

# IN KOENIGS' FOOTSTEPS: DIAGONALIZATION OF COMPOSITION OPERATORS

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ABSTRACT. Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map with a fixed point  $\alpha \in \mathbb{D}$  such that  $0 \leq |\varphi'(\alpha)| < 1$ . We show that the spectrum of the composition operator  $C_\varphi$  on the Fréchet space  $\text{Hol}(\mathbb{D})$  is  $\{0\} \cup \{\varphi'(\alpha)^n : n = 0, 1, \dots\}$  and its essential spectrum is reduced to  $\{0\}$ . This contrasts the situation where a restriction of  $C_\varphi$  to Banach spaces such as  $H^2(\mathbb{D})$  is considered. Our proofs are based on explicit formulae for the spectral projections associated with the point spectrum found by Koenigs. Finally, as a byproduct, we obtain information on the spectrum for bounded composition operators induced by a Schröder symbol on arbitrary Banach spaces of holomorphic functions.

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## 1. INTRODUCTION

Let  $\varphi$  be a holomorphic self-map of the open unit disc  $\mathbb{D}$  and let  $\text{Hol}(\mathbb{D})$  be the algebra of holomorphic functions on  $\mathbb{D}$  which is a Fréchet space endowed with the topology of uniform convergence on every compact subsets of  $\mathbb{D}$ .

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Denote by  $\text{Aut}(\mathbb{D})$  the group of all automorphisms on  $\mathbb{D}$ . It is a well-known fact that such functions have the form  $z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z}$  where  $a \in \mathbb{D}$  and  $\theta \in \mathbb{R}$ .

The functional equation  $f \circ \varphi = \lambda f$  where  $\lambda \in \mathbb{C}$  is called the *homogeneous Schröder equation*.

For those  $\varphi$  which are not automorphisms of  $\mathbb{D}$  and which admit a fixed point  $\alpha \in \mathbb{D}$ , the solution was found by G. Koenigs in 1884. Note that a fixed point in  $\mathbb{D}$  is unique whenever it exists.

By  $\mathbb{N}_0$  we denote the set of all nonnegative integers and let  $\mathbb{N} = \mathbb{N}_0 \setminus \{0\} = \{1, 2, \dots\}$ .

**Theorem 1.1** (Koenigs' theorem). *Let  $\varphi$  be a holomorphic map on  $\mathbb{D}$  such that  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ,  $\varphi \notin \text{Aut}(\mathbb{D})$  and assume that  $\varphi$  has a fixed point  $\alpha \in \mathbb{D}$  with  $\lambda_1 := \varphi'(\alpha)$ . Then the following holds:*

- *If  $\lambda_1 = 0$  the equation  $f \circ \varphi = \lambda f$  has a nontrivial solution  $f \in \text{Hol}(\mathbb{D})$  if and only if  $\lambda = 1$  and the constant functions are the only solutions.*
- *If  $\lambda_1 \neq 0$ , then:*
  - (a) *the equation  $f \circ \varphi = \lambda f$  has a nontrivial solution  $f \in \text{Hol}(\mathbb{D})$  if and only if  $\lambda \in \{\lambda_1^n : n \in \mathbb{N}_0\}$ ;*
  - (b) *there exists a unique function  $\kappa \in \text{Hol}(\mathbb{D})$  satisfying*

$$\kappa \circ \varphi = \lambda_1 \kappa \text{ and } \kappa'(\alpha) = 1;$$

- (c) *for  $n \in \mathbb{N}_0$  and  $f \in \text{Hol}(\mathbb{D})$ ,  $f \circ \varphi = \lambda_1^n f$  if and only if  $f = c\kappa^n$  for some  $c \in \mathbb{C}$ .*

The case where  $\varphi'(\alpha) \neq 0$  is the most interesting one. To be consistent with [23], we use the following terminology.

**Definition 1.2.** *A Schröder map is a holomorphic function  $\varphi$  satisfying the following conditions:*

$$\varphi(\mathbb{D}) \subset \mathbb{D}, \varphi \notin \text{Aut}(\mathbb{D}), \exists \alpha \in \mathbb{D} \text{ such that } \varphi(\alpha) = \alpha \text{ and } \varphi'(\alpha) \neq 0.$$

*The function  $\kappa$  associated to a Schröder map in Theorem 1.1 is called the Koenigs' eigenfunction of  $\varphi$ .*

As a consequence of the Schwarz lemma [19], a holomorphic self-map  $\varphi$  of  $\mathbb{D}$  with a fixed point  $\alpha \in \mathbb{D}$  is a Schröder map if and only if  $0 < |\varphi'(\alpha)| < 1$ . Moreover, Koenigs' eigenfunction  $\kappa$  is then obtained as the limit of  $\frac{\varphi_n}{\lambda_1^n}$  in  $\text{Hol}(\mathbb{D})$  as  $n \rightarrow \infty$ , where  $\varphi_n = \varphi \circ \dots \circ \varphi$ .

The aim of this paper is to study the non homogeneous Schröder equation

$$(1) \quad f \circ \varphi - \lambda f = g$$

where  $\lambda \in \mathbb{C}$  and  $g \in \text{Hol}(\mathbb{D})$  are given and  $f \in \text{Hol}(\mathbb{D})$  the solution.

As in Koenigs' work, we consider the case where  $\varphi \notin \text{Aut}(\mathbb{D})$  and  $\varphi$  has a fixed point  $\alpha$  in  $\mathbb{D}$ .

The study of the homogenous Schröder equation can be reformulated from an operator theory point of view in the following way: consider the composition operator  $C_\varphi : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$  given by  $C_\varphi(f) = f \circ \varphi$ . We denote by  $\sigma(C_\varphi)$  the spectrum and by  $\sigma_p(C_\varphi)$  the point spectrum of  $C_\varphi$ . Thus (1) has a unique solution for all  $g \in \text{Hol}(\mathbb{D})$  if and only if  $\lambda \notin \sigma(T)$ . Moreover, Koenigs' theorem implies that  $\sigma_p(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\}$ .

Our main result consists in finding "the spectral projections" associated with  $\lambda_n = \lambda_1^n$ . The difficulty is that these spectral projections are not defined since a priori we do not know that the  $\lambda_n$  are isolated in the spectrum. We define projections  $P_n$  of rank 1 such that  $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$ . Using these "spectral" projections we then show that actually the spectrum of the composition operator  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$  is given by

$$\sigma(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\}.$$

This looks very similar to spectral properties of compact operators. But we show that the operator  $C_\varphi$  is compact on  $\text{Hol}(\mathbb{D})$  only in very special situations.

Nevertheless, our results show that the operator  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$  is always a Riesz operator; i.e. its essential spectrum is reduced to  $\{0\}$ . This contrasts the situation where a restriction  $T = C_{\varphi|_{H^2(\mathbb{D})}}$  is considered. Indeed, in this case, the essential spectrum is a disc with  $r_e(T) > 0$  in many cases. Actually, much is known on such restrictions to spaces such as  $H^p(\mathbb{D})$ , Bergman space, Dirichlet space and others. See the monographs [6] of Cowen and MacCluer and of Shapiro [21], as well as the articles [5, 10, 11, 14, 15, 16, 22, 24, 26] to name a few.

Our results on  $\text{Hol}(\mathbb{D})$  allow us to prove some spectral properties of the restriction  $T$  of  $C_\varphi$  to some invariant Banach space  $X \hookrightarrow \text{Hol}(\mathbb{D})$ . For instance we will see that  $0 \in \sigma(T)$  if and only if  $\dim X = \infty$ , and in this case we show that the essential spectrum  $\sigma_e(T)$  is the connected component of 0 in  $\sigma_e(T)$ .

The paper is organized as follows. In Section 2 we characterize composition operators on  $\text{Hol}(\mathbb{D})$  as the non-zero algebra homomorphisms. This is also an interesting example of automatic continuity. We also characterize when  $C_\varphi$  is compact as operators on  $\text{Hol}(\mathbb{D})$  (which is much more restrictive than on  $H^2(\mathbb{D})$ , for example). Section 3 is devoted to the definition and investigation of the spectral projections. The main

theorem determining the spectrum of  $C_\varphi$  in  $\text{Hol}(\mathbb{D})$  is established in Section 4. Finally, we deduce spectral properties of restrictions to arbitrary invariant Banach spaces in Section 5.

## 2. COMPOSITION OPERATORS ON $\text{Hol}(\mathbb{D})$

Let  $\varphi$  be a holomorphic self-map of  $\mathbb{D}$ . We define the composition operator  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$  by  $C_\varphi(f) = f \circ \varphi$ . Then  $C_\varphi$  is in  $\mathcal{L}(\text{Hol}(\mathbb{D}))$  the algebra of linear and continuous operators on  $\text{Hol}(\mathbb{D})$ ; indeed the linearity is trivial and the continuity follows from the definition of the topology of the Fréchet space (uniform convergence on compact subsets of  $\mathbb{D}$ ) and the continuity of  $\varphi$ .

The next proposition is an algebraic characterization of composition operators. Note that  $\text{Hol}(\mathbb{D})$  is an algebra. An algebra homomorphism  $A : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$  is a linear map satisfying

$$A(f \cdot g) = A(f) \cdot A(g) \text{ for all } f, g \in \text{Hol}(\mathbb{D}).$$

**Proposition 2.1.** *Let  $A : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$  be linear. The following assertions are equivalent.*

- (i) *There exists a holomorphic map  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  such that  $A = C_\varphi$ ;*
- (ii)  *$A$  is an algebra homomorphism different from 0;*
- (iii)  *$A$  is continuous and  $Ae_n = (Ae_1)^n$  for all  $n \in \mathbb{N}_0$ .*

Here we define  $e_n \in \text{Hol}(\mathbb{D})$  by  $e_n(z) = z^n$  for all  $z \in \mathbb{D}$  and all  $n \in \mathbb{N}_0$ . For the proof we use the following well-known result.

