AN INVERSION OF THE LAMBERT TRANSFORM

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Introduction. A series introduced by J. L. Lambert [4; 507]* for use in his studies of the theory of numbers is the following,

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{1-x^n}. \tag{1}$$

It is natural to introduce an integral analogue of this series just as the Laplace integral may be presented as a continuous analogue of the power series. If in (1) we replace n by the continuous variable t, x by e^{-x} and Σ by $\int$ we obtain

$$F(x) = \int_0^{\infty} \frac{a(t)}{e^{xt}-1}\, dt. \tag{2}$$

If we wish to consider (1) and (2) together we may use a Stieltjes integral (compare A. Wintner [12; 75])

$$F(x) = \int_0^{\infty} \frac{d\alpha(t)}{e^{xt}-1}. \tag{3}$$

We shall refer to (2) or (3) as a Lambert transform. Since the denominator of the integrand vanishes at $t=0$, it is convenient to replace $a(t)$ by $t\,a(t)$ and $d\alpha(t)$ by $t\,d\alpha(t)$, thus introducing the continuous "kernal" $t/(e^{kt}-1)$. But for the heuristic description of the present section the forms (2) and (3) will be retained.

In §2 we discuss the convergence properties of the integral (3) and hence also of (1) and (2). They turn out to be just what one would expect from the known behavior of Lambert series (see, for example, [3]).

The remainder of the paper is devoted to the inversion of the transform (2). That of (3) could be derived therefrom in a familiar way [9]. Let us point out at once that a Lambert transform is a special case of a convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt. \tag{4}$$

For, if we replace x by e^{-x} and t by e^{t}, equation (2) takes the form (4) with

*Numbers enclosed in brackets refer to the bibliography at the end of the present paper; numbers following a semicolon refer to the page of the article cited.

$$(5) \qquad G(x) = (e^{e^{-x}} - 1)^{-1}$$

$$f(x) = F(e^{-x}), \qquad \varphi(x) = e^{x} a(e^{x}).$$

The present author with I. I. Hirschman, Jr. [6,7,8,10,11] has discussed the inversion of (4) for a large class of kernels $G(x)$. However, the Lambert kernel (5) is not among those studied and must be considered *ab initio*. But the same operational methods used in [6] or [10] may be employed here to conjecture an inversion formula for (3).

The bilateral Laplace transform of the Lambert kernel (5) is well known [5; 45],

$$\int_{-\infty}^{\infty} e^{-st} G(t)\, dt = \int_{0}^{\infty} \frac{t^{s-1}}{e^{t}-1} dt = \zeta(s)\Gamma(s).$$

Let D stand for differentiation with respect to x, and make the usual interpretation of $e^{aD}\varphi(x)$ as $\varphi(x+a)$. If s is replaced by D in the above equation it becomes natural to interpret $\zeta(D)\Gamma(D)\varphi(x)$ as

$$\zeta(D)\Gamma(D)\,\varphi(x) = \int_{-\infty}^{\infty} \varphi(x-t)\, G(t)\, dt.$$

By making an obvious change of variable in the integral (4) this is seen to be $f(x)$, so that we should expect the desired inversion to have the symbolic form

$$(6) \qquad \varphi(x) = \frac{1}{\Gamma(D)}\,\frac{1}{\zeta(D)}\, f(x).$$

But it is known [5; 315] that

$$(7) \qquad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}} = \sum_{n=1}^{\infty} \mu(n) e^{-s\log n},$$

where $\mu(n)$ is the Möbius function. That is,

$$(8) \qquad \frac{1}{\zeta(D)}\, f(x) = \sum_{n=1}^{\infty} \mu(n)\, f(x-\log n).$$

On the other hand it has been pointed out repeatedly [11] that $1/\Gamma(D)$ is the symbolic operator for the inversion of the Laplace transform (after an exponential change of variable). Hence if (6) is to be the inversion desired, we should expect the right-hand side of (8) to be effectively a Laplace transform. We shall show, in fact, that

$$\sum_{n=1}^{\infty} \mu(n) F(nx) = \int_0^{\infty} e^{-xt} a(t)\, dt,$$

after which any of the known inversions of the Laplace transform will be effective in obtaining $a(t)$. The details of this operational development are given in §4.

