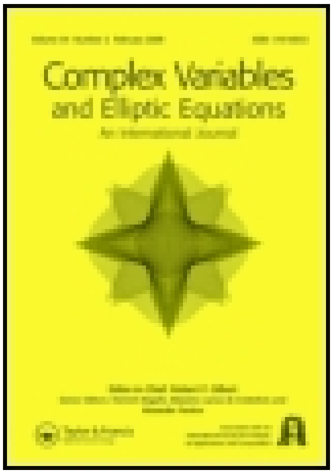

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A quadratic extremal problem on the dirichlet space

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A Quadratic Extremal Problem on the Dirichlet Space*

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It is shown that there is a unique solution F to the problem

$$\lambda = \sup\left\{\operatorname{Re}\int_\Delta \bar{f}f'\,dA : \int_\Delta |f'|^2\,dA \le 1\right\}.$$

The function F is entire with a number of special properties. The number λ is the reciprocal of the smallest zero of the 0th Bessel function of the first kind.

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INTRODUCTION

The Dirichlet space, D, on the open unit disc Δ consists of all analytic functions f

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \qquad |z| < 1, \quad f(0) = 0,$$

for which the quantity

$$\int_\Delta |f'(z)|^2\,dA(z) = \sum_{k=1}^{\infty} k|a_k|^2 =: \|f\|_D^2 \tag{1}$$

is finite. In connection with a generalization of Harnack's inequality, Boris Korenblum [2] has asked how large the quantity

$$\lambda =: \sup_{f\in D}\left\{\frac{\operatorname{Re}(\sum_1^\infty a_k\bar{a}_{k+1})}{\sum_1^\infty k|a_k|^2}\right\} \tag{2}$$

is and, if possible, to characterize all functions F which attain the value λ in (2). The expression in the numerator in (2) is not a linear function of f but rather quadratic; hence, the title of this paper.

It is simple to show that

$$\sum_1^\infty a_k\bar{a}_{k+1} = \int_\Delta \overline{f(z)}f'(z)\,dA(z) \tag{3}$$

*In memory of Ralph P. Boas, Jr. (1912–1992).

and therefore Korenblum's problem has this alternate form:

$$\lambda = \sup\left\{\operatorname{Re}\left(\int_{\Delta} \overline{f} f' \, dA\right) : \|f\|_D \leq 1\right\} \tag{4}$$

We show here that the extremal problem (2) or (4) has a unique solution F, up to multiplication by a constant; moreover, F is an entire function of exponential type with infinitely many zeros, all in the left half-plane, none of which lie in Δ or on the real axis, except for a first order zero at the origin. Moreover, the number λ is the reciprocal of the smallest positive zero of $J_0(x)$, the 0th Bessel function. Finally,

$$F(z) = C \sum_{n=1}^{\infty} J_n(1/\lambda) z^n$$

where J_n is the nth Bessel function and C is a certain constant.

The conclusions above are proved in Sections 1 and 2; Section 3 contains a number of results which generalize the extremal problem (2).

1. EXISTENCE AND UNIQUENESS

We begin by establishing simple bounds on λ.

PROPOSITION 1 $1/\sqrt{6} < \lambda \leq 1/2$.

Proof Since $2\operatorname{Re}(z\overline{w}) \leq |z|^2 + |w|^2$, we have

$$\begin{aligned} 2\operatorname{Re}(a_1\overline{a}_2 + a_2\overline{a}_3 + \cdots) &\leq |a_1|^2 + |a_2|^2 + |a_2|^2 + |a_3|^2 + \cdots \\ &= |a_1|^2 + 2|a_2|^2 + 2|a_3|^2 + \cdots \\ &\leq \sum k|a_k|^2 \end{aligned}$$

which implies that $\lambda \leq 1/2$. The lower bound is obtained by the specific choices

$$a_2 = \sqrt{\tfrac{2}{3}} a_1, \qquad a_3 = \tfrac{1}{3} a_1, \qquad a_4 = a_5 = \cdots = 0$$

which give

$$\lambda \geq \left\{\frac{a_1 a_2 + a_2 a_3}{a_1^2 + 2a_2^2 + 3a_3^2}\right\} = \frac{\sqrt{\frac{2}{3}} + \frac{1}{3}\sqrt{\frac{2}{3}}}{1 + 2\cdot\frac{2}{3} + 3\cdot\frac{1}{9}} = \frac{1}{\sqrt{6}}. \qquad \blacksquare$$

To prove the existence of a solution, we shall need the following Lemma.

