A Closed-Form Expression for the Inverse Compensator of a Hawkes Process Having a Multiexponential Kernel

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Abstract

A closed-form expression for the inverse of the compensator of a Hawkes process having a multiexponential kernel of the form $\nu(t) = \sum_{j=1}^{P} \alpha_j e^{-\beta_j t}$ is calculated, along with its expectation which allows a closed-form for the expectation of the counting function to be calculated as well.

1 The Multiexponential Hawkes Process

For a Hawkes process with a multi-exponential kernel of the form

$$\nu(t) = \sum_{j=1}^{P} \alpha_j e^{-\beta_j t} \tag{1}$$

the intensity function is given by

$$\lambda(t) = \mu + \sum_{i=1}^{N_t} \sum_{j=1}^{P} \alpha_j e^{-\beta_j (t - t_i)} \tag{2}$$

Compensator:

$$\Lambda(t) = \mu t + \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} (1 - e^{-\beta_j t}) * \sum_{i=1}^{N_t} e^{-\beta_j (t - t_i)} \tag{3}$$

1.1 Closed-Form Expressions for $\Lambda^{-1}(u)$, its Uniform Expectation and $\mathbb{E}[N(T)]$

To compute the expected number of events over the interval $[0, T]$, we first define the following functions:

$$\begin{aligned}
C_j(u) &= \frac{\beta_j \mu}{\alpha_j} \prod_{k=1}^{P} W_k^{-\frac{\alpha_k}{\alpha_j} e^{\beta_j u}} \\
F_j(u) &= W_j(C_j(u)) \\
J_{k,l} &= \left(1 - \frac{\beta_l F_l(0) C_l(0)}{\beta_k F_k(0) C_k(0) + \beta_l F_l(0) C_l(0)}\right) \\
Q_{j,k} &= \frac{\alpha_k \left(1 - \frac{1}{F_k(0)}\right)^{j-1}}{\beta_k + 1} \prod_{l=k+1}^{j-1} J_{k,l} \\
R_j &= \sum_{k=1}^{P} Q_{j,k}
\end{aligned} \tag{4}$$

Then, we have the following formula for the inverse of the compensator:

$$\Lambda^{-1}(u) = \sum_{j=1}^{P} \left(\frac{1}{F_j(u)} - \ln\left(F_j(u) + \frac{1}{\beta_j C_j(0)} \right) \right) \tag{5}$$

whose expectation can be calculated as

$$\mathbb{E}[\Lambda^{-1}(U)] = \int_0^\infty e^{-u} \Lambda^{-1}(u) \mathrm{d}u = \sum_{j=1}^{P} \frac{\alpha_j (1 - F_j(0))}{\beta_j + 1} \tag{6}$$

which allows the expected number of events over the interval $[0, T]$ to be expressed as:

$$\mathbb{E}[N(T)] = \mu T + \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} (T - e^{-\beta_j T} R_j) \tag{7}$$

which is independent of the exact occurance times of points of the process.