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METRIC SPACES AND COMPLETELY MONOTONE FUNCTIONS[1]

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INTRODUCTION

A set $\mathfrak{S}$ of elements, or points, $P, P', \cdots$, is said to be a semi-metric space if it is provided with a distance function PP' with the following two properties: 1. $PP' = P'P \geqq 0$, 2. $PP' = 0$ if and only if $P = P'$. Let now $F(t)$ be a real continuous function defined for $t \geqq 0$, $F(t) > 0$ if $t > 0$, $F(0) = 0$. According to L. M. Blumenthal we call the new semi-metric space which arises if we replace the metric PP', of $\mathfrak{S}$, by $F(PP')$ the *metric transform* of $\mathfrak{S}$ by $F(t)$ and denote it by the suggestive symbol $F(\mathfrak{S})$ ([2], p. 8).[2] This concept was also investigated by W. A. Wilson ([21], p. 64). Certain metric transforms have long been in use, for it is well known that if $\mathfrak{S}$ is metric, also the space with the new distance function $d(P, P') = PP'/(1 + PP')$ is metric. Moreover, these two metrics are topologically equivalent and the new space has a finite diameter. The new space is clearly the metric transform $F(\mathfrak{S})$ by the function

$$F(t) = \frac{t}{1+t}. \tag{0.1}$$

A general type of question suggested by the concept of metric transforms is whether functions $F(t)$ may be found such that $\mathfrak{S}$, if metrically transformed by $F(t)$, becomes more nearly comparable to certain simple spaces; or whether non-trivial functions $F(t)$ exist which will preserve certain simple characteristics of the original space $\mathfrak{S}$. To illustrate these points we shall state a few recent results.

If $\mathfrak{S}$ is a metric space it is readily seen that the metric transform $(\mathfrak{S})^\kappa$ (of $\mathfrak{S}$ by the function $F(t) = t^\kappa$) is also metric, provided the exponent κ is chosen in the range $0 < \kappa \leqq 1$. Blumenthal has shown that if this exponent is further restricted within the range $0 < \kappa \leqq \frac{1}{2}$, then $(\mathfrak{S})^\kappa$ has the 4-point property, i.e., any *four* points of $(\mathfrak{S})^\kappa$ may be imbedded[3] in a euclidean space. Hence $(\mathfrak{S})^\kappa$ $(0 < \kappa \leqq \frac{1}{2})$ is more "nearly euclidean" as compared with $\mathfrak{S}$, for $\mathfrak{S}$ enjoys only the 3-point property which is equivalent to the triangle inequality (see [2]).

As an example of a metric transformation which preserves certain properties

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[2] The numbers in square brackets refer to the list of references at the end of this paper.

[3] Here and throughout this paper the word *imbedding* is meant in the sense of *isometric imbedding*.

of a space we mention the result of Wilson ([21], p. 64) to the effect that the metric transform $(E_1)^{\frac{1}{2}}$, of E_1 [4] by the function $t^{\frac{1}{2}}$, which is not any more a euclidean space, is still isometrically imbeddable in $\mathfrak{H}$. This result was recently extended in two different directions. 1. It was shown by the author ([15]) that $(\mathfrak{H})^{\kappa}$ $(0 < \kappa \leqq 1)$ is imbeddable in $\mathfrak{H}$.[5] More significant is the following second result. 2. John von Neumann and the author ([12]) have determined *all* those functions $F(t)$ with the property that the metric transform $F(E_1)$ be imbeddable in $\mathfrak{H}$, or that it be imbeddable in E_n. This problem is equivalent with the problem of determining all screw lines in Hilbert space and in euclidean spaces.

These developments suggest the following general problem: *To determine all functions $F(t)$ such that the metric transform $F(E_m)$ be isometrically imbeddable in E_n, where the dimensions m and n are given in advance* $(1 \leqq m \leqq \infty, 1 \leqq n \leqq \infty)$.

We shall refer to this as the problem $\{E_m ; E_n\}$. The problems of the screw lines in E_n and $\mathfrak{H}$ are identical with the problems $\{E_1 ; E_n\}$ and $\{E_1 ; \mathfrak{H}\}$ in the present notation. The existence of *closed* screw lines in $E_n (n \geqq 2)$ and in $\mathfrak{H}$ may suggest that interesting solutions of a problem $\{E_m ; E_n\}$ are omitted by restricting the transforming function $F(t)$ to be > 0 if $t > 0$. However, if $F(t)$ does vanish for positive values of t, then $F(E_m)$ is by no means a semi-metric space. However, the problem of isometric imbedding of $F(E_m)$ in E_n retains also in this case its old meaning in an obvious way. In the present paper we are primarily interested in the case when $m \geqq 2$ and a simple lemma (Lemma 6, §5) will show that if we exclude the trivial solution $F(t) \equiv 0$, a problem $\{E_m ; E_n\}$, where m exceeds one, does admit, if any, *only* solutions $F(t)$ which are > 0 for $t > 0$.

The case when $m > n$ is immediately ruled out as trivial, for a moment's reflection shows that $F(t) \equiv 0$ is the only solution of $\{E_m ; E_n\}$. In the Appendix, §7, we solve the problem $\{E_m ; E_n\}$, where $2 \leqq m \leqq n < \infty$, with the rather negative result that the obvious solutions $F(t) = ct$ $(c \geqq 0)$, which correspond to similitude transformations of E_m, are the only possible ones. The problems $\{E_m ; \mathfrak{H}\}$, where $2 \leqq m < \infty$, will be partially solved in §8 of the Appendix where all "rectifiable" solutions of these problems are determined by a method suggested by recent investigations on homogeneous random processes in the calculus of probabilities. The chief objective of this paper is a complete solution of the problem $\{\mathfrak{H}; \mathfrak{H}\}$ (Part II, §5). The solutions $F(t)$ of this problem, as described by Theorems 6 and 6′ of §5, turn out to be essentially integrals of completely monotone functions. In particular it is found that (0.1) is a solution of this problem. Thus $F(\mathfrak{H})$, where $F(t) = t/(1 + t)$, is not only a bounded homeomorph of $\mathfrak{H}$, but is actually congruent to a subset of $\mathfrak{H}$.

A convenient way of attacking analytically these geometrical problems is afforded by combining an imbedding theorem of Menger ([9]) with a connection

[4] We denote by E_m the m-dim. euclidean space, while E_∞ or $\mathfrak{H}$ shall mean the real Hilbert space.

[5] This is a special case (for $p = 2$) of the following result: $(L_p)^{\kappa} (0 < \kappa < p/2; 0 < p \leqq 2)$ is imbeddable in $\mathfrak{H}$.

between euclidean simplices and quadratic forms, which goes back to Gauss, and which is so familiar in the geometry of numbers. The author had previously emphasized the usefulness of this combination in problems of isometric imbedding in Hilbert space ([14], [15], [16]). It allows us to attack our problems with such powerful tools as the Fourier-Stieltjes and Laplace-Stieltjes integrals.

Part I of this paper, on which the subsequent work is based, incidentally answers a question which has puzzled the author long before he ever thought of metric transforms of spaces, namely the connection between the class $\mathfrak{P}$ of Fourier-Stieltjes integrals (positive definite functions)

$$(0.2) \qquad g(t) = \int_{-\infty}^{\infty} e^{itu}\, d\alpha(u), \qquad (-\infty < t < \infty),$$

and the class $\mathfrak{M}$ of Laplace-Stieltjes integrals (completely monotone functions)

$$(0.3) \qquad f(t) = \int_{0}^{\infty} e^{-tu}\, d\beta(n), \qquad (0 \leqq t < \infty),$$

where $\alpha(u)$ and $\beta(u)$ are bounded non-decreasing functions. In spite of the entirely different analytical character of these two classes,[6] a certain kinship was to be expected for the following two reasons: 1. In both cases the defining kernel is the exponential function. 2. The less formal reason of the similarity of the closure properties of both classes, for both classes are *convex*, i.e., $a_1f_1 + a_2f_2$ ($a_1 \geqq 0$, $a_2 \geqq 0$) belongs to the class if f_1 and f_2 belong to it, *multiplicative*, i.e., also $f_1 \cdot f_2$ belongs to the class, and finally *closed* with respect to ordinary convergence to a continuous limit function. The answer, as given by Theorem 3 below, is based on a result of S. Bochner on characteristic functions of distribution functions in E_m([4], p. 407) and may best be stated in terms of such functions. A subclass of the class of characteristic functions $f(x_1, \cdots, x_m)$ in E_m (defined by (1.4) below) are those which are of radial symmetry. These are completely described by Theorem 1 below. While there is not as yet at present an explicit determination of the characteristic functions in Hilbert space $\mathfrak{H}$ by an appropriate Stieltjes integration process over $\mathfrak{H}$, these functions may easily enough be defined as positive definite vector functions in $\mathfrak{H}$. Those positive definite vector functions in $\mathfrak{H}$ which are of *radial symmetry* are now explicitly determined and turn out to be identical with the functions $f(t^2)$, where $f(t) \epsilon \mathfrak{M}$. Incidentally, Theorem 3 exhibits a new property (3.3) of completely monotone functions which is sufficient to characterize this class of functions.

The sixth section of Part II is devoted to further applications to the theory of completely monotone functions. It is shown that the solution of the problem $\{\mathfrak{H}\,;\mathfrak{H}\}$ (Theorem 6) is equivalent with a complete description of those trans-

[6] Indeed $f(t)$, defined by (0.3), is analytic in the half-plane $\Re t > 0$, while Weierstrass' nowhere differentiable function $g(t) = \sum_0^\infty a^n \cos b^n t$, $(0 < a < 1, ab \geqq 1)$, is readily representable as a Fourier integral (0.2) with an appropriate step function $\alpha(t)$. Thus $\mathfrak{P}$ contains "arbitrary" functions.

formations of the variable $t \mid \phi(t)$ which turn completely monotone functions $f(t)$ into functions $f(\phi(t))$ of the same class (Theorem 8). We venture to call them *inner transformations* of the class $\mathfrak{M}$. The fact that an essential part of this class of transformations does have this property was pointed out quite recently by S. Bochner ([5], p. 498).

I. Positive Definite Functions and Completely Monotone Functions

1. Positive definite functions in euclidean space E_m

1.1. Let $\mathfrak{S}$ be an abstract set of elements $P, P', Q, \cdots$. A real or complex-valued function $F(P, Q)$ of two arbitrary points of $\mathfrak{S}$ is called positive definite. if it enjoys the following two properties:

1. Hermitean symmetry

$$F(P, Q) = \overline{F(Q, P)}. \tag{1.1}$$

2. For any n points $P_1, \cdots, P_n$ of $\mathfrak{S}(n = 2, 3, \cdots)$ we have

$$\sum_{j,k=1}^{n} F(P_j, P_k)\rho_j\bar{\rho}_k \geqq 0 \tag{1.2}$$

for arbitrary ρ_j.[7]

If $\mathfrak{S}$ is also endowed with a metric PP', we would also require our function $F(P, Q)$ to be continuous in both points with respect to this metric. By imposing successive restrictions on $\mathfrak{S}$, as well as $F(P, Q)$, we shall now examine certain subclasses of Moore's general class of positive definite functions.

Let us assume first that $\mathfrak{S}$ is a linear vector space with the norm (metric) $P - P' \mid = PP'$. In this case we limit ourselves to positive definite functions $F(P, Q)$ which are functions of the vector $P - Q$ only:

$$F(P, Q) = f(P - Q). \tag{1.3}$$

Our continuous vector function $f(P - Q)$ is now subject to the conditions

$$f(P - Q) = \overline{f(Q - P)}, \tag{1.1'}$$

$$\sum_{j,k=1}^{n} f(P_j - P_k)\rho_j\bar{\rho}_k \geqq 0. \tag{1.2'}$$

If our underlying space is the m-dimensional euclidean space E_m, these functions $f(P - Q)$ are precisely the positive definite functions of m variables as defined by M. Mathias ($m = 1$) and S. Bochner (m arbitrary) ([4], p. 406). Let us refer E_m to rectangular coördinates and write $P = x = (x_1, \cdots, x_m)$, $f(P - 0) = f(x_1, \cdots, x_m) = f(x)$, $xy = x_1y_1 + \cdots + x_my_m$. Bochner established the identity of this class of positive definite functions with the class of characteristic functions of distribution functions in E_m:

$$f(x) = \int_{E_m} e^{ixy}\, d\phi(y), \tag{1.4}$$

[7] This general concept is due to E. H. Moore. See [11], pp. 3-4, 173, 181-190, 209-220. Moore calls $F(P, Q)$ a positive Hermitean matrix.

where $\phi(y)$ is the monotone point function of a non-negative, bounded, totally additive set function $\phi(S)$ defined for all Borel sets of E_m.

