*The Invariance of the Index in the Metric Space of Closed Operators**

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In this paper we attempt to give a systematic treatment of the properties of semi-Fredholm operators on a Hilbert space. Throughout this article, a semi-Fredholm operator will be a closed operator—not necessarily bounded—with a dense domain and a closed range, for which an index (*cf.* previous definitions of F. Noether [14] and F. V. Atkinson [1]) can be defined.

§1 is devoted to the definition and characterization of those operators.

§2 is concerned with the definition and algebraic properties of the index of semi-Fredholm operators. The main result of this paragraph was first established by Atkinson [1] for the special case of bounded operators, and then extended to unbounded operators (with finite indices) by Gokberg and Krein [8].

In §3, using the notion of the graph of a linear operator we introduce a metric d on the set of all closed operators on a given Hilbert space $\mathfrak{H}$, as has been previously done by Newburgh [13]. This is achieved by defining first the distance between two closed subspaces of a Hilbert space and then taking the distance between two closed operators to be the distance between their graphs. An equivalent metric is obtained by using the notion of gap between closed subspaces—introduced by Krein & Milmann [15]. Finally a third metric equivalent to the first two is defined in term of the operator norm of $R_A = (1 + A^*A)^{-1}$, R_{A^*}, AR_A.

On the subset of all bounded closed operators on $\mathfrak{H}$, there is a simple relation between the topology induced by $d(A, B)$ and that induced by the metric $s(A, B) = ||A - B||$.

In §4 & 5 an interesting relation between the distance of two semi-Fredholm operators and the distance of their null spaces is established, which together with a lemma proved by Nagy [16] leads to a relation similar to Atkinson's first stability theorem [1, 2] and its generalizations by Krein & Krasnosel'skii [7] and Nagy [9]. The set $\mathfrak{F}$ of all semi-Fredholm operators on $\mathfrak{H}$ is shown to be open in the set of all closed operators on $\mathfrak{H}$ with respect to the topology introduced in §3. Finally it is proved that there exists around each element

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A of $\mathfrak{F}$ a sphere, whose radius is explicitly given in terms of the lower bound of A on the range of A^* in which all elements have the same index as A. It follows from this that the index of the operators of $\mathfrak{F}$ is constant in each connected component of $\mathfrak{F}$.

In §6 we show that two operators in $\mathfrak{F}$ having the same index can always be connected by a continuous path in $\mathfrak{F}$. This essentially depends on the pathwise connectedness of the set of all linear operators with bounded inverse defined in the whole space and this in turn follows from the von Neumann decomposition of such operators [12]. Together with the results of §5, this shows in particular that every connected component of $\mathfrak{F}$ is pathwise connected.

It is worth remarking that in the case of a non-separable Hilbert space, the index of an operator can have any cardinality smaller or equal to that of $\mathfrak{H}$. In fact, most of the results of this paper extend to the case when both the null-space and the complement of the range of an operator are infinite dimensional, provided that their cardinality are different.

Also, while most of the results of this paper can be generalized to the case of operators on a Banach space, it is unknown to the authors whether the results of §6 are also correct in that case or not. In view of the recent developments in the domain of differential operators, it would be of great interest to know whether or not there exist other invariants under continuous deformations in the space $\mathfrak{F}$, in the case of the Banach spaces L^p, $1 < p < \infty$, or in the case of the space $C_\alpha(\Omega)$ of Hoelder continuous functions defined on a manifold Ω.

1. Semi-Fredholm operators. Let $\mathfrak{H}$ be an abstract Hilbert space of infinite dimension. In this section we want to outline some properties of a certain special class of closed linear operators of $\mathfrak{H}$ into itself, which we shall call "semi-Fredholm" operators, defined as follows.

Definition 1.1. *A linear operator A with domain $\mathfrak{D}(A)$ and range $\mathfrak{R}(A)$ is called semi-Fredholm if the following conditions are satisfied.*

(i) *A is closed*

(ii) *The range $\mathfrak{R}(A)$ of A is a closed subspace of $\mathfrak{H}$.*

(iii) *either* dim $\mathfrak{N}(A)$ *or* codim $\mathfrak{R}(A)$ *is finite, where $\mathfrak{N}(A)$ denotes the null-space of A, and where* codim $\mathfrak{R}(A)$ = dim $\mathfrak{R}(A)$. $\mathfrak{R}(A)^\perp$ *denotes the orthogonal complement of $\mathfrak{R}(A)$.*

The motivation for the term "semi-Fredholm" becomes clear after proving the following (well known)

Lemma 1.1. *A closed linear operator A is semi-Fredholm if and only if the following restricted Fredholm alternatives are satisfied.*

(ii′) *For any arbitrary $f \in \mathfrak{H}$ the equation $Au = f$, $u \in \mathfrak{D}(A)$, is solvable if and only if f is orthogonal to every solution ψ of $A^*\psi = 0$.*

(iii′) *At least one of the equations $Au = 0$, $A^*u = 0$ has only finitely many linearly independent solutions.*

Proof. We notice that $v \in \mathfrak{N}(A^*)$ if and only if $(v, Au) = (0, u) = 0$ for all $u \in \mathfrak{D}(A)$, or if and only if $v \perp \mathfrak{R}(A)$. Thus we have $\mathfrak{N}(A^*) = \mathfrak{R}(A)^{\perp}$, and accordingly $\mathfrak{N}(A^*) \oplus \mathfrak{R}(A) = \mathfrak{H}$ holds if and only if $\mathfrak{R}(A)$ is closed, which proves that (ii) and (ii′) are equivalent. Also it is now trivial that (iii) and (iii′) are equivalent, which proves the lemma.

Remark. From the above proof we record that more generally the following statements are true for all closed operators.

(1) $\mathfrak{N}(A^*) = \mathfrak{R}(A)^{\perp}$.
(2) dim $\mathfrak{N}(A^*) =$ codim $\mathfrak{R}(A)$ and dim $\mathfrak{N}(A) =$ codim $\mathfrak{R}(A^*)$.
(3) We have $\mathfrak{N}(A^*) \oplus \mathfrak{R}(A) = \mathfrak{H}$ if and only if $\mathfrak{R}(A)$ is closed.

In the following we derive a few (well known) properties for semi-Fredholm operators.

Lemma 1.2. *An operator A is semi-Fredholm if and only if the conditions* (i), (iii), *and* (ii″) *hold, where*

(ii″) $||Au|| \geqq c\ ||u||$ *for all* $u \in \mathfrak{N}(A)^{\perp} \cap \mathfrak{D}(A)$ *with a positive constant* c.

Proof. We more generally show

Lemma 1.3. *A closed operator A has closed range if and only if* (ii″) *is true.*

Proof. Let first condition (ii″) be satisfied. Then, if $f^n \to f$, $f^n \in \mathfrak{R}(A)$, let u^n be defined by $f^n = Au^n$, $u^n \in \mathfrak{D}(A) \cap \mathfrak{N}(A)$. We get $u^n - u^m \perp \mathfrak{N}(A)$, and thus

$$|| u^n - u^m || \leqq \frac{1}{c} || f^n - f^m || \to 0, \qquad n, m \to \infty .$$

Accordingly $u^n \to u$, $Au^n \to f$ which implies that $u \in \mathfrak{D}(A)$, $Au = f$, *i.e.* $f \in \mathfrak{R}(A)$, since A is closed. Thus $\mathfrak{R}(A)$ is closed. Now let $\mathfrak{R}(A)$ be closed. Then the mapping of $\mathfrak{D}(A) \cap \mathfrak{N}(A)^{\perp}$ into $\mathfrak{R}(A)$ defined by the correspondence $u \to Au$, $u \in \mathfrak{N}(A)^{\perp} \cap \mathfrak{D}(A)$, is "onto" and "one to one", for simple reasons, whereas both spaces $\mathfrak{R}(A)$ and $\mathfrak{N}(A)^{\perp}$ are closed subspaces of $\mathfrak{H}$, and thus can be interpreted as Hilbert spaces. The inverse of this mapping exists and is a well defined linear transformation of $\mathfrak{R}(A)$ into $\mathfrak{N}(A)^{\perp}$, with domain $\mathfrak{R}(A)$. Also this transformation is a closed linear operator of the Hilbert space $\mathfrak{R}(A)$ into the Hilbert space $\mathfrak{N}(A)^{\perp}$, as follows from the fact that A is closed. Thus it must be a bounded linear transformation, by a well known theorem (*c.f.* page 306 in [11]). However this boundedness clearly amounts to the condition (ii″) which completes the proof of Lemma 1.3.

Lemma 1.4. *A closed operator A is semi-Fredholm if and only if its adjoint A^* is.*

Proof. Since A is closed we get $A^{**} = A$. Hence it suffices to show that A^* is semi-Fredholm whenever A is. But A^* as adjoint is closed, i.e. satisfies (i). Also by statement 2) of our above remark it is clear that (iii) for the operator A implies (iii) for the operator A^*. Thus by Lemma 1.2. we merely need to show that (ii″) holds for A^* too. Suppose it does not then there exists $u^n \in \mathfrak{N}(A^*)^\perp \cap \mathfrak{D}(A^*)$ such that $A^*u^n \rightarrow 0$, $||u^n|| = 1$. However we must have $u^n = Av^n$, $v^n \in \mathfrak{N}(A)^\perp \cap \mathfrak{D}(A)$ due to statement 3) of the above remark. Also

$$1 = ||\, u^n \,||^2 = (v^n, A^*u^n) \leqq ||\, v^n \,||\; ||\, A^*u^n \,||, \qquad n = 1, 2, \cdots,$$

which implies that $||v^n|| \rightarrow \infty$. Then let $z^n = v^n/||v^n||$. Clearly we have

$$||\, z^n \,|| = 1, \qquad z^n \in \mathfrak{N}(A)^\perp \cap \mathfrak{D}(A), \qquad Az^n = \frac{u^n}{||\, v^n \,||} \rightarrow 0, \qquad n \rightarrow \infty.$$

Accordingly (ii″) cannot be true for A, *i.e.* A cannot be semi-Fredholm, a contradiction, and Lemma 1.4 is proved.

Definition 1.2. *Let A be semi-Fredholm, then we define the constant $c(A)$ by setting*

$$c(A) = \inf_{||u||=1,\, u \in \mathfrak{D}(A) \cap \mathfrak{N}(A)^\perp} ||\, Au \,||.$$

Lemma 1.5. *We have $c(A^*) = c(A)$ for every semi-Fredholm operator A.*

Proof. We already mentioned in the proof of Lemma 1.2. that the restriction $\hat{A}$ of A to the space $\mathfrak{R}(A^*) \cap \mathfrak{D}(A)$ can be considered as a closed linear operator of the Hilbert space $\mathfrak{R}(A^*)$ into the Hilbert space $\mathfrak{R}(A)$ having a bounded inverse $\hat{A}^{-1}$, and in particular we have $c(A) = ||\hat{A}^{-1}||^{-1}$. Clearly we also get $c(A^*) = ||\hat{A}^{*-1}||^{-1}$ using the corresponding restriction $\hat{A}^*$ of A^* to $\mathfrak{R}(A) \cap \mathfrak{D}(A^*)$ which is, a linear closed operator from $\mathfrak{R}(A)$ onto $\mathfrak{R}(A^*)$. But we now notice that $\hat{A}^* = \hat{A}^*$. Accordingly the statement of the lemma follows from the well known formula $||B|| = ||B^*||$.

