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Universal scaling of the tail of the density of eigenvalues in random matrix models

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Large random matrices have eigenvalue density distributions limited to a finite support. Near the endpoint of the support, when the size N of the matrices is large, one can study a scaling region of size $N^{-\mu}$ in which the cross-over from a non-zero density to a vanishing density takes place. This cross-over is shown to be universal (for random hermitian matrices with a unitary-invariant probability distribution), in the sense that it depends only on the order of multicriticality of the problem. For a multicritical point of odd order k, the large-N density vanishes as $|\lambda-\lambda_c|^{k-1/2}$ near λ_c. The cross-over function of the scaling variables $(\lambda_c-\lambda)N^{\mu}$ and $(g_c-g)N^{\nu}$ (where g is the coupling constant characterizing the potential or equivalently the cosmological constant of 2D quantum gravity) is related to the resolvent of the Schrödinger operator in which the potential is the scaling function which satisfies the string equation. The exponents are found to be $\mu=2/(2k+1)$ and $\nu=2k/(2k+1)$.

1. Introduction

Random matrix theory has been extensively studied recently as a powerful way of generating random surfaces [1–4]. Indeed an integral over an $N\times N$ hermitian matrix M of the type

$$Z=\int \mathrm{D}^{N^2}M\exp\left(-\frac{N}{g}\operatorname{tr} V(M)\right), \tag{1}$$

with $V(M)$ a polynomial potential, expanded in a double power series in $1/N^2$ and the coupling constant g, generates a set of Feynman diagrams which are dual to a discrete triangulation of a random manifold. The theory was originally solved in the large-N (planar) limit [5] and more recently in a "double scaling limit" [6–8] in which g approaches a critical value g_c and N tends to infinity with fixed scaling variable

$$x=(g_c-g)N^{\nu}. \tag{2}$$

The exponent ν and the singular part of the free energy $\log Z$ are universal functions of another scaling variable x, meaning that they are independent of the specific potential V within given universality classes.

Without any special tuning of the potential V the exponent ν is equal to $\frac{4}{5}$. By an appropriate adjustment of the coefficients which enter V, however, one can tune to multicritical points of various orders [9]. At the kth multicritical point

$$\nu=\frac{2k}{2k+1}, \tag{3}$$

and the results are universal in the sense that, as long as multicriticality of order k is reached, the theory is independent of the particular potential chosen. In the following we shall study in some detail the average density of eigenvalues

$$\rho(\lambda)=\frac{1}{N}\langle \operatorname{Tr}\delta(\lambda-M)\rangle, \tag{4}$$

where the expectation values refer to the probability weight

$$P(M)=\frac{1}{Z}\exp\left(-\frac{N}{g}\operatorname{Tr} V(M)\right). \tag{5}$$

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In the large-N limit $\rho(\lambda)$ has finite support, consisting of one or several intervals of the real line. The simplest case, and the most familiar, is that of a gaussian weight, which gives for $\rho(\lambda)$ Wigner's semi-circle distribution [10]. We shall assume here that the support of $\rho(\lambda)$ in the large-N limit is on the single interval (a, b). It will be clear later that for a multiple-well potential V, which may lead to support on several disconnected intervals for $\rho(\lambda)$, the considerations developed below apply mutatis mutandis.

We want to study how the large-N limit of $\rho(\lambda)$ is actually approached. Within the interval (a, b), at a finite distance from the two endpoints, it is well known that one can systematically compute a series of correction terms in powers of $1/N^2$. Outside the interval the density $\rho(\lambda)$ vanishes in general exponentially. We will show, however, that near the endpoints a or b, in a region of size $N^{-\mu}$, the approach to the large-N limit is given by a universal cross-over function of the scaling variable $(\lambda-\lambda_c)N^{\mu}$ (where λ_c is either a or b).

In fact the multicriticality to which we have alluded before can best be seen from the characteristic behavior of $\rho(\lambda)$ near its endpoints in the large-N limit. For an arbitrary potential, $\rho(\lambda)$ vanishes near a or b as a square root. Through an appropriate choice of the potential, however, one can obtain a density vanishing as $(\lambda-\lambda_c)^{k-1/2}$, with $k\geqslant 1$ [11]. The potential would then be called k-multicritical. At criticality, however, the integral (1) is well-defined only for odd integral values of k.

