BIMEASURES AND HARMONIZABLE PROCESSES

(Analysis, Classification, and Representation)

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Introduction. Bimeasures arise in studies such as multiparameter martingales, second order random processes, representation theory of linear operators, harmonic analysis, and are related to (often induced by) bilinear or sesquilinear forms on suitable function spaces. These play a key role in the theory of harmonizable random fields in their structural analyses as well as classifications. Although one may regard bimeasures often as extensions of noncartesian products of pairs of scalar measures, their integration, in the general case, departs significantly from the standard product integration and new techniques are needed for their employment in applications. If the underlying measure space has a group structure, then it is also possible to study extensions of the classical measure algebra theory for bimeasures. Thus the purpose of this article is to present a somewhat detailed description of some of these results, leading to a classification of harmonizable functions including certain new developments. For instance, the material in Section 2.2.3 below did not appear in print before. A brief account of the work, given in three parts, will be described here since then one gets a better perspective of the theory covered. It largely complements the recent detailed account presented in Chang and Rao (1986).

The first part is on bimeasure theory. Starting with a general concept, the bimeasure integrals in the sense of Morse and Transue (1955), termed MT-integrals hereafter, are introduced. The Lebesgue type limit theorems are not valid for them. So a subclass, termed strict MT-integrals, is isolated for use in stochastic theory. Specializing to groups, bimeasure algebras with a suitable convolution product are described. If these bimeasures are also positive definite, then the structure of function spaces on them are treated. A few extensions, if some of these objects are vector valued, are also included since such results are useful for multidimensional harmonizable random fields to be discussed later on in the paper.

The preceding work is applied, in Part II, for the stochastic theory. The primary class is the harmonizable random fields. It is used in classifying harmonizability into weak, strong, ultraweak, strict, and other types. Integral representations of harmonizable fields on LCA (= locally compact abelian) groups are given. For the nonabelian case, several new problems arise. Here a novel treatment is given for a class of the so-called type I groups. For them an integral representation is obtained, basing it on the structural analysis of these groups due to Mautner, Segal and others. This extends Yaglom's (1960) fundamental work on stationary random fields in many ways. Then stationary dilations and linear filtering problems are discussed. Further, some applications here lead to a study of harmonizability on hypergroups. This is briefly sketched. Also extensions to strictly harmonizable functions as a subclass of stable processes are included. Open problems suggested by this analysis abound, and several are pointed out at many places.

The final part discusses multidimensional extensions of the foregoing, and are motivated by applications. However, to keep the article in bounds, this part is unfortunately curtailed. Here weak and strong V-boundedness of Bochner's concept have to be separated, since one of them admits a dilation and the other does not. To see these distinctions clearly, a discussion of vector (and module) harmonizability is sketched. Also, the multiplicity problem of these processes and some related ones such as almost harmonizability are touched on. Let us now turn to the details. (Notation is given at the end of the paper.)

Part I: Bimeasure Theory

1.1 *The general concept.* A systematic study of bimeasures originated with the work of Fréchet (1915) and was continued, after a long lapse, in several papers by Morse and Transue (1949-56), and also Lévy (1946). The concept may be introduced as follows. Let $(\Omega_i, \Sigma_i), i = 1, 2$ be a pair of measurable spaces and $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ be a mapping such that $\beta(A, \cdot)$ and $\beta(\cdot, B)$ are (complex) measures for each $A \in \Sigma_1$ and $B \in \Sigma_2$. Then β is called a (complex) *bimeasure.* It should be noted that this definition does not assume (or imply) that β has an extension to be a (complex) measure on the generated product σ-algebra $\Sigma_1 \otimes \Sigma_2$.

The preceding remark is better understood if the concept is given an alternative form through tensor products and bilinear mappings on them. Since these products will come up again later on, let us recall them. Thus, if $\mathcal{X}_1, \mathcal{X}_2$ are Banach spaces, $\mathcal{X}_1 \otimes \mathcal{X}_2$ denotes the vector space of all formal sums of the form $x = \sum_{i=1}^{n} x_{1i} \otimes x_{2i}, x_{ji} \in \mathcal{X}_j, j = 1, 2$. If there is a norm α on this space such that $\|x_1 \otimes x_2\|_\alpha = \|x_1\|\|x_2\|$, termed a cross-norm α, let $\mathcal{X}_1 \otimes_\alpha \mathcal{X}_2$ be the completion for $\|\cdot\|_\alpha$. There exist several such norms, but those of interest below are the greatest and least cross-norms denoted $\|\cdot\|_\gamma$ and $\|\cdot\|_\lambda$, and are given by

$$\|x\|_\gamma = \inf\{\sum_{i=1}^{n} \|x_{1i}\|_{\mathcal{X}}\|x_{2i}\|_{\mathcal{X}} : x = \sum_{i=1}^{n} x_{1i} \otimes x_{2i}, n \geq 1\}, \tag{1}$$

and $\mathcal{X}_i^*$ denoting the adjoint space of $\mathcal{X}_i$,

$$\|x\|_\lambda = \sup\{|\sum_{i=1}^{n} \ell_1(x_{1i})\ell_2(x_{2i})| : \ell_i \in \mathcal{X}_i^*, \|\ell_i\| \leq 1, i = 1,2, n \geq 1\}. \tag{2}$$

The corresponding completed spaces are denoted by $\mathcal{X}_1 \otimes_\gamma \mathcal{X}_2$ or $\mathcal{X}_1 \widehat{\otimes} \mathcal{X}_2$, and $\mathcal{X}_1 \otimes_\lambda \mathcal{X}_2$ or $\mathcal{X}_1 \widehat{\widehat{\otimes}} \mathcal{X}_2$. These are termed the tensor product spaces. If $L(\mathcal{X}_1, \mathcal{X}_2)$ stands for the space of continuous linear operators on $\mathcal{X}_1$ into $\mathcal{X}_2$, the following relations are true and classical:

Proposition 1. *Let $\mathcal{X}_1, \mathcal{X}_2$ be Banach spaces. Then the following identifications hold:*

(a) $(\mathcal{X}_1 \widehat{\otimes} \mathcal{X}_2)^* \cong L(\mathcal{X}_1, \mathcal{X}_2^*), (\cong L(\mathcal{X}_2, \mathcal{X}_1^*))$,

(b) $(\mathcal{X}_1 \widehat{\widehat{\otimes}} \mathcal{X}_2)^* \hookrightarrow L(\mathcal{X}_1^*, \mathcal{X}_2)$

where "$\cong$" is an isometric isomorphism in (a) *and "$\hookrightarrow$" in* (b) *is an isometric imbedding into the second space.*

A specialization of (a) gives an alternative definition of bimeasures. Thus let Ω_i be locally compact and $C_0(\Omega_i)$ be the continuous scalar functions on Ω_i vanishing at "∞", $i = 1,2$. If $V(\Omega_1, \Omega_2) = C_0(\Omega_1) \widehat{\otimes} C_0(\Omega_2)$ where $\mathcal{X}_i = C_0(\Omega_i)$ in the above result, with uniform ($= \|\cdot\|_\infty$) norms, thus $V(\Omega_1, \Omega_2)^* \cong L(C_0(\Omega_1), C_0(\Omega_2)^*)$, then the correspondence is given by (the topology is always Hausdorff for the work here)

$$B(f,g) = F(f \otimes g) = T(f)g,\ f \in C_0(\Omega_1), g \in C_0(\Omega_2), \tag{2'}$$

with $F \in V(\Omega_1, \Omega_2)^*$ and $T : C_0(\Omega_1) \to C_0(\Omega_1)^*$, a bounded linear operator. Here $B : C_0(\Omega_1) \times C_0(\Omega_2) \to \mathbb{C}$ is the bounded bilinear form corresponding to F. Since $C_0(\Omega_2)^* = M(\Omega_2)$, the space of regular bounded scalar (= Radon) measures with total variation norm, so that $Tf \in M(\Omega_2)$, by the general Riesz-Markov theorem one has

$$(Tf)(\cdot) = \int_{\Omega_1} f(\omega_1)\mu(d\omega_1, \cdot) \in M(\Omega_2). \tag{3}$$

This holds since $M(\Omega_2)$ is weakly sequentially complete so that T is a weakly compact operator (cf. Dunford-Schwartz (1958), Theorems VI.7.3 and IV.9.9). Letting $\mu^f(\cdot) = (Tf)(\cdot)$ of

(3), (2′) becomes

$$\begin{aligned}F(f \otimes g) &= B(f,g) = (Tf)(g)\\ &= \int_{\Omega_2} g(\omega_2)\mu^f(d\omega_2)\\ &= \int_{\Omega_2} g(\omega_2)[\int_{\Omega_1} f(\omega_1)\mu(d\omega_1,\cdot)](d\omega_2),\\ &= \int_{\Omega_2}\int_{\Omega_1}(f,g)(\omega_1,\omega_2)\mu(d\omega_1,d\omega_2),\end{aligned} \tag{4}$$

by definition of this last symbol. Thus, $B(\cdot,\cdot)$ may be identified with $\mu(\cdot,\cdot)$, a bimeasure. For this reason Voropoulos (1967) calls each member of $V(\Omega_1,\Omega_2)^*$, also denoted $BM(\Omega_1,\Omega_2)$, a bimeasure. (Although he defined this for Ω_i compact, the local compactness presents no difficulty here.) Also

$$\begin{aligned}\|F\| = \|B\| &= \sup\{|B(f,g)| : \|f\|_\infty \le 1, \|g\|_\infty \le 1\}\\ &= \sup\{|\int_{\Omega_2}\int_{\Omega_1}(f,g)(\omega_1,\omega_2)\mu(d\omega_1,d\omega_2)| : \|f\|_\infty \le 1, \|g\|_\infty \le 1\}\\ &= \|\mu\|, \text{ the "semi-variation" of } \mu.\end{aligned} \tag{5}$$

For a bimeasure μ, the symbol $\|\mu\|$ should be termed Fréchet variation, following the work by Fréchet (1915). Since $V(\Omega_1,\Omega_2)^* \underset{\neq}{\supset} M(\Omega_1 \times \Omega_2)[= C_0(\Omega_1 \times \Omega_2)^*]$, in general μ is not a restriction of some scalar measure $\tilde{\mu}$ on the Borel σ-algebra $\mathcal{B}(\Omega_1 \times \Omega_2)$ to $\mathcal{B}(\Omega_1) \times \mathcal{B}(\Omega_2)$. This makes the last integration symbol of (4) different from the Lebesgue definition, so that a generalized concept of integration, retaining the property (5), is needed. This is provided in the next section and is used throughout the article.

1.2 *Bimeasure integrals.* Let (Ω_i, Σ_i) be a measurable space and $f_i : \Omega_i \to \mathbb{C}$ be measurable for $\Sigma_i, i = 1,2$. If $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a bimeasure, suppose that f_1 is $\beta(\cdot, B)$ and f_2 is $\beta(A,\cdot)$ - integrable for each $A \in \Sigma_1, B \in \Sigma_2$. Then

$$\beta^{f_1}(B) = \int_{\Omega_1} f(\omega_1)\beta(d\omega_1, B); \beta^{f_2}(A) = \int_{\Omega_2} f_2(\omega_2)\beta(A, d\omega_2), \tag{6}$$

exist and the β^{f_i} are σ-additive. Suppose moreover that f_2 is β^{f_1}-integrable, f_1 is β^{f_2}-integrable (Lebesgue's sense). Then (f_1, f_2) is said to be *β-integrable* (in the MT-sense) if the following equality holds:

$$\int_{\Omega_1} f_1(\omega_1)\beta^{f_2}(d\omega_1) = \int_{\Omega_2} f_2(\omega_2)\beta^{f_1}(d\omega_2)(= \int_{\Omega_1}\int_{\Omega_2}(f_1,f_2)\beta(d\omega_1,d\omega_2)), \tag{7}$$

where the double integral on the right of (7) (and in (4)) is, by definition, the common value of the first two terms. [The original definitions of MT-integral is slightly different but is equivalent to this one for the work below.] If $\Omega_1 = \Omega_2$, then f is termed *β-integrable* if (f, f) is such.

Regarding this definition, it must be noted that the left side integrals of (7) can exist and be not equal. In comparing (4) and (7) one can show that all bounded Borel functions (f, g) are β-integrable, although β is only a bimeasure. It was remarked that (5) is Fréchet variation. This can be written abstractly as follows, i.e., the *Fréchet variation* of μ is:

$$\begin{aligned}\|\mu\| &= \|\mu\|(\Omega_1, \Omega_2)\\ &= \sup\{|\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{b}_j\mu(A_i, B_j)| : \{A_i\}_1^n, \{B_j\}_1^n \text{ disjoint sets from } \Sigma_1, \Sigma_2, |a_i| \le 1,\\ &\quad |b_j| \le 1, n \ge 1\}. \end{aligned} \tag{8}$$

A more restricted concept is *Vitali's variation* which is given by

$$\begin{aligned}|\mu| &= |\mu|(\Omega_1, \Omega_2)\\ &= \sup\{\sum_{i=1}^{n}\sum_{j=1}^{n} |\mu(A_i, B_j)| : \{A_i\}_1^n, \{B_j\}_1^n \text{ are as in (8), } n \ge 1\}.\end{aligned} \tag{9}$$

It is clear that $\|\mu\| \le |\mu|$; and there is strict inequality when $|\mu| = +\infty$. Some properties of these variations and of the MT-integral of (7) will be included to distinguish it from the Lebesgue integral.

If a bimeasure μ has finite Vitali variation then the integral in (7) coincides with the standard Lebesgue concept, and μ can also be extended to a scalar measure $\tilde{\mu}$ onto $\Sigma_1 \otimes \Sigma_2$ uniquely. However, this statement fails if $|\mu|(\Omega_1, \Omega_2) = +\infty$ without further restrictions. It can be shown that $\|\mu\|(\Omega_1, \Omega_2) < \infty$ always, and the MT-integral of (7), in general, is not absolutely continuous in that (f, g) is μ-integrable does not imply the same of $(|f|, |g|)$. Also the dominated convergence theorem does not hold for the MT-integral. The following approximation and a sufficient condition for the existence of the MT-integral can be stated from the work of Morse and Transue, and it is useful in applications.