2. Product of semi-Fredholm operators. As a preparation let us prove the following lemmata.

Lemma 2.1. *Let $\mathfrak{D}$ be a dense subspace of $\mathfrak{H}$, and let $\mathfrak{M}_n$ be a closed subspace of $\mathfrak{H}$ with the finite co-dimension n. Then there exists a bounded projection operator P, not necessarily orthogonal, such that*

$$\mathfrak{R}(P) = \mathfrak{M}_n\,, \qquad \dim \mathfrak{N}(P) = n, \qquad \mathfrak{N}(P) \subset \mathfrak{D}.$$

Proof. Chose a basis $\varphi^1, \cdots, \varphi^n$ of $\mathfrak{N}_n = \mathfrak{M}_n^\perp$, and then select n elements $\psi^1, \cdots, \psi^n$ of $\mathfrak{D}$ such that $||\varphi^i - \psi^i|| < \delta$, where δ is chosen small enough to ensure that det $(((\psi^i, \varphi^l))) \neq 0$. This in particular implies linear independence of $\psi^1, \cdots, \psi^n$. Let $\mathfrak{Q}$ denote the space spanned by $\psi^1, \cdots, \psi^n$. Clearly $\mathfrak{Q} \subset \mathfrak{D}$. For every $u \in \mathfrak{H}$ there exists a unique decomposition $u = v + w$, $v \in \mathfrak{M}_n$, $w \in \mathfrak{Q}$. Indeed we simply set $w = \sum_{i=1}^n \alpha_i \psi^i$ where the α_i are uniquely determined by the system of linear equations

$$\left(u - \sum_{i=1}^n \alpha_i \psi^i, \varphi^l\right) = 0, \qquad l = 1, \cdots, n.$$

which has its determinant different from zero, by our assumption. Then define $Pu = v$, $u \varepsilon \mathfrak{H}$. This defines a projection operator satisfying all assumptions stated above. In particular P is bounded, since

$$\| Pu \| \leqq \| u \| + \| w \| = \| u \| + c \sum | \alpha_i | \leqq (1 + c') \| u \|.$$

This proves our lemma.

Lemma 2.2. *Under the assumptions of Lemma 2.1 the intersection $\mathfrak{M}_n \cap \mathfrak{D}$ is dense in $\mathfrak{D}$.*

Proof. If $u \varepsilon \mathfrak{M}_n$ and $\delta > 0$ are given, select a $v \varepsilon \mathfrak{D}$ such that

$$\|u - v\| \leqq \delta' = \delta(1 + \|1 - P\|)^{-1}.$$

Clearly then $w = Pv \varepsilon \mathfrak{D}$, and

$$\| u - w \| \leqq \| u - v \| + \| (1 - P)v \| = \| u - v \| + \| (1 - P)(v - u) \|$$
$$\leqq (1 + \| 1 - P \|)\delta' = \delta.$$

This proves the lemma.

We now recall the following

Definition 2.1. *If A, B are two linear operators of $\mathfrak{H}$ into itself, then we define the product AB with domain $\mathfrak{D}(AB)$ by setting*

$$\mathfrak{D}(AB) = \{u \mid u \varepsilon \mathfrak{D}(B), Bu \varepsilon \mathfrak{D}(A)\}, \qquad ABu = A(Bu).$$

Lemma 2.3. *If A and B are semi-Fredholm, if in addition*

$$\dim \mathfrak{N}(B^*) < \infty, \qquad \dim \mathfrak{N}(A) < \infty,$$

and if at least one of the dimensions $\dim \mathfrak{N}(B)$, $\dim \mathfrak{N}(A^*)$ *is also finite then AB is semi-Fredholm. Also B^*A^* is semi-Fredholm and $(AB)^* = B^*A^*$.*

Proof. We first show that $\mathfrak{D}(AB)$ is dense. Since by assumption $\dim \mathfrak{N}(B^*) = \dim \mathfrak{N}(B)^\perp < \infty$, the space $\mathfrak{R}(B) \cap \mathfrak{D}(A)$ is dense in $\mathfrak{R}(B)$ by the preceeding lemma. Let now $u \varepsilon \mathfrak{D}(B)$, then there exists $\dot{u} \varepsilon \mathfrak{N}(B)^\perp \cap \mathfrak{D}(B)$ with $B\dot{u} = Bu$, and we have $u - \dot{u} \varepsilon \mathfrak{N}(B) \subset \mathfrak{D}(AB)$. Since $\mathfrak{R}(B) \cap \mathfrak{D}(A)$ is dense in $\mathfrak{R}(B)$ for every $\delta' > 0$ there exists a $v \varepsilon \mathfrak{R}(B) \cap \mathfrak{D}(A)$ such that $\|v - Bu\| < \delta'$. Let $v = Bw$, $w \varepsilon \mathfrak{N}(B)^\perp$, then $w \varepsilon \mathfrak{D}(AB)$. We get $\dot{u} - w \varepsilon \mathfrak{N}(B)^\perp$, and thus $\|\dot{u} - w\| \leqq c \|Bu - v\| \leqq c\, \delta'$. Set $u' = w + u - \dot{u}$, and chose δ' such that $c\, \delta' = \delta$, then we have $\|u - u'\| < \delta$ and $u' \varepsilon \mathfrak{D}(AB)$, which means that $\mathfrak{D}(AB)$ is dense in $\mathfrak{D}(B)$. Since $\mathfrak{D}(B)$ is dense in $\mathfrak{H}$, we see that $\mathfrak{D}(AB)$ is dense in $\mathfrak{H}$.

Next we show that AB is closed. Suppose $u^n \to u$, $ABu^n \to v$. Let $Bu^n = w_1^n + w_2^n$, $w_1^n \perp \mathfrak{N}(A)$, $w_2^n \varepsilon \mathfrak{N}(A)$. We have $w_1^n \to w_1 \perp \mathfrak{N}(A)$ due to (ii'') of §1 for A. Also either $w_2^{n_k} \to w_2 \varepsilon \mathfrak{N}(A)$ for a suitable subsequence $w_2^{n_k}$ or $\|w_2^n\| \to \infty$ since $\dim \mathfrak{N}(A) < \infty$. In the first case we get $u^{n_k} \to u$, $Bu^{n_k} \to w_1 + w_2$, $ABu^{n_k} \to v$, *i.e.* $u \in \mathfrak{D}(AB)$, $ABu = v$, since A and B are closed. In the second case let $z^n = u^n/\|w_2^n\|$, then

$$z^n \to 0, \qquad Bz^n = \frac{w_1^n}{\| w_2^n \|} + \frac{w_2^n}{\| w_2^n \|}, \qquad ABz^n \to 0.$$

But $w_2^n/\|w_2^n\|$ must have a convergent subsequence, and we must get $z^{n_k} \to 0$, $Bz^{n_k} \to w$, $\|w\| = 1$, which is a contradiction, since B is closed. This proves that AB is closed. Now we show that AB satisfies condition (ii′′) of §1. Indeed let $u^n \in \mathfrak{D}(AB) \cap \mathfrak{N}(AB)^\perp$, $\|u^n\| = 1$, $ABu^n \to 0$ then we again write $Bu^n = w_1^n + w_2^n$, $w_1^n \perp \mathfrak{N}(A)$, $w_2^n \in \mathfrak{N}(A)$. We get $w_1^n \to 0$ by (ii′′) for A and again either $w_2^{n_k} \to w$ or $\|w_2^n\| \to \infty$. In the first case we get $ABu^{n_k} \to 0$, $Bu^{n_k} \to w$, $u^{n_k} \perp \mathfrak{N}(B)$, i.e. $u^{n_k} \to u$, $\|u\| = 1$, $u \in \mathfrak{D}(AB) \cap \mathfrak{N}(AB)^\perp$, $ABu = 0$, a contradiction. In the second case set again $z^n = u^n/\|w_2^n\|$. Bz^n must have a convergent subsequence, and thus we get

$$z^{n_k} \to 0, \qquad Bz^{n_k} \to w, \qquad \| w \| = 1$$

a contradiction, because B is closed. This proves (ii′′) for AB.

It is evident that B^*, A^* satisfy the same assumptions as A, B due to Lemma 1.4. Accordingly it is clear that B^*A^* too is closed, and has closed range. Let us show that $(AB)^* = B^*A^*$. Clearly we have $B^*A^* \subset (AB)^*$. To show that also $(AB)^* \subset B^*A^*$, let $f \in \mathfrak{D}((AB)^*)$, and $(AB)^*f = g$; then $(f, ABu) = (g, u)$ for all $u \in \mathfrak{D}(AB)$. We have $\mathfrak{N}(B) \subset \mathfrak{D}(AB)$, and thus $(g, u) = (f, ABu) = 0$ for all $u \in \mathfrak{N}(B)$. Thus $g \in \mathfrak{N}(B)^\perp = \mathfrak{R}(B^*)$, $g = B^*h$, $h \in \mathfrak{D}(B^*) \cap \mathfrak{N}(B^*)^\perp$. Accordingly we get

$$(f, ABu) = (B^*h, u) = (h, Bu), \qquad u \in \mathfrak{D}(AB)$$

or

$$(f, Av) = (h, v) \quad \text{for all} \quad v \in \mathfrak{R}(B) \cap \mathfrak{D}(A).$$

Now we apply Lemma 2.1. for $\mathfrak{D} = \mathfrak{D}(A)$, and $\mathfrak{M}_n = \mathfrak{R}(B)$. It follows that $(f, APu) = (h, Pu)$ for all $u \in \mathfrak{D}(A)$ which implies that

$$(f, Au) = (P^*h, u) + (f, A(1 - P)u) = (P^*h + (A(1 - P))^*f, u)$$

$$\text{for all} \quad u \in \mathfrak{D}(A),$$

where the adjoint $(A(1 - P))^*$ exists, because evidently $A(1 - P)$ is bounded, since $\dim \mathfrak{R}(1 - P) = \dim \mathfrak{N}(P) < \infty$. Now we see that this implies $f \in \mathfrak{D}(A^*)$. Accordingly $(A^*f, Bu) = (g, u)$, for all $u \in \mathfrak{D}(B)$, or

$$A^*f \in \mathfrak{D}(B^*), \qquad B^*A^*f = g = (AB)^*f, \qquad \text{q.e.d.}$$

Hence we also have shown that $(AB)^* = B^*A^*$. Finally it is clear that either

$$\dim \mathfrak{N}(AB) \leqq \dim \mathfrak{N}(A) + \dim \mathfrak{N}(B) < \infty$$

or

$$\begin{aligned}\operatorname{codim} \mathfrak{R}(AB) &= \dim \mathfrak{N}((AB)^*) = \dim \mathfrak{N}(B^*A^*) \\ &\leqq \dim \mathfrak{N}(B^*) + \dim \mathfrak{N}(A^*) < \infty,\end{aligned}$$

which establishes condition (iii) for the product operator AB, and therefore completes the proof of Lemma 2.3.

Definition 2.2. *If A is semi-Fredholm, and $r(A) = \dim \mathfrak{N}(A)$, $r^*(A) = \dim \mathfrak{N}(A^*)$ then we define $\rho(A) = r(A) - r^*(A)$ when both $r(A)$ and $r^*(A)$ are finite, and $\rho(A) = +\omega$ or $-\omega$ when $r(A)$ or $r^*(A)$ is infinite of power ω, respectively. $\rho(A)$ is called the index of A.*

We have the following product formula.

Theorem 2.1 (*cf*. Gokberg and Krein [8]). *If A, B satisfy the assumptions of Lemma* 2.3. *then*

$$\rho(AB) = \rho(A) + \rho(B).$$

Here we employ the rule that $k \pm \omega = \omega$ for all integers k and every infinite ω.

Proof. Let the two spaces $\mathfrak{M}$, $\mathfrak{M}^*$ be defined by $\mathfrak{M} = \mathfrak{N}(A) \cap \mathfrak{R}(B)$, $\mathfrak{M}^* = \mathfrak{N}(B^*) \cap \mathfrak{R}(A^*)$, and let $m = \dim \mathfrak{M} < \infty$, $m^* = \dim \mathfrak{M}^* < \infty$. Then it is clear that

$$r(AB) = m + \dim \mathfrak{N}(B) = m + r(B);$$

$$r^*(AB) = m^* + \dim \mathfrak{N}(A^*) = m^* + r^*(A).$$

Accordingly we get $\rho(AB) = m - m^* + r(B) - r^*(A)$. (At most one of the two numbers $r(B)$, $r^*(A)$ can be infinite). The statement of our theorem follows immediately if we can prove that $m - m^* = r(A) - r^*(B)$. Now let $a^1, \cdots a^{r(A)}$ and $b^1, \cdots b^{r^*(B)}$ be orthonormal bases of $\mathfrak{N}(A)$ and $\mathfrak{N}(B^*)$ respectively, and let σ denote the rank of the $r(A) \times r^*(B)$-matrix $(((a^k, b^i)))$, then we state that

$$m = r(A) - \sigma, \qquad m^* = r^*(B) - \sigma$$

This becomes evident if we observe that m and m^* are the numbers of linearly independent solutions of the two systems

$$\sum_{k=1}^{r(A)} \alpha_k(b^j, a^k) = 0 \qquad j = 1, \cdots r^*(B)$$

and

$$\sum_{k=1}^{r^*(B)} \beta_k(a^j, b^k) = 0 \qquad j = 1, \cdots r(A)$$

respectively. (Both $r(A)$ and $r^*(B)$ are finite by assumption). Therefore the theorem is proven.

