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A SIMPLE CLOSED-FORM FORMULA FOR PRICING DISCRETELY-SAMPLED VARIANCE SWAPS UNDER THE HESTON MODEL

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Abstract

We develop a simplified analytical approach for pricing discretely-sampled variance swaps with the realized variance, defined in terms of the squared log return of the underlying price. The closed-form formula obtained for Heston's two-factor stochastic volatility model is in a much simpler form than those proposed in literature. Most interestingly, we discuss the validity of our solution as well as some other previous solutions in different forms in the parameter space. We demonstrate that market practitioners need to be cautious, making sure that their model parameters extracted from market data are in the right parameter subspace, when any of these analytical pricing formulae is adopted to calculate the fair delivery price of a discretely-sampled variance swap.

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1. Introduction

Variance swaps are financial derivatives that are considered as forward contracts on annualized realized variance. In today's financial markets, variance swaps on stock indices are highly liquid and widely used by investors as an easy way to trade future realized variance against the current implied variance. Moreover, over-the-counter variance swaps can be linked to other types of underlying assets such as commodities or currencies. Hence, they can be very useful to hedge volatility risk exposure or to take positions on future realized volatility of the underlying assets.

Tremendous growth in trading variance swaps has been witnessed in recent years (http://cfe.cboe.com/education/finaleuromoneyvarpaper.pdf). As a result of increasing

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trading activities of variance swaps, many researchers have proposed various types of valuation approaches for pricing variance swaps with the realized variance defined in terms of either continuous sampling or discrete sampling. This paper focuses on a simplified analytical approach proposed by Rujivan and Zhu [19] to price discretely-sampled variance swaps with the realized variance defined in terms of the squared log return of the underlying price based on the Heston's two-factor stochastic volatility model [13].

In the literature on pricing variance swaps, the most influential pioneer works are those of Carr and Madan [7] and Demeterfi et al. [9], who have shown how to theoretically replicate a variance swap by a portfolio of standard options. Without specifying the function of volatility process, their models and analytical formulae are indeed very attractive. However, as pointed out by Carr and Corso [6], the replication strategy has a drawback that the sampling time of a variance swap needs to be assumed continuous rather than discrete. Such an assumption implies that the results obtained from a continuous model can only be viewed as an approximation for the real cases in financial practice, in which all contracts are written with the realized variance being evaluated on a set of discrete sampling points. Another drawback is that this strategy also requires options with a continuum of exercise prices, which is not actually available in the marketplace.

As reviewed by Zhu and Lian [25], there are two types of valuation approaches: numerical methods and analytical methods. The work presented in this paper belongs to the latter category which can be divided into two subcategories: the first of these shares a common assumption that the realized variance is approximated by a continuously-sampled one, which has greatly increased the mathematical tractability; while those in the second subcategory try to directly address the "discretely-sampled" nature of variance swaps. Since a comprehensive review of papers in the first subcategory has been given by Zhu and Lian [25], and the most recent one is presented by Swishchuk and Li [22], who studied valuation of variance swaps under stochastic volatility with delay and jumps and derived an analytic closed-form formula for pricing variance swaps under the assumption of continuous sampling, we shall mainly focus on a brief review of the literature in the second subcategory.

As pointed out by Zhu and Lian [25], developing analytical closed-form solutions in the second subcategory is generally much more difficult than is the case with the continuous sampling assumption. Broadie and Jain [4] presented a closed-form solution for volatility swaps as well as variance swaps with discrete sampling. They also examined the effects of jumps and stochastic volatility on the price of volatility and variance swaps by comparing calculated prices under various models such as the Black–Scholes model [3], the Heston stochastic volatility model [13], the Merton jump diffusion model [18], and the Bates [2] and Scott [20] stochastic volatility and jump model. Zhu and Lian [25] also presented an approach to obtain a closed-form formula for variance swaps with the realized variance being defined as the sum of the squared percentage increments of the underlying price. Unlike Broadie and Jain's approach [4], Zhu and Lian's approach [25] is based on Little and Pant's approach [17], and they found a closed-form formula by solving the governing partial differential equation (PDE) system. Moreover, their approach is more versatile in terms of dealing with different forms of realized variance, such as the case with realized variance defined in terms of the squared log return of the underlying price as demonstrated by Zhu and Lian [26].

However, Zhu and Lian's approach [25] is still too complicated. Rujivan and Zhu [19] pointed out that there is actually a simplified approach, and they applied this approach to the percentage return case, demonstrating that the exact same results can be produced without using the generalized Fourier transform. Very recently, Zheng and Kwok [24] proposed another analytical approach for pricing various types of discretely-sampled generalized variance swaps, including the same realized variance as used by Zhu and Lian [26]. In addition, they extended Zhu and Lian's work [25] to price variance swaps under the stochastic volatility models with simultaneous jumps in the asset price and variance processes. However, Zheng and Kwok's approach [24] relies on the availability of the analytical expression of the joint moment generating function of the underlying processes.

In this paper, we demonstrate that the analytical approach presented by Rujivan and Zhu [19] can be extended to derive a closed-form formula for the fair price of variance swaps with the realized variance defined in terms of squared log return of the underlying price. There are two major contributions of this paper. First, our closed-form formula is in a much simpler form than any of those presented by Broadie and Jain [4], Zheng and Kwok [24], and Zhu and Lian [26]. For example, our solution has completely avoided the requirement of the parameter functions being twice differentiable as in the results of Zheng and Kwok [24] and Zhu and Lian [26]. Second, we discuss the validity of our solution, as well as some other previous solutions in different forms in the parameter space, and conclude that all the solutions, although they may be of different forms, are only valid in a subspace of the original parameter space of the Heston model. This discussion has a practical implication in that market practitioners need to be cautious, making sure that their model parameters extracted from market data are in the right parameter subspace, when any of these analytical pricing formulae is adopted to calculate the fair delivery price of a discretely-sampled variance swap. In addition to these two major contributions, we demonstrate several applications of our closed-form formula in terms of deriving and investigating properties of the fair delivery prices of variance swaps. In particular, we provide a simple proof to show that our closed-form solution for the fair delivery price of a variance swap converges to the fair delivery price of the variance swap derived by using the continuous sampling realized variance.

The rest of the paper is structured as follows. In Section 2 we derive a simple closed-form formula for pricing variance swaps by using Rujivan and Zhu's approach [19]. An interesting discussion on the validity of all the analytical formulae shown in the literature is provided in Section 3. In Section 4 we present a couple of numerical examples demonstrating how easy it is to calculate numerical values using our formula. A brief conclusion is given in Section 5.

2. Our simple closed-form formula

In this section, for the sake of completeness, we shall briefly review the Heston model [13] which we adopt to describe the dynamics of the underlying asset first. Then we shall apply our previous approach [19] to derive a simple closed-form formula for pricing variance swaps with discretely-sampled realized variance defined in terms of the squared log return.

2.1. The Heston model In the Heston stochastic volatility model [13], the dynamics of the underlying price S_t defined on an original probability space $(\Omega, \mathcal{F}, P)$ is assumed to follow a diffusion process with a stochastic instantaneous variance v_t,

$$\begin{cases} dS_t = \mu S_t\,dt + \sqrt{v_t}S_t\,dB_t^S \\ dv_t = \kappa(\theta - v_t)\,dt + \sigma_V\sqrt{v_t}\,dB_t^V \end{cases} \tag{2.1}$$

where μ is the expected return of the underlying asset, θ is the long-term mean of variance, κ is a mean-reverting speed parameter of the variance, and σ_V is the so-called volatility of volatility. The two Wiener processes[1] dB_t^S and dB_t^V describe the random noises in asset and variance, respectively.

According to the risk-neutral pricing theory proposed by Harrison and Kreps [11] and Harrison and Pliska [12], we are able to change the original probability measure P to a so-called risk-neutral probability measure, denoted by Q, and describe the processes as

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{v_t}S_t\,d\widetilde{B}_t^S \\ dv_t = \kappa^*(\theta^* - v_t)\,dt + \sigma_V\sqrt{v_t}\,d\widetilde{B}_t^V \end{cases} \tag{2.2}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$ are the risk-neutral parameters. The new parameter r is a risk-free interest rate and λ is the premium of volatility risk [13]. The two transformed Wiener processes[2] $d\widetilde{B}_t^S$ and $d\widetilde{B}_t^V$ are assumed to be correlated with a constant correlation coefficient ρ, that is, $(d\widetilde{B}_t^S, d\widetilde{B}_t^V) = \rho dt$. For the rest of this paper, our analysis will be based on the probability space $(\Omega, \mathcal{F}, Q)$, and a filtration $(\mathcal{F}_t)_{t\geq 0}$. The conditional expectation with respect to $\mathcal{F}_t$ is denoted by $E_t^Q = E^Q[\cdot \mid \mathcal{F}_t]$.

In order to avoid dealing with degenerate cases in the parameter space, and to ensure that the Heston model (2.2) is a proper stochastic volatility model with the variance process reverting to a positive mean level, we make the following assumptions:

Assumption 2.1. *All parameters $r, \kappa^*, \theta^*, \sigma_V$ and an initial instantaneous variance v_0 are strictly positive.*

In addition, the stochastic volatility process is the so-called square-root process. Hence, to ensure that the variance is always positive, a further assumption, known as the Feller condition, is needed (see [8, 13]).

Assumption 2.2. *The parameters κ^*, θ^*, and σ_V satisfy the inequality $2\kappa^*\theta^* \geq \sigma_V^2$.*

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[1] With respect to the original probability measure P.

[2] With respect to the risk-neutral probability measure Q.

Due to Assumptions 2.1 and 2.2, we define the parameter space of the Heston model (2.2) as follows:

$$\Theta = \{p = (r, \kappa^*, \theta^*, \sigma_V, \rho) \in (\mathbb{R}^+)^4 \times [-1, 1] \mid 2\kappa^*\theta^* \geq \sigma_V^2\}.$$

The parameter space Θ will be referred to in Sections 3 and 4. Furthermore, for any given set $\mathcal{D}$ and a real-valued function $f : \mathcal{D} \times \Theta \to \mathbb{R}$, the value of f at $(x, p) \in \mathcal{D} \times \Theta$ is denoted by $f(x, p)$, or simply $f(x)$ when we do not consider the parameters.

2.2. Variance swaps Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The long position of a variance swap agrees to pay the short position a fixed delivery price at expiry, and receives the floating amounts of annualized realized variance. Therefore, variance swaps can be useful for investors to hedge volatility risk exposure or to take positions on future realized volatility.

For a given maturity, $T > 0$, the value of a variance swap can be written as $V_T = (\sigma_R^2 - K_{\text{var}}) \times L$, where σ_R^2 is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, and L is the notional amount of the swap in dollars per annualized volatility point squared.

