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A CONVERGENT PROCESS OF PRICE ADJUSTMENT AND GLOBAL NEWTON METHODS

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Section 1

One goal of this paper is to give a relation between such diverse parts of economic theory as the Arrow–Block–Hurwicz dynamics of price adjustment and Scarf's algorithm for finding economic equilibria. The underlying concept is an ordinary differential equation, which we call a 'Global Newton' one, associated to a system of n real functions, $f_1, \ldots, f_n$, of n real variables, $x_1, \ldots, x_n$.

The key feature of this differential equation is that, under suitable hypotheses, its solution will tend to a vector $(x_1^*, \ldots, x_n^*)$ satisfying

$$f_1(x_1^*, \ldots, x_n^*) = 0, \ldots, f_n(x_1^*, \ldots, x_n^*) = 0, \tag{1}$$

or in vector notation,

$$f(x^*) = 0. \tag{1'}$$

In fact in this way an algorithm for solving (1) is provided. The differential equation itself has the form

$$\mathrm{D}f(x)\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda f(x). \tag{2}$$

where the sign of λ is determined by the sign of Det $\mathrm{D}f(x)$. Here $\mathrm{D}f(x)$ is the linear transformation with matrix representation

$$\mathrm{D}f(x) = \left(\frac{\partial f_i}{\partial x_j}\right)(x),$$

and λ is a real valued function of x, unprescribed as yet.

*This work was done at Yale in the fall of 1974, and I would like to thank the Cowles Foundation and the Mathematics Department there for their hospitality.

We give now some background of our work. Suppose given a world of l commodities and corresponding prices each measured by non-negative real numbers. A *price system* is a vector $p = (p_1, \ldots, p_l)$, $p_i \geqq 0$, where p_i represents the price of a unit of the ith commodity. Let R_l denote l-dimensional real Cartesian space, and

$$R^l_+ = \{x \in R^l \,|\, x_i \geqq 0\}.$$

We suppose an economic setting which provides demand and supply functions $D, S: R^l_+ - 0 \to R^l_+$, with D, S functions of prices in R^l_+. The condition for economic equilibrium (or *price equilibrium*) is then 'supply equals demand' or $D(p) = S(p)$ as a condition on the price system p. The derived *excess demand* is the function $\xi: R^l_+ - 0 \to R^l$ given by $\xi(p) = D(p) - S(p)$. Thus $\xi_i(p)$ is the excess demand for the ith good at prices p. Natural hypotheses on ξ are:

(a) $\xi(\lambda p) = \xi(p)$ for $\lambda > 0$ (homogeneity),

(b) $p \cdot \xi(p) = 0$ (inner product), *Walras Law*
[for motivation see Quirk and Saposnik (1968)],

(c) if $p_i = 0$, then $\xi_i(p) > 0$.

Actually ξ can be taken as the primitive economic notion and will be assumed to satisfy (a), (b) and (c). Boundary conditions derived from a micro-economic setting, while technically more complex, are similar in principle. We hope to pursue this matter elsewhere.

By (a) it makes sense to normalize price systems or represent them in the space

$$S^l_+ = \{p \in R^l_+ \,|\, \Sigma(p_i)^2 = 1\}.$$

By (b) $\xi(p)$ is tangent to S^l_+ at p so ξ is a vector field (or ordinary differential equation) on S^l_+. By (c) this vector field ξ points in on the boundary so that the Hopf theorem [see Milnor (1965)] yields a price system p^* with $\xi(p^*) = 0$ or $D(p^*) = S(p^*)$. This is proof of the existence of economic equilibrium. A constructive, 'dynamic' proof of this existence result is given in section 4.

The question goes back to Walras; is there some process (economic and/or mathematical!) which leads to an equilibrium?

On price space S^l_+ the differential equation

$$\mathrm{d}p/\mathrm{d}t = \xi(p) \tag{3}$$

is well-defined and the equilibria (i.e., zeros) of this differential equation coincide with the price equilibria for the excess demand function ξ. Solutions of (3) can be thought of as price adjustments where a positive excess demand for some good raises the price of that good.

Do such solutions lead to equilibria?

The answer is yes according to Arrow, Hurwicz and Block (1958, 1959) provided a further substantial hypothesis 'Gross Substitutes' is made on ξ (see section 3).