3. A metric for closed operators. If x is an element and $\mathfrak{S}$ a closed linear sub-space of a Hilbert space $\mathfrak{H}$, then the distance $d(x, \mathfrak{S})$ between x and $\mathfrak{S}$ is well defined, we have $d(x, \mathfrak{S}) = \inf_{y \in \mathfrak{S}} ||x - y||$. If $P_{\mathfrak{S}}$ denotes the orthogonal projection onto $\mathfrak{S}$ then it is clear that

$$(1) \qquad d(x, \mathfrak{S}) = ||(1 - P_{\mathfrak{S}})x|| = ||x - P_{\mathfrak{S}}x||,$$

i.e. the above infimum is taken on at $y = P_{\mathfrak{S}}x$. Now if $\mathfrak{T}$ is another closed linear sub-space of $\mathfrak{H}$, then we define $d(\mathfrak{T}, \mathfrak{S})$ by setting

$$(2) \qquad d(\mathfrak{T}, \mathfrak{S}) = \sup_{x \in \mathfrak{T}, ||x||=1} d(x, \mathfrak{S}) + \sup_{x \in \mathfrak{S}, ||x||=1} d(x, \mathfrak{T})$$

(*c.f.* Gokberg and Krein [8] for a similar definition; also Newburgh [13]). This definition certainly gives a symmetric distance function $d(\mathfrak{T}, \mathfrak{S})$, *i.e.* we have $d(\mathfrak{T}, \mathfrak{S}) = d(\mathfrak{S}, \mathfrak{T})$, and we get $d(\mathfrak{T}, \mathfrak{S}) = 0$ if and only if $d(x, \mathfrak{S}) = 0$ for all $x \in \mathfrak{T}$, and $d(x, \mathfrak{T}) = 0$ for all $x \in \mathfrak{S}$, or if $\mathfrak{T} \subset \mathfrak{S}$ and $\mathfrak{S} \subset \mathfrak{T}$, *i.e.* $\mathfrak{T} = \mathfrak{S}$. If $P_{\mathfrak{S}}$, and $P_{\mathfrak{T}}$ are the orthogonal projections onto $\mathfrak{S}$ and $\mathfrak{T}$ respectively, then we notice that

$$(3) \qquad d(\mathfrak{T}, \mathfrak{S}) = ||(1 - P_{\mathfrak{S}})P_{\mathfrak{T}}|| + ||(1 - P_{\mathfrak{T}})P_{\mathfrak{S}}||.$$

Indeed from formulas (1) and (2) above it is clear that

$$\sup_{x \in \mathfrak{T}, ||x||=1} d(x, \mathfrak{S}) = \sup_{x \in \mathfrak{T}, ||x||=1} ||(1 - P_{\mathfrak{S}})x|| = \sup_{x \in \mathfrak{H}, ||x||=1} ||(1 - P_{\mathfrak{S}})P_{\mathfrak{T}}x||,$$

and analogously for the other supremum, which establishes (3). From (3) we obtain the triangle inequality:

$$\begin{aligned} d(\mathfrak{R}, \mathfrak{T}) &= ||(1 - P_{\mathfrak{R}})P_{\mathfrak{T}}|| + ||(1 - P_{\mathfrak{T}})P_{\mathfrak{R}}|| \\ &= ||(1 - P_{\mathfrak{R}})[(1 - P_{\mathfrak{S}}) + P_{\mathfrak{S}}]P_{\mathfrak{T}}|| + ||(1 - P_{\mathfrak{T}})[(1 - P_{\mathfrak{S}}) + P_{\mathfrak{S}}]P_{\mathfrak{R}}|| \\ &\leqq ||(1 - P_{\mathfrak{R}})P_{\mathfrak{S}}|| + ||(1 - P_{\mathfrak{S}})P_{\mathfrak{T}}|| + ||(1 - P_{\mathfrak{S}})P_{\mathfrak{R}}|| \\ &\quad + ||(1 - P_{\mathfrak{T}})P_{\mathfrak{S}}|| = d(\mathfrak{R}, \mathfrak{S}) + d(\mathfrak{S}, \mathfrak{T}). \end{aligned}$$

Accordingly we have the following result.

Lemma 3.1. *$d(\mathfrak{T}, \mathfrak{S})$ defines a metric for the totality of all closed linear sub-spaces of $\mathfrak{H}$.*

It turns out that the metric $d(\mathfrak{T}, \mathfrak{S})$ is closely connected to the so-called gap between two closed linear sub-spaces (*c.f.* Krein and Krasnesel'skii [6]) $\mathfrak{S}$, $\mathfrak{T}$ of $\mathfrak{H}$, defined by the formula

$$(4) \qquad g(\mathfrak{T}, \mathfrak{S}) = ||P_{\mathfrak{T}} - P_{\mathfrak{S}}||.$$

$g(\mathfrak{T}, \mathfrak{S})$ clearly is symmetric and satisfies the triangle inequality. Also we notice the inequality

$$(5) \qquad g(\mathfrak{T}, \mathfrak{S}) \leqq d(\mathfrak{T}, \mathfrak{S}) \leqq 2g(\mathfrak{T}, \mathfrak{S}).$$

Indeed we first get

$$\begin{aligned} g(\mathfrak{T}, \mathfrak{S}) &= \|P_{\mathfrak{T}} - P_{\mathfrak{S}}\| \\ &\leqq \|P_{\mathfrak{T}} - P_{\mathfrak{S}}P_{\mathfrak{T}}\| + \|P_{\mathfrak{S}} - P_{\mathfrak{S}}P_{\mathfrak{T}}\| \\ &= \|P_{\mathfrak{T}}(1 - P_{\mathfrak{S}})\| + \|P_{\mathfrak{S}}(1 - P_{\mathfrak{T}})\| = d(\mathfrak{T}, \mathfrak{S}). \end{aligned}$$

On the other hand

$$\begin{aligned} d(\mathfrak{T}, \mathfrak{S}) &= \sup_{x \varepsilon \mathfrak{T}, \|x\|=1} \|(P_{\mathfrak{T}} - P_{\mathfrak{S}})x\| + \sup_{x \varepsilon \mathfrak{S}, \|x\|=1} \|(P_{\mathfrak{S}} - P_{\mathfrak{T}})x\| \\ &\leqq 2 \sup_{x \varepsilon \mathfrak{H}, \|x\|=1} \|(P_{\mathfrak{T}} - P_{\mathfrak{S}})x\| = 2\|P_{\mathfrak{T}} - P_{\mathfrak{S}}\| = 2g(\mathfrak{T}, \mathfrak{S}). \end{aligned}$$

Here we were using the fact that $P_{\mathfrak{T}}x = x$ if $x \varepsilon \mathfrak{T}$ and $P_{\mathfrak{S}}x = x$ if $x \varepsilon \mathfrak{S}$. Summarizing we have the following.

Lemma 3.2. *$g(\mathfrak{T}, \mathfrak{S})$ is also a metric on the totality of all closed linear subspaces of $\mathfrak{H}$, and $d(\mathfrak{T}, \mathfrak{S})$ and $g(\mathfrak{T}, \mathfrak{S})$ are equivalent.*

Lemma 3.3. *We have*

$$d(\mathfrak{S}, \mathfrak{T}) = d(\mathfrak{S}^{\perp}, \mathfrak{T}^{\perp}); \qquad g(\mathfrak{S}, \mathfrak{T}) = g(\mathfrak{S}^{\perp}, \mathfrak{T}^{\perp}). \tag{6}$$

Proof. This follows immediately from the fact that $1 - P_{\mathfrak{S}}$, $1 - P_{\mathfrak{T}}$ are the orthogonal projections onto $\mathfrak{S}^{\perp}$, $\mathfrak{T}^{\perp}$ respectively, also we have to use that $\|(1 - P_{\mathfrak{S}})P_{\mathfrak{T}}\| = \|P_{\mathfrak{T}}(1 - P_{\mathfrak{S}})\|$, because $P_{\mathfrak{S}}$, $P_{\mathfrak{T}}$ are symmetric.

Lemma 3.4. *If* U *is a unitary transformation of $\mathfrak{H}$ into itself then*

$$d(\mathrm{U}\mathfrak{S}, \mathrm{U}\mathfrak{T}) = d(\mathfrak{S}, \mathfrak{T}), \qquad g(\mathrm{U}\mathfrak{S}, \mathrm{U}\mathfrak{T}) = g(\mathfrak{S}, \mathfrak{T}). \tag{7}$$

The proof is evident, after observing that

$$P_{\mathrm{U}\mathfrak{S}} = \mathrm{U}P_{\mathfrak{S}}\mathrm{U}^*, \qquad P_{\mathrm{U}\mathfrak{T}} = \mathrm{U}P_{\mathfrak{T}}\mathrm{U}^*.$$

Our above metrics $d(\mathfrak{S}, \mathfrak{T})$ and $g(\mathfrak{S}, \mathfrak{T})$ now will be used to introduce two equivalent metrics for the totality of all closed operators transforming $\mathfrak{H}$ into itself. For this we simply introduce the graph $\mathfrak{G}(A)$ of a closed operator A. $\mathfrak{G}(A)$ is the totality of all ordered pairs $\{u, Au\}$, $u \varepsilon \mathfrak{D}(A)$, considered as a linear subspace of the direct sum $\mathfrak{h} = \mathfrak{H} \oplus \mathfrak{H}$. It is well known that an operator A with dense domain is closed if and only if $\mathfrak{G}(A)$ is a closed subspace of $\mathfrak{h}$.

Definition 3.1. *If A, B are closed operators of $\mathfrak{H}$ into itself, then we define*

$$d(A, B) = d(\mathfrak{G}(A), \mathfrak{G}(B)), \qquad g(A, B) = g(\mathfrak{G}(A), \mathfrak{G}(B)).$$

The results established above immediately imply the following

Lemma 3.5. *$d(A, B)$, $g(A, B)$ both satisfy the axioms of a metric over the space of all closed operators of $\mathfrak{H}$ into itself. Both metrics $d(A, B)$ and $g(A, B)$ are equivalent; in particular we have*

$$g(A, B) \leqq d(A, B) \leqq 2g(A, B) \tag{8}$$

Also we have

$$d(A^*, B^*) = d(A, B), \qquad g(A^*, B^*) = g(A, B) \tag{9}$$

and

$$d(A^{-1}, B^{-1}) = d(A, B), \qquad g(A^{-1}, B^{-1}) = g(A, B) \tag{10}$$

whenever both operators A^{-1}, B^{-1} exist as closed operators.

Proof. We notice that

$$\mathfrak{G}(A^*) = V\{\mathfrak{G}(A)^{\perp}\}; \qquad \mathfrak{G}(A^{-1}) = W\mathfrak{G}(A) \tag{11}$$

with the two unitary operators V, W defined by

$$V\{u^1, u^2\} = \{-u^2, u^1\}; \qquad W\{u^1, u^2\} = \{u^2, u^1\} \tag{12}$$

After this observation all the statements of the lemma are simple consequences of the results established earlier in this paragraph.

