A NOTE ON THE BESSEL POLYNOMIALS

By L. Carlitz

1. Krall and Frink [10] defined the polynomial

$$y_n(x) = \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!} \left(\frac{x}{2}\right)^r, \tag{1.1}$$

which satisfies the differential equation

$$x^2y'' + (2x+2)y' - n(n+1)y = 0, \tag{1.2}$$

and proved a number of properties of $y_n(x)$ as well as of a generalized polynomial $y_n(x, a, b)$ which reduces to $y_n(x)$ for $a = b = 2$. Burchnall [3] and Grosswald [9] found additional properties of these polynomials; see also [13], [14]. Burchnall defined

$$\theta_n(x) = x^n y_n\left(\frac{1}{x}\right). \tag{1.3}$$

It is also convenient to let

$$y_{-n}(x) = y_{n-1}(x), \qquad \theta_{-n}(x) = x^{1-2n}\theta_{n-1}(x). \tag{1.4}$$

It follows from (1.4) that, for example, the recurrence formula satisfied by y_n and θ_n hold for all integral n.

In the present note we derive some more formulas satisfied by the polynomials. Some of the results are simpler when stated in terms of the polynomial $f_n(x)$ defined by

$$f_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right) = x\theta_{n-1}(x); \tag{1.5}$$

it is convenient to complete the definition by means of

$$f_{-n}(x) = x^{-1-2n}f_{n+1}(x), \qquad f_0(x) = 1. \tag{1.6}$$

2. Since [3; 64]

$$\theta_{n+1} = (2n+1)\theta_n + x^2\theta_{n-1}, \tag{2.1}$$

$$\theta_n' = \theta_n - x\theta_{n-1}, \tag{2.2}$$

it follows at once that

$$f_{n+1} = (2n-1)f_n + x^2 f_{n-1}, \tag{2.3}$$

$$f_n' = f_n - xf_{n-1}. \tag{2.4}$$

These formulas hold for all integral n.

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In the next place the formula [10; 106]

$$\exp\left(\frac{1-(1-2xt)^{\frac{1}{2}}}{x}\right) = \sum_{n=0}^{\infty} y_{n-1}(x)t^n/n!$$

implies

$$\exp\{x(1-(1-2t)^{1/2})\} = \sum_0^{\infty} f_n(x)t^n/n!, \tag{2.5}$$

which is equivalent to

$$e^{2xz} = \sum_0^{\infty} 2^n f_n(x)\,(z-z^2)^n/n!. \tag{2.6}$$

An immediate consequence of (2.5) is

$$f_n(u+v) = \sum_{r=0}^{n}\binom{n}{r} f_r(u)f_{n-r}(v). \tag{2.7}$$

(Indeed a formula like (2.7) holds for the sequence $\phi_n(x)$ defined formally by

$$e^{xg(t)} = \sum_0^{\infty} \phi_n(x)t^n/n!,$$

where $g(t)$ is an arbitrary power series in t.) Note in particular that for $v = -u$, (2.7) becomes

$$\sum_{r=0}^{n}\binom{n}{r} f_r(u)f_{n-r}(-u) = 0 \qquad (n \geq 1).$$

In the next place, if we expand each side of (2.6) in powers of z, we get

$$x^n = \sum_{2r\leq n}(-1)^r 2^{-r}\frac{n!}{r!(n-2r)!} f_{n-r}(x); \tag{2.8}$$

in terms of θ_n this is

$$x^n = \sum_{2r\leq n+1}(-1)^r 2^{-r}\frac{(n+1)!}{r!(n+1-2r)!}\theta_{n-r}(x). \tag{2.9}$$

Differentiating each side of (2.5) with respect to x we get

$$(1-(1-2t)^{\frac{1}{2}})e^{x(1-(1-2t)^{1/2})} = \sum_0^{\infty} f_n'(x)t^n/n!.$$

