Concerning Convolution on the Half-line

T. K. BOEHME

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0. We will say that a complex valued function on $[0, \infty)$ is locally integrable, locally p-integrable, locally of bounded variation, or locally absolutely continuous if it is integrable, p-integrable, of bounded variation, or absolutely continuous on each interval $[0, a]$, $a > 0$. The vector spaces of locally integrable and locally p-integrable functions will be denoted by $\mathscr{L}$ and $\mathscr{L}^p$, while the corresponding normed vector spaces for the finite interval $[0, a]$ will be denoted by $L[0, a]$ and $L^p[0, a]$. If there is no danger of confusion, the $[0, a]$ will be omitted.

The spaces $\mathscr{L}^p$, $\infty \geqq p \geqq 1$, will each be considered to have the topology defined by the countable collection of semi-norms $\|f\|_n = \|f\|_{L^p[0,n]}$. $\mathscr{L}$ can be considered to be a ring with convolution denoted by juxtaposition; thus, $k = fg$ is the function defined by the equation

$$k(t) = \int_0^t f(t-u)\, g(u)\, du, \qquad t \geqq 0.$$

Certain properties of the convolution are well known; for instance, if f is in $\mathscr{L}^p$ and g is in $\mathscr{L}^q$, $1 \leqq p \leqq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then fg is continuous. If f is in $\mathscr{L}^p$, $p \geqq 1$, and g is in $\mathscr{L}$, then fg is in $\mathscr{L}^p$. Other properties of the convolution on the half line have been studied by MIKUSIŃSKI, RYLL-NARDZEWSKI [*1*], [*2*] and others. J. D. WESTON has studied transformations on a subspace of $\mathscr{L}$ which commute with convolution and has based an operational calculus on them [*3*]. It is our purpose here to study transformations on $\mathscr{L}$ into $\mathscr{L}^p$, $1 \leqq p \leqq \infty$, which commute with convolution, transformations, that is, for which $T(fg) = fT(g)$ for each f and g in $\mathscr{L}$.

The convolution integral equation on the half-line $xf = k$ with f and k in $\mathscr{L}$ occurs frequently; however, there are no satisfactory existence theorems to show when there exists a solution x in $\mathscr{L}$. POLLARD & BLACKMAN [*4*] have demonstrated a method of solving the equation when there exists a solution in $\mathscr{L}$ (however, it may not be easy to carry out). We will show a comparison test for the existence of solutions (Corollary 3.3). If $g = s\,v\,f$, where s is MIKUSIŃSKI's differentiation operator and v is locally of bounded variation, then $xf = k$ has a solution in $\mathscr{L}$ if $xg = k$ has a solution in $\mathscr{L}$. In Theorem 4.1 a necessary and sufficient condition is given in order that $xf = k$ have a solution when f is locally of bounded variation and $f(0^+) \neq 0$. Namely, k must be locally absolutely continuous and zero at the origin.

We shall not explicitly require that our transformations be linear, but we shall see that the requirement that T commute with convolution implies that T is linear, and it implies that T is continuous also. Our main theorem is Theorem 3.1, which characterizes the transformations on $\mathscr{L}$ into $\mathscr{L}$ which commute with convolution as being of the form $T = s\,v$ where s is the differentiation operator of MIKUSIŃSKI's operational calculus and v is locally of bounded variation. That is, there is a v such that whenever g is in $\mathscr{L}$ $T(g) = s\,v\,g$. This result leads one to conjecture the result for transformations on $\mathscr{L}$ into $\mathscr{L}^p$, $1 < p \leqq \infty$, and Theorem 1.7 states that every transformation which commutes with convolution and takes $\mathscr{L}$ into $\mathscr{L}^p$, $1 < p \leqq \infty$, is such that there is an f in $\mathscr{L}^p$ and $T(g) = f g$ for every g in $\mathscr{L}$. Since the result for transformations on $\mathscr{L}$ into $\mathscr{L}$ depends in a minor way on the result for transformations on $\mathscr{L}$ into $\mathscr{L}^\infty$, the latter result is proved first. But the result for transformations on $\mathscr{L}$ into $\mathscr{L}$ depends in a major way on the result characterizing the transformations which commute with convolution and take $\mathscr{L}$ into the ring of functions which are locally of bounded variation. These transformations are characterized in Section 2. There it is shown that such a transformation is of the form $T = v$ (*i.e.* for every g in $\mathscr{L}$, $T(g) = v\,g$) for some v which is locally of bounded variation. It is also shown that a transformation of this type must take $\mathscr{L}$ not only into the ring of functions which are locally of bounded variation but, in fact, takes $\mathscr{L}$ into the ring of functions which are locally absolutely continuous and zero at the origin.