The above considerations lead us to the following inversion of the transform (2),

$$a(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} \sum_{n=1}^{\infty} \mu(n) n^k F^{(k)}\left(\frac{kn}{t}\right). \tag{9}$$

We illustrate its use by the simple example $a(t) = t$. From (2) it is clear that $F(x)$ is $\zeta(2)/x^2$. But easy computations reduce the right-hand side of (9) to

$$\zeta(2) \lim_{k\to\infty} \frac{k+1}{k} \sum_{n=1}^{\infty} \frac{\mu(n)\, t}{n^2},$$

and by (7) this is seen to be t as predicted.

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1. *Convergence*. We discuss first convergence of

$$F(x) = \int_0^{\infty} \frac{t\, d\alpha(t)}{e^{xt}-1} = \frac{1}{x} \int_0^{\infty} H(xt)\, d\alpha(t) \tag{1.1}$$

$$H(t) = \frac{t}{e^t - 1}, \quad 0 < t < \infty; \qquad H(0) = 1.$$

We assume that $\alpha(t)$ is a function of bounded variation in $0 \leqq t \leqq R$ for every positive R and that $\alpha(0) = 0$. In particular, if

$$\alpha(t) = \int_0^t a(u)\, du,$$

where $a(u)$ is a function which is integrable in the sense of Lebesque, then

$$F(x) = \frac{1}{x} \int_0^{\infty} H(xt)\, a(t)\, dt. \tag{1.2}$$

We shall refer to (1.2) as the *Lambert-Lebesque* transform to distinguish it from the *Lambert-Stieltjes* transform (1.1).

Since the kernel $H(xt)/x$, for positive x, differs very little from te^{-xt} when t is large it is natural to expect that the convergence behavior of (1.1) will be much the same as that of the Laplace integral

(1.3) $$\int_0^\infty e^{-xt}\, t\, d\alpha(t).$$

We shall show, in fact, that (1.1) and (1.3) converge for the same values of x if the abscissa of convergence σ_c for (1.3) is positive. For negative x, $H(xt)/x \sim -t$, and the similarity ends. Indeed when (1.3) converges at $x=0$, (1.1) converges for all $x\neq 0$. We prove the latter fact first.

Theorem 1.1. *If* (1.3) *converges at* $x=0$, *then* (1.1) *converges for all* $x\neq 0$.

Set

$$\beta(t) = \int_1^t u\, d\alpha(u), \qquad \beta(1) = 0,$$

so that $\beta(+\infty)$ exists by hypothesis. If $x\neq 0$, we have [9; 12]

$$\int_1^R \frac{t\, d\alpha(t)}{e^{xt}-1} = \int_1^R \frac{d\beta(t)}{e^{xt}-1} = \frac{\beta(R)}{e^{xR}-1} + x\int_1^R \frac{e^{xt}\beta(t)}{[e^{xt}-1]^2}\, dt \qquad R>1.$$

As $R\to+\infty$ the first term on the right tends to $-\beta(\infty)$ or to 0 according as $x<0$ or $x>0$. The integral on the right also tends to a limit since the integrand is $O(e^{-|x|t})$ as $t\to+\infty$. Hence we have established that (1.1) converges for all $x\neq 0$.

Theorem 1.2. *If* (1.3) *diverges at* $x=0$, *then the integrals* (1.1) *and* (1.3) *converge and diverge for the same values of* $x\neq 0$.

Let us first suppose that (1.3) converges at a point x_0 (which must be >0 by hypothesis). That is, if we set

$$\beta(t) = \int_1^t e^{-x_0 u}\, u\, d\alpha(u), \qquad \beta(1) = 0,$$

then $\beta(+\infty)$ exists, and

$$\int_1^R \frac{t\, d\alpha(t)}{e^{x_0 t}-1} = \int_1^R \frac{d\beta(t)}{1-e^{-x_0 t}} = \frac{\beta(R)}{1-e^{-x_0 R}} + x_0\int_1^R \frac{e^{-x_0 t}\beta(t)}{[1-e^{-x_0 t}]^2}\, dt \qquad R>1.$$

Clearly the right-hand side tends to a limit as $R\to+\infty$, so that (1.1) converges also at x_0.

