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A Carleman-Nevanlinna Theorem and Summation of the Riemann Zeta-Function Logarithm

Andriy A. Kondratyuk

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Abstract. A Carleman-Nevanlinna Theorem for a rectangle is proved. The theorem is applied to the summation of $\log|\zeta(s)|$ on the critical and other vertical lines, where $\zeta(s)$ is the Riemann zeta-function. In particular, let

$$I(\varepsilon) = \int_0^\infty e^{-\varepsilon t} \log\left|\zeta\left(\frac{1}{2} + it\right)\right| dt, \qquad \varepsilon > 0,$$

and let $\{\rho_j\}$ be non-trivial zeros of $\zeta(s)$, then

$$\frac{\pi}{2}\sum_j \left|\operatorname{Re}\rho_j - \frac{1}{2}\right| = I(+0) + \frac{\pi}{2},$$

where $I(+0) := \lim_{\varepsilon\to 0} I(\varepsilon)$. Thus, the Riemann hypothesis for $\zeta(s)$ holds if and only if $I(+0) = -\pi/2$.

Keywords. Meromorphic function, Carleman-Nevanlinna Theorem, summation, Riemann zeta-function, Euler product, Riemann hypothesis, Dirichlet series, Selberg class, L-function.

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1. Introduction and main result

In 1922–23 F. Nevanlinna and T. Carleman [10, 11, 2] (see also [5, p. 19]) established a relation between the distribution of zeros and poles of a meromorphic function f in a half-ring and its values on the boundary; more precisely, the values of some branch of $\log f$. R. Nevanlinna applied this relation to the value distribution theory for meromorphic functions in a half-plane [12] (see also [5, p. 37–43]). T. Carleman applied it to the polynomial approximation of holomorphic functions [2]. We prove a Carleman-Nevanlinna Theorem (Theorem 2 below) for a rectangle. Our proof is close to Littlewood's proof of a counterpart of the Jensen Theorem for a rectangle [7].

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The principal aim of this paper is an application of the above mentioned theorem to the summation of the logarithm of the Riemann zeta-function on the critical and other vertical lines. The zero distribution of the zeta-function plays an essential role in the results obtained.

We first recall some properties of the Riemann zeta-function $\zeta(s)$ [16, 4, 6]. This function is defined as

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}, \qquad \operatorname{Re} s > 1,$$

or

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad \operatorname{Re} s > 1, \tag{1}$$

where the product is taken over all primes. The function $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$ with a single pole at $s = 1$. The Euler product formula (1) gives a first impression at the close connection between $\zeta(s)$ and the distribution of prime numbers. In other words, relation (1) shows that $\zeta(s)$ is a generating function for the primes.

The Riemann hypothesis (RH) states that the non-real (non-trivial) zeros of $\zeta(s)$ all lie on the line $\operatorname{Re} s = 1/2$, named the *critical line*. It is known that they lie in the *critical strip* $\{s : 0 < \operatorname{Re} s < 1\}$. One reason for the great interest in RH is its connections with problems in number theory, algebraic geometry, topology, representation theory and perhaps even physics (see [8, 15]).

The distribution of non-trivial zeros of the function $\zeta(s)$ depends on its value distribution on the critical line [6].

The following results were obtained concerning the classical branch of $\log\zeta(s)$ in [13, 14, 16]:

$$\int_0^T \left|\log\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = T\log\log T + \mathcal{O}\left(T\sqrt{\log\log T}\right), \qquad T \to +\infty, \tag{2}$$

and for fixed σ, $1/2 < \sigma \le 1$, in [1]

$$\int_0^T |\log\zeta(\sigma + it)|^2\, dt = T\sum_{k=1}^{\infty} \frac{1}{k^2} \sum_p \frac{1}{p^{2k\sigma}} + o(T), \qquad T \to +\infty, \tag{3}$$

respectively.

For $\log|\zeta(s)|$ on the vertical lines we introduce a summing factor $\exp(-\varepsilon t)$ and consider the integrals

$$I(\varepsilon, c) = \int_0^{+\infty} e^{-\varepsilon t} \log|\zeta(c + it)|\, dt, \qquad \varepsilon > 0,\ c \ge \frac{1}{2},$$

setting $I(\varepsilon) = I(\varepsilon, 1/2)$.

Our main result is as follows.

