^2 = \sum_{n\geqslant 0} n^2 a_n^2 e^{nt} < \infty$ by the definition of t_G. □

7. *Gaussian entire functions with large variance — proof of Theorem 1.7*

In this section, we will prove Theorem 1.7. For an entire function G, we recall that

$$a_G(r) = r(\log G(r))', \quad b_G(r) = r a_G'(r),$$

where in this section, we will sometimes add the subscript G in order to distinguish between functions a, b associated with *different* entire functions G. Given a type II admissible covariance function G (see Section 7.2), we will construct a Gaussian entire function $\widetilde{f}$ with a transcendental covariance function $\widetilde{G}$, which is similar to G, that is

$$a_{\widetilde{G}}(r) \asymp_L a_G(r), \text{ and } b_{\widetilde{G}}(r) \asymp_L b_G(r),$$

moreover, $\widetilde{G}$ is a restriction of G (see Definition 1.6). We then prove

$$\mathrm{Var}(n_f(r)) \asymp_L b_{\widetilde{G}}(r^2), \tag{7.1}$$

thus showing that the bound in Theorem 1.2 is sharp up to a constant (and an exceptional set of values of r).

7.1. *Estimates for sums of Gaussians*

In this section, we fix the parameters $A, B, s > 0$ and p a positive integer.

Claim 7.1. Suppose $s \geqslant 1$, and $p \leqslant \sqrt{B}$. For $j \in \{0,1,2\}$, we have

$$\frac{1}{\sqrt{e}}\left(A - s\sqrt{B}\right)^j \leqslant \sum_{|kp-A|\leqslant s\sqrt{B}} (kp)^j \exp\left(-\frac{(kp-A)^2}{2B}\right) \leqslant e^2\left[\frac{\sqrt{2\pi B}}{p}\left(A^j + B\cdot \mathbf{1}_{\{j=2\}}\right) + 1\right],$$

where the sum runs over integers k, and

$$\mathbf{1}_{\{j=2\}} = \begin{cases} 0, & j \neq 2; \\ 1, & j = 2. \end{cases}$$

Proof. Since $p \leqslant \sqrt{B}$, the sum is non-empty and the lower bound follows. In the other direction, put $\phi(x) = \exp(-\frac{(x-A)^2}{2B})$. If $(k-1)p \geqslant A$, then we have

$$\begin{aligned}(kp)^j \phi(kp) &\leqslant \frac{1}{p}\left(\frac{k}{k-1}\right)^j \int_{(k-1)p}^{kp} x^j \phi(x)\, \mathrm{d}x \\ &\leqslant \frac{1}{p}\exp\left(\frac{j}{k-1}\right)\int_{(k-1)p}^{kp} x^j \phi(x)\, \mathrm{d}x \leqslant \frac{e^2}{p}\int_{(k-1)p}^{kp} x^j \phi(x)\, \mathrm{d}x,\end{aligned}$$

where in the last inequality we used $k \geqslant 2$. Thus,

$$\sum_{A+p \leqslant kp \leqslant s\sqrt{B}} (kp)^j \phi(kp) \leqslant \frac{e^2}{p}\int_A^{A+s\sqrt{B}} x^j \phi(x)\, \mathrm{d}x.$$

Using a similar argument for $(k-1)p \leqslant A$, we get (adding the integral from $-p$ to p twice)

$$\begin{aligned}\sum_{|kp-A| \leqslant s\sqrt{B}} (kp)^j \phi(kp) &\leqslant \frac{e^2}{p}\int_{A-s\sqrt{B}}^{A+s\sqrt{B}} x^j \phi(x)\, \mathrm{d}x + e^2 \\ &\overset{y=\frac{x-A}{\sqrt{B}}}{=} \frac{e^2\sqrt{B}}{p}\int_{-s}^{s}\left(y\sqrt{B}+A\right)^j e^{-\frac{1}{2}y^2}\, \mathrm{d}y + e^2 \\ &\leqslant e^2\left[\frac{\sqrt{2\pi B}}{p}\left(A^j + B\cdot \mathbf{1}_{\{j=2\}}\right) + 1\right].\end{aligned}$$

□

Claim 7.2. Suppose $s \geqslant 1$, $p \leqslant \sqrt{B}$. We have,

$$\frac{p^2}{e} \leqslant \sum_{\substack{|k_1p-A| \leqslant s\sqrt{B} \\ |k_2p-A| \leqslant s\sqrt{B}}} (k_1p - k_2p)^2 \exp\left(-\frac{(k_1p-A)^2}{2B} - \frac{(k_2p-A)^2}{2B}\right) \leqslant \frac{24e^4\pi B^2}{p^2},$$

where the double sum runs over integers k_1, k_2.

Proof. The lower bound follows since $p \leqslant \sqrt{B}$, so that the double sum is not empty. Put $\psi(x,y) = (x-y)^2\phi(x)\phi(y)$ so that

$$\sum_{\substack{|k_1p-A| \leqslant s\sqrt{B} \\ |k_2p-A| \leqslant s\sqrt{B}}} (pk_1 - pk_2)^2 \exp\left(-\frac{(k_1p-A)^2}{2B} - \frac{(k_2p-A)^2}{2B}\right) = \sum_{\substack{|k_1p-A| \leqslant s\sqrt{B} \\ |k_2p-A| \leqslant s\sqrt{B}}} \psi(k_1p, k_2p).$$

