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SURVEY PAPER

STOCHASTIC DYNAMICS AND FRACTIONAL OPTIMAL CONTROL OF QUASI INTEGRABLE HAMILTONIAN SYSTEMS WITH FRACTIONAL DERIVATIVE DAMPING

Lincong Chen [1], Fang Hu [2], Weiqiu Zhu [3]

Abstract

In the present survey, some progress in the stochastic dynamics and fractional optimal control of quasi integrable Hamiltonian systems with fractional derivative damping is reviewed. First, the stochastic averaging method for quasi integrable Hamiltonian systems with fractional derivative damping under various random excitations is briefly introduced. Then, the stochastic stability, stochastic bifurcation, first passage time and reliability, and stochastic fractional optimal control of the systems studied by using the stochastic averaging method are summarized. The focus is placed on the effects of fractional derivative order on the dynamics and control of the systems. Finally, some possible extensions are pointed out.

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Key Words and Phrases: fractional derivative damping, quasi integrable Hamiltonian system, stochastic averaging, stochastic stability, stochastic bifurcation, first-passage time, fractional optimal control

1. Introduction

Fractional calculus is the generalization of the classical calculus and has a wide range of applications in various fields of science and engineering in the past decades. Applications in mechanics mainly involve fractional

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derivatives, which are an adequate tool to model the frequency-dependent damping behavior of materials and physical systems exhibiting non-local and history-dependent properties. In this regard, original contribution is due to Gemant [34], who suggested a fractional derivative constitutive relationship to model cyclic-deformation tests performed on viscoelastic material specimens. Later, Caputo [15] reported experimental validity when he used fractional derivative for the description of the behavior of viscoelastic materials. Bagley and Torvik [9]-[11] provided the theoretical basis for the use of fractional derivative model to characterize viscoelasticity in the early 1980s. So far, many more researchers such as Adolfsson [1], Chen [16], Koeller [47], Koh and Kelly [48], Mainardi [56], Makris and Constantinou [59], Pritz [66], Papoulia and Kelly [63], Shen and Soong [76], Freed and Diethelm [32], Rossikhin and Shitikova [73] etc., gave further insight into the potential applications of fractional derivative when applied to the viscoelasticity modeling. Moreover, Gorenflo and Mainardi [35], Kempfle et al. [44], Makris and Constantinou [59], Rossikhin and Shitikova [70], [74], and Shimizu and Zhang [78] provided an excellent review of the research in this field. Recently, Chen et al. [18] published a monograph on the applications of fractional derivative in engineering. Mainardi [58] provided an historical perspective on fractional calculus in linear viscoelasticty. Mainardi and Gorenflo [57] gave a tutorial survey on the time-fractional derivatives in relaxation processes.

At the same time, many authors studied the dynamical systems with damping described by fractional derivatives. However, most of them are limited to the deterministic cases (see, e.g., [30], [52], [61], [62], [67], [71], [72], [85], [79], [77], [88]). Since stochastic perturbations are ubiquitous, it is necessary to investigate the fractionally damped systems subject to random excitations. To this end, a frequency-domain approach was pursued by Spanos and Zeldin [83] to study the random vibration of fractionally damped systems, and by Rüdinger [75] for tuned mass damper with fractional derivatives. Alternatively, a time domain Duhamel integral closed-form expression was obtained to analyze the stochastic response by using the Laplace-transform-based technique [2], [3], [40] or the Fourier-transform-based technique [90]. More recently, Spanos and Evangelatos [81] obtained the response of a single-degree-of-freedom (SDOF) nonlinear system with fractional derivative damping using time-domain simulation and statistical linearization technique. Di Paola et al. [29] studied the stationary and non-stationary stochastic response of linear fractional viscoelastic systems subjected to stationary and non-stationary random excitations, the key idea of which was generalized to fractionally damped Duffing oscillator subjected to a stochastic input [31].

In the last fifteen years, a stochastic averaging method for quasi Hamiltonian systems has been developed and effectively applied to study the stochastic dynamics and nonlinear stochastic optimal control of multi-degree-of -freedom (MDOF) strongly nonlinear stochastic systems by the present third author and his co-workers, [94]. In recent years, this stochastic averaging method was extended to the quasi integrable Hamiltonian systems with fractional derivative damping under excitations of Gaussian white noise [39], combined harmonic function and Gaussian white noise [20], wide-band real noise [27] and bounded narrow-band noise [37] and the generalized stochastic averaging method was applied to study the stochastic response [27], [37], [39], stochastic stability [17], [19], stochastic bifurcation [26], [36], [37], first passage time and reliability [20], [21], [25], [28], and fractional stochastic optimal control [38] of strongly nonlinear stochastic systems with fractional derivative damping.

In the present paper, the stochastic averaging method for quasi integrable Hamiltonian systems with fractional derivative damping and its applications in the stochastic dynamics and fractional optimal control of quasi integrable Hamiltonian systems with fractional derivative damping are reviewed. The effects of fractional derivative order on the dynamics and control are emphasized. Some possible extensions are pointed out.

2. Stochastic averaging method

Consider an nDOF quasi Hamiltonian system with fractional derivative damping governed by the following equations:

$$\begin{aligned}\dot{Q}_i =& \frac{\partial H}{\partial P_i}\\ \dot{P}_i =& -\frac{\partial H}{\partial Q_i} - \varepsilon d_{ij}(\mathbf{Q},\mathbf{P})\frac{\partial H}{\partial P_j} - \varepsilon C_i(\mathbf{Q},\mathbf{P}) D^{\alpha_i} Q_i(t)\\ &+ \varepsilon f_{ik}(\mathbf{Q},\mathbf{P})\cos\beta_k(t) + \varepsilon^{1/2} h_{ie}(\mathbf{Q},\mathbf{P})\xi_e(t),\\ i,j =& 1,2,...,n;\ k=1,2,...,m;\ e=1,2,...,w,\end{aligned} \tag{1}$$

where $\mathbf{Q} = [Q_1, Q_2, ..., Q_n]^T$ and $\mathbf{P} = [P_1, P_2, ..., P_n]^T$; $\beta_k = \Omega_k t + \varphi_k$; Q_i and P_i are generalized displacements and moments, respectively; $H(\mathbf{Q},\mathbf{P})$ is a twice differentiable Hamiltonian; ε is a small positive parameter; $d_{ij}(\mathbf{Q},\mathbf{P})$ denote the coefficients of quasi-linear damping; $\varepsilon C_i(\mathbf{Q},\mathbf{P})D^{\alpha_i}Q_i(t)$ is the small fractional damping term, in which $D^{\alpha_i}Q_i(t)$ denotes the fractional derivative defined by Riemann-Liouville, i.e.,

$$D^{\alpha_i}Q_i(t) = \frac{1}{\Gamma(l-\alpha_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \int_0^t \frac{Q_i(\tau)}{(t-\tau)^{\alpha_i - l + 1}}\mathrm{d}\tau, \quad (l-1) \le \alpha_i < l, \tag{2}$$

in which l is integer and $\Gamma(\bullet)$ is gamma function, in the present work, the value of fractional order α_i is restricted to 0$\sim$1, the case most relevant to structural damping; εf_{ik} denote the amplitudes of harmonic excitations; $\varepsilon^{1/2}h_{ie}$ denote the amplitudes of stochastic excitations; $\xi_e(t)$ are the independent wide-band real noises with zero mean and spectral densities $S_{el}(\omega)$. Specially, if $S_{el}(\omega) = S_{el0}$ is a costant, then $\xi_e(t)$ are Gaussian white noises.

The Hamiltonian system associated with system (1) is

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,. \tag{3}$$

System (3) can be characterized by the Hamiltonian $H = H(\mathbf{q}, \mathbf{p})$ (lower case is for deterministic quantity and capital for random quantity), which represents the total energy for mechanical/structural systems. A dynamic quantity $H_i = H_i(\mathbf{q}, \mathbf{p})$ is called first integral if $[H_i, H]$=0, and two first integrals are called in involution if $[H_i, H_j]$=0, where

$$[H_i, H_j] = \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k}, \quad i, j = 1, 2, ..., r; \quad k = 1, 2, ..., n, \tag{4}$$

is the Poisson bracket of H_i and H_j. System (3) is integrable if there exist n independent first integrals of the motion $H_1, H_2, ..., H_n$ which are in involution. In this case, system (1) is called quasi integrable Hamiltonian system.

Specifically, assume that the Hamiltonian H is separable, i.e.,

$$H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{n} H_i(q_i, p_i) \tag{5}$$

and $H_i(q_i, p_i)$ are of the form

$$H_i(q_i, p_i) = p_i^2/2 + U_i(q_i), \tag{6}$$

where

$$U_i(q_i) = \int_0^{q_i} g_i(u)\mathrm{d}u. \tag{7}$$

Moreover, if the functions $g_i(q_i)$ and $U_i(q_i)$ satisfy the following four conditions: (i) $g_i(b_i)$=0; (ii) there exists a point $q_{i0} > b_i$ such that $g_i(q_{i0}) \neq 0$ and $U_i(q_{i0}) > 0$; (iii) a point $q_{i1} < b_i$ can be found such that $g_i(q_{i1}) \neq 0$ and $U_i(q_{i0}) = U_i(q_{i1})$; (iv) for all $q_i \in (q_{i1}, q_{i0})$, $U_i(q_i) < U_i(q_{i1})$, system (3) has a family of periodic solutions

$$\begin{aligned} &q_i = a_i \cos\theta_i(t) + b_i \\ &p_i = \dot{q}_i = -a_i\upsilon_i(a_i, \theta_i)\sin\theta_i(t) \\ &\theta_i(t) = \phi_i(t) + \gamma_i, \end{aligned} \tag{8}$$

where

$$\upsilon_i(a_i, \theta_i) = \frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \sqrt{\frac{2[U(a_i + b_i) - U(a_i\cos\theta_i + b_i)]}{a_i^2\sin^2\theta_i}}\,, \tag{9}$$

in which a_i and b_i are defined by

$$U_i(a_i + b_i) = U_i(-a_i + b_i) = H_i \tag{10}$$

$a_i, b_i, \upsilon_i(a_i, \theta_i)$ and γ_i are the amplitude, symmetric center of q_i, instantaneous frequency and phase angle difference, respectively, of the i-th degree of freedom of system (3).

Expand $\upsilon_i(a_i, \theta_i)$ into a Fourier series as follows:

$$\upsilon_i(a_i, \theta_i) = \omega_{i0}(a_i) + \sum_{r=1}^{\infty} \omega_{ir}(a_i) \cos r\theta_i. \tag{11}$$

Integrating Eq. (11) with respect to θ_i from 0 to 2π leads to the following averaged frequency

$$\omega_i(a_i) = \frac{1}{2\pi} \int_0^{2\pi} \upsilon_i(a_i, \theta_i) \mathrm{d}\theta_i = \omega_{i0}(a_i) \tag{12}$$

of the i-th degree of freedom. Then, $\theta_i(t)$ in Eq.(8) can be approximated as

$$\theta_i(t) \approx \omega_i(a_i)t + \gamma_i. \tag{13}$$

When ε is small, the solution of system (1) is randomly periodic and of the form

$$\begin{aligned} Q_i &= A_i \cos \Theta_i(t) + B_i, \\ P_i &= \dot{Q}_i = -A_i \upsilon_i(A_i, \Theta_i) \sin \Theta_i(t), \\ \Theta_i(t) &= \Phi_i(t) + \Gamma_i(t), \end{aligned} \tag{14}$$

where

$$\upsilon_i(A_i, \Theta_i) = \frac{\mathrm{d}\Phi_i}{\mathrm{d}t} = \sqrt{\frac{2[U_i(A_i + B_i) - U_i(A_i \cos \Theta_i + B_i)]}{A_i^2 \sin^2 \Theta_i}} \tag{15}$$

in which A_i, $\Theta_i, \Phi_i, \Gamma_i$ are all random processes, A_i are related to H_i in a similar way as Eq. (10). The averaged frequency can be obtained in a way similar to that in Eqs. (11) and (12).

