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Abstract Harmonic Analysis on Locally Compact Abelian Groups

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ABSTRACT HARMONIC ANALYSIS ON LOCALLY COMPACT ABELIAN GROUPS

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CONTENTS

1 INTRODUCTION

The main purpose of this paper is to prove the decomposition theorem for unitary representations of locally compact abelian groups (abbreviated LCA), but we also intend to cover—what the author thinks are—the main results of abstract Harmonic analysis. The full corpus librorum can be found at the foot of the document. We follow G. Folland's book *A course in abstract harmonic analysis* throughout this paper.

Moreover, we assume that the reader is familiar with measure and integration theory, and functional analysis, in particular we assume the knowledge of the norm, strong, weak-* topologies and the topology of compact convergence, these topics are covered in any functional analysis book.

There are some results outside the scope of this article that we will come to need in certain situations, e.g. various Riesz representation theorems, the Gelfand-Raikov theorem, the Stone-Weierstrass theorem (locally compact spaces version), and the Banach-Alaoglu theorem.

2 TOPOLOGICAL GROUPS

The setting which concerns us in this paper will be that of the topological group, it is defined as follows. Suppose G is a group and a topological space with multiplication $\cdot : G \times G \to G$ and inversion $g^{-1} : G \to G$, then G is a topological group if the inversion and multiplication operations are continuous maps. Furthermore, a locally compact group is a topological group whose topology is locally compact and we implicitly assume the Hausdorff property. We follow Folland [1, Ch. 2] in this section.

Our most powerful tool will be a function called the Haar measure, it comes in two types, the left and the right variants, each giving the direction of translation invariance. More precisely, a left (resp. right) Haar measure μ is a nonzero Radon measure on G which satisfies $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for every Borel measurable set $E \subseteq G$ and for every $x \in G$. We note that $\mu(E)$ is a left invariant Haar measure if and only if $E \mapsto \mu(E^{-1})$ is a right invariant Haar measure, the proof of this is trivial, but it shows that the theory is essentially equivalent in either case. We will be working in the intersection of these cases, when the Haar measure is both left and right translation invariant, since our topological group will always be abelian. And lastly, the Haar measure is called normalized if $\mu(G) = 1$.

There is a relationship between left and right invariant measures called the modular function $\Delta : G \to \mathbb{R} \in \mathrm{Hom}(G, \mathbb{R})$, which in some sense measures the extent to which a left (right) invariant Haar measure is right (left) invariant. When G is abelian $\Delta \equiv 1$ (in which case we also call G unimodular) since the Haar measure is both left and right invariant, thus it will not be relevant to us, but it is worth mentioning.

It is a fundamental—and incredible—fact that there exists an essentially unique Haar measure on every locally compact group G.

Theorem 2.1. *Every locally compact group G has a Haar measure.*

We refer to Folland [1, p.37] for a complete proof of this theorem, we will just give a sketch of it here.

The idea is to utilize the Riesz representation theorem, so we want to construct a positive translation invariant functional T on $C_c(G)$, this is done as follows. We pick some function $\phi \in C_c(G)$ such that it is bounded by 1, equal to 1 on a small open subset of G and is supported on an slightly larger open set U. We also pick a function $f \in C_c(G)$ that is essentially constant on the translates of U, then f can be approximated by $f \approx \sum a_i L_{x_i}\phi$, such that we get $T(f) \approx \sum a_i T(\phi)$. This approximation will become exact as we shrink the support of ϕ to a point and by an appropriate normalization we can cancel the factor $T(\phi)$, so that $T(f)$ is approximated by the sums $\sum a_i$.

Regarding uniqueness we will just prove the case for abelian groups, since this is our main focus, but note that it is true for general locally compact groups.

Theorem 2.2. *If μ and λ are two Haar measures on the locally compact abelian group G then there exists a constant $c \in \mathbb{R}$ such that $\mu = c\lambda$.*

Proof. Fix $g \in C_c(G)$ with total integral 1 under λ, and define c by

$$\int_G g(x^{-1})\, d\mu = c.$$

Then for any $f \in C_c(G)$ we have

$$\begin{aligned}\int_G f\, d\mu &= \int_G g(x)\, d\lambda(x) \int_G f(y)\, d\mu(y)\\ &= \int_G g(x)\, d\lambda(x) \int_G f(xy)\, d\mu(y)\\ &= \int_G \int_G g(x) f(xy)\, d\mu(y) d\lambda(x)\end{aligned}$$

which by the substitution $(x, y) \mapsto (uv^{-1}, v)$ and by Fubini's theorem becomes

$$\begin{aligned}&= \int_G \int_G g(uv^{-1}) f(u)\, d\mu(v) d\lambda(uv^{-1})\\ &= \int_G g(uv^{-1})\, d\mu(v) \int_G f(u)\, d\lambda(u)\\ &= \int_G g(v^{-1})\, d\mu(v) \int_G f(u)\, d\lambda(u)\\ &= c \int_G f(u)\, d\lambda(u)\end{aligned}$$

thus we're done. ∎

We shall now present a few examples of different types of locally compact groups. First we have the groups $(\mathbb{R}, +)$, $(\mathbb{R}, \times)$, the first of which has the standard Lebesgue measure dx, and the second has the average $\frac{dx}{|x|}$.

There is also the discrete abelian group $(\mathbb{Z}, +)$, along with all the finite groups, these have the counting measure or the normalized counting measure in the finite case. Notice that the finite groups are both discrete and compact, there are many more cases when we release the condition on discreteness e.g. $(\mathbb{T}, \times)$, whose Haar measure is the pullback of the Lebesgue measure dx from $(\mathbb{R}, +)$ by the function $f(x) = e^{2\pi i x}$. Since $\mathbb{T}$ is compact, the whole space has finite measure (this follows by regularity) thus we can normalize this measure as well. If one does this one gets the measure $\frac{d(x \circ f^{-1})}{2\pi}$.

If one is familiar with the theory on smooth manifolds one might realize that they are all locally compact (since they are locally homeomorphic to

$\mathbb{R}^n$), thus one might suspect that there are smooth manifolds that have a group structure with smooth group operations (in particular locally compact groups). This structure is actually called a *Lie group* and has been studied extensively. In fact $GL(n, \mathbb{R})$ is a Lie group, the Haar measure on this group is the measure $|\det T|^{-n} \prod_{i<j} d\alpha_{ij}$ on $n \times n$ real invertible matrices $T = (\alpha_{ij})$. Moreover, the matrix groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $U(n)$, $SU(n)$, $O(n)$, $SO(n)$ are all lie groups.

3 THE MEASURE ALGEBRA & THE GROUP ALGEBRA

In this section we follow Folland [1, Ch. 2]. We define the *measure algebra*, denoted by $M(G)$, as the set of all complex Radon measures on a locally compact group G. This forms an associative Banach algebra under pointwise addition and convolution of measures over $\mathbb{C}$. To define what convolution of measures means let $\mu, \nu \in M(G)$, then the following map is clearly seen to be a linear functional on $C_0(G)$

$$\phi \longmapsto \int_G \phi(xy)\, d\mu(x) d\nu(y) \tag{3.1}$$

which satisfies

$$\Big| \int_G \phi(xy)\, d\mu(x) d\nu(y) \Big| \leq \|\phi\|_\infty \|\mu\| \|\nu\|$$

where $\|\mu\|$ is the total variation of μ over G. It follows that (3.1) must be given by some measure in $M(G)$ which we denote by $\mu * \nu$ (because of the Riesz representation theorem), it is also clear that $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

The following calculation shows that convolution is associative. Let $\mu, \nu, \delta \in M(G)$

$$\begin{aligned}\int_G \phi\, d((\mu * \nu) * \delta) &= \int_G \phi(xz)\, d(\mu * \nu)(x) d\delta(z) = \int_G \phi(xyz)\, d\mu(x) d\nu(y) d\delta(z) \\ &= \int_G \phi(xy)\, d\mu(x) d(\nu * \delta)(y) = \int_G \phi\, d(\mu * (\nu * \delta)).\end{aligned}$$

As our main focus is on locally compact abelian groups we note that $M(G)$ is commutative if and only if G is abelian, one direction follows immediately from (3.1), for the other direction, let $\delta_x \in M(G)$ be the point mass measure at the point $x \in G$, then

$$\int_G \phi(uv)\, d\delta_x(u) d\delta_y(v) = \phi(xy) = \int_G \phi\, d\delta_{xy}$$

thus $\delta_x * \delta_y = \delta_y * \delta_x$ if and only if $xy = yx$, thus $M(G)$ commutative means that G is abelian.