Writing $n_1 = k_1 + k_2$ and $n_2 = k_1 - k_2$, we find that

$$\sum_{\substack{|k_1p-A|\leqslant s\sqrt{B}\\|k_2p-A|\leqslant s\sqrt{B}}} \psi(k_1p,k_2p) \leqslant \sum_{\substack{|n_2p|\leqslant 2s\sqrt{B}\\|n_1p-2A|\leqslant 2s\sqrt{B}}} \psi\left(\frac{(n_2-n_1)p}{2},\frac{(n_2+n_1)p}{2}\right)$$

$$= \sum_{|n_2p|\leqslant 2s\sqrt{B}} (n_2p)^2\exp\left(-\frac{(n_2p)^2}{4B}\right)\cdot \sum_{|n_1p-2A|\leqslant 2s\sqrt{B}} \exp\left(-\frac{(n_1p-2A)^2}{4B}\right), \tag{7.2}$$

where we used the following identity in the last equality

$$\psi\left(\frac{x-y}{2},\frac{x+y}{2}\right) = y^2\exp\left(-\frac{y^2}{4B}\right)\exp\left(-\frac{(x-2A)^2}{4B}\right).$$

Bounding both factors in (7.2) using the previous claim, we conclude that

$$\sum_{\substack{|k_1p-A|\leqslant s\sqrt{B}\\|k_2p-A|\leqslant s\sqrt{B}}} \psi(k_1p,k_2p) \leqslant e^4\left(\frac{\sqrt{2\pi\cdot 2B}}{p}\cdot 2B+1\right)\left(\frac{\sqrt{2\pi\cdot 2B}}{p}+1\right)$$

$$\leqslant e^4\left(\frac{\sqrt{4\pi B}}{p}\right)^2\cdot(2B+1)\cdot 2 \leqslant \frac{24e^4\pi B^2}{p^2}.$$

□

7.2. *Type II admissible functions and their Taylor coefficients*

As before, we associate with G the functions

$$H(z) = G(e^z),\quad A(t) = \frac{H'(t)}{H(t)},\quad B(t) = A'(t) = \frac{H''(t)}{H(t)} - \left(\frac{H'(t)}{H(t)}\right)^2.$$

In this section, we will use a stronger version of the Hayman admissibility condition, which allows us to improve the error term in [**12**, Theorem I] (see the lemma below).

DEFINITION 7.3. We say that an *entire* function G is *type II admissible* if the function H has the following properties.

(1) $B\uparrow\infty$
(2) $A(t) = O(B^2(t))$ as $t\to\infty$.
(3) There exist constants $C_G > 2$ and $\varepsilon > 0$ such that

$$\log\frac{H(t+i\theta)}{H(t)} = i\theta A(t) - \frac{1}{2}\theta^2 B(t) + \Delta(t,\theta),\quad \forall|\theta|\leqslant\delta(t),$$

where

$$\delta(t) := \sqrt{C_G\frac{\log B(t)}{B(t)}},\quad |\Delta(t,\theta)|\leqslant B^{3/2-\varepsilon}(t)|\theta|^3.$$

(4) $|H(t+i\theta)| = O(\frac{H(t)}{B(t)})$ as $t\to\infty$, uniformly in $|\theta|\in[\delta(t),\pi]$.

We again put $r^2 = e^t$.

REMARK 7.1. Throughout this section, in order make the expressions shorter, we may suppress the dependence of H, A, B, and other parameters on t inside the proofs.

LEMMA 7.4 (cf. [12, Theorem 1]). *Suppose that $G(z) = \sum_{n=0}^{\infty} a_n^2 z^n$ is type II admissible, then there exists an $\varepsilon \in (0,1)$ such that for all t sufficiently large,*

$$a_n^2 e^{nt} = \frac{H(t)}{\sqrt{2\pi B(t)}}\left[\exp\left(-\frac{(n-A(t))^2}{2B(t)}\right) + O\left(\frac{1}{B^{\varepsilon}(t)}\right)\right], \quad \text{as} \quad t \to \infty,$$

uniformly in n.

Proof. Put $\delta = \delta(t)$. By Cauchy's formula

$$a_n^2 e^{nt} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(t+i\theta)e^{-in\theta}\,\mathrm{d}\theta,$$

which we write as $a_n^2 e^{nt} = I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi}\int_{|\theta|\leqslant\delta} H(t+i\theta)e^{-in\theta}\,\mathrm{d}\theta, \quad I_2 = \frac{1}{2\pi}\int_{\delta<|\theta|\leqslant\pi} H(t+i\theta)e^{-in\theta}\,\mathrm{d}\theta.$$

By Assumption 4, we have, uniformly in n,

$$I_2 = O\left(\frac{H(t)}{B(t)}\right).$$

By Assumption, we have

$$\begin{aligned}
I_1 &= \frac{H}{2\pi}\int_{|\theta|\leqslant\delta} \exp\left(-i\theta(n-A) - \frac{1}{2}\theta^2 B + \Delta(\theta)\right)\mathrm{d}\theta \\
&= \frac{H}{2\pi}\int_{|\theta|\leqslant\delta} (1+O(\Delta(\theta)))\exp\left(-i\theta(n-A) - \frac{1}{2}\theta^2 B\right)\mathrm{d}\theta \\
&= \frac{H}{2\pi}\int_{|\theta|\leqslant\delta} \exp\left(-i\theta(n-A) - \frac{1}{2}\theta^2 B\right)\mathrm{d}\theta + O\left(B^{3/2-\varepsilon}\delta^4\right) \\
&= \frac{H}{2\pi}\int_{|\theta|\leqslant\delta} \exp\left(-i\theta(n-A) - \frac{1}{2}\theta^2 B\right)\mathrm{d}\theta + O\left(\frac{(\log B)^2}{B^{\frac{1}{2}+\varepsilon}}\right).
\end{aligned}$$

