R. Basu *et al.* (2020) “Exponential Concentration for Zeroes of Stationary Gaussian Processes,”
International Mathematics Research Notices, Vol. 2020, No. 23, pp. 9769–9796
Advance Access Publication December 11, 2018
doi:10.1093/imrn/rny277

Exponential Concentration for Zeroes of Stationary Gaussian Processes

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We show that for any centered stationary Gaussian process of absolutely integrable covariance, whose spectral measure has compact support, or finite exponential moments (and some additional regularity), the number of zeroes of the process in $[0, T]$ is within ηT of its mean value, up to an exponentially small in T probability.

1 Introduction

The study of zeroes of stationary Gaussian processes goes back at least to Kac [11] and Rice [20]. Since then, much work on this topic appeared in the statistics, physics, mathematics and engineering literature. One of the earliest and most fundamental results in this area is the Kac–Rice formula, which calculates the mean number of zeroes in any interval. A similar formula may be written for the variance of the number of zeroes and for any higher factorial moment, but these are much harder to analyze. It was only many years after Kac and Rice formula that fluctuations and central limit theorems were better understood, with works by Cuzick [6], Slud [22], Azaïs-León [3], and others.

Received October 23, 2018; Revised May 6, 2018; Accepted November 7, 2018
Communicated by Prof. Misha Sodin

Questions of large deviations, that is, estimation of the rare event of having many more or much less zeroes than expected in a long interval, remained almost unexplored. One particular such event is that of having no zeroes at all in a long interval, which is also known by the name of "persistence". Results (and speculations) about this event were initiated by Slepian [21] and were better understood only recently [9].

In the meantime, complex zeroes of certain Gaussian analytic functions received much attention. Most notably, zeroes of the Fock–Bargmann model were introduced by Sodin–Tsirelson [23] and extensively studied by many authors since then. This model has a remarkable property; its zeroes form a point process in the plane with quadratic repulsion and invariance of distribution under all planar isometries. Sodin–Tsirelson proved the asymptotic normality of these zeroes in [23] and, moreover, an exponential concentration of the zeroes around the mean in [24] (see also [13, 18] for more about concentration and [17] for finer results on asymptotic normality). Exponential concentration was proved for other related models, such as nodal lines of spherical harmonics in [16]. Inspired by these works, the question of concentration for real zeroes got some attention (e.g. [26, Theorem 2c1] and personal communication with M. Sodin) but, until now, was not settled even for a single particular example.

The aim of this paper is to prove exponential concentration for *real zeroes* of certain Gaussian stationary functions on $\mathbb{R}$, which have an analytic extension to a strip in the complex plane and smooth spectral density. These conditions allow us to use tools from complex analysis, thus generalizing the mechanics of the aforementioned works on the Fock–Bargmann model.

We consider here centered Gaussian stationary processes $\{X(t) : t \in \mathbb{R}\}$ having a.s. *absolutely continuous* sample path. That is, random absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$, whose finite marginal distributions are mean-zero multi-normal, invariant to real shifts. Normalizing WLOG the process X to have variance one, its joint law is determined by $r(t-s) := \mathrm{Cov}\,[X(t), X(s)]$. Here $r : \mathbb{R} \to \mathbb{R}$ is a continuous, positive semi-definite function (with $r(0) = 1$). By Bochner's theorem, this yields the existence of a probability measure ρ on $\mathbb{R}$, called the *spectral measure*, such that

$$r(t) = \int_{\mathbb{R}} e^{-i\lambda t}\, \mathrm{d}\rho(\lambda)\,. \tag{1.1}$$

We further assume throughout that $\int \lambda^2 \mathrm{d}\rho < \infty$ or equivalently that $r(t)$ has finite 2nd derivative at $t = 0$ (in which case $r(t)$ is twice continuously differentiable and $-r''(0) = \int \lambda^2 d\rho$), and let $N_X(I) = |\{t \in I : f(t) = 0\}|$ count the number of zeroes, possibly infinite, of such a process in the interval $I \subset \mathbb{R}$. Since X is stationary, $\mathbb{E}N_X([0,T]) = \alpha T$

for $\alpha := \mathbb{E}N_X([0,1])$ and from the Kac–Rice formula, we have that in this case

$$\alpha = \frac{1}{\pi}\left(\int \lambda^2 \mathrm{d}\rho\right)^{\frac{1}{2}} < \infty\,. \tag{1.2}$$

Indeed, $\alpha/2$ is the expected number of 0-upcrossings by $X([0,1])$, all of which are strict (see [14, Theorem 7.3.2]), and $X^{-1}\{0\}\cap[0,1]$ has expected size α since it a.s. consists of only the (strict) 0-upcrossings and 0-downcrossings of $X([0,1])$ (see [14, Theorem 7.2.5]). Our objective is to prove the following exponential concentration of $N_X([0,T])$.

Theorem 1.1. Suppose the centered stationary Gaussian process $\{X(t) : t \in \mathbb{R}\}$ of a compactly supported spectral measure has an integrable covariance (i.e., $\int |r(t)|\mathrm{d}t < \infty$). Then, for some $C < \infty$ and $c(\cdot) > 0$,

$$\mathbb{P}\left(|N_X([0,T]) - \alpha T| \geq \eta T\right) \leq Ce^{-c(\eta)T}\,, \qquad \forall \eta > 0, \;\; T < \infty\,. \tag{1.3}$$

Further, if the spectral measure has only a finite exponential moment, namely, for some $\kappa > 0$

$$\int_{\mathbb{R}} e^{|\lambda|\kappa}\,\mathrm{d}\rho(\lambda) < \infty\,, \tag{1.4}$$

then (1.3) holds whenever $\int |r(t;\kappa_o)|\mathrm{d}t < \infty$ for some $\kappa_o \in (0,\kappa/2)$ and

$$r(t;\kappa_o) := \int_{\mathbb{R}} \cos(t\lambda)\cosh(2\kappa_o\lambda)\mathrm{d}\rho(\lambda)\,. \tag{1.5}$$

Remark 1.2. Theorem 1.1 applies for example to any spectral measure whose compactly supported density is in $W^{1,2}(\mathbb{R})$, as well as covariances such as $r(t) = e^{-t^2/2}$ or $r(t) = 1/(1+t^2)$ (of spectral densities $p(\lambda) = \frac{1}{\sqrt{2\pi}}e^{-\lambda^2/2}$ and $p(\lambda) = \frac{1}{2}e^{-|\lambda|}$, respectively), for which (1.4) holds and the LHS of (1.5) is integrable.

As shown next, the lower tail in (1.3) holds under much weaker regularity assumptions.

Proposition 1.3. Suppose the centered stationary Gaussian process $\{X(t) : t \in \mathbb{R}\}$ has an absolutely continuous sample path and bounded, continuous spectral density $p(\lambda)$ with $\int \lambda^2 p(\lambda)d\lambda < \infty$. Then, for some $C < \infty$ and $c(\cdot) > 0$ we have

$$\mathbb{P}\left(N_X([0,T]) - \alpha T \leq -\eta T\right) \leq Ce^{-c(\eta)T}\,, \qquad \forall \eta > 0, \;\; T < \infty\,. \tag{1.6}$$

Proposition 1.3 is a consequence of our next result, on exponential concentration of the number of sign changes in $[0,T]$, for any *discrete-time* centered stationary Gaussian process $\{Y_k : k \in \mathbb{Z}\}$ of continuous spectral density.

Theorem 1.4. Suppose $\{Y_k : k \in \mathbb{Z}\}$ is a centered stationary Gaussian process whose spectral measure ρ_Y has a continuous density $p_Y(\lambda)$ (supported within $[-\pi,\pi]$). Then, for

$$N_Y^\star(T) := \sum_{k=0}^{T-1} \mathbf{1}_{\{Y_k Y_{k+1} < 0\}}\,, \tag{1.7}$$

some $C < \infty$ and $c(\cdot) > 0$,

$$\mathbb{P}\left(|N_Y^\star(T) - \mathbb{E}N_Y^\star(T)| \geq \eta T\right) \leq Ce^{-c(\eta)T}\,, \qquad \forall \eta > 0,\ \ T \in \mathbb{N}\,. \tag{1.8}$$

Remark 1.5. In the setting of Theorem 1.4, one has that $T^{-1}\sum_{k=0}^{T-1} h(Y_k, Y_{k+1})$ satisfies the Large Deviation Principle for any fixed $h \in C_b(\mathbb{R}^2)$, with a convex, good rate function (see [4, Theorem 4.25]). While this may be extended to $N_Y^\star(T)$ by suitable approximations, we do not follow this route since the rate function is in any case not easily identifiable (see [4, Section 7(a)]). For any m-dependent process $\{Y_k : k \in \mathbb{Z}\}$, our proof of Theorem 1.4 provides the explicit $c(\eta) = \eta^2/(2m)$ and $C = 2m\sqrt{e}$ (see (2.6), where it suffices to consider $\eta \in [0,1]$). However, in general we rely on an approximation by m-dependent processes with unbounded $m = m(\eta)$, thereby having $\eta^{-2}c(\eta) \to 0$.

Proposition 1.3 will follow from Theorem 1.4, using the key observation that having few zeroes of a continuous time process implies few sign changes of its restriction to a lattice. This approach does not work for the more challenging upper tail in (1.3), since having many zeroes does not imply having many sign changes on a lattice, and to establish the upper tail we require the following decay and regularity assumptions about the spectral density of X.

Assumption A: The spectral measure is nonatomic and has finite exponential moment as in (1.4). Further, the covariance functions

$$r_\ell(x;y) := \int_{\mathbb{R}} e^{-i\lambda x}\varphi^\ell(\lambda y)\mathrm{d}\rho(\lambda)\,, \quad \ell = 1,2\,, \qquad \varphi(\lambda) := \sinh(\lambda)/\lambda\,, \tag{1.9}$$

and their x-derivatives, satisfy for some $\kappa' \in (0, \kappa/2)$ and finite $x_\star$,

$$\omega_\star(k) := \sum_{j \ge k} |r(jx_\star)| + \sup_{|y|<\kappa'} \left\{ \sum_{j \ge k} |r_1'(jx_\star; 2y)| + \sum_{j \ge k} |r_2''(jx_\star; y)| \right\} \to 0 , \quad \text{when} \quad k \to \infty . \tag{1.10}$$

Equipped with Assumption A, we state our main (technical) result.

Theorem 1.6. Subject to Assumption A we have for some $C < \infty$ and $c(\cdot) > 0$, the exponential upper tail

$$\mathbb{P}\left(N_X([0,T]) - \alpha T \ge 3\eta T\right) \le Ce^{-c(\eta)T} , \qquad \forall \eta > 0, \ \ T < \infty . \tag{1.11}$$

In particular, we recover Theorem 1.1 from Proposition 1.3 and Theorem 1.6, thanks to the following explicit sufficient condition for Assumption A.

Proposition 1.7. Assumption A is satisfied; the spectral measure $\rho(\cdot)$ has a bounded, continuous density, and a.s. the sample path $t \mapsto X(t) \in C^\infty(\mathbb{R})$, when either of the following holds:

(a) The support of $\rho(\cdot)$ is compact and $\int |r(t)| \mathrm{d}t < \infty$ for the covariance $r(t)$ of (1.1).

