# Decay of turbulence in the final period

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The final period of decay of a turbulent motion occurs when the effects of inertia forces are negligible. Under these conditions the instantaneous velocity distribution in the turbulence field may be solved as an initial value problem. It is shown that homogeneous turbulence tends to an asymptotic statistical state which is independent of the initial conditions. In this asymptotic state the energy of turbulence is proportional to $t^{-\frac{5}{2}}$ and the longitudinal double-velocity correlation coefficient for two points distance $r$ apart is $e^{-r^2/8\nu t}$, where $t$ is the time of decay. The asymptotic time-interval correlation coefficient is found to be different from unity for very large time intervals only, showing the aperiodic character of the motion. The whole field of motion comes gradually to rest, smaller eddies decaying more rapidly than larger eddies, and the above stable eddy distribution is established when only the largest eddies of the original turbulence remain.

Relevant measurements have been made in the field of isotropic turbulence downstream from a grid of small mesh. The above energy decay and space-interval correlation relations are found to be valid at distances from the grid greater than 400-mesh lengths and at a mesh Reynolds number of 650. The duration of the transitional period, in which the energy decay law is changing from that appropriate to the initial period of decay to the above asymptotic law, increases very rapidly with $R_M$. There is a brief discussion of the criterion for the existence of final period decay, although clarification must wait until the existence and termination of the initial period of decay are better understood.

## 1. Introduction

It has long been a standard practice in hydrodynamical research to seek conditions under which the Navier-Stokes equations become linear. For instance, the theories of slow motion of a viscous fluid, and of small perturbations to a basic flow, have each been well developed. It is thus natural to inquire if there are any circumstances under which the quadratic inertia terms play no part in the problem of turbulent motion. The immediate answer is that when the Reynolds number is low enough the inertia forces are negligible compared with the viscous forces, and the governing equations are effectively linear. Despite this tremendous gain in simplicity, considerations of low Reynolds states of turbulence are rare in the literature, and not always accurate. Theoretical investigators may have hesitated on account of the fact that no experimental evidence demonstrating clearly that inertia forces were negligible has yet been presented. It will be seen below that this is simply because measurements of turbulence have not been made at sufficiently low Reynolds numbers.

It is the essence of the phenomenon of turbulence that there is a continual transfer of energy from large to small eddies. This transfer, which maintains the high rate of dissipation characteristic of turbulent motion, is entirely a result of the action of inertia effects. Consequently it is slightly anomalous to speak of a low Reynolds number state of turbulent motion in which inertia effects play no part; local regions of the field will in fact have a stable *laminar* motion. The motion is only turbulent in the sense that it is the last stage of decay of a motion which was formerly turbulent in the precise sense. But it would be too confusing to term it a stable or laminar state of turbulence, and we propose that it be called the final period of decay of the turbulence. The term has already been suggested in a previous paper (Batchelor & Townsend 1948) in a consideration of the decay of isotropic turbulence. In that paper it was shown that there is an initial period in the decay of isotropic turbulence during which the energy decays according to the simple law

$$\overline{u^2} \propto t^{-1}, \tag{1·1}$$

and the viscous and inertia forces are of comparable importance, and that after a certain time there is a transitional period during which the law of energy decay is changing. The inference—it was not possible to obtain direct evidence by making measurements at greater times of decay—was that the law of energy decay was tending towards that which ultimately obtains in the final period, viz.

$$\overline{u^2} \propto t^{-\frac{5}{2}}, \tag{1·2}$$

the inertia forces then being negligible. The measurements to be presented below are concerned wholly with isotropic turbulence and do in fact confirm the law (1·2). But it should be noted that all decaying turbulent motions will have their final period, with certain characteristics in common. Some of these common features emerge from the analytical discussion to be given in §§ 3 and 4.

## 2. Recapitulation of previous discussions of isotropic turbulence at small Reynolds numbers

All previous investigators have considered the law of energy decay and the double-velocity correlation. The turbulent energy per unit mass of fluid is $\frac{3}{2}\overline{u^2}$, while the double-velocity correlation is completely specified by $\overline{u^2}$ and the function $f(r,t)$, where $\overline{u^2}f(r,t)$ is the correlation at time $t$ between parallel velocity components at two points distance $r$ apart along a line parallel to the velocities. With the neglect of any inertia forces, the Navier-Stokes equations supply the following equation for $f$ and $\overline{u^2}$:

$$\frac{\partial f\overline{u^2}}{\partial t} = 2\nu\left(\frac{\partial^2 f\overline{u^2}}{\partial r^2} + \frac{4}{r}\frac{\partial f\overline{u^2}}{\partial r}\right), \tag{2·1}$$

where $\nu$ is the kinematic viscosity of the fluid. When $r = 0$, (2·1) degenerates to the energy equation

$$\frac{d\overline{u^2}}{dt} = 10\nu\overline{u^2}f_0'' = -\frac{10\nu\overline{u^2}}{\lambda^2}, \tag{2·2}$$

where $\lambda$ is the dissipation length parameter.

Von Kármán & Howarth (1938) appear to be the first authors to have neglected deliberately the inertia effects in order to investigate isotropic turbulence at low Reynolds numbers. They considered only those solutions of (2·1) which are functions of $r/\lambda$ ($= \psi$, say) alone, and effectively assumed that the energy decays according to the law

$$\overline{u^2} \propto t^{-n}, \tag{2·3}$$

and thence that $\lambda^2 = (10/n)\,\nu t$, where $n$ is left undetermined. They showed that there is a singly infinite family of such solutions involving $n$ as parameter, viz.

$$f(\psi) = \left(\frac{4n}{5}\right)^{\frac{1}{4}} \psi^{-\frac{5}{2}} \exp\left[-\frac{5}{8n}\psi^2\right] M_{n-\frac{5}{4},\frac{3}{4}}\left(\frac{5}{4n}\psi^2\right), \tag{2·4}$$

where $M_{k,m}(z)$ is a confluent hypergeometric function. There does not appear to be any absolute reason why the solution (2·3) and (2·4) for an arbitrary value of $n$ should not represent the way in which the turbulence changes in practice provided that the appropriate correlation function can be produced at any instant. But as has already been pointed out (Batchelor 1948), it appears to be possible to do this only when $n = \frac{5}{2}$, so far as turbulence produced in the laboratory is concerned. When $n < \frac{5}{2}$, the function (2·4) is such that $\int_0^\infty \psi^4 f(\psi)\,d\psi$ is infinite, which is unlikely to be true under practical conditions, while when $n > \frac{5}{2}$, $\int_0^\infty \psi^4 f(\psi)\,d\psi = 0$ and $f(\psi)$ is negative for a certain range of values of $\psi$, unlike measured values of $f$. Since $\Lambda, = \overline{u^2}\int_0^\infty r^4 f(r)\,dr$, is independent of the time of decay (Loitsiansky 1939), if $\int_0^\infty r^4 f\,dr$ is neither zero not infinite at any instant, it will remain so. Thus the particular case $n = \frac{5}{2}$, which leads to

$$\overline{u^2} \propto t^{-\frac{5}{2}}, \quad f(r,t) = e^{-r^2/8\nu t}, \tag{2·5}$$

is the self-preserving solution which one might expect to find in an experiment.

