MATH 590: Meshfree Methods

Chapter 13: Reproducing Kernel Hilbert Spaces and Native Spaces for Strictly Positive Definite Functions

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Outline

In the next few chapters we will present some of the theoretical work on error bounds for approximation and interpolation with radial basis functions.

We focus on strictly positive definite functions (already technical enough), and only mention a few results for the conditionally positive definite case.

The following discussion follows mostly [Wendland (2005a)] where there are many more details.

Our first set of error bounds will come rather naturally once we associate with each (strictly positive definite) radial basic function a certain space of functions called its native space.

We will then be able to establish a connection to reproducing kernel Hilbert spaces, which in turn will give us the desired error bounds as well as certain optimality results for radial basis function interpolation (see Chapter 18).

Reproducing kernels are a classical concept in analysis introduced by Nachman Aronszajn (see [Aronszajn (1950)]).
We begin with

Definition

Let $\mathcal{H}$ be a real Hilbert space of functions $f : \Omega(\subseteq \mathbb{R}^s) \to \mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.
A function $K : \Omega \times \Omega \to \mathbb{R}$ is called reproducing kernel for $\mathcal{H}$ if

(1) $K(\cdot, \boldsymbol{x}) \in \mathcal{H}$ for all $\boldsymbol{x} \in \Omega$,

(2) $f(\boldsymbol{x}) = \langle f, K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$ and all $\boldsymbol{x} \in \Omega$.

Remark

The name reproducing kernel is inspired by the reproducing property (2) above.

It is known that

- the reproducing kernel of a Hilbert space is unique, and
- that existence of a reproducing kernel is equivalent to the fact that the point evaluation functionals $\delta_{\boldsymbol{x}}$ are bounded linear functionals on Ω, i.e., there exists a positive constant $M = M_{\boldsymbol{x}}$ such that

$$|\delta_{\boldsymbol{x}} f| = |f(\boldsymbol{x})| \leq M \|f\|_{\mathcal{H}}$$

  for all $f \in \mathcal{H}$ and all $\boldsymbol{x} \in \Omega$.

  - If K is the reproducing kernel of $\mathcal{H}$, then

$$\delta_{\boldsymbol{x}} f = f(\boldsymbol{x}) = \langle f, K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}$$

    which shows that $\delta_{\boldsymbol{x}}$ is linear. It is also bounded by Cauchy-Schwarz:

$$|\delta_{\boldsymbol{x}} f| = |\langle f, K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}| \leq \|f\|_{\mathcal{H}} \|K(\cdot, \boldsymbol{x})\|_{\mathcal{H}}.$$

  - The converse follows from the Riesz representation theorem.

Other properties of reproducing kernels are given by

Theorem

Suppose $\mathcal{H}$ is a Hilbert space of functions $f : \Omega \to \mathbb{R}$ with reproducing kernel K. Then we have

(1) $K(\boldsymbol{x}, \boldsymbol{y}) = \langle K(\cdot, \boldsymbol{y}), K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}$ *for* $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.

(2) $K(\boldsymbol{x}, \boldsymbol{y}) = K(\boldsymbol{y}, \boldsymbol{x})$ *for* $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.

(3) *Convergence in Hilbert space norm implies pointwise convergence, i.e., if we have*

$$\|f - f_n\|_{\mathcal{H}} \to 0 \quad \text{for } n \to \infty$$

then

$$|f(\boldsymbol{x}) - f_n(\boldsymbol{x})| \to 0 \quad \text{for all } \boldsymbol{x} \in \Omega.$$

Proof.

