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Continuous Newton's Method for Polynomials

In the early 1980s, many of us were surprised to hear of the fractal nature of domains of attraction for Newton's method applied to polynomials. To recall this shock, take

$$p(z) = z^3 - 1 \quad z \in C. \qquad (1)$$

For 1, ω, and ω^2, the three roots of p, color a point $z_0 \in C$ red if $\{z_n\}_{n=0}^{\infty}$ converges to 1,

$$z_{n+1} = z_n - p(z_n)/p'(z_n), \quad n = 0, 1, \ldots. \qquad (2)$$

Color z_0 blue if $\{z_n\}_{n=0}^{\infty}$ converges to ω, and color it green if $\{z_n\}_{n=0}^{\infty}$ converges to ω^2. Any remaining point gets colored black [e.g., if $p'(z_n) = 0$ for some non-negative integer n]. Figures 1 and 2 were generated by the code "croots.for" by Robert Renka. Figure 1 shows (with the lightest shading standing for red, the darkest for green, and the intermediate shading for blue) domains of attraction for Eq. (2).

Figure 2 indicates corresponding domains of attraction for the modified Newton's method (for $\delta = \frac{1}{2}$):

$$z_{n+1} = z_n - \delta * p(z_n)/p'(z_n), \quad n = 0, 1, \ldots. \qquad (3)$$

As δ is chosen smaller, one gets corresponding pictures with even smaller jewels. Encouraged by this, mathematicians naturally divided by δ and let $\delta \to 0$. This is the "continuous Newton's method." In its basic form, it consists of finding a function z on a subset of R so that

$$z'(t) = -p(z(t))/p'(z(t)), \quad t \in D(z). \qquad (4)$$

Then, $t \to \infty$ should give $p(z(t)) \to 0$.

For this note, I use an improved version of continuous Newton's method: For $z_0 \in C$, a continuous function z from *all* of R to C is sought so that

$$p(z(0)) = z_0, \qquad p(z)'(t) = -p(z(t)), \quad t \in R. \qquad (5)$$

The improvement is in the handling of singularities. Solutions of (5) may not be solutions of $p'(z)z' = -p(z)$, for they may be such that $p'(z(t)) = 0$ and $z'(t)$ does not exist. The analysis of (5) allows us to sail right through these singularities.

Now look how much better continuous Newton (4) or (5) does with the example $p(z) = z^3 - 1$. We wouldn't expect convergence starting from the rays $\theta = \pi/3$, $\theta = -\pi/3$, $\theta = \pi$. However, let us start in M, the complement of the union of these three rays. For z_0 in one of the three components of M and z satisfying (5), $u = \lim_{t\to\infty} z(t)$ exists and is the root of p in that component. These domains of attraction are just what any right-thinking person would (wrongly) suspect for Eq. (2).

This note gives a rather complete description of domains of attraction for continuous Newton's method for polynomials. Results are expressed in three theorems.

We are seeking roots of p, a nonconstant complex polynomial. The method is to seek continuous functions z from R to C which solve the differential equation

$$p(z)'(t) = -p(z(t)), \quad t \in R. \qquad (6)$$

Let Q be the set of such functions.

Theorem 1. *If $z \in Q$, then*

$$u = \lim_{t\to\infty} z(t) \textit{ exists and } p(u) = 0.$$

In short, continuous Newton's method does find roots!

The range of a member of Q will be called a trajectory.