In the same year, Reissner (1938) considered initial-value solutions of the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{2·6}$$

for the velocity $u(x,t)$ in a one-dimensional turbulence field, with the notion that this represents a simplified model of real turbulent motion. He finds that the correlation function at time $t$ is related to the velocity field at time $t_0$ by

$$\overline{u^2} f(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\nu\lambda^2(t-t_0)} \cos\lambda r \lim_{\alpha\to\infty} \frac{\left[\int_{-\alpha}^{\alpha} u(x,t_0)\cos\lambda x\,dx\right]^2}{2\alpha}\,d\lambda. \tag{2·7}$$

Letting $t\to\infty$ to obtain asymptotic expressions (which is a legitimate and useful operation in an inquiry into conditions in the final period of decay), Reissner shows that(2·7) becomes

$$\overline{u^2} \to \frac{1}{\sqrt{[8\pi\nu(t-t_0)]}} \lim_{\alpha\to\infty} \frac{\left[\int_{-\alpha}^{\alpha} u(x,t_0)\,dx\right]^2}{2\alpha}, \tag{2·8}$$

the correlation coefficient $f$ being effectively unity. He states an analogous result for three-dimensional turbulence in which each velocity component obeys the equation

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{2·9}$$

viz.

$$\overline{u^2} \to \frac{\text{const.}}{(t-t_0)^{\frac{3}{2}}} \lim_{\alpha,\beta,\gamma\to\infty} \frac{\left[\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\gamma}^{\gamma} u(x,y,z,t_0)\,dx\,dy\,dz\right]^2}{8\alpha\beta\gamma}. \tag{2·10}$$

This energy decay law of Reissner is not correct, even though equation (2·9) represents correctly the motion of the fluid in the final period. It can be shown that as a consequence of the condition of incompressibility of the fluid, the constant in (2·10) is identically zero, and a higher order term varying as $(t-t_0)^{-\frac{5}{2}}$ replaces the expression (2·10). It will be seen in the following sections that Reissner's idea of solving the equation for the instantaneous local velocity in the field is capable of yielding many useful results.

Loitsiansky (1939) has remarked that equation (2·1) is similar to the equation for the conduction of heat in a spherically symmetrical five-dimensional temperature field. Using this analogy, the relation between $\overline{u^2}f$ at times $t$ and $t_0$ is known to be

$$\overline{u^2}(t)f(r,t) = \frac{1}{[8\pi\nu(t-t_0)]^{\frac{5}{2}}}\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} \overline{u^2}(t_0)f(s,t_0)\exp\left[-\frac{\rho^2}{8\nu(t-t_0)}\right]dx_1\,dx_2\,dx_3\,dx_4\,dx_5, \tag{2·11}$$

where $$\rho^2 = \sum_{n=1}^{5}(\xi_n - x_n)^2, \quad r^2 = \sum_1^5 \xi_n^2, \quad s^2 = \sum_1^5 x_n^2.$$

The law of energy decay is found by putting $r = 0$ in (2·11), i.e.

$$\overline{u^2}(t) = \frac{8\pi^2\overline{u^2}(t_0)}{3[8\pi\nu(t-t_0)]^{\frac{5}{2}}}\int_0^\infty f(s,t_0)\exp\left[-\frac{s^2}{8\nu(t-t_0)}\right]s^4\,ds. \tag{2·12}$$

Thus, in general, the law of energy decay and the correlation function depend on, and are uniquely determined by, the conditions at time $t_0$. Loitsiansky points out that for the special initial conditions given by

$$\left.\begin{aligned} (\overline{u^2}f)_{t_0} &= \infty \quad \text{when} \quad r = 0,\\ &= 0 \quad \text{when} \quad r > 0,\\ \int_0^\infty (\overline{u^2}f)_{t_0} r^4\,dr &= \Lambda \text{ (finite)}, \end{aligned}\right\} \tag{2·13}$$

the integrals of (2·11) and (2·12) can be evaluated to give

$$\overline{u^2}(t) = \frac{\Lambda}{48\sqrt{(2\pi)}}[\nu(t-t_0)]^{-\frac{5}{2}}, \tag{2·14}$$

$$f(r,t) = \exp\left[-\frac{r^2}{8\nu(t-t_0)}\right]. \tag{2·15}$$

These special initial conditions are described as referring to a 'point source of strength $\Lambda$' in view of the corresponding problem in the heat analogy.

Millionshtchikov (1939) independently obtained this special solution in the same way; he interpreted the solution as describing the turbulence which exists subsequent to an initial random distribution of concentrated line eddies, provided that the effect of triple correlations is ignored. This is true in the sense that initially the velocity correlation is non-zero for $r = 0$ only, but the interpretation does not account for the third (and most important) of the initial conditions (2·13). In any case it is not practically possible to create a state of turbulence in which inertia forces are negligible during its whole history, so that the special initial conditions used by Loitsiansky and Millionshtchikov have no reference to real turbulence.

Sedov (1944) covered much the same ground as von Kármán & Howarth, Loitsiansky, and Millionshtchikov, and pointed out the relation between the parameter $n$ in von Kármán & Howarth's family of solutions and the value of $\overline{u^2}\int_0^\infty r^4 f\,dr$.

Finally, it was shown by one of us (Batchelor 1948) that, whatever the conditions at $t = t_0$, the energy decay and correlation function derived from equation (2·1) tend to the forms (2·14) and (2·15) as $t \to \infty$. This follows readily from a consideration of the general solutions (2·11) and (2·12) at large values of $t - t_0$ (provided that we accept the restrictions that $\Lambda$ is neither zero nor infinite). There is thus a unique asymptotic solution in the final period, the asymptotic correlation function being completely self-preserving in shape. It was also shown that such a self-preserving correlation function can *only* exist when inertia forces are negligible, and is necessarily an asymptotic solution under such conditions.

Mathematically, the position is this: that of the infinite family of self-preserving solutions derived by von Kármán & Howarth, a certain number only are asymptotic solutions for arbitrary initial conditions. Initial conditions for which $\int_0^\infty r^4 f\,dr$ is neither zero nor infinite (as will normally be the case in practice) give an asymptotic decay law such that $\overline{u^2} \propto t^{-\frac{5}{2}}$, corresponding to $n = \frac{5}{2}$ in the family of solutions. If the initial conditions are such that $\int_0^\infty r^4 f\,dr = 0$ but $\int_0^\infty r^6 f\,dr \neq 0$, the asymptotic decay law is obtained by taking the next term in the expansion of the exponential in (2·12) and is of the form $\overline{u^2} \propto t^{-\frac{7}{2}}$, corresponding to $n = \frac{7}{2}$. Likewise, if $\int_0^\infty r^4 f\,dr$ is infinite but $\int_0^\infty r^2 f\,dr$ is finite, the asymptotic decay law is $\overline{u^2} \propto t^{-\frac{3}{2}}$, corresponding to $n = \frac{3}{2}$; and so on, all half-integral values of $n$ providing possible asymptotic solutions. The corresponding asymptotic correlation functions are readily obtained from either (2·4) or (2·11).