Theorem 1. *The limit*

$$\lim_{\varepsilon\to 0} I(\varepsilon, c) =: I(+0, c), \qquad \frac{1}{2} \le c \le 1,$$

exists (not necessarily finitely). Furthermore,

(i) *the following equalities hold*

$$\pi \sum_{\operatorname{Re}\rho_j > c} (\operatorname{Re}\rho_j - c) = I(+0, c) + \pi(1-c), \qquad \frac{1}{2} < c \le 1, \tag{4}$$

$$\frac{\pi}{2} \sum_j \left|\operatorname{Re}\rho_j - \frac{1}{2}\right| = I(+0) + \frac{\pi}{2}, \tag{5}$$

where $I(+0) := I(+0, 1/2)$ and $\{\rho_j\}$ are non-trivial zeros of $\zeta(s)$;

(ii) *RH holds if and only if $I(+0) = -\pi/2$;*

(iii) *RH holds if and only if $I(+0, c) = -\pi(1-c)$ for each c, $1/2 < c < 1$;*

(iv) *if RH holds, then*

$$I(\varepsilon) = -\frac{\pi}{2} + K\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0,$$

where

$$K = \int_{1/2}^{+\infty} \left(\sigma - \frac{1}{2}\right) \log|\zeta(\sigma)|\, d\sigma.$$

So, RH may be expressed in terms of the behavior of $\log|\zeta(s)|$ only on the critical line.

In connection with Theorem 1 the notion of "weak RH" may be introduced, namely when

$$\sum_j \left|\operatorname{Re}\rho_j - \frac{1}{2}\right| < +\infty.$$

According to (5) this is equivalent to $I(+0) < +\infty$.

In fact, we prove our main result (Theorem 3 below) for a class of functions containing $\zeta(s)$ and, perhaps also functions from the Selberg class [13] (see also [3, 9]) represented by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

having an Euler product, analytic continuation and a functional equation. It is expected that for every function in the Selberg class an analogue of (RH) holds. Many L-functions belong to the Selberg class (see [3, 9]).

2. A Carleman-Nevanlinna Theorem for a rectangle

Let $f(s)$ be a meromorphic function on the closure of the rectangle

$$R_T=\{s=\sigma+it:\ \alpha<\sigma<\beta,\ 0<t<T\}.$$

Denote by $\{\rho_j\}$, $\rho_j=\beta_j+i\gamma_j$, and $\{\widetilde{\rho}_j\}$, $\widetilde{\rho}_j=\widetilde{\beta}_j+i\widetilde{\gamma}_j$, its zeros and poles in R_T respectively. Let $f(\beta)\neq 0,\infty$ and $\log f(\beta)$ be determined. Put

$$\log f(s)=\log f(\beta)+\int_\beta^s \frac{f'(\eta)}{f(\eta)}\,d\eta, \tag{6}$$

where the integral is taken along a path in R_T with the vertical slits

$$\left\{\beta_j+i\tau\gamma_j:\ 1\le\tau<\frac{T}{\gamma_j}\right\}\cup\left\{\widetilde{\beta}_j+i\tau\widetilde{\gamma}_j:\ 1\le\tau<\frac{T}{\widetilde{\gamma}_j}\right\}$$

the ends of which are β and s. Thus, $\log f(s)$ is determined on $\overline{R}_T$ with those slits except at zeros and poles on ∂R_T. Put also $\arg f(s)=\operatorname{Im}\log f(s)$.

We prove the following proposition which is close to the Carleman-Nevanlinna Theorem [10, 11, 2] for meromorphic functions in a half-ring.

Theorem 2. *Let $f(s)$ be a meromorphic function on the closure of the rectangle*

$$R_T=\{s=\sigma+it:\ \alpha<\sigma<\beta,\ 0<t<T\},\qquad \beta=\alpha+\pi\omega,\ \omega>0.$$

Then

$$\begin{aligned}
&2\pi\omega\left(\sum_{\rho_j\in R_T}\sinh\frac{T-\gamma_j}{\omega}\sin\frac{\beta_j-\alpha}{\omega}-\sum_{\widetilde{\rho}_j\in R_T}\sinh\frac{T-\widetilde{\gamma}_j}{\omega}\sin\frac{\widetilde{\beta}_j-\alpha}{\omega}\right)\\
&\quad=-\cosh\frac{T}{\omega}\int_\alpha^\beta\sin\frac{\sigma-\alpha}{\omega}\log|f(\sigma)|\,d\sigma\\
&\qquad-\sinh\frac{T}{\omega}\int_\alpha^\beta\cos\frac{\sigma-\alpha}{\omega}\arg f(\sigma)\,d\sigma\\
&\qquad+\int_0^T\sinh\frac{T-t}{\omega}\log|f(\alpha+it)|\,dt\\
&\qquad+\int_0^T\sinh\frac{T-t}{\omega}\log|f(\beta+it)|\,dt\\
&\qquad+\int_\alpha^\beta\sin\frac{\sigma-\alpha}{\omega}\log|f(\sigma+iT)|\,d\sigma,
\end{aligned}\tag{7}$$

where $\{\rho_j=\beta_j+i\gamma_j\}$ and $\{\widetilde{\rho}_j=\widetilde{\beta}_j+i\widetilde{\gamma}_j\}$ are zeros and poles of $f(s)$ respectively.