(b) Condition (1.4) holds and $\int |r(t; \kappa_o)| \mathrm{d}t < \infty$ for the covariance $r(t; \kappa_o)$ of (1.5).

It is reasonable when seeking the exponential concentration of $N_X([0,T])$ to require smoothness of the covariance $r(\cdot)$, such as having all spectral moments finite (or the stronger condition (1.4)). Indeed, such exponential concentration implies the finiteness of all moments $m_k := \mathbb{E}[N_X([0,T])^k]$, with $m_k = O((T \vee k)^k)$, and such conditions appear in previous studies concerning the finiteness of $\{m_k\}$. For instance, Nualart and Wschebor [19] show that $m_k < \infty$ for all k when $t \mapsto r(t)$ is real analytic (hence all spectral moments are finite), while when $\int \lambda^4 \mathrm{d}\rho = \infty$, Cuzick [7] can prove only the finiteness of m_k up to a certain order k_o. Similarly, Longuet-Higgins [15] shows that for $r(t)$ real analytic, $q_k(\tau) := \mathbb{P}(N_X(0,\tau) \ge k)$ decays, for $\tau \to 0$, as $c(k)\tau^{\frac{1}{2}k^2 + O(k)}$ (indicative of the mutual repulsion of zeroes), while with a discontinuity of $r^{(3)}$ at the origin the decay of $q_k(\tau)$ is merely $c(k)\tau^2$ for all k (so having a pair of nearby zeroes, the probability of k extra zeroes within the same short interval is $O_k(1)$).

A natural path toward proving the upper tail in our concentration result is to improve Cuzick's results on moments m_k [6] or Longuet-Higgins estimates on the tail of the number of zeroes q_k [15], so as to get accurate asymptotics of those quantities in k. Efforts in this direction were made by many authors (e.g., [3] and the references within). However, in our context it requires a lower bound on the determinant of nearly singular matrices (specifically, the covariance matrices for values of $X(t)$ at a short range), at a level of accuracy that seems out of reach. We bypass this difficulty by relating $N_X([0,T])$ to the count of zeroes within a suitable cover of $[0,T]$, for certain random analytic function $f : \mathbb{S} \to \mathbb{C}$ on a thin strip. Thereby, complex analytic tools allow us to replace exponential moments of zero counts by more regular integrals of $\log|f(z)|$. After this reduction, the core challenge of our strategy remains in the need to sharply estimate fractional moments of products of many dependent Gaussian variables. This highly nontrivial task (see [25], [27], and references within), requires our assumption (1.10), in order to get suitable diagonally dominant covariance matrices.

The paper is organized as follows. In Section 2 we prove Proposition 1.3, Theorem 1.4, and Proposition 1.7. The remainder of the paper is devoted to the proof of Theorem 1.6. In Section 3 this theorem is reduced to the key Proposition 3.6, concerning fractional moments of products of $\mathbb{C}$-valued Gaussian random variables. Proposition 3.6 is proved in Section 5, building on the auxiliary results about weakly correlated Gaussian variables that we establish in Section 4.

2 Proofs of Proposition 1.3, Theorem 1.4, and Proposition 1.7

2.1 Proof of Proposition 1.3

Assume WLOG that $c(\eta) \le 1$ and $C \ge e$. It suffices to consider $T \ge 1$. Fixing small $\delta > 0$, by the mean value theorem we have that $N_X([0,T]) \ge N_Y^\star([T/\delta]-1)$ for the stationary centered Gaussian sequence $Y_k := \delta^{-1}\int_0^\delta X(\delta k + t)\,\mathrm{d}t$. It is further easy to check that

$$\gamma_k := \mathbb{E}Y_0Y_k = \delta^{-2}\int_0^\delta\int_0^\delta r(\delta k + t - s)\mathrm{d}s\,\mathrm{d}t\,, \tag{2.1}$$

corresponds to the spectral density

$$p_Y(\lambda) = \frac{1}{\delta}\sum_{n\in\mathbb{Z}} p\left(\frac{\lambda+2\pi n}{\delta}\right)\mathrm{sinc}^2\left(\frac{\lambda}{2}+\pi n\right) \qquad \lambda \in [-\pi,\pi],$$

where $\mathrm{sinc}(\lambda) := \frac{\sin\lambda}{\lambda}$. Note that for $p(\cdot)$ bounded and continuous, p_Y is also continuous (by dominated convergence). Here $r(0) = 1$, $r'(0) = 0$ and $-r''(t)$, being the characteristic function of the finite measure $\lambda^2 p(\lambda)d\lambda$, is continuous at $t \to 0$. We thereby get from

(2.1) that $\gamma_0 \to 1$ and $2\delta^{-2}(\gamma_1 - \gamma_0) \to r''(0)$ when $\delta \downarrow 0$. Setting $\gamma_1 = \gamma_0 \cos\theta$, note that $Y_k = Y$ and $Y_{k+1} = X \sin\theta + Y \cos\theta$ for the i.i.d. Gaussian $(X, Y) = (R\cos U, R\sin U)$, where U is a Uniform($[0, 2\pi]$) variable. Thus, by elementary trigonometric identities,

$$\mathbb{P}(Y_k Y_{k+1} < 0) = \mathbb{P}(\cos\theta - \cos(2U + \theta) < 0) = \frac{\theta}{\pi} = \frac{1}{\pi} \arccos(\gamma_1/\gamma_0)\,,$$

from which we deduce that

$$\inf_{T\geq 1} \frac{1}{T} \mathbb{E} N_Y^\star([T/\delta] - 1) \geq (\delta^{-1} - 2)\mathbb{P}(Y_0 Y_1 < 0) \to \alpha\,.$$

As a result of the preceding, we get (1.6) by considering (1.8) for $\delta = \delta(\eta) > 0$ small enough.

2.2 Proof of Theorem 1.4

We shall use the following easy consequence of weak convergence.

Lemma 2.1. Let (Y_0, Y_1) be a zero mean jointly Gaussian, having $\mathbb{E}[Y_0^2] > 0$, $\mathbb{E}[Y_1^2] > 0$ and $\alpha_\xi := \mathbb{P}(Y_0 Y_1 < \xi)$. If the covariance matrices $\boldsymbol{\Sigma}^{(m)}$ of the zero mean Gaussian vectors $(W_0^{(m)}, W_1^{(m)})$ converge to the covariance matrix $\boldsymbol{\Sigma}$ of (Y_0, Y_1), then

$$\lim_{\xi\to 0} \lim_{m\to\infty} \alpha_\xi^{(m)} = \alpha_0\,, \qquad \alpha_\xi^{(m)} := \mathbb{P}(W_0^{(m)} W_1^{(m)} < \xi). \tag{2.2}$$

Proof. Since Y_0 and Y_1 have positive variances, the CDF of $Y_0 Y_1$ is continuous, so the weak convergence of $(W_0^{(m)}, W_1^{(m)})$ to (Y_0, Y_1) implies that for any ξ fixed, $\alpha_\xi^{(m)} \to \alpha_\xi$ as $m \to \infty$. Further, the monotone function $\xi \mapsto \alpha_\xi$ is then continuous and (2.2) holds. ■

We turn to the proof of Theorem 1.4. WLOG normalize to have $\mathbb{E}Y_0^2 = 1$. Let $\eta > 0$ be given. For any $m \geq 2$ we approximate Y by an $(m-1)$-dependent process using the following construction (see, e.g., [5, Proof of Theorem A]). Let $\{a_k : k \in \mathbb{Z}\}$ denote the Fourier coefficients of the continuous function $\sqrt{p_Y(\lambda)}$ on $[-\pi, \pi]$, and define $a_k^{(m)} := \left(1 - \frac{|k|}{m-1}\right)_+ a_k$. Then $Y_k = W_k^{(m)} + Z_k^{(m)}$, where $\{W_k^{(m)} : k \in \mathbb{Z}\}$ is an $(m-1)$-dependent, centered, stationary Gaussian sequence with covariance $\mathbb{E}[W_0^{(m)} W_n^{(m)}] = \sum_k a_k^{(m)} a_{k+n}^{(m)}$ and spectral measure $p_W^{(m)}(\lambda) = (\int F_m(\lambda - \lambda')\sqrt{p_Y(\lambda')}d\lambda')^2$, where $F_m(\lambda) = \sum_{|k|\leq m} \left(1 - \frac{|k|}{m-1}\right) e^{ik\lambda}$ is the Fejér Kernel. By Fejér's theorem, the spectral density $p_Z^{(m)}(\lambda) = \left(\sqrt{p_Y(\lambda)} - \sqrt{p_W^{(m)}(\lambda)}\right)^2$ of the centered, stationary Gaussian sequence

$\{Z_k^{(m)} : k \in \mathbb{Z}\}$ converges to zero as $m \to \infty$, uniformly on $[-\pi, \pi]$; namely, $\varepsilon_m := \sup_\lambda\{p_Z^{(m)}(\lambda)\} \to 0$ as $m \to \infty$.

By stationarity, $\mathbb{E}N_Y^\star(T) = \alpha_0 T$ for $\alpha_0 := \mathbb{P}(Y_0Y_1 < 0)$. Our assumption that the spectral measure ρ_Y has a continuous density implies that $|r_Y(1)| < 1$; hence the covariance matrix $\mathbf{\Sigma}$ of (Y_0, Y_1) is positive-definite. Further, by construction, the covariance matrices $\mathbf{\Sigma}^{(m)}$ of $(W_0^{(m)}, W_1^{(m)})$ converge to $\mathbf{\Sigma}$ when $m \to \infty$ and Lemma 2.1 applies. In particular, there exist $\xi \in (0, 1]$ and $m_\star < \infty$ so $\alpha_{3\xi}^{(m)} \le \alpha_0 + \eta$ whenever $m \ge m_\star$. Further, since

$$|(w+z)(\tilde{w}+\tilde{z}) - w\tilde{w}| \le |z||\tilde{z}| + |\tilde{z}||w| + |z||\tilde{w}|, \tag{2.3}$$

we have that for $\xi = \delta R \ge \delta^2$,

$$\{Y_kY_{k+1} < 0\} \subseteq \{W_k^{(m)}W_{k+1}^{(m)} < 3\xi\} \cup \{|W_k^{(m)}| \ge R\} \cup \{|W_{k+1}^{(m)}| \ge R\} \cup \{|Z_k^{(m)}| \ge \delta\} \cup \{|Z_{k+1}^{(m)}| \ge \delta\}.$$

