definite type (or, alternatively, since it is the limit of a sequence of correlation functions). Symbolically, we can write

$$\Delta(\xi) = \epsilon\delta(\xi),$$

$$\delta(\xi) = \frac{1}{\epsilon}\Delta(\xi),$$

where ϵ is an "infinitesimal positive constant" which adjusts $\delta(0) = \infty$ to $\Delta(0) = 1$. The symbolic meaning of the relations

$$\epsilon = \int_{-\infty}^{\infty} \exp(i\xi\eta)\Delta(\xi)\, d\xi,$$
$$\Delta(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\xi\eta)\epsilon\, d\eta, \tag{24}$$

obtained by formal multiplication of the relations (21) by ϵ is apparent. $\Delta(\xi)$ can be regarded as the correlation function of stationary white noise with unit average instantaneous power, just as $\delta(\xi)$ can be regarded as the correlation function of stationary white noise with infinite average instantaneous power.

The Solution of a Homogeneous Wiener-Hopf Integral Equation Occurring in the Expansion of Second-Order Stationary Random Functions*

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Summary—In many of the applications of probability theory to problems of estimation and detection of random functions an eigenvalue integral equation of the type

$$\phi(x) = \lambda \int_0^T K(x - y)\phi(y)\, dy, \qquad 0 \le x \le T,$$

is encountered where $K(x)$ represents the covariance function of a continuous stationary second-order process possessing an absolutely continuous spectral density.

In this paper an explicit operational solution is given for the eigenvalues and eigenfunctions in the special but practical case when the Fourier transform of $K(x)$ is a rational function of ω^2, *i.e.*,

$$K(x) \doteqdot G(s^2) = \frac{N(s^2)}{D(s^2)}, \qquad s = i\omega,$$

in which $N(s^2)$ and $D(s^2)$ are polynomials in s^2.

It is easy to show by elementary methods that the solutions are of the form[1]

$$\phi(x) = \sum_r C_r e^{-\alpha_r x}\cos(\beta_r x + \gamma_r),$$

the constants C_r, α_r, β_r, and γ_r being linked together by the integral equation. It is precisely the labor involved in their determination that in practice often causes the problem to assume awesome proportions. By means of the results given herein, this labor is diminished to the irreducible minimum—the solving of a transcendental equation.

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[1] Subject to the usual modifications if the roots are not distinct.

Introduction

The integral equation

$$\phi(x) = \lambda \int_0^T K(x, y)\phi(y)\, dy, \qquad 0 \le x \le T \tag{1}$$

makes its appearance most naturally when one attempts to expand a random function $n(t, \omega)$ in an infinite series

$$n(t, \omega) = \sum_{k=1}^{\infty} a_k(\omega)\phi_k(t), \qquad k = 1, 2, \cdots, \cdots, \tag{2}$$
$$0 \le t \le T,$$

where the $\phi_k(t)$ are orthonormal over $0 \le t \le T$ and the $a_k(\omega)$ uncorrelated random variables.

Denote ensemble average by $E\{\ \}$ and let

$$m(t) = E\{n(t, \omega)\}$$

and

$$K(s, t) = E\{[n(s, \omega) - m(s)][n(t, \omega) - m(t)]\}.$$

Of course, $K(s, t)$ is the familiar covariance function of the process $n(t, \omega)$; if $K(s, t)$ is continous in the square $0 \le t \le T$, $0 \le s \le T$, the process is said to be second-order continuous in $0 \le t \le T$, and if in addition $K(s, t) = K(|s - t|)$, it is said to be second-order stationary.

From a theorem of Karhunen[2] any second-order random

[2] K. Karhunen, "Uber Lineare Methoden in der Wahrscheinlichkeitsrechnung," *Ann. Acad. Sci. Fennicae, Helsinki*, vol. 37; 1947.

function $n(t, \omega)$ which is second-order continuous in $0 \leq t \leq T$ may be expanded as follows:

$$n(t, \omega) = m(t) + \sum_{k=1}^{\infty} \frac{E_k(\omega)\phi_k(t)}{\sqrt{\lambda_k}}, \tag{3}$$

with convergence in the mean for every t in $[0, T]$. The quantities λ_k and $\phi_k(t)$ are determined from the integral equation

$$\phi_k(t) = \lambda_k \int_0^T K(s, t)\phi_k(s)\, ds, \qquad k = 1, 2, \cdots, \cdots. \tag{4}$$

The $\phi_k(t)$ is orthonormal over $0 \leq t \leq T$ and the $E_k(\omega)$ normalized uncorrelated random variables, *i.e.*,

$$E\{E_k(\omega)E_j(\omega)\} = \delta_{kj},$$

$$\int_0^T \phi_k(t)\phi_j(t)\, dt = \delta_{kj}, \qquad k, j = 1, 2, \cdots, \cdots$$

in which δ_{kj} is the Kronecker delta.

