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Patterns and Structure in Systems Governed by Linear Second-Order Differential Equations

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Abstract. This article represents a survey of transmutation ideas and their interaction with typical physical problems. For linear second-order differential operators P and Q one deals with canonical connections $B: P \to Q$ (transmutations) satisfying $QB = BP$ and the related transport of 'structure' between the theories of P and Q. One can study in an intrinsic manner, e.g., Parseval formulas, eigenfunction expansions, integral transform, special functions, inverse problems, integral equations, and related stochastic filtering and estimation problems, etc. There are applications in virtually any area where such operators arise.

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1. Introduction

Given that many natural processes are governed by second-order linear differential equations, and related integral equations (at least to a first approximation) it is certainly no surprise to see that certain mathematical patterns and structures recur frequently in physics and applied mathematics. We will try to weave the theme of transmutation or intertwining of operators through certain areas in pure and applied mathematics (quantum scattering theory, linear stochastic estimation, special function theory, inverse problems in geophysics, orthogonal functions and integral transforms, etc.). Transmutation (cf. [81]) is of course a simple idea. One takes here two second-order ordinary differential operators P and Q (we usually work on $[O, \infty)$) and looks for B such that $BP = QB$, acting on suitable objects (we write $B: P \to Q$). Usually B will be an integral operator with a distribution kernel and there will be an inverse $\mathscr{B} = B^{-1}$ in the sense of Volterra operators in

C^0 (Schwartz topology) or L^p_{loc} (we are not dealing however with operator similarity in any L^p space). There are generally many transmutations $P \to Q$ characterized, e.g., by their action on suitable generalized eigenfunctions, or by domains, or by various minimization properties, or as solutions of certain Cauchy problems, etc. Given the 'categorical' nature of the matter (e.g., roughly one might think of the differential operators as functors and transmutations as natural transformations) it is perhaps not surprising to see transmutation methods occupy (implicitly or explicitly) a pervasive role in the study of the operators P and Q as above (or in the study of related first-order systems). In fact whole areas of work in, e.g., quantum scattering theory and linear stochastic estimation revolve around this theme in the sense that transmutation methods (often implicit) are one of the principal tools used in constructing the theory. On the other hand, the use of transmutation methods to study eigenfunction expansions (via what were called transformation operators) goes back to the Russian school in the early 1950s (cf. [99, 149, 150, 166, 173]). In particular, in these studies one finds that via transmutation methods a differential operator Q, with some unknown features can be investigated in terms of another operator P (e.g., $P = D^2$), about which everything is known. The transmutation machine not only provides a unifying structure and reveals significant patterns for such work, especially in relation to inverse problems, but it serves also as the vehicle to prove important theorems. We mention also a similar role of transmutation techniques in unifying and studying many results and patterns involving special functions (cf. [27, 29, 48, 49, 50, 53]). Although we have summarized and developed some of the transmutation material in book form (see [28, 29]) and in summary articles (see [30, 31]) new results and points of view have continued to emerge. There seems now to be some stabilization concerning the main lines of development and it appears that a systematic exposition of the transmutation theory should be possible and desirable (see [32] for what we hope will be, at least partially, a definitive treatment). We refer to [27–66] for the author's work on transmutation and in this article we will try to survey some of the main directions; in fact this also represents a survey of the principal material in [32].

Let us emphasize that we are trying to show the unifying structure and patterns in various kinds of problems without giving an exhaustive survey of the problems themselves. In fact, the use of transmutation methods has also been helpful as a guide and tool in developing new results about (for example) geophysical inverse problems (indicated later); thus the transmutation machine goes beyond mere unification in its usefulness. Let us remark that, relative to inverse problems, although transmutation methods are particularly suited to one-dimensional and certain three-dimensional problems, inverse problems in higher dimensions are only partially amenable to transmutation techniques and other methods are necessary (cf. [214–218, 175]); also this article is not designed to be a survey of inverse problems. Similarly, in dealing with singular differential equations and special functions we are concerned mainly with special functions arising from the radial Laplace–Beltrami operator in a noncompact Riemannian symmetric space of rank one – we have not explored the higher rank situations here nor have we treated special functions which arise, e.g., from combinatorics; even the general orthogonal 'polynomials' of Section 6 involve an underlying differential equation. Thus, to a specialist in any one area, the survey will seem incomplete and I can only ask that such a reader look at the whole picture. It is too much to expect that everyone's favorite result should be included. We generally refer to transmutation techniques as any of a variety of constructions and operations arising basically from a connection between generalized eigenfunctions (e.g., $\varphi_\lambda^Q(y) = \langle \beta(y, x), \varphi_\lambda^P(x)\rangle$) or from a spectral form of kernel (here, e.g., $\beta(y, x) = \langle \Omega_\lambda^P(x), \varphi_\lambda^Q(y)\rangle_\nu$). One emphasizes and isolates what is intrinsic and canonical; this leads to machinery which applies generally to any admissible P and Q and permits whole theories to be transported back and forth. There will be a lot of notation – things need names and we have tried to follow some standard notation where possible. In any one field however there will of course necessarily be deviations from standard notation. Given the large amount of interaction between various disciplines, in the hope of producing a readable report, we have decided to try to develop one theme at a time with only occasional connecting remarks. The connections will be evident as one continues reading and we have included some summary comments from time to time. Another feature of our presentation is to avoid frequently a precise description of the domain where various operators are acting. This gives some results a formal appearance but the alternative of specifying domains would bury the material in a maze of essentially trivial notation. Moreover, various operators are considered at different times to act on different objects (e.g., cosine transforms extend to distributions) so it is best to simply realize that the theory is certainly not vacuous and fill in the appropriate domain when needed or desired.

Let us remark that the transmutation theme might lend itself to some sort of categorical analysis of interpretation in a conceptually interesting manner. There are only a relatively small number of basic ingredients, namely the operators P and Q, the intertwining operators B and their kernels characterized by connections of generalized eigenfunctions, spectral pairings, minimization, Cauchy problems, Goursat problems, etc., certain critical structural equations such as the Gelfand-Levitan (GL) and Marčenko (M) equations, the role of analyticity, spectral asymptotics, and Paley–Wiener ideas in dealing with the eigenfunctions and triangularity of kernels (the latter being related to the underlying hyperbolic equations), and finally the connection to stochastic processes and filtering is an underlying feature. The theory is quite sensitive; via transmutation kernels it 'sees' the spectrum and the coefficients of the operators! Further examination of the precise nature of this sensitivity is probably worth pursuing. Thus, giving categorical notions linking the above concepts one might feel that the theory of second-order linear differential operators with regular singular points could be embalmed in a collection of arrows and diagrams. Although there seems to be little danger that this will happen, the connections between the various ingredients just indicated could be developed further. For example, the theory of deBranges spaces (cf. [84]) as used in [87–89, 7] binds together certain ideas (cf. Section 7). One should probably not expect one all-embracing guiding directive to emerge however; there seems to be too much going on. Also let us remark that the manner in which transmutation ideas are involved in the various applied areas indicated differs generally in nature and in depth. This is related to the kind of information involved in the theory, what is measured or sought, the 'level' (differential or integral equations for example) at which one works, etc.

We will organize this survey around the following main subthemes:

(A) *Parseval theorems*. A technique of Marčenko [165] is exhibited which connects operators $Q = D^2 - q$ with $P = D^2$ via a transmutation B_h in such a way that general Parseval formulas and eigenfunction expansion theorems for Q can be obtained via the known Fourier theory for P. Extensions to singular operators are indicated also following [29, 34].

(B) *Inverse problems*. Some formulas from quantum scattering theory are indicated following [69, 93] and some transmutation techniques for treating one-dimensional inverse problems in geophysics are developed following [36–42]. We discuss also some other techniques for geophysical inverse problems in 1 and 3 dimensions.

(C) *Stochastic estimation*. Some ideas from linear stochastic estimation involving smoothing and filterning are sketched. Various integral equations arising in filtering theory for example are of GL type and their structure can be studied in considerable detail with a view to computation; in particular one obtains partial differential equations for the filtering kernels. The spectral representation of observation processes leads to a differential equation $Q\gamma_\pm = -\lambda^2\gamma_\pm$ for the spectral innovations processes $\gamma_\pm$ and establishes further contact with scattering and transmutation theory (cf. here [32, 51, 52, 118, 158, 159]).

(D) *General transmutations*. Operators of the form $Qu = (\Delta_Q u')'/\Delta_Q$ arising from the radial Laplace–Beltrami operator in rank one noncompact symmetric spaces will be of inmportance for special functions. We indicate how Paley–Wiener techniques, contour integration, and hyperbolic differential equations lead to general triangularity results for kernels and this yields canonical connection formulas of Riemann–Liouville and Weyl-type between special functions. We discuss generalized translation, generalized convolution, spectral pairings for transmutation kernels, general GL and M equations, and show how transmutation kernels can be characterized in different ways. In particular, one can characterize such kernels via minimization (with the GL or M equations arising as 'Euler' equations). There is also an intrinsic connection between this minimization and least squares estimation when there is an underlying stochastic process. Relations between connection formulas and general Goursat problems are also indicated.

(E) *Special functions*. We consider first various special functions arising when transmuting operators of the form $x^2\varphi''+x^2[\lambda^2-\tilde{q}(x)]\varphi=(\nu^2-\frac{1}{4})\varphi$ with ν as a spectral variable (instead of $k\sim\lambda$ which is fixed here). Such operators arise in quantum scattering theory and in geophysics for example, and their study here involves as particular cases a study of the Bergman–Gilbert operator as a transmutation, general Kontorovič–Lebedev inversion theorems (e.g., a new Whittaker inversion theorem), some new results on generating functions via transmutation kernels, etc. Some further information on special functions of the type indicated in (D) is also included along with new results in the spirit of general 'orthogonal polynomials'. We also show via 'elliptic transmutation' how one can give for example a canonical (hence extendable) presentation of generalized axially symmetric potential theory (GASPT) with its related generalized Hilbert transforms and special functions.

(F) *Systems of equations.* We sketch briefly some developments involving 'canonical' systems of equations following [6, 7, 87, 88, 163, 164] (cf. also [94, 110, 148]). In particular, one deals with equations $JD_xU-VU=\lambda U$ in the context of deBranges spaces and reproducing kernels to produce expansion theorems, GL and M equations, scattering theory, etc. (see [32, 63, 156]).

(G) Let us mention two topics which we omit in this article but sketch briefly in [32]. These are first the ideas of random evolutions (going back to M. Kac) where, e.g., given $v_{tt}=Av$ and $T(t)$ a certain randomized time based on a Poisson process with intensity a, it follows that $u=E[v(T(t))]=\hat{v}(t)$ satisfies $u''+2a(t)u'=Au$ (cf. [133]). Thus, $E=$ expectation transmutes D^2 and D^2+2aD via a random time. The subject was developed further in [111, 195], for example, to which we refer for further references. Another area involving 'differential' transmutations $B=\beta D_x+\alpha$ related to Darboux transformations is partially developed in [235]. The motivation here involves the study of differential equations and special functions arising in the theory of black holes (cf. [73]). There are connections to isospectral transforms, Crum transforms, etc. (cf. also [205]).

2. Parseval Formulas

We sketch first a technique of Marčenko [165, 166] for obtaining Parseval formulas and (generalized eigenfunction expansion theorems for $Q=D^2-q$ on $[O,\infty)$. This shows how transmutation ideas enter into such investigations (cf. also [99, 149–151, 167, 173, 174]). In this connection we can use the following general result of transmutation (cf. [29, 33, 81, 115, 160, 167]).

THEOREM 2.1. *Let P and Q be second-order ordinary differential operators on $[O,\infty)$, A and C linear operators commuting with P (with action on suitable functions), and extend all functions to be even in x on $(-\infty,\infty)$. Assume the Cauchy problem* $(y\geq 0)$

$$P(D_x)\varphi=Q(D_y)\varphi;\qquad \varphi(x,0)=Af(x);\qquad \varphi_y(x,0)=Cf(x) \tag{2.1}$$

has unique solutions (in some suitable class of functions φ). Then defining $Bf(y)=\varphi(0,y)$ it follows that $BPf=QBf$.

This follows immediately since if $\psi(x,y)=P(D_x)\varphi(x,y)$ then $[P(D_x)-Q(D_y)]\psi=0$ with $\psi(x,0)=APf(x)$ and $\psi_y(x,0)=CPf(x)$. Hence,

$$BPf(y)=\psi(0,y)=P(D_x)\varphi(x,y)|_{x=0}=Q(D_y)\varphi(0,y)=QBf(y).$$

Let now $\varphi^Q_{\lambda,h}$ be the solution of

$$Q\varphi=-\lambda^2\varphi;\qquad Q=D^2-q;\qquad \varphi^Q_{\lambda,h}(0)=1;\qquad D_x\varphi^Q_{\lambda,h}(0)=h \tag{2.2}$$

where q can be complex. The corresponding eigenfunction of $D^2=P$ is $\varphi^P_{\lambda,h}=\cos\lambda x+h[\sin\lambda x/\lambda]$. One now constructs a Riemann function R for (2.1) with $q\in C^1$ (here $D_x^2\varphi=Q(D_y)\varphi$) by successive approximations for example. Setting $K(x,y,\xi)=-[R_\xi(\xi,0,x,y)+R_\eta(\xi,0,x,y)]$ one can pass to $q\in C^0$ and take $x=0$ to obtain

$$Bf(y)=f(y)+\int_{-y}^{y}K(y,\xi)f(\xi)\,\mathrm{d}\xi \tag{2.3}$$

(with $K(y,\xi)=K(0,y,\xi)$). Some further analysis then yields

THEOREM 2.2. *Set*

$$K_h(y,x)=h+K(y,x)+K(y,-x)+h\int_x^y[K(y,\xi)-K(y,-\xi)]\,\mathrm{d}\xi.$$

Then one has a transmutation B_h ($QB_h=B_hD^2$ acting say on f with $f'(0)=0$), with kernel $\beta_h(y,x)=\delta(x-y)+K_h(y,x)$, such that

$$\varphi^Q_{\lambda,h}(y)=\cos\lambda y+\int_0^y K_h(y,x)\cos\lambda x\,\mathrm{d}x. \tag{2.4}$$

B_h has a Volterra type (transmutation) inverse $\mathscr{B}_h$ ($D^2\mathscr{B}_h=\mathscr{B}_hQ$ acting say on functions f with $f'(0)=hf(0)$) with kernel $\mathscr{B}_h=\gamma_h(x,y)=\delta(x-y)+L_h(x,y)$. For $q\in C^n$ one has, e.g., $K_h\in C^{n+1}$ and the kernels K_h and L_h satisfy Goursat-type problems ($Q=D^2-q$)

$$D_x^2K_h(y,x)=Q(D_y)K_h;\qquad D_xK_h(y,0)=0;$$
$$K_h(y,y)=h+\tfrac12\int_0^y q(\eta)\,\mathrm{d}\eta; \tag{2.5}$$

$$D_x^2L_h(x,y)=Q(D_y)L_h(x,y);\quad D_yL_h(x,0)=hL_h(x,0);$$
$$L_h(x,x)=-h-\tfrac12\int_0^x q(\alpha)\,\mathrm{d}\alpha. \tag{2.6}$$

COMMENT 2.3. Transmutations can arise via Cauchy problems (Theorem 2.1).

The actual construction of kernels can be achieved via Riemann functions and the kernels satisfy Goursat problems (Theorem 2.2).

We note that this procedure for $P \to D^2 \to Q$ allows one to transmute $P \to Q$ with no concern for identity of spectra, etc. Now define for $f \in K^2 = L^2$ with compact support

$$\mathscr{Q}_h f(\lambda) = \int_0^\infty f(x)\varphi_{\lambda,h}^Q(x)\,dx; \qquad \mathscr{P}f(\lambda) = \int_0^\infty f(x)\cos\lambda x\,dx = Cf(\lambda);$$
$$\mathbb{P}F(x) = (2/\pi)\int_0^\infty F(\lambda)\cos\lambda x\,d\lambda. \tag{2.7}$$

Thus, $\mathbb{P} = \mathscr{P}^{-1}$ and one wants to determine a spectral 'function' R^Q, the Marčenko generalized spectral function (which is actually a distribution in general), such that one has a Parseval formula

$$\int_0^\infty f(x)g(x)\,dx = \langle R^Q, \mathscr{Q}_h f(\lambda)\mathscr{Q}_h g(\lambda)\rangle_\lambda \tag{2.8}$$

($f, g \in K^2$) from which an easy limiting argument for $g_n(x) \to \delta(x-y)$ yields

$$f(y) = \langle R^Q, \mathscr{Q}_h f(\lambda)\varphi_{\lambda,h}^Q(y)\rangle_\lambda. \tag{2.9}$$

Here one locates R^Q as follows. Write $K^2 = \bigcup K^2(\sigma)$ where $f \in K^2(\sigma)$ means $\operatorname{supp} f \subset [0, \sigma]$ and (by Paley–Wiener) $\mathscr{P}K^2(\sigma)$ consists of even entire functions $g(\lambda)$ with $g \in L^2$ for λ real and (*) $|g(\lambda)| \leq c\exp(\sigma|\mathrm{Im}\,\lambda|)$. Let $Z(\sigma)$ denote entire even functions satisfying (*) with $g \in L^1$ for λ real. Put inductive limit type topologies on $Z = \bigcup Z(\sigma)$ and $\mathscr{P}K^2 = \bigcup \mathscr{P}K^2(\sigma)$. Evidently $Z \subset \mathscr{P}K^2$ and g_1, $g_2 \in \mathscr{P}K^2$ implies $g_1 g_2 \in Z$. We will see that R^Q in (2.8)–(2.9) belongs to Z' and when q and h are real (and suitable) (2.8)–(2.9) reduce to the standard L^2 pairings with $R^Q \sim d\omega$ on $(-\infty, \infty)$ (spectral measure). The cosine transform is defined in Z' via a standard Parseval formula $\langle CT, f\rangle = \langle T, Cf\rangle$. We sketch now the construction of (2.8)–(2.9) in order to illustrate how transmutation ideas play an intrinsic and fundamental role. Thus, first define

$$B_h^* g(x) = \langle \beta_h(y,x), g(y)\rangle = g(x) + \int_x^\infty K_h(y,x)g(y)\,dy. \tag{2.10}$$

Then for, say $g \in K^2(\sigma)$, we see that $B_h^* g \in K^2(\sigma)$ and

$$(\blacklozenge)\,\mathscr{P}B_h^* g = \langle \cos\lambda x, \langle \beta_h(y,x), g(y)\rangle\rangle = \langle\langle \beta_h(y,x), \cos\lambda x\rangle, g(y)\rangle$$
$$= \langle \varphi_{\lambda,h}^Q(y), g(y)\rangle = \mathscr{Q}_h g.$$

Thus Paley–Wiener type results for $\mathscr{P}$ can be passed to $\mathscr{Q}_h$ – in particular, for $f, g \in K^2$ we see that $\mathscr{Q}_h f \mathscr{Q}_h g \in Z$.

The next ingredient concerns generalized translation. These special transmutations can be defined quite generally as solutions of

$$Q(D_x)U = Q(D_y)U; \qquad U(x,0) = Af(x); \quad U_y(x,0) = Cf(x) \tag{2.11}$$

with $(Af)'(0)=Cf(0)$ (cf. Theorem 2.1). Here, upon knowing (2.8)–(2.9), one might consider, e.g.,

$$U(x, y)=T_x^y f(x)=\langle \hat{f}(\lambda)\varphi_{\lambda,h}^Q(x)\varphi_{\lambda,h}^Q(y), R^Q\rangle_\lambda, \tag{2.12}$$

where $\hat{f}=\mathcal{Q}_h f$. Formally this U satisfies (2.11) with $U(x,0)=f(x)$ and $U_y(x,0)=hf(x)$ (so $f'(0)=hf(0)$ is indicated – note also $U(0,y)=f(y)$). Thus with $A=1$, $C=h$, and $f=\delta_n$ ($\delta_n\to\delta$ in $\mathscr{E}'$) one sets 'experimentally' in (2.11)

$$U_n(x, y)=\int_0^\infty R_n(\lambda)\varphi_{\lambda,h}^Q(x)\varphi_{\lambda,h}^Q(y)\,d\lambda \tag{2.13}$$

(where $R_n(\lambda)$ is to be determined). Such a generalized translation can be constructed also via Riemann functions as before and some routine calculation shows that $U_n(x,y)\to\delta(x-y)$ (cf. also (2.22) for an alternative proof). Now (2.13) leads to a determination of R_n via $\delta_n(x)=U_n(x,0)=\int_0^\infty R_n(\lambda)\varphi_{\lambda,h}^Q(x)\,d\lambda$. Indeed, since $\mathscr{B}_h\varphi_{\lambda,h}^Q=\cos\lambda y$ we have

$$\mathscr{B}_h\delta_n(y)=\delta_n(y)+\int_0^y L_h(y,x)\delta_n(x)\,dx=\int_0^\infty R_n(\lambda)\cos\lambda y\,d\lambda \tag{2.14}$$

and one obtains formally

$$R_n(\lambda)=(2/\pi)\int_0^\infty \cos\lambda y\mathscr{B}_h\delta_n(y)\,dy.$$

Hence, (2.13) is permitted and

$$\begin{aligned}\int_0^\infty f(x)g(x)\,dx&=\lim\langle U_n(x,y), f(x)g(y)\rangle\\ &=\lim\int_0^\infty R_n(\lambda)\mathcal{Q}_hf(\lambda)\mathcal{Q}_hg(\lambda)\,d\lambda\\ &=\lim\langle R_n(\lambda), \mathcal{Q}_hf\mathcal{Q}_hg\rangle_\lambda=\langle R^Q, \mathcal{Q}_hf\mathcal{Q}_hg\rangle.\end{aligned} \tag{2.15}$$

THEOREM 2.4. *$R_n\to R^Q$ in Z' weakly so that the Parseval formula* (2.8) *holds* (*with the inversion* (2.9)) *and explicitly $R^Q=(2/\pi)[1+CL_h(y,0)]$. When q and h are real R^Q is a spectral measure on $(-\infty,\infty)$ and one recovers the classical L^2 theory.*

COMMENT 2.5. Given the operator D^2 about which everything is known one can discover the spectral theory and expansion theorem for $Q=D^2-q$ with eigenfunctions $\varphi_{\lambda,h}^Q$ by explicit and fundamental use of transmutation methods. Paley–Wiener information is, in fact, related to the triangularity of kernels (which arise here also from the hyperbolic nature of the underlying partial differential equations). The spectrum of $(Q, \varphi_{\lambda,h}^Q)$ can be quite general and is determined by L_h. There are various hypotheses on q which will produce acceptable transmutation kernels L_h (cf. also Theorem 2.6) but the method is thoroughly transmutational in nature; the machinery runs via properties of eigenfunctions and transmutation kernels (e.g., $\varphi^Q_{\lambda,h}(x)$ is entire in λ of exponential type x, $L_h(y,0)$ is well defined, etc.) and, in fact, from (2.5) for example q is determined by $K_h(y,y)$. One is tempted to think of the 'connection' $D^2 \to Q$ as the natural habitat for hypotheses on Q (not q).

Let us indicate very briefly how this procedure can be extended to singular operators and special functions following [29, 34] (the notation is also needed later so we accomplish several goals). Thus, modeling operators Q on the radial Laplace–Beltrami operator in noncompact, rank one, Riemannian symmetric space we write ($x \in [0,\infty)$)

$$Qu = (\Delta_Q u')'/\Delta_Q + \rho_Q^2 u - q(x)u. \tag{2.16}$$

(Here $\rho_Q = \frac{1}{2}\lim \Delta_Q'/\Delta_Q$ as $x \to \infty$ is assumed to exist; its insertion in Q explicitly is basically a technical device to adjust spectra.) Typical (real) Δ_Q here are $\Delta_Q = x^{2m+1}(\rho_Q = 0)$ and $\Delta_Q = \mathrm{sh}^{2m+1}x$ ($\rho_Q = m+\frac{1}{2}$), and we refer to [16, 17, 29, 34, 70, 71, 95, 98, 140, 150, 210, 222] for various types of hypotheses and detailed analysis. Generally $\Delta Q = x^{2m+1}C_Q(x)$ for suitable positive C_Q and q is a suitable (complex) potential – for convenience we assume here that q has no strong singularities β/x^2 near $x=0$ in order that there should always exist spherical functions $\varphi^Q_\lambda(x)$ satisfying (●) $Q\varphi = -\lambda^2\varphi$ with $\varphi^Q_\lambda(0)=1$ and $D_x\varphi^Q_\lambda(0)=0$. Set $\Omega^Q_\lambda(x) = \Delta_Q(x)\varphi^Q_\lambda(x)$ and define

$$\mathfrak{Q}f(\lambda) = \hat{f}(\lambda) = \int_0^\infty f(x)\Omega^Q_\lambda(x)\,dx. \tag{2.17}$$

For comparison let $Pu = (\Delta_Q u')'/\Delta_Q + \rho_Q^2 u$ (i.e., $\Delta_P = \Delta_Q$ with $Q = P - q$) and define Jost solutions $\Phi^P_{\pm\lambda}(x)$ of $P\varphi = -\lambda^2\varphi$ which are asymptotic as $x\to\infty$ to $\Delta_Q^{-1/2}(x)\exp(\pm i\lambda x)$. One can write

$$\varphi^P_\lambda(x) = c_P(\lambda)\Phi^P_\lambda(x) + c_P(-\lambda)\Phi^P_{-\lambda}(x) \tag{2.18}$$

and define $\mathfrak{P}f$ as in (2.17) with $\Omega^P_\lambda = \Delta_P\varphi^P_\lambda$; one has for general classes of Δ_P ($=\Delta_Q$ here) an inversion $f = \mathfrak{P}\hat{f}$ in the form (cf. [29])

$$f(x) = \int_0^\infty \mathfrak{P}f(\lambda)\Phi^P_\lambda(x)\,\mathrm{d}\nu(\lambda); \qquad \mathrm{d}\nu = R_0\,\mathrm{d}\lambda = \mathrm{d}\lambda/2\pi|c_P(\lambda)|^2. \tag{2.19}$$

We assume R_0 is known now although if not there are techniques for 'discovering' it via a transmutation $B_P\colon D^2 \to P$ (cf. [29]). Let now $B\colon P \to Q$ be the transmutation characterized by $B\varphi^P_\lambda = \varphi^Q_\lambda$ (further details appear in Sections 4 and 5). Because of the identity $\Delta_P = \Delta_Q$, B will have a kernel $\beta(y,x) = \delta(x-y) + K(y,x)$ and $\mathcal{B} = B^{-1}$ has kernel $\gamma(x,y) = \delta(x-y) + L(x,y)$ (with triangularity as before). We will assume $\Delta_P(x)L(x,y)\Delta_Q^{-1}(y)$ is continuous for $0 \leq y \leq x$ and define $\hat{l}(x,y) = L(x,y)\Delta_Q^{-1}(y)$ with $\hat{l}(x,0) = \hat{l}(x)$. This is known to be realistic for important classes of q, Δ_Q, etc. (cf. [29]) and will be our substitute for more explicit hypotheses on q, Δ_Q, etc. One can define spaces analogous to

K^2, $\mathcal{P}K^2$, Z, etc. in which the transforms act and which are adapted to Paley–Wiener type interaction (here the space analogous to Z will be called W). The Parseval formula we want now is to have the form

$$\langle R^Q, \mathfrak{Q}f\mathfrak{Q}g\rangle = \int_0^\infty \Delta_Q^{1/2} f(x)\Delta_Q^{1/2} g(x)\,dx. \tag{2.20}$$

One uses the same ingredients as before for Theorem 2.4 – thus, with δ_n as before write $\delta_n^Q = \delta_n/\Delta_Q$ and experimentally set

$$T_x^y\delta_n^Q(x) = \langle R_n^\nu(\lambda), \varphi_\lambda^Q(x)\varphi_\lambda^Q(y)\rangle_\nu \tag{2.21}$$

where $\langle\ ,\ \rangle_\nu$ denotes the P-spectral pairing based on $d\nu$ in (2.19). Now via direct constructions as before or from the formula (cf. [29, 32, 150])

$$\langle T_x^y f(x), \Delta_Q(x)g(x)\rangle = \langle \Delta_Q(x)f(x), T_x^y g(x)\rangle \tag{2.22}$$

one sees that $T_x^y\delta_n^Q(x) \to \delta(x-y)/\Delta_Q(x)$. On the other hand for $y=0$ (2.21) gives

$$\delta_n^Q(x) = \langle R_n^\nu(\lambda), \varphi_\lambda^Q(x)\rangle_\nu \quad \text{so} \quad \mathcal{B}\delta_n^Q(y) = \langle R_n^\nu(\lambda), \varphi_\lambda^P(y)\rangle_\nu = \mathfrak{P}R_n^\nu\,(\mathfrak{P} = \mathfrak{P}^{-1}).$$

Hence,

$$R_n^\nu = \mathfrak{P}\mathcal{B}\delta_n^Q = \mathfrak{P}[\delta_n^Q(x) + \langle L(x,y), \delta_n^Q(y)\rangle] \to 1 + \mathfrak{P}\hat{l}.$$

Consequently, multiplying (2.21) by $\Delta_Q(x)f(x)\Delta_Q(y)g(y)$, integrating, and taking limits we have

THEOREM 2.6. $R_n^\nu R_0 \to R^Q = R_0[1+\mathfrak{P}\hat{l}]$ *weakly in a certain space* W' *and* (2.20) *holds with* $\mathfrak{Q}f\mathfrak{Q}g \in W$. *This leads to* $f(x) = \mathfrak{Q}(\mathfrak{Q}f) = \langle \mathfrak{Q}f(\lambda)\varphi_\lambda^Q(x), R^Q\rangle$ *for* f *as indicated.*

COMMENT 2.7. The transmutation machinery of Comment 2.5 can be extended to determine the spectral theory and expansion theorem for singular (Q, φ_λ^Q) in terms of a known singular operator (P, φ_λ^P) $(\Delta_P = \Delta_Q,\ Q = P - q)$. In fact the method extends to $\Delta_Q = \Delta_P C_Q$ for suitable positive C_Q (cf. Section 6) and prototypical are the known situations $\Delta_P = x^{2m+1}$, $\Delta_P = \mathrm{sh}^{2m+1}x$, etc.