Since

$$1-(1-2t)^{\frac{1}{2}} = \sum_1^{\infty}\frac{1\cdot 3\cdots(2m-3)}{m!}t^m$$

it follows that

$$f_n'(x) = \sum_{r=1}^{n}\binom{n}{r} 1\cdot 3\cdots(2r-3)f_{n-r}(x). \tag{2.10}$$

This result can be generalized as follows. We have [12; 35]

$$(1 - (1 - 2t)^{\frac{1}{2}})^m = \sum_{r=m}^{\infty} A_r x^r/r!,$$

where

$$A_r = \frac{(2r - m - 1)!m}{(r - m)!2^{r-m}}. \tag{2.11}$$

Hence differentiating (2.5) m times with respect to x we find that

$$f_n^{(m)}(x) = \sum_{r=m}^{n} \binom{n}{r} A_r f_{n-r}(x). \tag{2.12}$$

3. By means of (2.3) it is easy to show that

$$x^{2k} f_n(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (2n + 2k - 1)(2n + 2k - 3) \cdots (2n + 2k - 2r + 1) f_{n+2k-r}(x). \tag{3.1}$$

We shall show that more generally

$$x^k f_n(x) = \sum_{r=0}^{n+k} (-1)^r \binom{\frac{1}{2}k}{r} (2n + k - 1)(2n + k - 3) \cdots (2n + k - 2r + 1) f_{n+k-r}(x). \tag{3.2}$$

For k even, (3.2) coincides with (3.1). To prove (3.2) when $k = 1$, differentiate (2.5) with respect to t. This gives

$$x(1 - 2t)^{-\frac{1}{2}} \sum_0^{\infty} f_n(x) t^n/n! = \sum_0^{\infty} f_{n+1}(x) t^n/n!,$$

so that

$$x \sum_0^{\infty} f_n(x) \frac{t^n}{n!} = \sum_0^{\infty} f_{n+1}(x) \frac{t^n}{n!} \sum_0^{\infty} (-1)^r \binom{\frac{1}{2}}{r} 2^r t^r;$$

comparison of coefficients yields

$$x f_n(x) = \sum_{r=0}^{n} (-1)^r \binom{\frac{1}{2}}{r} n(n - 1) \cdots (n - r + 1) 2^r f_{n+1-r}(x),$$

which is the same as the case $k = 1$ of (3.2). Now assuming that (3.2) holds we have

$$\begin{aligned} x^{k+2} f_n &= \sum_r (-1)^r \binom{\frac{1}{2}k}{r} (2n + k - 1) \cdots (2n + k - 2r + 1) \\ &\qquad \cdot \{f_{n+k+2-r} - (2n + 2k + 1 - 2r) f_{n+k+1-r}\} \\ &= \sum_r (-1)^r \binom{\frac{1}{2}k}{r} (2n + k - 1) \cdots (2n + k - 2r + 1) f_{n+k+2-r} \end{aligned}$$

$$+ \sum_r (-1)^r \binom{\frac{1}{2}k}{r-1}(2n+k-1)\cdots(2n+k-2r+3)$$

$$(2n+2k-2r+3)f_{n+k+2-r}$$

$$= \sum_r (-1)^r \binom{\frac{1}{2}k}{r-1}(2n+k-1)\cdots(2n+k-2r+3)$$

$$\cdot\left\{\frac{\frac{1}{2}k-r+1}{r}(2n+k-2r+1)+(2n+2k-2r+3)\right\} f_{n+k+2-r}$$

$$= \sum_r (-1)^r \binom{\frac{1}{2}(k+2)}{r}(2n+k+1)\cdots(2n+k-2r+3)$$

$$f_{n+k+2-r}\ ,$$

so that (3.2) holds for $k+2$. This evidently completes the proof of the formula. We remark that for $n=0$, (3.2) reduces to (2.8), as is easily verified.