An explicit expression for the cross-over function of $(\lambda-\lambda_c)N^{2/(2k+1)}$ will be derived in terms of the resolvent of the Schrödinger operator for the string hamiltonian $p^2+f_k(x)$ of the double scaling limit. For the usual $k=1$ case this leads to an integral Fourier representation. We shall see that it is in fact universal for $k=1$ potentials. A multiple scaling region of size $N^{-2k/(2k+1)}$ in g_c-g, and $N^{-2/(2k+1)}$ in $\lambda_c-\lambda$, where g is a coupling constant which enters the potential, is characterized by the resolvent of the Schrödinger operator for the same string hamiltonian.

For simplicity of the algebra we generally restrict ourselves to even potentials, but we shall argue that nothing changes for non-even potentials and we illustrate this with a specific example.

The organization of the paper is the following: We first discuss the large-N limit in terms of the solution to the usual Riemann–Hilbert integral equation. We then summarize the orthogonal polynomial approach, which allows one to explore systematically the scaling limit. Then the finite N cross-over problem is related to a diffusion problem which is solved in terms of resolvents of Schrödinger operators. The $k=1$ resolvent is then explicitly computed with the help of the Gel'fand–Dikii [12] differential equation. Finally we discuss the issue of universality and the case of non-even potentials.

2. Large-N limit

The solution for the density of eigenvalues is well known [5] and we simply quote the results here. The density of states $\rho(\lambda)$ is obtained as the imaginary part of the function

$$f(z)=\int_{-a}^{a} \mathrm{d}x\,\frac{\rho(x)}{z-x}\,, \tag{6}$$

which is found to be

$$f(z)=\frac{1}{2g}\,V'(z)-P(z)\sqrt{z^2-a^2}\,, \tag{7}$$

in which P is a polynomial with real coefficients. From (6) one sees that $f(z)$ falls-off like $1/z$ at infinity [we have chosen $\rho(\lambda)$ such that its integral over its support is one]. If $V(z)$ is an even polynomial of degree $2k$, $P(z)$ has to be an even polynomial of degree $2k-2$. The vanishing of the coefficients of z^{2p-1} for $p=1, ..., k$ on the RHS of (7), together with the known residue of $f(z)$ at $z=\infty$, provide $(k+1)$ conditions which determine uniquely the coefficients of $P(z)$ as well as the parameter a. The density of eigenvalues is then

$$\rho(\lambda)=\lim_{\epsilon\to 0}\; -\frac{1}{\pi}\,\mathrm{Im}\, f(\lambda+\mathrm{i}\epsilon)$$

$$=\frac{P(\lambda)}{\pi}\sqrt{a^2-\lambda^2}\,. \tag{8}$$

For an arbitrary potential $P(a)$ is non-zero in general and $\rho(\lambda)$ has a $k=1$ square-root singularity at the endpoints. For particular choices of the coefficients, however, one can force $P(\lambda)$ to vanish at $\lambda^2=a^2$, up to $(k-1)$-times if the degree of V is $2k$.

The potential of lowest degree such that $\rho(\lambda)$ is proportional to $(a^2-\lambda^2)^{k-1/2}$ is the minimal k-multicritical potential, V_k. For this potential $P(\lambda) = c_k(a^2-\lambda^2)^{k-1}$ and

$$\rho(\lambda)=\frac{c_k}{\pi}(a^2-\lambda^2)^{k-1/2}. \tag{9}$$

The normalization of $\rho(\lambda)$ then fixes

$$c_k=\left(\frac{2}{a}\right)^{2k}\frac{(k!)^2}{(2k)!}. \tag{10}$$

By an appropriate rescaling of the matrix elements, one can always adjust a and from now on we will choose

$$a=2. \tag{11}$$

The minimal potential V_k, which is k-multicritical at $g=1$ (we take this value one for simplicity) is then entirely determined by the previous analysis and one finds

$$V_k(x)=\sum_{l=1}^{k}(-1)^{l+1}\frac{(l-1)!k!}{(2l)!(k-l)!}x^{2l}. \tag{12}$$

This potential is bounded below for odd k. For even k it is not and the integral (1) is ill-defined for finite N. For a non-minimal potential of the same kth class

$$P(\lambda)=(a^2-\lambda^2)^{k-1}Q(\lambda), \tag{13}$$

in which $Q(\lambda)$ is a polynomial with $Q(\pm a)\neq 0$.