The answer is no in general according to Scarf (1960) who constructs an example of excess demand derived from economically plausible individual demands with the property: almost all solutions oscillate for all time.

Subsequently, Sonnenschein, Mantel and Debreu [see Debreu (1974)] showed that ξ could be pactically arbitrary and still be derived from preference relations of classical types.

We find here a modification of the differential equation (3) whose solutions converge to equilibria under quite general conditions, in fact in situations with multiple equilibria (where previous methods failed to converge). This modification grew in part from a study of another development in mathematical economics, Scarf's algorithm (1960).

Scarf found an algorithm for finding fixed points of continuous maps of a simplex, which he applied in particular to the location of economic equilibria. This method is a simplicial method and turned out to be closely related to Hirsch's paper (1963) as can be seen especially clearly in the papers of Eaves and Eaves–Scarf (1975). Furthermore a differential analogue was already suggested in Hirsch's paper and this differential analogue was developed for algorithmic purposes (in a fixed point context) in the paper of Kellog, Li and Yorke (to appear).

What we do here is to look at the Scarf algorithm in the context of eq. (1). By using the differential point of view, a simple derivation leads to the differential equation (2). In the next section we will see how this goes. In section 3, these ideas are applied to the excess demand function ξ described above.

For a background on ordinary differential equations written in the style of this paper, see Hirsch–Smale (1974).

Many conservations with Curtis Eaves and especially Herb Scarf were very helpful for this paper. Also subsequent work with Moe Hirsch has had the effect of smoothing the account given here.

Section 2

We consider the purely mathematical problem here of solving a system of n non-linear equations in n unknowns through an associated ordinary differential equation. Suppose the domain M for these functions is given as a bounded open subset of real Cartesian space R^n together with its boundary ∂M which we suppose to be smooth (i.e., a submanifold). Thus ∂M is contained in M, and M is a closed set.

Let

$$f\colon M \to R^n, \quad f(x) = (f_1(x), \ldots, f_n(x)), \quad x = (x_1, \ldots, x_n) \in M,$$

be a map with continuous first and second derivatives (i.e., C^2). Let $\mathrm{D}f(x): R^n \to R^n$ be the first derivative of f at x as a linear map so that $\mathrm{D}f(x)$ has a matrix representation in terms of the partial derivatives,

$$\mathrm{D}f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right).$$

We seek a solution of

$$f(x^*) = 0, \qquad x^* \in M. \tag{4}$$

The 'Global Newton' equation is the ordinary differential equation on M given by

$$\mathrm{D}f(x)\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda f(x), \tag{5}$$

where λ is a real number unspecified for the moment, but the sign of λ is determined by the sign of Det $\mathrm{D}f(x)$ (Det A means the determinant of A).

In general one cannot expect solutions of (4) to exist; a boundary condition is necessary to assure such an existence. To see the ideas most clearly, we postulate a simple boundary condition, but one that can be substantially weakened and still have the results valid.

Boundary condition. For each $x \in \partial M$, $\mathrm{Det}\,\mathrm{D}f(x) \neq 0$, and there is a choice, (a) sign $\lambda(x) =$ sign Det $\mathrm{D}f(x)$, all $x \in \partial M$, or (b) sign $\lambda(x) = -$sign Det $\mathrm{D}f(x)$, all $x \in \partial M$, which make $-\lambda(x)\,\mathrm{D}f(x)^{-1}f(x)$ point into M at each $x \in \partial M$.

We specify the sign of λ accordingly. Thus we will assure that either (a) holds for all $x \in M$ with $\mathrm{Det}\,\mathrm{D}f(x) \neq 0$ or that (b) holds; and that $-\lambda \mathrm{D}f(x)^{-1}f(x)$ points into M at each $x \in \partial M$.

Define $E = f^{-1}(0)$ so that E is the solution set of (4).

We may now state the main result:

Theorem A. Let $f: M \to R^n$ be C^2 and satisfy the boundary condition. There exists a canonically defined subset Σ of measure 0 in ∂M such that if $x_0 \in \partial M$, $x_0 \notin \Sigma$, then there exists a unique C^1 solution φ: $[t_0, t_1) \to M$ of (5) *starting at x_0 [i.e., $\varphi(t_0) = x_0$] with $\|\mathrm{d}\varphi/\mathrm{d}t\,(t)\| = 1$, and t_1 maximal, $t_1 \leqq \infty$. This solution converges to E as $t \to t_1$.*

Almost all C^1 functions $f: M \to R^n$ have the property:

Genericity hypothesis. If $x \in E$, then $\mathrm{D}f(x)$ is non-singular.

