2 Basics on isotropic spaces

For later use, we provide a brief review of curves and surfaces in isotropic spaces from [16, 29–31].

An isotropic space based on the following group G_6 of affine transformations (so-called *isotropic congruence transformations* or *i-motions*) is a Cayley–Klein space:

$$\begin{aligned}\mathbf{x}' &= a + \mathbf{x}\cos\phi - \mathbf{y}\sin\phi,\\ \mathbf{y}' &= b + \mathbf{x}\sin\phi - \mathbf{y}\cos\phi,\\ \mathbf{z}' &= c + d\mathbf{x} + e\mathbf{y} + \mathbf{z}.\end{aligned}$$

This means that i-motions are indeed composed of an Euclidean motion in the **xy**-plane (i.e. translation and rotation) and an affine shear transformation in **z**-direction.

In general, the following terminology is used for the isotropic spaces. Consider the points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The projection in the **z**-direction onto $\mathbb{R}^2$, $(x_1, x_2, x_3) \mapsto (x_1, x_2, 0)$, is called the *top view*. In the sequel, many metric properties in isotropic geometry (invariants under G_6) are Euclidean invariants in the top view such as the isotropic distance, so-called i-distance. The *i-distance* of two points **x** and **y** is defined as the Euclidean distance of their top views, i.e.,

$$\|\mathbf{x} - \mathbf{y}\|_i = \sqrt{\sum_{j=1}^{2} (y_j - x_j)^2}.$$

Two points having the same top view are called *parallel points*. The i-metric is degenerate along the lines in the **z**-direction, and such lines are called *isotropic* lines. The plane containing an isotropic line is called an *isotropic plane*. Therefore, an *isotropic 3-space* $\mathbb{I}^3$ is the product of the Euclidean 2-space $\mathbb{R}^2$ and an isotropic line with a degenerate parabolic distance metric.

Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{I}^3$ be an admissible curve (i.e. without isotropic tangents) parametrized by arc-length $s \in I$. In the coordinate form, it can be written as

$$\gamma(s) = (x(s), y(s), z(s)),$$

where x, y and z are smooth functions of one variable. Denote the first derivative with respect to s by one prime, the second derivative by two primes, etc. Then the curvature and torsion functions of γ are respectively defined by

$$\kappa(s) = x'(s)y''(s) - x''(s)y'(s)$$

and

$$\tau(s) = \frac{1}{\kappa(s)} \det(\gamma'(s), \gamma''(s), \gamma'''(s)), \quad \kappa(s) \neq 0$$

for all $s \in I$. Moreover, the associated trihedron of γ is given by

$$\begin{aligned}\mathbf{T}(s) &= (x'(s), y'(s), z'(s)),\\ \mathbf{N}(s) &= \frac{1}{\kappa(s)}(x''(s), y''(s), z''(s)),\\ \mathbf{B}(s) &= (0, 0, 1).\end{aligned}$$

In the sequel, the Frenet formulas of such vectors are

$$\mathbf{T}' = \kappa\mathbf{N}, \quad \mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \mathbf{B}' = 0.$$

Let M^2 be a surface immersed in $\mathbb{I}^3$ which has no isotropic tangent planes. Such a surface M^2 is said to be *admissible* and can be parametrized by

$$X : D \subseteq \mathbb{R}^2 \to \mathbb{I}^3 : (u_1, u_2) \mapsto (X_1(u_1, u_2), X_2(u_1, u_2), X_3(u_1, u_2)),$$