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Shift-type invariant subspaces of contractions

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Abstract

Using the Sz.-Nagy–Foias functional model it was shown in [L. Kérchy, Injection of unilateral shifts into contractions with non-vanishing unitary asymptotes, Acta Sci. Math. (Szeged) 61 (1995) 443–476] that under certain conditions on a contraction T the natural embedding of a Hardy space of vector-valued functions into the corresponding L^2 space can be factored into the product of two transformations, intertwining T with a unilateral shift and with an absolutely continuous unitary operator, respectively. The norm estimates in the Factorization Theorem of this paper are sharpened to their best possible form by essential improvements in the proof. As a consequence we obtain that if the residual set of a contraction covers the whole unit circle then those invariant subspaces, where the restriction is similar to the unilateral shift with a similarity constant arbitrarily close to 1, span the whole space. Furthermore, the hyperinvariant subspace problem for asymptotically non-vanishing contractions is reduced to these special circumstances.

Keywords: Contraction; Functional model; Unilateral shift; Intertwining relation; Invariant subspaces; Hyperinvariant subspace problem

1. Introduction

One of the greatest achievements of the Sz.-Nagy–Foias theory of Hilbert space contractions is the functional model constructed in the completely non-unitary case. We use this model operator to prove a factorization theorem for asymptotically non-vanishing, absolutely continuous

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contractions. Namely, it is shown that if the spectral-multiplicity function of the unitary asymptote of the contraction T is at least $n(\in \mathbb{N} \cup \{\aleph_0\})$ on the Borel set $\gamma \subset \mathbb{T}$, then the natural embedding $J : f \mapsto \chi_\gamma f$ of the Hardy space $H^2(\mathfrak{G}_n)$ over the n-dimensional Hilbert space $\mathfrak{G}_n$ into the function space $\chi_\gamma L^2(\mathfrak{G}_n)$ can be factored into the product $J = ZY$, where Y intertwines the unilateral shift S_n on $H^2(\mathfrak{G}_n)$ with T, and Z intertwines T with the unitary operator $M_{n,\gamma}$ of multiplication by the independent variable on $\chi_\gamma L^2(\mathfrak{G}_n)$. Furthermore, the norms of the linear transformations Y and Z can be arbitrarily close to 1. This statement is sharpening of the main result in [14], where the norm conditions on Y and Z were weaker. This sharpening requires essential improvements in the proof given in [14].

In Section 2 we give a brief summary of the unitary asymptotes of contractions, with their representation in the functional model. The Factorization Theorem is formulated in Section 3. The first step in its proof is the construction of a vector-sequence in the space $\mathfrak{K}$ of the minimal unitary dilation, which is pointwise orthonormal, and which is transformed by a canonical intertwiner to a sequence which is also pointwise orthonormal. This is carried out in Section 4 relying on the connection of the defect fields. The results of Section 5 make possible to approximate the previous vectors in $\mathfrak{K}$ by vectors in the space $\mathfrak{K}_+$ of the minimal isometric dilation. The proof of the Factorization Theorem is completed in Section 6.

It turns out in Section 7 that the ranges of the possible intertwiners Y span the whole space $\mathfrak{H}$ of the contraction T. In the particular case $n = \aleph_0$ even the ranges of two intertwiners Y and $\widehat{Y}$ span $\mathfrak{H}$. As a consequence we obtain that if the unitary asymptote of T is of infinite spectral-multiplicity on the whole circle $\mathbb{T}$, then we can find two invariant subspaces of T which span the whole space $\mathfrak{H}$, and where the restrictions of T are similar to the infinite-dimensional unilateral shift; furthermore, the similarity constants can be chosen arbitrarily close to 1. Thus in this case we have a lot of information on the structure of T, in particular, T has plenty of invariant subspaces. It can be surprising that the hyperinvariant subspace problem for asymptotically non-vanishing contractions can be reduced to this particular situation, as shown in Section 8.

This statement is related to a result of H. Bercovici, C. Foias and C. Pearcy, achieved together with collaborators in a sequence of four recent papers [6], [8], [7] and [3]. Their theorem states that the hyperinvariant subspace problem for general operators on separable Hilbert spaces can be reduced to the special case, when T and its adjoint are asymptotically vanishing contractions with spectral radius 1, and with (left essential) spectrum coinciding with an annulus (having also some other good properties). (Furthermore, all possible hyperinvariant subspace lattices are represented in that special class.) The theory of dual algebras tells us that T has plenty of invariant subspaces in this case too (see [2]).

The Banach space of the bounded linear transformations from the Hilbert space $\mathfrak{A}$ to the Hilbert space $\mathfrak{B}$ will be denoted by $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$. The C^*-algebra of bounded linear operators acting on $\mathfrak{A}$ is denoted by $\mathcal{L}(\mathfrak{A}) = \mathcal{L}(\mathfrak{A}, \mathfrak{A})$. All Hilbert spaces considered in this paper are complex and separable.

2. Unitary asymptotes

Let T be a contraction acting on the Hilbert space $\mathfrak{H}$ (that is $T \in \mathcal{L}(\mathfrak{H})$ and $\|T\| \leqslant 1$). For every $x \in \mathfrak{H}$, the decreasing sequence $\{\|T^n x\|\}_n$ is convergent. Hence, by the polar identity the sequence $\{\langle T^n x, T^n y\rangle\}_n$ is also convergent $(x, y \in \mathfrak{H})$. The functional $w_T(x, y) := \lim_n \langle T^n x, T^n y\rangle$ is linear in x, conjugate linear in y, and bounded by 1. Thus, there exists a unique operator A_T on $\mathfrak{H}$ such that $\langle A_T x, y\rangle = w_T(x, y)$ holds for all $x, y \in \mathfrak{H}$. Since $w_T(x, x) \geqslant 0$ $(x \in \mathfrak{H})$, it follows that $0 \leqslant A_T \leqslant I$. Furthermore, the relation $w_T(Tx, Ty) =$

$w_T(x, y)$ yields $T^*A_TT = A_T$, whence $\|A_T^{1/2}Tx\| = \|A_T^{1/2}x\|$ $(x \in \mathfrak{H})$ follows. Introducing the transformation $X_T^+ : \mathfrak{H} \to \mathfrak{K}_T^+ := (A_T^{1/2}\mathfrak{H})^-$, $x \mapsto A_T^{1/2}x$, we obtain that there exists a unique isometry V_T on the space $\mathfrak{K}_T^+$ such that $X_T^+T = V_TX_T^+$. The isometry V_T is called the *isometric asymptote* of the contraction T. It is clear that the canonical intertwining transformation X_T^+ has dense range. Let W_T denote the minimal unitary extension of V_T acting on the Hilbert space $\mathfrak{K}_T$, determined uniquely up to isomorphism (see [19, Section I.2]). The operator W_T is the *unitary asymptote* of the contraction T. The transformation $X_T : \mathfrak{H} \to \mathfrak{K}_T$, $x \mapsto X_T^+x$ intertwines T with W_T: $X_TT = W_TX_T$. Furthermore,

$$\|X_Th\|^2 = \left\|X_T^+h\right\|^2 = \left\|A_T^{1/2}h\right\|^2 = \langle A_Th, h\rangle = \lim_n \left\|T^nh\right\|^2$$

is true, for every $h \in \mathfrak{H}$. Thus, the nullspace $\ker X_T$ coincides with the set of vectors whose orbits converge to zero under the action of T. The contraction T is called *asymptotically non-vanishing* if $\ker X_T \neq \mathfrak{H}$.

The pair (X_T, W_T) has an important *universal property*. For an operator A acting on a space $\mathfrak{A}$ and an operator B acting on a space $\mathfrak{B}$, the *intertwining set* $\mathcal{I}(A, B)$ consists of the (bounded linear) transformations $Y : \mathfrak{A} \to \mathfrak{B}$ satisfying the equation $YA = BY$. The *commutant* of A is defined by $\{A\}' := \mathcal{I}(A, A)$. Now, it is true that for any unitary operator G acting on a Hilbert space $\mathfrak{G}$, and for any transformation $Y \in \mathcal{I}(T, G)$, there exists a unique transformation $Z \in \mathcal{I}(W_T, G)$ such that $Y = ZX_T$. Furthermore, the commutants and spectra of T and W_T are closely related. For these properties and their extension to larger classes of operators we refer to [13] and [15]. (See also [1, Chapter XII] for the study of isometric asymptotes, called isometric extensions there.)

The unitary asymptote of the contraction T can be identified with the $*$-residual part of its minimal unitary dilation U acting on the Hilbert space $\mathfrak{K}$. We recall from [19, Chapter II] that the subspace $\mathfrak{L} = ((U - T)\mathfrak{H})^-$ is wandering, and the orthogonal sum $M(\mathfrak{L}) := \bigoplus_{k=-\infty}^{\infty} U^k\mathfrak{L}$ is reducing for U. The *$*$-residual part* R_{*T} is the restriction of U to its reducing subspace $\mathfrak{R}_{*T} := \mathfrak{K} \ominus M(\mathfrak{L})$. Let us consider the transformation $\widehat{X}_T : \mathfrak{H} \to \mathfrak{R}_{*T}$, $h \mapsto P_*h$, where P_* denotes the orthogonal projection in $\mathfrak{K}$ onto the subspace $\mathfrak{R}_{*T}$. Since $R_{*T}\widehat{X}_T = \widehat{X}_TT$ and $\|\widehat{X}_Th\| = \lim_n \|T^nh\|$ $(h \in \mathfrak{H})$ hold by [19, Section II.3], it can be easily verified that the pair $(\widehat{X}_T, R_{*T})$ *is equivalent to* (X_T, W_T), that is there exists a unitary transformation $Z \in \mathcal{I}(W_T, R_{*T})$ such that $\widehat{X}_T = ZX_T$.