In order to prove Theorem 2 we need three elementary lemmas.

Lemma 1. *If $f(s)$ is a holomorphic function on the closure of the rectangle R_T and no zero of f lies on ∂R_T, then* (7) *holds.*

Proof. By the Residue Theorem

$$\int_{\partial R_t} \frac{f'(s)}{f(s)}\, e^{i(s-\alpha)/\omega}\, ds = 2\pi i \sum_{\rho_j \in R_t} e^{i(\rho_j-\alpha)/\omega}. \tag{8}$$

Here $t < T$ and no zero of f lies on ∂R_t. The left side of (8) may be rewritten as follows:

$$\begin{aligned}
&\int_{\partial R_t} \frac{f'(s)}{f(s)}\, e^{i(s-\alpha)/\omega}\, ds \\
&\quad = \int_\alpha^\beta \frac{f'(\sigma)}{f(\sigma)}\, e^{i(\sigma-\alpha)/\omega}\, d\sigma - i\int_0^t \frac{f'(\alpha+i\tau)}{f(\alpha+i\tau)}\, e^{-\tau/\omega}\, d\tau \\
&\qquad -i\int_0^t \frac{f'(\beta+i\tau)}{f(\beta+i\tau)}\, e^{-\tau/\omega}\, d\tau - e^{-t/\omega}\int_\alpha^\beta \frac{f'(\sigma+it)}{f(\sigma+it)}\, e^{i(\sigma-\alpha)/\omega}\, d\sigma.
\end{aligned} \tag{9}$$

Multiplying (9) by $\exp(t/\omega)$, integrating over t from 0 to T, and taking into account (8), we obtain

$$\begin{aligned}
&2\pi i \int_0^T e^{t/\omega} \sum_{\rho_j \in R_t} e^{i(\rho_j-\alpha)/\omega}\, dt \\
&\quad = \int_0^T e^{t/\omega} \left(\int_\alpha^\beta \frac{f'(\sigma)}{f(\sigma)}\, e^{i(\sigma-\alpha)/\omega}\, d\sigma \right) dt \\
&\qquad -i\int_0^T e^{t/\omega} \left(\int_0^t \frac{f'(\alpha+i\tau)}{f(\alpha+i\tau)}\, e^{-\tau/\omega}\, d\tau \right) dt \\
&\qquad -i\int_0^T e^{t/\omega} \left(\int_0^t \frac{f'(\beta+i\tau)}{f(\beta+i\tau)}\, e^{-\tau/\omega}\, d\tau \right) dt \\
&\qquad -\int_0^T \left(\int_\alpha^\beta \frac{f'(\sigma+it)}{f(\sigma+it)}\, e^{i(\sigma-\alpha)/\omega}\, d\sigma \right) dt \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{10}$$

Integrating by parts, we have

$$\begin{aligned}
I_1 &= \omega\left(1 - e^{T/\omega}\right)\left(\log f(\beta) + \log f(\alpha)\right) + i\left(1 - e^{T/\omega}\right)\int_\alpha^\beta e^{i(\sigma-\alpha)/\omega} \log f(\sigma)\, d\sigma, \\
I_2 &= -i\omega e^{T/\omega} \int_0^T \frac{f'(\alpha+it)}{f(\alpha+it)}\, e^{-t/\omega}\, dt + i\omega \int_0^T \frac{f'(\alpha+it)}{f(\alpha+it)}\, dt \\
&= \omega e^{T/\omega}\left(\log f(\alpha) - e^{-T/\omega} \log f(\alpha+iT) - \frac{1}{\omega}\int_0^T e^{-t/\omega} \log f(\alpha+it)\, dt \right) \\
&\quad +\omega\left(\log f(\alpha+iT) - \log f(\alpha)\right) \\
&= \omega\left(e^{T/\omega} - 1\right) \log f(\alpha) - e^{T/\omega}\int_0^T e^{-t/\omega} \log f(\alpha+it)\, dt.
\end{aligned}$$

