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1981 Phys. Scr. 24 817

(http://iopscience.iop.org/1402-4896/24/5/002)

Physica Scripta, Vol. 24, 817–820, 1981

The Geometry of the Orbit Space for Non-Abelian Gauge Theories

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Received November 20, 1980

Abstract

We study the orbit space of vector potentials acted upon by gauge transformations. We describe its topology, its natural functional connection, and its curvature under a natural metric. A zeta function regularization is given for the functional Laplacian. When Sobolev metrics are used as an ultraviolet cutoff, the orbit space has a Gauss–Wiener measure.

1. Introduction

Gauge theories, in particular quantum chromodynamics (QCD), have led to a number of classical questions of special interest to mathematicians because they involve global geometry. The mathematicians' attention to these problems resulted in some successes, most notably the classification of the self-dual solutions to the Yang–Mills equations of motion [1].

It is unclear to us what effect these classical solutions have on the quantum theory. We wonder whether the techniques and insights of mathematics can contribute directly to the solution of quantum field theory problems, particularly the problems of continuous, nonperturbative QCD. We believe that a careful study of the space of orbits of vector potentials acted upon by the group of gauge transformations will be useful. This study was begun in [2] in order to analyze the Gribov ambiguity. See also [3] and [4]. We commented there, that when the gauge group is nonabelian, the orbit space is curved in a natural metric. It is our opinion that the curvature reflects important physical properties. Our purpose in this talk is to review what is known about the orbit space. We discuss its topology, its differential geometry, and finally some analysis on this space.

2. Notation and background

Let $\{P, M, G\}$ be a principal bundle over M with group G. The base M will be an oriented Riemannian manifold. It will be the 4-sphere S^4. or the 4-torus T^4, or $S^3 \times S^1$. In the canonical formalism it can be S^3, or T^3. With careful attention to growth conditions at ∞, M could be taken as Euclidean $\mathbb{R}^4$ (or $S^3 \times \mathbb{R}$ or $\mathbb{R}^3$). In pure QCD, it should suffice to let M be compact and let its volume go to ∞. The group G will be a compact group usually $\mathrm{SU}(N)$, the $N \times N$ unitary matrices with determinant 1.

Let $\mathfrak{A}$ denote the space of connections (vector potentials) on P. It is an affine space: if $A \in \mathfrak{A}$, then $\mathfrak{A} = [A + \tau; \tau \in \Lambda^1 \otimes g]$ where $\Lambda^1 \otimes g$ means the equivariant one forms on P with values in g, the Lie algebra of G. In particular, $\mathfrak{A}$ is an infinite dimensional Riemannian manifold. The tangent space of every point A is $\Lambda^1 \otimes g$ and the inner product is $\langle \alpha, \beta \rangle = \int_M \mathrm{tr}\,(\Sigma_\mu\, \alpha_\mu \beta_\mu)$, where at $m \in M$, $\alpha = \Sigma\, \alpha_\mu e_\mu$ and e_μ is an orthonormal base of $T^*(M, m)$.

Associated to $A \in \mathfrak{A}$ is its covariant differential D_A and its curvature field $F(A)$. Let $\mathscr{G}$ denote the group of automorphisms of P (the group of gauge transformations). That is, $\phi \in \mathscr{G}$ means (i) ϕ is a diffeomorphism of P, (ii) $\phi(pg) = \phi(p)\cdot g$ for $g \in G$ and (iii) ϕ induces the identity map on M.

Because $\mathscr{G}$ is the group of automorphisms of P, $\mathscr{G}$ acts on $\mathfrak{A}$. Specifically, $D_{\phi A} = \phi D_A \phi^{-1}$ and $F(\phi A) = \phi F(A) \phi^{-1} = Ad\,\phi(F(A))$. Let $\mathfrak{M}$ denote the orbit space of this action so that $\mathfrak{M} = \mathfrak{A}/\mathscr{G}$. The group $\mathscr{G}$ does not act freely on $\mathfrak{A}$. However, let $\mathscr{Z}$ denote the constant gauge transformations with values in the center of $\mathscr{G}$ and let $\tilde{\mathscr{G}} = \mathscr{G}/\mathscr{Z}$. Then $\tilde{\mathscr{G}}$ does act freely on $\mathscr{B} = [B \in \mathfrak{A}; B$ an irreducible connection$]$ and $\mathscr{B}/\tilde{\mathscr{G}}$ will be a nice infinite dimensional manifold [2]. Another possibility is to restrict gauge transformations to $\mathscr{G}_0 = [\phi \in G; \phi(m_0) =$ Identity$]$, with m_0 a fixed point of M. Frequently, m_0 is the north pole in S^4 (or S^3), i.e., the point of ∞ in $\mathbb{R}^4$ (or $\mathbb{R}^3$). Since $\mathscr{G}_0$ *does* act without fixed points on $\mathfrak{A}$, the space $\mathfrak{A}/\mathscr{G}_0 = \mathfrak{M}_0$ will turn out to be a nice manifold also.