The unitary asymptote has a particularly useful representation in the Sz.-Nagy–Foias functional model of completely non-unitary (c.n.u.) contractions. We recall the construction of the model operator given in [19, Chapter VI]. Let $\mathfrak{E}, \mathfrak{E}_*$ be (separable) Hilbert spaces, and let $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ be a purely contractive analytic function defined on the open unit disc $\mathbb{D}$. It is known that the radial limit $\Theta(\zeta) = \lim_{r\to 1-0} \Theta(r\zeta)$ exists in the strong operator topology for almost every ζ on the unit circle $\mathbb{T}$. Hence Θ can be considered also as a measurable function defined almost everywhere on $\mathbb{T}$, and taking values in $\mathcal{L}(\mathfrak{E}, \mathfrak{E}_*)$. We can extend Θ to the whole circle $\mathbb{T}$ defining its value by 0 on the exceptional set of measure zero. The defect operator-functions associated with Θ are defined by

$$\Delta(\zeta) := \left(I - \Theta(\zeta)^*\Theta(\zeta)\right)^{1/2} \quad \text{and} \quad \Delta_*(\zeta) := \left(I - \Theta(\zeta)\Theta(\zeta)^*\right)^{1/2} \quad (\zeta \in \mathbb{T}).$$

Let us consider the spaces $L^2(\mathfrak{E})$, $L^2(\mathfrak{E}_*)$ of vector-valued functions, defined with respect to the normalized Lebesgue measure m on $\mathbb{T}$, and the Hardy subspaces $H^2(\mathfrak{E})$, $H^2(\mathfrak{E}_*)$. Setting

$$\mathfrak{K} := L^2(\mathfrak{E}_*) \oplus \big(\Delta L^2(\mathfrak{E})\big)^-, \qquad \mathfrak{K}_+ := H^2(\mathfrak{E}_*) \oplus \big(\Delta L^2(\mathfrak{E})\big)^-,$$
$$\mathfrak{G} := \big\{\Theta u \oplus \Delta u\colon\ u \in L^2(\mathfrak{E})\big\}, \qquad \mathfrak{G}_+ := \big\{\Theta u \oplus \Delta u\colon\ u \in H^2(\mathfrak{E})\big\},$$

the model space $\mathfrak{H} = \mathfrak{H}(\Theta)$ is given by $\mathfrak{H} := \mathfrak{K}_+ \ominus \mathfrak{G}_+$. Let $U^\times$ denote the operator of multiplication by ζ on $L^2(\mathfrak{E}_*) \oplus L^2(\mathfrak{E})$. The subspaces $\mathfrak{K}$, $\mathfrak{G}$ are reducing for $U^\times$, while $\mathfrak{K}_+$, $\mathfrak{G}_+$ are invariant for $U^\times$. The model operator $T = S(\Theta)$ is defined by $T := P_+U_+|\mathfrak{H}$, where P_+ denotes the orthogonal projection onto $\mathfrak{H}$ in $\mathfrak{K}_+$, $U_+ := U^\times|\mathfrak{K}_+$ is the minimal isometric dilation of T, while $U := U^\times|\mathfrak{K}$ is the minimal unitary dilation of T.

Let us consider the restriction $\widetilde{R}_{*T}$ of $U^\times$ to the reducing subspace $\widetilde{\mathfrak{R}}_{*T} := (\Delta_* L^2(\mathfrak{E}_*))^-$, and the transformation $\widetilde{X}_T \in \mathcal{I}(T, \widetilde{R}_{*T})$ defined by

$$\widetilde{X}_T(u \oplus v) := -\Delta_* u + \Theta v \quad (u \oplus v \in \mathfrak{H}).$$

Since multiplication by the unitary operator-valued function

$$F(\zeta) = \begin{bmatrix} -\Delta_*(\zeta) & \Theta(\zeta) \\ \Theta(\zeta)^* & \Delta(\zeta) \end{bmatrix} \in \mathcal{L}(\mathfrak{E}_* \oplus \mathfrak{E}) \quad (\zeta \in \mathbb{T})$$

transfers the subspace $\mathfrak{R}_{*T}$ into $\widetilde{\mathfrak{R}}_{*T}$, we obtain that the pair $(\widetilde{X}_T, \widetilde{R}_{*T})$ is equivalent to the pair $(\widehat{X}_T, R_{*T})$, and so to the pair (X_T, W_T). (See [12] for details.) The great advantage of this representation of the unitary asymptote lies in the fact that the intertwining mapping $\widetilde{X}_T$ is multiplication by an operator-valued function.

3. Factorization Theorem

Let T be an absolutely continuous (a.c.) contraction on the Hilbert space $\mathfrak{H}$, that is we assume that the (spectral measure of the) unitary component of T is a.c. with respect to the normalized Lebesgue measure m on the unit circle $\mathbb{T}$. The minimal unitary dilation U of T is a.c. by [19, Theorem II.6.4]. It follows that the $*$-residual part R_{*T} is also a.c., and then so is the unitary asymptote W_T of T. The a.c. unitary operator W_T on the (separable) Hilbert space $\mathfrak{K}_T$ is uniquely determined—up to unitary equivalence—by its spectral-multiplicity function (see e.g. [4, Section IX.10]), which we will call the *asymptotic spectral-multiplicity function of the contraction T*, and denote by μ_T. We recall that μ_T is a measurable function defined on the unit circle $\mathbb{T}$, and taking values in the set $\mathbb{N} \cup \{0, \aleph_0\}$ of countable cardinals. It is determined by the decreasing sequence of Borel sets:

$$\rho_{T,n} := \big\{\zeta \in \mathbb{T}\colon\ \mu_T(\zeta) \geqslant n\big\} \quad \big(n \in \mathbb{N} \cup \{\aleph_0\}\big).$$

The set $\rho_T := \rho_{T,1}$ is the support of the spectral measure of W_T and is called the *residual set* of the a.c. contraction T. For its role in the study of T we refer to [17].

The Factorization Theorem establishes intertwining relations between the contraction T and unilateral shifts, exploiting the fine structure of W_T encoded in the spectral-multiplicity function μ_T.

For any cardinal number $1 \leqslant n \leqslant \aleph_0$, let $\mathfrak{G}_n$ be a fixed Hilbert space of dimension n. Let us consider the Hilbert space $L^2(\mathfrak{G}_n)$ of vector-valued functions. The σ-algebra of Borel subsets of $\mathbb{T}$ will be denoted by $\mathcal{B}_\mathbb{T}$. For any $\alpha \in \mathcal{B}_\mathbb{T}$, $\widetilde{M}_{n,\alpha}$ is the multiplication by $\chi(\zeta) = \zeta$

on the space $L^2(\mathfrak{G}_n,\alpha) := \chi_\alpha L^2(\mathfrak{G}_n)$. Clearly, $\widetilde{M}_{n,\alpha}$ is an a.c. unitary operator with spectral-multiplicity function $n\chi_\alpha$. (Here and in the sequel χ_ω stands for the characteristic function of the set ω.)

It is known that the Hardy space $H^2(\mathfrak{G}_n)$ of analytic vector-valued functions, defined on $\mathbb{D}$, can be identified with the subspace $L^2_+(\mathfrak{G}_n)$ of $L^2(\mathfrak{G}_n)$, consisting of the functions with zero Fourier coefficients of negative indices (see [19, Section V.1]). Let S_n be the multiplication by $\chi(\zeta)=\zeta$ on $H^2(\mathfrak{G}_n)$; S_n is clearly a unilateral shift of multiplicity n. For any $\alpha \in \mathcal{B}_{\mathbb{T}}$, let us consider the natural embedding

$$\widetilde{J}_{n,\alpha} : H^2(\mathfrak{G}_n) \to L^2(\mathfrak{G}_n,\alpha), \quad f \mapsto \chi_\alpha f,$$

of $H^2(\mathfrak{G}_n)$ into $L^2(\mathfrak{G}_n,\alpha)$. If $m(\alpha)=0$ then $L^2(\mathfrak{G}_n,\alpha)$ and $\widetilde{J}_{n,\alpha}$ reduce to zero. If $m(\alpha)>0$ then $\widetilde{J}_{n,\alpha}$ is one-to-one, $\|\widetilde{J}_{n,\alpha}\|=1$, and $\widetilde{J}_{n,\alpha}S_n = \widetilde{M}_{n,\alpha}\widetilde{J}_{n,\alpha}$. (See [9, Chapter 4] and [19, Chapter III].)

For the sake of brevity, we introduce the notation

$$M_{T,n} := \widetilde{M}_{n,\rho_{T,n}} \quad \text{and} \quad J_{T,n} := \widetilde{J}_{n,\rho_{T,n}} \quad (1 \leqslant n \leqslant \aleph_0)$$

in connection with the a.c. contraction T. The Factorization Theorem states that $J_{T,n}$ can be factored into the product of two mappings intertwining S_n and $M_{T,n}$ with T, with a control on the norms of the intertwiners.

Theorem 1. *Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$. For every cardinal number $1 \leqslant n \leqslant \aleph_0$, and for every $\varepsilon > 0$, there exist transformations $Y \in \mathcal{I}(S_n,T)$ and $Z \in \mathcal{I}(T,M_{T,n})$ satisfying the conditions*:

(i) *$ZY = J_{T,n}$, and*
(ii) *$\|Y\| < 1+\varepsilon$, $\|Z\| < 1+\varepsilon$.*

Notice that if $m(\rho_{T,n})=0$ then $J_{T,n}=0$, and so the transformations $Y=0$ and $Z=0$ evidently possess the required properties. The statement of the previous theorem becomes non-trivial when $m(\rho_{T,n})>0$.

4. Defect fields

The core of the proof of the Factorization Theorem (Theorem 1) is its verification in the functional model. So let us give a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$, and let us consider the model operator $T = S(\Theta) \in \mathcal{L}(\mathfrak{H} = \mathfrak{H}(\Theta))$ constructed at the end of Section 2. Since W_T is unitarily equivalent to $\widetilde{R}_{*T}$, we obtain that the asymptotic spectral-multiplicity function μ_T of T coincides with the function $\operatorname{rank} \Delta_*(\zeta)$ ($\zeta \in \mathbb{T}$). Thus

$$\rho_{T,n} = \{\zeta \in \mathbb{T}\colon \operatorname{rank} \Delta_*(\zeta) \geqslant n\}$$

holds for every $1 \leqslant n \leqslant \aleph_0$.

First we want to show that there exists a sequence $\{u_i \oplus v_i\}_{0\leqslant i<n}$ in the dilation space $\mathfrak{K}$, which is pointwise orthonormal on the set $\rho_{T,n}$, and whose transformed sequence $\{-\Delta_* u_i + \Theta v_i\}_{0\leqslant i<n}$ in $\widetilde{\mathfrak{R}}_{*T}$ is also pointwise orthonormal on $\rho_{T,n}$. (We remind the reader of the special form of the intertwining mapping $\widetilde{X}_T$.) In order to do so we have to make a closer look at the defect functions.

