# EMBEDDING A TRANSFORMATION GROUP IN AN AUTOMORPHISM GROUP[1]

MURRAY EISENBERG

1. **Introduction.** Using a construction of Baĭdosov [1], we show that a topological transformation group with completely regular phase space $X$ and locally compact phase group $T$ can be equivariantly embedded in a transformation group of automorphisms of a topological group $A$. The group $A$ in question is the free abelian topological group over $X$; some facts about $A$ are established in §2. In §4 several dynamical properties of $(A, T)$ are discussed as they relate to properties of $(X, T)$.

As general references to the notation and notions for transformation groups used here, see [4] and [5].

All topological spaces considered below, and in particular all topological groups, are assumed to be Hausdorff.

2. **The topological group** $A(X)$. Let $X$ be a completely regular space. Denote by $A(X)$ or simply $A$ the free abelian topological group over $X$ [6], [7, §8]. Algebraically $A$ is just the free abelian group generated by the set $X$; the topology of $A$ is the greatest separated topology compatible with the group structure and inducing on $X$ a topology weaker than the one initially given on $X$. We have:

(1) $A$ is a topological group containing $X$ as a closed subspace.

(2) If $f$ is a continuous map of $X$ into an abelian topological group $G$, then the unique extension of $f$ to a group homomorphism of $A$ into $G$ is continuous.

PROPOSITION 1. *Let $X$ be compact and infinite. Then $A$ is not a Baire space and a fortiori is not locally compact.*

PROOF. If $z\in A$ and $z\neq 0$, there exist distinct $x_1, \cdots, x_n\in X$ and nonzero integers $\alpha_1, \cdots, \alpha_n$ with $z=\sum_i \alpha_i x_i$, and we let $L(z) = \sum_i |\alpha_i|$. Set $L(0)=0$. For each positive integer $n$ let $A_n = \{z \mid z\in A,\ L(z)\leqq n\}$. Since $A=\bigcup_1^\infty A_n$, it is enough to show that each $A_n$ is compact and has vacuous interior.

Let $n>0$. Let $B_0=\{0\}$, and for each positive integer $p$ let $B_p$ be the set of all elements of $A$ of the form $\sum_1^p x_i$ where $x_1, \cdots, x_p\in X$, not necessarily distinct. Then $A_n=\bigcup\{B_p-B_q \mid 0\leqq p+q\leqq n\}$, where

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$B_p - B_q$ denotes the algebraic difference, so $A_n$ is compact.

Suppose $A_n$ contains a nonempty open subset $W$ of $A$. Choose $v \in W$ and let $U = -v + W$. Since $A$ is nondiscrete, there exists $z \in A$ with $z \neq 0$ and $(2n+1)z \in U$. Let $u = (2n+1)z$. Then $L(u) = (2n+1)L(z) \geqq 2n+1$. On the other hand, $u = -v + w$ for some $w \in W$, whence $L(u) \leqq L(v) + L(w) \leqq 2n$.

PROPOSITION 2. *Let $\phi$: $X \to Y$ be a continuous surjection of compact spaces. Then the canonical map $\phi^*$: $A(X) \to A(Y)$ induced by $\phi$ is a continuous-open group epimorphism.*

PROOF. Let $N = \ker \phi^*$, $G = A(X)/N$. The map $\psi$: $G \to A(Y)$ associated with $\phi^*$ is continuous and is an algebraic isomorphism. To show that $\phi^*$ is open, we prove that $\psi$ is homeomorphic.

Let $B = Y\psi^{-1}$, and let $B_1$ be the image of $X$ under the projection of $A(X)$ onto $G$, so that $B_1 \subset B$. Now $B_1$ generates $G$ since $X$ generates $A(X)$, and $B$ is free in $G$ since $Y$ is free in $A(Y)$. Then $B_1 = B$. Since $B$ is closed in $G$, it follows that $G = A(B)$. Moreover, $\psi$ maps $B$ homeomorphically onto $Y$, so $\psi$ is the canonical map of $A(B)$ into $A(Y)$ induced by $\psi | B$. Hence $\psi$ is homeomorphic.

