GENERATING FUNCTION METHOD FOR ORTHOGONAL POLYNOMIALS AND JACOBI-SZEGÖ PARAMETERS

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Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Suppose μ is not supported by a finite set of points. Then there exists a unique sequence $\{P_n(x)\}_{n=0}^{\infty}$ of orthogonal polynomials such that $P_n(x)$ is a polynomial of degree n with leading coefficient 1 and the equality $(x-\alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x)$ holds. The numbers $\{\alpha_n, \omega_n\}_{n=0}^{\infty}$ are called the Jacobi-Szegö parameters of μ. The family $\{P_n(x), \alpha_n, \omega_n\}_{n=0}^{\infty}$ determines the interacting Fock space of μ. In this paper we use the concept of generating function to give several methods for computing the orthogonal polynomials $P_n(x)$ and the Jacobi-Szegö parameters α_n and ω_n. We also describe how to identify the orthogonal polynomials in terms of differential or difference operators.

1. Accardi–Bożejko unitary isomorphism

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Assume that μ is not supported by a finite set of points and that the linear span of the monomials $\{x^n;\, n \geq 0\}$ is dense in the complex Hilbert space $L^2(\mu)$. Then we can apply the Gram-Schmidt orthogonalization procedure to the monomials $\{1, x, x^2, \ldots, x^n, \ldots\}$, in this order, to get orthogonal polynomials $\{P_0(x), P_1(x), \ldots, P_n(x), \ldots\}$. Here $P_n(x)$ is a polynomial of degree n with leading coefficient 1. It is well-known (see, e.g., the books by Chihara[5] and by Szegö[7]) that these orthogonal polynomials satisfy the recursion formula

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \quad n \geq 0, \tag{1}$$

where $\alpha_n \in \mathbb{R}$, $\omega_n > 0$ and by convention $\omega_0 = 1$, $P_{-1} = 0$. The numbers α_n and ω_n are called the Jacobi-Szegö parameters of μ.

Define a sequence $\{\lambda_n\}_{n=0}^{\infty}$ associated with the measure μ by

$$\lambda_n = \omega_0 \omega_1 \cdots \omega_n, \quad n \geq 0. \tag{2}$$

It can be easily checked that $\lambda_n = \int_{\mathbb{R}} |P_n(x)|^2 \, d\mu(x)$. Assume that the sequence satisfies the condition that $\inf_{n \geq 0} \lambda_n^{1/n} > 0$. Define a complex Hilbert space Γ_μ by

$$\Gamma_\mu = \left\{ (c_0, c_1, \ldots, c_n, \ldots) \,\middle|\, c_n \in \mathbb{C}, \sum_{n=0}^{\infty} \lambda_n |c_n|^2 < \infty \right\}$$

with norm $\| \cdot \|$ given by

$$\|(c_0, c_1, \ldots, c_n, \ldots)\| = \left(\sum_{n=0}^{\infty} \lambda_n |c_n|^2 \right)^{1/2}.$$

Let $\Phi_n = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the $(n+1)$st component. Define the *creation, annihilation,* and *neutral operators* a^+, a^-, and a^0 acting on Γ_μ, respectively, by

$$a^+ \Phi_n = \Phi_{n+1}, \quad a^- \Phi_n = \omega_n \Phi_{n-1}, \quad a^0 \Phi_n = \alpha_n \Phi_n, \quad n \geq 0,$$

where $\Phi_{-1} = 0$ by convention and α_n's and ω_n's are the Jacobi-Szegö parameters of μ. It can be easily shown that the operators a^+ and a^- are adjoint to each other.

The Hilbert space Γ_μ together with the operators $\{a^+, a^-, a^0\}$ is called the *interacting Fock space* associated with the measure μ. It has been shown by Accardi and Bożejko[1] that there exists a unitary isomorphism $U : \Gamma_\mu \to L^2(\mu)$ satisfying the conditions:

(1) $U\Phi_0 = 1$,
(2) $Ua^+U^*P_n = P_{n+1}$,
(3) $Ua^-U^*P_n = \omega_n P_{n-1}$,
(4) $U(a^+ + a^- + a^0)U^* = X$,

where the polynomials $P_n(x)$'s are given in Equation (1) and X is the multiplication operator by x.