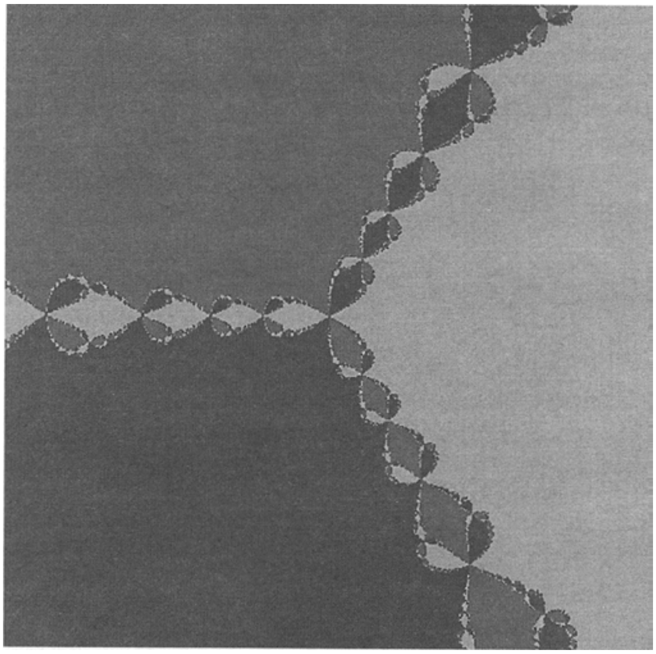


Figure 1. Newton's method for roots of p: $p(z) = z^3 - 1, z \in C$.

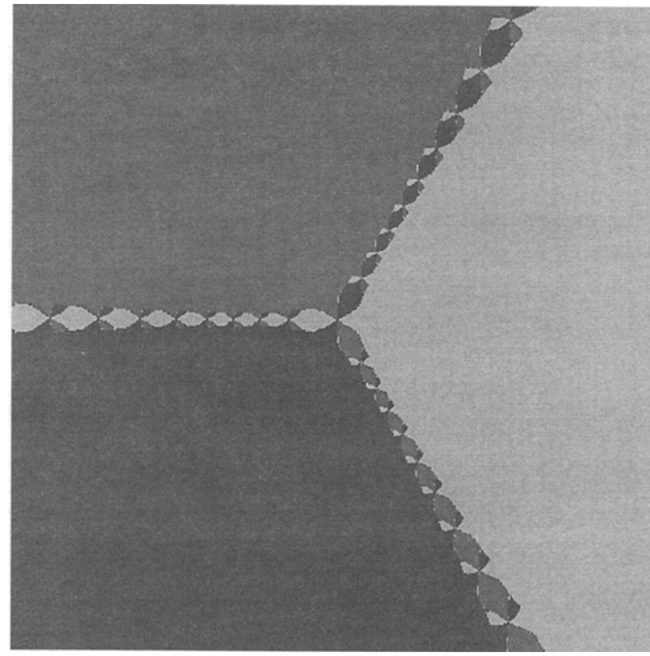


Figure 2. Damped Newton's method for p: $p(z) = z^3 - 1, z \in C$.

Theorem 2. *Every member of C is contained in some trajectory.*

A subset G of C is called an incoming trajectory if there are $x \in C$, $d \in R$, and $z \in Q$ so that

$$p(x) \neq 0, \qquad p'(x) = 0, \qquad z(d) = x, \quad \text{and} \quad G = z((-\infty, d]).$$

Such an incoming trajectory is said to end at x. It connects to an outgoing trajectory starting at x, namely $z([d, \infty))$.

Denote by M the set of all members of C which belong to no incoming trajectory.

Theorem 3. *Every component (maximal connected subset) of M contains just one root of p. If $z \in Q$ and $R(z)$ intersects S, a component of M, then*

$$u = \lim_{t \to \infty} z(t)$$

is the root of p which is in S.

The set of all continuous solutions z to Eq. (6) form a generalized flow in the sense of [1]. This concept helps in organizing situations like the present one, in which uniqueness under initial conditions doesn't hold.

I am grateful to John Mayer and Hartje Kriete for leading me to a number of recent references to Newton's methods. It is safe to say that any of the authors of [2, 3, 5, 6] could have written the present note had they decided to do so. All of these papers deal with discrete Newton's method (3), and most concern various aspects of continuous Newton's method as well. In [2], there is a treatment of how Julia sets connected with (3) converge to portions of the set M above. In [3, 5, 6] interesting connections between the two Newton's methods are given, for polynomials as well as for analytic functions more general than polynomials. Even though some of the present results can be gleaned from these cited papers, in the present note I try to give a rather self-contained treatment of the polynomial case.

