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Turning point properties as a method for the characterization of the ergodic dynamics of one-dimensional iterative maps

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Dynamical as well as statistical properties of the ergodic and fully developed chaotic dynamics of iterative maps are investigated by means of a turning point analysis. The turning points of a trajectory are hereby defined as the local maxima and minima of the trajectory. An examination of the turning point density directly provides us with the information of the position of the fixed point for the corresponding dynamical system. Dividing the ergodic dynamics into phases consisting of turning points and nonturning points, respectively, elucidates the understanding of the organization of the chaotic dynamics for maps. The turning point map contains information on any iteration of the dynamical law and is shown to possess an asymptotic scaling behaviour which is responsible for the assignment of dynamical structures to the environment of the two fixed points of the map. Universal statistical turning point properties are derived for doubly symmetric maps. Possible applications of the observed turning point properties for the analysis of time series are discussed in some detail. [S1054-1500(97)00402-3]

A new concept is introduced in order to analyze the ergodic and chaotic motion of discrete one-dimensional dynamical systems. This concept is based on the geometrical notion of a turning point and on the fact that the motion of the turning points yields important dynamical information of the system under consideration. Detailed analysis and comparisons are performed for random and chaotic systems. As a result we show that the chaotic motion possesses very complex and beautiful dynamical structures with respect to its oscillatory behaviour which are not present in random systems. Analytical as well as numerical investigations are performed in order to understand these structures. Our analysis goes beyond the usual description of the ergodic and chaotic dynamics in terms of, for example, Lyapunov exponents or invariant measures and provides a number of new insights which are revealed by considering, for example, the distribution or correlation of the turning points. The turning point map is shown to possess an appealing scaling property and contains information on arbitrarily long time scales (high iterates) of the underlying dynamical law. Statistical properties of the turning points are derived. Using the turning point concept we finally develop a method for the analysis of one-dimensional time series originating from one- or higher dimensional dynamical systems.

Iterated maps represent an important tool for the modeling and simulation of the nonlinear dynamics of more complex systems as they typically occur in physics, chemistry and biology. They are capable of describing characteristic nonlinear features and phenomena of physical systems as different as the irregular behaviour in electronic circuits, chemical reaction dynamics or certain properties of turbulent flows. Of particular relevance are the maps of the unit interval which possess a single maximum. Due to their noninvertibility they exhibit a very rich behaviour when a control parameter is changed continuously. As an example we mention the bifurcation route to chaos and its universal scaling laws which have been studied in detail in the literature [1–4] and are now well understood.

The purpose of the present paper is to investigate the dynamics of smooth and bounded maps for fully developed chaos, i.e., in their chaotic and ergodic state.[5–8] Relevant characterizing quantities for this state are the Lyapunov exponent λ and the invariant density ρ. Our goal is to show and analyze dynamical properties of the chaotic motion beyond these quantities. This will provide us with new insights into the underlying structure of the dynamical chaos of maps. We emphasize that our results are, apart from the statistical turning point properties, qualitatively valid for any smooth symmetric or nonsymmetric map. For our quantitative analysis we will however focus on so-called doubly symmetric maps f which possess both a symmetric map as well as a symmetric invariant density. Doubly symmetric maps are smoothly conjugate to each other and possess all the same maximal Lyapunov exponent $\lambda = \ln 2$. Their subsequent dynamical steps are uncorrelated.[6,7] They are, therefore, considered to have a high degree of randomness.

In the ergodic case of fully developed chaotic dynamics the measure of the chaotic trajectories is equal to one (unstable periodic orbits are dense in this chaotic ''sea'' but of measure zero). We begin our investigation by considering a typical, i.e., chaotic trajectory of a map in the ergodic limit. For reasons of illustration we hereby choose the logistic map, i.e., $x(n+1) = 4x(n)(1-x(n))$. Figure 1 shows such a trajectory for 1.5×10^4 iterations together with its density for 10^6 iterations. The invariant density for the logistic map in its ergodic limit is given by

$$\rho_L(x) = \frac{1}{\pi\sqrt{x(1-x)}} .$$

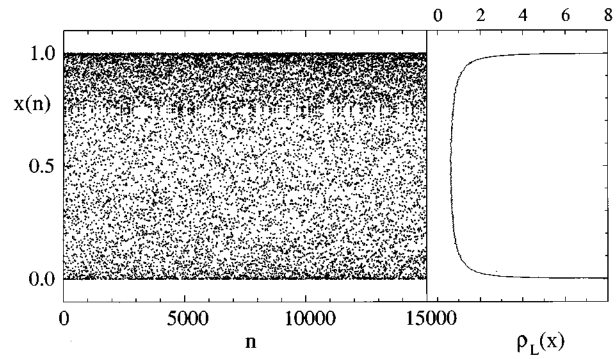

FIG. 1. A chaotic and ergodic trajectory of the logistic map for 1.5×10^4 iterations and the invariant density ρ_L for 10^6 iterations.

