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Geometric Approach to Quantum Statistical Mechanics and Application to Casimir Energy and Friction Properties

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Geometric Approach to Quantum Statistical Mechanics and Application to Casimir Energy and Friction Properties

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Abstract. A geometric approach to general quantum statistical systems (including the harmonic oscillator) is presented. It is applied to Casimir energy and the dissipative system with friction. We regard the (N+1)-dimensional Euclidean *coordinate* system (X^i,τ) as the quantum statistical system of N quantum (statistical) variables (X^i) and one *Euclidean time* variable (τ). Introducing paths (lines or hypersurfaces) in this space (X^i,τ), we adopt the path-integral method to quantize the mechanical system. This is a new view of (statistical) quantization of the *mechanical* system. The system Hamiltonian appears as the *area*. We show quantization is realized by the *minimal area principle* in the present geometric approach. When we take a *line* as the path, the path-integral expressions of the free energy are shown to be the ordinary ones (such as N harmonic oscillators) or their simple variation. When we take a *hyper-surface* as the path, the system Hamiltonian is given by the *area* of the *hyper-surface* which is defined as a *closed-string configuration* in the bulk space. In this case, the system becomes a O(N) non-linear model. We show the recently-proposed 5 dimensional Casimir energy (ArXiv:0801.3064,0812.1263) is valid. We apply this approach to the visco-elastic system, and present a new method using the path-integral for the calculation of the dissipative properties.

1. Introduction

In the quest for the fundamental structure of the space, time, and matter, the most advanced theories are the string theory, D-brane theory and M-theory[1]. They are beyond the quantum field theory in that the extended (in space-time) objects are treated as fundamental elements. Since the finding of AdS/CFT correspondence[2, 3, 4], various new ideas and techniques, developed for them, are imported into the *non-perturbative* analysis of the quantum field theories. In particular, the application to the material physics is marvelous: heavy ion collision physics and the viscosity in the quark-gluon plasma([5, 6, 7] for review), superconductivity and superfluidity [8, 9, 10, 11], baryon mass spectrum in QCD[12, 13]. In this circumstance, a *new standpoint* about the space-time quantization appear. One is proposed by Hořava[14, 15]. He introduced Lifshitz's higher-derivative scalar theory and its renormalization group behavior into his idea about the new quantum gravity. Another one is revively given by E. Verlinde[16]. He emphasizes the entropic force (rather than energetic force) and the thermodynamical behavior near the horizon (Hawking radiation). With this recent trend of the geometrical view, the statistical(thermal) view and the visco-elastic view , we present a *new* formalism where the quantum statistical system is treated purely in the geometrical way.

In the analysis of the dissipative material-system, due to friction, one familiar way is to treat it as the *hydrodynamic* system of a continuum medium. The incompressible viscous flow (Newton' flow is assumed) is discribed by Navier-Stokes equation.

$$
\begin{aligned}
&\frac{Dv^i}{Dt} \equiv (\frac{\partial}{\partial t} + v^j \partial_j) v^i = -\frac{1}{\rho}\partial_i P + \frac{\eta}{\rho}\Delta v^i + g^i \quad , \\
&i,j = 1,2,3 \quad , \quad \vec{x} \equiv (x^1, x^2, x^3) = (x, y, z) \quad , \quad \partial_i \equiv \frac{\partial}{\partial x^i} \quad , \quad \Delta \equiv \partial_1^2 + \partial_2^2 + \partial_3^2 \quad , \\
&v^i = v^i(t, \vec{x}) \quad , \quad P = P(t, \vec{x}) \quad , \qquad (1)
\end{aligned}
$$

where $\vec{g} \equiv (0, 0, g)$ is the gravitational acceleration constant. ρ and η is the mass density (constant) and the viscosity (constant) of the fluid, respectively. $\vec{v} \equiv (v^1, v^2, v^3)$ is the velocity field and $P(t, \vec{x})$ is the pressure field. This equation has been examined analytically and numericallly in various ways. The statistical quantities of the system has been obtained, for example, by introducing the random force (Langevin equation, the stochastic method, Fokker-Planck equation). In the present work, by introducing a *metric* in (t, x^i)-space, we treat this system geometrically. The statistical or global quantities (energy, entropy,...) are exprssed in the *path-integral* form.

As another application, we take Casimir energy. It is the vcuum (zero temperature) energy for the free (non-interacting) part of the quantum system. It is a basic quantity of the quantum field theory (QFT) such as the field of radiation (the electromagnetic field). Generally it is expressed by the boundary parameters of the system. The quantity is so delicately defined that we need careful regularization of divergences. The present approach gives a new regularization which has some characteristic points compared with the ordinary one. Especially the direction of the *renormalization group* flow determines the attractiveness or repulsiveness. We also apply the present approach to 5 dimensional (dim) field theories (flat and curved ones), and well-define the higher-dimensional QFTs.

The content is organized as follows. We start with the simple quantum statistical system of one harmonic oscillator in Sec.2. We see the geometric approach works well by regarding the *extra coordinate* as the *Euclidean time*. We generalize the harmonic oscillator potential (elastic system) to the general one in Sec.3. In Sec.4 the one variable system is generalized to the system of N variables. We analyze the quantum statistical system in the N+1 extra dimensional Euclidean *geometry*. The O(N) nonliner model naturally appears by taking the *closed-string* configuration. We stress that taking the area as Hamiltonian is one realization of the *minimal area principle*. In Sec.5, we apply the present approach to Casimir energy. The visco-elastic system is treated in Sec.6. We conclude in Sec.7. In Appendix A, some detail of Sec.5.2 (Casimir energy in 5D curved space) is explained.

2. Quantum Statistical System of Harmonic Oscillator

2.1. 'Dirac' Type

Let us consider 2 dim Euclidean space (X, τ) described by the following metric.

$$
\begin{aligned}
&ds^2 = dX^2 + \omega^2 X^2 d\tau^2 = G_{AB} dX^A dX^B \quad , \\
&(X^A) = (X^1, X^2) = (X, \tau) \quad , \quad (G_{AB}) = \mathrm{diag}(1, \omega^2 X^2) \quad , \\
&R_{AB} = 0 \quad , \quad R = G^{AB} R_{AB} = 0 \quad , \qquad (2)
\end{aligned}
$$

where $A, B = 1, 2$. The parameter ω is the 'spring' constant with the dimension of mass. We impose the periodicity (period: β) in the direction of the extra dimension τ.

$$
\tau \to \tau + \beta \quad . \qquad (3)
$$

This is a way to introduce the *temperature* $(1/\beta)$ in the system. Here we take a path $\{x(\tau),\ 0 \le \tau \le \beta\}$ in the 2D bulk space (X,τ) and the *induced* metric on the *line* is given by

$$X = x(\tau) \quad , \quad dX = \dot{x}d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \le \tau \le \beta \quad , \\ ds^2 = (\dot{x}^2 + \omega^2 x^2)d\tau^2 \quad . \tag{4}$$

See Fig.1.

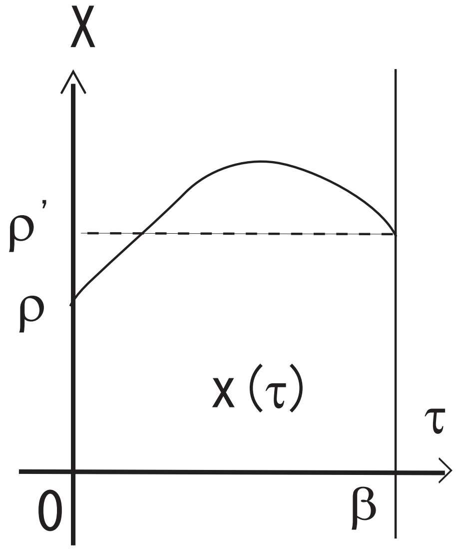

Figure 1. A path of line in 2D Euclidean space (X,τ). The path starts at x(0)=ρ and ends at x(β)=ρ'.