The formula

$$(3.3) \qquad f_{n+k} = \sum_{r=0}^{k} \binom{k}{r}(2n-1)(2n+1)\cdots(2n+2k-2r-3)x^{2r}f_{n-r}$$

may be noted in connection with (3.2). To prove (3.3), we have

$$f_{n+k+1} = (2n+2k-1)f_{n+k} + x^2 f_{n+k-1}$$

$$= (2n+2k-1)\sum_r \binom{k}{r}(2n-1)(2n+1)\cdots(2n+2k-2r-3)$$

$$x^{2r}f_{n-r}$$

$$+ \sum_r \binom{k}{r-1}(2n-3)(2n-1)\cdots(2n+2k-2r-3)x^{2r}f_{n-r}$$

$$= \sum_r \binom{k}{r-1}(2n-1)\cdots(2n+2k-2r-3)$$

$$\cdot\left\{(2n+2k-1)\frac{k-r+1}{r}+(2n-3)\right\} x^{2r}f_{n-r}$$

$$= \sum_r \binom{k+1}{r}(2n-1)\cdots(2n+2k-2r-1)x^{2r}f_{n-r}\ ,$$

which evidently completes the induction. In terms of θ_n, (3.3) becomes

$$(3.4) \qquad \theta_{n+k} = \sum_{r=0}^{k}(2n+1)(2n+3)\cdots(2n+2k-2r-1)x^{2r}\theta_{n-r}\ .$$

4. Returning to (1.1) and (1.3) we have

$$(4.1) \qquad 2^n\theta_n(\tfrac{1}{2}x) = \sum_{r=0}^{n}\frac{(n+r)!}{r!(n-r)!}x^{n-r} = n!\sum_{r=0}^{n}\binom{n+r}{r}\frac{x^{n-r}}{(n-r)!}.$$

On the other hand, if we recall that the Laguerre polynomial

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \binom{n+\alpha}{r} \frac{(-x)^{n-r}}{(n-r)!}, \tag{4.2}$$

comparison with (4.1) yields

$$(-2)^n \theta_n(\tfrac{1}{2}x) = n! L_n^{(-2n-1)}(x). \tag{4.3}$$

By using (2.5) we can obtain additional formulas involving the Laguerre polynomials. Thus if in (2.5) we put

$$t = \frac{2u}{(1+u)^2}, \qquad 1 - (1-2t)^{\frac{1}{2}} = \frac{2u}{1+u},$$

we get

$$\begin{aligned} e^{2xu/(1+u)} &= \sum_n \frac{f_n(x)}{n!} \frac{(2u)^n}{(1+u)^{2n}}, \\ (1-u)^{-\alpha-1} e^{-2xu/(1-u)} &= \sum_n \frac{f_n(x)}{n!} \frac{(-2u)^n}{(1-u)^{2n\alpha+1}} \\ &= \sum_n \frac{f_n(x)}{n!} (-2u)^n \sum_r \binom{2n+r+\alpha}{r} u^r \\ &= \sum_{k=0}^{\infty} u^k \sum_{n=0}^{k} (-2)^n \binom{k+n+\alpha}{k-n} \frac{f_n(x)}{n!}. \end{aligned}$$

But on the other hand

$$(1-u)^{-\alpha-1} e^{-2xu/(1-u)} = \sum_{k=0}^{\infty} L_k^{(\alpha)}(2x) u^k;$$

consequently

$$\begin{aligned} L_k^{(\alpha)}(2x) &= \sum_{r=0}^{k} (-2)^r \binom{k+r+\alpha}{k-r} \frac{f_r(x)}{r!} \\ &= \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} (-2)^r (k+\alpha+1)_r f_r(x), \end{aligned} \tag{4.4}$$

where $(\alpha)_r = \alpha(\alpha+1) \cdots (\alpha+r-1)$. In particular for $\alpha = -2k-1$ we get

$$L_k^{(-2k-1)}(2x) = (-1)^k \sum_{r=0}^{k} 2^{k-r} \binom{2r}{r} \frac{f_{k-r}(x)}{(k-r)!}. \tag{4.5}$$

Comparing (4.5) with (4.3) it is clear that

$$\begin{aligned} \theta_n(x) &= n! \sum_{r=0}^{n} 2^{-r} \binom{2r}{r} \frac{f_{n-r}(x)}{(n-r)!} \\ &= \sum_{r=0}^{n} \binom{n}{r} 1 \cdot 3 \cdots (2r-1) f_{n-r}(x), \end{aligned} \tag{4.6}$$

which can be proved directly from (2.5).

