

CERTAIN CLASSIFICATIONS ON SURFACES OF REVOLUTION IN A SEMI-ISOTROPIC SPACE

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ABSTRACT. A semi-isotropic space is a real affine 3-space endowed with the non-degenerate metric $dx^2 - dy^2$. The main purpose of this paper is to describe the surfaces of revolution in the semi-isotropic space that satisfy some equations in terms of the position vector and the Laplace operators with respect to the first and the second fundamental forms.

1. INTRODUCTION

The *isotropic 3-space* \mathbb{I}^3 is one of the Cayley-Klein 3-spaces and can be defined in the real projective 3-space. One can also appear as the affine 3-space \mathbb{R}^3 equipped with the semi-norm (see [1, 2, 28, 32, 33])

$$\|u\|_{\mathbb{I}} = \sqrt{(u_1)^2 + (u_2)^2}, \quad u = (u_1, u_2, u_3) \in \mathbb{I}^3.$$

The isotropic geometry has remarkable applications in Image Processing, architectural design and microeconomics, see [10, 13, 20], [29]-[31].

The fundamentals of curves and surfaces in \mathbb{I}^3 can be found in H. Sachs' monograph [32]. For further studies of these in \mathbb{I}^3 we refer to [6, 14, 18, 23, 24, 27].

A *semi-isotropic 3-space* \mathbb{SI}^3 is the product of the Lorentz-Minkowski 2-space \mathbb{E}_1^2 and the isotropic line equipped with a degenerate parabolic distance metric. More precisely, \mathbb{SI}^3 is the affine 3-space \mathbb{R}^3 endowed with the semi-norm

$$\|u\| = \sqrt{|(u_1)^2 - (u_2)^2|}, \quad u = (u_1, u_2, u_3) \in \mathbb{SI}^3.$$

The local theory of non-null curves and surfaces in \mathbb{SI}^3 was recently stated in [3]. This has caused that many (open or solved) problems of classic differential geometry can be treated to \mathbb{SI}^3 . For example, the surfaces of revolution in \mathbb{SI}^3 with constant curvature were classified in [3].

In this paper we aim to present the surfaces of revolution in \mathbb{SI}^3 that satisfy certain conditions in terms of the coordinate functions of the position vector and the Laplace operator.

These are natural, being related to the so-called *submanifolds of finite type*, introduced by B.-Y. Chen in the late 1970's (see [7]-[9],[11]).

Let r be an isometric immersion of a Riemannian manifold M into the Euclidean n -space \mathbb{E}^n and Δ denote the Laplace operator of M . Then it is said to be of *finite type* if its position vector field can be expressed as

$$r = c + r_0 + r_1 + \dots + r_k,$$

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where $c \in \mathbb{R}^n$ and x_i non-constant \mathbb{R}^n -valued maps such that

$$\Delta r_i = \lambda_i r_i, \quad 1 \leq i \leq k.$$

If $\lambda_1, \dots, \lambda_k$ are mutually different, then the immersion is said to be of k -type. If one of $\lambda_1, \dots, \lambda_k$ is zero, then the immersion is said to be of *null k -type*.

Additionally, O. Garay classified the surfaces of revolution in \mathbb{R}^3 satisfying the following relation ([15, 16])

$$(1.1) \quad \Delta r_i = \lambda_i r_i,$$

where r^i are component functions of the position vector. This idea was generalized to \mathbb{R}_1^3 in [5, 12, 17].

The results of this paper inherently resemble to these of existing ones in literature; however, the discrepancies also arise. For instance, in Theorem 4.2, we prove that a surface of revolution in $\mathbb{S}\mathbb{I}^3$ that satisfy (1.1) is formed by rotating the graph of a Bessel function.

Such functions which correspond to the solutions of Bessel's equation are named due to the German mathematician and astronomer Frederic Wilhelm Bessel (1784–1846) who first used them to analyze planetary orbits. The Bessel functions are widely used in classical physics, elasticity theory, heat conduction theory etc. (see [21]). We give a brief review of the Bessel functions in §3.