In order to find the asymptotic behavior of I_1, we make the change of variables

$$y = \theta\sqrt{\frac{B}{2}}, \quad \alpha := \frac{n-A}{\sqrt{B/2}},$$

and obtain (uniformly in n)

$$\begin{aligned}
I_1 &= \frac{H}{\pi\sqrt{2B}}\int_{|y|\leqslant\delta\sqrt{\frac{B}{2}}} \exp\left(-y^2 - i\alpha y\right)\mathrm{d}y + O\left(\frac{1}{B^{\frac{1}{2}(1+\varepsilon)}}\right) \\
&= \frac{H}{\pi\sqrt{2B}}\left[\int_{\mathbb{R}} - \int_{|y|\geqslant\delta\sqrt{\frac{B}{2}}}\right]\exp\left(-y^2 - i\alpha y\right)\mathrm{d}y + O\left(\frac{1}{B^{\frac{1}{2}(1+\varepsilon)}}\right) \\
&= \frac{H}{\sqrt{2\pi B}}\exp\left(-\frac{1}{4}\alpha^2\right) + O\left(\exp\left(-\left(\delta\sqrt{\frac{B}{2}}\right)^2\right)\right) + O\left(\frac{1}{B^{\frac{1}{2}(1+\varepsilon)}}\right)
\end{aligned}$$

$$= \frac{H}{\sqrt{2\pi B}} \exp\left(-\frac{1}{4}\alpha^2\right) + O\left(\frac{1}{B}\right) + O\left(\frac{1}{B^{\frac{1}{2}(1+\varepsilon)}}\right)$$
$$= \frac{H}{\sqrt{2\pi B}} \exp\left(-\frac{(n-A)^2}{2B}\right) + O\left(\frac{1}{B^{\frac{1}{2}(1+\varepsilon)}}\right).$$

□

7.3. *Construction of the covariance function $\widetilde{G}$*

Recall that in Section 4.1 we defined a sequence of intervals $\{T_\ell\}_{\ell=1}^{\infty}$ so that

$$T_\ell = [t_\ell, t_{\ell+1}], \quad |T_\ell| = t_{\ell+1} - t_\ell, \quad B(t_\ell) = \ell^6, \quad \ell \geqslant 1.$$

Moreover, the interval T_ℓ is *long* if $|T_\ell| \geqslant \frac{8}{\ell^2}$, and in that case

$$\mathring{T}_\ell = \left[t_\ell + \frac{2}{l^2}, t_{\ell+1} - \frac{2}{\ell^2}\right].$$

The set $\mathcal{X}$ of normal values is given by

$$\mathcal{X} := \bigcup_{T_\ell \text{ long}} \mathring{T}_\ell.$$

We also remind that the Lebesgue measure of the set $\mathbb{R}^+ \backslash \mathcal{X}$ is finite.

We now fix the sequences $p_\ell = \ell^3 = \sqrt{B(t_\ell)}$, and $s_\ell = c_1\sqrt{\log \ell}$ (see Proposition 7.5) and define the following sets

$$\mathcal{I}_\ell := [A(t_\ell), A(t_{\ell+1})) \cap \mathbb{N}, \quad \mathcal{I}_\ell^{\mathbf{p}} := \{n \in \mathcal{I}_\ell \, : \, p_\ell \mid n\},$$

and the corresponding exponential polynomials

$$P_\ell(t) = \sum_{n \in \mathcal{I}_\ell^{\mathbf{p}}} a_n^2 e^{nt}. \tag{7.3}$$

The function $\widetilde{G}$ is constructed as follows

$$\widetilde{G}(e^t) = \widetilde{H}(t) = \sum_{\ell=0}^{\infty} P_\ell(t) =: \sum_{n=0}^{\infty} \delta_n a_n^2 e^{nt}. \tag{7.4}$$

For $t \in \mathring{T}_\ell$, the main indices n corresponding to t are included in the following window

$$\mathcal{I}_\ell^{\mathbf{p}}(t) := \left\{n \in \mathcal{I}_\ell^{\mathbf{p}} \, : \, |n - A(t)| \leqslant s_\ell\sqrt{B(t)}\right\}.$$

Finally, we put

$$R_\ell^{(j)}(t) = \sum_{n \in \mathcal{I}_\ell^{\mathbf{p}}(t)} n^j a_n^2 e^{nt}, \quad \overline{R}_\ell^{[j]}(t) = \sum_{n \in \mathcal{I}_\ell^{\mathbf{p}}(t)} (n - A(t))^j a_n^2 e^{nt},$$

and

$$Q_\ell^{(j)}(\tau, t) = \sum_{|n-A(t)| > s_\ell\sqrt{B(t)}} n^j a_n^2 e^{n\tau}, \quad \overline{Q}_\ell^{[j]}(\tau, t) = \sum_{|n-A(t)| > s_\ell\sqrt{B(t)}} (n - A(t))^j a_n^2 e^{n\tau}.$$

PROPOSITION 7.5. *For $t \in \mathcal{X}$ and $|\tau - t| \leqslant \frac{1}{2\sqrt{B(t)}}$, we have*

$$\max_{j \in \{0,1,2\}} \left\{\left|Q_\ell^{(j)}(\tau)\right|, \left|\overline{Q}_\ell^{[j]}(\tau, t)\right|\right\} \leqslant \frac{H(\mathrm{Re}[\tau])}{B^3(t)}.$$

REMARK 7.2. The choice of the exponent 3 is arbitrary, it can be replaced by any large positive number.