Note that the Hilbert space is determined only by the numbers ω_n's, while the numbers α_n's and the polynomials $P_n(x)$'s are related to the unitary operator U. It is natural to ask the following question:

Question: Given a probability measure μ on $\mathbb{R}$, how to compute the associated orthogonal polynomials and the Jacobi-Szegö parameters $\{P_n, \alpha_n, \omega_n\}$?

In Section 2 we will explain the generating function method to derive the orthogonal polynomials. In Section 3 we will describe two ways for the computation of the Jacobi-Szegö parameters. In Section 4 we will discuss the computation of the orthogonal polynomials by differential and difference operators. In Section 5 we will list some important classical examples from the viewpoint of generating functions.

2. Pre-generating and generating functions

Let μ be a probability measure on $\mathbb{R}$ satisfying the conditions mentioned in Section 1. In a series of papers[2,3,4] we have introduced the generating function method to derive the associated orthogonal polynomials $\{P_n(x)\}$ and the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$.

A *pre-generating function* is a function $\varphi(t,x)$ which admits a power series expansion in t as follows:

$$\varphi(t,x) = \sum_{n=0}^{\infty} g_n(x) t^n,$$

where $g_n(x)$ is a polynomial of degree n and $\limsup_{n\to\infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty$. A *generating function* for μ is a pre-generating function of the form

$$\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n, \tag{3}$$

where $P_n(x)$'s are the orthogonal polynomials associated with μ as given in Equation (1). Note that a generating function for μ is not unique because we can always replace t in Equation (3) with ct for a nonzero constant $c \neq 1$ to get a different function $\psi(t,x)$. However, it is possible to have two essentially different generating functions for the same measure. For example, the following functions

$$\psi(t,x) = \frac{1}{1-2tx+t^2}$$
$$\psi(t,x) = \left(\frac{2}{(1-2tx+t^2)\big(1-tx+\sqrt{1-2tx+t^2}\,\big)}\right)^{1/2}$$

are generating functions for the measure $d\mu(x) = \frac{2}{\pi}\sqrt{1-x^2}\,dx$, $|x| \le 1$.

Suppose $\varphi(t,x)$ is a pre-generating function. Consider its multiplicative renormalization defined by

$$\psi(t,x) = \frac{\varphi(t,x)}{E_\mu \varphi(t,\cdot)}.$$

Theorem 2.1. *The multiplicative renormalization $\psi(t,x)$ is a generating function for μ if and only if $E_\mu[\psi(t,\cdot)\psi(t,\cdot)]$ is a function of ts.*

If we can check that $E_\mu[\psi(t,\cdot)\psi(t,\cdot)]$ is a function of ts, then by the above theorem $\psi(t,x)$ is a generating function. We can expand $\psi(t,x)$ as a power series in t to get

$$\psi(t,x) = \sum_{n=0}^{\infty} Q_n(x)t^n,$$

where $Q_n(x)$ is a polynomial of degree n. Let a_n be the leading coefficient of $Q_n(x)$ and let $P_n(x) = Q_n(x)/a_n$. Then the polynomials $\{P_n(x)\}$ are the orthogonal polynomials satisfying Equation (1) for the measure μ.

In the papers[2,3,4] we have applied the generating function method to pre-generating functions of the form

$$\varphi(t,x) = h\big(\rho(t)x\big),$$

where $h(x) = e^x$ or $h(x) = (1-x)^c$ and $\rho(t)$ is a function to be derived so that the condition in Theorem 2.1 is satisfied.

Case 1: $h(x) = e^x$

measure	polynomials
Gaussian	Hermite
Poisson	Charlier
gamma	Laguerre
negative binomial	Meixner

Case 2: $h(x) = (1-x)^c$

measure	polynomials
uniform	Legendre
arcsine	Chebyshev of 1st kind
semi-circle	Chebyshev of 2nd kind
beta-type	Gegenbauer

The above polynomials are derived from the power series expansion of the resulting generating functions. Consequently they are expressed in terms of sums of monomials. In Section 4 we will use differential and difference operators to identify these polynomials.