The method for measuring the realized variance is usually specified in the contract. Important factors contributing to the calculation of the realized variance include the underlying asset (or assets), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, and the method of calculating the variance. In the literature (see [17, 25, 26]), the methods of calculating realized variance can be categorised into two different definitions: the log-return realized variance defined by

$$\sigma_{R,d1}^2(0, N, T) = \frac{AF}{N} \sum_{i=1}^{N} \ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \times 100^2 = \frac{1}{T} \sum_{i=1}^{N} \ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \times 100^2, \tag{2.3}$$

and the actual return-based realized variance defined by

$$\sigma_{R,d2}^2(0, N, T) = \frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \times 100^2 = \frac{1}{T} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \times 100^2, \tag{2.4}$$

where S_{t_i} is the closing price of the underlying asset at the i^{th} observation time t_i, and there are N observations in all. AF is the annualized factor converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$, assuming that there are 252 trading days in one year; if every week, then $AF = 52$; if every month, then $AF = 12$; and so on. Typically, we set $T = N/AF$ and assume equally-spaced discrete observations so that the annualized factor is of a simple expression $AF = N/T = 1/\Delta t$.

In the risk-neutral world, the value of a variance swap at time t is the expected present value of the future payoff, $V_t = E_t^Q[e^{-r(T-t)}(\sigma_R^2 - K_{\text{var}})L]$. This should be zero at the beginning of the contract since there is no cost for either party to enter into a swap contract. Therefore, the fair delivery price of a variance swap can be defined as $K_{\text{var}} = E_0^Q[\sigma_R^2]$, after setting the value of $V_t = 0$ initially. The valuation problem for a variance swap is, therefore, reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

A simplified analytical approach for pricing variance swaps with the actual return-based realized variance $\sigma^2_{R,d2}(0, N, T)$ has been proposed by the authors [19]. In this paper, the same approach will be applied to obtain a simple closed-form solution for pricing variance swaps with the log-return realized variance $\sigma^2_{R,d1}(0, N, T)$, presented in the next section.

2.3. Our simplified analytical approach Following our approach in an earlier paper [19], we begin by taking the expectation of $\sigma^2_{R,d1}$ as follows:

$$E_0^Q[\sigma^2_{R,d1}(0, N, T)] = E_0^Q\Big[\frac{1}{T}\sum_{i=1}^{N} \ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] \times 100^2 = \frac{1}{T}\sum_{i=1}^{N} E_0^Q\Big[\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] \times 100^2. \tag{2.5}$$

Therefore, the problem of pricing variance swaps is reduced to calculating N conditional expectations such as

$$E_0^Q\Big[\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] \tag{2.6}$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t$ with $i = 0, 1, \ldots, N$. In the rest of this section, we will focus our main attention on calculating the expectation in (2.6), where both t_i and t_{i-1} are regarded as known constants.

Using the fact that $\mathcal{F}_0 \subset \mathcal{F}_{t_{i-1}}$ and $S_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$-measurable, we apply the tower property [5] to the conditional expectation as in (2.6), and this gives us a double conditional expectation as follows:

$$\begin{aligned} E_0^Q\Big[\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] &= E_0^Q\Big[E^Q_{t_{i-1}}\Big[\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big]\Big] \\ &= E_0^Q[E^Q_{t_{i-1}}[\ln^2(S_{t_i})] - 2\ln(S_{t_{i-1}})E^Q_{t_{i-1}}[\ln(S_{t_i})] + \ln^2(S_{t_{i-1}})]. \end{aligned} \tag{2.7}$$

The two conditional expectations with respect to $\mathcal{F}_{t_{i-1}}$ on the right-hand side of (2.7), that is $E^Q_{t_{i-1}}[\ln(S_{t_i})]$ and $E^Q_{t_{i-1}}[\ln^2(S_{t_i})]$, are computed by using the following proposition.

Proposition 2.3. *Suppose that S_t follows the dynamics described in (2.2). Let $X_t = \ln S_t$ and $\Delta t = t_i - t$. Then*

$$E^Q_{t_{i-1}}[X_t] = E^Q[X_t | (X_{t_{i-1}} = x, v_{t_{i-1}} = v)] = x + A_1(\Delta t) + A_2(\Delta t)v, \tag{2.8}$$

$$\begin{aligned} E^Q_{t_{i-1}}[X_t^2] &= E^Q[X_t^2 | (X_{t_{i-1}} = x, v_{t_{i-1}} = v)] \\ &= x^2 + 2A_1(\Delta t)x + 2A_2(\Delta t)xv + A_3(\Delta t) + A_4(\Delta t)v, + A_2^2(\Delta t)v^2 \end{aligned} \tag{2.9}$$

for all $t \in [t_{i-1}, t_i]$ and $(x, v) \in (-\infty, \infty) \times (0, \infty)$, where

$$A_1(\Delta t) = \frac{\theta^*(1 - e^{-\kappa^*\Delta t}) + (2r - \theta^*)\kappa^*\Delta t}{2\kappa^*}, \tag{2.10}$$

$$A_2(\Delta t) = \frac{e^{-\kappa^*\Delta t} - 1}{2\kappa^*}, \tag{2.11}$$

$$A_3(\Delta t) = \alpha_1 \Delta t + \alpha_2 (\Delta t)^2 + \alpha_3 (e^{-\kappa^*\Delta t} - 1) + \alpha_4 (e^{-2\kappa^*\Delta t} - 1) + \alpha_5 \Delta t e^{-\kappa^*\Delta t}, \tag{2.12}$$

$$A_4(\Delta t) = \beta_1 \Delta t + \beta_2 (e^{-\kappa^*\Delta t} - 1) + \beta_3 (e^{-2\kappa^*\Delta t} - 1) + \beta_4 \Delta t e^{-\kappa^*\Delta t}, \tag{2.13}$$

for all $\Delta t \geq 0$ and the constants $\alpha_1, \ldots, \alpha_5$ and $\beta_1, \ldots, \beta_4$ are given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(\kappa^*)^{-2}(\theta^*(2(2r - \theta^*)\kappa^* + 4\kappa^*(\kappa^* - \rho\sigma_V) + \sigma_V^2)) \\ \frac{1}{4}(2r - \theta^*)^2 \\ \frac{1}{2}(\kappa^*)^{-3}(\theta^*(2(\kappa^*)^2 - \kappa^*(\theta^* + 4\rho\sigma_V) + \sigma_V^2)) \\ \frac{1}{8}(\kappa^*)^{-3}(\theta^*(2\theta^*\kappa^* + \sigma_V^2)) \\ \frac{1}{2}(\kappa^*)^{-2}(\theta^*((\theta^* - 2r - 2\rho\sigma_V)\kappa^* + \sigma_V^2)) \\ \frac{1}{2}(\kappa^*)^{-1}(\theta^* - 2r) \\ (\kappa^*)^{-2}(\theta^* + \rho\sigma_V - \kappa^*) \\ -\frac{1}{4}(\kappa^*)^{-3}(2\theta^*\kappa^* + \sigma_V^2) \\ \frac{1}{2}(\kappa^*)^{-2}((2r - \theta^* + 2\rho\sigma_V)\kappa^* - \sigma_V^2) \end{pmatrix}.$$

The proof of this proposition is presented in Appendix A.

REMARK 2.4. It should be noted from Appendix A that if S_t follows a nonaffine model such as the 3/2 stochastic volatility model [15], the calculations for the conditional expectations $E^Q_{t_{i-1}}[X_t]$ and $E^Q_{t_{i-1}}[X_t^2]$ would be much more complicated than what we have done to obtain (2.8) and (2.9), respectively. This has limited our simplified analytical approach from being extended to nonaffine models, unless one can find explicit forms of $E^Q_{t_{i-1}}[X_t]$ and $E^Q_{t_{i-1}}[X_t^2]$.

REMARK 2.5. By using the method presented in Appendix A, however, we can derive the closed-form formulae for $E^Q_{t_{i-1}}[X_t]$ and $E^Q_{t_{i-1}}[X_t^2]$ when S_t follows the Heston model with jumps as studied by Broadie and Jain [4] and Zheng and Kwok [24]. Furthermore, our simplified analytical approach can also be extended to derive closed-form pricing formulae for generalized variance swaps as previously worked out by Zheng and Kwok [24]. The results of these two extensions will be presented in a forthcoming paper.

Also, note that from (2.8) and (2.9), $E^Q_{t_{i-1}}[X_t^2]$ can be expressed in a different form as

$$E^Q_{t_{i-1}}[X_t^2] = (E^Q_{t_{i-1}}[X_t])^2 + \{A_3(\Delta t) - A_1^2(\Delta t)\} + \{A_4(\Delta t) - 2A_1(\Delta t)A_2(\Delta t)\} v_{t_{i-1}}.$$

Therefore, the conditional variance of $X_t = \ln S_t$ with respect to $\mathcal{F}_{t_{i-1}}$ can be expressed as

$$\mathrm{Var}[X_t | \mathcal{F}_{t_{i-1}}] = \{A_3(\Delta t) - A_1^2(\Delta t)\} + \{A_4(\Delta t) - 2A_1(\Delta t)A_2(\Delta t)\} v_{t_{i-1}}.$$

We now prove the next proposition using Proposition 2.3.

PROPOSITION 2.6. *The conditional expectation in (2.6) can be written in terms of a quadratic form of the initial instantaneous variance v_0 as*

$$E_0^Q\left[\ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = \tilde{g}_i(\Delta t, v_0) = \tilde{A}_0(\Delta t, t_{i-1}) + \tilde{A}_1(\Delta t, t_{i-1})v_0 + \tilde{A}_2(\Delta t, t_{i-1})v_0^2 \tag{2.14}$$

for all $i = 1, 2, \ldots, N$ and $v_0 > 0$, where we set $\Delta t = t_i - t_{i-1}$ and

$$\begin{aligned} &\tilde{A}_0(\Delta t, t_{i-1}) \\ &\quad = A_3(\Delta t) - 2\kappa^*\theta^* A_2(t_{i-1})A_4(\Delta t) + \left(\kappa^*\theta^* + \frac{\sigma_V^2}{2}\right)\kappa^*\theta^*\{2A_2(t_{i-1})A_2(\Delta t)\}^2, \end{aligned} \tag{2.15}$$

$$\tilde{A}_1(\Delta t, t_{i-1}) = e^{-\kappa^* t_{i-1}}\{A_4(\Delta t) - 2(2\kappa^*\theta^* + \sigma_V^2)A_2(t_{i-1})A_2^2(\Delta t)\}, \tag{2.16}$$

$$\tilde{A}_2(\Delta t, t_{i-1}) = e^{-2\kappa^* t_{i-1}}A_2^2(\Delta t). \tag{2.17}$$

In addition,

$$E_0^Q\left[\ln^2\left(\frac{S_{t_1}}{S_{t_0}}\right)\right] = \tilde{g}_1(\Delta t, v_0) = A_3(\Delta t) + A_4(\Delta t)v_0 + A_2^2(\Delta t)v_0^2.$$

The proof of this proposition is provided in Appendix B.