We recall from [19, Section I.3] that the defect operator D_A of a contractive transformation $A \in \mathcal{L}(\mathfrak{G}, \mathfrak{G}_*)$ is the positive contraction defined by $D_A := (I - A^*A)^{1/2} \in \mathcal{L}(\mathfrak{G})$. The closure of its range is the defect space $\mathfrak{D}_A$ of A. Let D_{A^*} and $\mathfrak{D}_{A^*}$ be the analogous objects connected with the adjoint A^*. It is easy to check that $A^* D_{A^*} = D_A A^*$.

For any $\zeta \in \mathbb{T}$, let $\Delta(\zeta)$ and $\mathfrak{D}(\zeta)$ be the defect operator and the defect space of $\Theta(\zeta)$, respectively. Let $\Delta_*(\zeta)$ and $\mathfrak{D}_*(\zeta)$ stand for the analogous objects connected with the adjoint transformation $\Theta(\zeta)^*$. All these operator fields and subspace fields are measurable; see [14, Section 2]. Notice that the direct integrals $\int_\omega^\oplus (\mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta))\, dm(\zeta)$ and $\int_\omega^\oplus \mathfrak{D}_*(\zeta)\, dm(\zeta)$ can be viewed as subspaces in the space $\mathfrak{K}$ of the minimal unitary dilation U, and in $\widetilde{\mathfrak{R}}_{*T}$, respectively.

The following statement is an improvement of [14, Corollary 1]. (Its version formulated in the more general setting considered in [14] can be proved in a similar way. We note that only the pointwise orthogonality of the system $\{k_i\}_i$ below was shown in [14].)

Proposition 2. *For every cardinal number $1 \leqslant n \leqslant \aleph_0$, there exist sequences $\{u_i\}_{0 \leqslant i < n}$ and $\{v_i\}_{0 \leqslant i < n}$ of measurable vector fields in $\prod_{\zeta \in \rho_{T,n}} \mathfrak{D}_*(\zeta)$ and $\prod_{\zeta \in \rho_{T,n}} \mathfrak{D}(\zeta)$, respectively, such that*

- (i) *$\{u_i(\zeta) \oplus v_i(\zeta)\}_{0 \leqslant i < n}$ forms an orthonormal system in $\mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta)$ for every $\zeta \in \rho_{T,n}$, and*
- (ii) *$\{k_i(\zeta) := -\Delta_*(\zeta) u_i(\zeta) + \Theta(\zeta) v_i(\zeta)\}_{0 \leqslant i < n}$ is also an orthonormal system in $\mathfrak{D}_*(\zeta)$ for every $\zeta \in \rho_{T,n}$.*

Proof. Let $F : \mathbb{T} \to \mathcal{L}(\mathfrak{E}_* \oplus \mathfrak{E})$ be the unitary operator-valued, measurable function introduced in Section 2. The equation $\Theta(\zeta)^* \Delta_*(\zeta) = \Delta(\zeta)\Theta(\zeta)^*$ yields $F(\zeta)\mathfrak{D}_*(\zeta) \subset \mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta)$ $(\zeta \in \mathbb{T})$. Let us consider the isometry-valued, measurable transformation field $F_0(\zeta) : \mathfrak{D}_*(\zeta) \mapsto \mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta)$ defined by $F_0(\zeta)w := F(\zeta)w = -\Delta_*(\zeta)w \oplus \Theta(\zeta)^* w$ $(\zeta \in \mathbb{T},\ w \in \mathfrak{D}_*(\zeta))$. It is easy to see that the adjoint transformation field $F_0(\zeta)^* : \mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta) \to \mathfrak{D}_*(\zeta)$ is defined by $F_0(\zeta)^*(u \oplus v) = -\Delta_*(\zeta)u + \Theta(\zeta)v$.

Since $\dim \mathfrak{D}_*(\zeta) \geqslant n$ holds for every $\zeta \in \rho_{T,n}$, we can give measurable vector fields $\{k_i\}_{0 \leqslant i < n}$ in $\prod_{\zeta \in \rho_{T,n}} \mathfrak{D}_*(\zeta)$ so that $\{k_i(\zeta)\}_{0 \leqslant i < n}$ forms an orthonormal system for each $\zeta \in \rho_{T,n}$ (see [5, Chapter II.1]). Then the measurable vector fields $\{u_i\}_{0 \leqslant i < n}$ and $\{v_i\}_{0 \leqslant i < n}$, defined by

$$u_i(\zeta) \oplus v_i(\zeta) := F_0(\zeta) k_i(\zeta) \quad (0 \leqslant i < n,\ \zeta \in \rho_{T,n}),$$

satisfy all the required conditions. □

5. Approximation by analytic functions

The vector functions (or vector fields) $\{u_i \oplus v_i\}_i$ provided by Proposition 2 are contained in the space $\mathfrak{K}$ of the minimal unitary dilation. We want to approximate them with functions from the space $\mathfrak{K}_+$ of the minimal isometric dilation U_+. Hence we have to approximate measurable vector-valued functions by analytic ones. For any $u, \hat{u} \in L^2(\mathfrak{E}_*)$ the measurable function $[u, \hat{u}]$ is defined by $[u, \hat{u}](\zeta) := \langle u(\zeta), \hat{u}(\zeta) \rangle$ $(\zeta \in \mathbb{T})$. The norm-function of $u \in L^2(\mathfrak{E}_*)$ is denoted by $[u]$, that is $[u](\zeta) := \|u(\zeta)\|$ $(\zeta \in \mathbb{T})$. We recall from [14, Lemma 7] that if $u \in L^2(\mathfrak{E}_*)$ is a unimodular function, that is $[u] \equiv 1$, then for every $0 < \eta < 1$ there exists a function $u^\# \in H^2(\mathfrak{E}_*)$ such that $[u^\#] \equiv 1$ and $\big|[u^\#, u]\big| \geqslant \eta$. For the approximating purposes, mentioned above, we need the following lemma.

Lemma 3. *Let $u \oplus v \in L^2(\mathfrak{E}_*) \oplus L^2(\mathfrak{E}) \equiv L^2(\mathfrak{E}_* \oplus \mathfrak{E})$ be a function with the property that its norm-function $[u \oplus v] \equiv 1$. Then, for every $0 < \eta < 1$, there exist a function $u^\# \in H^2(\mathfrak{E}_*)$ and a measurable complex function ψ on $\mathbb{T}$ such that*

$$\left[u^\# \oplus \psi v\right] \equiv 1 \quad \textit{and} \quad \left|\left[u^\# \oplus \psi v, u \oplus v\right]\right| \geqslant \eta.$$

Proof. Let us give an arbitrary $0 < \eta < 1$, and let us choose positive numbers η_1 and ε satisfying the conditions

$$\eta < \eta_1^2 < \eta_1 < 1 \quad \text{and} \quad \left(\eta_1^2 - 2\varepsilon\right)(1 + 2\varepsilon)^{-1} > \eta. \tag{1}$$

Let us consider the decomposition $\mathbb{T} = \beta_1 \cup \beta_2$, where

$$\beta_1 := \left\{\zeta \in \mathbb{T}\colon \left\|u(\zeta)\right\| \geqslant \eta_1\right\} \quad \text{and} \quad \beta_2 := \mathbb{T} \setminus \beta_1.$$

Given any $e_* \in \mathfrak{E}_*$ with $\|e_*\| = 1$, the function $u_1 \in L^2(\mathfrak{E}_*)$ is defined by

$$u_1(\zeta) := \begin{cases} u(\zeta)/\|u(\zeta)\| & \text{if } \zeta \in \beta_1, \\ e_* & \text{if } \zeta \in \beta_2. \end{cases} \tag{2}$$

Since $[u_1] \equiv 1$, by [14, Lemma 7] there exists a function $u_1^\# \in H^2(\mathfrak{E}_*)$ such that

$$\eta_1 \leqslant \left|\left[u_1^\#, u_1\right]\right| \leqslant \left[u_1^\#\right] \equiv 1. \tag{3}$$

Let us give a positive η_2 so that $\eta_1 < \eta_2 < 1$. Applying [14, Lemma 7] for $u \oplus v$, we obtain a function $u_2^\# \oplus v_2^\# \in H^2(\mathfrak{E}_* \oplus \mathfrak{E}) \equiv H^2(\mathfrak{E}_*) \oplus H^2(\mathfrak{E})$ with the properties

$$\eta_2 \leqslant \left|\left[u_2^\# \oplus v_2^\#, u \oplus v\right]\right| \leqslant \left[u_2^\# \oplus v_2^\#\right] \equiv 1. \tag{4}$$

For every $\zeta \in \beta_2$, we have

$$\left\|v(\zeta)\right\| = \left(1 - \left\|u(\zeta)\right\|^2\right)^{1/2} \geqslant \left(1 - \eta_1^2\right)^{1/2} > 0. \tag{5}$$

Let us consider the decomposition

$$v_2^\#(\zeta) = \psi_2(\zeta)v(\zeta) + w(\zeta), \quad \text{where } w(\zeta) \perp v(\zeta)\ (\zeta \in \beta_2). \tag{6}$$

It is clear that the function

$$\psi_2(\zeta) = \left\|v(\zeta)\right\|^{-2}\left\langle v_2^\#(\zeta), v(\zeta)\right\rangle \quad (\zeta \in \beta_2) \tag{7}$$

is measurable. We want to show that the norm of $w(\zeta)$ is as small as we wish if η_2 is sufficiently close to 1.