Theorem 1. *Let Ω_i be locally compact, $C_c(\Omega_i)$ the space of scalar continuous functions with compact supports and Σ_i be Borel σ -algebras of $\Omega_i, i = 1, 2$. If $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a* *bimeasure and* $B : C_c(\Omega_1) \times C_c(\Omega_2) \to \mathbb{C}$ *is the bilinear form defined by* (4) *for this* β, *let*

$$B_*(p,q) = \sup\{|B(f_1,f_2)| : |f_1| \le p, |f_2| \le q, f_i \in C_c(\Omega_i)\}$$

$$B^*(u,v) = \inf\{B_*(p,q) : p \ge u \ge 0, q \ge v \ge 0\},$$

where p, q *are lower semicontinuous and* u, v *are Borel functions on* Ω_1, Ω_2. *Suppose that*

$$(i) B^*(u, f_2) < \infty \text{ and} (ii) B^*(f_1, v) < \infty, 0 \le f_i \in C_c(\Omega_i).$$

Then $\beta^u(\cdot)$ *and* $\beta^v(\cdot)$ *of* (6) *are* σ *-additive. If moreover* $B^*(|u|,|v|) < \infty$, *then the equality* (7) *holds so that* (u,v) *is* β *-integrable. In particular each bounded Borel pair* (u,v) *is* β*-integrable.*

Although the dominated convergence statement is not available, the following type of special approximation is nevertheless true.

Theorem 2. *Let* Ω_i, β *and* B *be as in the above theorem and* (f_1, f_2) *be* β*-integrable. If* $|f_i| \le p_i$ *with* p_i *lower semi-continuous (in particular if* $|f_i|$ *itself is lower semi-continuous,* $|f_i| = p_i$ *can be taken) and* $\epsilon > 0$ *is given, then there exist* $u_i \in C_c(\Omega_i)$ *such that* $|u_i| \le p_i, i = 1, 2,$ *and one has*

$$|B(f_1, f_2) - B(u_1, u_2)| < \epsilon. \tag{10}$$

This type of approximation from below by compactly based continuous functions, given by (10), is quite useful. Both these results, simple in form, are not easy. The details are given in Morse and Transue (1956). However, this approximation is not strong enough for applications in stochastic theory and elsewhere. Note that the MT-integral is weaker than the Lebesgue concept since the β-integrability of (f,g) does not imply that of $(|f|,|g|)$. It is of Riemann type and an additional uniformity condition can and should be added to obtain a dominated convergence statement. Also in the MT-integral it is significant that $(f,g)(\omega_1,\omega_2) = f(\omega_1)g(\omega_2)$ is a product of an ω_1 and an ω_2-function and not one of the form: $h : \Omega_1 \times \Omega_2 \to \mathbb{C}$, measurable relative to $\Sigma_1 \otimes \Sigma_2$. An extension of the integral to the latter type relative to a bimeasure has been given by Kluvanek (1981). A brief discussion of this will be included to illuminate the problem at hand, since the definition is patterned after that of Dunford and Schwartz [(1958), Sec. IV.10], the latter playing a pivotal role in our theory.

Let $f_n : \Omega = \Omega_1 \times \Omega_2 \to \mathbb{C}$ be a simple function so that $f_n = \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} a_{ij}^n \chi_{A_i^n} \chi_{B_j^n}, A_i^n \in \Sigma_1, B_j^n \in \Sigma_2$, disjoint. If $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a bimeasure, define

$$\int_\Omega f_n d\beta = \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} a_{ij}^n \beta(A_i^n, B_j^n). \tag{11}$$

If $f_n \to h$ pointwise and $\{\int_\Omega f_n d\beta, n \geq 1\} \subset \mathbb{C}$ is Canchy with a limit α_o, one can define $\alpha_o = \int_\Omega h d\beta$. It can be verified that $h \longmapsto \int_\Omega h d\beta$ is well-defined, and is linear. An equivalent form of this is given by Kluvanek (1981) even when the range $\mathbb{C}$ is replaced by a Banach space. He showed the usefulness of this extension with examples including the disintegration problem. However, the dominated or bounded convergence theorem does not hold for it. Thus a restriction of the MT-integral is needed for these limit operations to hold. Such a result will now be given since one needs it in the stochastic analysis below.

1.3 *The strict integral as a subclass.* If (Ω_i, Σ_i) is a measurable space, $f_i : \Omega_i \to \mathbb{C}$ is measurable for $\Sigma_i, i = 1, 2$, and $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a bimeasure, suppose that f_1 is $\beta(d\omega_1, \cdot)$ and f_2 is $\beta(\cdot, d\omega_2)$ -integrable in the D-S sense, treating these as vector measures. Then define

$$\tilde{\beta}_1^F : A \longmapsto \int_F f_2(\omega_2)[= (\int_F f_2(\omega_2)\beta(\cdot, d\omega_2))(A)], A \in \Sigma_1,$$

$$\tilde{\beta}_2^E : B \longmapsto \int_E f_1(\omega_1)\beta(d\omega_1, B)[= (\int_E f_1(\omega_1)\beta(d\omega_1, \cdot))(B)], B \in \Sigma_2,$$

for each $E \in \Sigma_1, F \in \Sigma_2$. $\tilde{\beta}_1^F$ and $\tilde{\beta}_2^E$ are complex measures. The pair (f_1, f_2) is termed *strictly β-integrable* if $(i) f_1$ is $\tilde{\beta}_1^F$ and f_2 is $\tilde{\beta}_2^E$-integrable for each pair E and F, and (ii) one has

$$\int_E f_1(\omega_1)\tilde{\beta}_1^F(d\omega_1) = \int_F f_2(\omega_2)\tilde{\beta}_2^E(d\omega_2). \tag{12}$$

The common value is denoted $\int_E \int_F^* f_1(\omega_1) f_2(\omega_2)\beta(d\omega_1, d\omega_2)$. Note that the difference between this and the original concept is that (12) must hold for each pair E, F now, whereas in (7) it should only hold for $E = \Omega_1$ and $F = \Omega_2$. This stregthening of the definition (motivated by Theorem 1.2.1 above) renders the validity of a useful dominated convergence theorem and restores the absolute continuity property, although it is still weaker than the Lebesgue concept. One should also observe that only the class of β-integrable functions is restricted here, but not the class of bimeasures. For instance, β need not admit a Jordan decomposition and thus the corresponding Lebesgue theory does not also hold for strict β-integrals.

The following two results will illuminate the structure:

Theorem 1. *Let $(\Omega_i, \Sigma_i), i = 1, 2$ and β be as above. If $f_{in} : \Omega_i \to \mathbb{C}$, is Σ_i-measurable, $n \geq 1$, and $f_{in} \to f_i$ pointwise as $n \to \infty$, suppose that $|f_{in}| \leq g_i, i = 1, 2$ where (g_1, g_2) is strictly β-integrable. Then both (f_{1n}, f_{2n}) and (f_1, f_2) are β-integrable and*

$$\int_A \int_B^* f_1(\omega_1) f_2(\omega_2)\beta(d\omega_1, d\omega_2) = \lim_{n\to\infty} \int_A \int_B^* f_{1n}(\omega_1) f_{2n}(\omega_2)\beta(d\omega_1, d\omega_2). \tag{13}$$

In particular, if $\{f_{in}, n \geq 1\}$ is bounded, then (13) *holds so that the bounded convergence criterion is valid.*

A proof of this result is given in Chang and Rao [(1986), p.45]. It is of interest to note that when β is also positive definite with $\Omega_1 = \Omega_2$ and $\Sigma_1 = \Sigma_2 = \Sigma$, so that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{a}_j\beta(A_i, A_j) \geq 0, A_i \in \Sigma, a_i \in \mathbb{C}, \tag{14}$$

one can obtain a (simpler) characterization of the strict β-integral through its relation to the D-S definition. This is given in Part II below. On the other hand, if $\beta : \Sigma_1 \times \Sigma_2 \to \overline{\mathbb{R}}^+$, then the variations of Vitali and Fréchet coincide, and the theory simplifies; and β essentially extends to a measure on $\Sigma_1 \otimes \Sigma_2$. More precisely one has:

Theorem 2. *Let $(\Omega_i, \Sigma_i), i = 1, 2$ be Borelian spaces where Ω_i is a Hausdorff space. If $\beta : \Sigma_1 \times \Sigma_2 \to \overline{\mathbb{R}}^+$ is a Radon bimeasure,* i.e. *$\beta(K_1, K_2) < \infty$ for each compact set $K_i \subset \Omega_i$ and is inner regular in the sense that*

$$\beta(A, B) = \sup\{\beta(K_1, K_2) : K_1 \subset A, K_2 \subset B, K_i \text{ compact}\}, \tag{15}$$

$A \in \Sigma_1, B \in \Sigma_2$, then β admits an extension to a Radon measure μ on $\Sigma_1 \otimes \Sigma_2$ so that $\beta(A, B) = \mu(A \times B)$.

The method of proof is standard but several details have to be filled in. These are available from Berg, Christensen, and Ressel [(1985), p.24]. This result allows an immediate extension of the Fubini-Tonelli type theorems for noncartesian product measures. In particular, this result gives a characterization of the *positive* elements of $V(\Omega_1, \Omega_2)^*$ of Section 1.1 above. The general set $V(\Omega_1, \Omega_2)^*$ will now be examined if the Ω_i have a group structure in addition.

1.4 *Bimeasure algebras on locally compact groups.* A refinement and specialization of the preceding work for locally compact groups Ω_i is of interest not only for applications of harmonizable random fields, but also because it generalizes the study of the classical group algebra $M(\Omega_1 \times \Omega_2)$ and unifies other results. Thus if β is a bimeasure on $\Sigma_1 \times \Sigma_2$ with Ω_i as LCA groups, one can define the Fourier transform $\widehat{\beta}$ of β by the formula (with the strict integrals on which "$*$" is dropped):

$$\widehat{\beta}(\gamma, \delta) = \int_{\widehat{\Omega}_1}\int_{\widehat{\Omega}_2} < \gamma, \gamma' >< \overline{\delta, \delta'} > \beta(d\gamma', d\delta'), \tag{16}$$

for $(\gamma,\gamma',)\in\Omega_1\times\Omega_2,<\gamma,\gamma\prime>$ being the duality pairing ($\widehat{\Omega}_i$ is the dual of Ω_i). It follows from (5) and (16) that

$$\|\widehat{\beta}\|_\infty=\sup\{|\widehat{\beta}(\gamma,\delta)|:\gamma\in\Omega_1,\delta\in\Omega_2\}\le\|\beta\|,\beta\in BM(\Omega_1,\Omega_2). \tag{17}$$

In a similar way one can define the *convolution* operation in the space of bimeasures $BM(\Omega_1,\Omega_2)$, denoted $\beta_1*\beta_2$, by

$$(\beta_1*\beta_2)(A,B)=\int_{\Omega_1}\int_{\Omega_2}\beta_1(A-\gamma,B-\delta)\beta_2(d\gamma,d\delta), \tag{18}$$

for each pair β_1,β_2 in $BM(\Omega_1,\Omega_2)$, and $A\in\Sigma_1,B\in\Sigma_2$. It is not hard to see that $\beta_1*\beta_2\in BM(\Omega_1,\Omega_2)$, and $\|\beta_1*\beta_2\|\le\|\beta_1\|\|\beta_2|$. However, a more refined analysis is possible only after proving a uniqueness theorem for the bimeasure Fourier transform and an employment of Grothendieck's inequality. One form of the latter states that for each $\beta\in BM(\Omega_1,\Omega_2)(\Omega_i$ need not be groups for this) there exists a pair of Radon probability measures μ_1,μ_2 on Ω_1,Ω_2 such that

$$\left|\int_{\Omega_1}\int_{\Omega_2}f(\omega_1)g(\omega_2)\beta(d\omega_1,d\omega_2)\right|^2\le C\int_{\Omega_1}|f(\omega_1)|^2\mu_1(d\omega_1)\int_{\Omega_2}|g(\omega_2)|^2\mu_2(d\omega_2), \tag{19}$$

where C is an absolute constant. If $\Omega_1=\Omega_2$, one may choose $\mu_1=\mu_2$. Using (19), Graham and Schreiber (1984) have made a detailed study of BM (Ω_1,Ω_2) for the LCA groups Ω_i, where $\widehat{\beta}$ and the convolution are defined differently. With the MT-integral one can show that both these definitions agree. Moreover, the work of these authors shows that $\|\beta_1*\beta_2\|\le C^2\|\beta_1\|\|\beta_2\|$, so that $BM(\Omega_1,\Omega_2)$ is a Banach algebra with this norm constant. Using still different techniques Ylinen (1987) recently showed that, with an *equivalent* norm, one can take $C=1$ in the last inequality so that $BM(\Omega_1,\Omega_2)$ is a (standard) Banach algebra. It may be noted that (18) extends to noncommutative groups without any change.

If $VM(\Omega_1,\Omega_2)$ is the subspace of $BM(\Omega_1,\Omega_2)$ consisting of those bimeasures of finite Vitali variation, then it is known (and easily verified) that $(BM(\Omega_1,\Omega_2),\|\cdot\|)$ and $(VM(\Omega_1,\Omega_2),|\cdot|)$ are Banach spaces. Since by (8) and (9), $\|\cdot\|\le|\cdot|$, a question of interest here is about the density of $VM(\Omega_1,\Omega_2)$ in the topology of the latter. This was raised in Chang and Rao [(1986), p. 33], but a negative solution is obtained from Graham and Schreiber [(1984), corollary 5.10], when Ω_1,Ω_2 are groups. This involves a delicate analysis.

To appreciate the structure of $BM(\Omega_1, \Omega_2)$, which is of interest even if $\Omega_1 = \Omega_2 = I\!R$ in stochastic theory, the above solution and a related result will be presented.

Let Ω_i be an LCA group with Γ_i as its dual group, $i = 1, 2$. Let $S(\Gamma_1, \Gamma_2) = BM(\Omega_1, \Omega_2)\hat{} = \{\widehat{\beta} : \beta \in BM(\Omega_1, \Omega_2)\}$, and $\Delta(\Omega) = \{(\omega, \omega) : \omega \in \Omega\}$ the diagonal set of $\Omega \times \Omega$. Then one has:

Theorem 1. *If $f : \widehat{G} \to \mathbb{C}$ is a uniformly continuous bounded function on the dual of an LCA group G, then there is a $\widehat{\beta} \in S(\widehat{G}, \widehat{G})$ such that $\widehat{\beta}|\Delta(\widehat{G}) = f$, i.e.,*

$$\widehat{\beta}(x, x) = f(x), x \in \widehat{G}. \tag{20}$$

If further G is nondiscrete, then $VM(G, G)$ is not dense in $BM(G, G)$ in the (norm) topology of the latter.

A proof of this result is based on several other propositions, and is given in the above authors' paper. In passing one should note another fact about $BM(\Omega_1, \Omega_2)$. An element $\beta \in BM(\Omega_1, \Omega_2)$ is termed *continuous* or *diffuse* if $\beta(A_1, A_2) = 0$ for all finite sets $A_i \subset \Omega_i$, and *discrete* if there are increasing sequences of finite sets A_{in} such that, letting $\beta_n = \beta|\Sigma_1(A_{1n}) \times \Sigma_2(A_{2n})$, then $\|\beta - \beta_n\| \to 0$ as $n \to \infty$. With these concepts the following result, from Graham and Schreiber (1984-88) and Gilbert, Ito, and Schreiber (1985), clarifies the structure of $BM(\Omega_1, \Omega_2)$ further.