With the conditional expectation expressed in (2.5), we can directly adopt Proposition 2.6 to obtain the pricing formula for log-return realized variance (2.3) as

$$K_{\text{var}} = E_0^Q[\sigma_{R,d1}^2(0, N, T)] = \frac{100^2}{T}\sum_{i=1}^{N}\tilde{g}_i(\Delta t, v_0), \tag{2.18}$$

which can also be written in terms of a quadratic form of the initial instantaneous variance v_0 as

$$\begin{aligned} &K_{\text{var}}(T, \Delta t, v_0) \\ &\quad = \frac{100^2}{T}\left\{\left(\sum_{i=1}^{N}\tilde{A}_0(\Delta t, t_{i-1})\right) + \left(\sum_{i=1}^{N}\tilde{A}_1(\Delta t, t_{i-1})\right)v_0 + \left(\sum_{i=1}^{N}\tilde{A}_2(\Delta t, t_{i-1})\right)v_0^2\right\}. \end{aligned} \tag{2.19}$$

It follows from (2.19) that the coefficient of v_0^2 is strictly positive unless Δt is zero. This nice property implies that K_{var} is convex with respect to v_0 on $(0, \infty)$. Moreover, K_{var} is strictly increasing with respect to v_0 on $(0, \infty)$ if the coefficient of v_0 in (2.19) is strictly positive or zero. While we shall leave a detailed discussion on these properties to Section 3.1, one should naturally expect a higher fair delivery price for a variance swap when the volatility of the underlying price is higher.

It should also be noted that our final formula (2.19) is in a much simpler form than those derived by Zheng and Kwok [24] and Zhu and Lian [26]. They are not all in exactly the same form, but it can be shown that the latter ones can be derived from the former. However, the current formula is derived using a much simpler approach and it is also in a much simpler form that can be exploited to explore some properties of the solution, which are discussed in the next section.

3. Validity of our closed-form formula

With the construction of the simple pricing formula, very interesting discussions, in terms of the validity of the solution in the parameter space, the determination of the required parameters, the verification of the newly derived formula against some previously presented analytical formulae and against the formula derived based on the continuous approximation, are presented in this section.

3.1. Validity of the solution in the parameter space In quantitative finance, when stochastic processes are used to simulate the random nature of the underlying price of a financial derivative, such as an option or a futures contract, all the model parameters in the original probability measure which is used to define the stochastic processes should be determined, in theory, from the market data of the underlying price alone. In other words, there is no need to use the market data of the derivative contract involved in the pricing exercise.

Under the risk-neutral pricing theory, however, derivatives such as futures and options are priced under a risk-neutral probability measure. Therefore, all model parameters in the original probability measure should be transformed to the risk-neutral parameters as demonstrated in Section 2.1. When all state variables of the adopted stochastic processes are observable, the risk-neutral parameters can be determined by the parameters in the original probability measure without the need to introduce the so-called risk premium parameters. For example, if the variance, v_t in the Heston model (2.1), were an observable quantity, this would imply that the premium of volatility risk would vanish, that is $\lambda = 0$, and thus we would have $\kappa^* = \kappa$ and $\theta^* = \theta$. However, v_t is actually an unobservable quantity. In this case, the premium of volatility risk might not be zero and using the market data of the underlying alone would not be sufficient to determine all the parameters involved in the pricing model as in the case of simple geometric Brownian motion adopted in the Black–Scholes model [3]. In other words, to completely determine the model parameters, market data of the derivative contract involved are also needed. In this sense, the extracted parameters are also contract-dependent, that is those parameters extracted for the purpose of pricing an option contract may not be suitable for pricing a futures contract. We assume that the risk-neutral parameters used for pricing variance swaps can be estimated by market data of a selected derivative contract, and they are all available at the time a pricing task needs to be carried out.

In view of this, the validity of the solution (2.19) in the parameter space needs to be examined. The purpose of such an examination is to ensure that one of the fundamental assumptions that the fair delivery price of a variance swap should be of a finite and positive value for a given set of parameters determined from market data, that is, $0 \le K_{\text{var}} < \infty$. The finiteness of K_{var} can be readily established. This is because, from (2.10)–(2.13), we can verify that the functions $A_k(\Delta t; p), k = 1, 2, \ldots, 4$, are finite for all $\Delta t \geqslant 0$, and $p = (r, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta$. Thus, from (2.14), we have $\tilde{g}_i(\Delta t, v_0; p) = E_0^Q[\ln^2(S_{t_i}/S_{t_{i-1}})] < \infty$ for all $i = 1, 2, \ldots, N$, and conclude immediately from (2.18) that $K_{\text{var}} < \infty$. On the other hand, the strict positiveness of K_{var} can only be ensured by a sufficient condition as shown in the following proposition, rather than a necessary and sufficient condition.

PROPOSITION 3.1. *Let* $p = (r, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta$ *be a parameter vector of the Heston model* (2.2)*. Set*

$$\Delta t_p^* = \min(\{\tau > 0 \mid A_3(\tau; p)A_4(\tau; p) = 0\} \cup \{\infty\}). \tag{3.1}$$

Then, the following assertions are true.

(1) Δt_p^* *is either strictly positive or infinite depending on* p*.*
(2) *For any* $\Delta t \in (0, \Delta t_p^*)$ *with* $T = N\Delta t$ *for some positive integer* N*,*

   (2.1) $0 < K_{\text{var}}(T, \Delta t, v_0; p) < \infty$ *for all* $v_0 > 0$ *and*
   (2.2) $K_{\text{var}}(T, \Delta t, v_0; p)$ *is strictly increasing with respect to* v_0 *on* $(0, \infty)$*.*

The proof of this proposition is given in Appendix C.

Next, we discuss some implications of Proposition 3.1. For a given variance swap contract, Δt is fixed. Once one has decided to adopt a particular set of parameter vectors $p_0 \in \Theta$, which have been determined by using an algorithm such as the one proposed by Aït-Sahalia and Kimmel [1] that is not discussed in this paper, Proposition 3.1 requires that $\Delta t_{p_0}^*$ be computed first[1], using a traditional numerical method such as Newton's method, before one can comfortably use formula (2.19). If the given Δt is such that $\Delta t < \Delta t_{p_0}^*$, then (2.19) can be used to price the variance swap without any problems, because the fair delivery price $K_{\text{var}}(T, \Delta t, v_0; p_0)$ of the variance swap is guaranteed to be finite and strictly positive for any initial instantaneous variance, by virtue of Proposition 3.1. Of course, if the given $\Delta t \geq \Delta t_{p_0}^*$, one now has to be careful with formula (2.19), because Proposition 3.1 cannot guarantee the strict positiveness and finiteness of the calculated fair delivery price. In Section 4 we shall show numerical examples of some computed Δt_p^* values for four different cases with detailed discussions.

Furthermore, if $T > 0$ and $\Delta t < \Delta t_{p_0}^*$ are fixed, assertion 2.2 of Proposition 3.1 implies that $K_{\text{var}}(T, \Delta t, v_0; p_0)$ is a monotonically increasing function of v_0 on $(0, \infty)$. From a financial point of view, this result supports our argument in Section 2.3 that market practitioners should naturally expect a higher fair delivery price for a variance swap when the volatility of the underlying is higher. Therefore, the condition $\Delta t < \Delta t_{p_0}^*$ should be fulfilled in order to obtain a financially meaningful formula for K_{var}.

In addition, assertion 2.2 of Proposition 3.1 implies that the minimum value of the delivery price of the variance swap defined by

$$K_{\text{var}}^{\min}(T, \Delta t; p_0) = \inf_{v_0 > 0} K_{\text{var}}(T, \Delta t, v_0; p_0)$$

[1] $\Delta t_{p_0}^*$ can be easily computed, as it is nothing but the smallest positive root of the product function $A_3(\Delta t; p_0)A_4(\Delta t; p_0)$.

is strictly positive. The minimum value can be easily derived by taking the limit as v_0 approaches zero in formula (2.19) as

$$K_{\text{var}}^{\min}(T, \Delta t; p_0) = \lim_{v_0 \to 0^+} K_{\text{var}}(T, \Delta t, v_0; p_0) = \frac{100^2}{T} \sum_{i=1}^{N} \tilde{A}_0(\Delta t, t_{i-1}; p_0) > 0 \tag{3.2}$$

for $T > 0$ and $\Delta t < \Delta t^*_{p_0}$.

It is interesting to note that the fair delivery price of a variance swap reaches a strictly positive value $K_{\text{var}}^{\min}$ when the spot variance of the underlying price approaches zero. It is even more interesting to provide a financial explanation for this rather peculiar feature of this type of futures contract on realized variance. Financially, the realized spot variance v_0 could never be zero. Even if it could reach zero, $K_{\text{var}}^{\min}$ is nonzero because the stochastic process (the Heston model in this case) we have assumed will always bear a nonzero realized variance for the period between now and any time in future. Therefore, unlike a futures contract written directly on an observable underlying asset, such as stocks, the fair delivery price of a variance swap starts immediately with a nonzero value given by $K_{\text{var}}^{\min}$. Of course, like a futures contract written on stocks, we still expect the monotonicity of the fair delivery price as a function of time. The main reason is that the discount factor in finance should not change when the underlying changes from an observable quantity to a nonobservable quantity.

Finally, it should be remarked that for the pricing formula presented in Rujivan and Zhu [19] for the case of the realized variance being defined in terms of the squared percentage return, that is $\sigma^2_{R,d2}(0, N, T)$ as in (2.4), there is no condition imposed in terms of the sampling frequency and the market-extracted model parameters. On the other hand, there is a different condition imposed in Proposition 2.1 of Rujivan and Zhu [19], the imposition of which forms a subspace of Θ, within which the solution is guaranteed to be finite. The subspace of Θ in which the pricing formula is valid for the payoff function of a contract provides evidence that parameters extracted from market data are contract-dependent, when a stochastic volatility model is adopted to price a derivative contract. Although Zheng and Kwok [24] did not explicitly mention this particular issue, that is, some restrictions need to be imposed in the parameter space, one can infer from their equation (2.6) that there is a subspace in Θ in which their solution is valid in the sense of guaranteeing a nonnegative and finite fair delivery price in some cases.

