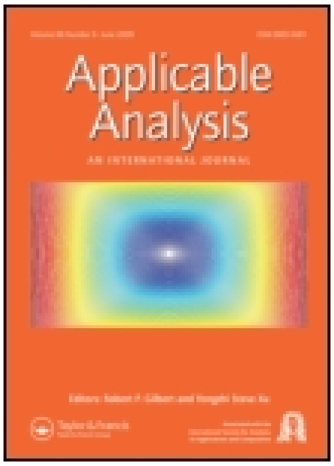

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Continuous newton method and its modification

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Continuous Newton Method and its Modification

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Abstract

The Continuous Newton method (CNM) and its modification (MCNM) are studied. MCNM permits one to replace the inversion of the derivative operator at every step of iterations by its inversion only at the initial approximation point. Then the extended system of the differential equations in Hilbert space, introduced in this paper, permits the realization of the iterative process with the simultaneous calculation of the inverse derivative operator. For CNM and MCNM the theorems establishing convergence with exponential rate are proved. The convergence theorem for MCNM is proved under almost the same conditions as for CNM.

AMS: 65J15, 58C15, 47H17

KEY WORDS: Continuous Newton method, Fréchet derivative, exponential convergence, integral inequality.

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1 Introduction

Let φ be some nonlinear operator from one Banach space to another. (Some results below are obtained under the assumption that φ is a nonlinear operator from a Hilbert space H to itself.) We consider a nonlinear equation

$$\varphi(x) = 0. \tag{1.1}$$

In the well-known Newton's method ([8, 6, 9]) one constructs a sequence $\{x_n\}$ for $n = 0, 1, \dots$ which converges to a solution of equation (1.1). The first term x_0 is an initial approximation point and the other terms are constructed by means of the following iterative process:

$$x_{k+1} = x_k - \gamma_k \varphi'(x_k)^{-1} \varphi(x_k), \tag{1.2}$$

where $\varphi'(x)$ is the Fréchet derivative of the operator φ and $\{\gamma_k\}$ is some sequence of positive numbers. According to the standart terminology one calls this method the damped Newton's method while the Newton's method corresponds to $\gamma_k = 1$ for all k.

Continuous Newton method is an alternative approach to the solution of (1.1). Instead of (1.2) one considers the following Cauchy problem for the nonlinear differential equation ([7, 14]):

$$\dot{x}(t) = -\varphi'(x(t))^{-1} \varphi(x(t)), \quad 0 \le t < \infty, \quad x(0) = x_0. \tag{1.3}$$

If one applies Euler's method with the step γ to (1.3) one gets the damped Newton's method. Continuous analogs of iterative methods have several advantages. Convergence theorems for continuous methods can usually be obtained easier. If a convergence theorem is proved for a continuous method, that is, for the Cauchy problem for a differential equation, for instance (1.3), one can construct various finite difference schemes for the solution of this Cauchy problem. These difference schemes give discrete methods for the solution of equation (1.1). For instance the methods of Euler and Runge-Kutta can be used. More information about the applications and modifications of continuous Newton methods can be found in ([7, 14, 4]).

In this paper continuous analogs of NM and some its modifications are studied and corresponding convergence theorems are proved. The sufficient conditions of convergence theorems for NM and CNM are much more restrictive than nessesary ones. The numerous applications of Newton's methods for numerical solution of nonlinear problems show that very often these methods work stably even if the initial approximation point is not close to the solution to equation (1.1) despite a violation of the assumptions of convergence theorems. By the same reason it is difficult to compare different iterative schemes or their continuous analogs from the point of view of convergence theorems. Thus a development of various methods for solving equation (1.1) is a problem of a practical interest.

A principal point of the realization of NM and CNM is the inversion of the operator $\varphi'(x)$. Such inversion is often difficult and could be a reason for a decreasing accuracy of calculations. To avoid this difficulty, some modifications of NM and CNM have been developed. The most well known of them is simplified NM of L. Kantorovich. In this method the operator of inverse derivative ${\varphi'}^{-1}(x_k)$ in (1.3) is replaced by ${\varphi'}^{-1}(x_0)$:

$$x_{k+1} = x_k - \varphi'(x_0)^{-1}\varphi(x_k). \tag{1.4}$$

A combination of CNM with the method of parameter variation ([5]) has been used in [10]. A similar algorithm, but in a framework of NM, has been earlier investigated in [13].

Below a different method (MCNM) of realization of a continuous process is developed (see also [2]). The main idea of the method is the construction of a process which converges to the solution to the nonlinear equation $\varphi(x) = 0$ with a simultaneous inversion of derivative operator $\varphi'(x(t))$. Such a construction permits one to replace the inversion of the derivative operator $\varphi'(x)$ on every step of iterations by its inversion only in the initial approximation point x_0. Iterative process "keeps" determined initial ${\varphi'}^{-1}(x_0) \sim Y_0 = {\varphi'}^{-1}(x_0)$, since $Y(x)$ remains nearby ${\varphi'}^{-1}(x)$ during the whole process. The convergence theorem for MCNM is proved under almost the same conditions as for CNM. At the same time the realizations of MCNM for the solution to a few difficult nonlinear problems have shown its practical efficiency. (The applications of MCNM are not included in this paper, but can be found in [1], [2], [3]).

The paper is organized as follows. In Sect. 2 the convergence theorems for CNM are formulated. In Sect. 3 MCNM is described and the convergence theorem is formulated. The convergence theorems are proved in Sect. 4.

2 Continuous Analogs of Newton Method

Let B_1 and B_2 be Banach spaces and $L(B_2, B_1)$ a space of linear bounded operators from B_2 to B_1. Denote by x_0 some initial approximation to the solution of (1.1) and by $B(x_0, r)$ the ball: $\{x \in B_1; ||x - x_0|| < r\}$. Denote also

$$\alpha_0 := ||[\varphi'(x_0)]^{-1}\varphi(x_0)||, \beta_0 := ||[\varphi'(x_0)]^{-1}||.$$

Condition A. *There exist some positive numbers $r \geq 2\alpha_0$ and k such that*

1. $\varphi : B(x_0, r) \to B_2$ is C^1-map and for every $x \in B(x_0, r)$ there exists Gâteaux derivative $\varphi''(x, \xi_1, \xi_2)$[1]*;*

2. $||\varphi''(x, \xi_1, \xi_2)|| \leq k||\xi_1||\cdot||\xi_2||$ for every $x \in B(x_0, r)$ and every $\xi_1, \xi_2 \in B_1$;

3. $\varphi'(x_0)^{-1} \in L(B_2, B_1)$, and

$$2k\alpha_0\beta_0 < 1. \tag{2.1}$$

Now let us consider the following Cauchy problems in B_1:

$$\dot{x}(t) = -\varphi'(x(t))^{-1}\varphi(x(t)), \quad 0 \leq t < \infty, \quad x(0) = x_0. \tag{2.2}$$

Definition 2.1 *Let us say that $x(t)$ converges to x^* exponentially if there exist positive constants c_1 and c_2 such that*

$$||x(t) - x^*|| \leq c_1 e^{-c_2 t}. \tag{2.3}$$

Theorem 2.2 *If Condition A holds then*

1) there exists a solution $x = x(t)$, $t \in [0, \infty)$ of the problem (2.2) and $x(t) \in B(x_0, r)$ for all $t \in [0, +\infty)$;

2) there exists

$$\lim_{t \to +\infty} x(t) = x^*, \tag{2.4}$$

x^* is the unique solution of the problem (1.1) in $B(x_0, 2\alpha_0)$, and $x(t)$ converges to x^* exponentially.