LEMMA *Given $\varepsilon > 0$ there is an R_0, $0 < R_0 < 1$, such that*

$$\int_R^1 \int_0^{2\pi} |f(re^{it})|^2 \, dt \, r \, dr < \varepsilon \|f\|_D^2 \tag{5}$$

whenever $R_0 \leq R < 1$ and $f(0) = 0$.

Proof Let $f(z) = \sum_1^\infty c_k z^k$. Then

$$\begin{aligned}\frac{1}{\pi}\int_R^1\int_0^{2\pi} |f(re^{it})|^2\,dt\,r\,dr &= \sum_1^\infty |c_k|^2 \frac{1-R^{2k+2}}{k+1}\\ &= \sum_1^\infty (k|c_k|^2)\left(\frac{1-R^{2k+2}}{k(k+1)}\right)\\ &\le \|f\|_D^2\left(\sum_1^\infty \frac{1-R^{2k+2}}{k(k+1)}\right).\end{aligned}$$

Here we used the simple inequality

$$k|c_k|^2 \le \|f\|_D^2, \qquad k = 1,2,\ldots .$$

The expression

$$\sum_1^\infty \frac{(1-R^{2k+2})}{k(k+1)}$$

goes to zero monotonically as R increases to 1. We are done. ■

THEOREM 1 *A solution to* (4) *exists.*

Proof Let $\{f_k\}$ be a sequence with $f_k(0) = 0$,

$$\|f_k\|_D = 1 \qquad \text{and} \qquad \operatorname{Re}\left(\int_\Delta \bar{f}_k f_k'\,dA\right) \to \lambda.$$

We may assume that $\{f_k\}$ converges weakly in the Hilbert space D to a function F, $F(0) = 0$, $\|F\|_D \le 1$. This implies that $f_k' \to F'$ uniformly on compact subsets of Δ and also that $f_k \to F$ uniformly on compact subsets of Δ. Thus,

$$\left|\int_\Delta \bar{f}_k f_k'\,dA - \int_\Delta \bar{F}F'\,dA\right| \le \left|\int_\Delta (\bar{f}_k f_k' - \bar{F} f_k')\,dA\right| + \left|\int_\Delta (\bar{F} f_k' - \bar{F}F')\,dA\right|.$$

The second term goes to zero since $f_k \to F$ weakly in D. The first term is no larger than

$$\|f_k\|_D\|f_k - F\|_{L^2} = \|f_k - F\|_{L^2}.$$

The latter goes to zero as $k \to \infty$ since

$$\begin{aligned}\left(\int_\Delta |f_k - F|^2\,dA\right)^{1/2} &\le \left(\int_{|z|\le R} |f_k - F|^2\,dA\right)^{1/2} + \left(\int_{R<|z|<1} |f_k - F|^2\,dA\right)^{1/2}\\ &\le \left(\int_{|z|\le R} |f_k - F|^2\,dA\right)^{1/2} + \left(\int_{R<|z|<1} |f_k|^2\,dA\right)^{1/2}\\ &\quad + \left(\int_{R<|z|<1} |F|^2\,dA\right)^{1/2}.\end{aligned}$$

Given $\varepsilon > 0$, we may employ the lemma to choose R so near 1 that the second and third terms on the right-hand side are each less than $\varepsilon/3$. For this R, the first term is less than $\varepsilon/3$ when k is large since $f_k \to F$ uniformly on compact subsets of Δ. ■

THEOREM 2 *A solution to* (4) *exists which has positive coefficients; every other solution is a unimodular constant multiple of this solution. Moreover, the coefficients of the solution satisfy the recursion relation*

$$2\lambda k a_k = a_{k+1} + a_{k-1}, \qquad k = 1,2,\ldots; \quad a_0 = 0. \tag{6}$$

