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Stochastic processes with orthogonal polynomial eigenfunctions

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ABSTRACT

Markov processes which are reversible with either Gamma, Normal, Poisson or Negative Binomial stationary distributions in the Meixner class and have orthogonal polynomial eigenfunctions are characterized as being processes subordinated to well-known diffusion processes for the Gamma and Normal, and birth and death processes for the Poisson and Negative Binomial. A characterization of Markov processes with Beta stationary distributions and Jacobi polynomial eigenvalues is also discussed.

1. Introduction

Bivariate probability distributions whose marginal distributions are in the Meixner class, which includes the Normal, Gamma, Poisson and Negative Binomial distributions, can be constructed as bilinear orthogonal polynomial series. These bivariate distributions have the form

$$g(x, y) = f(x)f(y)\left\{1 + \sum_{n=1}^{\infty} \rho_n P_n(x)P_n(y)\right\}, \tag{1}$$

where $f(x), f(y)$ are the marginal distributions and $\{P_n(x), n = 0, 1, \ldots\}$ are orthogonal polynomials, scaled to be orthonormal so that

$$\mathbb{E}\big[P_m(X)P_n(X)\big] = \delta_{mn}.$$

The orthogonal polynomials are eigenfunctions of the distribution and $\{\rho_n\}$ are eigenvalues in the sense that

$$\mathbb{E}\big[P_n(Y) \mid X = x\big] = \rho_n P_n(x), \qquad \mathbb{E}\big[P_m(X)P_n(Y)\big] = \delta_{mn}\rho_n, \tag{2}$$

for $m, n = 0, 1, \ldots,$ with $\rho_0 = 1$. $|\rho_n| \le 1$ because of (2) and in the Meixner distributions $\{|\rho_n|\}$ is a decreasing sequence. The bivariate density expansion (1) is a mean square expansion of $g(x, y)/f(x)f(y)$ in the product set of orthonormal polynomials which holds when $\sum_{n=0}^{\infty} \rho_n^2 < \infty$.

The problem of characterizing correlation sequences $\{\rho_n\}$ such that a bilinear series (1) is a non-negative sum and thus a bivariate probability distribution has become known as a *Lancaster problem* after Lancaster and colleagues who studied

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such bivariate distributions [1–5]. In the Gamma, Poisson, and Negative Binomial classes where the random variables are non-negative a necessary and sufficient condition that (1) is a probability distribution is that

$$\rho_n = \int_0^1 z^n \mu(\mathrm{d}z), \tag{3}$$

where μ is a probability measure on $[0, 1]$ [2–4,6]. The class of distributions is a convex set with extreme correlation sequence points $\{z^n\}$ for $z \in [0, 1]$. X and Y are independent if $z = 0$ and $X = Y$ if $z = 1$. In the Normal class

$$\rho_n = \int_{-1}^1 z^n \mu(\mathrm{d}z), \tag{4}$$

where μ is a probability measure on $[-1, 1]$ [7]. The extreme sequence when $z = -1$ then corresponds to $X = -Y$.

The main step in the necessity proof of (3) and (4) is to note that (2) implies

$$\mathbb{E}\big[Y^n \mid X = x\big] = \rho_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

for constants $a_{n-1}, \ldots, a_0$, so

$$\rho_n = \lim_{x\to\infty} \mathbb{E}\big[(Y/x)^n \mid X = x\big]$$

and μ is identified with the limit distribution of Y/x given $X = x$ as $x \to \infty$. Meixner distributions with extreme correlation sequence points are well known. The standard bivariate Normal distribution has correlation $z \in [-1, 1]$ with an expansion in Hermite–Chebycheff polynomials. The bivariate Gamma (α) distribution has correlation $z \in [0, 1]$ and a Laguerre polynomial expansion. The bivariate Poisson distribution with correlation $z \in [0, 1]$ has a construction with random elements in common as $X = U + V, Y = V + W$, where U, V, W are independent Poisson with means $\lambda(1 - z), \lambda z, \lambda(1 - z)$ $\lambda > 0$. The distribution has an expansion in Poisson–Charlier polynomials. This bivariate Poisson distribution is the only infinitely-divisible distribution with Poisson marginals [8]. The bivariate Negative Binomial has correlation $z \in [0, 1]$ and an expansion in Meixner polynomials. If (X, Y) is bivariate Normal with correlation ρ, then $(U = \frac{1}{2}X^2, V = \frac{1}{2}Y^2)$ is bivariate Gamma $(\frac{1}{2})$ with correlation $z = \rho^2$, and (U, V) is infinitely-divisible with distributions in the extreme correlation sequence class [9].