There is a very interesting subalgebra of $M(G)$ that turns out to be a Banach *-algebra, it consists of all $f(x)\, dx$ where dx is the Haar measure on G and f is some integrable function. This subalgebra is thus naturally isomorphic to $L^1(G)$ under the map $f \to f\, dx$. The involution is defined by

$$f^*(x) = \overline{f(x^{-1})} \Delta(x^{-1})$$

(which reduces to $f^*(x) = \overline{f(x^{-1})}$, in the case of G being abelian) and convolution is inherited from $M(G)$. Note also that Fubini's theorem implies that $f * g \in L^1(G)$ when $f, g \in L^1(G)$ since $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ and it also implies that

$$f * g(x) = \int_G f(y) g(y^{-1}x)\, dy$$

is absolutely convergent for a.e. $x \in G$. It is an easy check that convolution in $L^1(G)$ agrees with convolution in $M(G)$ when we embed L^1. Because this subalgebra is so important in Harmonic analysis we call it the L^1 *group algebra*.[1]

4 REPRESENTATION THEORY

We will need to consider two types of representations, both deeply connected—as we shall see—on the one hand we have representations of topological groups on Hilbert spaces and on the other hand we have representations of C* Algebras, in particular, representations of the group algebra. We start with the representations of topological groups, following Folland [1, Ch. 3].

In this text we will denote the space of bounded linear operators $V \to W$ by $B(V, W)$ and $B(V) = B(V, V)$.

Definition 4.1. Let G be a LCA group, and let $U(\mathcal{H}_\rho)$ denote the group of unitary operators on the Hilbert space $\mathcal{H}_\rho$, then a strongly continuous unitary representation is a homomorphism $\rho : G \to U(\mathcal{H}_\rho)$ from G to $U(\mathcal{H}_\rho)$ such that $g \mapsto \rho(g)u$ is continuous for every $u \in \mathcal{H}_\rho$.[2]

We will also quickly cover some more definitions that will be necessary. First, two representations $\rho : G \to U(\mathcal{H}_\rho)$, $\rho' : G \to U(\mathcal{H}_{\rho'})$ are called *unitarily equivalent* or intertwined if there is some $T \in B(\mathcal{H}_\rho, \mathcal{H}_{\rho'})$ such that $T\rho = \rho' T$, such T are called *equivariant* maps.

Furthermore if there is some closed subspace $\mathcal{M} \subset \mathcal{H}_\rho$ such that $\rho(g)\mathcal{M} \subseteq \mathcal{M}$ for all $g \in G$ then we call $\mathcal{M}$ a *ρ-invariant subspace* of $\mathcal{H}_\rho$. And if $\mathcal{M}$ is a ρ-invariant subspace then we define the *subrepresentation* of ρ as $\rho^{\mathcal{M}} : G \to U(\mathcal{M})$, $\rho^{\mathcal{M}}(g) = \rho(g)|_{\mathcal{M}}$.

If there exists a nontrivial ρ-invariant subspace $\mathcal{M} \subseteq \mathcal{H}_\rho$ then ρ is called *reducible*, otherwise ρ is called *irreducible*.

Moreover, the dimension of a representation ρ is the dimension of $\mathcal{H}_\rho$.

If we have a family $\{\rho_\alpha : G \to \mathcal{H}_{\rho_\alpha}\}_{\alpha \in A}$ of arbitrary unitary representations of G then we can naturally define their direct-sum unitary representation $\rho = \oplus_{\alpha \in A} \rho_\alpha : G \to \oplus_{\alpha \in A} \mathcal{H}_{\rho_\alpha}$ as

$$\rho(g)(\sum_{\alpha \in A} x_\alpha) = \sum_{\alpha \in A} \rho_\alpha(g) x_\alpha$$

[1] Convolution on L^1 actually extends to other L^p spaces, more information on this can be found in Folland [1, p.52]

[2] It should be noted that the weak-* topology and the strong topology coincide on $U(\mathcal{H}_\rho)$ (see Folland [1, p.68])

for $\sum_{\alpha\in A} x_\alpha \in \oplus_{\alpha\in A}\mathcal{H}_{\rho_\alpha}$ (Note that $\oplus_{\alpha\in A}\mathcal{H}_{\rho_\alpha}$ stands here for a Hilbert space direct sum see Folland [1, p.254]). Clearly every ρ_α is a subrepresentation of ρ corresponding to the ρ-invariant subspace $\mathcal{H}_{\rho_\alpha}$.

Now we are ready for some theory. As $\mathcal{H}_\rho$ is a Hilbert space we have a defined inner-product, thus it is interesting to consider the orthogonal complement of ρ-invariant subspaces, in fact:

Proposition 4.1. *Let $\mathcal{M}$ be a closed nontrivial subspace of $\mathcal{H}_\rho$, then $\mathcal{M}$ is ρ-invariant iff $\mathcal{M}^\perp$ is. Furthermore, $\rho = \rho^{\mathcal{M}} \oplus \rho^{\mathcal{M}^\perp}$.* [3]

Proof. For $u \in \mathcal{M}$, $v \in \mathcal{M}^\perp$ we have $\langle \rho(g)u, v\rangle = \langle u, \rho(g^{-1})v\rangle = 0$ so $\rho(g)v \in \mathcal{M}^\perp$. The first part is proved since $\mathcal{M} = \mathcal{M}^{\perp^\perp}$. And the second part is trivial as every $x \in \mathcal{H}_\rho$ can be decomposed into $x = u + v$ for some u, v as before, and ρ acts on these independently as the subrepresentations. ∎

Theorem 4.2 (Shur's lemma)**.** *1. A unitary representation $\rho : G \to U(\mathcal{H}_\rho)$ is irreducible if and only if all the maps in $B(\mathcal{H}_\rho)$ that commute with $\rho(g)(\forall g \in G)$ are the scalar multiples of I.*

2. Let $\rho : G \to U(\mathcal{H}_\rho)$, $\rho' : G \to U(\mathcal{H}_{\rho'})$ be two irreducible representations. If ρ and ρ' are unitarily equivalent then any equivariant map T is a constant multiple of a fixed isomorphism.

Proof. We start with (2). If T is equivariant then T^* is also equivariant, and TT^* is clearly a bounded self-adjoint operator. Any closed invariant subspace of TT^* is ρ-invariant, thus is trivial, and by the spectral theorem TT^* is a constant multiple of the identity. This also holds for T^*T, hence T is a constant multiple of a unitary map, in particular it is invertible, or the zero map.

For the first part, take $\rho = \rho'$ in (2). For the other direction assume ρ is reducible. Then there is a closed ρ-invariant subspace $\mathcal{M} \subseteq \mathcal{H}_\rho$, now let P be the orthogonal projection onto $\mathcal{M}$ and let $v \in \mathcal{M}^\perp$, $w \in \mathcal{M}$, then

$$\begin{aligned}\rho(x)Pv &= 0 = P\rho(x)v\\ \rho(x)Pw &= \rho(x)w = P\rho(x)w\end{aligned}$$

by prop. 4.1. But then $\rho(x)P = P\rho(x)$, and we're done. ∎

Corollary 4.3. *All irreducible unitary representations $\rho : G \to U(\mathcal{H}_\rho)$ of an LCA group G are 1-dimensional.*

Proof. All the $\rho(x)$ commute, hence by Shur's lemma $\rho(x) = \xi I$ for all $x \in G$ and some $\xi \in \mathbb{C}$, hence all one dimensional subspaces are invariant, therefore $\dim \mathcal{H}_\rho = 1$. ∎

4.1 REPRESENTATIONS OF THE GROUP ALGEBRA

In this section we consider the group algebra $L^1(G)$, not as embedded in $M(G)$ but as its own object. First we define representations of the group algebra:

[3]This is not true for non-unitary representations.

Definition 4.4. A **-representation* $\rho : L^1(G) \to B(\mathcal{H}_\rho)$ of the group algebra is a *-homomorphism. We call ρ *nondegenerate* if there is no $v \in \mathcal{H}_\rho$ such that $\rho(f)v = 0$ for all $f \in L^1(G)$.

Remark 4.5. As we will always require nondegeneracy of our *-representations in this text we will implicitly assume that they are defined as nondegenerate, and just refer to them as *-representations.

We will shortly present a result which says that we can identify the *-representations with the unitary representations of G by

$$\rho(f) = \int_G \rho(x) f(x)\, dx$$

this is defined to act on $\mathcal{H}_\rho$ in the sense that

$$\langle \rho(f)u, v\rangle = \int_G f(x)\langle \rho(x)u, v\rangle\, dx. \tag{4.1}$$

Clearly $|\langle \rho(f)u, v\rangle| \le \|f\|_1\|u\|\|v\|$ and since $\rho(f)$ is evidently linear we have $\rho(f) \in B(\mathcal{H}_\rho)$. But before we can present the result that gives this identification we must prove that $\rho(f)$ is actually a *-representation.