If f satisfies the genericity hypothesis then E must be a finite set. Furthermore:

Theorem B. If $f: M \to R^n$ is C^2, satisfies the boundary condition and genericity hypothesis, then Σ of Theorem A is a closed *set (of measure* 0) *and the solution $\varphi(t)$ starting at any $x_0 \in \partial M$, $x_0 \notin \Sigma$, converges to a single point x^* satisfying* $(x^*) = 0$.

Before proving Theorems A and B, we note other forms (5) may assume. For example one can simply take $\lambda = \mathrm{Det}\, \mathrm{D}f(x)$ [or $-\mathrm{Det}\, \mathrm{D}f(x)$ in case (b) of the boundary condition] to obtain

$$\mathrm{D}f(x) \frac{\mathrm{d}x}{\mathrm{d}t} = -(\mathrm{Det}\, \mathrm{D}f(x))f(x). \tag{6}$$

Moe Hirsch has suggested taking $B(x)$ so that with matrix multiplication one has $(\mathrm{Det}\, \mathrm{D}f(x))I = \mathrm{D}f(x)B(x)$, where I is the identity matrix. Then any solution of

$$\mathrm{d}x/\mathrm{d}t = -B(x)f(x) \tag{7}$$

is also a solution of (6) [apply $\mathrm{D}f(x)$ to both sides of (7)]. This gives a desingularized version of (5).

Note that since

$$\mathrm{D}f(x) \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}(x),$$

one can express (5) in the simple form

$$\mathrm{d}f/\mathrm{d}t = -\lambda f. \tag{8}$$

Eq. (8) is the 'target space' version of (5), and from it one can derive a geometric interpretation of the method.

In case $\mathrm{D}f(x)$ is singular for no x in M, we may choose $\lambda = 1$ [case (a) of boundary condition] and rewrite (5) in the form

$$\mathrm{d}x/\mathrm{d}t = -\mathrm{D}f(x)^{-1}f(x). \tag{9}$$

Theorems A and B apply, modified only by a different choice of parameterization of φ. Writing $\mathrm{d}x/\mathrm{d}t$ as a finite difference, (9) becomes

$$x_{n+1} - x_n = -\mathrm{D}f(x_n)^{-1}f(x_n), \tag{10}$$

or exactly Newton's method of iteration (which of course is always effective in some neighborhood of E). In this case (assuming the boundary condition) our proof demonstrates the existence of a unique solution x^* of $f(x^*) = 0$, and that *every* solution of (9) will tend to x^*. This special case is well-understood classically; see Ostrowski (1973).

For the proof of Theorems A and B, given $f: M \to R^n$ with $E = f^{-1}(0)$, we define an associated map $g: M - E \to S^{n-1}$ with target space S^{n-1}, the unit sphere in R^n. Let

$$g(x) = \frac{f(x)}{\|f(x)\|}, \qquad \|y\|^2 = \sum_{i=1}^{n} y_i^2,$$

then g is a C^2 map and we may apply the following proposition [implicit function - Sard, see Abraham-Robin (1967), Milnor (1965)]:

Proposition 1. For any C^2 map $g: M - E \to S^{n-1}$ (where M is n-dimensional), there is a set Λ (the 'exceptional set of critical values') of measure 0 in S^{n-1} with the property that if $y \in S^{n-1}$, $y \notin \Lambda$, then $g^{-1}(y)$ is a C^2 non-singular curve in $M - E$ (more properly, a 1-dimensional submanifold).

One can describe Λ as the image under g of the set C of critical points of g where $x \in C$ if the rank of $\mathrm{D}g(x)$ is less than $n-1$.

We also know:

Proposition 2. For each x in $M - E$, the kernel of the derivative $\mathrm{D}g(x)$ [*i.e.*, $\{v \in R^n | \mathrm{D}g(x)(v) = 0\}$] *is the set*

$$\mathrm{Ker}\,\mathrm{D}g(x) = \{v \in R^n | \mathrm{D}f(x)(v) = \lambda f(x), \text{ some } \lambda \in R\}.$$

For the proof one uses standard calculus techniques to obtain

$$\mathrm{D}g(x)(v) = \frac{\mathrm{D}f(x)(v)}{\|f(x)\|} - \frac{f(x)}{\|f(x)\|^3}\,(f(x) \cdot \mathrm{D}f(x)(v)),$$

for all $x \in M$, $v \in R^n$. From this formula one identifies $\mathrm{Ker}\,\mathrm{D}g(x)$ as in Proposition 2.