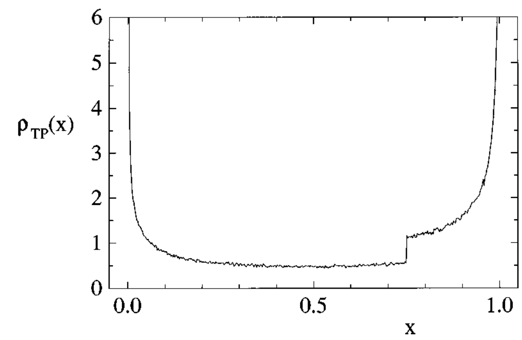

FIG. 2. The density of turning points ρ_{TP} of the logistic map. The step-like structure is exactly at the position of the fixed point $x_F=\frac{3}{4}$.

Both the symmetry around $\frac{1}{2}$ as well as the singular behaviour at $x=0,1$ are clearly reflected in the density of the chaotic trajectory given in Fig. 1. A closer look at the trajectory itself in Fig. 1 suggests that there apparently exists a structure around $x_F=\frac{3}{4}$ which is the position of the unstable fixed point (x_F refers in the following always to the fixed point different from zero). For the following we introduce two notions: a turning point $x(n)$ of a trajectory is a point which represents a local maximum or minimum of the trajectory [see Eq. (1) below for a quantitative criterion of a turning point]. We remark that the name ''turning point'' should not be confused with the same name that often refers to the local extrema of a map f. The center of an oscillation (CO) is defined to be the midpoint between subsequent local maxima and minima (and vice versa). Looking at the above-mentioned regime of the trajectory in Fig. 1 with a higher resolution reveals that the turning points and/or the center of oscillations contain relevant dynamical information and show a characteristic distribution around the position x_F of the fixed point.

In the following we analyze the dynamical and statistical properties of the turning points. This will lead us to a qualitative as well as quantitative understanding of the mechanisms which create, among others, the corresponding structures (see below) in the relevant densities. We begin by deriving a necessary and sufficient condition so that a point of an ergodic trajectory is a turning point. Our starting point is the geometric condition

$$(x(n+1)-x(n))(x(n)-x(n-1))<0 \tag{1}$$

for a turning point $x(n)$ $(x(n+1)=f(x(n)))$. x_1 is the inverse image of the fixed point x_F on the left branch of the map. By using the fact that f is single humped it is then possible to prove that for $0<x(n-1)<x_1$ the next iterated point $x(n)$ becomes a nonturning point whereas for $x_1<x(n-1)<1$ $x(n)$ becomes a turning point. For the logistic map this means $x(n-1)$ has to be larger than $\frac{1}{4}$ in order for $x(n)$ to become a turning point.

With this result we are now able to discuss and understand the occurrence of turning points in successive iterations of the maps, i.e., we gain insights into the dynamics of turning points. Successive iterations in the interval $[0,x_1]$ belong to a stretching phase of the iterative motion. Apart from the first point which is very important they consist according to the above condition only of nonturning points. Once we leave the interval $[0,x_1]$ the first point $x>x_1$ obeys $x\in[x_1,x_F]$ and is also a nonturning point. The next iterated points occur alternately on the right and left hand side of the fixed point, i.e., we obtain a stretching phase of motion around the fixed point x_F which consists exclusively of turning points. This phase ends in case we obtain $x>x^*$ where x^* is the inverse image of x_1 on the right branch of the map. The next iterated point x is then folded back to $[0,x_1]$ and represents a turning point. Subsequently the next stretching phase of nonturning points in the interval $[0,x_1]$ starts, etc. The ergodic dynamics of smooth one-dimensional single humped maps consists therefore of two alternating phases of motion: one phase contains only turning points whereas the other phase consists entirely of nonturning points.