Then the *length* L of the path $x(\tau)$ is given by

$$L = \int ds = \int_0^\beta \sqrt{\dot{x}^2 + \omega^2 x^2}d\tau \quad . \tag{5}$$

We take the half of the length $(\frac{1}{2}L)$ as the system Hamiltonian (*minimal length principle*). Then the free energy F of the system is given by

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_\tau \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sqrt{\dot{x}^2 + \omega^2 x^2}d\tau\right] \quad , \tag{6}$$

where the path-integral is done for all possible paths with the indicated boundary condition (b.c.). This quantum statistical system can be regarded as the square-root type ('Dirac' type) of the ordinary harmonic oscillator. [1]

2.2. Standard Type

Now we consider another type of 2 dim Euclidean space (X, τ) described by the following *line* element.

$$ds^2 = \frac{1}{d\tau^2}(dX^2)^2 + \omega^4 X^4 d\tau^2 + 2\omega^2 X^2 dX^2 = \frac{1}{d\tau^2}(dX^2 + \omega^2 X^2 d\tau^2)^2 \quad , \tag{7}$$

[1] The situation reminds us of the relation between Nambu-Goto action and Polyakov action in the string theory[17]. The introduction of an auxiliary variable helps to 'normalize' the square-root action (6). In this case, the geometric role of the auxiliary variable remains obscure.

where we put the following condition on the infinitesimal quantities, $d\tau^2$ and dX^2, in order to keep all terms of (7) in the same order.

$$\text{[Line Element Regularity Condition]} \ : \\ d\tau^2 \sim O(\epsilon^2) \quad , \quad dX^2 \sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2}dX^2 \sim O(1) \quad , \tag{8}$$

where ϵ is an arbitrary infinitesimal parameter with the dimension of length. [2] Note that we do *not* have 2D metric in this case. (We cannot define the bulk metric $G_{AB}(X)$.) We impose the periodicity (period: β).

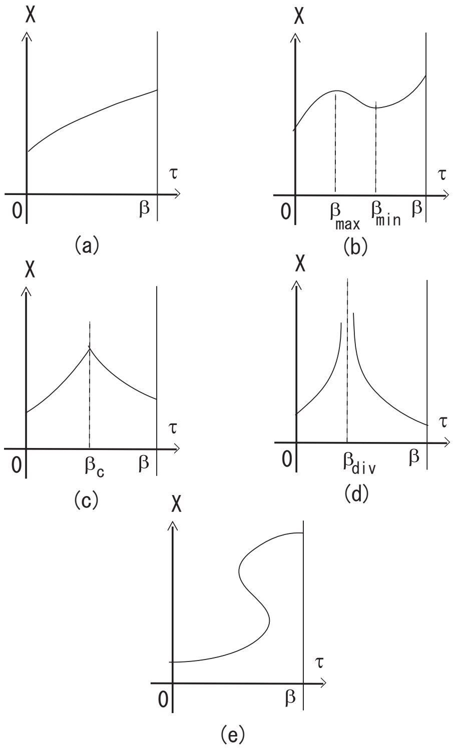

Figure 2. Singular and regular lines in 2D Euclidean space (X,τ). (a) regular line, simply increasing; (b) regular line, maximum at β_{max} and minimum at β_{min}; (c) singular line, different derivatives for $\beta \rightarrow \beta_c \pm 0$; (d) singular line, divergent at β_{div}; (e) singular line, multi-valued.

$$\tau \rightarrow \tau + \beta \quad . \tag{9}$$

Here we take a path $\{x(\tau),\ 0 \leq \tau \leq \beta\}$, and the *induced* metric on the line is given by

$$X = x(\tau) \quad , \quad dX = \dot{x}d\tau \quad , \quad \dot{x} \equiv \frac{dx}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad , \\ ds^2 = (\dot{x}^2 + \omega^2 x^2)^2 d\tau^2 \quad . \tag{10}$$

In the bulk we do *not* have the metric, but on the path, we *do* have this *induced* metric. Then the *length* L of the path $x(\tau)$ is given by

$$L[x(\tau)] = \int ds = \int_0^\beta (\dot{x}^2 + \omega^2 x^2) d\tau \quad . \tag{11}$$

[2] The condition (8) restricts the trajectory configuration (10) only to smooth-lines in the 2D bulk space, and excludes singular-lines which have some *singular* points (the derivative along τ can not be defined) between $0 \leq \tau \leq \beta$. See Fig.2 for singular and regular lines.

Hence, taking $\frac{1}{2}L$ as the Hamiltonian (*minimal length principle*), the free energy F of the system is given by

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_{\tau} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^\beta (\dot{x}^2 + \omega^2 x^2) d\tau \right] \quad , \tag{12}$$

where the path-integral is done for all possible paths with the indicated b.c.. This is exactly the free energy of the harmonic oscillator. See Feynman's textbook[18]. [3]

Note that the condition (8) is necessary for the *elastic* view to the path.

3. General Quantum Statistical System

We generalize the harmonic oscillator potential, $\frac{1}{2}\omega^2 X^2$, to the general one $V(X)$. As for $V(X)$, we have the following form in mind.

$$\frac{\omega^2}{2}X^2 + \frac{\lambda_3}{3!}X^3 + \frac{\lambda_4}{4!}X^4 + \cdots \quad , \tag{13}$$

where $\lambda_3, \lambda_4, \cdots$ are the coupling constants for additional terms.

3.1. 'Dirac' Type

We start with the following metric in 2 dim Euclidean space (X, τ).

$$\begin{gathered} ds^2 = dX^2 + 2V(X)d\tau^2 = G_{AB}dX^A dX^B \quad , \\ (X^A) = (X^1, X^2) = (X, \tau) \quad , \quad (G_{AB}) = \mathrm{diag}(1, 2V(X)) \quad , \\ (R_{AB}) = \begin{pmatrix} \frac{V''}{2V} - \frac{1}{4}\left(\frac{V'}{V}\right)^2, & 0 \\ 0 \quad , & V'' - \frac{1}{2}\frac{(V')^2}{V} \end{pmatrix} \quad , \quad R = G^{AB}R_{AB} = \frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2 \quad , \\ V' \equiv \frac{dV(X)}{dX} \quad , \quad V'' \equiv \frac{d^2V(X)}{dX^2} \quad , \end{gathered} \tag{14}$$

where $A, B = 1, 2$. Note that V(X) does *not* depend on τ. We impose the periodicity (period: β) in the direction of the extra dimension τ (3). On a path $\{x(\tau),\ 0 \leq \tau \leq \beta\}$, the *induced* metric is given by

$$ds^2 = (\dot{x}^2 + 2V(x))d\tau^2 \quad , \quad 0 \leq \tau \leq \beta \quad . \tag{15}$$

Hence the *length* L of the path $x(\tau)$ is given by

$$L = \int ds = \int_0^\beta \sqrt{\dot{x}^2 + 2V(x)}d\tau \quad . \tag{16}$$

Taking the half of the length ($\frac{1}{2}L$) as the Hamiltonian, we get the free energy F as

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_{\tau} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sqrt{\dot{x}^2 + 2V(x)}d\tau \right] \quad . \tag{17}$$

[3] $F = \frac{\omega}{2} + \frac{1}{\beta}\ln(1 - \mathrm{e}^{-\beta\omega}), E = < \frac{L}{2} > = \frac{\omega}{2}\coth(\frac{\omega\beta}{2}) = \frac{\omega}{2} + \frac{\omega}{\mathrm{e}^{\omega\beta}-1}, S = \frac{1}{T}(E - F) = k\{\frac{\beta\omega}{2}\coth\frac{\beta\omega}{2} - \frac{\beta\omega}{2} - \ln(1 - \mathrm{e}^{-\beta\omega})\}$

3.2. Standard Type

We start with the following line element.

$$ds^2 = \frac{1}{d\tau^2}(dX^2)^2 + 4V(X)^2 d\tau^2 + 4V(X)dX^2 = \frac{1}{d\tau^2}\left(dX^2 + 2V(X)d\tau^2\right)^2 \quad , \tag{18}$$

where we put the condition (8) on the infinitesimal quantities, $d\tau^2$ and dX^2, in order to keep all terms in the same order. The 2D bulk space do *not* have 2D metric. We impose the periodicity (period: β) (9). On a path $\{x(\tau),\ 0 \leq \tau \leq \beta\}$, we have the *induced* metric:

$$ds^2 = (\dot{x}^2 + 2V(x))^2 d\tau^2 \quad . \tag{19}$$

The *length* L is given by

$$L[x(\tau)] = \int ds = \int_0^\beta (\dot{x}^2 + 2V(x))d\tau \quad . \tag{20}$$

Taking $\frac{1}{2}L$ as the Hamiltonian, the free energy F is given by

$$\mathrm{e}^{-\beta F} = \int_{-\infty}^{\infty} d\rho \int_{\substack{x(0)=\rho \\ x(\beta)=\rho}} \prod_\tau \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^\beta (\dot{x}^2 + 2V(x))d\tau\right] \quad , \tag{21}$$

where the path-integral is done for all possible paths with the indicated b.c.. This is exactly the free energy of the quantum statistical system of one variable x in the general potential V(x).