Next suppose that (1.1) converges at a point x_0 (>0 or <0), and set

$$\gamma(t) = \int_1^t \frac{u\, d\alpha(u)}{e^{x_0 u}-1}, \qquad \gamma(1) = 0.$$

Then $\gamma(+\infty)$ exists, and

$$\int_1^R e^{-x_0 t} t\,d\alpha(t) = \int_1^R (1-e^{-x_0 t})d\gamma(t) = \gamma(R)(1-e^{-x_0 R}) - x_0\int_1^R e^{-x_0 t}\gamma(t)dt.$$

If $x_0 > 0$, the right-hand side approaches a limit as $R \to +\infty$. Hence (1.3) converges at x_0. If $x_0 < 0$, we have

$$\int_1^R t\,d\alpha(t) = \int_1^R (e^{x_0 t} - 1)d\gamma(t) = \gamma(R)(e^{x_0 R} - 1) - x_0\int_1^R e^{x_0 t}\gamma(t)dt. \qquad R > 1$$

The right-hand side tends to a limit as $R \to +\infty$. But this conclusion is untenable in the presence of the hypothesis that (1.3) diverges at $x = 0$. Hence (1.1) must diverge for negative x, and (1.1) and (1.3) converge and diverge together for positive x.

2. *Relation to the Laplace transform.* In subsequent work we deal only with the Lambert-Lebesque transform

$$F(x) = \frac{1}{x}\int_0^\infty H(xt)\,a(t)\,dt \tag{2.1}$$

where $a(t)$ is Lebesque integrable on $(0,\infty)$,

$$\int_0^\infty |a(t)|\,dt < \infty. \tag{2.2}$$

Of course assumption (2.2) permits $a(t)$ to become infinite as $t \to 0+$ but certainly not so strongly as $1/t$. For simplification of subsequent computations we make the following explicit assumption about the behavior of $a(t)$ near $t = 0$.

$$\lim_{t\to 0+} a(t)\,t^{1-\delta} = 0 \tag{2.3}$$

for some positive number δ. This condition is equivalent to

$$\overline{\lim_{t\to 0+}}\ \frac{\log|a(t)|}{\log(1/t)} < 1. \tag{2.4}$$

We consider also the corresponding Laplace-Lebesque transform

$$f(x) = \int_0^\infty e^{-xt}\,t\,a(t)\,dt, \tag{2.5}$$

and study first how $F(x)$ may be expressed in terms of $f(x)$.

Theorem 2.1. *If $F(x)$ and $f(x)$ are defined by* (2.1), (2.2), (2.3), (2.5), *then*

$$F(x) = \sum_{k=1}^{\infty} f(kx), \tag{2.6}$$

the series converging absolutely for $x > 0$.

Observe first that

$$H(t) = \frac{t}{e^t - 1} = t \sum_{k=1}^{\infty} e^{-kt} \qquad 0 < t < \infty$$

and that

$$0 < H(t) \leqq 1 \qquad 0 \leqq t < \infty. \tag{2.7}$$

If the series

$$\frac{1}{x} H(xt)\, a(t) = \sum_{k=1}^{\infty} t\, a(t) e^{-kxt}$$

may be integrated term by term with respect to t from 0 to ∞ we shall have equation (2.6) at once. This will be permissible if

$$\int_0^{\infty} t\, |a(t)| \sum_{k=1}^{\infty} e^{-kxt}\, dt < \infty.$$

But for $x > 0$ this integral is

$$\int_0^{\infty} \frac{t\,|a(t)|}{e^{xt} - 1}\, dt = \frac{1}{x} \int_0^{\infty} H(xt)\, |a(t)|\, dt,$$

and by (2.7)

$$\int_0^{\infty} H(xt)\, |a(t)|\, dt < \int_0^{\infty} |a(t)|\, dt.$$

By our hypothesis (2.2) the result is established.

