Speed, Arclength, and The Unit Tangent

Part 1: Properties of the Derivative

In the previous section, we showed that the velocity $\mathbf{v}(t)$ of a vector-valued function $\mathbf{r}(t)$ is *tangent* to the curve parameterized by $\mathbf{r}(t)$ at "time" t. In this section, we explore additional properties of the derivative of a vector-valued function.

To begin with, the following notations are commonly used to denote derivatives of vector values functions:

$$\mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t) = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}(t)$$

In particular, $\dot{\mathbf{r}}$ is used if t is interpreted to be "time", while $\mathbf{r}'(t)$ is used if no interpretation of t is to be inferred. Also, the operator notation allows us to state the following theorem.

Theorem 7.1: If $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are differentiable vector-valued functions over an interval I, if $f(t)$ is differentiable and in I for all t, and if a, b are scalars, then the following hold over that interval.

1. $\frac{d}{dt}[a\mathbf{p}(t) + b\mathbf{q}(t)] = a\frac{d\mathbf{p}}{dt} + b\frac{d\mathbf{q}}{dt}$
2. $\frac{d}{dt}[f(t)\,\mathbf{p}(t)] = f'(t)\,\mathbf{p}(t) + f(t)\,\mathbf{p}'(t)$
3. $\frac{d}{dt}\mathbf{p}(f(t)) = \mathbf{p}'(f(t))\ \frac{d}{dt}f(t)$
4. $\frac{d}{dt}[\mathbf{p}(t)\cdot\mathbf{q}(t)] = \mathbf{p}'(t)\cdot\mathbf{q}(t) + \mathbf{p}(t)\cdot\mathbf{q}'(t)$
5. $\frac{d}{dt}[\mathbf{p}(t)\times\mathbf{q}(t)] = \mathbf{p}'(t)\times\mathbf{q}(t) + \mathbf{p}(t)\times\mathbf{q}'(t)$

These properties follow directly from the definition of $\mathbf{r}'(t)$, and they are used frequently in elementary mechanics, as the next two examples illustrate.These properties follow directly from the definition of $\mathbf{r}'(t)$, and they are used frequently in elementary mechanics, as the next two examples illustrate.

EXAMPLE 1 Use theorem 7.1 to find $\mathbf{r}'(t)$ given that

$$\mathbf{r}(t) = t^2\ \mathbf{p}(t) \quad \textit{where} \quad \mathbf{p}(t) = \langle \cos(t), \sin(t), t\rangle$$

Then expand $\mathbf{r}(t)$ and find $\mathbf{r}'(t)$ directly, and show that the results are the same using either method.

Solution: Property 2 of theorem 7.1 implies that

$$\mathbf{r}'(t) = 2t\ \mathbf{p}(t) + t^2\ \mathbf{p}'(t)$$

where $\mathbf{p}'(t) = \langle -\sin(t), \cos(t), 1\rangle$. Thus,

$$\begin{aligned}
\mathbf{r}'(t) &= 2t\langle \cos(t), \sin(t), t\rangle + t^2\langle -\sin(t), \cos(t), 1\rangle \\
&= \left\langle 2t\cos(t), 2t\sin(t), 2t^2\right\rangle + \left\langle -t^2\sin(t), t^2\cos(t), t^2\right\rangle \\
&= \left\langle 2t\cos(t) - t^2\sin(t), 2t\sin(t) + t^2\cos(t), 3t^2\right\rangle
\end{aligned}$$

Alternately, $\mathbf{r}(t) = \left\langle t^2\cos(t), t^2\sin(t), t^3\right\rangle$ and

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle \frac{d}{dt}t^2\cos(t), \frac{d}{dt}t^2\sin(t), \frac{d}{dt}t^3\right\rangle \\
&= \left\langle 2t\cos(t) - t^2\sin(t), 2t\sin(t) + t^2\cos(t), 3t^2\right\rangle
\end{aligned}$$

which is the same as the result from using theorem 7.1.