3. **The transformation group** $(A(X), T)$. Now let the completely regular space $X$ be the phase space of a transformation group $(X, T, \pi)$. Following Baĭdosov [1], we extend $\pi$ to an action $\pi^*$ of $T$ on $A = A(X)$ as follows. Let $j$ be the inclusion map of $X$ into $A$. For each $t \in T$ let $\pi^{t*}$ be the continuous endomorphism of $A$ which extends the continuous map $\pi^t j$, where $\pi^t$ is the $t$-transition given by $x\pi^t = (x, t)\pi$ for $x \in X$. For $z \in A$ and $t \in T$ let $(z, t)\pi^* = z\pi^{t*}$. We call $\pi^*$ the *free extension of* $\pi$.

If $\phi$ is a homomorphism of $(X, T, \pi)$ into another transformation group $(Y, T, \sigma)$, with $Y$ completely regular, evidently the map $\phi^*$: $A(X) \to A(Y)$ induced by $\phi$ is equivariant with respect to $\pi^*$ and $\sigma^*$.

THEOREM 1. *Suppose $T$ is locally compact. Then $\pi^*$ is continuous, $(A, T, \pi^*)$ is a transformation group whose transitions are automorphisms of $A$, and $j$ is an isomorphism of $(X, T, \pi)$ into $(A, T, \pi^*)$.*

PROOF. The only nontrivial fact to be proved is the continuity of $\pi^*$. Denote by $C(T, X)$, $C(T, A)$ the sets of all continuous maps of $T$ into $X$, $A$ respectively, and endow these function spaces with their compact-open topologies. Define $\mu^*$: $A \to A^T$ by $t(z\mu^*) = (z, t)\pi^*$ for $t \in T$, $z \in A$. Since $T$ is locally compact, it suffices to show that $\mu^*$ maps $A$ continuously into $C(T, A)$.

We show that $A\mu^*\subset C(T, A)$. If $x\in X$, then $t\in T$ implies $t(x\mu^*) = (x, t)\pi$, so $x\mu^*\in C(T, A)$ by continuity of $\pi$. Now let $z\in A$. Choose $x_1, \cdots, x_n\in X$ and integers $\alpha_1, \cdots, \alpha_n$ with $z=\sum\alpha_i x_i$. Then $z\mu^*=\sum\alpha_i(x_i\mu^*)$, and $z\mu^*\in C(T, A)$.

Let $j^*$ be the canonical injection of $C(T, X)$ into $C(T, A)$, so that $j^*$ is continuous. Define $\mu: X\to X^T$ by $t(x\mu)= (x, t)\pi$ for $t\in T$, $x\in X$. Then $\mu$ is a continuous map of $X$ into $C(T, X)$, so $\psi=\mu j^*$ is a continuous map of $X$ into $C(T, A)$.

The addition on $C(T, A)$ defined pointwise makes this space into a separated abelian group. Hence $\psi$ extends to a continuous homomorphism $\psi^*$ of $A$ into $C(T, A)$. But $\mu^*$ is also a group homomorphism of $A$ into $C(T, A)$ extending $\psi$. It follows that $\mu^*=\psi^*$, and $\mu^*$ is continuous.

Corollary. *Suppose $X$ is compact and $(X, T, \pi)$ is equicontinuous. Then $\pi^*$ is continuous.*

Proof. Let $E$ be the enveloping semigroup [**4**] of $(X, T, \pi)$. Then $E$ is a compact group of homeomorphisms of $X$ onto $X$, and the evaluation map $\sigma: X\times E\to X$ is continuous and defines a transformation group $(X, E, \sigma)$. By the theorem the free extension $\sigma^*$ of $\sigma$ is continuous. The map $\mu$: $T\to E$ such that $t\in T$ implies $t\mu=\pi^t$ is continuous. The continuity of $\pi^*$ now follows from the factorization $\pi^*=(i\times\mu)\sigma^*$, where $i$ is the identity map of $X$.

Remark. Instead of the free abelian topological group $A$ over $X$, consider the free linear topological space $V$ over $X$ [**9**]. If $T$ is locally compact, then one can still show as above that $\pi$ has a continuous extension $\pi^*: V\times T\to V$ making $(V, T, \pi^*)$ a transformation group of linear automorphisms of $V$.

4. **Dynamical properties of** $(A(X), T)$. Again $(X, T, \pi)$ denotes a transformation group, where $X$ is completely regular. Continuity of $\pi^*$ is not needed below, and we suppress explicit mention of $\pi$ and $\pi^*$.