REMARK 2.8. Let us make a few comments here about Darboux–Christoffel-type formulas which play an important role in various contexts (cf. Theorems 4.8 and 4.15 for example as well as Section 7 for systems). Thus, let $Qu = (\Delta_Q u')'/\Delta_Q + \rho_Q^2 u = -\lambda^2 u$ for simplicity with φ_λ^Q, Ω_λ^Q, etc. as above and $d\omega_Q = d\lambda/2\pi|c_Q|^2 = \hat{\omega}_Q\,d\lambda$. For $W(\varphi_\lambda, \varphi_\mu) = \varphi_\lambda\varphi_\mu' - \varphi_\lambda'\varphi_\mu$ one easily obtains

$$\Delta_Q W(\varphi_\lambda, \varphi_\mu)(T) = \delta_T(\lambda,\mu) = (\lambda^2-\mu^2)\int_0^T \Omega_\lambda^Q(x)\varphi_\mu^Q(x)\,dx.$$

One defines a reproducing kernel

$$\Lambda^T(\lambda,\mu) = [\delta_T(\lambda,\mu)/(\lambda^2-\mu^2)] = \int_0^T \Omega_\lambda^Q(x)\varphi_\mu^Q(x)\,dx. \tag{2.23}$$

Thus, $d\omega_Q$ being known, one can set $\hat{f}_T(\lambda) = \int_0^T \varphi_\lambda^Q(x) f(x)\,dx$ for suitable f and it is easily checked that $\int_0^\infty \hat{f}_T(\lambda)\Lambda^T(\lambda,\mu)\,d\omega_Q = \hat{f}_T(\mu)$. Now note that from the inversion formulas $\mathfrak{Q}\mathfrak{Q} = I$ and $\mathfrak{Q}\mathfrak{Q} = I$ acting on suitable objects one has formally

$$\int_0^\infty \Omega_\lambda^Q(x)\varphi_\mu^Q(x)\,dx = \delta(\lambda-\mu)/\hat{\omega}_Q \quad \text{and} \quad \int_0^\infty \Omega_\lambda^Q(x)\varphi_\lambda^Q(y)\,d\omega_Q = \delta(x-y).$$

Thus, $\delta_T(\lambda,\mu)\hat{\omega}_Q \to (\lambda^2-\mu^2)\delta(\lambda-\mu)$ and, hence, in its action on suitable functions $\delta_T(\lambda,\mu)\to 0$ for λ, μ real. This is useful in establishing various results involving formulas of the type $\langle Q\varphi_\mu^Q, \Omega_\lambda^Q\rangle = \langle \varphi_\mu^Q, Q^*\Omega_\lambda^Q\rangle$ for example (note

$$\int_0^T Q\varphi_\mu^Q\Omega_\lambda^Q\,dx = \int_0^T \varphi_\mu^Q Q^*\Omega_\lambda^Q\,dx + \delta_T(\lambda,\mu)).$$

Further, let us show how $\Lambda^T(\lambda,\mu)\hat{\omega}_Q \to \delta(\lambda-\mu)$ in its action on analytic $\hat{f}$ of the type $\mathfrak{Q}f$ which arise naturally in the theory. Thus, use (2.18) with $\Phi_\lambda^Q \sim \Delta_Q^{-1/2}\exp(i\lambda x)$ to obtain ($c^- = c_Q(-\lambda)$ etc.)

$$\begin{aligned}\Lambda^T(\lambda,\mu)(\lambda^2-\mu^2) = \Delta_Q W(\varphi_\lambda,\varphi_\mu) \sim{}& c_\lambda c_\mu \exp[i(\lambda+\mu)T]i(\mu-\lambda) - \\ &- c_\lambda c_\mu^- \exp[i(\lambda-\mu)T]i(\lambda+\mu) + \\ &+ c_\mu c_\lambda^- \exp[i(\mu-\lambda)T]i(\lambda+\mu) + \\ &+ c_\lambda^- c_\mu^- \exp[-i(\lambda+\mu)T]i(\lambda-\mu).\end{aligned}$$

Now one need not assume *a priori* knowledge of $\hat{\omega}_Q$ but if we simply try $\Xi^T(\mu) = \int_0^\infty \Lambda^T(\lambda,\mu)f(\lambda)\,d\lambda/2\pi|c_Q|^2$ there follows

$$\Xi^T(\mu) \sim [ic_\mu e^{i\mu T}/2\pi]\int_{-\infty}^\infty \frac{f(\lambda)e^{-i\lambda T}\,d\lambda}{c_Q(\lambda)(\lambda-\mu)} - [ic_\mu^- e^{-i\mu T}/2\pi]\int_{-\infty}^\infty \frac{f(\lambda)e^{i\lambda T}\,d\lambda}{c_Q(-\lambda)(\lambda-\mu)}. \tag{2.24}$$

Now typical situations involve, e.g., $c_Q(-\lambda)$ analytic for $\operatorname{Im}\lambda > 0$ and $1/c_Q^-$ polynomially bounded so that, using distribution arguments when necessary, one obtains by contour integration $\Xi^T(\mu) \sim \hat{f}(\mu)$ (note $\hat{f}$ is even).

REMARK 2.9. In this paper we will not dwell extensively on various aspects of transmutation in connection with special functions arising from singular differential operators Q (see, however, Examples 5.1 and 5.7, Theorems 6.14 and 6.15, and assorted formulas in the text). Generally speaking, one is dealing with 'isomorphic' theories of integral transforms etc. based on the various spherical functions and Jost solutions and some of this is indicated or suggested in Section 5. Thus, the theories of the Fourier cosine transform, Hankel transform, generalized Mehler transform, Fourier–Jacobi transform, etc. are all basically equivalent and the ingredients can be transported back and forth via transmutation operators. Many connection formulas between special functions of the type involving Riemann–Liouville and Weyl type fractional integrals (or Erdélyi–Kober operators) are also transmutation formulas (cf. Example 5.7 and Theorem 6.18).

3. Inverse Problems in Quantum Scattering Theory and Geophysics

Since there are already available a number of excellent surveys of Gelfand–Levitan–Marčenko methods in quantum scattering theory we will only sketch this briefly (cf. [29, 69, 79, 93, 135, 165, 175–177]). Take q real with some sort of condition like $\int_0^\infty x|q(x)|\,dx<\infty$ (which can be relaxed) and denote by θ_λ^Q the solution (called regular in physics) of (†) $Qu=-\lambda^2 u$ satisfying $\theta_\lambda^Q(0)=0$ and $D_x\theta_\lambda^Q(0)=1$ $(Q=D^2-q)$. One has Jost solutions of (†) $\Phi_{\pm\lambda}^Q(x)\sim\exp(\pm i\lambda x)$ as $x\to\infty$. In physics $\lambda\sim k$ with $k^2=E$ (energy) and we omit for the moment the angular momentum term $l(l+1)/x^2$ (cf. Section 6); the equation (†) comes from a radial Schrödinger-type equation with a change of variables. We suppose q is unknown here and one performs experiments to determine the phase shift $\delta(\lambda)$ produced by the potential upon incident waves. This phase shift can then be used to determine the spectral measure for (Q,θ_λ^Q) (we assume here for simplicity that there are no bound states – discrete eigenvalues). Thus, first ones writes

$$\theta_\lambda^Q(x)=F(-\lambda)\Phi_\lambda^Q(x)-F(\lambda)\Phi_{-\lambda}^Q(x)/2i\lambda, \tag{3.1}$$

where $W(\Phi_\lambda^Q,\Phi_{-\lambda}^Q)=-2i\lambda$ and $F(\lambda)=-W(\theta_\lambda^Q,\Phi_\lambda^Q)$. Then

$$F(\lambda)=|F(\lambda)|\exp(-i\delta(\lambda))$$

and

$$F(\lambda)=\exp[-(2/\pi)\int_0^\infty \delta(\mu)\mu\,d\mu/(\mu^2-\lambda^2)].$$

Moreover, one can show with a routine contour integral argument that the spectral measure is

$$d\omega(\lambda)=2\lambda^2\,d\lambda/\pi|F(\lambda)|^2. \tag{3.2}$$

In various ways a connection formula can be derived in the form

$$\theta_\lambda^Q(y)=[\sin\lambda y/\lambda]+\int_0^y K(y,x)[\sin\lambda x/\lambda]\,dx \tag{3.3}$$

(e.g., using techniques of Section 2 or Paley–Wiener arguments – the latter is discussed below). Then write $d\omega=d\sigma+(2/\pi)\lambda^2 d\lambda$ and one obtains from (3.3) (cf. Section 5) a GL equation $(x<y)$

$$K(y,x)+\Omega(y,x)+\int_0^y K(y,\xi)\Omega(\xi,x)\,d\xi=0, \tag{3.4}$$

$$\Omega(x,y)=\int_0^\infty [\sin\lambda x/\lambda][\sin\lambda y/\lambda]\,d\sigma(\lambda). \tag{3.5}$$

We know K exists and one easily shows that the Fredholm equation (3.4) has unique solutions. Some further calculation then yields the potential via

$$q(x) = 2D_x K(x, x). \tag{3.6}$$

This is very neat and the formulas (3.3)–(3.4) have a canonical transmutational nature (cf. Section 5). Even neater is the manner in which the transmutation machine can be used in geophysical inverse problems. Thus, assume the earth is flat and stratified vertically downward ($x \geq 0$). One considers SH shear waves satisfying

$$\rho(x) v_{tt} = (\mu(x) v_x)_x, \tag{3.7}$$

where ρ is density and μ is a shear modulus. Given an impulsive input $v_t(0, x)$ (or $v_x(t, 0)$) the object is to determine ρ and μ from the reflection readout data $G(t) = v(t, 0)$. Somewhat more complicated input data will accomplish this (cf. below) but the best one can do from this experiment is determine the acoustic impedance $A = (\rho\mu)^{1/2}$ as a function of travel time $y = \int_0^x (\rho/\mu)^{1/2}\, d\xi$ (cf. [1, 2, 8, 12–14, 23, 26, 29, 36–42, 103, 113, 107, 209, 155–157, 206, 207, 214–217]). The treatment here is based on work of the author and F. Santosa [36–42]. Thus, change the variables in (3.7) to obtain

$$v_{tt} = (Av_y)_y / A = Q(D_y) v; \qquad v_t(0, y) = \delta(y); \qquad v(t, 0) = G(t) \tag{3.8}$$

and $v = 0$ for $t < 0$. We will assume $0 < \alpha \leq A \leq \beta < \infty$, $A \to A_\infty$ rapidly as $y \to \infty$, $A(0) = 1$ (which can always be achieved by a change of variables), and $A \in C^1$. It would be nice to be able to assume $A \in C^0$ or A piecewise continuous but this doesn't seem possible at all stages of the analysis with any of the known exact techniques. Let φ_λ^Q be the solution of (■) $Qu = -\lambda^2 u$ satisfying $\varphi_\lambda^Q(0) = 1$ and $D_x \varphi_\lambda^Q(0) = 0$. Let $\Phi_{\pm\lambda}^Q$ be 'Jost' solutions of (■) so that $\Phi_{\pm\lambda}^Q(y) \sim A_\infty^{-1/2} e^{\pm i\lambda y}$ as $y \to \infty$. We write as in (2.18)

$$\varphi_\lambda^Q(y) = c_Q(\lambda) \Phi_\lambda^Q(y) + c_Q(-\lambda) \Phi_{-\lambda}^Q(y). \tag{3.9}$$

Now φ_λ^Q and Φ_λ^Q can be constructed via formulas of the type ($q = -A'/A$)

$$\varphi_\lambda^Q(y) = \cos \lambda y + \int_0^y [\sin \lambda(y - \eta)/\lambda] q(\eta) D_\eta \varphi_\lambda^Q(\eta)\, d\eta, \tag{3.10}$$

$$\Phi_\lambda^Q(y) = A_\infty^{-1/2} e^{i\lambda y} + \int_y^\infty [\sin \lambda(\eta - y)/\lambda] q(\eta) D_\eta \Phi_\lambda^Q(\eta)\, d\eta. \tag{3.11}$$

Solutions can be found in the form $\varphi_\lambda^Q = \sum_0^\infty \varphi_n(\lambda, y)$ for example where the φ_n are determined recursively. One finds that $\varphi_\lambda^Q(y)$ is entire in λ of exponential type y with $|\varphi_\lambda^Q(y)| \leq c\, e^{y|\mathrm{Im}\, \lambda|} e^{\int_0^y |q|\, d\eta}$ while $\Phi_\lambda^Q(y)$ is analytic in the upper half plane with a bound there of the form $|\Phi_\lambda^Q(y)| \leq c\, e^{-y\, \mathrm{Im}\, \lambda} e^{c \int_y^\infty |q|\, d\eta}$. From $A(y) W(\Phi_\lambda^Q, \Phi_{-\lambda}^Q) = -2i\lambda$ we have $D_x \Phi_\lambda^Q(0) = 2i\lambda c_Q(-\lambda)$ and $\lambda c_Q(-\lambda)$ is analytic for $\mathrm{Im}\, \lambda > 0$ while in general $|1/c_Q(-\lambda)| \leq c$ for $\mathrm{Im}\, \lambda \geq 0$. There are no discrete eigenvalues and a variation of standard contour integral arguments gives the spectral measure

$$d\omega = d\lambda/2\pi|c_Q(\lambda)|^2 = \hat{\omega}\, d\lambda \quad (\lambda \geq 0). \tag{3.12}$$

(One remarks here that it is analyticity and asymptotic properties of eigenfunctions which lead to this kind of result and generally such properties along with properties of transmutation kernels make the machinery work.) One has now a Parseval formula

$$\int_0^\infty A(y)f(y)g(y)\, dy = \langle \hat{f}, \hat{g}\rangle_\omega; \qquad \hat{f} = \mathfrak{Q}f(\lambda) = \int_0^\infty A(y)f(y)\varphi_\lambda^Q(y)\, dy \tag{3.13}$$

and the corresponding inversion holds (cf. (2.19)).

Now for (3.8) the solution can be written as

$$v(t, y) = \langle \varphi_\lambda^Q(y), [\sin \lambda t/\lambda]\rangle_\omega \tag{3.14}$$

and it is interesting to note here that a Riemann function for (3.8) has the form $R(y, t, \eta, \tau) = \langle \varphi_\lambda^Q(y)\varphi_\lambda^Q(\eta), [\sin \lambda(t-\tau)/\lambda]\rangle_\omega$. From (3.14) $G(t) = v(t, 0) = \langle 1, \sin \lambda t/\lambda\rangle_\omega$ from which we obtain

THEOREM 3.1. *The spectral measure for (Q, φ_λ^Q) is determined directly from $G(t)$ via*

$$\hat{\omega}(\lambda) = (2\lambda/\pi)\int_0^\infty G(t) \sin \lambda t\, dt. \tag{3.15}$$

COMMENT 3.2. In inverse quantum scattering theory of one-dimensional inverse geophysical problems one performs an experiment to determine spectral information for the operator Q governing the response of the system. This spectral information will then be used to determine the coefficients of Q.

The result (3.15) appears to say that one must read for a long time in order to determine $\hat{\omega}(\lambda)$ but in fact one can expect fairly rapid stabilization. Next from the fact that $\chi(y, \lambda) = (2i/\lambda)[\varphi_\lambda^Q(y) - \cos \lambda y]$ is entire of exponential type y (and $\chi \in L_\lambda^2$ for λ real) we obtain via Paley–Wiener ideas

$$\chi(y, \lambda) = \int_{-y}^{y} K(y, \xi)\, e^{i\lambda\xi}\, d\xi. \tag{3.16}$$

Here $K(y, x)$ is odd in x with $K(y, 0) = 0$ and (3.16) can be written (for $\hat{K}(y, x) = K_x(y, x)$) (●)$\varphi_\lambda^Q(y) = [1 - K(y, y)] \cos \lambda y + \int_0^y \hat{K}(y, x) \cos \lambda x\, dx$. Using this with (3.10) one obtains after some calculation

$$1 - K(y, y) = A^{-1/2}(y). \tag{3.17}$$

Thus, knowledge of $K(y, y)$ will determine $A(y)$ and one is dealing here with a transmutation B: $\cos \lambda x \to \varphi_\lambda^Q(y)$ with kernel $\beta(y, x)$ of the form

$$\beta(y, x) = A^{-1/2}(y)\delta(x - y) + \hat{K}(y, x). \tag{3.18}$$

In order to determine $K(y, x)$ one uses an integral form of the canonical GL equation (cf. Section 5) which we simply display here (cf. Theorem 3.10) as $(x<y)$

$$K(y,x)+T(y,x)=\int_0^y K(y,\eta)T_\eta(\eta,x)\,\mathrm{d}\eta, \tag{3.19}$$

$$T(y,x)=\int_0^\infty [\sin\lambda x/\lambda]\cos\lambda y\,\mathrm{d}\sigma(\lambda) \tag{3.20}$$

$(\mathrm{d}\sigma=\mathrm{d}\omega-(2/\pi)\,\mathrm{d}\lambda)$. This equation (3.19) has a unique solution and in fact K can be obtained numerically using computer techniques (cf. [206, 207]). Then

THEOREM 3.3. *The inverse problem* (3.8) *for* $A(y)$ *can be solved uniquely via the GL equation* (3.19) *using* (3.17).

Now the GL equation has a version in the time domain which is very useful. Set

$$G(t)=\int_0^\infty [\sin\lambda t/\lambda][\mathrm{d}\sigma+(2/\pi)\,\mathrm{d}\lambda]=1+G_r(t)\quad(t>0). \tag{3.21}$$

Here $1=Y(t)$ (Heaviside function) represents the response when $A=1$ to a delta function input – i.e., $v(t,y)=Y(t-y)$ with $v_t=\delta(t-y)\to\delta(y)$ as $t\to 0$ (or $v_y=-\delta(t-y)\to-\delta(t)$ as $y\to 0$). This response is then present in all situations for variable A. It is now easily seen from (3.20)–(3.21) that

$$T_y(y,x)=\tfrac{1}{2}[G_r'(y+x)-G_r'(|y-x|)], \tag{3.22}$$

$$K(y,x)+\tfrac{1}{2}[G_r(y+x)-G_r(y-x)]=\tfrac{1}{2}\int_0^y K(y,s)[G_r'(x+s)-G_r'(|x-s|)]\,\mathrm{d}s \tag{3.23}$$

(the latter for $x<y$). This form of the GL equation can be used for a stability result. Thus if one has polluted data $G^*(t)=1+G_r^*(t)$ with corresponding kernel K^* satisfying (3.23) (and A^* determined by (3.17)) we write $\epsilon=G_r^*-G_r$ with $\Delta K=K^*-K$ and use (3.23) to obtain estimates of ΔK in terms of ϵ and ϵ'. Let $\|f\|_{\infty,y}=\sup|f(s)|$ for $0\leqslant s\leqslant y$ and then

THEOREM 3.4. *For* $\|\epsilon'\|_{L1(2y)}$ *sufficiently small one has* $\|\Delta K(y,\cdot)\|_{\infty,y}\leqslant c[\|\epsilon\|_{\infty,2y}+\|\epsilon'\|_{L1(2y)}]$. *Consequently an estimate on* $\Delta A=A^*-A$ *results from* (3.17).

COMMENT 3.5. The GL machinery for either quantum scattering or geophysics takes spectral information and produces coefficients. The geophysical inverse problem is in fact stable (with reasonable norms) and one remarks here that this is exceptional – most inverse problems are not well posed.

One considers next a transmission problem where the readout occurs at some $\tilde{x}>0$ corresponding to $\tilde{y}=y(\tilde{x})$. The model we treat here is a kind of subproblem arising from some problems in spherically symmetric scattering and one can envision also problems in one-dimensional tomography for example leading to transmission data. We assume $A(y) = A_\infty$ for $y \geq \tilde{y}$ and set $H(t) = v(\tilde{y}, t)$ for the transmission readout. If one can obtain $G(t) = v(0, t)$ of course it will lead directly to a solution of the inverse problem as before. If not we use $H(t)$ and in fact one obtains interesting mathematical information from $H(t)$ which is not available from $G(t)$. Thus from (3.14)

$$H(t) = \langle \varphi_\lambda^Q(\tilde{y}), [\sin \lambda t/\lambda] \rangle_\omega \tag{3.24}$$

from which $\varphi_\lambda^Q(\tilde{y})\hat{\omega}/\lambda = (2/\pi)\int_0^\infty H(t) \sin \lambda t \, \mathrm{d}t$ and consequently from (3.15)

$$\varphi_\lambda^Q(\tilde{y}) \int_0^\infty G(t) \sin \lambda t \, \mathrm{d}t = \int_0^\infty H(t) \sin \lambda t \, \mathrm{d}t. \tag{3.25}$$

The obvious deconvolution problem suggested by (3.25) is, however, not the best approach and we use the following splitting technique together with the transmutation machine ((3.25) is used later in another way – cf. (3.30)). First remove the $1/\lambda$ factor in (3.24) by working with

$$H'(t) = \langle \varphi_\lambda^Q(\tilde{y}), \cos \lambda t \rangle_\omega = \tfrac{1}{2} \int_{-\infty}^\infty \varphi_\lambda^Q(\tilde{y})\hat{\omega} \, \mathrm{e}^{i\lambda t} \, \mathrm{d}\lambda. \tag{3.26}$$

If we write now $\Psi_\lambda^Q(y) = \Phi_\lambda^Q(y)/c_Q(-\lambda)$ and

$$H_1(t) = (1/4\pi) \int_{-\infty}^\infty \Psi_\lambda^Q(\tilde{y}) \, \mathrm{e}^{-i\lambda t} \, \mathrm{d}\lambda \tag{3.27}$$

then $H'(t) = H_1(t) + H_1(-t)$ where $H_1(t) = 0$ for $t < 0$ (in fact $H_1(t) = 0$ for $t < \tilde{y}$ by a contour integral argument). Here H' is considered as an even function by virtue of the cosine representation in (3.26). Hence $\frac{1}{2}\Psi_\lambda^Q(\tilde{y}) = \mathscr{F}H_1 = \hat{H}_1$ (Fourier transform) and since $A(y) = A_\infty$ for $y \geq \tilde{y}$ with $\Phi_\lambda^Q(\tilde{y}) = A_\infty^{-1/2} \exp(i\lambda\tilde{y})$ one has

THEOREM 3.6. *The transmission readout $H_1 = H'$ for $t > 0$ determines $c_Q(-\lambda)$ via $[1/c_Q(-\lambda)] = 2A_\infty^{1/2} \exp(-i\lambda\tilde{y})\hat{H}_1(\lambda)$. This determines a splitting of the spectral measure ($\bar{c}_Q(\lambda) = c_Q(-\lambda)$ for λ real) in the form $\hat{\omega} = [1/2\pi|c_Q|^2] = (2/\pi)A_\infty|\hat{H}_1|^2$.*

REMARK 3.7. Although we have promised to refrain from intertwining remarks in the text, we feel compelled here to mention words like Wiener filter, Hardy spaces, etc. and refer to Sections 4, 7 for further details. Note also that one expects rapid stabilization of H_1 so computation is possible in a finite time.

