Proc. Natl. Acad. Sci. USA
Vol. 78, No. 9, pp. 5288–5289, September 1981
Mathematics

Representation of weakly harmonizable processes

(operator representation/V-boundedness/generalization of stationarity)

M. M. RAO

Department of Mathematics, University of California, Riverside, California 92521

Communicated by S. Bochner, May 29, 1981

ABSTRACT **Weakly harmonizable processes are represented by a family of positive definite contractive linear operators in a Hilbert space. This generalizes the known result on weakly stationary processes involving a unitary family. A characterization of the vector Fourier integral of a measure on $\mathbf{R} \to \mathcal{X}$, a reflexive space, is given, and this yields another characterization of weakly harmonizable processes when $\mathcal{X}$ is a Hilbert space. Also these processes are shown to have associated spectra, yielding a positive solution to a problem of Rozanov.**

Let (Ω,Σ,P) be a probability space, $L_0^2(P)$ be the space of (equivalence classes of) scalar square integrable functions f on (Ω,Σ,P) such that $E(f) = \int_\Omega f dP = 0$, and norm $\|\cdot\|$ where $\|f\|^2 = E(|f|^2)$. A process or mapping $X:\mathbf{R} \to L_0^2(P)$ is (weakly) stationary if the convariance function $r:(s,t) \leftrightarrow E(X(s)\overline{X(t)})$ is continuous and if $r(s,t) = \tilde{r}(s-t)$ for all s,t in $\mathbf{R}$. By the classical Bochner theorem,

$$\tilde{r}(s-t) = \int_{\mathbf{R}} e^{i(s-t)\lambda}F(d\lambda),\ s,t \in \mathbf{R}, \qquad [1]$$

for a unique, positive-bounded Radon measure F on $\mathbf{R}$. In the late 1940s, Loève (1) has introduced a generalization. A process $X:\mathbf{R} \to L_0^2(P)$ is (Loève or) strongly harmonizable if there is a bimeasure $F(\cdot,\cdot):\mathcal{B} \times \mathcal{B} \to \mathbb{C}$ ($\mathcal{B}$ = Borel σ-algebra of $\mathbf{R}$), such that:

$$(i)\ F(A,B) = \overline{F(B,A)},$$

$$(ii)\ \sum_{i,j=1}^{n} a_i\bar{a}_j F(A_i,A_j) \geq 0, a_i \in \mathbb{C},$$

and

$$(iii)\ V(F) = \sup\left\{\sum_{i,j=1}^{n} |F(A_i,B_j)|:A_i,B_j \in \mathcal{B}, \text{disjoint}\right\} < \infty,$$

in terms of which one has (the complex Lebesgue integral)

$$r(s,t) = \int_{\mathbf{R}}\int_{\mathbf{R}} e^{is\lambda - it\lambda'}F(d\lambda,d\lambda'),\ s,t \in \mathbf{R}. \qquad [2]$$

Note that $V(F)$ is the Vitali variation of F.

If $\{X(t),t \in \mathbf{R}\}$ is a stationary process, and Q is an orthogonal projection of $L_0^2(P)$, let $Y(t) = QX(t)$, $t \in \mathbf{R}$. Then it is easily seen that $Y = \{Y(t),t \in \mathbf{R}\}$ is strongly harmonizable if the range of Q is finite dimensional, or $Q = I$. However, when $Q \neq I$ but has infinite dimensional range, there exist simple examples showing that Y is *not* strongly harmonizable. In this context, the following concept, due essentially to Rozanov (2), is relevant. A process $X:\mathbf{R} \to L_0^2(P)$ is weakly harmonizable if there exists a bimeasure $F:\mathcal{B} \times \mathcal{B} \to \mathbb{C}$ satisfying i and ii above and

$$(iii)'\ \|F\| = \sup\left\{\sum_{i,j=1}^{n} a_i\bar{a}_j F(A_i,A_j):|a_i| \leq 1, A_i \in \mathcal{B}, \text{disjoint}\right\} < \infty, \qquad [3]$$

in terms of which $r(\cdot,\cdot)$ admits the representation (3) for this F, the symbol now being a Morse–Transue (or MT-) integral (4), which is nonabsolutely continuous in contrast to the Lebesgue concept. Note also that $\|F\|$ is the Fréchet variation of F and $\|F\| \leq V(F)$, so that every strongly harmonizable process is weakly harmonizable. (See refs. 5–7 on earlier work.)

