and *centered variogram*

$$\tilde{\gamma}(\boldsymbol{h}) = \tfrac{1}{2}\operatorname{var}(Z(\boldsymbol{x}) - Z(\boldsymbol{x}+\boldsymbol{h})), \qquad \boldsymbol{h} \in \mathbb{R}^d.$$

It follows readily that for a given process Z,

$$\gamma(\boldsymbol{h}) = Q(\boldsymbol{h}) + \tilde{\gamma}(\boldsymbol{h}), \qquad \boldsymbol{h} \in \mathbb{R}^d, \tag{5}$$

where Q is a nonnegative quadratic form, that is, $Q(\boldsymbol{h}) = \sum_{i=1}^{d} a_i h_i^2$ where $\boldsymbol{h} = (h_1, \dots, h_d)^{\mathrm{T}} \in \mathbb{R}^d$ and $a_i \geq 0$ for $i = 1, \dots, d$. Conversely, if $\tilde{\gamma}$ is a centered variogram and Q is a nonnegative quadratic form, then (5) is a noncentered variogram. The definitions coincide in the standard case when $\mathrm{E}(Z(\boldsymbol{x}) - Z(\boldsymbol{x}+\boldsymbol{h})) = 0$.

2.2. Characterization and decomposition theorems

We first review a fundamental characterization theorem for variograms. Recall that a function $\gamma : \mathbb{R}^d \to \mathbb{R}$ is said to be *conditionally negative definite* if the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma(\boldsymbol{x}_i - \boldsymbol{x}_j) \leq 0$$

holds for all finite systems of points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$ and coefficients $a_1, \dots, a_n$ for which $\sum_{i=1}^{n} a_i = 0$.

Theorem 1. *If γ is a real function in $\mathbb{R}^d$ satisfying $\gamma(\boldsymbol{0}) = 0$, the following properties are equivalent.*

(a) *There exists an intrinsically stationary Gaussian random function Z with variogram $\gamma(\cdot)$.*

(b) *The function $\gamma(\cdot)$ is conditionally negative definite.*

(c) *For all $t > 0$, $\exp(-t\gamma(\cdot))$ is a covariance function.*

This result is known, and the equivalence of (a) and (b) mimics the characterization of covariances as positive definite functions. However, in the statistical literature, Theorem 1 has been known only under the additional assumption of continuity of γ (cf. Cressie (1993, p. 87), Chilès and Delfiner (1999, pp. 66–67)). This excludes many practically important variogram models which are obtained by adding a *nugget effect*,

$$\gamma_0(\boldsymbol{h}) = \begin{cases} 0, & \boldsymbol{h} = \boldsymbol{0}, \\ \alpha, & \text{otherwise}, \end{cases} \qquad \alpha > 0,$$

to a continuous variogram. The general result presented here is immediate from Theorem 6.1.9 of Sasvári (1994).

The equivalence of (a) and (c) in Theorem 1 allows us to establish interesting and useful analogies between covariance functions and variograms, that is, between stationary and intrinsically stationary random fields. For instance, Davies and Hall (1999) show that if a stationary process in $\mathbb{R}^2$ satisfies the usual one-dimensional scaling laws, then the fractal dimensions of its line transect processes are the same in all directions, except possibly one, whose dimension may be less than in all others. By Theorem 1, the result carries over to intrinsically stationary processes in $\mathbb{R}^2$. We omit the technical statement and proof.