3.2. Comparison with other solutions There are three recently published papers [4, 24, 26] in which the authors proposed different closed-form formulae for pricing variance swap for the same payoff function as that presented in this paper. However, their approaches to the solution are completely different from the one we have presented here. Broadie and Jain [4] derived their formula by integrating the underlying stochastic processes directly. Zhu and Lian [26] obtained a formula in a different form from that of Broadie and Jain [4] by partially adopting Little and Pant's approach [17]. Unlike Little and Pant's approach [17], in which a numerical solution approach is adopted to solve the governing PDEs, Zhu and Lian [26] analytically solved the governing PDE by utilizing the generalized Fourier transform. Zheng and Kwok [24], on the other hand, proposed an analytic approach for pricing various types of discretely-sampled generalized variance swaps with a general payoff function including the payoff function specifically discussed in this paper. However, Zheng and Kwok's approach [24] relies on the availability of the analytical expression of the joint moment generating function of the underlying processes. Consequently, Zheng and Kwok's approach [24] produces a solution in closed form by solving a Riccati system of ordinary differential equations (ODEs) as written in their equation (2.4). Of course, a closed-form solution with the fair delivery prices written in terms of a quadratic form of the initial instantaneous variance shown in (2.19) of this paper is the simplest. Other solutions in different forms can also be simplified, which we discuss in this section.

It is interesting to verify whether the solutions presented in [4, 24, 26] match our solution in some way, although they all appear in different forms. With the payoff being exactly the same, we should be able to prove this as we verify the formula presented here. We have used Mathematica to show that both Zhu and Lian's [26] and Zheng and Kwok's [24] formulae reduce to our formula after some algebraic manipulations[1]. On the other hand, Broadie and Jain's formula [4] has coefficients $\sum_{i=1}^{N} \tilde{A}_0(\Delta t, t_{i-1})$ and $\sum_{i=1}^{N} \tilde{A}_1(\Delta t, t_{i-1})$ different from ours and Zhu and Lian's [26] and Zheng and Kwok's [24] formulae, while the coefficient $\sum_{i=1}^{N} \tilde{A}_2(\Delta t, t_{i-1})$ can be shown to be identical to ours and the formulae presented in [24, 26].

There is a clear advantage of our formula written as a quadratic function of v_0 as shown in (2.19). Both Zhu and Lian's [26] and Zheng and Kwok's [24] solutions are given in an "implicit" form in the sense that some differential operators remain in the final formula. For example, in Zhu and Lian's case [26], first-order and second-order derivatives of some complex-valued functions need to be worked out, while Zheng and Kwok's case [24] involves the calculation of a second-order derivative of a real-valued function. Of course, the calculation of these derivatives can easily be done with the aid of a symbolic package, such as Maple or Mathematica, but it is still much better to have a pricing formula that requires no further differentiations like the one we presented here in (2.19). This distinguishing feature of our solution with reduced computational time and effort makes our formula an exciting improvement on that previously presented in the articles [4, 24, 26].

3.3. Validity of the continuous model According to the definition of the log-return realized variance $\sigma^2_{R,d1}(0, N, T)$ defined in (2.3), Jacod and Protter [14] proved that when the sampling frequency increases to infinity, the discretely-sampled realized variance converges to a continuously-sampled realized variance defined by

$$\sigma^2_{R,c}(0, T) = \frac{1}{T} \int_0^T \sigma_t^2 \, dt \times 100^2$$

[1] Readers who are interested in the Mathematica code can contact the corresponding author.

where $\sigma_t = \sqrt{v_t}$ is the so-called instantaneous volatility of the underlying. That is,

$$\sigma^2_{R,c}(0,T) = \lim_{N\to\infty} \sigma^2_{R,d1}(0,N,T).$$

Therefore, for the convenience of calculation, many researchers [21, 23] have approximated $\sigma^2_{R,d1}(0,N,T)$ with $\sigma^2_{R,c}(0,T)$, when the sampling period is short enough (for example, daily sampling). In other words,

$$K_{\mathrm{var}}\Big(T,\frac{T}{N},v_0\Big) = E_0^Q[\sigma^2_{R,d1}(0,N,T)] \approx E_0^Q[\sigma^2_{R,c}(0,T)] \quad \text{as } N\to\infty.$$

Furthermore, Swishchuk [21] has shown that once the realized variance is defined in terms of an integral, the conditional expectation of this continuous integral can be easily found in explicit form as

$$E_0^Q[\sigma^2_{R,c}(0,T)] = \Big\{\Big(1-\frac{1-e^{-\kappa^*T}}{\kappa^*T}\Big)\theta^* + \Big(\frac{1-e^{-\kappa^*T}}{\kappa^*T}\Big)v_0\Big\}\times 100^2, \tag{3.3}$$

which can be interpreted as a weighted average of the initial instantaneous variance v_0 and the log-return mean of variance θ^*.

In Appendix D, we prove that our simple closed-form formula for K_{var} in (2.19) converges to the closed-form formula for $E_0^Q[\sigma^2_{R,c}(0,T)]$ as shown in (3.3), when the sampling frequency approaches infinity.

Proposition 3.2. *For any given $p\in\Theta$ and $T>0$, the functions $\tilde{A}_j(\Delta t,t;p)$ for $j=0,1,2$, defined in (2.15)–(2.17), have the following properties:*

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{T}\sum_{i=1}^{N}\tilde{A}_0\Big(\frac{T}{N},\frac{(i-1)T}{N};p\Big) &= \lim_{\Delta t\to 0^+}\frac{1}{T}\sum_{i=1}^{N}\tilde{A}_0(\Delta t,(i-1)\Delta t;p)\\ &= \Big(1-\frac{1-e^{-\kappa^*T}}{\kappa^*T}\Big)\theta^*,\end{aligned}\tag{3.4}$$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{T}\sum_{i=1}^{N}\tilde{A}_1\Big(\frac{T}{N},\frac{(i-1)T}{N};p\Big) &= \lim_{\Delta t\to 0^+}\frac{1}{T}\sum_{i=1}^{N}\tilde{A}_1(\Delta t,(i-1)\Delta t;p)\\ &= \frac{1-e^{-\kappa^*T}}{\kappa^*T},\end{aligned}\tag{3.5}$$

$$\lim_{N\to\infty}\frac{1}{T}\sum_{i=1}^{N}\tilde{A}_2\Big(\frac{T}{N},\frac{(i-1)T}{N};p\Big) = \lim_{\Delta t\to 0^+}\frac{1}{T}\sum_{i=1}^{N}\tilde{A}_2(\Delta t,(i-1)\Delta t;p) = 0. \tag{3.6}$$

Moreover,

$$\begin{aligned}\lim_{N\to\infty}K_{\mathrm{var}}\Big(T,\frac{T}{N},v_0;p\Big) &= \lim_{\Delta t\to 0^+}K_{\mathrm{var}}(T,\Delta t,v_0;p)\\ &= \Big\{\Big(1-\frac{1-e^{-\kappa^*T}}{\kappa^*T}\Big)\theta^* + \Big(\frac{1-e^{-\kappa^*T}}{\kappa^*T}\Big)v_0\Big\}\times 100^2\end{aligned}\tag{3.7}$$

for all $v_0>0$.

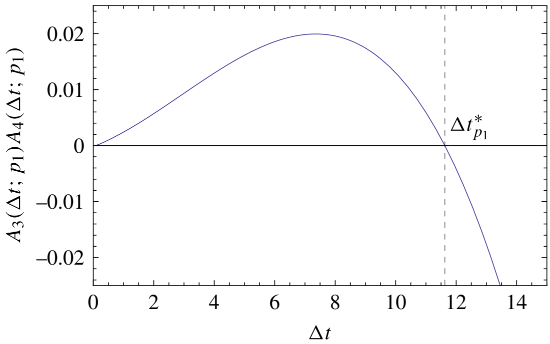

FIGURE 1. Variation of $A_3(\Delta t; p_1)A_4(\Delta t; p_1)$ against Δt with $\Delta t^*_{p_1} \approx 11.6249$.

4. Numerical examples

In this section we provide some numerical examples to illustrate one of the main contributions of this paper, that is Proposition 3.1 provides a sufficient condition to ensure that the extracted parameters can lead to a financially meaningful delivery price. In Examples 4.1 and 4.2, we show when the sampling period of a variance swap is less than the computed Δt^*_p defined in (3.1), from a given set of parameters, the calculated fair delivery price always satisfies assertions 2.1 and 2.2 of Proposition 3.1. In particular, Example 4.2 shows that Δt^*_p being infinite implies that the further requirement that only a subspace of Θ should be used for a finite K_{var} is no longer needed. In Example 4.3 we show that even when $\Delta t \geq \Delta t^*_p$, the calculated fair delivery prices can still be finite and positive, but the property claimed in assertion 2.2 of Proposition 3.1 is no longer there, showing that Proposition 3.1 is only a sufficient condition. Finally, Example 4.4 demonstrates that when $\Delta t \geq \Delta t^*_p$, it is indeed possible for K_{var} to be negative, stressing that it is important to use Proposition 3.1 to ensure that the calculated delivery price K_{var} is financially meaningful, that is $0 < K_{\mathrm{var}} < \infty$.

EXAMPLE 4.1. In this example we set the parameters used by Zhu and Lian [26]: $p_1 = (r, \kappa^*, \theta^*, \sigma_V, \rho) = (0.10, 11.35, 0.022, 0.618, -0.64) \in \Theta$ with $T = 1$ year. The variation of $A_3(\Delta t; p_1)A_4(\Delta t; p_1)$ against Δt is plotted in Figure 1. Also displayed in Figure 1 is the smallest positive zero of $A_3(\Delta t; p_1)A_4(\Delta t; p_1)$, that is $\Delta t^*_{p_1}$, which was obtained by using Mathematica. The calculated $\Delta t^*_{p_1}$ is approximately 11.6249. According to assertion 2.1 of Proposition 3.1, for any contract with a sampling frequency up to 1 year, that is $\Delta t = 1/252, 1/52, 1/12, 1$, this guarantees that K_{var} is finite and strictly positive. As displayed in Figure 2, K_{var} is a monotonically increasing function of volatility and the value of $K^{\min}_{\mathrm{var}}$ is approximately 201.094, calculated by using formula (3.2).

EXAMPLE 4.2. With θ^* increased by 10 times while the remaining parameters used in Example 4.1 are kept unchanged, we set $p_2 = (r, \kappa^*, \theta^*, \sigma_V, \rho) =$

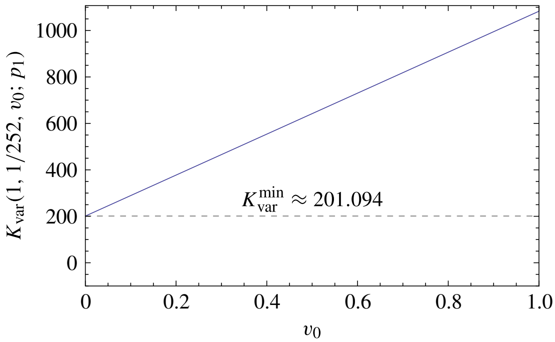

FIGURE 2. Variation of $K_{\text{var}}(1, \frac{1}{252}, v_0; p_1)$ against v_0.