In particular if $n(t, \omega)$ is stationary, $K(s, t) = K(|s - t|)$ and (4) becomes

$$\phi_k(t) = \lambda_k \int_0^T K(|s - t|)\phi_k(s)\, ds, \tag{5}$$
$$k = 1, 2, \cdots, \cdots, \qquad 0 \leq t \leq T.$$

For Gaussian processes the $E_k(\omega)$ are actually independent normal variables and (3) converges in the ordinary sense with probability one. These properties of the expansion have been exploited to great advantage in many theoretical applications, for example,[3-6] the results therein fully justifying the importance of (4) and (5). The interested reader may find alternative treatments of (5) in Slepian,[5] DeSobrina,[7] and Muller.[8] As far as the author is aware, none of these treatments yield an explicit eigenvalue equation.

Notation and Assumptions

If what follows, we restrict ourselves to a special class of stationary processes. It follows directly from a theorem of Bochner that if $K(x, y)$ is the covariance function of a second-order stationary process with absolutely continuous spectral density, then

$$K(x, y) = K(|x - y|) = \int_{-\infty}^{+\infty} e^{i\omega(x-y)}G(\omega)\, d\omega, \tag{6}$$

where

$$G(\omega) = G(-\omega), \qquad G(\omega) \geq 0.$$

It is not difficult to show[9] that the eigenfunctions generated by such kernels by means of (5) are complete in $0 \leq x \leq T$, and that the eigenvalues are positive, their sole possible limiting point being infinity.

As is well-known if stationary noise with spectral density $G_1(\omega)$ is passed through a linear filter whose frequency response is $H(i\omega)$, then the output noise has spectral density $G_1(\omega) \mid H(i\omega) \mid^2$. For lumped constant systems

$$H(s) = \frac{h_0 + h_1 s + h_2 s^2 + , \cdots, + h_m s^m}{g_0 + g_1 s + g_2 s^2 + , \cdots, + g_n s^n},$$
$$s = i\omega, \quad h_i, \quad g_i \quad \text{real.}$$

For white input noise with mean square unity, $G_1(\omega) \equiv 1$ and the spectral density $G(\omega)$ of the output is

$$G(\omega) = H(i\omega)H(-i\omega) = \frac{N(-\omega^2)}{D(-\omega^2)} = \frac{N(s^2)}{D(s^2)}.$$

$N(s^2)$ and $D(s^2)$ being polynomials of degrees m and n in s^2, *viz.*,

$$N(s^2) = \sum_{k=0}^{m} a_{2k}s^{2k}, \tag{7}$$

$$D(s^2) = \sum_{k=0}^{n} b_{2k}s^{2k}, \qquad b_0 \neq 0.$$

The total output power equals

$$\int_{-\infty}^{+\infty} \frac{N(-\omega^2)}{D(-\omega^2)}\, d\omega.$$

In order that it be finite the degree of $D(s^2)$ must exceed that of $N(s^2)$ by at least two and $D(s^2)$ must have no roots on the imaginary s axis. Secondly, the requirement $G(\omega) \geq 0$, ω real, implies that any purely imaginary zero of $N(s^2)$ must be of even multiplicity.

The inverse of $(s^2 - \alpha^2)^{-k}$, for k a positive integer and the real part of $\alpha > 0$ is

$$\frac{(-1)^k}{(k-1)!} e^{-\alpha|x|} \sum_{j=1}^{k} \frac{(k+j+2)!\, |x|^{k-j}}{(j-1)!\,(k-j)!\,(2\alpha)^{k+j-1}}.$$

By expanding $N(s^2)/D(s^2)$ in partial fractions it is seen that $K(|x - y|)$ must be the sum of products of polynomials in $|x - y|$ with exponentials of $|x - y|$.