Treating Eq. (14) as a generalized van der Pol transformation from Q_i, P_i to A_i, Θ_i, and then using the relation (13), one can obtain the following stochastic differential equations for A_i and Θ_i:

$$\begin{aligned} \frac{\mathrm{d}A_i}{\mathrm{d}t} =& \varepsilon[F_i^{(11)}(\mathbf{A}, \boldsymbol{\Theta}) + F_i^{(12)}(\mathbf{A}, \boldsymbol{\Theta}) + F_i^{(13)}(\mathbf{A}, \boldsymbol{\Theta}, \boldsymbol{\beta})] \\ & + \varepsilon^{1/2} G_{ie}^{(1)}(\mathbf{A}, \boldsymbol{\Theta}) \xi_e(t), \\ \frac{\mathrm{d}\Theta_i}{\mathrm{d}t} =& \omega_i(A_i) + \varepsilon[F_i^{(21)}(\mathbf{A}, \boldsymbol{\Theta}) + F_i^{(22)}(\mathbf{A}, \boldsymbol{\Theta}) + F_i^{(23)}(\mathbf{A}, \boldsymbol{\Theta}, \boldsymbol{\beta})] \\ & + \varepsilon^{1/2} G_{ie}^{(2)}(\mathbf{A}, \boldsymbol{\Theta}) \xi_e(t), \end{aligned} \tag{16}$$

where $\mathbf{A} = [A_1, A_2, ..., A_n]^T, \mathbf{\Theta} = [\Theta_1, \Theta_2, ..., \Theta_n]^T$,

$$\begin{aligned}
F_i^{(11)} &= -\frac{A_i \upsilon_i \sin\Theta_i}{g(A_i + B_i)(1 + h_i)}[d_{ij} A_i \upsilon_i \sin\Theta_i] \\
F_i^{(12)} &= \frac{A_i \upsilon_i \sin\Theta_i}{g(A_i + B_i)(1 + h_i)}[C_i D^{\alpha_i}(A_i \cos\Theta_i)] \\
F_i^{(13)} &= -\frac{A_i \upsilon_i \sin\Theta_i}{g(A_i + B_i)(1 + h_i)}[f_{ik} \cos\beta_k(t)] \\
F_i^{(21)} &= -\frac{\upsilon_i \sin\Theta_i(\cos\Theta_i + h_i)}{g(A_i + B_i)(1 + h_i)}[d_{ij} A_i \upsilon_i \sin\Theta_i] \\
F_i^{(22)} &= \frac{\upsilon_i \sin\Theta_i(\cos\Theta_i + h_i)}{g(A_i + B_i)(1 + h_i)}[C_i D^{\alpha_i}(A_i \cos\Theta_i)] \\
F_i^{(23)} &= -\frac{\upsilon_i \sin\Theta_i(\cos\Theta_i + h_i)}{g(A_i + B_i)(1 + h_i)}[f_{ik} \cos\beta_k(t)] \\
G_{ie}^{(1)} &= -\frac{A_i \upsilon_i \sin\Theta_i}{g(A_i + B_i)(1 + h_i)} h_{ie} \\
G_{ie}^{(2)} &= -\frac{\upsilon_i \sin\Theta_i(\cos\Theta_i + h_i)}{g(A_i + B_i)(1 + h_i)} h_{ie} \\
h_i &= \frac{\mathrm{d}B_i}{\mathrm{d}A_i} = \frac{g(-A_i + B_i) + g(A_i + B_i)}{g(-A_i + B_i) - g(A_i + B_i)}.
\end{aligned} \tag{17}$$

Eq.(16) shows that $\mathbf{A}$ are slowly varying processes and $\mathbf{\Theta}$ are rapidly varying processes. The form and dimension of averaged Itô equations and Fokker-Plank-Kolmogorov (FPK) equations depend on the resonance of the associated Hamiltonian system.

2.1. Non-resonant case

In this case, the harmonic excitations have no effect on the response in the first approximation. According to the stochastic averaging principle due to Stratonvich-Khasminskii [46], [84], $\mathbf{A}(t)$ in Eq. (16) converges weakly to an n-dimensional diffusion Markov process as $\varepsilon \to 0$, in a time interval $0 \leq t \leq T$, where $T \sim O(\varepsilon^{-1})$ (the time interval can be semi-infinite according to [13]), i.e.,

$$\mathrm{d}A_i = m_i(\mathbf{A})\mathrm{d}t + \sigma_{ie}(\mathbf{A})\mathrm{d}B_e(t), \tag{18}$$

where $B_e(t)$ are standard Wiener processes and

$$
\begin{aligned}
m_i(A) =& \varepsilon \left\langle F_i^{(11)} + F_i^{(12)} + \int_{-\infty}^{0} \left(\frac{\partial G_{ik}^{(1)}}{\partial A_j}\bigg|_t G_{jl}^{(1)}\bigg|_{t+\tau} \right.\right. \\
&+ \left.\left. \frac{\partial G_{ik}^{(1)}}{\partial \Theta_j}\bigg|_t G_{jl}^{(2)}\bigg|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right\rangle_t \\
b_{ij}(A) =& \sigma_{ie}\sigma_{je} = \varepsilon \left\langle \int_{-\infty}^{\infty} \left(G_{ik}^{(1)}\Big|_t G_{jl}^{(1)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right\rangle_t
\end{aligned}
\tag{19}
$$

in which $< \bullet >_t$ denotes the averaging operation, i.e.,

$$
\langle \bullet \rangle_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \bullet \rangle \mathrm{d}t = \frac{1}{(2\pi)^n} \int_0^{2\pi} \langle \bullet \rangle \mathrm{d}\boldsymbol{\Theta}. \tag{20}
$$

Since $A_i(t)$ varies slowly with time, the following approximate relation can be obtained from Eq. (13) if τ is small:

$$
\Theta_i(t-\tau) \approx \Theta_i(t) - \omega_i(A_i)\tau. \tag{21}
$$

Then, the averaging of $< F_i^{(12)} >_t$ can be simplified as follows:

$$
\begin{aligned}
\left\langle F_i^{(12)} \right\rangle_t =& \frac{1}{g(A_i+B_i)(1+h_i)} \\
& \times \lim_{T\to\infty} \frac{1}{T} \int_0^T C_i A_i \upsilon_i \sin\Theta_i D^{\alpha_i}(A_i\cos\Theta_i)\mathrm{d}t \\
=& \frac{1}{g(A_i+B_i)(1+h_i)\Gamma(1-\alpha_i)} \\
& \times \lim_{T\to\infty} \frac{1}{T} \int_0^T (C_i A_i \upsilon_i \sin\Theta_i)\mathrm{d}\left(\int_0^t \frac{X_i(t-\tau)}{\tau^{\alpha_i}}\mathrm{d}\tau \right) \\
=& \frac{1}{g(A_i+B_i)(1+h_i)\Gamma(1-\alpha_i)} \\
& \times \lim_{T\to\infty} \left\{ \frac{1}{T} \left[C_i A_i \upsilon_i \sin\Theta_i \int_0^t \frac{X_i(t-\tau)}{\tau^{\alpha_i}}\mathrm{d}\tau \right]\Bigg|_0^T \right. \\
& \left. - \frac{1}{T} \int_0^T \left(\int_0^t \frac{X_i(t-\tau)}{\tau^{\alpha_i}}\mathrm{d}\tau \right) \frac{\mathrm{d}}{\mathrm{d}t}(C_i A_i \upsilon_i \sin\Theta_i)\mathrm{d}t \right\}.
\end{aligned}
\tag{22}
$$

Introducing the following asymptotic integrals:

$$
\begin{aligned}
\int_0^t \frac{\cos(\omega\tau)}{\tau^q}\mathrm{d}\tau &= \omega^{q-1}\int_0^s \frac{\cos(u)}{u^q}\mathrm{d}u, \\
&= \omega^{q-1}\left[\Gamma(1-q)\sin\left(\frac{q\pi}{2}\right) + \frac{\sin(s)}{s^q} + \mathrm{O}(s^{-q-1})\right] \\
\int_0^t \frac{\sin(\omega\tau)}{\tau^q}\mathrm{d}\tau &= \omega^{q-1}\int_0^s \frac{\sin(u)}{u^q}\mathrm{d}u \\
&= \omega^{q-1}\left[\Gamma(1-q)\cos\left(\frac{q\pi}{2}\right) - \frac{\cos(s)}{s^q} + \mathrm{O}(s^{-q-1})\right] \\
&\qquad (u = \omega\tau, s = \omega t),
\end{aligned}
\tag{23}
$$

Eq. (22) can be further simplified to

$$
\begin{aligned}
\left\langle F_i^{(12)} \right\rangle_t \approx &-\frac{1}{2\pi g(A_i+B_i)(1+h_i)\omega_i^{1-\alpha_i}} \int_0^{2\pi} \frac{d}{\mathrm{d}\Theta_i}(C_i A_i^2 \upsilon_i \sin\Theta_i) \\
&\times \upsilon_i \left[\cos\Theta_i \sin(\alpha_i\pi/2) + \sin\Theta_i\cos(\alpha_i\pi/2)\right]\mathrm{d}\Theta_i.
\end{aligned}
\tag{24}
$$

In some situations, the averaged Itô stochastic differential equations for first integrals H_i are preferred. Since A_i is related to H_i in a similar way as Eq. (10), the averaged Itô equations for H_i can be obtained from Eq. (18) by using the Itô differential rule as follows:

$$
\mathrm{d}H_i = \bar{m}_i(\mathbf{H})\mathrm{d}t + \bar{\sigma}_{ie}(\mathbf{H})\mathrm{d}B_e(t), \tag{25}
$$

where $\mathbf{H} = [H_1, H_2, ..., H_n]^T$,

$$
\begin{aligned}
\bar{m}_i &= \{m_i g_i(A_i+B_i)(1+h_i) \\
&\quad + \frac{1}{2}\frac{\mathrm{d}[g_i(A_i+B_i)(1+h_i)]}{\mathrm{d}A_i}\sigma_{ie}\sigma_{ie}\}_{A_i=U_i^{-1}(H_i)-B_i}, \\
\bar{b}_{ij} &= \bar{\sigma}_{ie}\bar{\sigma}_{je} = [g_i(A_i+B_i)g_j(A_j+B_j) \\
&\quad \times (1+h_i)(1+h_j)\sigma_{ie}\sigma_{je}]_{A_i=U_i^{-1}(H_i)-B_i}.
\end{aligned}
\tag{26}
$$

The associated FPK equation is of the form

$$
\frac{\partial p}{\partial t} = \frac{\partial(\bar{m}_i p)}{\partial H_i} + \frac{1}{2}\frac{\partial^2(\bar{b}_{ij}p)}{\partial H_i \partial H_j}, \tag{27}
$$

where $p = p(\mathbf{H}, t|\mathbf{H}_0)$ is the transition probability density of $\mathbf{H}$. The initial condition is

$$
p(\mathbf{H}, 0|\mathbf{H}_0) = \delta(\mathbf{H} - \mathbf{H}_0) \tag{28}
$$

The FPK equation is usually subject to the following boundary conditions:

$$
\begin{aligned}
&p = \text{finite}, \quad \text{if} \quad \mathbf{H} \to 0, \\
&p = 0, \quad \partial p/\partial H_i = 0, \quad \text{if} \quad \mathbf{H} \to \infty.
\end{aligned}
\tag{29}
$$