We remark that from the chain rule it follows that $\mathrm{Ker}\,\mathrm{D}g(x) \supset \mathrm{Ker}\,\mathrm{D}f(x)$ and that if $\mathrm{D}f(x)$ is non-singular then $\mathrm{D}g(x)$ is surjective. Also if the corank of $\mathrm{D}f(x)$ is one, then the corank of $\mathrm{D}g(x)$ is one or two.

The tangent vectors at $x \in g^{-1}(y)$ to the curve $g^{-1}(y)$ in Proposition 1 are precisely the vectors in $\mathrm{Ker}\,\mathrm{D}g(x)$ in Proposition 2.

For the proof of Theorem A, apply Proposition 1 to the map g and let Λ be the exceptional set in S^{n-1}. Let $\Sigma = \{x \in \partial M | g(x) \in \Lambda\}$. Then since the restriction $g_0: \partial M \to S^{n-1}$ of g, is a local diffeomorphism, the measure of Σ is zero.

Note also that if the genericity hypothesis is made on f, then the set of critical points C of g is compact; then Λ is closed and $g_0^{-1}(\Lambda) = \Sigma$ is also closed.

Now let $x_0 \in \partial M$, $x_0 \notin \Sigma$, $y_0 = g(x_0)$, so that $g^{-1}(y_0)$ is a 1-dimensional submanifold in $M-E$. This curve has a component γ starting at x_0; by a simple orientation argument along γ, using the boundary condition, this curve γ cannot meet ∂M in any other point than x_0. Therefore since γ is a closed set and a 1-dimensional manifold in $M-E$, γ can be parameterized by t in $[t_0, t_1)$ and as $t \to t_1$, the limit points must all be 'at ∞' in $M-E$, or in E.

It remains for the proof of Theorems A and B to identify the solutions of the Global Newton equation (5) starting at x_0 with γ. But this is done by Proposition 2.

We finish section 2 with a sequence of remarks:

(A) I have been working with Moe Hirsch on the development of this algorithm and its implementation on a computer (in particular HP-65 and PDP 11). We hope to publish our results soon.

(B) There is a gradient version of the Global Newton, which goes as follows: For $x \in M$ define a bilinear symmetric form $\alpha_x(v, w) = \mathrm{D}f(x)(v) \cdot \mathrm{D}f(x)/(w)$. The form α defines a positive (though in general indefinite) metric on M.

Also let $\phi: M \to R$ be the function $\phi(x) = \|f(x)\|^2$. Then consider: $\mathrm{d}x/\mathrm{d}t = -\lambda \operatorname{grad} \phi(x)$, $\operatorname{sgn} \lambda = \operatorname{sgn} \operatorname{Det} \mathrm{D}f(x)$ [or $-\operatorname{sgn} \operatorname{Det} \mathrm{D}f(x)$ in (b) of boundary condition] where the gradient is taken with respect to α. This equation will agree with (5) whenever $\lambda \neq 0$.

(C) One can weaken the boundary condition of $f: M \to R^n$. For $x \in M$ let $K_x = \{v \in R^n | \mathrm{D}f(x)(v) = \lambda f(x), \text{ some } \lambda \in R\}$. Then one may relax the condition that $\operatorname{Det} \mathrm{D}f(x) \neq 0$ for $x \in \partial M$ by imposing $\dim K_x = 1$ for $x \in \partial M$, etc.

The theory still applies.

(D) One can also prove the theorems in case M has a piecewise smooth boundary. For example let $b_i: R^n \to R$, $i = 1, \ldots, N$, be C^1 functions which satisfy: if $b_{i_1}(x) = 0, \ldots, b_{i_k}(x) = 0$ then the vectors $\operatorname{grad} b_{i_1}(x), \ldots, \operatorname{grad} b_{i_k}(x)$ are linearly independent. Let M be the region defined by

$$M = \{x \in R^n | b_i(x) \geqq 0, \quad \text{all} \quad i = 1, \ldots, N\}.$$

Suppose that M is bounded. For such M an analogue of the boundary condition is:

Piecewise smooth boundary condition. For each $x \in \partial M$, $\operatorname{Det} \mathrm{D}f(x) > 0$ and $\mathrm{D}b_i(x)(\mathrm{D}f(x)^{-1} f(x)) < 0$ for each i, such that $b_i(x) = 0$. (Theorems A and B are valid with this boundary condition.)