Proof. Fix an interval $\mathring{T}_\ell \subset \mathcal{X}$. By Lemma 4.3, we will have

$$|Q_\ell(\tau)| \leqslant \sum_{|k-A(t)|>s\sqrt{B(t)}} a_k^2 e^{k\mathrm{Re}[\tau]} \leqslant 2H(\mathrm{Re}[\tau]) \exp\left(-\frac{1}{8}(s-4)^2\right),$$

as long as $4 < s < B^{1/6}(t_\ell)$. Note that in order to guarantee that

$$2\exp\left(-\frac{1}{8}(s-4)^2\right) \leqslant \frac{1}{\ell^{48}} = \frac{1}{B^8(t_\ell)} \leqslant \frac{1}{B^8(t)},$$

it is sufficient to take $s = 4 + 2\sqrt{2\log 2 + 96\log(\ell)}$. Thus, we may choose any $c_1 > 8\sqrt{6}$ in the definition of s_ℓ, and obtain

$$|Q_\ell(\tau)| \leqslant \frac{H(\mathrm{Re}[\tau])}{B^8(t)}.$$

For $\tau \in \mathbb{C}$ which satisfies $|\tau - t| < \frac{1}{2\sqrt{B(t)}}$, consider the contour $\Gamma = \{z : |z-\tau| = \frac{1}{2\sqrt{B(t)}}\}$. Using Cauchy's integral formula, we get

$$\left|Q_\ell^{(j)}(\tau)\right| = \left|\frac{1}{2\pi i}\int_\Gamma \frac{Q_\ell(z)}{(z-\tau)^{j+1}}\mathrm{d}z\right| \lesssim \frac{H(\mathrm{Re}[\tau])}{[B(t)]^{8-\frac{1}{2}j}}.$$

In addition, since the coefficients a_n^2 are non-negative, we also note that when $|\mathrm{Re}[\tau] - t| \leqslant \frac{1}{2\sqrt{B(t)}}$, and $j \in \{0,1,2\}$,

$$\left|Q_\ell^{(j)}(\tau)\right| \leqslant \left|Q_\ell^{(j)}(\mathrm{Re}[\tau])\right| \leqslant \frac{H(\mathrm{Re}[\tau])}{B^7(t)}. \tag{7.5}$$

To bound $|\overline{Q}_\ell^{[j]}(\tau,t)|$, note that $\overline{Q}_\ell^{[0]}(\tau,t) = Q_\ell(\tau,t)$,

$$\overline{Q}_\ell^{[1]}(\tau,t) = Q_\ell'(\tau,t) - A(t)\cdot Q_\ell(\tau,t),$$

$$\overline{Q}_\ell^{[2]}(\tau,t) = Q_\ell''(\tau,t) - 2A(t)\cdot Q_\ell'(\tau,t) + A(t)^2\cdot Q_\ell(\tau,t),$$

and use Assumption 2. □

PROPOSITION 7.6. *There are constants $c, C > 0$ such that for $t \in \mathcal{X}$, and $j \in \{0,1,2\}$, we have*

$$c\frac{H(t)(A(t))^j}{\sqrt{B(t)}} \leqslant R_\ell^{(j)}(t) \leqslant C\frac{H(t)(A(t))^j}{\sqrt{B(t)}},$$

and

$$\overline{R}_\ell^{[j]}(t) \leqslant C\frac{H(t)\left(s_\ell\sqrt{B(t)}\right)^j}{\sqrt{B(t)}}.$$

Proof. By Lemma 7.4, Claim 7.1, and (4.10), we obtain

$$R_\ell^{(j)}(t) = \sum_{|kp_\ell - A|<s_\ell\sqrt{B}} (kp)^j a_{kp}^2 e^{kpt}$$

$$
\begin{aligned}
&= \frac{H}{\sqrt{2\pi B}} \sum_{|kp_\ell - A| < s_\ell\sqrt{B}} (kp)^j \left[\exp\left(-\frac{(kp-A)^2}{2B}\right) + O\left(\frac{1}{B^\varepsilon}\right)\right] \\
&\lesssim \frac{H}{\sqrt{B}} \left[\frac{\sqrt{B}}{p_\ell}\left(A^j + B\cdot \mathbf{1}_{\{j=2\}}\right) + 1 + \frac{s_\ell\sqrt{B}}{p_\ell}\cdot \frac{\left(A + s_\ell\sqrt{B}\right)^j}{B^\varepsilon}\right] \\
&\lesssim \frac{H}{p_\ell}\cdot A^j\left[1 + s_\ell\cdot\frac{1}{B^\varepsilon}\right] \lesssim \frac{H(t)}{\sqrt{B(t)}}\cdot A(t)^j\left[1 + s_\ell\cdot \frac{1}{B^\varepsilon(t)}\right].
\end{aligned}
$$

The lower bound is obtained in a similar way, using the lower bound in Claim 7.1. To bound $\overline{R}_\ell^{[j]}(t)$, we simply use

$$
\overline{R}_\ell^{[j]}(t) \leqslant \sum_{n\in\mathcal{I}_\ell^{\mathbf{P}}(t)} |n - A(t)|^j a_n^2 e^{nt} \leqslant \left(s_\ell\sqrt{B}\right)^j \overline{R}_\ell(t) \lesssim \frac{H(t)\left(s_\ell\sqrt{B}\right)^j}{\sqrt{B(t)}}. \qquad \square
$$

Now we are ready to prove that G and $\widetilde{G}$ are similar.