Proof Let

$$\sum_1^\infty a_k z^k$$

be a solution to (4) and consider

$$F(z) = \sum_1^\infty |a_k| z^k.$$

Then $\|F\|_D = 1$ and

$$\lambda \geq \operatorname{Re}\left(\int_\Delta \overline{F} F' \, dA\right) = \sum_1^\infty |a_k||a_{k+1}|$$

$$\geq \left|\sum_1^\infty a_k \overline{a}_{k+1}\right| \geq \operatorname{Re}\left(\sum_1^\infty a_k \overline{a}_{k+1}\right) = \lambda.$$

Hence, F is also a solution and F has non-negative coefficients. Henceforth we shall assume that

$$F(z) = \sum_1^\infty a_k z^k, \qquad a_k \geq 0$$

is a solution. We shall show that any other solution

$$g(z) = \sum_1^\infty b_k z^k$$

is a multiple of F. Let ε be a (small) complex number and set

$$g_\varepsilon(z) = g(z) + \varepsilon z^k.$$

Then

$$\lambda \|g_\varepsilon\|_D^2 \geq \operatorname{Re}\left\{\int_\Delta \overline{g}_\varepsilon g'_\varepsilon \, dA\right\}$$

which yields

$$\lambda(\operatorname{Re} 2k\overline{b}_k\varepsilon + k|\varepsilon|^2) \geq \operatorname{Re}(\varepsilon\overline{b}_{k-1} + \varepsilon\overline{b}_{k+1}).$$

Since this holds for every (small) complex ε, we find that

$$2\lambda k b_k = b_{k-1} + b_{k+1}, \qquad k = 1,2,\ldots; \quad b_0 = 0. \tag{6'}$$

Applying (6′) repeatedly, we determine that b_{k+1} can be expressed in terms of λ and b_1, namely:

$$b_{k+1} = b_1 P_k(\lambda), \qquad k = 1,2,\ldots \tag{7}$$

where P_k is a polynomial of degree k which is an even function if k is even and an odd function if k is odd. The first three P_k are

$$P_1(x) = 2x; \qquad P_2(x) = 8x^2 - 1; \qquad P_3(x) = 8x(6x^2 - 1).$$

The formulas in (7) show that $b_1 \neq 0$ and hence

$$b_{k+1} = b_1 P_k(\lambda) = \frac{b_1}{a_1} a_1 P_k(\lambda) = \frac{b_1}{a_1} a_{k+1}$$

so that $g = (b_1/a_1)F$, as desired. ∎

Remark We shall assume henceforth that F is normalized so that $F'(0) = 1$; that is,

$$F(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{8}$$

where $a_2, a_3, \ldots$ are positive. (That all the a_k are positive follows directly from (6).)

2. PROPERTIES OF THE SOLUTION

PROPOSITION 2 *The coefficients* $\{a_k\}$ *are monotonically decreasing and satisfy*

$$0 < a_k \leq \left(\frac{1}{\lambda}\right)^{k-1} \frac{1}{k!}, \qquad k = 2,3,\ldots . \tag{9}$$

Proof First we note that

$$a_2 = 2\lambda \leq 1$$

and then that

$$2a_2 \geq 2\lambda 2a_2 = 1 + a_3 \geq a_2 + a_3$$

so that $1 \geq a_2 \geq a_3$.

Suppose N is some integer greater than or equal to 3 and $a_{N+1} < a_N$. Then

$$2a_N < 2\lambda N a_N = a_{N-1} + a_{N+1} < a_{N-1} + a_N$$

and hence $a_N < a_{N-1}$; now repeat this argument to conclude that $a_{N-1} < a_{N-2}$, etc. Since $Na_N^2 \to 0$, there are arbitrarily large N with $a_{N+1} < a_N$. Hence, $\{a_k\}$ decreases monotonically.

Since $a_1 \geq a_2 \geq \cdots > 0$, we then have

$$2\lambda k a_k = a_{k+1} + a_{k-1} < 2a_{k-1}$$

so that

$$a_k < \frac{1}{\lambda} \cdot \frac{1}{k} a_{k-1}, \qquad k = 2,3,\ldots$$

which gives (9) after iteration. ■

COROLLARY *The function F is entire of exponential type.*

THEOREM 3 *λ is the reciprocal of the smallest positive zero of J_0, the 0th Bessel function of the first kind. Moreover,*

$$F(z) = C\sum_{n=1}^{\infty} J_n(1/\lambda)z^n \tag{10}$$

where C is a constant and J_n is the nth Bessel function of the first kind.

