V. F. Gaposhkin, A theorem on the convergence almost everywhere of a sequence of measurable functions, and its applications to sequences of stochastic integrals,
Mathematics of the USSR-Sbornik, 1977, Volume 33, Issue 1, 1–17

https://www.mathnet.ru/eng/sm2933

Мат. Сборник
Том 104(146)(1977),Вып.1

MATH. USSR SBORNIK
Vol. 33(1977), No. 1

A THEOREM ON THE CONVERGENCE ALMOST EVERYWHERE OF A SEQUENCE OF MEASURABLE FUNCTIONS, AND ITS APPLICATIONS TO SEQUENCES OF STOCHASTIC INTEGRALS

UDC 519.2

V. F. GAPOŠKIN

ABSTRACT. It is shown that various problems on the convergence almost everywhere of sequences of stochastic integrals (the theorem of Kotel'nikov for stationary processes, estimates of the means of stationary processes and homogeneous fields) can be solved with the help of a general theorem on convergence of a sequence of measurable functions.

Bibliography: 9 titles.

Introduction

Many problems associated with random processes lead to the investigation of convergence almost everywhere as $n \to \infty$ of a sequence of stochastic integrals

$$\int\limits_{|\lambda|\leqslant A} R_n(\lambda)\, Z(d\lambda), \tag{1}$$

where $Z(d\lambda)$ is a stochastic measure with these or those properties, and $R_n(\lambda)$ is a sequence of real or complex functions of λ.

One such problem is the strong law of large numbers (s.l.l.n.) for a stationary process $\xi(t)$ or a stationary sequence ξ_n. In these cases the measure $Z(d\lambda)$ has orthogonal increments, and kernels $R_n(\lambda)$ of the form

$$\frac{e^{in\lambda}-1}{in\lambda} \quad \text{or} \quad \frac{\sin n\lambda}{n\lambda} \tag{2}$$

arise for the means $(1/n)\int_0^n \xi(t)\,dt$ or $(1/2n)\int_{-n}^n \xi(t)\,dt$, respectively, and

$$\frac{e^{in\lambda}-1}{n(e^{i\lambda}-1)}, \quad \frac{\sin\left(n+\frac{1}{2}\right)\lambda}{(2n+1)\sin\frac{\lambda}{n}} \tag{3}$$

for the respective means $(1/n)\sum_{k=0}^{n-1}\xi_k$ or $(1/(2n+1))\sum_{-n}^{n}\xi_k$.

In [1] the author gives necessary and sufficient conditions for the s.l.l.n., expressed in terms of the properties of the spectral measure $Z(d\lambda)$. These properties are obtained in the investigation of the integrals (1) by using special properties of the kernels $R_n(\lambda)$. Here we get a general theorem on convergence of integrals of the type (1) that is also useful in other problems. For example, application of this theorem completely solves the problem

AMS (MOS) subject classifications (1970). Primary 28A20, 60H05, 60F15, 60G10, 60G17; Secondary 28A40, 28A45, 40G05, 60G15.

of Kotel'nikov's theorem for almost all sample trajectories of a stationary process with bounded spectrum.

In §1 we establish the basic theorem on convergence of a sequence of measurable functions for majorizing bounds of a special form (Theorem 1). This theorem is then made specific for sequences of stochastic integrals (Theorem 2). Corollaries of these theorems for stationary processes or fields and their generalizations are contained in §2 and §3.

§1. Basic theorem and its corollary for stochastic integrals

Let $(X, \mathfrak{A}, \mu)$ be a measure space and $\{f_n(x)\}$ measurable functions on this space that belong to $L_r(X)$ for some $r > 0$. Let $(Y, \mathfrak{B}, \nu)$ be another measure space, with a finite measure ν.

Let $\varphi(y)$ be a measurable functional on Y, $\varphi(y) \geqslant 0$ $(y \in Y)$, and

$$g_m(y) = \min\{2^m\varphi(y);\ (2^m\varphi(y))^{-1}\}^b, \qquad m \geqslant 1; \tag{4}$$

$$g_{k,n}(y) = \min_{i=1,2}\{(n-k)^{a_i}(\varphi(y))^{b_i}n^{-c_i};\ (n\varphi(y))^{-a}\}, \qquad n > k. \tag{5}$$

THEOREM 1. *For a given sequence $\{f_n(x)\}$ suppose that there exist a functional $\varphi(y)$, a constant $C > 0$, and numbers a, b, a_i, b_i and c_i $(i = 1, 2)$ such that for the g_m and $g_{n,k}$ defined by* (4) *and* (5) *the following conditions hold*:

A) $\int\limits_X |f_{2^m}(x)|^r\,d\mu \leqslant C\int\limits_Y g_m(y)\,d\nu$; $\quad m = 1, 2, \ldots$;

B) $\int\limits_X |f_n(x) - f_k(x)|^r\,d\mu \leqslant C\int\limits_Y g_{k,n}(y)\,d\nu$ *for* $2^m \leqslant k < n < 2^{m+1}$; $m = 1, 2, \ldots$;

C) $a > 1$, $a_1 > 1$, $a_2 > 1$, $c_2 > 1$, $b > 0$; $b_1 + c_1 \geqslant a_1 > c_1$; $b_2 + c_2 \geqslant a_2$.

Then $\lim_{n\to\infty} f_n(x) = 0$ *almost everywhere, and* $\sup_n|f_n(x)| = g(x) \in L_r(X)$.

PROOF. We use the notation

$$\int\limits_{2^{-m}<\varphi(y)\leqslant 2^{-m+1}} d\nu = A_m,\ \ m = 1, 2, \ldots; \qquad \int\limits_{\varphi(y)>1} d\nu = A_0. \tag{6}$$

First, we show that

$$\sum_{m=1}^{\infty}\int\limits_X |f_{2^m}(x)|^r d\mu < \infty. \tag{7}$$

By the conditions A) and (4), we have

$$\begin{aligned}\int\limits_X |f_{2^m}(x)|^r\,d\mu &\leqslant C\left[\sum_{k=1}^{\infty}\ \int\limits_{2^{-k}<\varphi(y)\leqslant 2^{-k+1}} g_m(y)\,d\nu + \int\limits_{\varphi(y)>1} g_m(y)\,d\nu\right]\\ &\leqslant C_1\left[\sum_{k=1}^{m} 2^{b(k-m)}A_k + \sum_{k=m+1}^{\infty} 2^{b(m-k)}A_k + 2^{-bm}A_0\right],\end{aligned}$$

and the convergence of the series (7) follows from the estimates

$$\sum_{m=1}^{\infty}\sum_{k=0}^{m} 2^{(k-m)b}A_k \leqslant \sum_{k=0}^{\infty} 2^{kb}A_k \sum_{m=k}^{\infty} 2^{-mb} \leqslant C_1 \sum_{0}^{\infty} A_k < \infty,$$

$$\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty} 2^{(m-k)b}A_k \leqslant \sum_{k=2}^{\infty} A_k 2^{-kb} \sum_{m=1}^{k-1} 2^{mb} \leqslant C_1 \sum_{k=0}^{\infty} A_k < \infty.$$

It remains to show that we also have convergence of the series

$$\sum_{m=1}^{\infty} \int_X |\delta_m(x)|^r \, d\mu, \tag{8}$$

where

$$\delta_m(x) = \sup_{2^m < n < 2^{m+1}} |f_n(x) - f_{2^m}(x)|. \tag{9}$$

Let

$$\begin{gathered} f_{2^m + j2^{m-p}}(x) - f_{2^m + (j-1)2^{m-p}}(x) = \Delta_{m,p,j}(x), \\ p = 1, 2, \ldots, m;\ j = 1, 2, \ldots, 2^p;\ m = 1, 2, \ldots . \end{gathered} \tag{10}$$

For any n between 2^m and 2^{m+1} we have the representation

$$n = 2^m + \varepsilon_1^{(n)} \cdot 2^{m-1} + \ldots + \varepsilon_{m-1}^{(n)} \cdot 2 + \varepsilon_m^{(n)} \cdot 1, \quad \varepsilon_i^{(n)} = 0 \text{ or } 1. \tag{11}$$

According to this representation we can write

$$f_n(x) - f_{2^m}(x) = \sum_{p=1}^{m} \varepsilon_p^{(n)} \Delta_{m,p,j_p^{(n)}}, \tag{12}$$

where the indices $j_p^{(n)}$ are uniquely determined from (11).

