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Some nilpotent, tridiagonal matrices with a special sign pattern

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ABSTRACT

We prove that for each $n \geqslant 2$ there is a nilpotent $n \times n$ tridiagonal matrix satisfying

(a) The super-diagonal is positive.
(b) The sub-diagonal is negative.
(c) The diagonal is zero except that the $(1, 1)$ position is negative and the (n, n) position is positive.

1. Introduction

A *sign pattern* is an $n \times n$ matrix whose entries are chosen from $\{-, 0, +\}$. A real matrix A *has sign pattern* S if in positions corresponding to $-$ in S, the entry in A is negative, in positions corresponding to

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$+$ in S, the entry in A is positive and the rest of the entries in A are 0. A sign pattern S is called a *spectrally arbitrary pattern*, if for each real monic polynomial $r(x)$ of degree n, there exists a matrix with sign pattern S and with characteristic polynomial $r(x)$. For a summary of material on sign patterns, see [1–3].

In [1], Drew et al. introduced a sign pattern $\mathcal{T}_n$; this pattern was shown to be a spectrally arbitrary pattern for $2 \leqslant n \leqslant 7$ and was conjectured to be so for all n. In [2], Elsner et al. extended the range to $8 \leqslant n \leqslant 16$. The proof in [2] requires one to find a nilpotent matrix with sign pattern $\mathcal{T}_n$. Then one has to prove that a certain Jacobian is not zero. In this paper we establish the existence of a nilpotent matrix A_n with sign pattern $\mathcal{T}_n$ for $n \geqslant 2$.

This paper does not directly address the issue of whether or not $\mathcal{T}_n$ is a spectrally arbitrary pattern.

We prove that a certain matrix is nilpotent by constructing a change of basis matrix that makes it strictly upper triangular. We do this using Chebyshev polynomials of the first kind $T_n(x)$. Chebyshev polynomials of the second kind $U_n(x)$ also appear in the proof. These polynomials satisfy the following well-known (see, for example, [4]) recursion relations:

First kind	Second kind
$T_0(x) = 1$	$U_0(x) = 1$
$T_1(x) = x$	$U_1(x) = 2x$
$T_2(x) = 2x^2 - 1$	$U_2(x) = 4x^2 - 1$
$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x)$	$U_{r+1}(x) = 2xU_r(x) - U_{r-1}(x)$
for $r \geqslant 1$	for $r \geqslant 1$

It is also well-known (see, for example, [4]) that

$$T_r(\cos(t)) = \cos(rt), \tag{1}$$

$$U_r(\cos(t)) = \frac{\sin((r+1)t)}{\sin(t)}. \tag{2}$$

It can be shown, by a straightforward inductive argument using the recurrence relations satisfied by the $T_r(x)$ and the $U_r(x)$, that, for $r \geqslant 2$, $U_r(x) = 2T_r(x) + U_{r-2}(x)$ and so

$$U_r(x) = 2\left(T_r(x) + T_{r-2}(x) + T_{r-4}(x) + \cdots + \left\{T_1(x) \text{ or } \frac{1}{2}T_0(x)\right\}\right). \tag{3}$$

The last term is $T_1(x)$ when r is odd, and $\frac{1}{2}T_0(x)$ when r is even.

2. Proof of nilpotence

The sign pattern $\mathcal{T}_n$ is the n by n sign pattern that has positive super-diagonal, negative sub-diagonal, and the diagonal is zero except for the first entry which is negative and the last entry which is positive. The rest of the entries are zero. Let

$$A_n = \begin{pmatrix} -f_1 & f_1 & & & & & & & \\ -f_2 & 0 & f_2 & & & & & & \\ & & \ddots & & & & & & \\ & & -f_{k-1} & 0 & f_{k-1} & & & & \\ & & & -f_k & 0 & f_k & & & \\ & & & & -f_{k+1} & 0 & f_{k+1} & & \\ & & & & & & \ddots & & \\ & & & & & & -f_{n-1} & 0 & f_{n-1} \\ & & & & & & & -f_n & f_n \end{pmatrix},$$

where $f_k = \dfrac{1}{2\sin\left(\frac{(2k-1)\pi}{2n}\right)}$.

Henceforth we fix n and drop the subscript. Thus A will refer to the current matrix A_n under consideration.