Physically, the conditions under which turbulence is produced (at any rate, in a laboratory) will ensure that $\int_0^\infty r^4 f\,dr$ is neither zero nor infinite. Hence the solution which will be found when the inertia forces are negligible, and when the time of decay is sufficiently large, is

$$\overline{u^2} \propto t^{-\frac{5}{2}}, \quad f(r,t) = e^{-r^2/8\nu t}. \tag{2·16}$$

## 3. New approach to the problem

A new approach which, like Reissner's investigation, considers the local or unaveraged turbulent velocities will now be presented. The asymptotic decay law and correlation function already obtained, and other new results of interest, will be deduced ab initio. We confine attention to the case of zero mean motion.

When the Reynolds number of the turbulent motion is so low that inertia forces are negligible, the Navier-Stokes equation becomes

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u, \tag{3·1}$$

where $u$ is a typical velocity component. Using the equation of continuity, the pressure is found to satisfy the equation

$$\nabla^2 p = 0.$$

Since a potential function cannot have a maximum or a minimum in the interior of the field, the only possible bounded solution for an arbitrarily large fluid field not containing solid boundaries is that $p$ is independent of position. Hence the equation for $u$ reduces to

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u. \tag{3·2}$$

Each velocity component therefore changes with time in the final period of decay in the same way as a similar distribution of temperature in a homogeneous medium. The velocity components change independently of each other, and if the equation of continuity is satisfied at any time, it remains satisfied at all subsequent times. Since the velocity components do not interact with one another, symmetry conditions on the turbulence become of secondary importance and any component can be considered in isolation.

If the spatial distribution of $u$ is known at the instant from which the time is measured, the solution of (3·2) is

$$u(x,y,z,t) = \frac{1}{(4\pi\nu t)^{\frac{3}{2}}} \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} \int_{-\gamma}^{\gamma} u(X,Y,Z,0) \times \exp\left[-\frac{(X-x)^2-(Y-y)^2-(Z-z)^2}{4\nu t}\right] dX\,dY\,dZ, \tag{3·3}$$

as is well known from the study of other physical problems governed by the equation (3·2). The field of turbulence is here assumed to have (large) dimensions $2\alpha$, $2\beta$ and $2\gamma$ in the $x$, $y$ and $z$ directions. The restriction to a finite field is not significant and could be removed—at the expense of simplicity—by using Fourier transforms of the velocity distribution in a manner similar to Reissner's investigation.

The turbulence can thus be calculated if it is known at some instant. Naturally, only the statistical characteristics are likely to be known at the initial instant, so that (3·3) is not directly useful in its present form. The task is to use it to deduce the development with time of various mean values. The advantage of this explicit solution for the velocity field is that it provides a means (which may not always be practical!) of determining the development of *any* of the statistical characteristics of turbulence during the final period of decay.

The mean value of any quantity associated with the turbulence will be regarded as defined by a space integral, viz.

$$\bar{Q}(t) = \frac{1}{8\alpha\beta\gamma} \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} \int_{-\gamma}^{\gamma} Q(x,y,z,t)\,dx\,dy\,dz. \tag{3·4}$$

Implicit in this definition of a mean value is the assumption that the turbulence is approximately homogeneous.

## 4. Double velocity correlations in the final period

Consider first the correlation between two parallel simultaneous velocity components at two points $P$ and $P'$, where $PP'$ has the projection $\xi$ in the direction of the velocity components and projections $\eta$ and $\zeta$ in two other orthogonal directions. Using (3·4),

$$R(\xi,\eta,\zeta,t) = \overline{u(x,y,z,t)\,u(x+\xi,y+\eta,z+\zeta,t)}$$

$$= \frac{1}{8\alpha\beta\gamma}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\gamma}^{\gamma} u(x,y,z,t)\,u(x+\xi,y+\eta,z+\zeta,t)\,dx\,dy\,dz. \tag{4·1}$$



Substituting from (3·3) for the two velocities in the integrand gives a multiple integral for $R$ in terms of conditions at $t=0$. The integration connected with the $x$-direction is

$$\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} u(x,t)\,u(x+\xi,t)\,dx$$

$$= \frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{u(X,0)\,u(X',0)}{4\pi\nu t}\exp\left[\frac{-(X-x)^2-(X'-x-\xi)^2}{4\nu t}\right]dX\,dX'\,dx$$

$$\approx \frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{u(X,0)\,u(X',0)}{2\sqrt{(2\pi\nu t)}}\exp\left[-\frac{(X-X'+\xi)^2}{8\nu t}\right]dX\,dX'$$

if $\alpha$ is sufficiently large. Putting $X' = X+a$,

$$= \frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha-X}^{\alpha-X} \frac{u(X,0)\,u(X+a,0)}{2\sqrt{(2\pi\nu t)}}\exp\left[-\frac{(\xi-a)^2}{8\nu t}\right]dX\,da'$$

$$= \frac{1}{2\sqrt{(2\pi\nu t)}}\int_{-\alpha-X}^{\alpha-X} \overline{u(X,0)\,u(X+a,0)}\exp\left[-\frac{(\xi-a)^2}{8\nu t}\right]da. \tag{4·2}$$

The other two sets of integrations give a similar result, so that

$$R(\xi,\eta,\zeta,t) = \frac{1}{(8\pi\nu t)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(a,b,c,0)$$

$$\times\exp\left[-\frac{(\xi-a)^2-(\eta-b)^2-(\zeta-c)^2}{8\nu t}\right]da\,db\,dc, \tag{4·3}$$

where the limits of integration are again written as infinite without causing a significant error. Equation (4·3) is typical of the relations between mean values at two different instants in the final period of decay.

Our interest lies chiefly in the possibility that the mean values in the final period tend to asymptotic forms which are independent of the initial conditions. With this in mind, consider the expression (4·3) for large values of $t$. More precisely, let $\nu t$ be large compared with values of $a^2+b^2+c^2$ ($=s^2$ say) for which $R(a,b,c,0)$ is large enough to contribute to the integral. The exponential will then be approximately unity, and the right side will be independent of $\xi$, $\eta$ and $\zeta$, unless $\xi^2+\eta^2+\zeta^2$ ($=r^2$ say) is of order $\nu t$. That is, the correlation coefficient at time $t$ is approximately unity unless $r^2 = 0(\nu t)$. Hence $2(a\xi+b\eta+c\zeta)/8\nu t$ is of order $(\nu t)^{-\frac{1}{2}}$, $s^2/\nu t$ is of order $(\nu t)^{-1}$, and (4·3) can be written

$$R(\xi,\eta,\zeta,t) = \frac{e^{-r^2/8\nu t}}{(8\pi\nu t)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(a,b,c,0)\sum_n \frac{1}{n!}\left[\frac{2(a\xi+b\eta+c\zeta)-s^2}{8\nu t}\right]^n da\,db\,dc, \tag{4·4}$$

only the first few terms of the series being required.

There are two simple lemmas which will be used in evaluating the contributions from different terms in the series expansion. The first is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(a,b,c,0)\,db\,dc = 0, \tag{4·5}$$

since continuity requires the total flow across an infinite plane to be zero.* The second is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^i b^j c^k R(a,b,c,0)\,da\,db\,dc = 0, \tag{4·6}$$

when $i+j+k$ is an odd integer. This follows from the requirement of homogeneity that $R(a,b,c,0) = R(-a,-b,-c,0)$.