Proof The relation (6) implies that F satisfies the ordinary differential equation

$$2\lambda z^2 F'(z) = (1 + z^2)F(z) - z. \tag{11}$$

Moreover, because F is entire, (11) holds throughout the complex plane. In (11), replace z by $-1/z$ to obtain

$$2\lambda \frac{1}{z^2} F'\left(\frac{-1}{z}\right) = \left(1 + \frac{1}{z^2}\right) F\left(\frac{-1}{z}\right) + \frac{1}{z}. \tag{12}$$

Define $G(z) = F(-1/z)$. Then (12) may be written as

$$2\lambda G'(z) = \left(1 + \frac{1}{z^2}\right) G(z) + \frac{1}{z}$$

and hence

$$2\lambda z^2 G'(z) = (1 + z^2)G(z) + z. \tag{13}$$

Add (11) and (13) to conclude that $h = F + G$ satisfies

$$2\lambda z^2 h'(z) = (1 + z^2)h(z).$$

This equation is easily solved to give

$$h(z) = C\exp\left(\frac{1}{2\lambda}\left(z - \frac{1}{z}\right)\right)$$

where

$$C = h(1) = F(1) + F(-1) = 2(a_2 + a_4 + a_6 + \cdots) > 0.$$

That is

$$F(z) + F(-1/z) = C\exp\left(\frac{1}{2\lambda}(z - 1/z)\right)$$

$$= C\sum_{-\infty}^{\infty} J_n(1/\lambda)z^n. \tag{14}$$

This gives (10). Note, as well, that the left-hand side of (14) has no constant term. Therefore,

$$J_0(1/\lambda) = 0. \tag{15}$$

The first two zeros of J_0 are

$$\alpha_0 = 2.404826\ldots$$

$$\alpha_1 = 3.831706\ldots;$$

see [4; page 502]. The reciprocal of α_0 lies between $1/\sqrt{6}$ and $1/2$ while that of α_1 does not. Hence,

$$\lambda = 1/\alpha_0 = 0.4158305\ldots. \tag{16}$$

By the way, the value of λ gives

$$a_2 \doteq 0.8316, \qquad a_3 \doteq 0.3832, \qquad a_4 \doteq 0.1247, \qquad a_5 \doteq 0.0316$$

and

$$C \doteq 1.912. \qquad \blacksquare$$

THEOREM 4 *F has infinitely many zeros, all of which are simple and, except for the zero at the origin, all lie in the left half-plane. None of the zeros lie on the real axis or in the closed unit disc. The zeros in the second quadrant are asymptotic to the curve*

$$Cy = e^{-x/2\lambda}.$$

Finally, if P is the canonical product of the zeros of F, then

$$F(z) = ze^{2\lambda z}P(z). \tag{17}$$

Proof The formula (14) shows that F must have at least one zero away from the origin and that F is entire of order 1. Let $z_1, z_2, \ldots$ be the zeros of F; then cf. [1; Theorem 2.5.18],

$$\sum_m \frac{1}{|z_m|^2} < \infty$$

and P is given by

$$P(z) = \prod_m \left(1 - \frac{z}{z_m}\right) e^{z/z_m}.$$

At this point we do not know whether F has finitely many or infinitely many zeros. The Hadamard factorization theorem [1; page 22] implies that

$$F(z) = ze^{bz}P(z).$$

Hence,

$$\frac{zF'(z)}{F(z)} = 1 + zb - z^2 \sum_m \frac{1}{z_m(z_m - z)}$$

$$= 1 + zb - \sum_{j=0}^{\infty} z^{j+3} \sum_m \left(\frac{1}{z_m}\right)^{j+2},$$

the latter being valid for $|z|$ small. Comparing power series coefficients, we find that

$$b = a_2 = 2\lambda$$

and, incidentally,

$$\sum_m \left(\frac{1}{z_m}\right)^2 = -2(6\lambda^2 - 1). \tag{18}$$

To see that F has infinitely many zeros, multiply both sides of (14) by z and let $z = x \to -\infty$. If F had finitely many zeros, then

$$x^2 e^{2\lambda x} P(x) \to 0$$

and we would conclude that

$$\lim_{x \to -\infty} P(-1/x) = 0,$$

which is clearly incorrect since $P(0) = 1$.