If $r > 1$, we have

$$\begin{aligned} |f_n(x) - f_{2^m}(x)|^r &\leqslant \sum_{p=1}^{m} p^r |\Delta_{m,p,j_p^{(n)}}|^r \left(\sum_{p=1}^{m} p^{-r/(r-1)} \right)^{r-1} \\ &\leqslant C_r \sum_{p=1}^{m} p^r \sum_{j=1}^{2^p} |\Delta_{m,p,j}|^r. \end{aligned} \tag{13}$$

But if $0 \leqslant r \leqslant 1$, then it follows from (12) that

$$|f_n(x) - f_{2^m}(x)|^r \leqslant \sum_{p=1}^{m} |\Delta_{m,p,j_p^{(n)}}|^r,$$

i.e., all the more so, the estimate (13) holds.

Since the right-hand side of (13) does not depend on n, the estimate is valid also for $\delta_m^r = \sup_{2^m < n < 2^{m+1}} |f_n - f_{2^m}|^r$; integrating, we obtain

$$\int_X \delta_m^r(x) \, d\mu \leqslant C_r \sum_{p=1}^{m} p^r \sum_{j=1}^{2^p} \int_X |\Delta_{m,p,j}|^r \, d\mu. \tag{14}$$

We estimate the integral

$$\int_X |\Delta_{m,p,j}|^r \, d\mu = I_{m,p,j}. \tag{15}$$

Using the notation (5) and the assumption of the theorem, we have

$$I_{m,p,j} \leqslant C \int_Y g^*_{m,p}(y)\, d\nu \quad (j = 1, 2, \ldots, 2^p); \tag{16}$$

$$g^*_{m,p}(y) = \min_{i=1,2} \{2^{(m-p)a_i} (\varphi(y))^{b_i} 2^{-mc_i}; \ (2^m \varphi(y))^{-a}\}. \tag{17}$$

We represent the integral in (16) in the form

$$\int_Y g^*_{m,p}(y)\, d\nu = \int_{2^m \varphi(y) \leqslant 1} + \int_{1 < 2^m \varphi(y) \leqslant 2^p} + \int_{2^m \varphi(y) > 2^p} .$$

Using, in turn, the estimates (17) and the notation (6), we get

$$\int_Y g^*_{m,p}(y)\, d\nu \leqslant C_1 \left[\sum_{k=m+1}^{\infty} A_k 2^{(m-p)a_1 - kb_1 - mc_1} + \sum_{k=m-p+1}^{m} A_k 2^{(m-p)a_2 - kb_2 - mc_2} \right.$$
$$\left. + \sum_{k=0}^{m-p} A_k 2^{-am+ak} \right] \equiv C_1 (\Sigma_1^{(m;p)} + \Sigma_2^{(m;p)} + \Sigma_3^{(m;p)}).$$

Substituting these sums in (16) and (14) and summing over m, we find that

$$\sum_{m=1}^{\infty} \int_X |\delta_m|^r d\mu \leqslant C_1 (\Sigma_1 + \Sigma_2 + \Sigma_3),$$

$$\Sigma_j = \sum_{m=1}^{\infty} \sum_{p=1}^{m} p^r 2^p \Sigma_j^{(m;p)} \quad (j = 1, 2, 3).$$

Changing the order of summation and considering the arithmetic relations among a_i, b_i and c_i, we easily prove the finiteness of the sums Σ_j. Namely,

$$\Sigma_1 = \sum_{m=1}^{\infty} \sum_{p=1}^{m} p^r 2^p \sum_{k=m+1}^{\infty} A_k 2^{(m-p)a_1 - kb_1 - mc_1}$$
$$\leqslant \sum_{p=1}^{\infty} p^r 2^{-p(a_1-1)} \sum_{k=2}^{\infty} A_k 2^{-kb_1} \sum_{m=1}^{k-1} 2^{m(a_1-c_1)} \leqslant C_2 \sum_{k=2}^{\infty} A_k 2^{k(a_1-c_1-b_1)} \leqslant C_2 \sum_{k=2}^{\infty} A_k < \infty;$$

$$\Sigma_2 = \sum_{m=1}^{\infty} \sum_{p=1}^{m} p^r 2^p \sum_{k=m-p+1}^{m} A_k 2^{(m-p)a_2 - kb_2 - mc_2}$$
$$\leqslant \sum_{m=1}^{\infty} 2^{m(a_2-c_2)} \left(\sum_{k=1}^{m} A_k 2^{-kb_2} \sum_{p>m-k} p^r 2^{-p(a_2-1)} \right)$$
$$\leqslant C_2 \sum_{m=1}^{\infty} 2^{m(a_2-c_2)} \sum_{k=1}^{m} A_k 2^{-kb_2} (m-k+1)^r 2^{-(m-k)(a_2-1)}$$
$$\leqslant C_3 \sum_{k=1}^{\infty} A_k 2^{k(a_2-c_2-b_2)} \sum_{m=k}^{\infty} (m-k+1)^r 2^{(m-k)(1-c_2)} \leqslant C_4 \sum_{k=1}^{\infty} A_k < \infty;$$

$$\Sigma_3 = \sum_{m=1}^{\infty} 2^{-am} \sum_{p=1}^{m} p^r 2^p \sum_{k=0}^{m-p} A_k 2^{ka} \leqslant C_2 \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} A_k 2^{-a(m-k)} (m-k)^r 2^{m-k}$$
$$\leqslant C_3 \sum_{k=0}^{\infty} A_k \sum_{m=k+1}^{\infty} (m-k)^r 2^{(m-k)(1-a)} \leqslant C_4 \sum_{k=0}^{\infty} A_k < \infty.$$

From these estimates follow the convergence of the series (8), the integrability of the majorant $g(x) = (\Sigma_1^\infty(|f_{2^n}|^r + |\delta_n|^r))^{1/r}$ to the power $r > 0$, and the convergence of the sequence $\{f_n(x)\}$ to zero almost everywhere.

We now consider the sequence of stochastic integrals

$$I_n(\omega) = \int_{|\lambda|\leqslant A} Q_n(\lambda)\, Z(d\lambda),$$

where $Z(d\lambda)$ is a stochastic spectral measure with orthogonal increments and values in $L_2(\Omega)$, $\omega \in \Omega$, i.e.*

$$\begin{gathered}\mathbf{M}\,|Z(d\lambda)|^2 = F(d\lambda); \quad \int_{|\lambda|\leqslant A} F(d\lambda) < \infty, \\ \mathbf{M}\Big[\int f(\lambda)\, Z(d\lambda) \cdot \overline{\int g(\lambda)\, Z(d\lambda)}\Big] = \int f(\lambda)\overline{g(\lambda)}\, F(d\lambda), \\ \mathbf{M}\Big|\int f(\lambda)\, Z(d\lambda)\,\Big|^2 = \int |f(\lambda)|^2\, F(d\lambda)\end{gathered} \tag{18}$$

for any $f, g \in L_F^2$. Here $(\Omega, S, \mathbf{P})$ is some probability space. Such measures arise, in particular, for spectral representation of stationary processes.

THEOREM 2. *Let $Z(d\lambda)$ be a stochastic measure with orthogonal increments that is defined on $[-A, A]$ and satisfies the conditions* (18), *and suppose that for some λ_0, $-\infty < \lambda_0 < \infty$, the functions $Q_n(\lambda)$, $n = 1, 2, \ldots$, satisfy the following conditions*:

A) $$|(Q_n(\lambda)| \leqslant C \min\,[(n\,|\lambda - \lambda_0|)^\beta,\ (n\,|\lambda - \lambda_0|)^{-\beta}], \tag{19}$$

where $\beta > 0$; $n = 1, 2, \ldots$.