It follows immediately from (4·5) that the leading term, $n = 0$, makes no contribution to the integral (4·4). The next largest term in the series, of order $(\nu t)^{-\frac{1}{2}}$, is obtained from $n = 1$, but (4·6) shows that it too makes no contribution to the integral. The first non-zero contribution to the integral comes from terms of order $(\nu t)^{-1}$ in the series, and hence, to the first approximation,

$$R(\xi,\eta,\zeta,t) = \frac{\pi e^{-r^2/8\nu t}}{(8\pi\nu t)^{\frac{5}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(a,b,c,0)\left[\frac{(a\xi+b\eta+c\zeta)^2}{4\nu t}-s^2\right]da\,db\,dc. \tag{4·7}$$

Equation (4·7) describes the asymptotic behaviour of the most general space-interval double-velocity correlation and shows that the initial conditions are relevant. But there are certain special cases, which also happen to be important, in which the effect of the initial conditions is confined, at most, to a multiplicative constant. The most important special case is $r = 0$, giving the asymptotic law of energy decay as

$$\overline{u^2}(t) = \frac{-\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2 R(a,b,c,0)\,da\,db\,dc}{(8\pi\nu t)^{\frac{5}{2}}}, \tag{4·8}$$

* More rigorously, note that if $u'$, $v'$ and $w'$ are the velocity components at the point $(x+a, y+b, z+c)$, continuity requires

$$\overline{u(x,y,z)\left[\frac{\partial u'}{\partial a}+\frac{\partial v'}{\partial b}+\frac{\partial w'}{\partial c}\right]} = 0,$$

and hence
$$\frac{\partial}{\partial a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(a,b,c)\,db\,dc = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial\overline{uv'}}{\partial b}+\frac{\partial\overline{uw'}}{\partial c}\right)db\,dc$$
$$= 0.$$

Thus the integral on the left side is constant, and on taking the particular case in which $a$ is very large, the relation (4·5) follows.

in agreement with the deductions described in §2. When $\eta = \zeta = 0$, $R(\xi, \eta, \zeta, t)$ degenerates to the longitudinal correlation previously denoted by $\overline{u^2}(t) f(\xi, t)$; hence, making use of (4·5) again,

$$\overline{u^2}(t) f(\xi, t) = -\frac{\pi e^{-\xi^2/8\nu t}}{(8\pi\nu t)^{\frac{5}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2 R(a, b, c, 0)\, da\, db\, dc, \tag{4·9}$$

so that

$$f(\xi, t) = e^{-\xi^2/8\nu t}. \tag{4·10}$$

With the proviso that the integral in (4·8) and (4·9) is neither zero nor divergent, all longitudinal correlations should tend to the form (4·10) in the final period of decay.

It should be noticed that no assumptions about symmetry of the turbulence have yet been made, and the result (4·10) in no way depends on isotropy. The only assumption made is that the turbulence is approximately homogeneous. In point of fact, in a field of uniform mean flow a state of isotropy will usually have been attained before the decay is sufficiently far advanced for the laws (4·8) and (4·10) to apply, so that this generality may not have much practical use. When the turbulence at the initial instant $t = 0$ is isotropic, the triple integral in (4·8) and (4·9) can be reduced:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2 R(a, b, c, 0)\, da\, db\, dc \\
&\qquad = \overline{u^2}(0) \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} s^2 \left[ f(s, 0) + \frac{s}{2}\frac{\partial f(s, 0)}{\partial s} \sin^2\theta \right] s^2 \sin\theta\, ds\, d\theta\, d\phi \\
&\qquad = 4\pi\overline{u^2}(0) \int_0^{\infty} s^4 \left[ f(s, 0) + \frac{s}{3}\frac{\partial f(s, 0)}{\partial s} \right] ds \\
&\qquad = -\frac{8\pi}{3}\overline{u^2}(0) \int_0^{\infty} s^4 f(s, 0)\, ds \\
&\qquad = -\frac{8\pi}{3}\Lambda,
\end{aligned}$$

so that (4·8) becomes

$$\overline{u^2}(t) = \frac{\Lambda}{48\sqrt{(2\pi)}} (\nu t)^{-\frac{5}{2}}. \tag{4·11}$$

It is also possible to deduce the asymptotic form of the time-interval double-velocity correlation, i.e. the correlation between parallel velocity components at the same point but at two different times separated by an interval $\tau$. A suitable definition is

$$\begin{aligned}
R(\tau, t) &= \overline{u(x, y, z, t - \tfrac{1}{2}\tau)\, u(x, y, z, t + \tfrac{1}{2}\tau)} \\
&= \frac{1}{8\alpha\beta\gamma} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\gamma}^{\gamma} u(x, y, z, t - \tfrac{1}{2}\tau)\, u(x, y, z, t + \tfrac{1}{2}\tau)\, dx\, dy\, dz.
\end{aligned} \tag{4·12}$$

Substituting from (3·3), the integration connected with the $x$-direction is

$$\begin{aligned}
&\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{u(X, 0)\, u(X', 0)}{4\pi\nu\sqrt{(t^2 - \frac{1}{4}\tau^2)}} \exp\left[ -\frac{(X - x)^2}{4\nu(t - \frac{1}{2}\tau)} - \frac{(X' - x)^2}{4\nu(t + \frac{1}{2}\tau)} \right] dx\, dX\, dX' \\
&\qquad \approx \frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{u(X, 0)\, u(X', 0)}{2\sqrt{(2\pi\nu t)}} \exp\left[ -\frac{(X - X')}{8\nu t} \right] dX\, dX',
\end{aligned}$$

and $\tau$ has disappeared from the analysis. Hence, regarding the reference to the point $x, y, z$ as understood,

$$R(\tau, t) = \overline{u(t-\tfrac{1}{2}\tau)\, u(t+\tfrac{1}{2}\tau)} = \overline{u^2(t)}. \tag{4·13}$$

When $t$ is sufficiently large, (4·11) is valid so that the asymptotic form of the time-interval correlation coefficient is

$$\frac{\overline{u(t-\frac{1}{2}\tau)\, u(t+\frac{1}{2}\tau)}}{[\overline{u^2(t-\frac{1}{2}\tau)}\,\overline{u^2(t+\frac{1}{2}\tau)}]^{\frac{1}{2}}} = \frac{(t^2-\frac{1}{4}\tau^2)^{\frac{5}{4}}}{t^{\frac{5}{2}}} = \left(1-\frac{\tau^2}{4t^2}\right)^{\frac{5}{4}}. \tag{4·14}$$

Although (4·13) is valid for all values of $\tau$ such that $t-\frac{1}{2}\tau>0$, (4·14) requires $[\nu(t-\frac{1}{2}\tau)]^{\frac{1}{2}}$ to be large compared with the spread of the spatial correlation at $t=0$. In a practical experiment $\tau/t$ would normally be small enough for (4·14) to be written as $1-\frac{5}{16}(\tau^2/t^2)$.