A subclass of the class of vector functions $f(P - Q)$ associated with $\mathfrak{S}$ and enjoying the properties (1.1′) and (1.2′), is made up of those functions which are functions of the length $|P - Q|$ of the vector only:

$$f(P - Q) = g(|P - Q|) = g(PQ), \tag{1.5}$$

where $g(t)$ is a continuous function defined for $t \geqq 0$. They are necessarily real, for (1.5) implies the ordinary symmetry $f(P - Q) = f(Q - P)$, which together with the hermitean symmetry (1.1′) obviously implies the reality of $f(P - Q)$ and $g(t)$. These real functions $g(t)$ will be called positive definite "in $\mathfrak{S}$" throughout this paper. This final restriction implies on the other hand a gain in generality, for this restricted definition of positive definite functions may at once be extended to any semi-metric space $\mathfrak{S}$.

DEFINITION 1. *Let $\mathfrak{S}$ be a semi-metric space with the metric PP'. A real continuous function $g(t)$ defined in the range of values of $t = PP'(P, P' \in \mathfrak{S})$, is said to be positive definite in $\mathfrak{S}$, if for any n points $P_1, \cdots, P_n$ of $\mathfrak{S}(n = 2, 3, \cdots)$ we have*

$$\sum_{j,k=1}^{n} g(P_j P_k)\rho_j \rho_k \geqq 0, \tag{1.2″}$$

for arbitrary real ρ_j. We shall denote this class of functions by the symbol $\mathfrak{P}(\mathfrak{S})$.[8]

The class $\mathfrak{P}(\mathfrak{S})$ is never empty, for $g(t) \in \mathfrak{P}(\mathfrak{S})$, if $g(t) \equiv 1$. From (1.2″) we conclude that $|g(t)| \leqq g(0)$. It was pointed out before ([16], §2) that the class $\mathfrak{P}(\mathfrak{S})$ is convex, multiplicative and closed with respect to convergence to a continuous limit function.

1.2. Let us determine the functions $g(t)$ of the positive definite class $\mathfrak{P}(E_m)$. From (1.5) and Bochner's theorem we learn that our $g(t)$ are in a one-to-one correspondence with those characteristic functions (1.4) which are functions of $|x|$ only, i.e., of radial symmetry, and that the relationship valid both ways is

$$f(x) = g(|x|). \tag{1.5′}$$

Let $d\omega(\xi)$ denote the area element of the spherical shell $|\xi| = 1$ in E_m, and $\omega_m = \int d\omega(\xi)$ be its total area. The mean value of $e^{ix\xi}$ over $|\xi| = 1$ is obviously invariant with respect to rotations in E_m about the origin and hence is a function of $|x|$ only. We write for it

$$M_\xi\{e^{ix\xi}\} = \frac{1}{\omega_m}\int_{|\xi|=1} e^{ix\xi}\,d\omega(\xi) = \Omega_m(|x|). \tag{1.6}$$

By m-dimensional polar coördinates we readily find

$$\Omega_m(r) = \int_0^\pi e^{ir\cos\theta}\sin^{m-2}\theta\,d\theta \Big/ \int_0^\pi \sin^{m-2}\theta\,d\theta, \qquad (m \geqq 2), \tag{1.7}$$

[8] We see that $g(t) \in \mathfrak{P}(\mathfrak{S})$ if and only if $F(P, Q) = g(PQ)$ is positive definite in the sense of Moore.

whence the power series expansion and expression in terms of Bessel functions

$$
\begin{aligned}
\Omega_m(r) &= 1 - \frac{r^2}{2m} + \frac{r^4}{2\cdot 4\cdot m(m+2)} - \frac{r^6}{2\cdot 4\cdot 6m(m+2)(m+4)} + \cdots \\
&= \Gamma\left(\frac{m}{2}\right)\left(\frac{2}{r}\right)^{\frac{1}{2}(m-2)} J_{\frac{1}{2}(m-2)}(r), \qquad (m \geqq 1),
\end{aligned} \tag{1.8}
$$

showing that $\Omega_m(r)$ is an integral transcendental function.[9] We may now readily derive the following theorem.

THEOREM 1. *The class $\mathfrak{P}(E_m)$ of functions which are positive definite in E_m is identical with the class of functions of the form*

$$
g(t) = \int_0^\infty \Omega_m(tu)\,d\alpha(u), \tag{1.9}
$$

where $\alpha(u)$ is non-decreasing and bounded for $u \geqq 0$, and $\Omega_m(r)$ is the integral function defined above.

Indeed, let $f(x)$ of (1.4) be of radial symmetry: $f(x) = g(|x|)$. Since $f(x) = f(|x|\xi)$, for all unit vectors ξ, we have

$$
\begin{aligned}
g(|x|) = f(x) &= \frac{1}{\omega_m}\int_{|\xi|=1} f(|x|\xi)\,d\omega(\xi) \\
&= \frac{1}{\omega_m}\int_{|\xi|=1}\left(\int_{E_m} e^{i|x|\cdot\xi y}\,d\phi(y)\right)d\omega(\xi) \\
&= \int_{E_m}\left(\frac{1}{\omega_m}\int_{|\xi|=1} e^{i|x|\cdot y\xi}\,d\omega(\xi)\right)d\phi(y) \\
&= \int_{E_m}\Omega_m(|x|\cdot|y|)\,d\phi(y) = \int_0^\infty \Omega_m(|x|u)\,d\alpha(u),^{10}
\end{aligned} \tag{1.10}
$$

where we put

$$
\alpha(u) = \int_{|y|\leqq u} d\phi(y) = \phi(|y| \leqq u).
$$

Thus (1.10) shows that functions of $\mathfrak{P}(E_m)$ are of the form (1.9). The converse is obvious, for $\Omega_m(|x|u)$ is a characteristic function on account of (1.6).

2. Positive definite functions in Hilbert space $\mathfrak{H}$

2.1. Definition 1 implies a certain monotoneity property of the class $\mathfrak{P}(\mathfrak{S})$ in its dependence on the space $\mathfrak{S}$, for if $\mathfrak{S}$ is a subset of a space $\mathfrak{S}'$, or congruent to a subset of $\mathfrak{S}'$, then $\mathfrak{P}(\mathfrak{S}') \subset \mathfrak{P}(\mathfrak{S})$. Now as $E_1 \subset E_2 \subset \cdots \subset E_m \subset \cdots \subset \mathfrak{H}$, we see that

$$
\mathfrak{P}(E_1) \supset \mathfrak{P}(E_2) \supset \cdots \supset \mathfrak{P}(E_m) \supset \cdots \supset \mathfrak{P}(\mathfrak{H}),
$$

[9] The $\Omega_m(r)$ may be called Poisson functions as they were discovered by Poisson in connection with problems on heat conduction prior to Bessel's investigation of his functions. See G. N. Watson [18], p. 24.

[10] The interchange of integrations in (1.10) is justified by a general argument found in [4], p. 393.

with $\mathfrak{P}(\mathfrak{H})$ actually identical with the intersection (logical product) of this non-increasing sequence of classes $\{\mathfrak{P}(E_m)\}$.

It was pointed out elsewhere ([16], §2) that $g(t) = \exp\{-t^2u^2\}$ (u a real constant) is positive definite in E_m, for all m, hence also positive definite in $\mathfrak{H}$. This fact is immediately apparent from the elementary formula

$$(2.1)\qquad \begin{aligned} &e^{-(x_1^2+\cdots+x_m^2)u^2} \\ &= (2u)^{-m}\pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i(x_1y_1+\cdots+x_my_m)}\,e^{-(y_1^2+\cdots+y_m^2)/4u^2}\,dy_1\cdots dy_m, \end{aligned}$$

showing that $f(x) = \exp\{-|x|^2u^2\}$ is a characteristic function in E_m of radial symmetry about the origin. That we get all functions of the class $\mathfrak{P}(\mathfrak{H})$ as linear combinations of these particular functions, with non-negative coefficients, is stated by the following theorem.

THEOREM 2. *The class $\mathfrak{P}(\mathfrak{H})$ of functions which are positive definite in $\mathfrak{H}$ is identical with the class of functions of the form*

$$(2.2)\qquad g(t) = \int_0^\infty e^{-t^2u^2}\,d\alpha(u),$$

where $\alpha(u)$ is non-decreasing and bounded for $u \geqq 0$.[11]

Indeed, since $\mathfrak{P}(\mathfrak{H})$ is convex and $\exp\{-t^2u^2\} \in \mathfrak{P}(\mathfrak{H})$, we see that $g(t)$, of (2.2), belongs to $\mathfrak{P}(\mathfrak{H})$. There remains to prove the converse. Now let $g(t)$ be an element of $\mathfrak{P}(\mathfrak{H})$. As $g(t) \in \mathfrak{P}(E_m)$, we know by Theorem 1 that it is of the form

$$(2.3)\qquad g(t) = \int_0^\infty \Omega_m(tu)\,d\alpha_m(u), \qquad (m = 1, 2, 3, \cdots),$$

[11] Theorem 2 and Theorem 4 below (i.e. Theorem 1 of [16]) allow us to state the following result: *A separable semi-metric space $\mathfrak{S}$ is imbeddable in $\mathfrak{H}$ if and only if $\mathfrak{P}(\mathfrak{H}) \subset \mathfrak{P}(\mathfrak{S})$.* If we replace $\mathfrak{H}$ by E_m, then the theorem is no longer true without further restrictions on $\mathfrak{S}$. To show this, consider E_3, and let $\mathfrak{S}$ be a regular simplex σ_4 of E_4 of side 1, i.e., $\mathfrak{S}$ be the set of five points $P_0, \cdots, P_4$ with $P_iP_k = 1 (i \neq k)$. The statement $\mathfrak{P}(E_3) \subset \mathfrak{P}(\mathfrak{S})$ means (by Theorem 1 below for $m = 3$) that the family of functions $\Omega_3(tu)$ is positive definite in $\mathfrak{S} = \sigma_4$. As the $\Omega_3(tu)$ are known to be positive definite in E_3, our last statement amounts to the inequality

$$(*)\qquad \Delta(u) = \det \| \Omega_3(u\cdot P_iP_k) \|_{0,4} \geqq 0, \text{ for real } u.$$

Now $P_iP_k = \delta_{ik}$ implies $\Delta(u) = (1 + \Omega_3(u))^4(1 + 4\Omega_3(u))$. Hence (*) holds, since the inequality

$$(**)\qquad \Omega_3(u) = \frac{\sin u}{u} > -\tfrac{1}{4}$$

happens to be valid for all real u. Thus $\mathfrak{P}(E_3) \subset \mathfrak{P}(\sigma_4)$, without, of course, σ_4 being imbeddable in E_3. In order that $\mathfrak{P}(E_m) \subset \mathfrak{P}(\mathfrak{S})$ should imply that $\mathfrak{S}$ is imbeddable in E_m, it will probably be sufficient to assume that $\mathfrak{S}$ is made up of at least $m + 3$ points, an assumption suggested by certain results of Menger, [10].

$\{\alpha_m(u)\}$ being a certain sequence of uniformly bounded monotone functions. From the power series expansion (1.8) we derive the limiting relation

$$\lim_{m\to\infty} \Omega_m(r\sqrt{(2m)}) = e^{-r^2}.\tag{2.4}$$ [12]

As on the other hand (2.3) may be written in the form

$$g(t) = \int_0^\infty \Omega_m(tu\sqrt{(2m)})\,d\alpha_m(u\sqrt{(2m)}), \qquad (m = 1, 2, 3, \cdots),\tag{2.5}$$

we shall expect to be able to pass from (2.5) to the representation (2.2), of $g(t)$, by means of (2.4). Now it is easy to show that (2.4) holds uniformly within any finite interval of the r-axis (as well as in any finite domain of the complex r-plane). This, however, is insufficient to insure the applicability of Helly's well known convergence theorem for Stieltjes integrals, as our interval of integration is infinite. We shall have to show that (2.4) holds uniformly *for all real values of r*.[13]

2.2. The following proof of the uniformity of convergence in (2.4) is elementary.[14] For the sake of clarity we present it in the form of three lemmas.