We pause to describe the objects above in local coordinates. $A = \Sigma\, A_\mu\, \mathrm{d}x_\mu$ with $A_\mu(x)$ having values in the Lie algebra g so $A_\mu = \Sigma\, A_\mu^\beta\, X_\beta$, $\{X_\beta\}$ an orthonormal basis of g for an invariant inner product on g. $F = \Sigma\, F_{\mu\nu}\, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$ where $F_{\mu\nu}(A) = (\partial A_\nu/\partial x_\mu) - (\partial A_\mu/\partial x_\nu) + [A_\mu, A_\nu]$. If $B = \phi\cdot A$, then $B_\mu = \phi A_\mu \phi^{-1} - (\partial\phi/\partial x_\mu)\, \phi^{-1}$. Finally, the action $S(A) = ||F(A)||^2 = -\int_M \Sigma\, \mathrm{tr}\,(F_{\alpha\beta}F_{\gamma\delta})\, g_{\alpha\gamma} g_{\beta\delta}$. Note that $S(\phi\cdot A) = S(A)$ so that the action is gauge invariant.

The space $\mathfrak{M}_0$ differs from $\mathfrak{M}$ only by the action of the compact group $G = \mathrm{SU}(N)$ so that little is lost in replacing $\mathfrak{M}$ by $\mathfrak{M}_0$ in the infinite dimensional geometry. $\mathscr{B}/\tilde{\mathscr{G}}$ is an open dense set in $\mathfrak{A}/\mathscr{G}$ with the complement, the reducible vector potentials, a stratified closed, nowhere dense variety. Again, one expects that little will be lost in replacing $\mathfrak{A}/\mathscr{G}$ by $\mathscr{B}/\tilde{\mathscr{G}}$ which we denote by $\mathfrak{N}$. Many of the arguments given herein are the same for $\mathfrak{M}_0$ and $\mathfrak{N}$ but there are some differences.

3. The bundle structure and topology of the orbit space

Let π denote the projection of $\mathscr{B}$ onto $\mathfrak{N}$. The orbits can be identified with $\tilde{\mathscr{G}}$. Similarly, let π_0 be the projection of $\mathfrak{A}$ onto $\mathfrak{M}_0$ with the group $\mathscr{G}_0$ acting freely. We have principal bundles $(\mathscr{B}, \mathfrak{N}, \pi, \tilde{\mathscr{G}})$ and $(\mathfrak{A}, \mathfrak{M}_0, \pi_0, \mathscr{G}_0)$, for standard methods in elliptic analysis allow us to conclude that there are local cross-sections. These local sections also determine the topology of $\mathfrak{N}$ (or $\mathfrak{M}_0$). For example, an infinitesimal gauge transformation, an element of the Lie algebra of $\tilde{\mathscr{G}}$, is a function f with values in g. Its action on $\mathscr{B}$ gives the vector field $D_B f$ at B. So the tangent space to the orbit at B is $[D_B f,\ f \in \Lambda^0 \otimes g]$. The orthogonal complement of the "vertical" tangent space is the natural gauge condition $H_B = [\tau \in \Lambda^1 \oplus g; D_B^* \tau = 0]$. Locally, $B + \tau$, $\tau \in H_B$ hits each orbit exactly once and the topology of $\mathfrak{N}$ is determined by the C^∞ topology of H_B. See Fig. 1.