Theorem 2. *Let Ω_i be a locally compact space, $i = 1, 2$, and $BM(\Omega_1, \Omega_2)$ be the Banach space of bimeasures on (Σ_1, Σ_2) as before. Then one has:*

(i)

$$BM(\Omega_1, \Omega_2) = BM_c(\Omega_1, \Omega_2) \oplus BM_d(\Omega_1, \Omega_2) \tag{21}$$

where $BM_c(\Omega_1, \Omega_2)(BM_d(\Omega_1, \Omega_2))$ is the set of diffuse (discrete) bimeasures of $BM(\Omega_1, \Omega_2)$ which is a closed subspace. Further, the mapping $Q : \beta \longmapsto \beta_c, \beta \in BM(\Omega_1, \Omega_2)$ is a norm-decreasing projection whose kernel is $BM_d(\Omega_1, \Omega_2)$.

(ii) *If Ω_1, Ω_2 are also groups, then $BM_c(\Omega_1, \Omega_2)$ is a closed ideal, and if $BM_a(\Omega_1, \Omega_2) \subset BM_c(\Omega_1, \Omega_2)$ is the set of bimeasures of finite Vitali variation whose extensions are absolutely continuous relative to a (left) Haar measure on $\Omega_1 \times \Omega_2$ then $BM_a(\Omega_1, \Omega_2)$ is also a closed ideal in $BM(\Omega_1, \Omega_2)$ onto which there is no bounded projection.*

It may be observed that although $BM_a(\Omega_1, \Omega_2)$ is an $(AL-)$ space $BM(\Omega_1, \Omega_2)$ is not, and the decomposition (21) does not imply norm additivity. A simple counter example is given to this effect in the first of the above papers.

Using these ideas one may study algebras of multimeasures. In fact, Voropoulos (1968) considered the space $V(\Omega_1, \ldots, \Omega_n) = \widehat{\otimes}_{1 \leq i \leq n} C_0(\Omega_i)$, with $C_0(\Omega_i)$ replaced by $L^\infty(\Omega_i)$, under the name tensor algebras, and calling the members of $V(\Omega_1, \ldots, \Omega_n)^*$ the n-linear forms and multimeasures. Using a procedure that is similar to that of the bimeasure case discussed earlier, one can consider Fourier transforms of $u \in V(\Omega_1, \ldots, \Omega_n)^*$ where Ω_i is an LCA group. Thus $\widehat{u}$ is given by

$$\widehat{u}(\gamma_1, \ldots, \gamma_n) = < u, \gamma_1 \otimes \cdots \otimes \gamma_n >, \gamma_i \in C_b(\widehat{\Omega}_i), \tag{22}$$

$\widehat{\Omega}_i$ being the dual group of Ω_i. However, this extension brings in some unpleasant phenomenon. For instance, if $n = 3$, it was observed by Graham and Schreiber (1987) that $\widehat{u}$ need not be uniformly continuous, and that for $u, v \in V(\Omega_1, \Omega_2, \Omega_3)^*$, $\widehat{u}\widehat{v}$ need not be a Fourier transform, in contrast to the bimeasure case. Thus several new problems arise in this extension. These authors also considered questions of sets of interpolation, and many other problems of the classical harmonic analysis can be studied on these spaces.

1.5 *Positive definite bimeasures.* Although there are many applications, such as those in martingale theory and operator representations, of general bimeasures one finds the positive definite class playing a key role in second order stochastic processes. Such a bimeasure β satisfies the inequality (14). This restriction allows an introduction of inner product structure into the class of β-integrable functions, and the resulting space plays a useful role in the spectral analysis of these processes to be detailed in Part II below. Hence these measures are discussed here.

The following result of Grothendieck's is needed (cf., Pisier (1986), p. 55, which can be stated in an alternative form for locally compact spaces):

Theorem 1. *Let (Ω_i, Σ_i) be Borelian spaces with Ω_i locally compact, $i = 1, 2$, and $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ be a bimeasure. Then there exists a pair of Radon probability measures $\mu_i : \Sigma_i \to \mathbb{R}^+$ such that*

$$|B(f_1, f_2)| = \left| \int_{\Omega_1} \int_{\Omega_2} f_1(\omega_1) f_2(\omega_2) \beta(d\omega_1, d\omega_2) \right|$$

$$\leq K_G\|B\| \, \|f_1\|_{2,\mu_1}\|f_2\|_{2,\mu_2}, \tag{23}$$

for all $f_i \in C_0(\Omega_i), i = 1,2,$ *with* K_G *as an absolute constant. Here* $\|B\| = \|\beta\|$, *where* $B(\cdot,\cdot)$ *is the bilinear form determined by* β *through the* MT*-integral.*

As a consequence of (23), one can extend B to a bounded bilinear form on $L^2(\Omega_1, \Sigma_1, \mu_1) \times L^2(\Omega_2, \Sigma_2, \mu_2)$. Hence this extended B denoted by the same symbol, can be expressed as:

$$B(f,g) = (Tf, g) = \int_{\Omega_2} (Tf) g d\mu_2 \tag{24}$$

for a bounded linear operator $T : L^2(\mu_1) \to L^2(\mu_2)$, and then

$$\begin{aligned} \|T\| &= \sup\{|B(f,g)| : \|f\|_{2,\mu_1} \leq 1, \|g\|_{2,\mu_2} \leq 1\} \\ &\leq K_G\|B\|, \text{ by (23)}. \end{aligned} \tag{25}$$

In the following, let $\Omega = \Omega_1 = \Omega_2$ and $\mu = \mu_1 = \mu_2$ so that μ is a finite Radon measure ($\mu(\Omega) = C$), and $C_0(\Omega)$ is dense in $L^2(\Omega, \Sigma, \mu)$. So T is still a bounded linear operator on $L^2(\mu)$ with bound CK_G in (25). If β is moreover positive definite, then so is (B and) T. Hence T has a positive square root $T^{1/2}$ on $L^2(\mu)$. (The best known estimate of K_G, due to J.L. Krivine, is: $K_G \leq 1 \cdot 782$, to three places.) These facts are used in our applications.

Let $\mathcal{L}^2 = \{f \in \mathbb{C}^\Omega : f \text{ is } \mu\text{-measurable}, \|f\|^2 = B(f, \bar{f}) < \infty\}$ where B and β are positive definite. Then $\|f\|^2 = (T^{1/2}f, T^{1/2}f) = \|T^{1/2}f\|_2^2$ and $\|\cdot\|$ is a norm on $\mathcal{L}^2$. In fact, $\|f\| = 0$ implies $T^{1/2}f = 0$ a.e., and since T (hence $T^{1/2}$) is a bounded positive (hence invertible) operator, $f = 0$ a.e. $[\mu]$. Also $\|f_1 + f_2\| = \|T^{1/2}(f_1 + f_2)\|_2 \leq \|T^{1/2}f_1\|_2 + \|T^{1/2}f_2\|_2 = \|f_1\| + \|f_2\|$, and $\|af\| = \|aT^{1/2}f\|_2 = |a| \, \|T^{1/2}f\|_2 = |a| \, \|f\|$. Let $\{f_n, n \geq 1\} \subset \mathcal{L}^2$ be a Canchy sequence so that $\|f_m - f_n\| = \|T^{1/2}f_m - T^{1/2}f_n\|_2 \to 0$. By the completeness of $L^2(\mu)$, there is a $g \in L^2(\mu)$ such that $\|T^{1/2}f_n - g\|_2 \to 0$. Let $f = T^{-1/2}g$. Then f is in $\mathcal{L}^2$ since $\|f\| = \|T^{1/2}f\|_2 = \|g\|_2 < \infty$, the range of T being $L^2(\mu)$. Also

$$\|f_n - f\| = \|T^{1/2}f_n - T^{1/2}f\|_2 = \|T^{1/2}f_n - g\|_2 \to 0 \text{ as } n \to \infty. \tag{26}$$

Hence $f_n \to f$ in $\mathcal{L}^2$. This proves the following result:

Theorem 2. *Let* $\{\mathcal{L}^2, \|\cdot\|\}$ *be the space of scaler Borel functions on* Ω *with* β *as a positive definite bimeasure, introduced above. Then it is a complete inner product space, i.e., a Hilbert space in the usual sense.*

This fact will be of interest in connection with the covariance analysis of harmonizable processes, and $\mathcal{L}^2$ will be termed the spectral domain space there. It is somewhat curious that the result should depend on Grothendieck's theorem so intimately.

In the preceding argument the positive definiteness of β is used. If it is semi-definite, then one has to extend it using, for instance, the concept of a generalized inverse of operators to achieve a similar conclusion or by considering a suitable quotient space. If β is merely a bimeasure, then also (24) and (25) are available, but the argument for (26) does not work. Here one may perhaps use the polar decomposition of T in (24) and manipulate the rest of the analysis. This will not be considered here, since it is not necessary for the stochastic applications below.

1.6 *Vector bimeasures.* In some applications, including the analysis of noncommutative harmonizable fields in the next part, one needs to employ vector valued bimeasures also. As a straightforward extension of the scalar case, which however has some new problems to deal with, the concept may be introduced as follows. Thus let $(\Omega_i, \Sigma_i), i = 1, 2$, be measurable spaces, $\mathcal{X}$ a Banach space, and $\beta : \Sigma_1 \times \Sigma_2 \to \mathcal{X}$ be a mapping. Then β is termed a *vector* (or $\mathcal{X}$-valued) *bimeasure* if $\beta(\cdot, B)$ and $\beta(A, \cdot)$ are vector measures for each $A \in \Sigma_1, B \in \Sigma_2$. The β-integral can be given, generalizing Sections 1.2 and 1.3, as follows.

Definition 1. Let $(\Omega_i, \Sigma_i), i = 1, 2, \mathcal{X}$ and β be as above. If $f_i : \Omega_i \to \mathbb{C}$ is Σ_i-measurable, then (f_1, f_2) is said to be *strictly β-integrable* provided the following four conditions hold:

(i) for each $A \in \Sigma_1, B \in \Sigma_2, f_1$ is $\beta(\cdot, B)$-and f_2 is $\beta(A,\cdot)$-integrable (D-S sense for vector measures),

(ii) $\tilde{\beta}_1^F : A \longmapsto \int_F f_2(\omega_2)\beta(A, d\omega_2), \tilde{\beta}_2^E : B \longmapsto \int_E f_1(\omega_1)\beta(d\omega_1, B)$, exist as vector measures for each $E \in \Sigma_1, F \in \Sigma_2$,

(iii) f_1 is $\tilde{\beta}_1^F$-and f_2 is $\tilde{\beta}_2^E$-integrable in the D-S sense for each pair $E \in \Sigma_1, F \in \Sigma_2$; and

(iv) $\int_E f_1(\omega_1)\tilde{\beta}_1^F(d\omega_1) = \int_F f_2(\omega_2)\tilde{\beta}_2^E(d\omega_2)\Big[= \int_E \int_F^* f_1 f_2 \beta(d\omega_1, d\omega_2)\Big]$
where the common value is symbolically denoted by the double integral.

If (iv) is required to hold only for $E = \Omega_1, F = \Omega_2$, then (f_1, f_2) is termed simply β-*integrable.* This is the vector MT-integral. In case $\Omega_1 = \Omega_2, f$ is (strictly) β- *integrable* if (f, f) is such.

It should be noted that these concepts are of interest even in the integral (measure kernel) representation of bilinear operators with analogs in the multilinear theory. Some of these aspects are actively studied by abstract analysts (cf. e.g., Dobrakov (1987) with references given there to his and others' works). For instance, a vector bimeasure β has finite Fréchet variation in that

$$\|\beta\|(\Omega_1, \Omega_2) = \sup\{\| \sum_{i,j=1}^{n} a_j \bar{b}_j \beta(A_i, B_j)\|_{\mathcal{X}} : A_i \in \Sigma_1, B_j \in \Sigma_2, \text{ disjoint}, |a_i| \leq 1, |b_j| \leq 1, \text{ scalers }, n \geq 1\} < \infty. \tag{27}$$

This follows from the earlier case since $\ell o \beta$ is a scalar bimeasure with finite Fréchet variation for each $\ell \in \mathcal{X}^*$, and then the uniform boundedness principle applies. It follows that all bounded measurable functions on $(\Omega_i, \Sigma_i), i = 1, 2$, are β-integrable (in either sense). Some extensions of the scalar results for vector bimeasures have been given by Ylinen (1978), culminating in the following analog of the Riesz-Markov representation.

Theorem 2. *Let Ω_i be locally compact, $(\Omega_i, \mathcal{B}_i)$ be Borelian, $C_0(\Omega_i)$ be continuous scalar functions vanishing at "∞", and $\mathcal{X}$ be a reflexive Banach space. If $B : C_0(\Omega_1) \times C_0(\Omega_2) \to \mathcal{X}$ is a bounded bilinear form then there exists a unique vector bimeasure $\beta : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{X}$, regular in each component, such that*

$$B(f_1, f_2) = \int_{\Omega_1} \int_{\Omega_2} f_1(\omega_1) f_2(\omega_2) \beta(d\omega_1, d\omega_2), f_i \in C_0(\Omega_i), \tag{28}$$

and

$$\|B\| = \sup\{\|B(f_1, f_2)\|_{\mathcal{X}} : \|f_i\|_\infty \leq 1, i = 1, 2\} = \|\beta\|(\Omega_1, \Omega_2). \tag{29}$$

If $\mathcal{X}$ is not reflexive, then the same conclusion holds provided B is assumed to map bounded sets of $V(\Omega_1, \Omega_2) = C_0(\Omega_1) \widehat{\otimes} C_0(\Omega_2)$ into relatively weakly compact sets.

This is an extension of a classical theorem [cf. Dunford and Schwartz (1958), VI.7.3], and the details are given by Ylinen (1978). The result admits a generalization to the case that the $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Banach spaces with a bilinear mapping on $\mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$. Then $B : C_0(\Omega_1, \mathcal{Y}) \times C_0(\Omega_2, \mathcal{Z}) \to \mathcal{X}$ can be studied for a similar representation as in (28). In case $\Omega_1 = \Omega_2 = \Omega$ is a group one can also extend the theory of bimeasure algebras of Section 1.4. These are potential areas of research and some of these ideas will appear in Section 2.3 below.