The metric $d(\mathfrak{S}, \mathfrak{T})$ as introduced in (2), or (3) appears to be the sum of two terms $||(1 - P_{\mathfrak{S}})P_{\mathfrak{T}}|| = \delta(\mathfrak{T}, \mathfrak{S})$ and $||(1 - P_{\mathfrak{T}})P_{\mathfrak{S}}|| = \delta(\mathfrak{S}, \mathfrak{T})$. It will be of interest for the following to note a few elementary properties of this pseudo-distance $\delta(\mathfrak{S}, \mathfrak{T})$. The following is immediately clear.

Lemma 3.6.

$$\delta(\mathfrak{S}, \mathfrak{T}) \leqq \delta(\hat{\mathfrak{S}}, \hat{\mathfrak{T}}) \quad \textit{whenever} \quad \hat{\mathfrak{S}} \supset \mathfrak{S}, \qquad \hat{\mathfrak{T}} \subset \mathfrak{T};$$

$$\delta(\mathfrak{S}, \mathfrak{T}) = 0 \quad \textit{if and only if} \quad \mathfrak{S} \subset \mathfrak{T};$$

$$\delta(\mathfrak{S}, \mathfrak{T}) = \delta(\mathfrak{T}^{\perp}, \mathfrak{S}^{\perp});$$

$\delta(\mathrm{U}\mathfrak{S}, \mathrm{U}\mathfrak{T}) = \alpha(\mathfrak{S}, \mathfrak{T})$ *for every unitary transformation* U *of* $\mathfrak{H}$ *onto itself.*

Clearly we also can write

$$\begin{aligned} d(\mathfrak{S}, \mathfrak{T}) &= \delta(\mathfrak{S}, \mathfrak{T}) + \delta(\mathfrak{T}, \mathfrak{S}) \\ &= \delta(\mathfrak{S}, \mathfrak{T}) + \delta(\mathfrak{S}^{\perp}, \mathfrak{T}^{\perp}). \end{aligned} \tag{13}$$

The function $\delta(\mathfrak{S}, \mathfrak{T})$, of course, also may be used for the introduction of a non-symmetric pseudo-metric for closed linear operators. We set

$$\delta(A, B) = \delta(\mathfrak{G}(A), \mathfrak{G}(B)), \tag{14}$$

and then get

$$d(A, B) = \delta(A, B) + \delta(A^*, B^*) = \delta(A, B) + \delta(B, A), \tag{15}$$

and

$$\delta(A, B) = 0 \quad \text{if and only if} \quad A \subset B. \tag{16}$$

Now we prove the following result.

Lemma 3.7. *We have $\delta(\mathfrak{S}, \mathfrak{T}) = 1$ if the space $\mathfrak{S} \cap \mathfrak{T}^{\perp}$ contains elements different from zero.*

Proof. Clearly the existence of an element v different from zero in the space $\mathfrak{S} \cap \mathfrak{T}^{\perp}$ implies that $P_{\mathfrak{S}}v = v$, $(1 - P_{\mathfrak{T}})v = v$, and thus we have

$$||(1 - P_{\mathfrak{T}})P_{\mathfrak{S}}v|| = ||v||,$$

i.e. $\delta(\mathfrak{S}, \mathfrak{T}) = 1$, q.e.d.

The above simple lemma has the following interesting consequences.

Lemma 3.8. *If $\delta(\mathfrak{S}, \mathfrak{T}) < 1$, then* $\dim \mathfrak{S} \leqq \dim \mathfrak{T}$. *If also $g(\mathfrak{S}, \mathfrak{T}) < 1$, then* $\dim \mathfrak{S} = \dim \mathfrak{T}$.

The proof of Lemma 3.8. follows immediately from the formula

$$g(\mathfrak{S}, \mathfrak{T}) \geqq \text{Max}\,\{\delta(\mathfrak{S}, \mathfrak{T}),\ \delta(\mathfrak{T}, \mathfrak{S})\}, \tag{18}$$

from Lemma 3.7. and the following simple lemma.

Lemma 3.9. *If $\mathfrak{S}$ and $\mathfrak{T}$ are two closed subspaces of $\mathfrak{H}$ such that*

$$\dim \mathfrak{S} \geqq \dim \mathfrak{T} + k,\ \dim \mathfrak{T} < \infty$$

for a positive integer k, then the space $\mathfrak{S} \cap \mathfrak{T}^{\perp}$ is at least k-dimensional.

Proof of Lemma 3.9. We simply pick an orthonormal basis t^i, $j = 1, \cdots \dim \mathfrak{T} = r$, of $\mathfrak{T}$, and an orthonormal system of $r + k$ elements, s^i, $j = 1, \cdots, r + k$, in $\mathfrak{S}$. Then clearly every u of the form $u = \sum_{j=1}^{n+k} \alpha_j s^j$ with

$$\sum_{j=1}^{r+k} \alpha_j (t^l, s^j) = 0 \tag{19}$$

is in $\mathfrak{S} \cap \mathfrak{T}^{\perp}$. But this equation certainly has at least k linearly independent solutions, because the rank of its matrix is at most r. This proves the lemma.

For some applications it proves to be useful to have a third metric available also equivalent to the above $d(A, B)$ and $g(A, B)$. This third metric is denoted by $p(A, B)$, and defined by

$$p(A, B) = [||R_A - R_B||^2 + ||R_{A^*} - R_{B^*}||^2 + 2\,||AR_A - BR_B||^2]^{\frac{1}{2}} \tag{20}$$

with R_A defined by $R_A = (1 + A^*A)^{-1}$. We intend to prove the following result.

Lemma 3.10. *$p(A, B)$ is a metric, and we have*

$$g(A, B) \leqq \sqrt{2}\,p(A, B) \leqq 2g(A, B). \tag{21}$$

We prepare the proof of Lemma 3.10 by establishing the following preparatory lemmata.

Lemma 3.11. *If A is closed then $R_A = (1 + A^*A)^{-1}$ exists as a bounded self-adjoint operator with $\mathfrak{D}(R_A) = \mathfrak{H}$; the operator AR_A also is bounded and in particular we have $||R_A|| \leqq 1$, $||AR_A|| \leqq 1$.*

Proof. It is well known that the operator A^*A is self-adjoint for every closed linear operator A [12]. Also A^*A clearly is positive. Accordingly $(A^*A+1)^{-1}=R_A$ exists as a bounded operator defined in all of $\mathfrak{H}$, and $||R_A|| \leqq 1$, by well known theorems. Also $R_Au \in \mathfrak{D}(A^*A) \subset \mathfrak{D}(A)$ whenever $u \in \mathfrak{H}$, and

$$||AR_Au||^2 = (R_Au, A^*AR_Au) \leqq (R_Au, (A^*A + 1)R_Au) \leqq (R_Au, u) \leqq ||u||^2,$$

q.e.d.

Lemma 3.12. *If A is closed then*

$$AR_Au = R_{A^*}Au \quad \text{for all} \quad u \in \mathfrak{D}(A).$$

Proof. Let $u \in \mathfrak{D}(A)$, then $z = R_Au \in \mathfrak{D}(A)$, and we get $Au = A(1 + A^*A)z = Az + AA^*Az = (1 + AA^*)Az = (1 + AA^*)AR_Au$, which yields the statement if we multiply by $R_{A^*} = (1 + AA^*)^{-1}$.

Lemma 3.13. *If A is closed then $(AR_A)^* = A^*R_{A^*}$.*

Proof. Clearly we get $(AR_A)^* \supset R_AA^*$, and thus $(AR_A)^*u=R_AA^*u=A^*R_{A^*}u$ for all $u \in \mathfrak{D}(A^*)$, by Lemma 3.12 above. Since AR_A and $A^*R_{A^*}$ are bounded, by Lemma 3.11, the statement of Lemma 3.13 follows immediately by taking closure since $\mathfrak{D}(A^*)$ is dense in $\mathfrak{H}$.

Lemma 3.14. *Let A be closed, then*

$$P_{\mathfrak{G}(A)} = \begin{bmatrix} R_A, & A^*R_{A^*} \\ AR_A, & 1 - R_{A^*} \end{bmatrix}, \tag{22}$$

i.e. if we write the elements $\dot{u} = \{u, v\} \in \mathfrak{h}$ as a two-component column vector, then $P_{\mathfrak{G}(A)}$ acts on $\dot{u}$ like the 2×2-matrix above.

Proof. Assume first that $v \in \mathfrak{D}(A^*)$, and let $\dot{u} = \{u, v\}$ $P_{\mathfrak{G}(A)}\dot{u} = \{x, Ax\}$, $x \in \mathfrak{D}(A)$. Since $\mathfrak{G}(A)^\perp = V\mathfrak{G}(A^*)$ with the unitary transformation V of $\mathfrak{h}$ onto itself defined by $V\{u, v\} = \{-v, u\}$, we get $P_{\mathfrak{G}(A)}\dot{u}$ from the decomposition

$$\dot{u} = \{u, v\} = \{x, Ax\} + \{-A^*y, y\} \; y \in \mathfrak{D}(A^*); \qquad x \in \mathfrak{D}(A)$$

and we have x, y defined as solutions of the system

$$\begin{aligned} u &= x - A^*y \\ v &= Ax + y. \end{aligned} \tag{23}$$

Now we can solve (23) for x. Getting $x = R_Au + R_AA^*v = R_Au + A^*R_{A^*}v$. Accordingly

$$P_{\mathfrak{G}(A)}\dot{u} = \{R_Au + A^*R_{A^*}v, AR_Au + (1 - R_{A^*})v\}. \tag{24}$$

Here we used the fact that $AA^*R_{A^*}v = (1 - R_{A^*})v$ for all $v \in \mathfrak{H}$. (24) amounts to the desired formula (22), first for $v \in \mathfrak{D}(A^*)$, then, by taking closure, also for all $u, v \in \mathfrak{H}$, q.e.d.

We now notice that the expression $p(A, B)$, as defined in (20), is well defined

for arbitrary closed operators A, B. Also we now can prove Lemma 3.10. Indeed $p(A, B)$ is obviously symmetric and satisfies the triangle inequality. If we can establish (21) it automatically follows that $p(A, B)$ is a metric.
However from Lemma 3.14 it follows that

$$[g(A,B)]^2 = ||P_{(A)} - P_{\mathfrak{G}(B)}||^2 = \left\| \begin{Bmatrix} R_A - R_B, & A^*R_{A^*} - B^*R_{B^*} \\ AR_A - BR_B, & R_{B^*} - R_{A^*} \end{Bmatrix} \right\|^2$$

$$\leqq 2[||R_A - R_B||^2 + ||R_{A^*} - R_{B^*}||^2 + ||AR_A - BR_B||^2$$

$$+ ||A^*R_{A^*} - B^*R_{B^*}||^2] \leqq 2[p(A,B)]^2,$$

since Lemma 3.13 implies that

$$||A^*R_{A^*} - B^*R_{B^*}|| = ||AR_A - BR_B||.$$

Further for $\dot{u} = \{u, 0\}$ and $\dot{v} = \{o, v\}$ we get

$$[g(A,B)]^2\, ||u||^2 \geqq ||(P_{\mathfrak{G}(A)} - P_{\mathfrak{G}(B)})\dot{u}||^2 = ||(R_A - R_B)u||^2 + ||(AR_A - BR_B)u||^2$$

$$[g(A,B)]^2\, ||v||^2 \geqq ||(P_{\mathfrak{G}(A)} - P_{\mathfrak{G}(B)})\dot{v}||^2$$

$$= ||(A^*R_{A^*} - B^*R_{B^*})v||^2 + ||(R_{A^*} - R_{B^*})v||^2$$

respectively, which implies $[p(A, B)]^2 \leqq 4[g(A, B)]^2$. Accordingly (21) is established, which completes the proof of Lemma 3.10.

Remark**: It is easily seen that the topology induced by $d(A, B)$ on the set of bounded operators, is weaker than that induced by $s(A, B) = ||A - B||$ and that both topologies are equivalent on every ball $||A|| \leqq C$, C fixed.