Moreover we state that the surfaces of revolution in $\mathbb{S}\mathbb{I}^3$ that satisfy

$$\Delta^{II} r_i = \lambda_i r_i$$

are only (s-i)-minimal ones, where Δ^{II} is the Laplace operator with respect to the second fundamental form (see Theorem 4.3, Corollary 4.1).

2. PRELIMINARIES

We provide the basics of semi-isotropic space from [3].

The *semi-isotropic geometry* is based on the six-parameter group of affine transformations

$$(2.1) \quad \begin{cases} \mathbf{x}' = a_1 + \mathbf{x} \cosh a_2 + \mathbf{y} \sinh a_2, \\ \mathbf{y}' = a_3 + \mathbf{x} \sinh a_2 + \mathbf{y} \cosh a_2, \\ \mathbf{z}' = a_4 + a_5 \mathbf{x} + a_6 \mathbf{y} + \mathbf{z}, \end{cases} \quad a_i \in \mathbb{R}, \quad i = 1, \dots, 6.$$

These are called *semi-isotropic congruence transformations* or *(s-i)-motions*.

We notice that the (s-i)-motions are a composition of a Lorentzian motion in \mathbf{xy} -plane and an affine shear transformation in \mathbf{z} -direction. For detailed properties of the Lorentzian motions, see [22, 26].

The *semi-isotropic scalar product* between two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{S}\mathbb{I}^3$ is given by

$$\langle u, v \rangle = \begin{cases} u_3 v_3, & \text{if } u_1 = u_2 = v_1 = v_2 = 0, \\ u_1 v_1 - u_2 v_2, & \text{otherwise.} \end{cases}$$

The *vector product* in the sense of semi-isotropic space is

$$u \times v = \begin{vmatrix} (1, 0, 0) & (0, -1, 0) & (0, 0, 0) \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

For some vector $w = (w_1, w_2, w_3) \in \mathbb{S}\mathbb{I}^3$, denote \tilde{w} its canonical projection onto \mathbb{E}_1^2 , i.e. $\tilde{w} = (w_1, w_2, 0)$. It is then seen that

$$\langle u \times v, w \rangle = \det(u, v, \tilde{w}).$$

We call the vector $u \neq 0$ *isotropic* if \tilde{u} is a zero vector. If $u = 0$ or $\tilde{u} \neq 0$ it is called *non-isotropic*.

A non-isotropic vector $u \in \mathbb{S}\mathbb{I}^3$ is respectively called *spacelike*, *timelike* and *null* (or *lightlike*) if $\langle u, u \rangle > 0$ or $u = 0$, $\langle u, u \rangle < 0$ and $\langle u, u \rangle = 0$ ($u \neq 0$).

The *null-cone* and the *timelike-cone* of $\mathbb{S}\mathbb{I}^3$ are respectively given by

$$\mathcal{C} = \{(x, y, z) \in \mathbb{S}\mathbb{I}^3 \mid x^2 - y^2 = 0\} - \{0 \in \mathbb{S}\mathbb{I}^3\}.$$

and

$$\mathcal{T} = \{(x, y, z) \in \mathbb{S}\mathbb{I}^3 \mid x^2 - y^2 < 0\}.$$

The *semi-isotropic angle* between two timelike vectors $u, v \in \mathbb{S}\mathbb{I}^3$ is defined as

$$\langle u, v \rangle = -\|u\| \|v\| \cosh \phi.$$

Note that all isotropic vectors are orthonogonal to non-isotropic ones. Also, two non-isotropic vectors u, v in $\mathbb{S}\mathbb{I}^3$ are orthonogonal if $\langle u, v \rangle = 0$.

Let $\alpha(s)$ be a regular curve in $\mathbb{S}\mathbb{I}^3$, i.e. $\alpha'(s) = \frac{d\alpha}{ds} \neq 0$ for all s . Then it is said to be *admissible* if $\alpha(s)$ has no isotropic tangent vector, i.e. $\tilde{\alpha}'(s) \neq 0$. An admissible curve $\alpha(s)$ in $\mathbb{S}\mathbb{I}^3$ is called *spacelike* (resp. *timelike*, *null*) if $\alpha'(s)$ is spacelike (resp. timelike, null) for all s .