In this paper we are interested in the characterization and behaviour of reversible stochastic processes with given stationary distributions in the Meixner class and orthogonal polynomial eigenfunctions. Schoutens [10] describes the basic processes that the characterizations are built from: the Ornstein–Uhlenbeck, Laguerre and Jacobi diffusion processes, p. 27–29; the $M/M/\infty$ Queue, and the linear birth and death process. These processes have orthogonal polynomial eigenfunctions and eigenvalues e^{-cnt} where $c \geq 0$ is a constant, apart from the Jacobi diffusion which has eigenvalues $\mathrm{e}^{-\frac{1}{2}n(n+\theta-1)t}$, where $\theta > 0$ is a constant. If the backward generator of one of these reversible processes is $\mathcal{L}$, and the eigenfunction, eigenvalue pairs are $\{P_n(x), \mathrm{e}^{-\lambda_n t}\}$, then $\mathcal{L}P_n(x) = -\lambda_n P_n(x)$. $\mathcal{L}$ is a differential operator for the diffusion processes and a difference operator for the birth and death processes. Mazet [11] obtained a characterization of these mentioned diffusion processes with polynomial eigenfunctions. Earlier diffusion process characterizations are [12,13]. General processes in the class are obtained by time subordination of particular processes. A modern treatment of stochastic processes using eigenfunction expansions where the algebraic structure is explained in [14].

Bochner [15] was the original author to consider such problems for the ultraspherical polynomials, whose weight distribution is not in the Meixner class. Correlation sequences have a different form from moment sequences in his characterization. He was also interested in characterizing eigenvalues of reversible Markov processes with ultraspherical polynomial eigenfunctions. Eigenvalues of a general reversible time-homogeneous Markov process with countable spectrum must satisfy Bochner's consistency conditions:

(i) $\{c_n(t)\}$ is a correlation sequence for each $t \geq 0$,
(ii) $c_n(t)$ is continuous in $t \geq 0$,
(iii) $c_n(0) = c_0(t) = 1$, and
(iv) $c_n(t + s) = c_n(t)c_n(s)$ for $t, s \geq 0$.

If there is a spectrum $\{c_n(t)\}$ with corresponding eigenfunctions $\{\xi_n\}$ then

$$\begin{aligned}
c_n(t+s)\xi_n\big(X(0)\big) &= \mathbb{E}\Big[\xi_n\big(X(t+s)\big) \mid X(0)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[\xi_n\big(X(t+s)\big) \mid X(s)\big] \mid X(0)\Big] \\
&= c_n(t)\mathbb{E}\Big[\xi_n\big(X(s)\big) \mid X(0)\Big] \\
&= c_n(t)c_n(s)\xi_n\big(X(0)\big),
\end{aligned}$$

showing (iv). If a stationary distribution exists and $X(0)$ has this distribution then the eigenfunctions can be scaled to be orthonormal on this distribution and the eigenfunction property is then

$$\mathbb{E}\Big[\xi_m\big(X(t)\big)\xi_n\big(X(0)\big)\Big] = c_n(t)\delta_{mn}.$$

$\{X(t), t \geq 0\}$ is a Markov process such that the transition distribution of $Y = X(t)$ given $X(0) = x$ is

$$f(x, y; t) = f(y)\left\{1 + \sum_{n=1}^{\infty} c_n(t)\xi_n(x)\xi_n(y)\right\}. \tag{5}$$