By Property (1) of the definition of a RKHS above $K(\cdot, \boldsymbol{y}) \in \mathcal{H}$ for every $\boldsymbol{y} \in \Omega$.
Then the reproducing property (2) of the definition gives us

$$K(\boldsymbol{x}, \boldsymbol{y}) = \langle K(\cdot, \boldsymbol{y}), K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.
This establishes (1).
Property (2) follows from (1) by the symmetry of the Hilbert space inner product.
For (3) we use the reproducing property of K along with the Cauchy-Schwarz inequality:

$$|f(\boldsymbol{x}) - f_n(\boldsymbol{x})| = |\langle f - f_n, K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}| \leq \|f - f_n\|_{\mathcal{H}} \|K(\cdot, \boldsymbol{x})\|_{\mathcal{H}}.$$

$\square$

Remark

Property (1) of the previous theorem shows us what the bound $M_{\boldsymbol{x}}$ for the point evaluation functional is:

$$\begin{aligned}
|f(\boldsymbol{x})| = |\delta_{\boldsymbol{x}} f| \quad &\le \quad \|f\|_{\mathcal{H}} \|K(\cdot, \boldsymbol{x})\|_{\mathcal{H}} \\
&= \quad \|f\|_{\mathcal{H}} \sqrt{\langle K(\cdot, \boldsymbol{x}), K(\cdot, \boldsymbol{x})\rangle_{\mathcal{H}}} \\
&= \quad \|f\|_{\mathcal{H}} \sqrt{K(\boldsymbol{x}, \boldsymbol{x})} \\
&\Longrightarrow \quad M_{\boldsymbol{x}} = \sqrt{K(\boldsymbol{x}, \boldsymbol{x})}.
\end{aligned}$$

In particular, for radial kernels

$$K(\boldsymbol{x}, \boldsymbol{x}) = \kappa(\|\boldsymbol{x} - \boldsymbol{x}\|) = \kappa(0),$$

so $M = \sqrt{\kappa(0)}$ — independent of $\boldsymbol{x}$.

Now it is interesting for us that the reproducing kernel K is known to be positive definite.

Here we use a slight generalization of the notion of a positive definite function to a positive definite kernel.

Essentially, we replace $\Phi(\boldsymbol{x}_j - \boldsymbol{x}_k)$ in the definition of a positive definite function by $K(\boldsymbol{x}_j, \boldsymbol{x}_k)$.

We now show this.

Theorem

Suppose $\mathcal{H}$ is a reproducing kernel Hilbert function space with reproducing kernel $K : \Omega \times \Omega \to \mathbb{R}$. Then K is positive definite.

Moreover, K is strictly positive definite if and only if the point evaluation functionals $\delta_{\boldsymbol{x}}$ are linearly independent in $\mathcal{H}^*$.

Remark

Here the space of bounded linear functionals on $\mathcal{H}$ is known as its dual, and denoted by $\mathcal{H}^*$.

Proof.

Since the kernel is real-valued we can restrict ourselves to a quadratic form with real coefficients.
For distinct points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ and arbitrary $\boldsymbol{c} \in \mathbb{R}^N$ we have

$$\begin{aligned} \sum_{j=1}^{N} \sum_{k=1}^{N} c_j c_k K(\boldsymbol{x}_j, \boldsymbol{x}_k) &= \sum_{j=1}^{N} \sum_{k=1}^{N} c_j c_k \langle K(\cdot, \boldsymbol{x}_j), K(\cdot, \boldsymbol{x}_k) \rangle_{\mathcal{H}} \\ &= \langle \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j), \sum_{k=1}^{N} c_k K(\cdot, \boldsymbol{x}_k) \rangle_{\mathcal{H}} \\ &= \| \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j) \|_{\mathcal{H}}^2 \geq 0. \end{aligned}$$

Thus K is positive definite.

Proof (cont.).

To establish the second claim we assume K is not strictly positive definite and show that the point evaluation functionals must be linearly dependent.

If K is not strictly positive definite then there exist distinct points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ and nonzero coefficients c_j such that

$$\sum_{j=1}^{N} \sum_{k=1}^{N} c_j c_k K(\boldsymbol{x}_j, \boldsymbol{x}_k) = 0.$$

The same manipulation of the quadratic form as above therefore implies

$$\sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j) = 0.$$

Proof (cont.).