3. Diffusion equation

Now that we have defined k-multicriticality and the associated density of eigenvalues we shall proceed to rederive the same large-N limit result and the cross-over from the orthogonal polynomial method [10] which is, by far, the most appropriate method for dealing with the corrections to this limit.

Defining the orthogonal polynomials

$$\int d\lambda \exp\left(-\frac{N}{g}V(\lambda)\right)p_n(\lambda)p_m(\lambda)=\delta_{nm}, \tag{14}$$

with

$$p_n(\lambda)=\frac{1}{\sqrt{h_n}}\lambda^n+\text{lower degree}. \tag{15}$$

We introduce the (infinite) Jacobi matrix Q defined as

$$\lambda p_n(\lambda)=\sum_m Q_{nm}p_m(\lambda). \tag{16}$$

The only non-zero elements of Q are the two sub-diagonals

$$Q_{n,n+1}=Q_{n+1,n}=\sqrt{R_{n+1}}, \tag{17}$$

with

$$R_n=\frac{h_n}{h_{n-1}}. \tag{18}$$

It is straightforward to show that

$$\rho(\lambda)=\frac{1}{N}\sum_{n=0}^{N-1}[\delta(\lambda-Q)]_{n,n}. \tag{19}$$

Let us also introduce the resolvent

$$G_{n_1,n_2}(z)=\left(\frac{1}{z-Q}\right)_{n_1,n_2}, \tag{20}$$

which satisfies the diffusion-like equation [13]

$$zG_{n_1,n_2}(z)-\sqrt{R_{n_2+1}}G_{n_1,n_2+1}(z)-\sqrt{R_{n_2}}G_{n_1,n_2-1}(z)=\delta_{n_1,n_2}. \tag{21}$$

This equation is easily derived by calculating $[1/(z-Q)]Q$ either as the product of the two matrices $G(z)$ and Q or by replacing Q in the numerator by $(Q-z)+z$. The density of eigenvalues is then given by

$$\rho(\lambda)=-\frac{1}{\pi}\operatorname{Im}\sum_{n=0}^{N-1}G_{n,n}(\lambda+i0). \tag{22}$$

The coefficients R_n are known to satisfy the recursion formula [14,15]

$$\frac{n}{N}g=[V'(Q)]_{n,n-1}Q_{n,n-1}. \tag{23}$$

If the degree of V is $2k$, $V'(Q)$ is a matrix with $2k-1$ non-zero sub-diagonals above and below the principal diagonal. Let us consider one term of $V'(Q)$: $(Q^{2l-1}_{n,n-1})$ is a sum of $\binom{2l-1}{l}$ terms. Each term is a walk of length $2l-1$ starting at n and ending at $(n-1)$ on a lattice of integers, with a weight $\sqrt{R_{p+1}}$ for a move from p to $p+1$ and $\sqrt{R_p}$ for a move from p to $p-1$.

In the large-N limit, for n/N finite, $R_n(g, N)$ as determined from (23) is a function of $(n/N)g$ and one can neglect, in the leading large-N approximation, the variation of R_p with p along the walk. In other words

$$\lim_{N\to\infty} (Q^{2l-1})_{n,n-1} = \binom{2l-1}{l} R^{l-1/2}\left(\frac{n}{N}g\right). \tag{24}$$

In this limit the RHS of (23) is

$$[V'(Q)]_{n,n-1} Q_{n,n-1} = \sum_{l=1}^{k} (-1)^{l+1} \frac{(l-1)!k!}{(2l)!(k-l)!} 2l (Q^{2l-1})_{n,n-1} Q_{n,n-1} = 1-(1-R)^k. \tag{25}$$

For a non-minimal k-multicritical potential we would have in place of the RHS of (25) $h(R)[1-(1-R)^k]$ with $h(R)$ a polynomial non-vanishing at $R=1$. Restricting to the minimal multicritical potential we can solve (23) and (25), giving

$$R_n(g,N) = 1 - \left(1-\frac{n}{N}g\right)^{1/k}. \tag{26}$$

For k odd this solution is valid up to $g=1$, both for $n<N$ and $n>N$. For k even, however, the solution breaks down at $n=N$ and $g=1$. This is not unexpected since the problem itself is ill-defined for k even. We now return to the diffusion equation (21) and introduce the lattice translation operators

$$\exp(\pm \mathrm{i}p)\ |n\rangle = |n \mp 1\rangle, \tag{27}$$

and denote by A the diagonal matrix whose elements are

$$A_n = \sqrt{R_n}. \tag{28}$$

The solution to (21) is thus

$$G_{n,n}(z) = \langle n|\ \frac{1}{z - A\exp(-\mathrm{i}p) - \exp(\mathrm{i}p) A}\ |n\rangle. \tag{29}$$