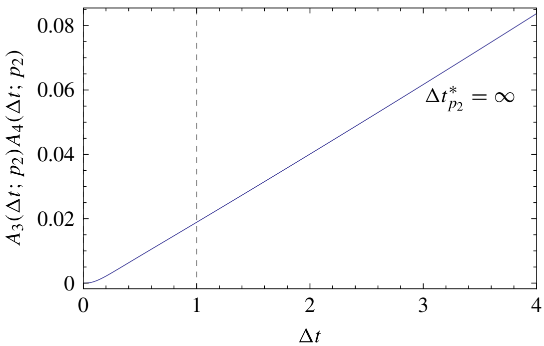

FIGURE 3. Variation of $A_3(\Delta t; p_2)A_4(\Delta t; p_2)$ against Δt.

$(0.10, 11.35, 0.22, 0.618, -0.64) \in \Theta$ with $T = 1$ year. Plugging these values into (2.12) and (2.13) yields

$$A_3(\Delta t; p_2) = -0.02068 + 0.00010e^{-22.7t} + 0.02058e^{-11.35t} + 0.22764t \\ + 0.00819e^{-11.35t}t + 0.00010t^2, \tag{4.1}$$

$$A_4(\Delta t; p_2) = 0.09038 - 0.00092e^{-22.7t} - 0.08947e^{-11.35t} + 0.00088t \\ - 0.03721e^{-11.35t}t. \tag{4.2}$$

From (4.1) and (4.2) one can easily show that $A_3(\Delta t; p_2)A_4(\Delta t; p_2)$ is strictly positive for all $\Delta t \in (0, \infty)$ as displayed in Figure 3. This implies that $\Delta t^*_{p_2} = \infty$. Hence, we immediately conclude that $K_{\text{var}}(T, \Delta t, v_0; p_2)$ satisfies assertions 2.1 and 2.2 of Proposition 3.1. In addition, we plot the variations of $K_{\text{var}}(T, \Delta t, v_0; p_2)$ against v_0 with $\Delta t = 1/12,\ 1/52,\ 1/252$, as displayed in Figure 4, where we can see that the fair delivery price of the variance swap tends to be higher as the sampling size increases.

EXAMPLE 4.3. Now, with another parameter r reset to $r = 300$ while the remaining parameters in Example 4.1 are kept unchanged, we use a set of new parameters

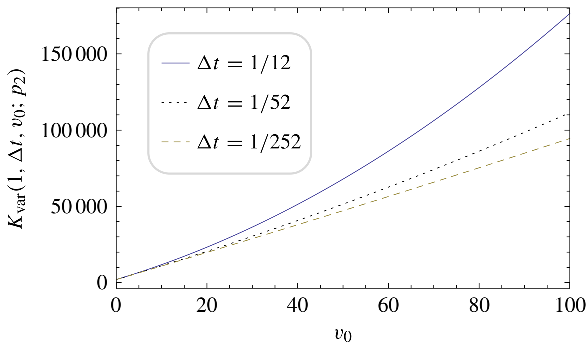

FIGURE 4. Variations of $K_{\text{var}}(1, \Delta t, v_0; p_2)$ against v_0 with three different Δt values.

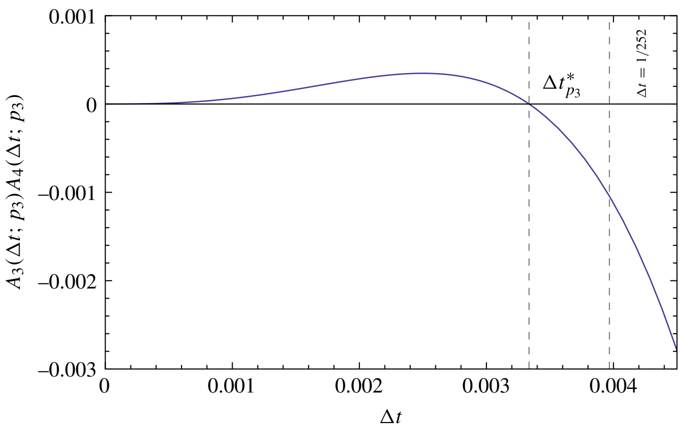

FIGURE 5. Variation of $A_3(\Delta t; p_3)A_4(\Delta t; p_3)$ against Δt with $\Delta t^*_{p_3} \approx 0.00334$.

$p_3 = (r, \kappa^*, \theta^*, \sigma_V, \rho) = (300, 11.35, 0.022, 0.618, -0.64) \in \Theta$ to show that there is a possibility that Δt^*_p is too small. Of course, setting the risk-free interest rate $r = 300$ makes no sense in reality. However, our purpose in doing so is to demonstrate that there is a possibility that meaningful (positive) strike prices can be obtained for a certain combination of model parameters even though Δt^*_p is less than the daily sampling size, that is $\Delta t = 1/252$. For some reasonable parameter vectors $p = (\kappa^*, \theta^*, \sigma_V, \rho)$, Δt^*_p is normally greater than or equal to the daily sampling size, which is why we took the unrealistic value of r, that is to demonstrate numerically that Proposition 3.1 is *only* a sufficient condition guaranteeing that $0 < K_{\text{var}} < \infty$.

In this case, Figure 5 displays the variation of $A_3(\Delta t; p_3)A_4(\Delta t; p_3)$ against $\Delta t \in (0, 0.0045)$. Using Mathematica, we obtain $\Delta t^*_{p_3} \approx 0.00334$, which is less than $1/252 \approx 0.00397$, that is we have a case of $\Delta t^*_{p_3} < \Delta t$ if the realized variance is calculated with daily sampling. Hence, Proposition 3.1 cannot guarantee $K_{\text{var}}(1, \Delta t, v_0; p_3)$ is finite and strictly positive for $\Delta t \geq \Delta t^*_{p_3}$ and $v_0 > 0$. However, we shall show that

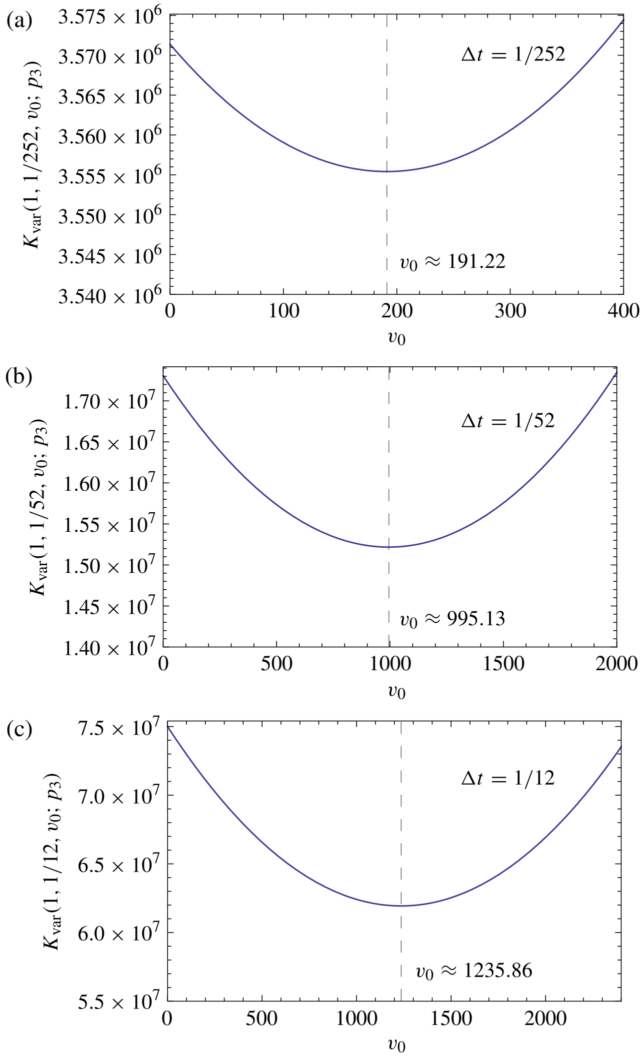

FIGURE 6. Variations of $K_{\mathrm{var}}(1, \Delta t, v_0; p_3)$ against v_0 with three different Δt values.

$K_{\mathrm{var}}(1, \Delta t, v_0; p_3)$, for $\Delta t = 1/252,\ 1/52,\ 1/12$, can still be finite and strictly positive for all $v_0 > 0$. Following (2.19), we have

$$K_{\mathrm{var}}(1, 1/252, v_0; p_3) = 3.57139 \times 10^6 - 167.11 v_0 + 0.43696 v_0^2, \tag{4.3}$$
$$K_{\mathrm{var}}(1, 1/52, v_0; p_3) = 1.73067 \times 10^7 - 4198.56 v_0 + 2.10956 v_0^2, \tag{4.4}$$
$$K_{\mathrm{var}}(1, 1/12, v_0; p_3) = 7.49920 \times 10^7 - 21132.30 v_0 + 8.54962 v_0^2. \tag{4.5}$$

Clearly, these K_{var} values are finite for all $v_0 > 0$. Applying the first and second derivative tests with respect to v_0 to (4.3)–(4.5), one can easily show that

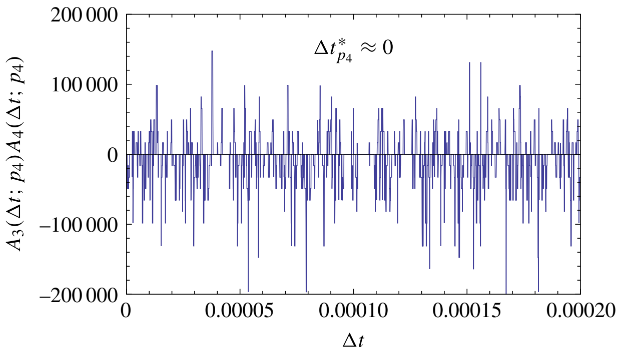

FIGURE 7. Variation of $A_3(\Delta t; p_4)A_4(\Delta t; p_4)$ against Δt.

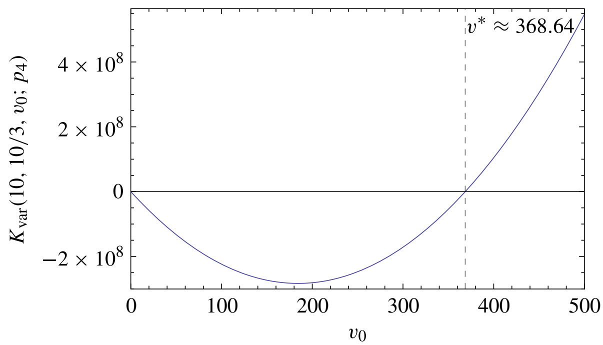

FIGURE 8. Variation of $K_{\mathrm{var}}(10, \frac{10}{3}, v_0; p_4)$ against v_0.