**Lemma 2.2.** *Let  $L : \text{Hol}(\mathbb{D}) \rightarrow \mathbb{C}$  be a continuous algebra homomorphism,  $L \neq 0$ . Then there exists  $z_0 \in \mathbb{D}$  such that  $Lf = f(z_0)$  for all  $f \in \text{Hol}(\mathbb{D})$ .*

*Proof.* Since  $Lf = L(f \cdot e_0) = L(f)L(e_0)$  for all  $f \in \text{Hol}(\mathbb{D})$  and since  $L \neq 0$ , it follows that  $L(e_0) = 1$ . Set  $z_0 := Le_1$ . Then  $z_0 \in \mathbb{D}$ . Indeed, otherwise  $g(z) = \frac{1}{z-z_0}$  defines a function  $g \in \text{Hol}(\mathbb{D})$  such that  $(e_1 - z_0e_0)g = e_0$ . Hence

$$1 = Le_0 = (Le_1 - z_0Le_0)Lg = 0,$$

a contradiction. For  $f \in \text{Hol}(\mathbb{D})$  such that  $f(z_0) = 0$ , we have  $Lf = 0$ . Indeed, since there exists  $g \in \text{Hol}(\mathbb{D})$  such that  $f = (e_1 - z_0e_0)g$ , it follows that  $L(f) = (L(e_1) - z_0L(e_0))L(g) = 0$ .

For an arbitrary  $f \in \text{Hol}(\mathbb{D})$ , note that  $h := f - f(z_0)e_0$  satisfies  $h(z_0) = 0$ . Hence  $0 = L(h) = L(f) - f(z_0)$ .  $\square$

**Remark 2.3.** *We are grateful to H.G. Dales and J. Esterle for helping us with Lemma 2.2. For much more information about automatic continuity, we refer to the monograph of Dales [7] and the survey article of Esterle [9].*

*Proof of Proposition 2.1. (ii)  $\Rightarrow$  (i):* since  $A \neq 0$ , it follows as in Lemma 2.2 that  $Ae_0 = e_0$ . Let  $z \in \mathbb{D}$ . Then  $L(f) := (Af)(z)$  is an algebra homomorphism and  $L(e_0) = 1$ . By Lemma 2.2, there exists  $\varphi(z) \in \mathbb{D}$  such that  $(Af)(z) = f(\varphi(z))$  for all  $f \in \text{Hol}(\mathbb{D})$ . In particular  $\varphi = Ae_1 \in \text{Hol}(\mathbb{D})$ .

*(iii)  $\Rightarrow$  (ii):* it follows from (iii) that  $A(fg) = A(f)A(g)$  if  $f$  and  $g$  are polynomials. Since the set of polynomials is dense in  $\text{Hol}(\mathbb{D})$  and since the multiplication is continuous, (ii) follows.

*(i)  $\Rightarrow$  (iii)* is trivial. □

For our purposes, the following corollary is useful.

**Corollary 2.4.** *Let  $X = \text{Hol}(\mathbb{D})$  and  $\varphi$  a holomorphic self-map of  $\mathbb{D}$ . The following assertions are equivalent:*

- (i)  $C_\varphi$  is invertible in  $\mathcal{L}(\text{Hol}(\mathbb{D}))$ ;
- (ii)  $\varphi$  is an automorphism of  $\mathbb{D}$ .

*Proof. (ii)  $\Rightarrow$  (i)* is clear since  $C_{\varphi^{-1}}C_\varphi = C_\varphi C_{\varphi^{-1}} = \text{Id}$ , where  $\text{Id}$  denotes the identity map on  $X$ .

*(i)  $\Rightarrow$  (ii):* let  $C_\varphi$  be invertible,  $A = C_\varphi^{-1}$ . Then  $A$  is an algebra homomorphism. By Proposition 2.1 there exists a holomorphic map  $\psi : \mathbb{D} \rightarrow \mathbb{D}$  such that  $A = C_\psi$ . Then

$$e_1 = C_\varphi(C_\psi e_1) = \psi \circ \varphi \text{ and } e_1 = C_\psi(C_\varphi e_1) = \varphi \circ \psi.$$

Thus  $\varphi$  is an automorphism and  $\psi = \varphi^{-1}$ . □

Next we want to characterize those  $\varphi$  for which  $C_\varphi$  is compact on  $\text{Hol}(\mathbb{D})$ . The reason of this investigation is the following. One of our main points in the article is to show that the spectral properties of a composition operator  $C_\varphi$  for  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  with interior fixed point looks very much to what one knows from compact operators. However, as we will show now, for composition operators on  $\text{Hol}(\mathbb{D})$ , compactness is a very restrictive condition.

Recall that  $\mathcal{V} \subset \text{Hol}(\mathbb{D})$  is a neighborhood of 0 if and only if there exist a compact subset  $K \subset \mathbb{D}$  and  $\varepsilon > 0$  such that

$$\mathcal{V}_{K,\varepsilon} := \{f \in \text{Hol}(\mathbb{D}) : |f(z)| < \varepsilon \text{ for all } z \in K\} \subset \mathcal{V}.$$

A linear mapping  $T : X \rightarrow X$  where  $X$  is a Fréchet space, is called *compact* if there exists a neighborhood  $\mathcal{V}$  of 0 such that  $T\mathcal{V}$  is relatively compact. Each compact linear mapping is continuous. We refer

to Kelley–Namioka [13] for these notions and properties of compact operators.

**Theorem 2.5.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic. The following assertions are equivalent:*

- (i)  $C_\varphi$  is compact as operator from  $\text{Hol}(\mathbb{D})$  to  $\text{Hol}(\mathbb{D})$ ;
- (ii)  $\sup_{z \in \mathbb{D}} |\varphi(z)| < 1$ .

*Proof.* (i)  $\Rightarrow$  (ii): assume that  $\varphi(\mathbb{D}) \not\subset r\overline{\mathbb{D}}$  for all  $0 < r < 1$ . Let  $\mathcal{V}$  be a neighborhood of 0. We show that  $C_\varphi(\mathcal{V})$  is not relatively compact. There exists  $0 < \varepsilon < 1$  and  $0 < r_0 < 1$  such that

$$\mathcal{V}_0 := \{f \in \text{Hol}(\mathbb{D}) : |f(z)| < \varepsilon, \forall z \in r_0\overline{\mathbb{D}}\} \subset \mathcal{V}.$$

Thus it is sufficient to show that  $C_\varphi(\mathcal{V}_0)$  is not relatively compact. By our assumption there exists  $w_0 \in \mathbb{D}$  such that  $z_0 := \varphi(w_0) \notin r_0\overline{\mathbb{D}}$ . Then there exist  $r_0 < r_1 < 1$  and  $\rho > 0$  such that  $r_1\overline{\mathbb{D}} \cap D(z_0, \rho) = \emptyset$ .

The set  $K := r_0\overline{\mathbb{D}} \cup \{z_0\}$  is compact and  $\mathbb{C} \setminus K$  is connected. Let  $n \in \mathbb{N}_0$  and define  $h_n$  by

$$h_n(z) = 0 \text{ for } z \in r_1\overline{\mathbb{D}} \text{ and } h_n(z) = n + 1 \text{ for } z \in D(z_0, \rho).$$

Set  $\Omega := r_1\overline{\mathbb{D}} \cup D(z_0, \rho)$ . Then  $K \subset \Omega$  and  $h_n : \Omega \rightarrow \mathbb{C}$  is holomorphic. By Runge's theorem, there exists a polynomial  $p_n : \mathbb{C} \rightarrow \mathbb{C}$  such that  $|p_n(z) - h_n(z)| < \varepsilon$  for all  $z \in K$ . This implies that  $p_n|_{\mathbb{D}} \in \mathcal{V}_0$  and  $|p_n(z_0)| \geq n$ . Since  $|C_\varphi(p_n)(w_0)| = |p_n(z_0)| \geq n$ , the sequence  $(C_\varphi p_n)_{n \in \mathbb{N}_0}$  has no convergent subsequence.

(ii)  $\Rightarrow$  (i): Assume that  $\sup_{z \in \mathbb{D}} |\varphi(z)| =: r_0 < 1$ . The set

$$\mathcal{V} := \{f \in \text{Hol}(\mathbb{D}) : |f(z)| < 1 \text{ if } |z| \leq r_0\}$$

is a neighborhood of 0. Let  $f \in \mathcal{V}$ . Since  $\varphi(\mathbb{D}) \subset r_0\overline{\mathbb{D}}$ , one has  $|f(\varphi(w))| < 1$  for all  $w \in \mathbb{D}$ . Now it follows from Montel's theorem that  $C_\varphi \mathcal{V}$  is relatively compact in  $\text{Hol}(\mathbb{D})$ .  $\square$

**Remark 2.6.** *The same characterization of compact composition operators is valid in some special Banach spaces of holomorphic functions, for example  $X = H^\infty(\mathbb{D})$  [25]. However on  $H^2(\mathbb{D})$ , the class of mappings  $\varphi$  such that  $C_\varphi$  is compact is much larger [6].*

### 3. DIAGONALIZATION OF COMPOSITION OPERATORS

In this section we show that composition operators  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$  can be diagonalized if the symbol  $\varphi$  is a Schröder map.

For the following we fix the holomorphic function  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ , with interior fixed point  $\alpha = \varphi(\alpha) \in \mathbb{D}$  and suppose that  $\varphi \notin \text{Aut}(\mathbb{D})$ ,  $\varphi'(\alpha) \neq 0$ . We set  $\lambda_n = \varphi'(\alpha)^n$  for  $n \in \mathbb{N}_0$ . Thus  $\lambda_1 \in \mathbb{D}$  by the

Schwarz lemma and  $|\lambda_n|$  tends to 0 as  $n \rightarrow \infty$ . The range of an operator  $T$  is denoted by  $\text{rg } T$ . We denote by  $\kappa$  Koenigs' eigenfunction associated with  $\varphi$ . The following properties of  $\kappa^n$  will be needed.