5. Burchnall and Chaundy [4; 127] have proved

$$(5.1)\qquad \begin{aligned} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y) &= \frac{\Gamma(1+\alpha+n)}{n!} \sum_{r=0}^{n} \frac{(xy)^r L_{n-r}^{(\alpha+2r)}(x+y)}{r!\Gamma(1+\alpha+r)} \\ &= \frac{1}{n!} \sum_{r=0}^{n} \frac{(xy)^{n-r}}{(n-r)!} (1+\alpha+n-r)_r \\ &\qquad\qquad L_r^{(\alpha+2n-2r)}(x+y), \end{aligned}$$

$$(5.2)\qquad \begin{aligned} L_n^{(\alpha)}(x+y) &= \sum_{r=0}^{n} (-1)^r \frac{(n-r)!(\alpha+2r)\Gamma(\alpha+r)}{r!\Gamma(\alpha+n+r+1)} \\ &\qquad\qquad x^r y^r L_{n-r}^{(\alpha+2r)}(x) L_{n-r}^{(\alpha+2r)}(y) \\ &= \sum_{r=0}^{n} (-1)^{n-r} \frac{r!(\alpha+2n-2r)}{(n-r)!(\alpha+n-r)_{n+1}} \\ &\qquad\qquad (xy)^{n-r} L_r^{(\alpha+2n-2r)}(x) L_r^{(\alpha+2n-2r)}(y); \end{aligned}$$

(5.1) had been proved earlier by Bailey [1; 216].

In (5.1) take $\alpha = -2n-1$, so that

$$(5.3)\qquad \begin{aligned} &L_n^{(-2n-1)}(x) L_n^{(-2n-1)}(y) \\ &\qquad = \frac{1}{(n!)^2} \sum_{r=0}^{n} (-1)^r \frac{(n+r)!}{(n-r)!} (xy)^{n-r} L_r^{(-2r-1)}(x+y); \end{aligned}$$

in view of (4.3) this becomes

$$(5.4)\qquad \theta_n(x)\,\theta_n(y) = \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!} 2^{-r} (xy)^{n-r} \theta_r(x+y).$$

Similarly (5.2) yields

$$(5.5)\qquad \begin{aligned} L_n^{(-2n-1)}(x+y) &= \sum_{r=0}^{n} (-1)^r \frac{r!r!(2r+1)}{(n-r)!(n+r+1)} \\ &\qquad\qquad (xy)^{n-r} L_r^{(-2r-1)}(x) L_r^{(-2r-1)}(y), \end{aligned}$$

or what is the same thing

$$(5.6)\qquad \begin{aligned} \theta_n(x+y) &= 2^n \sum_{r=0}^{n} (-1)^{n-r} \frac{n!(2r+1)}{(n-r)!(n+r+1)!} \\ &\qquad\qquad (xy)^{n-r} \theta_r(x)\,\theta_r(y). \end{aligned}$$

6. We recall that [2; 141]

$$(6.1)\qquad I_\alpha(2(xw)^{\frac{1}{2}}) = e^w\, xw)^{\alpha/2} \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)\,(-w)^n}{(1+\alpha+n)}.$$

In what follows we suppose that

$$(6.2)\qquad \alpha = k + \tfrac{1}{2},$$

where k is some fixed non-negative integer. Then by the definition of $K_\alpha(z)$,

$$K_\alpha(z) = (-1)^k \frac{\pi}{2}(I_{-\alpha}(z) - I_\alpha(z)), \tag{6.3}$$

when α satisfies (6.2). Thus by (6.1) and (6.3)

$$\begin{aligned}(xw)^\alpha K_\alpha(2xw) = (-1)^k \frac{\pi}{2} e^{w^2} \Bigg\{ \sum_0^\infty \frac{L_n^{(-\alpha)}(x^2)(-w^2)^n}{\Gamma(1-\alpha+n)} \\ - (xw)^{2k+1} \sum_0^\infty \frac{L_n^{(\alpha)}(x^2)(-w^2)^n}{\Gamma(1+\alpha+n)} \Bigg\}.\end{aligned} \tag{6.4}$$