For each of the theorems proved here concerning transformations on $\mathscr{L}$ to $\mathscr{L}^p$, $1 \leqq p \leqq \infty$, there is an analogous theorem concerning transformations on $L[-\infty, \infty]$ to $L^p[-\infty, \infty]$, but we shall not prove these here. The proofs directly imitate those proofs we give here, but to a certain extent are more readily carried out because of the fact that $L[-\infty, \infty]$ and $L^p[-\infty, \infty]$ are Banach spaces while $\mathscr{L}$ and $\mathscr{L}^p$ are not. The theorem analogous to Theorem 3.1 characterizes the transformations on L to L in the following manner. For each such transformation T there is a function v of bounded variation on $[-\infty, \infty]$ and with $v(-\infty) = 0$ such that $T(g) = k \Leftrightarrow k(t) = \int\limits_{-\infty}^{\infty} g(t-u)\,dv(u)$ for almost all t, and for all g in L. In order to prove this it is necessary to have the analogue on $[-\infty, \infty]$ of the Hardy-Littlewood theorem (our Theorem H-L in Section 2) which has recently been proved by P. L. BUTZER [5]. It is interesting to compare the results concerning transformations on $L[-\infty, \infty]$ to $L^p[-\infty, \infty]$ with the results of R. P. AGNEW [6].

1. Every f in $\mathscr{L}$ is also in $L[0, a]$, $a > 0$, and defines a continuous linear transformation on L to L by means of the equation $T_f(g) = f g$. We shall say that T_f is *the linear transformation determined by* f. By a simple application of Fubini's theorem on double integrals it is seen that T_f is bounded and

$$\|T_f\| \leqq \|f\|_L = \int\limits_0^a |f|\,du.$$

For $n = 1, 2, \ldots$ let g_n be the approximations to the delta function defined by $g_n(t) = n$ when $0 \leqq t \leqq \frac{1}{n}$ and $g_n(t) = 0$, $t > \frac{1}{n}$. The following theorem is well known, but for the sake of completeness the proof will be indicated.

Theorem 1.1. *Let f be in $\mathscr{L}$, and let g_n be defined as stated above for $\frac{1}{n} < a$ and $a > 0$. Then $f g_n \to f$ in $L[0, a]$ as $n \to \infty$. The convergence is convergence in norm.*

Proof. It is sufficient to prove the theorem for the case when f is a member of a dense subset of $\mathscr{L}$. Thus, let f be continuous and $f(0) = 0$. But for such an f direct calculation easily shows that $f g_n$ tends uniformly to f on $[0, a]$ and $f g_n$ tends to f also in norm.

Corollary 1.2. *If f is in $\mathscr{L}$ and $a > 0$, the norm of the linear transformation determined by f is $\|T_f\| = \|f\|_L$.*

Proof. Since we know that $\|T_f\| \leqq \|f\|_L$, it is only necessary to find for each $\varepsilon > 0$ a g such that $\|g\|_L = 1$ and $\|T_f(g)\| > \|f\|_L - \varepsilon$. But for the g_n specified above we have $\|g_n\| = 1$ for each n and $\|T_f(g)_n\|_L \to \|f\|_L$ as $n \to \infty$.

