ANNALS OF MATHEMATICS
Vol. 52, No. 3, November, 1950

VECTOR FIELDS ON COMPLEX AND REAL MANIFOLDS

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(Received January 6, 1950)

Previously, see [1]–[5], we have obtained statements linking Riemannian curvature to vector and tensor fields on a compact manifold, whether real or complex, and for the present some supplementary statements will be obtained in which curvature will not appear in the hypotheses.

1. On a compact complex V_{2k}, of real dimension $2k$, we consider a covariant vector field ξ_α, and a contravariant vector field η^α, both holomorphic, and we form the scalar contraction $\xi_\alpha\eta^\alpha$. It is again holomorphic, and if it is so on a compact space, then by the maximum modulus principle, it reduces to a constant

$$\xi_\alpha\eta^\alpha = c. \tag{1}$$

Thus for $k + 1$ vector fields

$$\xi_\alpha^{(j)}, \qquad j = 1, \cdots, k+1 \tag{2}$$

we obtain the system of relations

$$\xi_\alpha^{(j)}\eta^\alpha = c^j, \qquad j = 1, \cdots, k+1 \tag{3}$$

and from this we will deduce the following conclusion.

THEOREM 1. *If on a compact complex V_{2k} there exist $k + 1$ analytic vector fields* (2) *such that for no system of constants*

$$(c^1, \cdots, c^{k+1}) \neq (0, \cdots, 0) \tag{4}$$

the determinant

$$| \xi_1^{(j)}, \cdots, \xi_k^{(j)}, \qquad c^j |_{j=1,\cdots,k+1} \tag{5}$$

vanishes identically, then there exists on V_{2k} no contravariant analytic vector field η^α or tensor field $\eta^{\alpha_1\cdots\alpha_p}$ other than zero. In this statement, "covariant" and "contravariant" may be interchanged.

PROOF. If for a vector η^α we form the system (3) then, by our hypothesis, the constants c^j in it must all vanish. However our assumption also implies that the rank of the k by $k + 1$ matrix (2) must at some point have its maximal value k, and an analytic solution η^α of the system

$$\xi_\alpha^{(j)}\eta^\alpha = 0 \tag{6}$$

must then be zero, as claimed. For tensor fields $\eta^{\alpha_1\cdots\alpha_p}$ we may now apply induction on p. Consider for instance a tensor field $\eta^{\alpha\beta}$. Each entity

$$\zeta^{(j)\beta} = \xi_\alpha^{(j)}\eta^{\alpha\beta}$$

is a contravariant vector field, and hence it vanishes identically by what we have just proved, but any solution $\eta^{\alpha\beta}$ of the system

$$\xi_\alpha^{(j)}\eta^{\alpha\beta} = 0, \qquad j = 1, \cdots, k+1$$

must vanish identically too.

2. The hypothesis of Theorem 1 may be replaced by an alternate one in which general constants no longer occur. By differentiation of (3) we obtain

$$\xi_\alpha^{(j)} \frac{\partial\eta^\alpha}{\partial z_\beta} + \frac{\partial\xi_\alpha^{(j)}}{\partial z_\beta}\eta^\alpha = 0 \qquad \beta = 1, \cdots, k; j = 1, \cdots, k+1,$$

and if we consider this as a system of equations in the $k(k+1)$ variables

$$\eta^\alpha, \qquad \frac{\partial\eta^\alpha}{\partial z_\beta}$$

then it gives rise to a certain determinant

$$D_k\left(\xi_\alpha^{(j)}; \frac{\partial\xi_\alpha^{(j)}}{\partial z_\beta}\right) \tag{7}$$

which is a polynomial in the quantities inside the brackets. Now, an alternate hypothesis for Theorem 1 would be that this quantity (9) shall not vanish identically on V_{2k}, and we note that this would be an invariantive hypothesis in the sense that, for any point P_0, the non-vanishing of (9) is invariant with respect to a transformation of coordinates which leaves the point P_0 fixed. In fact, $D_k \neq 0$ at a point P_0 is a necessary and sufficient assumption for the conclusion that if for a system of power series

$$\eta^\alpha = a^\alpha + \textstyle\sum_{\beta=1}^k a_\beta^\alpha z_\beta + \text{(higher powers)}$$

with origin at P_0 we have

$$\zeta_\alpha^{(j)}\eta^\alpha = c^j + \text{(non-linear terms)}$$

then $a^\alpha = 0$, $a_\beta^\alpha = 0$.