Although the results are the same in both instances, the second method reveals a larger structure of $\mathbf{r}'(t)$ as the linear combination of $\mathbf{p}$ and $\mathbf{p}'$. Indeed, it is often advantageous to work with forms produced by properties 1-5 instead of direct calculations. For example, the *angular momentum* of a particle with position $\mathbf{r}(t)$ at time t and a constant mass m is

$$\mathbf{L}(t) = m\ (\ \mathbf{r}(t) \times \mathbf{v}(t)\ )$$

Correspondingly, the rate of change of $\mathbf{L}(t)$ with respect to time is

$$\begin{aligned}
\mathbf{L}'(t) &= m\left(\frac{d}{dt}\mathbf{r}(t)\right) \times \mathbf{v} + m\mathbf{r}(t) \times \frac{d}{dt}\mathbf{v} \\
&= m(\mathbf{v} \times \mathbf{v}) + m(\mathbf{r} \times \mathbf{a})
\end{aligned}$$

or equivalently, $\mathbf{L}' = m(\mathbf{r} \times \mathbf{a})$. The simplication $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ may not be obvious in a component-wise differentiation.

Also, when a vector is written in a bold typeface, the magnitude of a vector is often denoted by the same letter, only in an *italic typeface*. Thus, we often use r in place of $\|\mathbf{r}\|$, and likewise, we often write

$$v = \|\mathbf{v}(t)\|$$

That is, v is the magnitude of $\mathbf{v}$.

EXAMPLE 2 The *kinetic energy* of an object with a constant mass m and position $\mathbf{r}(t)$ at time t is defined to be

$$K = \frac{1}{2}mv^2$$

where $v^2 = \mathbf{v} \cdot \mathbf{v}$ and $\mathbf{v} = \mathbf{r}'(t)$. What is $K'(t)$?

Solution: Since $K = \frac{1}{2}m\,\mathbf{v}\cdot\mathbf{v}$, property 4 of theorem 7.1 implies that

$$\begin{aligned}
K'(t) &= \frac{m}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) \\
&= \frac{m}{2}\left[\left(\frac{d}{dt}\mathbf{v}\right)\cdot\mathbf{v}+\mathbf{v}\cdot\left(\frac{d}{dt}\mathbf{v}\right)\right] \\
&= \frac{m}{2}(\mathbf{a}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{a}) \\
&= \frac{m}{2}(2\mathbf{v}\cdot\mathbf{a}) \\
&= m\,\mathbf{v}\cdot\mathbf{a}
\end{aligned}$$

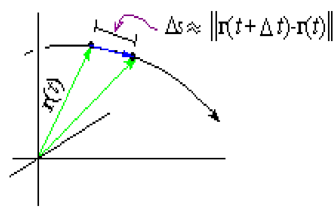
Check your Reading: What is $\mathbf{p}'(t)$ if $\mathbf{p}(t)=\mathbf{k}$ for all time t?

Speed and the Unit Tangent Vector

If $\mathbf{r}(t)$ denotes the position of an object at time t, then $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ is the *displacement* of the object between times t and $t+\Delta t$ for some small Δt. The distance Δs the object travels between t and $t+\Delta t$ satisfies

$$\Delta s \approx \|\mathbf{r}(t+\Delta t)-\mathbf{r}(t)\|$$

when $\mathbf{r}(t)$ is smooth and Δt is sufficiently close to 0.



That is, over the short period of time from t to time $t+\Delta t$, the rate of change of the object satisfies

$$rate \approx \frac{\Delta s}{\Delta t} \approx \frac{\|\mathbf{r}(t+\Delta t)-\mathbf{r}(t)\|}{\Delta t}$$

If we denote the rate by ds/dt, then applying the limit as Δt approaches 0 yields

$$\frac{ds}{dt} = \lim_{\Delta t\to 0}\left\|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right\| = \|\mathbf{v}\|$$

That is, ds/dt is the *speed* of the object, and the speed of the object is the magnitude of the velocity vector.