To investigate this further we have illustrated in Fig. 2 the density for the turning points, which we shall abbreviate by ρ_{TP}, for a typical chaotic trajectory of the logistic map. First of all we realize that the density ρ_{TP} exhibits a step-like structure at the position $x=\frac{3}{4}$ of the unstable fixed point. We observe an enhanced probability of finding turning points in the interval $[\frac{3}{4},1]$. In contrast to the invariant density ρ_L, the density of the turning points ρ_{TP} is not symmetric with respect to $\frac{1}{2}$. Its singular behaviour at $x=0$ and $x=1$ is, apart from a constant factor, the same as that for the density ρ_L (see below) and characteristic for the underlying dynamical law.

It is possible to derive the step-like structure of the turning point density ρ_{TP} in Fig. 2 analytically. Using the above condition $x(n-1)>x_1$ for $x(n)$ to become a turning point we obtain the density of the preimages of the turning points: $\mathcal{N}\rho_L(x)\Theta(x-x_1)$. Here Θ is the step function $\Theta(x)=\{1\ \mathit{for}\ x>0;0\ \mathit{for}\ x<0\}$ and $\mathcal{N}$ the corresponding normalization constant. Mapping this density forward with f we get the expression $(\mathcal{N}/2)\,\rho_L(x)(1+\Theta(x-x_F))$ for the density of the turning points. This short derivation clearly shows that the step-like structure of the turning point density at the position of the fixed point is a common property for all smooth maps. Looking at the turning point density therefore provides a useful tool for locating the fixed points of a given map. Going even one step further we suppose that the turning point dynamics provides a useful possibility for analyzing a time series of unknown dynamical origin (see below).

In order to get an impression of the dynamical information contained in Fig. 2 we have to compare the density ρ_{TP} of the turning points of the ergodic dynamics of the logistic map with the density of turning points which is generated by a corresponding weighted random map. The trajectories of this weighted random map possess the same density distribution $\rho_R=\rho_L$ as the logistic map. The density of the turning points of the weighted random map, which we shall abbreviate by ρ_{RTP}, is very similar to the density $\rho_R(x)$.

Indeed the density ρ_{RTP} can be derived in closed form via the following considerations. Given the tent map $g(y)=\{2y \;\; for \;\; y\leqslant \frac{1}{2};(2-2y) \;\; if \;\; y\geqslant \frac{1}{2}\}$ it is well-known that the conjugacy between g and the logistic map f is given by the homeomorphism $h(y)=[1-\cos(\pi y)/2]$, i.e., $h \circ g=f\circ h$. The invariant density of g is equal to one on the whole unit interval. If $y(n)=y$ is a point of the interval, the probability that it is a turning point is $y^2+(1-y)^2$. The normalized random turning point distribution σ_{RTP} belonging to the random map with unit density, i.e., $\sigma_R=1$, is therefore given by $\sigma_{\mathrm{RTP}}=\frac{3}{2}[y^2+(1-y)^2]$. The enhancement of the turning point density σ_{RTP} compared to σ_R at 0 and 1 are geometric boundary effects. Since the conjugacy h preserves the turning points and transports the invariant density for g to the invariant density of f we can treat the application of h as a coordinate change of the unit interval. As a result the weighted random turning point density ρ_{RTP} takes on the following appearance

$$\rho_{\mathrm{RTP}}(x)=\frac{\sigma_{\mathrm{RTP}}(y)}{\pi\sqrt{x(1-x)}}=\frac{6[(\arcsin\sqrt{x})^2+(\arccos\sqrt{x})^2]}{\pi^3\sqrt{x(1-x)}}. \tag{2}$$

ρ_{RTP} is symmetric with respect to $\frac{1}{2}$ and smoothly monotonically decreasing and increasing in $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, respectively. It possesses power singularities at $x=0$ and $x=1$ which are apart from a constant factor the same as for ρ_{R}.

The structures contained in the chaotic ergodic motion become even more obvious if we consider the density of the center of oscillations ρ_{CO} which is illustrated for a typical chaotic trajectory of the logistic map in Fig. 3. First of all we observe that the density ρ_{CO} is nonzero only in a subset of the unit interval (see below). ρ_{CO} possesses three main peaks which are located at $x=0.5, 0.58$ and in particular at $x=0.78$ close to the position of the fixed point. The distribution ρ_{CO} of the center of oscillations shows therefore a highly nontrivial structure which indicates the dynamical information contained in the chaotic motion. Part of these structures can be derived by the following considerations.