A more inclusive concept, called V-boundedness, was introduced in the early 1950s by Bochner (8) and he already had noted that it includes Loève harmonizability. A mapping $X:\mathbf{R} \to \mathcal{X}$, a Banach space, is V-bounded if (i) $\|X(t)\| \leq M < \infty$, (ii) X is strongly measurable, and (iii) the following set in $\mathcal{X}$ is relatively weakly compact,

$$\left\{\int_{\mathbf{R}} f(t)X(t)dt:\|\hat{f}\|_\infty \leq 1, f \in L^1(\mathbf{R})\right\}, \qquad [4]$$

where $\hat{f}(\lambda) = \int_{\mathbf{R}} f(t)e^{it\lambda}dt$, and $\int_{\mathbf{R}} f(t)X(t)dt$ is the Bochner integral. Note that if $\mathcal{X}$ is reflexive, the condition on the set in Eq. 4 is equivalent to its boundedness.

When $\mathcal{X} = L_0^2(P)$, these concepts are related as follows.

THEOREM 1. *A process* $X:\mathbf{R} \to L_0^2(P)$ *is weakly harmonizable iff* X *is V-bounded and weakly continuous. Moreover, if* X *is weakly harmonizable on* $\mathbf{R}$, *then there exists a sequence* $X_n:\mathbf{R} \to L_0^2$ *of strongly harmonizable processes such that* $X_n(t) \underset{n\to\infty}{\to} X(t)$, *in* $L_0^2(P)$, *uniformly in* t *belonging to the compact subsets of* $\mathbf{R}$.

Representation and other characterizations

If $X:\mathbf{R} \to L_0^2(P)$ is a stationary process, then there exists a unitary group $\{U_t,t \in \mathbf{R}\}$ of operators in $B(L_0^2(P))$, the algebra of bounded linear mappings on $L_0^2(P)$, such that $X(t) = U_tX(0)$, almost everywhere $t \in \mathbf{R}$, and conversely if $\{U_t,t \in \mathbf{R}\}$ is such a family, it defines a stationary process $X:t \leftrightarrow U_tX_0$, $t \in \mathbf{R}$, for each $X_0 \in L_0^2(P)$. A similar symmetric statement does not obtain for harmonizable processes. The precise result is given below. In order to include the discrete and continuous parameter versions and the corresponding random fields, it is convenient to consider X on a locally compact abelian (LCA) group G. It is clear that the definitions involving Eqs. 1–4 are valid if $\mathbf{R}$ is replaced by such G with 'dt' as a Haar measure, because the MT-integration is available on locally compact spaces (4).

Recall that if $\mathcal{H}$ is a Hilbert space and G is a group, then $T:G \to B(\mathcal{H})$ is positive definite whenever $T(g^{-1}) = T(g)^*$, the adjoint operator, and for each finite set $\{s_1, \ldots ,s_n\}$ of G and $\{h_{s_1}, \ldots ,h_{s_n}\}$ of $\mathcal{H}$, one has $\sum_{i,j=1}^{n}(T(s_j^{-1}s_i)h_{s_i},h_{s_j}) \geq 0$,

The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "*advertisement*" in accordance with 18 U. S. C. §1734 solely to indicate this fact.

$(\cdot,\cdot)$ being the inner product of $\mathcal{H}$. The main result on the operator representation is given by:

THEOREM 2. *Let* G *be an LCA group and* $X: G \to L_0^2(P) = \mathcal{H}$ *be weakly harmonizable. Then (i) there exists an enlarged probability space* $(\tilde\Omega, \tilde\Sigma, \tilde P)$ *such that* $\tilde\Omega \supset \Omega$, $\tilde\Sigma \supset \Sigma$, $\tilde P|\Sigma = P$ *so that* $\tilde{\mathcal{H}} = L_0^2(\tilde P) \supset \mathcal{H}$, *(ii) a random variable* Y_0 *in* $\tilde{\mathcal{H}}$, *and (iii) a weakly continuous family of linear contraction operators* $T(g): \tilde{\mathcal{H}} \to \mathcal{H}$, $T(0)|_{\mathcal{H}}$ = identity of $\mathcal{H}$, *with the property that* $\tilde T(g) = T(g)|_{\mathcal{H}}$, $g \in G$, *defines a positive definite family, in terms of which* $X(\cdot)$ *is representable as* $X(g) = T(g)Y_0$, $g \in G$.

Conversely, if $\{T(g), g \in G\} \subset B(\mathcal{H})$ *is a weakly continuous contractive family of positive definite operators with* $T(0)$ = identity, *then for each* $Y_0 \in \mathcal{H}$, *the process* $\{X(g) = T(g)Y_0, g \in G\} \subset \mathcal{H}$ *is weakly harmonizable.*

Note the asymmetry in the above statement regarding the forward and reverse directions in contrast to the stationary case. For the latter, no enlargement of $\mathcal{H}$ is needed, and $\{T(g), g \in G\}$ is unitary. In the harmonizable case, the enlargement is generally unavoidable.