Theorem 1. *There is an invertible matrix X and a strictly upper triangular matrix U such that $AX = XU$. Hence A is nilpotent.*

Proof. For $0 \leqslant k \leqslant n+1$, consider $\theta_k = \frac{(2k-1)\pi}{2n}$ and the points x_k defined by $x_k = \cos(\theta_k)$. Notice that since cosine is even, $x_0 = x_1$ and since $\cos(\pi - x) = \cos(\pi + x)$, $x_n = x_{n+1}$. For $0 \leqslant r \leqslant n-1$, let X_r be the n by 1 vector

$$X_r = \begin{pmatrix} T_r(x_1) \\ T_r(x_2) \\ \vdots \\ T_r(x_n) \end{pmatrix}.$$

Let X be the n by n matrix with columns $X_0, X_1, \ldots, X_{n-1}$.

Notice that the polynomials T_r are linearly independent and of degree less than n so that the multipoint evaluation map (on the n different points $x_1, \ldots, x_n$) is an isomorphism of vector spaces. Hence, the vectors X_r are linearly independent and the matrix X is invertible.

The first assertion of the theorem follows easily from the fact that X_0 is in the nullspace of A and the following claim.

Claim. For all $0 \leqslant r \leqslant n-2$

$$AX_{r+1} = -2\sin\left((r+1)\frac{\pi}{n}\right)\left(X_r + X_{r-2} + X_{r-4} + \cdots + \left\{X_1 \text{ or } \frac{1}{2}X_0\right\}\right)$$

Coordinatewise this says

$$AX_{r+1}(k) = -2\sin\left((r+1)\frac{\pi}{n}\right)\left(T_r(x_k) + T_{r-2}(x_k) + \cdots + \left\{T_1(x_k) \text{ or } \frac{1}{2}T_0(x_k)\right\}\right)$$

In this proof we use the formula

$$\cos(x) - \cos(y) = -2\sin\left(\frac{1}{2}(x+y)\right)\sin\left(\frac{1}{2}(x-y)\right) \tag{4}$$

For $2 \leqslant k \leqslant n-1$, we look at the kth entry of AX_{r+1}. The kth row of A has $-f_k$ on the sub-diagonal and f_k on the super-diagonal. The rest of the entries are zero. Therefore the kth entry in AX_{r+1} is

$$f_k\left[-X_{r+1}(k-1) + X_{r+1}(k+1)\right]$$

In order to simultaneously treat the first and last entries, we extend the use of $X_{r+1}(k)$ to $k = 0$ and $k = n+1$ and notice that $X_0 = X_1$ and $X_{n+1} = X_n$.

Therefore the kth entry $(1 \leqslant k \leqslant n)$ in AX_{r+1} is

$$\begin{aligned}
& f_k\left[-X_{r+1}(k-1) + X_{r+1}(k+1)\right] \\
&= f_k\left[-\cos((r+1)\theta_{k-1}) + \cos((r+1)\theta_{k+1})\right] \text{ by (1)} \\
&= f_k\left[\cos((r+1)\theta_{k+1}) - \cos((r+1)\theta_{k-1})\right] \text{ rearranging} \\
&= f_k\left[-2\sin((r+1)\theta_k)\sin\left((r+1)(+2)\frac{\pi}{2n}\right)\right] \text{ using (4)} \\
&= \frac{-2\sin((r+1)\theta_k)}{2\sin(\theta_k)}\sin\left((r+1)\frac{\pi}{n}\right) \text{ using definition of } f_k
\end{aligned}$$

$$= -U_r\left(\cos\left(\theta_k\right)\right)\sin\left(\left(r+1\right)\frac{\pi}{n}\right) \text{ by property (2)}$$

$$= -\sin\left(\left(r+1\right)\frac{\pi}{n}\right)U_r\left(\cos\left(\theta_k\right)\right) \text{ simplifying and rearranging}$$

$$= -\sin\left(\left(r+1\right)\frac{\pi}{n}\right)U_r\left(x_k\right)$$

$$= -2\sin\left(\left(r+1\right)\frac{\pi}{n}\right)\left(T_r(x_k)+T_{r-2}(x_k)+\cdots+\left\{T_1(x_k) \text{ or } \frac{1}{2}T_0(x_k)\right\}\right)$$

using relation (3)

Since A is similar to a strictly upper triangular matrix, we may conclude that $A^n = 0$. $\square$

3. Concluding remarks

Let R_n denote the $n \times n$ tridiagonal matrix with -1's on the sub-diagonal, 1's on the super-diagonal, -1 in the $(1, 1)$ entry, 1 in the (n, n) entry and zeros elsewhere. Note that R_n has sign pattern $\mathcal{T}_n$. Also note that $A_n = \text{Diag}(f_1, \ldots, f_n)R_n$. We believe that studying R_n will shed light on the proof that A_n is nilpotent.