Now we take the Hilbert space inner product with an arbitrary function $f \in \mathcal{H}$ and use the reproducing property of K to obtain

$$\begin{aligned} 0 &= \langle f, \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j) \rangle_{\mathcal{H}} \\ &= \sum_{j=1}^{N} c_j \langle f, K(\cdot, \boldsymbol{x}_j) \rangle_{\mathcal{H}} \\ &= \sum_{j=1}^{N} c_j f(\boldsymbol{x}_j) = \sum_{j=1}^{N} c_j \delta_{\boldsymbol{x}_j}(f). \end{aligned}$$

This, however, implies the linear dependence of the point evaluation functionals $\delta_{\boldsymbol{x}_j}(f) = f(\boldsymbol{x}_j)$, $j = 1, \ldots, N$, since the coefficients c_j were assumed to be not all zero.

An analogous argument can be used to establish the converse. $\square$

Remark

This theorem provides one direction of the connection between strictly positive definite functions and reproducing kernels.

However, we are also interested in the other direction.

Since the RBFs we have built our interpolation methods from are strictly positive definite functions, we want to know how to construct a reproducing kernel Hilbert space associated with those strictly positive definite basic functions.

In this section we will show that every strictly positive definite radial basic function can indeed be associated with a reproducing kernel Hilbert space — its native space.

First, we note that the definition of an RKHS tells us that $\mathcal{H}$ contains all functions of the form

$$f = \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j)$$

provided $\boldsymbol{x}_j \in \Omega$.

Using the properties of RKHSs established earlier along with the form of f just mentioned we have that

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= \langle f, f\rangle_{\mathcal{H}} = \langle \sum_{j=1}^N c_j K(\cdot, \boldsymbol{x}_j), \sum_{k=1}^N c_k K(\cdot, \boldsymbol{x}_k)\rangle_{\mathcal{H}} \\
&= \sum_{j=1}^N \sum_{k=1}^N c_j c_k \langle K(\cdot, \boldsymbol{x}_j), K(\cdot, \boldsymbol{x}_k)\rangle_{\mathcal{H}} \\
&= \sum_{j=1}^N \sum_{k=1}^N c_j c_k K(\boldsymbol{x}_j, \boldsymbol{x}_k).
\end{aligned}$$

So — for these special types of f — we can easily calculate the Hilbert space norm of f.

Therefore, we define the (possibly infinite-dimensional) space of all finite linear combinations

$$H_K(\Omega) = \operatorname{span}\{K(\cdot, \boldsymbol{y}) : \ \boldsymbol{y} \in \Omega\} \tag{1}$$

with an associated bilinear form $\langle \cdot, \cdot \rangle_K$ given by

$$\langle \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j), \sum_{k=1}^{M} d_k K(\cdot, \boldsymbol{y}_k) \rangle_K = \sum_{j=1}^{N} \sum_{k=1}^{M} c_j d_k K(\boldsymbol{x}_j, \boldsymbol{y}_k).$$

Remark

Note that this definition implies that a general element in $H_K(\Omega)$ has the form

$$f = \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j).$$

However, not only the coefficients c_j, but also the specific value of N and choice of points $\boldsymbol{x}_j$ will vary with f.

Theorem

If $K : \Omega \times \Omega \to \mathbb{R}$ is a symmetric strictly positive definite kernel, then the bilinear form $\langle \cdot, \cdot \rangle_K$ defines an inner product on $H_K(\Omega)$.

Furthermore, $H_K(\Omega)$ is a pre-Hilbert space with reproducing kernel K.

Remark

A pre-Hilbert space is an inner product space whose completion is a Hilbert space.

Proof.

$\langle\cdot,\cdot\rangle_K$ is obviously bilinear and symmetric.
We just need to show that $\langle f, f\rangle_K > 0$ for nonzero $f \in H_K(\Omega)$.
Any such f can be written in the form

$$f = \sum_{j=1}^{N} c_j K(\cdot, \boldsymbol{x}_j), \qquad \boldsymbol{x}_j \in \Omega.$$

Then

$$\langle f, f\rangle_K = \sum_{j=1}^{N}\sum_{k=1}^{N} c_j c_k K(\boldsymbol{x}_j, \boldsymbol{x}_k) > 0$$

since K is strictly positive definite.
The reproducing property follows from

$$\langle f, K(\cdot, \boldsymbol{x})\rangle_K = \sum_{j=1}^{N} c_j K(\boldsymbol{x}, \boldsymbol{x}_j) = f(\boldsymbol{x}).$$