In view of (4) and applying the Cauchy–Schwarz inequality we infer that

$$\begin{aligned}\eta_2 &\leqslant \bigl|\bigl\langle u_2^{\#}(\zeta)\oplus v_2^{\#}(\zeta), u(\zeta)\oplus v(\zeta)\bigr\rangle\bigr| \leqslant \bigl|\bigl\langle u_2^{\#}(\zeta), u(\zeta)\bigr\rangle\bigr| + \bigl|\bigl\langle v_2^{\#}(\zeta), v(\zeta)\bigr\rangle\bigr| \\ &\leqslant \bigl\|u_2^{\#}(\zeta)\bigr\|\bigl\|u(\zeta)\bigr\| + \bigl\|v_2^{\#}(\zeta)\bigr\|\bigl\|v(\zeta)\bigr\| =: \kappa(\zeta) \\ &\leqslant \bigl(\bigl\|u_2^{\#}(\zeta)\bigr\|^2 + \bigl\|v_2^{\#}(\zeta)\bigr\|^2\bigr)^{1/2}\bigl(\bigl\|u(\zeta)\bigr\|^2 + \bigl\|v(\zeta)\bigr\|^2\bigr)^{1/2} = 1 \end{aligned} \tag{8}$$

holds for every $\zeta \in \mathbb{T}$. Taking the decomposition of the ordered pairs

$$\bigl(\bigl\|u_2^{\#}(\zeta)\bigr\|, \bigl\|v_2^{\#}(\zeta)\bigr\|\bigr) = \kappa(\zeta)\bigl(\bigl\|u(\zeta)\bigr\|, \bigl\|v(\zeta)\bigr\|\bigr) + \bigl(a(\zeta), b(\zeta)\bigr),$$

we obtain that

$$\bigl|b(\zeta)\bigr|^2 \leqslant \bigl\|\bigl(a(\zeta), b(\zeta)\bigr)\bigr\|^2 = 1 - \kappa(\zeta)^2 \leqslant 1 - \eta_2^2,$$

whence

$$\bigl\|v_2^{\#}(\zeta)\bigr\| = \kappa(\zeta)\bigl\|v(\zeta)\bigr\| + b(\zeta) \geqslant \eta_2\bigl\|v(\zeta)\bigr\| - \bigl(1-\eta_2^2\bigr)^{1/2} \quad (\zeta \in \mathbb{T})$$

follows. Applying (5) we conclude that

$$\bigl\|v_2^{\#}(\zeta)\bigr\| \geqslant \eta_2\bigl(1-\eta_1^2\bigr)^{1/2} - \bigl(1-\eta_2^2\bigr)^{1/2} \tag{9}$$

is true for every $\zeta \in \beta_2$. Let us assume that η_2 is so close to 1 that

$$\eta_2\bigl(1-\eta_1^2\bigr)^{1/2} - \bigl(1-\eta_2^2\bigr)^{1/2} > 0 \tag{10}$$

is fulfilled. One can easily derive from (8) that

$$\bigl\|v_2^{\#}(\zeta)\bigr\|\bigl\|v(\zeta)\bigr\| - \bigl|\bigl\langle v_2^{\#}(\zeta), v(\zeta)\bigr\rangle\bigr| \leqslant 1 - \eta_2 \quad (\zeta \in \mathbb{T}).$$

It follows by (5)–(7) and (9) that

$$\begin{aligned}\bigl\|\psi_2(\zeta)v(\zeta)\bigr\| &= \bigl|\bigl\langle v_2^{\#}(\zeta), v(\zeta)/\bigl\|v(\zeta)\bigr\|\bigr\rangle\bigr| \geqslant \bigl\|v_2^{\#}(\zeta)\bigr\| - (1-\eta_2)\bigl(1-\eta_1^2\bigr)^{-1/2} \\ &\geqslant \eta_2\bigl(1-\eta_1^2\bigr)^{1/2} - \bigl(1-\eta_2^2\bigr)^{1/2} - (1-\eta_2)\bigl(1-\eta_1^2\bigr)^{-1/2} \end{aligned} \tag{11}$$

holds for every $\zeta \in \beta_2$. Choosing η_2 sufficiently close to 1 we can ensure that

$$\eta_2\bigl(1-\eta_1^2\bigr)^{1/2} - \bigl(1-\eta_2^2\bigr)^{1/2} - (1-\eta_2)\bigl(1-\eta_1^2\bigr)^{-1/2} > 0. \tag{12}$$

Applying (6), (11) and (12) we obtain

$$\begin{aligned}\bigl\|v_2^{\#}(\zeta) - \psi_2(\zeta)v(\zeta)\bigr\|^2 &= \bigl\|v_2^{\#}(\zeta)\bigr\|^2 - \bigl\|\psi_2(\zeta)v(\zeta)\bigr\|^2 \\ &\leqslant \bigl\|v_2^{\#}(\zeta)\bigr\|^2 - \bigl(\bigl\|v_2^{\#}(\zeta)\bigr\| - (1-\eta_2)\bigl(1-\eta_1^2\bigr)^{-1/2}\bigr)^2 \\ &\leqslant 2\bigl\|v_2^{\#}(\zeta)\bigr\|(1-\eta_2)\bigl(1-\eta_1^2\bigr)^{-1/2} \\ &\leqslant 2(1-\eta_2)\bigl(1-\eta_1^2\bigr)^{-1/2} \quad (\zeta \in \beta_2). \end{aligned} \tag{13}$$

Therefore, assuming that the positive number η_2 satisfies the conditions $\eta_1 < \eta_2 < 1$, (10), (12) and

$$2(1-\eta_2)\big(1-\eta_1^2\big)^{-1/2} < \varepsilon^2, \tag{14}$$

we conclude by (13) that

$$\big\| v_2^{\#}(\zeta) - \psi_2(\zeta)v(\zeta)\big\| < \varepsilon \tag{15}$$

holds for every $\zeta \in \beta_2$.

Let us give $\varphi_1, \varphi_2 \in H^\infty$ with absolute value

$$|\varphi_1| = \chi_{\beta_1} + \varepsilon\chi_{\beta_2}, \qquad |\varphi_2| = \varepsilon\chi_{\beta_1} + \chi_{\beta_2}, \tag{16}$$

and let us introduce the functions

$$\widetilde{u} := \varphi_1 u_1^{\#} + \varphi_2 u_2^{\#} \in H^2(\mathfrak{E}_*) \quad \text{and} \quad \widetilde{\psi}(\zeta) := \begin{cases} 0 & \text{for } \zeta \in \beta_1, \\ \varphi_2(\zeta)\psi_2(\zeta) & \text{for } \zeta \in \beta_2. \end{cases}$$

For every $\zeta \in \beta_1$, we have

$$\widetilde{u}(\zeta) \oplus \widetilde{\psi}(\zeta)v(\zeta) = \big(\varphi_1(\zeta)u_1^{\#}(\zeta) + \varphi_2(\zeta)u_2^{\#}(\zeta)\big) \oplus 0,$$

and so

$$1-\varepsilon \leqslant \big\|\widetilde{u}(\zeta) \oplus \widetilde{\psi}(\zeta)v(\zeta)\big\| \leqslant 1+\varepsilon \quad (\zeta \in \beta_1) \tag{17}$$

readily follows by (3), (4) and (16). Furthermore,

$$\begin{aligned} \big|\big\langle \widetilde{u}(\zeta) \oplus \widetilde{\psi}(\zeta)v(\zeta), u(\zeta) \oplus v(\zeta)\big\rangle\big| &= \big|\varphi_1(\zeta)\big\langle u_1^{\#}(\zeta), \|u(\zeta)\|u_1(\zeta)\big\rangle + \varphi_2(\zeta)\big\langle u_2^{\#}(\zeta), u(\zeta)\big\rangle\big| \\ &\geqslant \eta_1^2 - \varepsilon \quad (\zeta \in \beta_1) \end{aligned} \tag{18}$$

is clearly true by (2)–(4) and (16). On the other hand, for every $\zeta \in \beta_2$, we have

$$\begin{aligned} \widetilde{u}(\zeta) \oplus \widetilde{\psi}(\zeta)v(\zeta) = {} & \big(\varphi_1(\zeta)u_1^{\#}(\zeta) \oplus 0\big) + \varphi_2(\zeta)\big(u_2^{\#}(\zeta) \oplus v_2^{\#}(\zeta)\big) \\ & + \big(0 \oplus \varphi_2(\zeta)\big(\psi_2(\zeta)v(\zeta) - v_2^{\#}(\zeta)\big)\big). \end{aligned}$$

Hence, applying (15) together with (3), (4) and (16), one can easily verify that

$$1-2\varepsilon \leqslant \big\|\widetilde{u}(\zeta) \oplus \widetilde{\psi}(\zeta)v(\zeta)\big\| \leqslant 1+2\varepsilon \quad (\zeta \in \beta_2) \tag{19}$$

and

$$\big|\big\langle \widetilde{u}(\zeta) \oplus \widetilde{\psi}(\zeta)v(\zeta), u(\zeta) \oplus v(\zeta)\big\rangle\big| \geqslant \eta_2 - 2\varepsilon \geqslant \eta_1^2 - 2\varepsilon \quad (\zeta \in \beta_2). \tag{20}$$

Notice that $1 - 2\varepsilon > \eta_1^2 - 2\varepsilon > 0$ by (1). In virtue of (17) and (19) there exists an outer function $\varphi \in H^\infty$ with the property

$$|\varphi| = [\widetilde{u} \oplus \widetilde{\psi} v].$$

Defining $u^\# \in H^2(\mathfrak{E}_*)$ and the measurable function ψ by

$$u^\# := \varphi^{-1}\widetilde{u} \quad \text{and} \quad \psi := \varphi^{-1}\widetilde{\psi},$$

the equation $[u^\# \oplus \psi v] \equiv 1$ is clearly fulfilled. Finally, the relations (17)–(20) and (1) readily imply that

$$\left|\left\langle u^\#(\zeta) \oplus \psi(\zeta)v(\zeta), u(\zeta) \oplus v(\zeta)\right\rangle\right| \geqslant \left(\eta_1^2 - 2\varepsilon\right)\left|\varphi(\zeta)\right|^{-1} \geqslant \left(\eta_1^2 - 2\varepsilon\right)(1 + 2\varepsilon)^{-1} \geqslant \eta$$

is true for every $\zeta \in \mathbb{T}$. Thus the proof is complete. □

Since we shall work with vectors approximating an orthonormal system, we need a statement which describes how perturbation of an isometry on elements of an orthonormal basis affects the norm and the lower bound of the operator. Such a statement is the content of the following lemma taken from [14].