We prove that every Lambert transform (2.1), (2.2), (2.3) is also a Laplace transform.

Theorem 2.2. *If $F(x)$ is defined by* (2.1), (2.2), (2.3), *then*

$$F(x) = \int_0^{\infty} e^{-xt}\, t\, b(t)\, dt \qquad 0 < x < \infty,$$

where

$$b(t) = \sum_{k=1}^{\infty} \frac{1}{k^2}\, a\left(\frac{t}{k}\right),$$

the series converging absolutely for $0 < t < \infty$.

From the previous theorem we have

$$F(x) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kxt}\, t\, a(t)\, dt = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-xt} \frac{t}{k^2}\, a\left(\frac{t}{k}\right) dt.$$

Our result will be established if we may interchange integral and summation signs, and this will be permissible if

$$\sum_{k=1}^{\infty} \int_0^{\infty} e^{-xt} \frac{t}{k^2} \left|a\left(\frac{t}{k}\right)\right| dt < \infty.$$

This series is equal to

$$\sum_{k=1}^{\infty} \int_0^{\infty} e^{-kxt}\, t\, |a(t)|\, dt = \int_0^{\infty} \frac{t\,|a(t)|}{e^{xt}-1}\, dt$$

provided either side of the latter equation is finite. But we established this fact for the right-hand side in the proof of the previous theorem.

Conversely, the Laplace integral (2.5), (2.2), (2.3) can also be expressed in terms of the Lambert integral (2.1). To show this we need to introduce the Möbius function $\mu(n)$ mentioned in the introduction. It is defined to be 1 when $n = 1$, 0 when n is divisible by a square, and $(-1)^k$ when n is the product of k distinct primes. An alternative definition is

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

If we multiply these two series together we verify easily the familiar relation

$$\sum_{d/n} \mu(d) = 0 \qquad n = 2, 3, \cdots, \tag{2.8}$$

where the summation runs over all the divisors d of n (including 1 and n).

We shall need the following preliminary result.

Lemma 2.3. *If the double series*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mu(n)\, f(knx) \tag{2.9}$$

converges absolutely, it has the falue $f(x)$.

For then it may be summed in any manner. If we group all terms involving $f(mx)$, the coefficient will be $\sum_{d/m} \mu(d)$, and the double series

$$\sum_{m=1}^{\infty} f(mx) \sum_{d/m} \mu(d).$$

By (2.8) this is equal to $f(x)$.

Theorem 2.4. *If* $f(x)$ *is defined by* (2.5), (2.2), (2.3), *then*

$$f(x) = \int_0^{\infty} \frac{t\,c(t)}{e^{xt}-1}\,dt \qquad 0 < x < \infty,$$

where

$$c(t) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\, a\left(\frac{t}{n}\right),$$

the series converging absolutely for $0 < t < \infty$.

By the definition of $f(x)$ we have for any positive integers n and k

$$f(knx) = \int_0^{\infty} e^{-kxt} \frac{t}{n^2}\, a\left(\frac{t}{n}\right) dt \qquad 0 < x < \infty.$$

Since $|\mu(n)| \leqq 1$, the absolute convergence of the double series (2.9) will be established for the present function (2.5) if

$$(2.10) \qquad \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-knxt}\, t\,|a(t)|\,dt < \infty.$$

By hypothesis (2.3) there exist constants M, g and δ such that

$$|t\,a(t)| \leqq M\,t^{\delta} \qquad 0 \leqq t \leqq g.$$

Hence

$$(2.11) \quad \int_0^{\infty} e^{-xt}\, t\,|a(t)|\,dt \leqq M \int_0^{g} e^{-xt}\, t^{\delta}\,dt + \int_g^{\infty} e^{-xt}\, t\,|a(t)|\,dt$$

$$\leqq M \int_0^{\infty} e^{-xt}\, t^{\delta}\,dt + g\, e^{-gx} \int_g^{\infty} |a(t)|\,dt$$

when $x > 1/g$. That is, the integral on the left of (2.11) is $O(x^{-\delta-1})$ as $x \to +\infty$. Consequently the series (2.10) is dominated by