$K_{\mathrm{var}}(1, \Delta t, v_0; p_3)$, for $\Delta t = 1/252,\ 1/52,\ 1/12$, yields positive global minima at $v_0 \approx 191.22, 995.13, 1235.86$, respectively, as displayed in Figure 6. Therefore, the fair delivery prices are finite and strictly positive. However, as one can see from Figure 6, the fair delivery prices of the variance swaps are not monotonically increasing functions of volatility in this case.

EXAMPLE 4.4. This example shows that there exists a parameter vector in Θ in which the corresponding fair delivery price calculated with (2.19) reaches a negative value for some values of T, N and v_0. Let $p_4 = (r, \kappa^*, \theta^*, \sigma_V, \rho) = (0.02, 1 \times 10^{-10}, 0.10, 1.414 \times 10^{-6}, -0.50) \in \Theta$ with $T = 10$ and $N = 3$. Plugging these values into (2.12) and (2.13) yields

$$\begin{aligned} A_3(\Delta t; p_4) &= 1.24986 \times 10^{17} + 2.75 \times 10^{17} e^{(-1/5)\times 10^{-9}\Delta t} - 3.99986 \times 10^{17} e^{10^{-10}\Delta t} \\ &\quad - 2.49993 \times 10^{7}\Delta t - 4.00007 \times 10^{7} e^{-10^{-10}\Delta t}\Delta t + 0.0009(\Delta t)^2, \\ A_4(\Delta t; p_4) &= -4.49993 \times 10^{18} - 5.5 \times 10^{18} e^{-(1/5)\times 10^{-9}\Delta t} + 9.99993 \times 10^{18} e^{-10^{-10}\Delta t} \\ &\quad + 3 \times 10^{8}\Delta t - 4.00007 \times 10^{8} e^{-10^{-10}\Delta t}\Delta t. \end{aligned}$$

The variation of $A_3(\Delta t; p_4)A_4(\Delta t; p_4)$ against Δt is plotted in Figure 7. As displayed in Figure 7, the product function oscillates around zero when Δt approaches zero. This implies that $\Delta t^*_{p_4}$ is very small. For $\Delta t = 10/3$ set in this example, clearly it is greater than $\Delta t^*_{p_4}$. Following (2.19), we have

$$K_{\text{var}}(10, \tfrac{10}{3}, v_0; p_4) = -1.024 \times 10^{-4} - 3.072 \times 10^6 v_0 + 8.333 \times 10^3 v_0^2.$$

One can see from Figure 8 that $K_{\text{var}}(10,\ 10/3,\ v_0; p_4) < 0$ for all $v_0 \in (0, v^*)$, where $v^* \approx 368.64$. This demonstrates that when $\Delta t \geq \Delta t^*_p$, it is indeed possible for K_{var} to be negative.

5. Conclusions

In this paper, the simplified analytical approach proposed by the authors in the previous work [19] is extended to the case of pricing variance swaps, based on the Heston's two-factor stochastic volatility model [13] with the realized variance defined in terms of the squared log-return of the underlying price. Interestingly, we have obtained a closed-form formula for the fair delivery price of variance swaps, and it is in a much simpler form than those presented earlier in the literature. Another main contribution of the paper is that we have demonstrated that there may exist restrictions on model parameters, that is subspaces of the parameter space, that need to be imposed in order for the derived formula to lead to a financially meaningful fair delivery price. Of course, these restrictions are stronger than the Feller condition [8, 13] under which the stochastic processes in the Heston model (2.2) yield positive values of variance. We have provided an example of these restrictions in Proposition 3.1, which is shown to be only a sufficient condition to caution market practitioners, when the derived formula is used in their pricing exercises.

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Appendix A

Proof of Proposition 2.3. We first apply Itô's lemma [16] to the transformation $X_t = \ln S_t$. This gives

$$\begin{aligned} dX_t &= (r - \tfrac{1}{2}v_t)dt + \sqrt{v_t}d\tilde{B}^S_t, \\ dv_t &= \kappa^*(\theta^* - v_t)dt + \sigma_V \sqrt{v_t}d\tilde{B}^V_t. \end{aligned}$$

Consider the two-dimensional Itô process (X_t, v_t) and the two contingent claims (futures)

$$U_i^{(j)}(x, v, t) = E^Q[X_t^j | (X_{t_{i-1}} = x, v_{t_{i-1}} = v)], \quad \text{for } j = 1, 2,$$

with payoffs (terminal conditions) at expiry t_i being X_{t_i} and $X_{t_i}^2$, respectively. Utilizing the general asset valuation theory of Garman [10], $U_i^{(j)}$ satisfies the PDE

$$\begin{aligned}\frac{\partial U_i^{(j)}}{\partial t} + \frac{1}{2} v \frac{\partial^2 U_i^{(j)}}{\partial x^2} + \frac{1}{2} \sigma_V^2 v \frac{\partial^2 U_i^{(j)}}{\partial v^2} \\ + \rho \sigma_V v \frac{\partial^2 U_i^{(j)}}{\partial x \partial v} + \left(r - \frac{1}{2} v\right) \frac{\partial U_i^{(j)}}{\partial x} + \kappa^*(\theta^* - v) \frac{\partial U_i^{(j)}}{\partial v} = 0 \end{aligned} \tag{A.1}$$

for all $t \in [t_{i-1}, t_i)$ and $(x, v) \in \mathbb{R} \times \mathbb{R}^+$, subject to the terminal condition

$$U_i^{(j)}(x, v, t_i) = x^j, \quad \text{for } j = 1, 2, \tag{A.2}$$

and for all $(x, v) \in \mathbb{R} \times \mathbb{R}^+$. Let $\tau = t_i - t$. Next, we solve the PDE (A.1) subject to the terminal condition (A.2) for each $j = 1, 2$ separately.

Case $j = 1$. We assume that the solution can be expressed in the form

$$U_i^{(1)}(x, v, t) = x + A_1(t_i - t) + A_2(t_i - t)v \tag{A.3}$$

for all $t \in [t_{i-1}, t_i)$ and $(x, v) \in \mathbb{R} \times \mathbb{R}^+$ where $A_k(\tau)$, $k = 1, 2$, are deterministic functions defined for all $\tau \geq 0$ and to be determined later on. Calculating all partial derivatives of $U_i^{(j)}$ in (A.1) by using the solution form (A.3) yields the relations

$$\begin{gathered}\frac{\partial U_i^{(1)}}{\partial t} = -\left(\frac{dA_1}{d\tau} + \frac{dA_2}{d\tau} v\right), \quad \frac{\partial U_i^{(1)}}{\partial x} = 1, \quad \frac{\partial U_i^{(1)}}{\partial v} = A_2, \\ \frac{\partial^2 U_i^{(1)}}{\partial x^2} = \frac{\partial^2 U_i^{(1)}}{\partial v^2} = \frac{\partial^2 U_i^{(1)}}{\partial x \partial v} = 0. \end{gathered} \tag{A.4}$$

Inserting (A.4) into (A.1) gives

$$-\left(\frac{dA_1}{d\tau} + \frac{dA_2}{d\tau} v\right) + \left(r - \frac{1}{2} v\right) + \kappa^*(\theta^* - v)A_2 = 0.$$

This yields a system of linear ODEs

$$\frac{d}{d\tau}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa^*\theta^* \\ 0 & -\kappa^* \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} r \\ -1/2 \end{pmatrix} \tag{A.5}$$

subject to the zero initial conditions obtained by using the terminal condition (A.2) as

$$A_k(0) = 0, \quad \text{for } k = 1, 2. \tag{A.6}$$

The solution of (A.5) subject to (A.6), that is A_1 and A_2, can be found as expressed in (2.10) and (2.11), respectively.

Case $j = 2$. In this case we assume that

$$U_i^{(2)}(x, v, t) = x^2 + A_3(t_i - t) + A_4(t_i - t)v + A_5(t_i - t)v^2 + A_6(t_i - t)x + A_7(t_i - t)xv$$

for all $t \in [t_{i-1}, t_i)$ and $(x, v) \in \mathbb{R} \times \mathbb{R}^+$, where $A_k(\tau)$, $k = 3, 4, \dots, 7$, are deterministic functions defined for all $\tau \geq 0$, and to be determined later on. Following the same procedure as in the previous case, we obtain

$$\frac{\partial U_i^{(2)}}{\partial t} = -\Big(\frac{dA_3}{d\tau} + \frac{dA_4}{d\tau}v + \frac{dA_5}{d\tau}v^2 + \frac{dA_6}{d\tau}x + \frac{dA_7}{d\tau}xv\Big), \tag{A.7}$$

$$\frac{\partial U_i^{(2)}}{\partial x} = 2x + A_6 + A_7 v, \qquad \frac{\partial U_i^{(2)}}{\partial v} = A_4 + 2A_5 v + A_7 x, \tag{A.8}$$

$$\frac{\partial^2 U_i^{(2)}}{\partial x^2} = 2, \qquad \frac{\partial^2 U_i^{(2)}}{\partial v^2} = 2A_5, \qquad \frac{\partial^2 U_i^{(2)}}{\partial x \partial v} = A_7. \tag{A.9}$$

Substituting (A.7), (A.8), and (A.9) into the PDE (A.1) yields

$$\begin{aligned} &-\Big(\frac{dA_3}{d\tau} + \frac{dA_4}{d\tau}v + \frac{dA_5}{d\tau}v^2 + \frac{dA_6}{d\tau}x + \frac{dA_7}{d\tau}xv\Big) + v + \sigma_V^2 A_5 v + \rho\sigma_V A_7 v \\ &\quad + \Big(r - \frac{1}{2}v\Big)(2x + A_6 + A_7 v) + \kappa^*(\theta^* - v)(A_4 + 2A_5 v + A_7 x) = 0. \end{aligned} \tag{A.10}$$

Therefore, (A.1) can also be reduced to a system of linear ODEs

$$\frac{d}{d\tau}\begin{pmatrix} A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{pmatrix} = \begin{pmatrix} 0 & \kappa^*\theta^* & 0 & r & 0 \\ 0 & -\kappa^* & \sigma_V^2 + 2\kappa^*\theta^* & -1/2 & r + \rho\sigma_V \\ 0 & 0 & -2\kappa^* & 0 & -1/2 \\ 0 & 0 & 0 & 0 & \kappa^*\theta^* \\ 0 & 0 & 0 & 0 & -\kappa^* \end{pmatrix}\begin{pmatrix} A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2r \\ -1 \end{pmatrix}. \tag{A.11}$$