**Lemma 3.1.** *For all  $n \in \mathbb{N}$ ,  $(\kappa^n)^{(n)}(\alpha) = n!$  and  $(\kappa^n)^{(l)}(\alpha) = 0$  for  $l = 0, \dots, n-1$ .*

*Proof.* Since  $\kappa(\alpha) = 0$  and  $\kappa'(\alpha) = 1$ , we get that, as  $z \rightarrow \alpha$ ,

$$\kappa^n(z) = [(z - \alpha) + o(z - \alpha)]^n = (z - \alpha)^n + o((z - \alpha)^n).$$

Hence,  $(\kappa^n)^{(n)}(\alpha) = n!$  and  $(\kappa^n)^{(l)}(\alpha) = 0$  for  $l = 0, \dots, n-1$ .  $\square$

In the following theorem we define inductively a series of rank-one projections which diagonalize the operator  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$ .

**Theorem 3.2.** *Define iteratively rank-one projections  $P_n \in \mathcal{L}(\text{Hol}(\mathbb{D}))$  by*

$$(2) \quad P_0 f = f(\alpha)e_0 \text{ and for } n \in \mathbb{N}, P_n(f) = \frac{1}{n!} g^{(n)}(\alpha) \kappa^n,$$

where  $g = f - \sum_{k=0}^{n-1} P_k f$ . Then the following holds:

- (a)  $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$ .
- (b)  $f^{(l)}(\alpha) = (\sum_{k=0}^n P_k f)^{(l)}(\alpha)$  for  $l = 0, \dots, n$  and  $f \in \text{Hol}(\mathbb{D})$ .
- (c) There exist complex numbers  $c_{n,m}$  ( $n, m \in \mathbb{N}_0$ ) such that

$$P_n f = \left( \sum_{m=0}^n c_{n,m} f^{(m)}(\alpha) \right) \kappa^n.$$

- (d)  $P_n P_m = \delta_{n,m} P_n$  for all  $n, m \in \mathbb{N}_0$ .

We deduce the following decomposition property from Theorem 3.2. Let

$$\text{Hol}_n(\alpha) := \{f \in \text{Hol}(\mathbb{D}) : f(\alpha) = f'(\alpha) = \dots = f^{(n)}(\alpha) = 0\},$$

and  $Q_n = \sum_{k=0}^n P_k$ , where  $P_k$  is given in Theorem 3.2.

**Corollary 3.3.** *The mappings  $Q_n$  are projections commuting with  $C_\varphi$ . Moreover  $\{\kappa^l : l = 0, \dots, n\}$  is a basis of  $\text{rg } Q_n$  and  $\ker(Q_n) = \text{Hol}_n(\alpha)$ . Thus we have the decomposition*

$$\text{Hol}(\mathbb{D}) = \text{Span}\{\kappa^m : m = 0, \dots, n\} \oplus \text{Hol}_n(\alpha)$$

into two subspaces which are invariant by  $C_\varphi$ .

As a consequence,  $C_{\varphi|_{\text{rg } Q_n}}$  is a diagonal operator since  $C_\varphi(\kappa^l) = \lambda_l \kappa^l$  for  $l = 0, \dots, n$ . Of course, by definition  $\kappa^0$  is the constant function equal to 1.

*Proof of Corollary 3.3.* a) Let  $g = \sum_{m=0}^n a_m \kappa^m \in \text{Hol}_n(\alpha)$  where  $a_m \in \mathbb{C}$ . Then by Lemma 3.1,

$$0 = g(\alpha) = a_0, 0 = g'(\alpha) = a_1, \dots, 0 = g^{(n)}(\alpha) = n!a_n.$$

This shows that the functions  $\kappa^m, m = 0, \dots, n$  are linearly independent and that

$$\text{Span}\{\kappa^m : m = 0, \dots, n\} \cap \text{Hol}_n(\alpha) = \{0\}.$$

b) Let  $f \in \text{Hol}(\mathbb{D})$ . Then, by Theorem 3.2,  $f - Q_n f \in \text{Hol}_n(\alpha)$ . This shows that

$$\text{Hol}(\mathbb{D}) = \text{Span}\{\kappa^m : m = 0, \dots, n\} \oplus \text{Hol}_n(\alpha)$$

and that  $Q_n$  is the projection onto the first space along this decomposition. □

*Proof of Theorem 3.2.* We define  $P_n$  by the iteration equation (2).

At first we show (b) inductively. For  $n = 0$  it is trivial. Let  $n \geq 1$  and assume that (b) is true for  $n - 1$ . Let  $f \in \text{Hol}(\mathbb{D})$  and  $0 \leq l < n$ . Since  $\kappa(\alpha) = 0$ ,  $(\kappa^n)^{(l)}(\alpha) = 0$  for  $l < n$  (by Lemma 3.1), it follows that

$$\left( \sum_{k=0}^n P_k f \right)^{(l)}(\alpha) = \left( \sum_{k=0}^{n-1} P_k f \right)^{(l)}(\alpha) = f^{(l)}(\alpha),$$

by the inductive hypothesis. For  $l = n$ , we have

$$\begin{aligned} \left( \sum_{k=0}^n P_k f \right)^{(n)}(\alpha) &= \left( \sum_{k=0}^{n-1} P_k f \right)^{(n)}(\alpha) \\ &\quad + \frac{1}{n!} \left( f - \sum_{k=0}^{n-1} P_k f \right)^{(n)}(\alpha) (\kappa^n)^{(n)}(\alpha) \\ &= f^{(n)}(\alpha), \end{aligned}$$

since  $(\kappa^n)^{(n)}(\alpha) = n!$  (see Lemma 3.1). Thus (b) is proved.

It is clear that  $(P_n f) \circ \varphi = \lambda_n P_n f$  since  $\kappa^n \circ \varphi = \lambda_n \kappa^n$ . We show inductively that  $P_n(f \circ \varphi) = \lambda_n P_n f$ . For  $n = 0$  this is trivial. Let  $n \geq 1$  and assume now that  $P_l(f \circ \varphi) = \lambda_l P_l f$  for all  $l \leq n - 1$ . Note that

$$P_n(f \circ \varphi) = \frac{1}{n!} (\tilde{g})^{(n)}(\alpha) \kappa^n,$$

where

$$\tilde{g} = f \circ \varphi - \sum_{k=0}^{n-1} P_k(f \circ \varphi) = g \circ \varphi$$



by the inductive hypothesis, where  $g = f - \sum_{k=0}^{n-1} P_k f$ . It follows that  $P_n(f \circ \varphi) = \frac{1}{n!}(g \circ \varphi)^{(n)}(\alpha)\kappa^n$ . Now let us introduce some more notations in order to compute  $(g \circ \varphi)^{(n)}(\alpha)$ . For  $n \in \mathbb{N}$ , let

$$J_n = \{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \mid m_1 + 2m_2 + \dots + nm_n = n\}$$

and

$$K_n = J_n \setminus \{(n, 0, \dots, 0)\}$$

For  $\mathbf{m} = (m_1, \dots, m_n) \in J_n$ , set  $|\mathbf{m}| = m_1 + \dots + m_n$  and note that, for  $\mathbf{m} \in J_n$ ,  $\mathbf{m} \in K_n$  if and only if  $|\mathbf{m}| < n$ . For  $\mathbf{m} \in J_n$ , we also define the following coefficients

$$C_{\mathbf{m}}^n = \frac{n!}{m_1! m_2! \dots m_n!} \prod_{j=1}^n \left( \frac{\varphi^{(j)}(\alpha)}{j!} \right)^{m_j}$$

These coefficients are inspired by Faà di Bruno's Formula: indeed, if  $g \in \text{Hol}(\mathbb{D})$ , then, for every  $n \in \mathbb{N}$ ,

$$(g \circ \varphi)^{(n)}(\alpha) = \sum_{\mathbf{m} \in J_n} C_{\mathbf{m}}^n g^{(|\mathbf{m}|)}(\alpha) = (\varphi'(\alpha))^n g^{(n)}(\alpha) + \sum_{\mathbf{m} \in K_n} C_{\mathbf{m}}^n g^{(|\mathbf{m}|)}(\alpha)$$

Since  $g^{(|\mathbf{m}|)}(\alpha) = 0$  by (b), we get

$$(g \circ \varphi)^{(n)}(\alpha) = \varphi'(\alpha)^n g^{(n)}(\alpha) + \sum_{\mathbf{m} \in K_n} C_{\mathbf{m}}^n g^{(|\mathbf{m}|)}(\alpha) = \lambda_n g^{(n)}(\alpha).$$

Thus  $P_n(f \circ \varphi) = \lambda_n P_n f$  for all  $n \in \mathbb{N}_0$ , which implies that  $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$  for all  $n \in \mathbb{N}_0$ . Thus (a) is proved.

We show inductively that (c) holds for suitable coefficients. It is obvious for  $n = 0$  and assume that  $P_k$  has the property for all  $k \leq n-1$ . Then

$$\begin{aligned} P_n f &= \frac{1}{n!} \left( f - \sum_{k=0}^{n-1} P_k f \right)^{(n)}(\alpha) \kappa^n \\ &= \frac{1}{n!} \left( f^{(n)}(\alpha) + \sum_{k=0}^{n-1} \left( \sum_{l=0}^k c_{k,l} f^{(l)}(\alpha) \right) (\kappa^k)^{(n)}(\alpha) \right) \kappa^n, \end{aligned}$$

which proves the claim for  $n$ .

In order to prove (d), note that by the properties defining the projections and proved previously, for all  $k, l \in \mathbb{N}_0$ , we have:

$$\lambda_k P_l \kappa^k = P_l (\kappa^k \circ \varphi) = \lambda_l P_l \kappa^k.$$

Since  $\lambda_k \neq \lambda_l$  for  $l \neq k$ , it follows that  $P_l \kappa^k = 0$ . Hence  $P_l P_k = 0$  for  $k \neq l$ .

It remains to show that  $P_n^2 = P_n$ , which is equivalent to check that  $P_n \kappa^n = \kappa^n$ . We can show this easily and inductively since  $P_k \kappa^n = 0$  for  $k < n$  and  $(\kappa^n)^{(n)}(\alpha) = n!$ .  $\square$

We can now give explicit expressions of  $P_n$  for  $n = 0, 1, 2, 3$ .