(E) Solutions of our basic equation will cross smoothly much of the singularity set of f [where $Df(x)$ has corank 1 and f is a 'fold' singularity]. 'Branch points' give trouble to the solution curves of (5). These are points where $Dg(x)$ has corank 2. Also there can be families of periodic solutions of (5) (of course branch points and periodic solutions don't invalidate Theorems A and B).

(F) An abstract setting to the theory can be given for maps $f: M \to V$ when M is a compact oriented connected manifold with boundary and V is a vector space of the same dimension equipped with an inner product.

(G) One may replace (5) by the differential equation

$$Df(x)\frac{dx}{dt} = -(\lambda_1 f_1(x), \ldots, \lambda_n f_n(x)),$$

where each sgn λ_i is specific as before, and the rest of the section remains valid.

Section 3

Here the results of section 2 are applied to the case where the function f is the excess demand of some economy. A slightly different context than section 1 is chosen, because the excess demand there is not given as a map from a domain into a linear space of the same direction.

Following a certain amount of tradition in theoretical economics we suppose a distinguished commodity or a 'numeraire' [Quirk–Saposnik (1968)] so that prices of the other goods are measured in terms of this numeraire good.

Thus suppose there are $(l+1)$ commodities, with prices denoted by non-negative real numbers $p_0, \ldots, p_l$, where p_0 is the price of a unit of the numeraire good. We work in the domain of price systems $(p_0, \ldots, p_l)$ which satisfy $p_0 > 0$. In this domain, by the homogeneity property of price systems (see section 1), one can normalize a price system by dividing by p_0. Thus a price system $(p_0, \ldots, p_l)$ can be represented by a unique point

$$(p_1/p_0, \ldots, p_l/p_0) \text{ in } R^l_+.$$

With this interpretation, the space of price systems is R^l_+. We suppose C^2 excess demand functions $\xi_0(p_0, \ldots, p_l)$, $\xi_1(p_0, \ldots, p_l), \ldots, \xi_l(p_0, \ldots, p_l)$ are given from some economic setting, and that Walras Law is valid. Thus

$$\xi_0(p) = -\sum_{i=1}^{l} \frac{p_i}{p_0}\xi_i(p),$$

so that the equations

$$\xi_i(p^*) = 0, \qquad i = 1, \ldots, l,$$

are necessary and sufficient conditions for economic equilibrium.

Thus it may be supposed that the excess demand is represented by a map $\xi: R^l_+ \to R^l$ where source variables are normalized price systems and target variables are (excess demand for) commodities numbered one through l. The zeros of this map ξ are precisely the price equilibria.

We now state a boundary condition on ξ, which is slightly more subtle than that in section 1, and compares directly to that of section 2.

Heuristically, the condition says that as $p_i \to 0$, $\xi_i(p) \to \infty$ and $\partial\xi_i/\partial p_i(p) \to -\infty$ as in fig. 1. Of course one has to take into account the fact that ξ is a function of several variables.

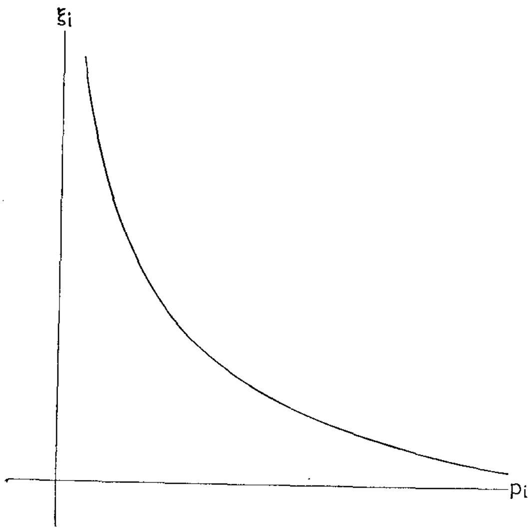

Fig. 1

Boundary condition on ξ. For $p \in \partial R^l_+$ let J_p be the set of indices such that $p_i = 0$. Then the system of linear equations

$$\sum_{j=1}^{l} \frac{\partial \xi_k}{\partial p_j} v_j = -\xi_k, \qquad k = 1, \ldots, l, \tag{11}$$

has a unique solution $(v_1, \ldots, v_l)$ and $v_i > 0$ for $i \in J_p$.