Remark. I have not treated the root structure of polynomial maps from C^k to C^k for $K > 1$. Continuous Newton's method might apply, but I expect both formulations and proofs would be different.

Next are some vector field pictures for five examples. These are done with Mathematica. The Mathematica commands for Fig. 3 are

$$\begin{aligned} p[z\_] &:= z^3 - 1, \\ n[z\_] &:= -p[z]/p'[z], \\ n1[x\_, y\_] &:= \text{Re}[n[x + Iy]] \\ n2[x\_, y\_,] &:= \text{Im}[n[x + Iy]] \end{aligned}$$

$$\text{PlotVectorField } [\{n1[x, y], n2[x, y]\}, \{x, -1.2, 2\}, \{y, -1.5,1.5\}].$$

Change the first line to get a vector field for another polynomial. Change limits on the last line to examine other regions in C. The reader is invited to sketch in the union of any incoming trajectories in each case.

Perhaps the main points of interest in these pictures are the points of attraction, i.e., the roots, and the hyperbolic points. Hyperbolic points are precisely the ends of incoming trajectories.

The example in Fig. 3 has three points of attraction and one hyperbolic point. Two of the roots are imaginary. The hyperbolic point is zero, the sole root of p'. It turns out that any incoming trajectory can be continued as a trajectory to converge to any of the three roots. This illustrates some typical behavior.

The example in Fig. 4 has three roots (two complex) and two hyperbolic points. Note that either of the incoming trajectories converging to the rightmost root of p' can be con-

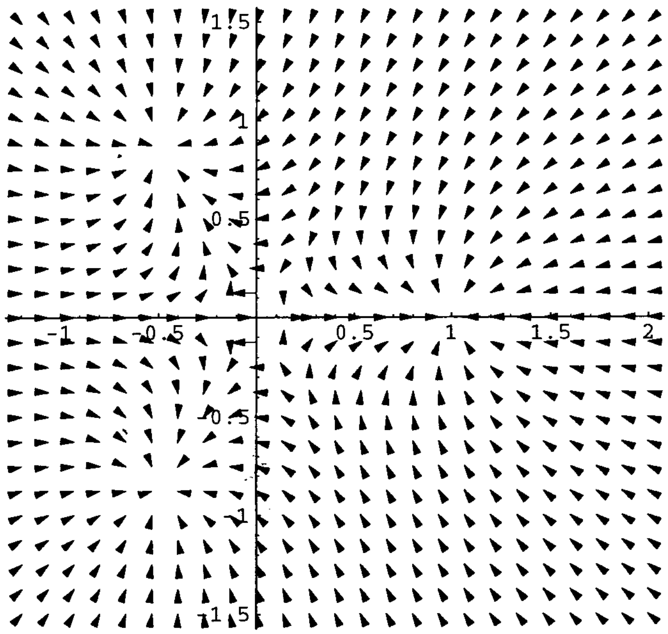


Figure 3. Vector field for p: $p(z) = z^3 - 1$, $z \in C$.

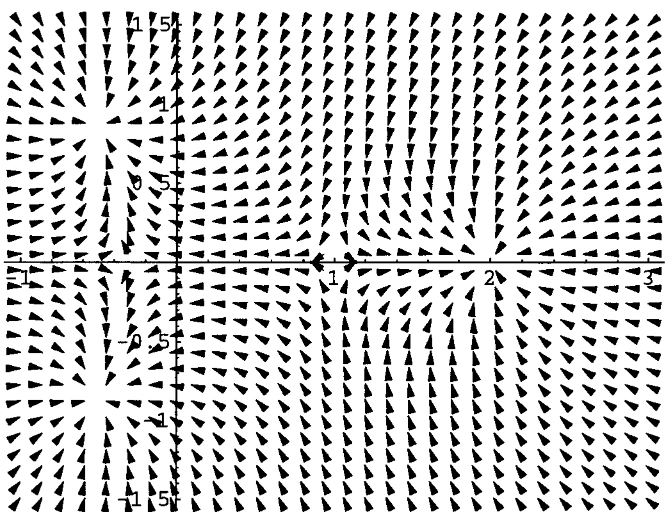


Figure 4. Vector field for p: $p(z) = (z^2 + z + 1)(z - 2)$, $z \in C$.

tinued to the right to converge to the real root of p. Either can also be continued to the left along the real axis to hit the other root of p'. From there, either can be continued to either of the complex roots. There is also an incoming trajectory coming in along the negative real axis. After arriving at the left critical point, the incoming trajectory can be continued to form a trajectory toward either imaginary root. This example shows nicely how an incoming trajectory can be extended nonuniquely to form a complete trajectory.