Lemma 7.7. *We have*

$$
a_{\widetilde{H}}(r) \asymp_L a(r), \quad b_{\widetilde{H}}(r) \asymp_L b(r).
$$

Proof. Let $c, C > 0$ denote constants. Fix $t \in \mathcal{X}$, it is sufficient to show that

$$
cA(t) \leqslant A_{\widetilde{H}}(t) \leqslant CA(t), \quad cB(t) \leqslant B_{\widetilde{H}}(t) \leqslant CB(t).
$$

It follows from the identities

$$
A_{\widetilde{H}}(t) = \frac{\widetilde{H}'(t)}{\widetilde{H}(t)}, \quad B_{\widetilde{H}}(t) = A'_{\widetilde{H}}(t) = \frac{\widetilde{H}''(t)}{\widetilde{H}(t)} - \left(\frac{\widetilde{H}'(t)}{\widetilde{H}(t)}\right)^2,
$$

and the definition of $\widetilde{H}$ in (7.4), that

$$
A_{\widetilde{H}}(t) = \frac{\sum_{n=0}^\infty n\delta_n a_n^2 e^{nt}}{\widetilde{H}(t)}, \quad B_{\widetilde{H}}(t) = \frac{\sum_{n,m=0}^\infty (n-m)^2 \delta_n\delta_m a_n^2 a_m^2 e^{(n+m)t}}{2\left(\widetilde{H}(t)\right)^2}.
$$

Note that for $j \in \{0,1,2\}$, we have

$$
\left|\widetilde{H}^{(j)}(t) - R_\ell^{(j)}(t)\right| \leqslant Q_\ell^{(j)}(t,t). \tag{7.6}
$$

Therefore, by Propositions 7.5 and 7.6, we have

$$
A_{\widetilde{H}}(t) = \frac{R'_\ell(t) + (H'(t) - R'_\ell(t))}{R_\ell(t) + (H(t) - R_\ell(t))} \lesssim \frac{C\frac{H(t)A(t)}{\sqrt{B(t)}} + \frac{H(t)}{B^3(t)}}{c\frac{H(t)}{\sqrt{B(t)}} - \frac{H(t)}{B^3(t)}} \lesssim A(t),
$$

and similarly for the lower bound. Put

$$
\Sigma_1 := \sum_{n,m\in\mathcal{I}_\ell^{\mathbf{P}}(t)} (n-m)^2 a_n^2 a_m^2 e^{(n+m)t},
$$

and note that clearly

$$
\Sigma_1 \leqslant 2B_{\widetilde{H}}(t)\left(\widetilde{H}(t)\right)^2.
$$

In addition,

$$
\begin{aligned}
2B_{\widetilde{H}}(t)\left(\widetilde{H}(t)\right)^2 &= \sum_{n,m=0}^{\infty} (n-m)^2\delta_n\delta_m a_n^2 a_m^2 e^{(n+m)t} \leqslant \sum_{n,m\in\mathcal{I}_\ell^{\mathrm{P}}(t)} (n-m)^2 a_n^2 a_m^2 e^{(n+m)t} \\
&+ 2\cdot \sum_{n\in\mathcal{I}_\ell^{\mathrm{P}}(t),\,|m-A(t)|>s_\ell\sqrt{B(t)}} (n-m)^2 a_n^2 a_m^2 e^{(n+m)t} \\
&+ \sum_{\substack{|n-A(t)|>s_\ell\sqrt{B(t)}\\ |m-A(t)|>s_\ell\sqrt{B(t)}}} (n-m)^2 a_n^2 a_m^2 e^{(n+m)t} \\
&=: \Sigma_1+\Sigma_2+\Sigma_3.
\end{aligned}
$$

By Claim 7.8 below, we have $cH^2(t)\leqslant \Sigma_1 \leqslant CH^2(t)$, so that again by (7.6) and Propositions 7.5 and 7.6, we have

$$
B(t)\widetilde{H}^2(t) \lesssim \Sigma_1 \lesssim B(t)\widetilde{H}^2(t).
$$

Writing

$$
(n-m)^2 \leqslant \frac{1}{2}\left((n-A)^2+(m-A)^2\right),
$$

we get, using Proposition 7.5,

$$
\begin{aligned}
\Sigma_3 &\leqslant \left(\sum_{|n-A(t)|>s_\ell\sqrt{B(t)}} (n-A)^2 a_n^2 e^{nt}\right)\left(\sum_{|m-A(t)|>s_\ell\sqrt{B(t)}} a_m^2 e^{mt}\right) = \overline{Q}_\ell^{[2]}(t,t)\overline{Q}_\ell^{[0]}(t,t) \\
&\leqslant \frac{H^2(t)}{B^6(t)} \lesssim \frac{\widetilde{H}^2(t)}{B^5(t)}.
\end{aligned}
$$

In addition, combining Propositions 7.5 and 7.6, we have

$$
\begin{aligned}
\Sigma_2 \leqslant &\left(\sum_{n\in\mathcal{I}_\ell^{\mathrm{P}}(t)} (n-A)^2 a_n^2 e^{nt}\right)\left(\sum_{|m-A(t)|>s_\ell\sqrt{B(t)}} a_m^2 e^{mt}\right) \\
&+ \left(\sum_{n\in\mathcal{I}_\ell^{\mathrm{P}}(t)} a_n^2 e^{nt}\right)\left(\sum_{|m-A(t)|>s_\ell\sqrt{B(t)}} (m-A)^2 a_m^2 e^{mt}\right) \\
=&\overline{R}_\ell^{[2]}(t)\overline{Q}_\ell^{[0]}(t,t)+\overline{R}_\ell^{[0]}(t)\overline{Q}_\ell^{[2]}(t,t) \\
\lesssim&\frac{H(t)\cdot s_\ell^2 B(t)}{\sqrt{B(t)}}\cdot\frac{H(t)}{B(t)^3}+\frac{H(t)}{\sqrt{B(t)}}\cdot\frac{H(t)}{B(t)^3} \\
\lesssim&\frac{H^2(t)}{B^2(t)} \lesssim \frac{\widetilde{H}^2(t)}{B(t)}.
\end{aligned}
$$