Another characterization is obtained from the next proposition, the general result being of independent interest. If $\mathcal{X}$ is a Banach space and G an LCA group, then a mapping $X: G \to \mathcal{X}$ is a vector Fourier integral whenever there is a regular vector measure ν on $\hat G$ to $\mathcal{X}$ such that $X(g) = \int_{\hat G} \langle g,s\rangle \nu(ds)$, $g \in G$, where $\langle g,\cdot\rangle$ is a character of the dual group $\hat G$ of G, and the integral is in the Dunford–Schwartz sense. (This is recalled with applications to stochastic theory in ref. 9.) Let $\hat L^1(G) = \{\hat f: \hat f(t) = \int_G f(s)\langle t,s\rangle ds, f \in L^1(G)\} \subset C_0(\hat G)$, and similarly $\hat L^1_{\mathcal{X}}(G)$ is defined for $\mathcal{X}$-valued functions using the Bochner integral, $\hat{\mathcal{M}}_{\mathcal{X}}(G)$ $(\supset \hat L^1_{\mathcal{X}}(G))$, $\mathcal{M}_{\mathcal{X}}(G)$ being the space of vector measures on G with semivariation norm.

THEOREM 3. *Let* G *be an LCA group and* $\mathcal{X}$ *be reflexive and separable. Then a mapping* $X: G \to \mathcal{X}$ *is a Fourier integral of a (regular) vector measure* ν *on* $\hat G$ *to* $\mathcal{X}$ *iff for each* $p \in \hat L^1(\hat G)$, $Y_p: g \mapsto (Xp)(g)$, $g \in G$, *is in* $\hat{\mathcal{M}}_{\mathcal{X}}(\hat G)$.

A comprehensive result on weakly harmonizable fields can be obtained from the above results with $\mathcal{X} = \mathcal{H}$, a Hilbert space, in the following manner. It may be noted that such an $\mathcal{H}$ can always be realized as $L_0^2(\mu)$ on some probability space $(S, \mathcal{S}, \mu)$ (9). This fact is used in proofs. Define the following classes on an LCA group G:

$\mathcal{W}$ = the set of all weakly harmonizable random fields $X: G \to \mathcal{H}$.

$\mathcal{P}$ = the class of all $X: G \to \mathcal{H}$ that are orthogonal projections of stationary random fields $Y: G \to \tilde{\mathcal{H}}$, an extension Hilbert space of $\mathcal{H}$, with possibly different $\tilde{\mathcal{H}}$s for different Ys.

$\mathcal{M} = \{X: G \to \mathcal{H}, X \cdot \hat L^1(\hat G) \subset \hat{\mathcal{M}}_{\mathcal{H}}(\hat G)\}$.

$\mathcal{F}$ = the class of all $X: G \to \mathcal{H}$ that are Fourier transforms of regular vector measures on $\hat G$ to $\mathcal{H}$.

$\mathcal{V}$ = the set of all weakly continuous V-bounded $X: G \to \mathcal{H}$. Then one has the following theorem when $\mathcal{H}$ is also separable.

THEOREM 4. $\mathcal{F} = \mathcal{M} = \mathcal{P} = \mathcal{V} = \mathcal{W}$.

The arguments employed for the vector Fourier integrals yield a positive solution to a problem of Rozanov (2). If $X: \mathbf{R} \to L_0^2(P)$ is a measurable function (or process), then it is said to be in *class* (KF) (after Kampé de Fériet and Frankiel who first introduced it), if the following limit exists:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T-|h|} (X(s+h), X(s))ds = r(h),\ h \in \mathbf{R}. \qquad [5]$$

Then $r(\cdot)$ is a measurable positive definite function so that by a result of F. Riesz (classical, 1933), r coincides with a continuous function outside of a Lebesgue null set. Hence with Bochner's theorem, there is a bounded positive Radon measure $F(\cdot)$ on $\mathbf{R}$, called the *associated spectral function* of X, so that

$$r(h) = \int_{\mathbf{R}} e^{ih\lambda} F(d\lambda), \text{ almost all } (h), \text{(Lebesgue)}. \qquad [6]$$

One has the following result.

