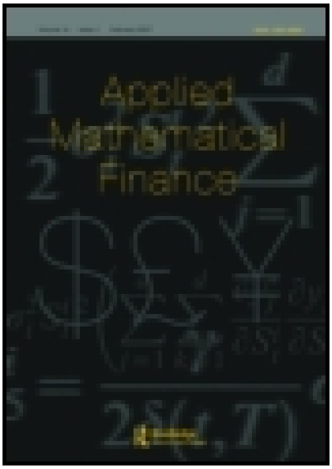


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# Estimation of stochastic volatility in the Hull-White model

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# *Estimation of stochastic volatility in the Hull–White model*

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Estimation of the stochastic volatility in the Hull–White framework is considered. Stock price is taken as the observation and the estimation problem is posed for the stochastic volatility. It is first shown that it is not possible to formulate this as the usual filtering problem, and an alternative formulation is proposed. A robust filtering equation is then derived suitable for real observation data.

**Keywords:** stochastic volatility, Hull–White model, robust filter



## 1. Introduction

There is a general consensus in the mathematical finance community that the assumption of constant volatility in the Black–Scholes model is not borne out by the actual data. Various alternative models have been proposed, including stochastic ones (Wilmott *et al.*, 1997). One popular model with stochastic volatility is due to Hull and White (1987). One immediate question that arises is: can one estimate the volatility (preferably online) as one gathers the stock price data? On a purely theoretical level, the solution to this problem is trivial, as the quadratic variation of the stock price over a time interval should be exactly equal to the integral of the volatility function over the same time interval. The purpose of this paper is to show that this theoretical result has no meaning when applied to real stock price data, and to propose a completely new formulation of the model leading to a well-defined estimation problem.

Several excellent surveys on the 'stochastic volatility', including the estimation problem, have appeared recently, for example, Bollerslev *et al.* (1994) and Ghysels *et al.* (1996). The usual approach to formulate the stochastic volatility estimation problem is to convert the continuous-time model to a discrete-time model and to apply normal statistical estimation techniques to the discrete-time model (see, for example, references in Ghysels *et al.*, 1996). The most relevant work related to this paper is a series of articles by Nelson. Nelson (1994a, 1994b) and Nelson and Foster (1994), first discretize the model (for example, ARCH, GARCH(1,1) and apply nonlinear filtering/smoothing algorithms (extended

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Kalman filter) to obtain the desired estimates. This automatically implies that the algorithms are merely sub-optimal. Secondly, the convergence property for the derived discrete-time algorithm to the continuous-time one is studied. In this paper, we propose an alternative approach to the stochastic volatility estimation, i.e.

- The estimation problem is formulated in the context of linear systems but the underlying statistical property is non-Gaussian.
- The derived filter is a continuous filter and the time evolution of the conditional probability density function is explicitly obtained from the results of Makowski (1986). The time evolution of this conditional density may be needed to apply the option-pricing problem.
- The observed data (stock price) are never continuous. The discrete-time observation data is used, however small the time interval, and the piecewise linear observation data is then constructed with respect to the time variable.
- Instead of using the continuous-time observation data, the above piecewise linear observation data is applied to the derived filter.
- As the sampling period for the observation data becomes smaller and smaller, convergence of the original optimal filter is guaranteed.



## 2. Problem formulation

Consider the adjusted Hull–White model for the stock market price:

$$\mathrm{d}S(t) = S(t)\{\mu\,\mathrm{d}t + \sqrt{V(t)}\,\mathrm{d}w_1(t)\},\ \ S(0) = S_o \tag{2.1}$$

with

$$\mathrm{d}V(t) = \xi V(t)\,\mathrm{d}t + \sigma(V(t)\wedge M)\,\mathrm{d}w_2(t)\},\ \ V(0) = V_o \tag{2.2}$$

where $S(t)$ denotes the stock market price at time $t$, $V(t)$ is stochastically varying volatility and $M$ is a positive constant. The original Hull–White model does not contain the constant $M$ and in this paper $M$ is taken to be a large positive number, so that there does not exist any practical difference between the original Hull–White model and the adjusted one.

In this model, (i) The stock market price is observable and our first objective is to estimate the stochastic volatility $V(t)$ based on the observation of stock prices $S(s)$; $0 \le s \le t$, (ii) $w_1$ and $w_2$ are mutually independent Brownian motion processes.

The observation process $S(t)$ can be explicitly solved as

$$S(t) \ = \ S_0 \exp\{\mu t \ - \frac{1}{2}\int_0^t V(s)\,\mathrm{d}s \ + \int_0^t \sqrt{V(s)}\,\mathrm{d}w_1(s)\} \tag{2.3}$$

We introduce the process $Z(t)$ defined by

$$Z(t) \ = \ \log\{S(t)/S_0\} \tag{2.4}$$

Then the modified observation process $Z(t)$ is the solution of

$$\mathrm{d}Z(t) \ = \ \mu\mathrm{d}t \ - \frac{1}{2}V(t)\,\mathrm{d}t \ + \ \sqrt{V(t)}\,\mathrm{d}w_1(t), \quad Z(0) \ = \ 0 \tag{2.5}$$

Now it seems that we have the normal filtering problem (Liptser and Shiryaev, 1974) for the signal process $V(t)$ and the observation process $Z(t)$:

$$\mathrm{d}V(t) = \xi V(t)\,\mathrm{d}t + \sigma(V(t) \wedge M)\,\mathrm{d}w_2(t), \quad V(0) = V_o \tag{2.6}$$

$$\mathrm{d}Z(t) = \mu\,\mathrm{d}t - \frac{1}{2}V(t)\,\mathrm{d}t + \sqrt{V(t)}\,\mathrm{d}w_1(t), \quad Z(0) = 0 \tag{2.7}$$

However, for the above system we cannot formulate the usual filtering problem because the observation noise depends on the signal process $V(t)$. In fact, for the theoretical model (2.6) and (2.7), we have the following result:

$$\lim_{n\to\infty} \sum_{i=1}^{N(n)} |\, Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})\,|^2 = \int_0^t V(s)\,\mathrm{d}s \text{ a.s.} \tag{2.8}$$

where $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N(n)}^{(n)} = t_f$ with

$$\Delta t^{(n)} = \max_{1\le j\le N(n)} |\, t_j^{(n)} - t_{j-1}^{(n)}\,| \le \frac{c_1}{n}, \quad N(n) \le c_2 n$$

More simply, by using the Ito formula, we also have the following filter:

$$Z^2(t) - 2\int_0^t Z(s)\,\mathrm{d}Z(s) = \int_0^t V(s)\,\mathrm{d}s \tag{2.9}$$

Noting that $V(t)$ is continuous, we can obtain the $V(t)$-process by differentiating $(Z^2(t) - 2\int_0^t Z(s)\,\mathrm{d}Z(s))$ with respect to $t$, i.e.

$$X(t, Z) = \frac{\mathrm{d}}{\mathrm{d}t}\left(Z^2(t) - 2\int_0^t Z(s)\,\mathrm{d}Z(s)\right) \tag{2.10}$$

is exactly equal to $V(t), \forall t > 0$. From this fact, we get

$$E\{f(V(t)) \mid \mathcal{Z}_t\} = f(V(t)), \quad \forall t > 0 \tag{2.11}$$

so that we do not have the usual filtering problem. Here $\mathcal{Z}_t$ is the smallest $\sigma$-algebra generated by $Z(s)$; $0 \le s \le t$.

**Remark 2.1** $X(t, Z)$ for $t \in (0, t_f]$ can be constructed from the observation data $\{Z(s); 0 \le s \le t\}$, and moreover,

$$\lim_{t\to 0} E\{f(V(t)) \mid \mathcal{Z}_t\} = E\{V(0)\}$$

In practice, however, the real data $Z(t)$ is never continuously available in $t$: we always have discrete observations, however small the time interval. Therefore, for real data, the formula (2.8) cannot be used to estimate the process $V(t)$ for any $t \in (0, t_f]$. Figure 1 shows the record of the one-day stock price of Aluminum Co. of America and Ford Motors Co. To check the formulae (2.8) and (2.9), square of the absolute value of the difference of the logarithm of stock price for the time difference between 30 min and 1 min is calculated. The result is shown in Figure 2 and from this result it seems that for a one day time interval we cannot draw any conclusion about estimating volatility during the day.

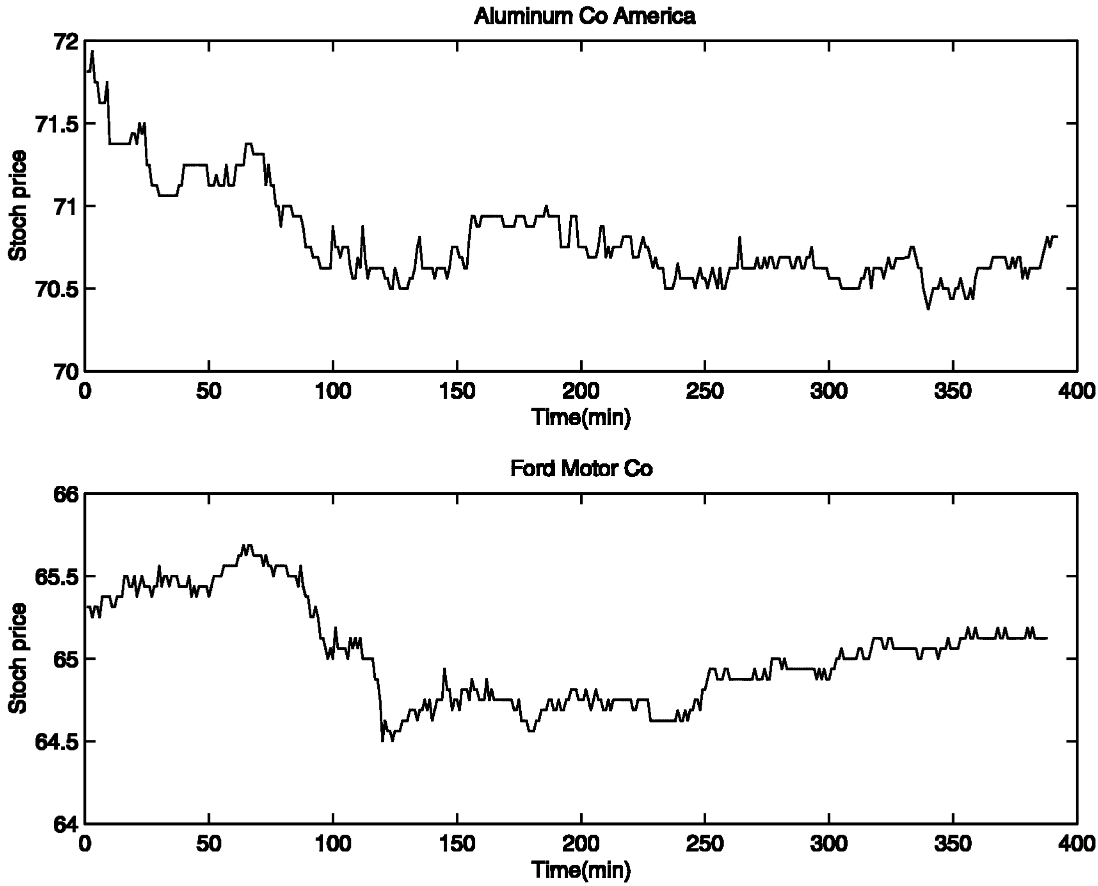


**Fig. 1.** New York Stock Exchange market (27 March 1998).



Moreover, to estimate $V(t)$ for each $t$, we must take a difference with respect to time. The discrete version of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Z^2(t) - 2\int_0^t Z(s)\,\mathrm{d}Z(s)\right)$$

becomes

$$\frac{|\, Z(t_{i+1}^{(n)}) - Z(t_i^{(n)}) \,|^2}{\Delta t_i^{(n)}}$$

It is very hard to say that the result shown in Figure 3 is the estimate of the process $V(t)$ because this is just a difference of the observation data at each time step.