Now split $G'(t)$ in the form $G'(t) = G_1(t) + G_1(-t)$ where $G_1(t) = 0$ for $t < 0$ is

$$G_1(t) = (1/4\pi) \int_{-\infty}^\infty \Psi_\lambda^Q(0) \, \mathrm{e}^{-i\lambda t} \, \mathrm{d}\lambda. \tag{3.28}$$

Putting $1/c_Q(-\lambda)$ into (3.28) from Theorem 3.4 and rearranging one obtains an interesting finite domain of dependence relation

$$G_1(t) = \int_{\tilde{y}}^{\tilde{y}+t} H_1(\tau)K(t-\tau)\,d\tau, \tag{3.29}$$

where K, however, involves the unknown $\Phi_\lambda^Q(0)$. This form suggests a procedure bo be indicated below where one obtains A by making data observations only on specific finite intervals of time. Even with stabilization of readouts one is not thrilled in practice by being obliged to use (3.15) or Theorem 3.4 to compute $\hat{\omega}$. In passing let us write $\mathcal{H}(t) = \int_{-\infty}^{\infty} H_1(\tau+t)H_1(\tau)\,d\tau$ so $\mathcal{F}\mathcal{H} = |\hat{H}_1|^2$. Then from $G'(t) = \int_0^\infty \hat{\omega}\cos\lambda t\,d\lambda$ and with an adjustment of 2 for the cosine representation one obtains (note also $A_\infty^{1/2}H_1(t) = \delta(t-y) + h_1(t)$)

THEOREM 3.8. *For $t > 0$ one has $G'(t) = G_1(t) = A_\infty \mathcal{H}(t)$ and stability can be obtained via Theorem* 3.4 *from estimates*

$$\left(h(t) = \int_{\tilde{y}}^{t} h_1(\tau)\,d\tau\right), \quad \|\epsilon'\|_{L^1(0,2T)} \leq c\|\Delta h_1\|_{L^1(\tilde{y},\infty)};$$

$$\|\epsilon\|_{\infty,2T} \leq \|\Delta h\|_{\infty,\tilde{y}+2T} + c[\|\Delta h_1\|_{L^1(\tilde{y},\infty)} + \|\Delta h\|_{L^1(\tilde{y},\infty)}].$$

REMARK 3.9. Returning to (3.25) one can multiply by $(2/\pi)\sin\lambda\tau$ and integrate to obtain after some calculation, for $\tau > \tilde{y}$

$$H(\tau) = \tfrac{1}{2}[G(\tilde{y}+\tau) + G(\tau-\tilde{y})] + \tfrac{1}{2}\int_0^{\tilde{y}} K(\tilde{y}, s)[G'(\tau-s) - G'(\tau+s)]\,ds. \tag{3.30}$$

This is actually part of the canonical extended GL equation (cf. Section 5) which can be derived canonically as follows. Recall (3.14) and use (3.15) to obtain

$$v(t, y) = \langle \varphi_\lambda^Q(y), [\sin\lambda t/\lambda]\rangle_\omega = \int_0^\infty G(\tau)\left[(2/\pi)\int_0^\infty \varphi_\lambda^Q(y)\sin\lambda t\sin\lambda\tau\,d\lambda\right]d\tau. \tag{3.31}$$

Now $\beta(y, x)$ in (3.18) has in fact the canonical form

$$\beta(y, x) = (2/\pi)\int_0^\infty \varphi_\lambda^Q(y)\cos\lambda x\,d\lambda.$$

We treat G as odd now in (3.31) because of the representation (3.21) and (3.31) becomes $v(t, y) = \frac{1}{2}\beta(y, \cdot) * G$. Then using (3.18) one obtains after some calculation

THEOREM 3.10. *The extended GL equation has the form* $(\tau > 0)$

$$H(\tau) - K(\tilde{y}, \tau) = \mathfrak{G}(\tilde{y}, \tau) + \tfrac{1}{2}K(\tilde{y}, \cdot) * G' \tag{3.32}$$

where $\mathfrak{G}(\tilde{y}, \tau) = \frac{1}{2}[G(\tau+\tilde{y}) + G(\tau-\tilde{y})]$ and we treat $K(\tilde{y}, \xi)$ as odd in ξ with $K(\tilde{y}, 0) = 0$, G as odd, and G' as even. Equation (3.32) yields (3.30) for $\tau > \tilde{y}$ and (3.23) for $\tau < \tilde{y}$.

Going back now to (3.29) we conjecture that the following procedure should lead to an effective computational scheme. First one proves an extension formula for $K(y, x)$ from $0 \leqslant x \leqslant y \leqslant \tilde{y}$ to say $2\tilde{y} \geqslant y \geqslant \tilde{y}$ (and appropriate x) (see [41]). We assume readout data H recorded for $\tilde{y} \leqslant t \leqslant 3\tilde{y}$. Pick some data G on $[0, 2\tilde{y}]$ as a first approximation to determine K on $0 \leqslant x \leqslant y \leqslant \tilde{y}$; extend K to the larger region; using the GL equation determine then G on $[2\tilde{y}, 4\tilde{y}]$; from this and the known H data use (3.30) to obtain G data $\Gamma(G)$ on $[0, 2\tilde{y}]$ again. A fixed point $G = \Gamma(G)$ is then the solution of the inverse problems and leads to A via the machinery indicated earlier. Theoretical estimates are possible for this procedure but very unpleasant so one hopes to check first on a computer that an iterative process converges (at least for $\tilde{y}$ small).

COMMENT 3.11. The transmission problem contains more structure intrinsically and leads to a splitting of the spectral measure. This will be seen later to be connected with situations arising in filtering theory. The general extended GL equation in the time domain gives a model (extended canonically in Section 5) where ideas of triangularity and causality interact nicely. Much of the interesting mathematics here has a physical interpretation and the physics and mathematics each serve as a guide for the other.

There are several other distinct techniques for treating the one-dimensional geophysical inverse problem with reflection data (cf. in particular [1, 2, 3, 12–14, 23, 26, 103, 107, 209, 113, 155–157, 208, 214–216]) and we sketch briefly here a technique of layer stripping (cf. [155–157]) which is related to some procedures in linear estimation which are discussed in Section 4 (where relations to transmutation are indicated). The application here to seismic problems is due to Levy and Yagle [155, 157]. One takes v as displacement and P as pressure so that the basic equation is $\rho v_{tt} = -P_x$ with $P = -\mu v_x$. Let y be travel time and A be impedance as before and set $w = v_t$ (velocity). Then one obtains a system $w_y = -A^{-1}P_t$; $P_y = -Aw_t$. Write now $\Psi = A^{-1/2}P$, $\Phi = A^{1/2}w$, $p = \frac{1}{2}(\Psi + \Phi)$, and $q = \frac{1}{2}(\Psi - \Phi)$. Then with 'reflectivity' (▲) $r = \frac{1}{2}D_y \log A(y)$ one has $p_y + p_t = -rq$; $q_y - q_t = -rp$. One notes the analogy here to a lossless transmission line problem with $\Psi = Z^{-1/2}V$ and $\Phi = Z^{1/2}i$ with i = current, V = voltage, Z = impedance = $(L/C)^{1/2}$, and $y = \int_0^x (LC)^{1/2}\, d\xi$ again represents a travel time. The input now is, e.g., $P(0, t) = P_0\delta(t)$ and one reads, say

$$w(0, t) = w_0[\delta(t) + 2\tilde{g}(t)Y(t)] \quad \text{with} \quad P_0/w_0 = A(0) = 1$$

(thus take $w_0 = 1$, $P_0 = 1$, and $G' = \delta(t) + 2\tilde{g}(t)Y(t)$ to connect this with (3.8)). The downgoing and upgoing waves $p(0, t)$ and $q(0, t)$ then have the form

$$p(0, t) = \delta(t) + \tilde{g}(t)Y(t); \; q(0, t) = -\tilde{g}(t)Y(t).$$

Further, write now $p(y, t) = \delta(y - t) + \tilde{p}(y, t)Y(t - y)$ and $q(y, t) = \tilde{q}(y, t)Y(t - y)$ to obtain the so-called fast Cholesky relations (or downward continuation recursions)

$$\tilde{p}_y + \tilde{p}_t = -r(y)\tilde{q}; \qquad \tilde{q}_y - \tilde{q}_t = r(y)\tilde{p}; \qquad r(x) = 2\tilde{q}(x, x). \tag{3.33}$$

When the measured waves $\tilde{p}(0, t) = \tilde{g}(t)$ and $\tilde{q}(0, t) = -\tilde{g}(t)$ are used as initial data in (3.33) one obtains, upon discretization, a rapid 'layer stripping' numerical procedure. If Fourier transforms in t are taken in, say, (3.33) one has a frequency domain counterpart referred to as the Shur recursions. There is also a variation of (3.33) arising in linear estimation theory called the Krein–Levinson recursions but the procedure is somewhat different (cf. Section 4, Theorem 4.6). If one considers an impulsive pressure wave obliquely incident to the medium, the layer stripping technique can also be used and data from two incident angles allows one to recover ρ and μ as functions of x (cf. also [14, 37, 177, 218, 209, 217]).

COMMENT 3.12. Layer stripping techniques are being energetically developed at the present time in various areas (see *loc. cit.*). They are numerically realistic and embody a nice interaction theoretically between physical and mathematical ideas (cf. also [63]).

We mention briefly also some results of the author and Santosa [37, 42] using transmutation methods for solving some three-dimensional inverse problems based on a formulation of Blagovščenskij [14] (whose results in the form of nonlinear integral equations seemed intractable). Our results are also a bit unrealistic for numerical computation but they have a number of interesting features. First let us note (since it is needed below) that if input in the one-dimensional problem (3.8) is taken in the equivalent form $v_y(t, 0) = \tilde{f}(t)$ (rather than $v_t(0, y) = -\tilde{f}(y)$) then it is easy to see that

$$v(t, y) = \int_0^t v^\delta(t - \tau, y)\tilde{f}(\tau)\, d\tau \tag{3.34}$$

where v^δ is the inpulse response wave to a delta function input. Then $\tilde{G}(t) = \int_0^t G^\delta(t - \tau)\tilde{f}(\tau)\, d\tau$ and if the input $\tilde{f}$ does not excite all frequencies, the recovery of the impulse response $G^\delta(t)$ (our $G(t)$ previously) from $\tilde{G}(t)$ may be impossible. Now the three-dimensional problem of [14] is a sort of Lamb wave problem for a stratified half-space $x_1 = x \geqslant 0$. The equations for displacements u_i, with impulsive stresses, are (ρ = density and (λ, μ) represent the Lamé moduli)

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + \lambda_1 u_{j,j}\delta_{1i} + \mu_1(u_{1,i} + u_{i,1}) = \rho u_{i,tt}; \tag{3.35}$$

$\tau_{1i} = \delta(x_2, x_3)\delta(t)$ at $x = 0$; $u_i(t, \mathbf{x}) = g_i(t, x_2, x_3)$ = readout at $x = 0$.

One defines new quantities $v_i(t, x) = \int\int u_i\, dx_2\, dx_3$ and $w(t, x) = \int\int x_2 u_1\, dx_2\, dx_3$ with travel times $y_1(x) = \int_0^x (\rho/\lambda + 2\mu)^{1/2}\, d\xi$ and $y_2(x) = \int_0^x (\rho/\mu)^{1/2}\, d\xi$. It is enough to consider two v_i equations in the form

$$v_{j,tt} = A_j^{-1}(A_j v_{j,y_j})_{y_j} \quad (j = 1, 2); \qquad v_j(t, 0) = h_j(t) \tag{3.36}$$

where $A_1 = [\rho(\lambda + 2\mu)]^{1/2}$, $A_2 = (\rho\mu)^{1/2}$, and $v_{j,j}(t, 0) = \delta(t)/A_j(0)$ ($v_{j,j}$ denotes the derivative in y_j here). The w equation is

$$w_{tt} = (A_1 w_{y_1})_{y_1}/A_1 - B(y_1)v_2(t, y_2) - D(y_1)v_{2,y_2}(t, y_2) \tag{3.37}$$

where D is known but not B and one knows (◆) $w_1(t, 0) = \lambda(0)h_2(t)/A_1(0)$ with $w(t, 0) = j(t)$. The v_i problems are the same as before and one finds $A_j(y_j)$ $(j = 1, 2)$ with $dy_2/dy_1 = A_1(y_1)/A_2(y_2)$ determining $y_2 = y_2(y_1)$. Also from the GL kernels $K_j(y_j, s)$ etc. one constructs $\varphi_\lambda^Q(y_j)$ etc. which lead to v_j explicitly. This allows one to determine $D(y_1)$ above but $B(y_1) = (\rho/A_1)[(A_1^2 - 2A_2^2)/\rho]_1$ is unknown. To solve (3.37) one thinks first of $w_1(t, 0)$ above as generating a solution w^h to the homogeneous equation (which is the v_1 equation) and then from (3.34)

$$w^h(t, y_1) = \lambda(0) \int_0^t h_2(\tau)v_1(t - \tau, y_1)\, d\tau. \tag{3.38}$$

Hence, $w^h(t, 0) = \lambda(0)(h_1 * h_2)(t)$ and we write $w = w^h + w^i$ where w^i satisfies the inhomogeneous w equation (3.37) with $w_1^i(t, 0) = 0$ and $w^i(t, 0) = j(t) - \lambda(0)(h_1 * h_2)(t)$. After some calculations (using the principle of no incoming waves from ∞), one arrives at the following (horrible!) integral equation for B ($F(\lambda)$ is known)

$$F(\lambda) = \int_0^\infty \Psi_\lambda^Q(\tilde{y})\Psi_\lambda^P(y)B(y)\Delta_Q\, d\tilde{y} \tag{3.39}$$

where, e.g., $\Psi_\lambda^Q(y) = \Phi_\lambda^Q(y)/c_Q(-\lambda)$ and $\Delta_Q(\tilde{y})\, d\tilde{y} = \Delta_P(y)\, dy$ determines $\tilde{y} = \tilde{y}(y)$ or $y = y(\tilde{y})$. Here $\Delta \sim A$ for the v_1 or v_2 equations. Equation (3.39) can actually be handled quite effectively by transmutation methods and motivates a certain amount of work on integral transforms with kernels Ψ_λ^Q (cf. [29, 43]). Indeed, taking Fourier transforms in (3.39) we get

$$f(t) = \int_0^\infty B(y)\hat{G}(y, t)\, dy = (1/2\pi) \int_0^\infty B(y)\Delta_P(y) \int_{-\infty}^\infty e^{-i\lambda t}\Psi_\lambda^Q(\tilde{y})\Psi_\lambda^P(y)\, d\lambda\, dy. \tag{3.40}$$

From general transmutation theory (cf. Section 5), there is a transmutation $\tilde{B}_Q: D^2 \to Q$ adjoint to $B_Q: \cos\lambda x \to \varphi_\lambda^Q$ such that $\tilde{B}_Q[2 e^{i\lambda x}] = \Psi_\lambda^Q(y)$. Also $\tilde{\beta}_Q(\tilde{y}, x) =$ kernel $\tilde{B}_Q$ has the form

(■) $\tilde{\beta}_Q(\tilde{y}, x) = \langle \cos\lambda x, \varphi_\lambda^Q(\tilde{y})\rangle_\omega$

$$= \Delta_Q^{-1/2}(0)\Delta_Q^{-1/2}(\tilde{y})\delta(x - \tilde{y}) + \tilde{K}_Q(\tilde{y}, x) \quad (\omega = \omega_Q)$$

with $\tilde{K}_Q(\tilde{y}, x) = 0$ for $x < \tilde{y}$ (anticausality). Thus, e.g.,

$$\Psi_\lambda^Q(\tilde{y}) = 2 \int_{\tilde{y}}^\infty \tilde{\beta}_Q(\tilde{y}, x)\, e^{i\lambda x}\, dx = \mathscr{F}[2\tilde{\beta}_Q(\tilde{y}, \cdot)]. \tag{3.41}$$

From (3.40) $\mathscr{F}\hat{G}(y, \cdot) = \Psi_\lambda^Q(\tilde{y})\Psi_\lambda^P(y)$ (recall $y = y(\tilde{y})$) so it follows that

$$\begin{aligned}\hat{G}(y,t)&=4\Delta_P(y)[\tilde{\beta}_Q(\tilde{y},\cdot)*\beta_P(y,\ )](t)\\&=4\Delta_P(y)\int_y^t \tilde{\beta}_P(y,s)\tilde{\beta}_Q(\tilde{y},t-s)\,\mathrm{d}s.\end{aligned} \tag{3.42}$$

Using (■), one arrives at a Volterra-type integral equation

$$\tilde{f}(\tau)=B(\tau)+\int_0^{T(\tau)} B(y)\check{K}(y,\tau)\,\mathrm{d}y. \tag{3.43}$$

where $T(\tau)$ is a known monotone function of τ. This can be solved in various contexts by variation of Volterra techniques and yields a unique solution B. Then given $B(y_1)$, $\rho(y_1)$ can be determined and, subsequently, ρ, μ, $\lambda(x)$ will be calculable from the information we already possess about A_1, A_2, $y_2(y_1)$, etc.

COMMENT 3.13. Three-dimensional inverse problems involve many new features and ill posedness arises naturally. For quantum scattering theory a recent technique of R. Newton (cf. [175]) is very effective. For general problems let us mention, e.g., [156, 157, 155, 215–218] for various points of view and considerable insight. The version given here for the geophysical problem seems to be an improvement over that of Blagoveščenskij (who arrives at a system of nonlinear integral equations) and it shows how transmutation methods are useful; however it is not recommended for computation.

REMARK 3.14. Referring back to the one-dimensional problem, from (3.14) one has

$$v_t(t,y)=\langle\varphi_\lambda^Q(y),\cos\lambda t\rangle_\omega=\tilde{\beta}(y,t)=A^{-1}(y)\gamma(t,y)$$

with

$$\text{kernel } B^{-1}=\gamma(t,y)=A^{1/2}(y)\delta(t-y)+L(t,y).$$

Hence,

$$G'(t)=v_t(t,0)=\delta(t)+L(t,0)$$

(recall $A(0)=1$) and one can write $\hat{\omega}(\lambda)=(2/\pi)[1+\mathscr{F}_cL(t,0)]$. This is exactly the same as the result in Theorem 2.4.

4. Linear Stochastic Estimation

We will sketch first some ideas in linear stochastic estimation which involve integral equations of GL type and lead to partial differential equations and ordinary differential equations of the type found in transmutation and scattering theory. There are many very strong connections between these theories and also the type of minimization which characterizes certain transmutation kernels has a direct correspondence to linear least squares estimation when there is an underlying stochastic model. We will extract from many sources; in particular the Kailath 'school' has been particularly influential. Thus let us cite [86, 89, 117–132, 144–146, 158, 159, 188–191, 196, 197, 212, 237, 239, 242] for major source material and in our survey we will not usually pinpoint a specific origin for a result or point of view. We do not touch upon nonlinear stochastic filtering or upon stochastic differential equations (beyond the most elementary considerations). Also neglected here is the subject of Kalman–Bucy filtering.

Let us first give a few background ideas and notation from probability theory, following [239]. Thus, generally we will want to deal with stochastic processes X_t ($= X_t(\omega)$, $\omega \in \Omega$, $t \in \tilde{T}$, etc.) with $EX_t = \int_\Omega X_t(\omega)\,dP(\omega) = 0$, $E|X_t|^2 < \infty$, and X_t mean square continuous in the sense that $E|X_{t+h} - X_t|^2 \to 0$ as $h \to 0$. We can assume $X_t(\cdot)$ is measurable and separable with autocorrelation $R(t, s) = EX_t\bar{X}_s$ continuous and nonnegative definite (for convenience, let us also assume the X_t to be Gaussian). We recall that X_t is called (wide sense) stationary if $R(t, s) = R(t-s)$ is a function of $t-s$ (with $E|X_t|^2 < \infty$). Let H_X be the Hilbert space generated by X_t, $t \in \tilde{T}$, via completion of finite sums $\sum \alpha_j X(t_j, \omega)$ in the topology determined by $\|Y\|^2 = E|Y|^2$ (here $X_t(\omega) = X(t, \omega)$). Then, e.g., $\int_a^b f(t)X_t\,dt$ (limit in H_X) exists if and only if $\int\int f(t)\bar{f}(s)R(t, s)\,dt\,ds$ exists as a Riemann integral. For stationary processes with, say, $R \in L^\infty$, there is a finite Borel measure such that

$$R(\tau) = (1/2\pi)\int_{-\infty}^{\infty} e^{-i\lambda\tau}\,dF(\lambda) \tag{4.1}$$

and for convenience here we will usually assume $F(d\lambda) = dF(\lambda) = \hat{R}(\lambda)\,d\lambda$ ($\hat{R} = \mathscr{F}R$). Associated with such a stationary process is a spectral process with orthogonal increments $\hat{X}_\lambda$ such that

$$\hat{X}_b - \hat{X}_a = (1/2\pi)\int_{-\infty}^{\infty}\left[\int_a^b e^{i\lambda t}\,d\lambda\right]X_t\,dt; \qquad E\,d\hat{X}_\lambda\,d\bar{\hat{X}}_\mu = \delta_{\lambda\mu}F(d\lambda)/2\pi, \tag{4.2}$$

$$X_t = \int_{-\infty}^{\infty} e^{-i\lambda t}\,d\hat{X}_\lambda. \tag{4.3}$$

Then $H_X \sim L^2(dF/2\pi)$ under the correspondence $Y = \int \eta(\lambda)\,d\hat{X}_\lambda$ with $Y \in H_X$ and $\eta \in L^2(dF/2\pi)$. A (Gaussian) white noise X_t is defined, e.g., via limiting procedures to satisfy $EX_{t+\tau}\bar{X}_t = \delta(\tau)$ and there will then be a related process ξ_t with orthogonal increments such that for $f \in L^2$

$$X(f) = \int_{-\infty}^{\infty} f(t)\,d\xi_t, \quad \left(X(f) \sim \int f(t)X_t\,dt;\ \xi_t \sim \int_0^t X_s\,ds\right). \tag{4.4}$$

Using the spectral processes indicated in (4.3) one can find another white noise $\hat{\xi}_\lambda$ such that for h and $\hat{h} = \mathscr{F}h$ belonging to L^2

$$(1/2\pi)X(\hat{h}) = (1/2\pi)\int \hat{h}(t)\,d\xi_t = \int h(\lambda)\,d\hat{\xi}_\lambda = \hat{X}(h). \tag{4.5}$$

We recall that a (real) Brownian motion (or Wiener process) is a Gaussian X_t with $EX_tX_s = R(t, s) = \min(t, s)$. Then X_t has orthogonal increments and one can define $\dot{X}(f) = \int f(t)\, dX_t$ directly (stationarity is not needed in this construction). We note that $\dot{X}_t$ is a white noise formally since $E\dot{X}_t\dot{\bar{X}}_s = D_sD_tR(t, s) = D_sD_t \min(t, s) = \delta(t-s)$.

Now given H_X generated by X_t $(t \in \tilde{T})$ the linear least squares estimator $\hat{Y} \in H_X$ of Y is determined by

$$E|\hat{Y} - Y|^2 = \min_{Z \in H_X} E|Z - Y|^2. \tag{4.6}$$

An equivalent characterization of $\hat{Y}$ follows from the orthogonality property characteristic of Hilbert space projections

$$E(\hat{Y} - Y)X_t = 0 \quad (t \in \tilde{T}). \tag{4.7}$$

In particular, suppose now we have a signal Z_t (of the type X_t above) perturbed by a white noise N_t with observations $Y_t = Z_t + N_t$ (assume real processes for convenience). One can develop a theory under various hypotheses here (e.g., $N_t \perp Z_s$ for $t > s$ will do) but for simplicity we assume $N_t \perp Z_s$ for all s, t (i.e., $EN_tZ_s = 0$). Since $EN_tN_s = \delta(t-s)$ we have

$$R(t, s) = EY_tY_s = \delta(t-s) + K(t, s); \quad K(t, s) = EZ_tZ_s \tag{4.8}$$

where $K(t, s)$ is continuous say on $[0, t] \times [0, T]$. Consider the problem of smoothing: Given H_Y^T = the Hilbert space generated by Y_t for $0 \leq t \leq T$, find $\check{Z}(t|T) \in H_Y^T$ such that $E|Z_t - \check{Z}(t|T)|^2$ represents a minimum as in (4.6); equivalently from (4.7) we want

$$E(Z_t - \hat{Z}(t|T))Y_s = 0 \quad (0 \leq s \leq T). \tag{4.9}$$

One shows that $(H(t, s) = H(t, s, T))$

$$\hat{Z}(t|T) = \int_0^T H(t, s) Y_s\, ds, \tag{4.10}$$

$$K(t, \tau) = H(t, \tau) + \int_0^T H(t, s)K(s, \tau)\, ds. \tag{4.11}$$

(Equation (4.11) being a consequence of the orthogonality (4.9).) Thus, $K = H(1+K)$ or $(1-H)(1+K) = 1$ (in obvious notation); $H(t, s, T)$ is called the Fredholm resolvant of K. When $T = t$ one refers to filtering and we write

$$\hat{Z}(t|t) = \int_0^t h(t, s) Y_s\, ds, \tag{4.12}$$

$$K(t, \tau) = h(t, \tau) + \int_0^t h(t, s)K(s, \tau)\, ds \tag{4.13}$$

(thus, $H(t, s, t) = h(t, s)$). Here $K(t, s)$ is symmetric and one sees that

$H(t, s, T) = H(s, t, T) = H(T-s, T-t, T)$ while from $H = K(1+K)^{-1}$ we can write (4.11) in the form (◆) $K(t, \tau) = H(t, \tau) + \int_0^T K(t, s)H(s, \tau)\, ds$.

THEOREM 4.1. *The smoothing and filtering kernels* $H(t, s, T)$ *and* $h(t, s)$, *which locate* $\hat{Z}(t \mid T)$ *and* $\hat{Z}(t \mid t)$ *in* H_Y^T *and* H_Y^t *respectively, are solutions of the integral equations* (4.11) (≡ (◆)) *and* (4.13).

COMMENT 4.2. After only minimal concern with probabilistic notions, one arrives at integral equations for the filtering and smoothing kernels. Much of the remaining analysis involves studying the structure of these equations and devising techniques to solve them. The connection of such integral equations with underlying differential equations will allow one to establish contact with transmutation techniques for differential operators.

The similarity of (4.13) to a GL equation is apparent and we will develop this point later. Generally speaking in linear estimation theory one wants to solve such integral equations practically by devising computational schemes to obtain the filtering and smoothing kernels.