Considering this set containment for all $0 \le k \le T-1$ yields that

$$N_Y^\star(T) \le N_{m,\xi}(T) + N_m^R(T) + \widetilde{N}_m^R(T) + N_{Z^{(m)}}(T) + \widetilde{N}_{Z^{(m)}}(T), \tag{2.4}$$

where

$$N_{m,\xi}(T) := \sum_{k=0}^{T-1} \mathbf{1}_{\{W_k^{(m)}W_{k+1}^{(m)} < 3\xi\}}, \quad N_m^R(T) := \sum_{k=0}^{T-1} \mathbf{1}_{\{|W_k^{(m)}| \ge R\}}, \quad N_Z(T) := \sum_{k=0}^{T-1} \mathbf{1}_{\{|Z_k| \ge \delta\}} \tag{2.5}$$

and $\widetilde{N}_m^R(T)$ and $\widetilde{N}_Z(T)$ denote the corresponding counts for $\{W_{k+1}^{(m)}\}$ and $\{Z_{k+1}\}$, respectively. By a union bound, we thus get for the upper tail of $N_Y^\star(T)$, with our choice of ξ that for any $m \ge m_\star$ and $R = \xi/\delta \ge 1$,

$$\mathbb{P}\big(N_Y^\star(T) - \alpha_0 T \ge 8\eta T\big) \le \mathbb{P}\left(N_{m,\xi}(T) - \alpha_{3\xi}^{(m)} T \ge \eta T\right) + 2\mathbb{P}\left(N_m^R(T) \ge 2\eta T\right) + 2\mathbb{P}\left(N_{Z^{(m)}}(T) \ge \eta T\right).$$

The zero mean, $[-1, 1]$-valued variables $I_k := \mathbf{1}_{\{W_k^{(m)}W_{k+1}^{(m)} < 3\xi\}} - \alpha_{3\xi}^{(m)}$ are m-dependent. Hence, setting $n_T := \lfloor T/m \rfloor$ we get by stationarity, followed by Hoeffding's inequality for the i.i.d. variables $\{I_{jm}\}_j$ that for $m \ge m_\star$

$$\mathbb{P}\left(N_{m,\xi}(T) \ge (\alpha_{3\xi}^{(m)} + \eta)T\right) \le m \max_{n \in \{n_T, n_T+1\}} \mathbb{P}\left(\sum_{j=0}^{n-1} I_{jm} \ge \eta n\right) \le m e^{-n_T \eta^2/2}. \tag{2.6}$$

Since $\mathbb{E}[(W_0^{(m)})^2] \le 1$, fixing $R < \infty$ with $\mathbb{P}(|Y_0| \ge R) \le \eta$, we have that $\widehat{\alpha}_R^{(m)} := \mathbb{P}(|W_0^{(m)}| \ge R) \le \eta$ for all $m \ge 1$. Hence, by stationarity and the m-dependence of the $[-1,1]$-valued zero mean $\widehat{I}_k := \mathbf{1}_{\{|W_k^{(m)}|>R\}} - \widehat{\alpha}_R^{(m)}$, applying once more Hoeffding's inequality, we get that

$$\mathbb{P}\left(N_m^R(T) \ge 2\eta T\right) \le \mathbb{P}\left(\sum_{k=0}^{T-1} \widehat{I}_k \ge \eta T\right) \le m \max_{n\in\{n_T, n_T+1\}} \mathbb{P}\left(\sum_{j=0}^{n-1} \widehat{I}_{jm} \ge \eta n\right) \le m e^{-n_T \eta^2/2}. \quad (2.7)$$

Finally, with $\mathbb{E}[(Z_0^{(m)})^2] \le 2\pi\varepsilon_m$, from Markov's inequality and [5, identity (7)] at $\theta_m = \varepsilon_m^{-1/2}$, we deduce that for all T large enough

$$\mathbb{P}\left(N_{Z^{(m)}}(T) \ge \eta T\right) \le e^{-\theta_m \delta \eta T} \mathbb{E}\left[e^{\theta_m \sum_{k=0}^{T-1} |Z_k^{(m)}|}\right] \le e^{-(\theta_m \delta\eta - 27)T}. \quad (2.8)$$

To complete the proof of the upper tail, combine (2.6)–(2.8) taking $m \ge m_\star$ so large that $\theta_m \delta \eta \ge 28$.

Turning to prove the lower tail, set $\xi \in (0,1]$ and $m_\star < \infty$ so $\alpha_{-3\xi}^{(m)} \ge \alpha_0 - \eta$ whenever $m \ge m_\star$ and deduce by yet another application of (2.3), that for any $R = \xi/\delta \ge 1$,

$$\{Y_k Y_{k+1} \ge 0\} \subseteq \{W_k^{(m)} W_{k+1}^{(m)} \ge -3\xi\} \cup \{|W_k^{(m)}| \ge R\} \cup \{|W_{k+1}^{(m)}| \ge R\} \cup \{|Z_k^{(m)}| \ge \delta\} \cup \{|Z_{k+1}^{(m)}| \ge \delta\}.$$

Thus, recalling from (1.7) and (2.5) that

$$N_Y^\star(T) = T - \sum_{k=0}^{T-1} \mathbf{1}_{\{Y_k Y_{k+1} \ge 0\}}, \qquad N_{m,-\xi}(T) = T - \sum_{k=0}^{T-1} \mathbf{1}_{\{W_k^{(m)} W_{k+1}^{(m)} \ge -3\xi\}},$$

we find, similarly to (2.4), that

$$T - N_Y^\star(T) \le T - N_{m,\xi}(T) + N_m^R(T) + \widetilde{N}_m^R(T) + N_{Z^{(m)}}(T) + \widetilde{N}_{Z^{(m)}}(T),$$

which in turn implies by the union bound and our choice of ξ and m that for any $\eta > 0$,

$$\mathbb{P}\left(\alpha_0 T - N_Y^\star(T) \ge 8\eta T\right) \le \mathbb{P}\left(\alpha_{-3\xi}^{(m)} T - N_{m,-\xi}(T) \ge \eta T\right) + 2\mathbb{P}\left(N_m^R(T) \ge 2\eta T\right) + 2\mathbb{P}\left(N_{Z^{(m)}}(T) \ge \eta T\right).$$

We have already established exponentially small in $T \ge T_0(\eta)$ upper bounds on the two left-most terms (in (2.7) and (2.8)), and re-running the derivation of (2.6) for

$I_k := \alpha^{(m)}_{-3\xi} - \mathbf{1}_{\{W^{(m)}_k W^{(m)}_{k+1} < -3\xi\}}$ yields such a bound on $\mathbb{P}(\alpha^{(m)}_{-3\xi} T - N_{m,-\xi}(T) \geq \eta T)$. Lastly, reducing to $c(\eta) \leq 1/T_0(\eta)$ and taking $C \geq e$, extends (1.8) to all $T \geq 0$.

2.3 Proof of Proposition 1.7

(a) Recall that $\int |r(t)|\mathrm{d}t < \infty$ for $r(\cdot)$ of (1.1) implies that ρ has a continuous, bounded density $p(\lambda)$. Assuming ρ (hence $p(\lambda)$) is supported on $[-K, K]$, fix a compactly supported even function $\psi(\cdot)$ such that $\psi(\lambda) \equiv 1$ on $[-K, K]$ and $\sum_{\ell=0}^{2} \|\psi^{(\ell)}\|_\infty \leq 1$. Setting $h_{\ell,y}(\lambda) := [\lambda\varphi(\lambda y)]^\ell \psi(\lambda)$ and $r_{\ell;\psi}(x; y)$ via (1.9) but with $\psi(\cdot)$ replacing $p(\cdot)$, we find that

$$r'_{1;\psi}(x; y) = \int \sin(\lambda x) h_{1,y}(\lambda)\mathrm{d}\lambda = \frac{1}{x^2} \int \sin(\lambda x) h''_{1,y}(\lambda)\mathrm{d}\lambda \tag{2.9}$$

(getting the RHS for $x \neq 0$ upon twice integrating by parts). One easily verifies that

$$c_\psi := 2 \max_{\ell=0,1,2} \sup_{\ell|y|<2\kappa'} \{\|h''_{\ell,y}\|_1 + \|h_{\ell,y}\|_1\} < \infty, \tag{2.10}$$

hence $|r'_{1;\psi}(x; 2y)| \leq c_\psi/(1+x^2)$ for any $|y| < \kappa'$ and all $x \in \mathbb{R}$. Since $\psi(\lambda)p(\lambda) = p(\lambda)$, it follows that for $g(x) := c_\psi \int \mathrm{d}t|r(t)|/[1+(x-t)^2]$, any $|y| < \kappa'$ and $x \in \mathbb{R}$,

$$|r'_1(x; 2y)| = \left| \int r'_{1;\psi}(x - t; 2y) r(t)\mathrm{d}t \right| \leq g(x)\,. \tag{2.11}$$

Likewise,

$$-r''_{2;\psi}(x; y) = \int \cos(\lambda x) h_{2,y}(\lambda)\mathrm{d}\lambda = \frac{1}{x^2} \int \cos(\lambda x) h''_{2,y}(\lambda)\mathrm{d}\lambda\,, \tag{2.12}$$

hence $|r''_{2;\psi}(x; y)| \leq c_\psi/(1 + x^2)$ for all $|y| < \kappa'$, yielding similarly to (2.11) that

$$|r''_2(x; y)| = \left| \int r''_{2;\psi}(x - t; y) r(t)\mathrm{d}t \right| \leq g(x)\,. \tag{2.13}$$

The same argument shows that

$$|r(x)| = \left| \int r_{0;\psi}(x - t) r(t)\mathrm{d}t \right| \leq g(x)\,. \tag{2.14}$$

Taking $x_\star = 1$ we thus find, in view of (2.11) and (2.13), that $\omega_\star(k) \le 3\sum_{j\ge k} g(j)$ and (1.10) follows from the finiteness of

$$\sum_{j=1}^{\infty} g(j) = c_\psi \int \mathrm{d}t|r(t)| \left[\sum_{j=1}^{\infty} \frac{1}{1+(j-t)^2}\right] \le 2c_\psi \left[\sum_{j\ge 0} \frac{1}{1+j^2}\right] \int |r(t)|\mathrm{d}t\,.$$

(b) If $\rho(\cdot)$ of unbounded support satisfies (1.4), then the covariance $r(\cdot)$ of (1.1) is real analytic and a.s. the sample path $t \mapsto X(t)$ is in $C^\infty(\mathbb{R})$. Suppose also that for some $\kappa_o \in (0, \kappa/2)$ the covariance $r(\cdot;\kappa_o)$ of (1.5) is integrable. The latter implies that the measure $\cosh(2\kappa_o\lambda)\mathrm{d}\rho(\lambda)$ has a continuous, bounded density $p_{\kappa_o}(\lambda)$; hence $\rho(\cdot)$ has the continuous, bounded density $p(\lambda) = \psi(\lambda)p_{\kappa_o}(\lambda)$ for the even, $(0,1]$-valued, integrable function $\psi(\lambda) := 1/\cosh(2\kappa_o\lambda)$. It is easy to verify that (2.10) remains valid for such choice of $\psi(\lambda)$, provided $\kappa' < \kappa_o$. Further, in this case $|h_{\ell,y}(\lambda)| \to 0$ and $|h'_{\ell,y}(\lambda)| \to 0$ as $|\lambda| \to \infty$, whenever $\ell|y| < 2\kappa'$, justifying the integration by parts that lead to the right-most equality in both (2.9) and (2.12). The convolution identities (2.11), (2.13), and (2.14) apply upon replacing $r(t)$ by $r(t;\kappa_o)$, as do the corresponding bounds, albeit with $g(x) := c_\psi \int \mathrm{d}t|r(t;\kappa_o)|/[1+(x-t)^2]$, so the integrability of $|r(\cdot;\kappa_o)|$ indeed suffices for (1.10).