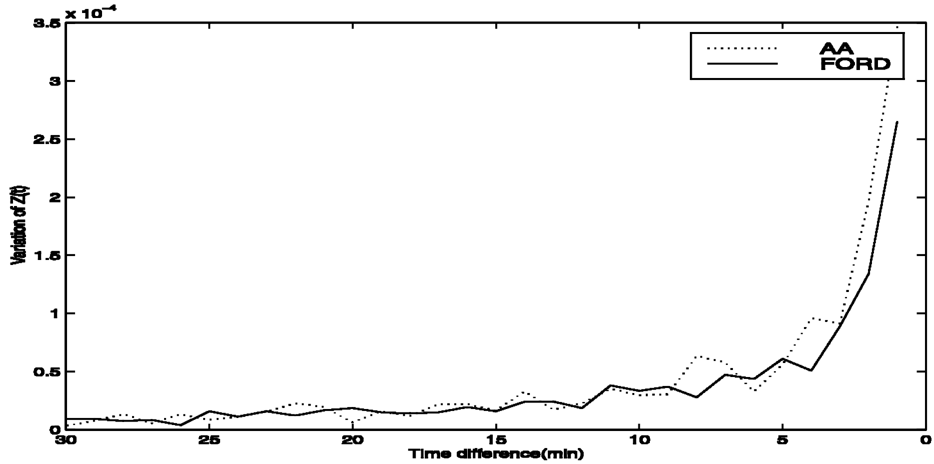


**Fig. 2.** $\Sigma_{i=1}^{N(n)}|z(t_{t+1}^{(n)}) - z(t_i^{(n)})|^2$

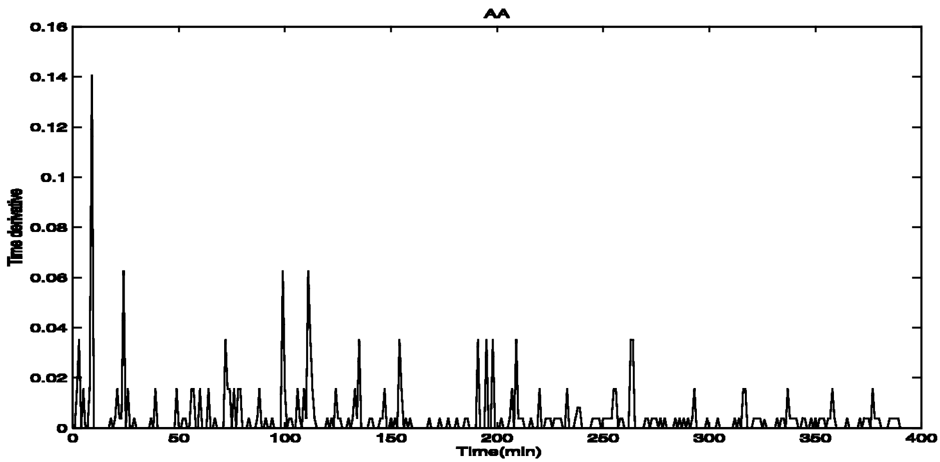


**Fig. 3.** $\frac{|z(t_{i+1}^{(n)}) - z(t_i^{(n)})|^2}{\Delta t_i^{(n)}}$

From these experiments, one rational model of the observation data can be described as follows:

- The observation $Z(t)$ is an ideal one. The real observation data (for example shown in Figure 1) is something like the piecewise deterministic approximation of $Z(t)$:

$$Z^{(n)}(t) = Z(t_i^{(n)}) + \frac{t - t_i^{(n)}}{\Delta t_i^{(n)}}(Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})) \quad \text{for} \quad \Delta t_i^{(n)} = t_{i+1}^{(n)} - t_i^{(n)} \tag{2.12}$$

- We can obtain the refined data $Z^{(n')}(t)$, $n' > n$ for better fitting of the ideal data $Z(t)$, but we can never take a limit as $n' \to \infty$.

We again note the discrepancy between the ideal model and the real data. From (2.9), using the data $Z^n(t)$, we should ideally get:

$$\lim_{n\to\infty} (Z^{(n)}(t))^2 - 2\int_0^t Z^{(n)}(s)\frac{dZ^{(n)}(s)}{ds}ds) = \int_0^t V(s)\,ds \text{ a.s.} \tag{2.13}$$

and consequently,

$$\frac{d}{dt}\lim_{n\to\infty} (Z^{(n)}(t))^2 - 2\int_0^t Z^{(n)}(s)\frac{dZ^{(n)}(s)}{ds}ds) = V(t) \tag{2.14}$$

However, for the real observation data $Z^{(n)}(t)$ we cannot take a limit as $n \to \infty$. Therefore we can apply the formula (2.9) to estimate the process $V(t)$ for large $n$:

$$\hat{V}(t) = \frac{d}{dt}(Z^{(n)}(t))^2 - 2\int_0^t Z^{(n)}(s)\frac{dZ^{(n)}(s)}{ds}ds)$$

But it is easy to see that

$$\frac{d}{dt}(Z^{(n)}(t))^2 - 2\int_0^t Z^{(n)}(s)\frac{dZ^{(n)}(s)}{ds}ds) = 0 \text{ a.s.} \tag{2.15}$$

and this method to estimate $V(t)$ is, indeed, meaningless. This is the reason why we have the poor result as shown in Figure 3.

Our point of departure from the conventional approach is based on the observation that the real data $Z(t)$ will be very close, but not necessarily identical to the solution of Equation 2.5. In fact, if the real data did follow from Equation 2.5, the actual path would be almost nowhere differentiable. In practice, $Z(t)$ is only available at discrete time-points, however small the time interval. Any plausible interpolation to construct continuous data will lead to a process whose sample paths are differentiable a.e. It is also reasonable to argue that the real data is, indeed, differentiable a.e. It is known that, for a function $K(\cdot)$ which is differentiable a.e. in $[0, T]$,

$$\lim_{m\to\infty} (\int_0^T |\dot{K}(t)|^m dt)^{1/m} = \sup_t |\frac{dK(t)}{dt}|$$

The absolute value of the difference of the logarithm of stock price is calculated for each $n$:

$$(N(n)\Sigma_{n=1}^{N(n)} | Z(t_{i+1}^{(n)}) - Z(t_i^n) |^n )^{1/n} \tag{2.16}$$

The result is shown in Figure 4 and it suggests that the left-hand side of (2.16) takes a finite value as $n \to \infty$. This strongly suggests that the real data $Z(t)$ is, in fact, differentiable a.e. This viewpoint is now well-established in the engineering literature and has led to the derivation of robust filtering formula (Davis, 1980; Balakrishnan, 1980). The starting point is that the real observation is very close to the ideal one in the C-norm, but is differentiable a.e. The filtering formula must be applicable to the real data and the result must be very close to the ideal nonlinear filter. One can then use discrete observation data without worrying about the possible discrepancy as one goes to the limit.

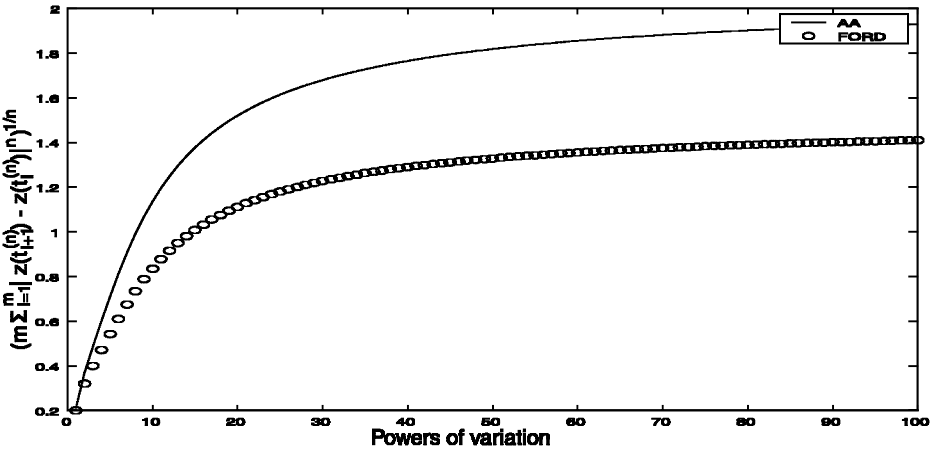


**Fig. 4.** $(N(n)\Sigma_{i=1}^{N(n)}|z(t_{i+1}^{(n)}) - z(t_i^{(n)})|^n)^{1/n}$

# 3. Transformation to a filtering problem

The important items of the filtering technique are summarized in the context of the volatility estimation problem.

- The relation (2.10) is used to convert the original filtering problem to the linear filtering problem with non-Gaussian initial condition. This problem has been solved by Makowski.
- Recalling the relation (2.11), set

$$X(s, Z) = \hat{V}(s)(\text{the filtered output})$$

- Replace the ideal observation data $Z(t)$ by the real data $Z^{(n)}(t)$ in the Makowski filter.
- Use a theorem which establishes convergence for the filtering output by using a $Z^{(n)}$-process at the output using the ideal data $Z$ as $n \to \infty$.

Assuming

$$V_o > 0 \text{ a.s.}$$

we find that the signal $V(t)$ is positive.

Noting that

$$V(t) = X(t, Z), \quad \forall t > 0 \tag{3.1}$$

$$\sqrt{V(t)} = X^{1/2}(t, Z), \quad \forall t > 0 \tag{3.2}$$

the original system can be rewritten

$$\mathrm{d}V(t) = \xi V(t)\mathrm{d}t + \sigma(X(t, Z) \wedge M)\mathrm{d}w_2(t) \tag{3.3}$$

$$\mathrm{d}Z(t) = \mu\,\mathrm{d}t - \frac{1}{2}V(t)\,\mathrm{d}t + X^{1/2}(t, Z)\,\mathrm{d}w_1(t) \tag{3.4}$$

Noting that $V_o$ is not Gaussian, the linear Kalman type filter cannot be derived. However, from Makowski's (1986) results, assuming that $\tilde{P}(t)$ defined by (3.9) below is inevitable for $t \in (0, T]$, the conditional probability density function $p(t, v|Z_t)$ can be derived explicitly:

$$p(t, v \mid \mathcal{Z}_t) = \frac{\int_0^\infty \exp[x\tilde{b}(t) - \frac{1}{2}\tilde{R}(t)x^2]\exp[-\frac{1}{2}(v - (\tilde{V}(t) + \tilde{Q}(t)x))^2\tilde{P}^{-1}(t)]\, p_o(x)\,\mathrm{d}x}{\sqrt{2\pi\tilde{P}(t)}\int_0^\infty \exp[x\tilde{b}(t) - \frac{1}{2}\tilde{R}(t)x^2]p_0(x)\mathrm{d}x} \tag{3.5}$$

where

$$p_o(.): \text{ probability density function of } V_o \tag{3.6}$$

and

$$\mathrm{d}\tilde{V}(t) = \xi\tilde{V}(t)\,\mathrm{d}t - \frac{\tilde{P}(t)}{2X(t, Z)}(\mathrm{d}Z(t) - (\mu - \frac{1}{2}\tilde{V}(t))\,\mathrm{d}t), \quad \tilde{V}(0) = 0 \tag{3.7}$$

$$\mathrm{d}\tilde{b}(t) = -\frac{\tilde{Q}(t)}{2X(t, Z)}(\mathrm{d}Z(t) - (\mu - \frac{1}{2}\tilde{V}(t))\,\mathrm{d}t), \quad \tilde{b}(0) = 0 \tag{3.8}$$

with

$$\frac{\mathrm{d}\tilde{P}(t)}{\mathrm{d}t} = 2\xi\tilde{P}(t) - \frac{\tilde{P}^2(t)}{4X(t, Z)} + \sigma^2(X(t, Z) \wedge M)^2, \quad \tilde{P}(0) = 0 \tag{3.9}$$

$$\frac{\mathrm{d}\tilde{Q}(t)}{\mathrm{d}t} = \left(\xi - \frac{\tilde{P}(t)}{4X(t, Z)}\right)\tilde{Q}(t), \quad \tilde{Q}(0) = 1 \tag{3.10}$$

$$\frac{\mathrm{d}\tilde{R}(t)}{\mathrm{d}t} = \frac{\tilde{Q}^2(t)}{4}, \quad \tilde{R}(0) = 0 \tag{3.11}$$

The normal state estimate $E\{V(t)|\mathcal{Z}_t\}$ is also given by the following simple form:

$$\begin{aligned}\hat{V}(t) &:= E\{V(t) \mid \mathcal{Z}_t\} \\ &= \tilde{V}(t) + \tilde{Q}(t)\gamma_{p_o}[\tilde{b}(t), \tilde{R}(t)]\end{aligned} \tag{3.12}$$

where

$$\gamma_{p_o}[b, \tilde{R}] = \frac{\int_0^\infty x\exp[bx - \frac{1}{2}\tilde{R}x^2]\,p_o(x)\,\mathrm{d}x}{\int_0^\infty \exp[bx - \frac{1}{2}\tilde{R}x^2]\,p_o(x)\,\mathrm{d}x} \tag{3.13}$$

In order to derive the robust forms of (3.7) and (3.9) for the $Z^{(n)}(t)$-process, one may set

$$X(t, Z^{(n)} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{(Z^{(n)}(t))^2 - 2\int_0^t Z^{(n)}(s)\mathrm{d}Z^{(n)}(s)\right\} \tag{3.14}$$

However from (2.15), we have

$$X(t, Z^{(n)}) = 0$$

So $Z(t)$ cannot be replaced by $Z^{(n)}(t)$ directly.