[1]For $x \in B_1$ Gâteaux derivative $\varphi''(x, \xi_1, \xi_2)$ is a bilinear operator on B_1 such that $\varphi'(x+\xi_1)\xi_2 - \varphi'(x)\xi_2 = \varphi''(x, \xi_1, \xi_2) + \eta\xi_2$, and $||\eta||\cdot||\xi_1||^{-1} \longrightarrow 0$ for $\xi_1 \longrightarrow 0, ||\xi_1|| > 0$.

Remark 2.3 *In contrast to the work [7], where the convergence of CNM is proved under the assumption that* $||[\varphi'(x)]^{-1}||$ *is boundedly invertible for all* $x \in B(x_0, r)$, *in Theorem 2.2 we assume only that* $||[\varphi'(x_0)]^{-1}|| \leq \beta_0 < \infty$, *and then prove that* $||[\varphi'(x)]^{-1}||$ *is uniformly bounded for* $x \in B(x_0, r)$ *(see estimate (4.1) below).*

Condition B. *There exist some positive numbers* r, γ, β, δ *and mapping* $A : B(x_0, r) \to L(B_1, B_2)$ *such that*

1. $\varphi : B(x_0, r) \to B_2$ *is* C^1*-map,*

2. for every $x \in B(x_0, r)$ *there exists* $A^{-1}(x)$,

$$||\varphi'(x) - A(x)|| \leq \delta \text{ and } ||A^{-1}(x)|| \leq \beta, \tag{2.5}$$

3. $A^{-1}(x)$ *is a Lipschitz mapping from* $B(x_0, r)$ *to* $L(B_2, B_1)$,

4. the following inequalities hold

$$\beta(\delta + \gamma) < 1 \text{ and } r \geq \frac{\beta||\varphi(x_0)||}{1 - \beta(\delta + \gamma)}, \tag{2.6}$$

where

$$\gamma := \sup_{x_1, x_2 \in B(x_0, r)} ||\varphi'(x_1) - \varphi'(x_2)||. \tag{2.7}$$

Now let us consider the following Cauchy problems in B_1:

$$\dot{x}(t) = -[A(f(x))]^{-1}\varphi(x), \quad 0 \leq t < \infty, \quad x(0) = x_0. \tag{2.8}$$

where $f : B(x_0, r) \to B(x_0, r)$ is an arbitrary Lipschitz mapping.

Theorem 2.4 *If Condition B holds then*

1) there exists a solution $x = x(t)$, $t \in [0, \infty)$ *of the problem (2.8) and* $x(t) \in B(x_0, r)$ *for all* $t \in [0, +\infty)$;

2) there exists

$$\lim_{t \to +\infty} x(t) = x^*, \tag{2.9}$$

x^* *is the unique solution of the problem (1.1) in* $B(x_0, r)$ *and* $x(t)$ *converges to* x^* *exponentially.*

Remark 2.5 *For instance if* $f(x) = x_0$ *for* $x \in B(x_0, r)$ *and* $A = \varphi'(x)$ *then (2.8) turns into continuous analog of simplified Newton method:*

$$x'(t) = -[\varphi'(x_0)]^{-1}\varphi(x(t)), \quad 0 \leq t < \infty, \quad x(0) = x_0. \tag{2.10}$$

The convergence of the simplified Newton method is also established under the following condition.

Condition C. *There exist some positive numbers* r *and* k *such that*

1. $\varphi : B(x_0, r) \to B_2$ *is a* C^1*-map.*

2. $||\varphi'(x_1) - \varphi'(x_2)|| \leq k||x_1 - x_2||$ *for every* $x_1, x_2 \in B(x_0, r)$,
3. $\varphi'(x_0)^{-1} \in L(B_1, B_2)$,

$$2k\alpha_0\beta_0 < 1, \text{ and } \quad \frac{2\alpha_0}{1+\sqrt{1-2k\alpha_0\beta_0}} \leq r. \tag{2.11}$$

Theorem 2.6 *If Condition C holds then*

1) there exists a solution $x = x(t), \quad t \in [0, \infty)$ *of the problem (2.10) and* $x(t) \in B(x_0, r)$ *for all* $t \in [0, +\infty)$;

2) there exists

$$\lim_{t \longrightarrow +\infty} x(t) = x^*, \tag{2.12}$$

x^* *is the unique solution of the problem (1.1) in* $B(x_0, 2\alpha_0)$ *and* $x(t)$ *converges to* x^* *exponentially.*

Remark 2.7 *Theorems 2.2, 2.4, 2.6 are continuous analogs of convergence theorems for NM and simplified NM ([6], [9]).*

Let H be a real or a complex Hilbert space, $L(H)$ be the space of linear operators in H, and A^* be the adjoint of A for any $A \in L(H)$. Let us consider the equation (1.1) for $\varphi : H \longrightarrow H$.

Condition D. *There exist some positive real numbers* r, c *and continuous operator-valued function* $F(x)$ *from* $B(x_0, r)$ *to* $L(H)$ *such that*

1) there exist Fréchet derivative $\varphi'(x)$ *and Gâteaux derivatives* $\varphi''(x)$ *and* $F'(x)$, *and they are locally bounded in* $B(x_0, r)$;

2) for every $x \in B(x_0, r)$ *the spectrum of the operator* $\varphi'(x)F(x) + (\varphi'(x)F(x))^*$ *belongs to the half-line* $[2c, +\infty)$;

3) for any $x \in B(x_0, r)$

$$||\varphi(x_0)|| \cdot ||F(x)|| \leq rc. \tag{2.13}$$

Now let us consider the following Cauchy problem in H:

$$\dot{x}(t) = -F(x(t))\varphi(x(t)), \quad 0 \leq t < \infty, \quad x(0) = x_0. \tag{2.14}$$

Theorem 2.8 *If Condition D holds then*

1) there exists a solution $x = x(t), \quad t \in [0, \infty)$ *of the problem (2.14) and* $x(t) \in B(x_0, r)$ *for all* $t \in [0, +\infty)$;

2) there exists

$$\lim_{t \longrightarrow +\infty} x(t) = x^*, \tag{2.15}$$

x^* *is the solution of the problem (1.1) and* $x(t)$ *converges to* x^* *exponentially.*

Remark 2.9 *a) If* $F(x) \equiv [\varphi'(x)]^{-1}$ *then we get CNM. In this case* $c = 1$ *and Theorem 1 yields the convergence theorem for CNM([7]);*

b) $F(x) = [\varphi'(x)]^*$ *corresponds to the gradient method.*

3 CNM with Simultaneous Inversion of φ' (MCNM)

Here we also consider a nonlinear operator φ from a Hilbert space H to itself. Our goal is to develop the Continuous Newton Method with Simultaneous Inversion of φ'.