REMARK 4.3. Referring back to Remark 3.7, we recall the splitting of $\hat{\omega} = (1/2\pi)(1/c_Q(\lambda)c_Q(-\lambda))$ where $c_Q(-\lambda)$ is analytic for $\mathrm{Im}\, \lambda > 0$ etc. This is somewhat more general but is clearly related to the classical situation in Wiener filtering. Thus one recalls first a theorem stating that if $S(\lambda) \geq 0$, $S \in L^1$, and $\int_{-\infty}^{\infty} |\log S(\lambda)|\, d\lambda/(1+\lambda^2) < \infty$ then there exists $\hat{c} \in L^2$, $|\hat{c}|^2 = S$, $\hat{c}(\lambda) = \bar{\hat{c}}(-\lambda)$ for λ real, $\hat{c}(\lambda) = \int_0^\infty c(t)\, e^{i\lambda t}\, dt$, and $\hat{c}(\lambda) \neq 0$ for $\mathrm{Im}\, \lambda > 0$ (with c real for S even). Let $S(\lambda) = S_X(\lambda) = \mathscr{F}R_X(\tau)$ where $R_X(\tau) = EX_{t+\tau}\bar{X}_t$ for some stationary Gaussian X_t. Let us think of X_t resulting from filtering a white noise N_t through a causal filter with transfer function $\hat{c}$. This means that if $\nu_t \sim N_t$ is the process with orthogonal increments associated with N_t as in (4.4) ($N(f) = \int f(t)\, d\nu_t$) then

$$X_t = \int_{-\infty}^{t} c(t-s)\, d\nu_s = \int_{-\infty}^{\infty} \hat{c}(\lambda)\, e^{-i\lambda t}\, d\nu_\lambda \tag{4.14}$$

(note here $c(t-s) = \mathscr{F}[\hat{c}(\lambda)\, e^{-i\lambda t}/2\pi]$ and cf. (4.5)). Given (4.14) with $E\, d\hat{\nu}_\lambda\, d\bar{\hat{\nu}}_\mu = \delta_{\lambda\mu}\, d\lambda/2\pi$ one obtains easily

$$EX_{t+\tau}\bar{X}_t = R_X(\tau) = (1/2\pi)\int |\hat{c}|^2\, e^{-i\lambda\tau}\, d\lambda$$

which is consistent with $S(\lambda) = \mathscr{F}R_X$. Moreover, given $\hat{c}$ with properties as indicated and suitable growth, we expect $\hat{\mathfrak{c}} = 1/\hat{c} = \mathscr{F}\mathfrak{c}$ with $\mathfrak{c}(t) = 0$ for $t < 0$ and then formally one is led to

$$\nu_t = \int_0^t N_s\, ds = \int_{-\infty}^{\infty} [1/\hat{c}(\lambda)] \int_0^t e^{-i\lambda s}\, ds\, d\hat{X}_\lambda. \tag{4.15}$$

Thus formally $N_t \sim \dot{\nu}_t$ is the result of filtering X_t by a filter with transfer function $\hat{\mathfrak{c}} = 1/\hat{c}$ which we express heuristically as $N_t = \int_{-\infty}^t \mathfrak{c}(t-s)X_s\, ds$ (again $\mathfrak{c}(t-s) =$

$\mathscr{F}[c\, e^{-i\lambda t}/2\pi])$. Equation (4.15) gives $EN_{t+\tau}\bar{N}_t = \delta(\tau)$ since $E\, d\hat{X}_\lambda\, d\bar{\hat{X}}_\mu = S(\lambda)\, d\lambda/2\pi$. Let now Y_t be another stationary Gaussian process and consider $\hat{Y}(t\,|\,t) \in H_X^t$ determined here, corresponding to (4.9), by the condition $E(Y_t - \hat{Y}(t\,|\,t))\bar{X}_s = 0$ for $-\infty < s \leq t$. It can be shown that

$$\hat{Y}(t\,|\,t) = \int_{-\infty}^{t} g(t-s)\, d\nu_s \tag{4.16}$$

and setting $G(\lambda) = \int_0^\infty g(t)\, e^{i\lambda t}\, dt$ (which is not the Fourier transform!) one has

$$\hat{Y}(t\,|\,t) = \int_{-\infty}^{\infty} [G(\lambda)/\hat{c}(\lambda)]\, e^{-i\lambda t}\, d\hat{X}_\lambda . \tag{4.17}$$

The filter with transfer function $(G/\hat{c})$ is called a Wiener filter and g has the form

$$g(t) = (1/2\pi) \int_{-\infty}^{\infty} [\bar{S}_{XY}(\lambda)/\bar{\hat{c}}(\lambda)]\, e^{-i\lambda t}\, d\lambda \tag{4.18}$$

where $S_{XY}(\lambda) = \mathscr{F}(EX_{t+\tau}\bar{Y}_t)$.

COMMENT 4.4. The interaction between integral equations, spectral measures, analytic functions in the Hardy space context, factorization of operators, etc. has led to many fascinating developments in mathematics and physics (cf. [87–89, 7] and Section 7).

One goes next to the concept of innovations process for our situation $Y_t = Z_t + N_t$ as above. Writing $\hat{Z}(t) = Z(t\,|\,t)$ and $Z_t = Z(t)$ etc., define

$$J(t) = Y(t) - \hat{Z}(t) = \tilde{Z}(t) + N(t) \tag{4.19}$$

where $\tilde{Z}(t) = Z(t) - \hat{Z}(t)$ represents the portion of $Z(t)$ that cannot be predicted from past observations of $Y(s)$ and $J(t) \perp Y(s)$ for $s \leq t$ (sometimes innovations or new information processes $J(t)$ are treated in an integral form at the martingale level - cf. [131]). In fact, $J(t)$ is a white noise $(EJ(t)J(s) = \delta(t-s))$ related to $Y(t)$ by a causal and causally invertible filter ($h \sim$ operator of (4.12))

$$J = (1-h)Y; \qquad Y = (1-h)^{-1}J. \tag{4.20}$$

Here $(1-h)^{-1}$ is the Volterra-type inverse and J contains the same statistical information as Y (note that white noise can be generated by Gaussian, Poisson, etc. processes and contains statistical information). Innovations processes will be used extensively below. First from (4.11)–(◆) and (4.13)

THEOREM 4.5. (Bellman–Krein–Siegert). *The kernels $H(t, s, T)$ and $h(t, s)$ are related by $H_T(t, s, T) = -h(T, t)h(T, s)$ with $H(t, s, T) = h(t, s) + h(s, t) - \int_0^T h(r, t)h(r, s)\, dr$. This equation is often written as $H = h + h^* - h^*h$.*

Suppose now that Z_t is stationary so that $K(t, s) = K(t-s)$ and define

$$a(T, t) = H(T, t, T) = h(T, t); \qquad b(T, t) = a(T, T-t) = \check{H}(t, 0, T). \tag{4.21}$$

THEOREM 4.6. *For Z_t stationary the $a(T, t)$ can be computed recursively using the Kerin–Levinson recursions*

$$(D_T + D_s)a(T, s) = -a(T, T-s)a(T, 0)$$

where, from (4.13),

$$a(T, 0) = \tilde{R}(T) = K(T) - \int_0^T a(T, s)K(s)\,\mathrm{d}s.$$

Note this can be written $(D_T + D_s)a(T, s) = -b(T, s)\tilde{R}(T)$ *with* $(D_T - D_s)b(T, s) = -a(T, s)\tilde{R}(T)$.

One calls an operator with kernel $K(t-s)$ as above a Toeplitz operator (while an operator with kernel $K(t+s)$ is called Hankel). The Fredholm resolvant $H(t, s, T)$ of a Toeplitz operator is not Toeplitz but one has

THEOREM 4.7 (Sobolev identity). *For Z_t stationary has*

$$(D_t + D_s)H(t, s, T) = a(T, t)a(T, s) - b(T, t)b(T, s).$$

The equation in Theorem 4.5 is actually a form of the Darboux–Christoffel formula for the $a(t, s)$ and this is made more specific later (see Theorem 4.8). One sees that $H(t, s, T)$ is completely determined by a knowledge of the $a(T, t)$ (cf. also Theorem 4.3). Write next $R(t-s) = \delta(t-s) + K(t-s)$ in (4.8) for Z_t stationary with $R(\tau)$ given as in (4.1). Write (4.3) as

$$Y_t = \int \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\hat{Y}_\lambda \quad (Y \sim \mathrm{e}^{-\mathrm{i}\lambda t}). \tag{4.22}$$

Set $J = (1-h)Y$ as in (4.20) with $J \sim P(\lambda, t)$ (Krein functions – cf. [146, 120]), $J(t) = \int P(\lambda, t)\,\mathrm{d}\hat{Y}_\lambda$, so that

$$P(\lambda, t) = \mathrm{e}^{-\mathrm{i}\lambda t} - \int_0^t h(t, s)\,\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s = \mathrm{e}^{-\mathrm{i}\lambda t} - \int_0^t b(t, \tau)\,\mathrm{e}^{-\mathrm{i}\lambda(t-\tau)}\,\mathrm{d}\tau \tag{4.23}$$

$(h(t, t-\tau) = a(t, t-\tau) = b(t, \tau))$. From $EJ(t)J(s) = \delta(t-s)$ we obtain from $E\,\mathrm{d}\hat{Y}_\lambda\,\mathrm{d}\bar{\hat{Y}}_\mu = \delta_{\lambda\mu}\,\mathrm{d}F(\lambda)/2\pi$ the equation

$$\delta(t-s) = (1/2\pi)\int_{-\infty}^{\infty} P(\lambda, t)\bar{P}(\lambda, s)\,\mathrm{d}F(\lambda). \tag{4.24}$$

Define now $P_*(\lambda, t) = \mathrm{e}^{-\mathrm{i}\lambda t}\bar{P}(\lambda, t)$; then

$$P_*(\lambda, t) = 1 - \int_0^t b(t, \tau)\,\mathrm{e}^{-\mathrm{i}\lambda\tau}\,\mathrm{d}\tau. \tag{4.25}$$

THEOREM 4.8. *The Krein functions P and P_* satisfy*

$$D_tP(\lambda, t) = -i\lambda P(\lambda, t) - b(t, t)P_*(\lambda, t) \quad \text{with} \quad D_tP_*(\lambda, t) = -b(t, t)P(\lambda, t).$$

Defining $\mathfrak{K}_T(\lambda, \mu) = \int_0^T P(\lambda, s)\bar{P}(\mu, s)\,\mathrm{d}s$ *it follows that* $\mathfrak{K}_T$ *is a reproducing kernel in the sense that* $P(\lambda, t) = (1/2\pi)\int_{-\infty}^{\infty} \mathfrak{K}_T(\lambda, \mu)P(\mu, t)\,\mathrm{d}F(\mu)$ *and one has a Darboux–Christoffel relation*

$$\mathfrak{K}_T(\lambda, \mu) = [P(\lambda, T)\bar{P}(\mu, T) - P_*(\lambda, T)\bar{P}_*(\mu, T)]/i(\mu - \lambda). \tag{4.26}$$

Further one can write $\mathfrak{K}_T(\lambda, \mu) = \int_0^T \int_0^T [\delta(t-s) - H(t, s, T)]\,\mathrm{e}^{i\lambda t}\,\mathrm{e}^{-i\mu s}\,\mathrm{d}s\,\mathrm{d}t.$

COMMENT 4.9. The structural equations (Bellman–Krein–Siegert–Sobolev–Levinson, etc.) are important theoretically and also lead to realistic computational techniques for determining the filtering and smoothing kernels. The Krein functions are generalizations of orthogonal polynomials and are discussed and extended further in Theorem 4.15 and in Section 6.

We go next to some work of Levy–Tsitsiklis [158–159] which displays very elegantly some relations between linear estimation and scattering theory. Let $Y_t = Z_t + N_t$ as before with Z_t stationary, $EZ_tN_s = 0$, etc. One defines even and odd processes $Z_\pm(t) = \frac{1}{2}[Z(t) \pm Z(-t)]$ (and similarly for $N_\pm$ and $Y_\pm$). Evidently

$$EZ_\pm(t)Z_\pm(s) = K_\pm(t, s) = \tfrac{1}{2}[K(t-s) \pm K(t+s)]; \qquad EN_\pm(t)N_\pm(s) = \tfrac{1}{2}\delta(t-s) \tag{4.27}$$

and $EZ_\pm(t)N_\pm(s) = 0$. Let $\hat{Z}_\pm(t) = \hat{Z}_\pm(t\,|\,t) \in {}^{+}H_Y^t =$ the Hilbert space generated by $Y_\pm(s)$ for $0 \leq s \leq t$, be the best least squares approximation to $Z_\pm(t)$ so that $\tilde{Z}_\pm(t) = Z_\pm(t) - \hat{Z}_\pm(t)$ is orthogonal to ${}^{\pm}H_Y^t$. One has, corresponding to (4.12)–(4.13)

$$\hat{Z}_\pm(t\,|\,t) = \hat{Z}_\pm(t) = \int_0^t g_\pm(t, s)\,Y_\pm(s)\,\mathrm{d}s, \tag{4.28}$$

$$K_\pm(t, s) = \tfrac{1}{2}g_\pm(t, s) + \int_0^t g_\pm(t, \tau)K_\pm(\tau, s)\,\mathrm{d}\tau. \tag{4.29}$$

From (4.29) one obtains now (cf. also Remark 5.19)

THEOREM 4.10. *If* $K_\pm \in C^2$ *one has* $[D_t^2 - V_\pm(t)]g_\pm(t, s) = D_s^2 g_\pm(t, s)$ *with* $D_s g_+(t, 0) = 0$ *and* $g_-(t, 0) = 0$ *where* $V_\pm(t) = -2D_t g_\pm(t, t)$.

Fredholm resolvants $H_\pm(t, s, T) = H_\pm(t, s)$ exist as before (cf. (4.11)) and satisfy

$$K_\pm(t, s) = \tfrac{1}{2}H_\pm(t, s) + \int_0^T K_\pm(t, u)H_\pm(u, s)\,\mathrm{d}u \tag{4.30}$$

(thus $(\frac{1}{2} + K_\pm)(1 - H_\pm) = \frac{1}{2}$). One obtains as before a Bellman–Krein–Siegert formula and a version of the Krein–Levinson recursions in the form (cf. Theorems 4.5, 4.6, and 4.7).

THEOREM 4.11. *Under the hypotheses indicated*

$$D_T H_\pm(t, s, T) = -g_\pm(T, t) g_\pm(T, s);$$

$$H_\pm(t, s) = g_\pm(t, s) + g_\pm(s, t) - \int_0^T g_\pm(r, t) g_\pm(r, s)\, dr;$$

$$D_T g_+(T, t) + D_t g_-(T, t) = -2\rho(2T) g_+(T, t);$$

$$D_T g_-(T, t) + D_t g_+(T, t) = 2\rho(2T) g_-(T, t);$$

$$2\rho(2T) = g_+(T, T) - g_-(T, T).$$

Further ρ satisfies the Riccati equation $4[\rho^2(2T) \mp \dot\rho(2T)] = V_\pm(T)$ *and thus ρ or $V_\pm$ serve as equivalent parameterizations of the $Y_\pm$ process. If $a(T, s)$ is defined as before* (*cf.* (4.21)), *one can show also that*

$$g_\pm(T, t) = a(2T, T + t) \pm a(2T, T - t) \tag{4.31}$$

so that it is equivalent to compute $a(T, t)$ or $g_\pm(T, t)$.

One defines now innovations $J_\pm = Y_\pm - \hat{Z}_\pm$ and evidently $EJ_\pm(t)J_\pm(s) = \frac{1}{2}\delta(t - s)$. We take Y_t represented as in (4.22) etc. and suppose $dF(\lambda) = \hat{R}(\lambda)\, d\lambda$ with $\hat{R} = 1 + \hat{K} = \mathcal{F}[\delta + K]$ (note $\hat{R}$ is even for real Y, Z etc.). Then $Y_+(t) \sim \cos \lambda t$, $Y_-(t) \sim -i \sin \lambda t$, and $J_\pm \sim \gamma_\pm(\lambda, t)$ where, e.g.,

$$\gamma_+(\lambda, t) = \cos \lambda t - \int_0^t g_+(t, s) \cos \lambda s\, ds, \tag{4.32}$$

$$(1/2\pi) \int_{-\infty}^{\infty} \gamma_+(\lambda, t) \bar{\gamma}_+(\lambda, s) \hat{R}(\lambda)\, d\lambda = \tfrac{1}{2}\delta(t - s). \tag{4.33}$$

THEOREM 4.12. *The spectral innovations functions satisfy*

$$[D_t^2 - V_\pm(t)]\gamma_\pm(\lambda, t) = -\lambda^2 \gamma_\pm(\lambda, t); \qquad \gamma_+(\lambda, 0) = 1;$$

$$D_t \gamma_+(\lambda, 0) = -2K(0); \qquad \gamma_-(\lambda, 0) = 0; \qquad D_t \gamma_-(\lambda, 0) = -i\lambda.$$

Using Theorem 4.11 *one can also show* $D_t\gamma_+ + 2\rho(2t)\gamma_+ = -i\lambda\gamma_-$ *and* $D_t\gamma_- - 2\rho(2t)\gamma_- = -i\lambda\gamma_+$ *with*

$$-i\lambda\gamma_-(\lambda, t) = W[\gamma_+(0, t), \gamma_+(\lambda, t)/\gamma_+(0, t)]. \tag{4.34}$$

REMARK 4.13. One can transform the equations of Theorem 4.12 into the vibrating string equations of Krein–Dym–McKean by writing $\xi(\lambda, t) = \gamma_+(\lambda, t)/\gamma_+(0, t) = y(x(t), \lambda)$ where $x(t) = \int_0^t ds/\gamma_+^2(0, s)$. Then $D_x^2 y + \lambda^2 \mu(x) y = 0$; $y(0, \lambda) = 1$; $D_x y(0, \lambda) = 0$ $(0 \leq x \leq l = x(\infty))$ and $\mu(x(t)) = \gamma_+^4(0, t)$. In general of course the 'string' theory involves singular measures and refers to 'Feller type' differential equations (cf. [89]).

THEOREM 4.14. *The Sobolev identity corresponding to that of Theorem* 4.7 *involves*

$$(H_+(t,s)=H_+(t,s,T))\,D_tH_+(t,s)+D_sH_-(t,s)=g_-(T,t)g_+(T,s)$$

and

$$D_tH_-(t,s)+D_sH_+(t,s)=g_+(T,t)g_-(T,s).$$

The structure of $H_\pm$ *is close to Hankel + Toeplitz in the sense that*

$$[D_t^2-D_s^2]H_\pm(t,s,T)=g_\pm(T,t)D_Tg_\pm(T,s)-g_\pm(T,s)D_Tg_\pm(T,t)$$

with $D_tH_+(0,s,T)=0$ *and* $H_-(0,s,T)=0$.

The $\gamma_\pm$ also play the role of Krein functions as in Theorem 4.8. Define

$$\hat{f}_\pm(\lambda)=\Gamma_\pm^T f(\lambda)=\int_0^T f_\pm(s)\gamma_\pm(\lambda,s)\,\mathrm{d}s, \tag{4.35}$$

$$\mathfrak{K}_\pm^T(\lambda,\mu)=\int_0^T \gamma_\pm(\mu,\tau)\gamma_\pm(\lambda,\tau)\,\mathrm{d}\tau. \tag{4.36}$$

THEOREM 4.15. $\mathfrak{K}_\pm^T(\lambda,\mu)$ *acts as a reproducing kernel on the* $\hat{f}_\pm$ *in the sense that* $\int_\infty^\infty \mathfrak{K}_\pm^T(\lambda,\mu)\hat{f}_\pm(\lambda)\hat{R}\,\mathrm{d}\lambda/\pi=\hat{f}_\pm(\mu)$. *Further,*

$$\mathfrak{K}_\pm^T(\lambda,\mu)=\pm\int_0^T\int_0^T[\delta(t-s)-$$

$$-H_\pm(t,s,T)]T_\pm(\lambda,s)T_\pm(\mu,t)\,\mathrm{d}s\,\mathrm{d}t$$

$$(T_+(\lambda,t)=\cos\lambda t,\ T_-(\lambda,t)=\sin\lambda t)$$

and one has a Darboux–Christoffel formula

$$\mathfrak{K}_\pm^T(\lambda,\mu)=W[\gamma_\pm(\lambda,T),\gamma_\pm(\mu,T)]/[\lambda^2-\mu^2]. \tag{4.37}$$

COMMENT 4.16. The spectral innovations $\gamma_+(\lambda,t)$ evidently play the role of $\varphi_{\lambda,h}^Q$ in Section 2 with $g_+\sim -K_h$ and $\mathrm{d}\omega=(2/\pi)\hat{R}\,\mathrm{d}\lambda$ (cf. also Section 5, Theorem 5.18). The techniques indicated here in estimation theory have also been useful in studying other aspects of integral connection formulas and the underlying differential problems (cf. Section 6).

We mention here also some results of [159] on random fields which can be phrased in terms of transmutation machinery (cf. also [239–241]). Thus, one speaks of random fields on R^n as a family of random variables X_z, $\mathbf{z}\in R^n$, say mean square continuous with $EX_\mathbf{z}=0$ and $E|X_\mathbf{z}|^2<\infty$, such that $X_\mathbf{z}$ is homogeneous and isotropic (so $EX_\mathbf{z}\bar{X}_{\mathbf{z}'}=R(\|\mathbf{z}-\mathbf{z}'\|)$ where $\|\ \ \|$ denotes the Euclidean norm). As in [159] we consider here only the case $n=2$ and let $Y(\mathbf{x})=Z(\mathbf{x})+N(\mathbf{x})$ be observations of a two-dimensional Gaussian random field Z with covariance ($\|\mathbf{x}-\mathbf{s}\|=l$)

$$EZ(\mathbf{x})\bar{Z}(\mathbf{s}) = k(l) \tag{4.38}$$

and $N(\mathbf{x})$ is a two-dimensional white Gaussian noise field with $EN(\mathbf{x})\bar{N}(\mathbf{s}) = \delta(l)/2\pi l$. Assume $EZ(\mathbf{x})\bar{N}(\mathbf{s}) = 0$ and that $k(\cdot) \in L^1(l^{1/2}\,\mathrm{d}l)$ so that

$$k(l) = \int_0^\infty J_0(\lambda l)\hat{k}(\lambda)\lambda\,\mathrm{d}\lambda; \qquad \hat{k}(\lambda) = \int_0^\infty J_0(\lambda l)k(l)l\,\mathrm{d}l \tag{4.39}$$

(this is a well-known version of (4.1) for higher dimensions). One observes now $Y(\mathbf{x})$ over a disc D_R of radius R and we let Y_R be the Hilbert space generated by $Y(\mathbf{x})$ with $\|\mathbf{x}\| \leq R$ so that elements of Y_R have the form

$$b = \textstyle\int_0^R \int_0^{2\pi} b(r, \theta)\, Y(r, \theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Write

$$Y(r, \theta) = \textstyle\sum_{-\infty}^{\infty} Y_n(r)\,\mathrm{e}^{in\theta},$$

$$Z(r, \theta) = \textstyle\sum_{-\infty}^{\infty} Z_n(r)\,\mathrm{e}^{in\theta},$$

and

$$N(r, \theta) = \textstyle\sum_{-\infty}^{\infty} V_n(r)\,\mathrm{e}^{in\theta}$$

where, e.g., $Y_n(r) = (1/2\pi)\int_0^{2\pi} Y(r, \theta)\,\mathrm{e}^{-in\theta}\,\mathrm{d}\theta$. One obtains then estimation problems for $Y_n(r) = Z_n(r) + V_n(r)$ and $EZ_n(r)\bar{Z}_m(s) = 0 = EV_n(r)\bar{V}_m(s)$ for $n \neq m$ with $EZ_n(r)\bar{V}_m(s) = 0$ for any (n, m). Let Y_R^n be the Hilbert space generated by $Y_n(r)$ for $0 \leq r \leq R$ and then one can show that

$$EZ_n(r)\bar{Z}_n(s) = k_n(r, s) = \int_0^\infty J_n(\lambda r)J_n(\lambda s)k(\lambda)\lambda\,\mathrm{d}\lambda \tag{4.40}$$

while $EV_n(r)\bar{V}_n(s) = \delta(l)/2\pi l$ ($l = |r - s|$). It follows that

THEOREM 4.17. *If $k \in C^2$ or equivalently if $Z(\cdot)$ is mean square differentiable, one has for*

$$P_n(D_r) = D^2 + (1/r)D - (n^2/r^2), \qquad P_n(D_r)k_n(r, s) = P_n(D_s)k_n(r, s);$$

$$D_r k_0(0, s) = 0; \qquad k_n(0, s) = 0 \quad (n \neq 0).$$

Next for filtering kernels one has

$$\hat{Z}_n(R \mid R) = E(Z_n(R) \mid Y_R^n) = \int_0^R g_n(R, s)\, Y_n(s)\,\mathrm{d}s. \tag{4.41}$$

From $\tilde{Z}_n(R \mid R) = Z_n(R) - \hat{Z}_n(R \mid R)$ being perpendicular to $Y_n(s)$ for $0 \leq s \leq R$ we have

$$k_n(R, r) = (1/2\pi)g_n(R, r) + \int_0^R g_n(R, s)k_n(r, s)\,\mathrm{d}s. \tag{4.42}$$

Thus $(1/2\pi) + k_n$ is invertible (assuming say $k_n \in L^2(r\,\mathrm{d}r)$ on $[0, R]$). One defines innovations processes $\mathcal{I}_n(r) = Y_n(r) - \hat{Z}_n(r \mid r)$ with $E\mathcal{I}_n(r)\bar{\mathcal{I}}_n(s) = (1/2\pi l)\,\delta(r-s)$ and the previous machinery all has a version here. Thus

THEOREM 4.18. *Under the hypotheses indicated one has for*

$$V_n(R) = -2D_R[Rg_n(R, R)], \quad [P_n(D_R) - V_n(R)]g_n(R, r) = P_n(D_r)g_n(R, r);$$

$$D_r g_0(R, 0) = 0; \qquad g_n(R, 0) = 0 \quad (n \neq 0).$$

The Fredholm resolvant $H_n(r, s, R)$ satisfies

$$(H_n(R, s, R) = g_n(R, s)),$$

$$k_n(r, s) = (1/2\pi)H_n(r, s, R) + \int_0^R k_n(r, u)H_n(u, s, R)\,du;$$

$$D_R H_n(r, s, R) = -Rg_n(R, r)g_n(R, s) \quad \text{and} \quad 1 - H_n = (1 - g_n^*)(1 - g_n).$$

Further, for $k \in C^1$ one has a new feature in the form (here, $g_n = g_n(R, r)$ and

$$\rho_n(R) = R[g_n(R, R) - g_{n+1}(R, R)])\ (D_r - (n/r))k_n(r, s) +$$

$$+ (D_s + [(n+1)/s])k_{n+1}(r, s) = 0;$$

$$(D_r + [(n+1)/r])k_{n+1} + (D_s - (n/s))k_n = 0;$$

$$(D_R - (n/R))g_n + (D_r + [(n+1)/r])g_{n+1} = -\rho_n(R)g_n;$$

and

$$(D_r - (n/r))g_n + (D_R + [(n+1)/R])g_{n+1} = \rho_n(R)g_{n+1}.$$

Further one can exhibit a mutual dependence between ρ_n and V_n via a Riccati equation. In the spectral domain we write $\hat{r}(\lambda) = (1/2\pi) + \hat{k}(\lambda)$ and one introduces spectral processes with orthogonal increments as before. The correspondence $Y(\mathbf{x}) \sim e^{-i\langle \boldsymbol{\nu}, \mathbf{x}\rangle}$ involves an isometry between the Hilbert space generated by the $Y(\mathbf{x})$ and $L^2(\hat{r}\lambda\,d\lambda\,d\theta/2\pi)$. In particular $(-\boldsymbol{\nu} = (\lambda, \theta))$

$$EY_n(r)\bar{Y}_m(s) = \delta_{mn}\int_0^\infty J_n(\lambda r)J_n(\lambda s)\hat{r}(\lambda)\,d\lambda \tag{4.43}$$

and there is an isometry $Y_n(r) \sim J_n(\lambda r)$: $Y_n \sim L^2(\hat{r}\lambda\,d\lambda)$ with $(\mathcal{I}_n \sim \mathcal{I}_n(r, \lambda))$

$$\mathcal{I}_n(r, \lambda) = J_n(\lambda r) - \int_0^r g_n(r, s)J_n(\lambda s)s\,ds. \tag{4.44}$$

Thus we make contact with the singular operators of Section 2. Further

$$\int_0^\infty \mathcal{I}_n(r, \lambda)\mathcal{I}_n(s, \lambda)\hat{r}\lambda\,d\lambda = (1/2\pi l)\,\delta(r-s). \tag{4.45}$$

THEOREM 4.19. *For $n > 0$, $\mathcal{I}_n$ satisfies $[P_n(D_r) - V_n(r)]\mathcal{I}_n(r, \lambda) = -\lambda^2\mathcal{I}_n(r, \lambda)$ with $\lim n!2^n(\lambda r)^{-n}\mathcal{I}_n(r, \lambda) = 1$ as $r \to 0$ and for $n < 0$ one sets $\mathcal{I}_n(r, \lambda) = (-1)^n\mathcal{I}_{-n}(r, \lambda)$.*

In terms of calculation the $\mathcal{J}_n$ can, in fact, all be generated from each other (cf. [1959]). To give an inverse scattering interpretation to the above set $\varphi_n(r, \lambda) = (r\lambda)^{1/2}\mathcal{J}_n(r, \lambda)$ so that

$$[D_r^2 - V_n(r) - (1/r^2)(n^2 - \tfrac{1}{4})]\varphi_n = -\lambda^2\varphi_n \tag{4.46}$$

with $n!2_n(\lambda r)^{-n-1/2}\varphi_n(r, \lambda) \to 1$ as $r \to 0$. This is the Schrödinger equation for a particle with angular momentum n and energy $E = \lambda^2$. There is then a connection to the standard GL machinery with $\hat{r} = 1/\pi^2|F(\lambda)|^2$, where F is the Jost function. Here the Jost function is the same for all n – with varying potential – whereas in physics the potential is constant while the Jost function varies.