Comparing with the well-known formulas that express the Hermite polynomials in terms of the Laguerre polynomials [2; 145], we define

$$\begin{aligned} H_{2n}^{(k)}(x) &= (-1)^{n+k} 2^k \pi^{\frac{1}{2}} \frac{(2n)!}{\Gamma(1-\alpha+n)} L_n^{(-\alpha)}(x^2), \\ H_{2n+1}^{(k)}(x) &= (-1)^n 2^k \pi^{\frac{1}{2}} \frac{(2n+1)!}{\Gamma(1+\alpha+n-k)} x^{2k+1} L_{n-k}^{(\alpha)}(x^2) \qquad (n \geq k). \end{aligned} \tag{6.5}$$

Thus (6.4) becomes

$$(xw)^\alpha K_\alpha(2xw) = \frac{\pi^{\frac{1}{2}}}{2} e^{w^2} \sum_{m=0}^\infty (-1)^m H_m^{(k)}(x) \frac{w^m}{m!}. \tag{6.6}$$

We have also [2; 24]

$$K_\alpha(x) = \pi^{\frac{1}{2}}(-1)^k k!(2x)^{-\alpha} e^{-x} L_k^{(-2k-1)}(2x);$$

using (4.3) this becomes

$$K_\alpha(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} x^{-\alpha} e^{-x} \theta_k(x). \tag{6.7}$$

Substituting from (6.7) in (6.6) we get

$$\sum_{m=0}^\infty H_m^{(k)}(x) \frac{w^m}{m!} = e^{2xw - w^2} \theta_k(-2xw). \tag{6.8}$$

In the next place it is clear from (6.8) and (5.4) that

$$\begin{aligned} &\sum_{m,n=0}^\infty H_m^{(k)}(x) H_n^{(k)}(x) \frac{u^m v^n}{m!n!} \\ &\quad = e^{2x(u+v)-(u_2+v_2)} \sum_{r=0}^k \frac{(k+r)!}{(k-r)!r!} 2^{-r}(4x^2uv)^{k-r} \theta_r(-2x(u+v)) \\ &\quad = e^{2uv} \sum_{r=0}^k \frac{(k+r)!}{(k-r)!r!} 2^{-r}(4x^2uv)^{k-r} \cdot e^{2x(u+v)-(u+v)^2} \theta_r(-2x(u+v)) \\ &\quad = e^{2uv} \sum_{r=0}^k \frac{(k+r)!}{(k-r)!r!} 2^{-r}(4x^2uv)^{k-r} \sum_{m=0}^\infty H_m^{(r)}(x) \frac{(u+v)^m}{m!} \end{aligned}$$

$$= e^{2uv} \sum_{r=0}^{k} \frac{(2k-r)!}{(k-r)!r!} 2^{r-k}(4x^2uv)^r \sum_{m,n=0}^{\infty} H_{m+n}^{(k-r)}(x) \frac{u^m v^n}{m!n!}$$

$$= \sum_{r=0}^{k} \frac{(2k-r)!}{(k-r)!r!} 2^{r-k}(4x^2uv)^r \sum_{s=0}^{\infty} \frac{(2uv)^s}{s!} \sum_{m,n=0}^{\infty} H_{m+n}^{(k-r)}(x) \frac{u^m v^n}{m!n!}.$$

Equating coefficients we get

$$(6.9) \quad H_m^{(k)}(x)H_n^{(k)}(x) = \sum_{r,s} \frac{m!n!(2k-r)!2^{3r-k}x^{2r}}{(k-r)!r!s!(m-r-s)!(n-r-s)!} H_{m+n-2r-2s}^{(k-r)}(x).$$