Lemma 4. *Let $1 \leqslant n \leqslant \aleph_0$ be a cardinal number, let $\{g_i\}_{0 \leqslant i < n}$ be an orthonormal basis in the Hilbert space $\mathfrak{G}_n$, and let $\{f_i\}_{0 \leqslant i < n}$ be an orthonormal system in a Hilbert space $\mathfrak{F}$. Let us give constants $0 < \delta < c < 1$ and a sequence $\{\delta_i\}_{0 \leqslant i < n}$ of positive numbers satisfying the condition $\sum_{0 \leqslant i < n} \delta_i^2 \leqslant \delta^2$. For any $0 \leqslant i < n$, let $f_i^\# \in \mathfrak{F}$ be a vector of the form*

$$f_i^\# = c_i f_i + s_i, \quad \text{where } c \leqslant |c_i| \leqslant 1 \text{ and } \|s_i\| \leqslant \delta_i.$$

Then there exists a uniquely determined transformation $A \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{F})$ such that $Ag_i = f_i^\#$ holds, for every $0 \leqslant i < n$. Furthermore, for this transformation A we have

$$c - \delta \leqslant \Lambda(A) \leqslant \|A\| \leqslant 1 + \delta.$$

We remind the reader that the lower bound $\Lambda(A)$ of A is defined by $\Lambda(A) := \inf\{\|Ax\|\colon x \in \mathfrak{G}_n,\ \|x\| = 1\}$.

6. Proof and supplement

Now we are ready to prove Theorem 1 for the model operator $T = S(\Theta)$.

Proposition 5. *The statement of Theorem 1 is true for the c.n.u. contraction $T = S(\Theta)$.*

Proof. Let us fix a cardinal number $1 \leqslant n \leqslant \aleph_0$, and let us assume that $m(\rho_{T,n}) > 0$. Recall that $\rho_{T,n} = \{\zeta \in \mathbb{T}\colon \operatorname{rank} \Delta_*(\zeta) \geqslant n\}$. For simplicity, we shall use the notation $\gamma := \rho_{T,n}$. Let us give an arbitrary $\varepsilon > 0$.

Let $\{u_i\}_{0 \leqslant i < n}$ and $\{v_i\}_{0 \leqslant i < n}$ be measurable vector-valued functions obtained by applying Proposition 2. We extend these functions to the whole circle $\mathbb{T}$ in the following way. Given any orthonormal system $\{e_{*i}\}_{0\leqslant i<n}$ in $\mathfrak{E}_*$, let $u_i(\zeta) := e_{*i}$ and $v_i(\zeta) := 0$, for every $\zeta \in \mathbb{T} \setminus \gamma$ and $0 \leqslant i < n$. It is clear that $u_i \oplus v_i \in \mathfrak{K}$ for every $0 \leqslant i < n$. Furthermore, $\{u_i(\zeta) \oplus v_i(\zeta)\}_{0\leqslant i<n}$ forms an orthonormal system for every $\zeta \in \mathbb{T}$, and

$$\big\{k_i(\zeta) := -\Delta_*(\zeta)u_i(\zeta) + \Theta(\zeta)v_i(\zeta)\big\}_{0\leqslant i<n} \subset \mathfrak{D}_*(\zeta) \tag{21}$$

is also an orthonormal system for every $\zeta \in \gamma$.

Let us give constants $0 < \delta < c < 1$ and a sequence $\{\delta_i\}_{0\leqslant i<n}$ of positive numbers with the property $\sum_{0\leqslant i<n} \delta_i^2 \leqslant \delta^2$. Applying Lemma 3 we obtain that, for every $0 \leqslant i < n$, there exist $u_i^\# \in H^2(\mathfrak{E}_*)$ and a measurable complex function ψ_i on $\mathbb{T}$ such that

$$\eta_i \leqslant \big|\big[u_i^\# \oplus \psi_i v_i, u_i \oplus v_i\big]\big| \leqslant \big[u_i^\# \oplus \psi_i v_i\big] \equiv 1 \tag{22}$$

holds with $\eta_i := \max(c, (1-\delta_i^2)^{1/2})$. Then $u_i^\# \oplus \psi_i v_i \in \mathfrak{K}_+$ is clearly true for every $0 \leqslant i < n$. In view of (22), these functions can be written in the form

$$u_i^\#(\zeta) \oplus \psi_i(\zeta)v_i(\zeta) = c_i(\zeta)\big(u_i(\zeta) \oplus v_i(\zeta)\big) + \big(r_i(\zeta) \oplus s_i(\zeta)\big), \tag{23}$$

where

$$c \leqslant \eta_i \leqslant \big|c_i(\zeta)\big| \leqslant 1 \quad (\zeta \in \mathbb{T},\ 0 \leqslant i < n) \tag{24}$$

and

$$\big\|r_i(\zeta) \oplus s_i(\zeta)\big\|^2 = 1 - \big|c_i(\zeta)\big|^2 \leqslant 1 - \eta_i^2 \leqslant \delta_i^2 \quad (\zeta \in \mathbb{T},\ 0 \leqslant i < n). \tag{25}$$

Let us fix an orthonormal basis $\{g_i\}_{0\leqslant i<n}$ in $\mathfrak{G}_n$. Given any $\zeta \in \mathbb{T}$, in virtue of (23)–(25), Lemma 4 implies the existence of a uniquely determined transformation $\Phi(\zeta) \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{E}_* \oplus \mathfrak{E})$ satisfying the condition

$$\Phi(\zeta)g_i = u_i^\#(\zeta) \oplus \psi_i(\zeta)v_i(\zeta) \quad \text{for every } 0 \leqslant i < n;$$

furthermore,

$$c - \delta \leqslant \Lambda\big(\Phi(\zeta)\big) \leqslant \big\|\Phi(\zeta)\big\| \leqslant 1 + \delta. \tag{26}$$

We shall write $\chi_{\mathbb{T}} g_i$ for the constant function in $H^2(\mathfrak{G}_n)$ with value g_i. Since $\Phi(\chi_{\mathbb{T}} g_i) = u_i^\# \oplus \psi_i v_i$ is a measurable vector-valued function for every $0 \leqslant i < n$, it follows that the bounded transformation-valued function Φ is measurable (see [5, Section II.2.1]). The transformation $M(\Phi)$ of multiplication by Φ maps $L^2(\mathfrak{G}_n)$ into $\mathfrak{K}$, and clearly $M(\Phi)H^2(\mathfrak{G}_n) \subset \mathfrak{K}_+$. Let us consider the restriction

$$Y_+ := M(\Phi) \,\big|\, H^2(\mathfrak{G}_n) \in \mathcal{L}\big(H^2(\mathfrak{G}_n), \mathfrak{K}_+\big).$$

It is evident that

$$Y_+ S_n = U_+ Y_+ \quad \text{and} \quad Y_+(\chi_{\mathbb{T}} g_i) = u_i^\# \oplus \psi_i v_i \ (0 \leqslant i < n). \tag{27}$$

Let $\widetilde{P}_+ \in \mathcal{L}(\mathfrak{K}_+, \mathfrak{H})$ stand for the transformation defined by $\widetilde{P}_+ x := P_+ x$ $(x \in \mathfrak{K}_+)$, where P_+ is the orthogonal projection onto $\mathfrak{H}$ in $\mathfrak{K}_+$. We know from [19, Theorem I.4.1] that

$$\widetilde{P}_+ U_+ = T \widetilde{P}_+. \tag{28}$$

Now the transformation $Y \in \mathcal{L}(H^2(\mathfrak{G}_n), \mathfrak{H})$ is defined by

$$Y := \widetilde{P}_+ Y_+. \tag{29}$$

The relations (27)–(29) result in that

$$Y S_n = T Y. \tag{30}$$

Furthermore, in view of (26) we obtain that

$$\|Y\| \leqslant \|Y_+\| \leqslant \|\Phi\|_\infty \leqslant 1 + \delta. \tag{31}$$

Let us consider the vector-valued functions

$$h_i := Y(\chi_{\mathbb{T}} g_i) \in \mathfrak{H} \quad \text{and} \quad k_i^{\#} := \widetilde{X}_T h_i \in \widetilde{\mathfrak{R}}_{*T} \quad (0 \leqslant i < n). \tag{32}$$

Let $\widetilde{\mathfrak{R}}_{*+}$ be the reducing subspace of $\widetilde{R}_{*T}$ generated by the vectors $\{\chi_\gamma k_i^{\#}\}_{0 \leqslant i < n}$, and let us consider the restriction $\widetilde{R}_{*+} := \widetilde{R}_{*T} | \widetilde{\mathfrak{R}}_{*+}$. Let $\widetilde{Q}_+ \in \mathcal{L}(\widetilde{\mathfrak{R}}_{*T}, \widetilde{\mathfrak{R}}_{*+})$ stand for the transformation defined by $\widetilde{Q}_+ x := Q_+ x$ $(x \in \widetilde{\mathfrak{R}}_{*T})$, where Q_+ is the orthogonal projection onto $\widetilde{\mathfrak{R}}_{*+}$ in $\widetilde{\mathfrak{R}}_{*T}$. Then clearly

$$\widetilde{Q}_+ \widetilde{R}_{*T} = \widetilde{R}_{*+} \widetilde{Q}_+ \quad \text{and} \quad \widetilde{Q}_+ k_i^{\#} = \chi_\gamma k_i^{\#} \quad (0 \leqslant i < n). \tag{33}$$

Introducing the transformation $\widetilde{X}_+ : \mathfrak{K}_+ \to \widetilde{\mathfrak{R}}_{*T}$ defined by

$$\widetilde{X}_+(u \oplus v) := -\Delta_* u + \Theta v \quad (u \oplus v \in \mathfrak{K}_+),$$

we can see from the equation $\Theta \Delta = \Delta_* \Theta$ that $\ker \widetilde{X}_+ \supset \mathfrak{G}_+$. Consequently

$$k_i^{\#} = \widetilde{X}_T \widetilde{P}_+ \big(u_i^{\#} \oplus \psi_i v_i\big) = \widetilde{X}_+ \big(u_i^{\#} \oplus \psi_i v_i\big) = -\Delta_* u_i^{\#} + \Theta(\psi_i v_i) \quad (0 \leqslant i < n). \tag{34}$$

We infer by (21) and (23) that for any $0 \leqslant i < n$ the function $k_i^{\#}$ is of the following form on the set γ:

$$k_i^{\#}(\zeta) = c_i(\zeta) k_i(\zeta) + y_i(\zeta), \quad \text{where } y_i(\zeta) := -\Delta_*(\zeta) r_i(\zeta) + \Theta(\zeta) s_i(\zeta) \ (\zeta \in \gamma). \tag{35}$$

Recalling that the operator $F(\zeta)$ in Section 2 is an isometry, it follows by (25) that

$$\big\|y_i(\zeta)\big\| \leqslant \big\|r_i(\zeta) \oplus s_i(\zeta)\big\| \leqslant \delta_i \quad (\zeta \in \gamma,\ 0 \leqslant i < n). \tag{36}$$