Proposition 4.2. *For every unitary representation $\rho : G \to U(\mathcal{H}_\rho)$ the map*

$$\rho(f) = \int_G \rho(x) f(x)\, dx$$

*defines a *-representation on $L^1(G)$.*

Proof. As we have stated before $f \to \rho(f)$ is clearly linear and bounded, so it suffices to show that it preserves involution, is a homomorphism and is nondegenerate. We start with involution:

$$\begin{aligned}\rho(f^*) &= \int_G \rho(x)\overline{f(x^{-1})}\Delta(x^{-1})\, dx = \int_G \overline{f(x)}\rho(x^{-1})\, dx \\ &= \int_G [f(x)\rho(x)]^*\, dx = \rho(f)^*\end{aligned}$$

And to show that its a homomorphism we just need to show that it preserves convolution since we already noted that it was linear:

$$\begin{aligned}\rho(f * g) &= \int_G\int_G f(x)g(x^{-1}y)\rho(y)\, dxdy \\ &= \int_G\int_G f(x)g(y)\rho(xy)\, dxdy \\ &= \int_G\int_G f(x)g(y)\rho(x)\rho(y)\, dxdy = \rho(f)\rho(g).\end{aligned}$$

These calculations are justified by equation (4.1), i.e. applying the operators to $u \in \mathcal{H}_\rho$ and taking the inner product with $v \in \mathcal{H}_\rho$. The last step is to show nondegeneracy, so suppose $u \neq 0 \in \mathcal{H}_\rho$ and pick a compact neighborhood V of 1 in G such that

$$\|\rho(x)u - u\| \le \|u\| \quad \text{for } x \in V.$$

Now set $f(x) = |V|^{-1}\chi_V(x)$ where χ_V is the indicator function of V, then

$$\|\rho(f)u - u\| = |V|^{-1}\|\int_V [\rho(x)u - u]\, dx\| \leq \|u\|$$

hence $\rho(f)u \neq 0$. ■

Theorem 4.6. *Let $\tilde{\rho} : L^1(G) \to \mathcal{H}_\rho$ be a nondegenerate *-representation. Then there is a unique unitary representation $\rho : G \to U(\mathcal{H}_\rho)$ of G such that*

$$\tilde{\rho}(f) = \int_G \rho(x) f(x)\, dx \qquad \forall (f \in L^1(G)).$$

Proof. See Folland [1, p.74] for a proof of this. ■

5 THE DUAL SPACE

Let G be a LCA group. We follow Folland [1, Ch. 4] in the remaining sections. Then every irreducible unitary representation $\rho : G \to U(H_\rho)$ is 1-dimensional by corollary 4.3, thus $H_\rho \cong \mathbb{C}$ and there exists $\xi(x) \in \mathbb{C}$ for every $x \in G$ such that $\rho(x)(z) = \xi(x)z$ in particular $|\xi(x)| = 1$ by unitarity, $\xi(x)$ varies continuously with x, and ρ is a homomorphism in x, so $\xi(x)$ satisfies $\xi(xy) = \xi(x)\xi(y)$. We call such maps ξ characters and the space of characters is called the dual group, denoted by $\widehat{G}$. We shall denote $\xi(x)$ by (x, ξ).

We will use the notation $\sigma(\mathcal{A})$ when $\mathcal{A}$ is a Banach algebra to denote the spectrum of $\mathcal{A}$ i.e. the set of nonzero algebra homomorphisms from $L^1(G)$ into $\mathbb{C}$.

Proposition 5.1. *Every $\Phi \in \sigma(L^1(G))$ is given by integration against a character $\xi \in \widehat{G}$, i.e. by*

$$\Phi(f) = \int_G (x, \xi) f(x)\, dx \quad (f \in L^1(G)). \tag{5.1}$$

*This gives a nondegenerate *-representation on $L^1(G)$.*

Proof. First we note that proposition 4.2 says that (5.1) is a nondegenerate *-representation on $L^1(G)$, so we only need to prove that every multiplicative functional of $L^1(G)$ is given by integration against a character, this we prove as follows:

Every $\Phi \in (L^1(G))^*$ is given by integration against some $\phi \in L^\infty(G)$, first we will recover such a ϕ, then show that it is in $\widehat{G}$. Let $f, g \in L^1(G)$ such that $\Phi(f) \neq 0$, then

$$\phi(f) \int_G \phi(y) g(y)\, dy = \Phi(f)\Phi(g) = \phi(f * g) = \int_G \Phi(L_y f) g(y)\, dy$$

hence $\phi(x) = \frac{\Phi(L_x f)}{\Phi(f)}$, locally a.e., but taking this as our definition of $\phi(x)$, we have everywhere continuity. We also see that $\phi(xy)\Phi(f) = \Phi(L_{xy} f) = \Phi(L_x L_y f) = \phi(x)\phi(y)\Phi(f)$, thus we also see that $\phi(x^n) = \phi(x)^n$ but since $|\phi| \leq M$ is bounded, hence it must be that $\phi(x) \in \mathbb{T}$. And we're done. ■

Corollary 5.1. *The dual group $\widehat{G}$ can be identified with $\sigma(L^1(G))$ by* (5.1).

Right now $\widehat{G}$ has no more structure than being a group, we remedy this by giving it the topology of compact convergence, i.e. the topology of uniform convergence on compact sets.

Theorem 5.2. *The dual group $\widehat{G}$ is a locally compact abelian group under the topology of uniform convergence on compact sets, which coincides with the weak-* topology inherited from $L^\infty(G)$.*

Remark 5.3. We will not prove the fact that this topology coincides with the weak-* topology, but we note that it follows from the theory of functions of positive type, and can be found in Folland [1, p.89].

Proof. We have already showed that $\widehat{G}$ is an abelian group, so it suffices to show that the topology is locally compact and that the group operations are continuous, we begin with the latter.

The topology is the topology of compact convergence which means that we have to show that for two sequences $\{\xi_n\}, \{\eta_n\}$ of elements of $\widehat{G}$ and any compact subset $K \subset G$ such that $\xi_n \to \xi$ and $\eta_n \to \eta$ converges uniformly on K then $\xi_n\eta_n^{-1} \to \xi\eta^{-1}$ converges uniformly on K, but this follows immediately by the (pointwise) triangle inequality:

$$\begin{aligned}
|\xi_n\eta_n^{-1} - \xi\eta^{-1}| &= |\xi_n\eta_n^{-1} - \xi\eta_n^{-1} + \xi_n\eta^{-1} - \xi_n\eta^{-1}| \\
&\leq |\xi_n(\eta_n^{-1} - \eta^{-1})| + |(\xi_n - \xi)\eta^{-1}| \\
&= |\xi_n||(\eta_n^{-1} - \eta^{-1})| + |(\xi_n - \xi)||\eta^{-1}| \\
&= |\eta_n^{-1} - \eta^{-1}| + |\xi_n - \xi| \\
&= |\overline{\eta_n} - \overline{\eta}| + |\xi_n - \xi|
\end{aligned}$$

This gives uniform convergence for $\xi_n\eta_n^{-1} \to \xi\eta^{-1}$.

Now we show local compactness. By theorem 5.1 $\widehat{G}\cup\{0\}$ equals $\mathrm{Hom}(L^1(G), \mathbb{C})$, which is a subset of the closed unit ball of $L^\infty(G)$ and is weak-* closed and hence weak-* compact by the Banach-Alaoglu theorem, hence $\widehat{G}$ is locally compact. ■

We now prove a result which gives us methods for decomposing the dual group $\widehat{G}$ into a direct sum of cartesian product when G is cartesian product of other locally compact groups. First we define $\oplus_{i\in I}G_i$ to be all $(x_i)_{i\in I} \in \prod_{i\in I} G_i$ such that all but finitely many $x_i = 1$.

Theorem 5.4. *Let $\{G_i\}_{i\in I}$ be LCA groups.*

1. *If $|I| < \infty$ then $\widehat{\prod_{i\in I} G_i} \cong \prod_{i\in I} \widehat{G_i}$.*
2. *Moreover, if G_i is compact for all $i \in I$ then $\widehat{\prod_{i\in I} G_i} = \oplus_{i\in I}\widehat{G_i}$ for arbitrary $|I|$.*

Proof. We prove them in numerical order. Note that the inner product is on $\mathbb{C}$. Each $\xi = (\xi_i)_{i\in I} \in \prod_{i\in I} \widehat{G_i}$ defines a character on $\prod_{i\in I} G_i$, by

$\langle (x_i)_{i\in I}, (\xi_i)_{i\in I}\rangle = \prod_{i\in I}\langle x_i, \xi_i\rangle$, furthermore all characters in $\chi \in \widehat{\prod_{i\in I} G_i}$ are of this form, where ξ_i is given by $\langle x_i, \xi_i\rangle = \langle (1, \dots, 1, x_i, 1, \dots, 1), \chi\rangle$.