**Corollary 3.4.** *For all  $f \in \text{Hol}(\mathbb{D})$ , we have:*

$$\begin{aligned} P_0 f &= f(\alpha)1 \\ P_1(f) &= f'(\alpha)\kappa \\ P_2(f) &= \frac{1}{2} \left( f''(\alpha) + \frac{\varphi''(\alpha)}{\lambda_2 - \lambda_1} f'(\alpha) \right) \kappa^2 \\ P_3(f) &= \frac{1}{3} \left( f'''(\alpha) + \frac{3\varphi''(\alpha)}{\lambda_2 - \lambda_1} f''(\alpha) + \left( \frac{\varphi'''(\alpha)}{\lambda_3 - \lambda_1} + \frac{3(\varphi''(\alpha))^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \right) f'(\alpha) \right) \kappa^3 \end{aligned}$$

A natural question concerns the density of  $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$  in  $\text{Hol}(\mathbb{D})$ . The following proposition gives the answer.

**Proposition 3.5.** *The space  $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$  is dense in the Fréchet space  $\text{Hol}(\mathbb{D})$  if and only if  $\varphi$  is univalent.*

*Proof.* The function  $\varphi$  is univalent if and only if  $\kappa$  is univalent (see [23]). Thus univalence of  $\varphi$  is necessary for the density of  $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$ . Conversely, assume that  $\kappa$  is univalent. Then  $\Omega := \kappa(\mathbb{D})$  is a simply connected domain. It follows from Runge's theorem (see [18, Chap. 13 § 1 Section 2]) that the algebra  $\mathcal{A}^{(\Omega)}$  of all polynomials on  $\Omega$  is dense in  $\text{Hol}(\Omega)$ . Composition by  $\kappa$  shows that  $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$  is dense in  $\text{Hol}(\mathbb{D})$ .  $\square$

We consider two illustrations.

**Example 3.6.** (a) *Consider the univalent Schröder symbol  $\varphi(z) = \frac{z}{2-z}$ . The Koenigs eigenfunction is  $\kappa(z) = \frac{z}{1-z}$  and  $\Omega = \kappa(\mathbb{D}) = \{z \in \mathbb{C} : \Re(z) > -1/2\}$ .*  
 (b) *Let  $\varphi(z) = z \frac{z+1/2}{1+z/2}$  which satisfies  $\varphi(0) = 0$  and  $\varphi'(0) = 1/2$ . Since  $\kappa \circ \varphi(z) = \kappa(z)/2$ , it follows that  $\kappa(0) = 0 = \kappa(-1/2)$ , which obviously contradicts the density of  $\text{Span}\{\kappa^n : n \in \mathbb{N}_0\}$  in the Fréchet space  $\text{Hol}(\mathbb{D})$ .*

#### 4. THE SPECTRUM OF COMPOSITION OPERATORS ON $\text{Hol}(\mathbb{D})$

In this section we determine the spectrum of  $C_\varphi$  on the Fréchet space  $\text{Hol}(\mathbb{D})$ . We suppose throughout that  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic map,  $\varphi \notin \text{Aut}(\mathbb{D})$ , with an interior fixed point  $\varphi(\alpha) = \alpha \in \mathbb{D}$ , and that  $0 < |\varphi'(\alpha)| < 1$ . The case where  $\varphi'(\alpha) = 0$  is treated at the very end of this section. We let  $\lambda_n = \varphi'(\alpha)^n, n \in \mathbb{N}_0$ . By  $\sigma(C_\varphi)$  (resp.  $\sigma_p(C_\varphi)$ ) we denote the *spectrum* (resp. *point spectrum*) of  $C_\varphi$ , that is the set  $\{\lambda \in \mathbb{C} : \lambda \text{Id} - C_\varphi \text{ is not bijective}\}$  (resp.  $\{\lambda \in \mathbb{C} : \lambda \text{Id} - C_\varphi \text{ is not injective}\}$ ). Note that for  $\lambda \notin \sigma(C_\varphi)$ ,  $(\lambda \text{Id} - C_\varphi)^{-1}$  is a continuous linear operator on  $\text{Hol}(\mathbb{D})$  (by the closed graph theorem).

Since  $\varphi \notin \text{Aut}(\mathbb{D})$ , we already know that  $0 \in \sigma(C_\varphi)$ , by Corollary 2.4. Moreover, by Koenigs' theorem,

$$\sigma_p(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\}.$$

Now we show that the entire spectrum  $\sigma(C_\varphi)$  is equal to  $\sigma_p(C_\varphi) \cup \{0\}$ . This is surprising for several reasons. First of all, the operator  $C_\varphi$  is not compact in general (see Theorem 2.5). Nonetheless its spectral properties on  $\text{Hol}(\mathbb{D})$  are exactly those of a compact operator (see [27] for the Riesz theory for compact operators on Fréchet spaces which is the same as for Banach spaces). The other surprise comes from the well developed spectral theory of  $C_{\varphi|_X}$  for invariant Banach space  $X \hookrightarrow \text{Hol}(\mathbb{D})$ , which shows in particular that, on  $X$ , the spectrum is much larger in general (see Section 5).

**Theorem 4.1.** *One has*

$$\sigma(C_\varphi) = \{0\} \cup \{\varphi'(\alpha)^n : n \in \mathbb{N}_0\}.$$

In order to prove the surjectivity of  $C_\varphi - \lambda \text{Id}$  on  $\text{Hol}(\mathbb{D})$  for a complex number  $\lambda \notin \{0\} \cup \{\varphi'(\alpha)^n : n \in \mathbb{N}_0\}$ , we will use the following lemma.

**Lemma 4.2.** *Let  $\psi : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic,  $\psi \notin \text{Aut}(\mathbb{D})$ , such that  $\psi(0) = 0$ . Let  $g \in \text{Hol}(\mathbb{D})$  and  $\lambda \in \mathbb{C} \setminus \{0\}$ . Assume that there exist  $0 < \varepsilon < 1$  and  $f \in \text{Hol}(\varepsilon\mathbb{D})$  such that*

$$\lambda f - f \circ \psi = g \text{ on } \varepsilon\mathbb{D}.$$

*Then  $f$  has an extension  $\tilde{f} \in \text{Hol}(\mathbb{D})$  such that*

$$\lambda \tilde{f} - \tilde{f} \circ \psi = g \text{ on } \mathbb{D}.$$

*Proof.* Let  $\rho := \sup\{r \in [\varepsilon, 1] : f \text{ has an analytic extension on } r\mathbb{D}\}$ . We show that  $\rho = 1$ . Assume that  $\rho < 1$ . Then there exists  $\tilde{f} \in \text{Hol}(\rho\mathbb{D})$ , a holomorphic extension of  $f$ , satisfying:

$$\lambda \tilde{f} - \tilde{f} \circ \psi = g \text{ on } \varepsilon\mathbb{D}.$$

Since both sides are holomorphic, by the uniqueness theorem, the identity remains true on  $\rho\mathbb{D}$ . Note that by the Schwarz lemma  $\psi(r\mathbb{D}) \subset r\mathbb{D}$  for all  $0 < r < 1$ . It follows also from the Schwarz lemma that there exists  $\rho < \rho' \leq 1$  such that  $\psi(\rho'\mathbb{D}) \subset \rho\mathbb{D}$ . Indeed, otherwise we find  $(z_n)_n \in \mathbb{D}$ ,  $|z_n| \downarrow \rho$  such that  $|\psi(z_n)| > \rho$ . Taking a subsequence we may assume that  $z_n \rightarrow z$  and then  $|z| = \rho$  and  $|\psi(z)| \geq \rho$ . This is not possible since  $\psi$  is not an automorphism. Now, since

$$\lambda \tilde{f} = \tilde{f} \circ \psi + g \text{ on } \rho\mathbb{D},$$

and since  $\psi(\rho'\mathbb{D}) \subset \rho\mathbb{D}$ , it follows that  $f$  has a holomorphic extension to  $\rho'\mathbb{D}$ , a contradiction to the choice of  $\rho$ .  $\square$

*Proof of Theorem 4.1. First case:*  $\alpha = 0$ . Let  $\lambda \in \mathbb{C}$  and  $\lambda \notin \{0\} \cup \{\lambda_n : n \in \mathbb{N}_0\}$ . From Koenigs' theorem we know that  $\lambda \text{Id} - C_\varphi$  is injective. Thus we only have to prove the surjectivity. Let  $g \in \text{Hol}(\mathbb{D})$  and choose  $n \in \mathbb{N}_0$  such that  $|\lambda_{n+1}| < |\lambda|$ . Since by Corollary 3.3,  $\text{Hol}(\mathbb{D}) = \text{rg } Q_n \oplus \text{Hol}_n(0)$  we can write  $g = g_1 + g_2$  where  $g_1 \in \text{rg } Q_n$  and  $g_2 \in \text{Hol}_n(0)$ . Since  $C_{\varphi|_{\text{rg } Q_n}}$  is a diagonal operator and  $\lambda \notin \sigma(C_{\varphi|_{\text{rg } Q_n}})$ , there exists  $f_1 \in \text{rg } Q_n$  such that  $\lambda f_1 - f_1 \circ \varphi = g_1$ . Next we look at  $g_2$ . Choose  $|\lambda_1| < q < 1$  such that  $q^{n+1} < |\lambda|$ . Since  $\lim_{z \rightarrow 0} \frac{\varphi(z)}{z} = \lambda_1$ , there exists  $0 < \varepsilon \leq 1$  such that  $|\varphi(z)| \leq q|z|$  for  $|z| < \varepsilon$ . Consider the iterates  $\varphi_k := \varphi \circ \dots \circ \varphi$  ( $k$  times) of  $\varphi$ . Then  $|\varphi_k(z)| \leq q^k|z| \leq q^k\varepsilon$  for  $|z| < \varepsilon$ . Since  $g_2 \in \text{Hol}_n(0)$ , there exists  $B \geq 0$  such that

$$|g_2(z)| \leq B|z|^{n+1} \text{ for } |z| < \varepsilon.$$

Hence for  $k \in \mathbb{N}_0$ ,  $|z| < \varepsilon$ ,

$$\begin{aligned} \left| \frac{g_2(\varphi_k(z))}{\lambda^{k+1}} \right| &\leq \frac{1}{|\lambda|} B \frac{|\varphi_k(z)|^{n+1}}{|\lambda|^k} \\ &\leq \frac{1}{|\lambda|} B \frac{q^{k(n+1)}}{|\lambda|^k} \varepsilon \\ &\leq \frac{B\varepsilon}{|\lambda|} \left( \frac{q^{n+1}}{|\lambda|} \right)^k. \end{aligned}$$