Similarly,

$$I_3 = \omega\left(e^{T/\omega} - 1\right)\log f(\beta) - e^{T/\omega}\int_0^T e^{-t/\omega}\log f(\beta + it)\,dt.$$

Using the Fubini Theorem, we obtain

$$I_4 = i\int_\alpha^\beta \left(\log f(\sigma + iT) - \log f(\sigma)\right) e^{i(\sigma-\alpha)/\omega}\,d\sigma.$$

Thus, the relation (10) can be rewritten in the form

$$\begin{aligned}
&2\pi\int_0^T e^{t/\omega}\sum_{\rho_j\in R_t} e^{i(\rho_j-\alpha)/\omega}\,dt \\
&\quad = e^{-T/\omega}\int_\alpha^\beta e^{i(\sigma-\alpha)/\omega}\log f(\sigma)\,d\sigma + ie^{T/\omega}\int_0^T e^{-t/\omega}\log f(\alpha+it)\,dt \\
&\quad +ie^{T/\omega}\int_0^T e^{-t/\omega}\log f(\beta+it)\,dt + \int_\alpha^\beta e^{i(\sigma-\alpha)/\omega}\log f(\sigma+iT)\,d\sigma.
\end{aligned}\tag{11}$$

Similarly, the Residue Theorem applied to the function $f'(s)/f(s)e^{i(\alpha-s)/\omega}$ yields

$$\begin{aligned}
&2\pi\int_0^T e^{-t/\omega}\sum_{\rho_j\in R_t} e^{-i(\rho_j-\alpha)/\omega}\,dt \\
&\quad = -e^{-T/\omega}\int_\alpha^\beta e^{-i(\sigma-\alpha)/\omega}\log f(\sigma)\,d\sigma - ie^{-T/\omega}\int_0^T e^{t/\omega}\log f(\alpha+it)\,dt \\
&\quad -ie^{-T/\omega}\int_0^T e^{t/\omega}\log f(\beta+it)\,dt + \int_\alpha^\beta e^{-i(\sigma-\alpha)/\omega}\log f(\sigma+iT)\,d\sigma.
\end{aligned}\tag{12}$$

Adding the conjugates of both sides of (12) to (11) and taking the imaginary parts, we obtain on the left side

$$2\pi\int_0^T\sum_{\rho_j\in R_t}\cosh\frac{t-\gamma_j}{\omega}\sin\frac{\beta_j-\alpha}{\omega}\,dt.$$

Integrating by parts this Stieltjes integral and taking into account the right side of the equality obtained, we have (7). Hence Lemma 1 holds. ∎

Lemma 2. *Let $\alpha \le \beta_0 \le \beta$, $\beta = \alpha + \pi\omega$, $\omega > 0$, $T > 0$. Then* (7) *holds for $f(s) = s - \beta_0$.*

Proof. Let $\alpha < \beta_0 < \beta$, let D_δ denote the disk of radius δ centered at $s = \beta_0$. Applying the Cauchy Theorem to the function $e^{i(s-\alpha)/\omega}/(s-\beta_0)$ in the domain

$R_t^\delta = R_t \setminus D_\delta$ where δ is sufficiently small we have

$$\begin{aligned} 0 &= \int_{\partial R_t^\delta} e^{i(s-\alpha)/\omega} \frac{ds}{s-\beta_0} \\ &= \int_\alpha^{\beta_0-\delta} e^{i(\sigma-\alpha)/\omega} \frac{d\sigma}{\sigma-\beta_0} + \int_{\beta_0+\delta}^{\beta} e^{i(\sigma-\alpha)/\omega} \frac{d\sigma}{\sigma-\beta_0} - \int_{\gamma_\delta} e^{i(s-\alpha)/\omega} \frac{ds}{s-\beta_0} + J \\ &= J_1 + J. \end{aligned}$$

Here $\gamma_\delta = \{s = \beta_0 + \delta e^{i\varphi} : 0 \le \varphi \le \pi\}$. Integrating by parts we obtain

$$\begin{aligned} J_1 &= e^{i(\beta_0-\delta-\alpha)/\omega} \log(-\delta) - \log(\alpha-\beta_0) \\ &\quad -\frac{i}{\omega}\int_\alpha^{\beta_0-\delta} e^{i(\sigma-\alpha)/\omega} \log(\sigma-\beta_0)\, d\sigma - \int_{\gamma_\delta} e^{i(s-\alpha)/\omega} \frac{ds}{s-1} \\ &\quad - \log(\beta-\beta_0) - e^{i(\beta_0+\delta-\alpha)/\omega} \log\delta - \frac{i}{\omega}\int_{\beta_0+\delta}^{\beta} e^{i(\sigma-\alpha)/\omega} \log(\sigma-\beta_0)\, d\sigma. \end{aligned}$$