Similarly, the zero initial conditions obtained by using the terminal condition (A.2) are

$$A_k(0) = 0, \quad \text{for } k = 3, 4, \dots, 7. \tag{A.12}$$

We solve (A.11) subject to (A.12) by using Mathematica to obtain A_3 and A_4 as expressed in (2.12) and (2.13), respectively, where A_5, A_6, A_7 satisfy the relations

$$\begin{cases} A_5(\tau) = A_2^2(\tau), \\ A_6(\tau) = 2A_1(\tau), \\ A_7(\tau) = 2A_2(\tau), \end{cases}$$

for all $\tau \geq 0$. This completes the proof of Proposition 2.3. □

Appendix B

Proof of Proposition 2.6. Utilizing Proposition 2.3, the conditional expectations with respect to $\mathcal{F}_{t_{i-1}}$ on the right-hand side of (2.7) can be written as

$$\begin{aligned} &E^Q_{t_{i-1}}[\ln^2(S_{t_i})] - 2\ln(S_{t_{i-1}})E^Q_{t_{i-1}}[\ln(S_{t_i})] + \ln^2(S_{t_{i-1}}) \\ &= E^Q_{t_{i-1}}[X_{t_i}^2] - 2X_{t_{i-1}}E^Q_{t_{i-1}}[X_{t_i}] + X_{t_{i-1}}^2 \\ &= X_{t_{i-1}}^2 + A_3(\Delta t) + A_4(\Delta t)v_{t_{i-1}} + A_2^2(\Delta t)v_{t_{i-1}}^2 + 2A_1(\Delta t)X_{t_{i-1}} \\ &\quad + 2A_2(\Delta t)X_{t_{i-1}}v_{t_{i-1}} - 2X_{t_{i-1}}(X_{t_{i-1}} + A_1(\Delta t) + A_2(\Delta t)v_{t_{i-1}}) + X_{t_{i-1}}^2. \end{aligned} \tag{B.1}$$

Since the terms $X_{t_{i-1}}, X^2_{t_{i-1}}$ and $X_{t_{i-1}} v_{t_{i-1}}$ on the right-hand side of (B.1) can be cancelled, it is simplified to

$$E^Q_{t_{i-1}}[\ln^2(S_{t_i})] - 2\ln(S_{t_{i-1}})E^Q_{t_{i-1}}[\ln(S_{t_i})] + \ln^2(S_{t_{i-1}}) = A_3(\Delta t) + A_4(\Delta t)v_{t_{i-1}} + A_2^2(\Delta t)v^2_{t_{i-1}} \tag{B.2}$$

where $\Delta t = t_i - t_{i-1}$. Inserting the right-hand side of (B.2) into the right-hand side of (2.7) gives

$$E^Q_0\Big[\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] = E^Q_0[A_3(\Delta t) + A_4(\Delta t)v_{t_{i-1}} + A_2^2(\Delta t)v^2_{t_{i-1}}]. \tag{B.3}$$

Notice that $A_k(\Delta t)$ for $k = 2, 3, 4$ do not depend on v_{i-1}. In fact, these functions are deterministic functions depending on Δt. Hence, using the linearity of conditional expectations, the conditional expectation on the right-hand side of (B.3) can be written as

$$E^Q_0\Big[\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] = A_3(\Delta t) + A_4(\Delta t)E^Q_0[v_{t_{i-1}}] + A_2^2(\Delta t)E^Q_0[v^2_{t_{i-1}}] \tag{B.4}$$

for all $i = 1, 2, \ldots, N$. The two conditional expectations on the right-hand side of (B.4) are, respectively, the first and second conditional moments of the square-root process, $dv_t = \kappa^*(\theta^* - v_t)dt + \sigma_V\sqrt{v_t}d\tilde{B}^V_t$. The formulae for these two conditional moments have been found in explicit form (see the proof of Broadie and Jain [4, Proposition 5]) as

$$\begin{aligned}
E^Q_0[v_{t_{i-1}}] &= \theta^*(1 - e^{-\kappa^* t_{i-1}}) + e^{-\kappa^* t_{i-1}}v_0,\\
E^Q_0[v^2_{t_{i-1}}] &= \frac{\theta^*\sigma_V^2}{2\kappa^*}(1 - e^{-\kappa^* t_{i-1}})^2 + \frac{\sigma_V^2}{\kappa^*}(e^{-\kappa^* t_{i-1}} - e^{-2\kappa^* t_{i-1}})v_0\\
&\quad + (\theta^*\{1 - e^{-\kappa^* t_{i-1}}) + e^{-\kappa^* t_{i-1}}v_0\}^2.
\end{aligned}$$

Next, we employ the relations

$$\begin{aligned}
\theta^*(1 - e^{-\kappa^* t_{i-1}}) &= -2\kappa^*\theta^* A_2(t_{i-1}),\\
\frac{\theta^*\sigma_V^2}{2\kappa^*}(1 - e^{-\kappa^* t_{i-1}})^2 &= 2\kappa^*\theta^*(\sigma_V A_2(t_{i-1}))^2,\\
\frac{\sigma_V^2}{\kappa^*}(e^{-\kappa^* t_{i-1}} - e^{-2\kappa^* t_{i-1}}) &= -2\sigma_V^2 A_2(t_{i-1})e^{-\kappa^* t_{i-1}}
\end{aligned}$$

to express the first and second conditional moments in terms of A_2 as

$$E^Q_0[v_{t_{i-1}}] = -2\kappa^*\theta^* A_2(t_{i-1}) + e^{-\kappa^* t_{i-1}}v_0, \tag{B.5}$$

$$E^Q_0[v^2_{t_{i-1}}] = 2\kappa^*\theta^*(\sigma_V A_2(t_{i-1}))^2 - 2\sigma_V^2 A_2(t_{i-1})e^{-\kappa^* t_{i-1}}v_0 + \{-2\kappa^*\theta^* A_2(t_{i-1}) + e^{-\kappa^* t_{i-1}}v_0\}^2. \tag{B.6}$$

By substituting (B.5) and (B.6) into (B.4) and collecting the coefficients of v_0^k for $k = 0, 1, 2$, we have succeeded in obtaining the coefficient $\tilde{A}_k$ of v_0^k for $k = 0$, $1, 2$, as expressed in (2.15), (2.16) and (2.17), respectively, The last assertion of

Proposition 2.6 is trivial due to the fact that $\tilde{A}_k(0,t) = 0$, $k = 0, 1, 2$, for all $t \geq 0$ and $\tilde{A}_0(\Delta t, 0) = A_3(\Delta t)$, $\tilde{A}_1(\Delta t, 0) = A_4(\Delta t)$, and $\tilde{A}_2(\Delta t, 0) = A_2^2(\Delta t)$ for all $\Delta t \geq 0$. This completes the proof of Proposition 2.6. □

Appendix C

Proof of Proposition 3.1. First, we show that Δt_p^* defined in (3.1) is either strictly positive or infinite. From the system of linear ODEs (A.11) with the initial conditions (A.12), we have $(dA_4/d\tau)|_{\tau=0} = 1 > 0$. This implies that A_4 is strictly increasing on $(0, \Delta t_p^{A_4})$ for some $\Delta t_p^{A_4} > 0$. Moreover, it is easy to show that $(dA_3/d\tau)|_{\tau=0} = \kappa^*\theta^* A_4(0) + rA_6(0) = 0$ and $(d^2A_3/d\tau^2)|_{\tau=0} = \kappa^*\theta^*\{(dA_4/d\tau)|_{\tau=0}\} + r\{(dA_6/d\tau)|_{\tau=0}\} = \kappa^*\theta^* + 2r^2 > 0$. These results imply that A_3 has a local minimum at $\tau = 0$. Hence, there exists $\Delta t_p^{A_3} > 0$ such that $A_3(\tau; p) > 0$ for all $\tau \in (0, \Delta t_p^{A_3})$. Consequently, A_3 and A_4 are strictly positive on $(0, \min(\Delta t_p^{A_3}, \Delta t_p^{A_4}))$. Clearly, a positive root of A_3 or A_4 may or may not exist depending on p. Suppose that either A_3 or A_4 has a positive root. Since Δt_p^* is the smallest positive root of the product function $A_3(\tau; p)A_4(\tau; p)$, it follows that $\Delta t_p^* > \min(\Delta t_p^{A_3}, \Delta t_p^{A_4}) > 0$. On the other hand, if neither A_3 nor A_4 has a positive root, we immediately conclude that $\Delta t_p^* = \infty$.

Second, we consider $K_{\text{var}}(T, \Delta t, v_0; p)$ expressed in (2.19). Since A_3 and A_4 are strictly positive on $(0, \Delta t_p^*)$ and $A_2(t; p) < 0$ for all $t > 0$, we have $\tilde{A}_k(\Delta t, t; p) > 0$ for all $k = 0, 1, 2$, $\Delta t \in (0, \Delta t_p^*)$ and $t \in [0, T]$. Moreover, for any given $\Delta t \in (0, \Delta t_p^*)$ and $t \in [0, T]$, Assumption 2.1 is a sufficient condition to ensure that $\tilde{A}_k(\Delta t, t; p) < \infty$ for all $k = 0, 1, 2$. Then we obtain assertion 2.1. Next, one can see that assertion 2.2 is a consequence of assertion 2.1, since for any given $\Delta t \in (0, \Delta t_p^*)$ and $t_i = i\Delta t \in [0, T]$, $i = 0, 1, \ldots, N$,

$$\frac{\partial K_{\text{var}}}{\partial v_0} = \frac{100^2}{T}\bigg[\bigg\{\sum_{i=1}^{N} \tilde{A}_1(\Delta t, t_{i-1}; p)\bigg\} + 2\bigg\{\sum_{i=1}^{N} \tilde{A}_2(\Delta t, t_{i-1}; p)\bigg\} v_0\bigg] > 0, \quad \text{for all } v_0 > 0.$$

The proof of Proposition 3.1 is thus complete. □

Appendix D

Proof of Proposition 3.2. We begin with some useful identities. Set $\Delta t = T/N$ and $t_{i-1} = (i-1)\Delta t = (i-1)T/N$. From (2.15)–(2.17),

$$\begin{aligned}\sum_{i=1}^{N} \tilde{A}_0\Big(\frac{T}{N}, \frac{(i-1)T}{N}\Big) &= NA_3\Big(\frac{T}{N}\Big) - 2\kappa^*\theta^* A_4\Big(\frac{T}{N}\Big)\sum_{i=1}^{N} A_2\Big(\frac{(i-1)T}{N}\Big) + \Big(\kappa^*\theta^* + \frac{\sigma_V^2}{2}\Big)\kappa^*\theta^*\Big(2A_2\Big(\frac{T}{N}\Big)\Big)^2 \\ &\quad \times \sum_{i=1}^{N} A_2^2\Big(\frac{(i-1)T}{N}\Big),\end{aligned} \tag{D.1}$$

$$\sum_{i=1}^{N}\tilde{A}_1\Big(\frac{T}{N},\frac{(i-1)T}{N}\Big)$$
$$= A_4\Big(\frac{T}{N}\Big)\sum_{i=1}^{N}e^{-\kappa^*(i-1)T/N} - 2(2\kappa^*\theta^* + \sigma_V^2)A_2^2\Big(\frac{T}{N}\Big)\sum_{i=1}^{N}e^{-\kappa^*(i-1)T/N}A_2\Big(\frac{(i-1)T}{N}\Big), \tag{D.2}$$

$$\sum_{i=1}^{N}\tilde{A}_2\Big(\frac{T}{N},\frac{(i-1)T}{N}\Big) = A_2^2\Big(\frac{T}{N}\Big)\sum_{i=1}^{N}e^{-2\kappa^*((i-1)T/N)}. \tag{D.3}$$