Behn et al. [5] determined the eigenvalues of R_n. They are 0 and $2i\cos(j\pi/n)$ for $j = 1, \ldots, n-1$. It is interesting to note that the angle π/n appears in the proof of the nilpotent theorem given above; in particular, the quantity $\sin((r+1)\pi/n)$ appears in the claim in that proof.

Let E_n denote the **exchange** (or **flip**) matrix which is defined by $E_n(i,j) := \delta(i+j, n+1)$. An $n \times n$ matrix H is **centro-symmetric** if $E_nHE_n = H$; an $n \times n$ matrix K is **centro-skew** if $E_nKE_n = -K$. (For more on such matrices, see Trench [6] and the references in that paper.) Note that A_n and R_n are centro-skew.

Using the theory of centro-symmetry, Driessel [7] showed that R_n is closely related to a certain circulant matrix and a certain skew-circulant matrix. More precisely, let E_+ (respectively, E_-) be the projection defined by $x \mapsto x + E_nx$ (respectively, $x \mapsto x - E_nx$). Then

$$R_n = (\pi_n - \pi_n^T)E_- + (\eta_n + \eta_n^T)E_+.$$

where π_n is the basic $n \times n$ circulant matrix and η_n is the basic $n \times n$ skew-circulant matrix. In other words, if x is a vector in the range of E_- then $R_nx = (\pi_n - \pi_n^T)x$ and if x is in the range of E_+ then $R_nx = (\eta_n + \eta_n^T)x$.

The eigenvalues of circulants and skew-circulants are well-known. (See, for example, Davis [8].) In particular, the eigenvalues of $\pi_n - \pi_n^T$ are $\omega_n^k - \bar{\omega}_n^k$ where $\omega_n := \exp(i2\pi/n) = \cos(2\pi/n) + i\sin(2\pi/n)$. Note $\omega_n^k - \bar{\omega}_n^k = 2i\sin(k2\pi/n)$. The eigenvalues of $\eta_n + \eta_n^T$ are $\sigma_n^{2k-1} + \bar{\sigma}_n^{2k-1}$ where $\sigma_n := \exp(i\pi/n) = \cos(\pi/n) + i\sin(\pi/n)$. Note that $\sigma_n^k + \bar{\sigma}_n^k = 2\cos((2k-1)\pi/n)$. Here we recognize angles that appear in the proof of the nilpotent theorem.

In the main part of this report, we noted one possible application of the fact that A_n is nilpotent – namely, to the spectrally arbitrary problem for the pattern $\mathcal{T}_n$. Here is another possible application. Note that R_n corresponds to a finite difference approximation of the linear operator d/dx. Also note that d/dx is nilpotent on the space of polynomials of degree less than $n-1$. The matrix A_n is another finite difference approximation. Since it is nilpotent, it is qualitatively a better approximation.

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References

[1] J.H. Drew, C.R. Johnson, D.D. Olesky, P. vanden Driessche, Spectrally arbitrary patterns, Linear Algebra Appl. 308 (1–3) (2000) 121–137.10.1016/S0024-3795(00)00026-4.
[2] L. Elsner, D.D. Olesky, P. vanden Driessche, Low rank perturbations and the spectrum of a tridiagonal sign pattern, Linear Algebra Appl. 374 (2003) 219–230.10.1016/S0024-3795(03)00580-9.
[3] M. Catral, D.D. Olesky, P. vanden Driessche, Allow problems concerning spectral properties of sign pattern matrices: a survey, Linear Algebra Appl. 430 (11–12) (2009) 3080–3094.10.1016/j.laa.2009.01.031.
[4] P.J. Davis, Interpolation and Approximation, Blaisdel Publishing Co., Ginn and Co., New York, Toronto, London, 1963.
[5] A. Behn, K. Driessel, I.R. Hentzel, The eigen-problem for some special near-Toeplitz centro-skew tridiagonal matrices, arXiv:1101.5788v1 [math.SP].
[6] W.F. Trench, Characterization and properties of matrices with generalized symmetry or skew symmetry, Linear Algebra Appl. 377 (2004) 207–218.10.1016/j.laa.2003.07.013.
[7] K. Driessel, A relation between some special centro-skew, near-Toeplitz, tridiagonal matrices and circulant matrices, arXiv:1102.1953v2 [math.SP].
[8] P.J. Davis, Circulant Matrices, John Wiley & Sons, New York, Chichester, Brisbane, 1979. (a Wiley-Interscience Publication, Pure and Applied Mathematics).