We consider the following system:

$$\begin{cases} \varphi(x) = 0 \\ \varphi'(x)Y - I = 0, \end{cases} \tag{3.1}$$

where $Y \in L(H)$ and I is an identity operator. Let Y_0 be an approximation to $\varphi'(x_0)^{-1}$ and ρ a positive number.

Instead of (2.14), we consider the problem

$$\begin{cases} \dot{x}(t) = -Y(t)\varphi(x(t)), \\ \dot{Y}(t) = -\rho^2((\varphi'(x)^*\varphi'(x)Y(t) + Y(t)\varphi'(x)(\varphi'(x))^*) + 2\rho^2(\varphi'(x))^*, \end{cases}$$
$$x(0) = x_0, \quad Y(0) = Y_0. \tag{3.2}$$

Condition E. *There exist some positive real numbers r and ϵ such that*

1) there exist Fréchet derivative $\varphi'(x)$ and Gâteaux derivative $\varphi''(x)$ in $B(x_0, r)$ and

$$||\varphi''||_r = \sup_{x\in B(x_0,r)} \sup_{\xi\in H, ||\xi||=1} ||\varphi''(x)\xi||_{L(H)} < \infty,$$

($\varphi''(x)$ is a bilinear operator from H to itself for fixed $x \in B(x_0, r)$, and $\varphi''(x)\xi$ is a linear operator from H to itself for any $\xi \in H$);

2) for any $x \in B(x_0, r)$ the operator $(\varphi'^*(x))$ is invertible and

$$||\varphi'^{*-1}||_r = \sup_{x\in B(x_0,r)} ||(\varphi'^*(x))^{-1}|| < \infty;$$

3) the following inequality holds

$$0 < \frac{max\{||Y_0||, ||\varphi'^{*-1}||_r\}}{1 - max\{||\varphi'(x_0)Y_0 - I||, \epsilon\}} ||\varphi(x_0)|| < r. \tag{3.3}$$

Let us denote

$$||\varphi||_r = \sup_{x\in B(x_0,r)} ||\varphi(x)||,$$

$$\rho_0 = \max\{||Y_0||, ||\varphi'^{*-1}||_r\} \, ||\varphi'^{*-1}||_r \sqrt{\frac{||\varphi''||_r ||\varphi||_r}{2\epsilon}}. \tag{3.4}$$

The following theorem establishes the convergence of the method.

Theorem 3.1 *If Condition E holds, then for $\rho > \rho_0$ and for $t \in [0, +\infty)$ one has:*

1) there exists the solution $(x(t), Y(t))$ of the problem (3.2) and

$$x(t) \in B(x_0, r), \tag{3.5}$$

$$||\varphi'(x(t))Y(t) - I|| \leq max\{||\varphi'(x_0)Y_0 - I||, \epsilon\}; \tag{3.6}$$

2) there exists

$$\lim_{t \to +\infty} x(t) = x^*,$$

x^* is the solution of the problem (1.1) and $x(t)$ converges to x^* exponentially.

4 Proofs

4.1 Proof of Theorem 2.2

The main part of the proof is to show that the solution to problem (2.2) exists for all $t \in [0, \infty)$ and does not leave the ball $B(x_0, r)$ (Lemma 4.3). In order to prove this statement one has to show that the operator $\varphi'(x(t))$ is invertible for all positive t. By the assumption of Theorem 2.2 the second derivative $\varphi''(x)$ exists for all $x \in B(x_0, r)$. So if $\varphi'(x)$ is boundedly invertible at an arbitrary point $\hat{x} \in B(x_0, r)$, then it is boundedly invertible in some open neighborhood $B(\hat{x}, \delta) \cap B(x_0, r)$. First it is shown that for all $\hat{x} \in B(x_0, r)$ such that $\hat{x} = x(\hat{t})$ for some $\hat{t}$ one can choose δ larger than a fixed positive number. In order to prove this statement the estimate for $||[\varphi'(x(t))]^{-1}||$ is established in Lemma 4.2. Then it is shown in Lemma 4.3 that $\varphi'(x(t))$ is boundedly invertible as long as $x(t) \in B(x_0, r)$ and the proof of Lemma 4.3 is completed by using the second estimate derived in Lemma 4.2.

Thus the proof of the Theorem 2.2 is based on the estimates of Lemma 4.2. The proof of this lemma consists of two steps: the derivation of nonlinear integral inequalities and the solution to these inequalities by means of Lemma 4.1.

For the convenience of the reader we reformulate now in a simplified form a statement from [12] about an integral inequality.

Let $Y = (y_1, \dots, y_n)$, $\tilde{Y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ be two points of the n-dimensional space. Then $Y \leq \tilde{Y}$ means that $y_j \leq \tilde{y}_j$ for $j = 1, \dots, n$. The following lemma is a simple corollary of Theorem 22.1 from [12].

Lemma 4.1 *Let $\sigma_j(t, y_1, \dots, y_n)$ $(j = 1, \dots, n)$ be continuous in the open region $(0, T) \times D$ and satisfies the following condition: for $t \in (0, T)$ and $Y, \tilde{Y} \in D$ if $Y \leq \tilde{Y}$ then $\sigma_j(t, Y) \leq \sigma_j(t, \tilde{Y})$ for $j = 1, \dots, n$. Let $Y_0 =$*

$(y_1^0,\dots,y_n^0)\in D$. Suppose that $F(t)=(f_1(t),\dots,f_n(t))$ is continuous and $F(t)\in D$ on $[0,T)$. Suppose also that the Cauchy problem

$$\dot\omega_j=\sigma(t,\omega_1,\dots,\omega_n),\quad \omega_j(0)=y_j^0,\quad j=1,\dots,n$$

has the unique solution $\Omega(t)=(\omega_1(t),\dots,\omega_n(t))$ on $[0,T)$.