Furthermore, p satisfies the normalization condition, i.e.,

$$
\int_0^\infty p\mathrm{d}\mathbf{H} = 1. \tag{30}
$$

2.2. Internal-resonant case

Assume that there are λ $(1 \le \lambda \le n-1)$ internal resonance relations in Eq. (1), i.e.,

$$L_i^s \omega_i = O(\varepsilon); \quad s = 1, 2, ..., \lambda; \quad i = 1, 2, ..., n, \tag{31}$$

where L_i^s are integers but not all zero. In this case, the terms containing $\cos \beta_k(t)$ in Eq. (1) can be neglected in the first approximation. In this case, λ combinations of angle variables $\Psi_s = L_i^s \Theta_i$ and amplitudes $\mathbf{A}$ are slowly varying processes. Then similar derivation yields the following averaged Itô stochastic differential equations:

$$\begin{aligned} \mathrm{d}A_i &= m_i^{(1)}(\mathbf{A}, \mathbf{\Psi})\mathrm{d}t + \sigma_{ie}^{(1)}(\mathbf{A}, \mathbf{\Psi})\mathrm{d}B_e(t) \\ \mathrm{d}\Psi_s &= m_s^{(2)}(\mathbf{A}, \mathbf{\Psi})\mathrm{d}t + \sigma_{se}^{(2)}(\mathbf{A}, \mathbf{\Psi})\mathrm{d}B_e(t) \end{aligned} \tag{32}$$

where $\mathbf{\Psi} = [\Psi_1, \Psi_2, ..., \Psi_\lambda]^T, \mathbf{\Theta'} = [\Theta_{1+\lambda}, \Theta_{2+\lambda}, ..., \Theta_n]^T$

$$\begin{aligned}
m_i^{(1)} &= \frac{\varepsilon}{(2\pi)^{n-\lambda}} \int_0^{2\pi} \Big[F_i^{(11)} + F_i^{(12)} \\
&+ \int_{-\infty}^{0} \left(\frac{\partial G_{ik}^{(1)}}{\partial A_j}\bigg|_t G_{jl}^{(1)}\Big|_{t+\tau} + \frac{\partial G_{ik}^{(1)}}{\partial \Theta_j}\bigg|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \Big] \mathrm{d}\mathbf{\Theta'}, \\
m_s^{(2)} &= \frac{\varepsilon L_i^s}{(2\pi)^{n-\lambda}} \int_0^{2\pi} \Big[F_i^{(21)} + F_i^{(22)} \\
&+ \int_{-\infty}^{0} \left(\frac{\partial G_{ik}^{(2)}}{\partial A_j}\bigg|_t G_{jl}^{(1)}\Big|_{t+\tau} + \frac{\partial G_{ik}^{(2)}}{\partial \Theta_j}\bigg|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \Big] \mathrm{d}\mathbf{\Theta'}, \\
\sigma_{ie}^{(1)}\sigma_{je}^{(1)} &= \frac{\varepsilon}{(2\pi)^{n-\lambda}} \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(1)}\Big|_t G_{jl}^{(1)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right] \mathrm{d}\mathbf{\Theta'}, \\
\sigma_{s_1 e}^{(2)}\sigma_{s_2 e}^{(2)} &= \frac{\varepsilon L_i^{s_1} L_i^{s_2}}{(2\pi)^{n-\lambda}} \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(2)}\Big|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right] \mathrm{d}\mathbf{\Theta'}, \\
\sigma_{ie}^{(1)}\sigma_{se}^{(2)} &= \frac{\varepsilon L_i^s}{(2\pi)^{n-\lambda}} \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(1)}\Big|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right] \mathrm{d}\mathbf{\Theta'}, \\
& i, j = 1, 2, ..., n; \; s, s_1, s_2 = 1, 2, ..., \lambda; \; k, l, e = 1, 2, ..., w,
\end{aligned} \tag{33}$$

in which the expression for $< F_i{}^{(12)} >_t$ is governed by Eq. (24) and the expression for $< F_i{}^{(22)} >_t$ can be similarly derived as $< F_i{}^{(12)} >_t$, i.e.,

$$\begin{aligned} \left\langle F_i^{(22)} \right\rangle_t &\approx \frac{-1}{2\pi g(A_i + B_i)(1 + h_i)\omega^{1-\alpha_i}} \int_0^{2\pi} \frac{d}{d\Theta_i}[C_i \upsilon_i(\cos\Theta_i + h_i)] \\ &\times \upsilon_i \left[\cos\Theta_i \sin(\alpha_i\pi/2) + \sin\Theta_i \cos(\alpha_i\pi/2)\right]\mathrm{d}\Theta_i. \end{aligned} \tag{34}$$

By using the relation (10) and Itô differential rule, Eq. (32) can be converted into

$$\begin{aligned} \mathrm{d}H_i &= \bar{m}_i^{(1)}(\mathbf{H},\mathbf{\Psi})\mathrm{d}t + \bar{\sigma}_{ie}^{(1)}(\mathbf{H},\mathbf{\Psi})\mathrm{d}B_e(t), \\ \mathrm{d}\Psi_s &= \bar{m}_s^{(2)}(\mathbf{H},\mathbf{\Psi})\mathrm{d}t + \bar{\sigma}_{se}^{(2)}(\mathbf{H},\mathbf{\Psi})\mathrm{d}B_e(t), \end{aligned} \tag{35}$$

where

$$\begin{aligned} \bar{m}_i^{(1)} &= \{m_i^{(1)} g_i(A_i+B_i)(1+h_i) \\ &\quad + 0.5\sigma_{ie}^{(1)}\sigma_{ie}^{(1)}\mathrm{d}[g_i(A_i+B_i)(1+h_i)]/\mathrm{d}A_i\}_{A_i=U_i^{-1}(H_i)-B_i} \\ \bar{m}_s^{(2)} &= m_s^{(2)}\Big|_{A_i=U_i^{-1}(H_i)-B_i} \\ \bar{b}_{ij}^{(1)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{je}^{(1)} = [g_i(A_i+B_i)g_j(A_j+B_j) \\ &\quad \times (1+h_i)(1+h_j)\sigma_{ie}^{(1)}\sigma_{je}^{(1)}]_{\substack{A_i=U_i^{-1}(H_i)-B_i \\ A_j=U_j^{-1}(H_j)-B_j}} \\ \bar{b}_{s_1s_2}^{(2)} &= \bar{\sigma}_{s_1e}^{(2)}\bar{\sigma}_{s_2e}^{(2)} = \left[\sigma_{s_1e}^{(2)}\sigma_{s_2e}^{(2)}\right]_{A_i=U_i^{-1}(H_i)-B_i} \\ \bar{b}_{is}^{(0)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{se}^{(2)} = \left[\sigma_{ie}^{(1)}\sigma_{se}^{(2)}g_i(A_i+B_i)(1+h_i)\right]_{A_i=U_i^{-1}(H_i)-B_i}. \end{aligned} \tag{36}$$

The averaged FPK equation corresponding with Eq. (35) is of the form

$$\frac{\partial p}{\partial t} = -\frac{\partial(\bar{m}_i^{(1)}p)}{\partial H_i} - \frac{\partial(\bar{m}_i^{(2)}p)}{\partial \Psi_s} + \frac{1}{2}\frac{\partial^2(\bar{b}_{ij}^{(1)}p)}{\partial H_i\partial H_j} + \frac{1}{2}\frac{\partial^2(\bar{b}_{s_1s_2}^{(2)}p)}{\partial\Psi_{s_1}\partial\Psi_{s_2}} + \frac{\partial^2(\bar{b}_{is}^{(0)}p)}{\partial H_i\partial\Psi_s}. \tag{37}$$

The initial condition is

$$p(\mathbf{H},\mathbf{\Psi},0|\mathbf{H}_0,\mathbf{\Psi}_0) = \delta(\mathbf{H}-\mathbf{H}_0)\delta(\mathbf{\Psi}-\mathbf{\Psi}_0). \tag{38}$$

The boundary conditions with respect to $\mathbf{H}$ are the same as Eq. (29). Since p is the periodic function of $\mathbf{\Psi}$, it satisfies the following periodic boundary condition with respect to $\mathbf{\Psi}$:

$$p(\mathbf{H},\mathbf{\Psi}+2\bar{n}\pi,t|\mathbf{H}_0,\mathbf{\Psi}_0) = p(\mathbf{H},\mathbf{\Psi},t|\mathbf{H}_0,\mathbf{\Psi}_0), \tag{39}$$

where $\bar{n}$ is arbitrary integer. The normalization condition is of the form

$$\int_0^\infty\int_0^{2\pi} p\mathrm{d}\mathbf{H}\mathrm{d}\mathbf{\Psi} = 1. \tag{40}$$

2.3. External-resonant case

Assume that there are η $(1 \le \eta \le m)$ external resonant relations but no internal resonance in Eq. (16), i.e.,

$$E_k^v\Omega_k + I_j^v\omega_j = \varepsilon\chi_v;\;\; k=1,2,...,m;\;\; v=1,2,...,\eta;\;\; j=1,2,...,n, \tag{41}$$

where E_k^v and I_j^v are integers but not all zero, $\varepsilon\chi_v$ are detuning parameters. Then η combinations $\Delta_v = E_k^v\beta_k + I_j^v\Theta_j$ of angle variables and amplitudes $\mathbf{A}$ are slowly varying processes. A similar derivation yields the following averaged Itô equations:

$$\begin{aligned} \mathrm{d}A_i &= m_i^{(1)}(\mathbf{A},\mathbf{\Delta})\mathrm{d}t + \sigma_{ie}^{(1)}(\mathbf{A},\mathbf{\Delta})\mathrm{d}B_e(t), \\ \mathrm{d}\Delta_v &= m_v^{(2)}(\mathbf{A},\mathbf{\Delta})\mathrm{d}t + \sigma_{ve}^{(2)}(\mathbf{A},\mathbf{\Delta})\mathrm{d}B_e(t), \end{aligned} \tag{42}$$

where $\mathbf{\Delta} = [\Delta_1, \Delta_2, ..., \Delta_\eta]^T, \mathbf{\Theta}'' = [\Theta_{1+\eta}, \Theta_{2+\eta}, ..., \Theta_n]^T, \boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_m]^T$,

$$\begin{aligned} m_i^{(1)} &= \frac{\varepsilon}{(2\pi)^{n-\eta+m}} \int_0^{2\pi}\int_0^{2\pi} \Bigg[F_i^{(11)} + F_i^{(12)} + F_i^{(13)} \\ &\quad + \int_{-\infty}^0 \left(\frac{\partial G_{ik}^{(1)}}{\partial A_j}\bigg|_t G_{jl}^{(1)}\Big|_{t+\tau} + \frac{\partial G_{ik}^{(1)}}{\partial \Theta_j}\bigg|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \Bigg] \mathrm{d}\mathbf{\Theta}''\mathrm{d}\boldsymbol{\beta}, \\ m_v^{(2)} &= \frac{\varepsilon}{(2\pi)^{n-\eta+m}} \int_0^{2\pi}\int_0^{2\pi} \Bigg\{ I_i^v \Bigg[F_i^{(21)} + F_i^{(22)} + F_i^{(23)} \\ &\quad + \int_{-\infty}^0 \left(\frac{\partial G_{ik}^{(2)}}{\partial A_j}\bigg|_t G_{jl}^{(1)}\Big|_{t+\tau} + \frac{\partial G_{ik}^{(2)}}{\partial \Theta_j}\bigg|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \Bigg]\Bigg\} \mathrm{d}\mathbf{\Theta}''\mathrm{d}\boldsymbol{\beta}, \end{aligned} \tag{43}$$

$$\begin{aligned} \sigma_{ie}^{(1)}\sigma_{je}^{(1)} &= \frac{\varepsilon}{(2\pi)^{n-\eta+m}} \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} \left(G_{ik}^{(1)}\Big|_t G_{jl}^{(1)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau\mathrm{d}\mathbf{\Theta}''\mathrm{d}\boldsymbol{\beta}, \\ \sigma_{ie}^{(1)}\sigma_{ve}^{(2)} &= \frac{\varepsilon I_i^v}{(2\pi)^{n-\eta+m}} \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} \left(G_{ik}^{(1)}\Big|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau\mathrm{d}\mathbf{\Theta}''\mathrm{d}\boldsymbol{\beta}, \\ \sigma_{v_1e}^{(2)}\sigma_{v_2e}^{(2)} &= \frac{\varepsilon I_i^{v_1} I_j^{v_2}}{(2\pi)^{n-\eta+m}} \int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} \left(G_{ik}^{(2)}\Big|_t G_{jl}^{(2)}\Big|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau\mathrm{d}\mathbf{\Theta}''\mathrm{d}\boldsymbol{\beta}, \\ & i,j = 1,2,...,n;\ v, v_1, v_2 = 1,2,...,\eta;\ k,l,e = 1,2,...,w, \end{aligned}$$

in which the expressions for $< F_i{}^{(12)} >_t$ and $< F_i{}^{(22)} >_t$ are governed by Eqs. (24) and (34), respectively.

