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STRONG LAW OF LARGE NUMBERS FOR WEAKLY HARMONIZABLE PROCESSES

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If $X:\mathbb{R}\to L^2_{\mathbb{C}}(\Omega, \mathcal{A}, P)$ is a weakly harmonizable process with spectral stochastic measure $\mu:\mathcal{B}_{\mathbb{R}}\to L^2_{\mathbb{C}}(\Omega, \mathcal{A}, P)$, we first prove that

$$\lim_{t\to+\infty}\frac{1}{2t}\int_{-t}^{t} X(s)\,\mathrm{d}s=\mu(0)\quad\text{a.s.}$$

if and only if there exists some integer $p\geq 2$ such that

$$\lim_{n\to+\infty}\mu(|u|<p^{-n})=\mu(0)\quad\text{a.s.}$$

As a consequence we then get criteria for the strong law of large numbers for the process X to hold, i.e.

$$\lim_{t\to+\infty}\frac{1}{2t}\int_{-t}^{t} X(s)\,\mathrm{d}s=0\quad\text{a.s.}$$

These are extensions to the weakly harmonizable case of results previously obtained by several authors and specially by Gaposhkin in the strongly harmonizable case.

harmonizable processes * stochastic measures * bimeasures * strong laws of large numbers

1. Introduction

1.1.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A weakly harmonizable process $X:\mathbb{R}\to L^2_{\mathbb{C}}(\Omega, \mathcal{A}, P)$ is the Fourier transform of a stochastic measure i.e. a σ-additive set function $\mu:\mathcal{B}_{\mathbb{R}}\to L^2_{\mathbb{C}}(\Omega, \mathcal{A}, P)$, which is called its spectral stochastic measure.

The spectral bimeasure of X is the complex function defined on $\mathcal{B}_{\mathbb{R}}\times\mathcal{B}_{\mathbb{R}}$ by

$$M(A\times B)=E(\mu(A)\cdot\overline{\mu(B)}),\quad A, B\in\mathcal{B}_{\mathbb{R}}.$$

The process X is called strongly harmonizable if its spectral bimeasure M is extendable to a measure (known as its spectral measure) on $\mathcal{B}_{\mathbb{R}}\otimes\mathcal{B}_{\mathbb{R}}$. More particularly, if M concentrates on the diagonal Δ of $\mathbb{R}\times\mathbb{R}$, i.e. if

$$M(B)=M(B\cap\Delta),\quad B\in\mathcal{B}_{\mathbb{R}}\otimes\mathcal{B}_{\mathbb{R}},$$

X is a continuous (in q.m.) (wide sense) stationary process, and conversely.

It is well known that there exist non extendable spectral bimeasures (e.g., [6, Example 1]). We will overcome the technical problems generated by this difficulty with the help of the following Miamee and Salehi's domination lemma [7]: for every spectral bimeasure $M:\mathcal{B}_{\mathbb{R}}\times\mathcal{B}_{\mathbb{R}}\to\mathbb{C}$ there exists a bounded non-negative measure m on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ such that for any bounded measurable function $f:\mathbb{R}\to\mathbb{C}$, one has

$$0\leqslant\int\int f(t)\cdot\overline{f(s)}M(\mathrm{d}t,\mathrm{d}s)\leqslant\int|f(t)|^2m(\mathrm{d}t),$$

where we use the concept of integration w.r.t. the bimeasure M as introduced by Moché [8, Chapter IV] (see also [11]).

1.2. *The mean process*

Let X be a weakly harmonizable process. Since it is continuous, we can suppose that it is measurable and has locally integrable sample paths (if X is separable, it actually has such a measurable modification). Therefore we can define a new second order process $\sigma_X:]0,+\infty[\to L^2_{\mathbb{C}}(\Omega,\mathcal{A},P)$ called the (time averaged) mean process of X such that

$$\sigma_X(t)=\frac{1}{2t}\int_{-t}^{t}X(s)\,\mathrm{d}s,\quad t>0\ \text{(strong } L^2\text{-integral)},$$

$$\sigma_X(t,\omega)=\frac{1}{2t}\int_{-t}^{t}X(s,\omega)\,\mathrm{d}s,\quad t>0,\ \omega\in\Omega.$$

1.3. *Convergence of the mean process—previous results*

We recall that one has [11, inversion formulae]

$$\sigma_X(t)=\int_{\mathbb{R}}\frac{\sin(tu)}{tu}\cdot\mu(\mathrm{d}u)\xrightarrow[t\to+\infty]{}\mu(0)\quad\text{(in q.m.)}.\tag{1}$$

Does this result remain true for the a.s. convergence?