Griffiths [2] shows that in the Gamma class with Laguerre polynomial eigenfunctions the eigenvalues are characterized as having the form

$$c_n(t) = \exp\left\{-d_n t\right\}, \quad \text{where}$$
$$d_n = \int_0^1 \frac{1-y^n}{1-y} G(\mathrm{d}y) = an + \int_0^{1-} \frac{1-y^n}{1-y} G(\mathrm{d}y), \quad a \geq 0, \tag{6}$$

with G a finite positive measure on [0,1] with an atom of a at 1. Although the proof is for the Gamma class, it also holds in the Poisson, Negative Binomial and Normal classes. In the Normal class there is a similar characterization theorem where G has support on $[-1, 1]$. The process $\{X(t), t \geq 0\}$ can be constructed in the following way. Let $\{X_k, k = 0, 1, \ldots\}$ be a Markov chain with stationary distribution $f(y)$ and transition distribution of Y given $X = x$ corresponding to (1), with (3) holding, and $\{N(t), t \geq 0\}$ be an independent Poisson process of rate λ. Then (X_0, X_k) has a correlation sequence $\{\rho_n^k\}$ and the transition functions of

$$X(t) = X_{N(t)} \tag{7}$$

have the form (5), with

$$d_n = \lambda \int_0^1 (1 - z^n)\mu(\mathrm{d}z). \tag{8}$$

The general form (6) is obtained by choosing a pair (λ, μ_λ) such that

$$d_n = \lim_{\lambda\to\infty} \lambda \int_0^1 (1 - z^n)\mu_\lambda(\mathrm{d}z) = \int_0^1 \frac{1-y^n}{1-y} G(\mathrm{d}z). \tag{9}$$

If μ_λ is concentrated at a single point $1 - \lambda^{-1}$ then

$$d_n = \lim_{\lambda\to\infty} \lambda\left(1 - (1 - \lambda^{-1})^n\right) = n,$$

and G is concentrated at the single point 1 with atom 1 so $c_n(t) = \mathrm{e}^{-nt}$.

2. Subordinated processes in the Meixner class

Let $\{Z(t), t \geq 0\}$ be a subordinator, that is a Lévy process which is a process starting at $Z(0) = 0$ with $Z(t)$ an increasing right continuous jump process such that $Z(t) \to \infty$ as $t \to \infty$. The Laplace transform of $Z(t)$ has the form

$$\mathbb{E}\left[\mathrm{e}^{-\lambda Z(t)}\right] = \exp\left\{-t \int_0^\infty \left(1 - \mathrm{e}^{-\lambda u}\right) H(\mathrm{d}u)\right\}, \tag{10}$$

where H is a positive measure with

$$\int_0^\infty (1 \wedge u) H(\mathrm{d}u) < \infty. \tag{11}$$

[16] is a reference book for Lévy processes. If $\{X(t), t \geq 0\}$ is a Markov process then (for $a \geq 0$) $\{\widetilde{X}(t) = X(at + Z(t)), t \geq 0\}$ is a Markov process. The behaviour of $\widetilde{X}(t)$ can be very different from $X(t)$. If $X(t)$ has continuous sample paths, such as in a diffusion process, then $\widetilde{X}(t)$ is continuous between jumps of $Z(t)$ and discontinuous at jump points. A subordinated birth and death process can have multiple births and deaths at an instant. If $f(x, y; t)$ is the transition density of $Y = X(t)$ given $X(0) = x$, then the transition density of $\widetilde{X}(t)$ is

$$\widetilde{f}(x, y; t) = \mathbb{E}\left[f(x, y; at + Z(t))\right]. \tag{12}$$

Common subordinators are the Poisson process, the Gamma process, and positive Stable processes with index in [0, 1]. Stable processes have an infinite number of points of increase in finite intervals. Subordinated processes occur naturally in studying eigenvalues of processes with given eigenfunctions because random time subordination does not change the stationary distribution or eigenfunctions of the process. If the eigenvalues of $\{X(t), t \geq 0\}$ are $\{c_n(t)\}$ then, straight from (12), the eigenvalues of $\{\widetilde{X}(t), t \geq 0\}$ are

$$\widetilde{c}_n(t) = \mathbb{E}\left[c_n(at + Z(t))\right]. \tag{13}$$

If the eigenvalues $\{c_n(t)\}$ are characterized as being in a particular class for given stationary distribution and eigenfunctions, then $\{\widetilde{c}_n(t)\}$ must also be in this class. In this paper we are interested in the behaviour of the processes. The extreme points of the processes are familiar as either diffusion or birth and death processes, with general processes obtained by time subordination. We illustrate this by a new theorem for reversible Markov processes with stationary distributions in the Gamma, Poisson, Negative Binomial and Normal classes which have orthogonal polynomial eigenfunctions.

Markov processes with Beta stationary distributions and Jacobi polynomials as eigenfunctions have a very different form [15,17].