Now assume all the G_i are compact and let $|I|$ be arbitrary. In the same way as before we get the characters so we only need to prove that all but finitely many of $\xi_i = 1$. By continuity, there is a neighborhood U of 1 in $\prod_{i\in I} G_i$ such that $|(x, \xi) - 1| < 1$ for $x \in U$. By definition U contains a set $\prod_{i\in I} U_i$ where $U_i = G_i$ for all but finitely many i. In which case $\xi_i(U_i) \subset \xi(U)$ hence $\xi_i(U_i)$ is a subgroup of $\mathbb{T}$ contained in $\{z \in \mathbb{T} : |t-1| < 1\}$ hence equals 1 hence $\xi_i = 1$. And we're done. ∎

Considering the case when G is finite we see that:

Corollary 5.5. *All finite abelian groups are self-dual.*

Proof. By the fundamental theorem of finite abelian groups and (1) of theorem 5.4 we only need to show that the cyclic groups $\mathbb{Z}_n$ are self-dual. The representation theory of finite groups (see Sagan [3]) says that all characters of cyclic groups are of the form $\chi(k) = e^{\frac{k\pi i}{n}}$ where $k = 0, \dots, n-1$, and are all 1-dimensional, it is obvious that $\widehat{\mathbb{Z}_n} = \{\chi_k : k = 0, \dots, n-1\}$ which is isomorphic to $\mathbb{Z}_n$ by $\chi_k \longmapsto k$, thus we're done. ∎

The following result is an amazing fact about the dual group of a compact abelian group.

Proposition 5.2. *Let G be a compact abelian group with normalized Haar measure. Then $\widehat{G}$ is an orthonormal set in $L^2(G)$.*

Proof. If $\xi, \eta \in \widehat{G}$ then $|\xi|^2 = 1$ so $\|\xi\|_2 = 1$ because the Haar measure is normalized on G. If $\xi \neq \eta$ then there is an $x_0 \in G$ such that $(x_0, \xi\eta^{-1}) \neq 1$ and

$$\int \xi\overline{\eta} = \int_G (x_0, \xi\eta^{-1})\, dx = (x_0, \xi\eta^{-1}) \int_G (x_0^{-1}x, \xi\eta^{-1})\, dx = (x_0, \xi\eta^{-1}) \int \xi\overline{\eta}$$

which implies that $\langle \xi, \eta\rangle_{L^2} = \int \xi\overline{\eta} = 0$. ∎

5.1 THE FOURIER-STIELTJES TRANSFORM

There is a more convenient homomorphism that identifies $\widehat{G}$ with the spectrum of $L^1(G)$ than the one we saw in section 5, its called the Fourier transform and is defined as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_G \overline{(x, \xi)} f(x)\, dx$$

where $f \in L^1(G)$, $\xi \in \widehat{G}$.

It is a notable fact that there is a version of the Riemann-Lebesgue lemma that is valid in this general context. It is a part of the following proposition.

Proposition 5.3. *The Fourier transform is a norm decreasing *-homomorphism with a dense image in $C_0(G)$.*

Proof. The proof that $\mathcal{F}$ is a *-homomorphism is almost identical to the proof of proposition 4.2. Norm-decreasing follows by a simple calculation. That the image lies in $C_0(G)$ is slightly less trivial, and takes some development of Gelfand theory of nonunital Banach algebras, for a proof that this is true see Folland [1, p.15]. The last part, i.e. the fact that the image of $\mathcal{F}$ is dense in $C_0(G)$ follows by the Stone Weierstrass theorem. ∎

If we recall the definition of the group algebra from section 3, i.e. we found a copy of $L^1(G)$ inside $M(G)$, one might wonder if there is a generalization of the Fourier transform to the whole space $M(G)$, indeed there is. It is called the Fourier-Stieltjes transform and is defined as follows. Let $\mu \in M(G)$, then the Fourier-Stieltjes transform is the bounded continuous function $\hat{\mu}$ on $\widehat{G}$ defined by

$$\hat{\mu}(\xi) = \int_G \overline{(x,\xi)}\, d\mu(x).$$

We check if it respects convolution:

$$\begin{aligned}\widehat{\mu * \nu}(\xi) &= \int_G \overline{(xy,\xi)}\, d\mu(x) d\nu(y)\\ &= \int_G \overline{(x,\xi)}\,\overline{(y,\xi)}\, d\mu(x) d\nu(y) = \widehat{\mu}(\xi)\widehat{\nu}(\xi)\end{aligned}$$

It turns out that the Fourier-Stieltjes transform on $M(\widehat{G})$ is more interesting in our setting and we will see it in Bochner's theorem in section 5.3. We define the Fourier-Stieltjes transform on $M(\widehat{G})$ as

$$\phi_\mu(x) = \int_{\widehat{G}} (x,\xi)\, d\mu(\xi) \tag{5.2}$$

for $\mu \in M(\widehat{G})$ and $x \in G$.

Proposition 5.4. *The Fourier-Stieltjes transform on $M(\widehat{G})$ is a norm-decreasing linear injective map to the space of bounded continuous functions on G with the uniform norm.*

Proof. All but the injectivity is trivial. Suppose $\phi_\mu = 0$ and $f \in L^1(G)$ then

$$0 = \int_G f(x)\phi_\mu(x)\, dx = \int_G \int_{\widehat{G}} f(x)(x,\xi)\, d\mu(\xi) dx = \int_{\widehat{G}} \widehat{f}(\xi^{-1})\, d\mu(\xi).$$

But then $\mu = 0$ since the image of $\mathcal{F}$ is dense in $C_0(G)$. ∎

5.2 FUNCTIONS OF POSITIVE TYPE

Let G be a locally compact abelian group. We define $\phi \in L^\infty(G)$ to be a function of positive type if

$$\int_G (f^* * f)\phi\, dx \geq 0 \quad \forall f \in L^1(G).$$

Note that we can rewrite this as

$$\begin{aligned}&\int_G (f^* * f)\phi\, dx = \int_G \int_G f^*(x) f(y^{-1}x)\phi(x)\, dydx\\ &= \int_G \int_G \overline{f(y^{-1})}\Delta(y^{-1}) f(y^{-1}x)\phi(x)\, dydx = \int_G \int_G \overline{f(y)} f(yx)\phi(x)\, dydx\end{aligned}$$

and finally

$$= \int_G \int_G \overline{f(y)} f(x) \phi(y^{-1}x) \, dy dx \geq 0 \quad \forall f \in L^1(G) \tag{5.3}$$

which will be a useful form for the proofs. Define

$$\mathcal{P}(G) = \{\phi \in L^\infty(G) : \phi \text{ is of positive type}\} \cap C(G)$$

Proposition 5.5. *If $\phi \in \mathcal{P}(G)$ then $\overline{\phi} \in \mathcal{P}(G)$.*

Proof. This essentially follows from the fact that the complex conjugate of an integral is the integral of the complex conjugate.

$$\int_G (f^* * f)\overline{\phi} \, dx = \overline{\int_G (\overline{f}^* * \overline{f})\phi \, dx} \geq 0 \quad \forall f \in L^1(G)$$

and we're done. ∎

Proposition 5.6. *If $\rho : G \to \mathcal{H}_\rho$ is a unitary representation and $u \in \mathcal{H}_\rho$, and if $\phi(x) = \langle \rho(x)u, u \rangle$ then $\phi \in \mathcal{P}(G)$.*

Proof. Clearly, continuity of ρ implies continuity of ϕ, and

$$\phi(y^{-1}x) = \langle \rho(x)u, \rho(y)u \rangle$$

thus for $f \in L^1(G)$ we find that (5.3) becomes

$$\begin{aligned} 0 &\leq \int_G \int_G \overline{f(y)} f(x) \phi(y^{-1}x) \, dy dx \\ &= \int_G \int_G \langle f(x)\rho(x)u, f(y)\rho(y)u \rangle \, dy dx = \|\rho(f)u\|^2_{\mathcal{H}_\rho}. \end{aligned}$$

∎

For the next result recall that $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ denotes involution of f, but here G is unimodular, i.e. $\Delta \equiv 1$. Furthermore $L^2(G) \subseteq L^1(G)$ follows by

$$\|f\|_{L^2} \leq \|f\|_{L^1} \quad (\forall f \in L^1(G))$$

thus involution is well-defined for $L^2(G)$ functions.