The form of this time interval correlation throws a little more light on the nature of the final period. Equation (4·13) shows that, roughly speaking, the only change in the velocity at any point is a gradual diminution corresponding to the decay of the whole pattern of turbulence. The shape of the spatial distribution of velocity remains approximately the same as $t$ increases, the steeper gradients being modified most. The turbulence may be described as aperiodic (in time), like other flow fields dominated by viscosity.

Finally, the most general double velocity correlation coefficient involving both space and time intervals may be found, by analysis similar to that leading to (4·13), to be a simple product of the correlation coefficients already discussed. Since

$$\overline{u(x,y,z,t-\tfrac{1}{2}\tau)\, u(x+\xi,y+\eta,z+\zeta,t+\tfrac{1}{2}\tau)} = \overline{u(x,y,z,t)\, u(x+\xi,y+\eta,z+\zeta,t)}$$

it follows that

$$\frac{\overline{u(x,y,z,t-\frac{1}{2}\tau)\, u(x+\xi,y+\eta,z+\zeta,t+\frac{1}{2}\tau)}}{[\overline{u^2(t-\frac{1}{2}\tau)}\,\overline{u^2(t+\frac{1}{2}\tau)}]^{\frac{1}{2}}} = \frac{R(\tau,t)}{[\overline{u^2(t-\frac{1}{2}\tau)}\,\overline{u^2(t+\frac{1}{2}\tau)}]^{\frac{1}{2}}}\,\frac{R(\xi,\eta,\zeta,t)}{\overline{u^2(t)}}, \tag{4·15}$$

and the asymptotic form may be obtained from the results already derived. In particular, the longitudinal correlation coefficient tends to the form

$$\left(1-\frac{\tau^2}{4t^2}\right)^{\frac{5}{4}} e^{-\xi^2/8\nu t}. \tag{4·16}$$

## 5. Measurements of turbulence in the final period

To produce isotropic turbulence whose decay may be observed over a sufficient interval for the establishment of the final period of decay, it is necessary to use a grid of very small mesh. It is known that the initial period of decay extends to about 150-mesh lengths from the grid, and a decay range much greater than this is required if a reasonable range of the final period is to be available after the transitional period. A mesh of 0·159 cm. was used in the experiments, allowing measurements to 1040-mesh lengths downstream from the grid. This grid is of the biplane type, and made of stretched eureka wires of diameter 0·030 cm. It is geometrically similar to grids previously used (Batchelor & Townsend 1948), with a spacing-diameter ratio of 5·33. Measurements were made of the turbulent intensity, of the dissipation parameter $\lambda$, and of the longitudinal double velocity correlation at an air speed of 615 cm.sec.$^{-1}$, and of the turbulent intensity at air speeds of 895 and 1280 cm.sec.$^{-1}$. The respective mesh Reynolds numbers, $R_M = UM/\nu$, are 650, 950 and 1360. Using a 0·635 cm. grid at 150 cm.sec.$^{-1}$ ($R_M = 635$), rather more accurate measurements in the initial period are possible and these are included in the results.

The technique for the measurement of the turbulent intensity and the length $\lambda$ has been described previously (Townsend 1947), the only significant change here being the upward restriction of the amplifier frequency response in order that an adequate signal-noise ratio should be available when measuring the differentiated signal.

The scale of the turbulence produced by a 0·159 cm. mesh is so small that the distance between two hot-wires with finite velocity correlation is difficult to measure accurately. It was therefore decided to measure the auto-correlation function of the output of a single hot-wire, and to compute from it the longitudinal double-velocity correlation. If $F(\tau)$ is the auto-correlation for a time interval $\tau$, then the longitudinal correlation is related to it by

$$f(r) = F(U\tau),$$

provided that the turbulence does not change appreciably (apart from its translation with velocity $U$) in a time $r/U$. This assumption is nearly true if the turbulence level is low, as has been shown by the experimental verification of the relation between the spectrum and the correlation function of isotropic turbulence (Taylor 1938). It is even more likely to be accurate under the conditions of these experiments, viz. very low turbulent intensities and weak diffusive properties.

To measure the auto-correlation, the hot-wire anemometer output must be compared at times separated by a constant but adjustable interval, which is done by delaying the transmission of the fluctuations. The anemometer output is recorded at two points on a magnetic tape driven at constant speed, and each recorded signal is reproduced after a time interval determined by the tape speed and the distance between each recording point and the corresponding reproducing point as measured along the tape. The correlation between these reproduced signals is measured, and the appropriate correlation interval is the difference of the time delays of the two reproduced signals. In this way, not only are effects of slight distortion minimized, but both positive and negative time delays can be used. A detailed description of the magnetic tape time delay equipment will be published separately.

A survey of the turbulence behind the 0·159 cm. mesh showed that, although the mesh appeared completely uniform to the eye, considerable spatial variations of intensity occurred across the tunnel, and it became evident that these were caused by slight non-uniformity of the mesh. The authors' notion is that at Reynolds numbers close to values which are critical for the instability of the laminar flow immediately behind the grid, small large-scale irregularities in the grid will cause the production of widely different levels of turbulence behind different parts of the grid. That the effect is due to large-scale irregularities is confirmed by the comparative insensitivity of the intensity distribution to quite large small-scale irregularities, such as that produced by omitting one element of the mesh. This intensity variation, although it is annoying, is not a serious obstacle, for the turbulence shows extremely weak diffusive properties, and the transverse intensity profile remains nearly similar during a decay of intensity in a ratio twenty to one. This is evidence that the turbulence consists of eddies nearly independent of one another. The effect of non-uniformity has been removed from the results by averaging the intensity over a considerable part of the tunnel section. It is probable that the mean intensity obtained in this way is a good approximation to the intensity behind an ideal mesh.

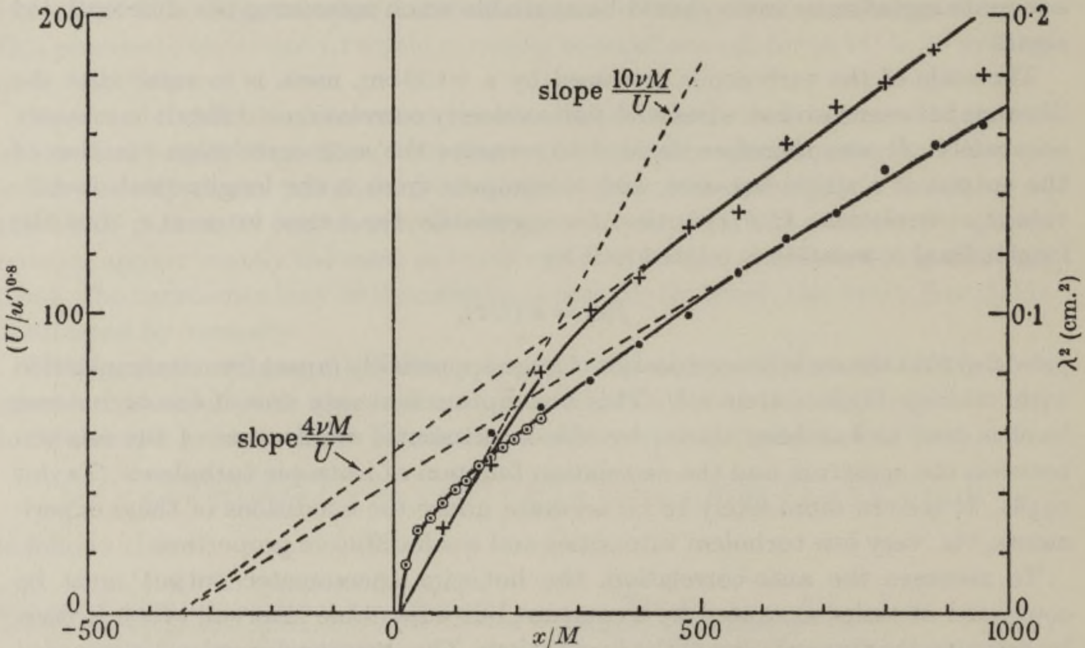


FIGURE 1. Variation of intensity and $\lambda$ ($R_M = 650$).