Noting that

$$\log(-\delta) - \log\delta = \log f(\beta_0-\delta) - \log f(\beta_0+\delta) = \int_{\gamma_\delta} \frac{ds}{s-\beta_0} = \pi i,$$

letting δ tend to 0 and multiplying both sides of the equality obtained by $\omega(\exp(T/\omega)-1)$ we obtain

$$\begin{aligned} \omega\left(e^{T/\omega}-1\right) J_1 &= \omega\left(1-e^{T/\omega}\right)\left(\log(\beta-\beta_0) + \log(\alpha-\beta_0)\right) \\ &\quad + i\left(1-e^{T/\omega}\right)\int_\alpha^\beta e^{i(\sigma-\alpha)/\omega} \log(\sigma-\beta_0)\, d\sigma. \end{aligned}$$

This is I_1 from the proof of Lemma 1 for $f(s) = s-\beta_0$. Further, we multiply J by $\exp(t/\omega)$ and integrate over t from 0 to T. The continuation of the proof is as in Lemma 1.

Let $\beta_0 = \alpha$. The Cauchy Theorem and integration by parts yield

$$\begin{aligned} 0 &= \int_{\partial R_t^\delta} e^{i(s-\alpha)/\omega} \frac{ds}{s-\alpha} \\ &= e^{-\delta/\omega}\log(i\delta) - e^{i\delta/\omega}\log\delta - i\int_0^{\pi/2} \exp(i\delta e^{i\varphi})\, d\varphi + B \\ &= A(\delta) + B. \end{aligned}$$

Since $\log(i\delta) = \log\delta + \pi i/2$, we have $A(\delta) \to 0$ as $\delta \to 0$ and proceed as in the proof of Lemma 1. We deal with the case $\beta_0 = \beta$ similarly. ■

Lemma 3. *Let* $0 < \gamma < T$, $\beta = \alpha + \pi\omega$, $\omega > 0$. *Then relation* (7) *holds for* $f(s) = s - \alpha - i\gamma$ *and for* $f(s) = s - \beta - i\gamma$.

Proof. Let $f(s)=s-\alpha-i\gamma$. If $0<t<\gamma$ we rewrite (9) as follows:

$$0=-i\int_0^t e^{\tau/\omega}\frac{d\tau}{\tau-\gamma}+J_3(t)=J_2(t)+J_3(t).$$

If $\gamma<t$ we apply the Residue Theorem to the function $f(s)e^{i(s-\alpha)/\omega}$ in the domain $R_t\setminus K_\delta$ where K_δ is the disk of radius δ centered at $\alpha+i\gamma$, $\delta<\min(t-\gamma,\gamma)$. In this case, relation (9) can by rewritten in the form

$$\begin{aligned}0&=-i\int_0^{\gamma-\delta} e^{-\tau/\omega}\frac{d\tau}{\tau-\gamma}-i\int_{\gamma+\delta}^t e^{-\tau/\omega}\frac{d\tau}{\tau-\gamma}\\ &\quad-\int_{\Gamma_\delta} e^{i(s-\alpha-i\gamma)/\omega}\frac{ds}{s-\alpha-i\gamma}+J_3(t)\\ &=J_2(t,\delta)+J_3(t),\end{aligned}\tag{13}$$

where $\Gamma_\delta=\{s:\ s=\alpha+i\gamma+\delta e^{i\varphi},\ 0\le\varphi\le\pi\}$.
If $t<\gamma$ then

$$J_2(t)=\log(-i\gamma)-e^{-t/\omega}\log(it-i\gamma)-\frac{1}{\omega}\int_0^t e^{-\tau/\omega}\log(i\tau-i\gamma)\,d\tau.$$

If $t>\gamma$ we have

$$\begin{aligned}J_2(t,\delta)\ &=\ -e^{-(\gamma-\delta)/\omega}\log(-i\delta)+\log(-i\gamma)-\frac{1}{\omega}\int_0^{\gamma-\delta} e^{-t/\omega}\log(it-i\gamma)\,dt\\ &\quad-e^{-t/\omega}\log(it-i\gamma)+e^{-(\gamma+\delta)/\omega}\log(i\delta)\\ &\quad-\frac{1}{\omega}\int_{\gamma+\delta}^t e^{-\tau/\omega}\log(i\tau-i\gamma)\,d\sigma\\ &\quad-\int_{\Gamma_\delta} e^{i(s-\alpha-i\gamma)/\omega}\frac{ds}{s-\alpha-i\gamma}.\end{aligned}\tag{14}$$