The boundary condition will be satisfied at p for example if the system (11) is dominated by terms of the form,

$$\frac{\partial \xi_i}{\partial p_i} v_i = -\xi_i, \qquad i \in J_p,$$

$$\frac{\partial \xi_i}{\partial p_i} < 0, \qquad \xi_i > 0$$

and hence the heuristic justification above.

Now R^l_+ is not bounded and so we are not yet in a position to apply section 2. Note that the *boundary condition on* ξ is still valid for p_i close enough to 0, $i \in J_p$, so that one does not need $p_i = 0$ literally. Now extend the boundary condition to the numeraire good or to all $l+1$ prices and commodities to obtain the *extended boundary condition* on ξ. Note that going to ∞ in R^l_+ corresponds to p_0 going to 0; so the extended boundary condition on ξ make the solutions of the Global Newton equation, which start near ∞ (or outside some bounded set), behave as if they started on the boundary.

Now the hypotheses of section 2 are satisfied for the excess demand and so Theorems A and B apply to yield:

Theorem C. Suppose as above, the excess demand $\xi : R^l_+ \to R^l$, defined in terms of the numeraire satisfies the extended boundary condition on ξ. Then (maximal) solutions of

$$\sum_{j=1}^{l} \frac{\partial \xi_i}{\partial p_j} \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\lambda \xi_i, \qquad \operatorname{sgn} \lambda = (-1)^l \operatorname{sgn} \operatorname{Det}\left(\frac{\partial \xi_i}{\partial p_j}\right), \tag{12}$$

starting on the boundary of R_+ (or near ∞) except for some set Σ of measure 0 *in the boundary of R^l_+ will converge to the set of price equilibria.*

It can be checked that ξ derived from a 'regular' economy satisfies the genericity hypothesis of section 2.

Theorem D. If ξ in Theorem C is derived from a regular economy in the sense of Debreu (1970) *or Smale* (1974) *then the set Σ will be closed and a solution starting in $\partial R^l_+ - \Sigma$ will converge to a single price equilibrium.*

Say that [e.g., as in Quirk–Saposnik (1968)] an excess demand $\xi_0, \ldots, \xi_l$ satisfies the *gross substitutes* condition or G.S. if $\partial \xi_i / \partial p_i < 0$ each i and $\partial \xi_i / \partial p_j > 0$ whenever $i \neq j$, $i, j = 0, \ldots, l$.

If ξ satisfies G.S. then $\mathrm{D}\xi(p)$ is non-singular for all price systems p. Here $\mathrm{D}\xi(p) = \partial \xi_i(p)/\partial p_i$, $i, j = 1, \ldots, l$. This follows from work of L. McKenzie since $\mathrm{D}\xi(p)$ has a 'quasidominant diagonal', see Quirk–Saposnik (1968, pp. 167 and 173). From section 2, the remark after (10) we have:

Theorem E. Let ξ be as above (Theorem C) and satisfy G.S. then there exists a unique price equilibrium p^* and every *solution of* (12) (*with, e.g.*, $\lambda = 1$) *converges to p^*.*

It remains an interesting question, to what extent and in what situations one can find a valid economic interpretation for (12). It is clearly more subtle than the classical equation of section 1, $\dot{p} = \xi(p)$.

Consider the simplest case where ξ satisfies G.S. and choose $\lambda = 1$. Then if the off diagonal terms of $\partial\xi_i/\partial p_j$ are neglected, the equations become the equations of Arrow, Hurwicz and Block.

Our equations contain more explicitly, relations between the various markets in their effect of price adjustment.

We end this section with a couple of remarks.

In Arrow–Hahn (1972, p. 303), a similar ordinary differential equation is discussed very briefly. Also this book has a general account of the Arrow, Hurwicz and Block results and related work.