Denote the roots of $D(s^2)$ by $\pm\mu_1, \pm\mu_2, \cdots, \pm\mu_n$ and assume them so ordered that $0 < \text{Re}\,\mu_1 \leq \text{Re}\,\mu_2 \leq, \cdots, \leq \text{Re}\,\mu_n$. Define the polynomials $D^+(s)$ and $D^-(s)$ by $D(s^2) = D^+(s)\, D^-(s)$, $D^+(-s) = D^-(s)$, the roots of $D^+(s)$ being $-\mu_j$ and those of $D^-(s)$, $+\mu_j$, $j = 1, 2, \cdots, n$. Symbolically,

$$D^+(s) = \sum_{k=0}^{n} d_k s^k, \qquad d_k \quad \text{real.}$$

[3] R. C. Davis, "The detectability of random signals in the presence of noise," IRE Trans., vol. IT-3, pp. 52-62; March, 1954.
[4] U. Grenander, "Stochastic processes and statistical inference," *Ark F. Mat.*, vol. 1, pp. 195-277; October, 1950.
[5] D. Slepian, "Estimation of signal parameter in the presence of noise," IRE Trans., vol. IT-3, pp. 68-89; March, 1954.
[6] D. C. Youla, "The use of the method of maximum likelihood in estimating continuous modulated intelligence which has been corrupted by noise," IRE Trans., vol. IT-3, pp. 90-105; March, 1954.
[7] R. DeSobrino, "Optimum signal detection with incompletely specified signal noise," Ph. D. dissertation, Columbia Univ., New York, N. Y.; April, 1953.
[8] F. A. Muller, "Communication in the Presence of Additive Gaussian Noise," Tech. Rep. No. 244, M.I.T. Res. Lab. of Electronics; May, 1953.
[9] D. C. Youla, "A Finite-Time Homogeneous Weiner Hopf Integral Equation," Tech. Rep. No. 367, Microwave Res. Inst., Polytechnic Inst. of Brooklyn, N.Y.; October, 1955.

The following properties of $K(x)$ may be derived without too much effort from its representation in (6) and the fact that its transform is a rational function of ω^2 possessing the properties enumerated above.

1) The highest order derivative possessed by $K(x)$ at the origin is $2n - 2m - 2$.

2) $$D^+\left(\frac{d}{dx}\right)K(x) = 0, \qquad x > 0.$$

3) $$D^-\left(\frac{d}{dx}\right)K(x) = 0, \qquad x < 0.$$

4) $$D\left(\frac{d^2}{dx^2}\right)K(x) = 0, \qquad x \neq 0.$$

5) $K(x)$ possesses continuous derivatives of all order for $x \neq 0$.

The Determination of the Eigenvalues and Eigenfunctions Generated by a Kernel $K(|x-y|)$ whose Transform is a Rational Function of s^2

We now direct our attention exclusively to the integral equation

$$\phi(x) = \lambda \int_0^T K(x - y)\phi(y)\, dy, \qquad 0 \leq x \leq T, \tag{8}$$

where

$$K(x) \doteqdot \frac{N(s^2)}{D(s^2)}, \qquad -\text{Re}\,\mu_1 < \text{Re}\, s < \text{Re}\,\mu_1$$

and $N(s^2)$ and $D(s^2)$ are as described in (6) and (7). The notation is that employed by van der Pol.[10] From the breakup

$$\phi(x) = \lambda \int_0^x K(x - y)\phi(y)\, dy + \lambda \int_x^T K(x - y)\phi(y)\, dy \tag{9}$$

of (8) and the continuity properties of $K(x)$, it follows that $\phi(x)$ possesses continuous derivatives of all orders for $0 < x < T$. Again by the continuity properties of $K(x)$ and its derivatives, it follows that the function

$$g(x) \equiv \int_0^T K(x - y)\phi(y)\, dy, \qquad -\infty < x < \infty,$$

where $\phi(x)$ is a solution of (8), possesses continuous derivatives of all orders everywhere except perhaps at $x = 0$ and $x = T$; for $x < 0$ or $x > T$ these derivatives are given by

$$\frac{d^r g}{dx^r} = \int_0^T \frac{\partial^r K(x - y)}{\partial x^r}\phi(y)\, dy, \qquad r = 1, 2, \cdots, \cdots.$$

Lastly,

$$\frac{d^r g}{dx^r} = 0[e^{-(\text{Re}\,\mu_1 + \delta)|x|}] \quad \text{as} \quad |x| \to \infty, \qquad (r = 0, 1, \cdots, \cdots), \qquad \delta > 0.$$

Define $\Phi(x)$ by

$$\Phi(x) = \phi(x), \qquad 0 \leq x \leq T,$$
$$= 0 \quad \text{otherwise}$$

and let

$$v(x) \equiv \Phi(x) - \lambda g(x),$$
$$= \Phi(x) - \lambda \int_0^T K(x - y)\Phi(y)\, dy, \qquad -\infty < x < \infty. \tag{10}$$

From the previous discussion, the properties 2) and 3) of $K(x)$ and the fact that $\phi(x)$ is a solution of (8), it follows that

1) $$D^+\left(\frac{d}{dx}\right)v(x) = 0, \qquad x > T,$$

2) $$D^-\left(\frac{d}{dx}\right)v(x) = 0, \qquad x < T,$$

3) $v(x) = 0, \quad 0 \leq x \leq T$ and

4) The left and right hand limits of $v(x)$ as $x \to 0$, T, exist and are finite.