The example in Fig. 5 has three real roots and two real hyperbolic points. The union of the incoming trajectories is not connected.

The example in Fig. 6 has just zero for a root. There are no incoming trajectories and hence there is no hyperbolic point.

The example in Fig. 7 has just two roots. There is just a single hyperbolic point, for one root of p' is also a root of p and, hence, doesn't qualify as an endpoint of an incoming trajectory.

With a little practice, one can get a good idea from the Mathematica plot of where the roots of a given p lie; these are points of attraction of the corresponding vector field. This provides excellent starting points for discrete Newton's method if one requires greater accuracy.

Proofs

Proof [Theorem 1]. Fix a solution z of (6), and denote $z(0)$ by x. then,

$$p(z(t)) = \exp(-t)p(x), \quad t \in R. \tag{8}$$

Note that $\{z(t)\colon t \geq 0\}$ is bounded. If $s \geq 0$, then $z(s)$ is a root of $p_{s,x}$, where

$$p_{s,x}(y) = p(y) - \exp(-s)p(x), \quad y \in C, \tag{9}$$

and by a classical estimate, the set of all roots of $\{p_{s,x}\colon s \geq 0\}$ is bounded.

Suppose $\lim_{t\to\infty} z(t)$ does not exist. Still by compactness z has ω-limit points (points v such that $\lim_{k\to\infty} z(t_k) = v$ for some sequence $t_k \to \infty$); and by (8), every ω-limit point

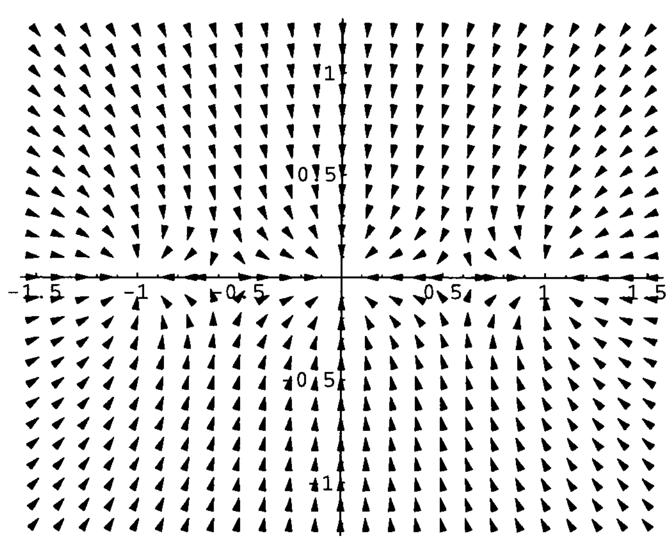


Figure 5. Vector field for p: $p(z) = z^3 - z$, $z \in C$.