The edge λ_c of the distribution of eigenvalues is given by setting $A=A_N$ and $p=0$ in the denominator

$$\lambda_c = 2A_N = 2[1-(1-g)^{1/k}]^{1/2}. \tag{30}$$

In the vicinity of λ_c we can expand the denominator of (29) for A_n close to A_N and p small, in the continuum limit in which n/N is finite:

$$z - A\exp(-\mathrm{i}p) - \exp(\mathrm{i}p) A = z - 2A_N - 2(A-A_N) + \mathrm{i}[A,p] + A_N p^2 + \dots \tag{31}$$

and we have to examine in some detail each term. For $n/N = 1-x$ and x finite between 0 and 1

$$A_n - A_N = [1-(1-g+gx)^{1/k}]^{1/2} - [1-(1-g)^{1/k}]^{1/2}, \tag{32}$$

and we have to distinguish three cases; namely g strictly smaller than the multicritical value one, for which we can expect an ordinary $k=1$ behavior, $g=1$, which is the k-multicritical point, and $1-g$ small enough to cross over from the $k=1$ to the multicritical behaviour.

(1) For $g<1$

$$2(A_n - A_N) = -\alpha x + \mathrm{O}(x^2), \tag{33}$$

in which α is the finite number $(g/k)[1-(1-g)^{1/k}]^{-1/2}(1-g)^{1/k-1}$. In these units p becomes the continuum operator $p \to -\mathrm{i}N^{-1}\partial/\partial x$, and $[A,p]$ is of order $1/N$. For N large, x finite, we can write in place of (29) (neglecting all higher powers of $1/N$),

$$G_{x,x}(z) = N^{-1}\langle x| \left(z - \lambda_c + \alpha x - \frac{\lambda_c}{2}\frac{1}{N^2}\frac{\partial^2}{\partial x^2}\right)^{-1} |x\rangle. \tag{34}$$

If N goes to infinity first, we can replace the RHS of (34) by its WKB approximation (with $1/N$ analogous to $\hbar$):

$$G_{x,x}(z) = \frac{1}{\sqrt{2\lambda_c}}\frac{1}{\sqrt{z-\lambda_c+\alpha x}}. \tag{35}$$

The imaginary part of (35) is zero for $\lambda>\lambda_c$ as expected and for $\lambda<\lambda_c$ we obtain in the vicinity of λ_c

$$\rho(\lambda) \propto \int_0^1 \frac{\mathrm{d}x}{\sqrt{\lambda_c - \lambda - \alpha x}}. \tag{36}$$

The integrand of (36) is of the form indicated for x small only, but it is this small x region which gives the leading singular behaviour of $\rho(\lambda)$ near λ_c

$$\rho(\lambda) \propto (\lambda_c - \lambda)^{1/2}. \tag{37}$$

If, instead of letting N go to infinity first, we explore a region of size $N^{-\nu}$ we can write

$$x=N^{-\nu}y \tag{38}$$

with an exponent ν to be fixed later. Then instead of (34) we find

$$G_{y,y}(z)=N^{\nu-1} \times \langle y|\left(z-\lambda_c+\alpha N^{-\nu}y-\frac{\lambda_c}{2}N^{-2+2\nu}\frac{\partial^2}{\partial y^2}\right)^{-1}|y\rangle\,. \tag{39}$$

Choosing now

$$\nu=\tfrac{2}{3}\,, \tag{40}$$

and using the scaling variable

$$\zeta=N^{2/3}(z-\lambda_c)\,, \tag{41}$$

we obtain the finite ζ cross-over resolvent

$$G_{y,y}(\zeta)=N^{1/3}\langle y|\left(\zeta+\alpha y-\frac{\lambda_c}{2}\frac{\partial^2}{\partial y^2}\right)^{-1}|y\rangle\,. \tag{42}$$

This expression shows that in a region of size $N^{-2/3}$ near the edge λ_c, there is a cross-over density of eigenvalues

$$\rho(\lambda)\propto N^{-1/3}\int_0^a \mathrm{d}y \times \operatorname{Im}\langle y|\left(N^{2/3}(\lambda-\lambda_c+\mathrm{i}0)+\alpha y-\frac{\lambda_c}{2}\frac{\partial^2}{\partial y^2}\right)^{-1}|y\rangle. \tag{43}$$

The eigenstates of the Schrödinger operator for the linear potential are related to Airy functions. The resolvent will be computed below from the Gel'fand–Dikii equation [12].