As an immediate consequence of these properties, it follows that $V(s)$, the bilateral Laplace transform of $v(x)$ is given by

$$V(s) = \frac{P(s)}{D^-(s)} - e^{-sT}\frac{Q(s)}{D^+(s)}, \qquad -\text{Re}\,\mu_1 < \text{Re}\, s < \text{Re}\,\mu_1, \tag{11}$$

$P(s)$ and $Q(s)$ being polynomials of degree $n - 1$ at most. The proof is obvious once it is realized that $v(x)$ is a sum of exponentials in both the regions $x > T$ and $x < 0$.

Denote the bilateral Laplace transform of $\Phi(x)$ by $\overline{\Phi}(s)$. Transforming both sides of (10) with respect to x (a common strip of convergence exists) yields

$$V(s) = \overline{\Phi}(s)\left[1 - \lambda\frac{N(s^2)}{D(s^2)}\right] \qquad -\text{Re}\,\mu_1 < \text{Re}\, s < \text{Re}\,\mu_1.$$

Using (11),

$$\overline{\Phi}(s) = \frac{D^+(s)P(s) - e^{-sT}D^-(s)Q(s)}{D(s^2) - \lambda N(s^2)}, \qquad -\text{Re}\,\mu_1 < \text{Re}\, s < \text{Re}\,\mu_1. \tag{12}$$

However, since $\overline{\Phi}(s)$ is an integral function of s,

$$\left(\frac{d\overline{\Phi}}{ds} = -\int_0^T x\phi(x)e^{-sx}\, dx\right)$$

[10] B. van der Pol and H. Bremmer, "Operational Calculus Based on the Two Sided Laplace Transform," Cambridge University Press, Cambridge, Eng.; 1955.

(12) is valid in the entire strip $-\infty < \text{Re } s < \infty$ and numerator and denominator must possess the same order zeros.

In (12) we have an explicit representation for the transform of an eigenfunction. It remains to establish the connection between $P(s)$ and $Q(s)$.

Let the roots of $D(s^2, \lambda) \equiv D(s^2) - \lambda N(s^2)$ be $\pm\, \omega_1(\lambda)$, $\pm\, \omega_2(\lambda)$, $\pm, \cdots, \pm\, \omega_n(\lambda)$ arranged so that $0 \leq \text{Re } \omega_1 \leq \text{Re } \omega_2 \leq, \cdots, \leq \text{Re } \omega_n$. In what follows we shall, in order to minimize the algebra, assume that λ is an eigenvalue for which the $\omega_k(\lambda)$ are distinct. In any case there exist at most a finite number of eigenvalues which violate this assumption and they are easily determined in any given problem.

By equating the numerator of (12) to zero for $s = \pm\, \omega_1(\lambda), \pm\, \omega_2(\lambda), \pm, \cdots, \pm\, \omega_n(\lambda)$ we get the n pairs of equations

$$D^+(\omega_r)P(\omega_r) = e^{-\omega_r T}D^-(\omega_r)Q(\omega_r),$$
$$D^-(\omega_r)P(-\omega_r) = e^{\omega_r T}D^+(\omega_r)Q(-\omega_r), \tag{13}$$
$$r = 1, 2, \cdots, n.$$

Multiplication of corresponding sides of (13) yields [any factor common to both $D(s^2)$ and $N(s^2)$ is assumed to have been cancelled],

$$P(\omega_r)P(-\omega_r) = Q(\omega_r)Q(-\omega_r),\quad r = 1, 2, \cdots, n. \tag{14}$$