Recently all measures of exponential type have been derived in the paper[6]. In particular, the probability law of the Lévy stochastic area is in this class.

3. Computation of orthogonal polynomials and Jacobi-Szegö parameters

Suppose $\psi(t,x)$ is a generating function for μ. The object is to compute the orthogonal polynomials $\{P_n(x)\}$ and the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ from $\psi(t,x)$. Recall that $\lambda_n = \omega_0\omega_1\cdots\omega_n$, $n \geq 0$, as defined by Equation (2). The following theorem has been proved in our paper[3].

Theorem 3.1. *Let $\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n$ be a generating function for μ. Then we have*

$$\lim_{t\to 0} \psi\Big(t, \frac{x}{t}\Big) = \sum_{n=0}^{\infty} a_n x^n, \tag{4}$$

$$E_\mu\big[\psi(t,\cdot)^2\big] = \sum_{n=0}^{\infty} a_n^2 \lambda_n t^{2n}, \tag{5}$$

$$E_\mu\big[x\psi(t,\cdot)^2\big] = \sum_{n=0}^{\infty} \Big(a_n^2 \lambda_n \alpha_n t^{2n} + 2a_n a_{n-1} \lambda_n t^{2n-1}\Big), \tag{6}$$

where $a_{-1} = 0$ by convention.

Thus once we have found a generating function for μ, then we can compute $\{P_n, \alpha_n, \omega_n\}$ as follows:

$$\psi(t,x) \;\longmapsto\; \{a_n, P_n\} \;\longmapsto\; \{\lambda_n\} \;\longmapsto\; \{\alpha_n, \omega_n\}$$

Namely, first expand $\psi(t,x)$ as a power series in t to get a_n and $P_n(x)$, which is expressed as a sum of monomials. If we are not interested in finding P_n, then we do not have to expand $\psi(t,x)$ as a power series in t. We can simply use Equation (4) to find a_n. Then we use Equation (5) to find λ_n, which can be used in turn to find ω_n since $\omega_n = \lambda_n/\lambda_{n-1}$, $n \geq 1$, $\omega_0 = 1$. Finally we can use Equation (6) to derive α_n.

In our paper[3] we have used this method to compute $\{P_n, \alpha_n, \omega_n\}$ for those measures listed in Section 2. However, the computation is somewhat complicated due to the following difficulties:

Difficulties: There are several difficulties in applying Theorem 3.1.

(1) It might be difficult to find the series expansion of $\psi(t,x)$ in t.
(2) The computation of $E_\mu\big[\psi(t,\cdot)^2\big]$ in Equation (5) might be very complicated.
(3) The computation of $E_\mu\big[x\psi(t,\cdot)^2\big]$ in Equation (6) is even more involved.
(4) The orthogonal polynomials are expressed as sums of monomials. How to find the close forms for them?

Resolution: Here are some ideas to overcome the above difficulties:

(a) Find a computation method without having to use the power series expansion of $\psi(t,x)$ in t.
(b) Find a system of linear equations for the Jacobi-Szegö parameters.
(c) Determine $P_n(x)$ by a differential or difference operator.
(d) Find a differential equation satisfied by $P_n(x)$. This equation is determined by the measure μ.
(e) Determine $\{\alpha_n, \omega_n\}$ from the eigenvalues of a differential operator.

We have made some progress regarding to Items (a), (b), and (c). The key idea is to use the series expansions of $\psi(t,0)$ and $\partial_x\psi(t,x)|_{x=0}$ in t. Then we can avoid the difficulty (1).

So, suppose $\varphi(t,x)$ is a pre-generating function and assume that its multiplicative renormalization $\psi(t,x) = \varphi(t,x)/E_\mu\varphi(t,\cdot)$ is a generating function for μ. Define three functions $A(x)$, $B(t)$, and $C(t)$ with their respective power series expansions by

$$A(x) = \lim_{t\to 0} \psi\Big(t, \frac{x}{t}\Big) = \sum_{n=0}^{\infty} a_n x^n, \tag{7}$$

$$B(t) = \psi(t,0) = \sum_{n=0}^{\infty} b_n t^n, \tag{8}$$

$$C(t) = \frac{\partial}{\partial x}\psi(t,x)\Big|_{x=0} = \sum_{n=0}^{\infty} c_n t^n. \tag{9}$$

The next theorem is from our paper[4].