EXAMPLE 3 Find the speed of an object with position $\mathbf{r}(t) = \langle 3\sin(2t), 5\cos(2t), 4\sin(2t)\rangle$ in feet at time t in seconds.

Solution: To do so, we first compute the velocity:

$$\mathbf{v} = \langle 6\cos(2t), -10\sin(2t), 8\cos(2t)\rangle$$

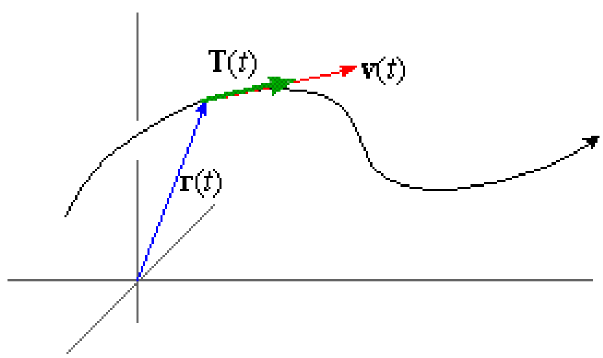
The speed is then the magnitude of the velocity:

$$\begin{aligned}
\frac{ds}{dt} &= \sqrt{[6\cos(2t)]^2 + [-10\sin(2t)]^2 + [8\cos(2t)]^2} \\
&= \sqrt{36\cos^2(2t) + 100\sin^2(2t) + 64\cos^2(2t)} \\
&= \sqrt{100\cos^2(2t) + 100\sin^2(2t)} \\
&= 10 \text{ feet per second}
\end{aligned}$$

Speed is also denoted by simply v (given that we typically use the italic to denote the magnitude of a vector denoted by the same letter in bold typeface). In addition, at points where the speed is non-zero, we define the *unit tangent vector* $\mathbf{T}(t)$ to be the unit vector in the direction of $\mathbf{v}$:

$$\mathbf{T} = \frac{1}{v}\mathbf{v}$$

(A curve is said to be *regular* if v is non-zero at every point.)



As a result, $\mathbf{v} = v\mathbf{T}$, which shows that velocity can be written as the product of its speed and direction:

EXAMPLE 4 Find the unit tangent vector to $\mathbf{r}(t) = \langle e^t, 2t, 2e^{-t}\rangle$.

Solution: To do so, we first compute the velocity:

$$\mathbf{v}(t) = \frac{d}{dt}\left\langle e^t, 2t, 2e^{-t}\right\rangle = \left\langle e^t, 2, -2e^{-t}\right\rangle$$

Then we find the magnitude of the velocity

$$\|\mathbf{v}\| = \sqrt{\left(e^t\right)^2 + (2)^2 + \left(-2e^{-t}\right)^2} = \sqrt{\left(e^t\right)^2 + 4 + 4\left(e^{-t}\right)^2}$$

Notice now that since $e^t e^{-t} = 1$, the quantity under the square root can be factored into a perfect square:

$$\|\mathbf{v}\| = \sqrt{\left(e^t + 2e^{-t}\right)^2} = e^t + 2e^{-t}$$

We then divide the velocity by the magnitude to obtain the unit tangent vector:

$$\mathbf{T}(t) = \frac{1}{e^t + 2e^{-t}}\,\mathbf{v} = \left\langle \frac{e^t}{e^t + 2e^{-t}}, \frac{2}{e^t + 2e^{-t}}, \frac{-2e^{-t}}{e^t + 2e^{-t}}\right\rangle$$

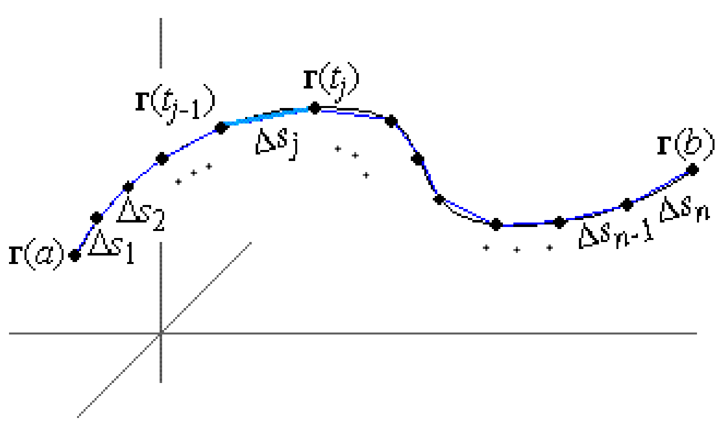
Check your Reading: What are the units of $\mathbf{T}(t)$?