The polynomial $L(s^2) \equiv P(s)\ P(-s) - Q(s)\ Q(-s)$ is of degree $n - 1$ in s^2 and vanishes for n distinct values of s^2, namely $\omega_1^2, \omega_2^2, \cdots, \omega_n^2$. Hence

$$P(s)P(-s) \equiv Q(s)Q(-s). \tag{15}$$

Let $P(s) = p_{n-1}\ (s - \alpha_1)(s - \alpha_2) \cdots (s - \alpha_{n-1})$. Then any solution $Q(s)$ of (15) is of the form ($j_1, j_2, \cdots, j_{n-1}$ is any permutation of $1, 2, \cdots, n - 1$)

$$Q(s) = \pm p_{n-1} \prod_{k=1}^{r} (s - \alpha_{j_k}) \prod_{l=1}^{n-r-1} (s + \alpha_{j_{r+l}}),\quad 0 \leq r \leq n - 1, \tag{16}$$

and are 2^n in number. But from (13) it is seen that the coefficients $q_k (k = 0, 1, \cdots, n - 1)$ of $Q(s)$ must depend linearly on the coefficients $p_k(k = 0, 1, \cdots, n - 1)$ of $P(s)$. This allows us to rule out all but four solutions of (16), namely, $P(s) = \pm\, Q(s)$ and $P(s) = \pm\, Q(-s)$.

The following example should make this clear. Supp

$$P(s) = p_0 + p_1 s + p_2 s^2$$

and

$$Q(s) = q_0 + q_1 s + q_2 s^2.$$

Then

$$P(s)P(-s) = p_0^2 + (2p_0p_1 - p_1^2)s + p_2^2 s^4$$

and

$$Q(s)Q(-s) = q_0^2 + (2q_0q_1 - q_1^2)s + q_2^2 s^4,$$

so that by (15)

$$p_0^2 = q_0^2, \qquad p_2^2 = q_2^2,$$
$$2p_0p_2 - p_1^2 = 2q_0q_2 - q_1^2\ .$$

The solutions are

$$p_0 = -q_0 \qquad p_0 = -q_0 \qquad p_0 = -q_0 \qquad p_0 = q_0$$
$$p_2 = -q_2 \qquad p_2 = q_2 \qquad p_2 = -q_2 \qquad p_2 = q_2$$
$$p_1 = \pm\sqrt{q_1^2 - 4q_0q_2} \qquad p_1 = \pm\sqrt{q_1^2 - 4q_0q_2} \qquad p_1 = \pm q_1 \qquad p_1 = \pm q_1\, .$$

To prove that $P(s) = \pm\, Q(s)$ and $P(s) = \pm\, Q(-s)$ the only linear solutions of (16), note first that they co spond to $r = n - 1$ and $r = 0$, respectively; the otl correspond to $0 < r < n - 1$. The coefficient of s^{n-} $Q(s)$ is, for any r, given by

$$q_{n-2} = \mp p_{n-1}(\alpha_{j_1} + \alpha_{j_2} +, \cdots, + \alpha_{j_r} - \alpha_{j_{r+1}} - \alpha_{j_{r+2}} -, \cdots, - \alpha_{j_{n-1}}).$$

Clearly if $r \neq 0$ or $n - 1$, q_{n-2} is not a symmetric func of the roots $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ of $P(s)$ and this in t implies, *a fortiori*, that it cannot be a linear combina of the p_k.

The two pairs of solutions $P(s) = \pm\, Q(-s)$, $P(s$ $\pm\, Q(s)$, coincide when $n = 1$. A glance at (13) reveals the former pair is consistent with both the upper lower sets of equations and that for $n > 1$ the latter is not. Thus the only possible relations between the p nomials P and Q are $P(s) = \pm\, Q(-s)$. Substituting $Q(s)$ in the top set in (13) we get,

$$\sum_{k=0}^{n-1} [1 \mp (-1)^k x_r]\omega_r^k p_k = 0,$$

where

$$x_r \equiv e^{-\omega_r T}\left[\frac{D^-(\omega_r)}{D^+(\omega_r)}\right], \qquad (r = 1, 2, \cdots, n).$$

In order that a nontrivial solution for the p_k exist, determinant of the system must equal zero. Hence,

$$\Delta = \begin{vmatrix} (1 \mp x_1), & (1 \pm x_1)\omega_1 & \cdots, & [1 \mp (-1)^{n-1}x_1]\omega_1^{n-1} \\ (1 \mp x_2), & (1 \pm x_2)\omega_2 & \cdots, & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ (1 \mp x_n), & (1 \pm x_n)\omega_n & \cdots, & [1 \mp (-1)^{n-1}x_n]\omega_n^{n-1} \end{vmatrix} = 0. \tag{18}$$

These two transcendental equations serve to determine the eigenvalues which are contained implicitly in the x's and ω's.