B) *If* $2^m \leqslant k < n < 2^{m+1}$, $m = 0, 1, \ldots$, *then*

$$|Q_n(\lambda) - Q_k(\lambda)| \leqslant C \min_{i=1,2}\,[(n-k)^{\alpha_i}\,|\lambda - \lambda_0|^{\beta_i}\, n^{-\gamma_i};\ (n\,|\lambda - \lambda_0|)^{-\alpha}]. \tag{20}$$

C) *The numbers α_i, β_i and γ_i satisfy the conditions*

$$\alpha_1 > \frac{1}{2},\quad \alpha_2 > \frac{1}{2},\quad \gamma_2 > \frac{1}{2},\quad \alpha > \frac{1}{2};\quad \gamma_1 + \beta_1 > \alpha_1 > \gamma_1,\ \gamma_2 + \beta_2 \geqslant \alpha_2. \tag{21}$$

Then

$$\lim_{n\to\infty} \int_{|\lambda|\leqslant A} Q_n(\lambda)\, Z(d\lambda) = 0 \quad a.s. \tag{22}$$

(*the abbreviation a.s. means convergence almost surely, i.e., almost everywhere with respect to the probability measure* $\mathbf{P}(d\omega)$ *in* Ω).

PROOF. The sequence $I_n(\omega)$ belongs to the space $L_2(\Omega)$ and satisfies the conditions of Theorem 1 for $r = 2$, $Y = \mathbf{R}^1$, $\nu(d\lambda) = F(d\lambda)$ and $\varphi(\lambda) = |\lambda - \lambda_0|$. This follows from the assumptions of the theorem and the isometric equalities:

$$\begin{gathered}\int_\Omega \Big|\int Q_n(\lambda)\, Z(d\lambda)\Big|^2 \mathbf{P}(d\omega) = \int_{|\lambda|\leqslant A} |Q_n(\lambda)|^2\, F(d\lambda); \\ \int_\Omega \Big|\int (Q_n(\lambda) - Q_k(\lambda))\, Z(d\lambda)\Big|^2 \mathbf{P}(d\omega) = \int_{|\lambda|\leqslant A} |Q_n(\lambda) - Q_k(\lambda)|^2\, F(d\lambda).\end{gathered}$$

The analogue of this theorem is true also for $Y = \mathbf{R}^m$.

* *Editor's note.* In the Russian literature, $\mathbf{M}X$ signifies the expectation of the random variable X.

§2. Kotel'nikov's theorem for the sample trajectories of a stationary process

The scheme for application of Theorem 2 to concrete problems is as follows: from the integral (1) whose convergence conditions we are investigating we separate the "principal part" determined by the behavior of the spectral measure on certain sets $\Delta_n \subset [-A, A]$, and apply Theorem 2 to the remainder.

Moreover, the convergence of the "principal part" is equivalent to the convergence of the orthogonal series $\Sigma \int_{\Delta_{n-1} \setminus \Delta_n} Z(d\lambda)$, and this enables us to obtain in a convenient form convergence conditions for integrals of the type (1) by using the methods of the theory of orthogonal series. As a first example we present an analogue of Kotel'nikov's theorem for sample trajectories of a stationary process.

Suppose that the stationary process has a bounded spectrum

$$\xi(t) = \int_{-\Lambda_0}^{\Lambda_0} e^{it\lambda} Z(d\lambda). \tag{23}$$

The well-known theorem of Kotel'nikov on the representation of signals with bounded spectrum by means of discrete readings of them has an analogue for processes $\xi(t)$: if (23) holds, then for $-\infty < t < \infty$

$$\xi(t) = \lim_{n\to\infty} \sum_{k=-n}^{n} \xi\left(\frac{k\pi}{\Lambda_0}\right) \frac{\sin(\Lambda_0 t - k\pi)}{\Lambda_0 t - k\pi}. \tag{24}$$

It is not hard to check that the series (24) converges in the mean square if and only if

$$Z(\{-\Lambda_0\}) = Z(\{+\Lambda_0\}) = 0 \quad \text{a.s.} \tag{25}$$

The more complicated problem of convergence of the series (24) for almost all sample trajectories of the process was considered by Beljaev in [2], where the following result was obtained: if any $\Lambda_1 > \Lambda_0$ is taken instead of Λ_0 in (24), then (24) is valid for almost all sample trajectories of the process (23) (and the series converges uniformly on $T_1 \leqslant t \leqslant T_2$ $(\forall T_1 < T_2)$).

This result is a particular case of the next theorem.

THEOREM 3. *The expansion* (24) *is valid for some* $t \neq r\pi/\Lambda_0$ *for almost all sample trajectories of the stationary process* (24) *if and only if*

$$\lim_{m\to\infty} \left[\int_{|\lambda+\Lambda_0| \leqslant 2^{-m}} Z(d\lambda) - \int_{|\lambda-\Lambda_0| \leqslant 2^{-m}} Z(d\lambda) \right] = 0 \quad a.s. \tag{26}$$

Moreover, if (26) *holds, then*

(*) *the expansion* (24) *converges uniformly for* $T_1 \leqslant t \leqslant T_2$ $(\forall T_1 < T_2)$ *for almost all sample trajectories of the process* (23).

LEMMA 1. *The condition* (25) *holds if and only if at any point* $t \neq r\pi/\Lambda_0$ *equation* (24) *holds in the mean square.*

The sufficiency of (25) is noted in [3], p. 191. We present a complete proof. Let $Z_- = Z(\{-\Lambda_0\})$ *and* $Z_+ = Z(\{+\Lambda_0\})$. From the spectral representation (23) it follows that

$$\xi(t) = \int_{|\lambda| < \Lambda_0} e^{i\lambda t} Z(d\lambda) + Z_+ e^{i\Lambda_0 t} + Z_- e^{-i\Lambda_0 t}; \tag{27}$$

in particular,

$$\xi\left(\frac{k\pi}{\Lambda_0}\right)=\int\limits_{|\lambda|<\Lambda_0} e^{i\lambda\frac{k\pi}{\Lambda_0}}Z(d\lambda)+Z_+e^{ik\pi}+Z_-e^{-ik\pi}. \tag{28}$$

As usual, we make use of the equality

$$e^{i\lambda t}=\lim_{n\to\infty}\sum_{k=-n}^{n} e^{i\frac{k\pi}{\Lambda_0}\lambda}\frac{\sin(\Lambda_0 t-k\pi)}{\Lambda_0 t-k\pi}, \tag{29}$$

where the series converges uniformly for $-\Lambda_0+\varepsilon<\lambda<\Lambda_0-\varepsilon\ \forall\varepsilon>0$.

Therefore,

$$\int\limits_{|\lambda|<\Lambda_0} e^{i\lambda t}Z(d\lambda)=\lim_{n\to\infty}\sum_{k=-n}^{n}\int\limits_{|\lambda|<\Lambda_0} e^{i\frac{k\pi}{\Lambda_0}\lambda}Z(d\lambda)\frac{\sin(\Lambda_0 t-k\pi)}{\Lambda_0 t-k\pi}. \tag{30}$$

We remark that for $\lambda=\pm\Lambda_0$ the series (29) also converges, though not to $e^{\pm i\Lambda_0 t}$, but to $(e^{i\Lambda_0 t}+e^{-i\Lambda_0 t})/2=\cos\Lambda_0 t$, i.e.,

$$\cos\Lambda_0 t=\sum_{-\infty}^{\infty} e^{ik\pi}\frac{\sin(\Lambda_0 t-k\pi)}{\Lambda_0 t-k\pi}. \tag{31}$$

Comparing (27) with (30) and (31), we get that the desired equality (24) holds in the mean square if and only if

$$e^{i\Lambda_0 t}Z_+ + e^{-i\Lambda_0 t}Z_- = (Z_+ + Z_-)\cos\Lambda_0 t,$$

almost surely, i.e., if and only if $i(Z_+-Z_-)\sin\Lambda_0 t=0$ a.s.