COMMENT 4.20. One sees that integral equations (e.g., (4.42)) whose kernels have underlying structure relative to P_n lead to solutions with 'related' differential structure. The k_n structure is, in fact, a kind of generalized translation and this picture is clarified further in Section 6. In terms of stochastic theory the presence of an underlying Fourier theory for stationary processes is an obvious and natural consequence of the 'displacement' kernels. Similarly the underlying P_n translation theory for k_n is related to a spectralization of the stochastic process Y_n via $J_n(\lambda r)$ etc.

5. General Transmutation Theory

We consider operators as in (2.16) with say $\Delta_Q = x^{2m+1}C_Q$ and q real (again strong singularities in q will be excluded for convenience and $m = -\frac{1}{2}$ will lead to operators as in Section 3). Since some of the material here already appears in [29] we will emphasize newer developments and proceed somewhat formally, referring to [29] for details and examples. Let us also base the development primarily on spherical function solutions φ_λ^Q of $Q\varphi = -\lambda^2\varphi$ with $\varphi_\lambda^Q(0) = 1$ and $D_x\varphi_\lambda^Q(0) = 0$ (for background information on spherical functions see, e.g., [70, 83, 95, 109, 140–143, 147, 171]). In a general way one can show that the transmutation B: $P \to Q$: $\varphi_\lambda^P \to \varphi_\lambda^Q$ can be represented by a kernel of the form

$$\beta(y, x) = \langle R^P, \varphi_\lambda^Q(y)\Omega_\lambda^P(x)\rangle_\lambda = \langle \varphi_\lambda^Q(y), \Omega_\lambda^P(x)\rangle_\nu \tag{5.1}$$

where $\Omega_\lambda^P(x) = \Delta_P(x)\varphi_\lambda^P(x)$ and $R^P \sim d\nu(R^Q \sim d\omega)$. One considers, e.g., the Cauchy problem (2.1) with $A = 1$ and $C = 0$ whose solution can in fact be written as

$$\varphi(x, y) = \langle \beta(y, \xi), T_\xi^x f(\xi)\rangle = \langle \mathfrak{P} f(\lambda), \varphi_\lambda^P(x)\varphi_\lambda^Q(y)\rangle_\nu \tag{5.2}$$

where T_ξ^x denotes the generalized translation which can be represented via

$$U(x, \xi) = T_\xi^x f(\xi) = \langle \mathfrak{P} f(\lambda), \quad \varphi_\lambda^P(x)\varphi_\lambda^P(\xi)\rangle_\nu = \langle f(\eta), \langle \Omega_\lambda^P(\eta), \varphi_\lambda^P(x)\varphi_\lambda^P(\xi)\rangle_\nu\rangle \tag{5.3}$$

One must naturally be careful in writing down general spectral pairings as in (5.1)–(5.3) but, in fact, the formal technique can be applied quite generally. Of course many pairings are distributional and represent distributions but this is expected and hopefully a routine matter today. Let us indicate a few typical pairings involving singular operators

EXAMPLE 5.1. Take $P = D^2$ and $Q = Q_m$ where $Q_m u = (x^{2m+1}u')'/x^{2m+1}$. Then $\varphi_\lambda^P(x) = \cos \lambda x$ and $B_Q[\cos \lambda x] = \varphi_\lambda^Q(y)$ where

$$\varphi_\lambda^Q(y) = 2^m \Gamma(m+1) J_m(\lambda y)/(\lambda y)^m, \tag{5.4}$$

$$\begin{aligned}\beta_Q(y, x) &= \langle \cos \lambda x, \varphi_\lambda^Q(y)\rangle_\nu = (2/\pi c_m)\int_0^\infty \cos \lambda x (\lambda y)^{-m} J_m(\lambda y)\,d\lambda \\ &= [2\Gamma(m+1)y^{-2m}/\Gamma(\tfrac{1}{2})\Gamma(m+\tfrac{1}{2})](y^2 - x^2)_+^{m-1/2}\end{aligned} \tag{5.5}$$

($c_m = \frac{1}{2}^m \Gamma(m+1)$ and $d\omega = \hat{\omega}\,d\lambda$ with $\hat{\omega} = c_m^2 \lambda^{2m+1}$). The inverse $\mathcal{B}_Q = B_Q^{-1}$ has a kernel $\gamma_Q(x, y) = \langle \cos \lambda x, \Omega_\lambda^Q(y)\rangle_\omega$ which is manifestly distributional, namely

$$\begin{aligned}\gamma_Q(x, y) &= c_m \int_0^\infty (\lambda y)^{m+1} J_m(\lambda y) \cos \lambda x\,d\lambda \\ &= [2\sqrt{\pi}\,\mathrm{sgn}\, x\, y^{2m+1} x/\Gamma(m+1)\Gamma(-m-\tfrac{1}{2})](x^2 - y^2)_+^{-m-3/2}.\end{aligned} \tag{5.6}$$

The generalized translation kernel $\beta(x, \xi, \eta) = \langle \Omega_\lambda^P(\eta), \varphi_\lambda^P(x)\varphi_\lambda^P(\xi)\rangle_\nu$ in (5.3) for $P = D^2$ is

$$\beta(x, \xi, \eta) = (2/\pi)\int_0^\infty \cos \lambda\eta \cos \lambda x \cos \lambda\xi\,d\lambda. \tag{5.7}$$

Thus

$$\beta(x, \xi, \eta) = \tfrac{1}{4}[\delta(x+\xi+\eta) + \delta(x+\xi-\eta) + \delta(x-\xi+\eta) + \delta(x-\xi-\eta)]$$

and acting on even functions f (which as we have seen in Theorem 2.1 is a standard procedure in transmuting with operators on $[0, \infty)$) one obtains $T_\xi^x f(\xi) = \frac{1}{2}[f(x+\xi) + f(x-\xi)]$; this is, of course, the d'Alembert solution of the wave equation $D_x^2 U = D_\xi^2 U$ with $U(x, 0) = f(x)$ and $U_\xi(x, 0) = 0$. Somewhat more exotic is the kernel of the generalized translation for Q_m, namely

$$\begin{aligned}\gamma(x, y, \eta) &= 2^m \Gamma(m+1)(xy)^{-m}\eta^{m+1/2} \times \\ &\quad \times \int_0^\infty \lambda^{-m+1/2}(\eta\lambda)^{1/2} J_m(\lambda x) J_m(\lambda y) J_m(\lambda\eta)\,d\lambda \\ &= \langle \varphi_\lambda^Q(x)\varphi_\lambda^Q(y), \Omega_\lambda^Q(\eta)\rangle_\omega \\ &= [\Gamma(m+1)/\sqrt{\pi}\,\Gamma(m+\tfrac{1}{2})](\eta/xy)(1-z^2)^{m-1/2}\end{aligned} \tag{5.8}$$

for $|x - y| < \eta < x + y$ while $\gamma = 0$ for $0 < \eta < |x - y|$ and $\eta > x + y$; here $z = (x^2 + y^2 - \eta^2)/2xy$. Then writing $\eta = (x^2 + y^2 - 2xyz)^{1/2}$ and $S_x^y f(x) = \langle \gamma(x, y, \eta), f(\eta)\rangle$ we have

$$S_x^y f(x) = [\Gamma(m+1)/\sqrt{\pi}\,\Gamma(m+\tfrac{1}{2})]\int_{-1}^1 (1-z^2)^{m-1/2} f(\eta)\,dz. \tag{5.9}$$

This gives an idea of typical situations (5.1)–(5.3) for singular operators.

Thus, $\beta(y,x)$ in (5.1) represents B: $\varphi_\lambda^P \to \varphi_\lambda^Q$ and $\mathcal{B} = B^{-1}$ has kernel (cf. Example 5.1)

$$\gamma(x,y) = \langle \varphi_\lambda^P(x), \Omega_\lambda^Q(y)\rangle_\omega. \tag{5.10}$$

Another important transmutation $\tilde{B} = \mathcal{B}^\#$ is obtained by adjointness

$$\langle \Delta_P(x)u(x), \mathcal{B}v(x)\rangle = \langle \Delta_Q(y)v(y), \tilde{B}u(y)\rangle, \tag{5.11}$$

$$\tilde{\beta}(y,x) = \langle \Omega_\lambda^P(x), \varphi_\lambda^Q(y)\rangle_\omega = \Delta_P(x)\Delta_Q^{-1}(y)\gamma(x,y). \tag{5.12}$$

In fact, $\tilde{B}$ can be characterized by means of a Cauchy problem (2.1) where $C=0$ and A is the operator with kernel $(x \to \xi)$

$$A(\xi,x) = \langle \varphi_\lambda^P(\xi), \Omega_\lambda^P(x)\rangle_\omega. \tag{5.13}$$

This kernel $A(\xi,x)$ has significance in another role as kernel in the general extended GL equation. Indeed, in $\varphi_\lambda^Q(y) = \langle \beta(y,\xi), \varphi_\lambda^P(\xi)\rangle$ take ω scalar products with $\Omega_\lambda^P(x)$ to obtain (cf. also Theorem 5.4).

THEOREM 5.2. *The general extended GL equation is*

$$\tilde{\beta}(y,x) = \langle \beta(y,\xi), A(\xi,x)\rangle.$$

COMMENT 5.3. The main theme here so far in Section 5 is 'spectral pairings'. The formal expressions usually have a distributional meaning and serve also as a guide in demonstrating formulas.

Under typical hypotheses one finds that the $\varphi_\lambda^Q(y)$ are entire functions of λ of exponential type y and, writing

$$\varphi_\lambda^Q(y) = \langle \beta(y,x), \varphi_\lambda^P(x)\rangle = \mathcal{P}[\beta(y,\cdot)](\lambda) \tag{5.14}$$

(where $\mathcal{P}f(\lambda) = \langle f(x), \varphi_\lambda^P(x)\rangle$) one is in the context of Paley–Wiener ideas; there results

$$\beta(y,x) = 0 \quad \text{for } x > y; \qquad \tilde{\beta}(y,x) = 0 \quad \text{for} \quad x < y. \tag{5.15}$$

The latter follows from (5.12) for example and we remark that triangularity results of the form (5.15) can also be established by contour integral arguments using analyticity properties of $\Phi_{\pm\lambda}^Q$ and $c_Q(\pm\lambda)$ (cf. below). Connection formulas between special functions (such as $\varphi_\lambda^Q = B\varphi_\lambda^P$) using B and $\tilde{B}$ now take on the character of Riemann–Liouville or Weyl-type fractional integrals and we will give examples below. Let us deviate, however, to give a new proof of triangularity for $\tilde{\beta}(y,x)$. We recall from (3.30) that the extended GL equation arises via $H'(t) = \langle \varphi_\lambda^Q(\tilde{y}), \cos\lambda t\rangle_\omega = \tilde{\beta}(\tilde{y},t)$ and we can give a general version of this as follows. Thus, consider for simplicity $R^Q \sim \mathrm{d}\omega \sim \hat{\omega}\,\mathrm{d}\lambda$ with $R^P \sim \mathrm{d}\nu = \hat{\nu}\,\mathrm{d}\lambda$ and recall (5.2). Thus, in Theorem 5.2 we write

$$(W(\lambda) = \hat{\omega}/\hat{\nu}), \qquad A(\xi,x) = \langle \varphi_\lambda^P(\xi), \varphi_\lambda^P(x)W(\lambda)\rangle_\nu \Delta_P(x).$$

If one defines then $\check{W}(t)$ formally by

$$\check{W}(t)=\int_0^\infty W(\lambda)\varphi_\lambda^P(t)\,d\nu \tag{5.16}$$

so that $W(\lambda)=(\mathfrak{P}\check{W})(\lambda)$ then

$$A(\xi,x)=T_\xi^x\check{W}(\xi)\Delta_P(x) \tag{5.17}$$

and the GL equation becomes

$$\tilde{\beta}(y,x)=\langle\beta(y,\xi),\,T_\xi^x\check{W}(\xi)\rangle\Delta_P(x) \tag{5.18}$$

Let us write $\tilde{\beta}(y,x)\Delta_P^{-1}(x)=\langle\varphi_\lambda^P(x),\,\varphi_\lambda^Q(y)\rangle_\omega=\varphi(x,y)$. Then formally (cf. (5.2)) $\varphi(x,y)=\langle\beta(y,\xi),\,T_\xi^x\check{W}(\xi)\rangle$ and

$$P(D_x)\varphi=Q(D_y)\varphi;\qquad \varphi_x(0,y)=0;\qquad \varphi(0,y)=\langle 1,\varphi_\lambda^Q(y)\rangle_\omega=\delta(y)/\Delta_Q(y) \tag{5.19}$$

(note $\mathfrak{Q}[\delta(y)/\Delta_Q(y)]=\langle\Omega_\lambda^Q(y),\,\delta(y)/\Delta_Q(y)\rangle=\varphi_\lambda^Q(0)=1$). Thus, $\varphi(x,y)=\tilde{\beta}(y,x)/\Delta_P(x)$ is the impulse response to $\varphi(0,y)=\delta(y)/\Delta_Q(y)$ and $\tilde{\beta}(y,x)=0$ for $y>x$ by domain of dependence arguments. Further,

$$\varphi(x,0)=\langle\beta(0,\xi),\,T_\xi^x\check{W}(\xi)\rangle=\check{W}(x). \tag{5.20}$$

THEOREM 5.4. *The extended GL equation of Theorem* 5.2 *can be expressed as the impulse response to initial data* $\delta(y)/\Delta_Q(y)$ *as in* (5.19) *from which* $\tilde{\beta}(y,x)=0$ *for* $x<y$ *results, and* (5.20) *holds, expressing* $\check{W}$ *as an impulse response.*

COMMENT 5.5. The interaction here between Paley–Wiener type results (or contour integral techniques) for triangularity and the triangularity arising from domain of dependence arguments in hyperbolic differential equations is probably worth investigating further. One hopes for a 'categorical' meaning for some of the ingredients.

For convenience again let us assume $R^Q\sim d\omega\sim\hat{\omega}\,d\lambda$ and $R^P\sim d\nu\sim\hat{\nu}\,d\lambda$ and set $W(\lambda)=(\hat{\omega}/\hat{\nu})$. Let $\Phi_{\pm\lambda}^Q(x)$ be Jost solutions of $Q\varphi=-\lambda^2\varphi$ satisfying $\Phi_{\pm\lambda}^Q(x)\sim\Delta_Q^{-1/2}(x)\,e^{\pm i\lambda x}$ as $x\to\infty$ $(\sim e^{(\pm i\lambda-\rho_Q)x})$. One writes, as in (3.9) for example, $\varphi_\lambda^Q(x)=c_Q(\lambda)\Phi_\lambda^Q(x)+c_Q(-\lambda)\Phi_{-\lambda}^Q(x)$ and in fairly general circumstances one obtains again (3.12), i.e., $d\omega=\hat{\omega}\,d\lambda=d\lambda/2\pi|c_Q(\lambda)|^2$.

THEOREM 5.6. *Let* $\Psi_\lambda^Q(x)=\Phi_\lambda^Q(x)/c_Q(-\lambda)$. *Under the hypotheses indicated*

$$\tilde{B}\varphi_\lambda^P=W(\lambda)\varphi_\lambda^Q;\qquad \tilde{B}\Psi_\lambda^P=\Psi_\lambda^Q. \tag{5.21}$$

The latter formula can be proved by analytic continuation as in [29] or via the inversion formula for $\mathfrak{Q}$ of (2.17) in Kontorovič–Lebedev form (cf. [66] for details)

$$f(x)=(1/2\pi)\int_{-\infty}^{\infty}\mathfrak{Q}f(\lambda)\Psi_\lambda^Q(x)\,d\lambda. \tag{5.22}$$

To show (5.22) one notes that $f(x) = \mathfrak{Q}\hat{f}(x) = \langle \hat{f}(\lambda), \varphi_\lambda^Q(x)\rangle_\omega$ can be written in the form (5.22) upon writing out φ_λ^Q in terms of $c_Q(\pm\lambda)$, $\Phi_{\pm\lambda}^Q$, and $d\omega$. Typical connection formulas of the type $\varphi_\lambda^Q = B\varphi_\lambda^P$ and $\Psi_\lambda^Q = \tilde{B}\Psi_\lambda^P$ are indicated below in Example 5.7. We emphasize that our derivation yields such connections for huge classes of special functions simultaneously, in a canonical and intrinsic manner, whereas previously one established such formulas individually, in a few cases, by special arguments using properties of hypergeometric functions etc. (cf. [4, 95, 140]).

EXAMPE 5.7. For $\Delta_Q = (e^x - e^{-x})^{2\alpha+1}(e^x + e^{-x})^{2\beta+1}$ one has $\rho_Q = \alpha + \beta + 1$ and (setting $\varphi_\lambda^Q \sim \varphi_\lambda^{\alpha,\beta}$ etc., with Γ denoting some appropriate combination of gamma functions)

$$\begin{aligned}&\varphi_\lambda^Q(x) = F(\tfrac{1}{2}(\rho + i\lambda), \tfrac{1}{2}(\rho - i\lambda), \alpha + 1, -\mathrm{sh}^2 x),\\ &\Phi_\lambda^Q(x) = (e^x - e^{-x})^{i\lambda-\rho}F(\tfrac{1}{2}(\beta - \alpha + 1 - i\lambda), \tfrac{1}{2}(\beta + \alpha + 1 - i\lambda), 1 - i\lambda, -\mathrm{sh}^{-2} x),\end{aligned} \tag{5.23}$$

$$\Gamma\Delta_{\alpha+\mu,\beta+\mu}(x)\varphi_\lambda^{\alpha+\mu,\beta+\mu}(x) = \mathrm{sh}\, 2x \int_0^x \Delta_{\alpha,\beta}(\xi)\varphi_\lambda^{\alpha,\beta}(\xi)[\mathrm{ch}\, 2x - \mathrm{ch}\, 2\xi]^{\mu-1}\, d\xi, \tag{5.24}$$

$$\Psi_\lambda^{\alpha,\beta}(s) = \Gamma \int_s^\infty \Psi_\lambda^{\alpha+\mu,\beta+\mu}(t)[\mathrm{ch}\, 2t - \mathrm{ch}\, 2s]^{\mu-1}\, \mathrm{sh}\, 2t\, dt. \tag{5.25}$$

We observe next that there are a number of factorization type formulas which follow formally immediately from the spectral representation of kernels and which play an important role in the general theory. For example, results of the type (◆) after (2.10) can be expressed quite generally in the form $\mathcal{P}\mathcal{B}^* = \mathcal{Q}$; $\mathcal{Q}\mathcal{B}^* = \mathcal{P}$ where $B \sim \beta(y, x)$, $\mathcal{B} = B^{-1}$, etc. Similarly for suitable f

$$\begin{aligned}\tilde{B}f(y) &= \langle \tilde{\beta}(y, x), f(x)\rangle\\ &= \langle\langle \Omega_\lambda^P(x), \Phi_\lambda^Q(y)\rangle_\omega, f(x)\rangle = \langle \varphi_\lambda^Q(y), \langle \Omega_\lambda^P(x), f(x)\rangle\rangle_\omega = \mathfrak{Q}\mathfrak{P}f\end{aligned} \tag{5.26}$$

and consequently $\tilde{\mathcal{B}} = \tilde{B}^{-1} = \mathfrak{P}\mathfrak{Q}$. Also one sees easily that (for suitable f) $\mathcal{B}^*(\Delta_P f) = \Delta_Q \tilde{B} f$ and $B^*(\Delta_Q f) = \Delta_P \tilde{\mathcal{B}} f$. Hence

THEOREM 5.8. *The factorizations* $\mathcal{P}B^* = \mathcal{Q}$, $\mathcal{Q}\mathcal{B}^* = \mathcal{P}$, $\mathcal{B}^*\Delta_P = \Delta_Q\tilde{B}$, $B^*\Delta_Q = \Delta_P\tilde{\mathcal{B}}$, *and* $\tilde{B} = \mathfrak{Q}\mathfrak{P}$ *with* $\tilde{\mathcal{B}} = \mathfrak{P}\mathfrak{Q}$ *all hold with action on suitable objects. For* $B_Q\colon D^2 \to Q$ *one defines an 'Abel' transform* $F_Q(f) = B_Q^*\Delta_Q f = \tilde{\mathcal{B}}_Q f$. *Then* $\mathcal{P}F_Q(f) = \mathcal{P}B_Q^*\Delta_Q f = \mathcal{Q}\Delta_Q f = \mathfrak{Q}f$ *for suitable* f.

REMARK 5.9. The last formula is an analogue of factoring the spherical transform into the composition of a Mellin and a Harish transform in the theory of Lie groups (cf. [147]). Thus, one reason the transmutation machine yields 'good' connection formulas etc. is that it embodies the natural 'symmetries' or structure intrinsic to radial Laplace–Beltrami operators (cf. also [141]).

Transmutation methods have also proved fruitful in dealing with integral transforms of various types related to the $\mathfrak{Q} - \mathfrak{Q}$ inversion (cf. also (3.39) and

(5.22)). For example one has an inversion (cf. [29, 43, 140, 171])

$$F(\lambda)=\int_0^\infty f(y)\Delta_Q(y)\Psi_\lambda^Q(y)\,dy;$$
$$f(x)=(1/2\pi)\int_0^\infty [F(\lambda)+F(-\lambda)]\varphi_\lambda^Q(x)\,d\lambda. \tag{5.27}$$

We will indicate some further examples later in Section 6 in connection with Kontorovič–Lebedev-type theorems. Let us shown now how one can arrive at a canonical intrinsic Marčenko (M)-type equation via techniques of possible later use in general operational calculus (the canonical intrinsic nature of the equation is also indicated when we show later that it is the natural minimizing criterion in the characterization of kernels via extremum properties). The M equation in classical scattering theory is an alternate way (to the GL procedure) of working experimental data (e.g., the one-dimensional scattering matrix $F(-\lambda)/F(\lambda)=S(\lambda)$ in the problem (3.1)–(3.6')) into an inversion technique yielding the potential $q(x)$ (see also [175] for an elegant extension). Using Fourier analysis one connects D^2 and $D^2-q=Q$ by formulas whose only Q input is $S(\lambda)$. We gave a preliminary generalization of this in [29, 44] and later obtained the canonical version sketched briefly here (cf. [31, 45, 46, 60]). Thus, let $Pu=u''-pu$ be a Fourier-type operator on $(-\infty,\infty)$, by which we mean that p is real, even, continuous, positive, and $p(x)\,e^{2Hx}\in L^1(0,\infty)$. These hypotheses are stronger than needed but suffice for illustration (cf. [114, 213, 220]). Let φ_λ^P (resp. χ_λ^P) satisfy (*) $P\varphi=-\lambda^2\varphi$ with $\varphi_\lambda^P(0)=1$ and $D_x\varphi_\lambda^P(0)=0$ (resp. $\chi_\lambda^P(0)=0$ and $D_x\chi_\lambda^P(0)=-1$ – thus $\chi_\lambda^P=-\theta_\lambda^P$). One defines $\Phi_{\pm\lambda}^P(x)\sim e^{\pm i\lambda x}$ as usual to be Jost solutions of (*) and we set $M_1(\lambda)=c_P(-\lambda)$ with

$$M(\lambda)=\frac{1}{2i\lambda F(\lambda)}$$

($c_P\sim\varphi_\lambda^P$ and $F\sim\chi_\lambda^P=-\theta_\lambda^P$). Then $\Phi_\lambda^P=2i\lambda[M(\lambda)\varphi_\lambda^P-M_1(\lambda)\chi_\lambda^P]$ and one defines

$$\Sigma_\lambda^P(x)=2i\lambda[M(\lambda)\varphi_\lambda^P(x)+M_1(\lambda)\chi_\lambda^P(x)]. \tag{5.28}$$

In terms of full line scattering theory (cf. [69, 93, 135]), we have a transmission coefficient ρ and a reflection coefficient $-A$ defined by

$$\left(\rho=\frac{1}{4i\lambda MM_1},\quad M^-=M(-\lambda)\right)$$
$$\rho=\tfrac12[M_1^-/M_1-M^-/M];$$
$$A(\lambda)=-\tfrac12[M_1^-/M_1+M^-/M]=-[MM_1^-+M^-M_1]/2MM_1 \tag{5.29}$$

such that $\Phi_{-\lambda}^P=\rho\Sigma_\lambda^P+A\Phi_\lambda^P$ and there is an inversion

$$F(\lambda)=\int_{-\infty}^\infty f(x)\Phi_\lambda^P(x)\,dx;\qquad f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty \rho(\lambda)F(\lambda)\Sigma_\lambda^P(x)\,d\lambda. \tag{5.30}$$

Now let Q be a 'general' operator of the form (2.16) with however $R^Q \sim d\omega = \hat{\omega}\, d\lambda$ on $[0, \infty)$. Our philosophy' of the M equation (following Fadeev [93] in part) is to exhibit it as a relation between three transmutations B, $\tilde{B}$, and $\check{B}$ where $\check{B}: \Phi_\lambda^P \to \Phi_\lambda^Q$ is determined formally via a spectral kernel

$$\check{\beta}(y, x) = (1/2\pi) \int_{-\infty}^{\infty} \Phi_\lambda^Q(y)\rho(\lambda)\Sigma_\lambda^P(x)\, d\lambda \tag{5.31}$$

(by contour integration one sees that $\check{\beta}(y, x) = 0$ for $x < y$ and hence we work with a full line theory for $P(\sim x)$ while $y \geq 0$ $(y \sim Q)$ – see [32, 66] for details – the presentation of [60] is improved in [32, 66]). The GL equation is now $B = \tilde{B}\tilde{A}$ where

$$\tilde{A}(x, y) = \langle \varphi_\lambda^P(x), \varphi_\lambda^P(y) \rangle_\mu; \qquad d\mu = (\hat{\nu}^2/\hat{\omega})\, d\lambda \tag{5.32}$$

(here $d\nu = \hat{\nu}\, d\lambda$ refers to the spectral measure for P relative to φ_λ^P). One can show then

THEOREM 5.10. *Under the hypotheses indicated $\tilde{B} = \check{B}\mathcal{H}$ where $\mathcal{H}(x, s) =$ kernel $\mathcal{H}$ $(s \to x)$ can be written $\mathcal{H}(x, s) = (1/2\pi) \int_{-\infty}^{\infty} [M_1(\lambda)/c_Q(-\lambda)]\Sigma_\lambda^P(s)\Phi_\lambda^P(x)\rho(\lambda)\, d\lambda$.*

REMARK 5.11. We mention here in connection with the calculations involved in developing the M equation that there is perhaps some independent interest in dealing more extensively with the generalized translation and generalized convolution determined by $(\Phi(f) = \int_{-\infty}^{\infty} f(x)\Phi_\lambda^P(x)\, dx)$

$$\begin{aligned} \mathfrak{T}_x^y f(x) &= (1/2\pi) \int_{-\infty}^{\infty} \Phi(f)\Phi_\lambda^P(y)\Sigma_\lambda^P(x)\rho(\lambda)\, d\lambda; \\ (g * f)(x) = (f * g)(x) &= (1/2\pi) \int_{-\infty}^{\infty} \Phi(f)\Phi(g)\Sigma_\lambda^P(x)\rho(\lambda)\, d\lambda. \end{aligned} \tag{5.33}$$

Thus, $\Phi(f * g) = \Phi(f)\Phi(g)$ etc.