2.1. Planes, circles and spheres in $\mathbb{S}\mathbb{I}^3$. The lines in \mathbf{z} -direction are called *isotropic lines*. The planes containing an isotropic line are called *isotropic planes*. Other planes are *non-isotropic*.

In the non-isotropic planes the Lorentzian metric is basically used. Let Γ be a non-isotropic plane and $c_0 = (x_0, y_0, z_0)$ a fixed point in Γ . The iso-distance set of c_0 in Γ is

$$\{p = (x, y, z) \in \mathbb{S}\mathbb{I}^3 : \|p - c_0\| = r, r \in \mathbb{R}^+\}.$$

The projection of such a set onto \mathbf{xy} -plane is a rectangular hyperbola. We call it a *semi-isotropic circle* (or *(s-i)-circle*) of *hyperbolic type*.

In the isotropic planes we have an isotropic metric. The *(s-i)-circle of parabolic type* is a parabola with \mathbf{z} -axis lying in an isotropic plane. An (s-i)-circle of parabolic type is not the iso-distance set of a fixed point.

We have two types of semi-isotropic spheres. One is the *semi-isotropic sphere* (or *(s-i)-sphere*) of *cylindrical type* which is the set of all points $p \in \mathbb{S}\mathbb{I}^3$ with $\|p - c_0\| = r$. This sphere is a right Lorentz hyperbolic cylinder with \mathbf{z} -parallel rulings from the Lorentz-Minkowski perspective.

Other one is the *(s-i)-sphere of parabolic type*,

$$z = r(x^2 - y^2) + ax + by + c, \quad r > 0.$$

These are indeed hyperbolic paraboloids.

The intersections of these (s-i)-spheres with non-isotropic (resp. isotropic) planes are (s-i)-circles of hyperbolic type (resp. of parabolic type).

2.2. Spacelike and timelike surfaces in $\mathbb{S}\mathbb{I}^3$. Let M be an admissible surface immersed in $\mathbb{S}\mathbb{I}^3$, i.e., without isotropic tangent planes. Denote g the metric on M induced from $\mathbb{S}\mathbb{I}^3$. The surface M is said to be *spacelike* (resp. *timelike*, *null*) if g is positive definite (resp. a metric with index 1, degenerate).

The following contains the formulas for only spacelike and timelike admissible surfaces in $\mathbb{S}\mathbb{I}^3$.

Assume that M has a local parameterization of the form

$$r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{S}\mathbb{I}^3 : (u, v) \longmapsto (x(u, v), y(u, v), z(u, v)).$$

The components of the first fundamental form I with respect to the basis $\{r_u, r_v\}$ are

$$E = \langle r_u, r_u \rangle, \quad F = \langle r_u, r_v \rangle, \quad G = \langle r_v, r_v \rangle,$$

where $r_u = \partial r / \partial u$. M is timelike (spacelike) when $W = EG - F^2 < 0$ (> 0).

Denote Δ the Laplace operator of M . For a smooth function $\psi : M \longrightarrow \mathbb{R}$, $(u, v) \longmapsto \psi(u, v)$ its Laplacian is defined as

$$(2.2) \quad \Delta \psi = -\frac{1}{\sqrt{|W|}} \left\{ \frac{\partial}{\partial u} \left(\frac{G\psi_u - F\psi_v}{\sqrt{|W|}} \right) - \frac{\partial}{\partial v} \left(\frac{F\psi_u - E\psi_v}{\sqrt{|W|}} \right) \right\}.$$

The unit normal vector field of M is the isotropic vector $(0, 0, 1)$ since it is perpendicular to all non-isotropic vectors.

The components of the second fundamental form II are given by

$$L = \frac{\det(r_{uu}, r_u, r_v)}{\sqrt{|W|}}, \quad M = \frac{\det(r_{uv}, r_u, r_v)}{\sqrt{|W|}}, \quad N = \frac{\det(r_{vv}, r_u, r_v)}{\sqrt{|W|}}$$

for $r_{uu} = \frac{\partial^2 r}{\partial u^2}$, etc.