Theorem 1. *Let $\{\widetilde{X}(t), t \geq 0\}$ be a positive reversible Markov process with a given stationary distribution which is in the Gamma, Poisson or Negative Binomial distribution classes. Then $\{\widetilde{X}(t), t \geq 0\}$ has orthogonal polynomial eigenfunctions if and only if there is a subordinator $\{Z(t), t \geq 0\}$ and process $\{X(t), t \geq 0\}$ with eigenvalues $\{\mathrm{e}^{-nt}, n = 0, 1, \ldots\}$ such that $\{\widetilde{X}(t) = X\big(at + Z(t)\big), t \geq 0\}$. The processes $\{X(t), t \geq 0\}$ which are extreme points are described below. The class $\{\widetilde{X}(t), t \geq 0\}$ with Normal stationary distribution has a different characterization and is also described.*

Gamma. *$\{X(t), t \geq 0\}$ is a Laguerre diffusion process, which is a diffusion process model of a sub-critical branching process with immigration having backward generator (with $\alpha > 0$) of $\mathcal{L} = x\frac{\partial^2}{\partial x^2} + (-x+\alpha)\frac{\partial}{\partial x}$, a stationary Gamma ($\alpha$) distribution, Laguerre polynomial eigenfunctions and eigenvalues e^{-nt}.*

Poisson. *A Markov process $\{X(t), t \geq 0\}$ with Poisson (λ) stationary distribution, Poisson–Charlier polynomial eigenfunctions and eigenvalues e^{-nt} is such that $X(t)$ given $X(0) = x$ is distributed as $X(t) = \xi(t) + \eta(t)$, where $\xi(t), \eta(t)$ are independent and distributed as Poisson $\big(\lambda(1 - \mathrm{e}^{-t})\big)$ and Binomial $\big(x, \lambda\mathrm{e}^{-t}\big)$. The process describes an $M/M/\infty$ queue, a birth and death process with rates $\lambda_n = \lambda$, $\mu_n = n$.*

Negative Binomial. *The Negative Binomial process is the discrete analogue of the Laguerre diffusion. $\{X(t), t \geq 0\}$ is a linear birth and death process with rates $\lambda_n = (n + \beta)\lambda$, $\mu_n = n(1 + \lambda)$, $\beta, \lambda > 0$ with stationary distribution $(1 - \gamma)^\beta \Gamma(\beta + n)\gamma^n/\Gamma(\beta)n!$, $n = 0, 1, \ldots,$ with $\gamma = \lambda/(1 + \lambda)$, Meixner polynomial eigenfunctions, and eigenvalues e^{-nt}.*

Normal. *A reversible Markov processes $\{\widetilde{X}(t), t \geq 0\}$ with standard Normal stationary distribution and Hermite–Cheycheff polynomial eigenfunctions has eigenvalues which are characterized as*

$$\begin{aligned}\widetilde{c}_n(t) &= \exp\left\{-t\int_{-1}^{1}\frac{1-y^n}{1-y}G(\mathrm{d}y)\right\}\\ &= \exp\left\{-t\int_{0}^{1}\frac{1-y^n}{1-y}G(\mathrm{d}y) + t\gamma\int_0^1\big(1-(-z)^n\big)\nu(\mathrm{d}z)\right\},\end{aligned} \tag{14}$$

where G is a finite positive measure on $[-1, 1]$, $\gamma \geq 0$ and ν is a probability measure on $[0, 1]$. If G has support only on $[0, 1]$ the process $\{\widetilde{X}(t), t \geq 0\}$ is subordinated to the Ornstein–Uhlenbeck diffusion process $\{X(t), t \geq 0\}$, with eigenvalues e^{-nt}. There is not a continuous time subordinated process with a representation similar to (16) when G has negative support. If G has only negative support the following construction holds. Let $\{X_k, k = 0, 1, 2, \ldots\}$ be a reversible Gaussian Markov chain with standard Normal stationary distribution and Hermite–Chebycheff polynomial eigenfunctions such that X_{k+1} and X_k have correlation sequence

$$\rho_n = \int_0^1 (-z)^n\nu(\mathrm{d}z),$$

$\{N(t), t \geq 0\}$ an independent Poisson process of rate γ, and $\widetilde{X}(t) = X_{N(t)}$. Then $\{\widetilde{X}(t), t \geq 0\}$ has correlation sequence (14) when G only has negative support. The general process can also be constructed as a limit similar to (7)–(9) with μ, μ_λ, G having support on $[-1, 1]$.