Let now $\mathcal{H}^*$ have kernel $\mathcal{H}^*(x, s) = \mathcal{H}(s, x)$. Then the general extended M equation is

$$B\mathcal{H}^* = \tilde{B}\tilde{A}\mathcal{H}^* = \check{B}(\mathcal{H}\tilde{A}\mathcal{H}^*) = \check{B}\mathfrak{K}. \tag{5.34}$$

One shows that kernel $B\mathcal{H}^* = 0$ for $x > y$ and we obtain

THEOREM 5.12. *The general extended* M *equation is $B\mathcal{H}^* = \check{B}\mathfrak{K}$ from* (5.34) *and for $x > y$ one obtains an integral equation for $\check{\beta}$ in the form $\check{\beta}(y, x) + \int_y^\infty \check{\beta}(y, t)[T(t, x) + S(t, x)]\, dt = 0$ where*

$$\begin{aligned} T(t, x) &= (1/2\pi) \int_{-\infty}^{\infty} A(\lambda)\Phi_\lambda^P(t)\Phi_\lambda^P(x)\, d\lambda; \\ S(t, x) &= (1/2\pi) \int_{-\infty}^{\infty} s_Q(\lambda)\Phi_\lambda^P(t)\Phi_\lambda^P(x)\, d\lambda \end{aligned} \tag{5.35}$$

and $s_Q(\lambda) = c_Q(\lambda)/c_Q(-\lambda)$ is the only Q input to the connection.

EXAMPLE 5.13. For $P = D^2$ and $Qu = (Au')'/A - qu$ with A as in Section 3 and q suitable (q known) one has $\check{\beta}(y, x) = A^{-1/2}(y)\delta(x - y) + \check{K}(y, x)$, $\beta(y, x) = A^{-1/2}(y)\delta(x - y) + K(y, x)$, $\tilde{\beta}(y, x) = A^{-1/2}(y)\,\delta(x - y) + \tilde{K}(y, x)$, and $\mathfrak{M}(y, x) = \delta(x - y) + k(y, x)$. Consequently the M equation (5.34) has the form $(x > y)$

$$A^{-1/2}(y)k(y, x) + \check{K}(y, x) + \int_y^\infty \check{K}(y, t)k(t, x)\,\mathrm{d}t = 0. \tag{5.36}$$

An integrated form of this would then be used to serve as a recovery vehicle for A (cf. (3.19), remarks later, and Section 6). For $A = 1$ with $Q = D^2 - q$ one has $0 = A(\lambda) = T(t, x)$ and $S(t, x) = (1/2\pi)\int_{-\infty}^{\infty} s_Q(\lambda)\,\mathrm{e}^{i\lambda(x+t)}\,\mathrm{d}\lambda$. The equation in Theorem 5.12 becomes then (for $\check{\beta}(y, x) = \delta(x - y) + \check{K}(y, x)$),

$$0 = \check{K}(y, x) + S(y, x) + \int_y^\infty \check{K}(y, t)S(t, x)\,\mathrm{d}t.$$

This is equivalent to the classical M equation since $c_Q(\lambda)F(\lambda) + c_Q(-\lambda)F(-\lambda) = 1$ and $S(t, x)$ will reduce to $-\int_{-\infty}^{\infty} (F(-\lambda)/F(\lambda))\,\mathrm{e}^{i\lambda(x+t)}\,\mathrm{d}\lambda$.

COMMENT 5.14. The idea of treating the GL and M equations via transmutation seems to go back to Fadeev [93] for the quantum scattering situation. We gave a preliminary generalization of this in [29, 44] and later extended this to the present formulation (cf. [31, 32, 45, 46, 60, 66]). We will see below that our version also arises via minimization and thus we believe it to be intrinsic and canonical. The M equation appears to be a greater depth than the GL equation (cf. [236]) but as indicated below they actually have a similar structure (see [32, 66] and cf. also [87, 88]).

REMARK 5.15. Let us show how one can write the M-data $\mathcal{H}\tilde{A}\mathcal{H}^*$ as an upper-lower factorization $\check{B}^{-1}\hat{B}$ ($\hat{B} = B\mathcal{H}^*$) and compare with the lower-upper factorization $B^{-1}\tilde{B}$ of GL data A. This is developed in [32, 66] and extends the presentation of [87, 88] (where $P = D^2$) in certain ways involving canonical ingredients (e.g., generalized translations) which do not arise explicitly in [87, 88]. One notes here that our M equation is based on Φ_λ^P and Σ_λ^P, for example, in a nontrivial generalization of structure and it cannot be treated by the Fourier transform as when $P = D^2$. There is also an interesting interaction between our full line theory for P and the corresponding half line KL theory but we refer for this to [32, 66]. Now one notes that

$$(1/2\pi)\int_{-\infty}^{\infty} \Sigma_\lambda^P(x)\Sigma_\mu^P(x)\,\mathrm{d}x = (1/\rho\rho^-)\delta(\lambda + \mu) - (A/\rho^2)\delta(\lambda - \mu)$$

(this was reported incorrectly in [60] and corrected in [66]). Then in dealing with $\tilde{\beta}(y, x)$ on the full axis we identify $\tilde{\beta}(y, x)$ with

$$\tilde{\beta}(y, x) = (1/2\pi)\int_{-\infty}^{\infty} \rho M_1 \Psi_\lambda^Q(y)\Sigma_\lambda^P(x)\,\mathrm{d}\lambda. \tag{5.37}$$

The action $\langle\tilde{\beta}(y,x), \Psi_\mu^P(x)\rangle = \Psi_\mu^Q(y)$ follows immediately from (5.37) and (5.30) in the full line theory and the factorization $\tilde{B} = \check{B}\mathcal{H}$ follows directly from (5.31) and (5.37) where $\mathcal{H}(x,s)$ $(s \to x)$ is given as above in Theorem 5.10. Now recall the M equation as $\hat{B} = B\mathcal{H}^* = \check{B}(\mathcal{H}\tilde{A}\mathcal{H}^*)$ $(\mathcal{H}^*(x,s) = \mathcal{H}(s,x))$ and write for $\hat{\beta}(y,\xi) = \ker \hat{B}(\xi \to y)$

$$\begin{aligned}\hat{\beta}(y,\xi) &= (1/2\pi)\int_{-\infty}^{\infty} [c_{\bar{P}}/c_{\bar{Q}}]\rho\Phi_\lambda^P(\xi)\int_{-\infty}^{\infty} \beta(y,x)\Sigma_\lambda^P(x)\,dx\,d\lambda \\ &= (1/2\pi)\int_{-\infty}^{\infty} [1/c_Q(-\lambda)]\Phi_\lambda^P(\xi)\varphi_\lambda^Q(y)\,d\lambda = \langle\beta(y,x), \mathcal{H}(\xi,x)\rangle\end{aligned} \tag{5.38}$$

and by contour integration $\hat{\beta}(y,\xi) = 0$ for $\xi > y$. Now an inverse $\check{\mathcal{B}}$ to $\check{B}$ can be produced via the kernel $\check{\gamma}(x,y) = (1/2\pi)\int_{-\infty}^{\infty} \Phi_\lambda^P(x)[\Omega_\lambda^Q(y)/c_Q(-\lambda)]\,d\lambda$ so that in particular $\check{\mathcal{B}}\Phi_\mu^Q = \Phi_\mu^P$ and $\check{\gamma}(x,y) = 0$ for $x > y$. There is no *a priori* full line theory for Q so the KL type development is needed here and one uses analytic continuation arguments to show that $\check{\Upsilon}(\lambda,\mu) = (1/2\pi)\int_0^\infty \Omega_\lambda^Q(y)\Psi_\mu^Q(y)\,dy \sim \delta(\lambda-\mu)$ acting on suitable objects. Next, one can explicitly write out the upper-lower factorization $\check{B}^{-1}\hat{B}$ of $\mathcal{H}\tilde{A}\mathcal{H}^*$. A direct calculation shows that the operator $\check{B}^{-1}\hat{B}$ has kernel $S+T+\delta$ (cf. Theorem 5.12). We also see that $\check{\gamma}(x,y) = \Delta_Q(y)\hat{\beta}(y,x)$ and, hence, formally $\check{\mathcal{B}}^* = \Delta_Q\hat{B}$ and $\check{\mathcal{B}}\hat{\mathcal{B}} = \check{\mathcal{B}}\Delta_Q^{-1}\check{\mathcal{B}}^*$. One can also calculate the kernel of $\mathcal{H}\tilde{A}\mathcal{H}^*$ more directly via (recall $W^{-1}(\lambda) = \hat{\nu}/\hat{\omega} = |c_Q/c_P|^2$)

$$\begin{aligned}\tilde{A}(\xi,\eta) &= \langle\gamma_\lambda^P(\xi), \varphi_\lambda^P(\eta)W^{-1}(\lambda)\rangle_\nu \\ &= (1/4\pi)\int_{-\infty}^{\infty} \rho^2[|c_Q|^2/(M_1^-)^2][\Sigma_\lambda^P(\xi)+\Phi_\lambda^P(\xi)][\Sigma_\lambda^P(\eta)+\Phi_\lambda^P(\eta)]\,d\lambda.\end{aligned}$$

This decomposition spreads the generalized translation over the whole axis – here one recalls that if $\check{W}^{-1}(\eta) = \int_0^\infty W^{-1}(\lambda)\varphi_\lambda^P(\eta)\,d\nu$ (so that $W^{-1}(\lambda) = \langle\check{W}^{-1}(\eta), \varphi_\lambda^P(\eta)\rangle = (\mathfrak{P}\check{W}^{-1})(\lambda)$) then $\tilde{A}(\xi,\eta) = T_\eta^\xi\check{W}^{-1}(\eta)$ for a certain generalized translation (cf. (5.17)). Thus the GL and M equations have a similar structure. For the GL equation one writes $B^{-1}\tilde{B} = A$ as a lower-upper factorization of GL data A given via generalized translation from $\check{W}$ as $A(t,x) = T_x^t\check{W}(x)$. The M equation represents an upper-lower factorization $\check{\mathcal{B}}\hat{B}$ of M data $\mathcal{H}\tilde{A}\mathcal{H}^*$ where the M data is 'canonically' related to the GL data via $\mathcal{H}$ and generalized translation $(\tilde{A}(\xi,\eta) = T_\eta^\xi\check{W}^{-1}(\eta))$. Note also that if $\Delta_Q = 1$ for illustrative purposes then $\tilde{B} = (B^{-1})^*$ so the GL equation is $A = \mathcal{B}\mathcal{B}^*$ for $\mathcal{B} = B^{-1}$ while the M equation is $\mathcal{H}\tilde{A}\mathcal{H}^* = \check{\mathcal{B}}\check{\mathcal{B}}^*$. The factorization point of view is useful for computational purposes.

REMARK 5.16. One can also derive formally (cf. [87, 88] and Section 7) the general GL equation as a factorization, using a Parseval formula as in Section 2, by the following heuristic calculation (with obvious notational abuses). Let

$$\begin{aligned}\hat{f}(\lambda) &= \langle f(x), \varphi_\lambda^Q(x)\rangle \\ &= \langle f(x), \langle \beta(x, \xi), \varphi_\lambda^P(\xi)\rangle\rangle \\ &= \langle f(x), (B\varphi_\lambda^P)(x)\rangle.\end{aligned}$$

For suitable f, g one has $\langle \hat{f}, \hat{g}\rangle_\omega = \langle f, g\rangle$ (take $\Delta_Q = 1$ here for illustration) and, hence,

$$\begin{aligned}&(A(\xi, \eta) = \langle \hat{\omega}, \varphi_\lambda^P(\xi)\varphi_\lambda^P(\eta)\rangle_\lambda), \\ &\langle f, g\rangle = \langle\langle f, B\varphi_\lambda^P\rangle, \langle g, B\varphi_\lambda^P\rangle\rangle_\omega \\ &\quad = \int f(x) \int B(x, \eta) \int A(\xi, \eta) \int B(y, \xi) g(y)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad = \langle f, BAB^* g\rangle.\end{aligned}$$

Consequently, $BAB^* = I$ or $BA = B^{*-1} = (B^{-1}) = \tilde{B}$. The same kind of procedure for the M equation as used in [88] relies heavily on the underlying Fourier transform (i.e., $P = D^2$ is needed) and will not work for more general P.

We turn next to the idea of minimization as a directive in characterizing transmutation kernels (cf. [41, 48, 51, 52, 64–65]). Some motivation comes here from [90, 91] where it is shown how GL equations can be obtained by minimizing a certain quadratic functional $Q(t, K)$ (cf. also [67]). However, a motivation within 'scattering theory' to consider $Q(t, K)$ was lacking and in [90] this was regarded as unsatisfactory. We will show here how such quadratic functionals $Q(t, K)$ arise as a criterion in characterizing transmutation kernels and, when there is an underlying stochastic problem as in Section 4, the procedure is essentially equivalent to linear least squares estimation. We will sketch the procedure for $P = D^2$ and $Qu = (Au')'/A$ as in Section 3 (a term $-qu$ could also be added in Qu) with B: $\varphi_\lambda^P(x) = \cos \lambda x = a(\lambda, x) \to \varphi_\lambda^Q(y) = s(\lambda, y)$. With minor modifications the same formulation and calculations apply for $Q = D^2 - q$ with $s(\lambda, y) = \varphi_{\lambda,h}^Q(y)$ (and $a(\lambda, x) = \cos \lambda x$) or for $P = D^2$ and $Q = D^2 - q$ with $a(\lambda, x) = \theta_\lambda^P(x) = \sin \lambda x/\lambda$ and $s(\lambda, y) = \theta_\lambda^Q(y)$ as in (3.3) (we assume for convenience that $\mathrm{d}\omega = \hat{\omega}\,\mathrm{d}\lambda$ on $[0, \infty)$). Thus, consider the expression ($T > 0$ arbitrary)

$$\Xi = \Xi(T, \hat{\mathfrak{K}}) = \int_0^T \int_0^\infty [A^{-1/2}a - s + \langle \hat{\mathfrak{K}}(y, x), a(\lambda, x)\rangle]^2\,\mathrm{d}\omega\,\mathrm{d}y \tag{5.39}$$

where $\hat{\mathfrak{K}}$ is to run over a suitable class of causal kernels (i.e., $\hat{\mathfrak{K}}(y, x) = 0$ for $x > y$) having the same properties as the GL kernel $\hat{K}$ ($= K_x(y, x)$ as in (3.17)–(3.19)). Recall here that $\beta(y, x) = A^{-1/2}\delta(x - y) + \hat{K}(y, x)$ from (3.18) so the GL kernel $\hat{K}$ is a minimizing kernel for (5.39). The point here is that $\hat{K}$ can in fact be characterized by this property via the GL equation. Now for an operator Λ with kernel $\Lambda(y, x)$ we write $\mathrm{Tr}\,\Lambda = \int_0^T \Lambda(y, y)\,\mathrm{d}y$ (this integral $\int_0^T$ in Ξ can actually be omitted – it is inserted in order to have a direct comparison to the $Q(t, K)$ of [90]). Then, writing $\hat{\Xi} = \int_0^T \int_0^\infty [A^{-1/2}a - s]^2\,\mathrm{d}\omega\,\mathrm{d}y$ (which makes sense) we have

$$\begin{aligned}\Xi = \hat{\Xi} + 2\int_0^T \int_0^y A^{-1/2}(y)\hat{\mathfrak{K}}(y, x)\mathfrak{A}(y, x)\,dx\,dy - \\ -2\int_0^T \int_0^y \hat{\mathfrak{K}}(y, x)\tilde{\beta}(y, x)\,dx\,dy + \\ + \int_0^T \int_0^y \int_0^y \hat{\mathfrak{K}}(y, x)\hat{\mathfrak{K}}(y, \xi)\mathfrak{A}(\xi, x)\,d\xi\,dx\,dy\end{aligned} \tag{5.40}$$

where $\tilde{\beta}(y, x) = \langle s(\lambda, y), a(\lambda, x)\rangle_\omega$ (cf. (5.12)) and (cf. (5.13)) $\mathfrak{A}(y, x) = \langle a(\lambda, x), a(\lambda, y)\rangle_\omega = \delta(x-y) + \Omega(y, x)$ (the splitting follows from $d\omega = d\sigma + (2/\pi)\,d\lambda$). Now $\tilde{\beta}(y, x) = A^{-1/2}(y)\,\delta(x-y) + \tilde{K}(y, x)$ with $\tilde{K}$ anticausal so the $\tilde{\beta}$ term in (5.40) only contributes $A^{-1/2}(y)\,\delta(x-y)$ which is cancelled by a contribution from $\mathfrak{A}$. It results that (■)$\Xi = \hat{\Xi} + \mathrm{Tr}\{A^{-1/2}(y)[\hat{\mathfrak{K}}\Omega + \Omega\hat{\mathfrak{K}}^*] + \hat{\mathfrak{K}}(1+\Omega)\hat{\mathfrak{K}}^*\}$. One writes $\hat{\mathfrak{K}} = \hat{\mathfrak{K}}_0 + \epsilon J$ for $\hat{\mathfrak{K}}_0$ a minimizing kernel and J an admissible causal kernel. An elementary variational procedure gives as the 'Euler' equation for $x < y$

$$\hat{\mathfrak{K}}_0(y, x) + A^{-1/2}(y)\Omega(y, x) + \int_0^y \hat{\mathfrak{K}}_0(y, \xi)\Omega(\xi, x)\,d\xi = 0 \tag{5.41}$$

which is the canonical GL equation (from which (3.19) follows). Since one has unique solutions of (5.41) it follows that

THEOREM 5.17. *The GL kernel $\hat{K}(y, x) = \hat{\mathfrak{K}}_0(y, x)$ is characterized via the GL equation as the solution of minimizing Ξ over a suitable class of admissible causal $\hat{\mathfrak{K}}$.*

COMMENT 5.18. The result in Theorem 5.17 seems to be essentially theoretical in nature and not adopted to computational procedures of use in practice. The fact that the GL kernel arises out of a minimizing procedure as a kind of Euler equation seems however to be of interest and when there is an underlying stochastic problem we show later that this minimization is essentially equivalent to linear least squares estimation in locating the best approximation in a certain Hilbert space. Some further minimization results in the spirit of approximation are indicated in Section 6 (see also Remark 5.24).

One can characterize the kernel $\gamma(x, y)$ for $\mathcal{B} = B^{-1}$ or the kernel $\tilde{\beta}(y, x)$ for $\tilde{B}$ in a similar manner. In fact, the latter characterization leads to a derivation of the general M equation as a minimizing criterion. Thus let $P = D^2 - p$ be a Fourier type operator with B: φ_λ^P and $\tilde{B}$: $\varphi_\lambda^P \to \varphi_\lambda^P \to W(\lambda)\varphi_\lambda^Q$ (say $Qu = (Au')'/A - qu$ as above for suitable A and q with $d\omega = \hat{\omega}\,d\lambda$). Then $\tilde{\beta}(y, x) = A^{-1/2}(y)\,\delta(x-y) + \tilde{K}(y, x)$ with K anticausal and we consider

$$\tilde{\Xi} = \int_0^T \int_0^\infty [A^{-1/2}a - Ws + \langle\tilde{\mathfrak{K}}(y, x), a(\lambda, x)\rangle]^2\,d\mu\,dy \tag{5.42}$$

where $d\mu = (\hat{\nu}^2/\hat{\omega})\,d\lambda$ as in (5.32). A direct calculation now as above for Ξ yields the canonical GL equation for $\tilde{K}(y, x) = \tilde{\mathfrak{K}}_0(y, x)$ as the Euler variational equation and thus characterizes $\tilde{K}$. However, somewhat more can also be done. Thus we think of $\tilde{B} = \check{B}\mathscr{H}$ as in Theorem 5.10 and, with suitable A, q, one has $\check{\beta}(y, x) = A^{-1/2}(y)\,\delta(x-y) + \check{K}(y, x)$ with $\check{K}$ anticausal. Then also $\mathscr{H}(x, s) = \delta(x-s) + h(x, s)$ (with $h(x, s) = 0$ for $s < x$) and, in (5.32), $\tilde{A}(x, y) = \delta(x-y) + \tilde{\Omega}(x, y)$. In particular, this leads to the equation

$$\tilde{K}(y, x) = \check{K}(y, x) + A^{-1/2}(y)h(y, x) + \int_y^x \check{K}(y, s)h(s, x)\, ds.$$

Now writing out $\tilde{\Xi}$ as in (5.40) and (■) we obtain first $\tilde{\Xi} = \hat{\hat{\Xi}} + \mathrm{Tr}\{A^{-1/2}(y)[\tilde{\mathfrak{K}}\tilde{\Omega} + \tilde{\Omega}\tilde{\mathfrak{K}}^*] + \tilde{\mathfrak{K}}(1 + \tilde{\Omega})\tilde{\mathfrak{K}}\}$ as expected (where $\hat{\hat{\Xi}}$ does not depend on $\tilde{\mathfrak{K}}$). Now think of the trial $\tilde{\mathfrak{K}}$ as arising from a construction as in (5.43) with trial anticausal $\check{\mathfrak{K}}$ (i.e., think of trial $\tilde{\mathbb{B}} = \check{\mathbb{B}}\mathscr{H}$ with kernel $\check{\mathbb{B}} = A^{-1/2}(y)\delta(x-y) + \check{\mathfrak{K}}(y, x)$ etc.). Then the operator $\mathfrak{K} = \mathscr{H}\tilde{A}\mathscr{H}^*$ of (5.34) with kernel $\mathfrak{K}(y, x) = \delta(x-y) + k(y, x)$ arises naturally in the calculation and one has eventually

$$\tilde{\Xi} = \dot{\Xi} + \mathrm{Tr}\, \check{\mathfrak{K}}(1 + k)\check{\mathfrak{K}}^* + 2\,\mathrm{Tr}\, A^{-1/2}(y)k\check{\mathfrak{K}}^* \tag{5.43}$$

($\dot{\Xi}$ independent of $\check{\mathfrak{K}}$). A variational argument as before leads now to an Euler equation for the minimizing $\check{\mathfrak{K}}_0$, namely, for $x > y$ (cf. (5.36))

$$A^{-1/2}(y)k(y, x) + \check{\mathfrak{K}}_0(y, x) + \int_y^\infty \check{\mathfrak{K}}_0(y, s)k(s, x)\, ds = 0.$$

THEOREM 5.19. *The kernel $\check{K}(y, x)$ of $\check{\beta}(y, x) = A^{-1/2}(y)\,\delta(x-y) + \check{K}(y, x)$ is characterized via minimization as the (unique) solution of the canonical M equation of Theorem* 5.12 *(i.e., kernel $\check{B}\mathfrak{K} = 0$ for $x > y$).*

Let us go now to the context of Section 4, Theorem 4.12, and suppose we have an underlying stochastic process $Y_+(t) = Z_+(t) + N_+(t)$ etc., with $\gamma_+(\lambda, t) = \varphi^Q_{\lambda,h}(t)$ satisfying Theorem 4.12 in the form $Q(D_t)\varphi^Q_{\lambda,h} = -\lambda^2\varphi^Q_{\lambda,h}$, $Q = D^2 - q$ with $q = V_+$, $h = -2K(0)$, and from (4.33) $d\omega = \hat{\omega}\, d\lambda$ with $\hat{\omega} = (2/\pi)\hat{R}$. The minimizing procedure of (5.39), for example, can be also applied to $B_h : \cos\lambda x \to \varphi^Q_{\lambda,h} : a(\lambda, x) \to s(\lambda, y)$ (cf. Theorem 2.2) where $\beta_h(y, x) = \delta(x-y) + K_h(y, x)$. From (4.32) we have $K_h(y, x) = -g_+(y, x)$ and the GL equation for K_h is then from (4.29), setting $K_+(t, s) = \frac{1}{2}\Omega(t, s)$, for $s < t$

$$\Omega(t, s) + K_h(t, s) + \int_0^t K_h(t, \tau)\Omega(\tau, s)\, d\tau = 0 \tag{5.44}$$

(which has the same form as (5.41) for example with $A = 1$). Here one wants then by analogy (recall from Section 4, $\mathscr{F}(\delta + K) = \hat{R}$)

$$\begin{aligned}\mathfrak{A}(y, x) &= \langle \cos\lambda x, \cos\lambda y\rangle_\omega = (2/\pi)\int_0^\infty \cos\lambda x \cos\lambda y \hat{R}\, d\lambda \\ &= R(t+s) + R(t-s) = \delta(t+s) + \delta(t-s) + 2K_+(t, s) \\ &= \delta(t-s) + \Omega(t, s)\end{aligned} \tag{5.45}$$

(for s, $t>0$ the $\delta(t+s)$ term can be dropped). Thus, everything fits together and we want to give the minimization of Ξ a meaning in the stochastic theory - one expects it is related to least squares estimation of course and we will spell this out precisely. In this connection we first observe that the discovery of the filtering kernel g_+ (i.e., $-K_h$) is equivalent in 'stochastic geometry' to locating $\hat{Z}_+(t|t)$ in ${}^+H_Y^t$ (cf. (4.28)). Next, using this information it is easy to see that in fact

$$\begin{aligned}\Xi &= (1/\pi)\int_0^T\int_0^\infty [\gamma_+ - \cos\lambda t + \langle \mathfrak{g}_+(t,\tau), \cos\lambda\tau\rangle]^2 \hat{R}\,\mathrm{d}\lambda\,\mathrm{d}t \\ &= 2E\int_0^T [J_+(t) - Y_+(t) + \langle \mathfrak{g}_+(t,\tau), Y_+(\tau)\rangle]^2\,\mathrm{d}t\end{aligned} \tag{5.46}$$

where $\mathfrak{g}_+ \sim -\mathfrak{K}_h$ for an admissible $\mathfrak{K}_h$. Hence, minimizing Ξ corresponds to locating $\hat{Z}_+ = J_+ - Y_+$ in stochastic geometry (i.e., in a least-squares sense). Recall here from (4.3) etc. one knows that $X = \int \xi(\lambda)\,\mathrm{d}\hat{Y}_\lambda$ gives an isometry between H_Y and $L^2(\hat{R}\,\mathrm{d}\lambda/2\pi)$.