□

Since we just showed that $H_K(\Omega)$ is a pre-Hilbert space, i.e., need not be complete, we now first form the completion $\widetilde{H}_K(\Omega)$ of $H_K(\Omega)$ with respect to the K-norm $\|\cdot\|_K$ ensuring that

$$\|f\|_K = \|f\|_{\widetilde{H}_K(\Omega)} \quad \text{for all } f \in H_K(\Omega).$$

In general, this completion will consist of equivalence classes of Cauchy sequences in $H_K(\Omega)$, so that we can obtain the native space $\mathcal{N}_K(\Omega)$ of K as a space of continuous functions with the help of the point evaluation functional (which extends continuously from $H_K(\Omega)$ to $\widetilde{H}_K(\Omega)$), i.e., the (values of the) continuous functions in $\mathcal{N}_K(\Omega)$ are given via the right-hand side of

$$\delta_{\boldsymbol{x}}(f) = \langle f, K(\cdot, \boldsymbol{x})\rangle_K, \qquad f \in \widetilde{H}_K(\Omega).$$

Remark

The technical details concerned with this construction are discussed in [Wendland (2005a)].

In summary, we now know that the native space $\mathcal{N}_K(\Omega)$ is given by (continuous functions in) the completion of

$$H_K(\Omega) = \operatorname{span}\{K(\cdot, \boldsymbol{y}) : \ \boldsymbol{y} \in \Omega\}$$

— a not very intuitive definition of a function space.

In the special case when we are dealing with strictly positive definite (translation invariant) functions $\Phi(\boldsymbol{x} - \boldsymbol{y}) = K(\boldsymbol{x}, \boldsymbol{y})$ and when $\Omega = \mathbb{R}^s$ we get a characterization of native spaces in terms of Fourier transforms.

We present the following theorem without proof (for details see [Wendland (2005a)].

Theorem

Suppose $\Phi \in C(\mathbb{R}^s) \cap L_1(\mathbb{R}^s)$ is a real-valued strictly positive definite function. Define

$$\mathcal{G} = \{f \in L_2(\mathbb{R}^s) \cap C(\mathbb{R}^s) : \frac{\hat{f}}{\sqrt{\hat{\Phi}}} \in L_2(\mathbb{R}^s)\}$$

and equip this space with the bilinear form

$$\langle f, g\rangle_{\mathcal{G}} = \frac{1}{\sqrt{(2\pi)^s}}\langle \frac{\hat{f}}{\sqrt{\hat{\Phi}}}, \frac{\hat{g}}{\sqrt{\hat{\Phi}}}\rangle_{L_2(\mathbb{R}^s)} = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} \frac{\hat{f}(\omega)\overline{\hat{g}(\omega)}}{\hat{\Phi}(\omega)} d\omega.$$

Then $\mathcal{G}$ is a real Hilbert space with inner product $\langle \cdot, \cdot\rangle_{\mathcal{G}}$ and reproducing kernel $\Phi(\cdot - \cdot)$. Hence, $\mathcal{G}$ is the native space of Φ on $\mathbb{R}^s$, i.e., $\mathcal{G} = \mathcal{N}_\Phi(\mathbb{R}^s)$ and both inner products coincide.
In particular, every $f \in \mathcal{N}_\Phi(\mathbb{R}^s)$ can be recovered from its Fourier transform $\hat{f} \in L_1(\mathbb{R}^s) \cap L_2(\mathbb{R}^s)$.

Another characterization of the native space (of an arbitrary strictly positive definite kernel on a bounded domain Ω) is given in terms of the eigenfunctions of a linear operator associated with the reproducing kernel.

This operator, $T_K : L_2(\Omega) \to L_2(\Omega)$, is given by

$$T_K(v)(\boldsymbol{x}) = \int_\Omega K(\boldsymbol{x}, \boldsymbol{y}) v(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \qquad v \in L_2(\Omega), \quad \boldsymbol{x} \in \Omega.$$

Remark

The eigenvalues λ_k, $k = 1, 2, \ldots$, and eigenfunctions ϕ_k of this operator play the central role in Mercer's theorem [Mercer (1909), Riesz and Sz.-Nagy (1955), Rasmussen and Williams (2006)].