Similarly using (6.8) and (5.6) we get

$$\sum_{m,n=0}^{\infty} H_{m+n}^{(k)}(x) \frac{u^m v^n}{m!n!} = \sum_{m=0}^{\infty} H_m^{(k)}(x) \frac{(u+v)^m}{m!}$$

$$= e^{2x(u+v)-(u+v)^2} \theta_k(-2x(u+v))$$

$$= e^{2x(u+v)-(u+v)^2} 2^k \sum_{r=0}^{k} (-1)^{k-r} \frac{k!(2k-2r+1)}{r!(2k-r+1)!} \cdot (4x^2uv)^r \theta_{k-r}(-2xu)\theta_{k-r}(-2xv)$$

$$= 2^k e^{-2uv} \sum_{r=0}^{k} (-1)^{k-r} \frac{k!(2k-2r+1)}{r!(2k-r+1)!} (4x^2uv)^r \cdot e^{2xu-u^2}\theta_{k-r}(-2xu) \cdot e^{2xv-v^2}\theta_{k-r}(-2xv)$$

$$= 2^k \sum_{s=0}^{\infty} \frac{(-2uv)^s}{s!} \sum_{r=0}^{k} (-1)^{k-r} \frac{k!(2k-2r+1)}{r!(2k-r+1)!} (4x^2uv)^r \cdot \sum_{m=0}^{\infty} H_m^{(k-r)}(x) \frac{u^m}{m!} \sum_{n=0}^{\infty} H_n^{(k-r)}(x) \frac{v^n}{n!},$$

which yields

$$(6.10) \quad H_{m+n}^{(k)}(x) = 2^k \sum_{r,s} (-1)^{k+r+s} \frac{m!n!k!(2k-2r+1)(2x)^{2r}}{r!s!(2k-r+1)!(m-r-s)!(n-r-s)!} \cdot H_{m-r-s}^{(k-r)}(x)H_{n-r-s}^{(k-r)}(x).$$

The following variants of (6.9) and (6.10) may be noted.

$$(6.11) \quad \sum_{r=0}^{\min(m,n)} (-2)^r \binom{m}{r}\binom{n}{r} r!H_{m-r}^{(k)}(x)H_{n-r}^{(k)}(x) = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{r} \frac{r!(2k-r)!}{(k-r)!} 2^{3r-k}x^{2k}H_{m+n-2r}^{(k-r)}(x),$$

$$(6.12) \quad \sum_{r=0}^{\min(m,n)} 2^r \binom{m}{r}\binom{n}{r} r!H_{m+n-2r}^{(k)}(x) = 2^k \sum_{r=0}^{k} (-1)^{k-r} \binom{m}{r}\binom{n}{r} \frac{k!r!(2k-2r+1)}{(2k-r+1)!} (2x)^{2r}H_{m-r}^{(k-r)}(x)H_{n-r}^{(k-r)}(x).$$

For $k = 0$, (6.11) and (6.12) reduce to the formulas

$$\sum_{r=0}^{\min(m,n)} (-2)^r \binom{m}{r}\binom{n}{r} r! H_{m-r}(x) H_{n-r}(x) = H_{m+n}(x), \tag{6.13}$$

$$\sum_{r=0}^{\min(m,n)} 2^r \binom{m}{r}\binom{n}{r} r! H_{m+n-2r}(x) = H_m(x) H_n(x) \tag{6.14}$$

due to Nielsen [11; 31-33] and rediscovered by Dhar [7] and Feldheim [8].

7. Returning again to (2.1), that is

$$\theta_{n+1}(x) = (2n+1)\theta_n(x) + x^2\theta_{n-1}(x), \tag{7.1}$$

we remark that $\theta_n(-x)$ is also a solution of (7.1); for $n \geq 1$ it is evident that $\theta_n(x)$ and $\theta_n(-x)$ are linearly independent. It is easy to write down various identities that are a direct consequence of (7.1).