By using the relation (10) and Itô differential rule, Eq. (42) can be converted into

$$\begin{aligned} \mathrm{d}H_i &= \bar{m}_i^{(1)}(\mathbf{H},\mathbf{\Delta})\mathrm{d}t + \bar{\sigma}_{ie}^{(1)}(\mathbf{H},\mathbf{\Delta})\mathrm{d}B_e(t), \\ \mathrm{d}\Delta_v &= \bar{m}_v^{(2)}(\mathbf{H},\mathbf{\Delta})\mathrm{d}t + \bar{\sigma}_{ve}^{(2)}(\mathbf{H},\mathbf{\Delta})\mathrm{d}B_e(t), \end{aligned} \tag{44}$$

where

$$
\begin{aligned}
\bar{m}_i^{(1)} &= \{m_i^{(1)} g_i(A_i+B_i)(1+h_i) \\
&\quad + 0.5\sigma_{ie}^{(1)}\sigma_{ie}^{(1)} \mathrm{d}[g_i(A_i+B_i)(1+h_i)]/\mathrm{d}A_i\}_{A_i=U_i^{-1}(H_i)-B_i}, \\
\bar{m}_v^{(2)} &= m_v^{(2)}\Big|_{A_i=U_i^{-1}(H_i)-B_i}, \\
\bar{b}_{ij}^{(11)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{je}^{(1)} = [g_i(A_i+B_i)g_j(A_j+B_j) \\
&\quad \times (1+h_i)(1+h_j)\sigma_{ie}^{(1)}\sigma_{je}^{(1)}]_{\substack{A_i=U_i^{-1}(H_i)-B_i \\ A_j=U_j^{-1}(H_j)-B_j}}, \\
\bar{b}_{v_1v_2}^{(22)} &= \bar{\sigma}_{v_1e}^{(2)}\bar{\sigma}_{v_2e}^{(2)} = \left[\sigma_{v_1e}^{(2)}\sigma_{v_2e}^{(2)}\right]_{A_i=U_i^{-1}(H_i)-B_i} \\
\bar{b}_{iv}^{(12)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{ve}^{(2)} = \left[\sigma_{ie}^{(1)}\sigma_{ve}^{(2)} g_i(A_i+B_i)(1+h_i)\right]_{A_i=U_i^{-1}(H_i)-B_i}.
\end{aligned} \tag{45}
$$

The FPK equation associated with Eq. (44) is of the form

$$
\frac{\partial p}{\partial t} = -\frac{\partial(\bar{m}_i^{(1)}p)}{\partial H_i} - \frac{\partial(\bar{m}_v^{(2)}p)}{\partial \Delta_v} + \frac{1}{2}\frac{\partial^2(\bar{b}_{ij}^{(11)}p)}{\partial H_i \partial H_j} + \frac{1}{2}\frac{\partial^2(\bar{b}_{v_1v_2}^{(22)}p)}{\partial \Delta_{v_1}\partial \Delta_{v_2}} + \frac{\partial^2(\bar{b}_{iv}^{(12)}p)}{\partial H_i \partial \Delta_v}. \tag{46}
$$

The initial condition is

$$
p(\mathbf{H},\boldsymbol{\Delta},0|\mathbf{H}_0,\boldsymbol{\Delta}_0) = \delta(\mathbf{H}-\mathbf{H}_0)\delta(\boldsymbol{\Delta}-\boldsymbol{\Delta}_0). \tag{47}
$$

The boundary conditions with respect to $\mathbf{H}$ are the same as Eq. (29). The periodic boundary condition with respect to $\boldsymbol{\Delta}$ is of the form

$$
p(\mathbf{H},\boldsymbol{\Delta}+2\bar{k}\pi,t|\mathbf{H}_0,\boldsymbol{\Delta}_0) = p(\mathbf{H},\boldsymbol{\Delta},t|\mathbf{H}_0,\boldsymbol{\Delta}_0), \tag{48}
$$

where $\bar{k}$ is arbitrary integer. The normalization condition is of the form

$$
\int_0^\infty \int_0^{2\pi} p\,\mathrm{d}\mathbf{H}\mathrm{d}\boldsymbol{\Delta} = 1. \tag{49}
$$

2.4. Both internal and external-resonant case

In this case, η combinations $\Delta_v = E_k^v\beta_k + I_j^v\Theta_j$ of angle variables, λ combinations $\Psi_s = L_i^s\Theta_i$ of angle variables and amplitudes $\mathbf{A}$ are slowly varying processes. Then, similar derivation yields the following averaged Itô equations:

$$
\begin{aligned}
\mathrm{d}A_i &= m_i^{(1)}(\mathbf{A},\boldsymbol{\Delta},\boldsymbol{\Psi})\mathrm{d}t + \sigma_{ie}^{(1)}(\mathbf{A},\boldsymbol{\Delta},\boldsymbol{\Psi})\mathrm{d}B_e(t), \\
\mathrm{d}\Delta_v &= m_v^{(2)}(\mathbf{A},\boldsymbol{\Delta},\boldsymbol{\Psi})\mathrm{d}t + \sigma_{ve}^{(2)}(\mathbf{A},\boldsymbol{\Delta},\boldsymbol{\Psi})\mathrm{d}B_e(t), \\
\mathrm{d}\Psi_s &= m_s^{(3)}(\mathbf{A},\boldsymbol{\Delta},\boldsymbol{\Psi})\mathrm{d}t + \sigma_{se}^{(3)}(\mathbf{A},\boldsymbol{\Delta},\boldsymbol{\Psi})\mathrm{d}B_e(t),
\end{aligned} \tag{50}
$$

where $\boldsymbol{\Theta}''' = [\Theta_{1+\eta+\lambda}, \Theta_{2+\eta+\lambda}, ..., \Theta_n]^T$,

$$
\begin{aligned}
m_i^{(1)} =& \frac{\varepsilon}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi} \int_0^{2\pi} \Big[F_i^{(11)} + F_i^{(12)} + F_i^{(13)} \\
&+ \int_{-\infty}^0 \left(\frac{\partial G_{ik}^{(1)}}{\partial A_j} \bigg|_t G_{jl}^{(1)} \Big|_{t+\tau} + \frac{\partial G_{ik}^{(1)}}{\partial \Theta_j} \bigg|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \Big] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
m_v^{(2)} =& \frac{\varepsilon}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi} \int_0^{2\pi} \Big\{ I_i^v \Big[F_i^{(21)} + F_i^{(22)} + F_i^{(23)} \\
&+ \int_{-\infty}^0 \left(\frac{\partial G_{ik}^{(2)}}{\partial A_j} \bigg|_t G_{jl}^{(1)} \Big|_{t+\tau} + \frac{\partial G_{ik}^{(2)}}{\partial \Theta_j} \bigg|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \Big] \Big\} \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
m_s^{(3)} =& \frac{\varepsilon}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi} \int_0^{2\pi} \Big\{ L_i^s \Big[F_i^{(21)} + F_i^{(22)} + F_i^{(23)} \\
&+ \int_{-\infty}^0 \left(\frac{\partial G_{ik}^{(2)}}{\partial A_j} \bigg|_t G_{jl}^{(1)} \Big|_{t+\tau} + \frac{\partial G_{ik}^{(2)}}{\partial \Theta_j} \bigg|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \Big] \Big\} \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta},
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\sigma_{ie}^{(1)} \sigma_{je}^{(1)} =& \frac{\varepsilon}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi}\int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(1)} \Big|_t G_{jl}^{(1)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \right] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
\sigma_{ie}^{(1)} \sigma_{ve}^{(2)} =& \frac{\varepsilon I_i^v}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi}\int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(1)} \Big|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \right] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
\sigma_{ie}^{(1)} \sigma_{se}^{(3)} =& \frac{\varepsilon L_i^s}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi}\int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(1)} \Big|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \right] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
\sigma_{v_1e}^{(2)} \sigma_{v_2e}^{(2)} =& \frac{\varepsilon I_i^{v_1} I_j^{v_2}}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi}\int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(2)} \Big|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \right] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
\sigma_{ve}^{(2)} \sigma_{se}^{(3)} =& \frac{\varepsilon I_i^v L_j^s}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi}\int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(2)} \Big|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \right] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta} \\
\sigma_{s_1e}^{(3)} \sigma_{s_2e}^{(3)} =& \frac{\varepsilon L_i^{s_1} L_j^{s_2}}{(2\pi)^{n-\eta+m-\lambda}} \int_0^{2\pi}\int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(G_{ik}^{(2)} \Big|_t G_{jl}^{(2)} \Big|_{t+\tau} \right) R_{kl}(\tau) \mathrm{d}\tau \right] \mathrm{d}\boldsymbol{\Theta}''' \mathrm{d}\boldsymbol{\beta}
\end{aligned}
$$

$i, j = 1, 2, ..., n;\ v, v_1, v_2 = 1, 2, ..., \eta;\ s, s_1, s_2 = 1, 2, ..., \lambda;\ k, l, e = 1, 2, ..., w,$ in which the expressions for $< F_i^{(12)} >_t$ and $< F_i^{(22)} >_t$ are governed by Eqs. (24) and (34), respectively.

By using the relation (10) and Itô differential rule, Eq.(50) can be converted into

$$\begin{aligned}
\mathrm{d}H_i &= \bar{m}_i^{(1)}(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi})\mathrm{d}t + \bar{\sigma}_{ie}^{(1)}(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi})\mathrm{d}B_e(t),\\
\mathrm{d}\Delta_v &= \bar{m}_v^{(2)}(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi})\mathrm{d}t + \bar{\sigma}_{ve}^{(2)}(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi})\mathrm{d}B_e(t),\\
\mathrm{d}\Psi_s &= \bar{m}_s^{(3)}(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi})\mathrm{d}t + \bar{\sigma}_{se}^{(3)}(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi})\mathrm{d}B_e(t),
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
\bar{m}_i^{(1)} &= \{m_i^{(1)} g_i(A_i+B_i)(1+h_i)\\
&\quad + 0.5\sigma_{ie}^{(1)}\sigma_{ie}^{(1)}\mathrm{d}[g_i(A_i+B_i)(1+h_i)]/\mathrm{d}A_i\}_{A_i=U_i^{-1}(H_i)-B_i}\\
\bar{m}_v^{(2)} &= m_v^{(2)}\Big|_{A_i=U_i^{-1}(H_i)-B_i}; \quad \bar{m}_s^{(3)} = m_s^{(3)}\Big|_{A_i=U_i^{-1}(H_i)-B_i}\\
\bar{b}_{ij}^{(11)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{je}^{(1)} = [g_i(A_i+B_i)g_j(A_j+B_j)\\
&\quad \times (1+h_i)(1+h_j)\sigma_{ie}^{(1)}\sigma_{je}^{(1)}]_{\substack{A_i=U_i^{-1}(H_i)-B_i\\ A_j=U_j^{-1}(H_j)-B_j}}\\
\bar{b}_{iv}^{(12)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{ve}^{(2)} = \left[\sigma_{ie}^{(1)}\sigma_{ve}^{(2)} g_i(A_i+B_i)(1+h_i)\right]_{A_i={U_i}^{-1}(H_i)-B_i}\\
\bar{b}_{is}^{(13)} &= \bar{\sigma}_{ie}^{(1)}\bar{\sigma}_{se}^{(3)} = \left[\sigma_{ie}^{(1)}\sigma_{se}^{(3)} g_i(A_i+B_i)(1+h_i)\right]_{A_i={U_i}^{-1}(H_i)-B_i}\\
\bar{b}_{v_1v_2}^{(22)} &= \bar{\sigma}_{v_1e}^{(2)}\bar{\sigma}_{v_2e}^{(2)} = \left[\sigma_{v_1e}^{(2)}\sigma_{v_2e}^{(2)}\right]_{A_i={U_i}^{-1}(H_i)-B_i}\\
\bar{b}_{vs}^{(23)} &= \bar{\sigma}_{ve}^{(2)}\bar{\sigma}_{se}^{(3)} = \left[\sigma_{r_1e}^{(2)}\sigma_{se}^{(3)}\right]_{A_i={U_i}^{-1}(H_i)-B_i}\\
\bar{b}_{s_1s_2}^{(33)} &= \bar{\sigma}_{s_1e}^{(3)}\bar{\sigma}_{s_2e}^{(3)} = \left[\sigma_{s_1e}^{(3)}\sigma_{s_2e}^{(3)}\right]_{A_i={U_i}^{-1}(H_i)-B_i}.
\end{aligned} \tag{53}$$