Theorem (Mercer)

Let $K(\cdot,\cdot)$ be a continuous positive definite kernel that satisfies

$$\int_\Omega \int_\Omega K(\boldsymbol{x},\boldsymbol{y})v(\boldsymbol{x})v(\boldsymbol{y})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \geq 0, \qquad \textit{for all } v \in L_2(\Omega),\ \boldsymbol{x},\boldsymbol{y} \in \Omega. \tag{2}$$

Then K can be represented by

$$K(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{\infty} \lambda_k \phi_k(\boldsymbol{x})\phi_k(\boldsymbol{y}), \tag{3}$$

where λ_k are the (non-negative) eigenvalues and ϕ_k are the (L_2-orthonormal) eigenfunctions of T_K.
Moreover, this representation is absolutely and uniformly convergent.

Remark

We can interpret condition (2) as a type of integral positive definiteness.
As usual, the eigenvalues and eigenfunctions satisfy

$$T_K \phi_k = \lambda_k \phi_k$$

or

$$\int_\Omega K(\boldsymbol{x}, \boldsymbol{y}) \phi_k(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \lambda_k \phi_k(\boldsymbol{x}), \qquad k = 1, 2, \ldots.$$

Mercer's theorem allows us to construct a reproducing kernel Hilbert space $\mathcal{H}$ for any continuous positive definite kernel K by representing the functions in $\mathcal{H}$ as infinite linear combinations of the eigenfunctions of T_K, i.e.,

$$\mathcal{H} = \left\{ f : \; f = \sum_{k=1}^{\infty} c_k \phi_k \right\}.$$

It is clear that the kernel $K(\boldsymbol{x}, \cdot)$ itself is in $\mathcal{H}$ since it has the eigenfunction expansion

$$K(\boldsymbol{x}, \cdot) = \sum_{k=1}^{\infty} \lambda_k \phi_k(\boldsymbol{x}) \phi_k$$

guaranteed by Mercer's theorem.

The inner product for $\mathcal{H}$ is given by

$$\langle f, g\rangle_{\mathcal{H}} = \langle \sum_{j=1}^{\infty} c_j\phi_j, \sum_{k=1}^{\infty} d_k\phi_k\rangle_{\mathcal{H}} = \sum_{k=1}^{\infty} \frac{c_k d_k}{\lambda_k},$$

where we used the $\mathcal{H}$-orthogonality

$$\langle \phi_j, \phi_k\rangle_{\mathcal{H}} = \frac{\delta_{jk}}{\sqrt{\lambda_j}\sqrt{\lambda_k}}$$

of the eigenfunctions.

We note that K is indeed the reproducing kernel of $\mathcal{H}$ since the eigenfunction expansion (3) of K and the orthogonality of the eigenfunctions imply

$$
\begin{aligned}
\langle f, K(\cdot,\boldsymbol{x})\rangle_{\mathcal{H}} &= \langle \sum_{j=1}^{\infty} c_j\phi_j, \sum_{k=1}^{\infty} \lambda_k\phi_k\phi_k(\boldsymbol{x})\rangle_{\mathcal{H}} \\
&= \sum_{k=1}^{\infty} \frac{c_k\lambda_k\phi_k(\boldsymbol{x})}{\lambda_k} \\
&= \sum_{k=1}^{\infty} c_k\phi_k(\boldsymbol{x}) = f(\boldsymbol{x}).
\end{aligned}
$$

Finally, one has (*c.f.* [Wendland (2005a)]) that the native space $\mathcal{N}_K(\Omega)$ is given by

$$\mathcal{N}_K(\Omega) = \left\{ f \in L_2(\Omega) : \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left| \langle f, \phi_k \rangle_{L_2(\Omega)} \right|^2 < \infty \right\}$$

and the native space inner product can be written as

$$\langle f, g \rangle_{\mathcal{N}_K} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle f, \phi_k \rangle_{L_2(\Omega)} \langle g, \phi_k \rangle_{L_2(\Omega)}, \qquad f, g \in \mathcal{N}_K(\Omega).$$

Remark

Since $\mathcal{N}_K(\Omega)$ is a subspace of $L_2(\Omega)$ this corresponds to the identification $c_k = \langle f, \phi_k \rangle_{L_2(\Omega)}$ of the generalized Fourier coefficients in the discussion above.