• $(U/u')^{0.8}$: $M = 0{\cdot}159$ cm., $U = 620$ cm.sec.$^{-1}$.
⊙ $(U/u')^{0.8}$: $M = 0{\cdot}635$ cm., $U = 150$ cm.sec.$^{-1}$.
+ $\lambda^2$: $M = 0{\cdot}159$ cm., $U = 620$ cm.sec.$^{-1}$.

In figure 1, $(U^2/\overline{u^2})^{0.4}$ and $\lambda^2$ measured at a Reynolds number of 650 are plotted as functions of $x/M$. The foregoing theory predicts the following asymptotic relations in the final period:

$$\left(\frac{U^2}{\overline{u^2}}\right)^{0.4} = A\frac{(x-x_0)}{M}, \quad \lambda^2 = \frac{4\nu}{U}(x-x_0),$$

where $A$ and $x_0$ are constants characteristic of the flow, and

$$A = \frac{4\nu}{MU}\left(\frac{U^2M^5}{\overline{u^2}\lambda^5}\right)^{\frac{2}{5}} = \frac{\nu}{MU}\left(\frac{48\sqrt{(2\pi)}\,U^2M^5}{\Lambda}\right)^{\frac{2}{5}}.$$

On the diagram, lines satisfying these equations are drawn to fit the experimental points, and good agreement for $x/M > 400$ is obtained if $A = 0{\cdot}129$ and $x_0 = -350M$.

For reference, figure 1 shows a line representing the known variation of $\lambda^2$ in the initial period (Batchelor & Townsend 1948), viz.

$$\lambda^2 = \frac{10\nu}{U}(x-10M),$$

and this adequately represents the range to $x/M = 160$. At this Reynolds number then, the asymptotic energy decay relation in the final period is established after 400-mesh lengths.

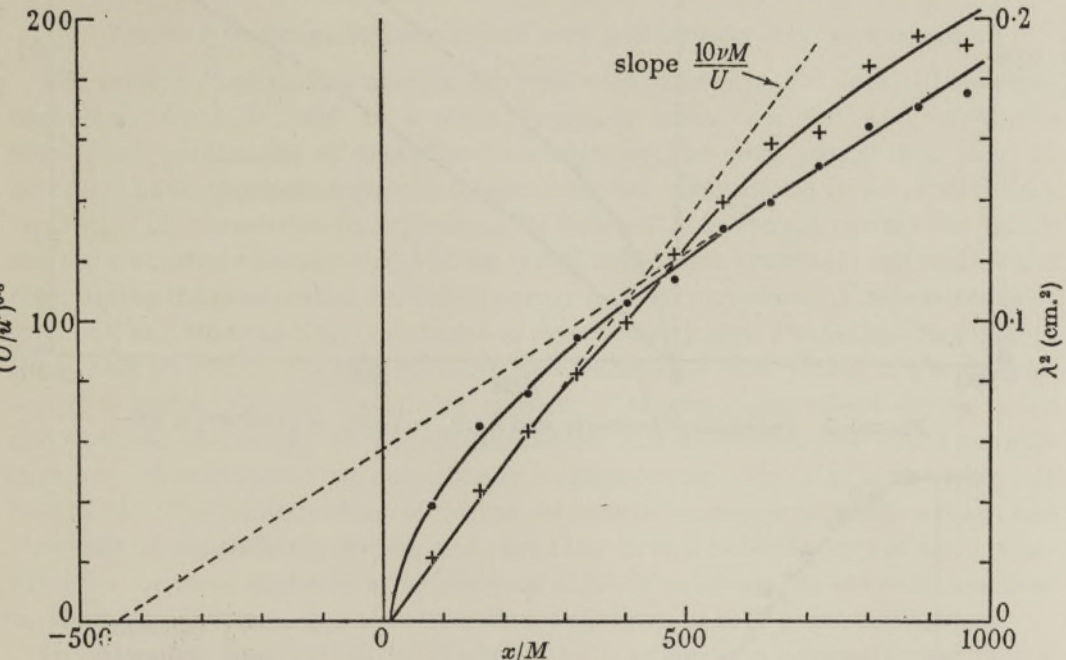


FIGURE 2. Variation of intensity and $\lambda$ ($R_M = 950$). $(U/u')^{0.8}$; + $\lambda^2$.

Measurements of intensity and of $\lambda^2$ at higher speeds are given in figures 2 and 3, but experimental difficulties in the measurements of $\lambda$ at high speeds and low intensities make the results for $\lambda^2$ at large $x/M$ rather irregular. It is evident from the intensity plots that the setting up of the final period asymptotic decay cannot occur here before $x/M = 600$ and is probably not complete before $x/M = 800$. It can be said that the slope of the $\lambda^2$ versus $x/M$ curve is less than $10\nu(M/U)$, except in the initial period, but the interesting values for $x/M > 700$ are subject to considerable uncertainty. Supposing the asymptotic decay law toward the limits of observation, we find the following values for $A$, $x_0/M$ and $\Lambda$.*

| mesh Reynolds number | $A$ | $x_0/M$ | $\Lambda$ (cm.⁷ sec.⁻²) |
|---|---|---|---|
| 650 | 0·13 | −350 | 0·072 |
| 950 | 0·13 | −440 | 0·058 |
| 1360 | 0·13 | −450 | 0·048 |

* The results for $\Lambda$ show that at a given value of $x/M$ lying in the initial period, the value of $\int_0^\infty r^4 f(r)\,dr$ diminishes as $U$ increases; if $A$ were accurately constant, this decrease would be as $U^{-\frac{1}{2}}$, which is of interest in connexion with the question of the turbulence set up immediately behind a grid.