Distance and Arclength

If $t_0, t_1, \ldots, t_n$ is a partition of a closed interval $[a, b]$, then the distance L traveled by an object with position $\mathbf{r}(t)$ over a time interval $[a, b]$ can be approximated by

$$L \approx \sum_{j=1}^{n} \Delta s_j$$

where Δs_j is the distance between $\mathbf{r}(t_{j-1})$ and $\mathbf{r}(t_j)$.



We can rewrite the summation in the form

$$L \approx \sum_{j=1}^{n} \frac{\Delta s_j}{\Delta t_j} \Delta t_j$$

and application of a limit results in a definite integral. Thus, the distance an object travels along a curve is given by

$$L = \int_a^b \|\mathbf{v}(t)\| \, dt$$

That is, total distance traveled is an integral of the speed.

EXAMPLE 5 Find the distance traveled along the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for t in $[0, 4\pi]$.

Solution: To do so, we first compute the velocity

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

The speed is then the magnitude of the velocity:

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

Finally, the length of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(4\pi)$ is

$$L = \int_0^{4\pi} \|\mathbf{v}(t)\| \, dt = \int_0^{4\pi} \sqrt{2} dt = 4\pi\sqrt{2}$$

The value of L in example 5 is the length of the helix for t in $[0, 4\pi]$ because each point on the curve corresponds to only one value of t in $[0, 4\pi]$. However, if $\mathbf{r}(t)$, t in $[a, b]$, covers a curve more than once, then the total distance L is more than the length of the curve (for example, $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, t in $[0, 6\pi]$, wraps around a circle 3 times, and correspondingly, L is 3 times the circumference of the unit circle).

However, if we are careful to use a parameterization $\mathbf{r}(t)$, t in $[a, b]$, in which there is a 1-1 correspondence between t and points on the curve, then we are correct in interpreting L as the *length of the curve between the point* $\mathbf{r}(a)$ *and* $\mathbf{r}(b)$.

EXAMPLE 6 Find the length of the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cosh(t) \rangle$ for t in $[0, \ln 2]$

Solution: To do so, we first compute the velocity:

$$\mathbf{v}(t) = \langle -\sin(t), \cos(t), \sinh(t) \rangle$$

The speed is then the magnitude of the velocity:

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + \sinh^2(t)}$$

The identities $\cos^2(t) + \sin^2(t) = 1$ and $\cosh^2(t) = \sinh^2(t) + 1$ imply that

$$\|\mathbf{v}(t)\| = \sqrt{1 + \sinh^2(t)} = \sqrt{\cosh^2(t)}$$

Thus, $\|\mathbf{v}(t)\| = \cosh(t)$ and as a result,

$$L = \int_0^{\ln 2} \|\mathbf{v}(t)\| \, dt = \int_0^{\ln 2} \cosh(t) \, dt$$

Evaluating the integral leads to

$$L = \sinh(t)|_0^{\ln(2)} = \sinh(\ln 2) - \sinh(0)$$

Since $\sinh(0) = 0$, this simplifies to

$$L = \frac{e^{\ln(2)} - e^{-\ln(2)}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$$

Check your Reading: How does the relationship between an odometer and a speedometer compare to the relationship between arclength and speed?

The Arclength Function

The *arclength function* for a smooth curve $\mathbf{r}(t)$, t in $[a, b]$, is the function $s(t)$ defined by

$$s(t) = \int_0^t \|\mathbf{v}(\tau)\| \, d\tau$$

where τ is the Greek letter "tau." It follows that the arclength function $s(t)$ measures the distance an object has traveled since the time $t = 0$ (i.e., it is the "odometer" for the object).