1.7 *Bimeasures originating from classical problems.* Applications of non-negative bimeasures appear frequently in classical analysis. Some of these are included in the book by Berg, Christensen and Ressel (1984). A natural place is the theory of (regular) conditional probability functions. For instance, if $b : \Omega \times \Sigma \to \mathbb{R}^+$ is such that $b(\omega, \cdot)$ is a probability and $b(\cdot, A)$ is measurable relative to a σ-algebra $\mathcal{B}$ of Ω (need not be related to Σ), then

$$\tilde{\beta}^\bullet(A, B) = \int_A b(\omega, B)\mu(d\omega), A \in \mathcal{B}, B \in \Sigma, \tag{30}$$

where μ is a probability on $\mathcal{B}$, defines $\tilde{\beta}^\bullet : \mathcal{B} \times \Sigma \to \mathbb{R}^+$ as a positive bimeasure. It may be identified as the restriction of a measure on $\mathcal{B} \otimes \Sigma$. If $\mathcal{B} \subset \Sigma$, then $\tilde{\beta}^\bullet(A, B) = \mu(A \cap B)$, and such a $\tilde{\beta}^\bullet$ is familiar. From another point of view (30) is of interest also. If ν denotes a measure on $\mathcal{B} \otimes \Sigma$ extending $\tilde{\beta}^\bullet$, of the form (30), it is said to be disintegrated into a family $\{b(\omega, \cdot), \omega \in \Omega\}$. An application of bimeasure theory to the disintegration problem has recently been considered by Calbrix (1981), and much further work remains to be done. Other applications of bimeasures, of finite Vitali variation, to planar semi-martingales have been considered by Merzbach and Zakai (1986), and earlier in a special case by Horowitz (1977). By specializing $(\Omega_i, \Sigma_i), i = 1, 2$, and restricting the bimeasures suitably, a detailed analysis was presented by M. Cotlar and his associates (cf. (1982), and references there). It has a close relation to Toeplitz matrices. An indication will be given here for comparison, and to show its potential.

Suppose $\Omega = \mathbb{Z}$ and $\mathbb{Z}_1 = \{n \in \mathbb{Z} : n \geq 0\} = \mathbb{Z}^+, \mathbb{Z}_2 = \mathbb{Z} - \mathbb{Z}^+$. Let $\beta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be defined by a self-adjoint matrix such that in each quadrant $\mathbb{Z}_i \times \mathbb{Z}_j, \beta$ can be expressed as

$$\beta(m, n) = \rho_{ij}(m - n), (m, n) \in \mathbb{Z}_i \times \mathbb{Z}_j, 1 \leq i, j \leq 2. \tag{31}$$

Then $\beta = (\beta(m, n) : (m, n) \in \mathbb{Z} \times \mathbb{Z})$ is called a *generalized Toeplitz kernel* (and it is the Toeplitz kernel if $\rho_{ij} = \rho$ all i, j). The studies based on stationary random sequences whose covariance functions turn out to be positive definite Toeplitz kernels, is a motivation for the present investigation. Thus a generalized Toeplitz kernel is said to be positive definite if for each vector $\underline{a}_i = (a_i(1), a_i(2), \ldots, a_i(k)), i = 1, 2$ with $a_i(k) \in \mathbb{C}$, the following quadratic form is positive for each $k \geq 1$:

$$[\underline{a}_1\ \underline{a}_2] \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} \underline{a}_1^* \\ \underline{a}_2^* \end{bmatrix} \geq 0, \tag{32}$$

where ρ_{ij} is given by (31). If M is the 2-by-2 block matrix of ρ_{ij}'s, then one can associate (non-uniquely) a matrix of complex measures $\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}$ on the torus. This representation leads to a semi-group theory and the corresponding convolution, with the Lebesgue-Stieltjes integration, has a parallel analysis developed by Professor M. Cotlar and his associates, who also have obtained an interesting stochastic application, generalizing the stationary sequences.

Thus the bimeasure theory is sufficiently flexible, and general enough, that a variety of applications are possible. Since the interest here is in probability, let us concentrate on that aspect and utilize the bimeasure theory which is developed thus far.

Part II: Harmonizability and Integral Representations

2.1 *Concepts and classification.* Let $L_0^2(P)$ be the usual Hilbert space of square integrable centered random variables on a probability space (Ω, Σ, P). A mapping $X : \mathbb{R} \to L_0^2(P)$ is called a second order process with covariance function $r : (s,t) \longmapsto E(X_s\overline{X}_t) = \int_\Omega X_s\overline{X}_t dP$. The process can be classified according to the structure of r. If r is continuous on $\mathbb{R}^2$ and is of the form $r(s,t) = \tilde{r}(s-t)$, then the process is called *weakly* (or *Khintchine*) *stationary.* But by a classical theorem of Bochner such $\tilde{r}$ is representable as

$$\tilde{r}(s-t) = \int_{\mathbb{R}} e^{i(s-t)\lambda}\mu(d\lambda), \tag{1}$$

for a unique bounded Borel measure μ on $\mathbb{R}$. Less restrictively, suppose only that the continuous covariance r can be expressed as:

$$r(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda - it\lambda'}\beta(d\lambda, d\lambda'), \tag{2}$$

for a (unique) positive definite bimeasure β on $\mathbb{R}^2$ which is of finite Vitali variation. A process whose covariance admits the representation (2) is called *strongly harmonizable.* This class was introduced by Loève (1947). If β in (2) concentrates on the diagonal $\lambda = \lambda'$ so that $\beta(A,B) = \mu(A\cap B)$, then it reduces to (1) so that each weakly stationary process is strongly harmonizable. However, there are simple strongly harmonizable processes that are not stationary. For instance, if $\hat{f}$ is the Fourier transform of a Lebesgue integrable function f on $\mathbb{R}$, then $r(s,t) = \hat{f}(s)\overline{\hat{f}}(t)$ defines the covariance of such a process.

Suppose now that β of (2) is a bimeasure which is thus only of Fréchet variation finite (but its Vitali variation is infinite). Then the integral in (2) cannot be given in the Lebesgue sense. It can be defined as the (even strict-) MT-integral. When this is adapted, then the corresponding process is termed *weakly harmonizable.* This concept was introduced by Bochner

(1956) in a more general form under the name "V-bounded" process, and by Rozanov (1959) under the name "harmonizable", a term which was used by Loève in the more restricted (the strong) sense. Thus weak stationarity is stronger than strong harmonizability which in turn is stronger than weak harmonizability.

The preceding concepts extend immediately if $\mathbb{R}$ is replaced by any LCA group G. Thus if $\widehat{G}$ is the dual group of G, then $X : G \to L_0^2(P)$ is *weakly (or strongly) harmonizable* accordingly as its covariance $r(g_1, g_2) = E(X_{g_1}\overline{X}_{g_2})$ is representable as ($< g, \cdot > \in \widehat{G}$ being a character of G):

$$r(g_1, g_2) = \int_{\widehat{G}} \int_{\widehat{G}} < g_1, \gamma_1 >< \overline{g_2, \gamma_2} > \beta(d\gamma_1, d\gamma_2), \tag{3}$$

where β is a positive definite bimeasure, on $\mathcal{B}(\widehat{G}) \times \mathcal{B}(\widehat{G})$, of Fréchet (or finite Vitali) variation. If $G = \mathbb{R}^n$ or $\mathbb{Z}^n, n > 1$. Then $\{X_g, g \in G\}$ is usually called a random field. The latter term is used here for any locally compact space G, indexing the family.

It is of interest to note a few related second order random functions before proceding to the integral representation of harmonizable fields, since this helps in a better understanding of the subject. Thus let T be an index set and $\{X_t, t \in T\} \subset L_0^2(P)$ be a family with covariance $r(s,t) = E(X_s\overline{X}_t)$. If $\mathcal{B}$ is a σ-algebra of subsets of T, then the family is said to be of *class* (C) (Cramér class), if there is a measurable space $(S, \mathcal{S})$ and a positive definite bimeasure $\beta : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$, of finite Vitali variation such that

$$r(s,t) = \int_S \int_S g_s(\lambda)\overline{g_t(\lambda')}\beta(d\lambda, d\lambda'), s, t \in T, \tag{4}$$

relative to a collection $\{g_s, s \in T\}$ of (Lebesgue) β-integrable scalar functions so that $r(s,s) < \infty$ for each $s \in T$. If β has only a finite Fréchet variation and the integral is the strict β-integral, then the corresponding family is of *weak class* (C). Clearly this reduces to the strong or weak harmonizability if $S = T = \mathbb{R}, g_s(\lambda) = e^{is\lambda}$. The class (4) (with $S = T = \mathbb{R}$) was introduced by Cramér (1951). If β in (4) concentrates on the diagonal of $S \times S$, so that $\beta(A, B) = \mu(A \cap B)$ for some positive finite (or σ-finite) measure μ on $\mathcal{S}$. Then it becomes

$$r(s,t) = \int_S g_s(\lambda)\overline{g_t(\lambda)}\mu(d\lambda), s, t \in T, \tag{5}$$

and the corresponding family is of *Karhunen class* introduced by him in 1947. Again if $S = T = \mathbb{R}, g_s(\lambda) = e^{is\lambda}$, then it reduces to the stationary class. Similarly, if $T = G$, an LCA group, $S = \widehat{G}$ and $g_s(\lambda) =< s, \lambda >$, then (3) is recovered. Beyond these identifications one has the following nontrivial result.

Proposition 1. *Every harmonizable random field $X : G \to L_0^2(P), G$ is an LCA group, belongs to a Karhunen class. More explicitly, if the given family is weakly harmonizable, then there is a finite Borel measure μ on $\widehat{G}$ and a suitable family $\{g_s, s \in G\} \subset L^2(\widehat{G}, \mu)$, such that* (5) *holds with $T = G$ and $S = \widehat{G}$ there.*

A proof of this result and certain other related extensions of strong harmonizability may be found in, e.g., Rao (1985).

2.2 *Integral representation of harmonizable fields on LCA groups.* For the integral representations it will be helpful to restate precisely the D-S integral of a scalar function relative to a vector measure. Thus if (Ω, Σ) is a measurable space, $f : \Omega \to \mathbb{C}$ is measurable for $\Sigma, Z : \Sigma \to \mathcal{X}$ (a Banach space) is a vector measure then f is D-S integrable relative to Z whenever the following two conditions hold:

(i) there is a sequence $f_n : \Omega \to \mathbb{C}$ of simple (measurable for Σ) functions such that $f_n \to f$ pointwise, and

(ii) if $f_n = \sum_{i=1}^{k_n} a_i^n \chi_{A_i^n}, \int_E f_n dZ = \sum_{i=1}^{k_n} a_i^n Z(E \cap A_i^n) \in \mathcal{X}$, then $\{\int_E f_n dZ, n \geq 1\}$ is a Canchy sequence in $\mathcal{X}, E \in \Sigma$.

Then the unique limit of this sequence in $\mathcal{X}$ is denoted $\int_E f dZ, E \in \Sigma$. It is standard (but not trivial) to show that the D-S integral is a uniquely defined element of $\mathcal{X}$, is linear, and the dominated convergence theorem is valid for it. However, if $\mathcal{X}$ is infinite dimensional, then the D-S integral should *not* be confused with the Lebesgue-Stieltjes integral, and the evaluation of $\int_E f dZ$ as a Stieltjes integral is generally *false*. Also the convergence in (i) is pointwise, and strengthening it to uniformity restricts the generality of the D-S integral. These points should be kept in mind in its applications.

Now let $L^1(Z)$ denote the space of scalar functions on $(S, \mathcal{S})$, D-S integrable relative to Z, and $\mathcal{L}_*^2(\beta)$ be the collection of strictly β-integrable (MT-integration) $f : S \to \mathbb{C}$ where $\beta : (A, B) \longmapsto E(Z(A)\overline{Z(B)})$ is the bimeasure associated with Z, when $\mathcal{X} = L_0^2(P)$ on a probability space (Ω, Σ, P). In this case $Z(\cdot)$ is called a *stochastic measure* and β its *spectral bimeasure* of a second order process related by the following result.

Theorem 1. *Let $(S, \mathcal{S})$ be a measurable space and $\beta : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ be a positive definite bimeasure. Then there exists a probability space (Ω, Σ, P) and a stochastic measure $Z : \Sigma \to L_0^2(P)$ such that*

(i) $E(Z(A)\overline{Z(B)}) = \beta(A, B)$ *for all* $A, B, \in \mathcal{S}$, *and*

(ii) $L^1(Z) = \mathcal{L}_*^2(\beta)$, *equality as sets of functions.*

This result can be established quickly by using the Aronszajn theory of reproducing kernels. Then it is used in representing second order random fields. A general form of the latter is obtained as follows. If $(S, \mathcal{S})$ is a Borelian space, S being a topological space, a bimeasure β on $\mathcal{S} \times \mathcal{S}$ is said to have *locally finite Fréchet* (or Vitali) variation if $\beta : \mathcal{S}(E) \times \mathcal{S}(E) \to \mathbb{C}$ has finite Fréchet (or respectively Vitali) variation for each bounded Borel set $E \subset S$ (i.e., E is included in a compact set). [Regarding the clear distinction of these concepts, see also Edwards (1955).] Then the following general representation, to be specialized later, holds:

Theorem 2. *Let $(S, \mathcal{S})$ be a Borelian space with S locally compact. Suppose that $\{X_t, t \in T\} \subset L_0^2(P)$, on a probability space (Ω, Σ, P), is a (locally) weakly class (C) process relative to a positive definite bimeasure $\beta : \mathcal{S}_0 \times \mathcal{S}_0 \to \mathbb{C}$ of (locally) finite Fréchet variation and a family $g_t : S \to \mathbb{C}, t \in T$, of functions each of which is (locally) strictly β-integrable, where $\mathcal{S}_0$ is the δ-ring of bounded (Borel) sets of $\mathcal{S}$. Then there exists a σ-additive $Z : \mathcal{S}_0 \to L_0^2(P)$ such that (T being an index set)*

(i) $X_t = \int_S g_t(\lambda) Z(d\lambda), t \in T$, (D-S *integral*)

(ii) $E(Z(A)\overline{Z(B)}) = \beta(A, B), A, B \in \mathcal{S}_0$.