THEOREM 5.20. *When there is an underlying stochastic process the corresponding minimization of Ξ for B_h: $\cos\lambda x \to \varphi^Q_{\lambda,h}$ is equivalent in stochastic geometry to linear least-squares estimation and serves to locate $\hat{Z}_+(t\,|\,t)$ in ${}^+H_Y^t$.*

REMARK 5.21. Let us show here how a connection formula $\varphi^Q_\lambda(y) = \langle\beta(y,x), \varphi^P_\lambda(x)\rangle$ plus, e.g., a spectral form for $\beta(y,x)$ leads to intertwining via the GL equation (see also Theorem 6.6 and cf. Theorem 4.10). We recall the construction in Section 2 where a kernel K_h was constructed via a Riemann function and shown to transmute because of the Cauchy problem characterization in Theorem 2.1. Then in addition it followed that K_h satisfied a Goursat problem as in Theorem 2.2 and provided a connection between $\cos\lambda x$ and $\varphi^Q_{\lambda,h}$. Here we start with a connection and for simplicity take the model $Qu = u'' - q(x)u$ with B: $D^2 \to Q$: $\cos\lambda x \to \varphi^Q_\lambda(y)$ and $\beta(y,x) = \text{kernel } B = \delta(x-y) + \hat{K}(y,x)$ where $\hat{K}(y,x) = 0$ for $x>y$. Thus, $\varphi^Q_\lambda(y) = \cos\lambda x + \int_0^y \hat{K}(y,\xi)\cos\lambda\xi\,\mathrm{d}\xi$ and one obtains a GL equation immediately in the form

$$\tilde{\beta}(y,x) = \langle\varphi^Q_\lambda(y), \cos\lambda x\rangle_\omega = \mathfrak{A}(y,x) + \int_0^y \hat{K}(y,\xi)\mathfrak{A}(\xi,x)\,\mathrm{d}\xi \tag{5.47}$$

where $\mathfrak{A}(y,x) = \langle\cos\lambda y, \cos\lambda x\rangle_\omega = \delta(x-y) + \Omega(y,x)$. Now by analysis such as that leading to (3.17) or that in Theorem 2.2 one knows that $\hat{K}(y,y) = \frac{1}{2}\int_0^y q(x)\,\mathrm{d}x$ (this is dependent only on the connection and on the differential equation $(D^2-q)\varphi^Q_\lambda = -\lambda^2\varphi^Q_\lambda$). Now formally if one deals with functions $f(x) = \langle\tilde{f}(\lambda), \cos\lambda x\rangle_\nu$ where $\tilde{f} = \mathscr{F}_C f$ then $QBf = Q\langle\tilde{f}(\lambda), \varphi^Q_\lambda(y)\rangle_\nu = \langle\tilde{f}(\lambda), -\lambda^2{}^Q_\lambda(y)\rangle_\nu$ while $BD^2 f = B\langle\tilde{f}(\lambda), -\lambda^2\cos\lambda x\rangle_\nu = \langle\tilde{f}(\lambda), -\lambda^2\varphi^Q_\lambda(y)\rangle_\nu$. Thus, for such f, given that $\langle\tilde{f}, -\lambda^2\cos\lambda x\rangle_\nu$ makes sense etc., B is a transmutation. We can arrive at this conclusion also in another way via (5.47) and a Goursat problem as in Theorem 4.10 (incidently, we will produce here a sketch of a proof of Theorem 4.10).

Thus, in (5.47) one can deduce that $\tilde{\beta}(y, x) = 0$ for $x < y$ by several arguments (e.g., $\tilde{\beta}(y, x) = \gamma(x, y)$ or a contour integral – Paley–Wiener argument based on the form of $\tilde{\beta}(y, x)$ as a spectral pairing). Hence, for $x < y$ (5.47) becomes

$$0 = \Omega(y, x) + \hat{K}(y, x) + \int_0^y \hat{K}(y, \xi)\Omega(\xi, x)\,d\xi. \tag{5.48}$$

Evidently, $D_x^2\Omega = D_\xi^2\Omega$, $\Omega_y(0, x) = 0$, and we can set $K_\xi(y, 0) = 0$. Writing $\blacksquare\hat{K} = \hat{K}_{yy} - \hat{K}_{xx}$ (recall here $\hat{K}'(y, y) = \frac{1}{2}q(y)$ and note that $\hat{K}'(y, y) = \hat{K}_y(y, y) + \hat{K}_\xi(y, y)$) there results $0 = \blacksquare\hat{K} + q(y)\Omega(y, x) + \int_0^y \blacksquare\hat{K}(y, \xi)\Omega(\xi, x)\,d\xi$. It follows that $\blacksquare\hat{K}(y, x)/q(y)$ satisfies (5.48) which we can assume to have a unique solution. Consequently $\blacksquare\hat{K}/q = \hat{K}$ and one obtains

THEOREM 5.22. *Given a connection* $\varphi_\lambda^Q(y) = \cos \lambda y + \int_0^y \hat{K}(y, x)\cos \lambda x\,dx$ *as indicated it follows that* B *with kernel* $\beta(y, x) = \delta(x - y) + \hat{K}(y, x)$ *is a transmutation* $B\colon D^2 \to Q$ *(acting on functions with* $f'(0) = 0$*). The kernel* $\hat{K}$ *satisfies the Goursat problem* $Q(D_y)\hat{K}(y, x) = D_x^2\hat{K}(y, x)$ *with* $q(y) = 2D_y\hat{K}(y, y)$ *and* $\hat{K}_x(y, 0) = 0$.

We note also that if in fact we express $\beta(y, x) = \langle\varphi_\lambda^Q(y), \cos \lambda x\rangle_\nu$ then $\hat{K}(y, x)$ automatically satisfies the differential equation in Theorem 5.22 for $y \neq x$ (and $\hat{K}_x(y, 0) = 0$). In any event we see that connections of the type with which we are dealing automatically are transmutation formulas.

COMMENT 5.23. We see that there are various ways of characterizing or thinking of transmutation kernels. One can begin with Cauchy problems as in Theorem 2.1 or one can begin with a suitable eigenfunction connection (obtained via Paley–Wiener ideas or wherever) and show via a GL equation and a Goursat problem for the kernel that the connection represents a transmutation acting on suitable objects. The Goursat problem is of course exactly what arises if one simply writes out formally the condition for $QBf = BPf$ acting on suitable f. We have seen also that our eigenfunction connection kernels are characterized by minimization. Formally we note also that if $f(x) = \langle Pf(\lambda), \varphi_\lambda^P(x)\rangle_\nu$ and $B\colon \varphi_\lambda^P \to \varphi_\lambda^Q$ then whenever the spectral integrals make sense $BPf = QBf$ (cf. Remark 6.19).

REMARK 5.24. One can give still another interpretation of the minimization procedure for Ξ. Thus, for simplicity take $Q = D^2 - q$ and $P = D^2$ with $B\colon P \to Q\colon \cos \lambda x \to \varphi_\lambda^Q$, $\beta(y, x) = \delta(x - y) + \hat{K}(y, x)$, $\tilde{\beta}(y, x) = \delta(x - y) + \tilde{K}(y, x)$, and $\gamma(x, y) = \delta(x - y) + L(x, y)$ where $L(x, y) = \tilde{K}(y, x)$. We think of $\hat{f}(\lambda, y) = \varphi_\lambda^Q(y) - \cos \lambda y - \langle\hat{\mathfrak{K}}(y, \xi), \cos \lambda\xi\rangle$ in an expression like (5.37) as $\hat{f}(\lambda, y) = \mathcal{Q}f(x, y)$ so that by Parseval relations of the form (2.20) for example (with $\mathcal{Q} = \mathfrak{Q}$ and $\Delta_Q = 1$)

$$\Xi = \int_0^T \int_0^\infty |\hat{f}(\lambda, y)|^2\,d\omega\,dy = \int_0^T \int_0^\infty |f(x, y)|^2\,dx\,dy. \tag{5.49}$$

Thus,

$$f(x, y) = \langle \hat{f}(\lambda, y), \varphi_\lambda^Q(x)\rangle_\omega$$
$$= \langle \varphi_\lambda^Q(y), \varphi_\lambda^Q(x)\rangle_\omega - \langle \cos\lambda y, \varphi_\lambda^Q(x)\rangle_\omega - \langle \hat{\mathfrak{K}}(y, \xi), \langle \cos\lambda\xi, \varphi_\lambda^Q(x)\rangle_\omega\rangle.$$

Since

$$\langle \cos\lambda y, \varphi_\lambda^Q(x)\rangle_\omega = \delta(x-y) + \tilde{K}(x, y)$$

one obtains

$$f(x, y) = -L(y, x) - \hat{\mathfrak{K}}(y, x) - \langle \hat{\mathfrak{K}}(y, \xi), L(\xi, x)\rangle.$$

But $(1+\hat{K})^{-1} = 1 + L$ which means that for $x < y$

$$0 = L(y, x) + \hat{K}(y, x) + \int_x^y \hat{K}(y, \xi) L(\xi, x)\, d\xi \tag{5.50}$$

(thus $f(x, y) = 0$ for $\hat{\mathfrak{K}} = \hat{K}$). Consider then formally, for

$$\langle \hat{\mathfrak{K}}(y, \xi), L(\xi, x)\rangle = \int_x^y \hat{\mathfrak{K}}(y, \xi) L(\xi, x)\, d\xi,$$

$$\Xi = \int_0^T \int_0^\infty [L + \hat{\mathfrak{K}} + \hat{\mathfrak{K}} L]^2\, dx\, dy$$
$$= 2\,\mathrm{Tr}[(L + LL^*) + \hat{\mathfrak{K}} L]\hat{\mathfrak{K}}^* + \mathrm{Tr}\, LL^* + \mathrm{Tr}[\hat{\mathfrak{K}} + \hat{\mathfrak{K}} LL^*]\hat{\mathfrak{K}}^*.$$

Setting $\hat{\mathfrak{K}} = \hat{\mathfrak{K}}_0 + \epsilon J$, and using the invertibility of $1 + L$, one obtains

THEOREM 5.25. *Minimization of* Ξ *via* (5.49) *leads to the criterion* (5.50) $\hat{\mathfrak{K}}_0 + L + \hat{K}_0 L = 0$, *which characterizes* K.

REMARK 5.26. In terms of classifying 'all' transmutations between say $P = D^2$ and $Q = D^2 - q$, one thinks first of the B_h in Theorem 2.2 with (*) $\varphi_{\lambda,h}^Q(y) = \cos\lambda y + \int_0^y K_h(y, x)\cos\lambda x\, dx$ $(h \neq \infty)$ and the transmutation B_∞ given via $\theta_\lambda^Q(y) = [\sin\lambda y/\lambda] + \int_0^y K_\infty(y, x)[\sin\lambda x/\lambda]\, dx$ (cf. (3.3)). The boundary conditions $\varphi_{\lambda,h}^Q(0) = 1$ and $D_x\varphi_{\lambda,h}^Q(0) = h$ $(h \neq \infty)$ plus $\theta_\lambda^Q(0) = 0$ with $D_x\theta_\lambda^Q(0) = 1$ exhaust the possible domains determined by boundary conditions. (Recall also that for $h \neq \infty$ the transmutation B_h is really taking $\cos\lambda x + h[\sin\lambda x/\lambda] \to \varphi_{\lambda,h}^Q - K_h$ contains a factor h.) On the other hand, if we think of $B_h : \cos\lambda x \to \varphi_{\lambda,h}^Q$ then we have an invertible transmutation for any $h \neq \infty$ with kernel $K_h + \delta$ which mixes the boundary conditions. However, one has a different kind of transmutation situation if it is required to mix $\cos\lambda x$ and θ_λ^Q or $[\sin\lambda x/\lambda]$ and $\varphi_{\lambda,h}^Q$. To see this take $\cos\lambda x = \varphi_\lambda^Q(x) + \int_0^x L(x, \xi)\varphi_\lambda^Q(\xi)\, d\xi$, for example, $(\varphi_\lambda^Q = \varphi_{\lambda,0}^Q$ and $L = L_0)$ and integrate to get

$$[\sin\lambda y/\lambda] = \int_0^y \varphi_\lambda^Q(\xi)\left[1 + \int_\xi^y L(x, \xi)\, dx\right] d\xi = \int_0^y \varphi_\lambda^Q(\xi) l(x, \xi)\, d\xi. \tag{5.51}$$

The kernel l does not have the form $\delta(x-\xi) + \ell(x, \xi)$; it is a smoothing kernel (cf. (5.5)) and its inverse will involve a derivative. To see a typical form for a related inverse look at $[\sin \lambda x/\lambda] = \theta_\lambda^Q(x) + \int_0^x L_\infty(x, \xi)\theta_\lambda^Q(\xi)\, d\xi$ and differentiate to get formally

$$\cos \lambda x = D_x\theta_\lambda^Q(x) + L_\infty(x, x)\theta_\lambda^Q(x) + \int_0^x D_x L_\infty(x, \xi)\theta_\lambda^Q(\xi)\, d\xi \tag{5.52}$$

(which should invert the smoothing transmutation $\cos \lambda x \to \theta_\lambda^Q(y)$). One can see what is happening from looking at spectral integrals. Thus, formally for (5.51) with

$$\check{\mathscr{B}}: [\sin \lambda x/\lambda] \to \varphi_\lambda^Q$$

and

$$\check{B} = \check{\mathscr{B}}^{-1}: \varphi_\lambda^Q \to [\sin \lambda x/\lambda],\ \check{\gamma}(x, y)$$
$$= [\sin \lambda y/\lambda], \varphi_\lambda^Q(x)\rangle_\mu\ (d\mu = (2/\pi)\lambda^2\, d\lambda$$

and

$$\check{\beta}(y, x) = \langle \varphi_\lambda^Q(x), [\sin \lambda y/\lambda]\rangle_\omega\ (d\omega = (2/\pi)\, d\lambda + d\sigma).$$

Taking $P = Q = D^2$ one has

$$\check{\gamma}(x, y) = (2/\pi)\int_0^\infty \lambda \sin \lambda y \cos \lambda x\, d\lambda \sim (2/\pi)\delta'(x - y);$$

$$\check{\beta}(y, x) = (2/\pi)\int_0^\infty \cos \lambda x[\text{Sin}\ \lambda y/\lambda)\, d\lambda = Y(y - x) \quad (= 0 \text{ for } x > y) \tag{5.53}$$

There are other transmutations of course, in particular those obtained by adjointness. Thus, e.g., given $B: \cos \lambda x \to \varphi_\lambda^Q(y)$ one defines $\tilde{B} = (B^{-1})^*$ and if the spectral theory for (Q, φ_λ^Q) involves a measure $d\omega = \hat{\omega}\, d\lambda$ on $[0, \infty)$ then $\tilde{B}[\cos \lambda x] = W(\lambda)\varphi_\lambda^Q(y)$ where $W(\lambda) = \hat{\omega}/\hat{\nu} = (\pi/2)\hat{\omega}$. The kernel for $\tilde{B}$ has the form $\tilde{\beta}(y, x) = \langle \cos \lambda x, \varphi_\lambda^Q(y)\, W(\lambda)\rangle_\nu$ $(d\nu = (2/\pi)\, d\lambda)$ and, more generally, any spectral pairing $\beta_\chi(y, x) = \langle \cos \lambda x, \varphi_\lambda^Q(y)\chi(\lambda)\rangle_\nu$ which makes sense gives rise formally to a transmutation $B_\chi: \cos \lambda x \to \chi(\lambda)\varphi_\lambda^Q(y)$. Indeed, take the domain to be, e.g., functions of the form $f(x) = \langle \hat{f}(\lambda)\chi(\lambda), \cos \lambda x\rangle_\nu$ for suitable $\hat{f} = \mathscr{F}_C f$ and then formally $Bf(y) = \langle \hat{f}\chi, \varphi_\lambda^Q(y)\rangle_\nu$ with $B[\cos \mu x] = \langle \delta(\lambda - \mu), \chi\varphi_\lambda^Q(y)\rangle_\lambda = \chi\varphi_\mu^Q(y)$. This is essentially the same as looking at solutions of Cauchy problems as in Theorem 2.1 with $\varphi(x, y) = \langle \hat{f}(\lambda)\chi(\lambda) \cos \lambda x, \varphi_\lambda^Q(y)\rangle_\nu$ and $\varphi(x, 0) = Af(x) = \langle \hat{f}(\lambda)\chi(\lambda), \cos \lambda x\rangle_\nu = \langle f(\xi), \langle \cos \lambda\xi, \cos \lambda x\chi(\lambda)\rangle_\nu\rangle = \langle f(\xi), A(x, \xi)\rangle$. For $f'(0) = 0$ and $f(x)$ vanishing for large x, one has formally commutativity of A and $P = D^2$. We leave open the question of characterizing all possible transmutations.

6. Special Functions

We have already shown in Section 5 how the transmutation machine can be used to generate information about certain special functions and integral transforms and some further comments about the corresponding operators will appar below. First, however, we go to another class of operators which arise both in physics and in the study of special functions. As prototypical equations here one considers ($\varphi = ru$)

$$\tilde{Q}_r u = r^2 u_{rr} + 2ru_r + r^2[k^2 - \tilde{q}(r)]u = \mu^2 u; \qquad r^2\varphi'' + r^2[k^2 - \tilde{q}(r)]\varphi = \mu^2\varphi; \tag{6.1}$$

$$r^2\Delta_n u + r^2 F(r^2)u = 0. \tag{6.2}$$

Equation (6.1) arises in quantum scattering theory at fixed energy k^2 from a one-dimensional Schrödinger operator and $\mu^2 = l(l+1) = \nu^2 - \frac{1}{4}$ involves (complex) angular momentum (cf. [15, 25, 69, 77, 79, 93, 97, 24, 9, 152, 162, 176, 178, 187, 198, 199, 200]). The equation (6.2) arises in many areas, in particular in three-dimensional scattering problems in acoustics and geophysics, and can be studied via integral operators of Bergman–Gilbert-type for example which involves transmutations in the radial variable cf. [9, 74–76, 104–106, 32, 116, 139, 223]). First consider (6.2) and write ($F = F(r^2)$)

$$\tilde{P}^n u = r^2(r^{n-1}u')'/r^{n-1}; \; \tilde{Q}^n u = \tilde{P}^n u + r^2 Fu. \tag{6.3}$$

The Laplace operator Δ_n has the property that $r^2\Delta_n u = \tilde{P}^n u + \Omega_n^s u$ where Ω_n^s does not depend on r. Hence, if one can transmute via B: $\tilde{P}^n \to \tilde{Q}^n$ it follows that B: $r^2\Delta_n \to r^2\Delta_n + r^2 F$ (since B will commute with Ω_n^s). The Bergman–Gilbert operator has this property in fact but it was originally designed to express some solutions of $(\Delta_n + F)w = 0$ in terms of a solution of $\Delta_n h = 0$. Thus, e.g.,

$$u(r, \cdot) = h(r, \cdot) + \int_0^1 \sigma^{n-1} G(r, 1-\sigma^2) h(\mathbf{x}\sigma^2)\, d\sigma \tag{6.4}$$

($\mathbf{x} = (x_1 \ldots x_n) - n \geq 2$) where G is the Bergman kernel. For our purposes it will be convenient to change variables and use the differential equation for G to express the radial action in the form (for suitable $h = h(r)$)

$$u(r) = h(r) + \langle \rho^{n-3}\check{K}(r, \rho), h(\rho)\rangle \tag{6.5}$$

where $\check{K}(r, \rho) = K(r, \rho)\, Y(r - \rho)$ and for $(\tilde{P}^n_\rho)^*\theta = (\rho^2\theta)'' - (n-1)(\rho\theta)'$,

$$[\tilde{Q}^n_r - (\tilde{P}^n_\rho)^*][\rho^{n-3}\check{K}] = -r^2F(r^2)\delta(r-\rho); \; K(r, r) = -\tfrac{1}{2}\int_0^r F(\rho^2)\rho\, d\rho / r^{n-2}. \tag{6.6}$$

We refer to the operator $\check{B}h = u$ in (6.5) with kernel $\check{B} = \check{\beta}(r, \rho) = \delta(r-\rho) + \rho^{n-3}K(r, \rho)$ as the Bergman–Gilbert (BG) operator and one sees easily via (6.6) that $\check{B}$ is a transmutation, acting on suitable h. We note also, as indicated above, that $\check{B}[\rho^2\Delta_n] = r^2[\Delta_n + F]\check{B}$.

THEOREM 6.1. *The BG operator $\check{B}$ determines a transmutation $\tilde{P}^n \to \tilde{Q}^n$ and the kernel $\check{\beta}(r, \rho)$ can be characterized by*

$$\check{\beta}(r,\rho)=(1/2\pi i)\int_{c-i\infty}^{c+i\infty}\rho^{-\sigma-1}\psi_\sigma^Q(r)\,d\sigma \tag{6.7}$$

where ψ_σ^Q is the solution of $\tilde{Q}^n u=\sigma(\sigma+n-2)u$ with $u\sim r^\sigma$ as $r\to 0$. Further $\check{B}(\rho^\sigma)=\psi_\sigma^Q(r)$ and the integral in (6.7) *arises from an inverse Mellin transform.*

EXAMPLE 6.2. For $n=3$, $\tilde{q}=0$, $\tilde{Q}^n=x^2D^2+2xD+k^2x^2$ (i.e., $F=k^2$) one has for $\nu=\sigma+\frac{1}{2}$ and $z=kx$ ($\nu^2-\frac{1}{4}=\sigma(\sigma+1)$)

$$\psi_\sigma^Q(x)=2^{\sigma+1/2}k^{-\sigma}\Gamma(\sigma+\tfrac{3}{2})z^{-1/2}J_\nu(z), \tag{6.8}$$

$$K(r,\rho)=-(k/2)(\rho/r)^{1/2}J_1[kr(1-\rho/r)^{1/2}]/[1-\rho/r]^{1/2}, \tag{6.9}$$

$$M[\rho K(r,\rho)]=-k^{-\sigma}(kr)^{-1/2}s_{\sigma+3/2,\sigma+1/2}, \tag{6.10}$$

where M denotes Mellin transform and $s_{\alpha,\beta}$ refers to the standard Lommel function. Expressions like (6.9) are related to the study of transmutation kernels as generating functions and will be discussed briefly later (cf. Remark 6.20).

The construction of $\check{K}$ as in (6.6) can be carried out directly without intervention of the Bergman theory. In the same spirit one constructs an analogous exterior transmutation in [74, 76] for example. Thus suppose one wants solutions of $\Delta_n u+k^2(1-q)u=0$ in some exterior region, satisfying the Sommerfeld radiation condition as $r\to\infty$ in the form $\lim r^{(n-1)/2}[D_r u-iku]=0$ as $r\to\infty$. A solution is sought in the form

$$u(r,\cdot)=B_e h(r,\cdot)=h(r,\cdot)+\int_r^\infty s^{n-3}K_e(r,s)h(s,\cdot)\,ds \tag{6.11}$$

where h satisfies the Helmholz equation $(\Delta_n+k^2)h=0$. The kernel K_e can be constructed, for example, by successive approximations as the solution of a Goursat type problem

$$\tilde{Q}_r K_e(r,s)=\tilde{Q}_s^1 K_e(r,s);\ 2r^{n-2}K_e(r,r)=\int_r^\infty k^2 sq(s)\,ds \tag{6.12}$$

where $\tilde{Q}_s^1 u=s^2u''+(n-1)su'+s^2k^2u$ and $\tilde{Q}=\tilde{Q}^n$ with $F=k^2(1-q)$.

THEOREM 6.3. *The kernel constructed from* (6.12) *represents a transmutation $B_e\colon \tilde{Q}^1\to\tilde{Q}$ with $u=B_e h+\langle\check{K}_e(r,s),h(s)\rangle$ (h suitable), and $\check{K}_e(r,s)=s^{n-3}K_e(r,s)Y(s-r)$. This kernel K_e then works in* (6.11).

COMMENT 6.4. The Bergman kernel has played an important role in pure and applied mathematics (see, e.g., [9, 74–76, 104–106, 223]). Kernels of the type indicated in Theorem 6.3 are useful in various acoustical scattering problems (cf. [74–76]). The role of Goursat problems in constructing transmutations is illustrated again in Theorems 6.1 and 6.3.

Consider now (6.1) ($\tilde{q}$ real) and we will follow a method of [25, 24] (note here that the technique of [152, 162] for example introduces an additional parameter, arising in describing self adjoint realizations of $\tilde{Q}$; the kind of results which we want would then contain an unnecessary parameter inextricably bound up in the formulas). We take $\mu^2 = \nu^2 - \frac{1}{4}$ and use ν as spectral parameter. Assumptions such as $\int_0^\infty x|\tilde{q}|\,dx < \infty$ and even $\int_0^\infty x^2|\tilde{q}|\,dx < \infty$ are made but this can be weakened (or strengthened in order to enlarge domains of analyticity). The desideratum here is to obtain the right asymptotic behavior of certain eigenfunctions in order to be able to perform certain calculations. One constructs 'regular' solutions $\varphi(\nu, k, x)$ of the φ equation in (6.1), $\varphi(\nu, k, x) \sim x^{\nu+1/2}$ as $x \to 0$, and Jost solutions $f(\nu, \pm k, x) \sim e^{\mp ikx}$ as $x \to \infty$ (we follow the 'Italian' notation here $f(\nu, -k, x) \sim e^{ikx}$, etc.). These can be constructed via integral equations as for φ_λ^Q, Φ_λ^Q in (3.10)–(3.11). One defines also $f(\nu, \pm k) = W(f(\nu, \pm k, x), \varphi(\nu, k, x))$.

EXAMPLE 6.5. *When* $\tilde{q} = 0$ we have

$$\begin{aligned} &\varphi_0(\nu, k, x) = 2^\nu \Gamma(\nu + 1) k^{-\nu} x^{1/2} J_\nu(kx); \\ &f_0(\nu, -k, x) = (\pi/2)^{1/2} i^{\nu+1/2} (kx)^{1/2} H_\nu^1(kx); \\ &f_0(\nu, -k) = 2^\nu (\pi/2)^{1/2} \Gamma(\nu + 1) k^{-\nu+1/2} i^{\nu-1/2} \end{aligned} \tag{6.13}$$

One can verify easily that f is entire in ν while φ is analytic for Re $\nu > 0$, $\varphi(\nu, -k, x) = \varphi(\nu, k, x)$, $\bar{\varphi}(\bar{\nu}, \bar{k}, x) = \varphi(\nu, k, x)$, $\bar{f}(\bar{\nu}, -\bar{k}, x) = f(\nu, k, x)$, $f(-\nu, k, x) = f(\nu, k, x)$, $f(\nu, -k) f(-\nu, k) - f(\nu, k) f(-\nu, -k) = 4i\nu k$, and setting $f_\pm = f(\nu, \pm k)$ with $f^\pm = f(\pm\nu, k)$

$$\begin{aligned} &\varphi(\nu, k, x) = [f_+ f(\nu, -k, x) - f_- f(\nu, k, x)]/2ik; \\ &f(\nu, k, x) = [f^- \varphi(\nu, k, x) - f^+ \varphi(-\nu, k, x)]/(-2\nu) \end{aligned} \tag{6.14}$$

Upon writing the Green's function as

$$G(\nu, k, r, r') = -\varphi(\nu, k, r_\langle) f(\nu, -k, r_\rangle)/f(\nu, -k) \tag{6.15}$$

($r_\langle = \min(r, r')$ and $r_\rangle = \max(r, r')$) and doing some contour integration, where the asymptotic properties of φ and f in ν for Re $\nu \geqslant 0$ come into play (the prototypical asymptotic behavior is exhibited with Example 6.5 in (6.13)), the spectral measure $d\rho$ for the functions $g = f/r$ can be expressed as

$$\begin{aligned} d\rho(\nu) &= 2i\nu^2\, d\nu / \pi f(\nu, -k) f(-\nu, -k) \quad (\nu \in [0, i\infty)); \\ &= \sum \delta(\nu - \nu_j)/M^2(\nu_j, k) \qquad (\nu \in Z) \end{aligned} \tag{6.16}$$

where Z denotes the zeros (if any) of $f(\nu, -k)$ in Re $\nu > 0$ (M is a normalization factor).