4. Quantum Statistical System of N Harmonic Oscillators and O(N) Nonlinear Model

4.1. 'Dirac' Type of N Harmonic Oscillators and O(N) nonlinear system

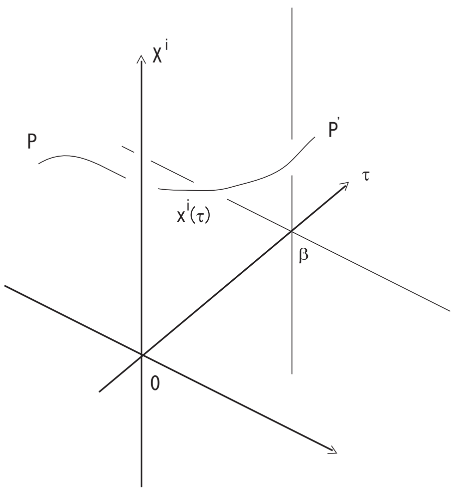

Figure 3. A path of line $\{x^i(\tau)|i = 1, 2, \cdots, N\}$ in N(=2)+1 dim space. It starts at P=$(\rho_1, \rho_2, \cdots, \rho_N, 0)$ and ends at P$'$=$(\rho'_1, \rho'_2, \cdots, \rho'_N, \beta)$.

Let us consider N+1 dim Euclidean space $(X^i, \tau), i = 1, 2, \cdots, N$ described by the following metric.

$$ds^2 = \sum_{i=1}^N (dX^i)^2 + \omega^2 d\tau^2 \sum_{i=1}^N (X^i)^2 = \sum_{i=1}^N (dX^i)^2 + \omega^2 r^2 d\tau^2 = G_{AB} dX^A dX^B \quad ,$$
$$A, B = 1, 2, \cdots, N, N+1; \quad X^{N+1} \equiv \tau \quad ,$$
$$(G_{AB}) = \mathrm{diag}(1, 1, \cdots, 1, \omega^2 r^2) \quad , \quad r^2 \equiv \sum_{i=1}^N (X^i)^2 \quad . \tag{22}$$

(Subsec.2.1 is the $N=1$ case.) The Ricci tensor and the scalar curvature are, for N=2, given by [4]

$$
\begin{gathered}
ds^2 = dx^2 + dy^2 + \omega^2(x^2+y^2)d\tau^2 \quad , \\
(R_{AB}) = \frac{1}{(r^2)^2}\left(\begin{array}{ccc} y^2 & -xy & 0 \\ -yx & x^2 & 0 \\ 0 & 0 & \omega^2(r^2)^2 \end{array}\right) \quad , \quad R = \frac{2}{r^2} > 0 \quad , \quad r^2 = x^2+y^2 \quad , \\
\sqrt{G} = \omega\sqrt{x^2+y^2} \quad , \quad \sqrt{G}R = \frac{2\omega}{\sqrt{x^2+y^2}}
\end{gathered} \tag{23}
$$

where $(X^1, X^2, X^3) = (x, y, \tau)$ is taken.

We impose the periodicity (3)(period: β), and take a path $\{X^i = x^i(\tau) |\ 0 \leq \tau \leq \beta,\ i = 1, 2, \cdots, N\}$(See Fig.3). The *induced* metric on the *line* is given by

$$
\begin{gathered}
X^i = x^i(\tau) \quad , \quad dX^i = \dot{x}^i d\tau \quad , \quad \dot{x}^i \equiv \frac{dx^i}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad , \\
i = 1, 2, \cdots, N \quad , \quad ds^2 = \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau^2 \quad .
\end{gathered} \tag{24}
$$

Then the *length* L of the path $\{x^i(\tau)\}$ is given by

$$
L = \int ds = \int_0^\beta \sqrt{\sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)}\ d\tau \quad . \tag{25}
$$

We take the half of the length ($\frac{1}{2}L$) as the system Hamiltonian(*minimal length principle*). Then the free energy F of the system is given by

$$
\mathrm{e}^{-\beta F} = (\prod_i \int_{-\infty}^{\infty} d\rho_i) \int_{\substack{x^i(0)=\rho_i \\ x^i(\beta)=\rho_i}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sqrt{\sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2 x^{i2})}d\tau\right] \quad , \tag{26}
$$

where the path-integral is done for all possible paths $\{x^i(\tau); i = 1, 2, \cdots N\}$ with the indicated b.c.. We can regard this as the free energy for a variation ('Dirac' type) of the N harmonic oscillators's. (See next subsection for the ordinary type of the N harmonic oscillators.)

Instead of the length L, we can take another geometric quantity. Let us consider the following N dim *hypersurface* in N+1 dim space (a closed-string configuration). See Fig.4 for the N=2 case.

$$
\sum_{i=1}^{N}(X^i)^2 = r^2(\tau) \quad , \quad \sum_{i=1}^{N} X^i dX^i = r\dot{r}d\tau \quad , \quad 0 \leq \tau \leq \beta \quad . \tag{27}
$$

The form of $r(\tau)$ describes a path (N dimensional hypersurface in the bulk) which is *isotropic* in the 'brane' at τ (the N dim space 'perpendicularly' standing at τ of the extra axis, not the hypersurface). The *induced* metric on the N dim hypersurface is given by

$$
\begin{gathered}
ds^2 = \sum_{i,j}(\delta_{ij} + \frac{\omega^2}{\dot{r}^2}x^i x^j)dx^i dx^j \equiv \sum_{i,j} g_{ij}dx^i dx^j \quad , \\
g_{ij} = \delta_{ij} + \frac{\omega^2}{\dot{r}^2}x^i x^j \quad , \quad r^2 = \sum_{i=1}^{N}(x^i)^2 \quad , \quad \det(g_{ij}) = 1 + \frac{\omega^2 r^2}{\dot{r}^2} \quad .
\end{gathered} \tag{28}
$$

[4] All curvature calculation in this work is checked by the algebraic calculation soft "Maxima"[19].

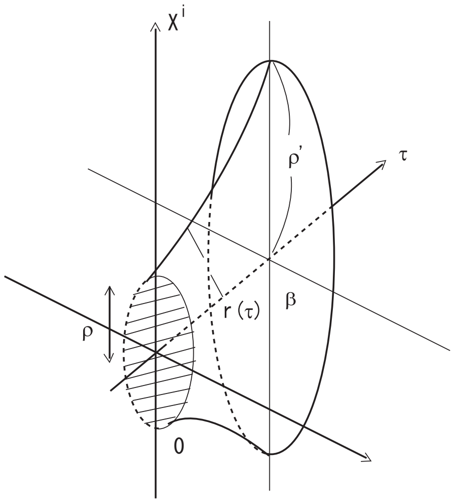

Figure 4. A path of hyper-surface. N(=2) dim hypersurface in N+1 dim space $(X^1, X^2, \cdots, X^N, \tau)$. S^{N-1} radius $r(\tau)$ starts with $r(0) = \rho$ and ends with $r(\beta) = \rho'$. We take this configuration as a path in the path integral (30) and (41). This is a closed-string configuration.

This is the metric of a O(N) nonlinear system and is the one dimensional *nonlinear sigma model* as the field theory. (The standard model (2 dim nonlinear sigma model) has often been used so far in order to show the *renormalization group* behavior of various systems. The background (effective action) formulation of the string theory heavily relies on the model.) Then the *area* of the N dim hypersurface is given by

$$A_N = \int \sqrt{\det g_{ij}}\ d^N x = \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + \omega^2 r^2}\, r^{N-1} d\tau \quad . \tag{29}$$

When we take $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy F is given by

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2} \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int \sqrt{\dot{r}^2 + \omega^2 r^2}\, r^{N-1} d\tau \right] \quad . \tag{30}$$

We should compare this result ($N = 4$) with the proposed 5D Casimir energy for the *flat* geometry (59). The component $\sqrt{\dot{r}^2 + \omega^2 r^2}$ in the integrand of (30) is replaced by $\sqrt{\dot{r}^2 + 1}$ in (59).