Conversely, if $\{X_t, t \in T\}$ is defined by (i) *for a stochastic measure Z, then the process is of (local) weak class* (C) *relative to a bimeasure β given by* (ii) *and the g_t of* (i) *being (locally) strictly β-integrable. The process is of (local) Karhunen class iff* (i) *and* (ii) *hold with $\beta(A, B) = \mu(A \cap B)$ for a σ-finite measure μ on $\mathcal{S}$.*

In fact if $K \subset S$ is a compact set, consider the trace $\mathcal{S}(K)$, of $\mathcal{S}$, on K which is a σ-algebra and $\beta : \mathcal{S}(K) \times \mathcal{S}(K) \to \mathbb{C}$ is a positive definite bimeasure for which the preceding theorem applies. If $\tilde{Z} : \mathcal{S}(K) \to L_0^2(P)$ is the representing stochastic measure then one has

$$\int_K g_t(\lambda) \tilde{Z}(d\lambda) = j\Big(\int_K g_t(\lambda) \beta(d\lambda, \cdot)\Big) \in L_0^2(P), \tag{6}$$

where j is the isometric isomorphism between $\tilde{Z}$ and β guaranteed by that result. By the local compactness of S, one can define a vector measure $Z : \mathcal{S} \to L_0^2(P)$ and extend (6) uniquely using a familiar procedure (cf. Hewitt and Ross (1963), pp. 133-134). Without local compactness of S this method of piecing together does not work. From here on the details are as in Chang and Rao [(1986), p. 53].

Since the functions $\{g_t, t \in T\}$ are not explicitly given in the above case, and are somewhat arbitrary, it will be interesting to specialize the result for harmonizable and stationary fields and show how these functions are naturally obtained in their representations.

Theorem 3. *Let G be an LCA group and $\{X_t, t \in G\} \subset L_0^2(P)$ be given. Then this family is weakly (resp. strongly) harmonizable relative to a positive definite bimeasure $\beta : \mathcal{B}(\widehat{G}) \times \mathcal{B}(\widehat{G}) \to \mathbb{C}$ (also of finite Vitali variation) iff there is a stochastic measure $Z : \mathcal{B}(\widehat{G}) \to L_0^2(P)$ such that*

$$\text{(i) } X_t = \int_{\widehat{G}} < t, \lambda > Z(d\lambda), t \in G, \text{ (D-S \textit{integral})} \tag{7}$$

where $< t, \cdot >$ is a character of G, and

$$\text{(ii) } E(Z(A)\overline{Z(B)}) = \beta(A, B), A, B \in \mathcal{B}(\widehat{G}). \tag{7'}$$

When these conditions are met the mapping $t \longmapsto X_t$ is strongly uniformly continuous in $L_0^2(P)$. Further the random field $\{X_t, t \in G\}$ is weakly stationary iff (i) *and* (ii) *hold with $\beta(A, B) = \mu(A \cap B)$ for a bounded Borel measure μ so that Z also has orthogonal increments.*

An obvious question is to extend this result when G is not necessarily abelian. However, this needs several new concepts. Let us start with a vector analog of the above theorem which will be useful in the desired extension. Thus if $\mathcal{X}$ is a reflexive Banach space and $X : T \to L_0^2(P; \mathcal{X})$ is an $\mathcal{X}$-valued strongly measurable process or field on (Ω, Σ, P) with $E(\|X_t\|^2) < \infty$, then it is termed of weakly class (C), Karhunen class, harmonizable or stationary accordingly as the scalar process or field $\ell(X) : T \to L_0^2(P)$ is of the corresponding class as defined before, for each $\ell \in \mathcal{X}^*$ and $\sup_t E(\|X_t\|^2) < \infty$. [If $\mathcal{X}$ is not reflexive, the last condition should be replaced by the relative weak compactness of $\{X_t, t \in T\}$ in $L_0^2(P; \mathcal{X})$, and the work extends. For simplicity the reflexive case is considered.]

To see how this is accomplished, let us discuss the harmonizable case, so that $\ell(X) : G \to$

$L_0^2(P), \ell \in \mathcal{X}^*$, admits a representation as in (7):

$$\ell(X_t) = \int_{\widehat{G}} < t, \lambda > Z_\ell(d\lambda), t \in G, \tag{8}$$

where Z_ℓ is a stochastic measure. The mapping $\ell \longmapsto Z_\ell$ is linear and Z_ℓ is a regular vector measure with semi-variation $\|Z_\ell\|(\widehat{G}) < \infty$. Moreover,

$$\|Z_\ell\|(\widehat{G}) = \sup_t \|\ell(X_t)\|_2 \leq \|\ell\| \sup_t \|X_t\|_2 < \infty,$$

since $X(G)$ is bounded. By the uniform boundedness, $\sup\limits_{\|\ell\|\leq 1} \|Z_\ell\|(\widehat{G}) < \infty$ and there is a $\tilde{Z}$ such that $Z_\ell = \ell(\tilde{Z})$, where $E(\tilde{Z}(A)) \in \mathcal{X}^{**} \cong \mathcal{X}$ by reflexivity, $A \in \Sigma$. Hence (8) can be expressed as

$$\ell(X_t) = \int_{\widehat{G}} < t, \lambda > \ell(\tilde{Z})(d\lambda) = \ell\Big(\int_{\widehat{G}} < t, \lambda > \tilde{Z}(d\lambda)\Big). \tag{9}$$

Since $\ell \in \mathcal{X}^*$ is arbitary, one gets

$$X_t = \int_{\widehat{G}} < t, \lambda > \tilde{Z}(d\lambda), t \in G, \text{ (D-S integral)} . \tag{10}$$

Thus one has

Theorem 4. *Let G be an LCA group, $\mathcal{X}$ a reflexive Banach space and $X : G \to L_0^2(P;\mathcal{X})$, a second order random function such that $X(G)$ is norm bounded (or $X(G)$ is relatively weakly compact if $\mathcal{X}$ is not reflexive). Then $\mathcal{X}$ is weakly harmonizable iff there is a stochastic measure $\tilde{Z}$ such that the representation* (10) *holds.*

This suggests that one may characterize weakly harmonizable random fields differently without using bimeasure integration. Such a procedure was given by Bochner (1956) with $\mathcal{X} = \mathbb{C}$. This will be employed when G is not necessarily abelian. The weakly harmonizable case when $G = \mathbb{R}$ and $\mathcal{X} = \mathbb{C}$ was first considered by Niemi (1975) who analyzed this class for certain other properties (cf. e.g., (1975-76)); and some special representations are given in Chang and Rao (1988).

2.3 *Noncommutative harmonizable random fields.* For a definition and integral representation of harmonizable functions in this case, one should define a suitable Fourier transform extending the LCA case above. A general form of the latter can be obtained through a use of C^*-algebras when G is any locally compact group. But an integral representation which usually depends on a Plancherel measure is then not possible since there is no dual group of G, and the analysis loses any resemblance with the previous theory. (See Ylinen (1975), (1984) and (1987) who has investigated the general case through C^*-algebra theory employing the techniques developed by Eymard (1964).) However, if we restrict G to be separable and (for simplicity here) unimodular, then the desired result can be derived, as shown below. Thus, in this section, G will be a *separable locally compact unimodular group.*

To proceed further, it is necessary to recall some results from the representation theory of such groups. Thus a locally compact group G is of type I if each unitary representation u of G into a Hilbert space $\mathcal{H}$ has the property that the weakly closed self-adjoint algebra $\mathcal{A}$ generated by $\{u_g, g \in G\}$ is isomorphic to some weakly closed self-adjoint subalgebra $\tilde{\mathcal{A}}$ of $L(\mathcal{H})$ such that $\tilde{\mathcal{A}}'$ is abelian. Here $\tilde{\mathcal{A}}'$ is the set of elements, of the algebra of bounded linear operations $L(\mathcal{H})$, that commute with $\tilde{\mathcal{A}}$. The group G is of type II if there is a normal semi-finite trace functional τ on $\mathcal{A}$ so that τ is linear, and for each $A \in \mathcal{A}$ there is a $B \leq A$ such that $|\tau(B)| < \infty$ and a monotone convergence theorem holds for it. One knows that each separable unimodular group is of type I or type II, and the following important facts are available (cf., Segal (1950), Mautner (1955) and especially Tatsuuma (1967); also Naĭmark (1964), Ch. 8):

(i) If $\widehat{G}$ denotes the set of all irreducible (strongly) continuous unitary representations of G into a Hilbert space, then one can endow $\widehat{G}$ a topology relative to which it becomes a locally compact Hausdorff space. And if μ is a Haar measure on G, then there is a unique Radon measure ν on $\widehat{G}$ such that $(\widehat{G}, \nu)$ becomes a dual object (or dual gauge) of (G, μ), and a Plancherel formula holds.

(ii) The representation Hilbert space $\mathcal{H}$ may be taken as $L^2(G,\mu) = L^2(G)$, and $\mathcal{H}$ can be expressed as a direct sum $\mathcal{H} = \bigoplus_{y \in \widehat{G}} \mathcal{H}_y$, with $\mathcal{H}_y$ as the representation space for each y in $\widehat{G}$. If $\mathcal{A}_y$ is the weakly closed self-adjoint subalgebra of $L(\mathcal{H}_y)$ generated by the strongly continuous unitary operators $\{u_y(g), g \in G\}$, then $\mathcal{A}_y$ is of type I or type II, and

$$L^2(G) = \int_{\widehat{G}}^{\oplus} \mathcal{H}_y \nu(dy), \text{ (direct integral).} \tag{11}$$

Moreover, if $(L_a f)(x) = f(a^{-1}x), x \in G$, then the weakly closed self-adjoint algebra $\mathcal{A}$ generated by $\{L_a, a \in G\}$ of $L(\mathcal{H})$, admits a direct sum decomposition of $\mathcal{A}_y, y \in \widehat{G}$, and for each

$f \in L^1(G) \cap L^2(G)$, the following (Bochner) integral exists

$$\hat{f}(y) = \int_G u_y(g) f(g) \mu(dg), u_y(g) \in \mathcal{A}_y, y \in \widehat{G}, \tag{12}$$

and defines a bounded linear mapping on $\mathcal{H}$. Also $\hat{f}(y)$ may be extended uniquely to a dense subspace of $\mathcal{H}$ containing $L^1(G) \cap L^2(G)$ so that it is closed and self-adjoint. This extended function $y \longmapsto \hat{f}(y)$, denoted by the same symbol, is the (generalized) *Fourier transform* of f.

(iii) There is a trace functional $\tau_y : \mathcal{A}_y \to \mathbb{C}$ which is positive, normal, semi-finite, and faithful, in terms of which one has the Plancherel formula for $f_i \in L^2(G), i = 1, 2$ ($\hat{f}_i^*$ denoting the adjoint of $\hat{f}_i$):

$$\int_G f_1(g)\overline{f_2(g)}\mu(dg) = \int_{\widehat{G}} \tau_y(\hat{f}_1(g)\hat{f}_2^*(y))\nu(dy). \tag{13}$$

The measurability of $\hat{f}_i$ as well as that of $y \longmapsto \tau_y(\hat{f}_1(y)\hat{f}_2^*(y))$ relative to ν are nontrivial facts and are established in the theory. An important result here is that there is a one-to-one correspondence between f and $\hat{f}$, and there is an inversion formula as well, (cf. Mautner, 1955). This is given next.

(iv) If $A(y) \in \mathcal{A}_y, y \longmapsto A(y)$ measurable, $y \longmapsto \|A(y)\|$ bounded and $\int_{\widehat{G}} \tau_y(A(y) A^*(y))\nu(dy) < \infty$, then there exists $f \in L^2(G)$ such that $\hat{f}(y) = A(y), y \in \widehat{G}$. On the other hand, if $h \in L^2(G)$ such that $h = f \star f$ for some $f \in L^2(G)$,then one has (the inversion formula):

$$h(g) = \int_{\widehat{G}} \tau_y(u_y(g)^* \hat{h}(y))\nu(dy), g \in G. \tag{14}$$

With these results, especially the (generalized) Fourier transform, the concept and a characterization of weak harmonizability for noncommutative groups can be given. The general concept is motivated by Bochner's classical notion of V-boundedness.

Definition 1. Let G be a separable locally compact unimodular group, and $X : G \to X_g \in L_0^2(P), g \in G$, be a random field. Then X is *weakly harmonizable* if it is weakly continuous and the set

$$\{\int_G X_g \varphi(g)\mu(dg) : \|\widehat{\varphi}\|_\infty = \sup_{y \in \widehat{G}} \|\widehat{\varphi}(y)\| \leq 1, \varphi \in L^1(G) \cap L^2(G)\},$$

is bounded in the Hilbert space $L_0^2(P), \widehat{\varphi}$ being the generalized Fourier transform of φ defined above.

With this concept at hand, the main integral representation of X is in:

Theorem 2. *Let $X : g \longmapsto X_g \in L_0^2(P), g \in G$, be a weakly harmonizable random field. Then there is a weakly σ-additive regular operator measure $\mathbf{m}(dy)$ on $\widehat{G}$, operating on $\mathcal{H}_y \to L_0^2(P)$, vanishing on ν-null sets and a trace functional $\tau_y : \mathcal{A}_y \to \mathbb{C}$, such that one has:*

$$X_g = \int_{\widehat{G}} \tau_y(u_g(y)\mathbf{m}(dy)), g \in G \text{ (Bartle integral)}, \tag{15}$$

and $X_{(\cdot)}$ is uniformly continuous in the strong topology of $L_0^2(P)$. On the other hand, a weakly continuous $X : g \to X_g$ defined by (15) *is weakly harmonizable. Further, the covariance function r of the weakly harmonizable X, satisfying* (15), *is given by (a corresponding MT-integral for vector functions):*

$$r(g_1, g_2) = \int_{\widehat{G}} \int_{\widehat{G}} \tau_{y_1} \otimes \tau_{y_2}\{(u_{g_1}(y_1) \otimes u_{g_2}(y_2))\beta(dy_1, dy_2)\}, \tag{16}$$

where β is an operator valued bimeasure (cf. Section 1.6) on $\mathcal{B}(\widehat{G}) \times \mathcal{B}(\widehat{G})$, with $\mathcal{B}(\widehat{G})$ as the Borel σ-algebra of $\widehat{G}$.