3. For any $p \geqq 1$, for any two analytic tensor fields $\xi_{\alpha_1\cdots\alpha_p}$, $\eta^{\alpha_1\cdots\alpha_p}$ we again have

$$\xi_{\alpha_1\cdots\alpha_p}\eta^{\alpha_1\cdots\alpha_p} = c;$$

and if the two tensor fields are skew symmetric then for $p = k$ they are determined by the one component corresponding to $\alpha_1 = 1, \cdots, \alpha_k = k$, so that we then have

$$\xi_{1\cdots k}\eta^{1\cdots k} = c.$$

Any skew-tensor $\xi_{\alpha_1\cdots\alpha_k}$ determines a k-tuple Abelian integral of the first kind, and in keeping with the nomenclature of algebraic geometry we denote by p_g

(geometric genus) the number of linearly independent ones for constant complex coefficients. Now, for $p_g \geqq 2$ we have

$$\xi^{(j)}_{1\cdots k}\, \eta^{1\cdots k} = c^j, \qquad j = 1, 2,$$

and this implies

$$\eta^{1\cdots k} = 0. \tag{8}$$

We are thus led to the following theorem which for algebraic surfaces has been given by Picard [7].

THEOREM 1. *If the geometric genus of V_{2k} exceeds one, $p_g \geqq 2$, then the total group of complex homeomorphisms of V_{2k} cannot be transitive, not even locally so in the neighborhood of any point.*

PROOF. In fact the total group of such homeomorphisms is a complex Lie group (compare the discussion in [3]), and if it is locally transitive in the neighborhood of some point then Lie's own local theory implies that there exist k contravariant analytic vector fields $\eta^{\alpha}_{(j)}$; $j = 1, \cdots, k$, such that in the neighborhood of that point the determinant

$$|\, \eta^{\alpha}_{(j)} \,| \qquad \alpha, j = 1, \cdots, k$$

is $\neq 0$. However this determinant has the same transformation law as a quantity $\eta^{1\cdots k}$ which defines a skew symmetric tensor, and by (8) it must therefore vanish, which is contradictory.

Another obvious theorem (compare Picard l.c. and Painlevé [8]) is as follows.

THEOREM 3. *If $p_g \geqq 1$, and if there exists a one-parameter group of complex homeomorphisms (at least) then there must exist a skew symmetric analytic tensor $\zeta_{\alpha_1\cdots\alpha_{k-1}}$ which does not vanish identically.*

In fact, if $\xi_{\alpha_1\cdots\alpha_k}$ is skew and $\not\equiv 0$ and $\eta^{\alpha} \not\equiv 0$, then

$$\zeta_{\alpha_1\cdots\alpha_{k-1}} = \xi_{\alpha_1\cdots\alpha_k}\eta^{\alpha_k}$$

is $\not\equiv 0$.

4. We will now describe a connection with curvature by means of a metric which was first introduced in [2], and then utilized in [6] for the evaluation of the Euler-Poincaré characteristic.

If there are given r covarient analytic vector fields

$$\xi^{(j)}_{\alpha}, \qquad j = 1, \cdots, r, \tag{9}$$

with $r \geqq k + 1$ (we will later explain why not $r \geqq k$), and if we take any numerical positive definite Hermitian matrix A_{jl}, for instance the unit matrix $A_{jl} = \delta_{jl}$, then the expression

$$g_{\alpha\beta^*} = \textstyle\sum_{j,l} A_{jl}\, \xi^{(j)}_{\alpha}\, \overline{\xi^{(l)}_{\beta}} \tag{10}$$

defines a positive semi-definite Hermitian tensor field on V_{2k}. If we assume that the rank of the matrix (9) has its maximal value k at all points *without exception* then (10) is positive definite and may be used as a metric tensor. Next, if each vector field (9) satisfies the relation

$$\frac{\partial \xi_\alpha^{(j)}}{\partial z_\beta} = \frac{\partial \xi_\beta^{(j)}}{\partial z_\alpha}, \tag{11}$$

that is if it gives rise to a simple Abelian integral of the first kind

$$w_j = \int^z \xi_\alpha^{(j)} \, dz_\alpha, \tag{12}$$

then our metric tensor enjoys the crucial Kaehler property

$$\frac{\partial g_{\alpha\beta^*}}{\partial z_\gamma} = \frac{\partial g_{\gamma\beta^*}}{\partial z_\alpha} \tag{13}$$

and it derives from a locally Euclidean imbedding of our manifold V_{2k} into the Abelian variety of the coordinates (12) with the line element $\sum A_{jl} \, dw_j \, d\overline{w_l}$. We have shown in [2] that the Ricci tensor is then non-positive, having the value

$$-R_{\alpha\delta^*} = -\sum_{jl} A_{jl} \, g^{\gamma\beta^*} \xi_{\alpha,\gamma}^{(j)} \overline{\xi_{\beta,\delta}^{(l)}}, \tag{14}$$

where $g^{\gamma\beta^*}$ is the inverse to (10) and $\xi_{\alpha,\gamma}^{(j)}$ is the covariant derivative of $\xi_\alpha^{(j)}$.