Proof. If $\{\widetilde{X}(t), t \geq 0\}$ has a subordinated process representation then it has the same stationary distribution and polynomial eigenfunctions as $\{X(t), t \geq 0\}$. The eigenvalues are

$$\begin{aligned}\widetilde{c}_n(t) &= \mathrm{e}^{-ant}\mathbb{E}\big[\mathrm{e}^{-nZ(t)}\big]\\ &= \exp\left\{-ant - t\int_0^\infty\big(1-\mathrm{e}^{-nu}\big)H(\mathrm{d}u)\right\},\end{aligned} \tag{15}$$

from (13) and (10).

Conversely if $c_n(t) = \exp\{-d_n t\}$, with d_n of the form (6) then a subordinated process $\{Z(t), t \geq 0\}$ with Laplace transform

$$\mathbb{E}\big[\mathrm{e}^{-\lambda Z(t)}\big] = \exp\left\{-t\int_0^{1-}\frac{1-y^\lambda}{1-y}G(\mathrm{d}y)\right\} \tag{16}$$

has these correct eigenvalues. The two expressions (15) and (16) are seen to be equivalent by a change of variable $u = -\log(y)$, then for $y \in [0, 1)$

$$G(\mathrm{d}y) = \frac{1-y}{y}H\big(-\log(\mathrm{d}y)\big),$$

and G has an additional atom of a at 1. The finite measure conditions are satisfied because

$$\int_0^{1-} G(\mathrm{d}y) = \int_0^{\infty} \left(1 - \mathrm{e}^{-u}\right) H(\mathrm{d}u) \tag{17}$$

and the right-hand side of (17) being finite is equivalent to (11), seen from the inequalities

$$\frac{1}{2}u < 1 - \mathrm{e}^{-u} < u, \quad 0 \le u \le 1.$$

This proof holds for the positive processes and for the Normal when G in (14) has support only on [0,1]. The Normal case when G has negative support is dealt with as a construction described in the statement of the Theorem. □

3. Subordinated Jacobi diffusion

Bivariate distributions and processes with Jacobi polynomial eigenfunctions have a different form from the Gamma, Normal, Negative Binomial and Poisson cases because the Beta weight distribution has finite support. Let $\left\{R_n^{(\alpha,\beta)}(x)\right\}$ be the Jacobi polynomials, scaled to be orthogonal on the Beta distribution

$$g_{\alpha\beta}(x) = B(\alpha,\beta)^{-1} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1, \tag{18}$$

with $\alpha, \beta > 0$, normalized so that $R_n^{(\alpha,\beta)}(1) = 1$. Denote

$$h_n^{-1} = \mathbb{E}\left[R_n^{(\alpha,\beta)}(X)^2\right], \quad n = 0, 1, \ldots.$$

Gasper [17] showed under the conditions

$$\alpha < \beta, \quad \text{and} \quad \text{either } \alpha \ge 1/2 \text{ or } \alpha + \beta \ge 2, \tag{19}$$

that for $x, y, z \in [0, 1]$

$$K(x,y,z) = \sum_{n=0}^{\infty} h_n R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y) R_n^{(\alpha,\beta)}(z) \ge 0.$$

This implies that extreme correlation sequences in exchangeable bivariate Beta distributions with Jacobi polynomial eigenfunctions are the scaled Jacobi polynomials $\left\{R_n^{(\alpha,\beta)}(z)\right\}$, $z \in [0, 1]$, and that $\{\rho_n\}$ is a correlation sequence if and only if

$$\rho_n = \mathbb{E}\left[R_n^{(\alpha,\beta)}(Z)\right]$$

for some random variable Z in [0, 1].

