series of functional equations of the form

$$\beta_0 H(x)+\beta_1 H(q^{k_1}x)+\cdots+\beta_s H(q^{k_s}x)=0,$$

where $1<k_1<k_2<\cdots<k_s$ are any integers (the β_k will of course be different here).

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ON A HYPOTHESIS PROPOSED BY B. V. GNEDENKO

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(*Summary*)

Several years ago Academician B. V. Gnedenko proposed the following:

Let $\zeta_n=(1/B_n)(\xi_1+\cdots+\xi_n)-A_n$ be a sequence of normed sums of independent stochastic quantities having a nondegenerate limit distribution $G(x)$ for appropriately selected constants A_n and B_n. If among the distributions of stochastic quantities ξ_i there are only s different ones, then the limit distribution $G(x)$ is a composition of not more than stable laws.

In the paper the hypothesis proposed by B. V. Gnedenko is proved for $s=2$ and an example is presented showing that the theorem by E. Lebedintseva [2] does not prove this hypothesis in its entirety.

ON THE MEAN NUMBER OF CROSSINGS OF A LEVEL BY A STATIONARY GAUSSIAN PROCESS

E. V. BULINSKAYA

(*Translated by D. Lieberman*)

In [1] and [2], formulas are given for the mean number of crossings of some level by a stationary Gaussian process, and in [3] a formula is given for the mean number of zeros, but without rigorous proof. In [4] a rigorous proof is given for a formula for the mean number of zeros in the rather special case of the Gaussian process

$$\xi(t)=\sum_{j=1}^{N} a_j(X_j\cos\lambda_j t+Y_j\sin\lambda_j t),$$

where the X_j and Y_j are independent Gaussian variables with mean 0 and dispersion 1.

We shall start by proving a general assertion which does not require the assumption that the process is Gaussian.

Theorem 1. *If the one-dimensional density of the process $\xi(t)$ is bounded and the derivative $\xi'(t)$ is continuous, with probability* 1, *then the number of crossings of the level u on the segment $[a, b]$ by the process $\xi(t)$ is finite, with probability* 1, *and moreover, the probability that $\xi(t)$ becomes tangent to the level u is zero.*

Proof. For simplicity we shall consider the segment $[0, 1]$ of values of t. Let $C^{(1)}[0, 1]$ be the space of functions which together with their derivatives are continuous on $[0, 1]$, and let $\mathbf{P}$ be the probability measure in $C^{(1)}[0, 1]$ corresponding to the process $\xi(t)$. Let $A_{h,n,k}$ denote the set of functions $x(t)\in C^{(1)}[0, 1]$ which have a zero derivative $x'(\tau_x)=0$ at at least one point τ_x in the interval $[(k-1)/n, k/n)$, $1\leqq k\leqq n$, and for which $|x(\tau_x)-u|\leqq h$. Then for $x(t)\in A_{h,n,k}$ we have

$$x\left(\frac{k}{n}\right)=x(\tau_x)+\left(\frac{k}{n}-\tau_x\right)x'\left[\tau_x+\theta\left(\frac{k}{n}-\tau_x\right)\right],$$

and consequently

$$\left| x\left(\frac{k}{n}\right) - u \right| \leqq h + \frac{1}{n}\,\omega_{x'}\left(\frac{1}{n}\right)$$

where $\omega_{x'}(\delta)$ is the modulus of continuity of the derivative, i. e.

$$\omega_{x'}(\delta) = \sup_{t', t'', |t'-t''| \leqq \delta} |x'(t') - x'(t'')|.$$

Furthermore, let $\omega(\delta) \downarrow 0$ for $\delta \downarrow 0$, and let B_ω be the set of functions $x(t) \in C^{(1)}[0, 1]$ for which the inequality $\omega_{x'}(\delta) \leqq \omega(\delta)$ is satisfied for all δ, $0 \leqq \delta \leqq 1$. We prescribe $\varepsilon > 0$. Then (see for example, [5], Chap. 1), we can choose the function $\omega_\varepsilon(\delta) \downarrow 0$, for $\delta \downarrow 0$, in such a way that $\mathbf{P}(B_{\omega_\varepsilon}) > 1 - \varepsilon/2$. If $A_h = \{x(t)$: at at least one point t_x of the segment $[0, 1]$, $x'(t_x) = 0$, $|x(t_x) - u| \leqq h\}$, then

$$A_h = \bigcup_{k=1}^{n} A_{h,n,k}, \tag{1}$$

$$\mathbf{P}(A_h) \leqq \sum_{k=1} \mathbf{P}(A_{h,n,k} \cap B_{\omega_\varepsilon}) + \mathbf{P}(\bar{B}_{\omega_\varepsilon}).$$