Thus,

$$
B(t) \lesssim B_{\widetilde{H}}(t) \lesssim B(t). \qquad \square
$$

Claim 7.8. There are constants $c, C > 0$, so that for $t \in \mathcal{X}$, we have

$$cH^2(t) \leqslant \Sigma_1 \leqslant CH^2(t),$$

where

$$\Sigma_1 := \sum_{n,m \in \mathcal{I}_\ell^{\mathbf{P}}(t)} (n-m)^2 a_n^2 a_m^2 e^{(n+m)t}.$$

Proof. By Lemma 7.4, we have

$$\Sigma_1 = \frac{H^2}{2\pi B} \sum_{n,m \in \mathcal{I}_\ell^{\mathbf{P}}(t)} (n-m)^2 \left(e^{-\frac{(n-A)^2}{2B}} + O\left(\frac{1}{B^\varepsilon}\right)\right)\left(e^{-\frac{(m-A)^2}{2B}} + O\left(\frac{1}{B^\varepsilon}\right)\right),$$

and therefore

$$|\Sigma_1 - S_1| \leqslant S_2,$$

where

$$S_1 := \frac{H^2}{B} \sum_{n,m \in \mathcal{I}_\ell^{\mathbf{P}}(t)} (n-m)^2 \exp\left(-\frac{(n-A)^2}{2B} - \frac{(m-A)^2}{2B}\right),$$

$$S_2 := \frac{H^2}{B^{1+\varepsilon}} \sum_{n,m \in \mathcal{I}_\ell^{\mathbf{P}}(t)} (n-m)^2.$$

By Claim 7.2, we have

$$S_1 \asymp \frac{H^2}{B} \cdot B = H^2.$$

For $n, m \in \mathcal{I}_\ell^{\mathbf{P}}(t)$, $|n-m| \leqslant 2s_\ell\sqrt{B}$, thus we have

$$S_2 \lesssim \frac{H^2}{B^\varepsilon} \sum_{n,m \in \mathcal{I}_\ell^{\mathbf{P}}(t)} s_\ell^2 \lesssim \frac{H^2}{B^\varepsilon} \cdot s_\ell^4 = o(H^2).$$

□

7.4. *Proof of Theorem 1.7*

Let

$$\widetilde{f}(z) = \sum_{\ell=0}^{\infty} \sum_{n \in \mathcal{I}_\ell^{\mathbf{P}}} \xi_n a_n z^n$$

be a Gaussian entire function with covariance function $\widetilde{G}$. In order to prove that $\mathrm{Var}(n(r)) \gtrsim p_\ell^2 \gtrsim B(t)$, it is enough to show that $n(r) = kp_\ell$ for two different values of k, with probability at least $c > 0$ each. By Rouché's theorem, it is enough to show that the term $\xi_{kp_\ell} a_{kp_\ell} z^{kp_\ell}$ dominates all other terms.

Proposition 7.9. *There is a constant $c > 0$, so that for every $t \in \mathcal{X}$ the probability of the event $\{n_{\widetilde{f}}(r) = kp_\ell\}$ is at least c for two different values $k \in \mathbb{N}$.*

Proof. Put

$$m_1 = \max\{m \in \mathcal{I}_\ell^{\mathbf{P}} : m < A\}, \quad m_2 = \min\{m \in \mathcal{I}_\ell^{\mathbf{P}} : m > A\},$$

and note that

$$\frac{1}{2}p_\ell \leqslant \max\{|m_1 - A|, |m_2 - A|\} \leqslant p_\ell, \quad p_\ell \leqslant |m_1 - m_2| \leqslant 2p_\ell.$$

Define the events

$$E_j = \left\{ \left|\xi_{m_j} a_{m_j} z^{m_j}\right| > \left|f(z) - \xi_{m_j} a_{m_j} z^{m_j}\right| \right\}, \quad j \in \{1,2\}.$$

We will prove $\mathbb{P}[E_1] > c$, the proof for E_2 is similar. The result will then follow by Rouché's theorem. We have the crude bound:

$$\mathbb{E}\left|f(z) - \xi_{m_j} a_{m_j} z^{m_j}\right| \leqslant \sum_{n \neq m_1} \mathbb{E}|\xi_n| \delta_n a_n r^n = \frac{\sqrt{\pi}}{2}[S_1 + S_2],$$

where

$$S_1 = \sum_{n \neq m_1, n \in \mathcal{I}_\ell^{\mathrm{P}}} a_n r^n, \quad S_2 = \sum_{|n-A| > s_\ell \sqrt{B}} \delta_n a_n r^n.$$

By Lemma 7.4, we have, uniformly in n,

$$a_n^2 e^{nt} = \frac{H}{\sqrt{2\pi B}} \left[\exp\left(-\frac{(n-A)^2}{2B} \right) + O\left(\frac{1}{B^\varepsilon} \right) \right].$$