Corollary 5.6. *If $f \in L^2(G)$ then $f * f^* \in \mathcal{P}(G)$.*

Proof. Let $\rho(x) = L_x$ be the left regular representation, then

$$\langle L_x f, f \rangle = \int_G f(x^{-1}y)\overline{f(y)} \, dy = \overline{\int_G \overline{f(x^{-1}y)} f(y) \, dy} = \overline{f * \overline{f(x^{-1})}}$$

hence $f * f^* \in \mathcal{P}(G)$ by propositions 5.5 and 5.6. ∎

5.3 BOCHNER'S THEOREM AND THE FIRST FOURIER INVERSION THEOREM

Bochner's theorem will give us the necessary correspondence between positive complex Radon measures and functions of positive type, so that we can prove the Fourier inversion theorem and the Plancherel theorem. The general case is due to André Weil, but was first proved by Gustav Herglotz for $\mathbb{T}$ and by Salomon Bochner for $\mathbb{R}$, whom was the first to establish a place for functions of positive type in harmonic analysis.

Theorem 5.7 (Bochner's Theorem)**.** *Let G be a LCA Group. A function ϕ on G is of positive type if and only if there is a (unique) non-negative measure $\mu \in M(\widehat{G})$ such that $\phi(x) = \phi_\mu(x)$*

Proof. ($\Longleftarrow$) Assume $\mu \in M(\widehat{G})$ is a non-negative measure and $f \in L^1(G)$ then

$$\begin{aligned}\int_G \int_G \overline{f(y)} f(x) \phi(y^{-1}x)\, dydx &= \int_G \int_G \int_{\widehat{G}} \overline{f(y)} f(x)(y^{-1}x, \xi)\, d\mu(\xi) dydx \\ &= \int_G \int_G \int_{\widehat{G}} \overline{f(y)(y,\xi)} f(x)(x,\xi)\, d\mu(\xi) dydx \\ &= \int_{\widehat{G}} |\widehat{f}(\xi^{-1})|^2\, d\mu(\xi) \geq 0\end{aligned}$$

i.e. the condition (5.3).
($\Longrightarrow$) We will manufacture a continuous linear functional on $C_0(\widehat{G})$ and apply the Riesz-Markov theorem.

Assume $\phi \in \mathcal{P}(G)$. Since $\|\phi\|_\infty = \phi(1)$ (see Folland [1, p.79]) for all functions of positive type it suffices to assume that $\phi(1) = 1$. Now, if $\phi \neq 0$ is a positive function then it induces a positive Hermitian form

$$\langle f, g \rangle_\phi = \int_G g^* * f \phi\, dx$$

on $L^1(G)$ which clearly satisfies

$$|\langle f, g \rangle_\phi| \leq \|\phi\|_\infty \|f\|_1 \|g\|_1$$

by applying the Schwarz-inequality. With our assumption on ϕ we get $|\langle f, g\rangle_\phi|^2 \leq \langle f, f\rangle_\phi \langle g, g\rangle_\phi$.

Now let g be an approximate identity $\{e_\lambda\}$. Then (by definition) $\lim_{\lambda\in\Lambda} \|e_\lambda * f - f\|_1 = 0$, hence $|\langle f, e_\lambda\rangle_\phi| \to \int_G \phi f\, dx$. If the support of e_λ is in U then the support of $e_\lambda^* * e_\lambda$ is in $U^{-1}U$, and $\langle e_\lambda, e_\lambda\rangle_\phi = |\int_G e_\lambda\, dx|^2 = 1$, hence $e_\lambda^* * e_\lambda$ is an approximate identity and $\langle e_\lambda, e_\lambda\rangle_\phi\, dx \to \|\phi\|_\infty = \phi(1)$ which is 1 by our assumption.

Applying the Schwarz-inequality and the previous discussion to $\langle f, e_\lambda\rangle_\phi$ and taking limits gives

$$\Big|\int_G \phi f\, dx\Big|^2 \leq \langle f, f\rangle_\phi. \tag{5.4}$$

Now define $h^{*n} = h * \cdots * h$ n times, and let $h = f^* * f$ then clearly $h^* = h$. Apply (5.4) to the sequence $f, h, h^{*2}, \ldots$, then we get

$$\Big| \int_G \phi h^{*2^{n-1}} \, dx \Big|^{\frac{1}{2^n}} \leq \Big| \int_G \phi h^{*2^n} \, dx \Big|^{\frac{1}{2^{n+1}}} \leq \|h^{*2^n}\|_1^{\frac{1}{2^{n+1}}}$$

and by the spectral radius formula and the fact that the range of the Fourier transform is the spectrum of the function, it follows that

$$\lim_{n\to\infty} \|h^{*2^n}\|_1^{\frac{1}{2^{n+1}}} = \|\hat{h}\|_\infty^{\frac{1}{2}} = \||\hat{f}|^2\|_\infty^{\frac{1}{2}} = \|\hat{f}\|_\infty.$$

Thus $f \to \int_G \phi f$ induces a linear functional $A : \hat{f} \to \int_G \phi f$ on $\mathcal{F}(L^1(G))$. And since $\mathcal{F}(L^1(G))$ is dense in $C_0(\widehat{G})$ by the Stone-Weierstrass theorem, it extends to a linear functional there, of norm ≤ 1. Thus by the Riesz-Markov representation theorem there is a measure $\hat{\mu} \in M(\widehat{G})$ such that

$$A(f) = \int_{\widehat{G}} \hat{f}(\xi) \, d\hat{\mu}(\xi) = \int_G \int_{\widehat{G}} f(x)\overline{(x,\xi)} \, d\hat{\mu}(\xi) dx$$

but then $\phi(x) = \int_{\widehat{G}} (x,\xi) \, d\hat{\mu}(\xi^{-1})$. Thus $\phi(1) = \mu(\widehat{G}) \leq \|\mu\| \leq 1$, it follows that $\mu \geq 0$. ■

Now define the Bochner space $\mathcal{B}(G) = \{\phi_\mu : \mu \in M(\widehat{G})\}$. Bochner's theorem says that $\mathcal{B}(G) = \mathrm{span}_{\mathbb{C}} \mathcal{P}(G)$. We also define $\mu_\phi \in M(G)$ to correspond to ϕ under (5.2), i.e. it is the inverse map $\phi \longmapsto \mu_\phi$ of (5.2).

Before we prove the Fourier inversion theorem we need a lemma

Lemma 5.8. *If $U \subseteq \widehat{G}$ is compact then there exists a compactly supported function $f \in \mathcal{P}(G)$ such that $\hat{f} \geq 0$ on $\widehat{G}$ and $\hat{f} > 0$ on U.*

Proof. Pick $h \in C_c(G)$ such that $\int_G h = 1$. Then $\widehat{h^* * h} = |\hat{h}|^2$, $\widehat{h^* * h} \geq 0$ and $\widehat{h^* * h}(1_{\widehat{G}}) = 1$, thus there is a neighborhood U' of $1_{\widehat{G}}$ in $\widehat{G}$ on which $\widehat{h^* * h} > 0$.

By compactness we can cover U with finitely many of these. Let the translates be given by $\xi_1, \ldots \xi_k$, and define $f(x) = \sum_{i \leq k} \xi_i h^* * h(x)$, thus $\hat{f}(\xi) = \sum_{i \leq k} \mathcal{L}_{\xi_i} \widehat{h^* * h}(\xi)$, which satisfies the conclusion. The convolution of two $L^2(G)$ functions is continuous, thus f is, and since

$$\int_G \psi^* * \psi f(x) \, dx = \sum_{i \leq k} \int_G (\xi_i \psi)^* * (\xi_i \psi) h^* * h(x) \, dx \geq 0$$

for all $\psi \in L^1(G)$, we have $f \in \mathcal{P}(G)$. ■

Theorem 5.9 (First Fourier Inversion Theorem)**.** *Suppose dual Haar measure $d\xi$ is suitably normalized relative to dx. If $f \in \mathcal{B}(G) \cap L^1(G)$ then $\hat{f} \in L^1(G)$ and*

$$f(x) = \int_{\widehat{G}} (x,\xi) \hat{f}(\xi) \, d\xi. \tag{5.5}$$

Proof. Again, we will construct a positive linear functional on $C_c(\widehat{G})$ where we can apply the Riesz representation theorem.

So let $\psi \in C_c(\widehat{G})$, then with $U = \operatorname{supp}(\psi)$ we apply lemma 5.8, and get some $f \in C_c(G) \cap \mathcal{P}(G) \subseteq L^1(G) \cap \mathcal{P}(G)$ with $\hat{f} > 0$ on U. If g is another such function (as f) and $h \in L^1(G)$ then

$$\int_G \hat{h}\, d\mu_g = \int_G \int_{\widehat{G}} h(x)\overline{(x,\xi)}\, dx d\mu_g(\xi) = \int_G h(x) f(x^{-1})\, dx = h * f(1)$$

substituting h for $h * g$ or $h * f$ we get $\int_G \hat{h} g\, d\mu_f = \int_G \hat{h} f\, d\mu_g$, now recalling that $\mathcal{F}(L^1(G))$ is dense in $C_0(\widehat{G})$ we ge that $f\, d\mu_g = g\, d\mu_f$.