When $n = 1$, (18) reduces to $x_1 = \pm 1$, or

$$e^{\omega_1 T} = \pm \frac{D^-(\omega_1)}{D^+(\omega_1)}. \tag{19}$$

A straightforward application of the Maximum Modulus theorem reveals that for any $n \geq 0$ all solutions of

$$e^{sT} = \pm \frac{D^+(s)}{D^-(s)} \tag{20}$$

lie on the purely imaginary s axis. Denote them by $\pm i\beta_k(k = 0, 1, \cdots, \cdots)$. After a little algebra we find that the roots of (20) may be determined from the two transcendentals

$$\tan \frac{\beta T}{2} = -\frac{d_1\beta - d_3\beta^3 + d_5\beta^5 - , \cdots}{d_0 - d_2\beta^2 + d_4\beta^4 - , \cdots},$$
$$\operatorname{ctn} \frac{\beta T}{2} = \frac{d_1\beta - d_3\beta^3 + d_5\beta^5 - , \cdots}{d_0 - d_2\beta^2 + d_4\beta^4 - , \cdots}. \tag{21}$$

Now when

$$n = 1, \quad N(s^2) = a_0 > 0,$$
$$D(s^2) = b_0 - cs^2, \quad b_0, c > 0,$$
$$\omega_1(\lambda) = i\sqrt{\frac{a_0}{c}}\left(\lambda - \frac{b_0}{a_0}\right), \tag{21}$$
$$D^+(s) = \sqrt{b_0} + \sqrt{c}\, s$$

and reduce to

$$\tan \frac{\beta T}{2} = -\sqrt{\frac{c}{b_0}},$$

and

$$\operatorname{ctn} \frac{\beta T}{2} = \sqrt{\frac{c}{b_0}}\, \beta. \tag{22}$$

The λ_k are calculated from

$$\lambda_k = \frac{b_0 + c\beta_k^2}{a_0}, \qquad (k = 0, 1, \cdots, \cdots). \tag{23}$$

For $n > 1$ (18) may be written in a more manageable form as (we choose $n = 3$ to avoid cumbersome determinants)

$$\Delta = \begin{vmatrix} 1 & \omega_1 \coth \theta_1 & \omega_1^2 \tanh \theta_1 \\ 1 & \omega_2 \coth \theta_2 & \omega_2^2 \tanh \theta_2 \\ 1 & \omega_3 \coth \theta_3 & \omega_3^2 \tanh \theta_3 \end{vmatrix} = 0, \tag{24}$$

where

$$\tanh \theta_r = \frac{(d_0 + d_2\omega_r^2 + \cdots) \tanh (\omega_r T/2) + (d_1\omega_r + d_3\omega_r^3 + \cdots)}{(d_0 + d_2\omega_r^2 + \cdots) + (d_1\omega_r + d_3\omega_r^3 + \cdots) \tanh (\omega_r T/2)}, \qquad r = 1, 2, \cdots, n. \tag{25}$$

The other equation is obtained by interchanging $\tanh \theta_r$ and $\coth \theta_r$ in (24).

Once the eigenvalues λ_r have been determined, the p_k are found from (17) and through them $\phi_r(x)$. Explicitly,

$$\phi_r(x) \doteqdot \frac{P(s, \lambda_r)D^+(s)}{D(s^2) - \lambda_r N(s^2)}, \tag{26}$$
$$\operatorname{Re} s > \omega_r(\lambda_r),$$
$$0 \leq x \leq T.$$

The ϕ_r defined by (26) are orthogonal but not orthonormal over $0 \leq x \leq T$. An expression for the normalizing constant can probably be derived but its undoubted complexity makes it useless for practical work.

The Semi-Infinite Range Singular Integral Equation

The solution of the homogeneous integral equation

$$\phi(x) = \lambda \int_0^\infty K(x - y)\phi(y)\, dy, \qquad 0 \leq x < \infty \tag{27}$$

has been given by Wiener and Hopf for a wide class of kernels and can be found reproduced in Titchmarch's book.[11] However, when the transform of $K(x)$ is a rational function of s^2 our elementary technique gives the solution immediately. Omitting the details and adhering to the same notation as before, we find that

[11] E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals," Clarendon Press, Oxford, Eng., 2nd ed.; 1948.

$$\phi''(0^+, \lambda) = \frac{b_{2n}(d_n p_{n-3} + d_{n-1}p_{n-2} + d_{n-2}p_{n-1}) - (b_{2n-2} - \lambda A_{2n-2})\, d_n p_{n-1}}{b_{2n}^2}$$

$$\bar{\Phi}(s) = \frac{P(s)D^+(s)}{D(s^2) - \lambda N(s^2)}, \tag{28}$$

$$\mathrm{Re}\, s > \mathrm{Re}\, \omega_n(\lambda),$$

the polynomial $P(s)$ being arbitrary and of degree $n - 1$. The eigenvalues are determined by the requirement that the integral in (29) converge. This leads to

$$\mathrm{Re}\, \omega_n(\lambda) < \mathrm{Re}\, \mu_1. \tag{29}$$

All λ satisfying (29) are eigenvalues.