Now recalling that

$$f(V(t)) = E\{f(V(t)) \mid \mathcal{Z}_t\}, \quad \forall t > 0 \tag{3.15}$$

we can reset $\forall s > 0$

$$X(s, Z) = \hat{V}(s) \tag{3.16}$$

Hence we get

$$\mathrm{d}\tilde{V}(t) = \xi\tilde{V}(t)\,\mathrm{d}t - \frac{\tilde{P}(t)}{2\hat{V}(t)}(\mathrm{d}Z(t) - (\mu - \frac{1}{2}\tilde{V}(t))\,\mathrm{d}t), \quad \tilde{V}(0) = 0 \tag{3.17}$$

$$\mathrm{d}\tilde{b}(t) = -\frac{\tilde{Q}(t)}{2\hat{V}(t)}(\mathrm{d}Z(t) - (\mu - \frac{1}{2}\tilde{V}(t))\,\mathrm{d}t), \ \tilde{b}(0) = 0 \tag{3.18}$$

where $\hat{V}(t)$ can be obtained from (3.12) and (3.13). Noting that $\hat{V}(t)$ is a function of $\tilde{V}(t)$ and $\tilde{b}(t)$, the derived filters (3.17) and (3.18) are not linear. Hence replacing $Z(t)$ by $Z^{(n)}(t)$ in (3.17) and (3.18), the well-known Wong–Zakai correction terms (Wong, 1971) are to be taken into account for terms of the form

$$\int_0^t \frac{\tilde{P}(s;\, n)}{\hat{V}(s;\, n)}\mathrm{d}Z^{(n)}(s) \quad \text{and} \quad \int_0^t \frac{\tilde{Q}(s;\, n)}{\hat{V}(s;\, n)}\mathrm{d}Z^{(n)}(s) \tag{3.19}$$

So in order to establish the desired limits:

$$\lim_{n\to\infty} \tilde{V}(t;\, n) = \tilde{V}(t)\,\text{a.s.}, \quad \forall t \in [0, T]$$

and

$$\lim_{n\to\infty} \tilde{b}(t;\, n) = \tilde{b}(t)\,\text{a.s.}, \quad \forall t \in [0, T]$$

we must *a priori* subtract the correction terms in $\tilde{V}(t;\, n)$ and $\tilde{b}(t;\, n)$-equations.

The robust filter equations for $Z^{(n)}$-process are listed below:

$$\begin{aligned}\frac{\mathrm{d}\tilde{V}(t;\, n)}{\mathrm{d}t} &= \xi\tilde{V}(t;\, n) - \frac{1}{2}\frac{\tilde{P}(t;\, n)}{\hat{V}(t;\, n)}\left\{\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t} - (\mu - \frac{1}{2}\tilde{V}(t;\, n))\right\}\\ &\quad + \frac{1}{4}\tilde{P}(t;\, n)\int_{t_i^{(n)}}^t \left[\frac{\tilde{P}(s;\, n)}{\hat{V}^3(s;\, n)} + \frac{F(\tilde{Q}, \eta_{p_o}, \gamma_{p_o})}{\hat{V}^3(s;\, n)}\right]\mathrm{d}s\left|\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t}\right|^2, \quad \tilde{V}(0;\, n) = 0\end{aligned} \tag{3.20}$$

$$\begin{aligned}\frac{\mathrm{d}\tilde{b}(t;\, n)}{\mathrm{d}t} &= -\frac{1}{2}\frac{\tilde{Q}(t;\, n)}{\hat{V}(t;\, n)}\left\{\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t} - (\mu - \frac{1}{2}\tilde{V}(t;\, n))\right\}\\ &\quad + \frac{1}{4}\tilde{Q}(t;\, n)\int_{t_i^{(n)}}^t \left[\frac{\tilde{P}(s;\, n)}{\hat{V}^3(s;\, n)} + \frac{F(\tilde{Q}, \eta_{po}, \gamma_{po})}{\hat{V}^3(s;\, n)}\right]\mathrm{d}s\left|\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t}\right|^2, \quad \tilde{b}(0;\, n) = 0\end{aligned} \tag{3.21}$$

$$\frac{\mathrm{d}\tilde{P}(t;\, n)}{\mathrm{d}t} = 2\xi\tilde{P}(t;\, n) - \frac{\tilde{P}^2(t;\, n)}{4\hat{V}(t;\, n)} + \sigma^2(\hat{V}(t;\, n) \wedge M)^2, \quad \tilde{P}(0;\, n) = 0 \tag{3.22}$$

$$\frac{\mathrm{d}\tilde{Q}(t;\, n)}{\mathrm{d}t} = \left(\xi - \frac{\tilde{P}(t;\, n)}{4\hat{V}(t;\, n)}\right)\tilde{Q}(t;\, n), \quad \tilde{Q}(0;\, n) = 1 \tag{3.23}$$

$$\frac{d\tilde{R}(t;\,n)}{dt} = \frac{\tilde{Q}^2(t;\,n)}{4}, \quad \tilde{R}(0;\,n) = 0 \tag{3.24}$$

$$\hat{V}(t;\,n) = \tilde{V}(t;\,n) + \tilde{Q}(t;\,n)\gamma_{p_o}[\tilde{b}(t;\,n), \tilde{R}(t;\,n)] \tag{3.25}$$

where

$$F(\tilde{Q}, \eta_{p_o}, \gamma_{po}) = \tilde{Q}^2(t;\,n)\{\eta_{p_o}[\tilde{b}(t;\,n), \tilde{R}(t;\,n)] - \gamma^2_{p_o}[\tilde{b}(t;\,n), \tilde{R}(t;\,n)]\} \tag{3.26}$$

and $\gamma_{p_o}$ is defined by (3.13) and

$$\eta_{p_o}[b, \tilde{R}] = \frac{\int_0^\infty x^2 \exp[bx - \frac{1}{2}\tilde{R}x^2]dp_o(x)}{\int_0^\infty \exp[bx - \frac{1}{2}\tilde{R}x^2]dp_o(x)} \tag{3.27}$$

The conditional probability density function is also given by

$$p^{(n)}(t, v) = \left\{\int_0^\infty \exp[x\tilde{b}(t;\,n) - \frac{1}{2}\tilde{R}(t;\,n)x^2] \right.$$
$$\left. \times \exp[-\frac{1}{2}(v - (\tilde{V}(t;\,n) + \tilde{Q}(t;\,n)x))^2\tilde{P}^{-1}(t;\,n)]\, p_o(x)dx \right\}$$
$$\Big/ \left\{\sqrt{2\pi\tilde{P}(t;\,n)}\int_0^\infty \exp[x\tilde{b}(t;\,n) - \frac{1}{2}\tilde{R}(t;\,n)x^2]\, p_o(x)dx\right\} \tag{3.28}$$

**Theorem 3.1** Under assumptions

*(A-1):* $\exists \ell > 0$ which is independent of $t$ and $n$:

$$\hat{V}(t;\,n) \geq \ell > 0 \text{ a.s.}, \forall t \quad \in [0, T]$$

and
*(A-2):*

$$\text{supp}\,[p_o] \subset [0, C_o]$$

we have

$$\lim_{n\to\infty} \tilde{V}(t;\,n) = \tilde{V}(t) \text{ a.s.}, \quad \forall t > 0 \tag{3.29}$$

$$\lim_{n\to\infty} \tilde{b}(t;\,n) = \tilde{b}(t) \text{ a.s.}, \quad \forall t > 0 \tag{3.30}$$

The proof of this theorem is presented in the Appendix.

**Remark 3.1** Solving Equations 3.20 to 3.25 simultaneously, the conditional probability density function (3.28) is obtained by using the data $Z^{(n)}$. The most important property is that convergence of the derived conditional density (3.28) to the original density for the ideal process $Z(t)$ can be proved. From (3.25) can be obtained the minimum mean square estimate $\hat{V}$ $(t;\,n)$ of the volatility process but this is not sufficient for option pricing. For the option pricing problem the density process (3.28) must be obtained.

# 4. Simulation studies

## 4.1 *Numerical results of filtering problem*

In the digital simulation, we *a priori* obtain the signal process $V(t)$. So one can check whether the proposed algorithm works or not. Originally for the ideal data $Z(t)$, we have filter (2.9) and hence it is necessary to verify that better results are obtained by using the derived algorithm (3.20)–(3.25) than by using (2.9).

Simulation studies set

$$\xi = \frac{1}{2}\sigma^2$$

and

$$\mu = 0.09, \quad S_o = 1, \quad V_o = 0.04$$

The following four cases are considered:

$$\begin{aligned}
&(\text{Case } 1) \quad \sigma = 0.001\\
&(\text{Case } 2) \quad \sigma = 0.005\\
&(\text{Case } 3) \quad \sigma = (0.001)^{1/2} \sim 0.0316\\
&(\text{Case } 4) \quad \sigma = 0.3
\end{aligned}$$

For initial data information:

$$p_o(x) = \begin{cases} [\exp(-\frac{1}{2}(x - x_o)^2/P_o) - cp]/NC & \text{for } \; 0 \leq x \leq xm \\ 0 & \text{for } \; xm < x \end{cases} \tag{4.1}$$

where

$$\begin{aligned}
cp &= \min(\exp(-\frac{1}{2}x_o^2/P_o), \exp(-\frac{1}{2}(xm - x_o)^2/P_o))\\
NC &= \text{Normalized constant}
\end{aligned}$$

Using the finite difference method and setting $dt = 0.01$, the data $V(t)$, $S(t)$ and $Z(t)$ are generated. To solve the filtering algorithm, the time difference $ddt = dt/2$ is chosen and the integral term is approximated as follows:

$$\int_{t_i^{(n)}}^{t} \left[\frac{\tilde{P}(s;\, n)}{\hat{V}^3(s;\, n)} + \frac{F(\tilde{Q}, \eta_{p_o}, \gamma_{p_o})}{\hat{V}^3(s;\, n)}\right] \mathrm{d}s \sim \left[\frac{\tilde{P}(t_i^{(n)};\, n)}{\hat{V}^3(t_i^{(n)};\, n)} + \frac{F(\tilde{Q}, \eta_{p_o}, \gamma_{p_o})}{\hat{V}^3(t_i^{(n)};\, n)}\right] \mathrm{dd}t$$