Now let $f = b$ be in $\mathscr{L}^\infty$. The linear transformation determined by b takes L into L^∞ (in fact $b g$ is continuous if g is in L). Again, it is clear that T_b as a transformation on L to L^∞ is continuous and $\|T_b\| \leqq \|b\|_{L^\infty} = \operatorname*{ess.\,sup}_{0 \leqq t \leqq a} |b(t)|$. But we have

Theorem 1.3. *If b is in $\mathscr{L}^\infty$ and $a > 0$, the norm of the linear transformation T_b on L to L^∞ is*

$$\|T_b\| = \|b\|_{L^\infty}.$$

Proof. Again it is only necessary to find a set of g_n such that $\|g_n\|_L = 1$ for each n and $\|b g_n\|_{L^\infty} \to \|b\|_{L^\infty} = K$ as $n \to \infty$. Define the sets

$$E_n = \left\{u \mid 0 \leqq u \leqq a, |b(a-u)| > K - \frac{1}{n}\right\}, \qquad n = 1, 2, \ldots .$$

The measure, $\mu(E_n)$, of each E_n is positive if K is different from zero. If g_n is defined by

$$\mu(E_n)\, g_n(u) = \mathrm{X}_{E_n}(u) \operatorname{sgn} b(a-u), \qquad n = 1, 2, \ldots, \qquad 0 \leqq u \leqq a,$$

where X_E is the characteristic function of the set E, then $\|g_n\|_L = 1$. Moreover,

$$(b g_n)(a) = \int_0^a b(a-u)\, g_n(u)\, du = \frac{1}{\mu(E_n)} \int_0^a \mathrm{X}_{E_n}(u)\, |b(a-u)|\, du > K - \frac{1}{n}.$$

Since $b\, g_n$ is continuous, $\|b\, g_n\|_{L^\infty} \geqq K - \frac{1}{n}$, and since $\|b\, g_n\|_{L^\infty} \leqq K$, we have $\|b\, g_n\|_{L^\infty} \to \|b\|_{L^\infty}$ as $n \to \infty$, which completes the proof.

Since $L^\infty[0, a]$ is a conjugate space, it has weak* topology, and strongly closed spheres are compact in this topology. If f_m is a sequence of functions in $\mathscr{L}^\infty$, and if b in $\mathscr{L}^\infty$ is such that for every $a > 0$ and every g in $L[0, a]$

$$\lim \int_0^a f_m(u)\, g(u)\, du = \int_0^a b(u)\, g(u)\, du,$$

we shall say that f_m tends weak* to b on each interval $[0, a]$.

Lemma 1.4. *Let f_n, $n = 1, 2, \ldots$, be a sequence of functions in $\mathscr{L}^\infty$, and suppose that for each $a > 0$ there is a $B_a < \infty$ such that $\|f_n\|_{L^\infty[0,a]} < B_a$. Then there is a b in $\mathscr{L}^\infty$ and a subsequence f_m of f_n such that f_m tends weak* to b on each interval $[0, a]$.*

Proof. Since strongly closed spheres in $L^\infty[0, a]$ are weak* compact, we can pick a set of subsequences $f_{K,n}$ of f_n, $K = 1, 2, \ldots$, such that $f_{K+1,n}$ is subsequence of $f_{K,n}$, and $f_{K,n}$ tends weak* to b_K on $[0, K]$ as $n \to \infty$. By the diagonal process a single subsequence f_m of f_n can be found which converges weak* on each $[0, K]$ to b_K. If $K_2 > K_1$ and g_1 is in $L[0, K_1]$, there is a corresponding function g_2 in $L[0, K_2]$ such that $g_2(t) = g_1(t)$ on $[0, K_1]$ and $g_2(t) = 0$ on $(K_1, K_2]$, and the fact that