3 Analytic Extension, Jensen's Formula and Decorrelation

3.1 An analytic extension and its properties

Under (1.4) the covariance kernel $r : \mathbb{R} \to \mathbb{R}$ of the process $X(t)$ analytically extends to the strip $\mathbb{S}_\kappa = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \kappa\}$, by plugging $t = z$ in (1.1). Utilizing this fact, we next construct a complex analytic, mean zero, Gaussian function $f : \mathbb{S} = \mathbb{S}_{\kappa/2} \to \mathbb{C}$ that is at the center of our proof of Theorem 1.6.

Proposition 3.1. For a real, stationary mean-zero, Gaussian process X that satisfies (1.4) there exist a complex analytic, zero mean, Gaussian $f : \mathbb{S} := \mathbb{S}_{\kappa/2} \to \mathbb{C}$ such that

(a) The function $f(\cdot)$ is conjugation equivariant, namely $f(\bar{z}) = \overline{f(z)}$.

(b) The covariances of $f(\cdot)$ are given by the formulas

$$K(z,w) := \mathbb{E}[f(z)\overline{f(w)}] = r(z-\bar{w}); \quad \mathbb{E}[f(z)f(w)] = r(z-w) \quad \forall z,w \in \mathbb{S}\,. \tag{3.1}$$

(c) The law of $z \mapsto f(z)$ is stationary under real translations and $\{f(t+i0)\}_{t\in\mathbb{R}} \overset{d}{=} \{X(t)\}_{t\in\mathbb{R}}$.

Proof. As ρ is an even real-valued measure, there exists an orthonormal basis (ONB) for $\mathcal{L}^2_\rho(\mathbb{R})$ composed of Hermitian functions $\{\varphi_n\}$ (i.e., with $\varphi_n(-\lambda) = \overline{\varphi_n(\lambda)}$). For such a basis and $e_z(\lambda) := e^{i\lambda\bar{z}}$, $z \in \mathbb{S}_{\kappa/2}$ let

$$\psi_n(z) := \langle \varphi_n, e_z \rangle_{\mathcal{L}^2_\rho(\mathbb{R})} = \int_{\mathbb{R}} \varphi_n(\lambda)\overline{e_z(\lambda)}\mathrm{d}\rho(\lambda) = \int_{\mathbb{R}} e^{-i\lambda z}\varphi_n(\lambda)\mathrm{d}\rho(\lambda)\,, \tag{3.2}$$

and for i.i.d. coefficients $\zeta_n \sim \mathcal{N}_{\mathbb{R}}(0,1)$ consider the random series

$$f(z) := \sum_n \zeta_n \psi_n(z)\,.$$

Having (1.4) hold for κ, standard arguments (see [12, Chapter 3, Theorem 2] or [10, Lemma 2.2.3]) yield that the series defining $f(\cdot)$ converges almost surely to a zero mean, complex analytic Gaussian function on $\mathbb{S} = \mathbb{S}_{\kappa/2}$, having there the covariance

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}] = \sum_n \psi_n(z)\overline{\psi_n(w)}\,.$$

Since $\{\varphi_n\}$ are Hermitian and ρ is even and real-valued, it follows that $\overline{\psi_n(z)} = \psi_n(\bar{z})$ and we get part (a) upon taking the conjugate in the defining series for $f(z)$. Further, since $\{\varphi_n\}$ is an ONB in $\mathcal{L}^2_\rho(\mathbb{R})$

$$K(z,w) = \sum_n \langle \varphi_n, e_z \rangle_{\mathcal{L}^2_\rho(\mathbb{R})} \langle e_w, \varphi_n \rangle_{\mathcal{L}^2_\rho(\mathbb{R})} = \langle e_w, e_z \rangle_{\mathcal{L}^2_\rho(\mathbb{R})} = r(z - \bar{w})\,,$$

as stated in (3.1) (and the RHS of (3.1) then follows from part (a)). The formulas in (3.1) are invariant to real shifts $(z,w) \mapsto (z+t, w+t)$, $t \in \mathbb{R}$; hence, the Gaussian function $f(\cdot)$ is stationary with respect to such real shifts. To complete the proof of part (c), note that by part (a) the function $f(z)$ is real-valued when $z \in \mathbb{R}$ and the covariance kernel of (3.1) coincides for $z, w \in \mathbb{R}$ with the original covariance $r : \mathbb{R} \to \mathbb{R}$ of the given real Gaussian process X. ∎

Remark 3.2. Recall that $\mathrm{Re}f(z) = [f(z) + \overline{f(z)}]/2$, $\mathrm{Im}f(z) = [f(z) - \overline{f(z)}]/(2i)$ with (3.1) determining the covariance between the real and imaginary parts of $f(z)$ and $f(w)$, $z, w \in \mathbb{S}$. By Proposition 3.1(c), when $\mathrm{Im}(z) = \mathrm{Im}(w)$ the latter depend only on $w - z$ so WLOG we may set $\mathrm{Re}(z) = 0$. Specifically, for $|y| < \kappa/2$ and $x \in \mathbb{R}$ we have

$$\mathbb{E}[\mathrm{Re}(f(iy))\mathrm{Re}(f(x+iy))] = \frac{1}{2}[\mathrm{Re}(r(x+2iy)) + r(x)] = \int_{\mathbb{R}} \cos(\lambda x)\cosh^2(\lambda y)\mathrm{d}\rho(\lambda) \tag{3.3}$$

$$\mathbb{E}[\mathrm{Im}(f(iy))\mathrm{Im}(f(x+iy))] = \frac{1}{2}[\mathrm{Re}(r(x+2iy)) - r(x)] = y^2 \int_{\mathbb{R}} \cos(\lambda x)\lambda^2\varphi^2(\lambda y)\mathrm{d}\rho(\lambda)\,, \quad (3.4)$$

$$\mathbb{E}[\mathrm{Re}(f(iy))\mathrm{Im}(f(x+iy))] = \frac{1}{2}\mathrm{Im}(r(x+2iy)) \qquad = -y \int_{\mathbb{R}} \sin(\lambda x)\lambda\varphi(2\lambda y)\mathrm{d}\rho(\lambda)\,, \quad (3.5)$$

where $\varphi(\lambda) := \sinh(\lambda)/\lambda$ and the RHS of (3.3)–(3.5) follows from (1.1) and the even symmetry of the spectral measure $\rho(\cdot)$.

We next utilize (3.3)–(3.5) to deduce from Assumption A the absolute summability of the corresponding correlations, uniformly in $\mathbb{S}_{\kappa'}$.

Lemma 3.3. For $f(\cdot)$ of Proposition 3.1, consider the vector $\widehat{\mathbf{r}}(z) \in [-1,1]^4$ of correlations between $[\mathrm{Re}(f(z)), \mathrm{Im}(f(z))]$ and $[\mathrm{Re}(f(iy)), \mathrm{Im}(f(iy))]$ when $y = \mathrm{Im}(z)$. Then, for $x_\star$ and $0 < \kappa' < \kappa/2$ of Assumption A,

$$\omega(k) := \sup_{|y|<\kappa'} \left\{ \sum_{|j|\geq k} \|\widehat{\mathbf{r}}(jx_\star + iy)\|_1 \right\} \to 0 \quad \text{for } k \to \infty \quad (3.6)$$

and in particular

$$\sup_{|y|<\kappa'} \left\{ \sum_{j\in\mathbb{Z}} |r(jx_\star + 2iy)| \right\} < \infty\,. \quad (3.7)$$

Proof. In view of (3.3) and (3.4), for any $|y| < \kappa/2$ and $x \in \mathbb{R}$,

$$v_{\mathrm{I}}(y) := y^{-2}\mathrm{Var}\ (\mathrm{Im}(f(x+iy)) = \int_{\mathbb{R}} \lambda^2\varphi^2(\lambda y)\mathrm{d}\rho(\lambda)\,, \quad (3.8)$$

$$v_{\mathrm{R}}(y) := \mathrm{Var}\ (\mathrm{Re}(f(x+iy)) = \int_{\mathbb{R}} \cosh^2(\lambda y)\mathrm{d}\rho(\lambda)\,, \quad (3.9)$$

are uniform in y, bounded away from zero and thanks to (1.4),

$$\mathrm{Var}\ (\mathrm{Im}(f(x+iy)) \leq \mathrm{Var}\ (\mathrm{Re}(f(x+iy)) \leq r(2iy) \quad (3.10)$$

is uniformly bounded over $|y| \leq \kappa' < \kappa/2$. From Proposition 3.1(b) at $z = x+iy$, $w = iy$, the definition of $\widehat{\mathbf{r}}(x+iy)$ and (3.10), we deduce that

$$|r(x+2iy)| = |\mathbb{E}[\overline{f(iy)}f(x+iy)]| \leq r(2iy)\ \|\widehat{\mathbf{r}}(x+iy)\|_1\,.$$

Thus, if (3.6) holds, then necessarily $\omega(0)$ is finite and (3.7) must hold as well. Turning to show (3.6), let $\widehat{\mathbf{r}}_{\mathsf{RR}}(z)$, $\widehat{\mathbf{r}}_{\mathsf{RI}}(z)$, $\widehat{\mathbf{r}}_{\mathsf{IR}}(z)$, and $\widehat{\mathbf{r}}_{\mathsf{II}}(z)$ denote the coordinates of $\widehat{\mathbf{r}}(z)$ (e.g., $\widehat{\mathbf{r}}_{\mathsf{IR}}(x+iy)$ stands for the correlation between $\mathrm{Im}(f(x+iy))$ and $\mathrm{Re}(f(iy))$, so $\sqrt{v_{\mathsf{R}}(y)y^2 v_{\mathsf{I}}(y)}\,\widehat{\mathbf{r}}_{\mathsf{IR}}(x+iy)$ is precisely the LHS of (3.5)). Then, from the LHS of (3.3) and (3.4), we see that

$$(r(2iy)+r(0))\,\widehat{\mathbf{r}}_{\mathsf{RR}}(x+iy) = \mathrm{Re}(r(x+2iy)) + r(x)\,,$$
$$(r(2iy)-r(0))\quad \widehat{\mathbf{r}}_{\mathsf{II}}(x+iy) = \mathrm{Re}(r(x+2iy)) - r(x)\,,$$

which since $r(0)=1$ yields for $\beta_y := 2r(0)/(r(2iy)+r(0)) \in (0,1)$, the identity

$$\widehat{\mathbf{r}}_{\mathsf{RR}}(x+iy) = (1-\beta_y)\,\widehat{\mathbf{r}}_{\mathsf{II}}(x+iy) + \beta_y r(x)\,. \tag{3.11}$$

Similarly, from the left-hand side of (3.5) we have that

$$\widehat{\mathbf{r}}_{\mathsf{RI}}(x+iy) = -\widehat{\mathbf{r}}_{\mathsf{IR}}(x+iy)\,, \tag{3.12}$$

whereas from the right-hand side of (3.4) and (3.5) we get in terms of the x-derivatives of $r_\ell(\cdot)$ of (1.9), that

$$v_{\mathsf{I}}(y)\,\widehat{\mathbf{r}}_{\mathsf{II}}(x+iy) = -r_2''(x;y)\,, \tag{3.13}$$

$$\sqrt{v_{\mathsf{R}}(y)v_{\mathsf{I}}(y)}\,\widehat{\mathbf{r}}_{\mathsf{IR}}(x+iy) = r_1'(x;2y)\,\mathrm{sgn}(y)\,. \tag{3.14}$$