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{N}{(knx)^{\delta+1}} = \frac{N\,\zeta^2(1+\delta)}{x^{\delta+1}},$$

where N is a suitable constant. By Lemma 2.3 the sum of the series (2.9) is $f(x)$, and by (2.10) the summation and integral signs may be interchanged in the right-hand side of

$$\sum_{n=1}^{\infty} \mu(n)\, f(knx) = \sum_{n=1}^{\infty} \mu(n) \int_0^{\infty} e^{-kxt} \frac{t}{n^2}\, a\left(\frac{t}{n}\right) dt$$

to obtain

$$\int_0^{\infty} e^{-kxt}\, t\, c(t)\, dt.$$

Thus

$$f(x) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kxt}\, t\, c(t)\, dt = \int_0^{\infty} \frac{t\, c(t)}{e^{xt}-1}\, dt \qquad 0 < x < \infty,$$

the final interchange of integral and summation signs being also justified by (2.10). This completes the proof of the theorem.

3. *Inversion formulas.* We observe first that the Möbius function $\mu(n)$ enables us to express the Laplace integral (2.5) in terms of the Lambert integral (2.1). We prove a companion theorem to Theorem 2.1.

THEOREM 3.1. *If $F(x)$ and $f(x)$ are defined by* (2.1), (2.2), (2.3), (2.5), *then*

$$f(x) = \sum_{n=1}^{\infty} \mu(n)\, F(nx) \qquad 0 < x < \infty, \tag{3.1}$$

the series converging absolutely.

This is the inversion of equation (2.6). It follows immediately from Theorem 2.1 and Lemma 2.3. The absolute convergence of the double series (2.9) is established by use of the relation (2.10).

THEOREM 3.2. *If $F(x)$ is defined by* (2.1), (2.2), (2.3), *and if $a(t)$ is continuous at a point $t_0 > 0$, then*

$$t_0\, a(t_0) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0}\right)^{k+1} \sum_{n=1}^{\infty} \mu(n)\, n^k F^{(k)}\left(\frac{nk}{t_0}\right). \tag{3.2}$$

By formal differentiation of series (2.6) we obtain for every positive integer k

$$F^{(k)}(x) = \sum_{p=1}^{\infty} f^{(k)}(px)\, p^k \qquad 0 < x < \infty. \tag{3.3}$$

The formal step will be justified if the series (3.3) converges uniformly in $c \leqq x < \infty$ for an arbitrary positive number c. Proceeding as in the proof of (2.11) or by use of [9; 182], we see easily that there exists a constant M such that

$$|f^{(k)}(x)| \leqq \frac{M}{x^{k+\delta+1}} \qquad c \leqq x < \infty. \tag{3.4}$$

Hence $$\sum_{p=1}^{\infty} f^{(k)}(px)p^k \ll \sum_{p=1}^{\infty} \frac{Mp^k}{(px)^{k+\delta+1}} \ll \frac{M}{c^{k+\delta+1}} \sum_{p=1}^{\infty} \frac{1}{p^{\delta+1}}.$$

This is sufficient to prove the desired uniform convergence.

Now

$$(3.5) \quad \sum_{n=1}^{\infty} \mu(n)n^k F^{(k)}(nx) = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \mu(n)(np)^k f^{(k)}(npx) = f^{(k)}(x)$$

by Lemma 2.3 provided the double series (3.5) converges absolutely. But in view of (3.4) it is dominated by the convergent double series

$$\frac{1}{x^{k+\delta+1}} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{M}{(np)^{\delta+1}}.$$

Hence (3.5) is established, and the right-hand side of (3.2) is seen to be

$$\lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0}\right)^{k+1} f^{(k)}\left(\frac{k}{t_0}\right).$$

But this is the familiar [9; 288] inversion formula for the Laplace transform (2.5) and consequently yields $t_0\, a(t_0)$, since $a(t_0)$ is assumed continuous at t_0.