LEMMA 1. *Let*

$$F(x) = \int_0^1 \cos(x\tau(u))\,du,\tag{2.6}$$

where $\tau(u)$ is continuous for $0 \leqq u \leqq 1$, $\tau(0) = 0$, and $\tau'(u)$ non-decreasing for $0 \leqq u < 1$, $\tau'(0) > k > 0$. We have

$$|F(x)| < \frac{1}{kx} \qquad \textit{for } x > 0.\tag{2.7}$$

[12] The author is indebted to J. von Neumann for pointing out to him this relation.

[13] To stress the decisive rôle of the uniformity of convergence in (2.4) for all real r, we point out the following fact. The expansion (1.8) shows that

$$\lim_{m\to\infty} \Omega_m(r) = 1,\tag{2.4'}$$

uniformly in any finite r-interval. From this and (2.3) we may be tempted to expect, as $m \to \infty$, that

$$g(t) = \int_0^\infty 1\cdot d\alpha(u),$$

and therefore conclude that $\mathfrak{P}(\mathfrak{H})$ is made up of constant functions only. This however, is not true, as e.g. $\exp\{-t^2\} \in \mathfrak{P}(\mathfrak{H})$. The point is that (2.4′) does not hold uniformly for all real r.

[14] None of the available asymptotic estimates of Bessel functions was found to apply in the present situation.

Indeed, as $\tau'(u) > k$ holds throughout [0, 1], the second mean value theorem gives

$$F(1) = \int_0^1 \cos(\tau(u))\,du = \int_0^1 \frac{1}{\tau'(u)} \cos(\tau(u))\tau'(u)\,du$$
$$= \frac{1}{\tau'(0)} \int_0^\xi \cos(\tau(u))\tau'(u)\,du = \frac{1}{\tau'(0)} (\sin\tau(\xi) - \sin\tau(0)) = \frac{\sin\tau(\xi)}{\tau'(0)},$$

which implies

$$|F(1)| = \left| \int_0^1 \cos(\tau(u))\,du \right| < \frac{1}{k}.$$

Applying this inequality to $x\tau(u)$ instead of $\tau(u)$ (for a fixed positive value of x), and therefore with kx, instead of k, we immediately get (2.7).

LEMMA 2. *We have*

$$(2.8) \qquad |\Omega_m(x\sqrt{(2m)})| < \frac{1}{x} \quad \textit{for} \quad x > 0, \qquad m = 3, 4, 5, \cdots,$$

Indeed, passing to the variable $t = \cos\theta$ in

$$\Omega_m(x\sqrt{(2m)}) = \int_0^{\pi/2} \cos(x\sqrt{(2m)}\cos\theta)\sin^{m-2}\theta\,d\theta \Big/ \int_0^{\pi/2} \sin^{m-2}\theta\,d\theta,$$

we get

$$\Omega_m(x\sqrt{(2m)}) = \int_0^1 \cos(x\sqrt{(2m)}t)(1 - t^2)^{(m-3)/2}\,dt \Big/ \int_0^1 (1 - t^2)^{(m-3)/2}\,dt, \qquad (m \geqq 3).$$

By means of the function

$$(2.9) \qquad u = \int_0^t (1 - t^2)^{(m-3)/2}\,dt \Big/ \int_0^1 (1 - t^2)^{(m-3)/2}\,dt, \qquad (0 \leqq t \leqq 1),$$

whose inverse we denote by $t = t(u)(0 \leqq u \leqq 1)$, this goes over into

$$\Omega_m(x\sqrt{(2m)}) = \int_0^1 \cos(x\sqrt{(2m)}t(u))\,du.$$

With

$$(2.10) \qquad \tau(u) = \sqrt{(2m)}t(u),$$

we finally have

$$(2.11) \qquad \Omega_m(x\sqrt{(2m)}) = \int_0^1 \cos(x\tau(u))\,du.$$

The change of variable and the inversion $t = t(u)$, of (2.9) are justified by the fact that the function (2.9) is continuous and strictly increasing for $0 \leqq t \leqq 1$, with $u(0) = 0$, $u(1) = 1$. Moreover

$$\tau'(u) = \sqrt{(2m)}t'(u) = \sqrt{(2m)}(1 - t^2)^{-(m-3)/2} \cdot \int_0^1 (1 - t^2)^{(m-3)/2} dt, \qquad (0 \leqq t < 1),$$

is an increasing function and

$$\tau'(0) = \sqrt{(2m)} \int_0^1 (1 - t^2)^{(m-3)/2} dt$$
(2.12)
$$= \sqrt{(2m)} \int_0^{\pi/2} \sin^{m-2} \theta \, d\theta \geqq \sqrt{(2m)} \frac{(m-3)!!}{(m-2)!!} > 1.^{15}$$

Now (2.11), (2.12) and Lemma 1 imply the required inequality (2.8).

Lemma 3. *The limiting relation*

$$\lim_{m \to \infty} \Omega_m(x\sqrt{(2m)}) = e^{-x^2} \tag{2.13}$$

holds uniformly for $-\infty < x < \infty$.

Indeed, the difference

$$| \Omega_m(x\sqrt{(2m)}) - e^{-x^2} |$$

will be $< \epsilon$ for real x with $| x | > A(\epsilon)$, and $m \geqq 3$, since it falls below $x^{-1} + e^{-x^2}$ by Lemma 2. Now for sufficiently large m it will likewise be $< \epsilon$ for $| x | \leqq A(\epsilon)$, since (2.13) holds uniformly in every finite interval as already mentioned.

2.3. We can now readily complete a proof of Theorem 2. Returning to the relation (2.5), Helly's compactness theorem insures the existence of a subsequence $\{\alpha_{m_\nu}(u\sqrt{(2m_\nu)})\}$ converging to the monotone function $\alpha(u)(\alpha(0) = 0)$ in all points of continuity of $\alpha(u)$ ([8], p. 286). Let t assume a fixed positive value. By Lemma 3 we know that $\Omega_{m_\nu}(tu\sqrt{(2m_\nu)})$ tends to $\exp\{-t^2u^2\}$ uniformly for all real values of u, hence (2.5) implies

$$g(t) = \int_0^\infty e^{-t^2u^2} d\alpha_{m_\nu}(u\sqrt{(2m_\nu)}) + \sigma,$$

[15] Here we write $k!! = k(k-2)(k-4) \cdots$. The last inequality (2.12) can be verified by Stirling's formula. I owe the following interesting direct proof to F. Bohnenblust. Let $m = 2n + 2$ be even. Then

$$\frac{(2n-1)!!}{(2n)!!}\sqrt{(4n+4)}$$
$$= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{\sqrt{(2n)}(\sqrt{(2n)}\sqrt{(2n-2)})(\sqrt{(2n-2)}\sqrt{(2n-4)})\cdots(\sqrt{4}\sqrt{2})\sqrt{2}}\sqrt{(4n+4)}$$
$$\geqq \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{\sqrt{(2n)}(2n-1)(2n-3)\cdots 3\cdot\sqrt{2}}\sqrt{(4n+4)} = \sqrt{\left(\frac{n+1}{n}\right)} > 1.$$

since $\sqrt{(2n(2n-2))} < 2n - 1$. Similar argument in case m is odd.

with $|\sigma| < \epsilon$ for sufficiently large values of ν. But this implies that

$$g(t) = \int_0^l e^{-t^2u^2}\, d\alpha_{m_\nu}(u\sqrt{(2m_\nu)}) + \rho,$$

with $|\rho| < 2\epsilon$, provided both l and ν are sufficiently large. If $u = l$ is a continuity point of $\alpha(u)$, letting $\nu \to \infty$, we get ([8], pp. 288–289)

$$g(t) = \int_0^l e^{-t^2u^2}\, d\alpha(u) + \rho_1. \qquad |\rho_1| \leqq 2\epsilon.$$

Now letting $l \to \infty$ and then $\epsilon \to 0$ we get the desired representation (2.2). This formula, established for $t > 0$, holds also if $t = 0$ in view of the continuity of its two sides.

3. Completely monotone functions

3.1. A real function $f(t)$ is said to be *completely monotone for* $t \geqq 0$, if

$$(3.1) \qquad (-1)^n f^{(n)}(t) \geqq 0 \quad \text{for} \quad 0 < t < \infty, \qquad (n = 0, 1, 2, \cdots),$$

and

$$(3.1') \qquad f(0) = f(+0),$$

the last condition expressing the continuity of $f(t)$ at the origin. A fundamental theorem[16] states the identity of this class with the class of functions representable as a Laplace-Stieltjes integral

$$(3.2) \qquad f(t) = \int_0^\infty e^{-tu}\, d\beta(u),$$

where $\beta(u)$ is non-decreasing and bounded for $u \geqq 0$. *We shall denote this class of functions throughout this paper by the symbol* $\mathfrak{M}$.

If $g(t)$ is positive definite in $\mathfrak{H}$, then we may write in view of Theorem 2

$$g(t) = \int_0^\infty e^{-t^2u^2}\, d\alpha(u) = \int_0^\infty e^{-t^2u}\, d\alpha(\sqrt{u}) = \int_0^\infty e^{-t^2u}\, d\beta(u) = f(t^2).$$

Thus Theorem 2 and the Laplace integral representation (3.2) immediately prove the following theorem.

THEOREM 3. *A function $f(t)$ is completely monotone for $t \geqq 0$, if and only if $f(t^2)$ is positive definite in Hilbert space $\mathfrak{H}$. In symbols: $f(t) \in \mathfrak{M}$ if and only if $f(t^2) \in \mathfrak{P}(\mathfrak{H})$.*

The elements $g(t)$ of $\mathfrak{P}(\mathfrak{H})$ and $f(t)$ of $\mathfrak{M}$ are therefore in a one-to-one correspondence expressed by the relation $g(t) = f(t^2)$, which is valid both ways.

Definition 1 of positive definite functions and Theorem 3 allow us to state that the conditions (3.1) and (3.1′) for complete monotoneity of a function $f(t)$ are entirely equivalent with the following two requirements:

1. *$f(t)$ be real and continuous for $t \geqq 0$.*

[16] See [7], [1], [19], and [13].

2. *If $P_1, P_2, \cdots, P_n$ are any n points of a euclidean space ($n = 1, 2, 3, \cdots$), we should have*

$$\begin{vmatrix} f(P_1P_1^2) & f(P_1P_2^2) & \cdots & f(P_1P_n^2) \\ f(P_2P_1^2) & f(P_2P_2^2) & \cdots & f(P_2P_n^2) \\ \vdots & \vdots & & \vdots \\ f(P_nP_1^2) & f(P_nP_2^2) & \cdots & f(P_nP_n^2) \end{vmatrix} \geqq 0.^{17} \tag{3.3}$$

A direct derivation of (3.3) as a consequence of (3.1) and (3.1′) would furnish a new proof of the theorem that (3.1) and (3.1′) imply the representation (3.2).

The connection between the class $\mathfrak{P}$ of characteristic functions (0.2) and the class $\mathfrak{M}$ is now clear. We have the infinite sequence of classes

$$\mathfrak{P} \supset \mathfrak{P}(E_1) \supset \mathfrak{P}(E_2) \supset \cdots \supset \mathfrak{P}(E_m) \supset \cdots \supset \mathfrak{P}(\mathfrak{H}). \tag{3.4}$$

The class $\mathfrak{P}(E_1)$ is made up of those elements of $\mathfrak{P}$ which are of radial symmetry in E_1, i.e., are *even*, or merely *real*. Their form is obviously

$$g(t) = \int_0^\infty \cos tu \, d\alpha(u),$$

which is in accord with Theorem 1, since $\Omega_1(r) = \cos r$. A function $g(t)$ of $\mathfrak{P}(E_1)$ belongs to $\mathfrak{P}(E_m)$ if and only if

$$g(\sqrt{(x_1^2 + \cdots + x_m^2)}) \text{ is a characteristic function of } E_m. \tag{3.5}$$

The completely monotone $f(t^2)(f(t) \in \mathfrak{M})$ are identical with those elements $g(t)$ of $\mathfrak{P}(E_1)$ which enjoy the property (3.5) for *all* values of m.

3.2. Further light is thrown on the gradual narrowing down of the classes (3.4) by the requirement of positive definiteness in E_m, for increasing m, by the following lemma.