Since $\sqrt{x+y} \leqslant \sqrt{x} + \sqrt{y}$, we get by Claim 7.1,

$$\begin{aligned} S_1 &\leqslant \sum_{n \in \mathcal{I}_\ell^{\mathrm{P}}} a_n e^{\frac{1}{2}tn} \lesssim \frac{\sqrt{H}}{B^{1/4}} \sum_{n \in \mathcal{I}_\ell^{\mathrm{P}}} \exp\left(-\frac{(n-A)^2}{4B} \right) + \frac{\sqrt{H}}{B^{1/4}} \cdot \frac{s_\ell \sqrt{B}}{p_\ell} \cdot O\left(\frac{1}{B^\varepsilon} \right) \\ &\lesssim \frac{\sqrt{H}}{B^{1/4}} \cdot \frac{\sqrt{B}}{p_\ell} + O\left(\frac{\sqrt{H}}{B^{1/4+\varepsilon}} \right) \lesssim \frac{\sqrt{H(t)}}{B^{1/4}(t)}. \end{aligned}$$

Now let $\eta = \frac{1}{A} \leqslant \frac{1}{2\sqrt{B}}$, where the inequality holds (for t sufficiently large) by (4.10). Writing

$$r = e^{\frac{1}{2}t}, \quad \tau = t + \eta, \quad r^n = e^{\frac{1}{2}tn} = e^{\frac{1}{2}(\tau-\eta)n} = e^{\frac{1}{2}\tau n} e^{-\frac{1}{2}\eta n},$$

and using the Cauchy–Schwarz inequality, Proposition 7.5, and (4.3), we have

$$\begin{aligned} S_2^2 &\leqslant \sum_{|n-A| > s_\ell \sqrt{B}} a_n^2 e^{\tau n} \cdot \sum_{|n-A| > s_\ell \sqrt{B}} e^{-\eta n} \leqslant Q_\ell^{(0)}(\tau, t) \cdot \sum_{n=0}^{\infty} e^{-\eta n} \\ &\leqslant \frac{H(\mathrm{Re}[\tau])}{B^3} \cdot \frac{2}{\eta} \lesssim H(t) \exp\left(\eta \cdot A + C\eta^2 B\right) \cdot \frac{A}{B^3} \\ &\lesssim H \cdot \exp\left(\frac{CB}{A^2} \right) \cdot \frac{A}{B^3} \lesssim H \cdot \frac{A}{B^3}, \end{aligned}$$

where we again used (4.10) in the last inequality. Therefore, by Assumption 2, we get

$$S_2 \lesssim \frac{\sqrt{H(t)}}{B^{1/4}(t)}.$$

We conclude by Markov's inequality that for $C > 0$ sufficiently large, we have

$$\mathbb{P}\left[\left|f(z) - \xi_{m_j} a_{m_j} z^{m_j}\right| > \frac{\sqrt{\pi}}{2}[S_1 + S_2] \right] \leqslant \mathbb{P}\left[\left|f(z) - \xi_{m_j} a_{m_j} z^{m_j}\right| > \frac{C\sqrt{H(t)}}{B^{1/4}(t)} \right] < \frac{1}{2}.$$

Finally, again by Lemma 7.4

$$\left(a_{m_j} r^{m_j}\right)^2 \gtrsim \frac{H(t)}{\sqrt{B(t)}},$$

and thus with probability at least $c > 0$ we have that $|\xi_{m_j} a_{m_j} z^{m_j}| > |f(z) - \xi_{m_j} a_{m_j} z^{m_j}|$. $\square$

Appendix. *Formulas for the expectation and variance*

For the convenience of the reader, here we give proofs for the formulas of the expected value and variance of the number of zeros in a disk from [16, p. 195]. Let $\Pi \subset \mathbb{C}$ be a compact subset of the plane, and denote by $n_f(\Pi)$ the number of zeros of the Gaussian analytic function f in Π (we assume that Π is contained inside the domain of convergence of f).

We denote by K_f the covariance kernel of f, and by J_f the normalized covariance kernel, given by

$$J_f(z, w) = \frac{K_f(z, w)}{\sqrt{K_f(z, z)}\sqrt{K_f(w, w)}}.$$

To avoid technicalities, we assume that $K_f(z, z) > 0$ for all $z \in \Pi$.

We first recall the Edelman–Kostlan formula [14, p. 25], which states

$$\mathbb{E}[n_f(\Pi)] = \frac{1}{4\pi} \int_{\Pi} \Delta_z \log K_f(z, z)\, \mathrm{d}m(z), \tag{A.1}$$

where m is the Lebesgue measure on $\mathbb{C}$ and $\Delta_z = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$ is the usual Laplace operator. In addition, we have (see [31, Theorem 3.1] or [24, Lemma 2.3])

$$\mathrm{Var}(n_f(\Pi)) = \tfrac{1}{16\pi^2} \iint_{\Pi\times\Pi} \Delta_z \Delta_w \mathrm{Li}_2\left(|J_f(z, w)|^2\right) \mathrm{d}m(z)\, \mathrm{d}m(w), \tag{A.2}$$

where

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is the dilogarithm function.

We now derive more explicit formulas when $\Pi = r\mathbb{D} := \{|w| \leqslant r\}$. Recall that $n(r) := n_f(r\mathbb{D})$, $K_f(z, w) = G(z\overline{w})$,

$$H(z) = G(e^z), \quad A(z) = \frac{H'(z)}{H(z)}, \quad B(z) = A'(z),$$

and that we put $e^t = r^2$.