Theorem 3.2. *Suppose $\psi(t,x) = \varphi(t,x)/E_\mu\varphi(t,\cdot)$ is a generating function for μ. Let a_n, b_n, c_n be the numbers defined in Equations (7)–(9). Assume that $b_n c_{n-1} \neq b_{n-1}c_n$. Then the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ are the unique solution of the system of the linear equations:*

$$\begin{cases} \dfrac{b_n}{a_n}\alpha_n + \dfrac{b_{n-1}}{a_{n-1}}\omega_n = -\dfrac{b_{n+1}}{a_{n+1}}, \\ \dfrac{c_n}{a_n}\alpha_n + \dfrac{c_{n-1}}{a_{n-1}}\omega_n = -\dfrac{c_{n+1}}{a_{n+1}} + \dfrac{b_n}{a_n}. \end{cases} \tag{10}$$

Thus we can compute the Jacobi-Szegö parameters as follows:

$$\psi(t,x) \longmapsto \{A(t), B(t), C(t)\} \longmapsto \{a_n, b_n, c_n\} \longmapsto \{\alpha_n, \omega_n\}$$

Now, consider the special case when the pre-generating function $\varphi(t,x)$ is of the form

$$\varphi(t,x) = h\big(\rho(t)x\big), \tag{11}$$

where $\rho(t) = \sum_{n=1}^{\infty} \rho_n t^n$ is an analytic function near $x=0$ with $\rho_1 \neq 0$ and $h(x) = \sum_{n=0}^{\infty} h_n x^n$ is an analytic function near $x=0$ with $h_0 = 1$, $h_n \neq 0$ for all $n \geq 1$ and there exists $t_1 > 0$ such that $\rho(tx)$ is analytic in x on the support of μ for all $|t| < t_1$, and $\limsup_{n\to\infty} \big(|h_n|\, \|x^n\|_{L^2(\mu)}\big)^{1/n} < \infty$. In this case the function $A(x)$ in Equation (7) is given by

$$A(x) = \lim_{t\to 0} \frac{h\Big(\rho(t)\dfrac{x}{t}\Big)}{E_\mu h\big(\rho(t)\,\cdot\big)} = \frac{h(\rho_1 x)}{h_0} = h(\rho_1 x).$$

Therefore, $a_n = h_n \rho_1^n$ and so Equation (10) becomes

$$\begin{cases} \dfrac{\rho_1 b_n}{h_n}\alpha_n + \dfrac{\rho_1^2 b_{n-1}}{h_{n-1}}\omega_n = -\dfrac{b_{n+1}}{h_{n+1}}, \\ \dfrac{\rho_1 c_n}{h_n}\alpha_n + \dfrac{\rho_1^2 c_{n-1}}{h_{n-1}}\omega_n = -\dfrac{c_{n+1}}{h_{n+1}} + \dfrac{\rho_1 b_n}{h_n}. \end{cases} \tag{12}$$

In particular, when μ is symmetric, we have $\alpha_n = 0$ for all n. Then Equation (12) yields the values of ω_n's as follows:

$$\begin{cases} \omega_{2m+1} = -\dfrac{b_{2m+2}h_{2m}}{\rho_1^2 b_{2m}h_{2m+2}}, \\ \omega_{2m} = -\dfrac{c_{2m+1}h_{2m-1}}{\rho_1^2 c_{2m-1}h_{2m+1}} + \dfrac{b_{2m}h_{2m-1}}{\rho_1 c_{2m-1}h_{2m}}. \end{cases} \tag{13}$$

4. Orthogonal polynomials in terms of differential or difference operators

Suppose $\psi(t,x)$ is a generating function for μ. Then we can expand $\psi(t,x)$ as a power series in t

$$\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n,$$

where $P_n(x)$'s are the orthogonal polynomials associated with μ. As pointed out in Section 3, these polynomials are expressed as sums of monomials. Since it might be difficult to compute the power series expansion of $\psi(t,x)$, it is desirable to determine or identify these polynomials in some other ways, namely, we raise the following

Question: How to determine or identify the orthogonal polynomials $\{P_n(x)\}$ without using the power series expansion of $\psi(t,x)$ in t?