4. Homotopy of semi-Fredholm operators; first stability theorem. Let us define the metric space $\mathfrak{F}$ of all semi-Fredholm operators with distance $d(A, B)$, or equivalently $g(A, B)$, or $p(A, B)$

Definition 4.1. *A, B ε $\mathfrak{F}$ will be called homotopic if there exists a continuous mapping A_t, $0 \leqq t \leqq 1$ of the unit interval I: $0 \leqq t \leqq 1$ into $\mathfrak{F}$ such that $A_0 = A$, $A_1 = B$.*

We shall use the symbol $A \sim B$ to express homotopy of A and B. Clearly $A \sim B$ if and only if they are in the same path-component of $\mathfrak{F}$.

It is natural to ask for a complete system of homotopy invariants, or for a characterization of the path-components of $\mathfrak{F}$. This question is completely answered by the following theorem. The invariance of $\rho(A)$ postulated in this theorem, is clearly related to a result of Atkinson [2], [3] which has been generalized in various ways by Nagy [9], Krein and Gokberg [8], Dieudonné [4], Kato [5], and others.

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* *Added in Proof:* For a stronger statement *cf.* addendum at the end of this paper.

Theorem 4.1. *The index $\rho(A)$, as introduced in definition 2.2, constitutes a complete system of homotopy invariants. In other words, we have*

$$A \sim B \quad \text{if and only if} \quad \rho(A) = \rho(B). \tag{25}$$

For every $|\rho| \leqq \dim \mathfrak{H}$ there exists operators $A \in \mathfrak{F}$ with $\rho(A) = \rho$.

The proof of Theorem 4.1. will be approached by proving a series of lemmata.

Lemma 4.1. *Let $A, B \in \mathfrak{F}$, and let*

$$\delta(B, A) < [2 + c^{-2}(A)]^{-\frac{1}{2}}, \tag{26}$$

then we have

$$\dim \mathfrak{N}(B) \leqq \dim \mathfrak{N}(A) \tag{27}$$

Furthermore, if

$$g(A, B) < [2 + c^{-2}(A)]^{-\frac{1}{2}}, \tag{28}$$

then we have

$$\dim \mathfrak{N}(B) \leqq \dim \mathfrak{N}(A) \quad \textit{and} \quad \dim \mathfrak{N}(B^*) \leqq \dim \mathfrak{N}(A^*) \tag{29}$$

Atkinson [3] has a similar result for bounded operators, which he calls "first stability theorem".

Proof. The second part of the lemma is an immediate consequence of the first part, because we have the inequality

$$g(A, B) \geqq \operatorname{Max} [\delta(B, A), \delta(B^*, A^*)]. \tag{30}$$

To prove the first part it suffices to establish the following lemma, in view of Lemma 3.8.

Lemma 4.2. *We have*

$$\delta(\mathfrak{N}(B), \mathfrak{N}(A)) \leqq [2 + c^{-2}(A)]^{\frac{1}{2}} \delta(B, A). \tag{31}$$

Proof. We first notice that the following lemma is true.

Lemma 4.3. *If A is semi-Fredholm, then*

$$\mathfrak{G}(A) = \mathfrak{G}(\hat{A}) \oplus \mathfrak{M}(A), \tag{32}$$

where $\hat{A}$ denotes the restriction of A to $\mathfrak{D}(A) \cap \mathfrak{N}(A)^{\perp} = \mathfrak{D}(\hat{A})$, and $\mathfrak{M}(A)$ denotes the set of all $\dot{u} = \{u, 0\} \in \mathfrak{G}(A)$, i.e. of all $\dot{u} = \{u, 0\}$ with $u \in \mathfrak{N}(A)$.

Proof. If $\dot{v} = \{v, Av\} \in \mathfrak{G}(A)$, then $v = u + w$, $u \in \mathfrak{N}(A)$, $w \in \mathfrak{N}(A)^{\perp}$, and $\hat{A}w = Aw = Av$. Accordingly $\dot{v} = \{u, 0\} + \{w, \hat{A}w\}$ with $\dot{u} = \{u, 0\} \in \mathfrak{M}(A)$, $\dot{w} = \{w, \hat{A}w\} \in \mathfrak{G}(\hat{A})$. Also it is evident that $(\dot{u}, \dot{w}) = (u, w) = 0$ for all $\dot{u} \in \mathfrak{M}(A)$, $\dot{w} \in \mathfrak{G}(\hat{A})$, because $\dot{u} = \{u, 0\} \in \mathfrak{M}(A)$ if and only if $u \in \mathfrak{N}(A)$, whereas $w \perp \mathfrak{N}(A)$, q.e.d.

Proof of Lemma 4.2. We have

$$\begin{aligned}\delta(\mathfrak{N}(B), \mathfrak{N}(A)) &= \sup_{\|u\|=1} \|(1 - P_{\mathfrak{N}(A)})P_{\mathfrak{N}(B)}u\| \\ &= \sup_{\|u\|=1, u\in\mathfrak{N}(B)} \|(1 - P_{\mathfrak{N}(A)})u\| \\ &= \sup_{\|\dot{u}\|=1, \dot{u}\in\mathfrak{M}(B)} \|(1 - P_{\mathfrak{M}(A)})\dot{u}\|.\end{aligned}$$

Now for $\dot{u} \in \mathfrak{M}(B)$ we get

$$\begin{aligned}\|(1 - P_{\mathfrak{M}(A)})\dot{u}\|^2 &= \|(1 - P_{\mathfrak{S}(A)})(1 - P_{\mathfrak{M}(A)})\dot{u}\|^2 \\ &\quad + \|P_{\mathfrak{S}(A)}(1 - P_{\mathfrak{M}(A)})\dot{u}\|^2.\end{aligned}$$

The first term is easily estimated as follows.

$$\begin{aligned}\|(1 - P_{\mathfrak{S}(A)})(1 - P_{\mathfrak{M}(A)})\dot{u}\|^2 &= \|(1 - P_{\mathfrak{M}(A)})(1 - P_{\mathfrak{S}(A)})\dot{u}\|^2 \\ &\leqq \|(1 - P_{\mathfrak{S}(A)})\dot{u}\|^2 = \|(1 - P_{\mathfrak{S}(A)})P_{\mathfrak{S}(B)}\dot{u}\|^2 \\ &\leqq [\delta(B, A)]^2 \|\dot{u}\|^2 = [\delta(B, A)]^2.\end{aligned}$$

Here we used the fact that $\mathfrak{M}(B) \subset \mathfrak{S}(B)$, $\mathfrak{M}(A) \subset \mathfrak{S}(A)$, by Lemma 4.3., which implies that $P_{\mathfrak{S}(B)}\dot{u} = \dot{u}$ for all $\dot{u} \in \mathfrak{M}(B)$, and that $P_{\mathfrak{S}(A)}$ and $P_{\mathfrak{M}(A)}$ commute. Concerning the second term we first notice that

$$\|P_{\mathfrak{S}(A)}(1 - P_{\mathfrak{M}(A)})\dot{u}\|^2 = \|P_{\mathfrak{S}(\hat{A})}\dot{u}\|^2$$

by Lemma 4.3. Let

$$P_{\mathfrak{S}(\hat{A})}\dot{u} = \{v, \hat{A}v\} = \dot{v}, \qquad v \in \mathfrak{N}(A)^{\perp} \cap \mathfrak{D}(A),$$

then

$$\|P_{\mathfrak{S}(\hat{A})}\dot{u}\|^2 = \|\dot{v}\|^2 = \|v\|^2 + \|\hat{A}v\|^2 \leqq (1 + c^{-2}(A)) \|\hat{A}v\|^2.$$

On the other hand let $(1 - P_{\mathfrak{M}(A)})\dot{u} = \dot{w}$ then we have

$$\dot{w} = \{(1 - P_{\mathfrak{N}(A)})u, 0\} = \{w, 0\},$$

and thus

$$\begin{aligned}\dot{w} - \dot{v} = \{w - v, \hat{A}v\}, \quad \|\hat{A}v\|^2 &\leqq \|w - v\|^2 + \|\hat{A}v\|^2 = \|\dot{w} - \dot{v}\|^2 \\ &= \|(1 - P_{\mathfrak{M}(A)})\dot{u} - P_{\mathfrak{S}(\hat{A})}\dot{u}\|^2 \\ &= \|(1 - P_{\mathfrak{M}(A)})(1 - P_{\mathfrak{S}(A)})\dot{u}\|^2 \leqq [\delta(B, A)]^2.\end{aligned}$$

In summary we get

$$\|P_{\mathfrak{S}(A)}(1 - P_{\mathfrak{M}(A)})\dot{u}\|^2 \leqq (1 + c^{-2}(A))[\delta(B, A)]^2$$

as an estimate for the second term above, and we thus have the desired estimate (31), and the lemma is proved.

It is clear from Lemma 4.1. that we must have

$$r(A) = r(B), \qquad r(A^*) = r(B^*) \tag{33}$$

whenever

$$g(A, B) < \mathrm{Min}\,\{[2 + c^{-2}(A)]^{-\frac{1}{2}}, [2 + c^{-2}(B)]^{-\frac{1}{2}}\}. \tag{34}$$

This means that for a continuous family A_t, $0 \leqq t \leqq 1$ of semi-Fredholm operators we must have $\rho(A_t)$ constant, independent of t, whenever only $c(A_t)$, $0 \leqq t \leqq 1$, is bounded below by a positive constant. In this case we even get $r(A_t) = \text{const.}$, $r^*(A_t) = \text{const.}$, individually.

5. Invariance of ρ(A). Reduction to the case of bounded operators. Our intention in this paragraph is to prove the following theorem.

Theorem 5.1. *If $A \in \mathfrak{F}$, and B is closed, and*

$$p(A, B) < c^2(A)(1 + c^2(A))^{-1} \tag{35}$$

then we have $B \in \mathfrak{F}$ and $\rho(A) = \rho(B)$.

In other words for every $A \in \mathfrak{F}$ there exists an open neighbourhood in the space of all closed operators under the the metric $p(A, B)$ such that all operators in this neighbourhood have the same index as A. This particularly implies that $\mathfrak{F}$ is an open subset of the space $\mathfrak{K}$ of all closed operators under the metric.

We prepare the proof of theorem 5.1. by a series of lemmata. If A is closed then the self-adjoint positive definite operator $R_A = (1 + A^*A)^{-1}$ has a unique positive definite self-adjoint square root, which we denote by S_A.

Lemma 5.1. *S_A, AS_A are bounded, and $||S_A|| \leqq 1$, $||AS_A|| \leqq 1$.*

Proof. The statement for S_A is trivial. But

$$||AR_A u||^2 \leqq (R_A u, u) = ||S_A u||^2$$

for all $u \in \mathfrak{H}$. Accordingly we get $\mathfrak{R}(S_A) \subset \mathfrak{D}(AS_A)$, and $||AS_A v|| \leqq ||v||$ for all $v \in \mathfrak{R}(S_A)$. But $\mathfrak{R}(S_A)$ clearly is dense. Therefore the remaining statement follows by taking closure, and the lemma is proved.