(2) For $g=1$, i.e. $\lambda_c=2$, we have again to expand A_n-A_N in the vicinity of $n/N=1$. From the study of the double scaling limit of matrix models it is found that in the regime of N large and y, defined as

$$y=N^{\nu}\left(1-\frac{n}{N}\right) \tag{44}$$

finite

$$R_n=1+N^{-\nu}f_k(y)\,. \tag{45}$$

The exponent ν is found to be

$$\nu=\frac{2k}{2k+1}\,, \tag{46}$$

and the function f_k is a solution of a non-linear differential equation of order $2k-2$, the string equation. For odd k the differential equation, together with boundary conditions, satisfied by f_k is known to determine it uniquely, whereas there is no real consistent solution for even k [16,17]. As an example we give here the first two equations:

$$f_1(x)=x\,, \tag{47}$$

and we recover for $k=1$ the linear potential of the previous discussion; $f_3(x)$ satisfies the fourth-order equation

$$f_3^3(x)+f_3f_3''+\tfrac{1}{2}(f_3')^2+\tfrac{1}{10}f_3^{(4)}=x\,. \tag{48}$$

One obtains then in place of (39)

$$G_{y,y}(z)=N^{\nu-1} \times\langle y|\left(z-2+N^{-\nu/k}f_k(y)-N^{2\nu-2}\frac{\partial^2}{\partial y^2}\right)^{-1}|y\rangle\,. \tag{49}$$

It follows that for ν again given by (46), the cross-over near λ_c is characterized by the scaling variable

$$\zeta=N^{2/(2k+1)}(z-2)\,, \tag{50}$$

and

$$G_{y,y}(\zeta)=N^{1/(2k+1)}\langle y|\left(\zeta+f_k(y)-\frac{\partial^2}{\partial y^2}\right)^{-1}|y\rangle\,. \tag{51}$$

In the large-N limit, in which y goes to infinity, $f_k(y)$ behaves asymptotically as $y^{1/k}$. Returning then to the variable

$$x=1-\frac{n}{N}\,, \tag{52}$$

the resolvent takes the form

$$G_{x,x}(z)=N^{-1}\langle x|\left(z-2+x^{1/k}-\frac{1}{N^2}\frac{\partial^2}{\partial x^2}\right)^{-1}|x\rangle. \tag{53}$$

Since N is large, we can use again WKB and write

$$G_{x,x}(z) \approx \frac{1}{2\sqrt{z-2+x^{1/k}}}, \tag{54}$$

and we recover the density of eigenvalues

$$\rho(\lambda) \propto \int_0^a \mathrm{d}x \frac{1}{(2-\lambda-x^{1/k})^{1/2}} \propto (2-\lambda)^{k-1/2}. \tag{55}$$

(3) Let us now determine the actual range of g near $g_c=1$ in which one goes from the $k=1$ regime, governed by the resolvent of h_1, to the k-multicritical point. We now work in a multiple scaling region with x small, g near one and λ near 2. Setting

$$x=N^{-\nu}y, \tag{56}$$

and

$$1-g=N^{-\nu}\gamma, \tag{57}$$

we return to the "string" double scaling limit in which it was shown that, in this regime,

$$R_n=1+N^{-\nu/k}f_k(y+\gamma), \tag{58}$$

with f the solution of the same string equation. The exponent ν is still

$$\nu=\frac{2k}{2k+1}. \tag{59}$$

The resolvent is now

$$G_{y,y}(z)=N^{-1/(2k+1)} \times \langle y|\left(z-2-N^{-2/(2k+1)}f_k(y+\gamma) -N^{-2/(2k+1)}\frac{\partial^2}{\partial y^2}\right)^{-1}|y\rangle, \tag{60}$$

and therefore as a function of

$$\zeta=(z-2)N^{2/(2k+1)}, \tag{61}$$

the "triple" scaling resolvent is given by

$$G_{y,y}(\zeta)=N^{1/(2k+1)}\langle y|\frac{1}{\zeta+h_k}|y\rangle, \tag{62}$$

in which

$$h_k=-\frac{\partial^2}{\partial y^2}+f_k(y+\gamma) \tag{63}$$

is the hamiltonian found in the double scaling region of 2D quantum gravity [8,18]. The region that we are exploring is thus of size $N^{-2/(2k+1)}$ for λ in the vicinity of $\lambda_c=2$ and $N^{-2k/(2k+1)}$ for g near $g_c=1$. For $\gamma=0$ we recover the formulae of the $g=1$ case as expected.