Now define $I : C_c(G) \to \mathbb{C}$, $\psi \longmapsto \int_G \frac{\psi}{f}\, d\mu_f$, clearly I is a positive linear functional, by the above discussion we get

$$\int_G \frac{\psi}{\widehat{f}}\, d\mu_f = \int_G \frac{\psi}{\widehat{f}\widehat{g}}\widehat{g}\, d\mu_f = \int_G \frac{\psi}{\widehat{f}\widehat{g}}\widehat{f}\, d\mu_g = \int_G \frac{\psi}{\widehat{g}}\, d\mu_g$$

therefore $I(\psi)$ is invariant of the choice of f. And by the same discussion we get that $I(\widehat{g}\psi) = \int_G \psi d\mu_g$, hence for appropriate ψ, f, $I \not\equiv 0$.

Now we show translation invariance. Let $\eta \in \widehat{G}$ be a character, then

$$\int_G (x,\xi)\, d\mu_f(\eta\xi) = \overline{(x,\eta)} f(x) = (\overline{\eta} f)(x)$$

hence $d\mu_f(\eta\xi) = d\mu_{\overline{\eta}f}(\xi)$. Hence by translation of $\mathcal{F}$ we have for f such that $\hat{f} > 0$ on $U \cap \operatorname{supp}(L_\eta(\psi))$ that

$$I(L_\eta\psi) = \int_G \frac{\psi(\eta^{-1}\xi)}{\widehat{f}} d\mu_f(\xi) = \int_G \frac{\psi(\xi)}{\widehat{f}(\eta\xi)} d\mu_f(\eta\xi) = \int_G \frac{\psi(\xi)}{\widehat{\overline{\eta}f}(\xi)} d\mu_{\overline{\eta}f}(\xi) = I(\psi).$$

Next, we note that it also follows that $I(\psi) = \int_G \psi(\xi)\, d\xi$ which is the regular dual Haar measure. And by our previous choice-independence of f we have that $I(\psi\hat{f}) = \int_G \psi\, d\mu_f$, hence $\hat{f}(\xi)\, d\xi = d\mu_f(\xi)$, thus $\hat{f} \in L^1(\widehat{G})$ and

$$\int_G (x,\xi)\hat{f}(\xi)\, d\xi = \int_G (x,\xi) d\mu_f(\xi) = f(x) \tag{5.6}$$

by Bochner's theorem, and we're done. ■

The Fourier Inversion theorem is essentially one-half of a duality statement about "sufficiently nice" functions $f : G \to \mathbb{C}$ determining precisely another "sufficiently nice" function $\widehat{f} : \widehat{G} \to \mathbb{C}$. If one has "very-sufficiently-nice" functions this relationship becomes exact and they determine one-another completely, we make this more precise in the next section.

5.4 PLANCHEREL THEOREM

The Plancherel theorem says that the $L^2(G)$ and $L^2(\widehat{G})$ are indistinguishable as Hilbert spaces, and later any decomposition of either gives a decomposition of the other. This is the fundamental L^2 duality theorem in other words and just one of many dualities in Harmonic Analysis.

Theorem 5.10 (Plancherel Theorem, 1910)**.** *The Fourier transform is extendable to a unitary isomorphism between $L^2(G)$ and $L^2(\widehat{G})$, in particular it is an isometry:*

$$\|f\|_{L^2(G)}^2 = \int_G |f(x)|^2\, dx = \int_{\widehat{G}} |\widehat{f}(x)|^2\, d\xi = \|\hat{f}\|_{L^2(\widehat{G})}^2.$$

Proof. Let $f \in L^1(G) \cap L^2(G)$ then $f * f^* \in L^1(G) \cap \mathcal{P}(G)$ by the same argument as in lemma 5.8, and $\widehat{f * f^*} = |\hat{f}|^2$, so by the inversion theorem

$$\int_G |f|^2\, dx = f * f(1) = \int_{\widehat{G}} \widehat{f * f^*}(\xi)\, d\xi = \int_{\widehat{G}} |\hat{f}|^2\, d\xi$$

hence the Fourier transform is an isometry in the L^2-norm and extends uniquely to an isometry $L^2(G) \to L^2(\widehat{G})$. We already have injectivity so it suffices to show surjectivity. Let $\psi \in L^2(\widehat{G})$ be orthogonal in the L^2-inner-product against all $\hat{f}$ with f as before, then

$$\langle \psi, \widehat{L_x f} \rangle = 0 = \int_{\widehat{G}} (x, \xi)\psi(\xi)\overline{\hat{f}(\xi)}\, d\xi.$$

But $\psi\overline{\hat{f}} \in L^1(\widehat{G})$, since $\psi, \hat{f} \in L^2(\widehat{G})$, hence $\psi(\xi)\overline{\hat{f}(\xi)}\, d\xi \in M(\widehat{G})$. It the follows by injection of $\mathcal{F}$ that $\psi\overline{\hat{f}} = 0$ a.e., but lemma 5.8 says $\hat{f} > 0$ hence $\psi = 0$ a.e. And we're done. ■

We have now established the Fourier transform on $L^1(G) + L^2(G)$ hence it is natural to ask what happens for general L^p-spaces. In the case of $1 \leq p \leq 2$ we have $L^p(G) \subseteq L^1(G) + L^2(G)$ which means that we can define $\hat{f}$ for $f \in L^p(G)$. Moreover, with the help of the Riesz-Thorin interpolation theorem one can prove the Hausdorff-Young inequality (see Folland [1, p.100]). In the case of $p > 2$, it becomes technical as one needs help from the theory of distributions, this exceeds the scope of this paper but is recommended for the curious reader as a continuance of this section.

6 PONTRYAGIN DUALITY

Pontryagin Duality is a special case of Stone-Gelfand duality, nevertheless it gives a dual relation on the category of locally compact abelian groups called the double dual functor, this functor is covariant and naturally isomorphic to the identity functor, i.e. given a locally compact group G the character group $\widehat{G} = \hom(G, \mathbb{T})$ is again locally compact given the topology of compact convergence, and Pontryagin duality states that

$$G \cong \widehat{\widehat{G}}.$$

There are further subcategory dualities which we will present soon, but for now let us prove the main result.

Consider the map $\Phi : G \to \widehat{\widehat{G}}$ defined by $(\xi, \Phi(x)) = (x, \xi)$ for all $x \in G$. This map Φ turns out to have some strong qualities, both topological and group theoretic.

Let $\mathcal{B}(G) \cap L^1(G) = \mathcal{B}^1(G)$.

Lemma 6.1. *The linear span of $\mathcal{P}(G) \cap C_c(G)$ contains all functions of the form $f * g$ for $f, g \in C_c(G)$, moreover it is dense in $C_c(G)$ in the uniform norm and dense in $L^p(G)$ in the L^p-norm for any $1 \leq p < \infty$.*

Proof. By corollary 5.6, $\mathcal{P}(G) \cap C_c(G)$ includes all functions of the form $f * \overline{f(x^{-1})}$ for $f \in C_c(G)$. By the polarization identity we get that

$$\begin{aligned} f * \overline{h(x^{-1})} = \frac{1}{4}[&(f+h) * \overline{(f+h)(x^{-1})} - (f-h) * \overline{(f-h)(x^{-1})} \\ &+ i((f+h) * \overline{(f+h)(x^{-1})} - (f-ih) * \overline{(f-ih)(x^{-1})})] \end{aligned}$$

is in $\mathcal{P}(G) \cap C_c(G)$ for all $f, h \in C_c(G)$, hence functions of the form $f * h$ are in $\mathcal{P}(G) \cap C_c(G)$. Now, $\{f * g : f, g \in C_c(G)\}$ is dense in $C_c(G)$ in the uniform norm and dense in $L^p(G)$ in the L^p-norm, thus concluding the proof. ■

From this lemma it clearly follows that $\mathcal{B}(G)$ contains all functions of the form $f * g$ for $f, g \in C_c(G)$, moreover $\mathcal{B}^1(G)$ is dense in $C_c(G)$ in the uniform norm and dense in $L^1(G)$ in the L^1-norm.