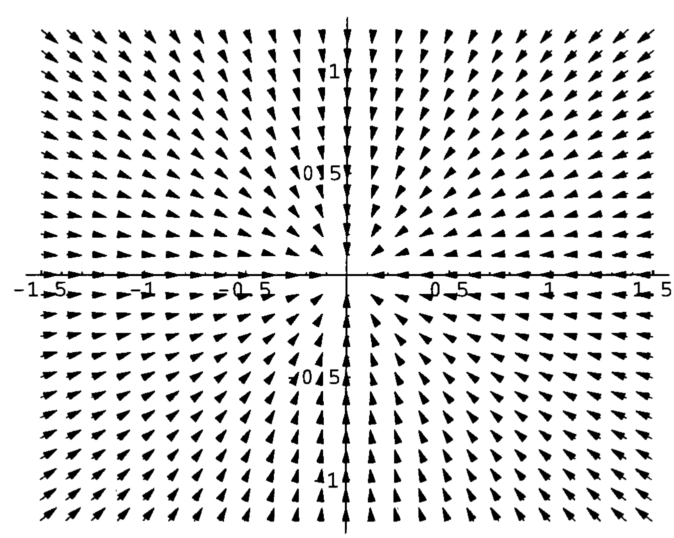


Figure 6. Vector field for p: $p(z) = z^3$, $z \in C$.

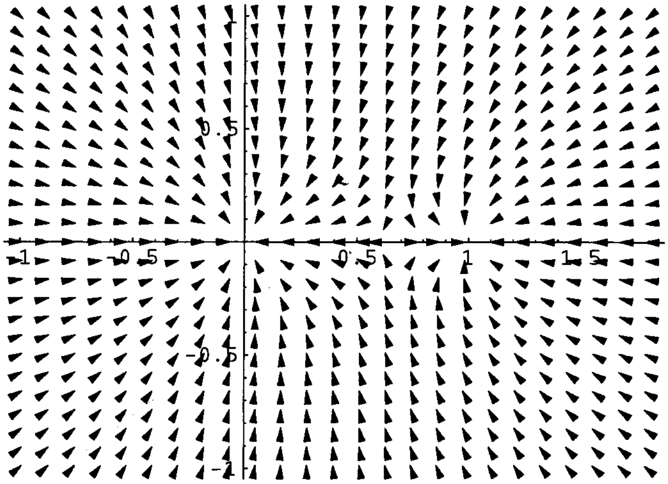


Figure 7. Vector field for p: $p(z) = z^2(z-1)$, $z \in C$.

is a root of p. This says the trajectory must shuttle infinitely often between distinct roots of p. Let U be a union of separated open disks each containing exactly one root. Continuity of z obliges it to have infinitely many values in $C \backslash U$. Then it must have ω-limit points there—a contradiction. □

Following are some lemmas on which arguments for Theorems 2 and 3 depend.

Lemma 1. *Suppose* $x \in C$, $q \in R$, and $p'(x) \neq 0$. *There is a maximal open interval* (a, b) *containing* q *on which there is a function* w *so that*

$$w(q) = x \quad \text{and} \quad p'(w(t)) \neq 0, \qquad p(w)'(t) = -p(w(t)), \quad t \in (a, b). \tag{10}$$

Moreover, such a function w *is unique on this maximal interval.*

Proof. Note that if (10) holds, then w is differentiable on (a, b) and, consequently,

$$w(q) = x \quad \text{and} \quad p'(w(t)) \neq 0, \qquad w'(t) = -p(w(t))/p'(w(t)), \quad t \in (a, b). \tag{11}$$

The result follows from classical existence and uniqueness theory for ordinary differential equations. □

Lemma 2. *Suppose* $x \in C$, $q \in R$, $p'(x) \neq 0$ *with* w *and* (a, b) *as in* (10). *If* $b < \infty$, *then*

$$\lim_{t \to b-} w(t) \text{ exists and } \lim_{t \to b-} p'(w(t)) = 0. \tag{12}$$

If $a > -\infty$, *then*

$$\lim_{t \to a+} w(t) \text{ exists and } \lim_{t \to a+} p'(w(t)) = 0. \tag{13}$$

Proof. First, assume that $b < \infty$. Then, $\{w(t): t \in [q, b)\}$ is bounded as noted in the argument for Theorem 1. Suppose that

$$\lim_{t \to b-} w(t)$$

does not exist. Still every ω-limit point v satisfies the polynomial equation

$$p(v) = \exp(-(t-q))p(x), \quad t \in (a, b);$$

we can reason as in Theorem 1 to get a contradiction.