Since Z_+ is orthogonal to Z_- and $\sin\Lambda_0 t\neq 0$, the last condition means that $Z_+=Z_-=0$ a.s.

Proof of Theorem 3. Let $t\neq r\pi/\Lambda_0$. In the proof of Lemma 1 it was shown that (24) always converges in the mean square (but its sum does not coincide with $\xi(t)$ if (25) does not hold). Hence, if (24) converges a.s. to $\xi(t)$, then (25) holds. On the other hand, (26) clearly implies (25). Therefore, we assume henceforth that (25) holds, and in all estimates the integrals over the region $|\lambda|\leqslant\Lambda_0$ are equal to the integrals over the region $|\lambda|<\Lambda_0$. We estimate the remainder of the series (24):

$$\begin{aligned}\rho_n(t)&\overset{L_2}{=}\xi(t)-\sum_{|k|\leqslant n}(-1)^k\xi\left(\frac{k\pi}{\Lambda_0}\right)\frac{\sin\Lambda_0 t}{\Lambda_0 t-k\pi}\\ &=-\frac{\sin\Lambda_0 t}{\pi}\sum_{|k|>n}\xi\left(\frac{k\pi}{\Lambda_0}\right)\frac{(-1)^k}{k}+r_n^{(1)}(t).\end{aligned} \tag{32}$$

Here

$$r_n^{(1)}(t)=\sin\Lambda_0 t\sum_{|k|>n}(-1)^k\xi\left(\frac{k\pi}{\Lambda_0}\right)\left[\frac{1}{\Lambda_0 t-k\pi}+\frac{1}{k\pi}\right].$$

If T_1 and T_2 are fixed, then for $n>(|T_1|+|T_2|)\Lambda_0$

$$r_n=\max_{T_1\leqslant t\leqslant T_2}|r_n^{(1)}(t)|\leqslant C\sum_{|k|>n}\left|\xi\left(\frac{k\pi}{\Lambda_0}\right)\right|k^{-2}. \tag{33}$$

From the convergence of $\Sigma\mathbf{M}|\xi(k\pi/\Lambda_0)|k^{-2}$ we get that $r_n\to 0$ a.s., and for an estimate of $\rho_n(t)$ it remains to estimate the expression

$$\rho_n^* \overset{L_2}{=} \sum_{|k|>n} \frac{(-1)^k}{k} \xi\left(\frac{k\pi}{\Lambda_0}\right) = 2i \int_{|\lambda|<\Lambda_0} \sum_{k=n+1}^{\infty} \frac{\sin \frac{k\pi}{\Lambda_0}(\lambda+\Lambda_0)}{k} z(d\lambda). \tag{34}$$

Let

$$S_n(\mu) = \sum_{k=n+1}^{\infty} \frac{\sin k\mu}{k}. \tag{35}$$

We estimate the error in the obvious relation $S_n(\mu) \cong \int_{n\mu}^{\infty}((\sin u)/u)\,du$. It is well known that

$$S_n(\mu) = \frac{\pi}{2} - \frac{\mu}{2} - K_n(\mu), \quad K_n(\mu) = \sum_{k=1}^{n} \frac{\sin k\mu}{k} \quad \text{for } 0 < \mu \leqslant 2\pi. \tag{36}$$

An estimate of $K_n(\mu)$ for $0 \leqslant \mu \leqslant \pi$ is obtained in the standard way. We have

$$K_n(\mu) = \int_0^{\mu} \sum_{k=1}^{n} \cos kx\,dx = \int_0^{M} \left(\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin(x/2)} - \frac{1}{2} \right) dx$$

$$= \int_0^{n\mu} \frac{\sin u\,du}{u} - \frac{\mu}{2} + t_n(\mu),$$

$$t_n(\mu) = \frac{\sin n\mu}{2n} + \int_0^{\mu} f(x)\sin nx\,dx, \quad f(x) = \frac{1}{2\,\mathrm{tg}\,(x/2)} - \frac{1}{x}.$$

Since $f(x)$ and $f'(x)$ are bounded for $0 \leqslant x \leqslant \pi$, we get

$$|t_n(\mu)| \leqslant \frac{C}{n} \quad \text{for} \quad 0 \leqslant \mu \leqslant \pi. \tag{37}$$

Thus, for $0 < \mu \leqslant \pi$ we can use any of the representations

$$S_n(\mu) = \frac{\pi}{2} - \int_0^{n\mu} \frac{\sin u\,du}{u} - t_n(\mu) = \int_{n\mu}^{\infty} \frac{\sin u\,du}{u} - t_n(\mu),$$

and for $\pi \leqslant \mu < 2\pi$ the representation $S_n(\mu) = -S_n(2\pi - \mu)$.

Substituting in (34) (here the role of μ is played by $\pi(\lambda + \Lambda_0)/\Lambda_0$, $-\Lambda_0 < \lambda < \Lambda_0$), we get for ρ_n^* the representation

$$\begin{aligned} \rho_n^* = 2i &\left[\frac{\pi}{2} \int_{-\Lambda_0 \leqslant \lambda \leqslant -\Lambda_0+2^{-m}} Z(d\lambda) - \frac{\pi}{2} \int_{\Lambda_0-2^{-m} \leqslant \lambda \leqslant \Lambda_0} Z(d\lambda) \right] \\ &+ 2i \left[\int_{-\Lambda_0<\lambda\leqslant 0} Q_n^{(1)}(\lambda) Z(d\lambda) + \int_{0<\lambda\leqslant\Lambda_0} Q_n^{(2)}(\lambda) Z(d\lambda) \right] + r_n^*, \end{aligned} \tag{38}$$

where the kernels $Q_n^{(1)}(\lambda)$, $Q_n^{(2)}(\lambda)$ have the form ($m = [\lg_2 n]$):

$$Q_n^{(1)}(\lambda) = \begin{cases} -\displaystyle\int_0^{n\frac{\pi}{\Lambda_0}(\lambda+\Lambda_0)} \frac{\sin u\,du}{u} & \text{for } -\Lambda_0 < \lambda \leqslant -\Lambda_0 + 2^{-m}, \\ \displaystyle\int_{n\frac{\pi(\lambda+\Lambda_0)}{\Lambda_0}}^{\infty} \frac{\sin u\,du}{u} & \text{for } -\Lambda_0 + 2^{-m} < \lambda \leqslant 0, \end{cases}$$

$$Q_n^{(2)}(\lambda) = \begin{cases} \displaystyle\int_0^{n\frac{\pi}{\Lambda_0}(\Lambda_0-\lambda)} \frac{\sin u\,du}{u} & \text{for } \Lambda_0 - 2^{-m} \leqslant \lambda < \Lambda_0, \\ -\displaystyle\int_{n\frac{\pi}{\Lambda_0}(\Lambda_0-\lambda)}^{\infty} \frac{\sin u\,du}{u} & \text{for } 0 \leqslant \lambda \leqslant \Lambda_0 - 2^{-m}. \end{cases}$$

Moreover, by (37), $\mathbf{M}|r_n^*|^2 \leqslant C/n^2$; hence $\lim_{n\to\infty} r_n^* = 0$ a.s.

It remains to verify that the kernels $Q_n^{(1)}$ and $Q_n^{(2)}$ satisfy the conditions of Theorem 2. For example, for $Q_n^{(1)}$ we have

$$|Q_n^{(1)}(\lambda)| \leqslant \frac{\pi}{\Lambda_0} n |\Lambda_0 + \lambda| \quad \text{for } |\Lambda_0 + \lambda| \leqslant 2^{-m},$$

$$|Q_n^{(1)}(\lambda)| \leqslant \frac{2\Lambda_0}{n\pi|\Lambda_0 + \lambda|} \quad \text{for } |\Lambda_0 + \lambda| > 2^{-m}.$$

From this we get condition A) of Theorem 2 with $\beta = 1$.