Consider the points $x_0=1$ and $x_k=(f|_{[0,1/2]})^{-k}(x_F)$ for $k>0$ which are the preimages of the fixed point on the left branch of the map f. If the point $x(n)=x$ is a turning point and $x\in(x_k,x_{k-1}]$ then the next turning point is $x(n+k)=f^k(x)$. Set

$$\varphi_k(x)=\frac{f^k(x)+x}{2} \tag{3}$$

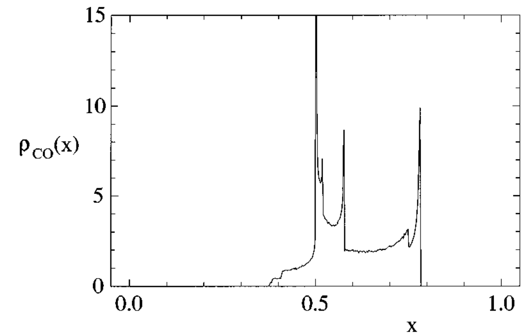

FIG. 3. The density ρ_{CO} for the center of oscillations of the ergodic trajectories of the logistic map. Three main peaks at the positions $x=0.5, 0.58$ and 0.78 as well as a dip at the position of the fixed point x_F are characteristics features.

which defines the center of an oscillation. Then

$$\rho_{\mathrm{CO}}=\sum_{k=1}^{\infty}(\varphi_k)_*(\rho_{\mathrm{TP}}\chi_{(x_k,x_{k-1}]}) \tag{4}$$

where χ_A denotes the characteristic function of A (1 on A and 0 outside A). The Perron–Frobenius operator $(\varphi_k)_*$ transforms the densities in the usual way

$$((\varphi_k)_*(\rho))(x)=\sum_{\varphi_k(y)=x}\frac{\rho(y)}{|\varphi_k'(y)|}. \tag{5}$$

In particular, the support of ρ_{CO} is the union of the supports of the densities $(\varphi_k)_*(\rho_{\mathrm{TP}}\chi_{(x_k,x_{k-1}]})$, that is the union of the sets $(\varphi_k)((x_k,x_{k-1}])$. We therefore have for the logistic map

$$\varphi_1(x)=\frac{5x-4x^2}{2} \tag{6}$$

and $(x_0,x_1]=(\frac{1}{4},1]$. The critical point of φ_1 is $\frac{5}{8}$. In addition $\varphi_1(\frac{1}{4})=\frac{1}{2}$, $\varphi_1(\frac{5}{8})=\frac{25}{32}$, and $\varphi_1(1)=\frac{1}{2}$. Therefore $\varphi_1((x_1,x_0])=[\frac{1}{2},\frac{25}{32}]$.

If $k>1$ and $x\in(x_k,x_{k-1}]$ then $f^{k-1}(x)\in(x_1,x_F]$ and $f^{k-1}>x$ so

$$\begin{aligned}\varphi_k(x)&=\frac{f^k(x)+x}{2}\\&<\frac{f^k(x)+f^{k-1}(x)}{2}=\varphi_1(f^{k-1}(x))\leqslant\frac{25}{32}.\end{aligned} \tag{7}$$

On the other hand, $f^k(x)\in[x_F,1]$, so

$$\varphi_k(x)=\frac{f^k(x)+x}{2}>\frac{f^k(x)}{2}\geqslant\frac{x_F}{2}=\frac{3}{8}. \tag{8}$$

Therefore the support of ρ_{CO} is contained in $[\frac{3}{8},\frac{25}{32}]$. Moreover, for $k>1$ there are points $a_k,b_k\in(x_k,x_{k-1}]$ with $f^k(a_k)=x_F$ and $f^k(b_k)=1$. We have

$$\left[\frac{a_k+x_F}{2},\frac{b_k+1}{2}\right]\subset\varphi_k((x_k,x_{k-1}]), \tag{9}$$

$$\frac{(a_k+x_F)}{2}\leqslant\frac{(x_{k-1}+x_F)}{2},\ \frac{b_k+1}{2}>\frac{1}{2},$$