THEOREM 5. *If* $X: \mathbf{R} \to L_0^2(P)$ *is weakly harmonizable, then the limit in Eq. 5 exists so* $X \in$ class(KF), *and* $r(\cdot)$ *is even a continuous positive definite function. Thus, Eq. 6 holds for all* h $\in \mathbf{R}$ *and* X *has an associated spectrum.*

Outline of proofs

Theorem 1 depends on an integral representation of X as (*) $X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda)$, for a vector measure Z on $\mathbf{R}$ to $\mathcal{H}$ with $F(A,B) = (Z(A), Z(B))$ satisfying *i–iii'*. The work uses the MT-integration and completes a result of ref. 2. Then a computation shows that X is V-bounded. The converse uses a theorem of Bartle–Dunford–Schwartz for the operator $T: \hat f \mapsto \int_{\mathbf{R}} f(t)X(t)dt$ to show that $T(\hat f) = \int_{\mathbf{R}} \hat f(\lambda) Z(d\lambda)$, and this implies that (*) holds so that X is weakly harmonizable. The last part is due to Niemi (10). *Theorem 2* is more involved. The construction of a super Hilbert space $\tilde{\mathcal{H}}$ uses some work on two absolutely summing operators by which one shows the existence of a bounded Radon measure μ on G, such that if $F(\cdot,\cdot)$ is the bimeasure of X (Eq. 3), then $\alpha(\cdot,\cdot): (A,B) \mapsto \mu(A \cap B) - F(A,B)$ is positive definite and satisfies Eq. 3. One shows $\mathcal{H} \subset \tilde{\mathcal{H}} \cong L^2(\mu)$, and there is a stationary $Y: G \to \tilde{\mathcal{H}}$, whose spectral measure is μ, $Q(\tilde{\mathcal{H}}) = \mathcal{H}$ with Q as orthogonal projection, and $QY(g) = X(g)$, $g \in G$. If $U_g: Y(s) \mapsto Y(s+g)$, then $T(g) = QU_g$, $g \in G$, satisfies the conditions. In the other direction, by a result of B. Sz.-Nagy (ref. 11, *Theorem I.7.1*), $T(\cdot)$ has a unitary dilation so that $T(g) = QU_g$ with $\{U_g, g \in G\}$ as a unitary group in $B(\tilde{\mathcal{H}})$ for some Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$, $Q(\tilde{\mathcal{H}}) = \mathcal{H}$. Then for any $Y_0 \in \mathcal{H}$, $X(g) = T(g)Y_0 = QU_gY_0$, $g \in G$ is V-bounded, and *Theorem 1* applies. In *Theorem 3*, if $X(\cdot)$ is a vector Fourier integral, then $pX \in \hat{\mathcal{M}}_{\mathcal{X}}(\hat G)$ results from a direct computation using elementary properties of Pettis integration. The sufficiency is more involved, however. It uses an idea from Helson (12). First one notes that $\tau: f \mapsto \hat f X$, $f \in L^1(\hat G)$ is a bounded linear operator. Next, it is shown that $T: \hat f \mapsto \int_G X(g)\hat f(g)dg$, $f \in L^1(G)$ is well defined, and (the hard part) $\|T(\hat f)\|_{\mathcal{X}} \le c\|\hat f\|_\infty$. This implies X is V-bounded, and then one deduces that X is a vector Fourier transform as in *Theorem 1*.

Finally, *Theorem 4* can be obtained from the preceding results, but *Theorem 5* only uses the ideas (not the results) of this work; one needs a suitably extended Fubini-type argument for vector integrals on product spaces.

This work is partially supported under the Office of Naval Research Contract N0014-79-C-0754 (and Modification P00001).

1. Loève, M. (1954) *Probability Theory* (Van Nostrand, Princeton), p. 474.
2. Rozanov, Yu. A. (1959) *Theory Probab. Its Appl.* (Moscow) **4**, 271–287.
3. Doob, J. L. (1953) *Stochastic Process* (Wiley, New York), p. 512.
4. Morse, M. & Transue, W. (1956), *Ann. Math.* **64**, 480–504.
5. Abreu, J. L. (1970) *Bol. Soc. Mat. Mex.* **15**, 48–51.
6. Niemi, H. (1977) *Ann. Acad. Sci. Fenn. Ser. A1* **3**, 43–52.
7. Miamee, A. G. & Salehi, H. (1978) *Indiana Univ. Math. J.* **27**, 37–50.
8. Bochner, S. (1956) *Proceedings Third Berkeley Symposium on Mathematics, Statistics, and Probability* (Univ. California Press, Berkeley, CA), Vol. 2, pp. 7–27.
9. Rao, M. M. (1979) *Stochastic Processes and Integration* (Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands).
10. Niemi, H. (1975) *Ann. Acad. Sci. Fenn. Ser. A1* **591**, 1–47.
11. Sz.-Nagy, B. & Foias, C. (1970) *Harmonic Analysis of Operators on Hilbert Space* (Akadémiai Kiadó, Budapest, Hungary).
12. Helson, H. (1953) *Ark. Mat.* **2**, 475–487.