In the first place since

$$\begin{aligned} \theta_{n+1}(x)\theta_n(-x) - \theta_n(x)\theta_{n+1}(-x) & \\ &= \{(2n+1)\theta_n(x) + x^2\theta_{n-1}(x)\}\theta_n(-x) \\ &\quad - \theta_n(x)\{(2n+1)\theta_n(-x) + x^2\theta_{n-1}(-x)\} \\ &= x^2\{\theta_{n-1}(x)\theta_n(-x) - \theta_n(x)\theta_{n-1}(-x)\} \end{aligned}$$

and $\theta_1(x)\theta_0(-x) - \theta_0(x)\theta_1(-x) = 2x$, it follows that

$$\theta_{n+1}(x)\theta_n(-x) - \theta_n(x)\theta_{n+1}(-x) = 2(-1)^n x^{2n+1}. \tag{7.2}$$

Again, since

$$\begin{aligned} y\theta_{n+1}(x)\theta_n(y) - x\theta_n(x)\theta_{n+1}(y) & \\ &= (2n+1)(y-x)\theta_n(x)\theta_n(y) - xy\{y\theta_n(x)\theta_{n-1}(y) - x\theta_{n-1}(x)\theta_n(y)\}, \end{aligned}$$

we get

$$\begin{aligned} (y-x)\sum_{r=0}^{n}(-1)^{n-r}(2r+1)(xy)^{n-r}\theta_r(x)\theta_r(y) & \\ &= y\theta_{n+1}(x)\theta_n(y) - x\theta_n(x)\theta_{n+1}(y) \\ &= \theta'_{n+1}(x)\theta_{n+1}(y) - \theta_{n+1}(x)\theta'_{n+1}(y), \end{aligned} \tag{7.3}$$

where at the last step we have used (2.2). In particular for $y = x$ we have

$$\begin{aligned} \sum_{r=0}^{n}(-1)^{n-r}(2r+1)x^{2n-2r}\theta_r^2(x) & \\ &= x\theta_{n+1}(x)\theta'_n(x) - x\theta_n(x)\theta'_{n+1}(x) + \theta_n(x)\theta_{n+1}(x) \\ &= \{\theta'_{n+1}(x)\}^2 - \theta_{n+1}(x)\theta''_{n+1}(x), \end{aligned} \tag{7.4}$$

while for $y = -x$, (7.3) becomes

$$2 \sum_{r=0}^{n} (2r + 1)x^{2n-2r}\theta_r(x)\theta_r(-x)$$

(7.5)
$$= \theta_{n+1}(x)\theta_n(-x) + \theta_n(x)\theta_{n+1}(-x)$$
$$= x^{-1}\{\theta_{n+1}(x)\theta'_{n+1}(-x) - \theta'_{n+1}(x)\theta_{n+1}(-x)\}.$$

Similarly, since

$$\begin{aligned}\theta_{n+1}(x)\theta_n(y) - \theta_n(x)\theta_{n+1}(y) &= x^2\theta_{n-1}(x)\theta_n(y) - y^2\theta_n(x)\theta_{n-1}(y) \\ &= (2n-1)(x^2 - y^2)\theta_{n-1}(x)\theta_{n-1}(y) \\ &\quad + x^2y^2\{\theta_{n-1}(x)\theta_{n-2}(y) - \theta_{n-2}(x)\theta_{n-1}(y)\},\end{aligned}$$

we get

$$(x^2 - y^2)\sum_{r=1}^{n}(4r-1)(xy)^{2n-2r}\theta_{2r-1}(x)\theta_{2r-1}(y) + (x-y)(xy)^{2n} \tag{7.6}$$
$$= \theta_{2n+1}(x)\theta_{2n}(y) - \theta_{2n}(x)\theta_{2n+1}(y),$$

$$(x^2 - y^2)\sum_{r=0}^{n}(4r+1)(xy)^{2n-2r}\theta_{2r}(x)\theta_{2r}(y) + (x-y)(xy)^{2n+1} \tag{7.7}$$
$$= \theta_{2n+2}(x)\theta_{2n+1}(y) - \theta_{2n+1}(x)\theta_{2n+1}(y).$$