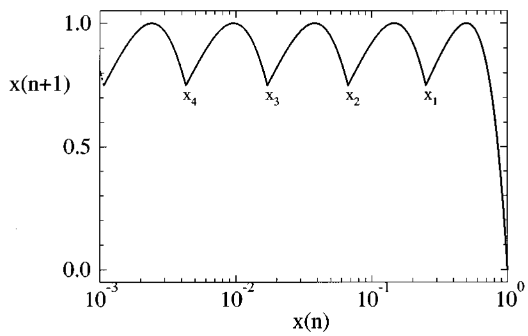

FIG. 4. The turning point map of the logistic map. Many of its characteristic properties are universal for all single humped smooth maps according to the discussion in the text.

so

$$\left[\frac{x_{k-1}+x_F}{2},\frac{1}{2}\right]\subset\varphi_k((x_k,x_{k-1}]). \tag{10}$$

Therefore the support of ρ_{CO} is equal to $[\frac{3}{8},\frac{25}{32}]$.

Apart from the support of the quantity ρ_{CO} also its singularity structure can be derived. ρ_{CO} possesses singularities whenever ρ_{TP} has a singularity or φ_k' is zero. There are two singularities of ρ_{TP}: at 0 and 1. The singularity at 1 gets carried by $(\varphi_1)_*$ to $\frac{1}{2}$, so ρ_{CO} has a singularity at $\frac{1}{2}$. The singularity at 0 gets ''washed out,'' since as $k\rightarrow\infty$, the derivative of φ_k is of order 2^k, whereas ρ_{TP} on $(x_k,x_{k-1}]$ is only of order $2^{-k/2}$. However, the zeros of $\varphi_k{}'$ produce additional singularities of ρ_{CO} (one singularity for every k). In particular, $\varphi_1'(\frac{5}{8})=0$, so we get a singularity of ρ_{CO} at $\frac{25}{32}$. For $k=2$ we have $\varphi_2'(t)=0$ for $t\approx0.163$ (t is a solution to $256t^3-384t^2+160t-17=0$) and we get a singularity of ρ_{CO} at $\varphi_2(t)\approx0.577$. To summarize, the distribution for the center of oscillations ρ_{CO} possesses a highly nontrivial structure on a finite subset of the unit interval and shows in particular infinitely many singularities which reflect the fact that the infinitesimal turning point neighbourhood of the origin 0 is mapped to arbitrary detail to the turning points in the finite neighbourhood of the fixed point x_F (see below in the context of the turning point map).

The corresponding density for the center of oscillations of the weighted random map, which we shall call ρ_{RCO}, is nonzero in the whole interval $]0,1[$ and symmetric with respect to $\frac{1}{2}$. ρ_{RCO} shows only one central sharp peak at $\frac{1}{2}$. Moving away from $x=\frac{1}{2}$, $\rho_{\mathrm{RCO}}(x)$ decays monotonically and eventually vanishes at $x=0,1$. The difference between ρ_{CO} and ρ_{RCO} provides evidence for the dynamical structures in the chaotic and ergodic motion of maps. We emphasize that these observations are in no way restricted to the logistic map.

Let us now consider the turning point map (TPM), i.e., the map of a turning point onto its subsequent turning point. Figure 4 shows the TPM belonging to the logistic map on a logarithmic scale. For reasons of illustration we have chosen the logistic map as a specific example. Apart from the first hump for $x_1<x(n)<1$ the eye-catching feature of the TPM is the extreme similarity of subsequent humps on a logarithmic scale. Approaching smaller and smaller values for $x(n)$ Fig. 4 suggests an asymptotically exact scaling law for subsequent humps. The TPM possesses an accumulation point of its oscillations at the origin and has dimension 1. In order to analyze and understand the structure of the TPM let us investigate how turning points are mapped onto each other. The monotonous part of the TPM for $x_F<x(n)<1$ occurs because the turning points in the interval $[x_F,1]$ are mapped via the TPM onto the interval $[0,x_F]$. The first hump represents the TPM-image of the interval $[x_1,x_F]$. Since turning points in the interval $[x_1,1]$ are mapped directly onto turning points by the map f itself the TPM in $[x_1,1]$ simply reflects the shape of f.