Thus, $\lim_{t \to b-} w(t)$ exists. Consequently, $\alpha = \lim_{t \to b-} p'(w(t))$ also exists. But $\alpha = 0$, otherwise the solution w to (11) could be extended past b, contradicting the maximality of (a, b). The second conclusion to the lemma follows with a similar argument. □

Next is the technique I promised for patching solutions together (nonuniquely) at a singularity.

Lemma 3. *Suppose* $x \in C$, $p'(x) = 0$ *and* $p(x) \neq 0$. *There is* $\delta > 0$ *so that if* $s \in R$ *and* $0 < |s| < \delta$, *then*

if $s > 0$, *there is* f: $[0, s] \to C$ *so that* $f(0) = x$ *and*

$$p(f)'(t) = -p(f(t)), \quad t \in [0, s], \tag{14}$$

and

if $s < 0$ *and* $v = -s$, *there is* g: $[0, v] \to C$ *so that* $g(v) = x$, $p(g)'(t) = -p(g(t))$, $t \in [0, v]$. (15)

Proof. Select $r_0 > 0$ so that if $y \in C$, $0 < |y - x| < r_0$, then $p(y) \neq p(x)$ and $p'(y) \neq 0$.

For s near 0, we will study $p_{s,x}$ as in (9). Denote by x_s a root of $p_{s,x}$ chosen to minimize $|x_s - x|$.

To see why

$$x = \lim_{s \to 0} x_s, \tag{16}$$

factor $p_{s,x}$ completely and note that if (16) failed, it could not be that

$$\lim_{s \to 0} p_{s,x}(x) = 0;$$

but that is true because $p_{s,x}(x) = p(x) - \exp(-s)p(x)$.

For $s \neq 0$ such that $|x_s - x| < r_0$, take a_s, b_s, and w_s so that

$$w_s(0) = x_s, \qquad p(w_s)'(t) = -p(w_s(t)), \quad t \in (a_s, b_s),$$

with $a_s < 0 < b_s$ and (a_s, b_s) maximal in the sense of (10). Choose $r > 0$ so that $r < r_0$.

Assertion. There is $\delta > 0$ so that if $0 < s < \delta$, then

$$|w_s(t) - x| < r \quad \text{if } a_s < t < 0.$$

To prove this assertion, assume it is not true. Denote by $\{s_k\}_{k=0}^{\infty}$ a decreasing sequence of positive numbers converging to 0 such that if k is a positive integer, there is

$$t \in (a_{s_k}, 0) \text{ such that } |w_{s_k}(t) - x| \geq r.$$

For each positive integer k, denote by t_k the largest number $t \in (a_{s_k}, 0)$ so that

$$|w_{s_k}(t) - x| = r,$$

and note that if $t_k \leq t \leq 0$, then $|w_{s_k}(t) - x| \leq r$ and so

$$p'(w_{s_k}(t)) \neq 0, \quad p(w_{s_k}(t)) \neq p(x).$$

Then for all k,

$$p(w_{s_k}(t)) = \exp(-t)p(x_{s_k}) = \exp(-(t + s_k))p(x), \quad a_{s_k} < t \leq 0.$$

If $t_k \leq -s_k$, then we could take $t + s_k = 0$ here, i.e., $p(w_{s_k}(t)) = p(x)$ even though $|w_{s_k}(t) - x| \leq r < r_0$, a contradiction. Thus, $t_k > -s_k$, $k = 1, 2, \ldots$. Hence, $\lim_{k\to\infty} (t_k + s_k) = 0$ since $\lim_{k\to\infty} s_k = 0$. Thus,

$$p(w_{sk}(t_k)) = \exp(-(t_k + s_s))p(x) \to p(x) \quad \text{as } k \to \infty.$$

However, $|w_{s_k}(t_k) - x| = r$, $k = 1, 2, \ldots$, so some subsequence of $\{w_{s_k}(t_k)\}_{k=1}^{\infty}$ converges to an element $y \in C$ so that $|y - x| = r$. But then, $p(y) = p(x)$, a contradiction. Thus, the assertion is true. Choose δ so that the assertion holds.

Observe that if $s \in (0, \delta)$, then $a_s = -s$; either of the assertions $a_s < -s$ and $a_s > -s$ leads to a contradiction.