The first summand on the right-hand side of (1) does not exceed

$$cn\left[h + \frac{1}{n}\,\omega_\varepsilon\left(\frac{1}{n}\right)\right],$$

where c is a constant, bounding the one-dimensional density of $\xi(t)$. It is easy to see that for any $\varepsilon > 0$ we can select n_0 and h_0 such that the above quantity does not exceed $\varepsilon/2$, and since $\mathbf{P}(\bar{B}_{\omega_\varepsilon}) \leqq \varepsilon/2$, then $\mathbf{P}(A_{h_0}) \leqq \varepsilon$. If $A = \{x(t) : x(t)$ does not become tangent to the level u on the segment $[0, 1]\}$, then $\bar{A} \subseteq A_h$ for any h, and therefore, $\mathbf{P}(\bar{A}) = 0$. This proves that the process $\xi(t)$, with probability 1, does not become tangent to the level u on the segment $[0, 1]$. Now, for arbitrary $x(t) \in \bar{A}_{h_0}$, we shall estimate the distance l between two neighboring crossings of the level u with abscissas t_1 and $t_2 > t_1$. Between t_1 and t_2 there is a zero derivative, i. e. a point t_0 such that $x'(t_0) = 0$. By the selection of $x(t)$, the inequality $|x(t_0) - u| > h_0$ is satisfied. We draw a secant through the points $(t_0, x(t_0))$ and (t_2, u). Its slope is $\tan\varphi \geqq h_0/l$. There is a point θ between t_0 and t_2 such that $\tan\varphi = |x'(\theta)|$. But $|x'(\theta)| = |x'(\theta) - x'(t_0)|$, which does not exceed $\omega_{x'}(l)$. This means that for $x(t) \in \bar{A}_{h_0} \cap B_{\omega_\varepsilon}$, with probability not less than $1 - 3(\varepsilon/2)$, the inequality

$$l\omega_\varepsilon(l) \geqq h_0 \tag{2}$$

is satisfied.

We now estimate l, taking into account that

$$cn_0\left[h_0 + \frac{1}{n_0}\,\omega_\varepsilon\left(\frac{1}{n_0}\right)\right] \leqq \frac{\varepsilon}{2}.$$

This relation will be satisfied if we define n_0 by the inequality

$$\frac{1}{n_0} \leqq \omega_\varepsilon^{-1}\left(\frac{\varepsilon}{4c}\right) < \left(\frac{1}{n_0 - 1}\right)$$

and take $h_0 = \varepsilon/4n_0c$. Clearly,

$$\frac{1}{n_0} > \frac{1}{2}\,\omega_\varepsilon^{-1}\left(\frac{\varepsilon}{4c}\right)$$

and consequently,

$$h_0 \geqq \frac{\varepsilon}{8c}\,\omega_\varepsilon^{-1}\left(\frac{\varepsilon}{4c}\right).$$

Using (2), we obtain

$$l \geqq \psi^{-1}\left(\frac{\varepsilon}{8c}\,\omega_\varepsilon^{-1}\left(\frac{\varepsilon}{4c}\right)\right),$$

where $\psi(l) = l\omega_\varepsilon(l)$. Thus, the number of crossings of the level u on the segment $[0, 1]$ by the function $x(t) \in \bar{A}_{h_0} \cap B_{\omega_\varepsilon}$ does not exceed $n(\varepsilon) = 1/l$, i. e. $\mathbf{P}\{x(t) : N_x(u) \geqq n(\varepsilon)\} \leqq 3(\varepsilon/2)$, where $N_x(u)$ is the number of times $x(t)$ crosses the level u on the segment $[0, 1]$. Whence it follows that $\mathbf{P}\{x(t) : N_x(u) = \infty\} = 0$.