The differential equation (11) satisfied by F shows that F and F' do not vanish simultaneously; that is, the zeros of F are simple.

To prove that F has no zeros in the right half-plane requires two steps. We first show that any zero in the right half-plane must have modulus no more than $(1 + C)/C$; we then show that any zero of F with modulus 4λ or less must lie in the left half-plane. Since $(1 + C)/C \doteq 1.52$ and $4\lambda \doteq 1.66$, we will be done.

From (14) (with $z = re^{i\theta}$)

$$\begin{aligned}\left|zF(z) - \frac{F(-1/z)}{(-1/z)}\right| &= rC\exp[(r - 1/r)\cos\theta] \\ &\geq rC \quad \text{if} \quad r \geq 1 \quad \text{and} \quad \cos\theta \geq 0.\end{aligned}$$

If $F(z) = 0$, then

$$\left|\frac{F(w)}{w}\right| \geq rC, \qquad w = -1/z.$$

Since $r \geq 1$, we have $|w| \leq 1$ so

$$\begin{aligned}\left|\frac{F(w)}{w}\right| &= |1 + a_2 w + a_3 w^2 + \cdots| \\ &\leq 1 + 2a_2 + 2a_4 + \cdots = 1 + C.\end{aligned}$$

Hence, $|z| \leq (1 + C)/C$, if z is a zero of F in the right half-plane.

As to the second step, we need the following result.

PROPOSITION 2 [3; page 129] *If $A_0, A_1, \ldots$ is a sequence of positive numbers satisfying*

$$\sum_0^\infty A_k < \infty \qquad \textit{and} \qquad A_{k+1} - 2A_k + A_{k+1} \geq 0 \quad (\textit{all } k)$$

then

$$\tfrac{1}{2}A_0 + \sum_{1}^{\infty} A_k \cos kx \geq 0 \qquad \textit{for all } x.$$

We consider

$$g(z) =: \frac{F(z) - z}{z^2} = a_2 + a_3 z + a_4 z^2 + \cdots.$$

We shall show that (with $R = 4\lambda$) the numbers

$$A_0 = 2a_2, \qquad A_k = a_{k+2}R^k, \qquad k = 1, 2, \ldots \tag{20}$$

satisfy the hypotheses of Proposition 2. Once this is proved, the Proposition implies that

$$\operatorname{Re}(g(\operatorname{Re}^{ix})) \geq 0, \qquad 0 \leq x \leq 2\pi$$

and hence $\operatorname{Re}(g(z)) > 0$ if $|z| < R = 4\lambda$. From this we see that if $F(z) = 0$ and $|z| < 4\lambda$, then

$$0 < \operatorname{Re} g(z) = -\operatorname{Re}(1/z)$$

and hence z lies in the left half-plane. Thus, to finish we need to establish (20).

First:

$$A_0 - 2A_1 + A_2 = 2a_2 - 2a_3 R + a_4 R^2 = 12\lambda(1 - 8\lambda^2)^2 > 0.$$

Next, for $k \geq 4$

$$\begin{aligned}
&a_{k+1}R^{k-1} - 2a_k R^{k-2} + a_{k-1}R^{k-3} \\
&\quad = R^{k-3}\{a_{k+1}R^2 - 2a_k R + a_{k-1}\} \\
&\quad = R^{k-3}\{(R^2 - 1)a_{k+1} + (2\lambda k - 2R)a_k\} \qquad \text{(from (6))} \\
&\quad > 0 \qquad \text{since} \quad \lambda k \geq 4\lambda = R \quad \text{and} \quad R > 1.
\end{aligned}$$

To show that F has no zeros in $0 < |z| < 1$, we use Proposition 2 for the function

$$f(z) =: \frac{F(z)}{z} = 1 + a_2 z + a_3 z^2 + \cdots.$$

Let $a_0 = 2$, $A_k = a_{k+1}$, $k = 1, 2, \ldots$. Then

$$A_0 - 2A_1 + A_2 = 2 - 2a_2 + a_3 > 0 \qquad \text{since} \quad a_2 = 2\lambda < 1.$$