Thus the *semi-relative curvature* and the *semi-isotropic mean curvature* of M are respectively defined by

$$K = -\varepsilon \frac{LN - M^2}{W} \quad \text{and} \quad H = -\varepsilon \frac{EN - 2FM + GL}{2W},$$

where $\varepsilon = \text{sgn}(W)$.

Assume that M has no parabolic points, i.e. $w = LN - M^2 \neq 0$. Then the Laplace operator with respect to II is given by

$$(2.3) \quad \Delta^{II} \psi = -\frac{1}{\sqrt{|w|}} \left\{ \frac{\partial}{\partial u} \left(\frac{N\psi_u - M\psi_v}{\sqrt{|w|}} \right) - \frac{\partial}{\partial v} \left(\frac{M\psi_u - L\psi_v}{\sqrt{|w|}} \right) \right\}.$$

3. BESSEL'S DIFFERENTIAL EQUATION

This section is devoted to recall the solutions of the following linear second-order ordinary differential equation (ODE)

$$(3.1) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2) y = 0$$

for a real positive constant p (see [4, 19, 25]). This is called *Bessel's equation of order p* . If the sign of the term $(x^2 - p^2)$ is negative, then it is called *Bessel's modified equation of order p* . Since the ODE (3.1) has a singular point at $x = 0$, a series solution for (3.1) via the method of Frobenius can be construct. Before this, it is useful to express the gamma function and the Pockhammer symbol.

The *gamma function* is defined as

$$\Gamma(x) = \int_0^\infty e^{-q} q^{x-1} dq, \quad x > 0.$$

If $x = n$ is a positive integer, then it can be seen that

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \dots = \\ &= n(n-1)\dots 2.1 = n!. \end{aligned}$$

The *Pockhammer symbol* provides a simplicity for the product of too many terms and is given by

$$(k)_n = k(k+1)(k+2)\dots(k+n-1).$$

In general $(k)_0 = 1$.

The indicial equation for (3.1) implies the roots $r_{1,2} = \pm p$. When $2p$ is not an integer, the method of Frobenius gives two linearly independent solutions as follows

$$y_{1,2}(x) = \frac{x^{\pm p}}{2^{\pm p}\Gamma(1 \pm p)} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (1 \pm p)_n n!}.$$

These are called *Bessel functions of order $\pm p$* denoted by $J_{\pm p}$. Thus the general solution of (3.1) becomes

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x), \quad c_1, c_2 \in \mathbb{R}.$$

In case $p = 0$, by applying again the method of Frobenius, we obtain that the first solution is

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

and second

$$Y_0(x) = \frac{2}{\pi} \left\{ \left(\ln \frac{x}{2} + \gamma \right) J_0 - \sum_{n=0}^{\infty} \frac{(-1)^n \phi(n) x^{2n}}{2^{2n} (n!)^2} \right\},$$

where $\phi(n) = \sum_{m=1}^n \frac{1}{m}$ and γ Euler-Mascheroni constant. Such functions are respectively called the *Bessel function of the first kind of order zero* and *Weber's Bessel function of order zero*, see Fig. 1(a).

For the Bessel's modified equation of order zero, the general solution is

$$y(x) = c_1 I_0(x) + c_2 K_0(x), \quad c_1, c_2 \in \mathbb{R},$$

where

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n} (n!)^2}$$

and

$$K_0(x) = - \left(\ln \frac{x}{2} + \gamma \right) I_0(x) + \sum_{n=0}^{\infty} \frac{\phi(n) x^{2n}}{2^{2n} (n!)^2},$$

which are respectively called *modified Bessel functions of first and second kind of order 0*, see Fig 1(b).