Under the above assumptions, if

$$f_j(t)\le y_j^0+\int\limits_0^t\sigma_j(\tau,f_1(\tau),\dots,f_n(\tau))d\tau \text{ for } 0\le t<T \text{ and } j=1,\dots,n$$

then

$$F(t)\le\Omega(t,Y_0) \text{ for } 0\le t\le T.$$

Denote

$$D(x_0,r)=\{x: x\in B(x_0,r) \text{ and } \varphi'(x) \text{ is boundedly invertible}\}.$$

Lemma 4.2 *Let condition A hold. If there exist a positive number T and a function $x(t)\in C^1[0,T]$, such that $x:[0,T]\to D(x_0,r)$ and $x(t)$ is the solution of problem (2.2) on $[0,T]$, then for $t\in[0,T]$ the following inequalities hold:*

$$||[\varphi'(x(t))]^{-1}||\le\frac{\beta_0}{\sqrt{1-2k\alpha_0\beta_0(1-e^{-t})}},\tag{4.1}$$

$$||[\varphi'(x(t))]^{-1}\varphi(x(t))||\le\frac{\alpha_0e^{-t}}{\sqrt{1-2k\alpha_0\beta_0(1-e^{-t})}}.\tag{4.2}$$

Proof. Let us denote

$$A(t)=[\varphi'(x(t))]^{-1},\quad \xi(t)=A(t)\varphi(x(t)),\quad \alpha(t)=||e^t\xi(t)||,\quad \beta(t)=||A(t)||.$$

Thus $\alpha(0)=\alpha_0$, $\beta(0)=\beta_0$ and

$$\varphi'(x(t))\xi(t)=\varphi(x(t)).\tag{4.3}$$

Differentiating this equality with respect to t one gets

$$\varphi'(x(t))(\dot\xi(t)+\xi(t))=\varphi''(x(t),\xi(t),\xi(t))$$

and therefore

$$\frac{d}{dt}\left(e^t\xi(t)\right)=e^tA(t)\varphi''(x(t),\xi(t),\xi(t)).$$

Thus the following integral inequality follows from Condition A:

$$\alpha(t) \leq \alpha_0 + k \int_0^t e^{-s}\beta(s)\alpha^2(s)ds.$$

In order to apply Lemma 4.1 one considers the corresponding Cauchy problem:

$$\dot{\omega}(t) - ke^{-t}\beta(t)\omega^2(t) = 0, \quad \omega(0) = \alpha_0.$$

This problem has the unique solution

$$\omega(t) = \frac{\alpha_0}{1 - k\alpha_0 \int_0^t e^{-s}\beta(s)ds}$$

on an interval $[0, T_1)$ if

$$k\alpha_0 \int_0^{T_1} e^{-s}\beta(s)ds < 1. \tag{4.4}$$

Thus from Lemma 4.1 one gets

$$||\xi(t)|| \leq \frac{\alpha_0 e^{-t}}{1 - k\alpha_0 \int_0^t e^{-s}\beta(s)ds}, \quad 0 \leq t < T_1. \tag{4.5}$$

Let $[0, T_1)$ be the maximal subinterval of $[0, T]$ for which (4.4) holds. Let us assume that $T_1 < T$, then it follows from the continuity of $\beta(t)$ that

$$k\alpha_0 \int_0^{T_1} e^{-s}\beta(s)ds = 1. \tag{4.6}$$

From $\varphi'(x(t))A(t) = I$ one gets the equation:

$$\dot{A}(t)\eta = A(t)\phi''(x(t), \xi(t), A(t)\eta) \text{ for any } \eta \in H. \tag{4.7}$$

Therefore

$$A(t)\eta = A(0)\eta + \int_0^t A(s)\phi''(x(s), \xi(s), A(s)\eta)ds \tag{4.8}$$

and for $t \in [0, T_1)$ one has

$$\beta(t) \leq \beta_0 + k\alpha_0 \int_0^t \frac{\beta^2(s)e^{-s}}{1 - k\alpha_0 \int_0^s e^{-\tau}\beta(\tau)d\tau} ds. \tag{4.9}$$

Let us denote

$$\gamma(t) = \int_0^t e^{-s}\beta(s)ds, \tag{4.10}$$

then $\dot\gamma(t) = e^{-t}\beta(t)$, $\beta = e^t\dot\gamma$, and

$$\dot\gamma(t) \le \beta_0 e^{-t} + k\alpha_0 e^{-t}\int_0^t \frac{\dot\gamma^2(s)e^s}{1-k\alpha_0\gamma(s)}ds. \tag{4.11}$$

Let $\omega(t)$ be the solution of the following Cauchy problem

$$\dot\omega(t) = \beta_0 e^{-t} + k\alpha_0 e^{-t}\int_0^t \frac{\dot\omega^2(s)e^s}{1-k\alpha_0\omega(s)}ds, \quad \omega(0) = 0. \tag{4.12}$$

Denote $u(\omega) = \dot\omega$. Since $= \dot\omega(0) = \beta_0$ and $\omega(0) = 0$ one gets the following Cauchy problem for $u(\omega)$:

$$\frac{du}{d\omega} + 1 = k\alpha_0 u(1-k\alpha_0\omega)^{-1}, \quad u(0) = \beta_0.$$

This equation has general solution

$$u(\omega) = \frac{1-k\alpha_0\omega}{2k\alpha_0} + \frac{C}{1-k\alpha_0\omega}.$$

So $C = \beta_0 - 1/(2k\alpha_0)$, and one gets

$$\dot\omega(t) = \frac{(1-k\alpha_0\omega)^2 + 2k\alpha_0\beta_0 - 1}{2k\alpha_0(1-k\alpha_0\omega)}, \quad \omega(0) = 0.$$

and

$$\omega(t) = (k\alpha_0)^{-1}[1 - \sqrt{1-2k\alpha_0\beta_0(1-e^{-t})}]. \tag{4.13}$$

It follows from (4.11),(4.12) and Lemma 4.1 that

$$\gamma(t) \le \omega(t) \text{ and } \dot\gamma(t) \le \dot\omega(t) \text{ for } 0 \le t < T_1.$$

Consequently

$$\beta(t) = e^t\dot\gamma(t) \le e^t\dot\omega(t) = \frac{\beta_0}{\sqrt{1-2k\alpha_0\beta_0(1-e^{-t})}} \tag{4.14}$$

and it follows from inequality (2.1) that

$$k\alpha_0\int_0^{T_1} e^{-s}\beta(s)ds \le k\alpha_0\beta_0\int_0^{T_1}\frac{e^{-s}ds}{\sqrt{1-2k\alpha_0\beta_0(1-e^{-t})}} =$$

$$= 1 - \sqrt{1-2k\alpha_0\beta_0(1-e^{-T_1})} < 1. \tag{4.15}$$

Since this contradicts to (4.6), it is proved that $T_1 = T$.