We recognize, if we start with

$$ds^2 = \sum_{i=1}^{N} (dX^i)^2 + d\tau^2 \quad \text{(N+1 dim Euclidean flat)} \quad , \tag{31}$$

instead of (22), the integration measure becomes *exactly* the same as (59).

4.2. Standard Type of N Harmonic Oscillators

Now we consider another type of N+1 dim Euclidean space $(X^i, \tau);\ i = 1, 2, \cdots N$ described by the following line element.

$$\begin{aligned} ds^2 &= d\tau^{-2}\{\sum_{i=1}^{N}(dX^i)^2\}^2 + \omega^4\{\sum_{i=1}^{N}(X^i)^2\}^2 d\tau^2 + 2\omega^2\{\sum_{i=1}^{N}(X^i)^2\}\{\sum_{j=1}^{N}(dX^j)^2\} \\ &= \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2 + \omega^2 r^2 d\tau^2\}^2 \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad , \end{aligned} \tag{32}$$

with the condition:

$$\text{[Line Element Regularity Condition]} \ :$$
$$d\tau^2 \sim O(\epsilon^2) \quad , \quad (dX^i)^2 \sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2\} \sim O(1) \ , \tag{33}$$

in order to keep all terms of (32) in the order of ϵ^2. [5] Again we note that, in the above case, we do *not* have N+1 dim (bulk) metric. We impose the periodicity (3): (period: β).

Here we take a path of Fig.3: $\{x^i(\tau)|\ 0 \leq \tau \leq \beta, i = 1, 2, \cdots, N\}$ and the *induced* metric on the path is given by

$$X^i = x^i(\tau) \quad , \quad dX^i = \dot{x}^i d\tau \quad , \quad \dot{x}^i \equiv \frac{dx^i}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$
$$ds^2 = [\sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)]^2 d\tau^2 \quad . \tag{34}$$

Then the *length* L of the path $\{x^i(\tau)\}$ is given by

$$L[x^i(\tau)] = \int ds = \int_0^\beta \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau \quad . \tag{35}$$

Hence, taking $\frac{1}{2}L$ as the Hamiltonian (*minimal length principle*), the free energy F of the system is given by

$$\mathrm{e}^{-\beta F} = \left(\prod_i \int_{-\infty}^{\infty} d\rho_i\right) \int_{\substack{x^i(0) = \rho_i \\ x^i(\beta) = \rho_i}} \prod_{i,\tau} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sum_{i=1}^{N}((\dot{x}^i)^2 + \omega^2(x^i)^2)d\tau\right], \tag{36}$$

where the path-integral is done for all possible paths with the indicated b.c.. This is *exactly* the free energy of N harmonic oscillators.

We note again the condition (33) is necessary for the *elastic* view to the hyper-surfaces.

4.3. Middle type of O(N) nonlinear system

Instead of (32), we can start from a slightly modified metric.

$$ds^2 = \omega^4\{\sum_{i=1}^{N}(X^i)^2\}^2 d\tau^2 + 2\omega^2\kappa\{\sum_{i=1}^{N}(X^i)^2\}\{\sum_{j=1}^{N}(dX^j)^2\}$$
$$= \omega^2 r^2 \left(\omega^2 r^2 d\tau^2 + 2\kappa\sum_{j=1}^{N}(dX^j)^2\right) \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad . \tag{37}$$

We drop the first term of (32), and add a free (real) parameter κ in the third one. We stress that, in this case, we need *not* the condition of (33). The line element is the ordinary type and

[5] As in (8), this condition restricts the trajectory configuration (34) only to *smooth* hyper-surfaces in the (N+1)-dim space.

we have the bulk metric G_{AB} in this case. The Ricci tensor and the scalar curvature, for $N = 2$, are given by

$$ds^2 = \omega^4(x^2+y^2)^2 d\tau^2 + 2\omega^2\kappa(x^2+y^2)(dx^2+dy^2) \quad ,$$
$$(R_{AB}) = \frac{1}{(r^2)^2}\left(\begin{array}{ccc} 4y^2 & -4xy & 0 \\ -4yx & 4x^2 & 0 \\ 0 & 0 & \frac{2\omega^2}{\kappa}(r^2)^2 \end{array}\right) \quad , \quad R = \frac{4}{\kappa\omega^2(r^2)^2} \quad , \quad r^2 = x^2+y^2 \quad ,$$
$$\sqrt{G} = 2\omega^4|\kappa|r^4 \quad , \quad \sqrt{G}R = 8\omega^2\cdot\mathrm{sign}(\kappa) \quad , \qquad (38)$$

where $(X^1, X^2, X^3) = (x, y, \tau)$ and $\mathrm{sign}(\kappa)$ is the sign of κ. [6] We consider the N dim hypersurface (27), or Fig.4, and the *induced* metric on it is given by

$$ds^2 = \sum_{i,j=1}^{N} 2\omega^2 r^2(\kappa\delta_{ij} + \frac{1}{2}\frac{\omega^2}{\dot{r}^2}x^i x^j)dx^i dx^j \equiv \sum_{i,j} g_{ij}dx^i dx^j \quad . \qquad (39)$$

Then the *area* of this hypersurface is given by

$$A_N = \int\sqrt{\det g_{ij}}\; d^N x = \frac{(2\pi\omega^2|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta r^N\sqrt{\dot{r}^2 + \frac{r^2\omega^2}{2|\kappa|}}\; r^{N-1}d\tau \quad . \qquad (40)$$

Taking $\frac{1}{2}A_N$ as the Hamiltonian (*minimal area principle*), the free energy, F, is given by

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i}\mathcal{D}x^i(\tau)\exp\left[-\frac{1}{2}\frac{(2\pi\omega^2|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)}\int_0^\beta r^N\sqrt{\dot{r}^2+\frac{r^2\omega^2}{2|\kappa|}}\; r^{N-1}d\tau\right] \quad .(41)$$

We should compare this result (N=4,κ=1/2) with the proposed 5D Casimir energy for the *warped* geometry (59). They are similar ($(\omega r)^4\sqrt{\dot{r}^2+r^2\omega^2}$ of (41) is replaced by $(1/\omega z)^4\sqrt{r'^2+1}$ of (59).).

4.4. Modified type of O(N) nonlinear system

Instead of (37), we take the following type of metric.

$$ds^2 = W(\tau)\left(V(r)d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\right) \quad , \quad r^2 = \sum_{i=1}^{N}(X^i)^2 \quad . \qquad (42)$$

Especially, if we start with $W(\tau) = 1/\tau^2, V(r) = 1$,

$$\text{Euclidean (AdS)}_{N+1}: \quad ds^2 = \frac{1}{\tau^2}\{d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\} \quad , \qquad (43)$$

we recognize the integration measure *exactly* becomes the same as that in (59): $\tau^{-N}\sqrt{\dot{r}^2+1}r^{N-1}d\tau$.

The content in this section can be generalized for the *general* isotropic potential.

[6] $R > 0$ for $\kappa > 0$, $R < 0$ for $\kappa < 0$.

5. Casimir Energy

One important application of the present approach is Casimir energy. As mentioned in the introduction, it requires carefull regularization to define the quantity rigorously. In order to deal with divergence properly, the IR and UV regularizations are important. The final (finite) result depends on the *boundary* parameters only.