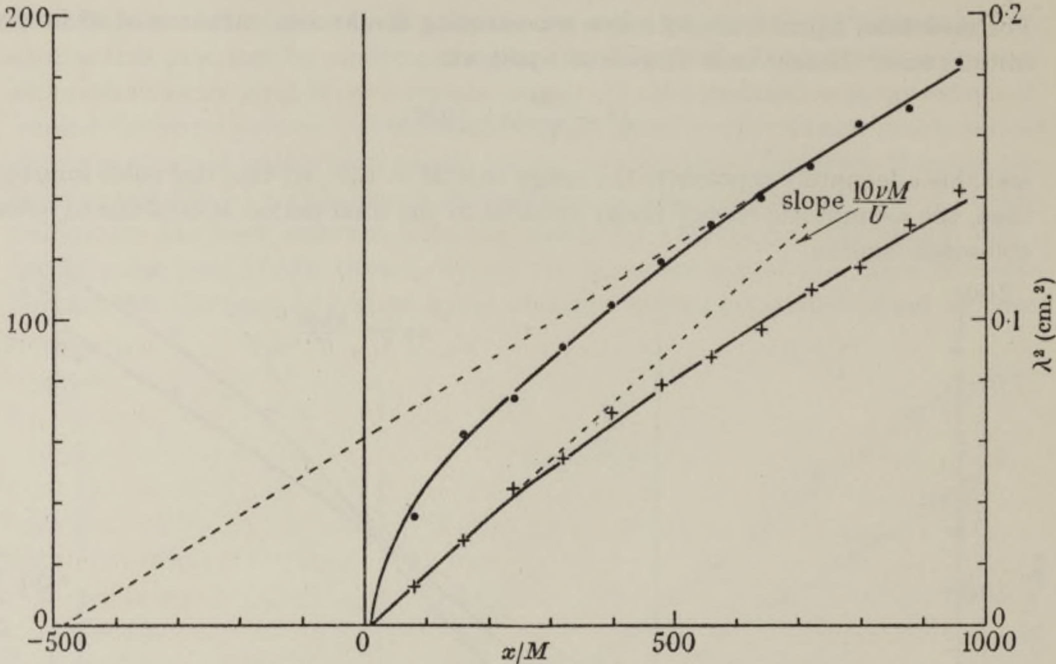


FIGURE 3. Variation of intensity and $\lambda$ ($R_M = 1360$). • $(U/u')^{0.8}$; + $\lambda^2$.

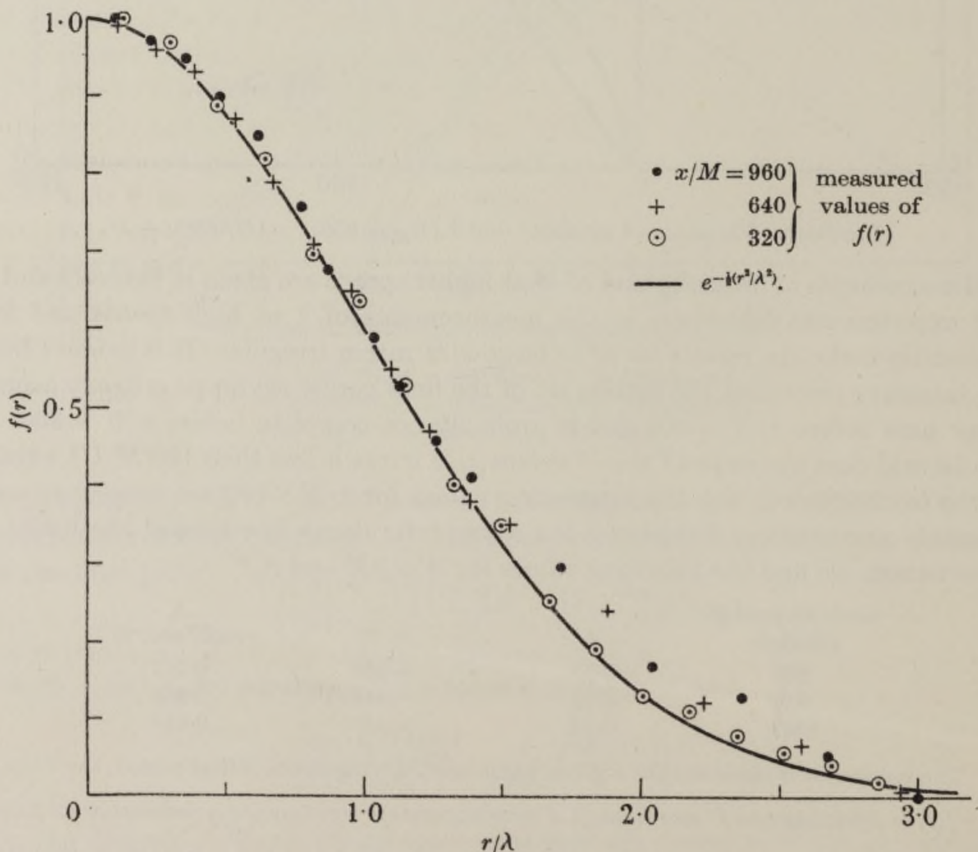


FIGURE 4. Double correlation function ($M = 0{\cdot}159$ cm., $U = 620$ cm.sec.$^{-1}$, $R_M = 650$).

The double correlation function is plotted in figure 4 as a function of $r/\lambda$ for several decay times, the value of $\lambda$ being taken from the radius of curvature at $r = 0$ in each case. Also shown is the theoretical asymptotic form of the function, viz.

$$f(r) = e^{-\frac{1}{2}r^2/\lambda^2}.$$

The agreement is good and adequately confirms the existence of the predicted correlation function.

## 6. Nature of final period decay and conditions for its existence

The present experiments confirm the existence of final period decay of isotropic turbulence for $x/M > 400$ at a mesh Reynolds number of 650, and verify the theoretical predictions of decay and correlation. The final period flow may be described as an aperiodic approach to complete rest. Asymptotic forms of the decay law and of the correlation functions may be inferred, even though neither the details nor the statistical characteristics of the initial motion are known. It has been found that, during the final period, turbulent energy diffuses very slowly down an intensity gradient, and that the time correlation is close to unity over a considerable range of time. This all points to a high stability and uniformity of structure of the motion, and it is likely that the turbulence consists of almost independent eddies. Such eddies would lose energy by viscous dissipation, and simultaneously would increase in size by viscous spreading, occasionally amalgamating with an adjacent eddy but only rarely. The limited extent of the spatial correlation suggests that the shape and structure of the eddies is simple, and that they have a limited range of sizes. Confirmation of these surmises would be very difficult to obtain, as the only practical method would seem to be the measurement of three- or four-point correlations.

In order that final period decay should be possible, it is necessary that inertia forces should play a negligible part in the turbulent motion. For the asymptotic relations to apply it is then necessary that decay shall have progressed so far that the initial conditions have been obliterated by viscous action. To express the first condition exactly requires a knowledge of the turbulent structure, but, if it can be assumed that $S_0, = \overline{\left(\frac{\partial u}{\partial x}\right)^3} \Big/ \left[\overline{\left(\frac{\partial u}{\partial x}\right)^2}\right]^{\frac{3}{2}}$, has the same value at *all* Reynolds numbers (Batchelor & Townsend 1947), then the condition that inertia plays no part in determining the vorticity balance is that

$$R_\lambda \ll -\frac{30}{7}\frac{1+n}{nS_0}$$
$$= 15$$

if the decay law is such that $\overline{u^2} \propto t^{-\frac{5}{2}}$, i.e. $n = \frac{5}{2}$. If the correlation function is of normal form, the second condition will include the restriction

$$4\nu t \gg L_0^2, \quad \text{i.e.} \quad \frac{x}{M} \gg \frac{R_M}{4}\left(\frac{L_0}{M}\right)^2,$$

where $L_0$ is the scale of the turbulence, possibly the integral scale $\int_0^\infty f(r)\,dr$, during the initial period. Taking $L_0/M = 0{\cdot}95$ as representative of conditions at $x/M = 100$, the condition becomes $x/M \gg 0{\cdot}23R_M$. At $R_M = 650$, this lower bound is 150, whereas observation shows the asymptotic relations to occur at $x/M = 400$. Both the above conditions are necessary rather than sufficient.