Theorem 3.3. *Under the conditions of Theorem* 3.2

$$(3.6) \quad t_0 a(t_0) = \sum_{n=1}^{\infty} \mu(n) \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0}\right)^{k+1} n^k F^{(k)}\left(\frac{nk}{t_0}\right).$$

Here we have interchanged the symbols for summation and limit in formula (3.2). To justify this we appeal to Theorem 2.2, which states that $F(x)$ may be regarded as a Laplace transform of the function $b(t)$ there defined. By the inversion of such a transform cited above we see that (3.6) becomes

$$(3.7) \quad t_0\, a(t_0) = \sum_{n=1}^{\infty} \mu(n) \frac{t_0}{n^2}\, b\left(\frac{t_0}{n}\right).$$

By the definition of $b(t_0)$ given in Theorem 2.2, equation (3.7) can now be established by another appeal to Lemma 2.3, the double series in question now being

$$\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\mu(n)}{(np)^2} a\left(\frac{t}{np}\right).$$

The function $f(x)$ of that lemma is here $x^{-2}a(1/x)$, and the absolute convergence required follows easily from the assumption (2.3).

4. *Operational considerations.* Let us show in detail how formulas (3.2) and (3.6) may be interpreted as $\frac{1}{\Gamma(D)}\frac{1}{\zeta(D)}F(e^{-x})$ and $\frac{1}{\zeta(D)}\frac{1}{\Gamma(D)}F(e^{-x})$, respectively. As we indicated in the introduction the Lambert transform (2.1) may be written as the convolution

$$F(e^{-x}) = \int_{-\infty}^{\infty} G(x-t)\, e^{2t}\, a(e^t)\, dt,$$

where $G(x)$ is the function (5). Using the definition (8), we have

$$\frac{1}{\zeta(D)} F(e^{-x}) = \sum_{n=1}^{\infty} \mu(n)\, F(ne^{-x}),$$

or by Theorem 3.1,

$$(4.1) \qquad \frac{1}{\zeta(D)} F(e^{-x}) = \int_{-\infty}^{\infty} K(x-t)\, e^{2t}\, a(e^t)\, dt$$

$$K(t) = e^{-e^{-t}}.$$

Since the right-hand side is a Laplace transform in which a change of variable has been made, we expect its inversion to be effected by the operator $1/\Gamma(D)$. We recall the formal details.

A familiar definition of $\Gamma(x)$ is

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\, e^{-t}\, dt = \int_{-\infty}^{\infty} e^{-xt}\, K(t)\, dt.$$

Using the interpretation of e^{-Dt} as a translation through distance $-t$, we see that

$$\Gamma(D)\, e^{2x}\, a(e^x) = \int_{-\infty}^{\infty} K(t)\, e^{2x-2t}\, a(e^{x-t})\, dt,$$

or by (4.1),

$$(4.2) \qquad e^{2x}\, a(e^x) = \frac{1}{\Gamma(D)}\frac{1}{\zeta(D)} F(e^{-x}).$$

Of course the actual realization of the operation $1/\Gamma(D)$ is best accomplished by use of one of the familiar inversions of the Laplace transform. It is thus that we have derived formula (3.2).

On the other hand, we saw in Theorem 2.2 that $F(x)$ is the Laplace transform of $tb(t)$. Hence

$$\frac{1}{\Gamma(D)} F(e^{-x}) = e^{2x}\, b(e^x),$$

and

$$\frac{1}{\zeta(D)}\frac{1}{\Gamma(D)} F(e^{-x}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\, e^{2x}\, b\left(\frac{e^x}{n}\right).$$

By Lemma 2.3 the right-hand side of this equation is $e^{2x}\, a(e^x)$.

We have thus permuted the two symbolic operators in (4.2). Since our interpretations of $1/\Gamma(D)$ and $1/\zeta(D)$ involve, respectively, the symbols *lim* and Σ, it is clear why these have been interchanged in (3.2) and (3.6).

In conclusion let us point out why our previous inversion [6] of the convolution (4) is not applicable here. In that general theory we were concerned with an inversion operator $E(D)$ in which the function $E(s)$ was entire. The corresponding function for the Lambert transform is the meromorphic function $[\zeta(s)\,\Gamma(s)]^{-1}$, which has poles at the complex zeros of $\zeta(s)$.

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