Now let $\xi(t)$ be a real stationary Gaussian process, and let $F(\lambda)$ be its spectral function and $\mathscr{B}(\tau)$ be its correlation function. As is shown in [6], the following condition is sufficient for the continuity of the derivative of a stationary Gaussian process: for any $\alpha > 0$,

$$\int_0^\infty \lambda^2[\log(1+\lambda)]^{1+\alpha} dF(\lambda) < \infty. \tag{3}$$

Therefore, it follows from Theorem 1 that when condition (3) is satisfied, the number of crossings of the level u by $\xi(t)$ on the segment $[a, b]$ is finite, with probability 1.

Theorem 2. *If condition* (3) *is satisfied for a real stationary Gaussian process* $\xi(t)$, *then the formula*

$$\mathbf{M}N_\xi(u) = \frac{1}{\pi}\left(-\frac{\mathfrak{B}''(0)}{\mathscr{B}(0)}\right)^{1/2} e^{-u^2/2B(0)}$$

is valid for the mean number of crossings by $\xi(t)$ *of the level* u *on the segment* $[0, 1]$.

Proof. We divide the segment $[0, 1]$ into 2^n equal parts, and on each of the portions $[k/2^n, (k+1)/2^n]$ we replace $\xi(t)$ by the segment of the straight line passing through the points

$$\left(\frac{k}{2^n}, \xi\left(\frac{k}{2^n}\right)\right) \quad \text{and} \quad \left(\frac{k+1}{2^n}, \xi\left(\frac{k+1}{2^n}\right)\right).$$

Let

$$\eta_n(t) = \xi\left(\frac{k}{2^n}\right) + 2^n\left[\xi\left(\frac{k+1}{2^n}\right) - \xi\left(\frac{k}{2^n}\right)\right]\left(t - \frac{k}{2^n}\right),$$

for $(k/2^n) \leqq t \leqq (k+1)/2^n$.

$N_\xi(u)$ is the number of times $\xi(t)$ crosses the level u on the segment $[0, 1]$. $N_{\eta_n}(u)$ is the number of times $\eta_n(t)$ crosses the level u on the segment $[0, 1]$. Clearly, for any n we have $N_{\eta_n}(u) \leqq N_\xi(u)$. Moreover, if $m > n$, then $N_{\eta_m}(u) \geqq N_{\eta_n}(u)$, and $N_{\eta_n}(u) \uparrow N_\xi(u)$ as $n \to \infty$. Since $N_\xi(u)$, and consequently also $N_{\eta_n}(u)$, is finite,

$$N_{\eta_n}(u) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^1 \varphi_\varepsilon(\eta_n(t))|\eta_n'(t)|dt,$$

where

$$\varphi_\varepsilon(\eta_n(t)) = \begin{cases} 1 & \text{for } |\eta_n(t) - u| \leqq \varepsilon, \\ 0 & \text{for } |\eta_n(t) - u| > \varepsilon, \end{cases}$$

$$\mathbf{M}N_{\eta_n}(u) = \mathbf{M}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^1 \varphi_\varepsilon(\eta_n(t))|\eta_n'(t)|dt = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mathbf{M}\int_0^1 \varphi_\varepsilon(\eta_n(t))|\eta_n'(t)|dt.$$

We can interchange the limit and mathematical expectation because for all $\varepsilon > 0$ we have

$$\frac{1}{2\varepsilon}\int_0^1 \varphi_\varepsilon(\eta_n(t))|\eta_n'(t)|dt \leqq 2^n.$$

Carrying out the computation,

$$\mathbf{M}N_{\eta_n}(u) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^1\int_{-\infty}^{+\infty}\int_{u-\varepsilon}^{u+\varepsilon} |y| p_n(x, y)\,dx\,dy\,dt, \tag{4}$$

where

$$p_n(x, y) = \frac{1}{2\pi\sqrt{D}}\exp\left\{-\frac{1}{2D}(B''x^2 - 2B'xy + By^2)\right\}$$

and

$$B = \mathbf{M}\eta_n^2(t), \quad B' = \mathbf{M}[\eta_n(t)\eta_n'(t)], \quad B'' = \mathbf{M}[\eta_n'(t)]^2, \quad D = BB'' - (B')^2.$$