Recalling that $\inf_y\{v_{\mathsf{I}}(y)\wedge v_{\mathsf{R}}(y)\} \ge c^{-1}$ we deduce from (3.11)–(3.14) that

$$\|\widehat{\mathbf{r}}(jx_\star + iy)\|_1 \le |r(jx_\star)| + 2c|r_2''(jx_\star;y)| + 2c|r_1'(jx_\star;2y)|\,,$$

hence $\omega(k) \le 8(c\vee 1)\omega_\star(k)$ and (3.6) follows from our assumption (1.10). ■

3.2 Relating real and complex zeroes

Thanks to the 2nd part of Proposition 3.1(c), for any $\mathbb{D}\subseteq\mathbb{S}$ containing $[0,T]$ we have

$$N_X([0,T]) \le N_f(\mathbb{D}) = |\{z\in\mathbb{D} : f(z)=0\}|\,. \tag{3.15}$$

For κ' of Assumption A and $\delta \in (0, \kappa'/2)$, let $\mathbb{B}_j(r)$ denote the ball of radius r centered at $x_j := (2j-1)\delta$. We shall use the bound (3.15) with the disjoint union of $n := \lceil T/(2\delta) \rceil$ balls

$$\mathbb{D} = \mathbb{D}_{n,\delta} := \bigcup_{j=1}^{n} \mathbb{B}_j(\delta)\,,$$

further estimating the value of $N_f(\mathbb{B}_j(\delta))$ by Jensen's enumeration formula for the zeroes of a complex analytic function (see [1, Section 5.3.1]). Specifically, for $\beta \in [0, \log 2]$ define the integral

$$\Gamma_j(\beta) := \int_{-1/2}^{1/2} \log |f(x_j + \delta e^{\beta} e^{i2\pi\theta})| \mathrm{d}\theta\,. \tag{3.16}$$

With such choices $\delta e^{\beta} < \kappa' < \kappa/2$, so $\mathbb{B}_j(\delta e^{\beta}) \subset \mathbb{S}$ and Jensen's formula tells us that for each j

$$\int_{\delta}^{\delta e^{\beta}} N_f(\mathbb{B}_j(r)) \frac{\mathrm{d}r}{r} = \Gamma_j(\beta) - \Gamma_j(0). \tag{3.17}$$

Since $r \mapsto N_f(\mathbb{B}_j(r))$ is nondecreasing, from (3.17) we deduce that

$$N_f(\mathbb{B}_j(\delta)) \le \frac{1}{\beta}\Big[\Gamma_j(\beta) - \Gamma_j(0)\Big] \le N_f(\mathbb{B}_j(\delta e^{\beta})). \tag{3.18}$$

Sum (3.18) over j to get

$$N_f(\mathbb{D}_{n,\delta}) \le \frac{1}{\beta}\sum_{j=1}^{n} \widehat{\Gamma}_j(\beta) - \frac{1}{\beta}\sum_{j=1}^{n} \widehat{\Gamma}_j(0) + \sum_{j=1}^{n} \mathbb{E}[N_f(\mathbb{B}_j(\delta e^{\beta}))]\,, \tag{3.19}$$

where $\widehat{\Gamma}_j(\cdot) := \Gamma_j(\cdot) - \mathbb{E}\Gamma_j(\cdot)$. The next lemma shows that for small positive δ and β the right-most (nonrandom) term in (3.19) is at most $(\alpha + \eta)T$.

Lemma 3.4. Suppose that (1.4) holds and the spectral measure $\rho(\cdot)$ is nonatomic. There exist $\delta_\star(\eta)$ and $\beta_\star(\eta)$ positive, such that for any $\delta \le \delta_\star(\eta)$, $\beta \le \beta_\star(\eta)$ and all $T \ge 1$,

$$\frac{1}{T}\sum_{j=1}^{n} \mathbb{E}[N_f(\mathbb{B}_j(\delta e^{\beta}))] \le \alpha + \eta\,. \tag{3.20}$$

Proof. Since the Gaussian function $f(z)$ has nonatomic spectral measure,

$$\mathbb{E}[N_f([0,1] \times [-r, r])] = \alpha + \mu_f([-r, r])\,, \qquad \forall r \ge 0\,,$$

where $\mu_f(\cdot)$ is some absolutely continuous, non-negative measure on $\mathbb{R}$ (this can be deduced from [8, Theorem 1] by noticing that the form of the limit of $\frac{1}{T}N_f([0,T]\times[-r,r])$ as $T\to\infty$, given in part (iii) of that theorem, must equal the expectation $\mathbb{E}[N_f([0,1]\times[-r,r])]$ when it is deterministic). Further, $z\mapsto f(z)$ is stationary under real translations; hence for any $x_j\in\mathbb{R}$ and $r\in[\delta e^{\beta},\frac{1}{2}]\cap\mathbb{Q}$

$$\mathbb{E}[N_f(\mathbb{B}_j(\delta e^{\beta}))]\le\mathbb{E}[N_f([-r,r]^2)]=2r(\alpha+\mu_f([-r,r]))\,.$$

With $n\le\frac{T}{2\delta}+1$, the LHS of (3.20) is thus for $T\ge1$ and $\delta<\frac{1}{4}$, at most

$$h(\delta,\beta):=(1+2\delta)e^{\beta}\left(\alpha+\mu_f([-\delta e^{\beta},\delta e^{\beta}])\right)$$

and we are done, since $h(\cdot,\cdot)$ is continuous with $h(0,0)=\alpha$. ■

3.3 Reducing Theorem 1.6 to the decorrelation of moments

Fixing $\eta>0$, in view of (3.15) and (3.19) it suffices for (1.11) to show that for $\beta=\beta^\star$ and $\delta^\star$ as in Lemma 3.4, there exist $\delta\in(0,\delta^\star]$ and $c=c(\eta,\beta,\delta)>0$ so that for all n large enough

$$\mathbb{P}\left(\sum_{j=1}^{n}\widehat{\Gamma}_j(\beta)\ge\eta\beta\delta n\right)+\mathbb{P}\left(\sum_{j=1}^{n}\widehat{\Gamma}_j(0)\le-\eta\beta\delta n\right)\le e^{-cn}\,. \tag{3.21}$$

To this end, let $x_\star$ be as in Assumption A and consider $\delta\in(0,\delta^\star]$ such that $x_\star/(2\delta):=\ell_\star\in\mathbb{N}$. Then, to utilize the decay of correlations in Lemma 3.3, fixing $\ell=k\ell_\star$ for some $k\in\mathbb{N}$, let $m=\lceil n/\ell\rceil\ge2$ and consider the disjoint sets $\mathcal{S}_\tau:=\{\ell-\tau,2\ell-\tau,\dots,m\ell-\tau\}$, for $\tau=0,\dots,\ell-1$, whose union $\{1,\dots,m\ell\}$ covers $\{1,\dots,n\}$. By stationarity of $f(\cdot)$ under real translations, the law of $\sum_{j\in\mathcal{S}_\tau}\widehat{\Gamma}_j(\cdot)$ is independent of τ. Setting $\xi:=\eta\beta/10$ (so $5\xi m\le\eta\beta(m-1)\le\eta\beta n/\ell$), by a union bound on the ℓ choices of τ, it suffices for (3.21) to show that some $c=c(\xi,\delta,\ell)>0$ and all m large enough

$$\mathbb{P}\left(\sum_{j\in\mathcal{S}_0}\widehat{\Gamma}_j(\beta)\ge5\xi\delta m\right)+\mathbb{P}\left(\sum_{j\in\mathcal{S}_0}\widehat{\Gamma}_j(0)\le-5\xi\delta m\right)\le2e^{-2c\ell m}\,. \tag{3.22}$$

A standard application of the exponential Markov inequality reduces this task for $c=\varepsilon\xi\delta/(2\ell)$, into showing that for some $\varepsilon=\varepsilon(\xi,\delta,\ell)>0$ and all large enough m,

$$\mathbb{E}\left[\exp(\varepsilon\sum_{j=1}^{m}\widehat{\Gamma}_{j\ell}(\beta))\right]\le e^{4\varepsilon\xi\delta m}\qquad\&\qquad\mathbb{E}\left[\exp(-\varepsilon\sum_{j=1}^{m}\widehat{\Gamma}_{j\ell}(0))\right]\le e^{4\varepsilon\xi\delta m}\,. \tag{3.23}$$

Upon setting $z_\beta(\theta) := \delta e^\beta e^{i\pi\theta} - \delta$, we get from (3.16) that

$$\sum_{j=1}^{m} \widehat{\Gamma}_{j\ell}(\beta) = \frac{1}{2}\int_{-1}^{1} S_m(z_\beta(\theta))\mathrm{d}\theta$$

where, $2\delta\ell = kx_\star$ thanks to our choice of ℓ, so

$$S_m(z) := \sum_{j=1}^{m} \left\{\log|f(jkx_\star + z)| - \mathbb{E}[\log|f(z)|]\right\} . \tag{3.24}$$

Thus, applying Jensen's inequality for the convex functions $\exp(\pm\varepsilon\,\cdot)$ and the uniform law of θ further reduces the task of proving (3.23) into showing that

$$\sup_{|\theta|\le 1} \mathbb{E}\left[e^{\varepsilon S_m(z_\beta(\theta))}\right] \le e^{4\varepsilon\xi\delta m} \qquad \& \qquad \sup_{|\theta|\le 1} \mathbb{E}\left[e^{-\varepsilon S_m(z_0(\theta))}\right] \le e^{4\varepsilon\xi\delta m} . \tag{3.25}$$

In view of the stationarity of $f(\cdot)$ under real translations, the law of $S_m(z)$ of (3.24) depends only on $\mathrm{Im}(z)$; hence in (3.25) we can WLOG re-set $z_\beta(\theta) = iy$ for $y = \sin(\pi\theta)\delta e^\beta$. Doing so, we consider for $|y| \le 2\delta$, the mean zero, Gaussian variables

$$G_j(y) := f(jx_\star + iy) , \tag{3.26}$$

and first relate $\mathbb{E}[\log|G_0(y)|]$, which is part of $S_m(iy)$ to $\mathbb{E}[|G_0(y)|^{\pm\varepsilon}]$.