It is easily verified that (7.6) and (7.7) imply the first half of (7.3). If we let $y \to x$, (7.6) and (7.7) become

$$2x\sum_{r=1}^{n}(4r-1)x^{4n-4r}\theta^2_{2r-1}(x) + x^{4n} = \theta_{2n}(x)\theta'_{2n+1}(x) - \theta_{2n+1}(x)\theta'_{2n}(x) \tag{7.8}$$
$$= x\{\theta_{2n-1}(x)\theta_{2n+1}(x) - \theta^2_{2n}(x)\},$$

$$2x\sum_{r=0}^{n}(4r+1)x^{4n-4r}\theta^2_{2r}(x) + x^{4n+2} = \theta_{2n+1}(x)\theta'_{2n+2}(x) - \theta_{2n+2}(x)\theta'_{2n+1}(x) \tag{7.9}$$
$$= x\{\theta_{2n}(x)\theta_{2n+2}(x) - \theta^2_{2n+1}(x)\}.$$

In passing we may note the identity

$$\theta_n(x)\theta'_{n+1}(x) - \theta_{n+1}(x)\theta'_n(x) = x\{\theta_{n-1}(x)\theta_{n+1}(x) - \theta^2_n(x)\}. \tag{7.10}$$

If we let $y \to -x$, (7.6) and (7.7) become

$$2x\sum_{r=1}^{n}(4r-1)x^{4n-4r}\theta_{2r-1}(x)\theta_{2r-1}(-x) \tag{7.11}$$
$$= \theta_{2n+1}(x)\theta'_{2n}(-x) - \theta_{2n}(x)\theta'_{2n+1}(-x) + (4n+1)x^{4n},$$

$$2x\sum_{r=0}^{n}(4r+1)x^{4n-4r}\theta_{2r}(x)\theta_{2r}(-x) \tag{7.12}$$
$$= \theta_{2n+2}(x)\theta'_{2n+1}(-x) - \theta_{2n+1}(x)\theta'_{2n+2}(-x) - (4n+3)x^{4n+2}.$$

We remark that (7.8) and (7.9) imply

$$\theta_n(x)\theta'_{n+1}(x) - \theta_{n+1}(x)\theta'_n(x) \geq 0 \qquad (x \geq 0) \tag{7.13}$$

$$\theta_{n-1}(x)\theta_{n+1}(x) - \theta_n^2(x) \geq 0 \qquad (x \geq 0). \tag{7.14}$$

For additional formulas like the above in the case of the classical orthogonal polynomials, see for example a recent paper by Danese [6].

8. In conclusion we wish to add a word about congruence properties of the polynomial $\theta_n(x)$; we remark that the coefficients of the polynomial are integers. With minor changes the proof of Theorem 1 of [5] applies. Let $2m + 1$ be an arbitrary odd integer. Then in the first place we find that

$$\theta_{m=2n+1} \equiv x^{2m+1}\theta_n \qquad (\mathrm{mod}\ 2m + 1) \tag{8.1}$$

for all n. In particular $\theta_{2m+1} \equiv x^{2m+1}$, $\theta_{2m} \equiv x^{2m}$ (mod $2m + 1$.)

Now define

$$\Delta^r\theta_n = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} x^{(2m+1)(r-s)}\theta_{n+s(2m+1)}. \tag{8.2}$$

It follows from (8.2) that

$$\Delta^r\theta_{n+1} = (2n + 1)\Delta^r\theta_n + x^2\Delta^r\theta_{n-1} + 2r(2m + 1)\Delta^{r-1}\theta_{n+s(2m+1)}. \tag{8.3}$$

We show next that

$$\Delta^{2r-1}\theta_{-r(2m+1)+m} \equiv \Delta^{2r-1}\theta_{-r(2m+1)+m+1} \equiv 0 \qquad (\mathrm{mod}\ (2m + 1)^r) \tag{8.4}$$

for all $r \geq 1$. It now follows from (8.2), (8.3) and (8.4) that

$$\Delta^{2r-1}\theta_n \equiv \Delta^{2r}\theta_n \equiv 0 \qquad (\mathrm{mod}\ (2m + 1)^r) \tag{8.5}$$

for $r \geq 1$ and all n.

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