The theorem characterizing the native spaces of translation invariant functions on all of $\mathbb{R}^s$ shows that these spaces can be viewed as a generalization of standard Sobolev spaces.

For $m > s/2$ the Sobolev space W_2^m can be defined as (see, e.g., [Adams (1975)])

$$W_2^m(\mathbb{R}^s) = \{f \in L_2(\mathbb{R}^s) \cap C(\mathbb{R}^s) : \hat{f}(\cdot)(1 + \|\cdot\|_2^2)^{m/2} \in L_2(\mathbb{R}^s)\}. \quad (4)$$

Remark

One also frequently sees the definition

$$W_2^m(\Omega) = \{f \in L_2(\Omega) \cap C(\Omega) : D^\alpha f \in L_2(\Omega) \text{ for all } |\boldsymbol{\alpha}| \le m, \ \boldsymbol{\alpha} \in \mathbb{N}^s\}, \quad (5)$$

which applies whenever $\Omega \subset \mathbb{R}^s$ is a bounded domain.

The former interpretation will make clear the connection between the natives spaces of Sobolev splines (or Matérn functions) and those of polyharmonic splines.

The norm of $W_2^m(\mathbb{R}^s)$ is usually given by

$$\|f\|_{W_2^m(\mathbb{R}^s)} = \left\{ \sum_{|\boldsymbol{\alpha}|\leq m} \|D^{\boldsymbol{\alpha}} f\|^2_{L_2(\mathbb{R}^s)} \right\}^{1/2}$$

or

$$\|f\|_{W_2^m(\mathbb{R}^s)} = \left\{ \int_{\mathbb{R}^s} \left|\hat{f}(\boldsymbol{\omega})\right|^2 (1+\|\boldsymbol{\omega}\|_2^2)^m \mathrm{d}\boldsymbol{\omega} \right\}^{1/2}.$$

According to (4), any strictly positive definite function Φ whose Fourier transform decays only algebraically has a Sobolev space as its native space.

Example

The Matérn functions

$$\Phi_\beta(\boldsymbol{x}) = \frac{K_{\beta-\frac{s}{2}}(\|\boldsymbol{x}\|)\|\boldsymbol{x}\|^{\beta-\frac{s}{2}}}{2^{\beta-1}\Gamma(\beta)}, \qquad \beta > \frac{s}{2},$$

of Chapter 4 with Fourier transform

$$\hat{\Phi}_\beta(\boldsymbol{\omega}) = \left(1 + \|\boldsymbol{\omega}\|^2\right)^{-\beta}$$

can immediately (from ▸ our earlier native space characterization theorem) be seen to have native space

$$\mathcal{N}_{\Phi_\beta}(\mathbb{R}^s) = W_2^\beta(\mathbb{R}^s) \quad \text{with } \beta > s/2$$

which is why some people refer to the Matérn functions as Sobolev splines.

Example

Wendland's compactly supported functions $\Phi_{s,k} = \varphi_{s,k}(\|\cdot\|_2)$ of Chapter 11 can be shown to have native spaces

$$\mathcal{N}_{\Phi_{s,k}}(\mathbb{R}^s) = W_2^{s/2+k+1/2}(\mathbb{R}^s)$$

(where the restriction $s \geq 3$ is required for the special case $k = 0$).

Remark

Native spaces for strictly conditionally positive definite functions can also be constructed.

However, since this is more technical, we limited the discussion above to strictly positive definite functions, and refer the interested reader to the book [Wendland (2005a)] or the papers [Schaback (1999a), Schaback (2000a)].