One can avoid choosing a numeraire by applying the theorems of section 2 to other functions derived from the excess demand. We give two explicit possibilities. First let

$$\Delta_1 = \{p \in R^l_+ \mid \textstyle\sum p_i = 1\},$$

$$\Delta_0 = \{y \in R^l \mid \textstyle\sum y_i = 0\}.$$

Define

$$f: \Delta_1 \to \Delta_0,$$

$$f(p) = l\xi(p) - \sum_{i=1}^{l} \xi_i(p)(1, 1, \ldots, 1),$$

where ξ is the excess demand of section 1. The price equilibria are solutions of $f(p) = 0$ and section 2 applies.

Finally, H. Scarf has suggested

$$f: \Delta_1 \to \Delta_0; \qquad f(p) = (p_1\xi_1(p), \ldots, p_l\xi_l(p)),$$

so that $p_i\xi_i$ are the values of the excess demand, for use of the Global Newton method. He has shown me that $\mathrm{D}f(p)$ is never singular in the G.S. case.

Section 4

The goal of this section is to present existence proofs of the main theorems of general equilibrium theory in the spirit of the preceding sections. Giving the existence proofs in this context achieves unity by bringing together existence theorems, algorithms, and dynamic questions. Also the proofs are simpler than those going through algebraic topology. The ideas are close to those of Hirsch's paper and to Li and Yorke, but we retain the equation approach rather than the fixed point approach. Roughly speaking these proofs work with simpler boundary questions than the preceding sections, but the process itself seems more complicated. Results are given in increasing generality.

Theorem F. Let $\Delta_1 = \{p \in R^l | p_i \geqq 0, \sum p_i = 1\}$ *be the price simplex and let* $\Delta_0 = \{p \in R^l | \sum p_i = 0\}$; *let* $\phi: \Delta_1 \to \Delta_0 \subset R^l$ *be a* C^2 *map,* $\phi(p) = (\phi_1(p_1, \ldots, p_l), \ldots, \phi_l(p_1, \ldots, p_l))$ *such that* $\phi_i(p) > 0$ *if* $p_i = 0$. *Then there is* $p \in \Delta_1$ *with* $\phi(p) = 0$.

Proof. Let $E = \{p \in \Delta_1 | \phi(p) = 0\}$ and $\partial\Delta_1 = \{p \in \Delta_1 | p_i = 0 \text{ some } i\}$. Then for each $p \in \Delta_1 - E$, there is a unique $\lambda = \lambda(p) \geqq 0$ such that $p + \lambda\phi(p)$ is in $\partial\Delta_1$. Define $h: \Delta_1 - E \to \partial\Delta_1$ by $h(p) = p + \lambda(p)\phi(p)$. Note that h is the identity on the boundary. Let ∂_i be the ∂_ith face of Δ_1, so $\partial_i = \{p \in \Delta_1 | p_i = 0\}$. Then h can be checked to be continuous and in fact will be C^2 at a point p such that $h(p)$ belongs to only one ∂_i.

Now by the Sard–implicit function theorems as in section 2, we can choose p_0 in just one face (to be a regular value of h) so that $h^{-1}(p_0)$ is a non-singular curve. Consider the component of $h^{-1}(p_0)$ which starts at p_0. This component must lead to E as one travels along it starting from P_0. Therefore E is not empty. Since E is the solution set of $\phi(p) = 0$, the theorem is proved.

Corollary 1. Let $z: R^l_+ - 0 \to R^l$ *an 'excess demand function', of class* C^2. *We suppose then z satisfies homogeneity, Walras Law and the boundary condition,* $z_i(p) > 0$ *if* $p_i = 0$ *(all as in section* 1*). Then there is some* $p^* \in \Delta_1$ *with* $z(p^*) = 0$. *There exists a price equilibrium.*

Proof. Let ϕ be the map $\phi: \Delta_1 \to \Delta_0$ given by $\phi(p) = z(p) - (\sum_{i=1}^l z_i(p))p$.

Then the following lemma is easily checked:

Lemma. $\phi_i(p) > 0$ *if* $p_i = 0$. *Furthermore if z satisfies,* $z_i(p) \geqq 0$ *if* $p_i = 0$' *instead of the stronger boundary condition, then* ϕ *satisfies the same.*

Apply Theorem F to obtain p^* with $\phi(p^*) = 0$. Then

$$z(p^*) = \left(\sum_{i=1}^{l} z_i(p^*)\right) p^*.$$

Take the dot product of both sides with p^*. This yields by Walras Law that $\sum_{i=1}^l z_i(p^*) = 0$, proving Corollary 1.