Lemma 5.2. *If A is semi-Fredholm, then AS_A also is semi-Fredholm, and $\mathfrak{N}(A) = \mathfrak{N}(AS_A)$, $\mathfrak{N}(A^*) = \mathfrak{N}((AS_A)^*)$.*

Proof. We notice that $\mathfrak{N}(A)$ is the space of all $u \in \mathfrak{H}$ satisfying $R_A u = u$. Indeed $R_A u = u$ is equivalent to

$$u \in \mathfrak{D}(A^*A), \qquad u = u + A^*Au, \quad \text{or} \quad A^*Au = 0 \quad \text{or} \quad Au = 0.$$

Now R_A is bounded self-adjoint, and leaves its eigen-space $\mathfrak{N}(A)$ as well as the orthogonal complement $\mathfrak{N}(A)^\perp$ invariant, and so does the square root S_A, as is well known. Accordingly $\mathfrak{N}(A) = \mathfrak{N}(AS_A)$. It is also clear that $u \in \mathfrak{N}((AS_A)^*)$

if and only if $(u, AS_A v) = 0$ for all $v \in \mathfrak{H}$, *i.e.* $(u, AR_A w) = 0$ for all $w \in \mathfrak{H}$, *i.e.* if $u \in \mathfrak{N}(A^*R_{A^*}) = \mathfrak{N}(A^*)$. Thus $\mathfrak{N}((AS_A)^*) = \mathfrak{N}(A^*)$. Finally

$$(AS_A)^*AS_A u = S_A A^* A S_A u = A^*AR_A u \quad \text{for all} \quad u \in \mathfrak{R}(S_A)$$

and thus for all $u \in \mathfrak{H}$. Hence

$$\|AS_A u\|^2 = (u, A^*AR_A u) = (u, u) - (u, R_A u),$$

and we get

$$(u, (A^*A + 1)u) = (1 + c^2(A))(u, u) \quad \text{for all} \quad u \in \mathfrak{D}(A) \cap \mathfrak{N}(A)^\perp,$$

and thus

$$(u, R_A u) \leqq (1 + c^2(A))^{-1}(u, u),$$

again, because R_A and S_A leave $\mathfrak{N}(A)^\perp$ invariant. Combining the above estimates we get

$$\|AS_A u\|^2 \geqq c^2(A)(1 + c^2(A))^{-1} \|u\|^2.$$

i.e. AS_A is semi-Fredholm, q.e.d. We notice that we also proved the following

Corollary to Lemma 5.2. *We have*

$$c(AS_A) \geqq c(A)(1 + c^2(A))^{-\frac{1}{2}}. \tag{36}$$

Definition 5.2. *If A, B are closed, then we define the operator*

$$V_{A,B} = S_A S_B + (AS_A)^* B S_B .$$

Lemma 5.3. *The operator $V_{A,B}$ satisfies the following relations*

$$V_{A,A} = 1, \qquad V_{B,A} = V_{A,B}^*. \tag{37}$$

and

$$\big|\, \|V_{A,B}u\|^2 - \|u\|^2 \,\big| \leqq p(A, B)\, \|u\|^2, \qquad u \in \mathfrak{H}. \tag{38}$$

Proof. We have

$$V_{A,A}u = R_A u + S_A A^* A S_A u = S_A S_A^{-2} S_A u = u, \qquad u \in \mathfrak{R}(S_A),$$

and thus $V_{A,A}u = u$ for all $u \in \mathfrak{H}$, by taking closure. Also

$$V_{A,B}^* = S_B S_A + (BS_B)^* A S_A = V_{B,A} .$$

Furthermore let $u \in \mathfrak{H}$, $x = S_B u$, $y = BS_B u$, then

$$\begin{aligned}
&\big|\, \|V_{A,B}u\|^2 - \|u\|^2 \,\big| = \big|\, \|V_{A,B}u\|^2 - \|V_{B,B}u\|^2 \,\big| = \big|\, \|S_A x + (AS_A)^* y\|^2 \\
&\quad - \|S_B x + (BS_B)^* y\|^2 \,\big| = |(x, (R_A - R_B)x) + (y, (R_{B^*} - R_{A^*})y) \\
&\quad + 2 \operatorname{Re}\, (x, (A^*R_{A^*} - B^*R_{B^*})y)| \leqq \|x\|^2\, \|R_B - R_A\| + \|y\|^2\, \|R_{B^*} - R_{A^*}\| \\
&\quad + 2\, \|x\|\, \|y\|\, \|B^*R_{B^*} - A^*R_{A^*}\| \leqq (\|x\|^4 + \|y\|^4 + 2\, \|x\|^2\, \|y\|^2)^{\frac{1}{2}} p(A, B) \\
&= (\|x\|^2 + \|y\|^2) p(A, B).
\end{aligned}$$

Also we get $||x||^2 + ||y||^2 = (u, V_{B,B}u) = ||u||^2$, which proves (38). Therefore the lemma is proved. In particular we notice the following.

Corollary to Lemma 5.3. *If $p(A, B) < 1$, then $V_{A,B}$ possesses a bounded inverse defined in all of $\mathfrak{H}$.*

Indeed we then get

$$||V_{A,B}u||^2 \geqq (1 - p(A, B))\ ||u||^2,$$

$$||V^*_{A,B}u||^2 = ||V_{B,A}u||^2 \geqq (1 - p(A, B))\ ||u||^2,$$

i.e.

$$\mathfrak{N}(V_{A,B}) = \mathfrak{N}(V^*_{A,B}) = \mathfrak{R}(V_{A,B})^\perp = \{0\},$$

q.e.d.

Lemma 5.4. *We have*

$$||BS_B V_{B,A} - AS_A|| \leqq p(A, B). \tag{39}$$

Proof. Let $u \in \mathfrak{H}$, then

$$\begin{aligned} BS_B V_{B,A}u - AS_A u &= BS_B V_{B,A}u - AS_A V_{A,A}u \\ &= BS_B(S_B S_A + (BS_B)^*(AS_A))u - AS_A(S_A S_A + (AS_A)^*(AS_A))u \\ &= (BR_B - AR_A)S_A u + (R_{A^*} - R_{B^*})AS_A u. \end{aligned}$$

Accordingly

$$\begin{aligned} ||BS_B V_{B,A}u - AS_A u|| &\leqq ||BR_B - AR_A||\ ||S_A u|| + ||R_{A^*} - R_{B^*}||\ ||AS_A u|| \\ &\leqq p(A, B)[||S_A u||^2 + ||AS_A u||^2]^{\frac{1}{2}} = p(A, B)[(u, V_{A,A}u)]^{\frac{1}{2}} = p(A, B)\ ||u||, \end{aligned}$$

which proves the lemma. (It suffices to use

$$\pi(A, B) = [||AR_A - BR_B||^2 + ||R_{A^*} - R_{B^*}||^2]^{\frac{1}{2}}$$

instead of $p(A, B)$. We then have

$$p^2(A, B) = \pi^2(A, B) + \pi^2(A^*, B^*).)$$

Lemma 5.5. *If A, B are closed then*

$$||V_{A,B}(BS_B)^*(AS_A) - (AS_A)^*(AS_A)|| \leqq p(A, B). \tag{40}$$

Proof evident.

Lemma 5.6. *If A is semi-Fredholm and B is closed, and if*

$$p(A, B) < c^2(A)(1 + c^2(A))^{-1}, \tag{41}$$

then

$$K = V_{A,B}(BS_B)^*(AS_A)$$

has closed range and $\mathfrak{N}(K) = \mathfrak{N}(A)$. *Also*

(42) $$\dim \mathfrak{N}(K) = \dim \mathfrak{N}(K^*).$$

Proof. Define the operator $H = (AS_A)^*(AS_A)$. Clearly H, K both are bounded. Moreover H is self-adjoint, and we have $\mathfrak{N}(K) \supset \mathfrak{N}(H) = \mathfrak{N}(A)$. Also it is clear that

(43) $$c(H) = c^2(A)(1 + c^2(A))^{-1},$$

as can be derived easily. Also we get

(44) $$||H - K|| \leqq p(A, B),$$

by Lemma 5.5.

Let us write $K = H - Z$ with $||Z|| \leqq p(A, B)$. Now

(45) $$||Ku|| = ||(H - Z)u|| \geqq (c(H) - p(A, B))\ ||u|| = c_0\ ||u||, \qquad u \in \mathfrak{N}(H)^\perp, \quad c_0 > 0.$$

This shows that we must have $\mathfrak{N}(K) = \mathfrak{N}(H)$, and that the range of K must be closed.

But $K^* = H - Z^*$ also allows the estimate

(46) $$||K^*u|| \geqq c_0\ ||u||, \qquad u \in \mathfrak{N}(H)^\perp,$$

for analogous reasons. We notice that K^* transforms $\mathfrak{N}(H)^\perp$ into itself, and moreover that the restriction L of K^* to $\mathfrak{N}(H)^\perp$ has a bounded inverse, defined in all of $\mathfrak{N}(H)^\perp$. Indeed we get $\mathfrak{R}(L) \subset \mathfrak{R}(K^*) \subset \mathfrak{N}(H)^\perp$, of course, and $Lu = 0$ if and only if $u = 0$, by (46). Also $\mathfrak{R}(L)$ must be closed, due to (46). Suppose $\varphi \in \mathfrak{N}(H)^\perp$ is orthogonal to $\mathfrak{R}(L)$. Then $(K\varphi, u) = (\varphi, Lu) = 0$ for all $u \in \mathfrak{N}(H)^\perp$, *i.e.* $K\varphi = H\varphi - Z\varphi \in \mathfrak{N}(H)$. But $H\varphi \perp \mathfrak{N}(H)$, thus $H\varphi = w$ where w is the component of $Z\varphi$ in $\mathfrak{N}(H)^\perp$. Now

$$c(H)\ ||\varphi|| \leqq ||H\varphi|| = ||w|| \leqq ||Z\varphi|| \leqq p(A, B)\ ||\varphi||,$$

and therefore $\varphi = 0$, because of $c(H) > p(A, B)$. Thus $\mathfrak{R}(L) = \mathfrak{N}(H)^\perp$. Now $K^*z = 0$ can be written as $K^*y = Z^*x$, where x, y are the components of z in $\mathfrak{N}(H)$ and $\mathfrak{N}(H)^\perp$ respectively. Also $Z^*x = -K^*x \perp \mathfrak{N}(H)$. Accordingly we get

(47) $$K\ z = 0 \quad \text{if and only if} \quad z = x + L^{-1}Z^*x \quad \text{for an} \quad x \in \mathfrak{N}(H).$$

This establishes a one to one correspondence between the elements of $\mathfrak{N}(H)$ and of $\mathfrak{N}(K^*)$ which is linear and therefore guarantees that (42) is true, q.e.d.

Proof of Theorem 5.1. (a) Let $|\rho(A)| < \infty$, then the two operators

$$V_{A,B}(BS_B)^*(AS_A), \qquad V_{A^*,B^*}(B^*S_{B^*})^*(A^*S_{A^*})$$

both are semi-Fredholm, and have index zero, by Lemma 5.6. Thus the adjoints

$$(AS_A)^*(BS_B)V_{B,A}\ , \qquad (A^*S_{A^*})^*(B^*S_{B^*})V_{B^*,A^*}$$

must have their null-spaces finite dimensional, which implies that the null-spaces $\mathfrak{N}(BS_B)$, $\mathfrak{N}(B^*S_{B^*})$, *i.e.* the null-spaces $\mathfrak{N}(B)$, $\mathfrak{N}(B^*)$ (by Lemma 5.2.), are also finite dimensional. Accordingly we can apply Theorem 2.1 to the above operators to obtain

$$\rho(K) = \rho(V_{A,B}) + \rho((BS_B)^*) + \rho(AS_A) = \rho(A) - \rho(B) = 0$$

particularly because Lemma 5.3. and its corollary imply that $V_{A,B}$ is semi-Fredholm with index zero. Thus the theorem is established in this case.

(b) Now let $\rho(A) = -\omega$, $\omega > 0$, infinite. Then we conclude as before that $\dim \mathfrak{N}(B) < \infty$. On the other hand the operator $(A^*S_{A^*})^*B^*S_{B^*} = \dot{K}$ now must satisfy the relation $\dim \mathfrak{N}(\dot{K}) = \omega$. Since $\dot{K} = AS_AB^*S_{B^*}$ and because $\dim \mathfrak{N}(AS_A) = \dim \mathfrak{N}(A) < \infty$, it follows that

$$\dim \mathfrak{N}(B^*S_{B^*}) = \dim \mathfrak{N}(B^*) = \omega,$$

i.e. that $\rho(B) = -\omega$, q.e.d.

6. Completeness of the index $\varrho(A)$. In this section we shall show that any two operators A, $B \in \mathfrak{F}$ having the same index ρ always can be connected by a continuous curve in $\mathfrak{F}$, *i.e.* that the open subsets of $\mathfrak{F}$ formed by all operators with a given index ρ are pathwise connected.

It suffices to accomplish this for the case $\rho \leqq 0$, because if $\rho > 0$ then $\rho(A^*) = -\rho(A) < 0$, and any connection between A^*, B^* will also define a connection between A and B, if we take the family of adjoint operators.

