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Linear recurrence relations for some generalized Pisot sequences

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Abstract

The sequences of integers which we consider are defined for arbitrary integers $0 < a_0 < a_1$ by $a_{n+2} = \left[\frac{a_{n+1}^2}{a_n} + r\right]$, where r is a fixed real number. We consider mainly the three special cases $r = 0, \frac{1}{2}$, and 1. For $r = \frac{1}{2}$, these are the Pisot sequences; when $r = 1$, they are the sequences recently suggested by Shallit. We show that, for $r = 0$ and 1, the possible recurrence relations for the sequences are more restricted than in the case $0 < r < 1$ and give some examples and experimental evidence for differences between the three types of sequences.

1 Introduction:

Let r be a given real number. For arbitrary integers $0 < a_0 < a_1$, define the sequence of integers $E_r(a_0, a_1)$ by the nonlinear recurrence relation

$$a_{n+2} = \left[\frac{a_{n+1}^2}{a_n} + r\right]. \tag{1}$$

For $r = \frac{1}{2}$, the right member of (1) is the "nearest" integer to $\dfrac{a_{n+1}^2}{a_n}$. These sequences were introduced by Pisot [Pi] in his study of the distribution of sequences of the form $\lambda\theta^n$ modulo one. Flor [Fl] showed that, if such a sequence satisfies a linear recurrence relation then the defining polynomial $Q(x)$ of that relation must either be $(x-1)^2$ or else have a single root $\theta > 1$ outside the unit circle. In the latter case, $Q(x)$ factors as $P(x)K(x)$ where P is the minimal polynomial of θ and K is a cyclotomic polynomial with simple roots (not necessarily irreducible) or else $K(x) \equiv 1$. If all the other conjugates of θ are strictly inside the unit circle, then θ is a Pisot number, otherwise θ is a Salem number. Throughout this paper,

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we will maintain the convention that P denotes the minimal polynomial of a Pisot or Salem number θ and that K denotes a cyclotomic polynomial with simple roots.

Not all polynomials of the form PK can occur as defining polynomials of a Pisot sequence. Flor showed that $Q(x) = P(x)$ can always occur but that $P(x)(x-1)$ can occur only if $|P(1)| > 2(\theta-1)^2$. Some generalizations of this were obtained in [B3] where criteria were given for deciding whether any particular combination PK can occur. It was shown that, if $\theta > 1 + 2^{d-2}$ then no nontrivial cyclotomic factor can be combined with P to give the defining polynomial of a recurrence for a Pisot sequence.

Recently, Shallit [Sh] asked about linear recurrence relations for the sequences (1) when $r = 1$. These sequences have the pleasant property that the ratios a_{n+1}/a_n form an increasing sequence and indeed, a_{n+2} is the minimal integer for which $a_{n+2}/a_{n+1} > a_{n+1}/a_n$. Although, at first glance, it would seem likely that the restrictions on recurrence relations in this case should be much the same as for Pisot sequences, it turns out that this is not the case as we will indicate below.

Another natural choice of r is $r = 0$, which was mentioned at the end of a paper of Cantor [Ca]. As Cantor pointed out in a conversation with Shallit, if one allows the a_n to take on negative values, then changing a_n to $(-1)^n a_n$ almost interchanges sequences with $r = 0$ and $r = 1$ (except that "floor" should now be changed to "ceiling", which is generally insignificant.) Thus, we would expect certain properties of Shallit sequences to be shared by sequences with $r = 0$, which we call "Truncate" sequences.

In contrast to the situation for Pisot sequences, if $r = 0$ or 1, if d is fixed and if $\theta > 1 + 2^{d-1}$ is a Salem number of degree d, then we show that there is no E_r–sequence for which $\lim \frac{a_{n+1}}{a_n} \to \theta$. If $r < 0$ or $r > 1$ and if $\theta > 1 + \max(r, 1-r)2^{d-1}$, and if θ is a Pisot number *or* a Salem number then the same conclusion holds. Thus, for any given degree there are only a finite number of Pisot or Salem numbers occurring as limiting ratios of E_r sequences in this case.

We showed in [B1],[B4] that there are Pisot sequences which satisfy no linear recurrence relation. Specific examples with a_0 small are given in [B2]. The method used there applies equally well to the sequences (1) for any value of r. We do not pursue this here.

For convenience, we shall write $E_r(a_0, a_1)$ as $T(a_0, a_1), E(a_0, a_1)$, and $S(a_0, a_1)$ in the three cases $r = 0, \frac{1}{2}$, and 1.