Next, if $2^m \leqslant k < n < 2^{m+1}$, then

$$|Q_k^{(1)}(\lambda) - Q_n^{(1)}(\lambda)| = \left| \int_{k\frac{\pi}{\Lambda_0}(\lambda+\Lambda_0)}^{n\frac{\pi}{\Lambda_0}(\lambda+\Lambda_0)} \frac{\sin u\,du}{u} \right|$$

$$\leqslant C \min\left\{|n-k||\lambda + \Lambda_0|,\ \frac{n-k}{n},\ (n|\lambda+\Lambda_0|)^{-1}\right\}.$$

Hence condition B) of Theorem 2 also holds, with $\beta_1 = 1$, $\gamma_1 = 0$, $\alpha = 1$, $\beta_2 = 0$ and $\gamma_2 = 1$. The conditions A) and B) for $Q_n(\lambda)$ are verified similarly.

Applying Theorem 2 and equations (32), (34) and (38), we conclude that the remainder of the series (24) converges a.s. to zero if and only if (26) holds. Theorem 3 is proved.

REMARK. The condition (26), generally speaking, is not equivalent to the pair of conditions

$$\int_{|\Lambda_0+\lambda|\leqslant 2^{-m}} Z(d\lambda) \to 0, \qquad \int_{|\lambda-\Lambda_0|\leqslant 2^{-m}} Z(d\lambda) \to 0 \quad \text{a.s.} \tag{39}$$

For example, we can take a discrete spectral measure having the form ($\Lambda_0 > 1$):

$$Z(-\Lambda_0 + 2^{-m}) = a_m(\Phi_m + \Psi_m),\ Z(\Lambda_0 - 2^{-m}) = a_m(\Phi_m - \Psi_m),$$

for $\Sigma a_m^2 < \infty$, $\{\Phi_m\}$ and $\{\Psi_m\}$ being orthonormal and orthogonal to each other, where

$\{\Psi_m\}$ is a system of convergence and $\{\Phi_m\}$ is a system of divergence, and $\{a_m\}$ is chosen so that the series $\Sigma_1^\infty a_m\Phi_m$ diverges a.s. Then (26) holds, but (39) does not.

It is convenient to have sufficient conditions for Kotel'nikov's theorem expressed in terms of simpler properties of the spectral measure or the correlation function of the process. Such conditions are easily obtained from Theorem 3.

COROLLARY 1. *If the spectrum of the process is concentrated on the segment* $[-\Lambda_1, \Lambda_1]$, $\Lambda_1 < \Lambda_0$, *then the assertion* $(*)$ *of Theorem* 3 *holds.*

This reformulation of the result of Beljaev cited above follows immediately from (26).

We use the notation

$$\Delta_k' = \{\lambda : 2^{-k-1} < \lambda + \Lambda_0 \leqslant 2^{-k}\},\ \Delta_k'' = \{\lambda : 2^{-k-1} < \Lambda_0 - \lambda \leqslant 2^{-k}\},$$

$$Z_{-k} = \int\limits_{\Delta_k'} Z(d\lambda),\ Z_k = -\int\limits_{\Delta_k''} Z(d\lambda),\ \Phi_k = Z_k + Z_{-k},\ Z_0 = \Phi_0 = 0.$$

The system $\{\Phi_k\}_0^\infty$ is orthogonal, and the condition (26) of Theorem 3 is equivalent to the convergence a.s. of the partial sums $S_n = \Sigma_1^n\Phi_k$ of the orthogonal series, for which $\Sigma_1^\infty\|\Phi_k\|^2 < \infty$, and to condition (25).

COROLLARY 2. a) *If*

$$F(\{\Lambda_0\}) = F(\{-\Lambda_0\}) = 0,\quad \int\limits_{-\Lambda_0}^{\Lambda_0}\left(\lg\lg\left(2 + \frac{1}{\Lambda_0^2 - \lambda^2}\right)\right)^2 F(d\lambda) < \infty, \tag{40}$$

then the assertion $(*)$ *of Theorem* 3 *holds.*

b) *If* $w(u) = v((\lg\lg u)^2)$, $w(u) \uparrow \infty$ *as* $u \to \infty$, *then there is a stationary random process* $\xi(t)$ *with spectrum in* $(-\Lambda_0, \Lambda_0)$ *such that for any* $t \neq r\pi/\Lambda_0$

$$\varlimsup_{n\to\infty}\left|\sum_{k=-n}^{n}\xi\left(\frac{k\pi}{\Lambda_0}\right)\frac{\sin(\Lambda_0 t - k\pi)}{\Lambda_0 t - k\pi}\right| = \infty \quad a.s. \tag{41}$$

although

$$F(\{\Lambda_0\}) = F(\{-\Lambda_0\}) = 0,\ \int\limits_{-\Lambda_0}^{\Lambda_0} w\left(\frac{1}{\Lambda_0^2 - \lambda^2}\right)F(d\lambda) < \infty. \tag{42}$$

PROOF. Since

$$\|\Phi_k\|^2 = \|Z_k\|^2 + \|Z_{-k}\|^2 = \int\limits_{\Delta_k'\cup\Delta_k''} F(d\lambda),$$

the condition (40) can be rewritten in the form $\Sigma_1^\infty\|\Phi_k\|^2\lg^2(k+2) < \infty$, and the first assertion follows from the Men'šov-Rademacher theorem. For the proof of b) we use a well-known counterexample of D. E. Men'šov: for the function $w(u)$ we construct an orthogonal series $\Sigma_1^\infty\Phi_k$ having the properties that

$$\varlimsup_{n\to\infty}\left|\sum_1^n \Phi_k\right| = \infty \quad \text{a.s.},\qquad \sum_1^\infty \|\Phi_k\|^2 w(k) < \infty,$$

and we define the stochastic measure $Z(d\lambda)$ by the equations

$$Z\{\Lambda_0 - 2^{-k+1}\} = \Phi_k \quad \left(k > \min\left\{1, \lg_2 \frac{1}{\Lambda_0}\right\}\right).$$

From the estimates given for the proof of Theorem 3 it follows that for the process

$$\xi(t) = \int_{-\Lambda_0}^{\Lambda_0} e^{i\lambda t} Z(d\lambda) \quad (-\infty < t < \infty)$$

the conditions (41) and (42) hold.

COROLLARY 3. *For any stationary Gaussian process the assertion* (*) *holds* (*if* (25) *holds*).

The series $\Sigma_1^\infty \Phi_k$ in this case consists of orthogonal (hence, independent) Gaussian variables and converges a.s. by Kolmogorov's theorem (by the condition $\Sigma\|\Phi_k\|^2 < \infty$).

COROLLARY 4. *If the process $\xi(t)$ has a continuous* (*or bounded*) *spectral density $f(\lambda)$, then the assertion* (*) *holds.*

This is a simple particular case of Corollary 2a.

A more general assertion is the following:

The condition

$$f(\lambda) = O((|\lambda^2 - \Lambda_0^2| \lg^2 |\lambda^2 - \Lambda_0^2| \lg^{3+\varepsilon} \lg |\lambda^2 - \Lambda_0^2|)^{-1}) \quad \text{for} \quad \lambda \to \pm\Lambda_0, \ \varepsilon > 0,$$

is sufficient for the assertion (*).

We remark that for $\varepsilon = 0$ this assertion becomes false. Corollaries 3 and 4 generalize the remark (made in [2] without proof) that Kotel'nikov's theorem is true for Gaussian processes with continuous spectral density.