Taking into account that $\{k_i(\zeta)\}_{0\leqslant i<n}$ is an orthonormal system for $\zeta \in \gamma$, Lemma 4 yields by (35), (36) and (24) that, for any $\zeta \in \gamma$, there exists a (unique) transformation $\Psi(\zeta) \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{E}_*)$ such that

$$\Psi(\zeta) g_i = k_i^{\#}(\zeta) \quad \text{for every } 0 \leqslant i < n; \tag{37}$$

furthermore,

$$c - \delta \leqslant \Lambda\big(\Psi(\zeta)\big) \leqslant \big\|\Psi(\zeta)\big\| \leqslant 1 + \delta. \tag{38}$$

For any $\zeta \in \mathbb{T} \setminus \gamma$ let us set $\Psi(\zeta) := 0 \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{E}_*)$. It can be easily verified, as before for Φ, that the bounded function Ψ is measurable. The transformation $M(\Psi)$ of multiplication by Ψ maps $L^2(\mathfrak{G}_n, \gamma)$ into $\widetilde{\mathfrak{R}}_{*+}$; let us consider the mapping

$$Z_+ := M(\Psi) \,\big|\, L^2(\mathfrak{G}_n, \gamma) \in \mathcal{L}\big(L^2(\mathfrak{G}_n, \gamma), \widetilde{\mathfrak{R}}_{*+}\big).$$

It is evident that

$$Z_+(\chi_\gamma g_i) = \chi_\gamma k_i^{\#} \quad \text{for every } 0 \leqslant i < n. \tag{39}$$

We infer by (38) that

$$c - \delta \leqslant \Lambda(Z_+) \leqslant \|Z_+\| \leqslant 1 + \delta. \tag{40}$$

Since Z_+ has dense range by (39), we obtain that Z_+ is invertible, and so (40) yields

$$\big\|Z_+^{-1}\big\| \leqslant (c - \delta)^{-1}. \tag{41}$$

Taking into account that $M_{T,n} = \widetilde{M}_{n,\gamma}$, we can see that

$$Z_+ M_{T,n} = \widetilde{R}_{*+} Z_+. \tag{42}$$

Now, the transformation $Z \in \mathcal{L}(\mathfrak{H}, L^2(\mathfrak{G}_n, \gamma))$ is defined by

$$Z := Z_+^{-1} \widetilde{Q}_+ \widetilde{X}_T. \tag{43}$$

The intertwining relations $\widetilde{X}_T T = \widetilde{R}_{*T} \widetilde{X}_T$, (33) and (42) yield that

$$ZT = M_{T,n} Z. \tag{44}$$

Taking into account that the mappings $\widetilde{Q}_+$ and $\widetilde{X}_T$ are contractions, it follows from (41) that

$$\|Z\| \leqslant (c - \delta)^{-1}. \tag{45}$$

Choosing the constants δ and c sufficiently close to 0 and 1, respectively, it can be achieved that $1 + \delta < 1 + \varepsilon$ and $(c - \delta)^{-1} < 1 + \varepsilon$ hold. Hence, by the inequalities (31) and (45) we conclude that

$$\|Y\| < 1 + \varepsilon \quad \text{and} \quad \|Z\| < 1 + \varepsilon. \tag{46}$$

Finally, in view of (32), (33), (39) and (43) we have for any $0 \leqslant i < n$ that

$$\begin{aligned} ZY(\chi_{\mathbb{T}} g_i) &= Zh_i = Z_+^{-1} \widetilde{Q}_+ \widetilde{X}_T h_i = Z_+^{-1} \widetilde{Q}_+ k_i^{\#} = Z_+^{-1} \chi_\gamma k_i^{\#} \\ &= \chi_\gamma g_i = \widetilde{J}_{n,\gamma}(\chi_{\mathbb{T}} g_i) = J_{T,n}(\chi_{\mathbb{T}} g_i). \end{aligned} \tag{47}$$

Since ZY and $J_{T,n}$ intertwine S_n with $M_{T,n}$, equalities (47) imply

$$ZY = J_{T,n}. \tag{48}$$

The relations (46) and (48) show that the mappings Y and Z possess all the required properties. $\square$

Now we complete the proof of the main result.

Proof of Theorem 1. Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$. Let us give a cardinal number $1 \leqslant n \leqslant \aleph_0$ and a positive ε.

The contraction T can be decomposed into the orthogonal sum $T = T_u \oplus T_c$, where T_u is an a.c. unitary operator and T_c is a c.n.u. contraction. It is known (see e.g. [4, Section IX.10]) that $T_u = W_{T_u}$ is unitarily equivalent to the orthogonal sum $\bigoplus_{k\in\mathbb{N}} M_{\alpha_k}$, where $\alpha_k := \rho_{T_u,k}$ and $M_{\alpha_k} = \widetilde{M}_{1,\alpha_k}$ $(k \in \mathbb{N})$. Let $Q_u \in \mathcal{I}(\bigoplus_{k\in\mathbb{N}} M_{\alpha_k}, T_u)$ be a unitary transformation; then

$$\widetilde{Q} := Q_u \oplus I \in \mathcal{I}\biggl(\biggl(\bigoplus_{k\in\mathbb{N}} M_{\alpha_k}\biggr) \oplus T_c, T_u \oplus T_c\biggr)$$

is also unitary. Given an arbitrary $0 < c < 1$, for every $k \in \mathbb{N}$, let $\vartheta_k \in H^\infty$ be an outer function with absolute value $|\vartheta_k| = c\chi_{\alpha_k} + \chi_{\mathbb{T}\setminus\alpha_k}$, and let us consider the c.n.u. contraction $T_k = S(\vartheta_k)$. By [19, Theorem IX.1.2] there exists an affinity $Q_k \in \mathcal{I}(T_k, M_{\alpha_k})$ satisfying the conditions

$$1 = \|Q_k\| \leqslant \|Q_k^{-1}\| \leqslant c^{-1} \quad (k \in \mathbb{N}).$$

The c.n.u. contraction $\widetilde{T} := (\bigoplus_{k\in\mathbb{N}} T_k) \oplus T_c$ is unitarily equivalent to a model operator $T' = S(\Theta)$ by [19, Theorem VI.2.3], let $Q' \in \mathcal{I}(T', \widetilde{T})$ be a unitary transformation. Then the affinity

$$Q := \widetilde{Q}\biggl(\biggl(\bigoplus_{k\in\mathbb{N}} Q_k\biggr) \oplus I\biggr)Q'$$

has the properties

$$QT' = TQ \quad \text{and} \quad 1 = \|Q\| \leqslant \|Q^{-1}\| \leqslant c^{-1}. \tag{49}$$

Clearly, $\mu_{T_k} = \chi_{\alpha_k}$ holds for every $k \in \mathbb{N}$, and so the asymptotic spectral-multiplicity functions of the contractions T and T' coincide: $\mu_T = \mu_{T'}$. Therefore

$$M_{T,n} = M_{T',n} \quad \text{and} \quad J_{T,n} = J_{T',n}. \tag{50}$$

Given an arbitrary $0 < \delta < 1$, Proposition 5 provides us with mappings $Y' \in \mathcal{I}(S_n, T')$ and $Z' \in \mathcal{I}(T', M_{T',n})$ satisfying the conditions

$$Z'Y' = J_{T',n} \quad \text{and} \quad \|Y'\| < 1 + \delta, \ \|Z'\| < 1 + \delta. \tag{51}$$

Then $Y := QY' \in \mathcal{I}(S_n, T)$, $Z := Z'Q^{-1} \in \mathcal{I}(T, M_{T,n})$, and we conclude by (49)–(51) that

$$ZY = Z'Y' = J_{T',n} = J_{T,n}$$

and

$$\|Y\| < 1 + \delta, \qquad \|Z\| < (1 + \delta)c^{-1}.$$

Choosing δ and c sufficiently close to 0 and 1, respectively, we can ensure that $1 + \delta < 1 + \varepsilon$ and $(1 + \delta)c^{-1} < 1 + \varepsilon$. The proof is complete. $\square$

We supplement the statement of Theorem 1 by showing that every factorization of any embedding $\widetilde{J}_{n,\alpha}$ through intertwining mappings with the contraction T is necessarily attached to the set $\rho_{T,n}$.

Proposition 6. *Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$. Let us give a cardinal number $1 \leqslant n \leqslant \aleph_0$ and a Borel set α on the unit circle $\mathbb{T}$. If there exist transformations $Y \in \mathcal{I}(S_n, T)$ and $Z \in \mathcal{I}(T, \widetilde{M}_{n,\alpha})$ with the property $ZY = \widetilde{J}_{n,\alpha}$, then α is a.e. contained in $\rho_{T,n}$, that is $m(\alpha \setminus \rho_{T,n}) = 0$.*

Proof. By the universal property of (X_T, W_T) there exists a unique transformation $L \in \mathcal{I}(W_T, \widetilde{M}_{n,\alpha})$ such that $Z = LX_T$. Since W_T and $\widetilde{M}_{n,\alpha}$ are unitaries it follows that $L \in \mathcal{I}(W_T^*, \widetilde{M}_{n,\alpha}^*)$ is also true. Hence the subspace $\mathfrak{K}_1 := \ker L$ is reducing for W_T, that is $W_T = W_0 \oplus W_1$ in the decomposition $\mathfrak{K}_T = \mathfrak{K}_0 \oplus \mathfrak{K}_1$. Taking into account that $L\mathfrak{K}_T \supset Z\mathfrak{H} \supset \widetilde{J}_{n,\alpha} H^2(\mathfrak{G}_n)$ and that $(L\mathfrak{K}_T)^-$ reduces $\widetilde{M}_{n,\alpha}$, we infer that L has dense range. Considering the polar decomposition $L = V_L|L|$ of L, one can easily check that the unitary transformation $V_0 := V_L|\mathfrak{K}_0 \in \mathcal{L}(\mathfrak{K}_0, L^2(\mathfrak{G}_n, \alpha))$ intertwines W_0 with $\widetilde{M}_{n,\alpha}$ (see the proof of [19, Proposition II.3.4]).