By Lemma 2, if $0 < s < \delta$, then $y = \lim_{t\to a_s+} w_s(t)$ exists, $p'(y) = 0$ and $|y - x| \leq r$ since $a_s = -s$. This implies that $y = x$.

For $0 < s < \delta$, define f on $[0, s]$ by

$$f(0) = x, \qquad f(t) = w_s(t-s), \quad t \in (0, s]$$

and observe that f satisfies (14). We proceed in an entirely similar way (possibly reducing our δ) to prove (15). □

The ideas in the proof of Lemma 3 can be extended to show that if $x \in C$, $p(x) \neq 0$, $p'(x) = 0$, and x as a root of p' has multiplicity k, then there are at least $k + 1$ incoming trajectories ending at x and at least $k + 1$ outgoing trajectories starting at x. This follows starting with the fact that if $s \neq 0$, x is a root of p, and x_s is a root of $p_{s,x}$, then $p_{0,x}(x_s) + p_{s,x}(x) = 0$, where $p_{s,x}$ and $p_{0,x}$ are defined using (9).

Proof [Theorem 2]. Take any $x \in C$.

Case 1. If $p(x) = 0$, then the function z so that $z(t) = x$, $t \in R$, is in Q.

Case 2. Suppose $p(x) \neq 0$ and $p'(x) \neq 0$. Using Lemma 1, choose w satisfying (10) where (a, b) is maximal and $q = 0$, i.e., $w(0) = x$. If $b < \infty$, then by Lemma 2, $y = \lim_{t\to\infty} w(t)$ exists and $p(y) \neq 0$ and $p'(y) = 0$. Using Lemma 3, pick f satisfying (14) and extend w so that $w(b)$ is the left limit of w at b and so that if $t \in (0, s)$, then $w(b + t) = f(t)$. Alternately, using Lemmas 1, 2, and 3, we arrive at w defined on (a, ∞) so that $p(w)'(t) = -p(w(t))$, $t \in (a, \infty)$. If $a = -\infty$, we are finished. If $a > -\infty$, repeat the extension process only going to the left. In any case the end result is a function z in Q.

Case 3. Finally, suppose that $p(x) \neq 0$ and $p'(x) = 0$. Pick f and g satisfying (14) and (15), respectively, and define w on $[-v, s]$ so that

$$w(0) = x,$$
$$w(t) = g(t + v) \quad (t \in [-v, 0)),$$
$$w(t) = f(t) \quad (t \in (0, s]).$$

Note that $p(w)'(t) = -p(w(t))$, $t \in [-v, s]$ (there is something to reflect upon concerning the differentiability of $p(w)$ at 0). Extend w to the left and right as needed using Lemmas 1, 2, and 3 to arrive at an extension $z \in Q$. □

Proof [Theorem 3]. By definitions, M contains every root of p. Also, note that if $z \in Q$, S a component of M, and $R(z)$ intersects S, then $R(z) \subset S$, so the root u of p such that $u = \lim_{t\to\infty} z(t)$ must also be in S. It follows that every component S of M contains at least one root of p.

Suppose a component S of M contains more than one root of p, say $u_1, \ldots, u_b$ for some integer $b > 1$. Partition S into $S_1, \ldots, S_b$ with

$$S_j = \{x \in S\colon z \in Q,\ x \in R(z),\ u_j = \lim_{t\to\infty} z(t)\}, \quad j = 1, \ldots, b,$$

and note that no two members of $S_1, \ldots, S_b$ intersect. Since S is connected, there are integers $m, n \in \{1, \ldots b\}$ such that $v = \lim_{k\to\infty} v_k$ for some $v \in S_m$ and $v_1, v_2, \ldots, \in S_n$. Choose $z, z_1, z_2, \ldots \in Q$ such that $z(0) = v$ and $z_k(0) = v_k$, $k = 1, 2, \ldots$. Then,

$$\lim_{t\to\infty} z(t) = u_m, \qquad \lim_{t\to\infty} z_k(t) = u_n, \quad k = 1, 2, \ldots.$$