EXAMPLE 7 Find the arclength function for the curve parameterized by

$$\mathbf{r}(t) = \left\langle t^2, \frac{8}{3}t^{3/2}, 4t \right\rangle, \quad t \geq 0$$

Solution: To begin with, $\mathbf{v} = \left\langle 2t, 4t^{1/2}, 4 \right\rangle$, so that the speed is given by

$$\|\mathbf{v}(t)\| = \sqrt{4t^2 + 16t + 16} = \sqrt{(2t+4)^2}$$

Thus, $v = 2t + 4$, so that $v(\tau) = 2\tau + 4$ and

$$s = \int_0^t (2\tau + 4)\, d\tau = t^2 + 4t$$

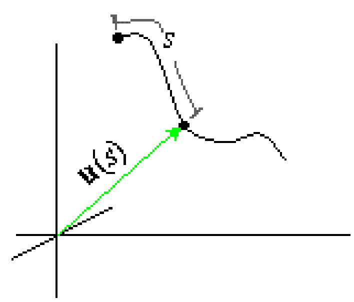
Since $v = \|\mathbf{v}(t)\| > 0$ for all t, the arclength function $s = f(t)$ is invertible, so that solving for t yields

$$t = f^{-1}(s)$$

where f^{-1} is the *inverse of f.* Substituting for t in the parameterization then yields a new parameterization

$$\mathbf{u}(s) = \mathbf{r}\left(f^{-1}(s)\right)$$

which is known as the *arclength parameterization* of the curve. The arclength parameterization allows us to parameterize the curve in terms of distance traveled along the curve (i.e., parameterized by an odometer reading), which makes it valuable in many applications.



EXAMPLE 8 Find the arclength parameterization of the curve parameterized by

$$\mathbf{r}(t) = \left\langle t^2, \frac{8}{3}t^{3/2}, 4t \right\rangle, \ t \geq 0$$

Solution: In example 7, we showed that $s = t^2 + 4t$, which is the same as

$$t^2 + 4t - s = 0$$

Solving for t (and using the positive root) yields

$$t = \frac{-4 \pm \sqrt{16 + 4s}}{2}$$

which implies that $t = \sqrt{s+4} - 2$. Finally, substituting for t in $\mathbf{r}(t) = \left\langle t^2, \frac{8}{3} t^{3/2}, 4t \right\rangle$ yields the arclength parameterization

$$\mathbf{u}(s) = \left\langle \left(\sqrt{s+4} - 2\right)^2, \frac{8}{3}\left(\sqrt{s+4} - 2\right)^{3/2}, \sqrt{s+4} - 2 \right\rangle$$

That is, $\mathbf{r}(t)$ and $\mathbf{u}(s)$ parameterize the same curve, the difference being that the speed of $\mathbf{r}(t)$ is **not** constant, but the speed of $\mathbf{u}(s)$ is always 1.

Exercises:

Let $\mathbf{v}(t)$ and $\mathbf{a}(t)$ denote the velocity and acceleration, respectively, of a vector-valued function $\mathbf{r}(t)$. Also, let $r = ||\mathbf{r}||$, $v = ||\mathbf{v}||$, and $a = ||\mathbf{a}||$; let

$$r^2 = \mathbf{r} \cdot \mathbf{r}, \quad v^2 = \mathbf{v} \cdot \mathbf{v}, \quad \text{and} \quad a^2 = \mathbf{a} \cdot \mathbf{a};$$

and let m, k, c, and $\mathbf{L}$ be constant. Use theorem 7.1 to find the specified derivative. Then expand and find the derivative directly. Show that both approaches produce the same result.