$$\int_0^{K_1} b_{K_1}(u)\, g_1(u)\, du = \lim_{m\to\infty} \int_0^{K_1} f_m(u)\, g_1(u)\, du = \lim_{m\to\infty} \int_0^{K_2} f_m(u)\, g_2(u)\, du$$
$$= \int_0^{K_2} b_{K_2}(u)\, g_2(u)\, du = \int_0^{K_1} b_{K_2}(u)\, g_1(u)\, du$$

makes it clear that $b_{K_2}(u) = b_{K_1}(u)$ almost everywhere on $[0, K_1]$. Thus the functions in $\mathscr{L}^\infty$ which are equal to b_K on $[0, K]$ and zero on (K, ∞) converge almost everywhere on $[0, \infty]$ to a function b which has the desired property.

Lemma 1.5. *Let f_m and b be as described in Lemma 1.4. Then if g is in $\mathscr{L}$, the sequence $f_m g$ tends weak* on each interval $[0, a]$ to $b g$.*

Proof. For any k in $L[0, a]$ and k_1 such that $k_1(a-u) = k(u)$, $0 \leqq u \leqq a$, $\int_0^a (f_m g)(u)\, k(u)\, du = \int_0^a (f_m g)(u)\, k_1(a-u)\, du = \int_0^a f_m(u)\, (g k_1)(a-u)\, du$ which tends to

$$\int_0^a b(u)\, (g\, k_1)\, (a-u)\, du = \int_0^a (b\, g)\, (u)\, k(u)\, du \quad \text{as} \quad m \to \infty.$$

Theorem 1.6. *Let T be a transformation which takes each element of $\mathscr{L}$ into $\mathscr{L}^\infty$. If T commutes with convolution, then T is a continuous linear transformation of $\mathscr{L}$ into $\mathscr{L}^\infty$, and there is a b in $\mathscr{L}^\infty$ such that $T(g) = b\, g$ for each g in $\mathscr{L}$. As a transformation on L to L^∞, $\|T\| = \|b\|_{L^\infty}$.*

Proof. The last statement of the theorem follows immediately from the preceding statement in view of Theorem 1.3.

Let g_n be as in Theorem 1.1, and let T_n be the linear transformations determined by the functions $T(g_n)$. Thus $S_n(g) = g\, T(g_n) = T(g_n g) = g_n T(g)$ for g in $\mathscr{L}$. Each T_n takes $L[0, a]$ into $L^\infty[0, a]$, and the norm of T_n as a transformation on L to L^∞ is $\|T_n\| = \|T(g_n)\|_{L^\infty[0,a]}$. For any fixed g in $L[0, a]$, and n sufficiently large

$$\|T_n(g)\|_{L^\infty} = \max_{0 \leqq t \leqq a} \left| \int_0^t T(g_n)\, (u)\, g(t-u)\, du \right|$$
$$= \max_{0 \leqq t \leqq a} \left| \int_0^t T(g)\, (u)\, g_n(t-u)\, du \right|$$
$$\leqq \|T(g)\|_{L^\infty} \int_0^a g_n(u)\, du = \|T(g)\|_{L^\infty}.$$

Since the T_n are bounded pointwise on L, they are bounded in norm (*i.e.* $a > 0$ implies there is a $B_a < \infty$ such that $\|T_n\| = \|T(g_n)\|_{L^\infty[0,a]} < B_a$). The functions $T(g_n)$ thus satisfy the conditions of Lemma 1.4. Consider then the subsequence $T(g_m)$ which converges weak* to b in $\mathscr{L}^\infty$. If g is in $\mathscr{L}$, the sequence $g\, T(g_m)$ converges weak* to $g\, b$ on each interval $[0, K]$ by Lemma 1.5. On the other hand $g\,T(g_m)=g_m\,T(g)$, which by Theorem 1.1 converges in L norm to $T(g)$. The strong limit $T(g)$ and the weak* limit $g\,b$ of the sequence $g\,T(g_m)$ must be the same, that is

$$T(g)=g\,b$$

for each g in $\mathscr{L}$.