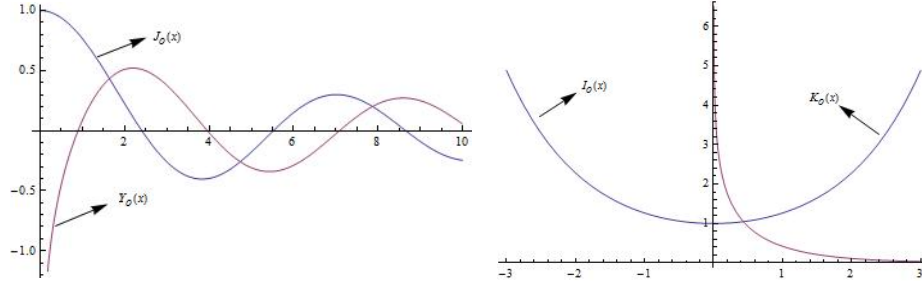


Figure1(a). Bessel functions of order zero, $x \in [0, 10]$. Figure1(b). Modified Bessel functions of order zero, $x \in [-3, 3]$.

4. CLASSIFICATIONS ON SURFACES OF REVOLUTION IN $\mathbb{S}\mathbb{I}^3$

Let $\gamma(u) = (0, u, f(u))$ (resp. $\gamma(u) = (u, 0, f(u))$) be a timelike (resp. spacelike) admissible curve in $\mathbb{S}\mathbb{I}^3$. By rotating them around \mathbf{z} -axis via the transformations (2.1) we derive

$$(4.1) \quad r(u, v) = (u \sinh v, u \cosh v, f(u))$$

and

$$(4.2) \quad r(u, v) = (u \cosh v, u \sinh v, f(u)).$$

These are called *surfaces of revolution* in $\mathbb{S}\mathbb{I}^3$. Also we call γ *profile curve*. Remark that such surfaces of $\mathbb{S}\mathbb{I}^3$ are timelike since $EG - F^2 = -u^2$.

The (s-r)-curvature K and (s-i)-mean curvature H of these surfaces in $\mathbb{S}\mathbb{I}^3$ are

$$K = \frac{f'f''}{u} \text{ and } H = \frac{1}{2} \left(\frac{f'}{u} + f'' \right),$$

where $f'(u) = \frac{df}{du}$ and so on.

The following classifies the surfaces of revolution in $\mathbb{S}\mathbb{I}^3$ with $K = \text{const.}$ and $H = \text{const.}$

Theorem 4.1. [3] *Let M be a surface of revolution in $\mathbb{S}\mathbb{I}^3$. Then the following statements hold:*

(i) *M has nonzero constant (s-r)-curvature K_0 if and only if its profile curve is of the form*

$$\begin{cases} (u, 0, f(u)) \text{ or } (0, u, f(u)) \\ f(u) = \frac{u}{2}\psi(u) + \frac{c_1}{2\sqrt{K_0}} \ln |2\sqrt{K_0}(\sqrt{K_0}x + \psi(u))|, \\ \psi(u) = \sqrt{c_1 + K_0 u^2}, c_1, c_2 \in \mathbb{R}. \end{cases}$$

In the particular case $K = 0$, f becomes a linear function.

(ii) *M has constant (s-im)-curvature H_0 if and only if its profile curve is given by*

$$\begin{cases} (u, 0, f(u)) \text{ or } (0, u, f(u)) \\ f(u) = \frac{H_0}{2}u^2 + c_1 \ln u + c_2, c_1, c_2 \in \mathbb{R}. \end{cases}$$

Now let M be a surface of revolution in $\mathbb{S}\mathbb{I}^3$ given by (4.1). Then we have

$$(4.3) \quad r_1(u, v) = u \sinh v, r_2(u, v) = u \cosh v, r_3(u, v) = f(u).$$

Denote Δ the Laplacian of M . It follows from (2.2) and (4.3) that

$$(4.4) \quad \Delta r_1 = \Delta r_2 = 0$$

and

$$(4.5) \quad \Delta r_3 = -f'' - \frac{1}{u}f'.$$

Assume that M is not (s-i) minimal and $\Delta r_i = \lambda_i r_i$, $i = 1, 2, 3$. Then from (4.4) and (4.5) we find that $\lambda_1 = \lambda_2 = 0$ and

$$(4.6) \quad f'' + \frac{1}{u}f' + \lambda_3 f = 0, \quad \lambda_3 \neq 0.$$

If $\lambda_3 > 0$ (resp. < 0) in (4.6), then it becomes a Bessel's (resp. modified) ODE of order zero. The general solutions for (4.6) turn to

$$f(u) = c_1 J_0(\sqrt{\lambda_3}u) + c_2 Y_0(\sqrt{\lambda_3}u)$$

and (in case $\lambda_3 < 0$)

$$f(u) = c_1 I_0(\sqrt{-\lambda_3}u) + c_2 K_0(\sqrt{-\lambda_3}u), \quad c_1, c_2 \in \mathbb{R}.$$