For absolutely continuous measures we have the next two theorems from our paper[4]. Recall that $A(x) = \sum_{n=0}^{\infty} a_n x^n$ from Equation (7) and that $\lambda_n = \omega_0\omega_1\cdots\omega_n$.

Theorem 4.1. *Let μ be a measure on (a,b) with a smooth density function $\theta(x)$ and assume that $\psi(t,x)$ is a generating function for μ. Suppose $q_n(x)$ is a smooth function satisfying the conditions:*

(a) $\dfrac{1}{\theta(x)}\, D_x^n\big[q_n(x)\theta(x)\big] \in L^2(\mu).$

(b) $D_x^k\big[q_n(x)\theta(x)\big] = 0$ *at* $x = a,\, b$ *for all* $0 \le k < n$.

Then the orthogonal polynomial $P_n(x)$ *associated with* μ *is given by*

$$P_n(x) = \frac{1}{k_n\theta(x)}\, D_x^n\big[q_n(x)\theta(x)\big]$$

with some constant k_n *if and only if*

$$\int_a^b \big[D_x^n\psi(t,x)\big]q_n(x)\, d\mu(x) = d_n t^n$$

with some constant d_n*. In this case,* d_n *and* k_n *are related by*

$$d_n = (-1)^n\lambda_n a_n k_n.$$

For the special case when $\varphi(t,x)$ is of the form in Equation (11), the generating function is given by

$$\psi(t,x) = \frac{\varphi(t,x)}{E_\mu\varphi(t,\cdot)}.$$

In this case, we can make use of the function $B(t)$ defined in Equation (8) and take $q_n(x) = \theta_n(x)/\theta(x)$ in Theorem 4.1 to get the next theorem.

Theorem 4.2. *Let* μ *be a measure on* (a,b) *with a smooth density function* $\theta(x)$ *and assume that*

$$\psi(t,x) = \frac{h\big(\rho(t)x\big)}{E_\mu\big[h\big(\rho(t)\cdot\big)\big]}$$

is a generating function for μ*. Suppose* $\theta_n(x)$ *is a smooth function with support in* $[a,b]$ *satisfying the conditions:*

(a) $\dfrac{1}{\theta(x)}\, D_x^n\big[\theta_n(x)\big] \in L^2(\mu)$.

(b) $D_x^k\big[\theta_n(x)\big] = 0$ *at* $x = a,\, b$ *for all* $0 \le k < n$.

Then the orthogonal polynomial $P_n(x)$ *associated with* μ *is given by*

$$P_n(x) = \frac{1}{k_n\theta(x)}\, D_x^n\big[\theta_n(x)\big]$$

with some constant k_n *if and only if*

$$\int_a^b h^{(n)}\big(\rho(t)x\big)\theta_n(x)\, dx = \frac{d_n}{B(t)}\Big(\frac{t}{\rho(t)}\Big)^n$$

with some constant d_n*. In this case,* d_n *and* k_n *are given by*

$$d_n = n!\rho_1^n h_n, \quad k_n = (-1)^n\frac{n!}{\lambda_n}.$$

For discrete measures on the set $\mathbb{N}_0 = \{0, 1, 2, \ldots, n, \ldots\}$ we have the analogous results in the following two theorems from our paper[4]. Define the right- and left-hand difference operators Δ_{x+} and Δ_{x-} acting on functions defined on $\mathbb{N}_0$ by

$$\Delta_{x+} f(x) = f(x+1) - f(x),$$
$$\Delta_{x-} f(x) = f(x) - f(x-1),$$

where $f(-1) = 0$ by convention. We have $\Delta_{x+}^n f(x-n) = \Delta_{x-}^n f(x)$.

Theorem 4.3. *Let μ be a measure on $\mathbb{N}_0$ and let $\theta(x) = \mu(\{x\})$, $x \in \mathbb{N}_0$. Assume that $\psi(t,x)$ is a generating function for μ. Suppose $q_n(x)$ is a function on $\mathbb{N}_0$ satisfying the conditions:*

(a) $\dfrac{1}{\theta(x)} \Delta_{x-}^n \big[q_n(x)\theta(x)\big] \in L^2(\mu)$.