5.1. 4D Flat Case (Ordinary Casimir Energy)

Before explaining the original motivation (the 5 dim Casimir energy), we discuss the ordinary case (1+3 dim electromagnetism) and the relation to the present approach. We consider the electromagnetism in Minkwski space:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad . \tag{44}$$

We place 2 perfectly-conducting plates parallel with the separation $2l$ in the x-direction. This configuration can be realized by taking the following boundary conditions. As for y- and z-directions, we impose the periodicity for the infra-red (IR) regularization.

$$\begin{aligned} \text{Periodicity} \ : \quad x \to x + 2l \quad , \quad y \to y + 2L \quad , \quad z \to z + 2L \quad , \\ L \gg l \quad , \end{aligned} \tag{45}$$

where L is the IR-regularization parameter. Then the eigen frequencies of the electromagnetic wave and Casimir energy are given by

$$\begin{aligned} \omega_{n,m_y,m_z} &= \sqrt{(n\frac{\pi}{l})^2 + (m_y\frac{\pi}{L})^2 + (m_z\frac{\pi}{L})^2} \quad , \\ E_{Cas} &= 2 \cdot \sum_{n,m_y,m_z \in \mathbf{Z}} \frac{1}{2}\omega_{n,m_y,m_z} \quad \geq 0 \quad , \end{aligned} \tag{46}$$

where $\mathbf{Z}$ indicates all integers. $\frac{1}{2}\omega_{n,m_y,m_z}$ is the zero-point oscillation energy. Introducing the cut-off function $g(x)$ ($= 1$ for $0 < x < 1$, 0 for otherwise), Casimir energy can formally be written as

$$E^{\Lambda}_{Cas} = \sum_{n,m_y,m_z \in \mathbf{Z}} \omega_{n,m_y,m_z} g\left(\frac{\omega_{n,m_y,m_z}}{\Lambda}\right) \quad \geq 0 \quad . \tag{47}$$

We take the continuum limit $L \to \infty,\ L \ll l \to \infty$.

$$\begin{aligned} E^{\Lambda 0}_{Cas} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_y dk_z}{(\frac{\pi}{L})^2} \int_{-\infty}^{\infty} \frac{dk_x}{\frac{\pi}{l}} \sqrt{k_x^2 + k_y^2 + k_z^2}\, g(\frac{k}{\Lambda}) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty |k| \leq \Lambda}^{\infty} \frac{dk_x dk_y dk_z}{(\frac{\pi}{L})^2 \frac{\pi}{l}} \sqrt{k_x^2 + k_y^2 + k_z^2} \quad \geq 0 \quad . \end{aligned} \tag{48}$$

Note that E_{Cas}, E^{Λ}_{Cas} and $E^{\Lambda 0}_{Cas}$ are all positive-definite.

In a familiar way, regarding $E^{\Lambda 0}_{Cas}$ as the origin of the energy scale, we consider the quantity $u = (E^{\Lambda}_{Cas} - E^{\Lambda 0}_{Cas})/(2L)^2$ as the physical Casimir energy and evaluate it with the help of the Euler-MacLaurin formula as $u = (\pi^2/(2l)^3)\ (B_4/4!) = -(\pi^2/720)(1/(2l)^3) < 0.$ [7] The final result is negative. In the present analysis we take a new regularization.

[7] $B_4 = -1/30$ is the 4-th Bernoulli number.

First we re-express $E^{\Lambda 0}_{Cas}$ using a simple identity : $l = \int_0^l dw$ (w: a regularization or 'extra' axis).

$$
\begin{aligned}
E^{\Lambda 0}_{Cas}/(2L)^2 &= \frac{1}{2^2\pi^3}\int_0^l dw \int_{k\leq\Lambda} P(k)2\pi k^2 dk \\
&= \frac{1}{2^2\pi^3}\int_0^l dw(-1)\int_{r\geq\Lambda^{-1}} P(1/r)(-1)2\pi r^{-4}dr \quad . \\
& P(k)\equiv k \quad , \quad r\equiv\frac{1}{k} \quad ,
\end{aligned}
\tag{49}
$$

where the integration variable changes from the momentum (k) to the coordinate ($r = \sqrt{x^2+y^2+z^2}$). The integration region in (R,w)-space is the infinite rectangular shown in Fig.5.

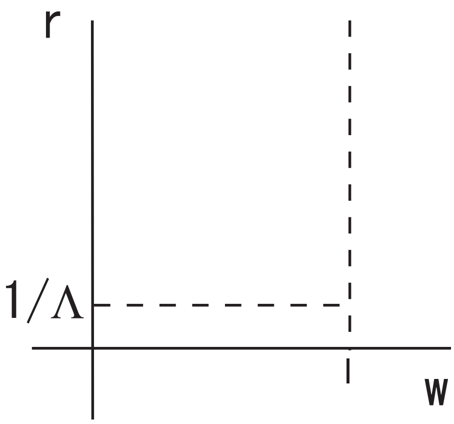

Figure 5. The integral region of (49).

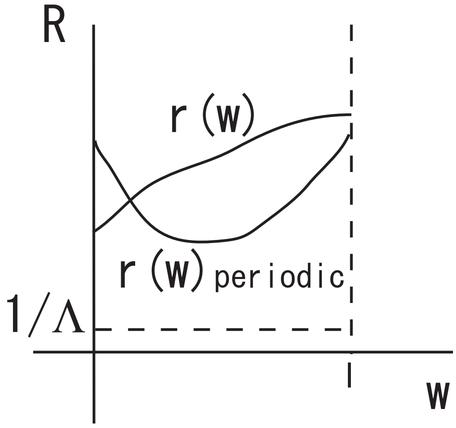

Figure 6. A general path $r(w)$ of (50) and a periodic path $r(w)$ of (51).

We regularize the above expression using the path-integral as

$$
E^{\mathcal{W}\ \prime}_{Cas}/(2L)^2 = \frac{1}{2^2\pi^3}(2\pi)\int_{\text{all paths } r(w)} \prod_w \mathcal{D}r(w)\left[\int dw' P(\frac{1}{r(w')})r(w')^{-4}\right]\exp\{-\mathcal{W}[r(w)]\} \quad , \tag{50}
$$

where the integral is over all paths $r(w)$ which are defined between $0\leq w\leq l$ and whose value is above Λ^{-1}, as shown in Fig.6.

$\mathcal{W}[r(w)]$ is some damping functional explained in the next paragraph. The case $\mathcal{W}[r(w)] = 0$ corresponds to (49). The slightly-more-restrictve regularization is

$$
E^{\mathcal{W}}_{Cas}/(2L)^2 = \frac{1}{2^2\pi^3}(2\pi)\int_{\Lambda^{-1}}^{\infty} d\rho \int_{r(0)=r(l)=\rho} \prod_w \mathcal{D}r(w)\left[\int dw' P(\frac{1}{r(w')})r(w')^{-4}\right]\exp\{-\mathcal{W}[r(w)]\} \quad \geq 0 \quad , \tag{51}
$$

where the integral is over all *periodic* paths. Note that the above regularization keep the positive-definite property.

Hence the present regularization mainly defined by the choice of $\mathcal{W}[r(w)]$. In order to specify it, we introduce the following metric in (R, w)-space.

$$ds^2 = \frac{1}{dw^2}(dR^2 + \Omega^2 R^2 dw^2)^2 \quad . \tag{52}$$

This is the same as that in Sec.2.2. On a path $R = r(w)$, the induced metric and the length L is given as follows. As the damping functional $\mathcal{W}[r(w)]$, we take the length L.

$$ds^2 = dw^2(r'^2 + \Omega^2 r^2)^2 \quad , \quad r' \equiv \frac{dr}{dw} \quad ,$$

$$L = \int ds = \int (r'^2 + \Omega^2 r^2)dw \quad , \quad \mathcal{W}[r(w)] \equiv \frac{1}{2\alpha}L = \frac{1}{2\alpha}\int (r'^2 + \Omega^2 r^2)dw \quad . \tag{53}$$

The two parameters α and Ω are considered as regularization ones. The limit $\alpha \to \infty$ corresponds to (49).

Numerical calculation can evaluate $E^{\mathcal{W}}_{Cas}$ (51), and we expect the following form[24, 25].

$$\frac{E^{\mathcal{W}}_{Cas}}{(2L)^2} = \frac{a}{l^3}(1 - 3c\ln\ (l\Lambda)) \quad , \tag{54}$$

where a and c are some constants. a should be positive because of the positive-definiteness of (51). The present regularization result has, like the ordinary renormalizable ones such as the coupling in QED, the log-divergence. The divergence can be renormalized into the boundary parameter l. This means l *flows* according to the renormalization group.

$$l' = l(1 - 3c\ln(l\Lambda))^{-\frac{1}{3}} \quad , \quad \beta \equiv \frac{d\ln(l'/l)}{d\ln\Lambda} = c \quad , \quad |c| \ll 1 \quad , \tag{55}$$

where β is the renormalization group function, and we assume $|c| \ll 1$. The sign of c determines whether the length separation increases ($c > 0$) or decreases ($c < 0$) as the measurement resolution becomes finer (Λ increases). In terms of the usual terminology, attractive case corresponds to $c > 0$, and repulsive case to $c < 0$.