Now, we have proved in [1] that if on a compact complex V_{2k} with an Hermite-Kaehler metric the Ricci curvature is everywhere non-positive and at some point strictly negative then no contravariant analytic vector or tensor field may exist, which is the same conclusion as in Theorem 1. Thus, to a certain extent at least, our present Theorem 1 may be viewed as a particular case of a more extensive theorem in which a curvature hypothesis is the decisive assumption on which everything hinges.

5. On the face of it we might have admitted the possibility $r = k$ in the definition of (10) but this would have been illusory as the following theorem will imply.

THEOREM 4. *For $r \geqq k$, if there are given r vector fields* (9) *which are linearly independent for constant complex coefficients; if the matrix* (9) *has everywhere rank k and if* (11) *holds; then for $r \geqq k + 1$ the Ricci tensor* (14) *is not identically zero for any A_{jl}, but for $r = k$ it is identically zero for every A_{jl} and V_{2k} is topologically a torus.*

PROOF. (14) vanishes at a point if we there have

$$\xi_{\alpha,\gamma}^{(j)} = 0. \tag{15}$$

For $r \geqq k + 1$ we pick a point P_0 and form a linear combination

$$\xi_\alpha = \sum_{j=1}^{r} a_j \xi_\alpha^{(j)} \tag{16}$$

with constants a_j not all zero, such that

$$\xi_\alpha(P_0) = 0.$$

Now (15) implies $\xi_{\alpha,\gamma} = 0$, that is

$$\frac{\partial \xi_\alpha}{\partial z_\beta} = \xi_\gamma \Gamma^\gamma_{\alpha\beta}, \tag{17}$$

and to this we may add the relations

$$\frac{\partial \xi_\alpha}{\partial \bar{z}_\beta} = 0. \tag{18}$$

But (17) and (18) taken together form a system of differential equations to which the classical uniqueness theorem applies and thus (16) would vanish identically, contrary to assumption.

For $r = k$, since, by assumption,

$$\det | \xi^{(j)}_\alpha | \not\equiv 0, \tag{19}$$

the following situation arises. The Abelian variety W as defined by (12) is topologically a torus and its complex analytic universal covering space is the whole open space. Also, V_{2k} maps into W locally topologically so that V_{2k} in its turn is an unbranched covering of W. Therefore V_{2k} itself is also a torus which is holomorphically covered by the total open space, and our assertion that its Ricci tensor (14) vanishes identically is an immediate consequence of the following lemma.

Every one-one analytic transformation

$$w_j = f_j(z_1, \cdots, z_k), \qquad j = 1, \cdots, k \tag{20}$$

of E_{2k} into itself which transforms a fundamental domain of multiperiodicity into another must be a linear transformation.

In fact, the functions (20) are such that for a period $(\omega_1, \cdots, \omega_k)$ we have

$$f_j(z_1 + \omega_1, \cdots, z_k + \omega_k) = f_j(z_1, \cdots, z_k) + \text{const.} \tag{21}$$

Therefore each partial derivative $\partial w_j / \partial z_l$ is multiperiodic in the ordinary sense and hence a constant.

6. We will now have a peculiar postscript to Theorem 1 which again will refer to the case $r = k$. As on previous occasions, see [1] and [6], we introduce the universal covering space $\tilde{V}$ of our compact manifold $V \equiv V_{2k}$ and the fundamental group $\Gamma = \tilde{V}/V$, and we consider on $\tilde{V}$ a set of vector fields

$$\xi^{(j)}_\alpha, \qquad j = 1, \cdots, k \tag{22}$$

which every element γ of Γ transforms into a linear combination

$$\xi'^{(j)}_\alpha = \textstyle\sum_l a^j_{l(\gamma)} \xi^{(l)}_\alpha \tag{23}$$

where each matrix

$$\{a^j_{l(\gamma)}\} \tag{24}$$

is a unitary one.

Now, if η^α is defined on V then we can extend it periodically to all of $\tilde{V}$, see [3], and if we form the scalar functions

$$\varphi^{(j)} = \xi_\alpha^{(j)}\eta^\alpha \tag{25}$$

then each of these holomorphic functions on V is almost periodic relative to the group Γ in the following sense: any infinite sequence $\{\gamma_p\}$ contains an infinite subsequence $\{\delta_p\}$ such that the sequence of functions $\varphi^{(j)}(\delta_p P)$ is convergent towards a limiting function $\psi^{(j)}(P)$, as $p \to \infty$, uniformly in all of $\tilde{V}_1$ see [1]; and the "reverse" sequence $\psi^{(j)}(\delta_p^{-1}P)$ converges back towards the function $\varphi^{(j)}(P)$ from which we started. Now if $\varphi^{(j)}$ is holomorphic then each limiting function $\psi^{(j)}$ is holomorphic, and one at least may be chosen to attain the maximum of its absolute value. Thus it is a constant and so is the original function itself,