Lemma 3.5. Given $\zeta > 0$, for any $\varepsilon \le \varepsilon_0(\zeta)$ positive and all $|y| \le \kappa'$,

$$\mathbb{E}[|G_0(y)|^\varepsilon] \le (1 + \varepsilon\zeta)\exp\left(\varepsilon\mathbb{E}\left[\log|G_0(y)|\right]\right) , \tag{3.27}$$

$$\mathbb{E}[|G_0(y)|^{-\varepsilon}] \le (1 + \varepsilon\zeta)\exp\left(-\varepsilon\mathbb{E}\left[\log|G_0(y)|\right]\right) . \tag{3.28}$$

Proof. Consider the non-negative function $g_\varepsilon(x) := |\varepsilon|^{-1}(e^{\varepsilon x} - \varepsilon x - 1)$. Setting $L(y) := \log|G_0(y)| - \mathbb{E}[\log|G_0(y)|]$, note that $\mathbb{E}L(y) = 0$ and hence elementary algebra transforms (3.27) and -(3.28) to the inequalities $\mathbb{E}[g_{\pm\varepsilon}(L(y))] \le \zeta$. We thus establish the lemma upon showing that

$$\lim_{\varepsilon\downarrow 0} \sup_{|y|\le\kappa'} \left\{\mathbb{E}[g_{\pm\varepsilon}(L(y))]\right\} = 0 . \tag{3.29}$$

To this end, since $|g_{\pm\varepsilon}(x)| \le \eta^{-1}e^{\eta|x|} := \widetilde{g}_\eta(x)$ whenever $|\varepsilon| \le \eta$ and $g_{\pm\varepsilon}(\cdot) \to 0$ uniformly on compact subsets of $\mathbb{R}$, the uniform in y convergence (3.29), is a consequence of having for some $\eta > 0$,

$$\sup_{|y|\le\kappa'} \left\{ \mathbb{E}[\widetilde{g}_\eta(L(y))\mathbf{1}_{\{|L(y)|\ge b\}}] \right\} \to 0 \quad \text{for} \quad b\to\infty\,. \tag{3.30}$$

Further, $\widetilde{g}_\eta(\cdot)$ diverges at infinity, so by Markov's inequality (3.30) follows from having $\sup_{|y|\le\kappa'}\left\{\mathbb{E}[\widetilde{g}^2_\eta(L(y))]\right\}$ finite, for which it suffices to verify that for some $\eta > 0$, $\sup_{|y|\le\kappa'}\{\mathbb{E}[|G_0(y)|^{\pm 2\eta}]\}$ is finite. For the latter, recall from (3.1) that $\mathbb{E}[|G_0(y)|^2] = r(2iy)$, which for $|y| \le \kappa'$ is uniformly bounded above (as $\kappa' < \kappa/2$). On the other hand, $\mathbb{E}[|G_0(y)|^{-1/2}] \le Cr(0)^{-1/4}$ for some universal $C < \infty$, since $|G_0(y)|^{-1/2} \le |X|^{-1/2}$ for the zero mean $\mathbb{R}$-valued Gaussian $X = \mathrm{Re}(G_0(y))$ of $\mathrm{Var}\,(X) = v_{\mathrm{R}}(y) \ge r(0)$ (see (3.9)). ■

The next proposition, which is our main technical statement, bounds small positive and negative fractional moments of the product of our Gaussian variables from (3.26), after a suitable dilution.

Proposition 3.6. For any $\zeta > 0$ there is $\varepsilon_\star(\zeta) > 0$ such that for $\varepsilon \le \varepsilon_\star$, $k \ge k_\star(\zeta,\varepsilon) \in \mathbb{N}$, any $m \ge 1$ and all $|y| < \kappa'$,

$$M_m(\varepsilon) := \mathbb{E}\left[\prod_{j=1}^m |G_{jk}(y)|^\varepsilon\right] \le e^{2\varepsilon\zeta m}\mathbb{E}[|G_0(y)|^{2\varepsilon}]^{m/2}\,, \tag{3.31}$$

$$M_m(-\varepsilon) := \mathbb{E}\left[\prod_{j=1}^m |G_{jk}(y)|^{-\varepsilon}\right] \le e^{2\varepsilon\zeta m}\mathbb{E}[|G_0(y)|^{-2\varepsilon}]^{m/2}\,. \tag{3.32}$$

We comment that the reverse inequality is conjectured to hold without the exponential correction, see [27] for partial results. Estimates in case of integer moments may be found in [25].

We proceed to obtain (3.25) from Proposition 3.6. In view of (3.24) and (3.26), the LHS of (3.25) amounts (after setting $z_\beta = iy$), to

$$\mathbb{E}\left[\prod_{j=1}^m |G_{jk}(y)|^\varepsilon\right] \le e^{4|\varepsilon|\xi\delta m}\exp\left(\varepsilon m\mathbb{E}\log|G_0(y)|\right)\,, \qquad \forall |y| \le 2\delta\,. \tag{3.33}$$

Proceeding to show (3.33), we set $\zeta := \xi\delta > 0$ and a positive $\varepsilon \le \varepsilon_\star(\zeta)\wedge\varepsilon_0(\zeta)/2$, so Lemma 3.5 applies at 2ε, then fix $k \ge k_\star(\zeta,\varepsilon)$ large enough as needed for Proposition 3.6. Combining now the bound (3.31) with (3.27) at 2ε and the elementary inequality

$(1 + 2\varepsilon\zeta) \leq e^{2\varepsilon\zeta}$ yields the bound (3.33). Similarly, the RHS of (3.25) amounts to the inequality (3.33) at $-\varepsilon < 0$, so having the control of (3.28) on the $-\varepsilon$-moment of $G_0(y)$ in terms of $\mathbb{E} \log |G_0(y)|$, we deduce that the RHS of (3.25) follows from the bound (3.32).

In conclusion, we have by now reduced the proof of Theorem 1.6 to the de-correlated moment computations of Proposition 3.6, to which we devote Sections 4 and 5.

4 Diagonally Dominant Gaussian Laws

We establish here a few preparatory results about weakly correlated, centered, $\mathbb{C}$-valued Gaussian vectors. Our results are phrased in terms of

$$f(jx_\star + iy) := G_j(y) := \sqrt{v_\mathsf{R}(y)} X_j(y) + i|y|\sqrt{v_\mathsf{I}(y)} Y_j(y)\,, \tag{4.1}$$

for $v_\mathsf{I}(y)$ and $v_\mathsf{R}(y)$ of (3.8) and(3.9), standard, $\mathbb{R}$-valued Gaussian $X_j(y)$, $Y_j(y)$ that are independent of each other (see (3.5)), and all absolute constants are independent of $y \in (-\kappa', \kappa')$. Such results apply whenever $\mathbb{E}[|G_j|^2]$ are uniformly bounded above and below, provided the covariance matrix of the Gaussian $\{G_j\}$ is diagonally dominant, in the sense that the correlations between $\{X_j, Y_j\}$ and $\{X_{j+k}, Y_{j+k}\}$ are absolutely summable (in k), with a uniform (in j), tail decay, as in (3.6).

Our first result (needed for proving (3.31)), is a uniform a priori control on the 2nd moment of the product of such Gaussian variables (assuming only that they have summable covariances, as in (3.7)).

Lemma 4.1. There exists a finite $C_\star \geq 1$ such that for all $|y| < \kappa'$ and any finite $J \subset \mathbb{N}$,

$$\mathbb{E}\left[\prod_{j \in J} |G_j(y)|^2\right] \leq C_\star^{2|J|}\,. \tag{4.2}$$

Proof. For centered Gaussian $(Z_1, \dots, Z_n) \in \mathbb{C}^n$, with $r_0(\ell, \ell') = \mathbb{E}[Z_\ell Z_{\ell'}]$ and $r_1(\ell, \ell') = \mathbb{E}[Z_\ell \overline{Z}_{\ell'}]$ one has that

$$M_n := \mathbb{E}[\prod_{\ell=1}^n |Z_\ell|^2\,] \leq \prod_{\ell=1}^n R_\ell\,, \tag{4.3}$$

$$R_\ell := 2\sum_{\ell'=1}^n |r_1(\ell, \ell')| \vee |r_0(\ell, \ell')|\,. \tag{4.4}$$

Indeed, setting $r_3 = \bar{r}_0$ and $r_2 = \bar{r}_1$ we have by Wick's formula (see [10, Lemma 2.1.7]), that

$$M_n = \sum_{\mathcal{P}} \prod_{j=1}^{n} r_{s_j}(a_j, b_j), \tag{4.5}$$

where we sum over pair partitions $\mathcal{P} = \{(a_j, b_j), j = 1, \ldots, n\}$ of $\mathcal{S}' := \{1, 1, 2, 2, \ldots, n, n\}$ (with the convention that a_j appears before b_j in the list $\mathcal{S}'$), and each $s_j \in \{0, 1, 2, 3\}$ is set according to whether a_j or b_j are taken from an odd or an even location in $\mathcal{S}'$. We bound M_n above by moving in the RHS of (4.5) to $\max_s |r_s(a_j, b_j)|$, so hereafter all terms there are non-negative and the choice of s_j made irrelevant. Next, let $\mathcal{S} := \{1, 2, \ldots, n\}$, observing that each pair partition $\mathcal{P}$ of $\mathcal{S}'$ induces a function $g : \mathcal{S} \to \mathcal{S}$ (mapping elements in odd locations in $\mathcal{S}'$ to their pairs from $\mathcal{S}$ according to $\mathcal{P}$), where each $g : \mathcal{S} \to \mathcal{S}$ corresponds to at most 2^n such pair partitions. Consequently,

$$M_n \le \sum_{g:\mathcal{S}\to\mathcal{S}} 2^n \prod_{\ell=1}^{n} |r_0(\ell, g(\ell))| \vee |r_1(\ell, g(\ell))| = \prod_{\ell=1}^{n} R_\ell\,.$$

Now apply (4.3) for the centered Gaussian $\{G_j(y), j \in J\}$ and bound R_ℓ of (4.4) by summing over all $\ell' \in \mathbb{Z}$. Setting $C_y := \sum_{j\in\mathbb{Z}} |r(jx_\star + 2iy)|$, we thus get form Proposition 3.1(b) that for any J and $|y| < \kappa'$,

$$\sup_\ell\{R_\ell\} \le 2(C_y + C_0) \le 4 \sup_{|y|<\kappa'} \{C_y\} := C_\star^2$$

that is finite by Lemma 3.3 (see (3.7)). ■

Let $\mathcal{J}_k$ denote the collection of all finite sets $\{j_1, j_2, \ldots, j_n\} \subset \mathbb{N}$, where $j_i \ge j_{i-1} + k$ for $j_0 := 0$ and any $i \in [1, n]$. Note that for the sequence $\omega(k) \to 0$ of (3.6), and $J \in \mathcal{J}_k$, the centered $\mathbb{R}$-valued Gaussian vector $\mathbf{Z} = (X_0(y), Y_0(y), \{X_j(y), Y_j(y)\}_{j\in J})$ has covariance matrix $\mathbf{\Sigma} := \mathbf{I} - \mathbf{S}$ such that for all $|y| < \kappa'$,

$$\|\mathbf{S}\|_{\infty\to\infty} := \sup_{\mathbf{x}\neq\mathbf{0}} \left\{ \frac{\|\mathbf{S}\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \right\} = \max_j \left\{ \sum_{j'} |\mathbf{S}_{jj'}| \right\} \le \omega(k)\,. \tag{4.6}$$

We next detail three elementary properties of Gaussian vectors having such a diagonally dominant covariance matrix.

Lemma 4.2. Suppose $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ is centered, n-dimensional $\mathbb{R}$-valued Gaussian vector and $\mathrm{Cov}\,(\mathbf{Z}) := \mathbf{I} - \mathbf{S}$ with $\|\mathbf{S}\|_{\infty\to\infty} \le \omega < 1$. Then, setting $\widehat{\omega}_i := \omega^i/(1-\omega)$, $i = 0, 1, 2$, we have that

(a) All entries of the PSD matrix $\mathrm{Cov}\,(\mathbf{Z}_1) - \mathrm{Cov}\,(\mathbf{Z}_1|\mathbf{Z}_2)$ are within $[-\widehat{\omega}_2, \widehat{\omega}_2]$.