1. $\mathbf{r}'(t)$ *if* $\mathbf{r}(t) = e^t\, \mathbf{u}(t)$ *and* $\mathbf{u}(t) = \langle \cos(t), \sin(t) \rangle$
2. $\mathbf{r}'(t)$ *if* $\mathbf{r}(t) = e^t\, \mathbf{u}(t)$ *and* $\mathbf{u}(t) = \langle \sin(2t), \sin(t), \cos(t) \rangle$
3. $\mathbf{r}'(t)$ *if* $\mathbf{r}(t) = \left\langle t^2, 2, 1 \right\rangle \times \left\langle t^3, 1, 2 \right\rangle$
4. $\mathbf{r}'(t)$ *if* $\mathbf{r}(t) = \left\langle t^2, e^t, 2t \right\rangle \times \mathbf{k}$
5. $\frac{dr}{dt}$ *if* $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ (*note* : $\frac{dr}{dt} = \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r})^{1/2}$)
6. $\frac{dr}{dt}$ *if* $\mathbf{r}(t) = \langle t, t, t \rangle$ (*note* : $\frac{dr}{dt} = \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r})^{1/2}$)

Find the speed and unit tangent vector for each $\mathbf{r}(t)$.

7. $\mathbf{r}(t) = \langle 3t, 4t + 3 \rangle$
8. $\mathbf{r}(t) = \langle 5t + 2, 12t + 3 \rangle$
9. $\mathbf{r}(t) = \left\langle t^2, 2t, \ln(t) \right\rangle$
10. $\mathbf{r}(t) = \left\langle t^3, 3t^2, 6t \right\rangle$
11. $\mathbf{r}(t) = \left\langle 3\sin\left(t^2\right), 4\sin\left(t^2\right), 5\cos\left(t^2\right) \right\rangle$
12. $\mathbf{r}(t) = \langle \sin(t), \cosh(t), \cos(t) \rangle$
13. $\mathbf{r}(t) = \left\langle e^{2t}, 2e^t, t \right\rangle$
14. $\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln|\sec(t)| \rangle$

Find the arclength of the given curve over the given interval.

15. $\mathbf{r}(t) = \langle \cos(2t), \sin(2t) \rangle$
t in $[0, \pi]$

16. $\mathbf{r}(t) = \langle 3\cos(\pi t), 3\sin(\pi t) \rangle$
t in $[0, 1]$

17. $\mathbf{r}(t) = \left\langle 2\cos^2(\theta), 2\sin(\theta)\cos(\theta) \right\rangle$
θ in $[0, \pi]$

18. $\mathbf{r}(t) = \left\langle \frac{\cos(\theta)}{\cos(\theta)+\sin(\theta)}, \frac{\sin(\theta)}{\cos(\theta)+\sin(\theta)} \right\rangle$
θ in $\left(0, \frac{\pi}{2}\right)$

19. $\mathbf{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$
t in $[0, \pi]$

20. $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$
t in $[0, \pi]$

21. $\mathbf{r}(t) = \langle 3t, 4t, 5\cosh(t) \rangle$
t in $[0, \ln 2]$

22. $\mathbf{r}(t) = \langle \cos(t),\ \cosh(t),\ \sin(t) \rangle$
t in $[0, \ln 2]$

23. $\mathbf{r}(t) = \left\langle \cos(t), \sin(t), 2t^{3/2} \right\rangle$
t in $(0, 1)$

24. $\mathbf{r}(t) = \left\langle \frac{\sin(t)}{\cosh(t)}, \frac{\cos(t)}{\cosh(t)}, \frac{\sinh(t)}{\cosh(t)} \right\rangle$
t in $[0, \pi]$

Find the arclength function for the given curve.