LEMMA 4. *The functions*

$$g(t) = \int_0^\infty \Omega_m(tu) \, d\alpha(u) \tag{3.6}$$

of the class $\mathfrak{P}(E_m)$ are $\left[\dfrac{m-1}{2}\right]$-times differentiable.

This fact shows clearly the smoothing effect on $g(t)$ of the requirement of positive definiteness in higher euclidean spaces. It implies in particular that the elements of $\mathfrak{P}(\mathfrak{H})$, and those of $\mathfrak{M}$, are ∞-often differentiable. Theorem 2, moreover, shows that the elements $g(t)$ of $\mathfrak{P}(\mathfrak{H})$ are even analytic in the sector $|\arg t| < \pi/4$ of the complex t-plane.

In order to prove Lemma 4 it suffices to consider

$$g^{(\nu)}(t) = \int_0^\infty \frac{\partial^\nu}{\partial t^\nu} \Omega_m(tu) \, d\alpha(u) \tag{3.7}$$

[17] See Theorem 3′ below which excludes the equality sign in (3.3), except in trivial cases.

and to see for what values of $\nu = 1, 2, 3, \cdots$ this integral converges for all values of t. The integrand is

$$\frac{\partial^\nu}{\partial t^\nu}\Omega_m(tu) = \Omega_m^{(\nu)}(tu)\cdot u^\nu. \tag{3.8}$$

From $\Omega_m(r) = c_m J_k(r) r^{-k}$, where $k = (m-2)/2$ and c_m is a positive constant, we get by the rule of Leibnitz

$$c_m^{-1}\Omega_m^{(\nu)}(r) = J_k^{(\nu)}(r)r^{-k} - \binom{\nu}{1} k J_k^{(\nu-1)}(r) r^{-k-1} + \cdots \pm k(k+1)\cdots(k+\nu-1)J_k(r)r^{-k-\nu}.$$

As $J_k^{(\nu)}(r) = O(r^{-\frac{1}{2}})$ we have

$$\Omega_m^{(\nu)}(r) = O(r^{-k-\frac{1}{2}}) = O(r^{-\frac{1}{2}(m-1)}),$$

and hence by (3.8)

$$\frac{\partial^\nu}{\partial t^\nu}\Omega_m(tu) = O(u^{\nu-\frac{1}{2}(m-1)}).$$

Thus (3.7) converges absolutely as long as $\nu \leqq (m-1)/2$, proving our lemma.

In view of Lemma 4 there are strong reasons to believe that the continuity assumption on $g(t)$ in Definition 1, section 1.1, may be dropped as soon as $\mathfrak{S}$ is an euclidean space E_m of dimension m exceeding one. More precisely: The inequality (1.2″) should imply the continuity of $g(t)$ everywhere, with the exception of the origin $t = 0$. The last discontinuity must obviously remain, for if $g(t)$ satisfies the inequalities (1.2″), it will still satisfy them if the value of $g(0)$ is increased.

3.3. We close this section with the following theorem.

THEOREM 3′. *If $P_1, P_2, \cdots, P_n$ are any n distinct points of E_{n-1} and $f(t) \in \mathfrak{M}$, then*

$$\det \| f(P_j P_k^2) \|_{1,n} > 0, \tag{3.9}$$

unless $f(t)$ reduces to a non-negative constant.

Indeed, let us show first that the quadratic form $\sum_{j,k=1}^{n} \exp\{-P_j P_k^2\}\rho_j\rho_k$ is positive definite. Transforming the integral (2.1) ($u = 1$) to polar coördinates and using again vector notation in $E_m(m = n-1)$, with $P_j = x^j = (x_1^j, \cdots, x_m^j)$, we get

$$e^{-|x|^2} = 2^{1-m}[\Gamma(m/2)]^{-1}\int_0^\infty M_\xi\{e^{i(x\xi)r}\}e^{-r^2/4}r^{m-1}\,dr.$$

Replacing the vector $x = x - 0$ by $P_j - P_k = x^j - x^k$, we get

$$e^{-P_jP_k^2} = c\cdot\int_0^\infty M_\xi\{e^{i(x^j\xi)r}\cdot e^{-i(x^k\xi)r}\}e^{-r^2/4}r^{m-1}\,dr,$$

Now

$$\sum_{j,k=1}^{n} e^{-P_jP_k^2}\rho_j\rho_k = c\cdot\int_0^\infty M_\xi\left\{\left|\sum_{j=1}^{n} e^{i(x^j\xi)r}\cdot\rho_j\right|^2\right\} e^{-r^2/4}r^{m-1}\,dr,$$

which proves the positive definite character of this form by an elementary argument previously used (see [15], p. 790). Hence

$$(3.10)\qquad \sum_{j,k=1}^{n} e^{-u\cdot P_jP_k^2}\rho_j\rho_k > 0, \qquad (u > 0,\ \rho_1^2 + \cdots + \rho_n^2 > 0).$$

Now (3.2) and (3.10) show that

$$\sum_1^n f(P_jP_k^2)\rho_j\rho_k = \int_0^\infty \left(\sum_1^n e^{-uP_jP_k^2}\rho_j\rho_k\right)d\beta(u)$$

is positive, if $\rho_1^2 + \cdots + \rho_n^2 > 0$, unless $\beta(u)$ reduces to a step function with a single saltus at the origin while $\sum \rho_j\rho_k = (\rho_1 + \cdots + \rho_n)^2 = 0$. But then $f(t)$ is a constant as was to be proved.

4. Integrals of completely monotone functions

4.1. We have denoted by $\mathfrak{M}$ the class of functions $f(t)$ completely monotone for $t \geqq 0$. If we drop the second condition (3.1′), which requires continuity at the origin, we get a larger class of functions $\psi(t)$ defined for $t > 0$ and satisfying the inequalities

$$(4.1)\qquad (-1)^n\psi^{(n)}(t) \geqq 0 \quad\text{for}\quad 0 < t < \infty, \qquad (n = 0, 1, 2, \cdots).$$

These functions are also representable as Laplace integrals

$$(4.2)\qquad \psi(t) = \int_0^\infty e^{-tu}\,d\gamma(u) \quad\text{for}\quad 0 < t < \infty,$$

where $\gamma(u)$ is non-decreasing. $\psi(0) = \psi(+0)$ exists if and only if $\gamma(u)$ is bounded, in which case $\psi(0) = \psi(+0) = \gamma(\infty)$. Integrating (4.2) between ϵ and $t(0 < \epsilon < t)$ we get

$$(4.3)\qquad \int_\epsilon^t \psi(t)\,dt = \int_0^\infty \frac{e^{-u\epsilon} - e^{-ut}}{u}\,d\gamma(u).$$

Letting here $\epsilon \to 0$ we get

$$(4.4)\qquad \phi(t) = \int_{+0}^t \psi(t)\,dt = \int_0^\infty \frac{1 - e^{-ut}}{u}\,d\gamma(u), \qquad (0 < t < \infty),$$

where both improper integrals converge or diverge simultaneously. We have convergence for all $t > 0$, if and only if

$$(4.5)\qquad \int_1^\infty \frac{d\gamma(u)}{u}$$

exists.

DEFINITION 2. *We denote by T the class of functions $\phi(t)$ defined by* (4.4), *and* $\phi(0) = 0$, *as integrals of those functions $\psi(t)$ completely monotone for $t > 0$ for which the improper integral $\int_{+0}\psi(t)dt$ converges. In terms of the Laplace representation* (4.2), *this last mentioned condition is equivalent with the assumption* (4.5).

This class T is fundamental in our subsequent work. In this section we only prove a certain closure property of this class expressed by the following lemma.

LEMMA 5. *The class T is closed with respect to ordinary convergence to a continuous limit function, that is, if $\{\phi_n(t)\}$ $(n = 1, 2, 3, \cdots)$ is a sequence of elements of T and $\phi_0(t)$ is continuous for $t \geqq 0$, then*

$$\lim_{n\to\infty} \phi_n(t) = \phi_0(t), \qquad (0 \leqq t < \infty), \tag{4.6}$$

implies that $\phi_0(t) \in T$.

Indeed, by assumption we have

$$\phi_n(t) = \int_{+0}^{t} \psi_n(t)\,dt, \qquad (n = 1, 2, 3, \cdots), \tag{4.7}$$

where $\psi_n(t)$ are completely monotone for $t > 0$ and all improper integrals converge. Now the $\psi_n(t)$ are analytic in the half-plane $\Re t > 0$ and therefore also their integrals $\phi_n(t)$ are of this nature. Let $t = \sigma + i\tau$. We want to show that the functions of the sequence $\{\phi_n(t)\}$ are uniformly bounded in the rectangular domain

$$D\colon\ \sigma_1 \leqq \sigma \leqq \sigma_2\,, \qquad |\tau| \leqq \tau_1\,, \qquad (0 < \sigma_1 < \sigma_2).$$

Indeed

$$\begin{aligned}\phi_n(t) = \phi_n(\sigma + i\tau) &= \int_0^\infty (1 - e^{-\sigma u - iu\tau})u^{-1}\,d\gamma_n(u)\\ &= \int_0^\infty (1 - e^{-\sigma u})u^{-1}\,d\gamma_n(u) + \int_0^\infty e^{-\sigma u}(1 - e^{-iu\tau})u^{-1}\,d\gamma_n(u).\end{aligned}$$

Hence

$$\begin{aligned}|\phi_n(t)| &\leqq \phi_n(\sigma) + |\tau| \int_0^\infty e^{-\sigma u} \left|\sin \frac{u\tau}{2} \Big/ \frac{u\tau}{2}\right| d\gamma_n(u) \leqq \phi_n(\sigma) + \tau_1 \int_0^\infty e^{-\sigma u}\,d\gamma_n(u)\\ &\leqq \phi_n(\sigma_2) + \tau_1 \int_0^\infty e^{-\sigma_1 u}\,d\gamma_n(u) \leqq \phi_n(\sigma_2) + \tau_1 \int_0^\infty (1 - e^{-\sigma_1 u})(\sigma_1 u)^{-1}\,d\gamma_n(u),\end{aligned}$$

since $e^{-x} < (1 - e^{-x})x^{-1} (x > 0)$. We therefore have

$$|\phi_n(t)| \leqq \phi_n(\sigma_2) + \tau_1\sigma_1^{-1}\phi_n(\sigma_1) \qquad (t \text{ in } D)$$

This proves, in view of (4.6), that the $\phi_n(t)$ are uniformly bounded in D. By Vitali's convergence theorem we conclude that (4.6) holds uniformly in any bounded domain inside the half-plane $\Re t > 0$ and that $\phi_0(t)$ is analytic. (See Titchmarsh, [17], p. 168.)

Let $\psi_0(t) = \phi_0'(t)$; now (4.6) implies

$$\lim_{n\to\infty} (-1)^k \psi_n^{(k)}(t) = (-1)^k \psi_0^{(k)}(t), \qquad (k = 0, 1, 2, \cdots),$$

if $\Re t > 0$, hence in particular for all positive values of t. Thus $(-1)^k\psi_n^{(k)}(t) \geqq 0$ implies $(-1)^k\psi_0^{(k)}(t) \geqq 0$ $(t > 0;\ k = 0, 1, 2, \cdots)$, hence $\psi_0(t)$ is completely monotone for $t > 0$. The proof is now immediately completed, for

$$\phi_0(t) - \phi_0(\epsilon) = \int_\epsilon^t \psi_0(t)\,dt, \quad (0 < \epsilon < t), \quad \text{and} \quad \phi_0(+0) = \phi_0(0) = 0,$$

imply, as $\epsilon \to 0$, the existence of $\int_{+0}\psi_0(t)dt$ and the relation

$$\phi_0(t) = \int_{+0}^t \psi_0(t)\,dt, \qquad (0 < t < \infty).$$

Hence $\phi_0(t) \in T$.

We conclude this section with two remarks concerning Lemma 5. The function theoretic argument shows that if 1. $\phi_n(t)$ are uniformly bounded in any finite interval $0 \leqq t \leqq \sigma_2$, 2. the limit (4.6) exists in a set of positive values of t with a positive limit point, then (4.6) holds for all t with positive real part, and $\phi_0(t)$ is analytic for $\Re t > 0$. In concluding, however, that $\phi_0(t) \in T$, it is essential to assume that $\phi_0(+0) = 0$, i.e., with $\phi_0(0) = 0$, $\phi_0(t)$ should be continuous at the origin. This fact was used in the last step of our proof. That $\phi_0(+0) = 0$ is not implied by (4.6), is seen by the following sequence: $\phi_n(t) = 1 - e^{-nt}$, $(\psi_n(t) = ne^{-nt})$, for which (4.6) holds for $t \geqq 0$ with $\phi_0(t) = 1 (t > 0)$, $\phi_0(0) = 0$. Our second remark is that a somewhat more involved but elementary proof of Lemma 5 (free of any appeal to analytic function theory) is available.