In order to complete the proof of Lemma 4.2 one derives inequality (4.2) from (4.15) and (4.5). □

Lemma 4.3 *If Condition A hold, then the solution $x(t)$ to problem (2.2) exists on $[0,\infty)$, $\varphi'(x(t))$ is boundedly invertible for all $t \in [0,\infty)$, and $x(t)$ does not leave the ball $B(x_0,r)$ for all $t \in [0,\infty)$.*

Proof. Let T be an arbitrary positive number. Denote by $[0,T_1)$ the maximal subinterval of $[0,T]$ such that the solution $x(t)$ to the problem (2.2) exists and $x(t) \in D(x_0,r)$.

For $t_1 < t_2 < T_1$ one has:

$$||x(t_2) - x(t_1)|| \leq \int_{t_1}^{t_2} ||\xi(s)||ds \leq \int_{t_1}^{t_2} \frac{\alpha_0 e^{-s}}{\sqrt{1-2k\alpha_0\beta_0(1-e^{-s})}}ds \leq$$

$$\leq \alpha_0 \int_{t_1}^{t_2} e^{-s/2}ds < \alpha_0(t_2 - t_1).$$

Thus for any sequence $\{t_n\} \to T_1 - 0$, $\{x(t_n)\}$ is a Cauchy sequence and there exists a finite limit:

$$x(T_1) := \lim_{t \to T_1 - 0} x(t).$$

It follows from the local existence theorem that the solution $x(t)$ to problem (2.2) exists as far as $x(t) \in D(x_0,r)$. It means that if $T_1 < T$, then $x(T_1) \notin D(x_0,r)$.

Assume that $x(T_1) \in B(x_0,r)$. For an arbitrary $\epsilon > 0$ there exists $\hat{t} \in [0,T_1)$ such that $\hat{x} = x(\hat{t}) \in D(x_0,r)$ and $||\hat{x} - x(T_1)|| < \epsilon$. It follows from Condition A that for $x \in B(x_0,r)$

$$||\varphi'(x) - \varphi'(\hat{x})|| \leq k||x - \hat{x}||.$$

One has the following representation for $\varphi'(x)$:

$$\varphi'(x) = \varphi'(\hat{x})\Big(I + [\varphi'(\hat{x})]^{-1}(\varphi'(x) - \varphi'(\hat{x}))\Big).$$

From (4.1) one gets the following estimates:

$$||[\varphi'(\hat{x})]^{-1}|| \leq \frac{\beta_0}{\sqrt{1-2k\alpha_0\beta_0(1-e^{-\hat{t}})}} < \frac{\beta_0}{\sqrt{1-2k\alpha_0\beta_0}},$$

$$||[\varphi'(\hat{x})]^{-1}(\varphi'(x) - \varphi'(\hat{x}))|| \leq \frac{k\beta_0}{\sqrt{1-2k\alpha_0\beta_0}}||x - \hat{x}||.$$

Thus for $x \in B(\hat{x}, \sqrt{1-2k\alpha_0\beta_0}/k\beta_0) \cap B(x_0,r)$ the operator $\varphi'(x)$ is boundedly invertible. If $\epsilon < \sqrt{1-2k\alpha_0\beta_0}/k\beta_0$, then $x(T_1) \in B(\hat{x}, \sqrt{1-2k\alpha_0\beta_0}/k\beta_0)$. Therefore the operator $\varphi'(x)$ is boundedly invertible in some neighborhood of $x(T_1)$. This contradicts the assumption that $x(T_1) \notin D(x_0,r)$. From this contradiction one gets that $x(T_1) \notin B(x_0, r)$. Thus $x(T_1)$ belongs to the boundary of the ball $B(x_0, r)$, that is $||x(T_1) - x_0|| = r$.

However

$$||x(T_1) - x_0|| \leq \int_0^{T_1} ||\xi(s)||ds \leq \int_0^{T_1} \frac{\alpha_0 e^{-s}}{\sqrt{1 - 2k\alpha_0\beta_0(1 - e^{-s})}} ds \leq$$

$$\leq \alpha_0 \int_0^{T_1} e^{-s/2} ds = 2\alpha_0(1 - e^{-T_1/2}) < 2\alpha_0 \leq r.$$

This contradiction proves that $T_1 = T$. Lemma 4.3 is proved. □

Thus the first statement of the Theorem 2.2 is proved. The existence of the strong limit x^* in the second statement of Theorem 2.2 follows from the estimate:

$$||x(t_2) - x(t_1)|| \leq \int_{t_1}^{t_2} ||\xi(s)||ds \leq \alpha_0 \int_{t_1}^{t_2} e^{-s/2} ds =$$

$$= 2\alpha_0(e^{-t_1/2} - e^{-t_2/2}) < 2\alpha_0 e^{-t_1/2} \text{ for any } 0 \leq t_1 < t_2.$$

After passing to limit for $t_2 \to \infty$ one gets the estimate

$$||x(t) - x^*|| \leq 2\alpha_0 e^{-t/2},$$

which proves the exponential convergence. In order to complete the proof of the theorem one has to prove the uniqueness of the solution of the problem (1.1) in the ball $B(x_0, 2\alpha_0)$. This is proved in [6]. We sketch below this proof for the reader's convenience.

If there is another solution x_1^* of equation (1.1) in $B(x_0, 2\alpha_0)$ then

$$||x^* - x_1^*|| \leq ||[\varphi'(x_0)]^{-1}|| \cdot ||\varphi(x^*) - \varphi(x_1^*) - \varphi'(x_0)(x^* - x_1^*)|| \leq$$

$$\leq \beta_0 ||[\int_0^1 \varphi'(x_1^* + t(x^* - x_1^*))dt - \varphi'(x_0)](x^* - x_1^*)|| \leq$$

$$\leq \beta_0 ||\int_0^1 dt \int_0^1 \varphi''(x_0 + s(x_1^* + t(x^* - x_1^*) - x_0), x_1^* + t(x^* - x_1^*) - x_0, x^* - x_1^*)||ds \leq$$

$$\leq k\beta_0 ||x^* - x_1^*|| \int_0^1 ||x_1^* + t(x^* - x_1^*) - x_0||dt \leq 2k\alpha_0\beta_0||x^* - x_1^*|| < ||x^* - x_1^*||.$$

Thus $x^* = x_1^*$. □

4.2 Proofs of Theorems 2.4, 2.6, 2.8

Remark 4.4 *The proof of the uniqueness of the solution in Theorems 2.4, 2.6, is similar to the proof of the uniqueness in Theorem 2.2, and is left to the reader by this reason.*