It is easy to describe the topology of $\mathfrak{M}_0$ when the base manifold is a sphere S^r of dimension r. Let $\{S^{r-1}, \mathrm{SU}(N)\}$ denote the space (Σ-model) of maps from the $r-1$ sphere to $G = \mathrm{SU}(N)$. Then $\mathfrak{M}_0$ is of the same homotopy type as $\{S^{n-1},$

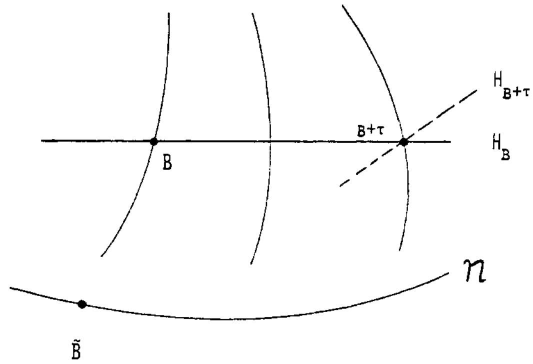

Fig. 1

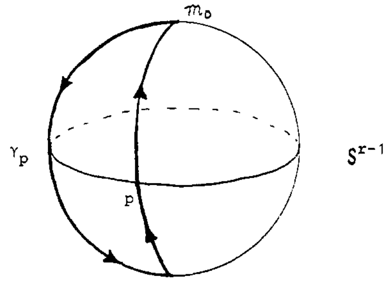

Fig. 2

$\mathrm{SU}(N)\}$. The map α which gives the homotopy equivalence is very geometrical.

Fix a great circle from the north pole to the south pole. Each point p on the equator S^{r-1} determines a great circle from the south pole to the north pole, giving the closed curve γ_p. For each $A \in \mathfrak{A}$, let $\alpha(A)$ be parallel transport around γ_p. Since $\phi \in \mathscr{G}_0$ means $\phi(m_0)$ is the identity, $\alpha(\phi \cdot A) = \alpha(A)$ so that α induces a map from $\mathfrak{M}_0$ to $\{S^{r-1}, \mathrm{SU}(N)\}$. See Fig. 2.

The Bott periodicity theorem implies that the space $\{S^{r-1}, \mathrm{SU}(N)\}$ becomes very simple as $N \to \infty$. Perturbative QCD also simplifies in the large N limit. One is compelled to ask whether there is a common underlying reason for these simplifications.

The self-dual solutions of the Yang–Mills equations of motion form a finite dimensional manifold of $\mathfrak{M}_0$. The relation between the topologies of these two spaces is discussed by Atiyah and Jones in [5].

The bundle $(\mathscr{B}, \mathfrak{N}, \pi, \mathscr{G})$ has a natural connection or functional vector potential [2]. In mathematical terms, this vector potential is defined by the horizontal space H_B for each $B \in \mathscr{B}$. Note that it is gauge invariant, i.e., $H_{\phi \cdot B} = \phi \cdot H_B$. The functional curvature field of this vector potential is the following skew symmetric bilinear function on tangent vectors of $\mathfrak{N}$ with values in $\Lambda^0 \otimes g$, the space of infinitesimal gauge transformations: $\mathscr{F}_{\mu,\tau}(B) = 2G_B \overset{*}{b}_\mu \tau$ where G_B is the Green's operator $(D_B^* D_B)^{-1}$ and $\overset{*}{b}_\mu \tau = \Sigma\, [\mu_i, \tau_i]$ at each point. Put differently, b_μ is the algebraic transformation from $\Lambda^0 \otimes g \to \Lambda^1 \otimes g$ given by $b_\mu (f) = [\mu, f]$ and $\overset{*}{b}_\mu$ is its adjoint transformation. From this curvature field one can compute the characteristic classes of the functional principal bundle.

Fermion fields are naturally associated to the bundle $(\mathfrak{A}, \mathfrak{M}_0, \pi_0, \mathscr{G}_0)$. Let E be the vector bundle over M given by the standard representation of $\mathrm{SU}(N)$ on $\mathbb{C}^N$ so that $E = P \underset{\mathrm{SU}(N)}{\times} \mathbb{C}^N$. Let S denote the bundle of spinors over M so that $C^\infty(S \otimes E)$ are the smooth fermion fields of QCD. Finally, let $\mathscr{H}$ denote its L_2 completion. The groups $\mathscr{G}$ and $\mathscr{G}_0$ act on $\mathscr{H}$ by acting on E. We obtain a bundle of Hilbert spaces $\mathfrak{A} \underset{\mathscr{G}_0}{\times} \mathscr{H} = \tilde{\mathscr{H}}$ over $\mathfrak{M}_0$*. The Dirac operators $\partial\!\!\!/_B$ coupled to vector potentials have the covariance property $\partial\!\!\!/_{\phi \cdot B} = \phi \partial\!\!\!/_B \phi^{-1}$ and that implies $\partial\!\!\!/_B$ is a field of operators indexed by $\mathfrak{M}_0$ which operates on the fibre of $\tilde{\mathscr{H}}$ at each point of $\mathfrak{M}_0$.