If $\rho(A) \leqq 0$, then $\dim \mathfrak{N}(A) < \infty$ and $\dim \mathfrak{N}(A) \leqq \dim \mathfrak{N}(A^*)$. Let φ^1, $\varphi^2, \cdots, \varphi^r$ be an orthonormal basis of $\mathfrak{N}(A)$ and let $\psi^1, \psi^2, \cdots, \psi^r$ be any orthonormal system of r elements in $\mathfrak{N}(A^*)$. Then we define

$$A_tu = Au + \sum_{k=1}^{r} t\psi^k(\varphi^k, u), \qquad u \in \mathfrak{D}(A), \qquad 0 \leqq t \leqq 1. \tag{48}$$

Lemma 6.1. *A_t is continuous in $0 \leqq t \leqq 1$, and $A_t \in \mathfrak{F}$, $0 \leqq t \leqq 1$, $A_0 = A$.*

Proof. Let $u \in \mathfrak{D}(A)$, $u = v + w$, $v \in \mathfrak{R}(A^*)$, $w \in \mathfrak{N}(A)$, then

$$||A_tu||^2 = ||Au||^2 + t^2 \sum_{k=1}^{r} |(\varphi^k, u)|^2 = ||Av||^2 + t^2 \sum_{k=1}^{r} |(\varphi^k, w)|^2$$

$$\geqq c^2(A)\, ||v||^2 + t^2\, ||w||^2 \geqq \text{Min}\, \{c^2(A), t^2\}\, ||u||^2.$$

This means that A_t satisfies condition (ii″) of section 1. We have $\mathfrak{N}(A_t) = \{0\}$, $t > 0$, and thus A_t will be semi-Fredholm, if it is closed. Suppose

$$u^n \in \mathfrak{D}(A_t) = \mathfrak{D}(A), \qquad u^n \to u, \qquad v^n = A_tu^n \to v$$

for any fixed $t > 0$. Decompose

$$u^n = x^n + y^n; \qquad u = x + y; \qquad x, x^n \in \mathfrak{N}(A); \qquad y, y^n \perp \mathfrak{N}(A);$$

$$v^n = \omega^n + \chi^n; \qquad v = \omega + \chi; \qquad \omega^n, \omega \in \mathfrak{N}(A^*); \qquad \chi^n, \chi \perp \mathfrak{N}(A^*).$$

Then we get

$$x^n \to x, \quad y^n \to y, \quad \omega^n \to \omega, \quad \chi^n \to \chi.$$

But $v^n = A_t u^n$ yields

$$\omega^n = t \sum_{j=1}^{r} \psi^j(\varphi^j, x^n), \quad \chi^n = Ay^n.$$

By passing to the limit we then conclude that

$$y \in \mathfrak{D}(A) \cap \mathfrak{N}(A)^\perp, \quad Ay = \chi; \quad x \in \mathfrak{N}(A), \quad \omega = t \sum_{j=1}^{r} \psi^j(\varphi^j, x),$$

and thus

$$u = x + y \in \mathfrak{D}(A); \quad v = \omega + \chi = Ay + t \sum_{j=1}^{r} \psi^j(\varphi^j, x)$$

$$= Au + t \sum_{j=1}^{r} \psi^j(\varphi^j, u) = A_t u.$$

Accordingly A_t is closed and therefore in $\mathfrak{F}$. We get

$$A_t^* u = A^* u + t \sum_{j=1}^{r} \varphi^j(\psi^j, u), \quad u \in \mathfrak{D}(A);$$

$$A_t^* A_t u = A^* A u + t^2 \sum_{j=1}^{r} \varphi^j(\varphi^j, u), \quad u \in \mathfrak{D}(A^* A),$$

in particular because

$$(\psi^j, Au) = 0, \quad u \in \mathfrak{D}(A), \quad A^* \psi^j = 0, \quad j = 1, \cdots, r.$$

Also we know that $\mathfrak{N}(A^*A) = \mathfrak{N}(A)$. Accordingly $R_{A_t} u = R_A u$ if $u \in \mathfrak{N}(A)^\perp$; $R_{A_t} u = (1 + t^2)^{-1} u$, if $u \in \mathfrak{N}(A)$, and thus

$$\|R_{A_t} - R_{A_{t'}}\| \leqq |t^2 - t'^2| \, [(1 + t^2)(1 + t'^2)]^{-1}.$$

Likewise

$$A_t R_{A_t} u = A R_A u \quad \text{if} \quad u \in \mathfrak{N}(A)^\perp; = t(1 + t^2)^{-1} \quad \text{if} \quad u \in \mathfrak{N}(A),$$

which means that

$$\|A_t R_{A_t} - A_{t'} R_{A_{t'}}\| \leqq tt'[(1 + t^2)(1 + t'^2)]^{-1}.$$

Finally

$$R_{A_t^*} u = AA^* u + t^2 \sum_{j=1}^{r} \psi^j(\psi^j, u)$$

which again yields

$$\|R_{A_t^*} - R_{A_{t'}^*}\| = |t^2 - t'^2| \, [(1 + t^2)(1 + t'^2)]^{-1}.$$

This means that A_t , $0 \leqq t \leqq 1$ is continuous and therefore Lemma 6.1. is proved. The essential result of Lemma 6.1. can be expressed as follows:

Lemma 6.2. *Any operator $A \in \mathfrak{F}$ such that $\rho(A) \leqq 0$ can be connected to an operator A_1 also of index $\leqq 0$ and with the additional property that $\mathfrak{N}(A_1) = \{0\}$.*

Lemma 6.3. (*cf.* J. v. Neumann [12]). *If $A \in \mathfrak{F}$, $\mathfrak{N}(A) = \{0\}$ then $A = T_A H_A$ where H_A denotes the positive square root of A^*A, and $T_A = AH_A^{-1}$ is an isometry of $\mathfrak{H}$ onto $\mathfrak{R}(A)$.*

Proof. From the definition of H_A we get:

$$\mathfrak{D}(H_A) = \mathfrak{D}(A) \quad \text{and} \quad ||Au|| = ||H_A u|| \quad \text{for all} \quad u \in \mathfrak{D}(A).$$

It is essential in the proof of these two facts that the closure of the restriction of A to $\mathfrak{D}(A^*A)$ coincides with A. Hence it follows that $\mathfrak{N}(H_A) = \mathfrak{N}(A) = \{0\}$. Since H_A is self adjoint, H_A^{-1} exists and is a bounded operator defined everywhere in $\mathfrak{H}$. Consequently $T_A = AH_A^{-1}$ is also defined everywhere in $\mathfrak{H}$ and satisfies the equation $||T_A u|| = ||u||$ for all $u \in \mathfrak{H}$. Besides $\mathfrak{R}(T_A) = \mathfrak{R}(A)$ and the lemma is proved.

Lemma 6.4. *Let $H \in \mathfrak{F}$ be a self-adjoint, positive definite operator on $\mathfrak{H}$. Then $H \sim I$*

Proof. Without loss of generality assume that $H \geqq I$. Then let

$$H_t = \int_0^\infty \mathrm{Min}\left(\lambda, \frac{1}{t}\right) dE_\lambda, \qquad 0 \leqq t \leqq 1$$

where E_λ denotes a spectral family of H. Clearly H_t is semi-Fredholm for $0 \leqq t \leqq 1$, and

$$H_1 = \int_1^\infty dE_\lambda = I, \qquad H_0 = H$$

In order to prove the continuity of H_t let $\varphi_t(\lambda) = \mathrm{Min}\ (\lambda, 1/t)$; then:

$$R_{H_t} = R_{H_t{*}} = \int_1^\infty [1 + \varphi_t^2(\lambda)]^{-1}\, dE_\lambda = \int_1^\infty \mathrm{Max}\ (t^2(1 + t^2)^{-1}, (1 + \lambda^2)^{-1})\, dE_\lambda$$

$$= \int_1^\infty \varphi_t(\lambda)[1 + \varphi_t(\lambda)]^{-1}\, dE_\lambda = \int_1^\infty \mathrm{Max}\ (t(1 + t^2)^{-1}, \lambda(1 + \lambda^2)^{-1}\, dE_\lambda.$$

A simple computation shows that $p(H_t , H_{t'}) < \epsilon$ whenever $|t - t'| < \delta(\epsilon)$ *i.e.* H_t is continuous in t and the lemma is proved.

Lemma 6.5. *If $A \in \mathfrak{F}$, $\mathfrak{N}(A) = \{0\}$ then $A \sim T_A$.*

Proof. Let H_t connect H_A to I (*cf.* previous lemma) and let $A_t = T_A H_t$, $0 \leqq t \leqq 1$; then $A_0 = A$ and $A_1 = T_A$ and clearly $A_t \in \mathfrak{F}$ for all t in [0, 1]. It remains to show that A_t is continuous in t. But $A_t^* A_t = H_t T_A^* T_A H_t = H_t^2$ and hence $R_{A_t} = R_{H_t}$ and $A_tR_{A_t} = T_AH_tR_{H_t}$ so that the continuity of R_{A_t} and of $A_tR_{A_t}$ is a consequence of the continuity of H_t.

Finally, since $T_AT_{A^*} = I - P_{\mathfrak{N}(A^*)}$ we have

$$A_tA_t^* + 1 = T_A(H_t^2 + 1)T_A^* + P_{\mathfrak{N}(A^*)}$$

and hence $R_{A^*_t} = T_AR_{H_t}T_A^* + P_{\mathfrak{N}(A^*)}$ which yields

$$||R_{A^*_t} - R_{A^*_{t'}}|| \leqq ||R_{H_t} - R_{H_{t'}}||$$

and again the continuity of $R_{A^*_t}$ follows from that of H_t. Putting these results together, we see that the continuity of A_t follows from the continuity of H_t and the lemma is proved.

Lemma 6.6. *Every unitary transformation* U *of* $\mathfrak{H}$ *onto* $\mathfrak{H}$ *can be connected to to the identity.*

Proof. The spectral theorem for unitary transformations yields $\mathrm{U} = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda$ where E_λ is a spectral family, $E_0 = 0$ and $E_{2\pi} = 1$. Then set: $\mathrm{U}_t = \int_0^{2\pi} e^{i(1-t)\lambda}\, dE_\lambda$. Clearly $\mathrm{U}_0 = \mathrm{U}$ and U_t is unitary for every t in $[0, 1]$ which implies that U_t is semi-Fredholm in that interval. Also $\mathrm{U}_1 = \int_0^{2\pi} dE_\lambda = I$ and

$$1 + \mathrm{U}_t^*\mathrm{U}_t = 1 + \mathrm{U}_t\mathrm{U}_t^* = 2$$

and therefore $R_{\mathrm{U}_t} = R_{\mathrm{U}^*_t} = \frac{1}{2}$ and $\mathrm{U}_tR_{\mathrm{U}_t} = \frac{1}{2}\mathrm{U}_t$. Consequently

$$p(\mathrm{U}_t, \mathrm{U}_{t'}) = \tfrac{1}{2}\sqrt{2}\, ||\mathrm{U}_t - \mathrm{U}_{t'}|| \leqq \sqrt{2}\pi\, |t - t'|$$

which implies that U_t is continuous in t for t in $[0, 1]$.

Corollary to Lemma 6.6. *Two unitary operators can always be connected by a family of unitary operators, continuous with respect to the* $|| \quad ||$ *norm.*

Lemma 6.7. *If* $A \varepsilon \mathfrak{F}$ *and* U_t *is a continuous family of unitary operators for* t *in* $[0, 1]$, *then* U_0A, $\mathrm{U}_1A \varepsilon \mathfrak{F}$ *and* U_tA *is a connection between them.*

Proof. Put $A_t = \mathrm{U}_tA$; then $A_t^*A_t = A^*A$, $A_tA_t^* = \mathrm{U}_tAA^*\mathrm{U}_t^*$. Hence

$$R_{A_t} = R_A,\ A_tR_{A_t} = \mathrm{U}_tAR_A,\ R_{A^*_t} = \mathrm{U}_tR_{A^*}\mathrm{U}_t^*.$$

This clearly shows that A_t is continuous in t. The fact that $A_t \varepsilon \mathfrak{F}$ follows at once from Lemma 2.3.