Proof. If $f \in L^1(G) \cap L^2(G)$, let $\hat{f}$ be defined by (12), which is a measurable operator function. To see that it is bounded, considering $\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_y \nu(dy)$, embed $\mathcal{H}_y$ in $\mathcal{H}$ and treat it as a closed subspace. Then $u_y(g) = u(g, y)$ in $L(\mathcal{H}_y)$ may be extended as $\tilde{u}(g, y) = u(g, y)$ on $\mathcal{H}_y$,= identity on $\mathcal{H}_y^{\perp}$ so that $\{\tilde{u}(g, y), g \in G\}$ is a family of unitaries in $L(\mathcal{H})$, and $\tilde{u}(g, \cdot) \in L(\mathcal{H}), g \in G$. If the corresponding operator of (12), obtained by replacing u by $\tilde{u}$, is again denoted by $\hat{f}$, then it is measurable. Let $\mathcal{A}(\mathcal{H}) = \{\hat{f} : \hat{f}(y) \in L(\mathcal{H}_y), y \in \widehat{G}\}$ which is identifiable with a subalgebra of $L(\mathcal{H})$. If $T : f \longmapsto \hat{f}$, then T is one-to-one and is a contraction. The former is a consequence of the general theory and the latter follows from the computation: ($\|\cdot\|_{op}$ denotes the operator norm)

$$\begin{aligned} \|\hat{f}(y)\|_{op} &= \|\int_G f(g)\tilde{u}(g, y)\mu(dg)\|_{op} \\ &\leq \int_G |f(g)| \, \|\tilde{u}(g, y)\|_{op} \, \mu(dg), \text{ by a property of the vector integral,} \\ &\leq \int_G |f(g)|\mu(dg) = \|f\|_1. \end{aligned} \tag{17}$$

Hence

$$\sup_{y \in \widehat{G}} \|\hat{f}(y)\|_{op} \leq \|f\|_1 < \infty. \tag{18}$$

Thus $T : L^1(G) \cap L^2(G) \to \mathcal{A}(\mathcal{H})$ is a contraction, and since X is weakly harmonizable, one has for each $f \in L^1(G) \cap L^2(G)$,

$$T_1(f) = \int_G f(g) X_g \, \mu(dg) \in L_0^2(P), \tag{19}$$

and T_1 is bounded. Let $\tilde{T} = T_1 \circ T^{-1}$ so that

$$\tilde{T}(\hat{f}) = T_1(T^{-1}(\hat{f})) = T_1(f), f \in L^1(G) \cap L^2(G). \tag{20}$$

Then $\tilde{T}$ is unambiguous and by Definition 1,

$$\|\tilde{T}(\hat{f})\|_2 \leq C\|f\|_2. \text{ (C is a constant.)} \tag{21}$$

Thus $\tilde{T}$ can be expressed as a direct sum of bounded operators from $\mathcal{A}(\mathcal{H}_y)$ into $L_0^2(P), y \in \widehat{G}$, by the general theory. Since the range of $\tilde{T}$ is reflexive, $\tilde{T}$ is weakly compact. Applying a suitable form of the Riesz-Markov theorem (cf. Dinculeanu (1967), p. 398, Thn. 9), and using the theory of direct integral for which the hypothesis on G and the separability of G are needed crucially at this point (cf. Naǐmark (1964), Ch. 8, Sec. 4), one gets a regular weakly σ-additive operator measure $\mathbf{m}$ on $\mathcal{B}(\widehat{G})$ into $L(\mathcal{A}(\mathcal{H}), L_0^2(P))$ such that

$$\tilde{T}(\hat{f}) = \int_{\widehat{G}} \tau_y(\hat{f}(y)\mathbf{m}(dy)), \hat{f} \in \mathcal{A}(\mathcal{H}), \tag{22}$$

where τ_y is a trace on $\mathcal{A}(\mathcal{H}_y)$ and where the integral is a suitable D-S (or Bartle (1956)) extension. Here $\mathbf{m}(\cdot)x : \mathcal{B}(\widehat{G}) \to L_0^2(P)$ is σ-additive and regular for each $x \in \mathcal{A}(\mathcal{H})$, and that property does not necessarily hold for $\mathbf{m}(\cdot)$ itself. It follows from (12), (19) and (22) that

$$\begin{aligned}
\int_G f(g) X_g \, \mu(dg) &= \int_{\widehat{G}} \tau_y(\hat{f}(y)\mathbf{m}(dy)) \\
&= \int_{\widehat{G}} \tau_y \Big[\int_G f(g)\tilde{u}(g,y)\mu(dg)\mathbf{m}(dy) \Big] \\
&= \int_G f(g) \int_{\widehat{G}} \tau_y[\tilde{u}(g,y)\mathbf{m}(dy)]\mu(dg),
\end{aligned}$$

since f is scalar and τ_y is linear and commutes with the integral over G, and a Fubini type argument applies. The above can be rearranged;

$$\int_G f(g)(X_g - \int_{\widehat{G}} \tau_y[\tilde{u}(g,y)\mathbf{m}(dy)])\mu(dg) = 0. \tag{23}$$

Since $f \in L^1(G) \cap L^2(G)$ is arbitrary and the latter is a dense set in $L^1(G)$ (as well as $L^2(G)$), the (continuous) function inside the parenthesis must vanish. Now replacing $\tilde{u}$ by u which is legitimate, (23) gives (15).

The reverse direction is obtained similarly. In fact, if (15) holds and $\varphi \in L^1(G) \cap L^2(G)$, then by the familiar reasoning one has

$$\begin{aligned} \int_G X_g \varphi(g)\mu(dg) &= \int_G \varphi(g)\Big[\int_{\widehat{G}} \tau_y(u(g,y)\mathbf{m}(dy))\Big]\mu(dg) \\ &= \int_G \int_{\widehat{G}} \tau_y[\varphi(g)u(g,y)\mathbf{m}(dy)]\mu(dg) \\ &= \int_{\widehat{G}} \tau_y\Big[\Big(\int_G \varphi(g)u(g,y)\mu(dg)\Big)\mathbf{m}(dy)\Big] \\ &= \int_{\widehat{G}} \tau_y[\widehat{\varphi}(y)\mathbf{m}(dy)]. \end{aligned}$$

From this it follows that

$$\Big\| \int_G X_g \varphi(g)\mu(dg)\Big\| \leq \|\widehat{\varphi}\|_{op}\|\mathbf{m}\|(\widehat{G}), \tag{24}$$

where $\|\mathbf{m}\|(\cdot)$ is the semi-variation of $\mathbf{m}$ (cf. Dinculeanu (1967), Sec. 19). Letting $C = \|\mathbf{m}\|(\widehat{G}) < \infty$ in (24) it follows that Definition 1 holds, and X is weakly harmonizable since it is clearly weakly continuous.

Finally to establish (16), one may calculate the covariance r of the $X(g)$'s using some properties of the extended MT-integral:

$$\begin{aligned} r(g_1, g_2) &= E(X_{g_1}\overline{X}_{g_2}) \\ &= E\Big(\int_{\widehat{G}} \tau_{y_1}(\tilde{u}(g_1,y_1)\mathbf{m}(dy_1)\cdot \int_{\widehat{G}} \tau_{y_2}(\tilde{u}(g_2,y_2)\mathbf{m}(dy_2))^*\Big) \\ &= E\Big[\int_{\widehat{G}}\int_{\widehat{G}} \tau_{y_1}\otimes\tau_{y_2}\{(\tilde{u}(g_1,y_2)\otimes\tilde{u}(g_2,y_2)^*)\mathbf{m}(dy_1)\otimes\mathbf{m}^*(dy_2)\}\Big] \end{aligned}$$

where one uses the properties of tensor products of

trace functionals (cf. Hewitt and Ross (1970), Appendix D),

$$\begin{aligned} &= \int_{\widehat{G}}\int_{\widehat{G}} \tau_{y_1}\otimes\tau_{y_2}\{(\tilde{u}(g_1,y_1)\otimes\tilde{u}(g_2,y_2)^*)E(\mathbf{m}(dy_1)\otimes\mathbf{m}(dy_2))\} \\ &= \int_{\widehat{G}}\int_{\widehat{G}} \tau_{y_1}\otimes\tau_{y_2}[(\tilde{u}(g_1,y_1)\otimes\tilde{u}(g_2,y_2)^*)\beta(dy_1,dy_2)], \end{aligned}$$

where $\beta(\cdot,\cdot)$ is an operator valued positive

definite bimeasure,

$$= \int_{\widehat{G}}\int_{\widehat{G}} \tau_{y_1}\otimes\tau_{y_2}[\tilde{u}(g_1,y_1)\beta(dy_1,dy_2)\tilde{u}(g_2,y_2)],$$

which is (16) written in a different form, and the result follows.

Remarks. Some complements to the above theorem will be included here in the form of remarks:

1. The integral representation (15) may be used for solving filter equations and in other applications. If G is abelian, then $\widehat{G}$ is a group, and each $\mathcal{H}_y = \mathbb{C}$ so that the result reduces to a previously known case (cf. Rao (1982)).

2. The corresponding representation for stationary random fields was first obtained by Yaglom (1960, 1961). For this class, several other interesting results for homogeneous spaces as well as multidimensional fields were also given there. I plan to consider the corresponding theory for weakly harmonizable fields later.

3. The measure $\mathbf{m}(\cdot)$ in (15) need not be σ-additive in the uniform operator topology, as known counter examples show. In the LCA case this difficulty disappears since $\mathcal{H}$ is $\mathbb{C}$, and by the classical Pettis's theorem weak and strong σ-additivities coincide.

4. If G is not a separable group, the decomposition theory runs into difficulties, and one may have to settle with the C^*-algebra approach, as was done by Ylinen (1975). Here the representation (15) is not available.

5. Using the Vitali variation of $\beta(\cdot\,,\cdot)$ in (16), one can present a result on *strongly harmonizable random* fields by a similar (and simpler) argument. A class intermediate to this and weak harmonizability is isolated by Ylinen (1988), who termed it "completely bounded". It coincides with weak harmonizability in case G is an LCA group.

6. If G is a compact group, then no separability assumptions are needed, and the Fourier transform can be derived through the Peter-Weyl theory. Thus a representation corresponding to (15) can be obtained using some classical computations as done, for instance in another context in Rao (1968), (cf. also the general theory on compact groups in Hewitt and Ross (1970)).

2.4 *The linear filter equation* $\Lambda X = Y$. Let G be an LCA group, and $X, Y : G \to L_0^2(P;\mathbb{C}^k)$ be k-dimensional random fields. If Λ is a linear operator on the class of such second order functions satisfying $T_h(\Lambda X)(g) = \Lambda(T_h X)(g)$ for all $g, h \in G$, where $(T_h X)(g) = X(g+h)$, then Λ is called a linear *filter* and $(\Lambda X)(g) = Y(g), g \in G$, is a *filter equation*. Thus Λ

commutes with translations. For instance if $G = \mathbb{R}^n$, then Λ can be an integro-difference-differential operator (with constant coefficients). In G the group operation is denoted "+". The problem now is that, given a random field Y (e.g., harmonizable or stationary) called an "output", find conditions on Λ in order that there is a solution X, called an "input", of the filter equation belonging to the same class. Here a solution is described if Λ is of the form:

$$Y(g) = (\Lambda X)(g) = \int_G A(s)X(g-s)ds \tag{25}$$

where 'ds' is an invariant measure on G, and A is a k-by-k matrix of scalar integrable Borel functions on G.

In the context of harmonizable functions one has:

Theorem 1. *Let the output Y be a k-dimensional weakly harmonizable random field with β_y as its k-by-k matrix spectral bimeasure. For the filter equation* (25), *there is a weakly harmonizable solution X iff*

$$\text{(i)} \quad \int_D \int_D^* (I - FF^{-1})(\lambda)\beta_y(d\lambda, d\lambda')(I - FF^{-1})^*(\lambda') = 0$$

for all Borel sets D of $\widehat{G}$, where $F = \widehat{A}$, the Fourier transform of A, F^{-1} is the generalized inverse of F and '' denotes the adjoint operation of the matrix, the integral being in the strict* MT-*sense, and where*

$$\text{(ii)} \quad \int_{\widehat{G}} \int_{\widehat{G}}^* F(\lambda)^{-1}\beta_y(d\lambda, d\lambda')(F(\lambda')^{-1})^* \text{ exists.}$$

When these conditions hold, the solution X can be given by:

$$X_t = \int_{\widehat{G}} < t, \lambda > F^{-1}(\lambda)Z_y(d\lambda), t \in G, \tag{26}$$

where Z_y is the stochastic measure representing Y. The solution is unique iff $F(\lambda)$ is nonsingular for each $\lambda \in \widehat{G}$.

Here $F(\lambda)$ is often called the *spectral characteristic* of the filter Λ. Under further restrictions on $A(\cdot)$ one can obtain a simpler condition, such as that given by the following:

Proposition 2. *Let F be the spectral characteristics of the filter Λ of* (25). *If conditions* (i) *and* (ii) *of Theorem* 1 *hold, and if there is an integrable k-by-k matrix function f whose*

Fourier transform $\hat{f}$ *satisfies* $\|F^{-1} - \hat{f}^*\|_{2,\beta_y} = 0$, *with the norm used in Thm.* 1.5.2 *before, then the solution can be given by*

$$X_t = \int_G f(s)Y(t-s)ds, t \in G. \tag{27}$$

When $G = \mathbb{R}^n$, but Λ is more general, similar problems were considered by Chang and Rao (1986), and their methods yield the last two results for LCA groups G. Since a stationary random field is also harmonizable, the preceding work implies that for stationary Y, (25) has a weakly harmonizable solution X under the given hypothesis. What else is needed to assert that X is also stationary? This was studied by Bochner (1956) who gave conditions for a positive solution. Those considerations have been analyzed in more detail and the corresponding results are given in Rao (1984). So further discussion of the problem will be omitted here.

2.5 *Harmonizability over hypergroups.* Some statistical applications such as sample means of stationary or harmonizable sequences can lead to classes of second order processes which are not of the same type but are closely related to the original family. Many of these can be described as second order processes not on topological groups but on objects which are a generalization of these, called hypergroups. The latter have an algebraic group structure, but the topology they are endowed with does not always make the group operation continuous. Since it has a potential for future developments in this area, harmonizability on such spaces will be defined and a result on its integral representation given here.

One of the origins of hypergroups K may be traced to a study of the double coset spaces $H \setminus G/H$, (also denoted $G//H$) of a locally compact group G with H as a compact subgroup. It is clear that such K are locally compact spaces which are not groups in general. However, a group operation through convolution can often be introduced in such a space and the corresponding representation theory developed. Thus the hypergroups may be considered as objects between topological groups and the homogeneous spaces G/H, with interesting structure, and hence they have applicational potential. Abstraction of this remark will now be stated, following Jewett (1975) and others, for further development:

Definition 1. A locally compact space K is called a *hypergroup* if the following conditions are met:

(i) There exists an operation $\star : K \times K \to M_1(K)$, called convolution, such that $(x,y) \to \delta_x \star \delta_y, (x,y \in K)$ where δ_x is the Dirac measure at $x, M_1(K)$ is the set of Radon probability measures on K endowed with the vague (or weak*-) topology when $M(K)$ is regarded as the dual space of $C_0(K)$, and $\delta_x \star (\delta_y \star \delta_z) = (\delta_x \star \delta_y) \star \delta_z$;

(ii) $\delta_x \star \delta_y$ has compact support;

(iii) There is an involution, denoted by "$\sim$", on K such that $x^{\approx} = x$ and $(\delta_x \star \delta_y)^{\sim} = \delta_{\tilde{y}} \star \delta_{\tilde{x}}, x, y \in K$, where for a measure $\mu \in M_1(K), \tilde{\mu}(A) = \mu(\tilde{A})$ with $\tilde{A} = \{\tilde{x} : x \in A\}$, and there is a unit e in K satisfying $\delta_e \star \delta_x = \delta_x \star \delta_e = \delta_x$; and

(iv) $e \in \text{supp}(\delta_x \star \delta_{\tilde{y}})$ iff $x = y$, and that $(x,y) \longmapsto \text{supp}(\delta_x \star \delta_y)$ is continuous when 2^K is given the Kuratowski topology.