If we put

$$\sigma_X(t)=\Psi_X(t)+\mu(|u|<2^{-n}),\quad 2^n+1<t\leqslant 2^{n+1}+1,$$

Gaposhkin proved [5] that in the strongly harmonizable case, one has

$$\Psi_X(t)\xrightarrow[t\to+\infty]{}0\quad\text{a.s.}\tag{2}$$

So he obtained that X obeys the strong law of large numbers (SLLN) if and only if one has

$$\mu(|u|<2^{-n})\xrightarrow[n\to+\infty]{}0\quad\text{a.s.}\tag{3}$$

He also deduced [4] that in the continuous stationary case, if M denotes the spectral bimeasure of X as well as its trace on Δ, X obeys the SLLN if the following conditions are both fulfilled:

$$M(0)=0 \quad \text{and} \quad \text{there exists a real number } u_0>0 \text{ such that}$$
$$\int_{(0<|u|<u_0)} \left(\operatorname{Log}\operatorname{Log}\frac{1}{|u|}\right)^2 M(\mathrm{d}u)<+\infty. \tag{4}$$

Various other criteria for the SLLN in the strongly harmonizable case had been previously settled [1, 2, 10]. All of them use the total variation measure of M and consequently are not applicable if M is not extendable to a spectral measure [9, 2. Harmonizability].

1.4. *The new results*

In Section 2 we extend (2) and (3) to the weakly harmonizable case while Section 3 is devoted to the SLLN. More particularly, theorem 3.2 is an extension of the criterion (4) from the continuous stationary case to the weakly harmonizable one.

2. Asymptotic behaviour of the mean process

2.1.

Let X be a weakly harmonizable process, p an integer $\geqslant 2$, and let us put

$$\sigma_X(t)=\Psi_X(p,t)+\mu(|u|<p^{-q}),$$
$$t>p+1,\ q\in\mathbb{N}\backslash(0),\ p^q+1<t\leqslant p^{q+1}+1, \tag{5}$$
$$\Psi_X(p,t)=(\sigma_X(t)-\sigma_X(n))+(\sigma_X(n)-\sigma_X(p^q))+(\sigma_X(p^q)-\mu(|u|<p^{-q})),$$
$$n,\ q\in\mathbb{N}\backslash(0),\ n<t\leqslant n+1,\quad p^q<n\leqslant p^{q+1}. \tag{6}$$

We are going to prove that each term of the right-hand side of (6) converges almost surely to 0 as t tends to infinity. This is already done for the first term since Rousseau has proved [10, Prop. 1] that

$$\operatorname{Sup}(|\sigma_X(t)-\sigma_X(n)|;\ n<t\leqslant n+1)\xrightarrow[t\to+\infty]{}0 \quad \text{a.s.} \tag{7}$$

2.2.

For the second term, we have

Proposition. $\operatorname{Lim}_{q\to+\infty}\operatorname{Max}(|\sigma_X(n)-\sigma_X(p^q)|;\ p^q<n\leqslant p^{q+1})=0$ *a.s.*

Proof. In order to simplify the proof, let us suppose that $p=2$ (for $p>2$, see [3, Chapter 3]).

For every integer $q \geq 1$, k such that $1 \leq k \leq q$ and every $e \in E(k) = \{0, 1\}^k$, we put

$$a(q, k, e) = 2^q + 1 + \sum_{j=1}^{k} e_j 2^{q-j},$$

$$b(q, k, e) = \begin{cases} 2^q + 1 + \sum_{j=1}^{k-1} e_j 2^{q-j} & \text{if } k \geq 2, \\ 2^q & \text{if } k = 1, \end{cases}$$

and let $(\alpha_k, k \geq 1)$ be a sequence of strictly positive numbers. Utilizing Rousseau's majorization lemma [10], we can deduce that

$$\begin{aligned} &\operatorname{Max}(|\sigma_X(n) - \sigma_X(2^q)|^2; 2^q < n \leq 2^{q+1}) \\ &\leq \left(\sum_{k=1}^{q} \alpha_k^{-1}\right)\left(\sum_{k=1}^{q} \alpha_k \left(\sum_{e \in E(k)} \left|\sum_{j=b(q,k,e)+1}^{a(q,k,e)} (\sigma_X(j) - \sigma_X(j-1))\right|^2\right)\right) \\ &= \left(\sum_{k=1}^{q} \alpha_k^{-1}\right)\left(\sum_{k=1}^{q} \alpha_k \left(\sum_{e \in E(k)} \left|\sigma_X(a(q, k, e) - \sigma_X(b(q, k, e))\right|^2\right)\right). \end{aligned}$$