Theorem 4.2. *Let M be a surface of revolution in $\mathbb{S}\mathbb{I}^3$ given by (4.1). If $\Delta r_i = \lambda_i r_i$ then $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, \lambda \neq 0)$ and its generating curve becomes either*

$$(0, u, c_1 J_0(\sqrt{\lambda}u) + c_2 Y_0(\sqrt{\lambda}u))$$

or

$$(0, u, c_1 I_0(\sqrt{-\lambda}u) + c_2 K_0(\sqrt{-\lambda}u)), \quad c_1, c_2 \in \mathbb{R}.$$

Remark 4.1. Theorem 4.2 asserts that a surface of revolution in $\mathbb{S}\mathbb{I}^3$ that satisfy $\Delta r_i = \lambda_i r_i$ is of null 2-type.

Example 4.1. Consider the surface of revolution M in $\mathbb{S}\mathbb{I}^3$ given by

$$(u \sinh v, u \cosh v, J_0(u)), \quad (u, v) \in [1, 4] \times [-1, 1].$$

Then M is of null 2-type and thus we draw it and its profile curve as in Fig 2.

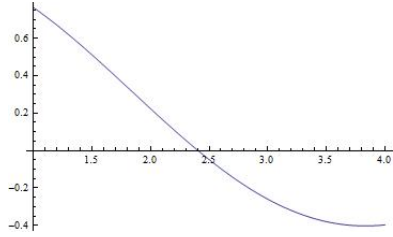


Figure2(a). Profile curve $(0, u, J_0(u))$.

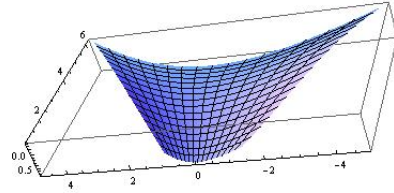


Figure2b. A surface of revolution of null 2-type.

Next suppose that M has no parabolic points. Then $LN - M^2 = -f'f''u \neq 0$, i.e. f is a non-linear function. In terms of the Laplace operator Δ^{II} we get

$$(4.7) \quad \begin{cases} \Delta^{II}(r_1(u, v)) = B(u) \sinh v - \frac{1}{f'(u)} \sinh v, \\ \Delta^{II}(r_2(u, v)) = B(u) \cosh v - \frac{1}{f'(u)} \cosh v, \\ \Delta^{II}(r_3(u, v)) = B(u) f'(u) + 1, \end{cases}$$

where

$$(4.8) \quad B(u) = \frac{1}{2f''(u)} \left(\frac{f'(u) + uf''(u)}{f'(u)u} - \frac{f'''(u)}{f''(u)} \right).$$

Taking $\triangle^{II}r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$, in (4.7) gives

$$(4.9) \quad \begin{cases} B(u) - \frac{1}{f'(u)} = \lambda u \\ B(u) f'(u) + 1 = \mu f. \end{cases}$$

where $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \mu$. In order to solve (4.9) we have to distinguish several cases.

Case 1. $\lambda = \mu = 0$. This immediately yields a contradiction from (4.9).

Case 2. $\lambda = 0$ and $\mu \neq 0$. From first line of (4.9), we get $B(u) = \frac{1}{f'(u)}$ and considering it into the second line of (4.9) we obtain a contradiction since f must be a non-linear function.

Case 3. $\lambda \neq 0$ and $\mu = 0$. The second line of (4.9) gives $B(u) = \frac{-1}{f'(u)}$ and putting it into the first line of (4.9) yields

$$f(u) = \frac{-2}{\lambda} \ln u + c, \quad c \in \mathbb{R}.$$

Note that in this case the surface of revolution M is (s-i)-minimal.