If (iv) is not assumed, then the object K for which (i)-(iii) hold is called a *weak hypergroup.* A number of concrete examples of these objects are given by Lasser (1983). For instance, several classical orthogonal polynomials on $K = \mathbb{Z}^+$, such as the Jocobi, Čebyšev, q-ultraspherical, Pollaczek, and certain Legendre polynomials are hypergroups. Also if K is a *commutative* hypergroup, i.e., $\delta_x \star \delta_y = \delta_y \star \delta_x$ holds in addition (the above examples are commutative hypergroups), its dual $\widehat{K}$ is defined as:

$$\widehat{K} = \{\alpha \in C_b(K) : (\delta_x \star \delta_y)(\alpha) = \int_K \alpha(t)(\delta_x \star \delta_y)(dt) = \alpha(x)\alpha(y), \\ x, y \in K \text{ and } \alpha(\tilde{x}) = \overline{\alpha}(x)\}. \tag{28}$$

Here $C_b(K)$ is the space of bounded continuous complex functions on K, with the topology of uniform convergence on compact sets. Then $\widehat{K}$ becomes a locally compact space which however need not be a hypergroup in general, the binary operation in $\widehat{K}$ being pointwise multiplication.

If K is a commutative hypergroup, then it admits an invariant (or Haar) measure, as shown by Spector (1978), and if $\widehat{K}$ its dual, also happens to be a hypergroup then $K \subset \widehat{\widehat{K}}$; and is termed a *strong hypergroup* provided $K = \widehat{\widehat{K}}$. A great deal of classical harmonic analysis is being extended to hypergroups (cf. e.g. Vren (1979), Lasser (1987), and references there). Our interest here is in the following stochastic application. For other developments of probability theory on these structures, one should refer to a detailed account in Heyer (1984).

If $X : K \to L_0^2(P)$ is a mapping such that its covariance function $\rho, \rho(a,b) = E(X_a\overline{X}_b)$, is bounded, continuous and representable as

$$\rho(a,b) = \int_K \rho(x,o)(\delta_a \star \delta_b)(dx), a, b \in K, \tag{29}$$

then X is termed a stationary random field on the commutative hypergroup K, or simply a *hyper-weakly stationary random field.* This concept is due to Lasser and Leitner (1988), except that they termed it "K-stationary". Since Bochner (1956) already used this term for Khintchine stationary, to avoid confusion the above term with the prefix "hyper" will be used here and below. It includes the sequences of symmetric Cesàro averages of ordinary stationary sequences, with $K = \mathbb{Z}^+$. For this concept the authors infer, via an analog of Bochner's theorem on positive definite functions, that X is hyper-weakly stationary on a commutative hypergroup K, iff

$$\rho(a,b) = \int_{\widehat{K}} \alpha(a)\overline{\alpha(b)}d\nu(\alpha), \tag{30}$$

for a unique bounded Borel measure ν on $\widehat{K}$. This allows an integral representation of X itself from the classical Karhunen-Cramér theorem.

The corresponding concept for harmonizability can be given as:

Definition 2. Let $X : K \to L_0^2(P)$ be a second order random field on a commutative hypergroup K whose dual object is denoted by $\widehat{K}$. If $\rho : (a,b) \longmapsto E(X_a\overline{X}_b), a, b \in K$, is its covariance function then X is called a *hyper-weakly (strongly) harmonizable* random field if ρ admits a representation

$$\rho(a,b) = \int_{\widehat{K}} \int_{\widehat{K}}^* \alpha_1(a)\overline{\alpha_2(b)}\beta(d\alpha_1, d\alpha_2), \tag{31}$$

where $\beta : \mathcal{B}(\widehat{K}) \times \mathcal{B}(\widehat{K}) \to \mathbb{C}$ is a positive definite bimeasure (of finite Vitali variation), and the integral is a strict MT-integral (a Lebesgue-Stieltjes integral).

It is well-known (cf. e.g., Chang and Rao (1986), p. 21) that β has always a finite Fréchet variation on the Borel σ-algebra $\mathcal{B}(\widehat{K})$. This definition reduces to the hyper-weakly stationary case if β concentrates on the diagonal of $\widehat{K} \times \widehat{K}$. The Fourier transform is well-defined, one-to-one and contractive, as in the LCA group case (cf. e.g., Heyer (1984), p. 491). Using these properties and the arguments of Sections 2.2 and 2.3, the following representation can be established.

Theorem 3. *Let $X : K \to L_0^2(P)$ be a hyper-weakly harmonizable random field in the sense of Definition 2. Then there is a stochastic measure $Z : \mathcal{B}(\widehat{K}) \to L_0^2(P)$ such that*

$$X_a = \int_{\widehat{K}} \alpha(a) Z(d\alpha), a \in K, \tag{32}$$

with $E(Z(A_1)\overline{Z}(A_2)) = \beta(A_1, A_2)$ defining the bimeasure β in (31). In fact, a second order weakly continuous random field on a commutative hypergroup K admits the representation (32), hence hyper-weakly harmonizable, iff the following set is norm bounded:

$$\{\int_K \varphi(a)X(a)d\mu(a) : \|\widehat{\varphi}\|_\infty \leq 1, \varphi \in L^1(K,\mu) \cap L^2(K,\mu)\} \subset L_0^2(P),$$

where μ is a Haar measure on K and $\widehat{\varphi}$ is the Fourier transform of φ so that $\widehat{\varphi}(\alpha) = \int_K \alpha(a)\varphi(a)\mu(a), \alpha \in \widehat{K}$, holds.

In the original talk at Oberwolfach this result was not included. It is a natural outgrowth of a study of some material on these objects. I should like to thank Drs. R. Lasser and R.C. Vrem for sending me their interesting published and some unpublished work on hypergroups which inspired it.

2.6 *Strict harmonizability and V-boundedness.* All the preceding study is based on the covariance properties of second order random functions. However, it is possible to analyze processes based on their distributional structure without asking for the existence of two moments, motivated by the strict stationarity concept. Since the study is linked to the Fourier transform of a stochastic measure eventually, the classical probability theory indicates that these stochastic measures be related to stability, so that one can define the corresponding integrals, if the values of the measures are independently scattered. Thus the desired concept can be presented as follows:

Definition 1. Let $(S, \mathcal{S})$ be a measurable space and $Z : \mathcal{S} \to L^p(P), 0 < p \leq 2$, be a mapping. Then $Z(\cdot)$ is an *independently scattered random stable measure of exponent p*, if the following conditions are met:

(i) $A_k \in \mathcal{S}, k = 1, \ldots, n, n \geq 1$, disjoint, implies $\{Z(A_k), 1 \leq k \leq n, n \geq 1\}$ is a mutually independent collection,

(ii) for each $A \in \mathcal{S}, Z(A)$ is a stable random variable of exponent p, and

(iii) $A_n \in \mathcal{S}$, disjoint, $A = \bigcup_n A_n$ implies $X(A) = \sum_{n=1}^{\infty} X(A_n)$, the series converging in probability (hence also with prob. 1 here).

For such a measure Z, a Wiener type stochastic integral can be defined, and if S is compact, ($\mathcal{S}$ Borel) then each continuous scalar function on S will be integrable relative to Z. Taking S as the torus, identified with $(-\pi, \pi)$ and $\mathcal{S}$ as its Borel σ-algebra, one says that a stochastic measure $Z : \mathcal{S} \to L^p(P)$ is *isotropic* if for each $A \in \mathcal{S}$ and each $\omega \in S$ (i.e., $-\pi < \omega < \pi), Z(A)$ and $e^{i\omega}Z(A)$ are identically distributed.

Recalling that a strictly stationary process is one whose (joint) finite dimensional distributions are invariant under translations of the index set, assumed an LCA group, one can present the following result when that index is the integers (considered as a group under addition and the torus as its dual):

Theorem 2. *Let $X : Z \to L^\alpha(P), 1 \le \alpha \le 2$ be a process. Then there is an independently scattered random stable measure Z of exponent α, which is isotropic on $\mathcal{S}$ (the Borel σ-algebra of $(-\pi, \pi)$) such that*

$$X_n = \int_{-\pi}^{\pi} e^{in\,t} Z(dt), n \in Z, \tag{33}$$

iff $\{X_n, n \in Z\}$ is strictly stationary and V*-bounded*, i.e. *for $1 < \alpha \le 2$*

$$\|\sum_{k=1}^{n} a_k X_k\|_\alpha \le C \sup\{|\sum_{k=1}^{n} a_k\, e^{-2\pi i t n_k}| : -\pi < t \le \pi\}, \tag{34}$$

and for $\alpha = 1, \{\sum_{k=1}^{n} a_k X_k, n \ge 1, a_k \in \mathbb{C}\}$ is relatively weakly compact in $L^1(P)$ in addition. When these conditions hold, the finite dimensional distributions, or equivalently their characteristic functions, are given by

$$\begin{aligned}\varphi_{n_1,\ldots,n_k}(u_1, \ldots, u_k) &= E\Big(\exp\Big[\sum_{j=1}^{k} iu_j X_{n_j}\Big]\Big)\\ &= \exp\Big\{-\int_{-\pi}^{\pi}\Big|\sum_{j=1}^{k} u_j\, e^{i\lambda n_j}\Big|^\alpha dG(\lambda)\Big\},\end{aligned} \tag{35}$$

for a non-negative, bounded, and non-decreasing function G.

The representation (33) is obtained from (34) through Bochner's criterion of V-boundedness for processes in a Banach space. The rest of the calculation through (35) is due to Hosoya (1982) who gives it for $0 < \alpha < 1$ also, when (33) is assumed. There are several extensions and related studies of processes given by the integral representation (33). Some of the references are Cambanis (1983), Weron (1985), Urbanik (1968), Kuelbs (1973), Okazaki (1979), Rosinski (1986) and Marcus (1987) among others.

Thus $X : Z \to L^{\alpha}(P), 0 < \alpha \leq 2$, admitting the representation (33) relative to a stable random measure of exponent α, and which is infinitely divisible, will be called *strictly harmonizable.* It follows from the above work and references, that there exist strictly stationary processes which are not strictly harmonizable and in fact an example is readily obtained from one in Sinaǐ (1963). Thus the hierarchy of the classes available for the second order processes is no longer valid for the strict sense concepts. Also a characterization of (33) for $0 < \alpha < 1$ is not available since the $L^{\alpha}(P)$ is not a Banach space and the V-boundedness theory has not been extended for these Fréchet (or invariant metric) spaces.

The next result gives an indication of how the Karhunen class undergoes a change for the strict sense case, with a Wiener type integral. It is due to Kuelbs (1973) and is an extension of an earlier "finite process" case by Schilder (1970):

Proposition 3. *Let $\{X_t, t \in T\}$ be a random field, with T as a second countable Hausdorff space, whose finite dimensional distributions belong to a symmetric stable class of index $\alpha, 0 < \alpha \leq 2$. Then there is an independently scattered stable random measure of index α on $[-\frac{1}{2}, \frac{1}{2}]$ and a family $\{f_t, t \in T\} \subset L^{\alpha}(-\frac{1}{2}, \frac{1}{2})$, the Lebesgue space, such that*

$$W_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_t(\lambda) Z(d\lambda), t \in T, \tag{36}$$

and the X_t and W_t- processes have the same finite dimensional distributions.

The integral (36) is a kind of Wiener integral. The replacement of $f_k(\lambda)$ by $e^{ik\lambda}$ is not generally possible without further hypothesis. It appears that some analog of the V-boundedness is needed in addition. Probabilistic studies here concentrate on the sample path behavior of the process. Detailed analysis of such families have been given by Okazaki (1979) and Marcus (1987), (see also references to earlier results on these problems). For some other applications and extensions see Pourahmadi (1984) and Rajput and Rama-Murthy (1987).

Part III: Vector Harmonizable Random Fields

3.1 *Multidimensional harmonizability.* If a second order process is vector valued, new questions and classifications arise even when the group $G = \mathbb{R}$. In this part the vector (and operator) valued casses will be discussed briefly to focus on the new problems. Suppose then $X : G \to L_0^p(P;\mathcal{X}), 1 \le p < \infty$, is an $\mathcal{X}$-valued process where $\mathcal{X}$ is a separable Banach space and G is an LCA group. First it is useful to present V-boundedness at two levels, since it is needed to classify vector harmonizability.

Definition 1. A mapping $X : G \to L_0^p(P;\mathcal{X}), 1 \le p < \infty$, is *weakly* V-*bounded* if for each $\ell \in \mathcal{X}^*$, the dual of $\mathcal{X}$, the scalar process $\ell \circ X : G \to L_0^p(P)$, is V-bounded in that the function $t \longmapsto (\ell \circ X)(t), T \in G$, is continuous in $L_0^p(P)$ and the set (with the sample path [or Lebesgue] integral)

$$\{\int_G (\ell \circ X)(t)\varphi(t)dt : \|\widehat{\varphi}\|_\infty \le 1, \varphi \in L^1(G)\} \subset L_0^p(P) \tag{1}$$

is relatively weakly compact and $\ell \circ X(G)$ is bounded in $L_0^p(P)$, where $\widehat{\varphi}$ is the Fourier transform of $\varphi \in L^1(G)$, and 'dt' is a Haar measure on G. The mapping X is *strongly* V-*bounded* if $t \longmapsto X(t)$ is continuous in $L_0^p(P;\mathcal{X})$ and the set (with the Bochner integral)

$$\{\int_G X(t)\varphi(t)dt : \|\widehat{\varphi}\|_\infty \le 1, \varphi \in L^1(G)\} \subset L_0^p(P;\mathcal{X}), \tag{2}$$

is relatively weakly compact.

It is seen that strong V-boundedness implies weak V-boundedness but not necessarily conversely. Taking $p = 2$ and $\mathcal{X} = \mathcal{H}$, a Hilbert space, the corresponding concept of interest here is given by:

Definition 2. A mapping $X : G \to L_0^2(P;\mathcal{H})$ is *weakly harmonizable* if it is strongly V-bounded (and continuous), and it is *ultra weakly harmonizable*, if it is weakly V-bounded or equivalently $\ell \circ X : G \to L_0^2(P)$ is a weakly harmonizable (scalar) random field for each $\ell \in \mathcal{H}^*$.