Using the identity

$$\sum_{i=1}^{N}e^{-x(i-1)T/N} = \frac{e^{-xT}(e^{xT}-1)}{1-e^{-xT/N}}$$

for $x > 0$, we can derive the identities

$$\sum_{i=1}^{N}e^{-\kappa^*(i-1)T/N} = \frac{e^{-\kappa^*T}(e^{\kappa^*T}-1)}{1-e^{-\kappa^*T/N}}, \tag{D.4}$$

$$\sum_{i=1}^{N}e^{-2\kappa^*(i-1)T/N} = \frac{e^{-2\kappa^*T}(e^{2\kappa^*T}-1)}{1-e^{-2\kappa^*T/N}}, \tag{D.5}$$

$$\sum_{i=1}^{N}A_2\Big(\frac{(i-1)T}{N}\Big) = \sum_{i=1}^{N}\Big(\frac{e^{-\kappa^*(i-1)T/N}-1}{2\kappa^*}\Big) = \frac{e^{\kappa^*(1-N)T/N} + (N-1)e^{\kappa^*T/N} - N}{2\kappa^*(1-e^{\kappa^*T/N})}, \tag{D.6}$$

$$\sum_{i=1}^{N}A_2^2\Big(\frac{(i-1)T}{N}\Big)$$
$$= \frac{2e^{(N+1)\kappa^*T/N} + 2e^{(N+2)\kappa^*T/N} + (N-1)e^{2(N+1)\kappa^*T/N} - Ne^{2\kappa^*T/N} - 2e^{\kappa^*(2N+1)T/N} - e^{2\kappa^*T/N}}{4(\kappa^*)^2e^{2\kappa^*T}(e^{2\kappa^*T/N}-1)}, \tag{D.7}$$

$$\sum_{i=1}^{N}e^{-\kappa^*(i-1)T/N}A_2((i-1)T/N) = \frac{(e^{\kappa^*T}-1)e^{\kappa^*T(1/N-2)}(e^{\kappa^*T/N}-e^{\kappa^*T})}{2\kappa^*(e^{2\kappa^*T/N}-1)}. \tag{D.8}$$

□

Proof of property (3.4). Inserting (D.6) and (D.7) into (D.1) gives

$$\sum_{i=1}^{N}\tilde{A}_0\Big(\frac{T}{N},\frac{(i-1)T}{N}\Big) = F_0(N)\times\{D_0(N) + D_1(N)N + D_2(N)N^2\} \tag{D.9}$$

where

$$F_0(N) = \frac{1}{8(\kappa^*)^3N(e^{\kappa^*T/N}-1)(e^{\kappa^*T/N}+1)},$$
$$D_0(N) = 2\kappa^*Te^{-\kappa^*T}(1+e^{\kappa^*T/N})\{\tilde{c}_1(e^{\kappa^*T/N}-1)+\tilde{c}_2\}, \tag{D.10}$$

$$D_1(N) = \theta^* e^{-2\kappa^* T}(e^{\kappa^* T/N} - 1) \times \{\tilde{c}_3(e^{\kappa^* T/N} - 1) - \sigma_V^2 e^{\kappa^* T/N} + \tilde{c}_4 e^{(1+N)\kappa^* T/N} + \tilde{c}_5 e^{(2+1/N)\kappa^* T} + \tilde{c}_6\}, \quad \text{(D.11)}$$

$$D_2(N) = 2\theta^* \sigma_V(4\kappa^* \rho - \sigma_V) e^{-\kappa^* T/N}(e^{\kappa^* T/N} - 1)^2 (e^{\kappa^* T/N} + 1) \quad \text{(D.12)}$$

and

$$\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \\ \tilde{c}_4 \\ \tilde{c}_5 \\ \tilde{c}_6 \end{pmatrix} = \begin{pmatrix} 4r^2(\kappa^*)^2 T e^{\kappa^* T} - 4r\kappa^* \theta^*(1 + e^{\kappa^* T}(\kappa^* T - 1)) + \kappa^*(\theta^*)^2(2 + e^{\kappa^* T}(\kappa^* T - 2)) \\ -2\theta^* \sigma_V(2\kappa^* \rho - \sigma_V)(e^{\kappa^* T} - 1) \\ 2\kappa^* \theta^*(e^{2\kappa^* T} - 1) \\ 4(2(\kappa^*)^2 - 2\kappa^* \rho \sigma_V + \sigma_V^2) \\ 8(\kappa^*)^2(\kappa^* T - 1) - 8\kappa\rho\sigma_V{}^*(\kappa^* T - 1) + (2\kappa^* T - 3)\sigma_V^2 \\ \sigma_V^2 + \tilde{c}_4 e^{\kappa^* T} + (\tilde{c}_5 - 2\sigma_V^2) e^{2\kappa^* T} \end{pmatrix}.$$

Next, we use L'Hôpital's rule to calculate

$$\lim_{N\to\infty} F_0(N) = \frac{1}{16(\kappa^*)^4 T}. \quad \text{(D.13)}$$

From (D.10) we have

$$\lim_{N\to\infty} D_0(N) = -8\kappa^* \theta^* \sigma_V(2\kappa^* \rho - \sigma_V)(e^{\kappa^* T} - 1) e^{-\kappa^* T} T, \quad \text{(D.14)}$$

and from (D.11) and (D.12) we can show that

$$\lim_{N\to\infty} D_j(N) = 0, \quad \text{for } j = 1, 2. \quad \text{(D.15)}$$

From (D.15), using L'Hôpital's rule, we get

$$\lim_{N\to\infty} D_1(N)N = 4\kappa^* \theta^* e^{-\kappa^* T} T[4(\kappa^*)^3 e^{\kappa^* T} T - 2(e^{\kappa^* T} - 1)\sigma_V^2 + \kappa^* \sigma_V\{4\rho(e^{\kappa^* T} - 1) + \sigma_V e^{\kappa^* T} T\} - 4(\kappa^*)^2\{e^{\kappa^* T}(\rho\sigma_V T + 1) - 1\}] \quad \text{(D.16)}$$

and

$$\lim_{N\to\infty} D_2(N)N^2 = 4(\kappa^*)^2 \theta^* \sigma_V(4\kappa^* \rho - \sigma_V) T^2. \quad \text{(D.17)}$$

From (D.9), using the limits (D.13), (D.14), (D.16) and (D.17), we obtain property (3.4). □

PROOF OF PROPERTY (3.5). Inserting (D.4) and (D.8) into (D.2) gives

$$\sum_{i=1}^{N} \tilde{A}_1\left(\frac{T}{N}, \frac{(i-1)T}{N}\right) = F_1(N) \times (D_3(N) + D_4(N)N) \quad \text{(D.18)}$$

where

$$F_1(N) = -\frac{e^{-2\kappa^* T}(e^{\kappa^* T} - 1)}{4N(\kappa^*)^3(e^{2\kappa^* T/N} - 1)}, \quad \text{(D.19)}$$

$$D_3(N) = 2\kappa^* T e^{\kappa^* T}(e^{\kappa^* T/N} + 1)\{-2r\kappa^* + \kappa^*(2r - \theta^*) e^{\kappa^* T/N} + \kappa^* \theta^* - 2\kappa^* \rho \sigma_V + \sigma_V^2\},$$

$$D_4(N) = (e^{\kappa^* T/N} - 1)\{\tilde{c}_7(e^{\kappa^* T/N} - 1) + \sigma_V^2 e^{\kappa^* T/N} + \tilde{c}_8 e^{(1+N)\kappa^* T/N} + \tilde{c}_9\} \quad \text{(D.20)}$$

and

$$\begin{pmatrix} \tilde{c}_7 \\ \tilde{c}_8 \\ \tilde{c}_9 \end{pmatrix} = \begin{pmatrix} 2\kappa^*\theta^*(1 + e^{\kappa^* T}) \\ -\{4(\kappa^*)^2 - 4\kappa^*\rho\sigma_V + \sigma_V^2\} \\ -\sigma_V^2 + \{-4(\kappa^*)^2 + 4\kappa^*\rho\sigma_V - 3\sigma_V^2\}e^{\kappa^* T} \end{pmatrix}. \quad \text{(D.21)}$$

Applying L'Hôpital's rule to (D.19) yields

$$\lim_{N\to\infty} F_1(N) = \frac{(1 - e^{\kappa^* T})}{8(\kappa^*)^4 e^{2\kappa^* T} T}. \quad \text{(D.22)}$$

In addition,

$$\lim_{N\to\infty} D_3(N) = -4\kappa^*\sigma_V(2\kappa^*\rho - \sigma_V)e^{\kappa^* T} T. \quad \text{(D.23)}$$

Bearing in mind that

$$\lim_{N\to\infty} D_4(N) = 0,$$

we have

$$\lim_{N\to\infty} D_4(N)N = -4\kappa^*(2(\kappa^*)^2 - 2\kappa^*\rho\sigma_V + \sigma_V^2)e^{\kappa^* T} T. \quad \text{(D.24)}$$

From (D.18), utilizing the limits (D.22), (D.23) and (D.24), we immediately obtain property (3.5). □

Proof of property (3.6). Inserting (D.5) into (D.3) gives

$$\sum_{i=1}^{N} \tilde{A}_2\Big(\frac{T}{N}, \frac{(i-1)T}{N}\Big) = \frac{(e^{2\kappa^* T} - 1)(e^{\kappa^* T/N} - 1)}{4(\kappa^*)^2 e^{2\kappa^* T}(e^{\kappa^* T/N} + 1)}. \quad \text{(D.25)}$$

From (D.25),

$$\lim_{N\to\infty} \sum_{i=1}^{N} \tilde{A}_2\Big(\frac{T}{N}, \frac{(i-1)T}{N}\Big) = \lim_{N\to\infty} \frac{(e^{2\kappa^* T} - 1)(e^{\kappa^* T/N} - 1)}{4(\kappa^*)^2 e^{2\kappa^* T}(e^{\kappa^* T/N} + 1)} = 0,$$

and this proves property (3.6). □

Proof of property (3.7). Using (2.19) and properties (3.4)–(3.6), we obtain (3.7). The proof of Proposition 3.2 is now complete. □

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