The averaged FPK equation corresponding with Eq. (52) is of the form

$$\begin{aligned}
\frac{\partial p}{\partial t} &= -\frac{\partial(\bar{m}_i^{(1)}p)}{\partial H_i} - \frac{\partial(\bar{m}_v^{(2)}p)}{\partial \Delta_v} - \frac{\partial(\bar{m}_s^{(3)}p)}{\partial \Psi_s}\\
&\quad + \frac{1}{2}\frac{\partial^2(\bar{b}_{ij}^{(11)}p)}{\partial H_i\partial H_j} + \frac{1}{2}\frac{\partial^2(\bar{b}_{v_1v_2}^{(22)}p)}{\partial\Delta_{v_1}\partial\Delta_{v_2}} + \frac{1}{2}\frac{\partial^2(\bar{b}_{s_1s_2}^{(33)}p)}{\partial\Psi_{s_1}\partial\Psi_{s_2}}\\
&\quad + \frac{\partial^2(\bar{b}_{iv}^{(12)}p)}{\partial H_i\partial\Delta_v} + \frac{\partial^2(\bar{b}_{is}^{(13)}p)}{\partial H_i\partial\Psi_s} + \frac{\partial^2(\bar{b}_{sv}^{(23)}p)}{\partial\Psi_s\partial\Delta_v}.
\end{aligned} \tag{54}$$

The initial condition is

$$p(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi},0|\mathbf{H}_0,\mathbf{\Delta}_0,\mathbf{\Psi}_0) = \delta(\mathbf{H}-\mathbf{H}_0)\delta(\mathbf{\Delta}-\mathbf{\Delta}_0)\delta(\mathbf{\Psi}-\mathbf{\Psi}_0). \tag{55}$$

The boundary conditions with respect to $\mathbf{H}$ are the same as Eq. (29). The periodic boundary condition with respect to $\mathbf{\Delta}$ and $\mathbf{\Psi}$ is of the form

$$p(\mathbf{H},\mathbf{\Delta}+2\bar{k}\pi,\mathbf{\Psi}+2\bar{n}\pi,t|\mathbf{H}_0,\mathbf{\Delta}_0,\mathbf{\Psi}_0) = p(\mathbf{H},\mathbf{\Delta},\mathbf{\Psi},t|\mathbf{H}_0,\mathbf{\Delta}_0,\mathbf{\Psi}_0). \tag{56}$$

The normalization condition is of the form

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{2\pi} p\mathrm{d}\mathbf{H}\mathrm{d}\boldsymbol{\Delta}\mathrm{d}\boldsymbol{\Psi} = 1. \tag{57}$$

Solving FPK equation (27), (37), (46) or (54) numerically together with its initial, boundary and normalization conditions yields the stochastic response of system (1) in the cases of non-resonance, internal resonance, external resonance, and both internal and external resonance, respectively. Note that the stationary response of SDOF nonlinear systems with fractional derivative damping driven by only Gaussian white noise excitations [39] or wide-band real noise excitations [27] is the special case of the stationary solutions of non-resonance. Fig.1 illustrates the stationary probability density of amplitude of the fractionally damped Duffing oscillator for different fractional derivative order under wide-band noise excitation [27]. A favorable agreement between the analytical results and the direct Monte Carlo simulation's data is found.

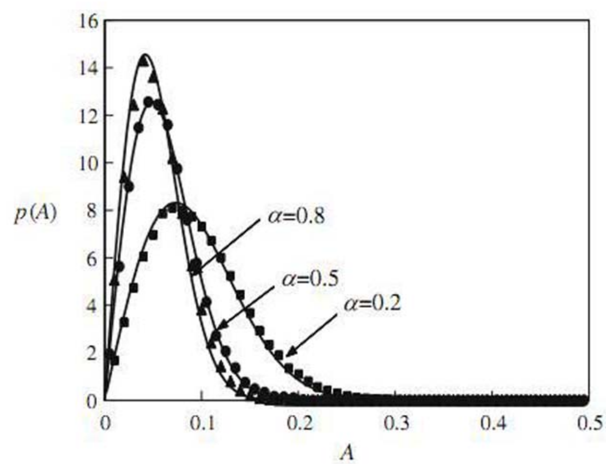

Fig. 1. The stationary probability density of amplitude. Solid lines denote the analytical results; ■, •, ▲ denote the results from Monte Carlo simulation of original system.

2.5. Case of narrow-band bounded noise excitation

In Secs. **2.1-2.4**, the stochastic averaging method for the quasi integrable Hamiltonian systems with fractional derivative damping under combined harmonic and random excitations is presented. In the case of narrow-band bounded noise excitation, the equations of system are of the form

$$\begin{aligned}
&\dot{Q}_i = \frac{\partial H}{\partial P_i}, \\
&\dot{P}_i = -\frac{\partial H}{\partial Q_i} - \varepsilon C_i(\mathbf{Q},\mathbf{P}) D^{\alpha_i} Q_i(t) + \varepsilon^{1/2} h_{ie}(\mathbf{Q},\mathbf{P}) \xi_e(t), \\
&i, j = 1, 2, ..., n;\ e = 1, 2, ..., w,
\end{aligned} \tag{58}$$

where $\xi_e(t)$ are independent bounded noises of the form

$$\xi_e = \cos(\Omega_e t + \sigma_e B_e(t) + \Lambda_e), \tag{59}$$

Ω_e and σ_e are constants representing center frequencies and strengths of frequency perturbations, respectively; B_e are standard Wiener processes; Λ_e are random phases uniformly distributed in $[0,2\pi]$. The spectral density of $\xi_e(t)$ can be found as

$$S_e(\omega) = \frac{\sigma_e^2}{4\pi} \frac{\omega^2 + \Omega_e^2 + \sigma_e^4/4}{(\omega^2 - \Omega_e^2 - \sigma_e^4/4)^2 + \sigma_e^4 \omega^2} \tag{60}$$

and their auto correlation functions are

$$R_e(\tau) = \frac{1}{2} \exp(-\frac{\sigma_e^2}{2}|\tau|) \cos \Omega_e(\tau). \tag{61}$$

In this case, the form and dimension of averaged Itô equations and FPK equations also depend on the resonance of the associated Hamiltonian system. In the cases of non-resonance and internal resonance, the bounded narrow-band excitations have no effect on the response in the first approximation. In the case of external resonance, the η combinations $\Delta_v = M_v(\Omega_v t + \sigma_v B_e(t) + \Lambda_v) + L_v \Theta_v, v = 1, 2, ..., \eta$ of angle variables and amplitudes $\mathbf{A}$ are slowly varying processes. In the case of both internal and external resonance, the η combinations $\Delta_v = M_v(\Omega_v t + \sigma_v B_e(t) + \Lambda_v) + L_v \Theta_v$ of angle variables, λ combinations $\Psi_s = L_i^s \Theta_i$ of angle variables and amplitudes $\mathbf{A}$ are slowly varying processes. Similar derivations as in Secs. **2.3** and **2.4** could yield the averaged Itô equations in the cases of external resonance, and both internal and external resonance. Then, establishing and solving the corresponding FPK equations, one can obtain the stochastic response of system (58). The stationary response of SDOF strongly nonlinear oscillator with fractional derivative damping under bounded noise excitations in the case of external resonance has been studied by Hu et al [37]. In Fig.2, the stationary probability density of amplitude of fractionally damped Duffing oscillator under bounded noise excitation [37] is shown. It is seen that the analytical results agree well with those from the Monte Carlo simulation of original system. The phenomenon of stochastic jump and bifurcation as the fractional orders change is found (the detail explanation on the stochastic jump and bifurcation is presented in Sec.4.3).

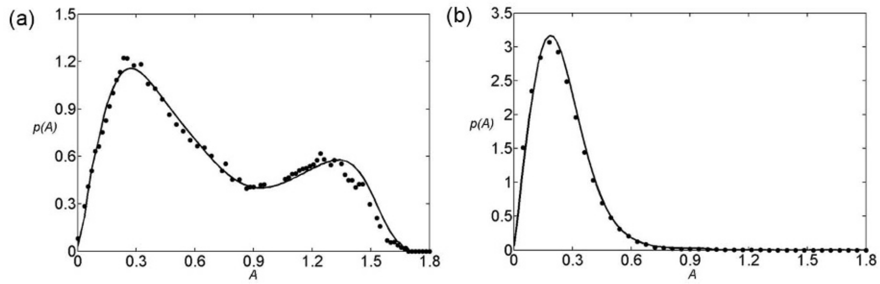

Fig. 2. Stationary probability density of amplitude, the fractional derivative order α=0.2 for (a) and α=0.9 for (b). Solid line denotes the analytical result; Symbol $\bullet$ denotes the results from Monte Carlo simulation of original system.

In addition, the stochastic averaging method for the quasi integrable Hamiltonian systems with fractional derivative damping under narrow-band bounded noise excitations proposed in this section can be also extended into the case of combined harmonic and narrow-band bounded noise excitations. However, the associated averaged Itô equations and their drift and diffusion coefficients are too tedious to be given here.

3. First passage time

The first passage problem is significant for structural reliability but it is very difficult to solve. So far, the known mathematical exact solutions are limited to the one-dimensional diffusion process. For a two-dimensional or higher system, the most feasible way is the combination of the stochastic averaging method and the diffusion process theory of the first passage time, which has been applied by Lin and Cai [53], Roberts [68], [69], Spanos [80], Spanos and Solomos [82] for the SDOF systems, and by Gan and Zhu [33], Zhu et al. [95], [100] for MDOF quasi Hamiltonian systems, and by Chen and Zhu [22], [23], [24] for the MDOF quasi generalized Hamiltonian systems. Recently, this combination technique has been generalized to the quasi integrable Hamiltonian systems with fractional derivative damping by Chen and his coworkers [20], [21], [25], [28].

Consider a quasi integrable Hamiltonian system with fractional derivative damping governed by Eq. (1). In the non-resonant case, the averaged Itô equations for H_i are of the form of Eq.(25). Suppose that H_i vary in some sub-interval of $[0,\infty)$. The first passage failure occurs when $\mathbf{H}(t)$ exceeds certain critical value H_c for the first time. The safety domain Ω_s is bounded with boundaries Γ_0 (at least one of H_i vanishes) and critical boundary Γ_c. The conditional reliability function

$$R(t|\mathbf{H}_0) = \text{Prob}\left\{\mathbf{H}(\tau) \in \Omega_s, \tau \in (0,t]|\mathbf{H}_0 \in \Omega_s\right\} \tag{62}$$

is governed by the following backward Kolmogorov equation:

$$\frac{\partial R}{\partial t} = \bar{m}_i(\mathbf{H}_0)\frac{\partial R}{\partial H_{i0}} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H}_0)\frac{\partial^2 R}{\partial H_{i0}\partial H_{j0}}\,, \tag{63}$$

where the drift and diffusion coefficients are defined by Eq.(26) with $\mathbf{H}$(t) replaced by $\mathbf{H}_0$. The initial condition is

$$R(0|\mathbf{H}_0) = 1, \quad \mathbf{H}_0 \in \Omega_s. \tag{64}$$

The boundary conditions are

$$\begin{aligned} &R(t|\mathbf{H}_0) = 0, \quad \text{at} \quad \Gamma_c, \\ &R(t|\mathbf{H}_0) = \text{finite}, \quad \text{at} \quad \Gamma_0. \end{aligned} \tag{65}$$

The conditional reliability function can be obtained by solving the backward Kolmogorov equation (63) together with its initial and boundary conditions (64) and (65) numerically. The conditional probability density of the first passage time T is obtained from

$$p(T|\mathbf{H}_0) = -\left.\frac{\partial R(t|\mathbf{H}_0)}{\partial t}\right|_{t=T}. \tag{66}$$

And the conditional mean of the first passage time is obtained from

$$\mu(\mathbf{H}_0) = \int_0^\infty Tp(T|\mathbf{H}_0)\mathrm{d}T = \int_0^\infty R(T|\mathbf{H}_0)\mathrm{d}T\,. \tag{67}$$

In addition, the conditional mean of the first passage time can be obtained by solving the Pontryagin equation as follows:

$$\bar{m}_i(\mathbf{H}_0)\frac{\partial \mu}{\partial H_{i0}} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H}_0)\frac{\partial^2 \mu}{\partial H_{i0}\partial H_{j0}} = -1, \tag{68}$$

with boundary conditions

$$\begin{aligned} &\mu(\mathbf{H}_0) = 0, \quad \text{at} \quad \Gamma_c, \\ &\mu(\mathbf{H}_0) = \text{finite}, \quad \text{at} \quad \Gamma_0. \end{aligned} \tag{69}$$

In the cases of internal resonance, external resonance, and both internal and external resonance, the first-passage time problem can be studied similarly. The main change is that the differential operator on the right-hand side of Eq. (63) or the left-hand side of Eq. (68) is replaced by that associated with Itô equation (35), (44) or (52), respectively. Obviously, the dimensions of backward Kolmogorov equation and Pontryagin equation in these cases are higher, but still much less than those of original system. Besides, since the conditional reliability function and mean first passage time are both the periodic function with respect to angle variables, the periodic boundary conditions should be added to solve these equations.