Case 4. $\lambda \neq 0 \neq \mu$. It follows from (4.9) that

$$(4.10) \quad \lambda u f' - \mu f = -2.$$

By solving (4.10) we derive

$$(4.11) \quad f(u) = \frac{2}{\mu} + c u^{\frac{\mu}{\lambda}}, \quad c \in \mathbb{R}.$$

Substituting (4.11) into (4.8) implies

$$(4.12) \quad B(u) = \frac{u^{\frac{-\mu}{\lambda}+1}}{c^{\frac{\mu}{\lambda}} \left(\frac{\mu}{\lambda} - 1 \right)}.$$

By considering (4.11) and (4.12) into the second line of (4.9), we conclude again a contradiction.

Therefore we have proved the following results.

Theorem 4.3. *Let M be a surface of revolution in $\mathbb{S}\mathbb{I}^3$ given by (4.1). If $\triangle^{II}r_i = \lambda_i r_i$ holds then $(\lambda_1, \lambda_2, \lambda_3) = (\lambda, \lambda, 0)$, $\lambda \neq 0$, and its generating curve is of the form*

$$\left(0, u, \frac{2}{\lambda} \ln u + c \right), \quad c \in \mathbb{R}.$$

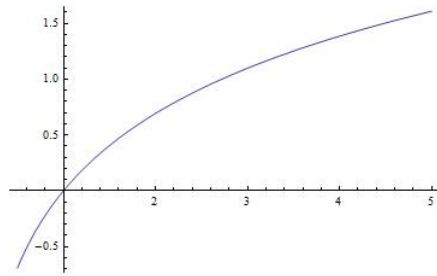
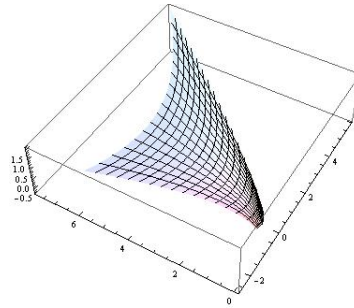
Theorem 4.1 and Theorem 4.3 immediately yields the following result.

Corollary 4.1. *A surface of revolution satisfies $\triangle^{II}r_i = \lambda_i r_i$ in $\mathbb{S}\mathbb{I}^3$ if and only if it is (s-i)-minimal.*

Example 4.2. Given the surface of revolution M in $\mathbb{S}\mathbb{I}^3$ by

$$(u \sinh v, u \cosh v, \ln u), \quad (u, v) \in [0.5, 5] \times [-0.5, 1].$$

Then M satisfies $\triangle^{II}r_i = \lambda_i r_i$ in $\mathbb{S}\mathbb{I}^3$ for $(\lambda_1, \lambda_2, \lambda_3) = (-2, -2, 0)$ and can be drawn as in Fig3.


 Figure 3(a). Profile curve, $f(u) = \ln u$.

 Figure 3(b). A surface of revolution satisfying $\Delta^{II} r_i = \lambda_i r_i$.