The TPM contains much more information if we turn to smaller values of $x(n)$. It can be shown[9] that the turning points x_i indicated in Fig. 4 up to x_4 are given by successive inverse images of the fixed point x_F on the left branch of the map f. The intervals $\Delta_i=[x_{i+1},x_i]$ each of which is mapped by the TPM onto the interval $[x_F,1]$ as well as the points x_i scale asymptotically with a factor κ which is equal to the derivative of the map f at the origin, i.e., we have $\kappa=f'(0)$. A numerical as well as analytical closer investigation of the ith hump of the TPM in the interval Δ_i reveals that it is identical to the hump of the ith iterated map f^i which is closest to the origin. More precisely: the ith hump of the TPM is identical to the intersection $x(n+1)>x_F$, i.e., the top of the first hump of the ith iterated map f^i. The TPM is therefore given by

$$f_{tp}=f^{i+1}(x),\quad \forall x\in\Delta_i,\quad \forall i\geqslant0 \tag{11}$$

with $\Delta_0=[x_1,1]$. The TPM can therefore be considered as the upper envelope of higher and higher iterated maps when approaching the origin. In addition it can be shown[9] that successive humps of the TPM for decreasing values of $x(n)$ converge, apart from the scaling, against an asymptotic final shape of the humps. This asymptotic form can be obtained by considering the scaling property of the humps $f^{n+1}(x)=f^n(\delta_n x)$ with $\delta_n=s_n/s_{n+1}$ being the ratio of the first roots of successive iterates of the map f. Introducing a new rescaled variable $z=x/s_{n+1}$ which is defined on the whole unit interval the asymptotic functional form is given by

$$h(z)=\lim_{n\rightarrow\infty}f^n(zs_n). \tag{12}$$

This equation provides a method for the systematic approximation of the asymptotic form $h(z)$ by calculating successive iterates of the map as well as their first zeros. The convergence properties of this procedure turn out to be excellent.

The TPM therefore contains valuable information on iterated maps f^i for all i and maps in particular arbitrarily small intervals in the neighbourhood of the origin onto the finite interval $[x_F,1]$. Due to its asymptotic scaling law and/or its convergence to a repeating functional form the TPM can be considered to extract dynamical properties and information from the chaotic and ergodic dynamics of the original map f.

We turn now to an investigation of the statistical turning point properties of the maps f. A measure for the frequency of the turning points is given by the ratio of the number of turning points N_{tp} and the total number of points N_{tot}, i.e., $P_{\mathrm{tp}}=N_{\mathrm{tp}}/N_{\mathrm{tot}}$. One half of that number (that is, only minima counted) has been called wave number[10] or over rotation number[11] in the literature. Since we are dealing with a fully chaotic and ergodic system the quantity P_{tp} is solely determined by the necessary and sufficient condition $x(n-1)>x_1$ for a turning point $x(n)$ and the frequency with which a certain region of the interval is visited. This frequency is however given by the invariant density $\rho_f(x)$ of the map f. We therefore obtain the following expression for the portion of turning points of the chaotic trajectories

$$P_{\mathrm{tp}}=\int_{x_1}^{1}\rho_f(x)\,dx, \tag{13}$$

where x_1 is the inverse image of the fixed point on the left branch of the map f. It can now be shown that the quantity P_{tp} is the same for all doubly symmetric maps, i.e., if we transform the map g into f via the conjugation $u(x)=1-u(1-x)$ we obtain

$$P_{\mathrm{tp}}=\int_{y_1}^{1}\rho_g(y)\,dy \tag{14}$$

with $y_1=(1-y_F)$. This proves the invariance of the statistics of turning points under conjugation. All doubly symmetric maps which are related by conjugation therefore possess the same ratio of turning points and nonturning points and P_{tp} is therefore a universal statistical quantity for this class. Calculating the integral for the specific case of the logistic map yields $P_{\mathrm{tp}}=\frac{2}{3}$ which simply means that statistically two out of three points of the trajectories are turning points.

The above dynamical as well as statistical turning point analysis represents an important tool for the investigation and characterization of the ergodic and chaotic motion of iterative maps and presumably also for other dynamical systems. It goes beyond the usual description of ergodic dynamics by the invariant density or the Lyapunov exponent and reveals universal properties in the dynamical motion as well as the statistical behaviour of the maps. It is well known that the dynamical image of the neighbourhood of the maximum of a map f is responsible for its bifurcation route to chaos and, in the ergodic limit, for the singular behaviour of the invariant density at the boundaries of the unit interval. We have shown that the neighbourhood of the origin (which corresponds to a stretching phase in the dynamical motion of the map) is the relevant regime for the dynamics of the turning points. It is mapped to any detail in an asymptotically converging and scaling way on the finite neighbourhood of the fixed point x_F thereby revealing information on any iterated map f^i. The turning point analysis therefore allows a characterization of different regions of configuration/phase space according to the dynamical properties of the ergodic motion.