$$\varphi^{(j)} = c^j. \tag{26}$$

But η^α is assumed invariant with regard to Γ and (23) and (26) together imply

$$c^j = \textstyle\sum_l a_l^j(\gamma)c^l. \tag{27}$$

Thus, the vector $(c^1, \cdots, c^k)$ is an eigenvector of the entire group (24). If all its components are zero, and the rank of (22) is k somewhere, then (26) implies $\eta^\alpha = 0$. If, however, the vector is not zero, then after a change of basis in the linear space based on the k vectors (22) we may assume $c^1 \neq 0$, $c^2 = \cdots = c^k = 0$. This implies $a_1^1(\gamma) = 1$ and since the matrix is unitary it becomes fully reducible and the vector $\xi^{(1)}$ of the new basis is invariant with regard to Γ. Hence the following theorem.

THEOREM 5. *If there are given k vector fields* (22) *on an unbranched covering space $\tilde{V}$ of V then either a linear combination of them is one-valued on V itself, or there exists no contravariant vector field η^α on V.*

6. Finally we will make a statement on real manifolds. On such a manifold the maximal modulus principle applies to a solution of a Laplacean; and in order to have a Laplacean a positive definite symmetric tensor g_{ij} must be available to start with, that is to say the manifold must be given as a Riemannian space, and we will assume that it is so given. But on a real Riemannian space, whether differentiable or even real analytic, there is no distinction of "holomorphy" as between "covariant" and "contravariant" and the substitute for it is the distinction between a harmonic vector and a Killing vector. A harmonic vector has the two properties

$$\xi_{i,j} = \xi_{j,i} \tag{28}$$

$$g^{ij}\xi_{i,j} = 0, \tag{29}$$

and a Killing vector has the property

$$\eta_{i,j} = -\eta_{j,i} \tag{30}$$

which automatically implies

$$g^{ij}\eta_{i,j} = 0. \tag{31}$$

We now claim that in consequence of these properties the scalar

$$\varphi = g^{ab}\xi_a\eta_b \equiv \xi_a\eta^a$$

satisfies the Laplace equation

$$\Delta\varphi = 0 \tag{32}$$

where

$$\Delta\varphi = g^{rs}\varphi_{,r,s}\,. \tag{33}$$

In fact, on carrying out differentiations, (33) will be the sum of

$$T_1 = g^{rs}g^{ab}(\xi_{a,r,s}\eta_b + \xi_a\eta_{b,r,s})$$

and

$$T_2 = 2g^{rs}g^{ab}\xi_{a,r}\eta_{b,s}$$

and these two terms vanish separately. For T_2 this follows from (28) and (30) since these two relations transform it into its negative

$$-2g^{rs}g^{ab}\xi_{r,a}\eta_{s,b}$$

For T_1 we first of all obtain

$$T_1 = g^{rs}g^{ab}(\xi_{r,a,s}\eta_b - \xi_a\eta_{r,b,s}), \tag{34}$$

and if we put

$$\xi_{r,a,s} = \xi_{r,s,a} + \xi_\lambda R^\lambda_{r,a,s}$$

$$\eta_{r,b,s} = \eta_{r,s,b} + \eta_\mu R^\mu_{r,b,s}$$

then (34) is the sum of the term

$$g^{rs}g^{ab}(\xi_{r,s,a}\eta_b - \xi_a\eta_{r,s,b}) \equiv g^{ab}\eta_b(g^{rs}\xi_{r,s})_a - g^{ab}\xi_a(g^{rs}\eta_{r,s})_{1b}$$

which is zero on account of (29) and (31), and of the term

$$g^{ab}\xi_\lambda\eta_b g^{rs}R^\lambda_{r,a,s} - g^{ab}\xi_a\eta_\mu g^{rs}R^\mu_{r,b,s}$$

which vanishes identically.

But if (32) holds on a compact M_n then $\varphi = c$, and thus

$$\xi_i\eta^i = c,$$

and this leads to the following theorem.

THEOREM 6. *If on a compact Riemannian space, M_n there are given $n + 1$ harmonic vector fields $\xi_i^{(j)}$, $j = 1, \cdots, n + 1$, such that for no system of constant $(c^1, \cdots, c^{n+1}) \neq (0, \cdots, 0)$ the determinant*

$$|\, \xi_1^{(j)}, \cdots, \xi_n^{(j)}, c^j \,|\; j = 1, \cdots, n + 1$$

vanishes identically, or if the determinant

$$D_n\left(\xi_i^{(j)};\ \frac{\partial \xi_i^{(\nu)}}{\partial \xi_l}\right)$$

is different from 0 *on a dense set; then there exists on* M_n *no Killing vector other than zero. In this statement "harmonic vector" and "Killing vector" may be interchanged.*

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