The same can be said of mesons. Let γ be a curve from x_0 to y_0. Let $G_A(x, y)$ denote the following Green's function: $G_A(x, y) = 0$ unless $x = x_0$, and $y = y_0$; $G_A(x_0, y_0)$ is parallel transport relative to the vector potential A along γ from y_0 to x_0. It is a linear transformation from $S_{y_0} \otimes E_{y_0}$ to $S_{x_0} \otimes E_{x_0}$ (the fibres at the corresponding points). Smooth this Green's function out and it is the kernel of a self-adjoint integral operator j_A satisfying the covariance condition $j_{\phi A} = \phi j_A \phi^{-1}$. Hence it too is a field of operators on the fibres of $\tilde{\mathscr{H}}$. We note that $\det(\partial\!\!\!/_A + j_A)$ is gauge invariant and occurs as an integrand in the Feynman path integral formulation of QCD.

4. The Riemannian geometry of $\mathfrak{N}$

Since the horizontal space H_B of the functional vector potential are gauge invariant. We can identify H_B by the projection π with the tangent space $T(\mathfrak{N}, \tilde{B})$ ($\tilde{B}$ denotes the orbit through B). Since the metric on $\mathfrak{A}$ is gauge invariant, $T(\mathfrak{N}, B)$ has an inner product ρ induced from H_B; that is, $\rho(\tau, \mu) = \langle \tau, \mu \rangle$ in $\Lambda_1 \otimes g$ [2]. We discuss the Riemannian geometry of $\{\mathfrak{N}, \rho\}$.

Its curvature tensor at each point of $\mathfrak{N}$ is a skew symmetric bilinear map into skew adjoint linear transformations on $T(\mathfrak{N}, \tilde{B})$. Here is its formula: $\mathscr{R}_{\tau,\mu} = 2b_{G_B b_\mu^*(\tau)} - (b_\tau G_B \overset{*}{b}_\mu - b_\mu G_B \overset{*}{b}_\tau)$, where $\tau, \mu \in H_B$ and $G_B = (D_B^* D_B)^{-1}$. The linear map $\mathscr{R}_{\tau,\mu}$ maps H_B into $\Lambda^1 \otimes g$ and one must project back into H_B.

The formula for the sectional curvature of the plane spanned by two orthonormal tangent vectors $\tau, \mu \in H_B$ is $3\langle G_B \overset{*}{b}_\mu \tau, \overset{*}{b}_\mu \tau \rangle$. Note that the sectional curvature is $\geqslant 0$ and that $\{\mathfrak{N}, \rho\}$ is curved when G is not abelian.

The Ricci tensor causes some problems and has to be regularized. The difficulty is this. In finite dimensions, the matrix for the Ricci operator is $R_{jl} = \Sigma_i R_{ijil}$. In our situation, the summation is an infinite sum and in general does not converge. Invariantly, the quadratic form for the Ricci tensor is $b(\mu, \tau) = 3 \operatorname{tr}(b_\tau G_B \overset{*}{b}_\mu)$. The trouble is that $b_\tau G_B \overset{*}{b}_\mu$ is not of trace class.

I emphasized in [6], that the orbit space $\mathfrak{N}$ is not just a Hilbert manifold with an additional hierarchy of Sobolev H_s norms. Since its tangent spaces can be identified with natural fields, each point of $\mathfrak{N}$ has a natural Laplacian which one can use to regularize.

Abstractly, suppose we have a Hilbert space with a Laplacian Δ. If T is an operator on the space, we could try and regularize $\operatorname{tr}(T)$ as $\operatorname{tr}(T\Delta^{-s})|_{s=0}$. That is, for a large class of operators T, the function $\operatorname{tr}(T\Delta^{-s})$ will be well defined and holomorphic for all $R\,ls \geqslant 0$ and will have a meromorphic extension to $s > 0$. Define $\operatorname{tr}(T) = \lim_{s\to 0} \operatorname{tr}(T\Delta^{-s})$ when it exists.

The regularized trace we have defined is natural in that it extends the usual notion of trace. That is if T is a compact operator of trace class, then it can be shown that $\operatorname{tr}(T\Delta^{-s})$ is holomorphic for all $R\,ls > 0$ and $\lim_{s\to 0} \operatorname{tr}(T\Delta^{-s}) = \operatorname{tr}(T)$.