The coefficients p_k of $P(s)$ serve to fix the initial values $\phi(0^+, \lambda)$, $\phi'(0^+, \lambda)$, $\cdots$, $\phi^{n-1}(0^+, \lambda)$. Since the number of linearly independent polynomials $P(s)$ is n, each eigenvalue has n-fold degeneracy. To give a formula for $\phi^r(0^+, \lambda)$, $(r = 0, 1, \cdots, n-1)$ in terms of the p_k, a_{2k}, and b_{2k}, it is first necessary to introduce some preliminary notation. Let

$$P(s)D^+(s) \equiv \sum_{k=0}^{2n-1} A_k s^k$$

and

$$D(s^2) - \lambda N(s^2) \equiv \sum_{k=0}^{n} B_{2k} s^{2k},$$

where

$$A_k = \sum_{i=0}^{k} p_{k-i}\, d_i$$

and

$$B_{2k} = b_{2k} - \lambda A_{2k}.$$

It is understood that $A_{2k} = 0$ for $k > m$ and any p or d whose subscript exceeds $n - 1$ and n, respectively, is to be taken equal to zero. Then (again omitting details),

$$\phi^r(0^+, \lambda) = (-1)^r \frac{\Omega^{r+3}}{b_{2n}^{r+3}}, \qquad r = 0, 1, 2, \cdots, \cdots, \tag{30}$$

where

$$\Omega^3 = \begin{vmatrix} 0 & B_{2n} & 0 \\ 0 & 0 & B_{2n} \\ A_{2n-1} & B_{2n-2} & 0 \end{vmatrix},$$

$$\Omega^4 = \begin{vmatrix} 0 & B_{2n} & 0 & 0 \\ 0 & 0 & B_{2n} & 0 \\ A_{2n-1} & B_{2n-2} & 0 & B_{2n} \end{vmatrix} \text{ etc.} \tag{31}$$

For example,

$$\phi(0^+, \lambda) = \frac{d_n p_{n-1}}{b_{2n}},$$

$$\phi'(0^+, \lambda) = \frac{d_n p_{n-2} + d_{n-1}p_{n-1}}{b_{2n}} \tag{32}$$

etc.

Illustrative Examples

Consider the Picard kernel

$$K(x) = \delta^2 e^{-k|x|} \doteq \frac{2k\delta^2}{k^2 - s^2}, \qquad k, \delta^2 > 0,$$

$$-k < \mathrm{Re}\, s < k.$$

Here $a_0 = 2k\delta^2$, $b_0 = k^2$, and $c = 1$. From (22) and (we get

$$\tan \frac{\beta T}{2} = -\beta/k,$$

$$\mathrm{ctn} \frac{\beta T}{2} = \beta/k,$$

$$\lambda_r = \frac{k^2 + \beta_r^2}{2k\delta^2}$$

and

$$\phi_r(x) = \frac{k}{\beta_r} \sin \beta_r x + \cos \beta_r x.$$

The $\phi_r(x)$ is not normalized.

A more complicated specimen is the kernel

$$K(x) \doteq \frac{1}{s^4 + 1}, \qquad -\frac{1}{\sqrt{2}} < \mathrm{Re}\, s < \frac{1}{\sqrt{2}}.$$

$$\therefore\ D^+(s) = s^2 + \sqrt{2}s + 1,$$

$$D^-(s) = s^2 - \sqrt{2}s + 1,$$

and

$$D(s^2) - \lambda N(s^2) = s^4 - (\lambda - 1).$$

Let

$$\omega_2(\lambda) = (\lambda - 1)^{\frac{1}{4}} \equiv \epsilon$$

and

$$\omega_1(\lambda) = i\epsilon$$

where Im. $\epsilon \leq 0 \leq$ Re. ϵ. From (24), the eigenva equations are found to be

$$\omega_2 \coth \theta_2 = \omega_1 \coth \theta_1,$$

and

$$\omega_2 \tanh \theta_2 = \omega_1 \tanh \theta_1$$

in which

$$\tanh \theta_1 = \frac{(1 + \omega_1^2) \tanh (\omega_1 T/2) + \sqrt{2}\omega_1}{(1 + \omega_1^2) + \sqrt{2}\omega_1 \tanh (\omega_1 T/2)},$$

$$\tanh \theta_2 = \frac{(1 + \omega_2^2) \tanh(\omega_2 T/2) + \sqrt{2}\omega_2}{(1 + \omega_2^2) + \sqrt{2}\omega_2 \tanh (\omega_2 T/2)}.$$