### 4.1.1 *Case 1*

In this case, we set

$$x_o = 0.04, \quad P_o = 2 \times 10^{-5}, \quad xm = 2x_o$$

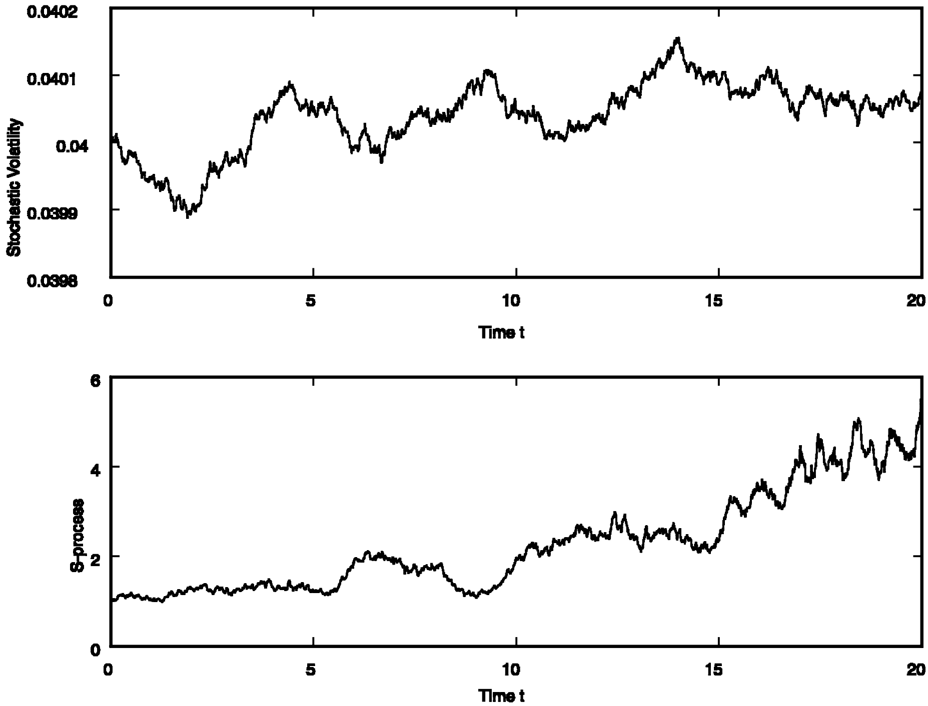


**Fig. 5.** System state $V(t)$ and stock price $S(t)$ (Case 1).

The simulated state $V(t)$ and stock price $S(t)$ are shown in Figure 5. The transformed observation data $Z(t)$ is also given in Figure 6. From Figure 7, it can be fairly said that the estimation algorithm works well rather than the formal use of the finite difference of the observation data showed in Figure 8. The time evolution of the probability density function $p(t, v|Z_t)$ is presented in Figure 9.

4.1.2 *Case 2*

In this case, setting

$$x_o = 0.04, P_o = 10^{-4}, xm = 2x_o$$

the following results are obtained:

| | |
|---|---|
| $V(t)$ and $S(t)$ processes | Figure 10 |
| $Z(t)$ process | Figure 11 |
| $V(t)$ and $\hat{V}(t)$ processes | Figure 12 |
| $p(t, v, \|\mathcal{Z}_t)$ | Figure 13 |

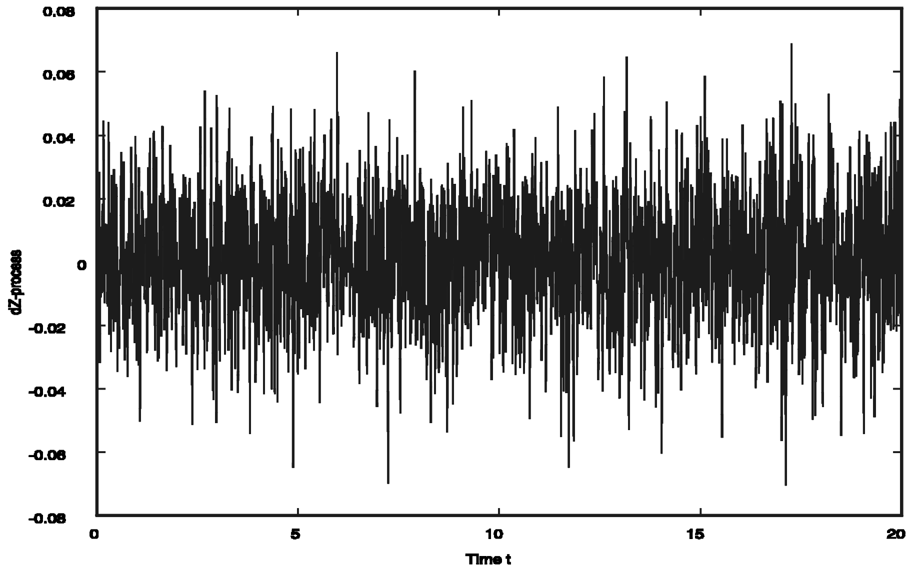


**Fig. 6.** Observational data $Z(t)$ (Case 1).

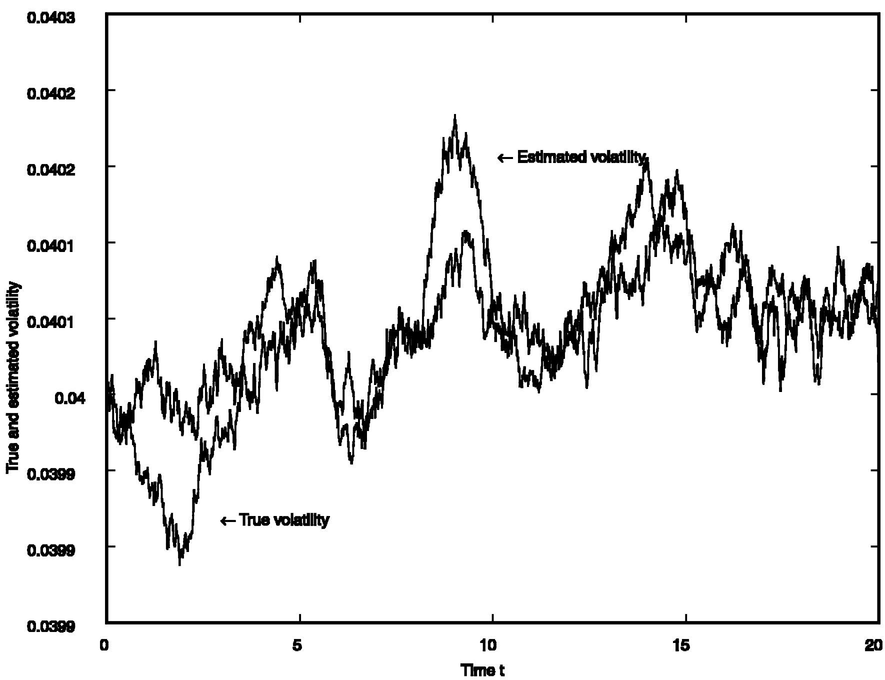


**Fig. 7.** True state $V(t)$ and estimated state $\hat{V}(t)$ (Case 1).

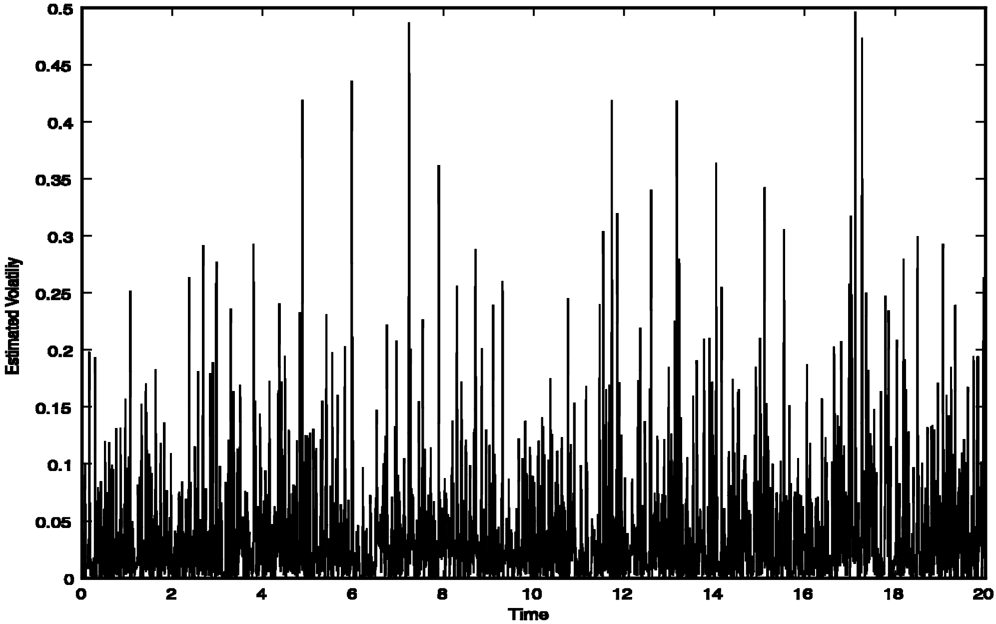


**Fig. 8.** Estimated state $\hat{V}(t)$ from $\frac{\mathrm{d}}{\mathrm{d}t}(Z^2(t) - 2\int_0^t Z(s)\,\mathrm{d}Z(s))$ (Case 1).