Since  $\frac{q^{n+1}}{|\lambda|} < 1$ , the series  $f_0(z) := \sum_{k=0}^{\infty} \frac{g_2(\varphi_k(z))}{\lambda^{k+1}}$  converges uniformly on  $\varepsilon\mathbb{D}$  and defines a function  $f_0 \in \text{Hol}(\varepsilon\mathbb{D})$ . Moreover, since  $\varphi(\varepsilon\mathbb{D}) \subset \varepsilon\mathbb{D}$ ,

$$\begin{aligned} f_0(\varphi(z)) &= \sum_{k=0}^{\infty} \frac{g_2(\varphi_{k+1}(z))}{\lambda^{k+1}} \\ &= \lambda \sum_{k=1}^{\infty} \frac{g_2(\varphi_k(z))}{\lambda^{k+1}} \\ &= \lambda f_0(z) - g_2(z) \end{aligned}$$

on  $\varepsilon\mathbb{D}$ . It follows from Lemma 4.2 that  $f_0$  has a holomorphic extension  $f \in \text{Hol}(\mathbb{D})$  satisfying  $\lambda f - f \circ \varphi = g_2$ . This shows that  $\lambda \notin \sigma(C_\varphi)$  in the case  $\alpha = 0$ .

*Second case:*  $\alpha \in \mathbb{D}$ ,  $\alpha \neq 0$ . Consider the Möbius transform  $\psi_\alpha : \mathbb{D} \rightarrow \mathbb{D}$  defined by  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  and note that  $\psi_\alpha(0) = \alpha$  and  $\psi_\alpha = \psi_\alpha^{-1}$ . Then  $\tilde{\varphi} := \psi_\alpha \circ \varphi \circ \psi_\alpha$  maps  $\mathbb{D}$  into  $\mathbb{D}$  and satisfies  $\tilde{\varphi}(0) = 0$ . Since

$$C_{\tilde{\varphi}} = C_{\psi_\alpha} C_\varphi C_{\psi_\alpha} = C_{\psi_\alpha} C_\varphi C_{\psi_\alpha}^{-1},$$

the operators  $C_{\tilde{\varphi}}$  and  $C_{\varphi}$  are similar. From the first case we deduce that

$$\sigma(C_{\varphi}) \setminus \{0\} = \sigma(C_{\tilde{\varphi}}) \setminus \{0\} = \sigma_p(C_{\tilde{\varphi}}) \setminus \{0\} = \sigma_p(C_{\varphi}) \setminus \{0\} = \{\lambda_n : n \in \mathbb{N}_0\}.$$

□

For later purposes we extract the following lemma from the previous proof.

**Lemma 4.3.** *Let  $n \in \mathbb{N}_0$ ,  $\lambda \in \mathbb{C}$  such that  $|\lambda| > |\lambda_{n+1}|$ . Then for each  $g \in \text{Hol}_n(\alpha)$ , there exists a unique  $f \in \text{Hol}_n(\alpha)$  solving the inhomogeneous Schröder equation  $\lambda f - f \circ \varphi = g$ .*

*Proof.* Since  $\kappa^k \notin \text{Hol}_n(\alpha)$  for  $k \in \{0, 1, \dots, n\}$ , uniqueness follows from Koenigs' theorem. In order to prove existence, we only have to show that there exists  $f \in \text{Hol}(\mathbb{D})$  such that  $\lambda f - f \circ \varphi = g$ . Then, since  $Q_n g = 0$ ,  $f_1 = f - Q_n f \in \ker Q_n = \text{Hol}_n(\alpha)$  satisfies  $\lambda f_1 - f_1 \circ \varphi = g$  as well.

In the case  $\alpha = 0$  the existence of  $f$  follows from the proof of Theorem 4.1. So let  $\alpha \neq 0$ . Consider the Möbius transform  $\psi_{\alpha}$  defined in the proof of Theorem 4.1 and let  $h = g \circ \psi_{\alpha}$ . Then  $h \in \text{Hol}_n(0)$ . Indeed,  $h(0) = g(\alpha) = 0$ . Moreover, for  $l \in \{1, \dots, n\}$ , using Faà di Bruno's formula (see the proof of Theorem 3.2 for notations), we get:

$$\begin{aligned} h^{(l)}(0) &= (g \circ \psi)^{(l)}(\alpha) \\ &= \psi'_{\alpha}(\alpha)^l g^{(l)}(\alpha) + \sum_{\mathbf{m} \in K_n} C_{\mathbf{m}}^m g_2^{(\mathbf{m})}(\alpha) \\ &= 0. \end{aligned}$$

Consider  $\tilde{\varphi} = \psi_{\alpha} \circ \varphi \circ \psi_{\alpha}$ . Since  $\tilde{\varphi}(0) = 0$  we can apply the first case and find  $\tilde{f} \in \text{Hol}(\mathbb{D})$  such that  $\lambda \tilde{f} - \tilde{f} \circ \tilde{\varphi} = h$ . Then  $f := \tilde{f} \circ \psi_{\alpha} \in \text{Hol}(\mathbb{D})$  and

$$g = h \circ \psi_{\alpha} = \lambda f - \tilde{f} \circ \tilde{\varphi} \circ \psi_{\alpha} = \lambda f - f \circ \psi_{\alpha} \circ \tilde{\varphi} \circ \psi_{\alpha} = \lambda f - f \circ \varphi.$$

□

Our next aim is to show that each  $P_n$  is the spectral projection associated with  $\lambda_n$  for each  $n \in \mathbb{N}$ . We use the following definition.

**Definition 4.4.** *Let  $Y$  be a Fréchet space and  $S : Y \rightarrow Y$  linear and continuous.*

- (1) *A number  $\lambda \in \sigma(T)$  is called a Riesz point if  $\lambda$  is isolated and if there exists a decomposition  $Y = Y_1 \oplus Y_2$  in closed subspaces which are invariant by  $S$  such that:*

$$\dim Y_1 < \infty, \sigma(S|_{Y_1}) = \{\lambda\} \text{ and } (\lambda \text{Id} - S)|_{Y_2} \text{ is invertible.}$$

*It is not difficult to see that this decomposition is unique. The projection  $P : Y \rightarrow Y$  onto  $Y_1$  along this decomposition is called the spectral projection associated with the Riesz point  $\lambda$ .*

- (2)  $\sigma_e(T) := \{\lambda \in \sigma(T) : \lambda \text{ is not a Riesz point}\}$  is the essential spectrum and  $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$  is the essential spectral radius.
- (3)  $T$  is a Riesz operator if  $r_e(T) = 0$ .

**Remark 4.5.** (1) *If  $X$  is a Banach space, the existence of the decomposition as in (1) of Definition 4.4 for  $\lambda \in \sigma(T)$  implies that  $\lambda$  is an isolated point since the set of all invertible operators is open in  $\mathcal{L}(X)$ . This last property is no longer true if  $X$  is a Fréchet space (see Example 4.9).*

- (2) *If  $X$  is a Banach space, then an isolated point  $\lambda \in \sigma(T)$  is a Riesz point if and only if  $\lambda$  is a pole of the resolvent whose residuum  $P$  has finite rank. In that case  $P$  is the spectral projection. Note that*

$$P = \frac{1}{2i\pi} \int_{|\mu-\lambda|=\varepsilon} R(\mu, T) d\mu.$$

- (3) *See [8], in particular [8, Theorem 3.19] for other equivalent definitions of Riesz operators on Banach spaces.*

Let  $P_n$  be the rank-one projections defined in Theorem 3.2 where  $n \in \mathbb{N}_0$ .

**Theorem 4.6.** *The operator  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$  is a Riesz operator. Moreover, for each  $n \in \mathbb{N}_0$ , the spectral projection associated with  $\lambda_n$  is  $P_n$ .*

*Proof.* Let  $n \in \mathbb{N}_0$ . We show that  $\lambda_n$  is a Riesz point with spectral projection  $P_n$ . Since  $P_n C_\varphi = C_\varphi P_n = \lambda_n P_n$  and  $\text{rg } P_n = \mathbb{C}\kappa^n$ , it follows that the decomposition

$$\text{Hol}(\mathbb{D}) = \mathbb{C}\kappa^n \oplus \ker P_n$$

is invariant under  $C_\varphi$ . Moreover,  $\sigma(C_{\varphi|_{\mathbb{C}\kappa^n}}) = \{\lambda_n\}$ . Thus it suffices to show that  $(\lambda_n \text{Id} - C_\varphi)|_{\ker P_n}$  is bijective. Since  $\kappa^n \notin \ker P_n$  injectivity follows from Koenigs' theorem. In order to prove surjectivity, let  $g \in \ker P_n$ . Then, by Corollary 3.3,  $g = g_1 + g_2$  where  $g_1 \in \text{Span}\{\kappa^m : m = 0, \dots, n\} =: Z$ ,  $g_2 \in \text{Hol}_n(\alpha)$ . Since  $P_n g_1 = P_n g - P_n g_2 = 0$  and since  $C_{\varphi|_Z}$  is diagonal, there exists  $f_1 \in Z$  such that  $\lambda_n f_1 - f_1 \circ \varphi = g_1$ . Note that  $|\lambda_n| > |\lambda_{n+1}|$ . Thus it follows from Lemma 4.3 that there exists  $f_2 \in \text{Hol}(\mathbb{D})$  such that  $\lambda_n f_2 - f_2 \circ \varphi = g_2$ . Therefore  $f := f_1 + f_2$  solves  $\lambda_n f - f \circ \varphi = g$ . This shows that  $\lambda_n$  is a Riesz point and  $P_n$  is the associated spectral projection.  $\square$

We want to prove that a version of the formula in (2), Remark 4.5, remains true for the operator  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$ .