Theorem 2. *The action of $T$ on $A$ is not topologically ergodic, that is, there exist nonempty open subsets $N$ and $M$ of $A$ with $Nt\cap M=\varnothing$ for all $t\in T$.*

Proof. Let $D_0$ be the constant map on $X$ with range $\{1\}$, let $D$ be the homomorphism of $A$ into the additive group of integers which extends $D_0$, and let $N=\ker D$. Then $N$ is a $T$-invariant subgroup of $A$, so it is enough to show that $N$ is open in $A$.

Let $\mathfrak{J}_0$, $\mathfrak{J}$ be the topologies of $X$, $A$ respectively. The topology $\mathfrak{S}$ of $A$ generated by $\{N\}$ and $\mathfrak{J}$ is separated and compatible with the group structure of $A$. Let $\mathcal{S}_0$ be the topology on $X$ induced by $\mathcal{S}$. It is now enough to show $\mathcal{S}_0 \subset \mathfrak{I}_0$, for then $N \in \mathcal{S} \subset \mathfrak{I}$.

Let $z \in A$ and $V \in \mathfrak{I}$ with $(z+N) \cap V \cap X \neq \varnothing$. It remains to show $(z+N) \cap V \cap X \in \mathfrak{I}_0$. Choose any $y \in (z+N) \cap X$. If $x \in X$, then $x = y + (x-y) \in z + N + N = z + N$. Hence $X \subset z + N$, and $(z+N) \cap V \cap X = V \cap X \in \mathfrak{I}_0$.

In case $X$ is connected, the group $N$ in the preceding proof is just the identity component of $A$.

We use below a theorem of Ellis [**3**] stating that if $T$ acts as a homeomorphism group of a space $Y$ in which each orbit is relatively compact, then $(Y, T)$ is distal if and only if $(Y^n, T)$ is pointwise almost periodic for some $n > 1$, or equivalently, $(Y^n, T)$ is pointwise almost periodic for every $n > 0$. This applies to $(A, T)$ when $X$ is compact, because $A$ is the union of the compact sets $A_n$ constructed in the proof of Proposition 1, and each $A_n$ is $T$-invariant.

To accommodate the commutativity of addition in $A$, it is convenient to use the symmetric product $X*X$ of $X$ with itself [**2**]. Here $X*X$ is the compact space obtained by identifying each $(x, y) \in X \times X$ with $(y, x)$. Let $p$: $X \times X \to X*X$ be the projection, and for $(x, y) \in X \times X$ let $x*y$ denote $(x, y)p$. The map $(x*y, t) \mapsto xt*yt$ of $X \times T$ into $X*X$ is well defined and makes $(X*X, T)$ a transformation group and $p$ a homomorphism.

THEOREM 3. *The following statements are equivalent when $X$ is compact*:

(1) $(X, T)$ *is distal.*

(2) $(A, T)$ *is distal.*

(3) $(A, T)$ *is pointwise almost periodic.*

PROOF. Assume (1). We show (2). Let $z \in A$ with $z \neq 0$. It is enough to show that $z$ is distal from 0. The compact $T$-invariant set $A_1 = -X \cup X \cup \{0\}$ is distal under $T$, so $(A_1^n, T)$ is distal for each $n$. But for sufficiently large $n$ both 0 and $z$ belong to the range of the homomorphism $(z_1, \cdots, z_n) \mapsto \sum z_i$ of $(A_1^n, T)$ into $(A, T)$.

By Ellis' theorem, (2) implies (3).

Assume (3). We show (1). We have an obvious isomorphism of $(X*X, T)$ with $(X+X, T)$, so $(X*X, T)$ is pointwise almost periodic. Let $x, y \in X$ with $x \neq y$. Choose disjoint compact neighborhoods $U$, $V$ of $x$, $y$ in $X$. Then $(U \times V)p$ is a neighborhood of $x*y$ in $X*X$, and $(x*y)T \subset (U \times V)pK$ for some compact subset $K$ of $T$. Then

$$\alpha = (X \times X) \backslash ((U \times V) \cup (V \times U))K$$

is an index of the uniformity of $X$, and $(x, y)T$ is disjoint from $\alpha$.

The following lemma concerning lifting of minimality is of interest in its own right (cf. [**0**, p. 27], [**8**, 2.1]).