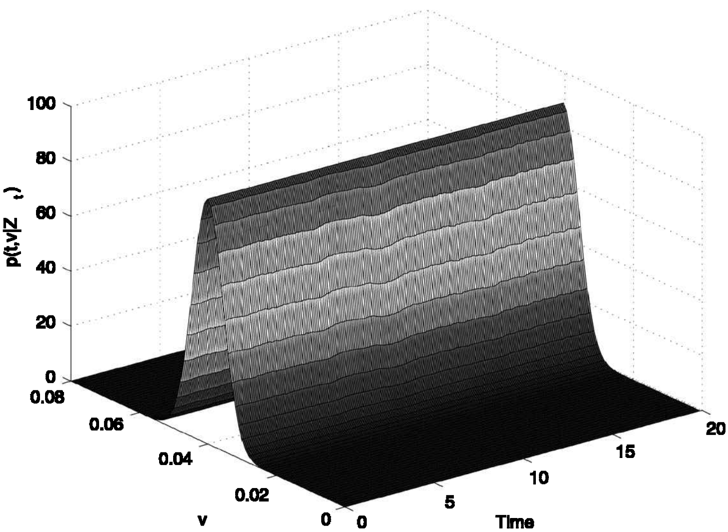


**Fig. 9.** The probability density function $P(t, v\mathcal{Z}_t)$ (Case 1).

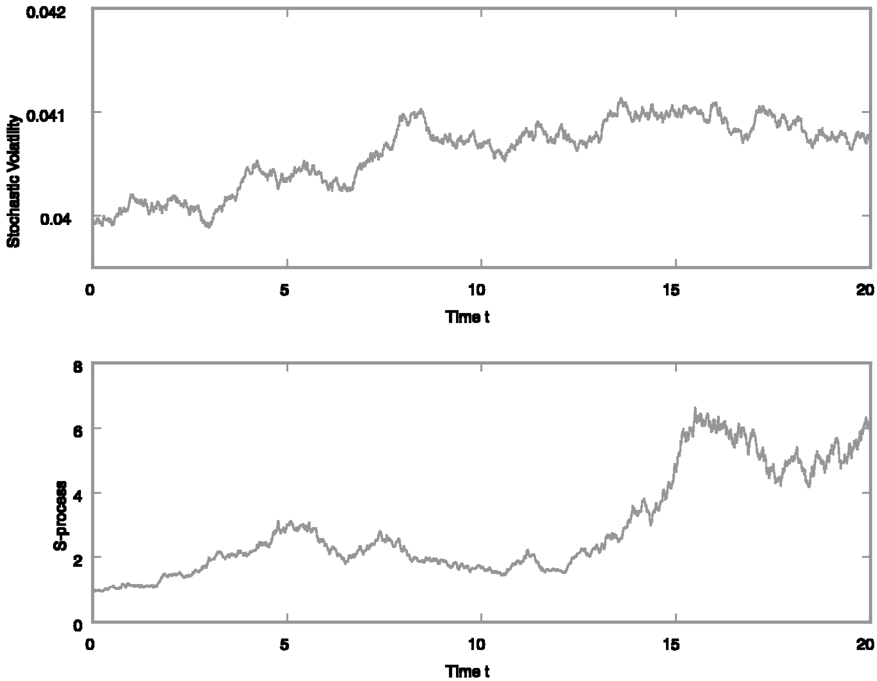


**Fig. 10.** System state $V(t)$ and stock price $S(t)$ (Case 2).

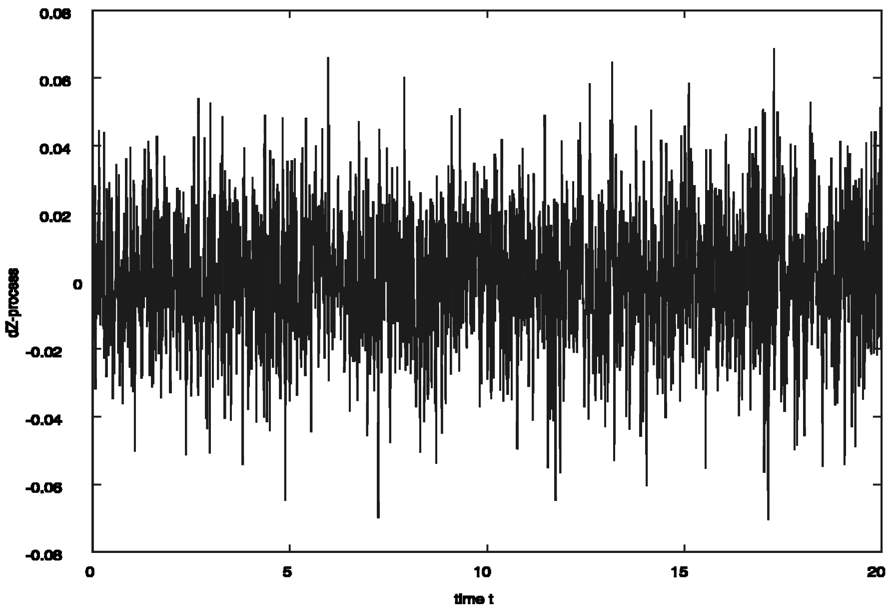


**Fig. 11.** Observational data $Z(t)$ (Case 2).

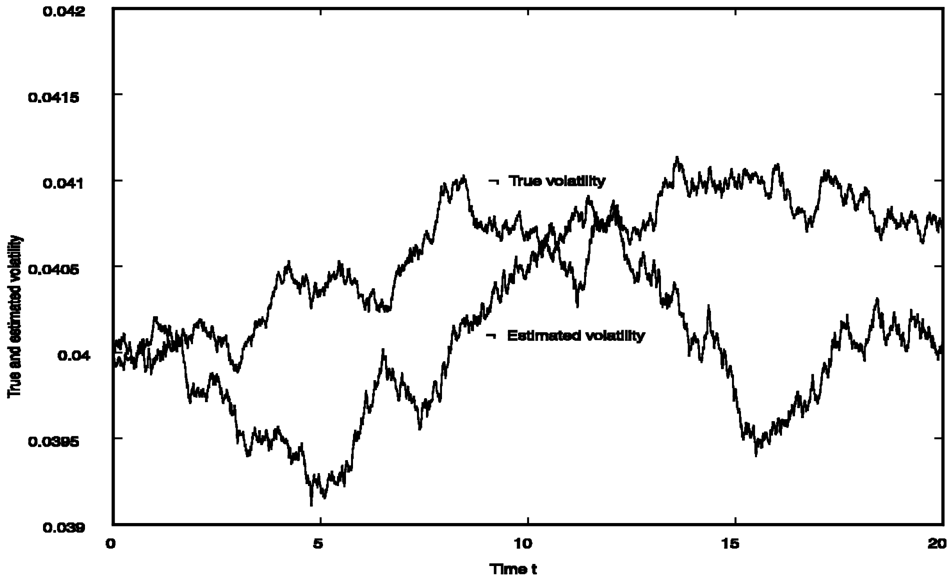


**Fig. 12.** True state $V(t)$ and estimated state $\hat{V}(t)$ (Case 2).

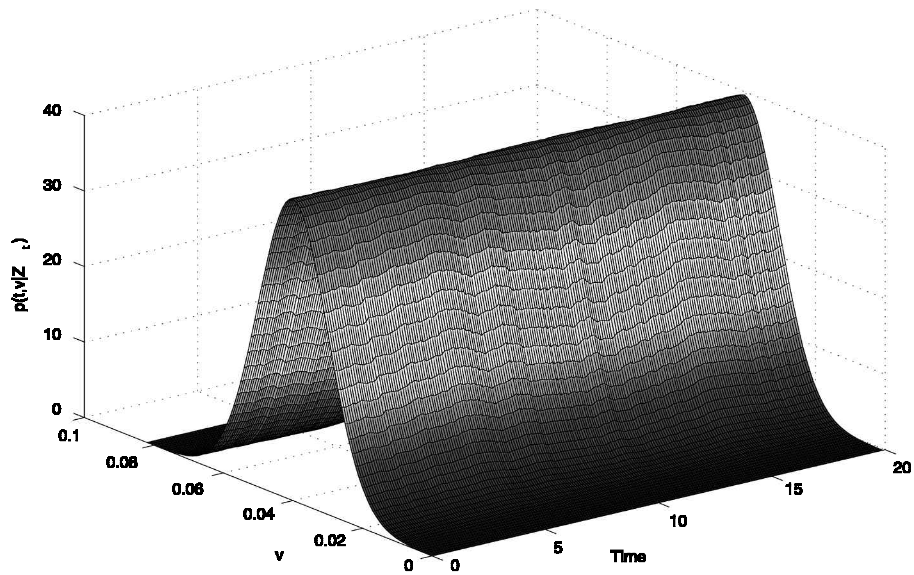


**Fig. 13.** The probability density function $P(t, v|\mathcal{Z}_t)$ (Case 2).

4.1.3 *Case 3*

Here setting

$$x_o = 0.035, \quad P_o = 10^{-3}, \quad xm = 4x_o$$

we get

| | |
|---|---|
| $V(t)$ and $S(t)$ processes | Figure 14 |
| $Z(t)$ process | Figure 15 |
| $V(t)$ and $\hat{V}(t)$ processes | Figure 16 |
| $p(t, v, \vert\mathcal{Z}_t)$ | Figure 17 |

4.1.4 *Case 4*

The same initial data is set as in Case 3.

Hence one obtains:

| | |
|---|---|
| $V(t)$ and $S(t)$ processes | Figure 18 |
| $Z(t)$ process | Figure 19 |
| $V(t)$ and $\hat{V}(t)$ processes | Figure 20 |
| $p(t, v, \vert\mathcal{Z}_t)$ | Figure 21 |