In our case, we want to regulate $\operatorname{tr}(b_\tau G_B \overset{*}{b}_\mu) = \operatorname{tr}(\overset{*}{b}_\mu b_\tau G_B)$. Set $T = b_\mu^* b_\tau G_B$ and $\operatorname{tr}(T\Delta_B^{-s}) = \operatorname{tr}(b_\mu^* b_\tau \Delta_B^{-(s+1)})$. Since $b_\mu^* b_\tau$ is algebraic, $\operatorname{tr}(b_\mu^* b_\tau \Delta_B^{-(s+1)})$ is meromorphic in the s plane. Its residue at $s = 0$ turns out to be $\int_M \operatorname{tr} b_\mu^* b_\gamma(x) R(x)$ where $R(x)$ is the scalar curvature of M at x. In particular, there is no

* Note that $\tilde{\mathscr{G}}$ does not act on E because $\mathrm{SU}(N)/Z^N$ does not act on $\mathbb{C}^N$ where Z^N is the center of $\mathrm{SU}(N)$. Hence we do not obtain a Hilbert bundle over $\mathfrak{N}$ unless the principle bundle $(\mathscr{B}, \mathfrak{N}, \pi, \tilde{\mathscr{G}})$ can be lifted to a bundle with group $\mathscr{G}$. The obstruction to doing so lies in $H^2(\mathfrak{N}, Z^N)$. It need not be zero and has been computed by M. F. Atiyah (oral communication).

residue when $R \equiv 0$. Then the Ricci tensor is well defined as 3 tr $(b_\mu^* b_\tau \Delta_B^{-s})|_{s=1}$.

The exponential mapping is easy to describe. We have identified the tangent space $T(\mathfrak{N}, \tilde{B})$ with H_B; the exponential map is the projection $\pi|_{H_B}$ into $\mathfrak{N}$. The geodesics emanating from $\tilde{B}$ are just the projections of the straight lines in H_B, i.e., the orbits $B + t\tau$, where $D_B^* \tau = 0$. One important feature of this infinite dimensional geometry is that the exponential mapping from $T(\mathfrak{N}, \tilde{B})$ to $\mathfrak{N}$ is not locally one to one. Because the sectional curvature is positive, there are conjugate points and this fact reflects the Gribov ambiguity [2]. In a neighborhood of $\tilde{B}$, the exponential map *is* one to one so that the Gribov ambiguity does not affect perturbative QCD. It is natural to ask how many local exponential maps are necessary to cover $\mathfrak{N}$. Does one need a countable number or will a finite number suffice? Can one get away with two?*

5. The Laplacian on $\mathfrak{N}$

In my Cargese lectures [6], I defined the Laplacian $\mathscr{L}$ on functions of $\mathfrak{N}$ in the following way. Suppose f is a C^2 function on $\mathfrak{N}$. Then because $\mathfrak{N}$ has a Riemannian connection, the Hessian of f, H_f, is well defined at every $\tilde{B} \in \mathfrak{N}$. Specifically, if $\tilde{\tau} \in T(\mathfrak{N}, \tilde{B})$, then choose a representative $\tau \in H_B$ and $H_f(\tilde{\tau}, \tilde{\tau}) = (\mathrm{d}^2/\mathrm{d}t^2) f(B + t\tau)|_{t=0}$. In terms of the inner product on $T(\mathfrak{N}, \tilde{B})$, the Hessian can be written as $H_f(\tilde{\tau}, \tilde{\mu}) = \langle T_f \tau, \mu \rangle$ with T_f a bounded symmetric operator. Then $\mathscr{L}f(\tilde{B})$ should be the trace of T_f because in terms of an orthonormal base $\{e_i\}$ of $T(\mathfrak{N}, \tilde{B})$, $\mathscr{L}f(\tilde{B})$ should be $\Sigma_i H_f(e_i, e_i) = \Sigma_i \langle T_f e_i, e_i \rangle$. Since the trace is not necessarily defined for T_f, we regularize as we did the Ricci tensor. At B, we have a natural Laplacian $\Delta_B = D_B^* D_B + I$; define tr $(T_f) = \lim_{s\to 0}$ tr $(T_f \Delta_B^{-s})$ if $T_f \Delta_B^{-s}$ has a trace for all $R\,ls \gg 0$ which can be meromorphically continued to $R\,ls > 0$, if the limit exists.