Proposition 3. The theorem is valid under the weaker hypotheses:

(a) ϕ *is continuous (not necessarily* C^2*).*

(b) $\phi_i(p) \geqq 0$ *if* $p_i = 0$ *(rather than strict inequality).*

Proof. Let a ϕ be given as in Proposition 3. Given $\varepsilon > 0$, using the Weierstrass approximation theorem, one can easily construct a $\psi : \Delta_1 \to \Delta_0$ satisfying the hypotheses of the theorem with ψ uniformly approximating ϕ on Δ_1 within ε. Now let $\varepsilon_i \to 0$. Choose $\psi^{(i)}$ as above with $\varepsilon = \varepsilon_i$ and let $p^{(i)}$ satisfy $\psi^{(i)}(p)^{(i)} = 0$ according to the theorem. Then the $p^{(i)}$ will have a subsequence converging to $p^* \in \Delta_1$ with $\phi(p^*) = 0$.

Then following the proof of Corollary 1 we have:

Corollary 2. Corollary 1 remains true if the excess demand is merely supposed continuous and to satisfy the weaker boundary condition, $z_i(p) \geqq 0$ *if* $p_i = 0$.

In all the above, the excess demand with minor changes could have been derived from a micro-economic setting. We hope to pursue this in a future account.

In the very general existence theorems of economic equilibria, e.g., as in Debreu's 'Theory of Value', one is given the possiblilty of production with constant returns and preferences which are not *strictly* convex. In these cases only an excess demand *correspondence* is obtained. However there exist theorems which yield continuous functions approximating such correspondences. Debreu has given me a reference for this: Cellina (1969). Thus even in these cases one can obtain existence via the above procedure.

We end by quoting from Scarf's book (1973, p. 29) concerning Brouwer's fixed point theorem:

> 'The transformation of an analytical question concerning the solution of equations into a geometrical statement about continuous mappings of the simplex is in many instances quite artificial, but it may be necessary for the application of this powerful technique.'

We hope that we have shown in this paper that this 'transformation', at least in some sense and in some ways is not so necessary.

References

Abraham, R. and J. Robbin, 1967, Transversal mappings and flows (Benjamin, New York).
Arrow, K. and F. Hahn, 1972, General competitive analysis (Holden-Day, San Francisco, CA).
Arrow, K. and L. Hurwicz, 1958, The stability of the competitive equilibrium I, Econometrica 26, 522–552.

Arrow, K., H. Block and L. Hurwicz, 1959, The stability of the competitive equilibrium II, Econometrica 27, 82–109.

Cellina, A., 1969, Approximation of set functions and fixed point theorems, Annali di Matematica, Pura et Applicata 82, 17–24.

Debreu, G., 1970, Economies with a finite set of equilibria, Econometrica 38, 387–392.

Debreu, G., 1974, Excess demand functions, Journal of Mathematical Economics 1, 15–22.

Eaves, C. and H. Scarf, 1975, The solution of systems of piecewise linear equations, Cowles Foundation Discussion Paper no. 390 (Yale, New Haven, CT).

Hirsch, M., 1963, A proof of the non-retractibility of a cell onto its boundary, Proceedings of the American Mathematical Society 14, 364–465.

Hirsch, M. and S. Smale, 1974, Differential equations, dynamical systems, and linear algebra (Academic Press, New York).

Kellog, B., T.Y. Li and J. Yorke, forthcoming, A method of continuation for calculating a Brouwer fixed point, in: S. Karamadiar, ed., Computing fixed points with applications (Academic Press, New York).

Milnor, J., 1965, Topology from the differentiable viewpoint (University Press of Virginia, Charlottesville, VA).

Quirk, J. and R. Saposnik, 1968, Introduction to general equilibrium theory and welfare economics (McGraw-Hill, New York).

Ostrowski, A., 1973, Solutions of equations in Euclidean and Banach spaces (Academic Press, New York).

Scarf, H., 1960, Some examples of global instability of the competitive equilibrium, International Economic Review 1, 157–172.

Scarf, H., 1973, The computation of economic equilibria (in collaboration with T. Hansen; Yale University Press, New Haven, CT).

Smale, S., 1974, Global analysis and economics IV, Journal of Mathematical Economics 1, 119–127.