2 Restrictions on possible linear recurrences.

It is not hard to show, as in [Fl], that for sequences defined by (1), $\lim \frac{a_{n+1}}{a_n} = \theta \geq 1$ always exists, and if $a_1 > a_0 + c\sqrt{a_0}$ (where c depends on r), then

$\theta > 1$. We will always assume that $\theta > 1$ here, since otherwise the sequence has defining polynomial $(x-1)^2$. When $\theta > 1$, we also have $\lim \frac{a_n}{\theta^n} = \lambda > 0$. We may rewrite the definition (1) as

$$a_{n+2} - \frac{a_{n+1}^2}{a_n} \in I_r = (r-1, r]. \tag{2}$$

If we write $a_n = \lambda\theta^n + \epsilon_n$ and substitute this into (2), we find that

$$(E-\theta)^2 a_n - \frac{((E-\theta)a_n)^2}{a_n} \in I_r, \tag{3}$$

where E denotes Boole's shift operator for which $Ea_n = a_{n+1}$. From (3), it follows that $(E-\theta)^2 a_n = (E-\theta)^2 \epsilon_n$ is bounded, and hence that ϵ_n is bounded. In fact, if $\overline{I}_r = [r-1, r]$, then the following are easily proved:

$$\operatorname{LIM}(E-\theta)^2 \epsilon_n \subseteq \overline{I}_r, \tag{3}$$

$$\operatorname{LIM}(E-\theta)\epsilon_n \subseteq \tfrac{1}{\theta-1}\overline{I}_r, \tag{4}$$

and

$$\operatorname{LIM}\ \ \epsilon_n \subseteq \tfrac{1}{(\theta-1)^2}\overline{I}_r, \tag{5}$$

where LIM denotes the set of limit points of the sequence in question.

From (4) it immediately follows that if a_n satisfies a linear recurrence relation with defining polynomial $Q(x)$, then Q has the single root θ outside the unit circle, the other roots being in $|x| \le 1$, with those on $|x| = 1$ being simple roots. These are the restrictions discovered by Flor for the case $r = \frac{1}{2}$. As in [Fl], one can show that, if $0 < r < 1$ and if $P(x)$ is the minimal polynomial of any Pisot or Salem number, then $Q(x) = P(x)$ occurs as the defining polynomial for some E_r–sequence.

Following [B3], we have the following simple result:

Proposition 1. *Suppose that a_n is an E_r–sequence with $\lim \frac{a_{n+1}}{a_n} = \theta > 1$, where θ is a Pisot or Salem number. Let d be the degree of θ. If*

$$\theta > 1 + \max(r, 1-r)2^{d-1} \tag{6}$$

then the sequence a_n satisfies a linear recurrence relation whose defining polynomial is the minimal polynomial $P(x)$ of θ (i.e. no cyclotomic factor $K(x)$ occurs.)

Proof. From (4),

$$\limsup |(E-\theta)a_n| \le \frac{\max(r, 1-r)}{\theta - 1}. \tag{7}$$

Write $P(x) = (x-\theta)P_\theta(x)$. Then all roots of P_θ lie in the unit circle, so $P_\theta(x)$ is dominated by $(x+1)^{d-1}$ and hence the length of P_θ (the sum of the absolute values of its coefficients) satisfies $L(P_\theta) \leq 2^{d-1}$. Combining this with (7) shows that, if $b_n = P(E)a_n$, then

$$\limsup |b_n| \leq \limsup L(P_\theta)|(E-\theta)a_n| \leq \frac{\max(r, 1-r)}{\theta - 1} 2^{d-1} < 1. \qquad (8)$$

But b_n is a sequence of integers so (7) implies $b_n = 0$ eventually, and hence $P(E)a_n = 0$ eventually, showing that a_n satisfies a linear recurrence relation with $P(x)$ as defining polynomial. **QED**

Now let us consider the cases $r = 0$ and $r = 1$. If $r = 0$, then (3) implies that

$$\limsup (E-\theta)^2 \epsilon_n \leq 0, \qquad (9)$$

while if $r = 1$ then

$$\liminf (E-\theta)^2 \epsilon_n \geq 0. \qquad (10)$$

If $Q(x)$ is the defining polynomial of a linear recurrence for a_n, and we let the roots of Q be θ_k, $k = 1, \ldots, m$, with $|\theta_k| \geq |\theta_{k+1}|$ for all k, then we can write $\epsilon_n = \lambda_2 \theta_2^n + \cdots + \lambda_m \theta_m^n$. Then (9) or (10) shows that θ_2 must be real and positive and that $\lambda_2 < 0$ if $r = 0$, while $\lambda_2 > 0$ if $r = 1$.

Thus, if θ is a Pisot number, then one possibility is that θ_2 is a conjugate of θ, in which case $\theta_2 > |\theta_k|$, for $k \geq 3$, which defines a proper subset of the Pisot numbers, which we will denote by PVD. The other possibility is that $\theta_2 = 1$. In case θ is a Salem number, only this latter possibility can occur.