4. A differential equation for the resolvent

The density of states has been related to the resolvent of the Schrödinger operators h_k at coincident points. For an arbitrary potential Gel'fand and Dikii [12] have shown that

$$F_\epsilon(x)=\langle x|\left(\epsilon-\frac{\partial^2}{\partial x^2}+V(x)\right)^{-1}|x\rangle \tag{64}$$

satisfies a second-order nonlinear differential equation

$$-2FF''+(F')^2+4(V+\epsilon)F^2=1, \tag{65}$$

in which the primes denote derivatives with respect to x. Differentiating with respect to x leads to the linear equation

$$-F'''+4(V+\epsilon)F'+2V'F=0. \tag{66}$$

The solution should have a Seeley expansion [19] of the form

$$F(x)=\sum_0^\infty \frac{c_k(x)}{(-\epsilon)^{k+1/2}}, \tag{67}$$

with $c_0=\frac{1}{2}\mathrm{i}$, etc. For the multicritical hamiltonian h_k (with $V=x^{1/k}$) this equation is not elementary. For $k=1$, and thus $V=\alpha x$, however, we find from (66) a simple differential equation for

$$\tilde{F}(p)=\int \exp(-\mathrm{i}px)\,F(x)\,\mathrm{d}x, \tag{68}$$

namely

$$(-2\alpha+4\epsilon\mathrm{i}p+\mathrm{i}p^3)\tilde{F}-4\alpha p\frac{\partial\tilde{F}}{\partial p}=0, \tag{69}$$

which leads to

$$F(x)=C\int\frac{\mathrm{d}p}{2\pi}\frac{1}{\sqrt{|p|}}\exp\left(\mathrm{i}px+\mathrm{i}\frac{\epsilon}{\alpha}p+\frac{\mathrm{i}}{12\alpha}p^3\right). \tag{70}$$

A priori the constant C in (70) could depend on ϵ; one can easily verify, however, that C is indeed a constant that is fixed by demanding that (67) is satisfied. This gives

$$C=\sqrt{\frac{\pi}{\alpha}}\exp(-i\pi/4)\,. \tag{71}$$

Indeed the differential equation (69) determines unambiguously the coefficients $\tilde{c}_{k+1}(p)$ of the Seeley expansion of

$$\tilde{F}=\sum_0^\infty \frac{\tilde{c}_k(p)}{(-\epsilon)^{k+1/2}}\,,$$

in term of $\tilde{c}_k$. The coefficient $\tilde{c}_0$, which is independent of the potential, is given by (67). Since (70) has an expansion in powers of $(-\epsilon)^{-k-1/2}$, fixing c_0 indeed determines the resolvent.

From the representation (70) it is easy to study the properties of the resolvent. The variable ϵ is proportional to $N^{2/3}(\lambda-\lambda_c)$. If we move away from the cross-over region we have to consider large values of ϵ in (70). The behaviour of $F_\epsilon(x)$ for large ϵ is given by the saddle-point method. For ϵ large and positive the saddle-point is purely imaginary and $F_\epsilon(x)$ falls off as $\exp[-(4/3\alpha)\epsilon^{3/2}]$. For ϵ large and negative the behaviour of $F_\epsilon(x)$ is oscillatory and proportional to $\cos((4/3\alpha)(-\epsilon)^{3/2})$. This oscillatory nature reproduces the zeroes of the orthogonal polynomials of high order. If we go back to N the exponential fall-off is in fact of the type $\exp(-aN)$. This exponential of n in a problem in which the coupling constant of the topological expansion is $1/N^2$ is presumably another manifestation of the strong non-perturbative effects of the string equations. Shenker [20] has discussed these effects and pointed out that the leading non-perturbative terms in the string coupling constant $1/N^2$ come from one eigenvalue tunneling outside the support of $\rho(\lambda)$. In an ordinary field theory these tunneling effects would go as $\exp(-N^2)$, but we do find here an analogous $\exp(-N)$ dependence. Our discussion is limited here to the $k=1$ case (or $c=-2$ in the language of 2D gravity), but presumably it applies to all k-multicritical points.

