On the existence and closure of sets of characteristic functions.

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The following lines give a brief and simple proof of the existence of characteristic functions for the symmetric kernel[1]), and then point out that the complete sets of such functions which arise from the usual self-adjoint differential equations with self adjoint boundary conditions are closed, not only with respect to continuous functions, but also with respect to summable functions.

The existence proof is based on a theorem due to Ascoli[2]), to the effect that if an infinite sequence of functions $[f_i(x)]$, or $f_1(x)$, $f_2(x)$, $f_3(x), \ldots$ is bounded and "equicontinuous", it contains an infinite subsequence which approaches uniformly a continuous limit function. By "equicontinuous", we mean characterized by the existence of a positive number δ, corresponding to each positive ε, such that for all n, $|f_n(x_2) - f_n(x_1)| \leqq \varepsilon$ whenever $|x_2 - x_1| \leqq \delta$. The bound also is to be independent of n.

Let $K(x, y)$ be symmetric in the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, and let $\int_a^b K^2(x, y)\,dy$ exist and be somewhere positive, but nowhere greater than some constant B^2. Further, we assume that to any positive ε there corresponds a positive δ, independent of $\psi(y)$, such that $\int_a^b \psi(y) K(x, y)\,dy$ varies by less than ε when x assumes any two values differing by less than δ, $\psi(y)$ being any summable function with sum-

[1]) For ~~the terminology and for his elegant~~ proof of the theorem, see E. Schmidt: Entwickelung willkürlicher Funktionen nach Systemen vorgeschriebener, Diss. Göttingen 1905, p. 18.

[2]) Memorie della R. Acc. dei Lincei 18 (1883), pp. 581—586.

mable square and norm unity; and finally we suppose $K(x, y)$ such that the order of integrations in $\int_a^b\int_a^b \varphi(x) K(x, y) \psi(y) dy\, dx$ may be inverted, $\varphi(x)$ being any continuous function. These hypotheses may be verified in the case of any continuous kernel, or also for kernels with certain integrable singularities distributed along a finite number of regular curves nowhere parallel to the axes.

Let $f_0(x)$ be one of the functions $\psi(x)$ above, such that $f_1(x) = \int_a^b f_0(y) K(x, y) dy$ is not identically zero — such functions evidently exist. We form then the infinite sequence:

$$(1) \qquad \begin{aligned} \overline{f_i}(x) &= \frac{f_i(x)}{\sqrt{\int_a^b f_i^2(x)\, dx}}, \\ f_{i+1}(x) &= \int_a^b \bar{f_i}(y) K(x, y) dy. \end{aligned}$$

The hypotheses imply that the sequence so defined satisfies the conditions of Ascoli's theorem, so that there exists a sub-sequence $[f_j(x)]$, $j = n_1, n_2, n_3, \ldots$, converging uniformly to a limit, $f(x)$. It follows that the sequences $[f_{j+1}(x)]$ and $[f_{j+2}(x)]$ also converge uniformly to limits which we denote by $g(x)$ and $h(x)$ respectively. We shall show that $h(x) = f(x)$.

From (1) follows

$$\int_a^b f_{i+1}(x) \bar{f}_{i-1}(x) dx = \int_a^b f_i(x) \bar{f}_i(x) dx \qquad (i \geqq 1),$$

or, denoting by n_i the norm of $f_i(x)$, and employing Schwarz' inequality,

$$(2) \qquad n_{i-1} n_i = \int_a^b f_{i-1}(x) f_{i+1}(x) dx \leqq n_{i-1} n_{i+1}.$$

Hence the numbers n_i, which never decrease nor exceed B, approach a limit n, while the integral in (2) approaches n^2. Thus $\int_a^b [f_{j+2}(x) - f_j(x)]^2 dx$ approaches 0, whence the continuous function $h(x) - f(x) = 0$.