Proof of Theorem 2.4. Let $x(t)$ be the solution to problem (2.8). Denote $\lambda(t) = \varphi(x(t))$, then $\dot{\lambda}(t) = \varphi'(x(t))\dot{x}(t)$. As long as $x(t) \in B(x_0, r)$ one has the following representation:

$$\varphi'(x)[A(f(x))]^{-1} = I - R_1 - R_2,$$

where

$$R_1 = [\varphi'(f(x)) - \varphi'(x)][A(f(x))]^{-1}, \quad R_2 = [A(f(x)) - \varphi'(f(x))][A(f(x))]^{-1}.$$

Thus from (2.8) one gets

$$\dot{\lambda}(t) = -\lambda(t) + (R_1 + R_2)\lambda(t).$$

So for $\xi(t) = e^t \lambda(t)$ one has

$$\dot{\xi}(t) = (R_1 + R_2)\xi(t), \quad \xi(0) = \varphi(x_0)$$

and

$$||\xi(t)|| \leq ||\varphi(x_0)|| + \int_0^t ||R_1 + R_2|| \cdot ||\xi(s)||ds.$$

Then the following integral inequality follows from (2.7) and (2.5):

$$||\xi(t)|| \leq ||\varphi(x_0)|| + \beta(\delta + \gamma) \int_0^t ||\xi(s)||ds. \tag{4.16}$$

So one gets the estimates:

$$||\xi(t)|| \leq ||\varphi(x_0)||e^{\beta(\delta+\gamma)t} \tag{4.17}$$

and

$$||\lambda(t)|| \leq ||\varphi(x_0)||e^{-(1-\beta(\delta+\gamma))t}. \tag{4.18}$$

For $0 \leq t_1 \leq t_2$ one has:

$$||x(t_2) - x(t_1)|| \leq \int_{t_1}^{t_2} ||[A(f(x(s)))]^{-1}|| \cdot ||\lambda(s)||ds.$$

Thus it follows from (4.18) that

$$||x(t_2) - x(t_1)|| \leq \frac{\beta||\varphi(x_0)||}{1 - \beta(\gamma + \delta)} \left(e^{-(1-\beta(\gamma+\delta))t_1} - e^{-(1-\beta(\gamma+\delta))t_2} \right).$$

So

$$||x(t) - x(0)|| \le \frac{\beta||\varphi(x_0)||}{1-\beta(\gamma+\delta)}\left(1 - e^{-(1-\beta(\gamma+\delta))t}\right) \tag{4.19}$$

and

$$||x(t_2) - x(t_1)|| \le \frac{\beta||\varphi(x_0)||}{1-\beta(\gamma+\delta)}e^{-(1-\beta(\gamma+\delta))t_1}. \tag{4.20}$$

From (4.19) one gets that $x(t)$ does not leave the ball $B(x_0, r)$ and so $x(t)$ exists for all positive t. Then it follows from (4.20) that $x(t)$ has strong limit for $t \to \infty$. Finally one shows exponential convergence using the following estimate:

$$||x(t) - x^*|| \le \frac{\beta||\varphi(x_0)||}{1-\beta(\gamma+\delta)}e^{-(1-\beta(\gamma+\delta))t}.$$

□

Proof of Theorem 2.6. Let us denote $\lambda(t) := \varphi(x(t))$, then from equation (2.10) one has

$$\dot{\lambda} = -\lambda + (\varphi'(x(t)) - \varphi'(x_0))[\varphi'(x_0)]^{-1}\lambda.$$

So for

$$\xi(t) := e^t[\varphi'(x_0)]^{-1}\lambda$$

one has

$$\dot{\xi}(t) = [\varphi'(x_0)]^{-1}(\varphi'(x(t)) - \varphi'(x_0))\xi(t).$$

Thus one gets the following integral equation:

$$\xi(t) = \xi(0) + \int_0^t [\varphi'(x_0)]^{-1}(\varphi'(x(s)) - \varphi'(x_0))\xi(s)ds.$$

From condition C and the initial condition $||\xi(0)|| = ||[\varphi'(x_0)]^{-1}\varphi(x_0)|| = \alpha_0$ the following integral inequality follows:

$$||\xi(t)|| \le \alpha_0 + k\beta_0 \int_0^t ||x(s) - x(0)|| \cdot ||\xi(s)||ds. \tag{4.21}$$

From equation (2.10) one also has the following integral inequality:

$$||x(t) - x_0|| \le \int_0^t ||\dot{x}(s)||ds \le \int_0^t e^{-s}||\xi(s)||ds. \tag{4.22}$$

It follows from Lemma 4.1 that if the functions $||\xi(t)||$ and $||x(t) - x_0||$ satisfy the (4.21) and (4.22), then they are majorized by the functions $u(t)$ and $v(t)$ respectively, where $u(t)$ and $v(t)$ are the solutions of the following problem:

$$\begin{cases} \dot{u} = k\beta_0 uv, & u(0) = \alpha_0, \\ \dot{v} = e^{-t}u, & v(0) = 0. \end{cases} \tag{4.23}$$

Lemma 4.5 *If* $2k\beta_0\alpha_0 < 1$*, then the solution of the problem (4.23) exists for all positive* t *and is given by the formulas:*

$$u(t) = \frac{4(1-2k\beta_0\alpha_0)}{2k\beta_0}\frac{\mu(t)}{(1-\mu(t))^2}e^t, \quad v(t) = \frac{1}{k\beta_0}\left(1 + \frac{1+\mu(t)}{1-\mu(t)}\frac{\dot{\mu}}{\mu}\right), \quad (4.24)$$

where

$$\mu(t) = \frac{1-\sqrt{1-2k\beta_0\alpha_0}}{1+\sqrt{1-2k\beta_0\alpha_0}}e^{-t\sqrt{1-2k\beta_0\alpha_0}}.$$

Proof. Denote $w = e^{-t}u$, then

$$\frac{d}{dt}\left(\frac{\dot{w}}{w}\right) = k\beta_0 w, \quad w(0) = \eta, \quad \dot{w}(0) = -\alpha_0.$$

So one easily gets

$$w(t) = \frac{4(1-2k\beta_0\alpha_0)}{2k\beta_0}\frac{\mu(t)}{1-\mu^2(t)}.$$

□

From this Lemma and Condition C one has the following estimates:

$$||x(t) - x_0|| < \frac{2\alpha_0}{1+\sqrt{1-2k\beta_0\alpha_0}} \leq r, \quad (4.25)$$

$$||x(t_2) - x(t_1)|| \leq \int_{t_1}^{t_2} e^{-s}||\xi(s)||ds \leq$$

$$\beta_0 \int_{t_1}^{t_2} \frac{4(1-2k\beta_0\alpha_0)}{2k\beta_0}\frac{\mu(s)}{(1-\mu(s))^2} \leq \frac{4(1-2k\beta_0\alpha_0)}{2k\beta_0}\frac{\mu(t_1)}{(1-\mu(t_1))^2}.$$

It follows from the last estimate that $x(t)$ strongly converges and that

$$||x(t) - x^*|| < \frac{4(1-2k\beta_0\alpha_0)}{2k\beta_0}\frac{\mu(t)}{(1-\mu(t))^2},$$

so $x(t)$ converges exponentially. □

Proof of Theorem 2.8 (see [7]).