Lemma 6.2. *If $\phi, \psi \in C_c(\widehat{G})$ then $\phi * \psi = \widehat{f}$ for some $f \in \mathcal{B}^1(G)$. And $\mathcal{F}(\mathcal{B}^1(G))$ is dense in $L^p(\widehat{G})$ for all $p < \infty$.*

Proof. Let $f(x) = \widehat{\phi}(x)$ and $g(x) = \widehat{\psi}(x)$ and $h(x) = \widehat{\phi * \psi}(x)$, then clearly,

$$\begin{aligned} h(x) &= \int\int (x, \xi)\phi(\xi\eta^{-1})\psi(\eta)\, d\xi d\eta \\ &= \int\int (x, \xi\eta)\phi(\xi)\psi(\eta)\, d\xi d\eta \\ &= \int\int (x, \xi)(x, \eta)\phi(\xi)\psi(\eta)\, d\xi d\eta = f(x)g(x) \end{aligned}$$

Now $f, g, h \in \mathcal{B}(G)$ since $\phi, \psi, \phi * \psi \in L^1(G)$. If $k \in L^1(G) \cap L^2(G)$, then it follows by the Plancherel theorem 5.10 that

$$\left| \int_G f\overline{k} \right| = \left| \int_G \int_{\widehat{G}} (x, \xi)\phi(x)\overline{k}(x)\, d\xi dx \right| = \left| \int_G \phi\overline{\overline{\widehat{k}}} \right| \leq \|\phi\|_2 \|\widehat{k}\|_2 = \|\phi\|_2 \|k\|_2.$$

Thus $f \in L^2(G)$, and by the same argument one notes that $g \in L^2(G)$. This implies that $h = fg \in L^1(G)$, in particular $h \in \mathcal{B}^1(G)$. Now the Fourier inversion theorem 5.9 gives

$$h(x) = \int_{\widehat{G}} (x, \xi)\widehat{h}(\xi)\, d\xi$$

thus injectivity of the Fourier transform gives $\widehat{h} = \phi * \psi$ and we're done. ■

We're now ready to prove Pontryagin's theorem.

Theorem 6.3 (The Pontryagin Duality Theorem)**.** *The map $\Phi : G \to \widehat{\widehat{G}}$ is a group isomorphism and a homeomorphism.*

The general case is due to Egbert van Kampen, while the compact and second countable case was proved by Lev Pontryagin.

Proof. Let $x, y \in G$ and $\xi \in \widehat{G}$, then

$$(\xi, \Phi(xy)) = (xy, \xi) = (x, \xi)(y, \xi) = (\xi, \Phi(x))(\xi, \Phi(y)) = (\xi, \Phi(x)\Phi(y))$$

thus $\Phi(xy) = \Phi(x)\Phi(y)$. Injectivity follows by the Gelfand-Raikov theorem (see Folland [1, p.84]) immediately. Thus Φ is an isomorphism into $\widehat{\widehat{G}}$. There are three more things to prove:

1. Φ is a homeomorphism.
2. $\Phi(G)$ is closed in $\widehat{\widehat{G}}$.
3. $\Phi(G)$ is dense in $\widehat{\widehat{G}}$.

Next let $x \in G$ and $\{x_\alpha\}_{\alpha \in A}$ be a net in G, then clearly $x_\alpha \to x$ implies that $f(x_\alpha) \to f(x)$ for all $f \in \mathcal{B}^1(G)$, on the other hand if $x_\alpha \not\to x$ then there is a neighborhood U of x and a set $B \subseteq A$, such that for every $\alpha \in A$ there is a $\beta \in B$ such that $\alpha \leq \beta$,[4] for which $x_\beta \notin U$ for $\beta \in B$. By lemma 6.1 there is a $f \in \mathcal{B}^1(G)$ such that $supp(f) \subset U$ and $f(x) \neq 0$, hence $f(x_\alpha) \not\to f(x)$.

Now by the Fourier inversion theorem 5.9 we get that $f(x_\alpha) \to f(x)$ is equivalent to

$$\int_{\widehat{G}} (x_\alpha, \xi)\widehat{f}(\xi)\, d\xi \to \int_{\widehat{G}} (x, \xi)\widehat{f}(\xi)\, d\xi \tag{6.1}$$

thus this is equivalent to $x_\alpha \to x$ by our previous discussion. But equation (6.1) clearly says that

$$\int_{\widehat{G}} \Phi(x_\alpha)\widehat{f}(\xi)\, d\xi \to \int_{\widehat{G}} \Phi(x)\widehat{f}(\xi)\, d\xi \quad (\forall f \in \mathcal{F}(\mathcal{B}^1(G))). \tag{6.2}$$

Now $\|\Phi(x_\alpha)\|_\infty = 1$ for all $\alpha \in A$ since the topology of $\widehat{\widehat{G}}$ is the weak-* topology of $L^\infty(\widehat{G})$. Moreover lemma 6.2 says that $\mathcal{F}(\mathcal{B}^1(G))$ is dense in $L^1(\widehat{G})$, thus it follows that equation (6.1) is equivalent to equation (6.2) and by transitivity of equivalence equation (6.2) is equivalent to $x_\alpha \to x$. In conclusion, Φ is a homeomorphism.

By (1) $\Phi(G)$ is locally compact in the relative topology. Let $\widehat{\xi_0} \in \overline{\Phi(G)}$, and let U be a neighborhood of $\widehat{\xi_0}$ whose closure is compact. Then $\Phi(G) \cap \overline{U}$ is compact and hence closed in $\widehat{\widehat{G}}$, but $\widehat{\xi_0} \in \overline{\Phi(G) \cap \overline{U}}$ and it follows that $\widehat{\xi_0} \in \Phi(G)$. Thus $\Phi(G)$ is closed.

Suppose $\Phi(G)$ isn't dense in $\widehat{\widehat{G}}$, then there is a function $F \in \mathcal{F}(\widehat{G})$ which is 0 at every point of $\Phi(G)$ but not identically 0 (see theorem 1.6.4 in Rudin [2, p.27]). So there exists $\phi \in L^1(\widehat{G})$ such that

$$F(\widehat{\xi}) = \int_{\widehat{G}} \phi(\xi)\overline{(\xi, \widehat{\xi})}\, d\mu(\xi).$$

Since $F(\alpha(x)) = 0$ for all $x \in G$ we get

$$\int_{\widehat{G}} \phi(\xi)\overline{(\xi, \Phi(x))}\, d\mu(\xi) = \int_{\widehat{G}} \phi(\xi)\overline{(x, \xi)}\, d\mu(\xi) = 0$$

from which one sees that $\phi = 0$ by proposition 5.4. This implies that $F = 0$, and this contradiction completes the proof. ■

[4]A set B with such a property is called cofinal

The following results are essentially corollaries of Pontryagin duality, but their importance render them theorems. They can be seen as completing the picture that we have been drawing so far.

Theorem 6.4 (Second Fourier Inversion Theorem). *If $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$, then*

$$f(x) = \int_{\widehat{G}} (x,\xi)\widehat{f}(\xi)\, d\xi \quad \text{for a.e. } x \in G.$$

Proof. We have $\widehat{f} \in \mathcal{B}(G) \cap L^1(G)$ because

$$\widehat{f}(\xi) = \int_G \overline{(x,\xi)} f(x)\, dx = \int_G (x,\xi) f(x^{-1})\, dx = \int_G (x,\xi)\, d\mu_{\widehat{f}}(x).$$

Thus by the first Fourier inversion theorem

$$f(x^{-1}) = \int_{\widehat{G}} (x^{-1},\xi)\widehat{f}(\xi)\, d\xi \quad \text{for a.e. } x \in G.$$

■

Theorem 6.5 (Uniqueness theorem). *If $\mu, \nu \in M(G)$ such that $\widehat{\mu} = \widehat{\nu}$ then $\mu = \nu$.*

Proof. By proposition 5.4 and by Pontryagin duality we get that μ is completely determined by $\phi_\mu(\xi) = \widehat{\mu}(\xi^{-1})$. ■

A nice fact about Pontryagin duality is that it gives a dual relationship between two topological notions,this is the subcategory duality we mentioned above.

Proposition 6.1. *The dual $\widehat{G}$ of G is discrete if and only if G is compact. And G is discrete if and only if the dual group $\widehat{G}$ is compact.*

Proof. Assume G is discrete, then $L^1(G)$ has a unit δ_1 (the point mass measure at $x = 1$). Hence its spectrum $\widehat{G}$ is compact. Now assume G is compact then the constant function 1 is in $L^1(G)$ so

$$\{f \in L^\infty(G) : \left|\int_G f\, dx\right| > \frac{1}{2}\}$$

is a weak* open set. Now, it follows by proposition 5.2 that for every $\xi \in \widehat{G}$ we have that

$$\int \xi = 1 \quad (if\, \xi = 1), \qquad\qquad \int \xi = 0 \quad (if\ \xi = 0).$$

Thus $\{1\}$ is an open set in $\widehat{G}$, so $\widehat{G}$ is discrete. The other directions follows by Pontryagin duality. ■

7 THE CULMINATION: DECOMPOSITION OF UNITARY REPRESENTATIONS

In this section we will describe a way of decomposing unitary representations of locally compact first countable abelian groups, but first we will need to cover some theory on projection-valued measures.