(b) The inequality $\|\mathbb{E}[\mathbf{Z}_1\,|\mathbf{Z}_2]\|_\infty \le \widehat{\omega}_1 \|\mathbf{Z}_2\|_\infty$ holds.

(c) The density $f_{\mathbf{Z}}(\cdot)$ of $\mathbf{Z}$ with respect to i.i.d. standard normal variables is such that

$$f_{\mathbf{Z}}(\mathbf{z}) \le (\widehat{\omega}_0)^{n/2} \exp(\widehat{\omega}_1 \|\mathbf{z}\|_2^2/2)\,. \tag{4.7}$$

Proof.

(a) Our assumption that $\|\mathbf{S}\| \le \omega < 1$ implies that $\boldsymbol{\Sigma}^{-1} = \sum_{n\ge 0} \mathbf{S}^n$ satisfies

$$\|\mathbf{I} - \boldsymbol{\Sigma}^{-1}\|_{\infty\to\infty} \le \sum_{n=1}^{\infty} \omega^n = \widehat{\omega}_1\,, \quad \|\boldsymbol{\Sigma}^{-1}\|_{\infty\to\infty} \le \sum_{n=0}^{\infty} \omega^n = \widehat{\omega}_0\,. \tag{4.8}$$

With $\boldsymbol{\Sigma}_{11} := \mathrm{Cov}\,(\mathbf{Z}_1)$, $\boldsymbol{\Sigma}_{22} := \mathrm{Cov}\,(\mathbf{Z}_2)$, $\boldsymbol{\Sigma}_{12} = (\boldsymbol{\Sigma}_{21})^\star = \mathrm{Cov}\,(\mathbf{Z}_1, \mathbf{Z}_2)$, and $\boldsymbol{\Sigma}_{1|2} := \mathrm{Cov}\,(\mathbf{Z}_1|\mathbf{Z}_2)$, recall that (see [10, Exercise 2.1.3]),

$$\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{1|2} = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\,. \tag{4.9}$$

The L_1 norm of each column of $\boldsymbol{\Sigma}_{21}$ is by assumption at most ω. Further, the RHS of (4.8) applies to $\boldsymbol{\Sigma}_{22}^{-1}$, which by (4.9) implies that all entries of $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{1|2}$ are indeed within $[-\widehat{\omega}_2, \widehat{\omega}_2]$.

(b) Since $\boldsymbol{\mu} := \mathbb{E}[\mathbf{Z}_1\,|\mathbf{Z}_2] = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{Z}_2$ and the RHS of (4.8) applies for $\boldsymbol{\Sigma}_{22}$, we deduce as in part (a) that necessarily $\|\boldsymbol{\mu}\|_\infty \le \omega\widehat{\omega}_0\|\mathbf{Z}_2\|_\infty$.

(c) The matrix norm of (4.6) dominates the spectral norm. In particular, from the LHS of (4.8) we deduce that

$$\langle \mathbf{z}, (\mathbf{I} - \boldsymbol{\Sigma}^{-1})\mathbf{z}\rangle \le \widehat{\omega}_1 \|\mathbf{z}\|_2^2\,.$$

Further, the RHS of (4.8) implies that all eigenvalues of $\boldsymbol{\Sigma}^{-1}$ are within $[0, \widehat{\omega}_0]$, hence the density

$$f_{\mathbf{Z}}(\mathbf{Z}) = |\boldsymbol{\Sigma}^{-1}|^{1/2} \exp\left(\frac{1}{2}\langle \mathbf{z}, (\mathbf{I} - \boldsymbol{\Sigma}^{-1})\mathbf{z}\rangle\right)\,,$$

satisfies the bound (4.7), as claimed. ∎

Relying on diagonal dominance to lower bound the conditional variances, as in Lemma 4.2(a), we get the following negative moment bound (which will later be useful when proving (3.32)).

Lemma 4.3. For some finite $k_o, C_o \geq 1$ and $\varepsilon_o > 0$, all $\varepsilon \leq \varepsilon_o$, $|y| < \kappa'$ and any $J \in \mathcal{J}_{k_o}$

$$\mathbb{E}\Big[|G_0(y)|^{-4\varepsilon} \;\mid\; \{G_j(y) : j \in J\}\Big] \leq C_o\,. \tag{4.10}$$

Proof. Since $|z| \geq |\mathrm{Re}(z)|$ it suffices to show that (4.10) holds when $\mathrm{Re}(G_0(y))$ replaces $|G_0(y)|$. Further, we only need to do so for say $\varepsilon_o = 1/8$, as it thereafter extends by Jensen's inequality (and the convexity of $g(x) = x^p$ on $\mathbb{R}_+$ when $p = \varepsilon_o/\varepsilon \geq 1$), to all $\varepsilon \leq \varepsilon_o$. To this end, recall that the conditional law of $\mathrm{Re}(G_0(y))$, given the finite $\mathbb{C}$-valued Gaussian collection $\{G_j(y), j \in J\}$, is Gaussian of some nonrandom (conditional) variance $v = v_{\mathsf{R}}(y; J)$ and random mean $\widehat{\mu}\sqrt{v}$ (see [10, Exercise 2.1.3]). With $\phi(\cdot)$ denoting the standard normal density, we thus have by scaling, that the conditional expectation of $|\mathrm{Re}(G_0(y))|^{-1/2}$ is at most

$$v^{-1/4} \sup_{\widehat{\mu} \in \mathbb{R}} \left\{ \int_{\mathbb{R}} (|x - \widehat{\mu}| \wedge 1)^{-1/2} \phi(x) \mathrm{d}x \right\} := v^{-1/4} C_1\,,$$

for some finite constant $C_1 \leq 1 + \phi(0) \int_{-1}^{1} |x|^{-1/2} \mathrm{d}x$. With $v_{\mathsf{R}}(y)$ uniformly bounded below and $\omega(k_o) \leq 1/3$ for some k_o finite (see (3.6)), it follows that

$$u(k_o) := (1 - \omega(k_o)) \inf_{|y| < \kappa'} \{v_{\mathsf{R}}(y)\} > 0$$

and we get (4.10) with $C_o = u(k_o)^{-1/4} C_1$, upon showing that

$$\inf_{J \in \mathcal{J}_{k_o}, |y| < \kappa'} \{v_{\mathsf{R}}(y; J)\} \geq u(k_o)\,. \tag{4.11}$$

To this end, as $\mathbb{E}[X_0(y)^2] = 1$ and $\omega(k_o) \leq 1/2$, from Lemma 4.2(a) we then have that

$$\frac{v_{\mathsf{R}}(y; J)}{v_{\mathsf{R}}(y)} = \mathbb{E}[X_0(y)^2 \mid \{X_j(y), Y_j(y) : j \in J\}] \geq \mathbb{E}[X_0(y)^2] - \frac{\omega(k_o)^2}{1 - \omega(k_o)} \geq 1 - \omega(k_o)\,,$$

and (4.11) follows. ∎

Next, for fixed $k \geq k_o$, y, m, and any threshold $\Delta > 0$, we define the collection

$$B_\Delta := \{1 \leq j \leq m : |X_{jk}(y)| \vee |Y_{jk}(y)| > \Delta\}, \tag{4.12}$$

of "bad" indices, and use diagonal dominance (specifically, Lemma 4.2(c)), to show that for large Δ it is exponentially highly unlikely to have many bad indices.

Lemma 4.4. There exists $c(\Delta) \to \infty$ as $\Delta \to \infty$ such that for any $k \geq k_o$, all $|y| < \kappa'$ and nonrandom $B \subset \{1, \ldots, m\}$,

$$\mathbb{P}(B \subseteq B_\Delta) \leq e^{-4c(\Delta)|B|}. \tag{4.13}$$

Proof. Since the event $\{B \subseteq B_\Delta\}$ is the union of

$$\{|X_{jk}(y)| > \Delta, \ \forall j \in J\} \cap \{|Y_{jk}(y)| > \Delta, \ \forall j \in B \setminus J\},$$

over the $2^{|B|}$ possible $J \subseteq B$, by Cauchy–Schwartz it suffices for (4.13) to show that for some $b(\Delta) := 8c(\Delta) + \log 4 \to \infty$ as $\Delta \to \infty$, both

$$p_J(\Delta) := \mathbb{P}(|X_{jk}(y)| > \Delta, \ \forall j \in J) \leq e^{-b(\Delta)|J|}, \tag{4.14}$$

$$q_J(\Delta) := \mathbb{P}(\,|Y_{jk}(y)| > \Delta, \ \forall j \in J) \leq e^{-b(\Delta)|J|}. \tag{4.15}$$

To this end, recall that $1 - \omega(k) \geq 2/3 \geq 2\omega(k)$, whenever $k \geq k_o$. Hence, from Lemma 4.2(c) we have the bound

$$p_J(\Delta)^{1/|J|} \leq (1-\omega(k))^{-1/2}\mathbb{E}\left[e^{\frac{\omega(k)X_0^2}{2(1-\omega(k))}}\mathbf{1}_{\{|X_0|>\Delta\}}\right] \leq \frac{2\sqrt{3/2}}{\sqrt{2\pi}}\int_\Delta^\infty e^{-x^2/4}\mathrm{d}x := e^{-b(\Delta)}$$

for which (4.14) holds. Exactly the same argument applies for $q_J(\Delta)$, yielding the bound (4.13). ∎

We conclude the section by showing that, thanks to Lemma 4.2(b), for large $k = k(\Delta, \varepsilon)$ and any $J \in \mathcal{J}_k$, the conditional expectation of $|G_0(y)|^{\pm\varepsilon}$ given a realization of $\{X_j(y), Y_j(y) : j \in J\}$, all of whom are in a specified range $[-\Delta, \Delta]$, is within error $(1+o(\varepsilon))$ of the unconditional expectation.