From the Condition D we get the local existence of a solution of the problem (2.14). Then from (2.14)

$$\varphi'(x(t))x'(t) = -\varphi'(x(t))F(x(t))\varphi(x(t)). \quad (4.26)$$

So from (2.14) one gets the following equation for $\lambda(t) = \varphi(x(t))$:

$$\lambda'(t) = -\varphi'(x(t))F(x(t))\lambda(t), \quad \lambda(0) = \varphi(x_0). \quad (4.27)$$

Therefore

$$\frac{d}{dt}||\lambda(t)||^2 = (\dot{\lambda}(t), \lambda(t)) + (\lambda(t), \dot{\lambda}(t)) = -((\varphi' F + F^* \varphi'^*))\lambda, \lambda) \le -2c||\lambda(t)||^2.$$

So the following estimate holds

$$||\lambda(t)|| \le ||\varphi(x_0)||e^{-ct}. \tag{4.28}$$

Thus for $0 < t_1 \le t_2$

$$||x(t_2) - x(t_1)|| \le ||\int_{t_1}^{t_2} \dot{x}(s)ds|| \le \int_{t_1}^{t_2} ||F(x)||\,||\lambda(s)||ds \le$$

$$\le \frac{||F|| \times ||\varphi(x_0)||}{c}(e^{-ct_1} - e^{-ct_2}) \le re^{-ct_1}. \tag{4.29}$$

In order to complete the proof one has to repeat the last part of the proof of Theorem 2.2. □

4.3 Proof of Theorem 3.1

Denote

$$\begin{pmatrix} \lambda(t) \\ \Lambda(t) \end{pmatrix} = \begin{pmatrix} \varphi(x(t)) \\ \varphi'(x(t))Y(t) - I \end{pmatrix},$$

and consider the Cauchy problem

$$\begin{cases} \dot{\lambda}(t) = -\varphi'(x(t))Y(t)\lambda(t), \\ \dot{\Lambda}(t) = -\rho^2(\varphi'(x(t))(\varphi'(x(t)))^*\Lambda(t) + \Lambda(t)\varphi'(x(t))(\varphi'(x(t)))^*) + \\ +(\varphi''(x(t))\dot{x}(t))Y(t), \end{cases} \tag{4.30}$$

$$\lambda(0) = \varphi(x_0), \quad \Lambda(0) = \varphi'(x_0)Y_0 - I.$$

First we prove some lemmas. Let us consider the Cauchy problem

$$\dot{Y}(t) + A^*(t)A(t)Y(t) + Y(t)A(t)A^*(t) - 2\rho A^*(t) = 0, \tag{4.31}$$

$$Y(0) = Y_0,$$

where $A(t)$ is some given continuous operator-valued function on $[0, +\infty)$. This is a linear equation, so there exists the unique solution to the problem on $[0, +\infty)$.

Lemma 4.6 *Let the operator $A^*(t)$ be invertible for every $t \in [0, T]$, $T \le +\infty$ and the following estimates be valid*

$$||Y_0|| \le C_1,$$

and

$$\rho||(A^*(t))^{-1}|| \le C_1, \quad \text{for} \quad t \in [0, T]. \tag{4.32}$$

Then

$$||Y(t)|| \le C_1 \quad \text{for} \quad t \in [0, T]. \tag{4.33}$$

Proof. As it is done for Riccati matrix equations in [11] , we consider the auxiliary system

$$\begin{cases} \dot{U}(t) - A(t)A^*(t)U(t) = 0, \quad U(0) = I, \\ \dot{V}(t) - 2\rho A^*(t)U(t) + A^*(t)A(t)V(t) = 0, \quad V(0) = Y_0. \end{cases} \tag{4.34}$$

$U(t)$ is invertible on $[0, T]$. Indeed, one can construct $U^{-1}(t)$ as the solution of the following Cauchy problem:

$$\frac{d}{dt}(U^{-1}(t)) + U^{-1}(t)A(t)A^*(t) = 0, \quad U^{-1}(0) = I.$$

Thus if one defines

$$Y(t) = V(t)U^{-1}(t). \tag{4.35}$$

then Y solves (4.31), as one can check using (4.34). Now we introduce the operator-valued function

$$\Psi(t) = V^*(t)V(t) - C_1^2U^*(t)U(t) = U^*(t)(Y^*(t)Y(t) - C_1^2E)U(t). \tag{4.36}$$

Denote by $(,)$ the scalar product in H, then, for any $\xi \in H$, one gets:

$$\frac{d}{dt}\Big(\xi, \Psi(t)\xi\Big) = \Big(\xi, [2\rho U^*(t)A(t)V(t) - V^*(t)A^*(t)A(t)V(t) +$$

$$+2\rho V^*(t)A^*(t)U(t) - V^*(t)A^*(t)A(t)V(t) - 2C_1^2U^*(t)A(t)A^*(t)U(t)]\xi\Big) =$$

$$= 4\rho \mathrm{Re}\Big(U(t)\xi, A(t)V(t)\xi\Big) - 2||A(t)V(t)\xi||^2 - 2C_1^2||A^*(t)U(t)\xi||^2.$$

From the inequality

$$2\rho\Big(U(t)\xi, A(t)V(t)\xi\Big) \leq \rho^2||U(t)\xi||^2 + ||A(t)V(t)\xi||^2,$$

$$||U(t)\xi|| = ||(A^*(t))^{-1}A^*(t)U(t)\xi|| \leq ||(A^*(t))^{-1}|| \cdot ||A^*(t)U(t)\xi||$$

and from (4.32) we get

$$\frac{d}{dt}\Big(\xi, \Psi(t)\xi\Big) \leq 0.$$

So

$$\Big(\xi, \Psi(t)\xi\Big) \leq \Big(\xi, \Psi(0)\xi\Big)$$

and

$$\Big(\xi, U^*(t)(Y^*(t)Y(t) - C_1^2E)U(t)\xi\Big) \leq \Big(\xi, U^*(0)(Y_0^*Y_0 - C_1^2E)U(0)\xi\Big) \leq 0.$$

Therefore

$$||Y(t)U(t)\xi||^2 - C_1^2||Y_0||^2||U(t)\xi||^2 \le ||Y_0\xi||^2 - C_1^2||\xi||^2 \le 0.$$

Estimate (4.33) follows from the last inequality and from the invertibility of $U(t)$. □

Now we consider the Cauchy problem

$$\Lambda'(t) + A(t)A^*(t)\Lambda(t) + \Lambda(t)A(t)A^*(t) - D(t) = 0, \quad \Lambda(0) = \Lambda_0, \quad (4.37)$$

where $\Lambda(t), A(t), D(t)$ are the given continuous operator-valued functions on $[0, +\infty]$.