However, by Proposition 1, if (6) holds, then the defining relation for a_n can have no cyclotomic factor, in particular no factor $x - 1$. Thus, if θ is a Pisot or Salem number satisfying (6), then θ must be in PVD.

If $r > 1$ or $r < 0$, then we must have $\liminf |(E-\theta)^2 \epsilon_n| > 0$ and hence $\theta_2 = 1$. Thus, if (6) holds, then there is no E_r–sequence with $r > 1$ or $r < 0$ having limiting ratio θ. So, for any fixed degree, there is only a finite number of such E_r sequences satisfying a recurrence relation of that degree.

Examples.

Given a Pisot sequence, one can check for linear recurrence relations of small degree (say ≤ 200), by using continued fractions in $Q\{\frac{1}{x}\}$, as in [C1], or by the Berlekamp-Massey algorithm, both of which are efficient methods of finding some of the Padé approximants to the generating function $\sum a_n x^n$. Using such methods, Cantor and his student Galyean [Ga] found many apparently non-recurrent Pisot sequences, among them $E(4, 13)$, $E(6, 16)$ and $E(8, 10)$. In some cases these sequences follow "false" recurrences for many terms, e.g $E(8, 10)$ follows $x^6 - x^5 - 1$ for 37 terms. This polynomial has three roots outside the unit circle, but two of them are very close to the unit circle and it is easy to see how this can permit the sequence to satisfy the recurrence relation for a large number of terms.

For $r = 0$ or 1, a similar but different phenomenon can occur. A false recurrence can be due to the recurrence having a second root $\theta_2 < 1$ for which there are other roots θ_k with $|\theta_k| > \theta_2$, but close to θ_2. The most extreme example of this which has been found in our computations is $S(8, 55)$ which seems to have the rational generating function

$$\frac{8 + 7x - 7x^2 - 7x^3}{1 - 6x - 7x^2 + 5x^3 + 6x^4}. \tag{11}$$

However, the reciprocals of the roots of the denominator are:

$$\theta = 6.892070726276\ldots$$

$$\alpha = .95484560059400\ldots$$

$$\beta, \overline{\beta} = .95484787673885 \exp \pm i(165.268\ldots)^\circ.$$

Here θ is certainly a Pisot number, but is not in PVD since $\alpha < |\beta|$. Note that the ratio $|\beta|/\alpha = 1.00000238378\ldots$ is very close to 1. If one writes c_n for the sequence given by (11) and a_n for $S(8, 55)$, then $a_n = c_n$ for $n \leq 11055$, but $a_n = c_n + 1$ for $n = 11056$, as Jeff Shallit calculated in 27 hours on a Sparc SLC. This can also be verified by writing

$$c_n = \lambda\theta^n + \mu\alpha^n + \gamma\beta^n + \overline{\gamma}\overline{\beta}^n,$$

where the coefficients are calculated from (11) to, say, 20 decimal places. Then one needs to find the first n for which

$$(E - \theta)^2\epsilon_n = \mu(\theta - \alpha)^2\alpha^n + 2\Re(\gamma(\theta - \beta)^2\beta^n) < 0.$$

This occurs first for $n = 11054$ which agrees with the integer calculation.

3 Families of E_r–sequences.

Experiment shows that E_r–sequences seem to fall into families whose properties depend on $a_1 \bmod a_0^2$. That is, the sequences $E_r(a_0, ka_0^2 + c)$, $k = 0, 1, \ldots$ tend to satisfy similar linear recurrences or else to be (apparently) non-recurrent. Cantor [C2] gave an explanation of this by treating k as a formal parameter and treating a_n as a polynomial in k. Although the emphasis there was on Pisot sequences, his considerations apply equally well to E_r–sequences.

According to that theory, the linear recurrence for $E(a_0, ka_0^2 + c)$, if it exists, should be of the form $F(x)/(A(x) - kxF(x))$, for $k \geq k_0$.

Following this lead, we did some experiments with sequences of the form $S(a_0, ka_0^2 + c)$ and $T(a_0, ka_0^2 + c)$, mainly for $a_0 \leq 20$. Remarkably, it seems that $S(a_0, ka_0^2 + c)$ is recurrent for $k \geq 1$ if $0 \leq c \leq a_0$, but not for $c = -1$ or $c = a_0 + 1$. Formally, if $T(a_0, a_1) = \{a_n\}$, then $S(a_0, -a_1) = \{(-1)^n a_n\}$, at least if no ratio a_{n+1}^2/a_n is an integer. This would suggest that the behaviour of the family $T(a_0, ka_0^2 - c)$ should be similar to that of $S(a_0, ka_0^2 + c)$. However, although we find that $T(a_0, ka_0^2 - c)$ does seem to be recurrent for a large interval of the form $0 \leq c \leq \kappa a_0$, for some $\kappa < 1$, it does not seem to be recurrent for all values of $c < a_0$ (we consider the extreme case $c = a_0 - 1$ below.) However, in contrast to S-sequences, $T(a_0, ka_0^2 - c)$ does seem to be recurrent for $c = -1$ or $c = a_0 + 1$.