Then we obtain, by integration,

$$\begin{aligned} &E(\operatorname{Max}(|\sigma_X(n) - \sigma_X(2^q)|^2; 2^q < n \leq 2^{q+1})) \\ &\leq \left(\sum_{k=1}^{q} \alpha_k^{-1}\right)\left(\sum_{k=1}^{q} 2^k \alpha_k \cdot \operatorname{Max}(E(|\sigma_X(a(q, k, e)) - \sigma_X(b(q, k, e))|^2); e \in E(k))\right) \\ &= \left(\sum_{k=1}^{q} \alpha_k^{-1}\right)\left(\sum_{k=1}^{q} 2^k \alpha_k \operatorname{Max}\left(\iint f_{q,k,e}(u) \cdot f_{q,k,e}(v) M(\mathrm{d}u, \mathrm{d}v); e \in E(k)\right)\right). \end{aligned}$$

from (1), where

$$f_{q,k,e}(u) = \frac{\sin(a(q, k, e) \cdot u)}{a(q, k, e) \cdot u} - \frac{\sin(b(q, k, e) \cdot u)}{b(q, k, e) \cdot u}, \quad u \in \mathbb{R}.$$

Now we use the key idea of the proof i.e. we reduce the problem to the classical stationary case through the domination lemma: there exists a bounded non-negative measure m on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ associated to the spectral bimeasure M such that

$$0 \leq \iint f_{q,k,e}(u) \cdot f_{q,k,e}(v) M(\mathrm{d}u, \mathrm{d}v) \leq \int f_{q,k,e}^2(u) m(\mathrm{d}u).$$

So we have

$$\begin{aligned} &\sum_{q=1}^{+\infty} E(\operatorname{Max}|\sigma_X(n) - \sigma_X(2^q)|^2; 2^q < n \leq 2^{q+1}) \\ &\leq \sum_{q=1}^{+\infty} \left(\sum_{k=1}^{q} \alpha_k^{-1}\right)\left(\sum_{k=1}^{q} 2^k \alpha_k \operatorname{Max}\left(\int f_{q,k,e}^2(u) m(\mathrm{d}u); e \in E(k)\right)\right). \end{aligned} \tag{8}$$

The end of the proof is not new: if we divide the integration domain of the last integral into the following four parts:

$$(|u|<2^{-q-1}),\quad (2^{-q-1}\leqslant|u|<2^{-q+k}),\quad (2^{-q+k}\leqslant|u|<1),\quad (1\leqslant|u|),$$

and if $\alpha_k=\alpha^k$, $k\geqslant 1$, $1<\alpha<2$, it appears four convergent series [4, Theorem 1; 10, Prop. 4] so that the series (8) is also convergent. We can obviously conclude that

$$\max(|\sigma_X(n)-\sigma_X(2^q)|;\ 2^q<n\leqslant 2^{q+1}) \xrightarrow[q\to+\infty]{} 0 \quad \text{a.s.} \tag{9}$$

2.3.

It is easy to prove that

$$\sigma_X(p^q)-\mu(|u|<p^{-q}) \xrightarrow[q\to+\infty]{} 0 \quad \text{a.s.,} \tag{10}$$

but one will need once again the domination lemma:

$$\sigma_X(p^q)-\mu(|u|<p^{-q})$$
$$=\int_{(p^{-q}\leqslant|u|)}\frac{\sin(p^q u)}{p^q u}\,\mu(\mathrm{d}u)+\int_{(|u|<p^{-q})}\left(\frac{\sin(p^q u)}{p^q u}-1\right)\mu(\mathrm{d}u);$$
$$E(|\sigma_X(p^q)-\mu(|u|<p^{-q})|^2)$$
$$\leqslant 2\iint_{(p^{-q}\leqslant|u|,|v|)}\frac{\sin(p^q u)}{p^q u}\cdot\frac{\sin(p^q v)}{p^q v}\,M(\mathrm{d}u,\mathrm{d}v)$$
$$+2\iint_{(|u|,|v|<p^{-q})}\left(\frac{\sin(p^q u)}{p^q u}-1\right)\left(\frac{\sin(p^q v)}{p^q v}-1\right)M(\mathrm{d}u,\mathrm{d}v)$$
$$\leqslant 2\int_{(p^{-q}\leqslant u)}\left(\frac{\sin(p^q u)}{p^q u}\right)^2 m(\mathrm{d}u)+2\int_{(|u|<p^{-q})}\left(\frac{\sin(p^q u)}{p^q u}-1\right)^2 m(\mathrm{d}u).$$

So we are now in the stationary case from which [4, Theorem 1] we can prove that

$$\sum_{q=1}^{+\infty} E(|\sigma_X(p^q)-\mu(|u|<p^{-q})|^2)<+\infty$$

so that (10) is true.

At last we can summarize (5), (6), (7), (9) and (10) by the following theorem.

2.4.

Theorem. *For every weakly harmonizable process X and every integer $p\geqslant 2$, one has*

$$\Psi_X(p,t)\xrightarrow[t\to+\infty]{} 0 \quad \textit{a.s.,}$$

so that the following two conditions are equivalent:

(i) *$\sigma_X(t)$ converges a.s. as t tends to infinity.*

(ii) *there exists an integer $p \geq 2$ such that $\mu(|u| < p^{-q})$ converges a.s. when q tends to infinity.*

Moreover, one then has, for every integer $p \geq 2$,

$$\lim_{t \to +\infty} \sigma_X(t) = \lim_{q \to +\infty} \mu(|u| < p^{-q}) = \mu(0) \quad a.s.$$

3. Criteria for the SLLN

3.1.

The next statement is an obvious corollary of the theorem in 2.4.

Theorem. *Let X be a weakly harmonizable process: it obeys the SLLN if and only if there exists an integer $p \geq 2$ such that:*

$$\lim_{q \to +\infty} \mu(|u| < p^{-q}) = 0 \quad a.s.$$

3.2.

We can now give an extension of Gaposhkin's criterion (4):

Theorem. *Let X be a weakly harmonizable process. If there exists a bounded non-negative measure M_0 on $(\mathbb{R}^2, \mathscr{B}_{\mathbb{R}^2})$ such that*

(i) *for every event A of the ring generated by the intervals, one has*

$$M(A \times A) \leq M_0(A \times A),$$

(ii) *there exists a real number $u_0 > 0$ such that*

$$\int\int_{(0<|u|,|v|<u_0)} \left(\operatorname{Log}\operatorname{Log}\frac{1}{|u|}\right)\left(\operatorname{Log}\operatorname{Log}\frac{1}{|v|}\right) M_0(\mathrm{d}u, \mathrm{d}v) < +\infty,$$

then one has

$$\sigma_X(t) \xrightarrow[t \to +\infty]{} \mu(0) \quad a.s.,$$

and X obeys the SLLN if and only if $M(0,0) = 0$.

Proof. (a) The theorem in 2.4 shows that we have only to prove that

$$\lim_{n \to +\infty} \mu(|u| < 2^{-n}) = \mu(0) \quad \text{a.s.}$$

More particularly, since we have

$$\mu(|u| < 2^{-n}) = \mu(0) + \mu(0 < |u| < 2^{-2^q}) - \mu(2^{-n} \leq |u| < 2^{-2^q}), \quad 2^q < n \leq 2^{2q+1},$$

we have only to prove that the last two terms converge to 0 a.s. as n tends to infinity.

(b) Let q_0 be an integer such that $2^{-2^{q_0}} < u_0$. Putting $B_q = (0 < |u| < 2^{-2^q})$ and utilizing (i) and (ii), we obtain

$$\begin{aligned}
\sum_{q=q_0}^{+\infty} E(|\mu(B_q)|^2) &\leqslant \sum_{q=q_0}^{+\infty} M_0(B_q \times B_q) \\
&\leqslant \sum_{q=q_0}^{+\infty} q^{-2} \iint_{B_q \times B_q} \left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|u|}\right)\left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|v|}\right) M_0(\mathrm{d}u, \mathrm{d}v) \\
&\leqslant \left(\sum_{q=q_0}^{+\infty} q^{-2}\right) \iint_{(0<|u|,|v|<u_0)} \left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|u|}\right)\left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|v|}\right) M_0(\mathrm{d}u, \mathrm{d}v) \\
&< +\infty,
\end{aligned}$$

