Classical Origins

The theta correspondence is a subject in the theory of automorphic forms. To begin with, I'll briefly discuss the theory of theta functions, and then move on to the more modern theory.

Modular Forms

Since most number theorists now are familiar with modular forms, I won't say much more than

A modular form $f : \mathcal{H} \to \mathbb{C}$ of weight k is a holomorphic function defined on the upper half plane $\mathcal{H} := \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$, satisfying a growth condition, and the functional equation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. More generally, given a finite index subgroup $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$, we can ask only for the functional equation to hold for elements of Γ, and then we get a modular form of weight k and level Γ.

Modular forms can be vastly generalised to automorphic forms, which are highly symmetric functions on higher dimensional spaces.

Theta Functions

Theta functions are a special class of automorphic forms of interest in the geometry of elliptic curves and abelian varieties. The good reference for these is (Mumford 1983). The classical theta series is defined as the analytic function in two variables

$$\vartheta(x, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

where $z \in \mathbb{C}, \tau \in \mathcal{H}$, the upper half plane. For a fixed $\tau \in \mathcal{H}$, it is easy to see that there is a quasi-periodicity in z according to translation by the lattice $\mathbb{Z}\langle 1, \tau \rangle$. Since Liouville's Theorem tells us that we cannot have a non-constant period function in this lattice, the theta function is in some sense the closest that a non-constant entire function can get to being periodic.

It turns out that the theta function, for fixed τ, is the unique (up to scalars) function which is invariant under the action of a specific discrete subgroup of the Heisenberg group, $\mathcal{G}$, via the representation of $\mathcal{G}$ defined by τ. By taking congruence subgroups we can define a number of other theta functions *with characteristics*. This connection to representation theory seems to be the basis of the word 'theta' arising in so many representation theoretic topics, in particular the theta correspondence. The representation of the Heisenberg group in another guise is called the Weil representation, and form the fundamental basis of the theta correspondence as we will see.

Let $\mathcal{G} = \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$ be the Heisenberg group with group structure

$$(\lambda, a, b)(\lambda', a', b') = (\lambda\lambda' e^{2\pi i b a'}, a + a', b + b'),$$

and fix some $\tau \in \mathcal{H}$. Then $\mathcal{G}$ has an action on the space of holomorphic functions on $\mathbb{C}$ defined by

$$\begin{aligned} U_{(\lambda,a,b)} : \mathrm{Hol}_{\mathbb{C}} &\longrightarrow \mathrm{Hol}_{\mathbb{C}} \\ f &\longmapsto \lambda e^{\pi i a^2 \tau + 2\pi i a z} f(z + a\tau + b). \end{aligned}$$

In fact, if we restrict to entire functions that satisfy a certain growth property (a 'twisted' version of being L^2), then this is an irreducible unitary representation of the Heisenberg group. A fundamental theorem of Stone-Von Neumann tells us that this is in fact the unique equivalence class of irreducible unitary representation of $\mathcal{G}$ in which $\mathbb{S}^1$ acts as the by multiplication (i.e. the identity central character). We will see later that this can be viewed in a general context of symplectic groups, and this places theta functions in a unique place in the representation theory of symplectic groups.

What is the Theta Correspondence?

Local Correspondence

The starting point of the entire theory of the theta correspondence in the Weil representation. The Weil representation is an infinite dimensional representation of the metaplectic group Mp_{2n}, which is the double cover of the symplectic group Sp_{2n}. In fact it is the only faithful irreducible linear representation of Mp_{2n}. It can be constructed as follows:

1. Begin with a symplectic vector space (V, ω).
2. From this, we construct the Heisenberg group similarly to the above construction by endowing the set $V \times \mathbb{R}$ with a group structure twisted via the symplectic form ω.
3. The Stone-von-Neumann theorem tells us that there is a unique irreducible unitary representation of the Heisenberg group, say $(\rho, \mathcal{H})$.
4. The symplectic group Sp_{2n} acts as automorphisms of the Heisenberg group (acting on V)
5. For any such automorphism, pre-composing this with the representation ρ gives a new representation of $\mathbb{H}$ on $\mathcal{H}$.
6. Therefore, for each element of the symplectic group, we get an automorphism of $\mathcal{H}$, defined up to scalars (the entwining map between these two representations of $\mathbb{H}$ on $\mathcal{H}$).
7. Thus we have a projective representation of Sp_{2n} on $\mathcal{H}$, and this lifts to a representation of Mp_{2n}.

This special representation can be made quite explicit, and clearly is related to the representation that we considered above when discussing theta functions. Now we have a nice situation, a large class of groups with a distinguished representation of arithmetic and geometric importance. One natural thing to start doing is considering subgroups of the metaplectic group and seeing how the Weil representation breaks up under restriction. The notion of dual reductive pairs gives a formalisation of this idea in which the restriction is tractable to study.

Let W be a symplectic vector space over a local field. Then a pair of reductive subgroups $G, H \leq \mathrm{Sp}(W)$ are called a dual reductive pair if they are each the centraliser of the other.

There is a classification of dual reductive pairs, which I believe is due to Howe. Given a dual reductive pair, we can consider restricting the Weil representation to the product

$\widetilde{G} \cdot \widetilde{H}$, where the tilde represents lifting these groups to the metaplectic group.

The local theta correspondence says that if we choose an irreducible representation ρ of $\widetilde{G}$, and then restrict the Weil representation to $\widetilde{G} \cdot \widetilde{H}$, and take the subspace on which $\widetilde{G}$ acts by ρ, then what we are left with is an irreducible representation of $\widetilde{H}$ and in fact this correspondence between representations is a bijection between irreducible admissible representations of these two groups.

Mumford, David. 1983. *Tata lectures on theta*. Progress in Mathematics ; Vol. 28, 43, 97. Boston: BirkhÃ¤user.