II. Determination of All Metric Transforms of Hilbert Space Which Are Isometrically Imbeddable in Hilbert Space. Applications to Completely Monotone Functions

5. Isometric imbedding in $\mathfrak{H}$ and positive definite functions

5.1. Before we enter into a discussion of our specific topic, a few general remarks concerning metric transforms are necessary. Let $\mathfrak{S}$ be a semi-metric space with the metric PP'. In a study of its metric transforms $F(\mathfrak{S})$ by a transforming function $F(t)$, the following distinction is of interest. We shall say that $F(t)$ is a *proper* transforming function if $F(t) > 0$ for all positive values of t; if, however, $F(t)$ has positive zeros, we call $F(t)$ an *improper* transforming function. The reason for this distinction is as follows. If $F(t)$ is proper, then $F(\mathfrak{S})$ is obviously semi-metric. This, however, is not always the case if $F(t)$ is improper, for the only properties of the metric $F(PP')$ of $F(\mathfrak{S})$ are 1. $F(PP') = F(P'P) \geqq 0$, 2. $F(PP) = 0$.[18] Operating with $\mathfrak{S}$, rather than $F(\mathfrak{S})$, we are

[18] Of course *some* transforms $F(\mathfrak{S})$ may still be semi-metric. Thus if $F(t) = |\sin t|$ $(t \geqq 0)$, we see that $F(\mathfrak{S})$ is semi-metric if the diameter of $\mathfrak{S}$ is $< \pi$.

thus led to consider spaces $\mathfrak{S}$ whose distance function PP' enjoys only the following weaker properties

$$(5.1) \qquad 1.\ PP' = P'P \geqq 0, \qquad 2.\ PP = 0.$$

We shall refer to such spaces as *quasi-metric* spaces.

A quasi-metric space $\mathfrak{S}$ can not, in general, be turned into a semi-metric space by the identification of points P, Q for which $PQ = 0$ (similar to the identification of measurable functions equal almost everywhere), for this identification does not enjoy the necessary transitive property. Indeed, if $PQ = 0$ and $QR = 0$, there is no reason why $PR = 0$ should hold. Moreover, even if this were always true, then $PQ = 0$ need not imply $PS = QS$, where S is a third point of $\mathfrak{S}$. It becomes now apparent that this identification is possible and turns $\mathfrak{S}$ into a semi-metric space $\mathfrak{R}$, if and only if $\mathfrak{S}$ is isometrically imbeddable in a semi-metric space $\mathfrak{R}$.

For the sake of generality we include also *improper* transforming functions $F(t)$ in our discussion, even $F(t) \equiv 0$ not being excluded. Accordingly we are bound to deal with *quasi-metric* spaces and their imbedding in $\mathfrak{H}$.

5.2. Our imbedding problem $\{E_m ; \mathfrak{H}\}$ (see Introduction) is of the following general type: *Given a separable semi-metric space* $\mathfrak{S}$, *determine all functions* $F(t)$, *proper or improper, such that the metric transform* $F(\mathfrak{S})$ *be imbeddable in* $\mathfrak{H}$. *We shall denote by* $\Pi(\mathfrak{S})$ *the class of functions* $F(t)$ *which are solutions of this problem* $\{\mathfrak{S}; \mathfrak{H}\}$. In particular $\Pi(E_m)$ and $\Pi(\mathfrak{H})$ represent the totality of solutions of the problems $\{E_m ; \mathfrak{H}\}$ and $\{\mathfrak{H} ; \mathfrak{H}\}$, respectively.

A characteristic property of a function $F(t)$ in $\Pi(\mathfrak{S})$ is as follows. If $P_0, P_1, \cdots, P_n$ are any $n + 1$ points of $\mathfrak{S}$, then the distances $F(P_iP_k)$ are the edges of a euclidean simplex, i.e., there exist $n + 1$ points $Q_0, \cdots, Q_n$ in E_n, such that $F(P_iP_k) =$ the euclidean distance between Q_i and Q_k. As $F(t)$ thus turns every finite set of points of $\mathfrak{S}$ into a euclidean simplex, $F(t)$ may be called a *simplicial function* of $\mathfrak{S}$[19], and the whole class $\mathfrak{P}(\mathfrak{S})$ may be referred to as the *simplicial class* of $\mathfrak{S}$. Note that $\Pi(\mathfrak{S})$ is never empty, for $F(t) \in \Pi(\mathfrak{S})$, if $F(t) \equiv 0$.

5.3. A connecting link between problems of isometric imbedding in $\mathfrak{H}$ and the theory of positive definite functions is furnished by the following theorem.

THEOREM 4. *A necessary and sufficient condition that a separable quasi-metric space* $\mathfrak{S}$ *be isometrically imbeddable in* $\mathfrak{H}$ *is that the family of functions* $\exp\{-\lambda t^2\}$ (*where* λ *is a positive parameter*) *be positive definite in* $\mathfrak{S}$.[20]

[19] The term "simplicial function" was suggested to the author by L. M. Blumenthal. Blumenthal had previously called $F(t)$ a "tetrahedral function" of $\mathfrak{S}$, if $F(\mathfrak{S})$ enjoys the 4-point property. His theorem mentioned in the Introduction may be stated as follows: $F(t) = t^\kappa (0 < \kappa \leqq \frac{1}{2})$ is a tetrahedral function of any *metric* space $\mathfrak{S}$. See [3].

[20] Definition 1, §1.1, of functions $g(t)$ which are positive definite in a semi-metric space $\mathfrak{S}$, applies unchanged in case $\mathfrak{S}$ is only quasi-metric.

A proof of this theorem is found in [16], §3. Let now $\mathfrak{S}$ be semi-metric and let $\mathfrak{P}(\mathfrak{S})$ be its positive definite class and $\Pi(\mathfrak{S})$ its simplicial class. An immediate consequence of Theorem 4 is the following characterization of the class $\Pi(\mathfrak{S})$ in terms of the class $\mathfrak{P}(\mathfrak{S})$.

THEOREM 5. *A function $F(t)(F(0) = 0)$ belongs to the simplicial class $\Pi(\mathfrak{S})$ of a semi-metric space $\mathfrak{S}$, if and only if the family of functions $\exp\{-\lambda F^2(t)\}$ $(\lambda > 0)$ belongs to the positive definite class $\mathfrak{P}(\mathfrak{S})$.*

Indeed, $F(t) \in \Pi(\mathfrak{S})$ is equivalent with the statement that $F(\mathfrak{S})$ is imbeddable in $\mathfrak{H}$. By Theorem 4 this is the case if and only if $\exp\{-\lambda t^2\}$ is positive definite in $F(\mathfrak{S})$, for all $\lambda > 0$, or, which is the same thing, if $\exp\{-\lambda F^2(t)\}$ is positive definite in $\mathfrak{S}$. This proves our theorem.

This general theorem will be used in this paper only in case $\mathfrak{S}$ is either a euclidean space or Hilbert space. For convenience we restate it in each of these cases as special corollaries.

COROLLARY 1. *$F(t) \in \Pi(E_m)$ if and only if $\exp\{-\lambda F^2(t)\} \in \mathfrak{P}(E_m)$ for all $\lambda > 0$.*

COROLLARY 2. *$F(t) \in \Pi(\mathfrak{H})$ if and only if $\exp\{-\lambda F^2(t)\} \in \mathfrak{P}(\mathfrak{H})$ for all $\lambda > 0$.*

Corollary 1 was already discussed, in case $m = 1$, in [16], §4, in connection with the class $\Pi(E_1)$ of "screw functions" of $\mathfrak{H}$; here it will be applied in §8 of the Appendix.

5.4. We shall presently use Corollary 2 in establishing the following theorem which determines the class $\Pi(\mathfrak{H})$, i.e., the totality of solutions of the problem $\{\mathfrak{H} ; \mathfrak{H}\}$.

THEOREM 6. *The simplicial class $\Pi(\mathfrak{H})$, that is, the totality of continuous functions $F(t)$ such that the metric transform $F(\mathfrak{H})$ be isometrically imbeddable in $\mathfrak{H}$, is identical with the class of functions of the form*

$$F(t) = \left\{ \int_0^\infty \frac{1 - e^{-t^2 u}}{u} \, d\gamma(u) \right\}^{1/2}, \qquad (t \geqq 0), \tag{5.2}$$

where $\gamma(u)$ is non-decreasing for $u \geqq 0$ and such that

$$\int_1^\infty \frac{d\gamma(u)}{u} \text{ exists.} \tag{5.3}$$

The proof that (5.2) furnishes functions $F(t)$ belonging to $\Pi(\mathfrak{H})$ is very simple (see [16], 2nd footnote of §3). For let $P_0, \cdots, P_n$ be any $n + 1$ points of $\mathfrak{H}$. In order to prove that $F(t) \in \Pi(\mathfrak{H})$, i.e., $F(\mathfrak{H})$ is imbeddable in $\mathfrak{H}$, it suffices to show that $F(P_iP_k)$ are distances of $n + 1$ points of E_n. As $\exp\{-ut^2\}(u \geqq 0)$ belongs to $\mathfrak{P}(E_m)$, we have

$$\sum_0^n e^{-uP_iP_k^2} \rho_i \rho_k \geqq 0 \qquad \text{for} \qquad u \geqq 0. \tag{5.4}$$

Hence if

$$\sum_0^n \rho_i = 0, \tag{5.5}$$

we conclude from (5.2), (5.4) and (5.5) that

$$(5.6) \qquad \sum_0^n F^2(P_iP_k)\rho_i\rho_k = -\int_0^\infty \left(\sum_0^n e^{-uP_iP_k^2}\rho_i\rho_k\right) u^{-1}\, d\gamma(u) \leqq 0.$$

The fact that $\sum F^2(P_iP_k)\rho_i\rho_k \leqq 0$ for any set of real ρ_i subject to the relation (5.5), is equivalent to the statement that $F(P_iP_k)$ are distances of $n + 1$ points of E_n.[21] This completes the first part of the proof.

Let us now prove the converse to the effect that an element $F(t)$ of $\Pi(\mathfrak{H})$ is necessarily of the form (5.2). From Corollary 2 we learn that $\exp\{-\lambda F^2(t)\}$ is positive definite in $\mathfrak{H}$ for all positive values of λ. By Theorem 2 it is therefore of the form

$$(5.7) \qquad e^{-\lambda F^2(t)} = \int_0^\infty e^{-t^2u}\, d\alpha(u, \lambda), \qquad (\lambda > 0),$$

where $\alpha(u, \lambda)$ is a family of non-decreasing functions of $u \geqq 0$, defined for all $\lambda > 0$, with

$$(5.8) \qquad \alpha(0, \lambda) = 0, \qquad \alpha(\infty, \lambda) = 1.$$

For convenience we introduce the function

$$(5.9) \qquad \phi(t) = F^2(\sqrt{t})$$

in terms of which (5.7) becomes

$$(5.10) \qquad e^{-\lambda\phi(t)} = \int_0^\infty e^{-tu}\, d\alpha(u, \lambda), \qquad (\lambda > 0).$$

In view of (5.2) and (5.9) our objective is to show that $\phi(t)$ is of the form

$$(5.11) \qquad \phi(t) = \int_0^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u),$$

with $\gamma(u)$ subject to the restriction (5.3). In different words: That $\phi(t)$ belongs to the class T of §4. This fact is now readily established. Indeed (5.10) and the relation $1 = \int_0^\infty d\alpha(u, \lambda)$ imply

$$\frac{1 - e^{-\lambda\phi(t)}}{\lambda} = \int_0^\infty \frac{1 - e^{-tu}}{\lambda}\, d\alpha(u, \lambda) = \int_0^\infty \frac{1 - e^{-tu}}{u} \cdot \frac{u}{\lambda}\, d\alpha(u, \lambda),$$

whence

$$(5.12) \qquad \frac{1 - e^{-\lambda\phi(t)}}{\lambda} = \int_0^\infty \frac{1 - e^{-tu}}{u}\, d\beta(u, \lambda), \qquad (t \geqq 0, \lambda > 0),$$

[21] Here we use the following theorems: 1. *The non-negative quantities* $a_{ik}(a_{ik} = a_{ki}$; $i, k = 0, 1, \cdots n$; $a_{ii} = 0)$ *are distances of* $n + 1$ *points of* E_n, *if and only if* (5.5) *implies* $\sum_0^n a_{ik}^2\rho_i\rho_k \leqq 0$. (See [16], §3). 2. *A separable quasi-metric space is imbeddable in* $\mathfrak{H}$, *if and only if any* $n + 1$ *of its points* $(n = 2, 3, \cdots)$ *are congruent with* $n + 1$ *points of* $\mathfrak{H}$. (See Menger, [9].)

where the new family of monotone functions $\beta(u, \lambda)$ is defined by

$$\beta(u, \lambda) = \frac{1}{\lambda}\int_0^u v\, d\alpha(v, \lambda), \qquad (\lambda > 0). \tag{5.13}$$

From (5.12) we see that

$$(1 - e^{-\lambda\phi(t)})\lambda^{-1}$$

belongs to the class T for all $\lambda > 0$. Letting $\lambda \to 0$ we find this function converging to the continuous limit function $\phi(t)$. On the basis of the closure property expressed by Lemma 5 (§4) we conclude that $\phi(t)$ belongs to T. This completes the proof of Theorem 6.