If $1<p<\infty$, a similar theorem can be proved by the same methods. Indeed since there exist sets dense both in $\mathscr{L}$ and in $\mathscr{L}^p$, $1<p<\infty$, the theorem corresponding to Theorem 1.3 can be proved by the method of Theorem 1.1. The important point in Lemma 1.4 is that the strongly closed unit sphere in L^∞ is weak* compact. This is true also of L^p, $1<p<\infty$. Thus we state

Theorem 1.7. *Let T be a transformation which takes each element of $\mathscr{L}$ into $\mathscr{L}^p$, $1<p\leqq\infty$. If T commutes with convolution, then T is a continuous linear transformation of $\mathscr{L}$ into $\mathscr{L}^p$, and there is an f in $\mathscr{L}^p$ such that $T(g)=f\,g$ for each g in $\mathscr{L}$. As a transformation on L to L^p, $\|T\|=\|f\|_{L^p}$.*

2. The next theorem depends on a theorem of HARDY & LITTLEWOOD.

Theorem H-L. *Suppose that $a>0$ and $f(u)=0$ for $u>a$. Then*

$$\int_0^a |f(u+h)-f(u)|\,du=O(h) \tag{1}$$

if and only if f is equal almost everywhere to a function which is of bounded variation on $[0, a]$.

We shall need this theorem in the following form.

Corollary H-L. *Suppose that f is in $\mathscr{L}$. A necessary and sufficient condition that f be equal almost everywhere on $[0,\infty)$ to a function which is locally of bounded variation is that equation (1) holds for each $a>0$.*

Theorem 2.1. *Let f be in $\mathscr{L}$. If for each g in $\mathscr{L}$, $f\,g$ is equal almost everywhere to a function which is locally of bounded variation, then f itself is equal almost everywhere to a function which is locally of bounded variation.*

Proof. Take $a>0$ Suppose g is in $\mathscr{L}$, and let $k=f\,g$. By corollary H-L

$$\int_0^a \left|\frac{k(u+h)-k(u)}{h}\right| du=O(1)$$

as $h\to 0$. Now

$$\begin{aligned}
I_g(h)&=\int_0^a \left|\int_0^u \frac{f(u+h-\xi)-f(u-\xi)}{h}\,g(\xi)\,d\xi\right| du\\
&=\int_0^a \left|\frac{k(u+h)-k(u)}{h}-\frac{1}{h}\int_u^{u+h} f(u+h-\xi)\,g(\xi)\,d\xi\right| du\\
&\leqq O(1)+\frac{1}{h}\int_0^a\int_u^{u+h} |f|\,(u+h-\xi)\,|g|\,(\xi)\,d\xi\,du.
\end{aligned}$$

Now $f\,g$ is certainly essentially bounded on $[0, a]$, and by Theorem 1.6 f is essentially bounded on $[0, a]$, which implies that the second term on the right in the above inequality is $O(1)$ as $h\to 0$. Thus $I_g(h)=O(1)$ as $h\to 0$. Applying the uniform boundedness principle and Corollary 1.2, we have

$$\int_0^a \left|\frac{f(u+h)-f(u)}{h}\right| du = O(1)$$

for each positive a. Thus f is equal almost everywhere to a function which is locally of bounded variation.

The converse of the above theorem is well known.

Mikusiński & Ryll-Nardzewski [1] have shown that if f is locally of bounded variation and g is continuous, then $f g$ is locally absolutely continuous. However, more is true.

Theorem 2.2. *If f is locally of bounded variation and g is in $\mathscr{L}$, then $f g$ is locally absolutely continuous.*

Proof. The proof is straightforward. For $\varepsilon>0$ we will show that the variation of $f g$ on subintervals of $[0, a]$ the sum of whose lengths are sufficiently small is less that ε.