According to the definition of $\log f(s)$, we find as above

$$\log f(\alpha+i(\gamma+\delta))-\log f(\alpha+i(\gamma-\delta))=\pi i.$$

Hence, as $\delta\to0$, relation (14) yields

$$\begin{aligned}J_2(t)\ &=\ J_2(t,0)\\ &=\log(-i\gamma)-e^{-t/\omega}\log(it-i\gamma)-\frac{1}{\omega}\int_0^t e^{-\tau/\omega}\log(i\tau-i\gamma)\,d\tau,\qquad \gamma<t.\end{aligned}$$

Thus, as $\delta\to0$, relation (13) takes the from

$$0=J_2(t)+J_3(t),\qquad \gamma<t.$$

Note that

$$\int_0^T J_2(t)e^{t/\omega}\,dt=\omega\left(e^{T/\omega}-1\right)\log f(\alpha)-e^{T/\omega}\int_0^T e^{-t/\omega}\log f(\alpha+it)\,dt,$$

which is I_2 from the proof of Lemma 1. The continuation of the proof is as before. The proof for $f(s) = s - \beta - i\gamma$ is similar. ■

Proof of Theorem 2. If neither zero nor pole of $f(s)$ lies on ∂R_T then the proof of (7) needs a few evident changes with respect to that of Lemma 1. Moreover, while proving this lemma we applied the Residue Theorem to the domain R_t and integrated over t from 0 to T. Hence, (7) remains valid also for meromorphic functions admitting zeros or poles on $\{s : \operatorname{Im} s = T\}$.

If $f(s)$ has a zero ρ on ∂R_T, $\operatorname{Im}\rho \neq T$, we apply Lemma 1 to the function $f(s)/(s-\rho)$. Choosing $\log((f(\beta)/(\beta-\rho)) = \log f(\beta) - \log(\beta-\rho)$ which implies $\log(f(s)/(s-\rho)) = \log f(s) - \log(s-\rho)$, $s \in \overline{R}_T \setminus \{\rho\}$, we use Lemma 2 or Lemma 3. The case when $f(s)$ has a pole $\widetilde{\rho}$ on ∂R_T, $\operatorname{Im}\widetilde{\rho} \neq T$, is similar to the previous one. We obtain (7) by recurrences for an arbitrary function $f(s)$ satisfying the conditions of Theorem 2. ■

3. Summation of the Riemann zeta-function logarithm on the vertical lines

Let $\varepsilon \geq 0$, $R_T(c,\varepsilon) = \{s : c < \operatorname{Re} s < c + \pi/\varepsilon,\ 0 < \operatorname{Im} s < T \leq +\infty\}$, $R_\infty(c) = R_\infty(c,0)$. Denote by $\mathcal{F}$ the class of functions $f(s)$ possessing the following properties:

(a) there exist real numbers $c, c_0, c_0 \geq c$, and a non-negative integer m such that the function $(s-c_0)^m f(s)$ is holomorphic on $\overline{R}_\infty(c)$ and $(\sigma - c_0)^m f(\sigma) > 0$, $c_0 < \sigma$;

(b) $\displaystyle\int_c^\sigma \log|f(\eta + iT)|\, d\eta = o(e^{\varepsilon T})$, $T \to +\infty$, for every fixed $\varepsilon > 0$ uniformly over σ, $\sigma \geq c$;

(c) $\displaystyle\int_0^T |\log|f(\sigma+it)||\, dt = o(e^{\varepsilon T})$, $T \to +\infty$, for every fixed $\varepsilon > 0$, $\sigma = c$ and $\sigma = c + \pi/\varepsilon$;

(d) $\displaystyle\int_0^{+\infty} e^{-\varepsilon t} \log\left|f\left(c + \frac{\pi}{\varepsilon} + it\right)\right| dt = o(\varepsilon)$, $\varepsilon \to 0$;

(e) the integral $\displaystyle\int_c^{+\infty} (\sigma - c)|\log|f(\sigma)||\, d\sigma$ converges.

For $f \in \mathcal{F}$ denote

$$I(\varepsilon, c) = \int_0^\infty e^{-\varepsilon t} \log|f(c+it)|\, dt, \qquad \varepsilon > 0.$$

The existence of this integral follows from (c).