What has just been established may also be stated as follows: *There is a one-to-one correspondence between the classes T and $\Pi(\mathfrak{H})$ expressed by the relation*

$$\phi(t) = F^2(\sqrt{t}) \quad \text{or} \quad F(t) = \sqrt{\phi(t^2)}, \qquad (\phi \in T, F \in \Pi(\mathfrak{H})). \tag{5.14}$$

It is likewise interesting to connect $F(t)$ directly with the completely monotone functions. From (5.14) and (4.4) we get

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}F^2(\sqrt{t}) = \psi(t). \qquad (t > 0).$$

We may therefore state the following theorem.

THEOREM 6'. *A necessary and sufficient condition that the non-negative continuous function $F(t)$, vanishing at the origin, have the property that $F(\mathfrak{H})$ be isometrically imbeddable in $\mathfrak{H}$, is that*

$$\frac{d}{dt}F^2(\sqrt{t}) = \psi(t) \tag{5.15}$$

be completely monotone for $t > 0$.

5.5. Let us apply the criterion expressed by Theorem 6' to a few examples. Let $F_1(t) = t^\kappa (\kappa > 0)$. By (5.15) we get $\psi(t) = \kappa t^{\kappa-1} = \kappa/t^{1-\kappa}$ and this function is completely monotone if $1 - \kappa \geqq 0$, or $\kappa \leqq 1$. Hence

$$F_1(t) = t^\kappa, \qquad (0 < \kappa \leqq 1). \tag{5.16}$$

belongs to $\Pi(\mathfrak{H})$. Let now

$$F_2(t) = \frac{t}{1+t}, \tag{5.17}$$

which was mentioned in the Introduction. In this case $\psi(t) = (1 + \sqrt{t})^{-3}$ is likewise completely monotone. This fact may be verified directly by successive differentiations or more elegantly as follows: $(1 + t)^{-1}$ is obviously in $\mathfrak{M}$ and therefore also its cube $(1 + t)^{-3}$. Finally it will follow from the results of §6 (Theorem 8) that if $f(t)$ is in $\mathfrak{M}$, then so is $f(\sqrt{t})$. Our last two examples are the functions

(5.18) $$F_3(t) = (1 - e^{-t^2})^{\frac{1}{2}}, \qquad F_4(t) = (\log (1 + t^2))^{\frac{1}{2}}.$$

We find that $\psi_3(t) = e^{-t}$, $\psi_4(t) = (1 + t)^{-1}$ are both in $\mathfrak{M}$, hence F_3, F_4 are elements of $\Pi(\mathfrak{H})$.

A glance at either (5.2) or (5.15) will show that, if we exclude from consideration the trivial function $F(t) \equiv 0$, all elements of $\Pi(\mathfrak{H})$ are *proper* transforming functions, that is, $F(t) > 0$ for $t > 0$. Moreover, if $F(t)$ is not of the form $F(t) = ct$ ($c \geqq 0$), we actually have

(5.19) $$F^2(\sqrt{t}) > 0, \qquad \frac{d}{dt} F^2(\sqrt{t}) > 0,$$
$$\frac{d^2}{dt^2} F^2(\sqrt{t}) < 0, \qquad \frac{d^3}{dt^3} F^2(\sqrt{t}) > 0, \cdots \qquad (t > 0).$$

Hence: *The continuity of $F(t)$ for $t \geqq 0$, $F(0) = 0$, and the inequalities* (5.19) *characterize completely all elements of $\Pi(\mathfrak{H})$, with the exception of the elements $F(t) = ct$ ($c \geqq 0$).*

Ruling out the case $F(t) \equiv 0$, we see that $\mathfrak{H}$ and $F(\mathfrak{H})$ are homeomorphs of each other, since both metrics PP' and $F(PP')$ are topologically equivalent.

The class $\Pi(\mathfrak{H})$ contains bounded as well as unbounded functions. Thus $F_2(t)$ and $F_3(t)$, defined above, are bounded. In terms of the function $\gamma(u)$ of (5.2) it is readily seen that $F(t)$ is bounded, if and only if

(5.20) $$\gamma(0) = \gamma(+0) \qquad \text{and} \qquad \int_{+0}^{1} \frac{d\gamma(u)}{u} \qquad \text{exists.}$$

5.6. In [15] it was shown that the function (5.16) ($0 < \kappa < 1$) enjoys the following further property: If $P_0, \cdots, P_n$ are any $n + 1$ *distinct* points of $F_1(\mathfrak{H})$, the distances $F_1(P_iP_k) = (P_iP_k)^\kappa$ are the edges of a *non-degenerate* simplex in E_n. This property extends as follows to the whole class $\Pi(\mathfrak{H})$. *If $F(t)$ is any element of $\Pi(\mathfrak{H})$, which is not of the form $F(t) = ct$, then any $n + 1$ distinct points of $F(\mathfrak{H})$ are congruent with a non-degenerate n-dim. euclidean simplex.*

Indeed, it suffices to show that if $P_0, \cdots, P_n$ are distinct points of $\mathfrak{H}$, then

$$\sum_0^n \rho_i = 0, \qquad \sum_0^n \rho_i^2 > 0, \qquad \text{imply} \qquad \sum_0^n F^2(P_iP_k)\rho_i\rho_k < 0.$$

Now (5.2) gives

(5.21) $$F^2(t) = c^2t^2 + \int_{+0}^{\infty} \frac{1 - e^{-t^2u}}{u}\, d\gamma(u), \qquad (c^2 = \gamma(+0) - \gamma(0)),$$

and therefore

(5.22) $$\sum_0^n F^2(P_iP_k)\rho_i\rho_k = c^2 \sum_0^n (P_iP_k)^2\rho_i\rho_k - \int_{+0}^{\infty} \left(\sum_0^n e^{-uP_iP_k^2}\rho_i\rho_k\right) u^{-1}\, d\gamma(u).$$

But $\sum (P_iP_k)^2\rho_i\rho_k \leqq 0$ and the integrand in (5.22) is positive by (3.10). Therefore the vanishing of the quadratic form (5.22) implies that $\gamma(u)$ is constant for $u > 0$, hence $F(t) = ct$.

5.7. It was pointed out in 5.5 that $\Pi(\mathfrak{H})$ contains only one improper transforming function, namely $F(t) \equiv 0$. That the same conclusion may be reached directly, independently of Theorem 6, and for all classes $\Pi(E_m)(2 \leqq m \leqq \infty)$, is shown by the following elementary lemma.

LEMMA 6. *Let $F(t)$ be continuous for $t \geqq 0$, $F(t) \geqq 0$, $F(0) = 0$, $F(t) \not\equiv 0$. If $F(E_m)(2 \leqq m \leqq \infty)$ is imbeddable in a semi-metric space $\mathfrak{R}$, then $F(t) > 0$ for $t > 0$.*

Indeed, let $P \to P'(P \epsilon F(E_m), P' \epsilon \mathfrak{R})$ indicate the mapping of $F(E_m)$ in $\mathfrak{R}$. Denoting by $[P', Q']$ the distance in $\mathfrak{R}$, the isometricity of this imbedding is expressed by the identical relation

$$[P', Q'] = F(PQ), \qquad (P, Q \epsilon E_m). \tag{5.23}$$

Assume that for $t = \tau > 0$ we have $F(\tau) = 0$. Take in E_m an ordinary circle Γ of radius τ, center O, and let A and B be any two points on Γ. Now, by (5.23) we have

$$[O', A'] = F(OA) = F(\tau) = 0, \qquad [O', B'] = F(OB) = F(\tau) = 0,$$

hence ($\mathfrak{R}$ being semi-metric) $O' = A'$, $O' = B'$, and therefore $A' = B'$ or $[A', B'] = 0$. But this implies

$$F(AB) = [A', B'] = 0.$$

As A, B were arbitrary points of Γ, we conclude that $F(t) = 0$ in the interval $0 \leqq t \leqq 2\tau$. Repeating the argument with 2τ, instead of τ, we conclude that $F(t)$ vanishes in the interval $(0, 4\tau)$, and so forth. Hence $F(t) \equiv 0$, in contradiction to our assumption that $F(t) \not\equiv 0$.

Notice that the proof as well as the result fail if $m = 1$. This is already shown by the simplest *closed* euclidean "screw line", namely the circle; for $F(E_1)$, where $F(t) = |\sin t|$ $(0 \leqq t < \infty)$, is imbeddable in E_2. Indeed, taking E_1 to be the real θ-axis with the euclidean metric $|\theta_1 - \theta_2|$, the imbedding of $F(E_1)$ in E_2 is performed by mapping the point $P = \theta$ onto the point P': $x = \frac{1}{2} \cos 2\theta$, $y = \frac{1}{2} \sin 2\theta$, of E_2. For we have identically

$$F^2(P_1P_2) = |\sin|\theta_1 - \theta_2||^2 = \sin^2(\theta_1 - \theta_2) = \tfrac{1}{4}(\cos 2\theta_1 - \cos 2\theta_2)^2 + \tfrac{1}{4}(\sin 2\theta_1 - \sin 2\theta_2)^2 = [P_1', P_2']^2, \text{ hence } [P_1', P_2'] = F(P_1P_2).$$

6. The inner transformations of completely monotone functions

6.1. The simplicial class $\Pi(\mathfrak{H})$ enjoys a further analytical property expressed by the following theorem.

THEOREM 7. *The class of non-negative continuous functions $F(t)$, $(F(0) = 0)$, with the property that if $g(t)$ is an element of $\mathfrak{P}(\mathfrak{H})$, also $g(F(t))$ should be in $\mathfrak{P}(\mathfrak{H})$, coincides with the simplicial class $\Pi(\mathfrak{H})$.*

Indeed, let $F(t) \epsilon \Pi(\mathfrak{H})$ and $g(t) \epsilon \mathfrak{P}(\mathfrak{H})$. In order to prove that $g(F(t)) \epsilon \mathfrak{P}(\mathfrak{H})$, we have to show that if $P_1, \cdots, P_n$ are any points of $\mathfrak{H}$, then

$$\sum_1^n g(F(P_iP_k))\rho_i\rho_k \geqq 0 .$$

But this is evident, since there are points Q_i of $\mathfrak{H}$, with $F(P_iP_k) = Q_iQ_k$ (since $F(t) \epsilon \Pi(\mathfrak{H})$), and $\sum g(Q_iQ_k)\rho_i\rho_k \geqq 0$ (since $g(t) \epsilon \mathfrak{P}(\mathfrak{H})$).

To prove the converse, let $F(t)$ be such that $g(t) \epsilon \mathfrak{P}(\mathfrak{H})$ implies $g(F(t)) \epsilon \mathfrak{P}(\mathfrak{H})$. As in particular $g(t) = \exp\{-\lambda t^2\} \epsilon \mathfrak{P}(\mathfrak{H})$, we have

$$e^{-\lambda F^2(t)} \epsilon \mathfrak{P}(\mathfrak{H}), \qquad \text{for all positive } \lambda.$$

By Corollary 2 we now conclude that $F(t) \epsilon \Pi(\mathfrak{H})$, which proves the theorem.