After replacing ω_1 and ω_2 by their values in terms of ϵ they simplify to

$$\frac{\sqrt{2}\epsilon + (1 + \epsilon^2)\tanh(\epsilon T/2)}{(1 + \epsilon^2) + \sqrt{2}\epsilon \tanh(\epsilon T/2)} = \pm\frac{\sqrt{2}\epsilon + (1 - \epsilon^2)\tan(\epsilon T/2)}{(1 - \epsilon^2) - \sqrt{2}\epsilon \tan(\epsilon T/2)}. \tag{33}$$

Denote their solutions by $\epsilon_1, \epsilon_2, \cdots, \cdots$, the odd subscripts corresponding to the plus sign. Then, $\lambda_r = \epsilon_r^4 + 1$, $(r = 1, 2, \cdots, \cdots)$; from (17),

$$p_{1r} = -\frac{p_0}{\epsilon_r}\tanh \theta_1(\epsilon_r), \qquad r \text{ odd}$$

$$= -\frac{p_0}{\epsilon_r}\coth \theta_1(\epsilon_r), \qquad r \text{ even}$$

and

$$\phi_r(x) \doteqdot \frac{(s^2 + \sqrt{2}s + 1)\left[1 - \frac{s}{\epsilon_r}\tanh \theta_1(\epsilon_r)\right]}{s^4 - \epsilon_r^4}, \qquad r \text{ odd},$$

$$\doteqdot \frac{(s^2 + \sqrt{2}s + 1)\left[1 - \frac{s}{\epsilon_r}\coth \theta_1(\epsilon_r)\right]}{s^4 - \epsilon_r^4}, \qquad r \text{ even},$$

$$\operatorname{Re} s > \epsilon_r,$$

$$0 \leq x \leq T.$$

If this same kernel is used over the semi-infinite range the eigenfunctions are most simple, *viz.*,

$$\phi(x, \lambda) \doteqdot \frac{P(s)(s^2 + \sqrt{2}s + 1)}{s^4 + (1 - \lambda)}, \qquad 0 \leq x \leq \infty.$$

To find the eigenvalues set $\lambda - 1 = r^4 e^{i\rho}$. Then, $\omega_2(\lambda) = r\, e^{i\rho/4}(0 \leq \rho < 2\pi)$ and from (29) $r \cos(\rho/4) < 1/\sqrt{2}$; a point (r, ρ) in this region determines the eigenvalue $\lambda(r, \rho) = 1 + r^4 e^{i\rho}$. Hence the strip of covergence of the above transform is Re $s > r \cos(\rho/4)$. Note that now the eigenvalues are no longer discrete but form a continuum. This is not surprising since K is not square-integrable over $0 \leq x, y < \infty$ and, consequently, the theory of bounded linear symmetric operators is inapplicable.

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The Correlation Function of Smoothly Limited Gaussian Noise*

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Summary—**The correlation function of "smoothly" limited Gaussian noise is calculated and compared with the correlation function of "extremely" clipped Gaussian noise. The limiting function is assumed to have the shape of the error integral curve. The output spectrum is calculated for the case of noise passed through an RC filter.**

I. Introduction

The effect of clipping on Gaussian noise has been investigated in detail.[1-3] Van Vleck has analyzed the case of an amplitude limiter with an output vs input characteristic $y(x)$ described by

$$\begin{aligned} y(x) &= x && \text{for } -x_0 < x < +x_0 \\ &= x_0 && \text{for } x > x_0 \\ &= -x_0 && \text{for } x < -x_0 \end{aligned}$$

and for varing ratios of rms noise to clipping threshold x_0. As this ratio approaches infinity the case of extreme clipping is reached.

Instead of clipping suddenly, when x exceeds the threshold x_0, smooth or gradual limiting may be preferred. The limiter characteristic then will have the aspect of Fig. 1(a). Van Vleck has calculated the autocorrelation function of the output noise for a limiter of the shape

$$y(x) = c_1 \tanh^{-1}(c_2 x).$$

Unfortunately the result cannot be expressed in closed form. Another example is discussed in Laning and Battin.[3]

The present paper assumes a limiter of the form of an error integral curve. This curve has considerable practical importance because:

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[2] S. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N.Y., p. 58; 1950.
[3] J. H. Laning and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., p. 163; 1956.