Clearly it is no restriction to take f non-decreasing, and even to suppose $f(0)=0$. Suppose that f is non-decreasing, $f(0)=0$, and that the variation of f on $[0, a]$ equals K. Now g can be split into a bounded part and a part whose norm is as small as we please. Let $g=g_B+g_\alpha$ where $|g_B(t)|<B$ and $\int_0^a |g_\alpha|\, du<\alpha$. Suppose x_n, y_n are such that

$$0\leqq x_n<y_n\leqq x_{n+1}<y_{n+1}\leqq a, \qquad n=0, 1, \ldots, N-1.$$

Let

$$h_n=y_n-x_n, \qquad I_n=[x_n, y_n], \qquad n=0, 1, \ldots, N,$$

and let

$$\varDelta=\sum_{n=0}^{N}\left|\int_0^{y_n} f(y_n-u)\, g(u)\, du-\int_0^{x_n} f(x_n-u)\, g(u)\, du\right|.$$

Let $S=\bigcup_1^N I_n$. We choose $\delta_1>0$ so small that $\mu(S)<\delta_1$ implies $\int_S |g|\, du<\frac{\varepsilon}{3K}$. We can take α to be as small as we please, and B depends only on α. Take $\alpha<\frac{\varepsilon}{3K}$. We now choose $\delta<\min\left[\delta_1, \frac{\varepsilon}{6BK}\right]$. Then suppose $\mu(S)<\delta$. Since K is a bound for f on $[0, a]$,

$$\varDelta\leqq\sum_{n=0}^{N}\left|\int_0^{x_n}[f(y_n-u)-f(x_n-u)]\, g(u)\, du\right|+K\int_S |g|\, du,$$

and the last term on the right is less than $\frac{1}{3}\varepsilon$. Further,

$$\sum_{n=0}^{N}\left|\int_0^{x_n}[f(y_n-u)-f(x_n-u)]\, g(u)\, du\right|\leqq\sum_{n=0}^{N}\left|\int_0^{x_n}[f(y_n-u)-f(x_n-u)]\, g_B(u)\, du\right|+$$
$$+\sum_{n=0}^{N}\left|\int_0^{x_n}[f(y_n-u)-f(x_n-u)]\, g_\alpha(u)\, du\right|=S_1+S_2,$$

where S_1 and S_2 are the first and second sums on the right in the above inequality.

Since f is increasing,

$$S_2 \leqq K \int_0^a |g_\alpha| \, du \leqq K \alpha < \tfrac{1}{3} \varepsilon.$$

Since f is increasing and g_B is bounded,

$$\begin{aligned} S_1 &\leqq B \sum_{n=0}^{N} \int_0^{x_n} [f(y_n - u) - f(x_n - u)] \, du \\ &= B \sum_{n=0}^{N} \int_0^{x_n} [f(t + h_n) - f(t)] \, du = B \sum_{n=0}^{N} \left[\int_{x_n}^{x_n + h_n} f \, dt - \int_0^{h_n} f \, dt \right] \\ &\leqq 2BK \sum_{n=0}^{N} h_n = 2BK \mu(S) < 2BK\delta < \tfrac{1}{3} \varepsilon. \end{aligned}$$

Thus $\Delta < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$.

3. Let h be the function whose value is 1 for every non-negative t, and let s be the differentiation operator of MIKUSIŃSKI's operational calculus (*i.e.* $s = h^{-1}$). We can now characterize transformations on $\mathscr{L}$ to $\mathscr{L}$ which commute with convolution.