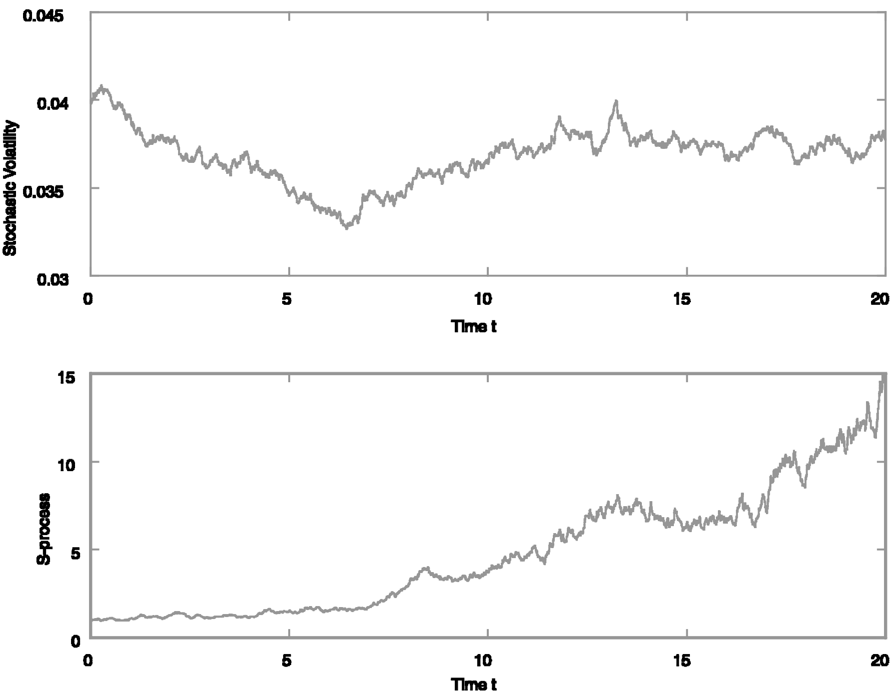


**Fig. 14.** System state $V(t)$ and stock price $S(t)$ (Case 3).

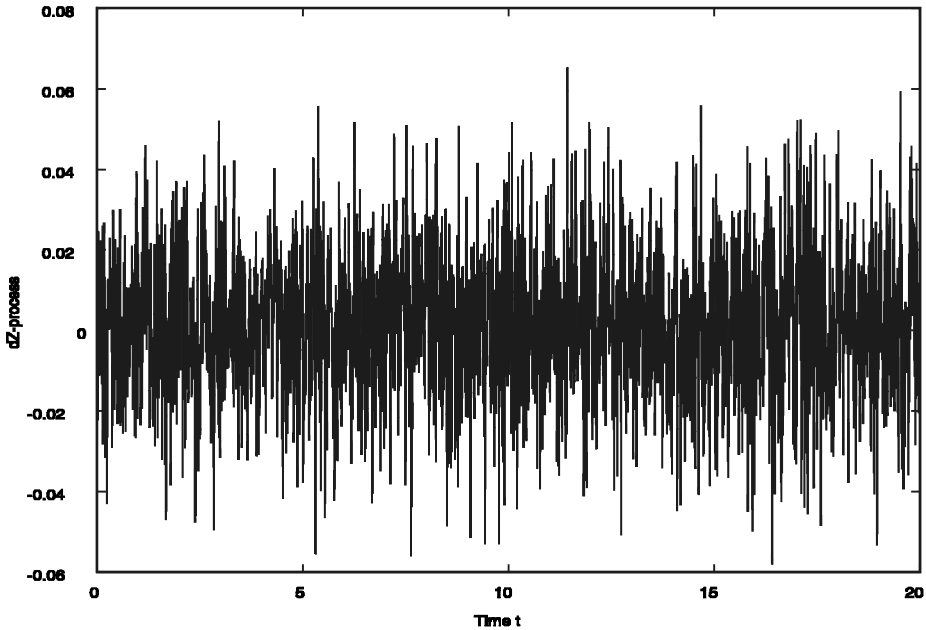


**Fig. 15.** Observational data $Z(t)$ (Case 3).

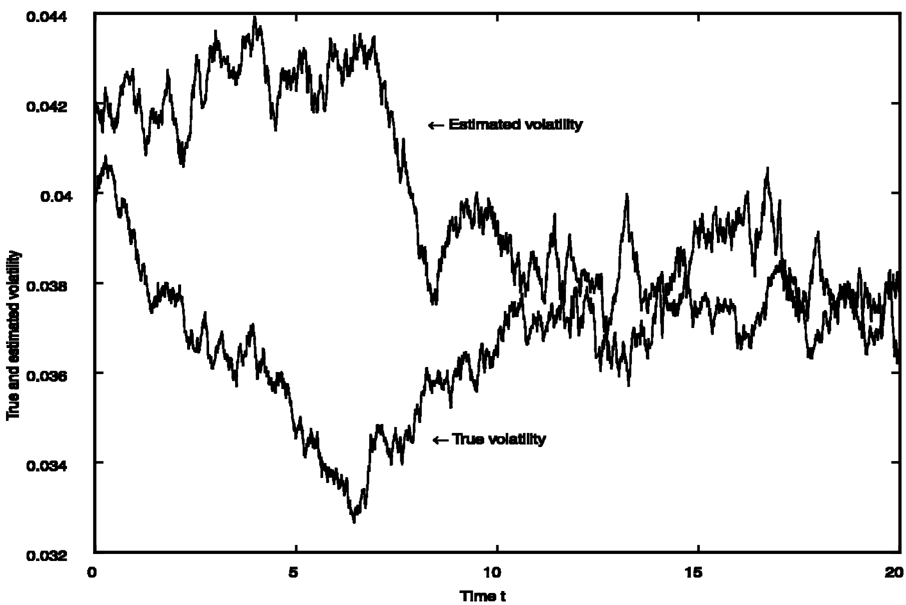


**Fig. 16.** True state $V(t)$ and estimated state $\hat{V}(t)$ (Case 3).

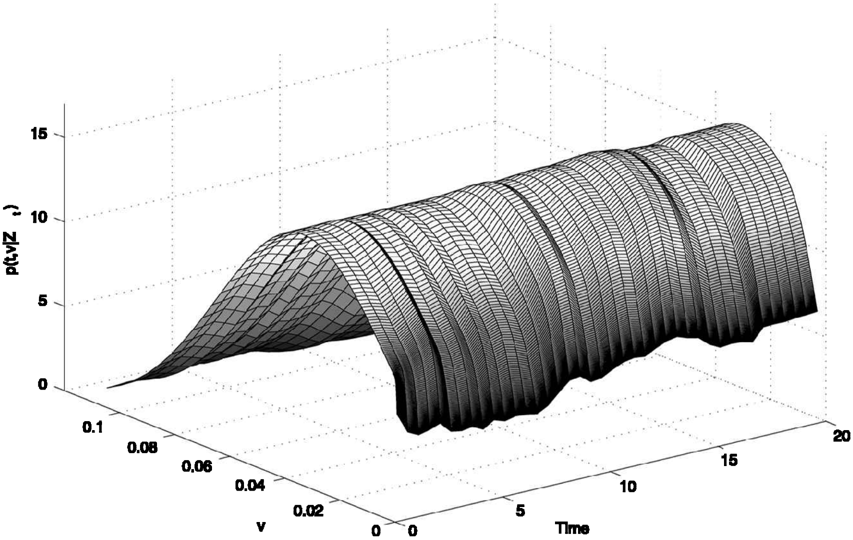


**Fig. 17.** The probability density function $P(t, v|\mathcal{Z}_t)$ (Case 3).

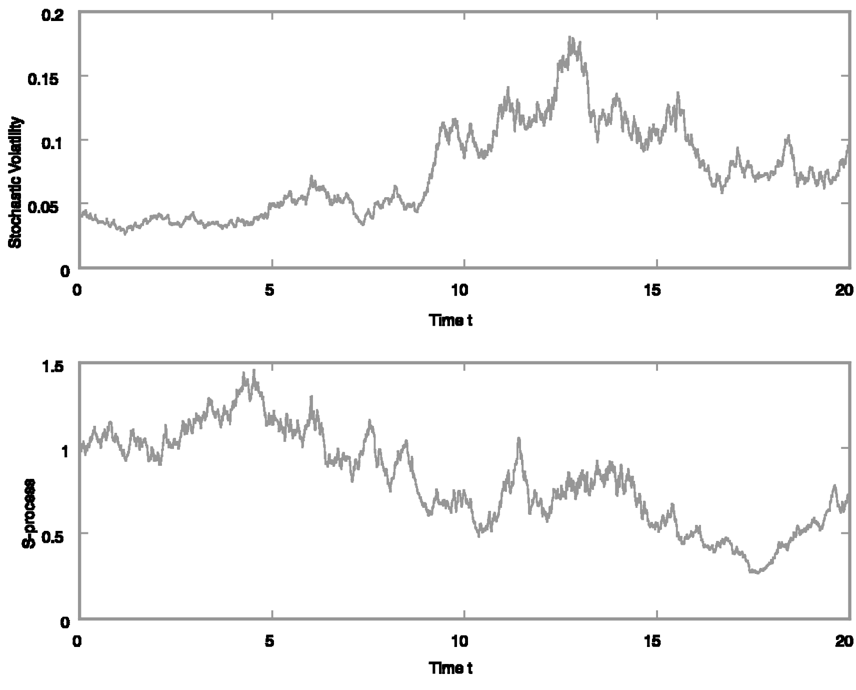


**Fig. 18.** System state $V(t)$ and stock price $S(t)$ (Case 4).

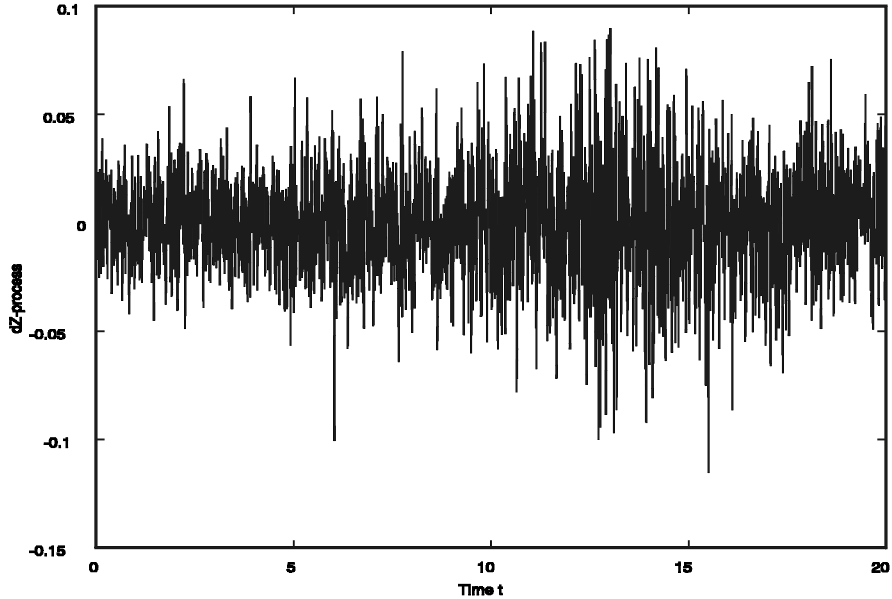


**Fig. 19.** Observational data $Z(t)$ (Case 4).

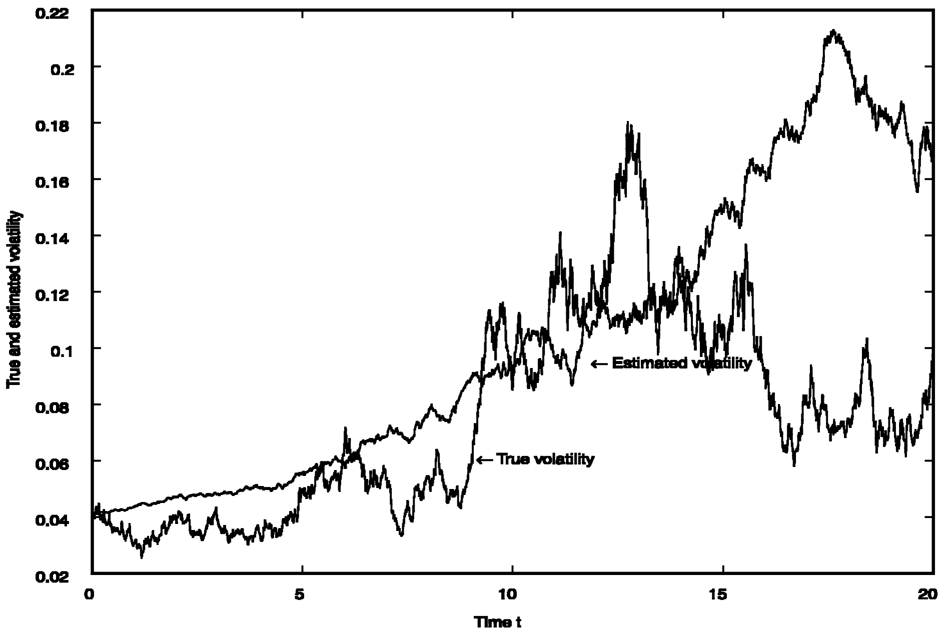


**Fig. 20.** True state $V(t)$ and estimated state $\hat{V}(t)$ (Case 4).

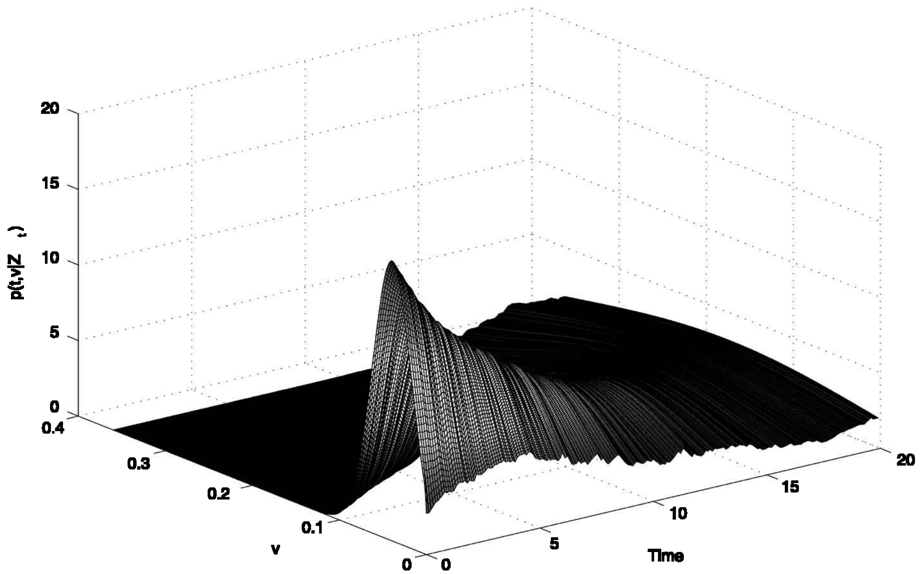


**Fig. 21.** The probability density function $P(t,\ v|\mathcal{Z}t)$ (Case 4).

**Remark 4.1** As stated in 4.1.1, the estimation procedure proposed here works better than the formal use of the finite difference method as shown in Figure 8. The same results were obtained for cases 2 to 4.

Finally, we comment on the estimates of the volatility process for each case. In every case, the estimate $\hat{V}(t)$ has some time lag to fit the true process. This phenomenon always exists in the stochastic filtering problem and this depends on the choice of the initial condition (4.1) and the parameters used. At present we do not have any theoretical way to determine the initial information. The use of historical or implied volatility in Hull (1993) may work well in determining the initial condition. In these digital simulations, also note that the important feature is the density process shown in Figures 9, 13, 17 and 21 rather than the volatility estimate. For cases 1 to 3, the initial density keeps its shape for some time interval. However for case 4, the density function becomes flat and then the estimate of the volatility seems to be too bad to apply this result to the option pricing problem.

# 5. Conclusions

In this paper, a robust filtering algorithm for estimating the stochastic volatility process used in the Hull–White model has been presented. From the simulation studies performed, it can be concluded that this algorithm is applicable to real data. The next important step is to apply the filtering algorithm to option pricing, and this will be studied in a subsequent paper.