Theorem 3. *Let $f \in \mathcal{F}$, and let $\{\rho_j\}$ be the zero sequence of f. Then the limit*

$$\lim_{\varepsilon\to 0} I(\varepsilon, c) =: I(+0, c),$$

exists (not necessarily finitely). Furthermore, we have

(i) $I(+0,c) = 2\pi \sum_{\rho_j \in R_\infty(c)} (\operatorname{Re}\rho_j - c) + m\pi(c - c_0)$ *where m and c_0 satisfy* (a)*;*
(ii) *no zero of f lies in $R_\infty(c)$ if and only if $I(+0,c) = m\pi(c-c_0)$;*
(iii) *if no zero of f lies in $R_\infty(c)$, then*

$$I(\varepsilon, c) = m\pi(c - c_0) + K_c\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0,$$

where

$$K_c = \int_c^{+\infty} (\sigma - c) \log|f(\sigma)|\, d\sigma.$$

Proof. Let $c_0 > c$. Let $F(s)$ denote $(s-c_0)^m f(s)$. With regard to (a), $F(\sigma) > 0$ as $\sigma > c$. Hence, $f(\sigma) > 0$, $\sigma > c_0$. Define $\log f(s)$ and $\log F(s)$ by (6) choosing $\arg F(\beta) = 0$ and $\arg f(\beta) = 0$, $\beta = c + \pi/\varepsilon$, $0 < \varepsilon < \pi/(c_0 - c)$. Thus, $\arg f(\sigma) = 0$, $\sigma > c_0$.

Further we will apply Theorem 2 to $f(s)$.

Removing the half-disk of radius δ centered at c_0 and applying the Cauchy Theorem we have as $\delta \to 0$

$$\arg f(\sigma) - m \operatorname{Im} \int_\sigma^\beta \frac{d\sigma}{\sigma - c_0} + m\pi = \arg F(\sigma) = 0, \qquad c < \sigma < c_0,$$

where the principal value of the integral on $[\sigma, \beta]$ is considered. Consequently, $\arg f(\sigma) = -m\pi$, $c < \sigma < c_0$. Therefore,

$$\begin{aligned} \int_c^{c+\pi/\varepsilon} \cos(\varepsilon(\sigma - c)) \arg f(\sigma)\, d\sigma &= -m\pi \int_c^{c_0} \cos(\varepsilon(\sigma - c))\, d\sigma \\ &= \frac{m\pi}{\varepsilon} \sin(\varepsilon(c - c_0)). \end{aligned} \tag{15}$$

If $\arg f(\beta) = 2k\pi i$, $k \in \mathbb{Z}$, we obtain the same value, because

$$2k\pi i \int_c^{c+\pi/\varepsilon} \cos(\varepsilon(\sigma - c))\, d\sigma = 0.$$

Thus, the second integral of (7) with $\alpha = c$, $\omega = 1/\varepsilon$ is calculated.

Integrating by parts, rewrite the last integral of (7) in the form

$$-\varepsilon \int_c^{c+\pi/\varepsilon} \left(\int_c^\sigma \log|f(\eta + iT)|\, d\eta \right) \cos(\varepsilon(\sigma - c))\, d\sigma. \tag{16}$$

Fix ε, $0 < \varepsilon < \pi/(c_0 - c)$. According to (b) the integral of (16) is $\mathcal{O}(e^{\varepsilon T})$, $T \to +\infty$. With regard to (c),

$$\begin{aligned} & e^{-\varepsilon T} \int_0^T \sinh \frac{\varepsilon(T-t)}{\omega} \log|f(\sigma+it)|\,dt \\ & = \int_0^T e^{-\varepsilon t} \log|f(\sigma+it)|\,dt + \mathcal{O}(1), \qquad T \to \infty, \end{aligned} \tag{17}$$

for $\sigma = c$ and $\sigma = c + \pi/\varepsilon$.

Dividing both sides of (7) by $\exp(\varepsilon T)$, taking into account (15), (17) and the value of integral (16) we obtain as $T \to +\infty$

$$\begin{aligned} & 2\pi \sum_{\rho_j \in R_\infty(c,\varepsilon)} e^{-\varepsilon\gamma_j} \frac{\sin(\varepsilon(\beta_j - c))}{\varepsilon} + \frac{m\pi}{\varepsilon} \sin(\varepsilon(c - c_0)) \\ & = \int_0^{+\infty} e^{-\varepsilon t} \log|f(c+it)|\,dt + \int_0^{+\infty} e^{-\varepsilon t} \log\left|f\left(c + \frac{\pi}{\varepsilon} + it\right)\right| dt \\ & - \int_c^{c+\pi/\varepsilon} \sin(\varepsilon(\sigma - c)) \log|f(\sigma)|\,d\sigma. \end{aligned} \tag{18}$$