Lemma 6.8. *Let* T_1 *and* T_2 *be two isometries such that* codim $\mathfrak{R}(T_1) =$ codim $\mathfrak{R}(T_2)$ *then* $T_1 \sim T_2$.

Proof. Let U be the unitary transformation defined as follows:

$$\mathrm{U}u = \begin{cases} T_1T_2^{-1}u & \text{for } u \varepsilon \mathfrak{R}(T_2), \\ Wu & \text{for } u \perp \mathfrak{R}(T_2), \end{cases}$$

where W is any unitary transformation of $\mathfrak{R}(T_2)^\perp$ onto $\mathfrak{R}(T_1)^\perp$. (The existence of W follows from codim $\mathfrak{R}(T_1) =$ codim $\mathfrak{R}(T_2)$.) Clearly U is unitary, and hence can be connected with I as was shown in Lemma 6.6. Also, by Lemma 6.7. $T_1 = UT_2 \sim T_2$ which proves the lemma.

Theorem 6.1. *If $A, B \varepsilon \mathfrak{F}$ and $\rho(A) = \rho(B)$ then $A \sim B$.*

Proof. From the previous lemmata:

$$B \sim B_1 \sim T_{B_1} \sim T_{A_1} \sim A_1 \sim A$$

where B_1 , A_1 are defined as in Lemma 6.2. and T_{B_1} , T_{A_1} as in Lemma 6.3.

Lemma 6.9. *For every ρ such that $|\rho| \leqq \omega = \dim \mathfrak{H}$ there exists an operator $A \varepsilon \mathfrak{F}$ such that $\rho(A) = \rho$.*

Proof. Without a loss of generality we can assume that $\rho \leqq 0$. Let $\{\varphi^\lambda\}_{\lambda \varepsilon \mathfrak{J}}$ be any orthogonal base of $\mathfrak{H}$ and let $\mathfrak{J}'$ be a subset of $\mathfrak{J}$ with power $\omega = -\rho$. If $\omega < \dim \mathfrak{H}$ it is clear that the complement of $\mathfrak{J}'$ in $\mathfrak{J}$ must have the same power as $\mathfrak{J}$; if $\omega = \dim \mathfrak{H}$, $\mathfrak{J}'$ can be chosen such that its complement in $\mathfrak{J}$ again has the same power as $\mathfrak{J}$. Hence, in either case, there exists a mapping α: $\mathfrak{J} \rightarrow \mathfrak{J} - \mathfrak{J}'$ which is onto and one-to-one. Then we can define an isometry of $\mathfrak{H}$ into itself, say T, by putting:

$$T\varphi^\lambda = \varphi^{\alpha(\lambda)}.$$

Clearly then, $\mathfrak{N}(T) = \{0\}$ and $\mathfrak{R}(T)^\perp$ is the space spanned by $\{\varphi^\lambda\}_{\lambda \varepsilon \mathfrak{J}'}$. Hence $\rho(T) = \rho$. q.e.d.

Let us consider briefly the possibility of a further generalization of the results obtained above.

Define $\mathfrak{F}^\infty$ as the class of closed operators A with closed range, which satisfy the following property:

Either $A \varepsilon \mathfrak{F}$ or, if $A \not\varepsilon \mathfrak{F}$ then $\dim \mathfrak{N}(A) \neq \operatorname{codim} \mathfrak{R}(A)$. When $A \varepsilon \mathfrak{F}^\infty$ and $A \not\varepsilon \mathfrak{F}$ we define its index ρ in the following manner:

$$\rho(A) = \begin{cases} \dim \mathfrak{N}(A) & \text{if } \dim \mathfrak{N}(A) > \operatorname{codim} \mathfrak{R}(A), \\ -\operatorname{codim} \mathfrak{R}(A) & \text{if } \dim \mathfrak{N}(A) < \operatorname{codim} \mathfrak{R}(A). \end{cases}$$

We state without proof the following theorem.

Theorem 6.2. *$\mathfrak{F}^\infty$ is a metric space under any of the equivalent norms d, g, p of paragraph 3; two operators contained in $\mathfrak{F}^\infty$ lie in the same component if and only if they have the same index.*

All the proofs given in this paper can easily be extended to cover this more general case.

Addendum

After submitting our paper for publication, we found that on the subset of all bounded closed operators, the topology induced by the metric $d(A, B)$ coincided with the topology induced by the metric $s(A, B) = ||A - B||$. What follows is a proof of that fact, and should be considered as the last part of §3 of our paper.

Lemma 1. *If A and B are bounded operators we have*:

$$p(A,B) \leqq 4\,||A - B||.$$

Proof. Consider the identity:

$$||(AR_A - BR_B)u||^2 + ||(R_A - R_B)u||^2$$
$$= ((AR_A - BR_B)u, (A - B)R_Bu) - (BR_Bu, (A - B)(R_A - R_B)u).$$

Then:

$$||AR_A - BR_B||^2 \leqq \{||AR_A - BR_B|| + ||R_A - R_B||\}\,||A - B||$$

and similarly:

$$||R_A - R_B||^2 \leqq \{||AR_A - BR_B|| + ||R_A - R_B||\}\,||A - B||$$
$$||R_A^* - R_B^*||^2 \leqq \{||A^*R_A^* - B^*R_B^*|| + ||R_A^* - R_B^*||\}\,||A - B||.$$

Hence

$$p^2(A,B) \leqq 2\,||A - B||\,2p(A,B)$$

and therefore

$$p(A,B) \leqq 4\,||A - B||.$$

It remains only to prove the identity. We have:

$$||(AR_A - BR_B)u||^2 + ||(R_A - R_B)u||^2 = (AR_Au, AR_Au) + (BR_Bu, BR_Bu)$$
$$- (AR_Au, BR_Bu) - BR_Bu, AR_Au) + (R_Au, R_Au) + (R_Bu, R_Bu)$$
$$- (R_Au, R_Bu) - (R_Bu, R_Au) = (u, R_Au) + (u, R_Bu)$$
$$- (AR_Au, BR_Bu) - (BR_Bu, AR_Au) - (u, R_Bu) + (A^*AR_Au, R_Bu)$$
$$- (u, R_Au) + (B^*BR_Bu, R_Au) = (AR_Au, (A - B)R_Bu)$$
$$- (BR_Bu, (A - B)R_Au) = ((AR_A - BR_B)u, (A - B)R_Bu)$$
$$- (BR_Bu, (A - B)(R_A - R_B)u)$$

which proves the identity.

Lemma 2. *If A is a bounded operator, and if B is a closed operator, such that*

$$p(A,B) < \frac{1}{1 + ||A||^2}$$

then:

(i) *B is bounded* (*in fact* $||B|| \leqq 2\,||A|| + (1 + 2\,||A||^2)^{\frac{1}{2}}$),
(ii) $||A - B|| \leqq \frac{1}{4}(2 + ||A||^2 + ||B||^2)^{\frac{1}{2}}\,p(A, B)$.

Proof. Let $u \in \mathfrak{D}(B)$; then:

$$\|Bu\| \leqq \|Au\| + \|(B - A)u\|.$$

Moreover:

$$\frac{1}{1 + \|A\|^2} \|(B - A)u\| \leq \|R_{A^*}(B - A)u\|$$
$$\leq \|R_{A^*} - R_{B^*}\| \, \|Bu\| + \|(R_{B^*}B - R_{A^*}A)u\|$$

and hence:

$$\|(B - A)u\| \leqq (1 + \|A\|^2)\{\|R_{A^*} - R_{B^*}\| \, \|Bu\| + \|AR_A - BR_B\| \, \|u\|\}$$
$$\leqq (1 + \|A\|^2)p(A, B)\{\|Bu\| + \|u\|\}.$$

Since $p(A, B) < 1/[1 + \|A\|^2]$ implies that there exists a positive ε such that $p(A, B)(1 + \|A\|^2) < 1 - \varepsilon$, it follows that

$$\|Bu\| < \|A\| \, \|u\| + (1 - \varepsilon) \, \|Bu\| + \|u\|$$

and hence that B is bounded since $\|Bu\| < (1/\varepsilon)(1 + \|A\|) \, \|u\|$. But since B is bounded, so is B^* and $\|B^* - A^*\| = \|B - A\|$. We had:

$$\|B - A\| \leqq (1 + \|A\|^2)\{\|R_{A^*} - R_{B^*}\| \, \|B\| + \|AR_A - BR_B\|\}$$

and therefore also:

$$\|B^* - A^*\| \leqq (1 + \|A\|^2)\{\|R_A - R_B\| \, \|B\| + \|A^*R_{A^*} - B^*R_{B^*}\|\}.$$

Adding both inequalities, and using Schwarz inequality:

$$2 \, \|A - B\| \leqq (1 + \|A\|^2)2^{1/2}(1 + \|B\|^2)^{1/2}p(A, B) < 2^{1/2}(1 + \|B\|^2)^{1/2}$$

i.e.

$$\|B\| < \|A\| + \frac{1}{2^{1/2}} (1 + \|B\|^2)^{1/2}$$

or

$$\|B\|^2 < \|A\|^2 + \frac{1 + \|B\|^2}{2} + 2^{1/2} \, \|A\| \cdot (1 + \|B\|^2)^{1/2}$$

or finally

$$\left(\frac{(1 + \|B\|^2)^{1/2}}{2} - \|A\|^2\right)^2 < 1 + 2 \, \|A\|^2$$

which establishes (i).

We have just seen that:

$$\|A - B\| \leqq (1 + \|A\|^2)(1 + \|B\|^2)^{1/2} \frac{1}{2^{1/2}} p(A, B).$$

By symmetry:

$$||A - B|| \leqq (1 + ||B||^2)(1 + ||A||^2)^{1/2} \frac{1}{2^{1/2}} p(A, B).$$

Adding both inequalities:

$$||A - B|| \leqq \frac{1}{2(2)^{1/2}} (1 + ||A||^2)^{1/2}(1 + ||B||^2)^{1/2} \cdot((1 + ||A||^2)^{1/2} + (1 + ||B||^2)^{1/2})p(A, B)$$

which can also be written (using two well known inequalities)

$$||A - B|| \leqq \tfrac{1}{4}(2 + ||A||^2 + ||B||^2)^{3/2} p(A, B)$$

Theorem 1. *The set of all bounded operators is open in the set of all closed operators. The topology induced by $p(A, B)$ on that subset of bounded operators is equivalent to that given by the norm $s(A, B) = ||A - B||$.*

Proof. The theorem follows at once from the two preceding lemmas.

Bibliography

[1] F. V. Atkinson, *Math. Sbornik*, **28** (70) (1951) 3–14.
[2] ——, *Acta Math. Sci. Hung.*, **3** (1952) 53–60.
[3] ——, *Acta Sci. Math.*, **XV** (1953) 38–56.
[4] J. Dieudonne, *Bull. Sci. Math.*, (2) **67** (1943) 72–84.
[5] T. Kato, *Journal d'Amal. Math.*, **6** (1958) 261–322.
[6] Krein & Krasnosel'skii, *Uspek. Math. Nauk*, **2**,**3**(19) (1947) 60–107.
[7] ——, *Mat. Sbornik*, **30**(72) (1952) 219–224.
[8] Gokberg & Krein, *Uspek. Math. Nauk*, **2**(74) (1957) 43–118.
[9] B. v. Sz. Nagy, *Acta Math. Sci. Hung.*, **3** (1952) 49–52.
[10] ——, *Acta Math. Sci. Hung.*, **3** (1952) 61–66.
[11] Riesz & Nagy, *Functional Analysis*, New York 1960.
[12] J. v. Neumann, *Ann. of Math.*, **33** (1932) 294–310.
[13] Newburgh, *Ann. of Math.*, **53** (1951) 250–255.
[14] F. Noether, *Math. Annalen*, **82** (1921) 42–63.
[15] Krein, Milmann & Krasnosel'skii, *Sb. Prat. Inst. Mat A. N. URSR* No. 11 97–112.
[16] B. v. Sz. Nagy, *Comm. Math. Helvet.*, **19** (1946–47) 347–366.

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