Unless $\mathcal{H}$ is finite dimensional, the ultra weak class properly contains the weak class. This fact can be demonstrated by means of an example due to O.E. Lanford, discussed for a related purpose in the paper of R.I. Loebl (1976). This example has again been detailed for a similar purpose in a recent survey of dilation problems in Makagon and Salehi (1987). The distinction can be anticipated as the following remarks indicate.

In both cases of V-boundedness, standard arguments of abstract analysis imply that X is representable as a suitable D-S integral:

$$X(t) = \int_{\widehat{G}} < t, \lambda > Z(d\lambda), t \in G \tag{3}$$

where $Z : \mathcal{B}(\widehat{G}) \to (L_0^2(P;\mathcal{H}))^* \supseteq L_0^2(P)\widehat{\widehat{\otimes}}\mathcal{H}$, the tensor product space with the least cross-norm and $L_0^2(P)\widehat{\otimes}\mathcal{H} \cong (L_0^2(P)\widehat{\widehat{\otimes}}\mathcal{H})^*$. If $\mathcal{H}$ or $L_0^2(P)$ is not finite dimensional, then the above containment is proper (cf. e.g., Rao (1975), p. 1178 with $L^\rho = L_0^2$ and $\mathcal{X} = \mathcal{H}$, and the general exposition of Gilbert and Leih (1980)). In any case the semi-variation of Z in (3) can be defined in different senses:

$$\|Z\|(A) = \sup\{\|\ell \circ Z\|(A) : \|\ell\| \leq 1\}, \tag{4}$$

where $\|\ell \circ Z\|(\cdot)$ is the ordinary semi-variation of $\ell \circ Z : \mathcal{B}(\widehat{G}) \to L_0^2(P)$. On the other hand, let

$$|||Z|||(A) = \sup\{|\sum_{i=1}^{n}(\ell_i \circ Z)(A_i)| : A_i \in \mathcal{B}(A), \text{ disjoint}, \|\ell_i\| \leq 1, n \geq 1,$$
$$\ell_i \in (L_0^2(P;\mathcal{H})^*\}. \tag{5}$$

Restricting ℓ_i, one gets (4) from (5), so that $\|Z\|(A) \leq |||Z|||(A)$ always, with strict inequality sometimes. [In the above mentioned Lanford example, $|||Z|||(\widehat{G}) = +\infty$, with strict inequality here.] Since by hypothesis $\sup_t \|X_t\|_2 < \infty$, (4) is always finite even if $A = \widehat{G}$. Thus the distinction is necessary. Viewing $L_0^2(P;\mathcal{H})$ differently, a general form of an integral representation of weak vector harmonizability will be presented, and then the dilation problem will be discussed.

3.2 *More on vector integral representations.* The range space of the random field X considered above has an alternative description: if f, g are in $L_0^2(P;\mathcal{H}), A \in L(\mathcal{H})$, then $Af \in L_0^2(P;\mathcal{H})$ and if one defines

$$[f,g] = \int_\Omega f \otimes \overline{g}\, dP, \text{ then } [f,g] \in L(\mathcal{H}), |\text{ trace } [f,g]| < \infty.$$

Hence letting

$$\tau([f,g]) = \int_\Omega < f, g > dP, \|f\|_{2,\tau}^2 = \tau([f,f]) \tag{6}$$

one has $(L_0^2(P;\mathcal{H}), \|\cdot\|_{2,\tau})$ to be a Hilbert space where $f \otimes \overline{g}$ is the formal tensor product of the vectors f, g and $< f, g > (\omega) \in \mathcal{H}$. These properties motivate the next abstraction.

If $\mathcal{H}$ is a Hilbert space and $\mathcal{T}(\mathcal{H})$ is the set of trace class operators in $L(\mathcal{H})$, let $\mathcal{X}_0$ be the vector space on which is given a mapping: $[\cdot,\cdot] : \mathcal{X}_0 \times \mathcal{X}_0 \to \mathcal{T}(\mathcal{H})$ with the following properties:

(i) $[x,x] \geq 0, = 0$ iff $x = 0$, (ii) $[x+y,z] = [x,z] + [y,z]$, (iii) $[Ax,y] = A[x,y]$, for each $A \in L(\mathcal{H})$, and (iv) $[x,y]^* = [y,x]$, where '*' is the adjoint operation in $L(\mathcal{H})$.

The mapping $[\cdot,\cdot]$ is called a Gramian on $\mathcal{X}_0$. If $\|x\|_\tau^2 = \tau([x,x]) = \text{trace}\,([x,x])$, let $\mathcal{X}$ be the completion of $\mathcal{X}_0$ under the norm $\|\cdot\|_\tau$. The space $(\mathcal{X}, \|\cdot\|_\tau)$ is termed the *normal Hilbert $L(\mathcal{H})$-module*, and $L_0^2(P;\mathcal{H})$ is an example. If $\mathcal{H}, \mathcal{K}$ are two Hilbert spaces over the same scalar field, then Kakihara (1985), who studied harmonizability on these spaces, shows that the space $HS(\mathcal{K},\mathcal{H})$ of Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$ is a normal $L(\mathcal{H})$-module if $[x,y] = xy^*$ and $A \cdot x = Ax$, for $x,y \in HS(\mathcal{K},\mathcal{H})$ and $A \in L(\mathcal{H})$. Another example is $\mathcal{K}^q, 1 \leq q \leq \infty$, the cartesian product of Hilbert spaces $\mathcal{K}$, which can be made a normal $L(\mathcal{H}_q)$-module if $\mathcal{H}_q = \ell_q^2$, the coordinate Hilbert space.

Thus let $X : G \to \mathcal{X}$ be a mapping on an LCA group G into a normal $L(\mathcal{H})$-module $\mathcal{X}$. It is *weakly harmonizable* if one has

$$X(t) = \int_{\widehat{G}} < t, \lambda > Z(d\lambda), t \in G, \tag{7}$$

where the integral is in the D-S sense and $Z(\cdot)$ is a bounded regular σ-additive function on $\mathcal{B}(\widehat{G})$ into $\mathcal{X}$. The regularity here is in the strong sense, i.e., for each $A \in \mathcal{B}(\widehat{G})$ and $\varepsilon > 0$, there exist compact F and open O of $\widehat{G}$ such that $F \subset A \subset O$ and $\|Z\|(O - F) < \varepsilon, \|Z\|$ being the semi-variation of Z. Considering other variations one gets other harmonizabilities. The V-boundedness is defined similarly. Then one has the following:

Theorem 1. *A random field $X : G \to \mathcal{X}$, a normal $L(\mathcal{H})$-module, is weakly harmonizable iff it is* V*-bounded and continuous in the norm topology of $\mathcal{X}$.*

Although the statement is familiar in view of the earlier work, there is considerable technical machinary to be developed for its proof. Kakihara (1985, 1986) has done this and obtained other extensions.

3.3 *Dilation of harmonizable processes.* The dilation problem in the present context is the statement that (under minimal conditions) a given harmonizable process in $L_0^2(P)$ is the orthogonal projection of some stationary process from a super Hilbert space containing $L_0^2(P)$. That every such projection defines a weakly harmonizable process is the easy part. The reverse direction, depending on a suitable construction is hard and depends generally on the Grothendieck inequality, given as Theorem 1.5.1. [The details of this construction can be found, e.g., in Rao (1982), p. 326.] The corresponding result can be continued for a normal $L(\mathcal{H})$-module valued harmonizable process under some restrictions resulting in the finiteness of the Fréchet variation of the bimeasure of the representing stochastic measure. But such a construction fails for ultra weakly harmonizable processes since the corresponding bimeasure has infinite Fréchet variation. This is also verifiable with Lanford's example.

It is a surprizing fact that the Grothendieck inequality should play a vital role in the dilation problem. [For the strongly harmonizable case, one does not need this inequality, cf. Abreu (1970).] On the other hand, given the existence of a stationary dilation, one can prove Grothendieck's inequality for positive definite bimeasures by considering its Fourier transform, through the MT-integration, which qualifies to be a covariance function. Then one can construct a centered Gaussian harmonizable process with this covariance function via the Kolmogorov existence theorem and dilate it. The desired inequality follows from this. It is also observed by Chatterji (1982) and others. However, a general form of Grothendieck's inequality for not necessarily positive definite bimeasures does not seem possible in this way. The simplest known proof of the general inequality, due to Blei (1987), uses a probabilistic argument in its key parts. On the other hand, the Lanford example shows that there can be no infinite dimensional analog of Grothendieck's inequality. Some special types of dilations weaker than the above are possible. To understand this situation, a problem with the noncommutative harmonizable random field will be indicated here (cf., Rosenberg (1982), for a related study).

If $X : G \to L_0^2(P)$ is a random field with covariance r given by $r(g_1, g_2) = E(X_{g_1}\overline{X}_{g_2})$, then there is a right, a left, and a two sided stationary concepts available, and so one has to discuss the dilation problem for each class. Thus X is *left* [*right*] *stationary* iff

$$r(gg_1, gg_2) = \tilde{r}(g_2^{-1}g_1), [r(g_1g, g_2g) = \tilde{r}(g_1g_2^{-1})], \tag{8}$$

and it is *two sided stationary* if it is both right and left stationary. Thus for a reasonable dilation problem one restricts the class of dilations admitted. A weaker condition is obtained from a combination of the left-right properties. Thus X is termed *hemihomogeneous*, by Ylinen (1986), if its covariance r can be expressed as:

$$r(g_1, g_2) = \rho_1(g_2^{-1}g_1) + \rho_2(g_1g_2^{-1}), g_1, g_2 \in G, \tag{9}$$

where ρ_1, ρ_2 are positive definite covariances on G. Then Ylinen's result implies the following statement wherein the weak harmonizability of Definition 2.3.1 is used.

Theorem 1. *Let G be a separable unimodular group and $X : G \to L_0^2(P)$ be a continuous random field in $L_0^2(P)$. Then X is weakly harmonizable iff it has a hemihomogeneous dilation $Y : G \to L_0^2(\tilde{P}) \supset L_0^2(P)$, so that $X(g) = (QY)(g), g \in G$, where Q is an orthogonal projection of $L_0^2(\tilde{P})$ onto $L_0^2(P)$.*

Actually the result was given by Ylinen (1987) for all locally compact groups using his treatment of Fourier transforms through Eymard's (1964) approach and C^*algebras. It reduces to the present case, and the treatment simplifies slightly for G as given here. Thus the dilation problem has additional difficulties to consider for vector valued random fields.

3.4 *Multiplicity and least squares prediction.* The problem is usually considered in two stages. First, one wants to predict a future value of the process or field based on the past and present, and this assumes that the indexing group G must have a partial order (or a cone) in its structure. The most natural examples are $G = \mathbb{R}$ or $\mathbb{Z}$, and in this case one proceeds as follows.

Let $X : \mathbb{R} \to L_0^2(P)$ be a process, with a continuous covariance, and be nondeterministic in that $\bigcap_{t \in \mathbb{R}} \overline{\mathrm{sp}}\{X(s) : s \leq t\} = \{0\}$. This is not a serious restriction in view of Wold's decomposition. Then there is a minimal integer $N \geq 1$, called the *multiplicity* of the process, jointly Borel measurable functions $F_n : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and orthogonally scattered measures Z_n such that

$$X(t) = \sum_{n=1}^{N} \int_{-\infty}^{t} F_n(t, \lambda) Z_n(d\lambda), t \in \mathbb{R}, \tag{10}$$

with $F_n(t, \lambda) = 0$ for $t < \lambda$ and $\sum_{n=1}^{N} \int_{-\infty}^{\infty} |F_n(t, \lambda)|^2 \mu_n(d\lambda) < \infty$ where $\mu_n(A) = E(|Z_n(A)|^2)$, $A \in \mathcal{B}(\mathbb{R})$. Even when X is strongly harmonizable it is possible that $1 \leq N \leq \infty$. If $N = 1$, one has a simple Karhunen process. For stationary processes $N = 1$ always. Here harmonizability and bimeasure theory play a secondary role. Also in any given problem, the F_n's are not unique. They arise from the Hellinger-Hahn theory and are not easily obtained.

The second approach is to study the (simpler) strongly harmonizable case when its bimeasure has also a (spectral) density that is rational. Then one may extend the classsical theory of multivariate prediction in analogy with various results in, e.g., Rozanov's (1967) monograph. For a subclass of these processes having "factorizable spectral measures" the corresponding analysis, worked out in a preliminary study, is promising and nontrivial. The sample path behavior of the harmonizable process is another avenue, and most of it is a potentially interesting area.

3.5 *A final remark.* The main idea underlying the analysis of harmonizable processes and fields is a use of the powerful Fourier analytic methods. It is thus true that on an LCA group G, one has the representation of a harmonizable function X as:

$$X(g) = \int_{\widehat{G}} < g, \lambda > Z(d\lambda), g \in G. \tag{11}$$

Now the function $< \cdot, \cdot >: G \times \widehat{G} \to \mathbb{C}$ is jointly continuous, bounded and $< \cdot, \lambda >$ is periodic uniformly relative to λ in relatively compact open sets. But this fact motivates a study of $X(\cdot)$ in (11) in which $< \cdot, \cdot >$ is replaced by $f : G \times \widehat{G} \to \mathbb{C}$, such that $f(\cdot, \lambda)$ is almost periodic uniformly (in λ) relative to $D \subset \widehat{G}$, bounded open sets. Thus the resulting random field X becomes

$$X(g) = \int_{\widehat{G}} f(g, \lambda) Z(d\lambda), g \in G, \tag{12}$$

where $Z : \mathcal{B}(\widehat{G}) \to L_0^2(P)$ is a stochastic measure. Such a random field may be termed *almost weakly harmonizable*; and if the bimeasure induced by Z has finite Vitali variation then one has the case of an *almost strongly harmonizable* family. These form a subfamily of class (C) of Section 2.1, but have a better structure than the general members. A few properties of the latter class when $G = \mathbb{R}$, have been discussed in Rao (1978). It has a good potential for further study because there is a considerable amount of available results on almost periodic functions with important applications both when $G = \mathbb{R}$ and general locally compact groups. These and many of the (vector) extensions, having interesting structure, present a rich source of problems for research.

Notation: Throughout the paper, definitions, propositions, theorems are serially numbered. Thus m.n.p. denotes that object in part m, section n, and name p. In a given part, m is omitted and in a section n is also dropped. All unexplained symbols, if any, are as in Dunford and Schwartz (1958). Also $\mathbb{R}$ denotes reals, $\mathbb{C}$ - complex numbers, and $\mathbb{Z}$ for the integers. Almost all the notation used is standard.

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