At first we deduce from [27, Lemma 3.2] that the following holds.

**Lemma 4.7.** *Let  $z \in \mathbb{D}$ ,  $f \in \text{Hol}(\mathbb{D})$ . The functions*

$$\lambda \mapsto ((\lambda \text{Id} - C_\varphi)^{-1}f)(z) : \mathbb{C} \setminus \sigma(C_\varphi) \rightarrow \mathbb{C}$$

*is holomorphic.*

This can also be seen directly from our proof of Theorem 4.1.

The following contour formula for  $P_n$  will be useful in Section 5.

**Lemma 4.8.** *Let  $n \in \mathbb{N}$ ,  $\varepsilon > 0$  such that  $\lambda_k \notin D(\lambda_n, 2\varepsilon)$  for all  $k \in \mathbb{N} \setminus \{n\}$ . Then, for all  $f \in \text{Hol}(\mathbb{D})$*

$$\frac{1}{2i\pi} \int_{|\lambda - \lambda_n| = \varepsilon} ((\lambda \text{Id} - C_\varphi)^{-1}f)(z) d\lambda = (P_n f)(z).$$

*Proof.* Write  $f = (\text{Id} - P_n)f + P_n f$ . The function

$$\lambda \mapsto ((\lambda \text{Id} - C_\varphi)^{-1}(\text{Id} - P_n)f)(z)$$

is holomorphic on  $D(\lambda_n, 2\varepsilon)$  whereas  $(\lambda \text{Id} - C_\varphi)^{-1}P_n f = \frac{1}{\lambda - \lambda_n} P_n f$ . From this the claim follows.  $\square$

The spectrum in a Fréchet space may be neither closed nor bounded. Indeed, here is an example of a composition operator on  $\text{Hol}(\mathbb{D})$ .

**Example 4.9.** *Let  $r \in (0, 1)$  and the automorphism*

$$\psi(z) = \frac{z + r}{1 + rz}.$$

*By Corollary 2.4,  $0 \notin \sigma(C_\psi)$  but  $\sigma_p(C_\psi) = \mathbb{C} \setminus \{0\}$  since for all  $\lambda \in \mathbb{C} \setminus \{0\}$  we have*

$$g_\lambda \circ \psi = \left( \frac{1+r}{1-r} \right)^\lambda g_\lambda \text{ where } g_\lambda(z) = \left( \frac{1+z}{1-z} \right)^\lambda.$$

*So, for  $\mu = se^{i\theta}$  with  $s > 0$  and  $\theta \in \mathbb{R}$ ,  $g_\lambda \circ \psi = \mu g_\lambda$  when*

$$\Re(\lambda) = \frac{\ln s}{\ln((1+r)/(1-r))} \text{ and } \Im(\lambda) = \frac{\theta + 2k\pi}{\ln((1+r)/(1-r))}, k \in \mathbb{Z}.$$

For this reason one defines a larger spectrum, the Waelbroeck spectrum  $\sigma_w(T)$  in the following way (see [27]).

Let  $T \in \mathcal{L}(\text{Hol}(\mathbb{D}))$ . Then

$$\mathbb{C} \setminus \sigma_w(T) := \{ \lambda \in \mathbb{C} \setminus \sigma(T) : \text{there exists a neighborhood } \mathcal{V} \text{ of } \lambda \text{ such that the family } ((\lambda \text{Id} - T)^{-1})_{\lambda \in \mathcal{V}} \text{ is bounded} \}.$$

Here a subset  $A$  of  $\text{Hol}(\mathbb{D})$  is called bounded if

$$\sup_{f \in A} \sup_{z \in K} |f(z)| < \infty$$

for all compact subsets  $K$  of  $\mathbb{D}$ .

From the proof of Theorem 4.1 one sees that, in our case,  $\sigma(C_\varphi) = \sigma_w(C_\varphi)$ . Now we can use Fréchet theory (see Theorem 3.11 in [27]). It tells us in particular that for the isolated point  $\lambda_n \in \sigma_w(C_\varphi)$ , there exists a unique projection  $R_n$  commuting with  $C_\varphi$  such that

$$\sigma(C_{\varphi|_{\text{rg } R_n}}) = \{\lambda_n\} \text{ and } \sigma(C_{\varphi|_{\ker R_n}}) = \sigma(C_\varphi) \setminus \{\lambda_n\}.$$

It is clear that  $R_n = P_n$ . Anyhow, we needed to define them differently since a priori it is not clear at all that  $\lambda_n$  is an isolated point.

Finally, we determine the spectrum of composition operators in the case where the symbol is not Schröder but has an interior fixed point.

**Theorem 4.10.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic,  $\alpha \in \mathbb{D}$  such that  $\varphi(\alpha) = \alpha$ . Assume that  $\varphi'(\alpha) = 0$ . Then*

$$\sigma(C_\varphi) = \sigma_w(C_\varphi) = \{0, 1\}.$$

*Proof.* Since  $\varphi'(\alpha) = 0$ ,  $\varphi \notin \text{Aut}(\mathbb{D})$  and thus,  $0 \in \sigma(C_\varphi)$  by Corollary 2.4. Since the constant functions are in the kernel of  $C_\varphi - \text{Id}$ ,  $1 \in \sigma_p(C_\varphi) \subset \sigma(C_\varphi)$ . By Theorem 1.1, for  $\lambda \notin \{0, 1\}$ ,  $C_\varphi - \lambda \text{Id}$  is injective. It remains to check that it is also surjective.

*First case:*  $\alpha = 0$ . Then  $\varphi_n(z) = z^{2^n} \tau_n(z)$  where  $\tau_n$  is a holomorphic self-map of the unit disc ( $\tau_n(\mathbb{D}) \subset \mathbb{D}$  follows from the Schwarz lemma). Let  $g \in \text{Hol}(\mathbb{D})$ ,  $g(z) = g(0) + g_1(z)$  where  $g_1 \in \text{Hol}(\mathbb{D})$  and  $g_1(z) = zg_2(z)$  with  $g_2 \in \text{Hol}(\mathbb{D})$ . Note that  $(\lambda \text{Id} - C_\varphi)(\frac{g(0)}{\lambda-1} \mathbf{1}) = g(0) \mathbf{1}$ . Moreover, the series

$$f(z) := \frac{1}{\lambda} \sum_{n \geq 0} \frac{g_1(\varphi_n(z))}{\lambda^n} = \frac{1}{\lambda} \sum_{n \geq 0} \frac{z^{2^n} \tau_n(z) g_2(\varphi_n(z))}{\lambda^n}$$

uniformly converges on every compact  $K \subset \mathbb{D} \cap \{|z| < \sqrt{|\lambda|}\}$  (since  $|z^{2^n}| \leq |z^{2^n}|$  for  $z \in \mathbb{D}$ ). Note also that  $\lambda f - f \circ \varphi = g_1$  on such  $K$ . The surjectivity of  $\lambda \text{Id} - C_\varphi$  follows from Lemma 4.2.

*Second case:*  $\alpha \neq 0$ . We proceed as in the proof of Theorem 4.1.  $\square$

## 5. SPECTRAL PROPERTIES ON ARBITRARY BANACH SPACES

In this section we study spectral properties of composition operators on arbitrary Banach spaces which are continuously injected in  $\text{Hol}(\mathbb{D})$ .

Throughout this section we assume that  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is holomorphic,  $\varphi \notin \text{Aut}(\mathbb{D})$  and that there exists  $\alpha \in \mathbb{D}$  such that  $\varphi(\alpha) = \alpha$  and



$\varphi'(\alpha) \neq 0$ ; i.e.  $\varphi$  is a Schröder function. By  $\kappa$  we denote Koenigs' eigenfunction. Let  $X$  be a Banach space such that  $X \hookrightarrow \text{Hol}(\mathbb{D})$  (which means that  $X$  is a subspace of  $\text{Hol}(\mathbb{D})$  and the injection is continuous; see [1] for equivalent formulations). Assume that  $C_\varphi X \subset X$  and define  $T : X \rightarrow X$  by  $T = C_{\varphi|_X}$ . Then  $T \in \mathcal{L}(X)$  by the closed graph theorem.

As before we will consider the spectral projections  $P_n$  on  $\text{Hol}(\mathbb{D})$  and let  $\lambda_n = \varphi'(\alpha)^n$ ,  $n \in \mathbb{N}_0$ . By Theorem 3.2,  $P_n f = \langle \Psi_n, f \rangle \kappa^n$ , where  $\Psi_n$  is a functional given by

$$\langle \Psi_n, f \rangle = \sum_{m=0}^n c_{nm} f^{(m)}(\alpha).$$

It follows from Theorem 3.2 (a) that,

$$\langle C_\varphi f, \Psi_n \rangle = \lambda_n \langle f, \Psi_n \rangle,$$

for all  $f \in \text{Hol}(\mathbb{D})$ . This implies that  $T' \Psi_{n|X} = \lambda_n \Psi_{n|X}$ . Thus, if  $\Psi_{n|X} \neq 0$ , then  $\lambda_n \in \sigma_p(T') \subset \sigma(T)$ . We note this as a first result.

**Proposition 5.1.** *Let  $n \in \mathbb{N}_0$ . Assume that  $\Psi_{n|X} \neq 0$ . Then*

$$\lambda_n \in \sigma_p(T') \subset \sigma(T).$$

The following corollary concerns all the classical Banach spaces  $X$  of holomorphic functions on the unit disc.

**Corollary 5.2.** *Assume that  $e_n \in X$  for all  $n \in \mathbb{N}_0$ . Then  $\lambda_n \in \sigma(T') \subset \sigma(T)$  for all  $n \in \mathbb{N}_0$ .*

*Proof.* We know that  $\Psi_n \neq 0$  for all  $n \in \mathbb{N}_0$ . Since the polynomials are dense in  $\text{Hol}(\mathbb{D})$ , it follows that  $\Psi_{n|X} \neq 0$ .  $\square$

It follows from the decomposition result, Corollary 3.3, that the  $\Psi_n$  separate points in  $\text{Hol}(\mathbb{D})$ , i.e. for  $f \in \text{Hol}(\mathbb{D})$ ,  $\langle \Psi_n, f \rangle = 0$  for all  $n \in \mathbb{N}_0$  implies  $f = 0$ .