5. Universality

For an arbitrary k-multicritical even potential eq. (25) reads

$$\frac{n}{N}g=h(R)[1-(1-R)^k]\,, \tag{72}$$

in which $h(R)$ is a polynomial in R non-vanishing at $R=1$. The only effect of this polynomial is to shift the critical value of g from one to

$$g_c=h(1)\,. \tag{73}$$

We then find in place of (26)

$$R_n(g,N)=1-\left(1-\frac{n}{N}\frac{g}{g_c}\right)^{1/k}, \tag{74}$$

and the subsequent calculations of section 3 still hold.

If the potential has no parity the situation is similar except that $h(R)$ is a non-singular function rather than a polynomial. To be more specific let us consider the model

$$V(z)=\tfrac{1}{2}z^2+\tfrac{1}{3}gz^3\,. \tag{75}$$

In spite of the fact that this potential is unbounded below and leads to a $k=2$ critical point, we shall use this simple example to illustrate the universal nature of the recursion formulae near criticality. A direct solution of the large-N limit, based on (7), leads to a distribution of eigenvalues of the type

$$\rho(\lambda)=(\alpha\lambda+\beta)\sqrt{(\lambda-a)(b-\lambda)}\,, \tag{76}$$

with in general square-root singularities at the end-points a and b. For

$$g_c=\tfrac{1}{2}\cdot 3^{-3/4} \tag{77}$$

$(\alpha\lambda+\beta)$ vanishes at

$$\lambda_c=b_c=3^{5/4}-3^{3/4} \tag{78}$$

and the density now has a power $\frac{3}{2}$ singularity at λ_c. The diffusion equation (21) is slightly modified because the matrix Q now has non-zero diagonal elements

$$Q_{n,n}=S_n\,. \tag{79}$$

Hence (21) is replaced by

$$(z-S_{n2})G_{n1,n2}(z)-\sqrt{R_{n2+1}}G_{n1,n2+1}(z)-\sqrt{R_{n2}}G_{n1,n2-1}(z)=\delta_{n1,n2}. \tag{80}$$

The coefficients R_n and S_n satisfy coupled recursion formulae which read

$$\frac{n}{N}=R_n+gR_n(S_n+S_{n-1}) , \tag{81}$$

and

$$0=S_n+gS_n^2+g(R_n+R_{n+1}) . \tag{82}$$

In the limit g near g_c, n/N near 1, again $g^2R_n=r_n$ and $gS_n=s_n$ are functions of $(n/N)g^2$. Neglecting as before the variations between n and $n\pm 1$, we can solve (82) for s_n and obtain for r_n the relation

$$\frac{n}{N}g^2=r_n(1-8r_n)^{1/2} . \tag{83}$$

This defines r_n as a function of g^2n/N, and away from $g_c^2=1/12\sqrt{3}$ and $r_c=\frac{1}{12}$ the derivative of the RHS is non-zero. Near $r_c=\frac{1}{12}$, however, the relation (83) becomes

$$(r_c-r_n)^2=\frac{1}{216}\left(1-\frac{n}{N}\frac{g^2}{g_c^2}\right) \tag{84}$$

and we recover a $k=2$ critical point.

In the solution to the new diffusion equation (80) the only difference is that A in (31) is replaced by $A+\frac{1}{2}S$. This gives a critical value for the eigenvalue

$$\lambda_c=\frac{1}{g_c}(2\sqrt{r_c}+s_c)=3^{3/4}(\sqrt{3}-1) , \tag{85}$$

which agrees with (78). The rest of the analysis of the cross-over of section 4 is not modified. Hence the cross-over functions for $g<g_c$ and $g=g_c$ are unchanged.

6. Conclusions

We have shown the universality of the behaviour of the density of eigenvalues near the tail λ_c. At a multicritical point of odd order k, the large-N density vanishes as $|\lambda-\lambda_c|^{k-1/2}$ near λ_c. In the double scaling limit of string theory it is the resolvent of $p^2+f_k(x)$, in which f_k satisfies the kth string equation, which gives the cross-over in terms of the two variables $(\lambda_c-\lambda)N^{2/(2k+1)}$ and $(g_c-g)N^{2k/(2k+1)}$ (with g being the cosmological constant).

This study has been limited here to unitary-invariant ensembles. It would be interesting to perform a similar study for orthogonal or symplectic ensembles, but the technique is much more cumbersome in this case.

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