Lemma 4.5. Let $H_J(y) := \max_{j\in J}\{|X_j(y)|, |Y_j(y)|\}$. There exist $k_\star(\Delta,\zeta,\varepsilon) : \mathbb{R}_+^3 \to [k_o,\infty)$ and $\varepsilon_\star > 0$, such that for any $\Delta,\zeta > 0$, $\varepsilon \le \varepsilon_\star$, $J \in \mathcal{J}_{k_\star}$ and $|y| < \kappa'$

$$\mathbb{E}\Big[|G_0(y)|^\varepsilon \mid \{G_j(y)\}_{j\in J}\Big]\mathbf{1}_{\{H_J(y)\le\Delta\}} \le e^{\varepsilon\zeta}\,\mathbb{E}[|G_0(y)|^\varepsilon], \tag{4.16}$$

$$\mathbb{E}\Big[|G_0(y)|^{-\varepsilon} \mid \{G_j(y)\}_{j\in J}\Big]\mathbf{1}_{\{H_J(y)\le\Delta\}} \le e^{\varepsilon\zeta}\,\mathbb{E}[|G_0(y)|^{-\varepsilon}]. \tag{4.17}$$

Proof. We use the representation (4.1), dividing (4.16) and (4.17) by $v_{\mathsf{R}}(y)^{\pm\varepsilon}$. Then, with $u := y^2 v_{\mathsf{I}}(y)/v_{\mathsf{R}}(y) \in [0,1]$ and $g_{u,\pm\varepsilon}(\mathbf{x}) := (x_1^2 + ux_2^2)^{\pm\varepsilon/2}$, the stated inequalities amount to

$$\mathbf{1}_{\{H_J(y)\le\Delta\}} \int_{\mathbb{R}^2} g_{u,\pm\varepsilon}(\mathbf{x}) f_J(\mathbf{x}) \mathrm{d}\gamma(\mathbf{x}) \le e^{\varepsilon\zeta} \int_{\mathbb{R}^2} g_{u,\pm\varepsilon}(\mathbf{x}) \mathrm{d}\gamma(\mathbf{x}), \tag{4.18}$$

where $f_J(\cdot)$ is the Radon–Nikodym density of the conditional law of $(X_0(y), Y_0(y))$ with respect to the standard two-dimensional Gaussian law γ. Recall Lemma 4.2(a) that for any $J \in \mathcal{J}_k$, $k \ge k_o$, the two-dimensional covariance matrix $\mathbf{\Sigma}_{1|2} := \mathbf{I}_2 - \mathbf{S}$ of $(X_0(y), Y_0(y))$, given $\{G_j(y), j \in J\}$, satisfies $\|\mathbf{S}\|_{\infty\to\infty} \le \omega(k)^2/(1-\omega(k)) \le \omega(k)$. Further, by Lemma 4.2(b), the conditional mean $\boldsymbol{\mu}$ of $(X_0(y), Y_0(y))$ must satisfy $\|\boldsymbol{\mu}\|_\infty \le 2\omega(k)H_J(y)$. Here $\omega = \omega(k) \le 1/3$, so similarly to the derivation of (4.7), we have for the (random) two-dimensional Radon–Nikodym density $f_J(\cdot)$ that

$$f_J(\mathbf{x}) = |\mathbf{\Sigma}_{1|2}|^{-1/2} \exp\left\{\frac{1}{2}\Big(\langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{x}-\boldsymbol{\mu}, \mathbf{\Sigma}_{1|2}^{-1}(\mathbf{x}-\boldsymbol{\mu})\rangle\Big)\right\} \le \widehat{f}_{\omega(k),H_J(y)}(\mathbf{x}), \tag{4.19}$$

where for any fixed $\Delta < \infty$,

$$\widehat{f}_{\omega,\Delta}(\mathbf{x}) := (1-\omega)^{-1} \exp\left\{\omega(x_1^2 + x_2^2 + 3\Delta|x_1| + 3\Delta|x_2|)\right\} \searrow 1, \quad \text{when} \quad \omega \searrow 0.$$

Note that $g_{u,\varepsilon} \le g_{1,\varepsilon}$ and $g_{u,-\varepsilon} \le g_{0,-\varepsilon}$. Further, both $g_{1,\varepsilon}\cdot(1+\widehat{f}_{\omega,\Delta})$ and $g_{0,-\varepsilon}\cdot(1+\widehat{f}_{\omega,\Delta})$ are in $\mathcal{L}^1_\gamma(\mathbb{R}^2)$ as soon as $\varepsilon \le \varepsilon_\star < 1$ and $\omega < 1/2$. Consequently, per $\varepsilon \le \varepsilon_\star$ and $\Delta < \infty$, the functions $\mathbf{x} \mapsto g_{u,\pm\varepsilon}(\mathbf{x})|\widehat{f}_{\omega,\Delta}(\mathbf{x}) - 1|$ are uniformly in u (and $\omega \le 1/3$), integrable with respect to γ, and converge pointwise to zero as $\omega \searrow 0$. Thus,

$$\lim_{\omega\searrow 0} \sup_{u\in[0,1]} \left| \int_{\mathbb{R}^2} g_{u,\pm\varepsilon}(\mathbf{x})\widehat{f}_{\omega,\Delta}(\mathbf{x})\mathrm{d}\gamma(\mathbf{x}) - \int_{\mathbb{R}^2} g_{u,\pm\varepsilon}(\mathbf{x})\mathrm{d}\gamma(\mathbf{x}) \right| = 0,$$

which together with (4.19) and (3.6) imply the existence of finite $k_\star(\Delta,\zeta,\varepsilon) \ge k_o$ such that (4.18) holds whenever $J \in \mathcal{J}_{k_\star}$ and $|y| < \kappa'$. ■

5 Moment Computations: Proof of Proposition 3.6

5.1 Proof of (3.31)

Since $c(\Delta)$ of Lemma 4.4 is unbounded, for any $\zeta > 0$ and $\varepsilon \le \varepsilon_\star < 1$ (where $\varepsilon_\star$ is from Lemma 4.5), we can take $\Delta = \Delta(\zeta, \varepsilon)$ so large that

$$C_\star e^{-c(\Delta)} \le \varepsilon\zeta e^{\varepsilon\zeta}\mathbb{E}\left[|G_0(y)|^{2\varepsilon}\right]^{1/2}, \tag{5.1}$$

where $C_\star$ is the finite constant from Lemma 4.1. Given such Δ, let $h_{\varepsilon,\Delta}(G) := |G|^\varepsilon \mathbf{1}_{\{|X|\vee|Y|\le\Delta\}}$ (for $G(y)$, $X(y)$, and $Y(y)$ related as in (4.1)). Then, fix $k \ge k_\star$ (also from Lemma 4.5), and partition the expression $M_m(\varepsilon)$ of (3.31) according to B_Δ of (4.12), to get that

$$M_m(\varepsilon) = \sum_B \mathbb{E}\left[\prod_{j\in B}|G_{jk}(y)|^\varepsilon \mathbf{1}_{\{B\subseteq B_\Delta\}}\prod_{j\in B^c} h_{\varepsilon,\Delta}(G_{jk}(y))\right], \tag{5.2}$$

where the sum is over all $B \subseteq \{1,\ldots,m\}$. For $(1-\varepsilon)/2 \ge 1/4$, using Hölder's inequality we bound the generic summand on the RHS of (5.2) by

$$\mathbb{E}\left[\prod_{j\in B}|G_{jk}(y)|^2\right]^{\varepsilon/2}\mathbb{P}(B\subseteq B_\Delta)^{1/4}\,\mathbb{E}\left[\prod_{j\in B^c}h_{2\varepsilon,\Delta}(G_{jk}(y))\right]^{1/2}. \tag{5.3}$$

Enumerating $B^c = \{j_1, j_2, \ldots\}$ with $1 \le j_1 < j_2 < \cdots \le m$ and utilizing the stationarity of $\{G_j(y)\}_j$, we appeal sequentially for $s = 1, \ldots,$ to (4.16) with $k^{-1}J = B^c \setminus \{j_1,\ldots,j_s\}$ shifted backward by j_s, to deduce that (since $k \ge k_\star$),

$$\mathbb{E}\left[\prod_{j\in B^c}h_{2\varepsilon,\Delta}(G_{jk}(y))\right] \le \left(e^{2\varepsilon\zeta}\mathbb{E}\left[|G_0(y)|^{2\varepsilon}\right]\right)^{|B^c|}. \tag{5.4}$$

Further bounding the left term of (5.3) via Lemma 4.1 and the middle one via Lemma 4.4, we complete the proof by deducing from (5.2) and (5.4) that

$$\begin{aligned} M_m(\varepsilon) &\le \sum_B C_\star^{|B|\varepsilon}e^{-c(\Delta)|B|}\left(e^{2\varepsilon\zeta}\mathbb{E}\left[|G_0(y)|^{2\varepsilon}\right]\right)^{|B^c|/2} \\ &= \left\{C_\star^\varepsilon e^{-c(\Delta)} + e^{\varepsilon\zeta}\mathbb{E}\left[|G_0(y)|^{2\varepsilon}\right]^{1/2}\right\}^m \le e^{2\varepsilon\zeta m}\mathbb{E}\left[|G_0(y)|^{2\varepsilon}\right]^{m/2} \end{aligned} \tag{5.5}$$

(with the last inequality holding thanks to having chosen Δ that satisfies (5.1)).

5.2 Proof of (3.32)

Here Lemma 4.3 replaces Lemma 4.1, so upon further reducing ε to satisfy $\varepsilon \le \varepsilon_o$ we set $\Delta = \Delta(\zeta, \varepsilon)$ so large that

$$C_o e^{-c(\Delta)} \le \varepsilon\zeta e^{\varepsilon\zeta} \mathbb{E}\left[|G_0(y)|^{-2\varepsilon}\right]^{1/2}, \tag{5.6}$$

where C_o and ε_o are the finite constants from Lemma 4.3. Proceeding as in the proof of (3.31), for $k \ge k_\star \ge k_o$ we partition the expression $M_m(-\varepsilon)$ of (3.32) according to B_Δ to get the identity (5.2) at $-\varepsilon$. Then, analogously to (5.3), we apply Hölder's inequality to bound the generic summand on the RHS of that identity (now at $-\varepsilon$), by

$$\mathbb{E}\left[\prod_{j\in B} |G_{jk}(y)|^{-4\varepsilon}\right]^{1/4} \mathbb{P}(B \subseteq B_\Delta)^{1/4}\, \mathbb{E}\left[\prod_{j\in B^c} h_{-2\varepsilon,\Delta}(G_{jk}(y))\right]^{1/2}. \tag{5.7}$$

The only difference WRT (5.3) is the 1st exponent -4ε instead of -2 (as $\mathbb{E}[|G_0(y)|^{-2}] = \infty$). The middle and last terms of (5.7) are handled precisely as in the proof of (3.31), upon appealing to Lemma 4.4 and (4.17), respectively. With Δ satisfying (5.6), the proof of (3.32) is thus complete upon establishing that

$$\mathbb{E}\left[\prod_{j\in B} |G_{jk}(y)|^{-4\varepsilon}\right] \le C_o^{|B|}. \tag{5.8}$$

Similarly to the derivation of (5.4), upon enumerating $B = \{j_1 < j_2 < \ldots\}$ we get the bound (5.8) by repeated conditioning and using (4.10) for $s = 1, 2, \ldots$ with $k^{-1}J = B \setminus \{j_1, \ldots, j_s\}$ shifted backward by j_s.

Open problem: Does the exponential upper tail of (1.11) hold in case of covariance $r(t) = \operatorname{sinc}(t)$ (with spectral density $p(\lambda) = \frac{1}{2}\mathbf{1}_{[-1,1]}(\lambda)$)? Note that this covariance satisfies Assumption A (for $x_\star = 2\pi$), apart from the lack of summability of the $r_1'(\cdot; 2y)$ term in (1.10). Motivation for this example comes, for instance, from scaling limits of random trigonometric polynomials, and some partial results on concentration were proved in this context, see for example,[2, Lemma 16].

Funding

This work was supported by a Ramanujan Fellowship from Government of India and a Simons Junior Faculty Fellowship [to R.B.]; National Science Foundation postdoctoral fellowship held at the Stanford University [to N.F.]; National Science Foundation [DMS-1613091to A.D.]; and United States–Israel Binational Science Foundation [2014019 to A.D. and O.Z.].

Acknowledgments

We thank Mikhail Sodin for motivating this research (in discussions with N.F.) and Manjunath Krishnapur for suggesting (to R.B.) that complex zeroes may be useful for showing concentration.

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