Projection valued-measures have broader applications in functional analysis, in particular in a generalized version of the spectral theorem to infinite dimensional spaces, there is more on this topic in Folland [1, p.16-26]. Let $\mathcal{H}$ be a Hilbert space and let X be a topological group with Borel σ-algebra $B(X)$, furthermore let $\mathbb{P}(\mathcal{H})$ denote the orthogonal projections onto subspaces of $\mathcal{H}$. Then a map $P : B(X) \to \mathbb{P}(\mathcal{H})$ is called a $\mathcal{H}$-projection valued measure if it satisfies:

1. (P1) $P(X) = I$
2. (P2) If $\{E_i\}_{i\in A} \subset B(X)$ are pairwise disjoint then
$$P(\bigcup_{i\in A} E_i) = \sum_{i\in A} P(E_i) \quad (A \text{ countable})$$
with convergence in the strong operator topology.

The interesting thing about these maps P is that they give rise to families of complex measures $\mu_{u,v}$, we prove this fact.

Proposition 7.1. *Let $P : B(X) \to \mathbb{P}(\mathcal{H})$ be a $\mathcal{H}$-projection valued measure, then $\mu_{u,v}(E) = \langle P(E)u, v\rangle$ is a complex measure for all $u, v \in \mathcal{H}$.*

Proof. Let $u, v \in \mathcal{H}$. Nonnegativity is clearly induced by the inner-product. Countable additivity follows from the calculation:

$$\begin{aligned}\mu_{u,v}(\bigcup_{i\in A} E_i) &= \langle P(\bigcup_{i\in A} E_i)u, v\rangle = \langle \sum_{i\in A} P(E_i)u, v\rangle \\ &= \sum_{i\in A}\langle P(E_i)u, v\rangle = \sum_{i\in A} \mu_{u,v}(E_i).\end{aligned}$$

And since $P(X) = I$ implies that $P(X) = P(X \cup \emptyset \cup \cdots) = I + P(\emptyset) + \cdots$ we get $P(\emptyset) = 0$ (projection onto $\{0\}$) and $\mu_{u,v}(\emptyset) = 0$. ∎

A projection valued measure P is called regular if the corresponding complex measure is regular. The projection valued measures share many properties with normal complex measures, one particularly interesting property is that of having a well defined notion of integration, i.e. one can integrate functions with projection valued measures, we shall define this more precisely now. Let f be a bounded and measurable function on $(X, B(X))$, then we have for all $v \in \mathcal{H}$

$$\left| \int_X f \, d\mu_{v,v} \right| \leq \|f\|_\infty \|\mu_{v,v}\| = \|f\|_\infty \mu_{v,v}(X) = \|f\|_\infty \|v\|^2_{\mathcal{H}}$$

thus by polarization, if $u, v \in \mathcal{H}$ such that $\|u\| = \|v\| = 1$, then

$$\left| \int_X f \, d\mu_{v,v} \right| \leq 4\|f\|_\infty.$$

It follows that

$$\left| \int_X f \, d\mu_{v,v} \right| \leq 4\|f\|_\infty \|u\| \|v\|.$$

Thus the Fréchet-Riesz representation theorem gives that there is some bounded operator T on $\mathcal{H}$ such that

$$\langle Tu, v \rangle = \int_X f \, d\mu_{u,v}.$$

We symbolically represent T by $\int_X f \, dP$. This is how we define integration of $\mathcal{H}$-projection valued measures. One could also define them by approximating f by simple functions since every Borel measurable function f can be approximated as a uniform limit by simple functions, just as one does in ordinary integration theory.

Before we can prove the main theorem of this section we need to define the Gelfand transform and prove the following theorem on projection-valued measures.

Let $x \in G$ and let $\mathcal{A}$ be a Banach algebra. Define the function $\widehat{x} : \sigma(\mathcal{A}) \to \mathbb{C}$ by $\widehat{x}(f) = f(x)$ for every $f \in \sigma(\mathcal{A})$. Then the map $x \to \widehat{x}$ is called the Gelfand map.

Theorem 7.1. *If $\mathcal{A}$ is a commutative nonunital Banach *-algebra and ϕ a nondegenerate *-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, then there is a unique regular projection-valued measure P on the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ such that*

$$\phi(x) = \int_X \widehat{x} \, dP \quad (\forall x \in \mathcal{A})$$

where $\widehat{x}$ is the Gelfand transform.

Proof. Let $\mathcal{B}$ be the norm-closure of $\phi(\mathcal{A})$ in $B(\mathcal{H})$, this is a C^* subalgebra of $B(\mathcal{H})$ since ϕ is a *-representation. There are two cases we need to consider, either $I \in \mathcal{B}$ (the identity operator) or not, we consider the former first. Now, ϕ induces a map $\phi^* : \sigma(\mathcal{B}) \to \sigma(\mathcal{A})$ called the pullback of ϕ, which is defined as $\phi^* h = h(\phi)$ for all $h \in \mathcal{B}$. If $\phi^* H_1 = \phi * h_2$ then h_1, h_2 agree on $\sigma(\mathcal{A})$ and therefore everywhere, thus ϕ^* is injective. Since $\sigma(\mathcal{B})$ is a compact Hausdorff space, ϕ^* is a homeomorphism onto its range, which then is a compact subset of $\sigma(\mathcal{A})$. And then the Spectral theorem (see Folland [1, p.22 (top)] for a reference) associates a unique regular projection valued measure P_0 to $\mathcal{B}$ on $\sigma(\mathcal{B})$ such that

$$T = \int \widehat{T} \, dP_0 \quad (\forall T \in \mathcal{B}).$$

The pullback of P_0 by ϕ^* induces a projection valued measure on $\sigma(\mathcal{A})$, more precisely $P(E) = P_0(\phi^{*-1}(E))$ (it is an easy check to see that P is regular, it follows by the properties of ϕ). And finally the Gelfand transforms on $\mathcal{A}$ and $\mathcal{B}$ are related by $\widehat{\phi(x)}(h) = \widehat{x}(\phi^* h)$, and therefore

$$\phi(x) = \int \widehat{x}(\phi^* h)\, dP_0(h) = \int \widehat{x}\, dP \quad (\forall x \in \mathcal{A}).$$

For the second case when $I \notin \mathcal{B}$, the proof is next to identical, with the only difference being that one considers the unital augmentation $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$ where $\tilde{\phi}(x, z) = \phi(x) + zI$ and $\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}I$. For a more detailed discussion on this see Folland [1, p.27 (bottom)]. ∎

Theorem 7.2. *Let $\mathcal{H}$ be some Hilbert space, G a locally compact first countable abelian group, moreover let $\rho : G \to U(\mathcal{H}_\rho)$ be a unitary representation G, then there is a unique regular $\mathcal{H}_\rho$-projection-valued measure P on $\widehat{G}$ such that*

$$\rho(x) = \int_{\widehat{G}} (x, \xi)\, dP(\xi) \quad (x \in G)$$
$$\rho(f) = \int_{\widehat{G}} \xi(f)\, dP(\xi) \quad (f \in L^1(G))$$

Proof. Since $\widehat{G}$ can be identified with the spectrum of $L^1(G)$ we get by theorem 7.1 (with $\mathcal{A} = L^1(G)$) that there is a unique $\mathcal{H}_\rho$-projection valued measure such that

$$\rho(f) = \int_{\widehat{G}} \xi(f)\, dP(\xi) \quad (f \in L^1(G)).$$

By the proof of theorem 4.6 $\rho(x)$ is the strong limit of $\rho(L_x \psi_\alpha)$ for some approximate identity $\{\psi_\alpha\}_{\alpha \in A}$ (taken as a sequence since G is first countable), thus

$$\rho(L_x \psi_\alpha) = \int_{\widehat{G}} \xi(L_x \psi_\alpha)\, dP(\xi) = \int_{\widehat{G}} (x, \xi) \xi(\psi_\alpha)\, dP(\xi)$$

which converges to $\int_{\widehat{G}} (x, \xi)\, dP(\xi)$ by the dominated convergence theorem since $|\xi(\psi_\alpha)| \leq 1$ and $\xi(\psi_\alpha) \to 1$ for every $\xi \in \widehat{G}$, thus we're done.

∎

As a finale to our discussion on abstract harmonic analysis we decompose the left regular representation on $L^2(G)$. Let $\rho(x)f(y) = f(x^{-1}y)$, then

$$\mathcal{F}^{-1}[\rho(x)f](\xi) = \int_G (y, \xi) f(x^{-1}y),\, dy = \int_G (xy, \xi) f(y),\, dy = (x, \xi)\mathcal{F}^{-1}[f](\xi).$$

Thus theorem 7.2 gives that $\mathcal{F}^{-1}[P(E)f](\xi) = \chi_E \mathcal{F}^{-1}[f]$, which implies that

$$P(E) = \mathcal{F}(\chi_E x)\mathcal{F}^{-1}.$$

Moreover,

$$\rho(f) = \int_{\widehat{G}} \xi(f) d[\mathcal{F}(\chi_E x)\mathcal{F}^{-1}].$$

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