5.2. 5D Flat and Curved Case

Let us mention the space-time quantization (quantum gravity) in relation to the motivation of the present work. The space-time geometry is specified by the metric tensor field $g_{\mu\nu}(x)$ which appears in the definition of the line element $(ds^2)_{4D} = g_{\mu\nu}(x)dx^\mu dx^\nu$ $(\mu, \nu = 0, 1, 2, 3)$. One of most important problems of the present theoretical physics is the clarification of the *quantum role* of the metric (gravitational) field $g_{\mu\nu}$. We already have a long (nearly half century) history of the quantum gravity since Feynman[20] and DeWitt [21] pioneered. About one decade ago, inspired by the development of the string theory and the D-brane theory, a fascinating model of unification of forces was proposed. It is a 5 dimensional(dim) model with AdS_5 geometry and is called "Randall-Sundrum model" or the "warped model"[22]. This is a representative of the extra dimensional models. An important purpose of the present work is to make this 5 dim model *legitimate* as the *quantum field theory.*

In the warped space-time , the geometry is described as

$$\text{Warped Metric (y-expression)} \quad ds^2 = \mathrm{e}^{-2\omega|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \ -l \le y \le l \ , \tag{56}$$

where $\{\mu, \nu = 0, 1, 2, 3\}$, $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$. The coordinate y is called the extra coordinate. The parameter ω is the 5 dim (bulk) scalar curvature. l is the size parameter of the extra coordinate. We respect the periodicity: $y \to y + 2l$, and Z_2-parity: $y \leftrightarrow -y$. Instead of y, another coordinate z is also used. [8]

$$\text{Warped Metric (z-expression)} \quad ds^2 = \frac{1}{\omega^2 z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) = G_{MN}dX^M dX^N \ ,$$
$$|z| = \frac{1}{\omega}\mathrm{e}^{\omega|y|} \quad , \quad \frac{1}{\omega} < |z| < \frac{1}{T} \quad , \quad T \equiv \omega \mathrm{e}^{-\omega l} \quad ,$$
$$R_{MN} = 4\omega^2 G_{MN} \quad , \quad R = 20\omega^2 > 0 \quad , \quad \sqrt{-G} = \sqrt{-\det G_{MN}} = \frac{1}{(\omega|z|)^5} \quad , \tag{57}$$

where $(X^M) \equiv (x^\mu, z)$, $\{M, N = 0, 1, 2, 3, 5\}$. [9] The flat (5D Minkowski) limit is obtained by $\omega \to 0$ in the y-expression (56).

$$\text{Flat Metric} \quad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \quad , \quad (X^M) = (x^\mu, y) \quad , \quad -l \leq y \leq l \quad , \tag{58}$$

Traditional calculation[23, 24, 25] gives the Λ^5-divergent result for Casimir energy, on the above geometries, of 5D models. In the calculation, Casimir energy is expressed as the 5D space-momentum integral ($\int d^4p_E dy$ or $\int d^4p_E dz$) of some energy (density) function $F(\tilde{p}, y)$ or $F(\tilde{p}, z)$. (See Appendix for detail.) In ref.[24, 25], we claim the Λ^5-divergence comes from this 'naive' integration measure and should be replaced by some proper measure, based on close numerical calculation using some trial integration measures. Finally, Casimir energy of the free fields (electromagnetism, free scalar theory) is *proposed* to be replaced by the following *path-integral.*

$$\begin{aligned}
&\text{For Flat Geometry :}\\
&-\mathcal{E}_{Cas}(l, \Lambda) = \int_{1/\Lambda}^{l} d\rho \int_{r(0)=r(l)=\rho} \prod_{a,y} \mathcal{D}x^a(y) \left[\int dy' F_1(\frac{1}{r(y')}, y')\right]\\
&\qquad \times \exp\left[-\frac{1}{2\alpha'}\int_0^l \sqrt{r'^2 + 1}\ r^3 dy\right] , \ r' = \frac{dr}{dy},\\
&\text{For Warped Geometry :}\\
&-\mathcal{E}_{Cas}(\omega, T, \Lambda) = \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(1/\omega)=r(1/T)=\rho} \prod_{a,z} \mathcal{D}x^a(z) \left[\int dz' F_2(\frac{1}{r(z')}, z')\right]\\
&\qquad \times \exp\left[-\frac{1}{2\alpha'}\int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4}\sqrt{r'^2 + 1}\ r^3 dz\right] \quad , \quad r' = \frac{dr}{dz} \quad ,
\end{aligned} \tag{59}$$

where $r = \sqrt{\sum_{a=1}^4 (x^a)^2}$. [10] ($\{x^a|a = 1, 2, 3, 4\}$ is the *Euclideanized* coordinates of $\{x^\mu|\ \mu = 0, 1, 2, 3\}$, $x^0 = ix^4$.) F_1 and F_2 are some energy density functions and will appear in Appendix ((A.2) and (A.4)). Λ is the UV-cutoff parameter, $\mu \equiv \Lambda T/\omega$ is the IR-cutoff one and l is the

[8] z is defined by y as

$$z = \begin{cases} \frac{1}{\omega}\mathrm{e}^{\omega y} & y > 0 \\ 0 & y = 0 \\ -\frac{1}{\omega}\mathrm{e}^{-\omega y} & y < 0 \end{cases}$$

[9] T is *not* a temperature parameter but a IR parameter like l ($T = \omega \mathrm{e}^{-\omega l}$). The temperature appears as β^{-1}. See eq.(3).

[10] The case $\alpha' \to \infty$ in (59) is essentially the traditional definition of Casimir energy.

periodicity(IR) one. Note that the above expressions of $-\mathcal{E}_{Cas}$ are positive-definite. The above path-integrals are over all paths of 4 dim *hypersurfaces* defined by

$$\text{Flat Geometry:} \qquad \sqrt{\sum_{a=1}^{4}(x^a)^2} = r(y) \quad , \quad -l \leq y \leq l \quad ,$$
$$\text{Warped Geometry:} \qquad \sqrt{\sum_{a=1}^{4}(x^a)^2} = r(z) \quad , \quad \frac{1}{\omega} \leq |z| \leq \frac{1}{T} \quad . \tag{60}$$

See Fig.7 for the case of the N+1 dim space. This is a *closed string* configurartion. The *area* (4D volume) plays the role of *Hamiltonian* of the quantum statistical system $\{x^a\}$. F_i comes from the *matter-field* quantization and plays a role of the *energy 'operator'* in the path-integral over the 4D hyper-surface $r(y)$ or $r(z)$. The *string (surface) tension* parameter $1/2\alpha'$ is introduced. The new point, compared with the 5D Casimir energy calculation so far[23], is the introduction of the 'minimal area' factor $\exp(-\frac{1}{2\alpha'}\text{Area}) = \exp(-\frac{1}{2\alpha'}\int\sqrt{\det(g_{ab})}d^4x)$ where g_{ab} is the *induced* metric on the hyper-surface (60). We have shown, in this paper, the above-type *path-integral* very naturally appears in many quantum-statistical systems when we view them *geometrically*. The proposed quantities (59) are shown to be valid.

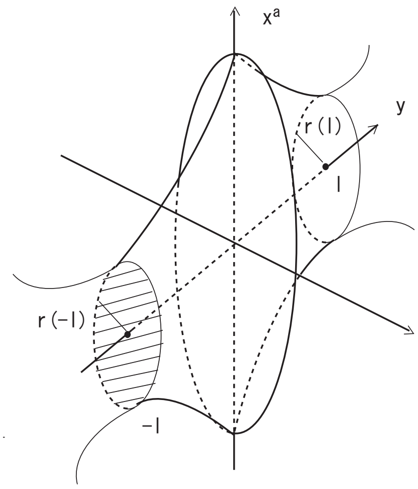

Figure 7. N(=2) dim hypersurface in N+1 dim (Euclidean flat) space $(x^1, x^2, \cdots, x^N, y)$. Sphere S^{N-1}(circles in the figure) at y has the radius $r(y)$.