Further, for $k \geq 2$

$$\begin{aligned}
A_{k+1} - 2A_k + A_{k-1} &= a_{k+2} - 2a_{k+1} + a_k \\
&= (2\lambda(k+1) - 2)a_{k+1} > 0.
\end{aligned}$$

Hence,

$$\operatorname{Re}\left(\frac{F(z)}{z}\right) > 0 \qquad \text{if} \quad |z| < 1. \tag{21}$$

To see that $F(x) \neq 0$ if $-\infty < x < 0$, suppose to the contrary that

$$F(c) = 0 \quad \text{and} \quad F(x) < 0 \quad \text{and} \quad c < x < 0.$$

From (11), we find $2\lambda c^2 F'(c) = -c > 0$ and so $F(x) > 0$ if $c < x < c + \delta$. This contradiction establishes that $F(x) < 0$ if $-\infty < x < 0$.

The asymptotic behavior of $\{z_m\}$ follows from (14). Let $\{z_m\}$ be the zeros of F in the second quadrant. Then

$$|z_m||F(-1/z_m)| = C|z_m|e^{1/2\lambda(x_m - \mathrm{Re}(1/z_m))}.$$

The left-hand side of this expression approaches 1 as $m \to \infty$. Since the right-hand side exceeds

$$C|z_m|e^{(1/2\lambda)x_m}$$

it follows that $x_m \to -\infty$ as $m \to \infty$. This, in turn, implies that

$$x_m e^{(1/2\lambda)x_m} \to 0 \quad \text{as} \quad m \to \infty.$$

Therefore,

$$\frac{1}{C} = \lim_{m\to\infty} y_m e^{(1/2\lambda)x_m}.$$

which is the desired conclusion.

3. EXTENSIONS

In this section we take up two extensions of the extremal problem (2). The first is to replace the Hilbert space D with the more general Hilbert space $\mathcal{H}$ determined by the condition

$$\sum_{k=0}^{\infty} \varepsilon_k |a_k|^2 < \infty \tag{22}$$

where $\varepsilon_0, \varepsilon_1, \ldots$ are given positive numbers. Here we determine a necessary and sufficient condition that the corresponding functional

$$\phi(a_0, a_1, \ldots) =: \frac{\mathrm{Re}\left(\sum_{k=0}^{\infty} a_k \overline{a}_{k+1}\right)}{\sum_{k=0}^{\infty} \varepsilon_k |a_k|^2} \tag{23}$$

be bounded; that is, that $\phi(a_0, a_1, \ldots) \leq A < \infty$ for all sequences $\{a_k\}$ satisfying (22). We also determine a sufficient condition that there is a solution in $\mathcal{H}$ to the extremal problem

$$\sup\{\phi(a_0, a_1, \ldots) : \{a_k\} \in \mathcal{H}\}. \tag{24}$$

This condition is general enough to provide an alternate proof of Theorem 1.

The other extension of (2) which we take up is to replace the expression $\sum a_k \overline{a}_{k+1}$ in (2) by $\sum a_k \overline{a}_{k+m}$ where m is some positive integer. Here we shall find that the extremal problem

$$\mu =: \sup_{f \in D} \frac{\mathrm{Re}(\sum_1^{\infty} a_k \overline{a}_{k+m})}{\sum_1^{\infty} k |a_k|^2} \tag{25}$$

is easily solved, given that we have already solved (24), and its solution is directly connected to that of (24).

We note first that if (24) has a solution, then there is a sequence $\{b_k\}$ with

$$\sum_{k=0}^{\infty} \varepsilon_k |b_k|^2 = 1$$

and

$$\phi(b_0, b_1, \ldots) = \sup\{\phi(a_0, a_1, \ldots) : \{a_k\} \in \mathcal{H}\}.$$

This, in turn, obviously implies that ϕ is bounded.