One can also interpret $\mathscr{L}(f)(\tilde{B})$ in the following way. $H_f(\tilde{\tau}, \tilde{\mu}) = \langle T_f \tau, \mu \rangle$ with T_f a bounded operator on H_B. It can be thought of as a bounded operator on $L_2(\Lambda^1 \otimes g) = H_B \oplus H_B^\perp$ with T_f equal to 0 on $H_B^\perp$. The Schwartz kernel theorem tells us that T_f is an integral operator whose kernel function $K_f(x, y)$ is a distribution on $M \times M$ with values in $\Lambda_x^1 \otimes g \otimes \Lambda_y^1 \otimes g$. Formally, $\mathscr{L}(f)(\tilde{B}) = \int_M$ tr $K_f(x, y)$. In general, this integral does not make sense. But suppose we let $G_s(x, y)$ denote the kernel of Δ^{-s}. Then for $R\,ls \gg 0$, $\int_M$ tr $G_s(x,y)K_f(y,x)\,\mathrm{d}y$ is integrable and equals tr $(\Delta^{-s}T_f)$. The domain of $\mathscr{L} = [f \in C^2;$ tr $(\Delta^{-s}T_f)]$ has a meromorphic extension to $R\,ls > 0$ and the limit exists as $s \to 0$.

We describe the Laplacian in still another way. The function f pulls up to $\mathscr{B}$ as a function which is constant on orbits. Hence $\mathscr{L}f(\tilde{B})$ should equal the Laplacian of $\mathfrak{N}$ applied to f at B. Rather than taking an orthonormal base of $T(\mathfrak{N}, B) = \mathscr{L}_2(\Lambda^1 \otimes g)$, one can take the second variation at each point $m \in M$. In the sense of P. Levy, $\mathscr{L}f(B)$ should be $\int_M \delta^2 f/\delta\tau(m)^2$. Again, the integral need not exist and our procedure regularizes it. Namely $\lim_{s\to 0} \int_{M\times M} (\delta^2 f/\delta\tau(x)\delta\tau(y))\, G_s(x, y)$. That is, T_f is a distribution on $M \times M$ in $H^{-\dim M/2}$ represented by $\delta^2 f/\delta\tau(x)\lambda\tau(y)$. $G_s(x,y)$ is the kernel of Δ^{-s} and lies in $H^{2s-\dim M/2}$ $(M \times M)$. The intregral $\int_{M\times M} (\delta^2 f/\delta\tau(x)\delta\tau(y))\, G_s(x,y)$ is the application of the distribution T_f to G_s and is well defined for $R\,ls > \dim M/2$.

* The topology of the orbit space implies that a countable number is necessary, as pointed out to me by a countable number of topologists.

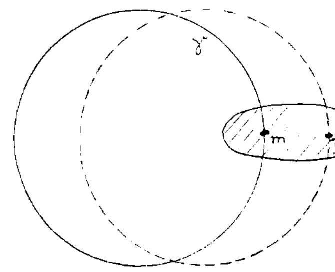

Fig. 3

For example, the Wilson loop functional is not in the domain of $\mathscr{L}$, but a smoothed out version is. Suppose γ is a smooth closed unknotted curve in M starting at m, and $W_\gamma(B) = \chi$ (parallel transport around γ); χ, a character of SU(N). Choose a tubular neighborhood of γ and let $\mathrm{d}\alpha$ be a smooth measure on the normal disc D at m with small compact support. Let γ_n be the closed curves near γ indexed by the normal disc. Then $B \to \int_D W_{\gamma_n}(B)\,\mathrm{d}\alpha$ will be in the domain of $\mathscr{L}$.

6. A cut-off theory

The Wilson loop example shows that the curve γ, for vector potentials, is a current, i.e., analogous to a Dirac δ-function and must be smoothed out. A convenient measure of smoothness is the set of Sobolev H_s spaces. They are in fact the foundation of the C^∞ theory [3, 4].