By (d) the second integral of relation (18) is $\mathcal{O}(1)$ as $\varepsilon \to 0$. Consider its last integral. If c_1 is fixed, $c < c_1$, then

$$\begin{aligned} & \int_c^{c_1} \sin(\varepsilon(\sigma - c)) \log|f(\sigma)|\,d\sigma \\ & = \varepsilon \int_c^{c_1} (\sigma - c) \frac{\sin(\varepsilon(\sigma - c))}{\varepsilon(\sigma - c)} \log|f(\sigma)|\,d\sigma \\ & = \varepsilon \int_c^{c_1} (\sigma - c) \log|f(\sigma)|\,d\sigma + o(\varepsilon), \qquad \varepsilon \to 0, \end{aligned} \tag{19}$$

in view of the fact that $\sin(\varepsilon(\sigma - c))/(\varepsilon(\sigma - c))$ tends to unity uniformly over σ on $[c, c_1]$ as $\varepsilon \to 0$.

Further, as $|\sin x| \le |x|$ using (e) we have

$$\begin{aligned} & \left| \int_{c_1}^{c+\pi/\varepsilon} \sin(\varepsilon(\sigma - c)) \log|f(\sigma)|\,d\sigma \right| \\ & \le \varepsilon \int_{c_1}^{+\infty} (\sigma - c) |\log|f(\sigma)||\,d\sigma = o(\varepsilon), \qquad \varepsilon \to 0. \end{aligned}$$

Since c_1 may be chosen sufficiently large, this and (19) ensure that

$$\int_c^{c+\pi/\varepsilon} \sin \varepsilon(\sigma - c) \log|f(\sigma)|\,d\sigma = K_c \varepsilon + o(\varepsilon), \qquad \varepsilon \to 0,$$

where

$$K_c = \int_c^{+\infty} (\sigma - c) \log|f(\sigma)|\,d\sigma.$$

Thus, taking into account (d) we can rewrite (18) as follows:

$$
\begin{aligned}
(20)\qquad & 2\pi \sum_{\rho_j \in R_\infty(c,\varepsilon)} e^{-\varepsilon\gamma_j}(\beta_j - c)\frac{\sin(\varepsilon(\beta_j - c))}{\varepsilon(\beta_j - c)} + m\pi\frac{\sin(\varepsilon(c - c_0))}{\varepsilon} \\
& = I(\varepsilon, c) - K_c\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0.
\end{aligned}
$$

Each term of the sum above is non-negative and increases as ε decreases. Hence, the limit of the left side of (20), finite or $+\infty$, exists as $\varepsilon \to 0$, and we obtain (i), (ii), and (iii). The case $c_0 = c$ is simpler because the second integral of (7) vanishes. ∎

Proof of Theorem 1. Verify that Riemann's zeta-function meets the conditions of Theorem 3.

Condition (a) is fulfilled by [16] for $\zeta(s)$ with $c_0 = 1$, $m = 1$ and $1/2 \le c \le 1$.

Properties (b) and (c) for $f(s) = \zeta(s)$ in a sharper form were proved in [7] (see also [16] and relations (2) and (3)).

Finally, the Euler product formula (1) yields

$$
\begin{aligned}
(21)\qquad \log\zeta(s) &= \log\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_p \log\left(1 - \frac{1}{p^s}\right)^{-1} \\
&= \sum_{k=1}^{\infty}\sum_p \frac{1}{kp^{ks}}, \qquad \operatorname{Re} s > 1.
\end{aligned}
$$

where p denotes primes. This proves (d) and (e).

Taking into account that if ρ is a non-trivial zero of $\zeta(s)$ then by [16] it follows that $\overline{\rho}$, $1-\rho$, and $1-\overline{\rho}$ are also such non-trivial zeros, we obtain all the conclusions of Theorem 1 immediately from Theorem 3. ∎

Remark 1. Since no zero of $\zeta(s)$ lies in $\{s : \operatorname{Re} s > 1\}$, relation (4) yields $I(+0, 1) = 0$.

Remark 2. In view of (21) the constant K may be represented in the form

$$
K = \int_{1/2}^{1} \left(\sigma - \frac{1}{2}\right) \log|\zeta(\sigma)|\, d\sigma + \frac{1}{2}\sum_{k=1}^{\infty}\sum_p \frac{2 + k\log p}{k^3 p^k \log^2 p}.
$$

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Andriy A. Kondratyuk E-MAIL: kond@franko.lviv.ua
ADDRESS: *Lviv National University, Department of Mathematics, Universitetska Str. 1, 79000 Lviv, Ukraine.*