It is easy to compute that for $(k/2^n) \leqq t \leqq (k+1)/2^n$,

$$B = \mathscr{B}(0)[(1-2^n t+k)^2 + (2^n t-k)^2] + 2\mathscr{B}\left(\frac{1}{2^n}\right)(2^n t-k)(1-2^n t+k),$$

$$B' = 2^n[(2^n t-k) - (1-2^n t+k)]\left[\mathscr{B}(0) - \mathscr{B}\left(\frac{1}{2^n}\right)\right],$$

$$B'' = 2^{2n+1}\left[\mathscr{B}(0) - \mathscr{B}\left(\frac{1}{2^n}\right)\right].$$

Since

$$\int_{u-\varepsilon}^{u+\varepsilon} p_n(x, y)dx \leqq 2\varepsilon \max_x p_n(x, y),$$

and

$$\max_x p_n(x, y) = e^{-y^2/2B''}, \qquad \int_{-\infty}^{+\infty} |y|e^{-y^2/2B''}dy < \infty,$$

it is possible to pass to the limit with respect to ε under the integral sign in (4), and

$$\mathbf{M}N_{\eta_n}(u) = \int_0^1 \int_{-\infty}^{+\infty} |y|p_n(u, y)dy\,dt.$$

Performing the integration over y, we obtain

$$\mathbf{M}N_{\eta_n}(x) = \frac{1}{\pi}\int_0^1 \left[\frac{\sqrt{D}}{B} e^{-u^2\{(B'^2/2DB)+(1/2B)\}} + \frac{B'}{B^{3/2}} e^{-u^2/2B} \int_0^{B'u/\sqrt{DB}} e^{-z^2/2}dz\right] dt. \tag{5}$$

Using the relation

$$\left|\int_0^b f(t)dt\right| \leqq \max_{a\leqq t\leqq b} |f(t)|(b-a),$$

it is easy to show that $\mathbf{M}N_{\eta_n}(u)$ is bounded by the same constant for any n. But since $N_{\eta_n}(u) \uparrow N_\xi(u)$ and $\mathbf{M}N_{\eta_n}(u)$ is bounded, $\mathbf{M}N_\xi(u) = \lim_{n\to\infty}\mathbf{M}N_{\eta_n}(u)$.

We can rewrite

$$\mathbf{M}N_{\eta_n}(u) = \sum_{k=0}^{2^n-1} \frac{1}{2^n} f\left(\frac{k}{2^n}\right) + o(1),$$

where $f(t)$ is the integrand in (5). Expanding $\mathscr{B}(1/2^n)$ in a Taylor series

$$\mathscr{B}\left(\frac{1}{2^n}\right) = \mathscr{B}(0) + \mathscr{B}'(0)\frac{1}{2^n} + \mathscr{B}''(t_1)\frac{1}{2^{2n+1}}, \qquad 0 \leqq t_1 \leqq \frac{1}{2^n},$$

and taking into account that since the process $\xi(t)$ is real, $B'(0) = 0$, we obtain

$$\lim_{n\to\infty} \mathbf{M}N_{\eta_n}(u) = \frac{1}{\pi}\left(-\frac{\mathscr{B}''(0)}{\mathscr{B}(0)}\right)^{1/2} e^{-u^2/2\mathscr{B}(0)}.$$

Thus, we have proven that

$$\mathbf{M}N_\xi(u) = \frac{1}{\pi}\left(-\frac{\mathscr{B}''(0)}{\mathscr{B}(0)}\right)^{1/2} e^{-u^2/2\mathscr{B}(0)}.$$

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ON THE MEAN NUMBER OF CROSSINGS OF A LEVEL BY A STATIONARY GAUSSIAN PROCESS

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(*Summary*)

Let $\xi(t)$ be a stationary Gaussian process and $N_\xi(u)$ denote the number of solutions of $\xi(t) = u$, $0 \leqq t \leqq 1$. We prove the well-known formula for $\mathbf{M}_\xi(u)$ under conditions that are very close to the necessary ones.