As an example, the mean first passage time for different fractional derivative order of Duffing oscillator with fractional derivative damping subjected to both additive and parametric excitations of Gaussian white noises

[21] are shown in Fig.3. A good agreement between the analytical and simulation's results is found and the larger fractional derivative order yields the higher reliability of system.

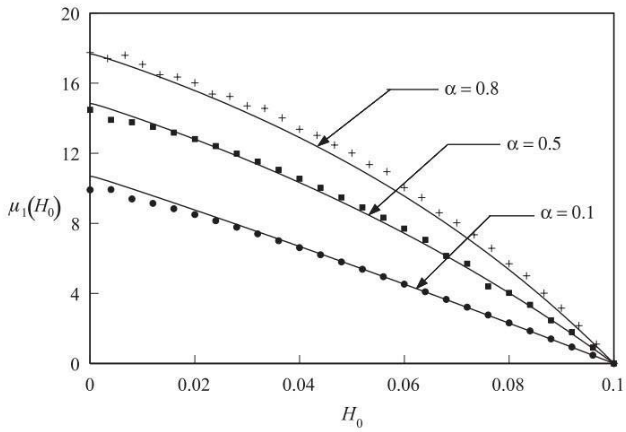

Fig. 3. The mean first passage time as function of initial energy H_0 for different fractional derivative order. Solid lines denote the analytical results; symbols ■, •, + denote the simulation results.

4. Stochastic Stability and Bifurcation

4.1. Stochastic stability of quasi integrable Hamiltonian systems with fractional derivative damping

The theory of stochastic stability deals with the stability of the trivial solution of dynamical systems under random parameter perturbations. There are many definitions of stochastic stability [49], among which the asymptotic stability with probability one is widely accepted. According to the Oseledec multiplicative ergodic theorem [60], the asymptotic stability with probability one can be determined by using the largest Lyapunov exponent. In 1967, Khasminskii [45] gave a general procedure for evaluating the largest Lyapunov exponent of linear stochastic systems. Later, the procedure was generalized to the nonlinear stochastic systems with homogeneous drift and diffusion coefficients of order one, [50]. However, the direct application of Khasminskii's procedure to the systems of dimension higher than two has not met with much success principally due to the difficulty of studying diffusion processes occurring on surface of unit hyperspheres in higher dimensional spaces. Since the stochastic averaging method can be used to reduce the dimension of stochastic dynamical systems, a combination of the stochastic averaging method and the Khasminskii's procedure becomes a powerful technique to evaluate the largest Lyapunov exponent of higher dimensional stochastic dynamical systems. This combination method has been used by many authors [7], [8], [41], [93], [96], [97], [101]. Recently, the combination method was generalized to the quasi integrable Hamiltonian systems with fractional derivative damping [17], [19], [39].

In the non-resonant case, the averaged Itô equation for H_i is in the form of Eq. (25). Suppose that the drift and diffusion coefficients of the averaged Itô equations are homogeneous in H_i of order one. Otherwise, linearize the averaged Itô equations with respect to H_i at the trivial solution $\mathbf{H}$=0. Define the norm in the definitions of stochastic stability and Lyapunov exponent as

$$\|Z\| = \sqrt{\sum_{r=1}^{n} H_r} \,. \tag{70}$$

By using a procedure similar to one proposed in [97], the following approximate expression for the largest Lyapunov exponent can be derived

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \int_0^t Q\left(\boldsymbol{\mu}'(\tau)\right) \mathrm{d}\tau = \int Q(\boldsymbol{\mu}') p(\boldsymbol{\mu}') \mathrm{d}\boldsymbol{\mu}', \tag{71}$$

where

$$\begin{aligned}
&\boldsymbol{\mu}' = [\mu_1, \mu_2, ..., \mu_{n-1}]^T \\
&\mu_r = H_r/\|Z\|; \quad r = 1, 2, ..., n-1 \\
&\mu_n = 1 - \textstyle\sum_{i=1}^{n-1} \mu_i \\
&Q(\boldsymbol{\mu}') = \frac{1}{2}\sum_{r=1}^{n} F_r - \frac{1}{4}\sum_{i,j=1}^{n}\sum_{e=1}^{s} G_{ie}G_{je},
\end{aligned} \tag{72}$$

F_r and G_{ie} are the linearized drift and diffusion coefficients of the averaged Itô equation (25), $P(\boldsymbol{\mu}')$ is the stationary probability density of $\boldsymbol{\mu}'$.

In the cases of internal resonance, external resonance, or both internal and external resonance, the averaged Itô equations are of the form of Eqs.(35), (44) or (52), respectively. The expression for the largest Lyapunov exponent can derived similarly. For example, in the case of both internal and external resonance, a similar derivation yields the following approximate expression for the largest Lyapunov exponent:

$$\lambda = \int Q(\boldsymbol{\mu}', \boldsymbol{\Delta}, \boldsymbol{\Psi}) p(\boldsymbol{\mu}', \boldsymbol{\Delta}, \boldsymbol{\Psi}) \mathrm{d}\boldsymbol{\mu}' d\boldsymbol{\Delta} \mathrm{d}\boldsymbol{\Psi} \,. \tag{73}$$

The asymptotic Lyapunov stability with probability one of Duffing oscillators with fractional derivative damping under parametric excitation of Gaussian white noise [39] and both harmonic function and Gaussian white noise [19] are studied as special examples. The stable region of Duffing oscillator with fractional derivative damping in plane $(E, \Omega/2\omega_0)$ for different values of fractional derivative order α [19] are shown in Fig.4. It is seen that the unstable region decreases as α increases. Besides, the analytical results and those from Monte Carlo simulation of original system are in good agreement.

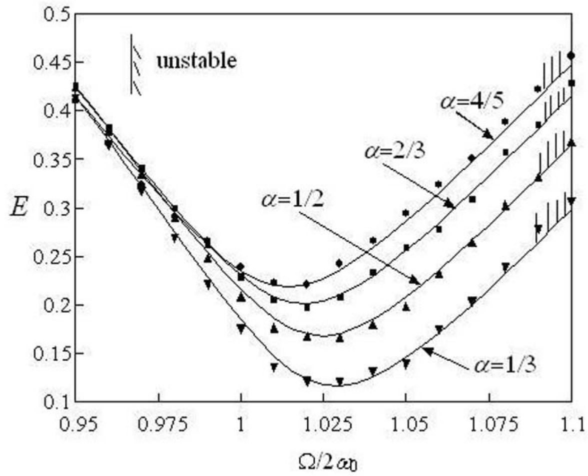

Fig. 4. The unstable region in plane $(E, \Omega/2\omega_0)$. Solid lines denote the analytical results; ■, •, ▲, ▼ denote the results from Monte Carlo simulation of original system.

4.2. Stochastic Hopf bifurcation of quasi integrable Hamiltonian systems with fractional derivative damping

Stochastic bifurcation theory studies the qualitative changes in parameterized families of random dynamical systems. In general, the stochastic bifurcation can be classified into dynamical bifurcation (D-bifurcation) and phenomenological bifurcation (P-bifurcation). D-bifurcation is related to the sign change of the Lyapunov exponent and it reduces to the deterministic bifurcation in the absence of random noise. P-bifurcation is associated with the qualitative change of the stationary probability density of system response. The stochastic Hopf bifurcation consists of a D-bifurcation and a P-bifurcation. Correspondingly, the stationary joint probability density of displacement and velocity transits from a delta function to a uni-model function with the peak at the origin at the D-bifurcation and then to a crater-like function at the P-bifurcation. Before the D-bifurcation, the trivial solution is asymptotically stable with probability one, and after that it is unstable. The stochastic Hopf bifurcation of quasi Hamiltonian systems has been studied by Zhu and his co-workers [55], [98]. Recently, the stochastic Hopf bifurcation of quasi integrable Hamiltonian with fractional derivative damping was studied by Hu et al [36].

In the non-resonant case, the averaged Itô equations for H_i is governed by Eq. (25). Introducing the new variables

$$\bar{\mu}_i = H_i/H; \quad H = \sum_{i=1}^{n} H_i; \quad i = 1, 2, ..., n, \tag{74}$$

the Itô equations for H and $\bar{\mu}_i$ can be obtained from Eq. (25) by using the Itô differential rule as follows:

$$\mathrm{d}H = Q(\bar{\boldsymbol{\mu}}', H)\mathrm{d}t + \Xi_e(\bar{\boldsymbol{\mu}}', H)\mathrm{d}B_e(t), \tag{75}$$

$$\begin{aligned} &\mathrm{d}\bar{\mu}_r = \bar{\bar{m}}_r(\bar{\boldsymbol{\mu}}', H)\mathrm{d}t + \bar{\bar{\sigma}}_{re}(\bar{\boldsymbol{\mu}}', H)\mathrm{d}B_e(t), \\ &r = 1, 2, ..., n-1, \end{aligned} \tag{76}$$

where

$$\begin{aligned} &\bar{\boldsymbol{\mu}}' = [\bar{\mu}_1, \bar{\mu}_2, ..., \bar{\mu}_{n-1}]^T, \quad \bar{\mu}_n = 1 - \sum\nolimits_{i=1}^{n-1} \bar{\mu}_i \\ &Q(\bar{\boldsymbol{\mu}}', \mathrm{H}) = \sum_{r=1}^{n} \bar{m}_i(\bar{\boldsymbol{\mu}}', \mathrm{H}); \quad \Xi_e(\bar{\boldsymbol{\mu}}', H) = \sum_{r=1}^{n} \bar{\sigma}_{ie}(\bar{\boldsymbol{\mu}}', \mathrm{H}) \\ &\bar{\bar{m}}_r(\bar{\boldsymbol{\mu}}', H) = -\frac{\mu_r}{H} \sum_{i=1}^{n} \bar{m}_i(\bar{\boldsymbol{\mu}}', \mathrm{H}) - \frac{1}{2H^2} \sum_{j=1}^{n} \sum_{e=1}^{s} \bar{\sigma}_{re}(\bar{\boldsymbol{\mu}}', \mathrm{H}) \bar{\sigma}_{je}(\bar{\boldsymbol{\mu}}', \mathrm{H}) \\ &\qquad + \frac{\mu_r}{2H^2} \sum_{i,j=1}^{n} \sum_{e=1}^{s} \bar{\sigma}_{ie}(\bar{\boldsymbol{\mu}}', \mathrm{H}) \bar{\sigma}_{je}(\bar{\boldsymbol{\mu}}', \mathrm{H}) + \frac{\bar{m}_r(\bar{\boldsymbol{\mu}}', \mathrm{H})}{H} \\ &\bar{\bar{\sigma}}_{re}(\bar{\boldsymbol{\mu}}', H) = \bar{\sigma}_{re}(\bar{\boldsymbol{\mu}}', \mathrm{H}) - \mu_r \sum_{j=1}^{n} \bar{\sigma}_{je}(\bar{\boldsymbol{\mu}}', \mathrm{H}). \end{aligned} \tag{77}$$