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## Appendix: Proof of Theorem 3.1

Each of the filtering equations (3.12) to (3.28) needs to be evaluated:

### A.1 $\tilde{P}(t;\, n)$-*process*

From the first assumption (A-1) of theorem 3.1, we have

$$\begin{aligned}\sup_{t\in[0,T]} |\tilde{P}(t;\, n)| &\le \text{Const.} \int_0^T (\hat{V}(t;\, n) \wedge M)^2 \mathrm{d}t \\ &\le CM^2 T \text{ a.s.}\end{aligned} \tag{A.1}$$

Hence it is easy to show that

$$\begin{aligned}&\tilde{P}(t_{i+1}^{(n)};\, n) - \tilde{P}(t_i^{(n)};\, n) \\ &= \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (2\xi\tilde{P}(t;\, n) - \tilde{P}^2(t;\, n)/\hat{V}(t;\, n) + \sigma^2(\hat{V}(t:\, n) \wedge M)^2 \mathrm{d}t\end{aligned}$$

It follows from (A.1) that

$$| \tilde{P}(t;\, n) - \tilde{P}(t_i^{(n)};\, n) | \leq \text{Const.}\, | t - t_i^{(n)} | \text{ a.s.} \tag{A.2}$$

## A.2 $\tilde{Q}$ $(t;\ n)$ *and* $\tilde{R}$ $(t;\ n)$*-processes*

Using the procedure used to derive (A.1) and (A.2) we have

$$\sup_t | \tilde{Q}(t;\, n) | \leq \text{Const.}, \quad \sup_t | \tilde{R}(t;\, n) | \leq \text{Const. a.s.} \tag{A.3}$$

and

$$|\tilde{Q}(t;n) - \tilde{Q}(t_i^{(n)};n)| \leq \text{Const.}|t - t_i^{(n)}| \text{ a.s.} \tag{A.4}$$

$$|\tilde{R}(t;\, n) - \tilde{R}(t_i^{(n)};\, n)| \leq \text{Const.}|t - t_i^{(n)}| \text{ a.s.} \tag{A.5}$$

## A.3 $\gamma_{p_o}[\tilde{b},\ \tilde{R}]$ *and* $\eta_{p_o}[\tilde{b},\ \tilde{R}]$*-processes*

From the second assumption (A-2) of theorem 3.1, we have

$$\gamma_{p_o}[\tilde{b},\tilde{R}] \leq C_o, \quad \eta_{p_o}[\tilde{b},\tilde{R}] \leq C_o^2 \text{ a.s.} \tag{A.6}$$

Hence we also get

$$\sup_t |\tilde{Q}(t;\, n)\gamma_{p_o}[\tilde{b},\tilde{R}]| \leq \text{Const. a.s.} \tag{A.7}$$

and

$$\sup_t |F(\tilde{Q}(t;\, n),\tilde{b},\tilde{R})| \leq \text{Const. a.s.} \tag{A.8}$$

## *A.4* $\tilde{V}$ $(t;\ n)$*-process*

It follows from (3.21) that

$$\begin{aligned}
\tilde{V}(t;\, n) = &-\frac{1}{2}\int_0^t \Phi(t,s,\tilde{P})\frac{\tilde{P}}{\hat{V}(s;\, n)}\left(\frac{\mathrm{d}Z^{(n)}(s)}{\mathrm{d}s} - \mu\right)\mathrm{d}s \\
&+ \frac{1}{4}\int_0^t \Phi(t,s,\tilde{P})\tilde{P}(s;\, n)\left\{\int_{t_i^{(n)}}^s \left[\frac{\tilde{P}(\tau;\, n)}{\hat{V}^3(\tau;\, n)} + \frac{F(\tilde{Q},\eta_{p_o},\gamma_{p_o})}{\hat{V}^3(\tau;\, n)}\right]\mathrm{d}\tau\left|\frac{\mathrm{d}Z^{(n)}(s)}{\mathrm{d}s}\right|^2\right. \\
&\left.+ \Sigma_{\hat{d}=1}^{i-1}\int_{t_d^{(n)}}^{t_{j+1}^{(n)}} \left[\frac{\tilde{P}(\tau;\, n)}{\hat{V}^3(\tau;\, n)} + \frac{F(\tilde{Q},\eta_{p_o},\gamma_{p_o})}{\hat{V}^3(\tau;\, n)}\right]\mathrm{d}\tau\left|\frac{\mathrm{d}Z^{(n)}(t_{j+1}^{(n)})}{\mathrm{d}t}\right|^2\right\}\mathrm{d}s
\end{aligned}$$

where

$$\Phi(t,s;\,\tilde{P}) = \exp\left\{\int_s^t \left(\xi - \frac{\tilde{P}(\tau;\, n)}{\hat{V}(\tau;\, n)}\right)\mathrm{d}\tau\right\}$$

with $|\Phi\ (t, s, \tilde{P})\ | \leq |\xi|(t-s)$. Hence we obtain

$$\sup_t |\ \tilde{V}(t;n)\ | \leq \text{Const.} \int_0^T \left\{ 1 + \left| \frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t} \right| + \left| \frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t} \right|^2 \Delta t^{(n)} \right\} \mathrm{d}t$$

$$\leq \text{Const.}\{T + \Sigma_{i=1}^{N(n)}[\ |\ Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})\ |\ +\ |\ Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})\ |^2\ ]\}$$

It is easy to show that

$$E\{(\sup_t |\ \tilde{V}(t;n)\ |^2\} \leq \text{Const.}[T^2 + E\{\int_0^T V(t)\mathrm{d}t\}] + O(n^{-1}) \leq \text{Const. independent of } n \tag{A.9}$$

Noting that $\exists\delta = \delta(\omega) > 0$ a.e. $\omega$; for $|t - t'| < \delta$,

$$|\ Z(t) - Z(t')\ | \leq C\ |\ t - t'\ |^{\alpha}, \quad 0 < \alpha < 1/2$$

we have

$$\sup_t |\ \tilde{V}(t;n)\ | \leq \text{Const.}\{T + N(n)((\Delta t^{(n)})^{\alpha} + (\Delta t^{(n)})^{2\alpha})\}$$

$$\leq \text{Const.}(\omega)\{T + n^{1-\alpha} + n^{1-2\alpha}\} \text{ a.s.} \tag{A.10}$$

Consequently

$$\sup_{t_i^{(n)} \leq t \leq t_{i+1}^{(n)}} \{ \int_{t_i^{(n)}}^t |\ \tilde{V}(s;n)\mathrm{d}s\}$$
$$\leq \text{Const.}(\omega)\{T + n^{1-\alpha} + n^{1-2\alpha}\}\Delta t^{(n)}$$
$$\leq \text{Const.}(\omega)\{T/n + 1/n^{\alpha} + 1/n^{2\alpha}\}$$
$$\leq O(n^{-\alpha}) \to 0, \quad \text{as} \to \infty \text{ a.s.}$$

Now we obtain

$$\tilde{V}(t;n) - \tilde{V}(t_i^{(n)};n)$$
$$= \int_{t_i^{(n)}}^t \left\{ \left. \xi - \frac{\tilde{P}(s;n)}{4\hat{V}(s;n)} \right) \tilde{V}(s;n) + \frac{\mu}{2\hat{V}(s;n)} \tilde{P}(s;n) \right\} \mathrm{d}s$$
$$- \frac{1}{2} \int_{t_i^{(n)}}^t \frac{\tilde{P}(s;n)}{4\hat{V}(s;n)} \left[ \frac{\mathrm{d}Z^{(n)}(s)}{\mathrm{d}s} \right] \mathrm{d}s$$
$$+ \frac{1}{4} \int_{t_i^{(n)}}^t \tilde{P}(s;n) \int_{t_i^{(n)}}^s \left[ \frac{\tilde{P}(\tau;n)}{4\hat{V}^3(\tau;n)} + \frac{F(\tilde{Q}, \xi_{p_o}, \gamma_{p_o})}{\hat{V}^3(\tau;n)} \right] \mathrm{d}\tau \left| \frac{\mathrm{d}Z^{(n)}(s)^2}{\mathrm{d}s} \right| \mathrm{d}s$$

Hence for $t_i^{(m)} \leq t \leq t_{i+1}^{(n)}$

$$|\ \tilde{V}(t;n) - \tilde{V}(t_i^{(n)};n)\ | \leq C[|t - t_i^{(n)}\ |\ (1 + \sup_{0 \leq t \leq T} |\ \tilde{V}(t;n)|)$$
$$+\ |\ Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})\ |\ +\ |\ Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})\ |^2 \leq O(n^{-\alpha}) \to 0, \quad \text{as } n \to \infty \tag{A.11}$$

## A.5 $\tilde{b}(t)$-*process*

By using the same procedure to derive the estimates of $\tilde{V}(t;\, n)$-process, we have

$$\sup_t |\, \tilde{b}(t;\, n)\,| \leq \text{Const.}(\omega)\{T + n^{1-\alpha} + n^{1-2\alpha}\}\ \text{a.s.}$$

$$E\{(\sup_t |\, \tilde{b}(t;\, n)|)^2\} \leq \text{Const. independent of } n$$

and

$$|\, \tilde{b}(t;\, n) - \tilde{b}(t_i^{(n)};\, n)\,| \leq O(n^{-\alpha})\ \text{a.s.}$$

## A.6 $\tilde{Q}\ (t;\, n)\ \gamma_{p_o}\ [\tilde{b},\, \tilde{R}]$-*process*

It is easy to show that

$$\tilde{Q}(t;\, n)\gamma_{p_o}[\tilde{b}(t;\, n), \tilde{R}(t;\, n)] - \tilde{Q}(t^{(n)};\, n)\gamma_{p_o}[\tilde{b}(t^{(n)};\, n), \tilde{R}(t^{(n)};\, n)]$$
$$= \int_{t_i^{(n)}}^{t} \left\{ \frac{\partial \tilde{Q}(s;\, n)}{\partial s}\gamma_{p_o} + \tilde{Q}(s;\, n) \left( \frac{\partial \gamma_{p_o}}{\partial \tilde{b}} \frac{\mathrm{d}\tilde{b}(s;\, n)}{\mathrm{d}s} + \frac{\partial \gamma_{p_o}}{\partial \tilde{R}} \frac{\mathrm{d}\tilde{R}(s;\, n)}{\mathrm{d}s} \right) \right\} \mathrm{d}s$$

We have

$$\frac{\partial \gamma_{p_o}}{\partial \tilde{b}} = \eta_{p_o}[\tilde{b}, \tilde{R}] - \gamma_{p_o}^2[\tilde{b}, \tilde{R}]$$

and

$$\frac{\partial \gamma_{p_o}}{\partial \tilde{R}} = -\frac{\int_0^\infty x^3 \exp[\tilde{b}x - \frac{1}{2}\tilde{R}x^2]\mathrm{d}p_o}{2\int_0^\infty \exp[\tilde{b}x - \frac{1}{2}\tilde{R}x^2]\mathrm{d}p_o} + \frac{1}{2}\gamma_{p_o}[\tilde{b}, \tilde{R}]\eta_{p_o}[\tilde{b}, \tilde{R}]$$

From the second assumption (A-2), we get

$$\sup_{0 \leq t \leq T} \left| \frac{\partial \gamma_{p_o}}{\partial \tilde{b}} \right| \leq \text{Const. and } \sup_{0 \leq t \leq T} \left| \frac{\partial \gamma_{p_o}}{\partial \tilde{R}} \right| \leq \text{Const.}$$

It is also possible to derive

$$\sup_{0 \leq t \leq T} \left| \frac{\partial \tilde{Q}(t;\, n)}{\partial t} \right| \leq \text{Const. and } \sup_{0 \leq t \leq T} \left| \frac{\partial \tilde{R}(t;\, n)}{\partial t} \right| \leq \text{Const.}$$

Hence

$$|\tilde{Q}(t; n)\gamma_{p_o}[\tilde{b}(t;\, n), \tilde{R}(t;\, n)] - \tilde{Q}(t^{(n)};\, n)\gamma_{p_o}[\tilde{b}(t^{(n)};\, n)\tilde{R}(t^{(n)};\, n)]|$$
$$\leq \text{Const.}|t - t_i^{(n)}| + \left| \int_{t_i^{(n)}}^{t} \tilde{Q}(s;\, n) \frac{\partial_{\gamma_{p_o}}}{\partial \tilde{b}} \frac{\mathrm{d}\tilde{b}(s;\, n)}{\mathrm{d}s} \mathrm{d}s \right|$$

Repeating the same argument used to derive (A.11), we also have for $t_i^{(n)} \le t \le t_{i+1}^{(n)}$

$$\left|\int_{t_i^{(n)}}^{t} \tilde{Q}(s;\, n)\frac{\partial \gamma_{p_o}}{\partial \tilde{b}}\frac{\mathrm{d}\tilde{b}(s;\, n)}{\mathrm{d}s}\mathrm{d}s\right| \le C[|t - t_i^{(n)}|(1 + \sup_{0\le t\le T}|\tilde{b}(t;\, n)|) + |Z(t_{i+1}^{(n)} - Z(t_i^{(n)})| + |Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|^2] \le O(n^{-\alpha})\,\text{a.s.}$$

## A.7 *Main proof*

In order to prove Theorem 3.1, we need to evaluate the following term for the $\tilde{V}(t;n)$-process

$$\Sigma_{i=1}^{N(n)}\left\{-\frac{1}{2}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \frac{\tilde{P}(s;\, n)}{\hat{V}(s;\, n)}\frac{\mathrm{d}Z^{(n)}(s)}{\mathrm{d}s}\mathrm{d}s + \frac{1}{4}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(s;\, n)\int_{t_i^{(n)}}^{s} \frac{[\tilde{P}(\tau;\, n) + F(\tilde{Q}, \eta_{p_o}, \gamma_{po})]}{\hat{V}(\tau;\, n)}\mathrm{d}\tau\right\}\left|\frac{dZ^{(n)}(s)}{ds}\right|^2 \mathrm{d}s$$