Theorem 3.1. *Let T be a transformation on $\mathscr{L}$ to $\mathscr{L}$ which commutes with convolution. Then T is a continuous linear transformation on $\mathscr{L}$ to $\mathscr{L}$. If $T(h) = f$, then f is equal almost everywhere to a function v which is locally of bounded variation and $T(g) = s\,v\,g$ for every g in $\mathscr{L}$. Furthermore, for any v which is locally of bounded variation the transformation $g \to s\,v\,g$ takes $\mathscr{L}$ into $\mathscr{L}$ and is a continuous linear transformation. If v is normalized by $v(t) = v(t^-)$ for $t > 0$ and $v(0) = v(0^+)$, the norm of T as a linear transformation on L to L is $\|T\| = \operatorname*{Variation}_{[0,a]} v + |v(0)|$.*

Proof. Let $T(h) = f$; then for every g in $\mathscr{L}$ $g\,f = g\,T(h) = h\,T(g)$ is locally absolutely continuous, and by Theorem 2.1 f is equal almost everywhere to a function v which is locally of bounded variation. Thus T is a linear transformation on $\mathscr{L}$ to $\mathscr{L}$ and $T(g) = s\,v\,g$ for all g. On the other hand any linear transformation on $\mathscr{L}$, which is such that $T(g) = s\,v\,g$ for all g with some fixed function which is locally of bounded variation, must have its range contained in $\mathscr{L}$, since $v\,g$ is locally absolutely continuous (by Theorem 2.2) and zero at the origin (since v is bounded in a neighborhood of the origin). That is, $v\,g = h\,k$ for k in $\mathscr{L}$ and $s\,v\,g$ is in $\mathscr{L}$.

To find the norm of T as a transformation on L to L, let $k_\varepsilon(t) = \frac{v\,g(t - \varepsilon) - v\,g(t)}{\varepsilon}$ when $t \geqq \varepsilon$ and $k_\varepsilon(t) = 0$ when $t < \varepsilon$. Then $k_\varepsilon \to T(g)$ almost everywhere as $\varepsilon \to 0^+$, and

$$k_\varepsilon = I_{1,\varepsilon} + I_{2,\varepsilon}$$

with

$$\begin{aligned} I_{1,\varepsilon}(t) &= \int_0^{t-\varepsilon} g(u) \frac{v(t-u-\varepsilon) - v(t-u)}{\varepsilon} \, du, && t \geqq \varepsilon, \\ &= 0, && t < \varepsilon, \\ I_{2,\varepsilon}(t) &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} g(u) \, v(t-u) \, du, && t \geqq \varepsilon, \\ &= 0, && t < \varepsilon. \end{aligned}$$

Since

$$\lim_{\varepsilon\to 0^+}\int_0^a |I_{1,\varepsilon}(t)|\,dt \leqq \|g\| \lim_{\varepsilon\to 0^+}\int_\varepsilon^a \left|\frac{\nu(u-\varepsilon)-\nu(u)}{\varepsilon}\right| du$$

$$\leqq \|g\| \operatorname*{Var}_{[0,a]} \nu$$

and

$$\lim_{\varepsilon\to 0^+}\int_0^a |I_{2,\varepsilon}(t)|\,dt \leqq \|g\|\,|\nu(0)|,$$

by Fatou's Lemma

$$\|T(g)\| \leqq \|g\|\,(\operatorname{Var}\nu + |\nu(0)|),$$

and

$$\|T\| \leqq \operatorname{Var}\nu + |\nu(0)|.$$

On the other hand if g_n, $n=1, 2, \ldots$, are as in Theorem 1.1, the L norm of $s\,g_n\,\nu$ tends to $\operatorname{Var}\nu + |\nu(0)|$ as n tends to infinity if ν is normalized by $\nu(t)=\nu(t^-)$ for $t>0$ and $\nu(0)=\nu(0^+)$.

It can also be shown that the Lebesgue-Stieltjes integral $\int_0^t g(t-u)\,d\nu(u)$ exists for almost every $t\geqq 0$, is in $\mathscr{L}$, and is equal almost everywhere to $s\,\nu\,g$. The proof follows the same line as in showing that $g\,k$ is in $\mathscr{L}$ whenever g and k are in $\mathscr{L}$. This yields an alternative method of proving Theorem 2.2.