The proposed expressions (59) of 5D Casimir energy can be evaluated numerically. We confirmed, not using the path-integral but using some effective approach (weight-function method), that they are given by as follows.

$$\text{Flat} \ : \ \frac{\mathcal{E}_{Cas}(l,\Lambda)}{\Lambda l} = -\frac{a}{l^4}(1-4c\ln(l\Lambda)) \ , \quad a \sim 2.5 \quad , \quad c>0 \quad , \quad c \sim O(10^{-3}) \quad ,$$
$$\text{Warped} \ : \ \frac{\mathcal{E}_{Cas}(\omega,T,\Lambda)}{\Lambda T^{-1}} = -a\omega^4(1-4c_1\ln(\frac{\Lambda}{\omega})-4c_2\ln(\frac{\Lambda}{T})) \ ,$$
$$a \sim 1.2 \ , \ c_1 \sim -0.11 < 0 \ , \ c_2 \sim 0.10 > 0 \ . \tag{61}$$

The boundary parameters *flow* as

$$\text{Flat} \ : \ \frac{\mathcal{E}_{Cas}(l,\Lambda)}{\Lambda l} = -\frac{a}{l'^4} \quad , \quad \beta = \frac{\partial}{\partial(\ln\Lambda)}\ln\frac{l'}{l} = c > 0 \quad ,$$
$$\text{Warped} \ : \ \frac{\mathcal{E}_{Cas}(\omega,T,\Lambda)}{\Lambda T^{-1}} = -a\omega'^4 \quad , \quad \beta = \frac{\partial}{\partial(\ln\Lambda)}\ln\frac{\omega'}{\omega} = -c_1 - c_2 \quad . \tag{62}$$

$\beta = c > 0$ in the flat case means the size of the extra world (periodicity l) shrinks as the measurement resolution becomes coarse. This image fits with the compactification of the extra axis in the higher-dimensional unified models.

6. Visco-Elastic System

The present approach gives a new method for the study of the visco-elastic system. Let us explain it using the harmonic oscillator with friction.

$$m\ddot{x} = -kx - \eta\dot{x}\ ,\ k:\ \text{spring constant},\quad \eta:\ \text{viscosity}\quad . \tag{63}$$

From this we obtain the "energy" at $t = t_0$.

$$\mathcal{E} = \left(\frac{1}{2}\dot{x}^2 + \frac{{\omega_1}^2}{2}x^2\right)|_{t_0} = \frac{1}{2}\dot{x}^2 + \frac{{\omega_1}^2}{2}x^2 + \eta'\int_{t_0}^{t}\left(\frac{dx(\tilde{t})}{d\tilde{t}}\right)^2 d\tilde{t}\ ,\quad {\omega_1}^2 = \frac{k}{m},\ \eta' = \frac{\eta}{m}. \tag{64}$$

This quantity is conserved (independent of t). The hysteresis term in the above expression represents the energy from the friction force. We can read the line elements in (X, t) space for this system.

$$\begin{aligned}
&\text{'Dirac' type}:\quad ds^2 = dX^2 + dt^2\left({\omega_1}^2X^2 + 2\eta'\int_{t_0}^{t}(dX)^2\frac{1}{d\tilde{t}}\right)\quad ,\\
&\text{Standard type}:\quad ds^2 = \frac{1}{dt^2}\left\{dX^2 + dt^2\left({\omega_1}^2X^2 + 2\eta'\int_{t_0}^{t}dX^2\frac{1}{d\tilde{t}}\right)\right\}^2
\end{aligned} \tag{65}$$

On a path $X = x(t)$, with the standard type, the *length* $L[x(t)]$ and the free energy F is given by $(t_0 = 0)$

$$\begin{aligned}
L[x(t)] = \int ds = \int_0^{\beta} dt\{\dot{x}^2 + {\omega_1}^2x^2 + 2\eta'\int_0^t(\frac{dx(\tilde{t})}{d\tilde{t}})^2 d\tilde{t}\}\quad ,\\
\mathrm{e}^{-\beta F(l,\beta)} = \int_{-l}^{l} d\rho\int_{x(0)=\rho,x(\beta)=\rho}\mathcal{D}x(t)\mathrm{e}^{-\frac{1}{2}L[x(t)]}\quad ,
\end{aligned} \tag{66}$$

where $2l$ is the periodicity, with which we impose X periodic.

$$X \to X + 2l\quad . \tag{67}$$

The energy E, the entropy S and the force f are given by

$$\begin{aligned}
&\text{Energy}\quad E(l,\beta) =< \frac{L}{2} >= \int_{-l}^{l} d\rho\int\mathcal{D}x(t)\frac{L[x(t)]}{2}\exp\{-\frac{1}{2}L[x(t)]\}\quad ,\\
&\text{Entropy}\quad S(l,\beta) = k_B\beta(E(l,\beta) - F(l,\beta))\quad ,\quad \text{Force}\quad f(l,\beta) = -\frac{\partial E(l,\beta)}{\partial l}\quad ,
\end{aligned} \tag{68}$$

where k_B is Boltzmann's constant.

The simple model (63) can be generalized as

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x} - \eta_1\dot{x}W_1(x) - \eta_2\dot{x}^2W_2(x)\quad , \tag{69}$$

where we assume the visco-elastic system is in the relatively slow motion. This assumption makes the velocity($\dot{x}$)-expansion of (69) valid. $V(x), W_1(x), W_2(x)$ are general functions. There are many choices:

$$\begin{aligned} V(x) &= \frac{1}{2}kx^2 \text{ (spring)},\ gx \text{ (rain-drop)},\ \cdots \\ W_1(x) &= 1 \quad , \quad x^2-1 \text{ (van der Pol eq.)},\ \cdots \\ W_2(x) &= 1 \quad , \quad x \quad , \quad x^2 \quad , \quad x^3 \quad , \quad \cdots \end{aligned} \tag{70}$$

The line elements in (X, t) space is given by

$$\begin{aligned} &\text{'Dirac' type :} \\ ds^2 = m\ dX^2 + dt^2 &\left\{ 2V(X) + 2\eta_1 \int_{t_0}^{t} (dX)^2 \frac{1}{d\tilde{t}} W_1(X) + 2\eta_2 \int_{t_0}^{t} (dX)^3 \frac{1}{(d\tilde{t})^2} W_2(X) \right\}, \\ &\text{Standard type :} \\ ds^2 = \frac{1}{dt^2} &\left[m\ dX^2 + dt^2 \left\{ 2V(X) + 2\eta_1 \int_{t_0}^{t} (dX)^2 \frac{1}{d\tilde{t}} W_1(X) \right.\right. \\ &\left.\left. +2\eta_2 \int_{t_0}^{t} (dX)^3 \frac{1}{(d\tilde{t})^2} W_2(X) \right\} \right]^2 . \end{aligned} \tag{71}$$

The above formulas are valid for the spacially-circling 1-dim visco-elastic fluid (Fig.8). The

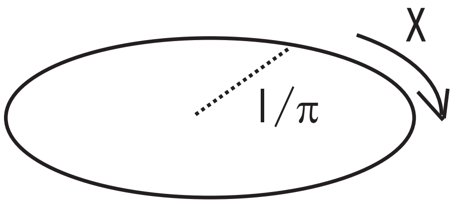

Figure 8. One dimensional, circling, visco-elastic flow. The radius l/π is sufficiently large.

global (statistical) physical quantities can be expressed by the path-integral in the same way as eq.(68).