where Log_2 is the log function to the base 2. Therefore we have

$$\mu(0 < |u| < 2^{-2^q}) \xrightarrow[q \to +\infty]{} 0 \quad \text{a.s.}$$

(c) Using the previous notations, we put

$$A(q, k, e) = (2^{-a(q,k,e)} \leqslant |u| < 2^{-b(q,k,e)})$$

and

$$C_q = (2^{-2^{q+1}} \leqslant |u| < 2^{-2^q}).$$

By means of (i), (ii) and the Rousseau's majorization lemma, one has

$$\begin{aligned}
&\sum_{q=q_0}^{+\infty} E(\mathrm{Max}(|\mu(2^{-n} \leqslant |u| < 2^{-2^q})|^2;\ 2^q < n \leqslant 2^{q+1})) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} q\left(\sum_{k=1}^{q}\left(\sum_{e \in E(k)} E(|\mu(A(q, k, e))|^2)\right)\right) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} q\left(\sum_{k=1}^{q}\left(\sum_{e \in E(k)} M_0(A(q, k, e) \times A(q, k, e))\right)\right) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} q\left(\sum_{k=1}^{q} M_0(C_q \times C_q)\right) = \sum_{q=q_0}^{+\infty} q^2 M_0(C_q \times C_q) \\
&\quad \leqslant \sum_{q=q_0}^{+\infty} \iint_{C_q \times C_q} \left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|u|}\right)\left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|v|}\right) M_0(\mathrm{d}u, \mathrm{d}v) \\
&\quad \leqslant \iint_{(0<|u|,|v|<u_0)} \left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|u|}\right)\left(\mathrm{Log}_2\, \mathrm{Log}_2 \frac{1}{|v|}\right) M_0(\mathrm{d}u, \mathrm{d}v) \\
&\quad < +\infty.
\end{aligned}$$

Therefore we have

$$\mathrm{Max}(|\mu(2^{-n} \leqslant |u| < 2^{-2^q})|;\ 2^q < n \leqslant 2^{q+1}) \xrightarrow[q \to +\infty]{} 0 \quad \text{a.s.},$$

and this completes the proof.

3.3. Remarks

(a) In the strongly harmonizable case, the both conditions (i) and (ii) of the theorem in 3.2 can be replaced by the following single one: there exists a real number u_0 such that

$$\iint_{(0<|u|,|v|<u_0)} \left(\operatorname{Log}\operatorname{Log}\frac{1}{|u|}\right)\left(\operatorname{Log}\operatorname{Log}\frac{1}{|v|}\right)|M|(\mathrm{d}u, \mathrm{d}v) < +\infty,$$

where $|M|$ is the total variation measure of M. Moreover, if X is continuous and stationary, it reduces exactly to the Gaposhkin's criterion (4).

In the general weakly harmonizable case, the condition (ii) can be replaced by more practical ones [3, Lemma 4.2.6 and Remark 4.2.7] which are also extensions of the corresponding result of Gaposhkin [4, Corollary 3].

(b) At last, once again as Gaposhkin [4] we have got some information about the rate of convergence of $\sigma_X(t)$ towards $\mu(0)$:

Theorem (see [3, Lemma V.5.2.] for the proof). *Let X be a weakly harmonizable process. Suppose that there exists a non-decreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(i) *there exist integers $p \geq 2$, q_0 and a real A, $1 < A < \sqrt{p}$ for which we have*

$$g^2(p^{p^{q+1}}) \leq A g^2(p^{p^q}), \quad q \text{ integer}, \ q \geq q_0$$

(ii) *there exist a bounded non-negative measure m on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ which dominates M (in the sense of the domination lemma) and a real number u_0 such that*

$$\iint_{(0<|u|<u_0)} \left(\left(\operatorname{Log}\operatorname{Log}\frac{1}{|u|}\right) g\left(\frac{1}{|u|}\right)\right)^2 m(\mathrm{d}u) < +\infty.$$

Then one has

$$\lim_{t\to+\infty} g(t)\cdot(\sigma_X(t) - \mu(0)) = 0 \quad \text{a.s.}$$

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