The above-discussed characteristics of the turning point dynamics in the neighbourhood of the unstable fixed points of the chaotic system can in fact be used to analyze chaotic time series. Finding the unstable fixed points of a given chaotic time series is an important issue (see Ref. 12 and references therein). We will use in the following the turning point dynamics in order to locate the fixed points (and analogously the higher periodic orbits) of the underlying dynamical system.

Since we are interested in fully chaotic and ergodic systems we encounter only hyperbolic fixed points (HFP). However, we have to distinguish between HFP with and without reflection. For HFP with reflection subsequent iterations of the map in the vicinity of the fixed point oscillate around the fixed point whereas for HFP without reflection a turning point in the neighbourhood of the fixed point is always followed by a stretching phase away from the fixed point. We will describe below an algorithm which allows us to determine the positions of the fixed points (periodic orbits) of a chaotic time series using these properties. The most important feature of our turning point method for the location of the fixed points is the fact that the above-described properties *are not restricted to the linear neighbourhood of the fixed point*. Therefore the position of the fixed point can be approximately determined even in the case when the linear neighbourhood of the fixed point is not visited by the finite trajectory, i.e., time series, at all.

As a first step of our algorithm we select those points of a given time series $\{x(i)|i=1,\ldots,N\}$ which are turning points, i.e., which obey Eq. (1). The next step distinguishes between HFP with and without reflection. To determine the HFPs with reflection we select those points $\{x_t(i)\}$ of the turning point trajectory (resulting from the first step) which have as a next iteration point in the original time series also a turning point $\{x_t(i+1)\}$. We group these points according to the values of their center of oscillations defined as $x_{\mathrm{CO}}(i)=\frac{1}{2}(x_t(i)+x_t(i+1))$. Thereby points with approximately the same values for $x_{\mathrm{CO}}(i)$ belong to the same group. Using the amplitudes of oscillations $d(i)=|x_t(i+1)-x_t(i)|$ we look subsequently within each group for the turning point pair $(\overline{x_t(i)},\overline{x_t(i+1)})$ which corresponds to the minimal amplitude of oscillations. The position of the corresponding HFP is then given by the corresponding center of oscillation $x_F=\frac{1}{2}(\overline{x_t(i)}+\overline{x_t(i+1)})$. By using the minimal amplitude we get rid of the big oscillations which occur due to the effects of the boundaries of the finite interval.

We emphasize that the minimal amplitude of oscillations needs not to be smaller than the size of the linear neighbourhood of the fixed point in order to obtain a good estimation of the position of the fixed point. Our method therefore possesses an advantage compared to methods relying on the properties of the linear regime around the fixed point or recurrence methods. This is due to the fact that the turning around the fixed point is *not* a strongly localized property, and in particular not restricted to the linear regime.

For the HFP without reflection we collect those points of the turning point trajectory which have as subsequent iteration points in the original trajectory nonturning points, i.e., points belonging to a stretching phase of motion. If the stretching phase is increasing the value of the coordinate, the corresponding points belong to the group $\{x_t^+(i)\}$ and in the opposite case to $\{x_t^-(i)\}$. As a next step we look for the extrema defined by $x_F^+ = \min\{x_t^+(i)\}$ and $x_F^- = \max\{x_t^-(i)\}$. The position of the fixed point is finally given by $x_F = \frac{1}{2}(x_F^+ + x_F^-)$.

In order to demonstrate the efficiency of our algorithm we have applied it to a time series of the second iterate of the logistic map $f^2(x) = f(f(x))$ with $f(x) = rx(1-x)$ for $r = 3.9$. Using a times series of only 200 points we are able to determine the positions of the fixed points of the system with a relative accuracy of 0.004. For the positions of the two HFP with reflection we get the values $x_{F1} = 0.36052$ and $x_{F2} = 0.89876$ compared to the exact values $0.35897\ldots$ and $0.89744\ldots$, respectively. For the HFP without reflection we obtain $x_{F3} = 0.74421$ compared to the exact value 0.74359. An analysis of the same time series with, for example, recurrence methods yields a relative accuracy of 0.05 for the position of the fixed points and is therefore much less accurate. Our algorithm for the turning point analysis of time series yields reliable results even if the number of points of the time series is comparatively small.

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