(b) $\Delta_{x+}^k \big[q_n(x)\theta(x)\big] = 0$ *at* $x = 0, \infty$ *for all* $0 \le k < n$.

Then the orthogonal polynomial $P_n(x)$ associated with μ is given by

$$P_n(x) = \frac{1}{k_n \theta(x)} \Delta_{x-}^n \big[q_n(x)\theta(x)\big]$$

with some constant k_n if and only if

$$\sum_{x=0}^{\infty} \big[\Delta_{x+}^n \psi(t,x)\big] q_n(x)\theta(x) = d_n t^n$$

with some constant d_n. In this case, d_n and k_n are related by

$$d_n = (-1)^n \lambda_n a_n k_n.$$

The next theorem is analogous to Theorem 4.2. It is a special case of Theorem 4.3 for the multiplicative renormalization of the pre-generating function $\varphi(t,x) = e^{\rho(t)x}$.

Theorem 4.4. *Let μ be a measure on $\mathbb{N}_0$ and let $\theta(x) = \mu(\{x\})$, $x \in \mathbb{N}_0$. Assume that the multiplicative renormalization*

$$\psi(t,x) = B(t) e^{\rho(t)x} \quad \left(B(t) = \frac{1}{E_\mu\big[e^{\rho(t)\cdot}\big]} \right)$$

is a generating function for μ, Here $\rho(t) = \sum_{n=1}^{\infty} \rho_n t^n$ is analytic with $\rho_1 \neq 0$. Suppose $\theta_n(x)$ is a probability mass function on $\mathbb{N}_0$ for each n satisfying the conditions:

(a) $\dfrac{1}{\theta(x)}\,\Delta_{x-}^{n}\big[\theta_n(x)\big] \in L^2(\mu)$.

(b) $\Delta_{x+}^{k}\big[\theta_n(x)\big] = 0$ *at* $x = 0, \infty$ *for all* $0 \le k < n$.

Then the orthogonal polynomial $P_n(x)$ *associated with* μ *is given by*

$$P_n(x) = \frac{1}{k_n\theta(x)}\,\Delta_{x-}^{n}\big[\theta_n(x)\big]$$

with some constant k_n *if and only if*

$$\sum_{x=0}^{\infty} e^{\rho(t)x}\theta_n(x) = \frac{d_n t^n}{B(t)\big(e^{\rho(t)}-1\big)^n}$$

with some constant d_n. *In this case,* d_n *and* k_n *are given by*

$$d_n = \rho_1^n, \quad k_n = (-1)^n\frac{n!}{\lambda_n}.$$

5. Classical examples

In our paper[3] we have used our method to derive generating functions $\psi(t,x)$ and $\{P_n(x), \alpha_n, \omega_n\}$ for those measures listed in Section 2. But some computations, for example in identifying the polynomials $P_n(x)$, are rather complicated. In the paper[4] the computations are simplified by using Equations (12) and (13) and Theorems 4.2 and 4.4. Recall that $\omega_0 = 1$.

Example 5.1. (*Gaussian measure*) $\theta(x) = \frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2}$, $x \in \mathbb{R}$.

$$\begin{aligned}
&\psi(t,x) = e^{tx-\sigma^2 t/2},\\
&P_n(x) = (-\sigma^2)^n e^{x^2/2\sigma^2} D_x^n\big[e^{-x^2/2\sigma^2}\big] \quad \text{(Hermite polynomial)},\\
&\alpha_n = 0, \quad n \ge 0,\\
&\omega_n = \sigma^2 n, \quad n \ge 1.
\end{aligned}$$

Example 5.2. (*Gamma distribution*) $\theta(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}$, $x > 0$. Here $\alpha > 0$.

$$\begin{aligned}
&\psi(t,x) = (1+t)^{-\alpha}e^{\frac{t}{1+t}x},\\
&P_n(x) = (-1)^n x^{-\alpha+1}e^x D_x^n\big[x^{n+\alpha-1}e^{-x}\big] \quad \text{(Laguerre polynomial)},\\
&\alpha_n = 2n+\alpha, \quad n \ge 0,\\
&\omega_n = n(n+\alpha-1), \quad n \ge 1.
\end{aligned}$$