A Wright–Fisher two-allele diffusion process where there are two types of genes a, A with mutation $a \to A$ at rate $\alpha/2$ and mutation $A \to a$ at rate $\beta/2$ has a transition density

$$f(x,y;t) = g_{\alpha\beta}(y)\left\{1 + \sum_{n=1}^{\infty} c_n(t) h_n R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y)\right\}, \quad 0 < x, y < 1, \tag{20}$$

where $\{X(t), t \ge 0\}$ is the relative frequency of type A genes, [18,22]. The eigenvalues are

$$c_n(t) = \exp\left\{-\frac{1}{2}n(n+\theta-1)t\right\}, \quad n = 0, 1, \ldots, \theta = \alpha + \beta.$$

Diffusion processes in Mathematical Genetics were first studied in [19]. The backward generator is

$$\mathcal{L} = \frac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2} + \frac{1}{2}(\alpha - \theta x)\frac{\partial}{\partial x}.$$

This process is also known as the Jacobi diffusion process.

A characterization of reversible Markov processes with stationary Beta distribution and Jacobi polynomial eigenfunctions, from [17], under (19), is that they have transition functions of the form (20) with $c_n(t) = \exp\{-d_n t\}$ where

$$d_n = \sigma n(n + \alpha + \beta - 1) + \int_0^{1-} \frac{1 - R_n^{(\alpha,\beta)}(z)}{1 - z} \mathrm{d}\nu(z),$$

$\sigma \ge 0$ and ν is a finite measure on [0, 1).

The Poisson kernel in orthogonal polynomial theory is

$$1 + \sum_{n=1}^{\infty} r^n h_n R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y),$$

which is non-negative for all $\alpha, \beta > 0, x, y \in [0, 1]$, and $0 \le r \lesssim 1$ [20]. A Markov process analogy to the Poisson Kernel is when the eigenvalues $c_n(t) = \exp\{-nt\}$. Following [15,21] let $\widetilde{X}(t) = X\big(Z(t)\big)$, where $\{Z(t), t \ge 0\}$ is a Lévy process with Laplace transform

$$\begin{aligned}&\exp\left\{-t\left[\sqrt{2\lambda + (\theta-1)^2/4} - \sqrt{(\theta-1)^2/4}\right]\right\}\\ &\quad = \exp\left\{-\frac{t}{\sqrt{2\pi}}\int_0^\infty \frac{\mathrm{e}^{-x(\theta-1)^2/8}}{x^{3/2}}\big(1-\mathrm{e}^{-x\lambda}\big)\mathrm{d}x\right\}.\end{aligned} \tag{21}$$

$Z(t)$ is a tilted positive stable process with index $\frac{1}{2}$ and density

$$\frac{t}{2\pi z^3}\exp\left\{-\frac{1}{2}\frac{t}{\sqrt{z}} - \frac{1}{8}(\theta-1)^2 z + \frac{1}{2}|(\theta-1)|t\right\}, \quad z > 0.$$

The usual stable density is obtained when $\theta = 1$. The eigenvalues of $\widetilde{X}(t)$ are, for $\theta \ge 1$,

$$\widetilde{c}_n(t) = \mathbb{E}\left[\exp\left\{-\frac{1}{2}n(n+\theta-1)Z(t)\right\}\right] = \exp\{-nt\}.$$

The process $\{\widetilde{X}(t), t \ge 0\}$ is a jump diffusion process, discontinuous at the jumps of $\{Z(t), t \ge 0\}$. Jump sizes increase as θ decreases. If $\theta < 1$ then for $n \ge 1$,

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}n(n+\theta-1)Z(t)\right\}\right] = \exp\{-nt\} \times \exp\{t(1-\theta)\}.$$

Let $\widetilde{f}(x, y; t)$ be the transition density of $\widetilde{X}(t)$, then the transition density with eigenvalues $\exp\{-nt\}$, $n \ge 0$ is

$$\mathrm{e}^{-t(1-\theta)}\widetilde{f}(x, y; t) + \big(1 - \mathrm{e}^{-t(1-\theta)}\big)g_{\alpha\beta}(y).$$

The subordinated process with this transition density is $X(\widehat{Z}(t))$, where $\widehat{Z}(t)$ is a similar process to $Z(t)$ but has an extra state ∞. $Z(t)$ is killed by a jump to ∞ at a rate $(1-\theta)$. Another possible construction does not kill the process $\widetilde{X}$, but restarts it in a stationary state drawn from the Beta distribution (18).

The transition density (20), where $c_n(t)$ has the general form $\widetilde{c}_n(t)$ of (15) can then be obtained by a composition of subordinators from the Jacobi diffusion with any $\alpha, \beta > 0$.

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