The sample behaviors of the processes $H(\mathrm{t})$ near the boundaries H=0 and $H \to \infty$ can be characterized by the diffusion exponent, the drift exponent and the character value [54]. For a left singular boundary of the first kind, i.e., $\Xi_e(0) = 0$, the conditional diffusion exponent α_l, the conditional drift exponent β_l and the conditional character value c_l are defined as follows:

$$\begin{aligned} &\Xi^2(H|\bar{\boldsymbol{\mu}}') = \sum_{k=1}^{s} \Xi_k^2 = O(H^{\alpha_l}), \quad \alpha_l > 0 \quad \text{as} \quad H \to 0, \\ &Q(H|\bar{\boldsymbol{\mu}}') = O(H^{\beta_l}), \quad \beta_l > 0 \quad \text{as} \quad H \to 0, \\ &c_l(H|\bar{\boldsymbol{\mu}}') = \lim_{H \to 0^+} \frac{2Q(H|\bar{\boldsymbol{\mu}}')H^{\alpha_l - \beta_l}}{\Xi^2(H|\bar{\boldsymbol{\mu}}')}. \end{aligned} \tag{78}$$

For a right singular boundary of the first kind, $Q(\infty) \to \infty$, the conditional diffusion exponent α_r, the conditional drift exponent β_r and the conditional character value c_r are defined as follows:

$$\Xi^2(H|\bar{\boldsymbol{\mu}}') = \sum_{k=1}^{s} \Xi_k^2 = O(H^{\alpha_r}), \quad \alpha_r > 0 \quad \text{as} \quad H \to \infty,$$
$$Q(H|\bar{\boldsymbol{\mu}}') = O(H^{\beta_r}), \quad \beta_r > 0 \quad \text{as} \quad H \to \infty, \tag{79}$$
$$c_r(H|\bar{\boldsymbol{\mu}}') = \lim_{H\to\infty} \frac{2Q(H|\bar{\boldsymbol{\mu}}')H^{\alpha_r-\beta_r}}{\Xi^2(H|\bar{\boldsymbol{\mu}}')} .$$

The unconditional diffusion exponent, the unconditional drift exponent and the unconditional character value can be obtained from the weighted integrals of α_l, β_l, c_l, α_r, β_r and c_r with weighting function $p(\bar{\boldsymbol{\mu}}')$, where $p(\bar{\boldsymbol{\mu}}')$ is the stationary probability density of $\bar{\boldsymbol{\mu}}'$ obtained from solving the FPK equation associated with Itô Eq. (76).

In order to make sure that a stationary probability density exists, the boundary $H \to \infty$ must be either an entrance or repulsively natural after D-bifurcation and P-bifurcation. As for left boundary, two cases can be identified.

Case 1. $\beta_l - \alpha_l = -1$.

In this case the unconditional probability density of $H(\mathrm{t})$ as $H \to 0$ is of the form

$$p(H) = O(H^{\upsilon}), \quad \text{as} \quad H \to 0, \tag{80}$$

where

$$\upsilon = \int_{\Omega} (c_l - \alpha_l)p(\bar{\boldsymbol{\mu}}')\mathrm{d}\bar{\boldsymbol{\mu}}', \quad \Omega = \{\bar{\boldsymbol{\mu}}' | \sum_{i=1}^{n} \bar{\mu}_i = 1, 0 \le \bar{\mu}_i \le 1\} \tag{81}$$

is called average bifurcation parameter. D-bifurcation occurs at $\upsilon = -1$ and the P-bifurcation at υ=0 provided that the right boundary $H \to \infty$ is either an entrance or repulsively natural.

Case 2. $\beta_l - \alpha_l \ne -1$.

In this case, the conditional probability density of $H(\mathrm{t})$ as $H \to 0$ is of the form

$$p(H|\bar{\boldsymbol{\mu}}') = O\{H^{-\alpha_l} \exp[\frac{c_l}{1+\beta_l-\alpha_l} H^{(\beta_l-\alpha_l+1)}]\} \quad \text{as} \quad H \to 0. \tag{82}$$

Although D-bifurcation may occur, it is impossible for the P-bifurcation to occur. So, there will be no stochastic Hopf bifurcation.

Similarly, the stochastic Hopf bifurcation of system (1) in the cases of internal resonance, external resonance, and both internal and external resonance can be studied.

Two coupled Rayleigh oscillators with fractional derivative damping subject to parametric excitations of Gaussian white noise has been investigated in [36]. The D-bifurcation and P-bifurcation curves for different fractional derivative order α are shown in parameter plane (β_{10}, β_{20}) in Fig. 5, where β_{10} and β_{20} are linear coefficients of fractional dampings. As α increases, D-bifurcation curve moves toward upright while P-bifurcation curve toward downleft, and thus, the bifurcation interval decreases.

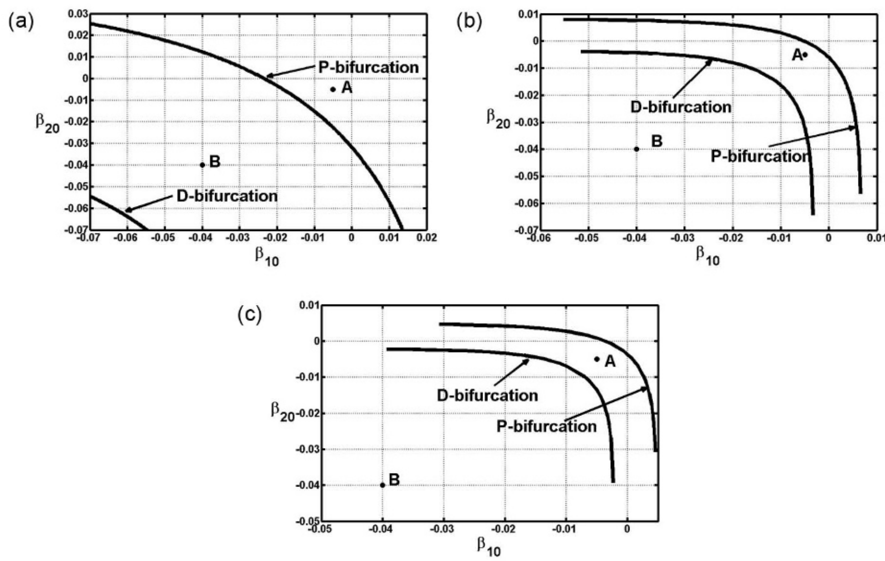

Fig. 5. D-bifurcation and P-bifurcation curves in plane (β_{10}, β_{20}), (a)$\alpha_1 = \alpha_2 = \alpha$ =0.1. (b)$\alpha_1 = \alpha_2 = \alpha$ =0.5. (c)$\alpha_1 = \alpha_2 = \alpha$ =1.0.

4.3. Stochastic jump and its bifurcation of Duffing oscillator with fractional derivative damping

The deterministic jump of a Duffing oscillator under harmonic excitation occurs only at the two extreme values of the frequency interval of the triple-valued amplitude-frequency curve and only in one direction: from larger amplitude to smaller amplitude for increasing frequency and from smaller amplitude to larger amplitude for decreasing frequency. However, the stochastic jump may occur at any frequency in the frequency interval and in both two directions. Based on the joint stationary probability density of amplitude and phase obtained by using the stochastic averaging method, the phenomenon of stochastic jump has been observed in the Duffing oscillator and coupled Duffing-van der Pol oscillators under combined harmonic and Gaussian white noise excitations or under bounded noise excitations [14], [42], [43], [89], [102]. As the system parameters such as frequency ratio, intensity of nonlinearity, intensity of excitation, or damping coefficient changes, the system may transits from one with stochastic jump to one without stochastic jump or vise versa. This is essentially a typical P-bifurcation and it is called the bifurcation of stochastic jump. The procedure has been extended to the Duffing oscillator with fractional derivative damping subjected to combined harmonic and white excitations [26] or bounded noise excitation [37]. Fig.6 shows the stationary probability density of amplitude for different fractional derivative order [26]. It is seen that as the fractional derivative order changes, the system transfers from having jump to having no jump. That is, P-bifurcation may occur.

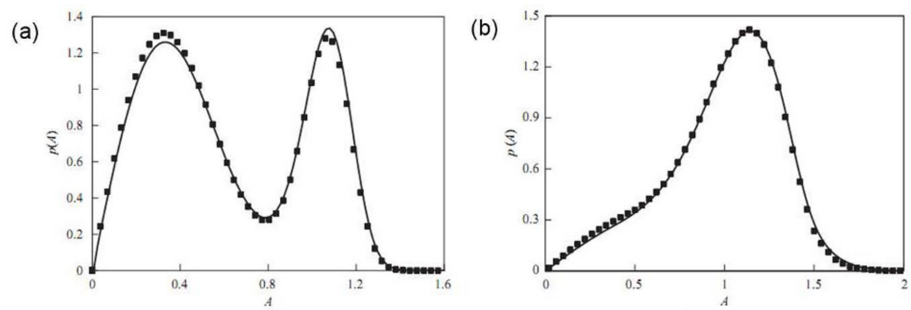

Fig. 6. The stationary probability density of amplitude, (a) α=0.8; (b)α=0.15. The solid lines denote the analytical results; symbol ■ denotes the Monte Carlo simulation of original system.

5. Stochastic fractional optimal control

Fractional optimal control problem (FOCP) is an optimal control problem in which the criterion and/or the differential equations governing the dynamics of the system contain at least one fractional derivative term. In recent years, significant research effort has been devoted to study the FOCP of deterministic systems. Podlubny [65] proposed the fractional order PID controller, as a generalization of traditional PID controller. Agrawal and his co-workers [4], [6], [12] developed formulations and numerical schemes for FOCPs based on the fractional variational principles. Petrāš [64] presented methods of tuning and implementation of Fractional-Order Controllers. Machado discussed the design of fractional-order discrete-time controllers [86] and proposed a new method for optimal tuning of fractional controllers by using genetic algorithms [87]. However, publications on the FOCP of stochastic systems are very limited. Agrawal [5] proposed a general scheme for stochastic analysis of FOCPs using the fractional variational calculus. In the last decade, a nonlinear stochastic optimal control strategy of quasi Hamiltonian systems was developed based on the stochastic averaging method for quasi Hamiltonian systems [94] and stochastic dynamical programming principle [91] and extended to stochastic stabilization, etc. [92], [99]. Recently, the control strategy is extended to quasi integrable Hamiltonian systems with fractional derivative damping by the present authors [38].