Let us assume that W_1 is unitarily equivalent to the model operator $\bigoplus_{k \in \mathbb{N}} M_{\beta_k}$, where $\{\beta_k\}_{k \in \mathbb{N}}$ is a decreasing sequence of Borel subsets of $\mathbb{T}$. Then it is easy to verify the following unitary equivalence relations:

$$W_T \simeq \widetilde{M}_{n,\alpha} \oplus \left(\bigoplus_{k \in \mathbb{N}} M_{\beta_k} \right) \simeq \bigoplus_{k \in \mathbb{N}} M_{\alpha_k},$$

where the decreasing sequence $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{B}_{\mathbb{T}}$ is defined by

$$\alpha_k := \alpha \cup \beta_k \quad \text{for all } k \in \mathbb{N}$$

if $n = \aleph_0$, while in the case $n < \aleph_0$ we have

$$\alpha_k := \begin{cases} \alpha \cup \beta_k & \text{for } 1 \leqslant k \leqslant n, \\ \beta_k \cup (\beta_{k-n} \cap \alpha) & \text{for } n < k. \end{cases}$$

(See e.g. the proof of [19, Theorem II.7.4].) We conclude that $\mu_T(\zeta) \geqslant n$ holds for a.e. $\zeta \in \alpha$, and so $m(\alpha \setminus \rho_{T,n}) = 0$. $\square$

7. Shift-type invariant subspaces

If T is an a.c. contraction then, for any $1 \leqslant n \leqslant \aleph_0$ and $\varepsilon > 0$, let $\mathcal{Y}(T, n, \varepsilon)$ stand for the set of those mappings $Y \in \mathcal{I}(S_n, T)$ which satisfy the conditions

$$ZY = J_{T,n}, \qquad \|Y\| < 1 + \varepsilon, \qquad \|Z\| < 1 + \varepsilon$$

with an appropriate $Z \in \mathcal{I}(T, M_{T,n})$ (depending on Y). We know that $J_{T,n}$ is one-to-one, and then so is every $Y \in \mathcal{Y}(T, n, \varepsilon)$, whenever $m(\rho_{T,n}) > 0$. The following proposition states that the ranges of the transformations in $\mathcal{Y}(T, n, \varepsilon)$ together span the whole space of T. (Though a modified version of this section is contained in [14], we present here a more streamlined discussion for the sake of completeness.)

Proposition 7. *Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$, and let us assume that $m(\rho_{T,n}) > 0$ holds for a cardinal number $1 \leqslant n \leqslant \aleph_0$.*

(a) *For every $\varepsilon > 0$ we have*

$$\bigvee \big\{ Y H^2(\mathfrak{G}_n)\colon\ Y \in \mathcal{Y}(T, n, \varepsilon) \big\} = \mathfrak{H}.$$

(b) *If $n = \aleph_0$ then, for every $\varepsilon > 0$, there exist $Y, \widehat{Y} \in \mathcal{Y}(T, n, \varepsilon)$ such that*

$$Y H^2(\mathfrak{G}_n) \vee \widehat{Y} H^2(\mathfrak{G}_n) = \mathfrak{H}.$$

Proof. It is sufficient to verify the statement for the model operator $T = S(\Theta)$. (See the analogous reduction in the proof of Theorem 1.)

For every $0 \leqslant i < n$, setting an arbitrary vector $e_{*i} \in \mathfrak{E}_*$ with $\|e_{*i}\| \leqslant \sqrt{2}/2$, a vector-valued function $v_i \in (\Delta L^2(\mathfrak{E}))^-$ with $[v_i] \leqslant \sqrt{2}/2$, a non-negative integer k_i and an integer l_i, let us consider the vectors

$$\hat{u}_i := \chi^{k_i} e_{*i} \in H^2(\mathfrak{E}_*) \quad \text{and} \quad \hat{v}_i := \chi^{l_i} v_i \in \big(\Delta L^2(\mathfrak{E})\big)^-,$$

where $\chi(\zeta) = \zeta$ ($\zeta \in \mathbb{T}$). It is clear that $\hat{u}_i \oplus \hat{v}_i \in \mathfrak{K}_+$ and $[\hat{u}_i \oplus \hat{v}_i] \leqslant 1$ ($0 \leqslant i < n$).

Fixing a positive ε, let us give the constants $0 < \delta < c < 1$ and the sequences $\{\delta_i\}_{0 \leqslant i < n}$, $\{\eta_i\}_{0 \leqslant i < n}$ as in the proof of Proposition 5. For every $0 \leqslant i < n$, one can choose positive numbers $\widehat{\eta}_i$ and ε_i so that

$$\eta_i < \widehat{\eta}_i < 1 \quad \text{and} \quad \eta_i < (1 - \varepsilon_i)\widehat{\eta}_i - \varepsilon_i.$$

Let $\{u_i^\# \oplus \psi_i v_i\}_{0 \leqslant i < n} \subset \mathfrak{K}_+$ be a sequence satisfying the condition

$$(\eta_i <)\ \widehat{\eta}_i \leqslant \big|\big[u_i^\# \oplus \psi_i v_i, u_i \oplus v_i\big]\big| \leqslant \big[u_i^\# \oplus \psi_i v_i\big] \equiv 1 \quad (0 \leqslant i < n)$$

instead of (22). Since

$$\big[(1-\varepsilon_i)\big(u_i^{\#}\oplus\psi_i v_i\big)+\varepsilon_i(\hat{u}_i\oplus\hat{v}_i)\big]\leqslant 1$$

and

$$\eta_i<(1-\varepsilon_i)\widehat{\eta}_i-\varepsilon_i\leqslant\big|\big[(1-\varepsilon_i)\big(u_i^{\#}\oplus\psi_i v_i\big)+\varepsilon_i(\hat{u}_i\oplus\hat{v}_i),u_i\oplus v_i\big]\big|\leqslant 1,$$

we infer that

$$\begin{aligned}&(1-\varepsilon_i)\big(u_i^{\#}(\zeta)\oplus\psi_i(\zeta)v_i(\zeta)\big)+\varepsilon_i\big(\hat{u}_i(\zeta)\oplus\hat{v}_i(\zeta)\big)\\&\quad=\hat{c}_i(\zeta)\big(u_i(\zeta)\oplus v_i(\zeta)\big)+\big(\hat{r}_i(\zeta)\oplus\hat{s}_i(\zeta)\big)\end{aligned}$$

holds for every $\zeta\in\mathbb{T}$ and $0\leqslant i<n$, where

$$c\leqslant\eta_i\leqslant|\hat{c}_i(\zeta)|\leqslant 1\quad\text{and}\quad\big\|\hat{r}_i(\zeta)\oplus\hat{s}_i(\zeta)\big\|\leqslant\big(1-\eta_i^2\big)^{1/2}\leqslant\delta_i.$$

We note that the relations (22)–(25) are also valid. The procedure described in the proof of Proposition 5 yields transformations Y and $\widehat{Y}$ in $\mathcal{Y}(T,n,\varepsilon)$ such that

$$Y(\chi_{\mathbb{T}}g_i)=P_+\big(u_i^{\#}\oplus\psi_i v_i\big)$$

and

$$\widehat{Y}(\chi_{\mathbb{T}}g_i)=(1-\varepsilon_i)P_+\big(u_i^{\#}\oplus\psi_i v_i\big)+\varepsilon_i P_+(\hat{u}_i\oplus\hat{v}_i)$$

are true for every $0\leqslant i<n$. Thus

$$\widehat{Y}(\chi_{\mathbb{T}}g_i)-(1-\varepsilon_i)Y(\chi_{\mathbb{T}}g_i)=\varepsilon_i P_+(\hat{u}_i\oplus\hat{v}_i)\quad(0\leqslant i<n).$$

If $n=\aleph_0$ then we can choose the sequence $\{\hat{u}_i\oplus\hat{v}_i\}_{0\leqslant i<n}$ to be total in $\mathfrak{K}_+$, and so (b) is obviously fulfilled. If $n<\aleph_0$ then a sequence of finite sequences

$$\big\{\big\{\hat{u}_i^{(k)}\oplus\hat{v}_i^{(k)}\big\}_{0\leqslant i<n}\colon k\in\mathbb{N}\big\}$$

can be chosen to be total in $\mathfrak{K}_+$. Denoting by $\widehat{Y}_k$ $(k\in\mathbb{N})$ the resulting transformations in $\mathcal{Y}(T,n,\varepsilon)$, we obtain that the subspaces $\{YH^2(\mathfrak{G}_n)\vee\widehat{Y}_kH^2(\mathfrak{G}_n)\}_{k\in\mathbb{N}}$ together span the whole space $\mathfrak{H}$, which proves (a). $\square$

If $\rho_{T,n}$ coincides with the whole circle $\mathbb{T}$, or more precisely, if $m(\rho_{T,n})=1$ $(=m(\mathbb{T}))$ then the embedding $J_{T,n}$ is an isometry, and so the conditions

$$ZY=J_{T,n},\qquad\|Y\|<1+\varepsilon,\qquad\|Z\|<1+\varepsilon$$

imply that

$$\Lambda(Y)>(1+\varepsilon)^{-1}.$$

Therefore, the restriction $T|YH^2(\mathfrak{G}_n)$ is similar to the unilateral shift S_n, and the intertwining affinity $Y_0 \in \mathcal{I}(S_n, T|YH^2(\mathfrak{G}_n))$, defined by $Y_0 g := Yg$, is close to unitary if ε is small. For an a.c. contraction T, for any $1 \leqslant n \leqslant \aleph_0$ and $\varepsilon > 0$, $\mathrm{Lat}(T, n, \varepsilon)$ stands for the set of those invariant subspaces $\mathfrak{M}$ of T, where the restriction $T|\mathfrak{M}$ is similar to S_n, and the similarity can be implemented by an affinity $Q \in \mathcal{I}(S_n, T|\mathfrak{M})$ with the properties

$$(1+\varepsilon)^{-1} < \Lambda(Q) \leqslant \|Q\| < 1 + \varepsilon.$$

We have seen that

$$\big\{YH^2(\mathfrak{G}_n)\colon\ Y \in \mathcal{Y}(T, n, \varepsilon)\big\} \subset \mathrm{Lat}(T, n, \varepsilon) \tag{52}$$

provided $m(\rho_{T,n}) = 1$. In view of Proposition 7 and (52) we obtain the following statement.