**Corollary 5.3.** *If  $X \neq \{0\}$ , then  $r(T) > 0$ , where  $r(T)$  is the spectral radius of  $T$ .*

*Proof.* Since the  $\Psi_n$ ,  $n \in \mathbb{N}_0$ , separate the functions of  $\text{Hol}(\mathbb{D})$ , there exists  $n \in \mathbb{N}_0$  such that  $\Psi_{n|X} \neq 0$ . Hence  $\lambda_n \in \sigma(T)$  by Proposition 5.1.  $\square$

We need the following characterization of the finite dimension also for further arguments.

**Proposition 5.4.** *The following assertions are equivalent:*

- (i)  $0 \notin \sigma(T)$ ;

- (ii) *for only finitely many  $n \in \mathbb{N}_0$  one has  $\Psi_{n|X} \neq 0$ ;*
- (iii)  *$\exists J \subset \mathbb{N}_0$  finite such that  $X = \text{Span}\{\kappa^l : l \in J\}$ ;*
- (iv)  *$\dim X < \infty$ .*

*Proof.* (i)  $\Rightarrow$  (ii): Since  $\lambda_n \rightarrow 0$  as  $n \rightarrow \infty$ , this follows from Proposition 5.1.

(ii)  $\Rightarrow$  (iii): Let  $J := \{n \in \mathbb{N}_0 : \Psi_{n|X} \neq 0\}$ . It follows from Corollary 3.3 that the  $\Psi_n$ ,  $n \in \mathbb{N}_0$ , separate  $\text{Hol}(\mathbb{D})$ . Thus the mapping

$$f \mapsto (\langle \Psi_n, f \rangle)_{n \in J} : X \rightarrow \mathbb{C}^d,$$

with  $d = |J|$  is injective and linear. It follows that  $\dim X \leq d$ . It follows from Proposition 5.1 that  $\{\lambda_n : n \in J\} \subset \sigma_p(T)$ . Since all  $\lambda_n$  are different, it follows that  $\dim X \geq d$ . We have shown that  $\dim X = d$  and  $\sigma_p(T) = \{\lambda_n : n \in J\}$ . Now it follows from Koenigs' theorem that  $X = \text{Span}\{\kappa^l : l \in J\}$ .

(iii)  $\Rightarrow$  (iv) is trivial.

(iv)  $\Rightarrow$  (i): Since  $\dim X < \infty$ , by Koenigs' theorem,

$$\sigma(T) = \sigma_p(T) \subset \{\lambda_n : n \in \mathbb{N}_0\} \subset \mathbb{C} \setminus \{0\}.$$

□

We note the following corollary which will be useful later.

**Corollary 5.5.** *Assume that  $\dim X = \infty$ . Then there exist infinitely many  $n \in \mathbb{N}_0$  such that  $\lambda_n \in \sigma(T)$ .*

This follows from Proposition 5.1 and Proposition 5.4.

Next we show that each isolated point in the spectrum of  $T$  is necessarily a simple pole, and thus equal to some  $\lambda_n$ .

Recall that if  $\mu$  is an isolated point of the spectrum, for the resolvent  $R(\lambda, T)$  we have the Laurent development

$$R(\lambda, T) = \sum_{k=-\infty}^{\infty} A_k(\lambda - \mu)^k,$$

which is valid for  $0 < |\lambda - \mu| < \delta$  and  $\delta = \text{dist}(\mu, \sigma(T) \setminus \{\mu\})$ . Here  $A_k \in \mathcal{L}(X)$  are the coefficients and  $A_{-1} = P$  is the spectral projection associated with  $\mu$ . Thus the spectral projection is equal to the residuum.

One says that  $\mu \in \sigma(T)$  is a *simple pole* if it is isolated in  $\sigma(T)$  and if  $\dim(\text{rg } P) = 1$ . This implies that  $A_k = 0$  for  $k \leq -2$ . Moreover  $\text{rg } P = \ker(\mu \text{Id} - T)$  and  $PT = TP = \mu P$ .

**Theorem 5.6.** *Let  $\mu \in \mathbb{C} \setminus \{0\}$  be an isolated point of  $\sigma(T)$ . Then there exists  $n \in \mathbb{N}_0$  such that  $\mu = \lambda_n$  and  $\mu$  is a simple pole. Moreover  $P_n X \subset X$  and  $P_{n|X}$  is the spectral projection associated with  $\mu$ . Here  $P_n$  is the projection from Theorem 3.2.*

*Proof.* Assume that  $\mu \notin \{\lambda_n : n \in \mathbb{N}_0\}$ ,  $\lambda_n = \varphi'(\alpha)^n$ . Let  $\varepsilon > 0$  such that  $D(\mu, 2\varepsilon) \cap \{\lambda_n : n \in \mathbb{N}_0\} = \emptyset$ . Denote by

$$P = \frac{1}{2i\pi} \int_{|\lambda - \mu| = \varepsilon} (\lambda \text{Id} - T)^{-1} d\lambda$$

the spectral projection associated with  $\mu$ . Let  $f \in X, z \in \mathbb{D}$ . Since for  $|\lambda - \mu| = \varepsilon$ ,  $\lambda \in \rho(C_\varphi) \cap \rho(T)$ , one has:

$$((\lambda \text{Id} - T)^{-1} f)(z) = ((\lambda \text{Id} - C_\varphi)^{-1} f)(z),$$

it follows from Lemma 4.7 and Cauchy's theorem that  $(Pf)(z) = 0$ . Since  $f \in X, z \in \mathbb{D}$  are arbitrary, it follows that  $P = 0$ , a contradiction.

Thus  $\mu = \lambda_{n_0}$  for some  $n_0 \in \mathbb{N}$ . Let  $\varepsilon > 0$  such that  $\lambda_n \notin D(\lambda_{n_0}, 2\varepsilon)$  for all  $n \neq n_0$ . Let

$$P = \frac{1}{2i\pi} \int_{|\lambda - \lambda_{n_0}| = \varepsilon} (\lambda \text{Id} - T)^{-1} d\lambda$$

be the spectral projection. It follows from Lemma 4.8 that  $P = P_{n_0}$ . Thus  $P$  has rank 1 and this means by definition that  $\mu = \lambda_{n_0}$  is a simple pole.  $\square$

**Remark 5.7.** *In [1] it was proved that each pole is necessarily simple. Now we know more: each isolated point in the spectrum is a simple pole.*

We note the following consequence of Theorem 5.6

**Corollary 5.8.** *If the spectrum of  $T = C_{\varphi|X}$  is countable, then  $\sigma(T) \subset \{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\}$ .*

*Proof.* Let  $\mathcal{U}$  be an open neighborhood of  $\{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\}$  and  $K := \sigma(T) \setminus \mathcal{U}$ . Then  $K$  is compact and countable. If  $K \neq \emptyset$ , then, by Baire's theorem,  $K$  has an isolated point. This is impossible by Theorem 5.6. Thus  $K = \emptyset$ . Since  $\mathcal{U}$  is arbitrary, the claim follows.  $\square$

Our next aim is to describe the connected component of 0 in  $\sigma(T)$ . Assume that  $X \hookrightarrow \text{Hol}(\mathbb{D})$  is invariant under  $C_\varphi$  and let  $T = C_{\varphi|X}$ , as before, where  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is the given Schröder map. Let us assume that  $\dim X = \infty$ . Then  $0 \in \sigma(T)$  and the set

$$J := \{n \in \mathbb{N}_0 : \Psi_{n|X} \neq 0\}$$

is infinite by Proposition 5.4

We let  $\lambda_n = \varphi'(\alpha)^n$  where  $\alpha$  is the interior fixed point of  $\varphi$ . By Proposition 5.1,  $\lambda_n \in \sigma(T)$  for  $n \in J$ . Moreover, let

$$J_0 := \{n \in \mathbb{N}_0 : \lambda_n \text{ is an isolated point of } \sigma(T)\}.$$

We know that for  $n \in J_0$ ,  $\kappa^n \in X$  and  $T\kappa^n = \lambda_n\kappa^n$ .

Our main result in this section is the following quite precise description of the spectrum of  $T$ .

**Theorem 5.9.** *Assume that  $\dim X = \infty$ . Denote by  $\sigma_0(T)$  the connected component of 0 in  $\sigma(T)$ . Then*

$$\sigma(T) = \sigma_0(T) \cup \{\lambda_n : n \in J_0\}.$$

*In particular,  $\sigma_0(T)$  is the essential spectrum of  $T$ .*

Of course it may happen that  $J_0 = \emptyset$ . This is the case if and only if  $\sigma(T)$  is connected.

For the proof, we need the following.

**Lemma 5.10.** *Let  $\sigma_1$  be an open and closed subset of  $\sigma(T)$ . If  $0 \notin \sigma_1$ , then there exists a finite set  $J_1 \subset J_0$  such that*

$$\sigma_1 = \{\lambda_n : n \in J_1\}.$$

*Proof.* Assume that  $0 \notin \sigma_1$ . Let  $\Gamma$  be a rectifiable Jordan curve such that  $\sigma_1 \subset \text{int } \Gamma$  and  $\sigma(T) \setminus \sigma_1 \subset \text{ext } \Gamma$ . We can choose  $\Gamma$  such that  $\lambda_n \notin \Gamma$  for all  $n \in \mathbb{N}_0$ , and  $0 \in \text{ext } \Gamma$ . Consequently  $J_2 := \{n \in \mathbb{N}_0 : \lambda_n \in \text{int } \Gamma\}$  is finite. Denote by

$$P = \frac{1}{2i\pi} \int_{\Gamma} R(\lambda, T) d\lambda$$

the spectral projection with respect to  $\sigma_1$ . Let  $Y := PX$ . Then  $TY \subset Y$  and  $\sigma(T|_Y) = \sigma_1$ . Let  $f \in X, z \in \mathbb{D}$ . Then

$$(R(\lambda, T)f)(z) = ((\lambda \text{Id} - C_{\varphi})^{-1}f)(z)$$

for all  $\lambda \in \Gamma$ . Now we choose  $\varepsilon > 0$  small enough and deduce from Lemma 4.7 and 4.8 that

$$\begin{aligned}
(Pf)(z) &= \frac{1}{2i\pi} \int_{\Gamma} (R(\lambda, T)f)(z) d\lambda \\
&= \frac{1}{2i\pi} \int_{\Gamma} ((\lambda \text{Id} - C_{\varphi})^{-1}f)(z) d\lambda \\
&= \sum_{k \in J_2} \frac{1}{2i\pi} \int_{|\lambda - \lambda_k| = \varepsilon} ((\lambda \text{Id} - C_{\varphi})^{-1}f)(z) d\lambda \\
&= \sum_{k \in J_2} \langle \Psi_k, f \rangle \kappa^k(z) \\
&= \sum_{k \in J_1} \langle \Psi_k, f \rangle \kappa^k(z),
\end{aligned}$$

where  $J_1 := J_2 \cap J$ .