Let us now state Theorem 7 in terms of completely monotone functions by means of Theorem 3. Let $\phi(t)$ be a continuous non-negative function vanishing at the origin. Let us determine all such functions with the property that if $f(t) \epsilon \mathfrak{M}$, also $f(\phi(t))$ should be an element of $\mathfrak{M}$. But if $\phi(t)$ is of this nature and $g(t) \epsilon \mathfrak{P}(\mathfrak{H})$, we may conclude successively that the following relations hold: $g(\sqrt{t}) \epsilon \mathfrak{M}$, $g(\sqrt{\phi(t)}) \epsilon \mathfrak{M}$, $g(\sqrt{\phi(t^2)}) \epsilon \mathfrak{P}(\mathfrak{H})$. Therefore $\sqrt{\phi(t^2)} = F(t) \epsilon \Pi(\mathfrak{H})$, by Theorem 7. In view of (5.14) we thus see that $\phi(t)$ belongs to the class T of §4. This proves the following theorem.

THEOREM 8. *The class of non-negative continuous functions $\phi(t)$, ($\phi(0) = 0$), with the property that if $f(t)$ is completely monotone for $t \geqq 0$, also $f(\phi(t))$ should be completely monotone for $t \geqq 0$, is identical with the class T of integrals of completely monotone functions*:

$$\phi(t) = \int_{+0}^t \psi(t)\, dt, \qquad (t > 0). \tag{6.1}$$

In view of this theorem we may call these functions $\phi(t)$ *inner transformations* of the class $\mathfrak{M}$ of completely monotone functions. In a recent paper ([5], p. 498) S. Bochner has already pointed out this property of the functions (6.1) with the restriction that $\psi(t)$ itself belong to the class $\mathfrak{M}$. This restriction, which amounts to assuming the existence of $\psi(+0)$ rules out such simple elements of T as $\phi(t) = t^\kappa (0 < \kappa < 1)$ for $\psi(t) = \phi'(t) = \kappa/t^{1-\kappa}$ is not bounded near the origin, although $\int_{+0}\psi(t)dt$ obviously exists. However, the principal new contribution of Theorem 8 is not the *enlargement* of Bochner's class of inner transformations, but the statement that the enlarged class T contains *all* such transformations.

6.2. The class T enjoys a further closure property which we state as a corollary.

COROLLARY 3. *If $\phi_1(t)$ and $\phi_2(t)$ are any two elements of T, i.e., of the form (6.1) or (4.4), also $\phi_1(\phi_2(t))$ belongs to T.*

For if $f(t)$ is completely monotone for $t \geqq 0$, then so is $f(\phi_1(t))$ and therefore also $f(\phi_1(\phi_2(t)))$. Hence $\phi_1(\phi_2(t))$ must belong to T by Theorem 8.

This closure property of T appears here as a consequence of virtually all previous results of this paper. It seems of sufficient interest to deserve a direct proof to which we now proceed.[22]

We prove first: *If* $\phi(t) \in T$, *then*

$$1 - e^{-\phi(t)} \in T. \tag{6.2}$$

It suffices to show that

$$\psi(t) = \frac{d}{dt}(1 - e^{-\phi(t)}) = e^{-\phi(t)}\phi'(t)$$

is completely monotone for $t > 0$, that is, verifies the inequalities (4.1). Now

$$\psi'(t) = e^{-\phi}(\phi'' - \phi'^2), \qquad \psi''(t) = e^{-\phi}(\phi''' - 3\phi'\phi'' + \phi'^3),$$

and generally

$$\psi^{(n-1)}(t) = \sum (-1)^{0\cdot k_1 + 1\cdot k_2 + \cdots + (n-1)k_n + n-1} \cdot c_{(k)} e^{-\phi}(\phi')^{k_1}(\phi'')^{k_2} \cdots (\phi^{(n)})^{k_n}, \qquad (c_{(k)} > 0), \tag{6.3}$$

where all terms of this sum have constant weight $k_1 + 2k_2 + \cdots + nk_n = n$. This last statement as well as the fact that the sign before each term has the expression indicated, is readily proved (by induction) by differentiating once both sides of (6.3). Now $\phi'(t)$ being completely monotone for $t > 0$, we have $(-1)^{m-1}\phi^{(m)}(t) \geqq 0$, which, in view of (6.3), clearly implies $(-1)^{n-1}\psi^{(n-1)}(t) \geqq 0$.

We can now complete a proof of Corollary 3. For let

$$\phi_1(t) = \int_0^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u) \tag{6.4}$$

be a second arbitrary element of T. This integral is the limit, for any fixed $t \geqq 0$, of an appropriate sequence of functions of the form

$$\phi_n^*(t) = \sum_{\nu=1}^{n} \frac{1 - e^{-tu_\nu}}{u_\nu}(\gamma(u_\nu) - \gamma(u_{\nu-1})) \qquad (0 = u_0 < u_1 < \cdots < u_n)$$

where the u_ν depend on n as well. Hence

$$\phi_n^*(\phi(t)) \to \phi_1(\phi(t)) \qquad \text{as} \qquad n \to \infty.$$

As $\phi_n^*(\phi(t)) \in T$, by (6.2), we conclude that also the limit functions $\phi_1(\phi(t))$ is in T, on the basis of the closure property of Lemma 5, §4.

6.3. We conclude this section with two further theorems on the class $\mathfrak{M}$ of functions $f(t)$ which are completely monotone for $t \geqq 0$. We find it convenient to normalize our functions by requiring that $f(0) = 1$. Let $\mathfrak{M}_0$ denote the subclass of elements of $\mathfrak{M}$ thus normalized.

[22] This direct proof was suggested to the author by von Neumann.

THEOREM 9. *The class of elements $f(t)$ of $\mathfrak{M}_0$ with the property that $(f(t))^\lambda$ also belongs to $\mathfrak{M}_0$, for all $\lambda > 0$, is identical with the class of functions of the form*

$$f(t) = e^{-\phi(t)} \tag{6.5}$$

where $\phi(t) \in T$.

THEOREM 10. *If to the previous theorem's assumptions on $f(t)$ we add the further assumption that $f(t)$ be bounded away from zero, $f(t) \geqq \epsilon_0 > 0$, then this new class of functions is identical with the class of functions of the form*

$$f(t) = e^{f_1(t) - f_1(0)}, \tag{6.6}$$

where $f_1(t) \in \mathfrak{M}$.

These theorems are immediate consequences of Corollary 2 and Theorems 3 and 6. Indeed, in order to prove Theorem 9, define $\phi(t)$ by (6.5). Now $(f(t))^\lambda = \exp\{-\lambda\phi(t)\} \in \mathfrak{M}_0$, i.e., $\exp\{-\lambda\phi(t^2)\} \in \mathfrak{P}(\mathfrak{H})$ (Theorem 3), if and only if $\sqrt{\phi(t^2)} \in \Pi(\mathfrak{H})$ (Corollary 2), hence $\phi(t) \in T$ (Theorem 6 and (5.14)). To prove Theorem 10, notice that $f(t)$ is bounded away from zero if and only if $\phi(t)$, of (6.5), is bounded. Now

$$\phi(t) = \int_0^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u)$$

is bounded, if and only if $\gamma(+0) = \gamma(0)$ and $\int_{+0}^\infty u^{-1}\, d\gamma(u)$ exists. But then

$$\phi(t) = \int_{+0}^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u) = \int_{+0}^\infty u^{-1}\, d\gamma(u) - \int_{+0}^\infty e^{-tu}u^{-1}\, d\gamma(u) = f_1(0) - f_1(t),$$

where $f_1(t) \in \mathfrak{M}$. Now (6.5) goes over into (6.6).

Incidentally we have proved the following result: *The elements $\phi(t)$ of T which are bounded are of the form*

$$\phi(t) = f(0) - f(t), \qquad (f(t) \in \mathfrak{M}), \tag{6.7}$$

and conversely.

Theorem 9 may also be stated in the following suggestive form: *The elements $f(t)$ of $\mathfrak{M}_0$ which belong to $\mathfrak{M}_0$ together with all their positive powers $(f(t))^\lambda (\lambda > 0)$ are identical with those elements of $\mathfrak{M}_0$ which arise out of the particular function*

$$f(t) = e^{-t} \tag{6.8}$$

by the general inner transformation $t \mid \phi(t)$, where $\phi(t) \in T$.

APPENDIX

7. On metric transforms of euclidean spaces which are isometrically imbeddable in euclidean spaces

7.1. The solution of the problem $\{E_m ; E_n\}$, $(2 \leqq m \leqq n < \infty)$, is given by the following theorem.

THEOREM 11. *If the metric transform $F(E_m)$, of $E_m (m \geqq 2)$ by $F(t)$, is isometrically imbeddable in the euclidean space $E_n (m \leqq n < \infty)$, then necessarily $F(t) = ct\ (c > 0)$, unless $F(t) \equiv 0$.*

In the proof of this theorem we shall use the solution of the problem $\{E_1 ; E_n\}$ $(n < \infty)$ (von Neumann-Schoenberg, [12]) which is as follows: *$F(E_1)$ is imbeddable in E_n if and only if $F^2(t)$ is of the form*

$$F^2(t) = c^2 t^2 + \sum_{\nu=1}^{r} A_\nu^2 \sin^2 (k_\nu t), \tag{7.1}$$

where $2r + 1 \leqq n$ if $c > 0$, or $2r \leqq n$ if $c = 0$, and c, A_ν, k_ν are non-negative constants.

Let E_1 be the real x-axis with the metric $| x_1 - x_2 |$. The statement that $F(E_1)$ is imbeddable in E_n means that there is a continuous curve $P' = \phi(x)$ $(P' \in E_n, -\infty < x < \infty)$ such that we have identically

$$F^2(| x_1 - x_2 |) = (P_1' P_2')^2 = | \phi(x_1) - \phi(x_2) |^2, \qquad (-\infty < x_1, x_2 < \infty), \tag{7.2}$$

in standard vector notation. The curve $P' = \phi(x)$, of E_n, is what was called a *screw line* of E_n, while the corresponding $F(t)$ is a *screw function* of E_n. Thus whenever a continuous function $F(t)$ satisfies an identity of the form $F^2(| t - t' |) = | \phi(t) - \phi(t') |^2$, $(\phi \in E_n)$, for all real t, t', we conclude that $F^2(t)$ is of the form (7.1).

We can now readily prove Theorem 11. Ruling out the trivial case $F(t) \equiv 0$, by Lemma 6, §5.7, we may assume that $F(t) > 0$ for $t > 0$. Consider now a straight line E_1 in E_n. The assumption of Theorem 11 clearly implies that $F(E_1)$ is imbeddable in E_n, hence $F^2(t)$ is of the form (7.1). Now consider in E_n an ordinary circle Γ, of radius $\frac{1}{2}$, referred to the angular parameter θ. Its image in E_n is a simple closed curve $P' = \phi(\theta)$ (of period 2π, $P' \in E_n$). The isometricity of the imbedding of $F(\Gamma)$ in E_n gives

$$F^2(P_1 P_2) = F^2\left(\left| 2 \sin \frac{\theta_1 - \theta_2}{2} \right|\right) = (P_1' P_2')^2 = | \phi(\theta_1) - \phi(\theta_2) |^2,$$

for any real θ_1, θ_2. This shows that also $F\left(\left| 2 \sin \frac{\theta}{2} \right|\right)$ is a screw function of E_n. By (7.1) we therefore have

$$F^2\left(\left| 2 \sin \frac{\theta}{2} \right|\right) = \sum_{\mu=1}^{s} B_\mu^2 \sin^2 (h_\mu \theta), \qquad (2s \leqq n), \tag{7.3}$$

this time without a term in θ^2, since the left side of (7.3) is bounded. From (7.1) and (7.3) we now derive the identity

$$\sum_{\mu=1}^{s} B_\mu^2 \sin^2 (h_\mu \theta) = 4c^2 \sin^2 \frac{\theta}{2} + \sum_{\nu=1}^{r} A_\nu^2 \sin^2 \left(2k_\nu \sin \frac{\theta}{2}\right),$$

from which we readily conclude that for each value of $\nu (\nu = 1, \cdots, r)$ we must have either $A_\nu = 0$, or else $k_\nu = 0$. Now (7.1) reduces to $F^2(t) = c^2 t^2$ and the theorem is established.