Theorem 8. *Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$, and let us assume that $m(\rho_{T,n}) = 1$ holds for a cardinal number $1 \leqslant n \leqslant \aleph_0$.*

(a) *For every $\varepsilon > 0$, the subspaces in $\mathrm{Lat}(T, n, \varepsilon)$ span the whole space $\mathfrak{H}$:*

$$\bigvee \mathrm{Lat}(T, n, \varepsilon) = \mathfrak{H}.$$

(b) *If $n = \aleph_0$ then, for every $\varepsilon > 0$, there exist two subspaces $\mathfrak{M}, \widehat{\mathfrak{M}} \in \mathrm{Lat}(T, n, \varepsilon)$ such that*

$$\mathfrak{M} \vee \widehat{\mathfrak{M}} = \mathfrak{H}.$$

8. Hyperinvariant subspace problem

We recall that the hyperinvariant subspace lattice $\mathrm{Hlat}\, A$ of an operator $A \in \mathcal{L}(\mathfrak{A})$ consists of those subspaces which are invariant for every operator in the commutant $\{A\}'$ of A. The *hyperinvariant subspace problem* asks whether every Hilbert space operator $A \in \mathcal{L}(\mathfrak{A})$, which is not scalar multiple of the identity, has a proper hyperinvariant subspace, that is $\mathrm{Hlat}\, A \neq \{\{0\}, \mathfrak{A}\}$ holds. The positive answer is known only under additional assumptions, for example, in the class of normal operators because of the Spectral Theorem, or in the class of compact operators by the celebrated Lomonosov theorem (see e.g. [4, Section VI.4]). Existence of proper hyperinvariant subspaces was proved in [16] under an orbit condition for asymptotically non-vanishing operators of regular norm-sequence.

Let T be an arbitrary asymptotically non-vanishing contraction on the Hilbert space $\mathfrak{H}$. It is known that T can be decomposed into the orthogonal sum $T = T_a \oplus U_s$ of an a.c. contraction T_a and a singular unitary operator U_s. Taking into account that the minimal unitary dilation of T_a is a.c., we infer by the Lifting theorem (see [19, Theorem II.2.3]) that the intertwining sets $\mathcal{I}(T_a, U_s)$ and $\mathcal{I}(U_s, T_a)$ consist only of the zero transformation. Hence the commutant of T splits into the direct sum of the commutants of T_a and U_s: $\{T\}' = \{T_a\}' \oplus \{U_s\}'$, and then the same is true for the hyperinvariant subspace lattices too: $\mathrm{Hlat}\, T = \mathrm{Hlat}\, T_a \oplus \mathrm{Hlat}\, U_s$. Thus, in the quest for proper hyperinvariant subspaces we may assume that the asymptotically non-vanishing contraction T is absolutely continuous.

Let us consider the residual set $\rho_T = \rho_{T,1}$ of T. Since ρ_T is of positive Lebesgue measure, there exists a point $\zeta_0 \in \mathbb{T}$ which is of full density for ρ_T. Replacing T by $\bar{\zeta}_0 T$, we may assume that $\zeta_0 = 1$. (We recall that $\lim_{n\to\infty} m(E_n \cap \rho_T)/m(E_n) = 1$ holds whenever the sequence $\{E_n\}_{n=1}^{\infty} \subset \mathcal{B}_{\mathbb{T}}$ shrinks to 1 nicely, see [18, Chapter 8].) Let us consider the singular inner function $\vartheta \in H^\infty$ defined by

$$\vartheta(\lambda) = \exp\bigl[(\lambda+1)/(\lambda-1)\bigr] \quad (\lambda \in \mathbb{D}),$$

and let us form the operator $A := \vartheta(T)$. We know from [19, Section III.2] that A is also an a.c. contraction. Furthermore, by [17, Theorem 14] the residual set of A is $\rho_A = \vartheta(\rho_T)$. The following lemma ensures us that ρ_A essentially covers the whole circle $\mathbb{T}$.

Lemma 9. *If the point* $1 \in \mathbb{T}$ *is of full density for the set* $\alpha \in \mathcal{B}_{\mathbb{T}}$, *then* $m(\vartheta(\alpha)) = 1$.

Proof. Notice that ϑ is analytic on $\mathbb{C} \setminus \{1\}$ and

$$\vartheta\bigl(e^{it}\bigr) = \exp\bigl[-i\cot(t/2)\bigr], \quad t \in (0, 2\pi).$$

For any integer $n \in \mathbb{Z}$, let $t_n \in (0, 2\pi)$ be defined by $\cot(t_n/2) = 1 + n \cdot 2\pi$. It is clear that $\mathbb{T} \setminus \{1\}$ is the union of the disjoint arcs $\omega_n := \{e^{it}\colon\ t_{n+1} < t \leqslant t_n\}$ $(n \in \mathbb{Z})$, and that $\vartheta_n := \vartheta|\omega_n : \omega_n \to \mathbb{T}$ is a continuous bijection for every $n \in \mathbb{Z}$. So the set $\vartheta(\alpha) = \bigcup_{n\in\mathbb{Z}} \vartheta_n(\alpha \cap \omega_n)$ is measurable.

Let us consider the complement $\gamma = \mathbb{T} \setminus \vartheta(\alpha)$ and the Borel sets $\beta_n = \vartheta_n^{-1}(\gamma)$ $(n \in \mathbb{N})$. Taking into account that for any $(0 <)t_{n+1} < s_1 < s_2 \leqslant t_n (< t_0 = \pi/2)$ the inequality

$$\cot(s_1/2) - \cot(s_2/2) = (\sin s_*)^{-2}(s_2/2 - s_1/2) \leqslant 8t_{n+1}^{-2}(s_2 - s_1)$$

is valid, we can easily infer that

$$m(\beta_n) \geqslant \bigl(m(\gamma)/8\bigr)t_{n+1}^2 \quad (n \in \mathbb{N}). \tag{53}$$

Since the arcs $\widetilde{\omega}_n := \bigcup_{k=n}^{\infty} \omega_k = \{e^{it}\colon\ 0 < t \leqslant t_n\}$ $(n \in \mathbb{N})$ shrink to 1 nicely,

$$\lim_{n\to\infty} \frac{m(\widetilde{\omega}_n \cap \alpha)}{m(\widetilde{\omega}_n)} = 1 \tag{54}$$

must hold by the assumption. On the other hand, in view of (53) we have

$$\begin{aligned} \frac{m(\widetilde{\omega}_n \cap \alpha)}{m(\widetilde{\omega}_n)} &= \frac{1}{m(\widetilde{\omega}_n)} \sum_{k=n}^{\infty} m(\omega_k \cap \alpha) \leqslant 1 - \frac{1}{m(\widetilde{\omega}_n)} \sum_{k=n}^{\infty} m(\beta_k) \\ &\leqslant 1 - \frac{m(\gamma)}{2t_n} \sum_{k=n+1}^{\infty} t_k^2 \quad (n \in \mathbb{N}). \end{aligned} \tag{55}$$

Starting from the inequalities $1/(2s) \leqslant \cot s \leqslant 2/s$ $(s \in (0, \pi/4))$, one can easily derive that

$$1/(8n) \leqslant t_n \leqslant 1/n \quad (n \in \mathbb{N}). \tag{56}$$

The relations (55) and (56) together imply

$$\frac{m(\widetilde{\omega}_n \cap \alpha)}{m(\widetilde{\omega}_n)} \leqslant 1 - \frac{m(\gamma)}{128} \cdot n \sum_{k=n+1}^{\infty} k^{-2} \leqslant 1 - \frac{m(\gamma)}{128} \frac{n}{n+1} \quad (n \in \mathbb{N}). \tag{57}$$

Tending n to infinity in (57), we conclude that $m(\gamma) \leqslant 0$, that is $m(\gamma) = 0$. □

Thus Lemma 9 yields that $m(\rho_A) = 1$. For every $r \in (0, 1)$, we set $\vartheta_r(\lambda) := \vartheta(r\lambda)$ $(\lambda \in \mathbb{D})$. Since $\vartheta_r(T)$ is the norm-limit of polynomials of T, and $\vartheta_r(T)$ converges to $\vartheta(T)$ in the strong operator topology as r tends to 1, we obtain that every operator commuting with T will commute with $A = \vartheta(T)$ as well. Therefore $\{T\}' \subset \{A\}'$, whence

$$\operatorname{Hlat} T \supset \operatorname{Hlat} A$$

follows. Let us form the inflation $B = A^{(\aleph_0)}$ of A acting on the orthogonal sum $\mathfrak{H}^{(\aleph_0)}$ of infinitely many copies of $\mathfrak{H}$. Clearly, B is an a.c. contraction with $m(\rho_{B,\aleph_0}) = 1$. Furthermore, it can be easily verified that

$$\operatorname{Hlat} B = \bigl\{\mathfrak{M}^{(\aleph_0)}\colon\ \mathfrak{M} \in \operatorname{Hlat} A\bigr\}.$$

Thus $\operatorname{Hlat} T$ contains a sublattice which is isomorphic to $\operatorname{Hlat} B$, and so we have arrived at the following reduction theorem.

Theorem 10. *If every absolutely continuous contraction B with $m(\rho_{B,\aleph_0}) = 1$ has a proper hyperinvariant subspace, then so does every asymptotically non-vanishing* (*non-scalar*) *contraction T too.*

We note that the subspace $\ker X_T$ of vectors with vanishing orbits is clearly hyperinvariant for the asymptotically non-vanishing contraction T on $\mathfrak{H}$. Hence we may assume that $\ker X_T$, and $\ker X_{T^*}$ as well, are trivial subspaces. Since $\ker X_T \neq \mathfrak{H}$ we obtain that $\ker X_T = \{0\}$, and so T is a $C_{1\cdot}$-contraction according to the classification in [19, Section II.4]. If $\ker X_{T^*} = \{0\}$ holds true also, that is when T is a C_{11}-contraction, then a subset of $\operatorname{Hlat} T$ is isomorphic to $\operatorname{Hlat} W_T$ by [19, Propositions II.3.5 and II.5.1]. (For a more complete description of the hyperinvariant subspace lattices of C_{11}-contractions we refer to [19, Section VII.5], [10,13].) Thus we may assume that $\ker X_{T^*} = \mathfrak{H}$, and so T is a C_{10}-contraction. Then $A = \vartheta(T)$ is also a C_{10}-contraction (see [11, Lemma 5]), and so is $B = A^{(\aleph_0)}$ too. Therefore, we can concentrate on C_{10}-contractions.

Corollary 11. *If every C_{10}-contraction B with $m(\rho_{B,\aleph_0}) = 1$ has a non-trivial hyperinvariant subspace, then so does every asymptotically non-vanishing* (*non-scalar*) *contraction T too.*

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