Lemma 4.7 *Let an operator $A^*(t)$ be invertible for every $t \in [0,T]$, $T \le +\infty$ and the following estimates be valid*

$$||\Lambda_0|| \le C_2,$$

$$||D(t)||\,||(A^*(t))^{-1}||^2 \le 2C_2, \quad \textit{for} \quad t \in [0,T]. \quad (4.38)$$

Then

$$||\Lambda(t)|| \le C_2 \quad \textit{for} \quad t \in [0,T]. \quad (4.39)$$

Proof. The proof is very similar to the previous one. We consider

$$\begin{cases} U'(t) - A(t)A^*(t)U(t) = 0, \quad U(0) = I, \\ V'(t) - D(t)U(t) + A(t)A^*(t)V(t) = 0, \quad V(0) = \Lambda_0. \end{cases} \quad (4.40)$$

Then

$$\Lambda(t) = V(t)U^{-1}(t). \quad (4.41)$$

and

$$\begin{aligned}\Psi(t) &= V^*(t)V(t) - C_2^2U^*(t)U(t) = \\ &= U^*(t)(\Lambda^*(t)\Lambda(t) - C_2^2E)U(t). \quad (4.42)\end{aligned}$$

For any $\xi \in H$

$$\frac{d}{dt}\Big(\xi, \Psi(t)\xi\Big) = \Big(\xi, [U^*(t)D^*(t)V(t) - 2V^*(t)A(t)A^*(t)V(t) +$$

$$+V^*(t)D(t)U(t) - 2C_2^2U^*(t)A(t)A^*(t)U(t)]\xi\Big) =$$

$$= 2\mathrm{Re}\Big(D(t)U(t)\xi, V(t)\xi\Big) - 2||A^*(t)V(t)\xi||^2 - 2C_2^2||A^*(t)U(t)\xi||^2.$$

$A^*(t)$ is invertible so one has

$$2\Big(D(t)U(t)\xi, V(t)\xi\Big) \leq$$

$$\leq 0.5||(A^*(t))^{-1}||^2||D(t)||^2||U(t)\xi||^2 + 2||(A^*(t))^{-1}||^{-2}||V(t)\xi||^2 \leq$$
$$\leq 0.5||(A^*(t))^{-1}||^4||D(t)||^2||A^*(t)U(t)\xi||^2 + 2||A^*(t)V(t)\xi||^2 \leq$$
$$\leq 2||A^*(t)V(t)\xi||^2 + 2C_2^2||\Lambda_0||^2||A^*(t)U(t)\xi||^2$$

Thus the inequality (4.39) is proved. □

Then we consider the problem

$$\lambda'(t) + \lambda(t) + \Lambda(t)\lambda(t) = 0, \quad \lambda(0) = \lambda_0, \tag{4.43}$$

where $\lambda(t)$ is H-valued function on $[0, +\infty)$, and $\Lambda(t)$ is a given continuous operator-valued function on $[0, +\infty)$.

Lemma 4.8 *Let*

$$||\Lambda(t)|| \leq \epsilon_1, \quad for \quad t \in [0, T], \quad \epsilon_1 \in (0, 1). \tag{4.44}$$

Then

$$||\lambda(t)|| \leq \lambda_0 e^{-(1-\epsilon_1)t} \quad for \quad t \in [0, T]. \tag{4.45}$$

Proof. (4.45) immediately follows from (4.43) and from the following inequalities

$$\frac{d}{dt}\Big(\lambda(t), \lambda(t)\Big) = -2\Big(\lambda(t), \lambda(t)\Big) - 2\mathrm{Re}\Big(\Lambda(t)\lambda(t), \lambda(t)\Big)$$
$$\leq -2(1-\epsilon_1)\Big(\lambda(t), \lambda(t)\Big).$$

□

Now we can prove Theorem 3.1. From Condition E we get the existence of a local continuous solution of (3.2) near every t if $x(t) \in B(x_0, r)$. Let us show that $x(t) \in B(x_0, r)$ for all $[0, +\infty)$. In fact let $x(t_1)$ be the first intersection point with the boundary of the ball $B(x_0, r)$ for the curve $x(t)$. So

$$||x(t) - x_0|| < r, \quad \text{for} \quad t \in [0, t_1),$$

and

$$||x(t_1) - x_0|| = r.$$

Denote $A(t) = \rho\varphi'(x(t))$, $C_1 = \max\{||Y_0||, ||\varphi'^{*-1}||_r\}$. From Lemma 4.6 we get the estimate

$$||Y(t)|| \quad \leq C_1 \quad \text{for} \quad t \in [0, t_1],$$

for the solution of the problem (3.2), where

$$C_1 = \max\{||Y_0||, ||\varphi'^{*-1}||_r\}.$$

The last term in (4.30) satisfies the estimate

$$\begin{aligned} ||(\varphi''(x(t))x'(t))Y(t)|| &\leq ||\varphi''(x(t))x'(t)||\,||Y(t)|| \leq \\ &\leq ||\varphi''||_r||x'(t)||\,||Y(t)|| \leq C_1^2||\varphi''||_r||\varphi||_r. \end{aligned}$$

It follows from (3.4) that if $\rho > \rho_0$, then the conditions of Lemma 4.7 are valid for $C_2 = \max\{||\varphi'(x_0)Y_0 - I||, \epsilon\}$. At the same time, from (3.3) we obtain $C_2 < 1$. Further from Lemmas 4.7,4.8 and from (3.3) we get that for $\rho > \rho_0$ the solution of the problem (4.30) satisfies the inequalities

$$||\Lambda(t)|| \leq C_2, \quad ||\lambda(t)|| \leq \lambda_0 e^{-(1-C_2)t} \quad \text{for} \quad t \in [0, t_1].$$

So we have the estimate for the solution of the problem (3.2)

$$||x(t_1) - x_0|| \leq ||\int_0^{t_1} x'(s)ds|| \leq$$

$$\leq C_1\,||\varphi(x_0)||\int_0^{t_1} e^{-(1-C_2)s}ds = \frac{C_1}{(1-C_2)}\,||\varphi(x_0)||(1 - e^{-(1-C_2)t_1}) < r.$$

The last one is in contradiction with the assumption that $x(t_1)$ belongs to the boundary of $B(x_0, r)$. Thus $x(t) \in B(x_0, r)$ for all $t \in [0, +\infty)$.

Consequently,

$$\lim_{t\to\infty} \varphi(x(t)) = \lim_{t\to\infty} \lambda(t) = 0$$

and $\lim_{t\to\infty} x(t)$ is the solution of the problem (1.1). □

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