By considering the behaviour of $R_\lambda$ during decay, it is possible to make some deductions about the length of the transitional period. If the decay is represented as proceeding in the initial period (with constant $R_\lambda$) for a time $t_0$, and then in the transitional period with a decay law intermediate between that in the initial and final periods, say

$$\lambda^2 = 7\nu(t+\tfrac{3}{7}t_0), \quad \overline{u^2} \propto (t+\tfrac{3}{7}t_0)^{-\frac{10}{7}},$$

the value at any time in the transition period ($t > t_0$) will be

$$R_\lambda = (R_\lambda)_{\text{initial period}}\left(\frac{7}{10}\frac{t}{t_0}+\frac{3}{10}\right)^{-\frac{3}{14}}.$$

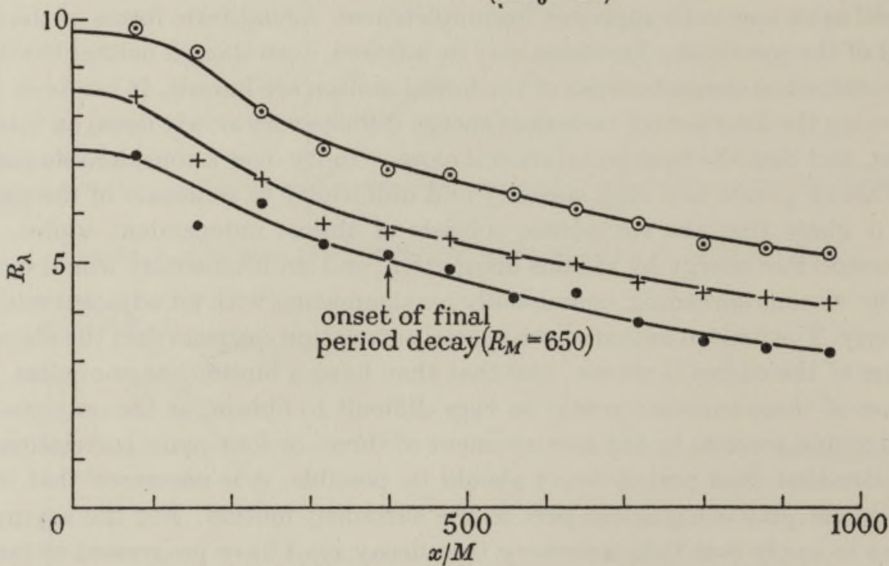


FIGURE 5. Variation of $R_\lambda$ with $x/M$. Values of $R_M$: • 650, + 950, ⊙ 1360.

This means that the decrease of $R_\lambda$ with time is extremely slow, and if $(R_\lambda)_{\text{initial period}}$ is much greater than 15, it is most improbable that the final period can be attained in a practicable decay time, e.g. a reduction of $R_\lambda$ by a factor of 4 would require decay for a time of order $1000t_0$. Since the initial period usually extends for about 150 mesh lengths, the reason why previous wind tunnel experiments have not detected final period decay relations is apparent.

In figure 5, values of $R_\lambda$ calculated from the results in figures 1 to 3 are plotted against $x/M$. The expected behaviour is an initial range where $R_\lambda$ is constant followed by a more or less gradual decrease. As nearly as can be made out, at the lowest Reynolds number in the experiments ($R_M = 650$), the final period decay law sets in at about $x/M = 400$ where $R_\lambda \approx 5$. At the next higher Reynolds number, $R_M = 950$, final period decay was not certainly observed, although the value, $R_\lambda = 5$, is attained when $x/M = 650$. Although the use of a simple Reynolds number criterion for the existence of final period flow is in general unjustifiable, there is nothing in the present results against the empirical, necessary criterion, $R_\lambda < 5$.

The experiments described in this paper were carried out in the Cavendish Laboratory. The wind tunnel and hot-wire equipment were acquired with the aid of a grant from the Aeronautical Research Council. During the course of the work one of us (G.K.B.) held a Senior Studentship given by the Royal Commissioners for the Exhibition of 1851.

## References

Batchelor, G. K. 1948 *Quart. Applied Math.* **6**, 97–116.
Batchelor, G. K. & Townsend, A. A. 1947 *Proc. Roy. Soc.* A, **190**, 534–550.
Batchelor, G. K. & Townsend, A. A. 1948 *Proc. Roy. Soc.* A, **193**, 539.
Kármán, T. von & Howarth, L. 1938 *Proc. Roy. Soc.* A, **164**, 192–215.
Loitsiansky, L. 1939 *Cent. Aero. and Hydro. Inst.* (*Moscow*), *Rep.* no. 440. (Translated as *N.A.C.A. Tech. Memo.* no. 1079.)
Millionshtchikov, M. 1939 *C.R. Acad. Sci. U.R.S.S.* **22**, 231–237.
Reissner, E. 1938 *Proc. 5th Int. Congr. Appl. Mech.* pp. 359–361.
Sedov, L. 1944 *C.R. Acad. Sci. U.R.S.S.* **42**, 116–119.
Taylor, G. I. 1938 *Proc. Roy. Soc.* A, **164**, 476.
Townsend, A. A. 1947 *Proc. Camb. Phil. Soc.* **43**, 560.

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# On the classical theory of particles

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A set of classical relativistic equations of motion of an electron in an electromagnetic field is postulated. These equations are free from 'run-away' solutions, and give the same results as the Maxwell-Lorentz theory for non-relativistic motions when the external electromagnetic field does not vary too rapidly. For the scattering of light by an electron, the scattering cross-section is independent of the frequency and is a universal constant. This brings out a point of difference from the Lorentz-Dirac equations according to which the scattering cross-section varies inversely as the square of the frequency of the incident light, for large frequencies. For the motion of an electron towards a fixed proton, the equations allow a collision, unlike the Lorentz-Dirac equations according to which the electron is brought to rest before it reaches the proton.

## 1. Introduction

There has been in recent years a marked revival of interest in the classical theories of particles. After the pioneering work of Abraham and Lorentz in this field some forty or fifty years ago, comparatively little attention was paid until very recently to the development of the classical theory, the centre of research in theoretical physics being occupied by the rapidly expanding field of the quantum theory. However, the various difficulties that have now arisen in quantum electrodynamics, particularly that of infinite self-energies of point particles, have led to the view that some of these difficulties may be of classical origin, and that a promising method of approach to eliminate these difficulties would be to study and improve the classical theory first, before passing on to the quantum theory.

A substantial step in this direction was made by Dirac (1938) when he showed that the infinite self-energy, ascribed to the point electron by the Maxwell-Lorentz