REFERENCES

- [1] M.E. Aydin, A. Mihai, *Ruled surfaces generated by elliptic cylindrical curves in the isotropic space*, Georgian Math. J., accepted for publication.
- [2] M.E. Aydin, I. Mihai, *On certain surfaces in the isotropic 4-space*, Math. Commun., in press.
- [3] M.E. Aydin, *Surfaces of constant curvature in a semi-isotropic space*, arXiv:1609.02437v1, 2016.
- [4] A. Baricz, *Generalized Bessel Functions of The First Kind*, Springer-Verlag Berlin Heidelberg, 2010.
- [5] M. Bekkar, H. Zoubir, *Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x^i = \lambda^i x^i$* , Int. J. Contemp. Math. Sci. **3**(24) (2008), 1173-1185.
- [6] B. Bukcu, D.W. Yoon, M.K. Karacan, *Translation surfaces in the 3-dimensional simply isotropic space \mathbb{I}_3^1 satisfying $\Delta^{III} x_i = \lambda_i x_i$* , Konuralp J. Math. **4**(1) (2016), 275-281.
- [7] B.-Y. Chen, *On submanifolds of finite type*, Soochow J. Math. **9** (1983), 65-81.
- [8] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapor-New Jersey-London, 1984.
- [9] B. Y. Chen, S. Ishikawa, *On classification of some surfaces of revolution of finite type*, Tsukuba J. Math., **17**(1) (1993), 287-298.
- [10] B. Y. Chen, S. Decu, L. Verstraelen, *Notes on isotropic geometry of production models*, Kragujevac J. Math. **7**(2) (2013), 217-220.
- [11] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type: recent development*, Tamkang. J. Math. **45**(1) (2014), 87-108.
- [12] M. Choi, Y. H. Kim, D. W. Yoon, *Some classification of surfaces of revolution in Minkowski 3-space*, J. Geom. **104** (2013), 85-106.
- [13] S. Decu, L. Verstraelen, *A note on the isotropical geometry of production surfaces*, Kragujevac J. Math. **38**(1) (2014), 23-33.
- [14] Z. Erjavec, B. Divjak, D. Horvat, *The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean, simple isotropic and double isotropic space*, Int. Math. Forum **6**(17) (2011), 837-856.
- [15] O. J. Garay, *An extension of Takahashi's theorem*, Geom. Dedicata **34** (1990), 105-112.
- [16] O. J. Garay, *On a certain class of finite type surfaces of revolution*, Kodai Math. J. **11**(1) (1998), 25-31.
- [17] G. Kaimakamis, B. Papantoniou, *Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta^{II} \vec{r} = A \vec{r}$* , J. Geom. **81** (2004), 81-92.
- [18] M.K. Karacan, D.W. Yoon, B. Bukcu, *Translation surfaces in the three dimensional simply isotropic space \mathbb{I}_3^1* , Int. J. Geom. Methods Mod. Phys. **13**, 1650088 (2016) (9 pages) DOI: <http://dx.doi.org/10.1142/S0219887816500882>.
- [19] A.C. King, J. Billingham, S.R. Otto, *Differential equations: Linear, Nonlinear, Ordinary, Partial*, Cambridge University Press, 2003.
- [20] J.J. Koenderink, A. van Doorn, *Image processing done right*, Lecture Notes in Computer Science **2350** (2002), 158-172.
- [21] B.G. Korenev, *Bessel Functions and their Applications*, CRC Press LLC, 2002.
- [22] R. Lopez, *Differential Geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Electron. J. Geom. **7** (2014), 44-107.

- [23] Z. Milin-Sipus, B. Divjak, *Mappings of ruled surfaces in simply isotropic space I_3^1 that preserve the generators*, Monatsh. Math. **139** (2003), 235–245.
- [24] Z. Milin-Sipus, *Translation surfaces of constant curvatures in a simply isotropic space*, Period. Math. Hung. **68** (2014), 160–175.
- [25] R. K. Nagle, E. B. Saff, A. D. Snider, *Fundamentals of Differential Equations* (8th Edition), Pearson Education, 2011.
- [26] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
- [27] A.O. Ogrenmis, *Rotational surfaces in isotropic spaces satisfying Weingarten conditions*, Open Physics 14(9) (2016), 221–225.
- [28] D. Palman, *Sphärische quartiken auf dem torus im einfach isotropen raum*, Glasnik Matematički **14(34)** (1979), 345–357.
- [29] H. Pottmann, K. Opitz, *Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces*, Comput. Aided Geom. Design **11** (1994), 655–674.
- [30] H. Pottmann, Y. Liu, *Discrete surfaces of isotropic geometry with applications in architecture*, In: Martin, R., Sabin, M., Winkler, J. (eds.), *The Mathematics of Surfaces*, pp. 341–363. Lecture Notes in Computer Science 4647, Springer (2007).
- [31] H. Pottmann, P. Grohs, N.J. Mitra, *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math. **31** (2009), 391–419.
- [32] H. Sachs, *Isotrope Geometrie des Raumes*, Vieweg Verlag, Braunschweig, 1990.
- [33] K. Strubecker, *Differentialgeometrie des isotropen Raumes III*, Flächentheorie, Math. Zeitsch. **48** (1942), 369–427.

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