Claim A.1. We have

$$\mathbb{E}[n(r)] = a\left(r^2\right) = A(t).$$

Proof. Writing the Laplace operator in polar coordinates and differentiating the covariance kernel, we get

$$\Delta \log K_f(z, z) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right] \log G\left(r^2\right).$$

Now, by (A.1) we have,

$$\mathbb{E}[n(r)] = \frac{1}{2}\int_0^r \frac{\partial}{\partial s}\left[s\frac{\partial}{\partial s}\right]\log G(s^2)\,\mathrm{d}s = \frac{1}{2}\left[s\frac{\partial}{\partial s}\right]\log G(s^2)\bigg|_{s=r} = r^2\frac{G'(r^2)}{G(r^2)} = a(r^2) = A(t).$$

□

Claim A.2 (cf. [4, Lemma 5]). We have

$$\begin{aligned}\mathrm{Var}(n_f(r)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|G(r^2)G'(r^2e^{i\theta})r^2e^{i\theta} - G(r^2e^{i\theta})G'(r^2)r^2\right|^2}{G^2(r^2)(G^2(r^2) - |G^2(r^2e^{i\theta})|)}\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|H(t)H'(t+i\theta) - H(t+i\theta)H'(t)|^2}{H^2(t)(H^2(t) - |H^2(t+i\theta)|)}\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|A(t+i\theta) - A(t)|^2}{\exp\left(2\cdot\mathrm{Im}\left[\int_0^\theta A(t+i\varphi)d\varphi\right]\right) - 1}\,\mathrm{d}\theta.\end{aligned}$$

Proof. Applying Stokes' Theorem to (A.2), we get

$$\mathrm{Var}(n(r)) = -\tfrac{1}{4\pi^2}\oint_{\partial(r\mathbb{D})}\oint_{\partial(r\mathbb{D})}\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial\overline{w}}\mathrm{Li}_2\left(|J_f(z,w)|^2\right)\mathrm{d}\bar{z}\,\mathrm{d}\overline{w}. \tag{A.3}$$

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\mathrm{Li}_2(\zeta) = \frac{1}{\zeta}\log\frac{1}{1-\zeta},$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial\overline{w}}\mathrm{Li}_2\left(|J_f(z,w)|^2\right) &= \frac{\partial}{\partial\overline{w}}\mathrm{Li}_2\left(\frac{K_f(z,w)K_f(w,z)}{K_f(z,z)K_f(w,w)}\right) = \frac{\partial}{\partial\overline{w}}\mathrm{Li}_2\left(\frac{G(z\overline{w})G(\bar{z}w)}{G(z\bar{z})G(w\overline{w})}\right)\\ &= \log\left(1 - \frac{G(z\overline{w})G(\bar{z}w)}{G(z\bar{z})G(w\overline{w})}\right)\frac{(wG(z\overline{w})G'(w\overline{w}) - zG(w\overline{w})G'(z\overline{w}))}{G(w\overline{w})G(z\overline{w})},\end{aligned}$$

hence, after some simplifications

$$\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial\overline{w}}\mathrm{Li}_2\left(|J_f(z,w)|^2\right) = \frac{(wG(z\bar{z})G'(\bar{z}w) - zG(\bar{z}w)G'(z\bar{z}))(zG(w\overline{w})G'(z\overline{w}) - wG(z\overline{w})G'(w\overline{w}))}{G(z\bar{z})G(w\overline{w})[G(z\bar{z})G(w\overline{w}) - G(z\overline{w})G(\bar{z}w)]}.$$

Using the parametrization $z = re^{i\theta_1}$, $w = re^{i\theta_2}$ in (A.3) (note the contour $\partial(r\mathbb{D})$ is oriented *clockwise*) and after some additional simplifications, we get

$$\mathrm{Var}(n_f(r)) = \tfrac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\left|G(r^2)G'(r^2e^{i(\theta_1-\theta_2)})r^2e^{i(\theta_1-\theta_2)} - G(r^2e^{i(\theta_1-\theta_2)})G'(r^2)r^2\right|^2}{G(r^2)^2\left[G(r^2)^2 - \left|G(r^2e^{i(\theta_1-\theta_2)})\right|^2\right]}\,\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Making a change of variables $\theta = \theta_1 - \theta_2$ and integrating out the other variable, we get

$$\mathrm{Var}(n_f(r)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|G(r^2)G'(r^2e^{i\theta})r^2e^{i\theta} - G(r^2e^{i\theta})G'(r^2)r^2\right|^2}{G^2(r^2)(G^2(r^2) - |G^2(r^2e^{i\theta})|)}\,\mathrm{d}\theta.$$

Now using $r^2 = e^t$, we find that

$$\frac{\left|G(r^2)G'(r^2e^{i\theta})r^2e^{i\theta} - G(r^2e^{i\theta})G'(r^2)r^2\right|^2}{G^2(r^2)(G^2(r^2) - |G^2(r^2e^{i\theta})|)} = \frac{|H(t)H'(t+i\theta) - H(t+i\theta)H'(t)|^2}{H^2(t)(H^2(t) - |H^2(t+i\theta)|)}$$
$$= \frac{|A(t+i\theta) - A(t)|^2}{\exp\left(-2\cdot \mathrm{Re}\Big[i\int_0^\theta A(t+i\varphi)d\varphi\Big]\right) - 1},$$

and since $\mathrm{Im}(z) = -\mathrm{Re}(iz)$, we get the required result. □

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Avner Kiro
Weizmann Institute of Science
Rehovot
Israel

Current Address:
Department of Mathematics
University of Toronto
Toronto, ON
Canada

akiro@math.toronto.edu

Alon Nishry
School of Mathematical Sciences
Tel Aviv University
Tel Aviv
Israel

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