With the extension of the theory to strictly conditionally positive definite functions the native spaces of the radial powers and thin plate (or surface) splines of Chapter 8 can be shown to be the so-called Beppo-Levi spaces of order k (on a bounded $\Omega \subset \mathbb{R}^s$)

$$\mathrm{BL}_k(\Omega) = \{f \in C(\Omega) : \; D^{\alpha} f \in L_2(\Omega) \text{ for all } |\alpha| = k, \; \alpha \in \mathbb{N}^s\},$$

where D^{α} denotes a generalized derivative of order α (defined in the same spirit as the generalized Fourier transform, see Appendix B).

Remark

In fact, the intersection of all Beppo-Levi spaces $\mathrm{BL}_k(\Omega)$ of order $k \leq m$ yields the Sobolev space $W_2^m(\Omega)$.

In the literature the Beppo-Levi spaces $\mathrm{BL}_k(\Omega)$ are sometimes referred to as homogeneous Sobolev spaces of order k.

Alternatively, the Beppo-Levi spaces on $\mathbb{R}^s$ are defined as

$$\mathrm{BL}_k(\mathbb{R}^s) = \{f \in C(\mathbb{R}^s) : \ \hat{f}(\cdot)\|\cdot\|_2^m \in L_2(\mathbb{R}^s)\},$$

and the formulas given in Chapter 8 for the Fourier transforms of radial powers and thin plate splines show immediately that their native spaces are Beppo-Levi spaces.

The semi-norm on BL_k is given by

$$|f|_{\mathrm{BL}_k} = \left\{ \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \ldots \alpha_d!} \|D^\alpha f\|^2_{L_2(\mathbb{R}^s)} \right\}^{1/2}, \qquad (6)$$

and its (algebraic) kernel is the polynomial space Π^s_{k-1}.

Remark

- *For more details see [Wendland (2005a)].*
- *Beppo-Levi spaces were already studied in the early papers [Duchon (1976), Duchon (1977), Duchon (1978), Duchon (1980)].*

The native spaces for Gaussians and (inverse) multiquadrics are rather small.

Example

According to the Fourier transform characterization of the native space, for Gaussians the Fourier transform of $f \in \mathcal{N}_\Phi(\Omega)$ must decay faster than the Fourier transform of the Gaussian (which is itself a Gaussian).

The native space of Gaussians was recently characterized in [Ye (2010)] in terms of an (infinite) vector of differential operators. In fact, the native space of Gaussians is contained in the Sobolev space $W_2^m(\mathbb{R}^s)$ for any m.

It is known that, even though the native space of Gaussians is small, it contains the important class of so-called band-limited functions, i.e., functions whose Fourier transform is compactly supported.

Band-limited functions play an important role in sampling theory.

Shannon's famous sampling theorem [Shannon (1949)] states that any band-limited function can be completely recovered from its discrete samples provided the function is sampled at a sampling rate at least twice its bandwidth.

Theorem (Shannon Sampling)

Suppose $f \in C(\mathbb{R}^s) \cap L_1(\mathbb{R}^s)$ such that its Fourier transform vanishes outside the cube $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^s$. Then f can be uniquely reconstructed from its values on $\mathbb{Z}^s$, i.e.,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\xi} \in \mathbb{Z}^s} f(\boldsymbol{\xi}) \operatorname{sinc}(\boldsymbol{x} - \boldsymbol{\xi}), \qquad \boldsymbol{x} \in \mathbb{R}^s.$$

Here the sinc function is defined for any $\boldsymbol{x} = (x_1, \ldots, x_s) \in \mathbb{R}^s$ as $\operatorname{sinc} \boldsymbol{x} = \prod_{d=1}^{s} \frac{\sin(\pi x_d)}{\pi x_d}$.

Remark

- *The content of this theorem was already known much earlier (see [Whittaker (1915)]).*
- *The sinc kernel* $K(x,z) = \frac{\sin(x-z)}{(x-z)}$ *is the reproducing kernel of the Paley-Wiener space*

$$PW(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \mathit{supp}\, \hat{f} \subset [-\frac{1}{2}, \frac{1}{2}] \right\}.$$

  It is positive definite since its Fourier transform is the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$*, i.e., the B-spline of order 1 (degree 0).*
- *For more details on Shannon's sampling theorem see, e.g., Chapter 29 in the book [Cheney and Light (1999)] or the paper [Unser (2000)].*

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