In addition, they give a good cut-off theory (similar to dimensional regularization) when the Fourier transform is not available (say, when M is not flat). For example, in ϕ^4 theory on a curved (compact) manifold M, the action: $\langle \Delta^{s+1}\phi, \phi \rangle + (m^2/2)\langle \phi, \phi \rangle + (\lambda/4!)\int \phi^4 = S(s, \phi)$ will make $\int \mathrm{e}^{-S(s,\phi)}\, \phi(x_1) \ldots \phi(x_{2l})\mathscr{D}\phi$ well defined as a Gauss–Wiener integral on the Sobolev space H_{s+1}, the Hilbert space with inner product $\langle \phi, \psi \rangle_{s+1} = \langle \Delta^{s+1}\phi, \psi \rangle$. (For all $R\,ls + 1 > \dim M/2$).

By analogy, we introduce the Sobolev metrics ρ_s on $\mathfrak{N}$. Actually, this can be done in two different ways.

Method 1. Introduce the H_s metric on $T(\mathscr{B}, B) = \Lambda^1 \otimes g$ and take the orthogonal compliment to the vertical as the representative tangent space. Since the H_s metric on $\mathscr{B}$ is gauge invariant, this procedure works as it does for $s = 0$.

Method 2. The representative H_B for $T(\mathfrak{N}, \tilde{B})$ has a natural Laplacian on it, namely $\Delta_B = D_B^* D_B + I$. Use it to define the H_s norm: $\langle \tau, \mu \rangle_s = \langle \Delta_B^s \tau, \mu \rangle$. It is not hard to show that these two metrics are uniformly equivalent on $\mathfrak{N}$. They give the same regularization, but of course have different curvatures. To be specific, let $(\mathfrak{N}, \rho_s)$ denote the second metric defined above.

Using s as cut-off parameter is at present the only gauge invariant nonperturbative regularization scheme available for continuous Euclidean QCD. Our program is to study QCD for the cutoff theory. Then remove the cut-off, which here means continue analytically from $R\,ls \gg 0$ to $s = 0$. For example, if f is in C^2 for $(\mathfrak{N}, \rho_s)$, then $H_f(\tilde{\tau}, \tilde{\mu}) = \langle T_f(s)\, \tau, \mu \rangle_s$ with $T_f(s)$ bounded in H_s. Since the theory now depends on s, rather than regulate tr $(T_f(s))$ as $\lim_{s'\to 0}$ tr $(\Delta^{-s'}T_f(s))$, one can use the cruder regularization tr $(\Delta^{-s}T_f(s))$. When $R\,ls > \dim M/2$, this trace is well defined because Δ^{-s} is of trace class and $T_f(s)$ is bounded. Equivalently (via the Mellin transform) one could use tr $(\mathrm{e}^{-t(s)\Delta}\, T_f(s))$.

Let then $\mathscr{L}_s f = \mathrm{tr}\,(\Delta^{-s}T_f(s))$, and let $\mathscr{H}_s$ be $-\mathscr{L}_s + g(s)V_s$, where $V_s(B) = \|F(B)\|_s^2$. $\mathscr{H}_s$ can be considered as the crudely regularized functional Hamiltonian operator (in the canonical formalism, with dim $M = 3$), and is well defined on C^2 functions. One step in our program would be to study the "spectrum" of $\mathscr{H}_s$, for $Rls \gg 0$. If our zeta function cut-off abides by our intuition, the spectrum should be discrete. Though no measure exists on $(\mathfrak{N}, \rho_s)$ relative to which $\mathscr{H}_s$ is self-adjoint, one can still study the spectrum on smooth functions decaying at ∞. Is there a countable set of such functions f_n with $\mathscr{H}_s f_n = \lambda_n f_n$, λ_n discrete in $[0, \infty)$ and the linear span of $\{f_n\}$ dense? If so, one hopes that $g(s)$ can be chosen so that $\mathscr{H}_s$ can be analytically continued to $s = 0$ with $\mathscr{H}_0$ having a positive mass gap. Some progress has been made in this program in [1] and [8] where, using K. Ito's stochastic integral equation, $e^{-t\mathscr{H}_s}$ is exhibited as a semigroup of transformations on bounded measurable functions on $(\mathfrak{N}, \rho_s)$ with generator $\mathscr{H}_s$ on smooth functions.

The path integral formulation of QCD requires a Gauss–Wiener measure on $(\mathfrak{N}, \rho_s)$ for M four dimensional. Such measures exist following the method of [9] and [10]. The corresponding Feynman–Kac formula for the cut-off theory has been announced in [7]. The next step in the program would be to find a nonperturbative renormalization in terms of the Gauss–Wiener measures $\mu_{g(s)}$.

Acknowledgement

This work was supported by DOE contract DOE AM 03-76SF00034.

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