LEMMA. *Let $(X, T)$, $(Y, T)$ be transformation groups, where $X$, $Y$ are compact, and let $\phi$ be a locally one-to-one homomorphism of $(X, T)$ onto $(Y, T)$. Suppose $(Y, T)$ is minimal and $(X, T)$ has a dense orbit. Then $(X, T)$ is minimal.*

PROOF. Suppose $(X, T)$ is not minimal. Choose $x_0 \in X$ with $x_0T$ dense in $X$, and set $y_0 = x_0\phi$. There exists some minimal subset $M$ of $X$. The fiber $y_0\phi^{-1}$ over $y_0$ is finite and, since $\phi$ maps $M$ onto $Y$, meets $M$. Let $y_0\phi^{-1} = \{x_0, x_1, \cdots, x_n\}$ with $y_0\phi^{-1} \cap M = \{x_m, x_{m+1}, \cdots, x_n\}$.

Choose pairwise disjoint open neighborhoods $W_0, \cdots, W_n$ of $x_0, \cdots, x_n$ with $\phi$ one-to-one on each $W_i$ and with $M$ disjoint from the closure of $W_i$ for $0 < i < m$. For each $z \notin \bigcup_0^n W_i$ there exist disjoint neighborhoods of $y_0\phi^{-1}$ and $z\phi\phi^{-1}$ which are saturated by $\phi$. A standard compactness argument produces a saturated neighborhood $U$ of $y_0\phi^{-1}$ such that $U \subset \bigcup_0^n W_i$. For $i = 0, \cdots, n$ let $U_i = U \cap W_i$, and let $V = U\phi$, whence $V$ is a neighborhood of $y_0$.

Because $(Y, T)$ is minimal it is discretely almost periodic at $y_0$, and there exist subsets $S$, $K$ of $T$ with $K$ finite, $T = SK$, and $y_0S \subset V$. Then $x_iS \subset U$ for each $i$.

There exists a net $(s_j, k_j)_j$ in $S \times K$ such that $\lim x_0s_jk_j = x_m$. By passing to a subnet if necessary, we may assume $k_j = k$ for some $k$ and all $j$. Then $\lim x_0s_j = x_mk^{-1} \in M$, so $x_0S \not\subset \bigcup_0^{m-1} U_i$, and $x_0s \in U_p$ for some $s \in S$ and some $p \geqq m$. It follows that some two of the $n - m + 2$ points $x_0s, x_ms, \cdots, x_ns$ belong to the same one of the $n - m + 1$ sets $U_m, \cdots, U_n$. This is impossible since $x_is\phi = y_0s$ for all $i$ and $\phi$ is one-to-one on each $U_i$.

THEOREM 4. *Let $x, y \in X$ with $x \neq y$. Then $(x, y)$ is almost periodic under $(X \times X, T)$ if and only if $x + y$ is almost periodic under $(A, T)$.*

PROOF. The restriction to $X \times X$ of addition in $A$ is a homomorphism of $(X \times X, T)$ into $(A, T)$. Hence $x + y$ is almost periodic if $(x, y)$ is.

Conversely, assume $x + y$ is almost periodic. We first show that $x$ is distal from $y$ in $X$. Choose a symmetric index $\alpha$ of the uniformity of $X$ with $(x, y) \notin \alpha^3$. Since $x * y$ is almost periodic under $(X * X, T)$, the set

$$S = \{s \mid s \in T, (xs, ys) \in (x\alpha \times y\alpha) \cup (y\alpha \times x\alpha)\}$$

is left syndetic in $T$, and $T=SK$ for some compact set $K$. Choose an index $\beta$ of $X$ with $\beta K^{-1}\subset\alpha$. Then $(x, y)T\cap\beta=\emptyset$.

Since $x$ is distal from $y$, the orbit-closure $B$ of $(x, y)$ in $X\times X$ is disjoint from the diagonal of $X\times X$. Then $p|B$ is a locally one-to-one (in fact, locally homeomorphic) homomorphism of $B$ onto the orbit-closure of $x*y$ in $X*X$. The almost periodicity of $(x, y)$ now follows from that of $x*y$ by means of the lemma.

A similar but more direct argument shows that $(x, y)$ is almost periodic if and only if $x-y$ is.

ADDED IN PROOF. Mr. Leonard Shapiro has kindly pointed out that a proposition equivalent to our lemma appears in a paper of R. Ellis [Amer. J. Math **87** (1965), 564–574]. Ellis' proof is entirely different from ours in that it employs the enveloping semigroup of a transformation group.

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UNIVERSITY OF MASSACHUSETTS, AMHERST