Now let $R(\tau)$ be the correlation function of the process $\xi(t)$;

$$K_1(n) = (-1)^n R\left(\frac{n\pi}{\Lambda_0}\right); \quad K_2(n) = \frac{1}{n}\sum_{\nu=0}^{n-1} K_1(\nu); \quad K_3(n) = \frac{1}{n}\sum_{\nu=0}^{n-1} K_2(\nu).$$

It is clear that $K_1(n)$ is the correlation function of the stationary sequence $\eta_n = (-1)^n \xi(n\pi/\Lambda_0)$, $n = 1, 2, \ldots$, and it is not hard to verify that

$$K_3(n) = \left\| \frac{1}{n} \sum_{\nu=0}^{n-1} \eta_\nu \right\|^2 = \int_0^{2\pi} \frac{\sin^2(n\mu/2)}{n^2 \sin^2(\mu/2)} \, dF_1(d\mu),$$

where $F_1(d\mu)$ is obtained from the spectral measure of the process $F(d\lambda)$ by the transformation $\mu = \lambda\pi/\Lambda_0 + \pi$, $-\Lambda_0 < \lambda < \Lambda_0$.

The condition (40) for $F(d\lambda)$ is equivalent to the condition

$$\sum_{n=3}^{\infty} \frac{K_3(n)}{n \lg n} \lg \lg n < \infty \tag{43}$$

(cf. [1]); in turn, the convergence of either of the series

$$\sum_{n=3}^{\infty} \frac{K_i(n)}{n \lg n} \lg \lg n \quad (i = 1, 2) \tag{44}$$

implies the convergence of the series (43). From these remarks we get the following propositions.

COROLLARY 5. *The convergence of any of the series* (44) ($i = 1, 2, 3$) *is sufficient for Kotel'nikov's theorem to hold* (*in the form* (*) *of Theorem* 3).

COROLLARY 6. *If* $R(\tau)$ *is the correlation function of the process* $\xi(t)$ *and* $R(n\pi/\Lambda_0)$ *is monotonically decreasing, then the assertion* (*) *holds.*

COROLLARY 7. *If for* $i = 1, 2$, *or* 3 *we have that* $K_i(n) = O((\lg\lg n)^{-2-\varepsilon})$ ($\varepsilon > 0$), *then the assertion* (*) *holds for the process* $\xi(t)$.

We note that for $\varepsilon = 0$ Corollary 7 fails (for $i = 1, 2, 3$ there are appropriate counterexamples).

§3. Other applications of the basic theorem

1. *The strong law of large numbers.* It is not hard to verify that if $R_n(\lambda)$ is any of the kernels (2), (3), then the kernel

$$Q_n(\lambda) = R_n(\lambda) - \chi_m(\lambda), \tag{45}$$

where $m = [\lg_2 n]$ ($n \geqslant 1$) and

$$\chi_m(\lambda) = \begin{cases} 1, & |\lambda| \leqslant 2^{-m}, \\ 0, & |\lambda| > 2^{-m}, \end{cases}$$

satisfies the conditions of Theorem 2 ($\Lambda_0 = 0$, $\alpha = \alpha_1 = \alpha_2 = 1$; $\beta = 1$, $\beta_1 = 1$, $\gamma_1 = 0$, $\beta_2 = 0$, $\gamma_2 = 1$).

From this we get Theorem 2 of [1]: *The strong law of large numbers holds for the stationary process* $\xi(t)$ *or the stationary sequence* ξ_n *if and only if the corresponding stochastic spectral measure* $Z(d\lambda)$ *satisfies the relation*

$$\lim_{m\to\infty} \int_{|\lambda|\leqslant 2^{-m}} Z(d\lambda) = 0 \quad a.s.. \tag{46}$$

2. *The rate of convergence in the strong law of large numbers.* For definiteness, we present formulations for the case of a stationary sequence $\{\xi_n\}$ (for processes they are completely analogous).

Let $w(u)$ be a fixed function that increases monotonically as $u \to \infty$. We consider the following conditions on $w(u)$:

$$\exists C < 2: \ w(2u) \leqslant Cw(u) \quad (u > 0), \tag{47}$$

$$\exists C_1 > 1: C_1 w(u) \leqslant w(2u) \quad (u > 0). \tag{48}$$

We consider stationary sequences $\{\xi_k\}$ for which

$$F(\{0\}) = 0, \qquad \int_{-\pi}^{\pi} w\left(\frac{1}{|\lambda|}\right) F(d\lambda) < \infty. \tag{49}$$

THEOREM 4. *If the conditions* (47) *and* (49) *hold and*

$$\sigma_n = \frac{1}{n} \sum_{k=0}^{n-1} \xi_k,$$

then the condition $\lim_{n\to\infty} \sigma_n (w(n))^{1/2} = 0$ *a.s. is equivalent to the condition*

$$\lim_{m\to\infty} \left[\int_{|\lambda|\leqslant 2^{-m}} Z(d\lambda) \right] \sqrt{w(2^m)} = 0 \quad a.s.. \tag{50}$$

PROOF. Let $Q_n(\lambda)$ have the form (45), and

$$Q_n^*(\lambda) = a(m;\ \lambda)\, Q_n(\lambda) \quad \text{for} \quad 2^m \leqslant n < 2^{m+1}, \quad a(m;\ \lambda) = \left[\frac{w(2^m)}{w\left(\frac{1}{|\lambda|}\right)}\right]^{1/2}. \tag{51}$$

We estimate $a(m; \lambda)$ from above. It follows from monotonicity that $a(m; \lambda) \leqslant 1$ for $2^m|\lambda| \leqslant 1$; but if $2^m|\lambda| = R > 1$, then, taking $\varepsilon = (\lg C)/2 < \frac{1}{2}$, we have, by (47),

$$a(m;\ \lambda) = \left(\frac{w(2^m)}{w(2^m/R)}\right)^{1/2} \leqslant 2R^{\varepsilon} = 2(2^m|\lambda|)^{\varepsilon}. \tag{52}$$

From (51), (52), and the properties of the kernels $Q_n(\lambda)$ it follows that $Q_n^*(\lambda)$ satisfies the conditions of Theorem 2 with $\beta = \alpha = \alpha_1 = \alpha_2 = 1$, $\gamma_1 = 0$, $\beta_2 = \varepsilon$ and $\gamma_2 = 1 - \varepsilon > \frac{1}{2}$. But for $2^m \leqslant n < 2^{m+1}$ we have

$$\sigma_n \sqrt{w(2^m)} = \int Q_n^*(\lambda)\, Z_1(d\lambda) + \sqrt{w(2^m)} \int\limits_{|\lambda| \leqslant 2^{-m}} Z(d\lambda), \tag{53}$$

where $Z_1(d\lambda) = (w(1/|\lambda|))^{1/2} Z(d\lambda)$ is a spectral measure with orthogonal increments, and $\int \mathbf{M}|Z_1(d\lambda)|^2 < \infty$, by (49).

By Theorem 2, the first integral on the right-hand side of (53) converges to zero a.s., and Theorem 4 is proved.

3. *The rate of convergence in Kotel'nikov's theorem.* If the spectral measure of the process satisfies certain restrictions in neighborhoods of the points $\pm\Lambda_0$, then we can give a sharper estimate of the remainder of the series (24). We specify the restrictions on the spectral measure $F(d\lambda)$ in the form

$$F(\{-\Lambda_0\}) = F(\{\Lambda_0\}) = 0, \quad \int\limits_{-\Lambda_0}^{\Lambda_0} w\left(\frac{1}{\Lambda_0^2 - \lambda^2}\right) F(d\lambda) < \infty, \tag{54}$$

where $w(u) \uparrow \infty$ is a certain function.

The next theorem is reduced to Theorem 2 in the same way as Theorem 4.

THEOREM 5. *If a stationary process has the representation* (23) *and the conditions* (47) *and* (54) *are fulfilled, then the following assertion holds for the remainder* $\rho_n(t)$ *of the series* (24): *for any* $T_1 < T_2$

$$\lim_{n\to\infty} \left\{ \max_{T_1 \leqslant t \leqslant T_2} |\rho_n(t)| \sqrt{w(n)} \right\} = 0 \quad a.s. \tag{55}$$

if and only if

$$\lim_{m\to\infty} \left[\int\limits_{-\Lambda_0 \leqslant \lambda \leqslant -\Lambda_0 + 2^{-m}} Z(d\lambda) - \int\limits_{\Lambda_0 - 2^{-m} \leqslant \lambda \leqslant \Lambda_0} Z(d\lambda) \right] \sqrt{w(2^m)} \to 0 \quad a.s. \tag{56}$$

Representing the expressions in the square brackets in (50) or (56) as remainders of orthogonal series, the conditions (50) and (56) can be formulated in terms of the rate of convergence of orthogonal series.