THEOREM 5

(i) *ϕ is bounded if and only if*

$$\liminf_{k\to\infty}(\varepsilon_k \varepsilon_{k+1}) > 0 \tag{26}$$

(ii) *ϕ has a solution if*

$$\lim_{k\to\infty} \varepsilon_k \varepsilon_{k+1} = +\infty. \tag{27}$$

Proof (i) Suppose that $\varepsilon_{k+1}\varepsilon_k \geq \delta^2$ for all k. Then

$$2\delta \operatorname{Re}(a_k \bar{a}_{k+1}) \leq 2 \operatorname{Re}\sqrt{\varepsilon_k \varepsilon_{k+1}} a_k \bar{a}_{k+1} \leq \varepsilon_k |a_k|^2 + \varepsilon_{k+1}|a_{k+1}|^2$$

and hence

$$\phi(a_0, a_1, \ldots) \leq 2\frac{1}{2\delta} = \frac{1}{\delta}.$$

Conversely, if ϕ is bounded, say $\phi(a_0, a_1, \ldots) \leq A$, and

$$f(z) = z^k + \sqrt{\varepsilon_k/\varepsilon_{k+1}} z^{k+1},$$

then

$$\sqrt{\varepsilon_k/\varepsilon_{k+1}} \leq A(\varepsilon_k + \varepsilon_k) = 2A\varepsilon_k$$

which gives (26).

(ii) Define

$$\mu = \sup\left\{\phi(a_0, a_1, \ldots) : \sum \varepsilon_k |a_k|^2 < \infty\right\}$$

and assume that

$$\lim \varepsilon_k \varepsilon_{k+1} = +\infty.$$

Let $\{a_k^{(n)}\}$ be a maximizing sequence:

$$\sum_k \varepsilon_k |a_k^{(n)}|^2 = 1 \qquad \text{and} \qquad \lim_n \left(\operatorname{Re}\sum_k a_k^{(n)} \overline{a}_{k+1}^{(n)}\right) = \mu.$$

We may assume with no loss that $\{a_k^{(n)}\} \to \{a_k\}$, as $n \to \infty$, weakly in the Hilbert space

$$\mathcal{H} = \left\{ (a_0, a_1, \ldots) : \sum_k \varepsilon_k |a_k|^2 < \infty \right\}.$$

In particular, $a_k^{(n)} \to a_k$ as $n \to \infty$ for each k and $\sum_k \varepsilon_k |a_k|^2 \le 1$. We then have

$$\left| \sum_k a_k \overline{a}_{k+1} - \sum_k a_k^{(n)} \overline{a}_{k+1}^{(n)} \right| \le \left| \sum_k (a_k - a_k^{(n)}) \overline{a}_{k+1} \right| + \left| \sum_k a_k^{(n)} (\overline{a}_{k+1} - \overline{a}_{k+1}^{(n)}) \right|$$
$$= I + II.$$

We shall assume (with no loss of generality) that $\varepsilon_k \varepsilon_{k+1} \ge 1$ for all k. Then

$$\begin{aligned}
I &\le \sum_{k=1}^{\infty} |a_k - a_k^{(n)}| \, |a_{k+1}| \\
&\le \sum_1^N |a_k - a_k^{(n)}| \, |a_{k+1}| + \sum_{N+1}^{\infty} \sqrt{\varepsilon_k \varepsilon_{k+1}} |a_k - a_k^{(n)}| \, |a_{k+1}| \\
&\le \sum_1^N |a_k - a_k^{(n)}| \, |a_{k+1}| + \left(\sum_{N+1}^{\infty} \varepsilon_k |a_k - a_k^{(n)}|^2 \right)^{1/2} \left(\sum_{N+1}^{\infty} \varepsilon_{k+1} |a_{k+1}|^2 \right)^{1/2} \\
&\le \sum_1^N |a_k - a_k^{(n)}| \, |a_{k+1}| + 2 \left(\sum_{N+1}^{\infty} \varepsilon_{k+1} |a_{k+1}|^2 \right)^{1/2}.
\end{aligned}$$

Both of these terms are small: the second for N large and the first for $n \ge n_0$, once N is fixed. Next,

$$\begin{aligned}
II &\le \sum_1^{\infty} |a_k^{(n)}| \, |a_{k+1} - a_{k+1}^{(n)}| \\
&\le \left(\sum_1^{\infty} \varepsilon_k |a_k^{(n)}|^2 \right)^{1/2} \left(\sum_1^{\infty} \frac{1}{\varepsilon_k} |a_{k+1} - a_{k+1}^{(n)}|^2 \right)^{1/2} \\
&\le \left(\sum_1^{N} \frac{1}{\varepsilon_k} |a_{k+1} - a_{k+1}^{(n)}|^2 \right)^{1/2} + \left(\sum_{N+1}^{\infty} \left(\frac{1}{\varepsilon_k \varepsilon_{k+1}} \right) \varepsilon_{k+1} |a_{k+1} - a_{k+1}^{(n)}|^2 \right)^{1/2}.
\end{aligned}$$