There are $t_1, t_2, \ldots \in R$ such that

$$\lim_{k\to\infty} z_k(t_k) = u_m \text{ and, consequently, } \lim_{k\to\infty} p(z_k(t_k)) = 0.$$

However, for fixed positive integer k,

$$\lim_{t\to\infty} z_k(t) = u_n, \qquad \lim_{t\to\infty} p(z_k(t)) = 0.$$

Because of the continuity of each of $\{z_k\}_{k=1}^{\infty}$ and the local compactness of C, we arrive at infinitely many roots of p, a contradiction. Thus, S contains only one root of p. □

Using similar arguments but without using Lemma 3, one can prove the weaker result that through every x with $p(x) \neq 0$ and $p'(x) \neq 0$ goes a solution of $z'(t) = -p(z(t))/p'(z(t))$ converging either to a root of p or to a root of p', and that for some $x \in C$, the first alternative holds. Now Lemma 3 is the only place in the development which uses the Fundamental Theorem of Algebra, so one can deduce that result from the one just stated.

Ulterior Motive

The above development gives a way to tag a solution u to $p(u) = 0$ with a region in C, roughly its domain of attraction in the descent process

$$p(z)' = -p(z).$$

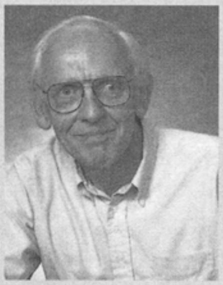
This problem is analogous to the following. Suppose each of H and K is a Hilbert space, F is a $C^{(2)}$ function from H to K, and

$$\phi(u) = \|F(u)\|_K^2/2, \quad u \in H.$$

I have in mind, for example, cases in which H is a Sobolev space of functions on some region in Euclidean space, K is an L_2 space on that region, and the problem of finding $u \in H$ so that

$$F(u) = 0 \qquad (17)$$

AUTHOR



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John Neuberger is a graduate of the University of Texas and a former Sloan Fellow. He has long advocated a more central position for partial differential equations. Aside from his University duties, he is a consultant in mathematics and computation. He enjoys bicycling and swimming, and traveling with his wife Barbara.

represents a system of partial differential equations with some but perhaps not enough boundary conditions to imply existence of one and only one solution. It is this kind of root-finding which has been a major focus of attention for me recently [4].

$$\phi'(u)h = \langle F'(u)h, F(u)\rangle_K = \langle h, F'(u)^*F(u)\rangle_H, \quad u, h \in H,$$

where $F'(u)^* \in L(K, H)$ is the Hilbert space adjoint of $F'(u)$, $u \in H$. This leads to a Sobolev gradient $\nabla\phi$ for ϕ satisfying the identity

$$\phi'(u)h = \langle h, (\nabla\phi)(u)\rangle_H, \quad u, h \in H,$$

by taking $(\nabla\phi)(u) = F'(u)^*F(u)$, $u \in H$.

Seek $u \in H$ such that $F(u) = 0$ by means of continuous steepest descent, i.e., consider z: $[0, \infty) \to H$ so that

$$z(0) = x \in H, \qquad z'(t) = -(\nabla\phi)(z(t)), \quad t \geq 0, \quad (18)$$

in the hope that

$$u = \lim_{t\to\infty} z(t) \text{ exists and } F(u) = 0. \qquad (19)$$

In analogy with continuous Newton's method for polynomials, one says that $x, y \in H$ are equivalent relative to (18) provided that they lead, through (18) and (19), to the same element $u \in H$. Granted that the limit exists for each $x \in H$ and z in (18), one has H partitioned in such a way that each leaf in the partition contains precisely one solution. These leaves are analogous to the components of M in the first section above. Numerical, geometric, and algebraic studies of these leaves may provide an approach to the general boundary value problem for the system (17). There are some results in this direction in [4].

The study of partial differential equations, however, is characterized by a constant struggle for a few crumbs of compactness. In contrast, in arguments for continuous Newton's method for polynomials I could relax: there was plenty of compactness. I felt like a kid again.

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