Moreover, if in addition to (47) we have also (48) for $w(u)$ in Theorem 4 (for example, if $w(u) = u^{\alpha}$, $0 < \alpha < 1$), then the condition (50) follows from (49), i.e., for the class of stationary processes (49) in this case we have that

$$\lim_{n\to\infty} \sigma_n \sqrt{w(n)} = 0 \quad \text{a.s.} \tag{57}$$

The following assertion is proved similarly: *If $w(u)$ satisfies* (47) *and* (48), *then under the assumption of* (54) *the rate of convergence in Kotel'nikov's theorem is estimated by* (55).

In these cases it can also be shown that the estimates (55) and (57) cannot be improved in the class of processes satisfying (49) and (54), respectively (cf. [4], where there are precise formulations for σ_n).

REMARKS. a) If the process has a spectral measure with "very nice" properties as $\lambda \to \pm\Lambda_0$, then (57) can be sharpened. Thus, it can be shown that if the condition (49) holds (respectively, (54)), where $\int_1^\infty (w(u))^{-1}\,du < \infty$ (say, $w(u) = u^\alpha$, $\alpha > 1$), then σ_n (or $\max_{T_1 \leqslant t \leqslant T_2} |\rho_n(t)|$) has order $o(n^{-1/2}(\lg n)^{1/2+\varepsilon})$ as $n \to \infty$ (or, more generally, $o((\psi(n)/n)^{1/2})$ for any ψ for which $\Sigma(n\psi(n))^{-1} < \infty$). In the particular case when the spectrum of the process lies in $(-\Lambda_1, \Lambda_1)$, $\Lambda_1 < \Lambda_0$ (here we can take any increasing $w(u)$ in the condition (54)), a similar estimate was obtained in [2].

b) The "boundary" case when $w(u) = u$ or $w(u) = u\varphi(u)$, $\varphi(u)$ a slowly varying function, is interesting (and more difficult).

The corresponding results for σ_n that are formulated in [4] can be carried over to the $\rho_n(t)$.

4. *Summation of stationary sequences and processes by Cesàro methods* (C, r). The law of large numbers for a sequence (ξ_n) means that the arithmetic means of the sequence (ξ_n) converge in some sense. Instead of the arithmetic means (the method $(C, 1)$) we can consider other Cesàro means: the methods (C, r).

If $\{\xi_n\}$ is a sequence, then its (C, r)-means σ^r have the form

$$\sigma_n^r = (A_n^r)^{-1} \sum_{k=0}^{n} A_{n-k}^{r-1}\xi_k,$$

$$A_n^r = \frac{(r+1)(r+2)\dots(r+n)}{n!} \simeq \frac{n^r}{\Gamma(r+1)} \qquad (r > -1).$$

THEOREM 6. *If $\{\xi_n\}$ is an arbitrary stationary sequence, $Z(d\lambda)$ is its stochastic spectral measure, and $r > \frac{1}{2}$, then the means σ_n^r converge a.s.* (*necessarily to $Z(\{0\})$*) *if and only if*

$$\lim_{m\to\infty} \int_{|\lambda| \leqslant 2^{-m}} Z(d\lambda) = Z(\{0\}) \quad a.s. \tag{58}$$

COROLLARY 8. *For $r > \frac{1}{2}$ all the Cesàro methods (C, r) for summation a.s. of stationary sequences are equivalent to the method $(C, 1)$.*

PROOF. Substituting $\xi_k = \int_{-\pi}^{\pi} e^{ik\lambda} Z(d\lambda)$, we have that

$$\sigma_n^r = \int_{-\pi}^{\pi} R_n^{(r)}(\lambda)\, Z(d\lambda), \quad R_n^{(r)}(\lambda) = \sum_{k=0}^{n} A_{n-k}^{r-1} (A_n^r)^{-1} e^{ik\lambda}. \tag{59}$$

Let $\chi_m(\lambda)$ be the same function as in (45), and $Q_n^{(r)}(\lambda) = R_n^{(r)}(\lambda) - \chi_m(\lambda)$ for $2^m \leqslant n < 2^{m+1}$. We have the formulas (see [5], Chapter III, §5)

$$R_n^{(r)}(\lambda) = e^{in\lambda}\left[(1 - e^{-i\lambda})^{-r} - \sum_{\nu=n+1}^{\infty} A_\nu^{r-1} e^{-i\nu\lambda}\right](A_n^r)^{-1},$$

$$\left|\sum_{\nu=n+1}^{\infty} A_\nu^{r-1} e^{-i\nu\lambda}\right| \leqslant 2A_{n+1}^{r-1} |1 - e^{-i\lambda}|^{-1},$$

$$|R_n^{(r)}(\lambda) - 1| \leqslant (A_n^r)^{-1} \sum_{\nu=0}^{n} A_{n-\nu}^{r-1} |e^{i\nu\lambda} - 1| \leqslant n|\lambda|.$$

From this it follows that

$$|Q_n^{(r)}(\lambda)| \leqslant C_r \min\left\{n|\lambda|;\ \frac{1}{n|\lambda|} + \frac{1}{(n|\lambda|)^r}\right\}. \tag{60}$$

Using the notation $b_{n,\nu} = (A_n^r)^{-1}A_{n-\nu}^{r-1}$, for $2^m \leqslant k < n < 2^{m+1}$ we have

$$|Q_k^{(r)}(\lambda) - Q_n^{(r)}(\lambda)| \leqslant \left[\sum_{\nu=0}^{k} |b_{n,\nu} - b_{k,\nu}| + \sum_{\nu=k+1}^{n} |b_{n,\nu}|\right] \min\{1;\ n|\lambda|\}. \tag{61}$$

The expression in the square brackets does not exceed

$$C(n-k)^r n^{-r} \quad \text{for} \quad 0 < r \leqslant 1, \quad C(n-k)n^{-1} \quad \text{for} \quad r \geqslant 1. \tag{62}$$

From (60)–(62) it follows that the kernels $Q_n^{(r)}(\lambda)$ for $r > \frac{1}{2}$ satisfy the conditions of Theorem 2, where for $\frac{1}{2} < r \leqslant 1$ we can set $\beta = \alpha = \alpha_1 = \alpha_2 = \gamma_2 = r > \frac{1}{2}$, $\beta_1 = 1$, $\gamma_1 = -(1-r)$ and $\beta_2 = 0$, and for $r \geqslant 1$ we can set $\alpha = \alpha_1 = \alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = 1$ and $\beta = 1$. From this the assertion of Theorem 6 follows immediately.

REMARKS. 1) Corollary 8 generalizes a theorem of Lorentz about orthonormal sequences (see [9]) to any stationary sequences (a particular result in this direction was obtained in [6]).

2) For $r = \frac{1}{2}$ Theorem 6, as is well known, becomes false for certain orthonormal systems. For example, for the Haar system (and any other system for which $\overline{\lim}_{n\to\infty}|\xi_n|n^{-1/2} > 0$ a.s.) the $(C, \frac{1}{2})$-means diverge a.s., while the $(C, 1)$-means converge a.s.

3) The same example shows that the conditions $\alpha > \frac{1}{2}$, $\alpha_1 > \frac{1}{2}$ and $\alpha_2 > \frac{1}{2}$ in Theorem 2 cannot be replaced by the condition $\alpha = \alpha_1 = \alpha_2 = \frac{1}{2}$.