Again, from (1) follow the equations satisfied by $f(x)$ and $g(x)$:

$$g(x) = \frac{1}{n} \int_a^b f(y) K(x, y) dy$$

$$f(x) = \frac{1}{n} \int_a^b g(y) K(x, y) dy,$$

from which it appears that $f(x)+g(x)$ is a characteristic function corresponding to the characteristic number $\frac{1}{n}$, and $f(x)-g(x)$ is a characteristic function corresponding to $-\frac{1}{n}$. Both functions cannot be identically zero, since $f(x)$ and $g(x)$ have the positive norms n. *Thus the existence of at least one characteristic function is established.*

As to further characteristic functions, we may state: *there exists always an additional characteristic function in case there is a summable function with summable square which is orthogonal to each of the characteristic functions already determined, but not orthogonal to $K(x, y)$.* For if such a function is taken as $f_0(x)$, and the sequence (1) established, it will be seen that $f_i(x)$ is orthogonal to all characteristic functions to which $f_0(x)$ is. It follows that all summable functions with summable squares which are orthogonal to *all* the characteristic functions of $K(x, y)$ are orthogonal to $K(x, y)$. The converse is obvious.

It should be noted that the summability of the squares of the functions $f_0(x)$ need not be postulated if $K(x, y)$ is bounded, and such that the integral with respect to y of its product by a summable function of y is a summable function of x. For since a summable function is absolutely summable, $|f_1(x)| \leqq \int_a^b |f_0(y)| dy \cdot \operatorname{Max} |K(x, y)|$. Thus $f_1(x)$ is bounded and hence has a summable square, and the properties of the members of the sequence (1) obtain at most one step later. The generalizations indicated by Schmidt (l. c.) as attained by the use of the iterated kernel manifestly have their analogues in the present case.

An infinite set of functions $[\varphi_i(x)]$ is said to be closed, or as we shall express it, closed with respect to continuous functions, provided there exists no continuous function $f(x)$ other than zero, such that $\int_a^b \varphi_i(x) f(x) dx = 0$ for all i. Closure of such sets have been studied by a number of investigators, and established in the case of the Sturm-Liouville differential equations and boundary conditions by Stekloff[3]), while Hilbert[4]) shows this closure to be a simple corrolary of his theory of integral equations in a large category of cases. As the integrals involved in the definition of closure admit a meaning in the case of a much broader class of func-

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[3]) Annales de la Faculté des Sciences de Toulouse (2) **3** (1901). Cf. also Westfall: Bull. Amer. Math. Soc. **15** (1908), pp. 76—78; Stèckloff: Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg **30** (1911), Nr. 4.

[4]) Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Zweite Mitteilung, Nachrichten d. K. Ges. d. Wiss. zu Göttingen 1904, Heft 3, p. 222.

tions than continuous ones, it is of interest to inquire whether such function sets are not also closed with respect to summable functions not null-functions. That such is indeed the fact, I have shown[5]) in the case of ordinary linear homogeneous self-adjoint differential equations of second order with self-adjoint boundary conditions. The method used does not involve integral equations. It consists, however, essentially, in establishing the equivalence of a non-homogeneous differential equation and boundary conditions, the differential equation being satisfied at all points of the interval (a, b) with exception of a set of measure 0, and an integral equation of the first kind (see Hilbert, l. c.):

$$(3) \qquad \text{and} \qquad \begin{aligned} L(w) &= \frac{d}{dx}(kw') + (\mu g - l)w = fg \\ w(x) &= \int_a^b G(x, y) f(y) g(y)\, dy. \end{aligned}$$

Here, it appears that if $f(x)$ is summable, $w(x)$ is continuous, and orthogonal to all characteristic functions to which $f(x)$ is. I should like to point out, in closing, that the methods and results admit of extension to a wide class of sets of characteristic functions of differential equations, in the direction a). of higher orders, b). of more independent variables, and with suitable understandings, c). of differential equations with singularities on the boundaries. It appears that the set of characteristic functions is closed with respect to all functions $f(x)$ which yield, in the integral equation (3), a function $w(x)$ which is continuous, the situation being similar in the case of more independent variables.

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[5]) Note on closure of orthogonal sets, Bull. Amer. Math. Soc. 27 (1921), pp. 165—169.

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