It is a result of RYLL-NARDZEWSKI [2] that an operator of the form $s\,\nu$ has an inverse of the form $s\,\nu_1$, with ν and ν_1 locally of bounded variation if and only if $\nu(0^+)\neq 0$. That is, we know precisely which of the operators on $\mathscr{L}$ to $\mathscr{L}$ that commute with convolution have inverses.

If two elements of $\mathscr{L}$, f and g, are such that the range of the linear transformation determined by f includes the range of the linear transformation determined by g, we shall say $f\geqq g$. If the containment is proper, we shall say $f>g$. Then

Theorem 3.2. (i) *$f\geqq g$ if and only if $g=s\,v\,f$ where v is locally of bounded variation.* (ii) *$f>g$ if and only if $g=s\,v\,f$ where v is locally of bounded variation and $v(0^+)=0$.*

Proof. (i) Suppose that $f\geqq g$. Then every element of $\mathscr{L}$ which has g as a factor also has f as a factor. Let k be in $\mathscr{L}$. Then g/f is in the field of Mikusiński operators, and

$$\frac{g}{f}\,k = \frac{g\,k}{f} = \frac{f\,k_1}{f} = k_1$$

where k_1 is in $\mathscr{L}$. Thus the transformation $k\to\frac{g}{f}\,k$ takes $\mathscr{L}$ into $\mathscr{L}$ and commutes with convolution. Thus $g/f=s\,v$ with v locally of bounded variation. On the other hand if $g=s\,v\,f$ and v is locally of bounded varitaion, any function of the form $g\,k$ is also of the form $(s\,v\,f)\,k=f\,k_1$ so that $f\geqq g$. (ii) If $g=s\,v\,f$ and $v(0^+)\neq 0$, then $\frac{1}{s\,v}=s\,v_1$ so that $f=s\,v_1\,g$. Thus the range of T must equal the range of T, and we cannot have $f>g$. On the other hand if the ranges of T_f and T_g are equal, we must have $g=s\,v\,f$ and $f=s\,v_1\,g$ with v and v_1 locally of bounded variation. Thus $\frac{1}{s\,v}=s\,v_1$, and by RYLL-NARDZEWSKI's theorem $v(0^+)\,v(0^+)\neq 0$.

Theorem 3.2 can be viewed as a comparison theorem for the existence of functional solutions to the convolution integral equation $xf=k$. Thus a restatement of Theorem 3.2 (i) is

Corollary 3.3. *Let f and g be in $\mathscr{L}$. The integral equation $xf=k$ has a solution in $\mathscr{L}$ for every k in $\mathscr{L}$ such that $xg=k$ has a solution in $\mathscr{L}$ if and only if $g=s\,v\,f$ where v is locally of bounded variation.*

4. A simple application is given by the functions which are locally of bounded variation. Any function f which is locally of bounded variation and for which $f(0^+)\neq 0$ is equivalent to h in the sense that both $h\geqq f$ $\left(\text{since } f=(s\,f)\,h\right)$ and $f\geqq h$ $\left(\text{since } h=\frac{1}{s\,f}\,f=s\,v\,f\right)$. Thus the integral equations $xh=k$ and $xf=k$ have solutions for exactly the same set of k.

Theorem 4.1. *If f is locally of bounded variation and $f(0^+)\neq 0$, a necessary and sufficient condition that the convolution integral equation $xf=k$ has a solution x in $\mathscr{L}$ is that k be locally absolutely continuous and zero at the origin.*

Proof. The condition on k is that k be the integral of a function in $\mathscr{L}$, and this is a necessary and sufficient condition that $xh=k$ have a solution x in $\mathscr{L}$.

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University of Washington
Seattle

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