For the spacially 3 dim visco-elastic system, we treat Navier-Stokes equation (1). We obtain the following relation.

$$\frac{D}{Dt}\left(\frac{\rho}{2}\vec{v}^2 + P - \rho\vec{x}\cdot\vec{g} \right) = \frac{\partial P}{\partial t} + \eta v^i \Delta v^i \quad . \tag{72}$$

From this, we obtain, for the case $\frac{\partial P}{\partial t} = 0$,

$$\mathcal{E} = \left(\frac{\rho}{2}\vec{v}^2 + P - \rho\vec{x}\cdot\vec{g} \right)|_{t_0} = \frac{\rho}{2}\vec{v}^2 + P - \rho\vec{x}\cdot\vec{g} - \eta\int_{t_0}^{t} v^i \Delta v^i d\tilde{t} \quad . \tag{73}$$

We can read, from the above result, the following line elements.

$$\begin{aligned} &\text{Dirac type :} \quad ds^2 = \rho\ dX^{i\ 2} + dt^2 \left\{ 2P(X) - 2\rho X^i g^i - 2\eta \int_{t_0}^{t} dX^i \frac{1}{d\tilde{t}} \Delta(dX^i \frac{1}{d\tilde{t}})\ d\tilde{t} \right\}, \\ &\text{Standard type :} \\ &ds^2 = \frac{1}{dt^2}\left[\rho\ dX^{i\ 2} + dt^2 \left\{ 2P(X) - 2\rho X^i g^i - 2\eta \int_{t_0}^{t} dX^i \frac{1}{d\tilde{t}} \Delta(dX^i \frac{1}{d\tilde{t}})\ d\tilde{t} \right\} \right]^2 . \end{aligned} \tag{74}$$

On a path $X^i = x^i(t)$, with the standard type, the *length* $L[\vec{x}(t)]$ and the free energy F are given by $(t_0 = 0)$

$$L[\vec{x}(t)] = \int ds = \int_0^\beta dt \left\{ \rho(\dot{x}^i)^2 + 2P(\vec{x}) - 2\rho\vec{x}\cdot\vec{g} - 2\eta\int_0^t \dot{x}^i\Delta\dot{x}^i\, d\tilde{t} \right\} \quad ,$$
$$\mathrm{e}^{-\beta F(l,\beta)} = \int_{-l}^{l} d\rho_1 \int_{-l}^{l} d\rho_2 \int_{-l}^{l} d\rho_3 \int_{x^i(0)=x^i(\beta)=\rho^i} \mathcal{D}x^i(t)\mathrm{e}^{-\frac{1}{2}L[\vec{x}(t)]} \quad , \qquad (75)$$

where $2l$ is the periodicity, with which we impose X^i periodic.

$$X^i \to X^i + 2l \qquad (i = 1, 2, 3) \quad . \qquad (76)$$

7. Discussion and Conclusion

Inspired by the recent new views on the quantum gravity, we have presented a geometrical approach to the general quantum statistical system. The idea lies in the introduction of the metric in the space of Feynman's path-integral. The length or the area gives the system Hamiltonian. In the application to 5D Casimir energy, the inverse temperature (or proper time) axis is played by the extra (5th) one. In the 4D case (ordinary Casimir energy), the regularization axis plays the role. Casimir force is explained from the *renormalization group flow*. The attractive force or the repulsive one corresponds to the positive β-function or the negative one. This geometrical approach also gives a new method to examine the dissipative system caused by friction. Taking simple visco-elastic models, we elaborate on how to choose the metric and present the path-integral expressions of the statistical quantities such as energy and entropy.

Appendix A. Traditional Definition of Casimir Energy in 5D Theories

The traditional definition of Casimir Energy of the 5D electromagnetic field theory is, for the flat case (58),

$$\mathrm{e}^{-l^4 E_{Cas}} = \int \mathcal{D}A \exp\left[i\int d^4x dy (\mathcal{L}^{5D}_{EM} + \mathcal{L}_{gauge}) \right]\Big|_{\mathrm{Euclid}} \quad ,$$
$$\mathcal{L}^{5D}_{EM}[A_M(X)] = -\frac{1}{4}F_{MN}F^{MN} \ ,\ F_{MN} = \partial_M A_N - \partial_N A_M \ ,\ \mathcal{L}_{gauge}[A_M(X)] = -\frac{1}{2}(\partial_M A^M)^2 \qquad (A.1)$$

The expression of E_{Cas} defined above, is given by

For Flat Geometry (5 dim elctromagnetism) :

$$E_{Cas}(l) = \int_{\tilde{p}\leq\Lambda} \frac{d^4p}{(2\pi)^4} \int_0^l dy (F_f^-(\tilde{p}, y) + 4F_f^+(\tilde{p}, y)) \quad ,$$
$$F_f^{\mp}(\tilde{p}, y) = -\int_{\tilde{p}}^{\infty} d\tilde{k} \frac{\mp\cosh\tilde{k}(2y - l) + \cosh\tilde{k}l}{2\sinh(\tilde{k}l)} \quad . \qquad (A.2)$$

The plus-minus symbol, $\mp$, indicates the contribution from Z_2-parity odd (-) and even (+) components. $\tilde{p}$ is the maginitude of 4D momentum $(p_a) = (p_1, p_2, p_3, p_4)$. The coincidence with the previous result[23] was confirmed[24]. As for the warped case (57), the traditional definition, for the 5D free scalar theory, is given by

$$\mathrm{e}^{-T^{-4}E_{Cas}} = \int \mathcal{D}\Phi\ \exp\left[i\int d^5X\sqrt{-G}\mathcal{L}^{5D}_s \right]\Big|_{\mathrm{Euclid}}$$
$$= \int \mathcal{D}\Phi(X) \exp\left[\int d^4xdz \frac{1}{(\omega z)^5}\frac{1}{2}\Phi\{\omega^2 z^2 \partial_a\partial^a\Phi + (\omega z)^5 \hat{L}_z\Phi\} \right] \quad ,$$

$$
\mathcal{L}_s^{5D}[\Phi(X);X] = -\frac{1}{2}\nabla^M\Phi\nabla_M\Phi - \frac{1}{2}m^2\Phi^2 \quad ,
$$
$$
\frac{1}{\omega} < |z| < \frac{1}{T} \quad , \quad \hat{L}_z = \frac{d}{dz}\frac{1}{(\omega z)^3}\frac{d}{dz} - \frac{m^2}{(\omega z)^5} \quad , \quad (m^2 = -4\omega^2) \quad . \tag{A.3}
$$

where $\hat{L}_z$ is the kinetic operator in the extra space (Bessel differential operator). Casimir energy E_{Cas} defined in (A.3) is explicitly given by

For Warped Geometry (5 dim Free Scalar, $m^2 = -4\omega^2$):

$$
-E^{\mp}_{Cas}(\omega,T) = \int \frac{d^4p_E}{(2\pi)^4}\bigg|_{\tilde{p}\leq\Lambda} \int_{1/\omega}^{1/T} dz\ F^{\mp}_w(\tilde{p},z) \quad , \quad F^{\mp}_w(\tilde{p},z) = \frac{1}{(\omega z)^3}\int_{\tilde{p}^2}^{\infty}\{G^{\mp}_k(z,z)\}dk^2 \quad ,
$$
$$
G^{\mp}_p(z,z') = \mp\frac{\omega^3}{2}z^2z'^2\frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z) \mp \mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\}\{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z') \mp \mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega}) - \mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})},
$$
$$
(\hat{L}_z - p^2 s(z))G^{\mp}_p(z,z') = \begin{cases} \epsilon(z)\epsilon(z')\hat{\delta}(|z|-|z'|) & \text{for P}=-1 \\ \hat{\delta}(|z|-|z'|) & \text{for P=1} \end{cases} \quad , \quad s(z) = \frac{1}{(\omega z)^3} \tag{A.4}
$$

where $\mathbf{I}_0$ and $\mathbf{K}_0$ are the modified Bessel functions of 0-th order.

Casimir energy defined above, which has been traditionally calculated, gives Λ^5-divergence. The integral $\int \frac{d^4p_E}{(2\pi)^4}dz$ ($\int \frac{d^4p}{(2\pi)^4}dy$) appearing in eq.(A.4) ((A.2)) corresponds to the summation over all positions in 5 dim bulk space $\int d^4xdz$ ($\int d^4xdy$). The above expression says E_{Cas} is the total sum of $F(r^{-1},z)$ ($F(r^{-1},y)$) over the bulk space positions. We notice here the Λ^5 divergence comes from the fact that we have overlooked some proper *integration measure*. The summation, or the *averaging* procedure (of F) should be properly defined at this stage. In the present standpoint we regard the *coordinate* system (x^a,z) ((x^a,y)) as the *quantum statistical* system and consider that the coordinate x^a is the *quantum mechanical* variable with the extra one z (y) as *Euclidean time*. The traditional treatment (simple summation over the set of positions) should be corrected by the present quantum (geometric) approach. We have proposed it should be done by the *path-integral* over all hypersurfaces in the bulk space (x^a,z) ((x^a,y)), as described in Sec.5. Hence the right expression of Casimir energy is given by (59).

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