This suggests looking at some different families than those considered by Cantor, where, rather than fixing a_0, we take $a_0 = a$ to be a formal parameter (to be thought of as an infinitely large integer.) For example, looking at the sequences $S(a, a^2 - 1)$, it is natural to define the formal S-sequences with initial terms $a_0 = a$ and $a_1 = a^2 - 1$ so that a_n is a polynomial of degree $n + 1$ in a which satisfies: $a_{n+1}^2 = a_{n+2}a_n + b_n$, where $\deg b_n \leq \deg a_n$, and where the leading coefficient of b_n is negative. So, a_{n+2} is either the usual quotient in the standard division of the polynomial a_{n+1}^2 by a_n or else this quotient minus a_n, if this is necessary to make the leading coefficient of b_n negative.

We find then that a_n satisfies a degree 4 linear recurrence with generating function:

$$\frac{a - x + x^2(1-a) + x^3}{1 - xa - x^3(1-a) - x^4}. \tag{12}$$

However, except for $S(2,3)$ which satisfies a linear recurrence of degree 2, all of $S(3,8)$, $S(4,15)$, etc, seem to be non-recurrent. In fact, for finite a, $S(a, a^2 - 1)$ seems to satisfy (12) for exactly $2a$ terms. The roots of the reciprocal of the denominator of (12) are $a - \frac{1}{a} + O(\frac{1}{a^2})$, $1 - \frac{1}{2a} - \frac{1}{a^2} + O(\frac{1}{a^3})$, $\frac{1}{a} + \frac{1}{a^2} + O(\frac{1}{a^3})$, and $-1 + \frac{1}{2a} - \frac{1}{a^2} + O(\frac{1}{a^3})$. Thus, the negative root dominates the second positive root so this is not PVD for finite a.

On the other hand, $T(a, a^2 + 1)$ satisfies a degree 7 linear recurrence:

$$\frac{a + x - x^3 - x^4 - x^5 - x^6}{1 - ax - x^2 + x^4 + x^5 + x^6 + x^7} = \frac{F(x)}{1 - xF(x)}, \tag{13}$$

which is PVD for all large a. (Here we define a_n as above except so that the leading coefficient of b_n is required to be positive.) This suggests that $T(a, a^2 + 1)$ does satisfy (13) for all large a, which experimentally is the case. However, it seems that $T(a, ka^2 + 1)$ does *not* satisfy the expected recurrence $F(x)/(1 - kxF(x))$ for large k but rather another recurrence.

Another interesting family is $T(a, a^2 - a + 1)$. Here the linear recurrence is of the form $F(x)/(1 + x - xF(x))$, where $F(x) = a + x - x^2 - x^4 - x^{22}$. The denominator is PVD for $a \geq 2849$ and this seems to be the correct recurrence for $T(a, a^2 - a + 1)$ for such a. For smaller a, we find that

$T(4,13)$ satisfies the recurrence $F(x)/(1+x-xF(x))$, with $F(x) = 4+x-x^2-x^4-x^{36}$.

The next member of the family, $T(5,21)$ satisfies a remarkable recurrence of this form, with $F(x) = 5+x-x^2-x^4-x^{26}-x^{2048}$. (Note that the rational function $F(x)/(1+x-xF(x))$ is not in lowest terms since both numerator and denominator are divisible by $1+x$. However, it is best left in this form!) The computations to verify that this is the correct recurrence are quite lengthy since they involve power sums of 2047 of the roots of the reciprocal of $G(x) = 1+x-xF(x)$ to check (3) for n up to 200,000. It was necessary to calculate the roots and other constants to 60 decimal places. Fortunately, it is easy to solve $G(x) = 0$ by Newton's method, since the roots are very close to the 2048^{th} roots of 4.

It seems that the next few members of this family, for $6 \leq a \leq 20$ are probably not recurrent. We note, however, that the example of $T(5,21)$ shows the danger of concluding that a sequence is non-recurrent by finding possible recurrences up to a given degree. If one were to calculate $T(5,21)$ up to, say, 1000 terms, and compute Padé approximants, the naïve conclusion would be that $T(5,21)$ satisfies the linear recurrence given by $F(x)/(1+x-xF(x))$, where $F(x) = 5+x-x^2-x^4-x^{26}$. Less naïvely, since this is not PVD and hence not a legitimate recurrence for a T-sequence, one might then conclude that the sequence is non-recurrent. However, this sequences *is* recurrent, it is just that the degree of the recurrence is rather larger than expected. Going back to the sequence $S(8,55)$ discussed earlier, we can only conclude from the calculations done for that sequence that it satisfies no linear recurrence relation of degree ≤ 11055.

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