Consider an n-DOF weakly controlled quasi integrable Hamiltonian system with light fractional derivative damping subjected to weak stochastic excitations. The equations of system are of the form

$$\begin{aligned}
&\dot{Q}_i = \frac{\partial H}{\partial P_i},\\
&\dot{P}_i = -\frac{\partial H}{\partial Q_i} - \varepsilon C_i(\mathbf{Q},\mathbf{P}) D^{\alpha_i} Q_i(t) + \varepsilon u_i + \varepsilon^{1/2} h_{ie}(\mathbf{Q},\mathbf{P})\xi_e(t),\\
&i, j = 1, 2, ..., n; \quad e = 1, 2, ..., w,
\end{aligned} \tag{83}$$

For simplicity, $\xi_e(t)$ are supposed to be Gaussian white noises with correlation functions $E[W_e(t)\ W_l(t+\tau)] = 2D_{el}\delta(\tau)$. For other stochastic excitations, the procedure is similar. Applying the stochastic averaging method given in Sect. **2** to system (83) except terms εu_i, the partially averaged Itô equations can be obtained. As indicated in Sect. **2**, the dimension and form of the averaged Itô equations depend upon the resonance of the associated Hamiltonian system. For example, in the case of non-resonant, the partially averaged Itô equations are of the form

$$\mathrm{d}H_i = \left(\bar{m}_i(\mathbf{H}) + \varepsilon < \frac{\partial H_i}{\partial p_i} u_i >_t \right)\mathrm{d}t + \bar{\sigma}_{ie}(\mathbf{H})\mathrm{d}B_e(t), \tag{84}$$

where $\bar{m}_i(\mathbf{H})$ and $\bar{\sigma}_{ie}(\mathbf{H})$ are defined by Eq. (26), in which $m_i(\mathbf{A})$ and $\sigma_{ie}(\mathbf{A})$ are of the form

$$\begin{aligned}
&m_i(\mathbf{A}) = \varepsilon \left\langle F_i^{(12)} + D_{el}\frac{\partial G_{ie}^{(1)}}{\partial A_j} G_{el}^{(1)} + D_{el}\frac{\partial G_{ie}^{(1)}}{\partial \Theta_j} G_{jl}^{(2)} \right\rangle_t,\\
&b_{ij}(\mathbf{A}) = \sigma_{ie}\sigma_{je} = \varepsilon \left\langle 2D_{el} G_{ie}^{(1)} G_{jl}^{(1)} \right\rangle_t.
\end{aligned} \tag{85}$$

The objective of control is to minimize the response of uncontrolled system (83), which can be expressed in terms of minimizing a performance index depending on control time interval. For finite time-interval control, the partially averaged performance index is of the form

$$\mathrm{J} = E\left[\int_0^{t_f} f(\mathbf{H}(s), < \mathbf{u}(s) >_t)\mathrm{d}s + g(\mathbf{H}(t_f))\right], \tag{86}$$

where $\mathbf{u} = [u_1, u_2, ..., u_n]^T$; $E[\bullet]$ denotes an expectation operation; t_f is the terminal time of control; f is called the cost function and $g(H(t_f))$ represents the final cost.

Applying the stochastic dynamical programming principle [91] to system (84) with performance index (86), the following dynamical programming equation can be formulated:

$$\begin{aligned}\frac{\partial V}{\partial t} = &- \min_{\mathbf{u}} \left\{ \frac{\partial V}{\partial H_i} \left[\bar{m}_i(\mathbf{H}) + \varepsilon < \frac{\partial H_i}{\partial p_i} u_i >_t \right] \right. \\ &+ \left. \frac{\bar{b}_{ij}(\mathbf{H})}{2} \frac{\partial^2 V}{\partial H_i \partial H_j} + f(\mathbf{H}, < \mathbf{u} >_t) \right\},\end{aligned} \tag{87}$$

where

$$V(\mathbf{H}, t) = \min_{\mathbf{u}} E \left[\int_t^{t_f} f(\mathbf{H}(s), < \mathbf{u}(s) >_t) ds + g(\mathbf{H}(t_f)) \right] \tag{88}$$

is the called value function. Note that

$$V(\mathbf{H}, t_f) = g(\mathbf{H}(t_f)). \tag{89}$$

The optimal control law $\mathbf{u}^*$ is determined by minimizing the right-hand side of Eq. (87) with respect to $\mathbf{u}$,

$$\left[\frac{\partial f(\mathbf{H}, < \mathbf{u} >_t)}{\partial u_i} + \varepsilon \frac{\partial H_i}{\partial p_i} \frac{\partial V}{\partial H_i} \right] |_{\mathbf{u} = \mathbf{u}^*} = 0, \quad i = 1, 2, ..., n. \tag{90}$$

Let the cost function be of the following form of fractional order:

$$f(\mathbf{H}, < \mathbf{u} >_t) = l(\mathbf{H}) + \varepsilon < (|\mathbf{u}|^\beta \mathrm{sign}(\mathbf{u}))^T \mathbf{R} (|\mathbf{u}|^\beta \mathrm{sign}(\mathbf{u})) >_t, \tag{91}$$

where $l(\mathbf{H}) \geq 0$ is a convex function of $\mathbf{H}$, $|\mathbf{u}|^\beta \mathrm{sign}(\mathbf{u}) = [|u_1|^\beta \mathrm{sign}(u_1), |u_2|^\beta \mathrm{sign}(u_2), \ldots, |u_n|^\beta \mathrm{sign}(u_n)]^T$, and $\mathbf{R}$ is a positive definite matrix. Then the optimal control forces are

$$u_i^* = -\left(\frac{R_{ij}^{-1}}{2\beta} \frac{\partial V}{\partial H_j} \right)^{\frac{1}{2\beta - 1}} \left| \frac{\partial H_j}{\partial p_j} \right|^{\frac{1}{2\beta - 1}} sign \left(\frac{\partial H_j}{\partial p_j} \right), \tag{92}$$

where R_{ij}^{-1} are the elements of $\mathbf{R}^{-1}$. If $\mathbf{R}$ is a diagonal matrix with positive elements R_i, then Eq. (92) can be reduced to

$$u_i^* = -\left(\frac{1}{2\beta R_i} \frac{\partial V}{\partial H_i} \right)^{\frac{1}{2\beta - 1}} \left| \frac{\partial H_j}{\partial p_j} \right|^{\frac{1}{2\beta - 1}} sign \left(\frac{\partial H_j}{\partial p_j} \right). \tag{93}$$

Substituting Eqs. (92) or (93) into Eq. (87) leads to the final dynamical programming equation. In the case of diagonal $\mathbf{R}$, the equation is

$$\begin{aligned}&\frac{\partial V}{\partial t} + \varepsilon \left[R_i \left(\frac{1}{2\beta R_i} \frac{\partial V}{\partial H_i} \right)^{\frac{2\beta}{2\beta - 1}} - \left(\frac{1}{2\beta R_i} \right)^{\frac{1}{2\beta - 1}} \left(\frac{\partial V}{\partial H_i} \right)^{\frac{2\beta}{2\beta - 1}} \right] < \left(\frac{\partial H_i}{\partial p_i} \right)^{\frac{2\beta}{2\beta - 1}} >_t \\ &\qquad + \bar{m}_i(\mathbf{H}) \frac{\partial V}{\partial H_i} + \frac{1}{2} \bar{b}_{ij}(\mathbf{H}) \frac{\partial^2 V}{\partial H_i \partial H_j} + l(\mathbf{H}) = 0.\end{aligned} \tag{94}$$

The stochastic optimal control force $\mathbf{u}^*$ can be obtained by solving the final dynamical programming equation with final condition (89) and then substituting the resultant $V(H, t)$ into the expression for u_i^*.

For semi-infinite time-interval control, the partially averaged performance index is usually the average cost per unit time

$$J_1 = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} f(\mathbf{H}(s), <\mathbf{u}(s)>_t)\mathrm{d}s, \tag{95}$$

and the corresponding optimal stochastic control is called ergodic control. The dynamic programming equation in this case is of the form [51]

$$\begin{aligned} &\min_{\mathbf{u}} \left\{ \frac{\partial V}{\partial H_i} \left[\bar{m}_i(\mathbf{H}) + \varepsilon < \frac{\partial H_i}{\partial p_i} u_i >_t \right] \right. \\ &\left. + \frac{\bar{b}_{ij}(\mathbf{H})}{2} \frac{\partial^2 V}{\partial H_i \partial H_j} + f(\mathbf{H}, <\mathbf{u}>_t) \right\} = \gamma, \end{aligned} \tag{96}$$

where

$$\gamma = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} f(\mathbf{H}(s), <\mathbf{u}^*(s)>_t)\mathrm{d}s \tag{97}$$

is the optimal average cost. The optimal control forces u_i^* for the ergodic control problem are still determined by Eq. (90). For cost function (91), the optimal control forces are still of the form of Eqs. (92) or (93).

After inserting u_i^* into Eq. (84) to replace u_i and averaging the terms involving u_i^*, the responses of uncontrolled and controlled quasi integrable Hamiltonian systems with fractional derivative damping can then be predicted by solving the FPK equations associated with the fully averaged Itô Eq. (84) with $\mathbf{u} = 0$ and $\mathbf{u} = \mathbf{u}^*$, respectively.

To evaluate the performance of the control strategy, two criteria are introduced. One is control effectiveness K_i and the other is control efficiency μ_i. They are defined as follows:

$$\begin{aligned} &K_i = \frac{E[Q_i^2]_u - E[Q_i^2]_c}{E[Q_i^2]_u}, \quad \mu_i = \frac{K_i}{E[u_i^2]/2D_{ii}}, \\ &i = 1, 2, ..., n, \end{aligned} \tag{98}$$

where $E[Q_i^2]_u$ and $E[Q_i^2]_c$ denote the mean-square displacements of the uncontrolled and controlled systems, respectively; $E[u_i^2]$ denote the mean-square control forces. Obviously, the higher K_i and μ_i are, the better the control strategy is. The control strategy has been applied to two coupled Van der Pol oscillators with fractional derivative dampings [38]. The numerical results show that the proposed control strategy is effective and efficient, and the system will be more robust by taking the fractional order control than integer order control, see Fig.7.

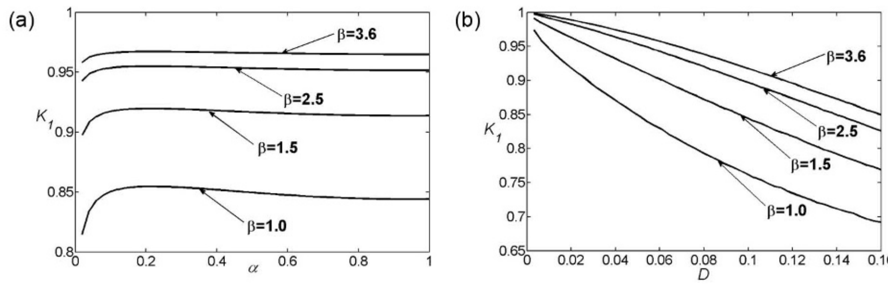

Fig. 7. (a) Control effectiveness K_1 for displacement of the first DOF versus fractional derivative order $\alpha_1 = \alpha_2 = \alpha$ for different β. (b) Control effectiveness K_1 for displacement of the first DOF versus excitation intensity $D_1 = D_2 = D$ for different β.

Consider system (83) with pure parametric stochastic excitations. Assume that the trivial solution of uncontrolled system is unstable i.e., the largest Lyapunov exponent is positive. Then the objective of control is to design a feedback control to stabilize the system, i.e., to make the largest Lyapunov exponent negative. To stabilize system (83) by using fractional optimal feedback control, an ideal performance index should be the analytical expression for the largest Lyapunov exponent of the controlled system. However, it is hardly possible to have such an expression. To overcome this difficulty, the stochastic stabilization problem is formulated as a fractional ergodic control with undetermined cost function, since both stabilization and ergodic control are conducted in a semi-infinite time-interval. The cost function is determined later by the requirement of minimizing the largest Lyapunov exponent. The stabilization of quasi integrable Hamiltonian systems with fractional derivative damping by using fractional optimal control has also been down.

6. Conclusion

In the present paper, some progress in the stochastic dynamics and fractional optimal control of quasi integrable Hamiltonian systems with fractional derivative damping, including stochastic averaging method, stochastic stability, stochastic bifurcation, first-passage time and stochastic fractional optimal control are reviewed. Specifically, the effects of fractional derivative orders in system and control have been shown clearly. Although significant progress has been made, many problems are still needed to be solved. For example, the solutions of higher-dimensional FPK equation, backward Kolmogorov equation, and dynamical programming equation associated with averaged Itô equations, and the fractional optimal control of stochastic systems with partial observation, with uncertain parameters and(or) time-delayed feedback, etc. Since quasi non-integrable Hamiltonian system and quasi integrable Hamiltonian system are both the special cases of quasi partially integrable Hamiltonian system, to develop a stochastic averaging method for quasi partially integrable Hamiltonian systems with fractional derivative damping is one of the most important tasks in near future.

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[1] *College of Civil Engineering*
Huaqiao University
Xiamen, 361021, CHINA
e-mail: lincongchen@yahoo.com

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[2] *Departments of Mechanics*
Zhejiang University
Hangzhou, 310027, CHINA
e-mail: hufang9360@zju.edu.cn

[3] *Departments of Mechanics*
State Key Laboratory of Fluid Power Transmission and Control
Zhejiang University
Hangzhou, 310027, CHINA
e-mail: wqzhu@yahoo.com

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