Example 5.3. (*Beta-type distribution*)

$$\theta(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})} (1-x^2)^{\beta-1/2}, \quad |x| < 1,$$

where the parameter $\beta > -1/2$, $\beta \neq 0$.

$$\psi(t,x) = \frac{1}{(1-2tx+t^2)^\beta}, \tag{14}$$

$$P_n(x) = (-1)^n \frac{\Gamma(n+2\beta)}{\Gamma(2n+2\beta)} (1-x^2)^{-\beta+1/2} D_x^n \left[(1-x^2)^{n+\beta-1/2}\right]$$

$$\text{(Gegenbauer polynomial)},$$

$$\alpha_n = 0, \quad n \geq 0,$$

$$\omega_n = \frac{n(n-1+2\beta)}{4(n+\beta)(n-1+\beta)}, \quad n \geq 1.$$

We have two special cases. When $\beta = 1/2$, the measure μ is the *uniform measure* on $[-1,1]$ and $P_n(x)$ is the Legendre polynomial. When $\beta = 1$, the measure μ is the *semi-circle distribution* on $[-1,1]$ and $P_n(x)$ is the Chebyshev polynomial of the second kind, which can be verified to equal

$$P_n(x) = \frac{1}{2^n} \frac{\sin\left[(n+1)\cos^{-1} x\right]}{\sin\left[\cos^{-1} x\right]}, \quad n \geq 0.$$

Note that we have to exclude $\beta = 0$ in this example since the function in Equation (14)) is not a generating function when $\beta = 0$. The next example takes care of this case.

Example 5.4. (*Arcsine distribution*) $\theta(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$.

$$\psi(t,x) = \frac{1-t^2}{(1-2tx+t^2)},$$

$$P_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{1}{2^{n-1}} \cos\left[n \cos^{-1} x\right], & \text{if } n \geq 1. \end{cases}$$

$$\text{(Chebyshev polynomial of the first kind)},$$

$$\alpha_n = 0, \quad n \geq 0,$$

$$\omega_n = \begin{cases} 1/2, & \text{if } n = 1, \\ 1/4, & \text{if } n \geq 2. \end{cases}$$

Example 5.5. (*Poisson measure*) $\theta(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $x \in \mathbb{N}_0$, $\lambda > 0$.

$$\begin{aligned}
&\psi(t,x) = e^{-\lambda t}(1+t)^x, \\
&P_n(x) = (-1)^n \lambda^{-x}\Gamma(x+1)\Delta_{x-}^n\Big[\frac{\lambda^{x+n}}{\Gamma(x+1)}\Big] \\
&\qquad\quad = (-1)^n \lambda^{-x}\Gamma(x+1)\Delta_{x+}^n\Big[\frac{\lambda^{x}}{\Gamma(x-n+1)}\Big] \\
&\qquad\qquad\quad \text{(Charlier polynomial)}, \\
&\alpha_n = n+\lambda, \quad n \geq 0, \\
&\omega_n = \lambda n, \quad n \geq 1.
\end{aligned}$$

Example 5.6. (*Negative binomial measure*)

$$\theta(x) = p^r\binom{-r}{x}(-1)^x(1-p)^x, \quad x \in \mathbb{N}_0,$$

where $r > 0$ and $0 < p < 1$.

$$\begin{aligned}
&\psi(t,x) = (1+t)^x\big(1+(1-p)t\big)^{-x-r}, \\
&P_n(x) = (-1)^n\frac{1}{p^n}\frac{\Gamma(x+1)}{\Gamma(x+r)}(1-p)^{-x}\Delta_{x+}^n\Big[\frac{\Gamma(x+r)}{\Gamma(x-n+1)}(1-p)^x\Big] \\
&\qquad\qquad \text{(Meixner polynomial)}, \\
&\alpha_n = \frac{(2-p)n + r(1-p)}{p}, \quad n \geq 0, \\
&\omega_n = \frac{n(n+r-1)(1-p)}{p^2}, \quad n \geq 1.
\end{aligned}$$

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