4) Theorem 6 and Corollary 8 admit an obvious generalization to stationary processes; instead of Cesàro means we can consider the Riesz means

$$\hat{\sigma}_T^r = rT^{-r}\int_0^T (T-t)^{r-1}\xi(t)\,dt, \qquad r > \frac{1}{2}.$$

The applications we have listed up to this point have dealt with the case when the space Y in Theorem 1 is $\mathbf{R}^1$. The multi-dimensional case is used in the following examples.

5. *The s.l.l.n. for homogeneous random fields.* Let $\xi(s)$, $s \in \mathbf{R}^k$, be a homogeneous random field, $\mathbf{M}\xi(s) = 0$,

$$\xi(s) = \int_{\mathbf{R}^k} e^{i\langle\lambda, s\rangle} Z(d\lambda), \qquad s = (s_1, \ldots, s_k),$$

and σ_ρ its spherical means, $|V_\rho|$ the volume of the sphere of radius ρ, and $\sigma_\rho = |V_\rho|^{-1}\int_{V_\rho}\xi(s)\,ds$. If $\chi_m(\lambda)$ has the form (45) ($|\lambda| = (\lambda_1^2 + \cdots + \lambda_k^2)^{1/2}$), then

$$\sigma_n - \int_{|\lambda| \leqslant 2^{-m}} Z(d\lambda) = \int_{\mathbf{R}^k} Q_n(\lambda) Z(d\lambda),$$

and it is easy to verify that $f_n = \int_{\mathbf{R}^k} Q_n(\lambda) Z(d\lambda)$ satisfies the conditions of Theorem 1 ($Y = \mathbf{R}^k$, $\nu(d\lambda) = F(d\lambda) = \mathbf{M}|Z(d\lambda)|^2$, $\varphi(\lambda) = |\lambda|$, etc.). From this follows the s.l.l.n. in the form obtained in [1]. We can also get estimates of the rate of convergence under restrictions of the type (49).

6. *The s.l.l.n. for harmonizable processes and fields.* According to [8], a process $\xi(t)$ is *harmonizable* if it has a representation

$$\xi(t) = \int_{\mathbf{R}^1} e^{i\lambda t} Z(d\lambda), \tag{63}$$

where $Z(d\lambda)$ is a stochastic spectral measure (not necessarily orthogonal) for which $\mathbf{M}[Z(d\lambda)\cdot\overline{Z(d\mu)}] = F(d\lambda, d\mu)$, F a finite measure in $\mathbf{R}^1 \times \mathbf{R}^1$. Using again the notation

$$\sigma_T = \frac{1}{T}\int_0^T \xi(t)\,dt = \int_{\mathbf{R}^1} \frac{e^{i\lambda T}-1}{i\lambda T} Z(d\lambda),$$

we observe that

$$\mathbf{M}\left|\sigma_{2^m} - \int_{|\lambda|\leqslant 2^{-m}} Z(d\lambda)\right|^2 = \int_{\mathbf{R}^1\times\mathbf{R}^1} Q_{2^m}(\lambda)\overline{Q_{2^m}(\mu)}F(d\lambda, d\mu),$$

$$\mathbf{M}|\sigma_n - \sigma_k|^2 = \int_{\mathbf{R}^1\times\mathbf{R}^1} [Q_n(\lambda) - Q_k(\lambda)]\,\overline{[Q_n(\mu) - Q_k(\mu)]}\,F(d\lambda, d\mu),$$

where the kernel $Q_n(\lambda)$ has the form (45).

Using the properties of the kernel $Q_n(\lambda)$ established earlier, we see that the conditions of Theorem 1 hold, where $Y = \mathbf{R}^1 \times \mathbf{R}^1$, $d\nu = F(d\lambda, d\mu)$ and $\varphi(y) = |\lambda\mu|$; hence, *the s.l.l.n. holds for a harmonizable process* (63) *if and only if*

$$\lim_{m\to\infty} \int_{|\lambda|\leqslant 2^{-m}} Z(d\lambda) = 0 \quad \text{a.s.}$$

This is a direct generalization of a result established earlier by us for stationary processes.

REMARKS. 1) A similar theorem holds for harmonizable fields (having the representation (63), where the integration is carried out over $\mathbf{R}^k$).

2) A similar result is true also for more general processes having a Karhunen-Loève representation:

$$\xi(t) = \int_{\mathbf{R}^1} f(t, \lambda) Z(d\lambda). \tag{64}$$

Suppose that $(1/n)\int_0^n f(t, \lambda)\,dt = R_n(\lambda)$ and there exists a functional $\varphi(\lambda) \geqslant 0$ such that if

$$\chi_m(\lambda) = \begin{cases} 1, & 2^m\varphi(\lambda) \leqslant 1, \\ 0, & 2^m\varphi(\lambda) > 1, \end{cases}$$

then $Q_n(\lambda) = R_n(\lambda) - \chi_m(\lambda)$ $(2^m \leqslant n < 2^{m+1})$ satisfies conditions of the type (19)–(21), where it is necessary to take $\varphi_1(\lambda) \geqslant 0$ in place of $|\lambda - \lambda_0|$. Then for any process of the form (64) we have that

$$\lim_{n\to\infty}\left[\frac{1}{n}\int_0^n \xi(t)\,dt - \int_{2^m\varphi(\lambda)\leqslant 1,\, m=[\lg_2 n]} Z(d\lambda)\right] = 0 \quad \text{a.s.}$$

3) If a harmonic process has the form

$$\xi(t) = \int_K e^{i\lambda t} Z(d\lambda), \quad -\infty < t < \infty,$$

and $\sup_{\lambda\in K}|\lambda| = \Lambda_0 < \infty$, then Kotel'nikov's theorem holds in the form (24) for almost all sample trajectories of this process if and only if

$$\lim_{m\to\infty}\left[\int\limits_{K\cap[-\Lambda_0,-\Lambda_0+2^{-m}]} Z(d\lambda) - \int\limits_{K\cap[\Lambda_0-2^{-m},\Lambda_0]} Z(d\lambda)\right] = 0 \ \text{a.s.} \tag{65}$$

The proof is carried out by the same scheme.

In the particular case when $\sup_{\lambda\in K}|\lambda| = \Lambda_1 < \Lambda_0$, the condition (65) automatically holds, and a result of [7] is obtained.

It seems that Theorems 1 and 2 can find also other applications in problems on the convergence of various sequences almost everywhere.

Received 23/DEC/76

BIBLIOGRAPHY

1. V. F. Gapoškin, *Criteria for the strong law of large numbers for classes of stationary processes and homogeneous random fields*, Dokl. Akad. Nauk SSSR **223** (1975), 1044–1047; English transl. in Soviet Math. Dokl. **16** (1975).

2. Ju. K. Beljaev, *Analytic random processes*, Teor. Verojatnost. i Primenen. **4** (1959), 437–444; English transl. in Theor. Probability Appl. **4** (1959).

3. B. R. Levin, *The theoretical foundations of statistical radio engineering*, "Sovetskoe Radio", Moscow, 1974. (Russian)

4. V. F. Gapoškin, *Exact bounds for the rate of convergence in the strong law of large numbers for classes of stationary sequences and processes in the wide sense*, Uspehi Mat. Nauk **31** (1976), no. 5 (191), 233–234. (Russian)

5. A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol. I, Cambridge Univ. Press, New York, 1959.

6. A. M. Jurčenko, *Summability of quasi-orthogonal sequences by Cesàro methods*, Anal. Math. **1** (1975), 231–247. (Russian)

7. Z. A. Piranašvili, *The problem of interpolation of random processes*, Teor. Verojatnost. i Primenen. **12** (1967), 708–717; English transl. in Theor. Probability Appl. **12** (1967).

8. Ju. A. Rozanov, *Spectral analysis of abstract functions*, Teor. Verojatnost. i Primenen. **4** (1959), 271–287; English transl. in Theor. Probability Appl. **4** (1959).

9. Georg Gunther Lorentz, *Über die Mittelwerte der Funktionen eines Orthogonalsystems*, Math. Z. **49** (1944), 724–733.

Translated by H. McFADEN