The second term is less than

$$2 \max_{k \ge N+1} \left(\frac{1}{\varepsilon_k \varepsilon_{k+1}} \right)^{1/2} < \varepsilon \qquad \text{if} \quad N \text{ is large}$$

and the first term goes to zero as $n \to \infty$ for N fixed. ■

THEOREM 6 *The number μ defined in* (25) *is equal to number γ defined by*

$$\gamma =: \sup\left\{\frac{\operatorname{Re}\left(\sum_0^\infty a_k \bar{a}_{k+1}\right)}{\sum_0^\infty (mk+1)|a_k|^2}\right\}. \tag{28}$$

Proof Let $\{a_0, a_1, \ldots\}$ be any sequence for which

$$\sum_0^\infty (mk+1)|a_k|^2 < \infty$$

and define

$$b_j = \begin{cases} a_k & \text{if } \quad j = km+1, \quad k = 0,1\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mu \geq \frac{\operatorname{Re}\left(\sum b_j \bar{b}_{j+m}\right)}{\sum j|b_j|^2} = \frac{\operatorname{Re}\left(\sum_0^\infty a_k \bar{a}_{k+1}\right)}{\sum_0^\infty (mk+1)|a_k|^2}$$

so that $\mu \geq \gamma$.

To prove the reverse inequality, we note that

$$\begin{aligned} \frac{\operatorname{Re}\left(\sum_{j=1}^\infty b_j \bar{b}_{j+m}\right)}{\sum_{j=1}^\infty j|b_j|^2} &= \frac{\sum_{r=1}^m \operatorname{Re}\left(\sum_{k=0}^\infty b_{km+r}\bar{b}_{(k+1)m+r}\right)}{\sum_{r=1}^m \sum_{k=0}^\infty (km+r)|b_{km+r}|^2} \\ &\leq \frac{\sum_{r=1}^m \operatorname{Re}\left(\sum_{k=0}^\infty b_{km+r}\bar{b}_{(k+1)m+r}\right)}{\sum_{r=1}^m \sum_{k=0}^\infty (km+1)|b_{km+r}|^2} \\ &\leq \gamma \end{aligned}$$

■

EXAMPLE *The extremal problem* (24) *has no solution when $\varepsilon_k = 1$ for all k; that is, when $\mathcal{H} = H^2$, the Hardy space.*

Let μ be the supremum in (24). *To see that $\mu = 1$, note that the Schwarz inequality gives*

$$\phi(a_0, a_1, \ldots) \leq \frac{\left(\sum_0^\infty |a_k|^2\right)^{1/2}\left(\sum_1^\infty |a_k|^2\right)^{1/2}}{\sum_0^\infty |a_k|^2} \leq 1 \tag{29}$$

while the choice $a_k = 1$, $k = 0, \ldots, N$, $a_k = 0$ for $k \geq N+1$, gives

$$\mu \geq \frac{N}{N+1}. \tag{30}$$

(29) *and* (30) *give $\mu = 1$. If $F(z) = \sum_0^\infty a_k z^k$ were a solution to* (24), *then it would necessarily follow that*

$$2\mu\varepsilon_k a_k = a_{k+1} + a_{k-1}, \qquad k = 0,1,\ldots; \quad a_{-1} = 0.$$

That is

$$2a_k = a_{k+1} + a_{k-1}.$$

Hence

$$\begin{aligned} a_1 &= 2a_0 \\ a_2 &= 2a_1 - a_0 = 3a_0 \\ a_3 &= 2a_2 \quad a_1 = 4a_0 \\ &\text{etc.} \end{aligned}$$

which yields $a_k = (k+1)a_0$ *and so*

$$F(z) = a_0 \sum_0^{\infty} (k+1)z^k = a_0 \frac{1}{(1-z)^2}.$$

But this F *does not lie in* H^2 (*unless* $a_0 = 0$).

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