Since  $T'\Psi_k|_X = \lambda_k \Psi_k|_X$  for all  $k \in J_1$ , and since the  $\lambda_k$  are all different, it follows that the  $\Psi_k$ ,  $k \in J_1$ , are linearly independent. Consequently we find  $f_l \in X$  such that  $\langle \Psi_k, f_l \rangle = \delta_{kl}$  for  $k, l \in J_1$ . It follows that the  $\kappa^k$ ,  $k \in J_1$ , form a basis of  $Y$  consisting of eigenvectors of  $T$ ,  $T\kappa^k = \lambda_k \kappa^k$ ,  $k \in J_1$ . Thus

$$\sigma_1 = \sigma(T|_Y) = \{\lambda_k : k \in J_1\}.$$

□

*Proof of Theorem 5.9.* Let  $K = \sigma(T) \setminus \{\lambda_n : n \in J_0\}$ . Then  $K$  is compact and  $0 \in K$ . It suffices to show that  $K$  is connected.

Let  $\sigma_1 \subset K$  be open and closed. We have to show that  $\sigma_1 = \emptyset$  or  $\sigma_1 = K$ .

*First case:*  $0 \notin \sigma_1$ .

We show that  $\sigma_1$  is open in  $\sigma(T)$ . Let  $z_0 \in \sigma_1$ . Then  $z_0 \neq 0$  and  $z_0 \neq \lambda_n$  for all  $n \in J_0$ . Thus there exists  $\varepsilon > 0$  such that  $z \neq \lambda_n$  for all  $n \in J_0$  if  $|z - z_0| < \varepsilon$  and, if  $z \in K$ , then  $z \in \sigma_1$ . Hence also  $D(z_0, \varepsilon) \cap \sigma(T) \subset \sigma_1$ . This proves that  $\sigma_1$  is open in  $\sigma(T)$ . It is trivially closed. By Lemma 5.10 there exists a finite set  $J_1 \subset J_0$  such that  $\sigma_1 \subset \{\lambda_n : n \in J_1\}$ . Since  $\sigma_1 \subset K$ , it follows that  $\sigma_1 = \emptyset$ .

*Second case:*  $0 \in \sigma_1$ .

Then  $K \setminus \sigma_1 = \emptyset$  by the first case. Thus  $\sigma_1 = K$ .

By Corollary 5.5, the point 0 is not isolated in  $\sigma(T)$ . Thus  $0 \in \sigma_e(T)$ . It follows that  $\sigma_0(T) \subset \sigma_e(T)$ . Since (by Theorem 5.6)  $\sigma(T) \setminus \sigma_0(T)$  consists of Riesz points, we conclude that  $\sigma_0(T) = \sigma_e(T)$ . □

Of course, it can happen that  $\sigma_0(T) = \{0\}$ . Here is a situation where it is bigger.

**Corollary 5.11.** *Let  $n_0 \in \mathbb{N}_0$ . Assume that  $\lambda_{n_0} \in \sigma(T)$  but  $\kappa^{n_0} \notin X$ . Then  $\lambda_{n_0} \in \sigma_0(T)$ . In particular  $r_e(T) \geq |\lambda_{n_0}|$ .*

*Proof.* It follows from Theorem 5.6 that  $\lambda_{n_0}$  is not an isolated point of  $\sigma(T)$ . Thus  $\lambda_{n_0} \notin \{\lambda_n : n \in J_0\}$ . Hence  $\lambda_{n_0} \in \sigma_0(T)$ .  $\square$

We cannot expect in the general situation we are considering here to prove more precise results on the geometry of  $\sigma_0(T)$ . However, for several concrete Banach spaces  $X$ , it is known that  $\sigma_0(T)$  is a disc. Moreover one can estimate the essential radius  $r_e(T) := \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$  by knowing whether  $\kappa^n \in X$  or  $\kappa^n \notin X$ . We explain this in the following examples.

**Example 5.12.** (1) *Let  $X = H^2(\mathbb{D})$  and let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a Schröder map with fixed point  $\alpha \in \mathbb{D}$  and Koenigs eigenfunction  $\kappa$ . Then  $C_\varphi(X) \subset X$ . Let  $T = C_{\varphi|_X}$ . Then it is known that  $\sigma_0(T) (= \sigma_e(T))$  is a disc (see [6, Theorem 7.30]). A question which has been investigated is for which  $n$  the eigenfunction  $\kappa^n$  of  $C_\varphi$  lies in  $X$ , which means, for which  $n$  actually  $\lambda_n = \varphi'(\alpha)^n \in \sigma_p(T)$ . This is related to the essential spectral radius  $r_e(T)$ .*

(a) *One has  $\sigma_0(T) = \{0\}$  if and only if  $\kappa^n \in X$  for all  $n \in \mathbb{N}_0$  (see [23, Section 6]). To say that  $\sigma_0(T) = \{0\}$  is the same as saying that  $T$  is a Riesz operator. We had seen in Section 4 that  $C_\varphi$  is always a Riesz operator on  $\text{Hol}(\mathbb{D})$ . So the situation is very different if we restrict  $C_\varphi$  to a Banach space.*

(b) *Let  $n \in \mathbb{N}$ . Then  $\kappa^n \in X$  if and only if  $|\lambda_n| > r_e(T)$ .*

*Proof.* If  $|\lambda_n| > r_e(T)$ , then  $\lambda_n \in \sigma_p(T)$ . Thus  $\kappa^n \in X$  by Koenigs' theorem (Theorem 1.1). The converse implication follows from deep results by Poggi-Corradini [17] and Bourdon–Shapiro [3]. We follow the survey article [23] by Shapiro. Assume that  $\kappa^n \in X$ . Then  $\kappa \in H^{2n}(\mathbb{D})$ . Using the notation of [23, Section 7], this implies that  $h(\kappa) \geq 2n$ . By [23, (12) in Section 6], one has  $r_e(T) = |\lambda_1|^{h(\kappa)/2}$ . Since  $|\lambda_1| < 1$  this implies  $r_e(T) \leq |\lambda_1|^n = |\lambda_n|$ . We need the strict inequality. Assume that  $|\lambda_n| = r_e(T)$ . Then  $h(\kappa) = 2n$ . By the "critical exponent result" in [23, Section 8], this implies that  $\kappa^n \in X$ . thus  $|\lambda_n| > r_e(T)$ .  $\square$

- (c) *The result of (b) can be reformulated by saying that there are no "hidden eigenvalues" besides possibly  $\lambda_0$ . More precisely, if  $\lambda_n$  is an eigenvalue, then  $\lambda_n \notin \sigma_e(T)$ . For  $\lambda_0 = 1$  the situation is different. In Example 3.6 an inner function  $\varphi$  is defined which is Schröder and 0 at 0. Thus  $T = C_{\varphi|_{H^2}}$  is isometric and non invertible (see [2, Section 1] for further informations on isometric composition operators on various Banach spaces). Hence  $\sigma(T) = \overline{\mathbb{D}} = \sigma_e(T)$ , and thus the eigenvalue  $\lambda_0 = 1$  is in the essential spectrum.*
- (2) *Also on some weighted Hardy spaces the essential spectrum is a disc. However it can happen that  $\lambda_{n+1} < r_e(T) < \lambda_n$  for some  $n$ . In fact Hurst [10] considered Schröder symbols  $\varphi$  which are linear fractional maps with a fixed point  $\alpha \in \mathbb{D}$  such that  $\varphi'(\alpha) \in (0, 1)$  and a second fixed point of modulus one. The Banach spaces on which the composition operators are defined are the weighted Hardy spaces*

$$H^2(\beta) := \left\{ f(z) = \sum_{n \geq 0} a_n z^n : \|f\|^2 = \sum_{n \geq 0} |a_n|^2 \beta(n)^2 < \infty \right\}$$

where  $\beta(n) = (n+1)^a$ ,  $a < 0$ . Recall that for  $a = -1/2$ , the Banach space is the classical Bergman space  $\mathcal{B}$ . In this case the spectrum is

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \varphi'(\alpha)^{(2|a|+1)/2}\} \cup \{\varphi'(\alpha)^n : n \in \mathbb{N}_0\},$$

and  $\kappa^p \in H^2(\beta)$  if and only if  $p < |a| + 1/2$ . For  $a = -1$ , it follows that  $\lambda_2 < r_e(T) < \lambda_1$ , whereas, for  $a = -1/2$ , we get  $r_e(T) = \lambda_1$ .

**Remark 5.13.** *By Proposition 5.1 we have seen that the composition operators associated with a Schröder symbol  $\varphi$  are not quasilinear on a large class of Banach spaces of holomorphic functions. Note that if  $\varphi$  has a fixed point  $\alpha \in \mathbb{D}$  such that  $\varphi'(\alpha) = 0$ , the description of the spectrum of  $T := C_{\varphi|_X}$  may be very different. For example, if  $\varphi(z) = z^2$  and  $X = zH^2(\mathbb{D})$ , then the spectrum of  $T$  is the closed unit disc ( $T$  is a non-invertible isometry) whereas for  $X = z\mathcal{B}$ ,  $T$  is quasilinear (see [2, Theorem 2.9]).*

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