7.2. In conclusion we remark that Theorem 11 may be easily proved without reference to the result (7.1) about screw functions of E_n, *if we limit ourselves to the case when* $m = n < \infty$. Omitting details we shall merely sketch the proof. The case $m = n = 1$ is easily disposed of in a manner similar to the treatment of Cauchy's functional relation $F(x + y) = F(x) + F(y)$. Now assume the theorem proved up to the dimension $m - 1$, and let $F(E_m)$ be imbeddable in E'_m. Take two distinct points A and B in E_m and let E_{m-1} be their "perpendicular bisector" in E_m, i.e., the locus of points equidistant from A and B. Now if the imbedding of $F(E_m)$ in E'_m maps A, B into A', B', it is clear that it will map E_{m-1} into the perpendicular bisector E'_{m-1} of A', B' Therefore $F(t) = ct (c > 0)$.

8. Determination of the rectifiable solutions of the problem $\{E_m ; \mathfrak{H}\}$, $(1 \leqq m < \infty)$

8.1. The class of solutions of the problem $\{E_m ; \mathfrak{H}\}$ may be described in the terminology of §5.2 as the simplicial class $\Pi(E_m)$. As $E_m \subset E_{m+1}$ clearly implies $\Pi(E_m) \supset \Pi(E_{m+1})$, we have before us the following non-increasing sequence of classes

$$\Pi(E_1) \supset \Pi(E_2) \supset \cdots \supset \Pi(E_m) \supset \cdots \supset \Pi(\mathfrak{H}). \tag{8.1}$$

In view of the inequality (5.6) it now becomes apparent that all functions $F(t)$ given by the formula

$$F^2(t) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2}\, d\gamma(u), \tag{8.2}$$

where $\gamma(u)$ is non-decreasing for $u \geqq 0$ and such that

$$\int_1^\infty \frac{d\gamma(u)}{u^2} \qquad \text{exists}, \tag{8.3}$$

belong to the simplicial class $\Pi(E_m)$. For $\sum_0^n \rho_i = 0$ and (8.2) imply

$$\sum_0^n F^2(P_iP_k)\rho_i\rho_k = -\int_0^\infty \left\{\sum_0^n \Omega_m(P_iP_k \cdot u)\rho_i\rho_k\right\} u^{-2}\, d\gamma(u) \leqq 0, \qquad (P_i \in E_m),$$

since $\Omega_m(tu)$ is positive definite in E_m for $u \geqq 0$.

Conversely, let $F(t)$ be an element of $\Pi(E_m)$, that is, such that the metric transform $F(E_m)$ be imbeddable in $\mathfrak{H}$. A closure property for integrals of the form (8.2), analogous to the closure property of Lemma 5 for the class T, would allow us to conclude by the method of §5.4 that $F^2(t)$ is of the form (8.2). This required closure property has not been established as yet. We shall show very simply, however, that a significant subclass of the class $\Pi(E_m)$ is necessarily given by (8.2) for *bounded* functions $\gamma(u)$. Let us explain first what we mean by a *rectifiable* simplicial function $F(t)$.

We need the following lemma (see Wilson, [21], p. 65, Property II)

LEMMA 7. *If* $F(t) \in \Pi(E_m)$, $F(t) \not\equiv 0$, *then* $F'(0)$ *exists and*

$$0 < F'(0) \leqq \infty. \tag{8.4}$$

We know by Lemma 6, §5.7, that $F(t) > 0$ for $t > 0$, provided $m \geqq 2$. In any case ($m \geqq 1$) it is readily seen that $F(t)$ can not possess arbitrarily small positive zeros, for this would again imply that $F(t) \equiv 0$. This secures a positive quantity τ such that $F(t) > 0$ if $0 < t \leqq \tau$. Now take for E_1 the real x-axis and consider in $F(E_1)$ the simple arc Γ: $0 \leqq x \leqq \tau$. Denote its length in the ordinary sense by s ($s \leqq \infty$), that is, the least upper bound of the length of inscribed polygonal lines. Note that $s \geqq F(\tau) > 0$, hence $s > 0$. Let now δ be a small positive quantity and choose the integer n such that $n\delta \leqq \tau < (n+1)\delta$. The length of the inscribed polygonal line of vertices $x = 0, \delta, 2\delta, \cdots, n\delta, \tau$, is $nF(\delta) + F(\tau - n\delta)$. As $\delta \to 0$, this length converges to the arc-length s; moreover $F(\tau - n\delta) \to 0$, for $\tau - n\delta \to 0$. Hence $nF(\delta) \to s$. From this and the fact that $n\delta \to \tau$, we get

$$\lim_{\delta \to 0} \frac{F(\delta)}{\delta} = \frac{s}{\tau} \qquad (> 0 \text{ and } \leqq \infty), \tag{8.5}$$

which proves our lemma.

The arc Γ of $F(E_1)$ is rectifiable ($s < \infty$) if and only if $F'(0)$ is finite and (8.5) gives $s = \tau F'(0)$. In fact *any* rectifiable arc Γ of E_m of length τ is also rectifiable in $F(E_m)$ and its length in $F(E_m)$ is $s = \tau F'(0)$. For this reason we call $F(t)$ *rectifiable* if $F'(0) < \infty$. Let $\Pi'(E_m)$ denote the subclass of rectifiable elements of $\Pi(E_m)$.

8.2. The rectifiable simplicial functions are determined by the following theorem.

THEOREM 12. *The class $\Pi'(E_m)$ of rectifiable simplicial functions $F(t)$ of $E_m (m \geqq 1)$ is identical with the class of functions such that $F^2(t)$ is of the form* (8.2) *with bounded $\gamma(u)$.*

Indeed, if $\gamma(\infty)$ is finite, then (8.2) and (1.8) give

$$\lim_{t \to 0} \left(\frac{F(t)}{t}\right)^2 = \lim_{t \to 0} \int_0^\infty \frac{1 - \Omega_m(tu)}{t^2 u^2}\, d\gamma(u) = \frac{1}{2m}\gamma(\infty) = \sigma^2, \tag{8.6}$$

where we write $2m\sigma^2 = \gamma(\infty)$. Hence $F(t) \in \Pi'(E_m)$.

Conversely, let $F(t) \in \Pi'(E_m)$. By Corollary 1, §5.3, we have $\exp\{-\lambda F^2(t)\} \in \mathfrak{P}(E_m)$ for all $\lambda > 0$, hence by Theorem 1

$$e^{-\lambda F^2(t)} = \int_0^\infty \Omega_m(tu)\, d\alpha(u, \lambda), \qquad (t, \lambda > 0),$$

where $\alpha(u, \lambda)$ is a family of monotone functions with $\alpha(0, \lambda) = 0$, $\alpha(\infty, \lambda) = 1$. As in §5.4 we may now write

$$\frac{1 - e^{-\lambda F^2(t)}}{\lambda} = \int_0^\infty \frac{1 - \Omega_m(tu)}{\lambda}\, d\alpha(u, \lambda) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2}\, d\beta(u, \lambda), \tag{8.7}$$

where

$$\beta(u, \lambda) = \frac{1}{\lambda}\int_0^u u^2\, d\alpha(u, \lambda). \tag{8.8}$$

On the other hand the finiteness of $\sigma = F'(0)$ implies $F(t) < (\sigma + \epsilon)t$, for t sufficiently small, and therefore

$$t^2(\sigma + \epsilon)^2 > F^2(t) \geqq \frac{1 - e^{-\lambda F^2(t)}}{\lambda}.$$

By (8.7) we now have

$$(\sigma + \epsilon)^2 > \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2 t^2}\, d\beta(u, \lambda),$$

for t sufficiently small. As $t \to 0$ we now get $\beta(\infty, \lambda) \leqq 2m(\sigma + \epsilon)^2$, showing that the family of monotone functions (8.8) is bounded uniformly for all $\lambda > 0$. For a suitable sequence $\lambda_n \to 0$ we shall therefore have $\beta(u, \lambda_n) \to \gamma(u)$ in all continuity points of $\gamma(u)$. Now, if t is kept fast in (8.7) while $\lambda = \lambda_n \to 0$, we get in the limit the desired representation (8.2).[23]

Let $F(t)(\not\equiv 0)$ be a rectifiable element of $\Pi(E_m)$ and let us assume $m \geqq 2$. The space $F(E_m)$ is a homeomorph of E_m and its geodesics are the straight lines of E_m. The geodesic distance, or intrinsic distance, $l(P, Q)$ between two points of $F(E_m)$ (in the terminology and notation of Wilson, [20], p. 420) is $l(P, Q) = F'(0) \cdot PQ$. Thus by passing in $F(E_m)$ from its metric $F(PQ)$ to the intrinsic metric $l(P, Q)$ which it generates, we merely get a similar image of the original space E_m, the ratio of similitude being $F'(0)$. Notice furthermore that an arc of a geodesic line in $F(E_m)$ (i.e., a straight segment of the underlying E_m) never loses its minimizing property of an absolute minimum no matter how far it is extended in both directions. In the isometric image Σ of $F(E_m)$ in $\mathfrak{H}$, the geodesics of $F(E_m)$ go over in screw lines of $\mathfrak{H}$, all of which are congruent among themselves. In fact any two congruent figures in E_m are mapped in congruent figures of Σ. To any rigid motion of E_m in itself corresponds a rigid motion of Σ in itself.

8.3. We close the paper with the following analogue of Theorem 6′.

THEOREM 12′. *A necessary and sufficient condition that the non-negative continuous function $F(t)$, vanishing at the origin, be an element of $\Pi'(E_m)$, is that $F^2(\sqrt{t})$ be continuously differentiable for $t \geqq 0$ and, on setting*

$$\psi(t) = \frac{d}{dt} F^2(\sqrt{t}), \tag{8.9}$$

the function $\psi(t^2)$ be positive definite in E_{m+2}.

Indeed, denoting by c_m, c_m', c_m'' positive constants and setting $k = (m - 2)/2$, we have by (1.8) (see [18], p. 45)

$$\Omega_m'(r) = c_m(d/dt)(J_k(r)r^{-k}) = -c_m J_{k+1}(r)r^{-k},$$

hence

$$-\Omega_m'(r)r^{-1} = c_m' \Omega_{m+2}(r).$$

[23] The basic idea of this proof was suggested by the proof of Theorem 27, p. 91, in H. Cramér's monograph [6]. References are there found to the work of Kolmogoroff, De Finetti, and P. Lévy.

An element $F(t)$ of $\Pi'(E_m)$ is given by (8.2) with a bounded $\gamma(u)$. Hence

$$F^2(\sqrt{t}) = \int_0^\infty \frac{1 - \Omega_m(\sqrt{(tu)})}{u}\, d\gamma(\sqrt{u}).$$

By differentiation we get

$$\psi(t) = \frac{d}{dt} F^2(\sqrt{t}) = -\int_0^\infty \frac{\Omega_m'(\sqrt{(tu)})}{2\sqrt{(tu)}}\, d\gamma(\sqrt{u}) = c_m'' \int_0^\infty \Omega_{m+2}(\sqrt{(tu)})\, d\gamma(\sqrt{u}),$$

or

$$\psi(t^2) = c_m'' \int_0^\infty \Omega_{m+2}(tu)\, d\gamma(u).$$

Hence $\psi(t^2) \in \mathfrak{P}(E_{m+2})$, by Theorem 1. The converse is proved by retracing our steps.

We discuss only one particular example, namely

$$F_5(t) = (1 - \Omega_3(t))^{\frac{1}{2}} = (1 - (\sin t/t))^{\frac{1}{2}}. \tag{8.10}$$

By (8.2) we have $F_5(t) \in \Pi'(E_3)$. As above we get

$$\psi(t) = \frac{d}{dt} F_5^2(\sqrt{t}) = c_3'' \Omega_5(\sqrt{t})$$

which is *not* completely monotone for $t > 0$, since $\Omega_5(r)$ has infinitely many positive zeros. Hence $F_5(t)$ is not an element of $\Pi(\mathfrak{H})$, by Theorem 6′. Thus $F_5(E_3)$ is imbeddable in $\mathfrak{H}$, but $F_5(\mathfrak{H})$ is not imbeddable in $\mathfrak{H}$. A further argument, which we omit, would show that already $F_5(E_4)$ is not congruent to a subset of $\mathfrak{H}$.

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