25. $\mathbf{r}(t) = \left\langle \sin\left(t^3\right), \cos\left(t^3\right) \right\rangle$

26. $\mathbf{r}(t) = \left\langle t, t^{3/2} \right\rangle$

27. $\mathbf{r}(t) = \langle \sin(t), \cosh(t), \cos(t) \rangle$

28. $\mathbf{r}(t) = \left\langle 3\sin\left(t^2\right), 4\sin\left(t^2\right), 5\cos\left(t^2\right) \right\rangle$

29. $\mathbf{r}(t) = \left\langle t^2, 2t^3, 2\left(1-t^2\right)^{3/2} \right\rangle$

30. $\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln|\sec(t)| \rangle$

31. $\mathbf{r}(t) = \left\langle e^{2t}, 2e^t, t \right\rangle$

32. $\mathbf{r}(t) = \left\langle 3t, 4t, 5\sqrt{1-t^2} \right\rangle$

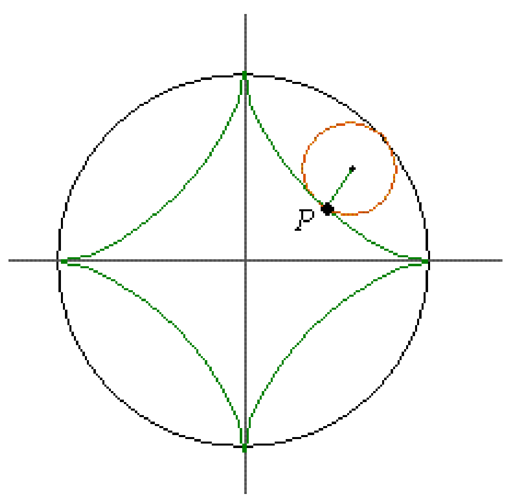
33. Find the arclength function and the arclength parameterization of the circle

$$\mathbf{r}(t) = \langle \cos(\ln t), \sin(\ln t) \rangle,\ t > 0$$

34. Find the arclength function and the arclength parameterization of the helix

$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$$

35. If a wheel of radius 0.25 is rolled around the inside of the unit circle,



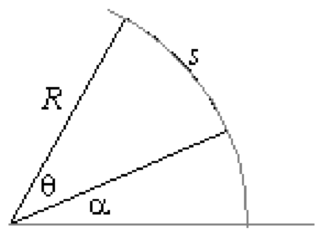
then the point P which is initially at $(1,0)$ traces out a curve known as an *astroid.* Plot the parameterization of the astroid, which is

$$\mathbf{r}(t) = \left\langle \cos^3(t), \sin^3(t) \right\rangle, \quad t \ in \ [0, 2\pi]$$

Then find the arclength of this astroid curve.

36. The line $y = mx + b$ is parameterized by $\mathbf{r}(t) = \langle t, mt + b \rangle$. What is the arclength parameterization of the line?

37. Show that if θ is the angle implied by an arc of length s on a circle of radius R,



then $s = R\theta$. (Hint: use the parameterization $\mathbf{r}(t) = \langle R\cos(t), R\sin(t) \rangle$ for t in $[\alpha, \alpha + \theta]$.

38. Show that if $\mathbf{r}(t)$ has constant speed v, then its arclength parameterization is

$$\mathbf{r}\left(\frac{s}{v}\right)$$

39. Suppose that $\mathbf{r}(t)$ satisfies the *harmonic oscillator equation*, which is

$$m\mathbf{a} = -\omega^2 \mathbf{r}$$

where m and ω are constant and $\mathbf{a} = \mathbf{r}''$. Show that if

$$h = \frac{1}{2}mv^2 + \frac{1}{2}\omega^2 r^2$$

then h is constant (hint: what is $h'(t)$)?

40. Suppose that $\mathbf{r}(t)$ satisfies the *inverse square law of attraction*, which is

$$\mathbf{a} = \frac{-k}{r^3}\mathbf{r}$$

where k is constant and $\mathbf{a} = \mathbf{r}''$. Show that if

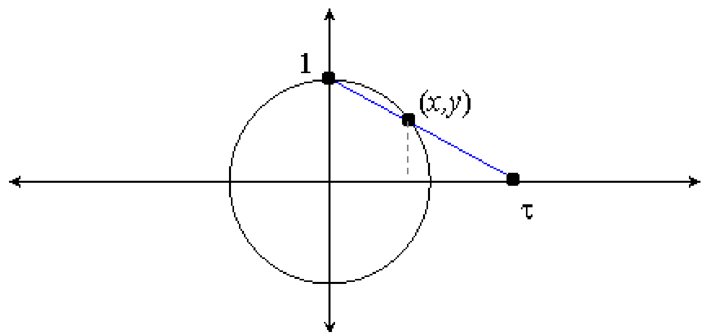
$$h = \frac{1}{2}v^2 - \frac{k}{r}$$

then h is constant (hint: what is $h'(t)$)?