Now

$$\begin{aligned}
I_1(T) &= -\frac{1}{2}\int_0^T \frac{\tilde{P}(t;\, n)}{\hat{V}(t;\, n)}\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t}\mathrm{d}t \\
&= -\frac{1}{2}\Sigma_{i=1}^{N(n)}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \frac{\tilde{P}(t;\, n)}{\hat{V}(t;\, n)}\frac{Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})}{\Delta t_i^{(n)}}\mathrm{d}t \\
&= -\frac{1}{2}\Sigma_{i=1}^{N(n)}\frac{\tilde{P}(t_i^{(n)};\, n)}{\hat{V}(t_i^{(n)};\, n)}(Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})) \\
&\quad -\frac{1}{2}\Sigma_{i=1}^{N(n)}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (\tilde{P}(t;\, n) - \tilde{P}(t_i^{(n)};\, n))\mathrm{d}t\frac{1}{\hat{V}(t_i^{(n)};\, n)}\frac{Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})}{\Delta t_i^{(n)}}\mathrm{d}t \\
&\quad -\frac{1}{2}\Sigma_{i=1}^{N(n)}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t;\, n)\left(\frac{1}{\hat{V}(t;\, n)} - \frac{1}{\hat{V}(t_i^{(n)};\, n)}\right)\mathrm{d}t\frac{Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})}{\Delta t_i^{(n)}} \\
&= I_{11}(T) + I_{12}(T) + I_{13}(T)
\end{aligned}$$

It is easy to show that

$$\lim_{n\to\infty} I_{11}(T) = -\frac{1}{2}\int_0^T \frac{\tilde{P}(t)}{\hat{V}(t)}\mathrm{d}Z(t)$$

and

$$|I_{12}(T)| \le \frac{1}{2}\Sigma_{i=1}^{N(n)}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} |\tilde{P}(t;\, n) - \tilde{P}(t_i^{(n)};\, n)|\mathrm{d}t\frac{1}{\ell}\frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}$$

$$\leq \text{ Const. } \Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} |t - t_i^{(n)}| \mathrm{d}t \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}$$
$$\leq \text{ Const. } \Sigma_{i=1}^{N(n)} |Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})| \Delta t_i^{(n)}$$
$$\leq O(n^{-\alpha}) \to 0, \quad \text{as } n \to \infty \text{ a.s.}$$

Furthermore

$$I_{13}(T) = \frac{1}{2} \Sigma_{i=1}^{N(n)} \{ \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t_i^{(n)}}^{t} \frac{1}{\hat{V}^2(s; n)} \frac{\mathrm{d}\tilde{V}(s; n)}{\mathrm{d}s} \mathrm{d}s^{\downarrow} \mathrm{d}t \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta \mathrm{t}_\mathrm{i}^{(\mathrm{n})}} \}$$
$$+ \frac{1}{2} \Sigma_{i=1}^{N(n)} \left\{ \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t_i^{(n)}}^{t} \frac{1}{\hat{V}^2(s; n)} \frac{\mathrm{d}\tilde{Q}(s; n)\gamma_{p_o}}{\mathrm{d}s} \mathrm{d}s\, \mathrm{d}t \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}} \right\}$$
$$= I_{131}(T) + I_{132}(T)$$

It follows from $\tilde{V}(s; n)$ equation that

$$I_{131}(T) = \frac{1}{2} \Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t_i^{(n)}}^{t} \frac{-1}{2} \frac{\tilde{P}(s; n)}{\hat{V}^3(s; n)} \mathrm{d}s \frac{\mathrm{d}Z^{(n)}(\mathrm{t})}{\mathrm{d}t} \mathrm{d}t \frac{|Z(t_{i+1}^{(n)}) - Z(t_i{}^{(n)})|}{t_i{}^{(n)}}$$
$$+ \frac{1}{2} \Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t_i^{(n)}}^{t} \frac{1}{\hat{V}^2(s; n)} \left\{ \xi \tilde{V}(s; n) - \frac{1}{2} \frac{\tilde{P}(s; n)}{\hat{V}(s; n)} (-(\mu - \frac{1}{2} \tilde{V}(s; n))) \right.$$
$$\left. + \frac{1}{4} \tilde{P}(s; n) \int_{t_i^{(n)}}^{s} \left[ \frac{\tilde{P}(\tau; n)}{\hat{V}^3(\tau; n)} + \frac{F(\tilde{Q}, \eta_{p_o}, \gamma_{p_o})}{\hat{V}^3(\tau; n)} \right] \mathrm{d}\tau \left( \frac{\mathrm{d}Z^{(n)}(s)}{d} s \right)^2 \right\} \mathrm{d}s\, \mathrm{d}t$$
$$\times \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}$$
$$= -\frac{1}{4} \Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t_i^{(n)}}^{t} \frac{\tilde{P}(s; n)}{\hat{V}^3(s; n)} \mathrm{d}s \left( \frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t} \right)^2 \mathrm{d}t$$
$$+ \frac{1}{2} \Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t_i^{(n)}}^{t} \frac{1}{\hat{V}^2(s; n)} \left\{ \xi - \frac{1}{4} \frac{\tilde{P}(s; n)}{\hat{V}(s; n)} \right\} \mathrm{d}s\, \mathrm{d}t$$
$$\times \tilde{V}(t_i^{(n)}; n) \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}$$
$$+ \frac{1}{2} \Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t; n) \int_{t^{(n)}}^{t} \frac{1}{\hat{V}^2(s; n)} \left\{ \xi - \frac{1}{4} \frac{\tilde{P}(s; n)}{\hat{V}(s; n)} \right\} \mathrm{d}s\, \mathrm{d}t$$
$$\times (\tilde{V}(s; n) - \tilde{V}(t_i^{(n)} j n)) \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}$$
$$+ O(n^{-\alpha})$$

Now

$E\{|$ The second term of the R.H.S. of $I_{132(T)}|^2\}$

$$\leq \text{Const.}\, E\left\{(\Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{t+1}^{(n)}} |\tilde{P}(t;\, n)| \int_{t_i^{(n)}}^{t} \frac{1}{\ell^2}\left(|\xi| + \frac{1}{4}\frac{\tilde{P}(s;\, n)}{\ell}\right) \mathrm{d}s\, \mathrm{d}t\right.$$

$$\left. \times |\tilde{V}(t_i^{(n)};\, n)| \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}\right)^2\right\}$$

$$\leq \text{Const.}\, E\left\{ \Sigma_{i=1}^{(n)} (\Delta t_i^{(n)})^2 |\tilde{V}(t_i^{(n)};\, n)| \frac{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}\right)^2 \right\}$$

$(\tilde{V}(t^{(n)};\, n)$ and $|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|$ are independent)

$$\leq \text{Const.}\, \Sigma_{i=1}^{(n)} (\Delta t_i^{(n)})^2 E\{|\tilde{V}(t_i^{(n)};\, n)|^2\} E\{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|^2\}$$

(from $(A.9)$)

$$\leq \text{Const.}\, \Sigma_{i=1}^{(n)} (\Delta t_i^{(n)})^2 E\{|Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|^2\}$$

$$\leq O(n^{-2}) \to 0 \quad \text{as } n \to \infty$$

By using the Borel–Cantelli lemma, it is also shown that the second term of the R.H.S. of $I_{131}(T)$ converges to 0 as $n \to \infty$ a.s. Furthermore

$|$The third term of the R.H.S. of $I_{131}(T)|$

$$\leq \text{Const.}\, \Sigma_{i=1}^{N(n)} (\Delta t_i^{(n)})^2 \sup_{t_i^{(n)} \leq t \leq t_{i+1}^{(n)}} |\tilde{V}(t;\, n) - \tilde{V}(t_i^{(n)};\, n)| \times \frac{|Z(t_{t+1}^{(n)} - Z(t_i^{(n)})|}{\Delta t_i^{(n)}}$$

$$\leq \text{Const.}\, (\omega) \Sigma_{i=1}^{N\ (n)} \Delta t_i^{(n)}.O(n^{-\alpha}).(\Delta t_i^{(n)})^{\alpha}$$

$$\leq O(n^{-2\alpha}) \to 0, \quad \text{as} \quad n \to \infty$$

That is

$$I_{131}(T) + \frac{1}{4}\Sigma_{i=1}^{N(n)} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \tilde{P}(t;\, n) \int_{t_i^{(n)}}^{t} \frac{\tilde{P}(s;\, n)}{\hat{V}^3(s;\, n)} \mathrm{d}s \left(\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t \to 0, \quad \text{as} \quad n \to \infty \text{ a.s} \tag{A.12}$$

Now we shall evaluate the $I_{132}(T)$-term. This term becomes

$$I_{132}(T) = \frac{1}{2}\Sigma_{i=1}^{N(n)} \int_{t_i^n}^{t_{i+1}^{(n)}} \tilde{P}(t;\, n) \quad \int_{t_i^n}^{t} \frac{1}{\hat{V}^2(s;\, n)} \left[\frac{\mathrm{d}\tilde{Q}(s;\, n)}{\mathrm{d}s}\gamma_{p_o} + \tilde{Q}(s;\, n)\frac{\partial_{\gamma_{po}}}{\partial \tilde{R}}\frac{\mathrm{d}\tilde{R}(s)}{\mathrm{d}s} + \tilde{Q}(s;\, n)\frac{\partial_{\gamma_{po}}}{\partial \tilde{b}}\frac{\mathrm{d}\tilde{b}(s)}{\mathrm{d}s}\right]\mathrm{d}s\,\mathrm{d}t\ \frac{Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})}{\Delta t_i^{(n)}}$$

From the estimates obtained for $\tilde{b}$ and $\tilde{Q}\gamma_{p_o}$-processes it is easy to show that the first and second terms of the $I_{132}(T)$ converge to zero as $n \to \infty$ a.s. So we need to evaluate the following term;

$$I_{1323}(T) = \frac{1}{2}\Sigma_{i=1}^{N(n)} \int_{t_i^n}^{t_{i+1}^{(n)}} \tilde{P}(t;\, n) \quad \int_{t_i^n}^{t} \frac{1}{\hat{V}^2(s;\, n)}\tilde{Q}(s;\, n)\frac{\partial_{\gamma_{po}}}{\partial b}\frac{\mathrm{d}\tilde{b}(s)}{\mathrm{d}s}\Bigg]\mathrm{d}s\,\mathrm{d}t\ \frac{Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})}{\Delta t_i^{(n)}}$$

By using the same procedure used to derive the estimate (A.12), we have

$$I_{1323}(T) + \frac{1}{4}\Sigma_{i=1}^{N(n)} \int_{t_i^n}^{t_{i+1}^{(n)}} \tilde{P}(t;\, n)\left(\int_{t_i^n}^{t} \frac{\tilde{Q}^2(s;\, n)}{\hat{V}^3(s;\, n)}\frac{\partial_{\gamma_{po}}}{\partial b}\mathrm{d}s\right)\left(\frac{\mathrm{d}Z^{(n)}(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t \to 0, \quad \text{as} \quad n \to \infty \text{ a.s.}$$

i.e. from $F(Q, \eta_{p_o}, \gamma_{p_o}) = \tilde{Q}^2(s;\, n)\partial_{\gamma_{p_o}}/\partial \tilde{b}$ we get

$$I_{132}(T) + \frac{1}{4}\Sigma_{i=1}^{N(n)} \int_{t_i^n}^{t_{i+1}^{(n)}} \tilde{P}(t;\, n)\left(\int_{t_i^n}^{t} \frac{F(\tilde{Q}, \eta_{p_o}, \gamma_{p_o})}{\hat{V}^3(s;\, n)}\mathrm{d}s\right. \quad \left.\frac{\mathrm{d}Z^{(n)}(\mathrm{t})}{\mathrm{d}t}\right)^2 \mathrm{d}t \to 0, \quad \text{as} \quad n \to \infty \text{ a.s}$$

This implies that

$$\lim_{n\to\infty}\{I_1(T) + I_2(T)\} = -\frac{1}{2}\int_0^T \frac{\tilde{P}(t)}{\hat{V}(t)}\mathrm{d}Z(t) \text{ a.s.}$$

For the $\tilde{b}$-process we can obtain the same results as above, which completes the proof.