41. The *Cayley transform* (i.e., stereographic projection of the circle) is the mapping of the real line into the unit circle given by

$$\mathbf{r}(\tau) = \left\langle \frac{\tau^2 - 1}{\tau^2 + 1}, \frac{2\tau}{\tau^2 + 1} \right\rangle$$

1. (a) Derive the parameterization by using similar triangles to associate τ with x and y (see diagram below):



(b) Show that $\mathbf{r}(\tau)$ is a parameterization of the unit circle by showing that it has constant length.

(c) Explain why all but one point on the unit circle is the image of a point on the real line. What is that point?

(d) Find the speed and arclength function of $\mathbf{r}(\tau)$ for τ in $(-\infty, \infty)$.

42.* Prove that the Cayley transform in exercise 41 is a 1-1 mapping.

43. The following is a parameterization of a circle in $\mathbb{R}^3$ for t in $[0, \pi]$:

$$\mathbf{r}(t) = \left\langle \frac{\sin^2(t)}{1+\sin^2(t)}, \frac{\sin(t)\cos(t)}{1+\sin^2(t)}, \frac{\cos^2(t)}{2+2\sin^2(t)} \right\rangle$$

What is its radius? (Hint: divide the length by 2π).

44. Computer Algebra System: The following is the parameterization of a circle in $\mathbb{R}^3$ for t in $[0, 2\pi]$.

$$\mathbf{r}(t) = \left\langle \frac{\cos(t)}{\sqrt{2\cos^2(t)+2\cos(t)+1}}, \frac{\sin(t)}{\sqrt{2\cos^2(t)+2\cos(t)+1}}, \frac{1+2\cos(t)}{\sqrt{2\cos^2(t)+2\cos(t)+1}} \right\rangle$$

What is its radius? (Hint: divide the length by 2π).

45. Write to Learn: Write a short essay which explain why if a curve $\mathbf{r}$ is parameterized in terms of its arclength variable s, then its velocity is also its unit tangent vector for all s. (That is, the speed of the parameterization is identically 1).

46. Write to Learn: Suppose that $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, t in $[a, b]$ parameterizes a curve C and suppose that ϕ is a differentiable, 1-1 function. In section 1-6, exercise 44, we showed that

$$\mathbf{r}(u) = \langle f(\phi(u)), g(\phi(u)), h(\phi(u)) \rangle, \ u \ in \ [c, d]$$

parameterizes the same curve C when $c = \phi^{-1}(a)$ and $d = \phi^{-1}(b)$. Write an essay explaining why the length of C is the same for both parameterizations.

47. Write to Learn: In section 1-5, we learned that circles can be parameterized by

$$\mathbf{r}\left(t\right)=\left\langle p+R\cos\left(\omega t\right),q+R\sin\left(\omega t\right)\right\rangle$$

where ω is the angular speed of the parameterization. What is the arclength parameterization of $\mathbf{r}\left(t\right)$? What is the significance of the arclength parameter? How is the arclength parameterization related to the parameterization

$$\mathbf{r}\left(\theta\right)=\left\langle p+R\cos\left(\theta\right),q+R\sin\left(\theta\right)\right\rangle$$

48. Suppose that $\mathbf{r}\left(t\right)$ is any curve that does not pass through the origin, and let

$$\mathbf{q}\left(t\right)=\frac{-k}{r}\,\mathbf{r}\left(t\right)$$

where k is a positive constant.

1. (a) Show that $\mathbf{q}'\cdot\mathbf{q}=0$ for all t. What is the signficance of this result?
   (b) What is the speed of $\mathbf{q}\left(t\right)$?