THEOREM 6.6. *Relative to* $d\rho$ *of* (6.16) *and* $g = f/r$ *one has* $\int g(\nu, -k, r) g(\nu, -k, s)\, d\rho = \delta(r - s)$. *Thus* (*for suitable* h) *from* $\mathfrak{G} h(\nu) = \hat{h}(\nu) = \int_0^\infty h(s) g(\nu, -k, s)\, ds$ *one recovers* h *by* $h(r) = \langle \hat{h}(\nu), g(\nu, -k, r) \rangle_\rho$.

Now let $\tilde{Q}_1 = \tilde{Q}^1$ be an operator of type (6.1) with potential $\tilde{q}_1$

THEOREM 6.7. *Define* $\beta(r, s) = \langle g(\nu, -k, r), \; g_1(\nu, -k, s)\rangle_\rho$ *and* $\tilde{\beta}(r, s) = \langle g(\nu, -k, r), \; g_1(\nu, -k, s)\rangle_{\rho^1}$. *Then* $\tilde{B}\colon \tilde{Q}_1 \to \tilde{Q}$ *and* $\tilde{\mathscr{B}} = \tilde{B}^{-1}\colon \tilde{Q} \to \tilde{Q}_1$ *are transmutations with*

$$\tilde{B}[g_1](r) = \langle \tilde{\beta}(r, s), g_1(\nu, -k, s)\rangle = g(\nu, -k, r)$$

and

$$\tilde{\mathscr{B}}[g](s) = \langle \beta(r, s), g(\nu, -k, r)\rangle = g_1(\nu, -k, s).$$

Further, $\beta(r, s) = 0$ *for* $s > r$, $\tilde{\beta}(r, s) = 0$ *for* $r > s$, *and, writing* $\tilde{\beta}(r, s) = \delta(r - s) + \tilde{K}(r, s)$ *for example,* $\tilde{K}(r, s)$ *satisfies a Goursat-type problem (from which intertwining follows):*

$$\tilde{Q}_r \tilde{K}(r, s) = \tilde{Q}_s^1 \tilde{K}(r, s); \quad (D_r + 1/r)\tilde{K}(r, r) = \tfrac{1}{2}[\tilde{q}_1(r) - \tilde{q}(r)].$$

Define also transmutations $B = \tilde{\mathscr{B}}^*$, *so* $Bf(r) = \langle \beta(r, s), f(s)\rangle$, *and* $\mathscr{B} = \tilde{B}^* = B^{-1}$, *so* $\mathscr{B}f(s) = \langle \tilde{\beta}(r, s), f(r)\rangle$. *Then with* $\mathfrak{G}$ *as in Theorem* 6.6 *one has for suitable* f, g: $\mathfrak{G}Bf = \mathfrak{G}_1 f$ *and* $\mathfrak{G}g = \mathfrak{G}_1 \mathscr{B}g$.

Suppose next that $f(\nu, -k)$ has no zeros for $\operatorname{Re} \nu > 0$ so that $d\rho(\nu) = \hat{\rho}(\nu)\, d\nu$ and from properties of $f(\pm \nu, -k, x)$ and $\varphi(\pm\nu, k, x)$ one obtains

THEOREM 6.8. *Given* $f(\nu, -k) = 0$ *for* $\operatorname{Re} \nu > 0$ *with* $d\rho = \hat{\rho}(\nu)\, d\nu$ *one can invert* $\hat{h}(\nu) = \mathfrak{G}h(\nu)$ *from Theorem* 6.6 *in the form* (h *suitable*)

$$h(r) = -(i/\pi) \int_{-i\infty}^{i\infty} \nu \hat{h}(\nu) \Psi(\nu, k, r)\, d\nu \tag{6.17}$$

where $\Psi(\nu, k, r) = \varphi(\nu, k, r)/f(\nu, -k)r$. *If* $\tilde{Q}$ *and* $\tilde{Q}_1$ *have both absolutely continuous spectral measures as above then* $B[g_1(\nu, -k, \cdot)](r) = (\hat{\rho}/\hat{\rho}_1) g(\nu, -k, r)$ *and* $B[\Psi_1(\nu, k, s)](r) = \langle \beta(r, s), \Psi_1(\nu, k, s)\rangle = \Psi(\nu, k, r)$.

There are obvious analogies here with Theorem 5.6 for example. The inversion (6.17) is a generalized Kontorovič–Lebedev (KL) theorem (cf. here [116, 139]) and for $\tilde{Q}$ with $\tilde{q} = 0$ we see from (6.13) that $f(\nu, -k)$ has no zero for $\operatorname{Re} \nu > 0$ while $d\rho(\nu) = -(\nu/\pi k) \sin \pi\nu \, d\nu$. The version of the KL theorem obtained from (6.17) can then be written as

$$\tilde{G}(\nu) = \int_0^\infty G(s) H_\nu^1(ks)\, ds; \qquad rG(r) = \int_{-i\infty}^{i\infty} \tfrac{1}{2}\nu \tilde{G}(\nu) J_\nu(kr)\, d\nu. \tag{6.18}$$

It turns out that the general KL procedure can also be extended to Coulomb potentials, for example, even though the asymptotic behavior used in calculating with (6.15) is somewhat different (see [47] and cf. also [238]). Thus take $\tilde{Q}$ as in (6.1) with $\tilde{q} = a/x$ and set $\nu = l + \frac{1}{2}$ with $\kappa = -a/2ik = in$. Then, with $M_{\kappa,\nu}$ and $W_{\kappa,\nu}$ denoting the standard Whittaker functions, it follows that for the φ equation in (6.1)

$$\varphi_l = (-2ik)^{-l-1} M_{-in,\nu}(-2ikx); \qquad f_l = W_{-in,\nu}(-2ikx). \tag{6.19}$$

The appropriate methods above can be used, with some care required for asymptotic estimates (cf. [136]), and we have a version of (6.17) as follows.

THEOREM 6.9. *Define, for suitable f, a Whittaker transform $\mathfrak{W}f$ and then the inversion* (6.17) *takes the form*

$$\mathfrak{W}f(\nu) = \hat{f}(\nu) = \int_0^\infty f(s)\, W_{-in,\nu}(-2iks)\, ds/s;$$
$$rf(r) = (1/2\pi k) \int_{-i\infty}^{i\infty} \Gamma(\nu + \tfrac{1}{2} + in)/\Gamma(2\nu + 1)\nu \hat{f}(\nu) M_{-in,\nu}(-2ikr)\, d\nu. \tag{6.20}$$

One can also extend the techniques of generalized translation, generalized convolution, Cauchy problems, GL equations, etc. as in Section 5 to the $\tilde{Q}$-type operators and we refer to [50, 55, 62] for details. Let us refer to operators $\tilde{Q}$ and $\tilde{Q}_1$ of the form (6.1), with prototypical asymptotic behavior as in (6.13), and in order to transplant the method of Theorem 2.1 we treat $r = 1$ or $s = 1$ as lines of evaluation (note that 0 is not suitable for evaluation of $\varphi/r \sim r^{\nu - 1/2}$). A change of variables $\eta = \log r$ and $\xi = \log s$ with $rD_r = D_\eta$ etc., transforms the problem $(\blacklozenge)\tilde{Q}^1_s\varphi(r, s) = \tilde{Q}_r\varphi(r, s)$; $\varphi(1, s) = Af(s)$; $D_r\varphi(1, s) = Cf(s)$ $(0 < r, s < \infty)$ into a Cauchy problem for $\eta \geqslant 0$ (or $\eta \leqslant 0$) with data on $-\infty < \xi < \infty$. Thus,

THEOREM 6.10. *Let A and C be linear operators commuting with $\tilde{Q}_1$ and φ be the unique solution of $(\blacklozenge)$. Then $Bf(r) = \varphi(r, 1)$ determines a transmutation $\tilde{Q}_1 \to \tilde{Q}$. Further, given $g(\nu, -k, 1) \neq 0$, this procedure with $\tilde{Q}_1 = \tilde{Q}$ yields a generalized translation $\varphi(x, y) = U(x, y)$, with $A = 1$ and $Cf(s) = \langle \Gamma(s, \sigma), f(\sigma)\rangle$ where*

$$(\hat{f} = \mathfrak{G}f)\, U(r, s) = \langle [\hat{f}(\nu)/g(\nu, -k, 1)],\ g(\nu, -k, r) g(\nu, -k, s)\rangle_\rho$$

and

$$\Gamma(s, \sigma) = \langle g(\nu, -k, s) g(\nu, -k, \sigma), Dg(\nu, -k, 1)/g(\nu, -k, 1)\rangle_\rho$$

with $Cf(1) = f'(1)$.

REMARK 6.11. One can show explicitly how B and $\tilde{B}$ arise in this way via Cauchy problems and produce a GL equation in the form $\beta(r, t) = \langle \tilde{\beta}(r, s), A(s, t)\rangle$ where $A(s, t) = \langle g_1(\nu, -k, s), g_1(\nu, -k, t)\rangle_\rho$. Other machinery and results are also available, corresponding to the previous development in Section 5. For example, one proves (cf. (2.22)), for $T^r_s f(s) = U(r, s)$ as in Theorem 6.10, $\langle T^r_s f(s), h(s)\rangle = \langle f(s), T^r_s h(s)\rangle$ and, setting $(f * h)(r) = \langle T^r_s f(s), h(s)\rangle$, it follows that $(f * h)\hat{} = \hat{f}\hat{h}/g(\nu, -k, 1)$ where $\hat{f} = \mathfrak{G}f$.

COMMENT 6.12. We emphasize again that the theory is based on analyticity and asymptotic properties of various eigenfunctions and the existence of certain transmutation kernels. Potentials are recovered again as in Theorem 6.7 and should probably be relegated to a secondary role. The role of Jost solution and 'regular' solution is interchanged here with the ν spectral variable but the constructions are otherwise basically the same. One is able however to deal with different classes of integral transforms and special functions as in the KL theory (the integral transforms arising from the special functions in Section 5 are, e.g., the Hankel or generalized Mehler transform). After some early interest in scattering at fixed energy in physics it was apparently not pursued too extensively because of the difficulty in extracting physical information from the complex momentum plane (cf. [176].

Another area involving special functions, whose development was only begun in [29, 58], concerns 'elliptic transmutation' and we will sketch some results.

EXAMPLE 6.13. If $P = D^2$ and $Q = -D^2$, one cannot deal with a Cauchy problem as in Theorem 2.1 but a half-plane Dirichlet problem (for suitable data) makes sense. Thus, consider $P(D_x)\varphi = Q(D_y)\varphi$ in the form $\Delta\varphi = 0$ with $\varphi(x, 0) = f(x)$. For $D^2u = -\lambda^2 u$ take $u = \varphi_\lambda^P(x) = \cos\lambda x$ while for $-D^2u = -\lambda^2 u$ one tries, e.g., an 'eigenfunction' $\hat{\varphi}_\lambda^Q(y) = \mathrm{e}^{-\lambda y}$ so at least $\hat{\varphi}_\lambda^Q(0) = 1$. The natural inversion for $\hat{\varphi}_\lambda^Q$ is, of course, the Laplace transform which introduces a 'disparity' in spectra for P and Q. However, given that our eigenfunctions are entire and that identity of spectra is of no importance for transmutation one is only concerned with being able to shift contours appropriately in spectral integrals for kernels etc. in order to utilize the appropriate inversion formulas when needed. In the present situation we try for B: $D^2 \to -D^2$: $\varphi_\lambda^P \to \hat{\varphi}_\lambda^Q$

$$\beta(y, x) = \langle \varphi_\lambda^P(x), \hat{\varphi}_\lambda^Q(y)\rangle_\nu = (2/\pi)\int_0^\infty \cos\lambda x\, \mathrm{e}^{-\lambda y}\,\mathrm{d}\lambda = 2y/\pi(x^2 + y^2) \quad (6.21)$$

Since $T_x^y f(x) = \frac{1}{2}[f(x+y) + f(x-y)]$ is a standard generalized translation for D^2 for the half-plane Dirichlet problem we extend f to be even and consider (cf. (5.2))

$$\varphi(x, y) = \langle \beta(y, \xi), T_\xi^x f(\xi)\rangle = \langle T_\xi^x \beta(y, \xi), f(\xi)\rangle = (4/\pi)\int_{-\infty}^\infty \frac{f(\xi)\,d\xi}{(x-\xi)^2 + y^2} \quad (6.22)$$

$(y > 0)$. This is, of course, the well-known Poisson integral formula. In this connection we recall also that the conjugate harmonic function to φ is

$$\psi(x, y) = (1/\pi)\int_{-\infty}^\infty \frac{(x-\xi)f(\xi)\,\mathrm{d}\xi}{(x-\xi)^2 + y^2} \quad (6.23)$$

and as $y \to 0$, $\psi(x, y) \to (1/\pi)[Pf(1/\xi) * f](x) = -\mathfrak{H}f$ where $\mathfrak{H}$ denotes the Hilbert transform (cf. [22, 180, 181, 224] for Hilbert transforms in a distribution context).

The above 'formulation' can in fact be generalized in a canonical way and one obtains for example most of generalized axially symmetric potential theory (GASPT) in a transmutational (and hence extendable!) manner (cf. [104, 85, 182, 183, 227–234]).

Thus take $P = P_m = D^2 + [(2m+1)/x]D(m > -\frac{1}{2})$ and $Q = -D^2$ with

$$\beta(y, x) = \langle \Omega_\lambda^P(x)\hat{\varphi}_\lambda^Q(y)\rangle_\nu = c_m \int_0^\infty (\lambda x)^{m+1} J_m(\lambda x)\, e^{-\lambda y}\, d\lambda = \frac{k_m y x^{2m+1}}{(x^2+y^2)^{m+3/2}} \tag{6.24}$$

$T_x^y \sim P_m$ is known (cf. (5.8)) and one obtains

THEOREM 6.14. *With $\beta(y, x)$ given by* (6.24) *and T_x^y by* (5.8) *the function $\varphi(x, y) = \langle \beta(y, \xi), T_\xi^x f(\xi)\rangle$ satisfies $P_m(D_x)\varphi = -D_y^2\varphi$ with $\varphi(x, 0) = f(x)$ and φ can be written as $\varphi(x, y) = \int_0^\infty P(y, x, \xi)\xi^{2m+1} f(\xi)\, d\xi$ where $P(y, x, \xi) = T_\xi^x[\beta(y, \xi)\xi^{-2m-1}] = (\xi x)^{-m}. \int_0^\infty e^{-yt} J_m(xt) J_m(\xi t) t\, dt$.*

One is led now to GASPT and pseudoanalytic functions in a canonical transmutational framework (cf. [10, 85, 104, 182, 183, 223, 227–234]). Thus conjugate to $\varphi(x, y)$ in Theorem 6.14 is a function

$$\psi(x, y) = x^{2m+1} \int_0^\infty Q(y, x, \xi) f(\xi)\xi^{2m+1}\, d\xi$$

(Q is constructed below canonically) and (φ, ψ) satisfy the generalized Cauchy–Riemann equations

$$\psi_x = x^{2m+1}\varphi_y; \qquad \psi_y = -x^{2m+1}\varphi_x. \tag{6.25}$$

To produce the 'intrinsic' linking between $P(y, x, \xi)$ and $Q(y, x, \xi)$ we set $\hat{\chi}_\lambda^Q(y) = e^{-\lambda y}/\lambda$ so $\hat{\chi}_\lambda^Q(0) = 1/\lambda$ and $D_y\hat{\chi}_\lambda^Q(0) = -1$. Define a formal transmutation B: $P = P_m \to -D^2 = Q$ via

$$\hat{\beta}(y, x) = \langle \Omega_\lambda^P(x), \hat{\chi}_\lambda^Q(y)\rangle_\nu = \Gamma(m+\tfrac{1}{2})x^{2m+1}/\sqrt{\pi}\,\Gamma(m+1)(x^2+y^2)^{m+1/2} \tag{6.26}$$

$(Bf(y) = \langle \beta(y, x), f(x)\rangle$; $B\varphi_\lambda^P = \hat{\chi}_\lambda^Q)$. Then one has

THEOREM 6.15. *Define*

$$\mathfrak{G}(\xi, x, y) = T_\xi^x[\beta(y, \xi)\xi^{-2m-1}] = (\xi x)^{-m} \int_0^\infty J_m(\xi t) J_m(xt)\, e^{-yt}\, dt.$$

Then $P(y, x, \xi) = -D_y\mathfrak{G}(\xi, x, y)$ and $Q(y, x, \xi) = D_x\mathfrak{G}(\xi, x, y)$. One has $P_x\mathfrak{G} = Q_y\mathfrak{G}$ and $\mathfrak{G}$ is a 'fundamental' solution with a logarithmic singularity at $(y = 0, x = \xi)$. Explicitly (Q_μ denotes the Legendre function of second kind) $\mathfrak{G}(\xi, x, y) = (1/\pi)(\xi x)^{-m-1/2} Q_{m-1/2}[(x^2+y^2+\xi^2)/2\xi x]$.

REMARK 6.16. The conjugate kernel leads to a general Hilbert transform which has been studied extensively in connection with Erdélyi–Kober (EK) operators (cf. [172, 192–194, 224]). Thus, the so-called conjugate Hankel transform is

$$H_m f(x) = \lim_{y \to 0} \int_0^\infty Q(y, x, \xi) f(\xi)\xi^{2m+1}\, d\xi. \tag{6.27}$$

The canonical transmutational form of φ and ψ above for $P = P_m$ suggests the development of other such Hilbert transforms as P varies. Another type of generalized Hilbert transform studied in [10, 28, 81, 138] arises when transmuting $Q = -D^2$ into $P = P_m$ for $m < -\frac{1}{2}$; the canonical formulation is indicated in [29].

COMMENT 6.17. GASPT was an important area of work in the 50s and 60s under the aegis of A. Weinstein (cf. [10, 85, 104, 182, 183, 223, 227–234]). There are numerous applications to physics and one imagines similar applications available from our canonical formulation in various Riemannian symmetric spaces. There should also be considerable value in further adapting transmutation methods by the study of EK type operators.

Let us mention here in connection with special functions some transmutation type operators of EK type which arises when studying certain singular pseudo-differential (psdo) operators in a canonical way (cf. [29, 92, 134, 137, 169, 211]). Thus, let $a(x, \lambda)$ be a classical symbol acting by Fourier transform (i.e., $A(x, D)u = (1/2\pi)\int_{-\infty}^{\infty} e^{i\lambda x} a(x, \lambda)\mathcal{F}u(\lambda)\, d\lambda$ where $\mathcal{F}u = \langle u(\xi), e^{-i\lambda\xi}\rangle$ here – cf. [221]). In view of the half-line (radial) nature of our operators, take $a(x, \lambda)$ even in x (and for convenience say even in λ – odd in λ is also a canonical situation). In a standard way one can also assume $a(x, \lambda)$ of compact support in x and 0 for say $|\lambda| \le \frac{1}{2}$. Let $\mathcal{F}_C$ denote the cosine transform and $B_Q: D^2 \to Q$: $\cos \lambda x \to \varphi_\lambda^Q$ our standard transmutation (here Q can be as in (2.16) with $R^Q \sim d\omega = \hat{\omega}\, d\lambda$). Define now $\check{a}(\zeta, \lambda) = (1/\pi)\mathcal{F}_C a(x, \lambda)(x \to \zeta)$ and $\tilde{a}(\zeta, \lambda) = \check{a}(\xi - \lambda, \lambda) + \check{a}(\zeta + \lambda, \lambda)$. One sets then ($\tilde{B}_Q = (B_Q^{-1})^*$ as in (5.11))

$$\hat{A}(y, Q) = B_Q A(x, DB_Q^{-1}; \qquad \tilde{A}(y, Q) = \tilde{B}_Q A(x, D)\tilde{B}_Q^{-1}. \tag{6.28}$$

THEOREM 6.18. *Let $\tilde{u} = \mathfrak{Q}u$ and $d\omega_Q = \hat{\omega}\, d\lambda$. Then*

$$\hat{A}(y, Q) = \int_0^\infty \int_0^\infty \tilde{a}(\zeta, \lambda)\hat{\omega}(\lambda)\varphi_\zeta^Q(y)\tilde{u}(\lambda)\; d\zeta\, d\lambda;$$

$$\tilde{A}(y, Q) = \int_0^\infty \int_0^\infty \tilde{a}(\zeta, \lambda)\hat{\omega}(\zeta)\varphi_\zeta^Q(y)\tilde{u}(\lambda)\; d\zeta\, d\lambda.$$

Further if one takes $Q = Q_m$ ($\Delta_Q = x^{2m+1}$, $q = 0$, and $\hat{\omega}(\lambda) = c_m^2\lambda^{2m+1}$) then operators of the form $A_\mu(y, Q) = c_m^2 \int_0^\infty \int_0^\infty \tilde{a}(\zeta, \lambda)\varphi_\zeta^Q(y)\tilde{u}(\lambda)\lambda^\mu \zeta^{2m+1-\mu}\, d\zeta\, d\lambda$ correspond to operators $\Pi_\mu A(y, D)\Pi_\mu^{-1}$ where $\Pi_\mu = \mathfrak{Q}\lambda^{-\mu}\mathcal{F}_C$ ($\Pi_{2m+1} = (\pi c_m^2/2)B_Q$ and $\Pi_0 = \tilde{B}_Q$). Thus, Π_μ 'interpolates' between B_Q and $\tilde{B}_Q$ and $\Pi_{m+1/2}$, for example, is an EK operator which is a sum of a Riemann–Liouville and a Weyl-type fractional integral.

REMARK 6.19. In connection with interpolation let P and Q be general operators with $d\nu = \hat{\nu}\, d\lambda$ and $d\omega = \hat{\omega}\, d\lambda$ and $W(\lambda) = \hat{\omega}/\hat{\nu}$. Define say $B_\alpha: P \to Q$ via

$$\beta_\alpha(y, x) = \langle \Omega_\lambda^P(x),\; W(\lambda)^\alpha \varphi_\lambda^Q(y)\rangle_\nu. \tag{6.29}$$

Thus $\beta_0(y, x) = \beta(y, x)$ and $\beta_1(y, x) = \tilde{\beta}(y, x)$. If we take $f(x) = \langle \hat{f}(\lambda), \varphi_\lambda^P(x)\rangle_\nu$

with say $\hat{f}(\lambda) = \mathfrak{P}f(\lambda)$ then since formally $\langle \Omega_\lambda^P(x), \varphi_\mu^P(x)\rangle = \delta(\lambda-\mu)/\hat{v}(\lambda)$ we have first

$$(B_\alpha \varphi_\mu^P)(y) = \langle W(\lambda)^\alpha \varphi_\lambda^Q(y), \delta(\lambda-\mu)/\hat{v}(\lambda)\rangle_\nu = W(\mu)^\alpha \varphi_\mu^Q(y)$$

and then formally (cf. Remark 5.19)

$$QB_\alpha f = QB_\alpha \langle \hat{f}(\lambda), \varphi_\lambda^P(x)\rangle = Q\langle \hat{f}(\lambda), W^\alpha \varphi_\lambda^Q(y)\rangle = \langle \hat{f}(\lambda)\, W^\alpha(\lambda), -\lambda^2 \varphi_\lambda^Q(y)\rangle_\nu$$

while evidently

$$B_\alpha Pf = B_\alpha \langle \hat{f}(\lambda), -\lambda^2 \varphi_\lambda^P(x)\rangle_\nu = \langle \hat{f}(\lambda), -\lambda^2 W^\alpha \varphi_\lambda^Q(y)\rangle_\nu.$$

Thus, formally B_α is a transmutation, acting on f as indicated such that the pairings make sense. Generally, however, W^α will not have any useful analyticity to exploit for triangularity-type results – except for $\alpha = 0$ or 1. For $P = D^2$ with $\hat{v} = 2/\pi$ and $Q = Q_m$ with $\hat{\omega} = c_m^2 \lambda^{2m+1}$ we have $W(\lambda) = (c_m^2 \pi/2)\lambda^{2m+1} = a_m \lambda^{2m+1}$. Now generally $B_\alpha = \mathfrak{Q} W^\alpha \mathfrak{P} = \mathfrak{O} W^{\alpha-1}\mathfrak{P}$ and, in particular, for D^2 and Q_m, $B_\alpha = \mathfrak{O}(a_m \lambda^{(2m+1)(\alpha-1)})\mathfrak{P}$ so (since $\mathfrak{P} = \mathscr{F}_C$ here) up to a constant $B_\alpha \sim \Pi_\mu$ in Theorem 6.18 with $-\mu = (2m+1)(\alpha-1)$ (thus, $\mu = m + \frac{1}{2} \sim \alpha = \frac{1}{2}$). We note also here that the objects on which $\tilde{B}$ and $\check{B}$ act as transmutations can be represented in the form $f(x) = (1/2\pi)\int_{-\infty}^{\infty} \hat{f}(\lambda)\Psi_\lambda^P(x)\, d\lambda$ where $\hat{f}(\lambda) = \mathfrak{P}f(\lambda)$ (recall here (5.22)). Thus, $\tilde{B}\Psi_\lambda^P = \Psi_\lambda^Q$ and $\check{B}\Phi_\lambda^P = \Phi_\lambda^Q$ so as above (cf. also Remark 5.19)

$$(\tilde{B}f)(y) = (1/2\pi)\int_{-\infty}^{\infty} \hat{f}(\lambda)\Psi_\lambda^Q(y)\, d\lambda$$

with

$$(\check{B}f)(y) = (1/2\pi)\int_{-\infty}^{\infty} \hat{f}(\lambda)[\Phi_\lambda^Q(y)/c_P(-\lambda)]\, d\lambda$$

and when everything makes sense $Q\tilde{B}f = \tilde{B}Pf$ with $Q\check{B}f = \check{B}Pf$.

REMARK 6.20. We indicate next a few results (cf. [53]) about generating functions arising in a transmutation context as transmutation kernels. Thus, in Example 6.2 one can write for example (cf. [162, 198])

$$K(r,\rho) = -(r\rho)^{-1/2}\sum_{1}^{\infty} (k\rho/2)^m J_m(kr)/(m-1)! \tag{6.30}$$

which exhibits K as a generating function for the Bessel functions. This sort of situation will often occur when using the Mellin transform and is related to interpolation formulas in complex analysis. As an analogue of (6.30), we consider the Whittaker inversion of Theorem 6.9. One proceeds formally via contour integration and residue theorems to obtain an apparently new generating function for certain Whittaker functions. Thus, one has a transmutation kernel ($\tilde{P} = D(r^2D)$, $\tilde{Q} = \tilde{P} + r^2(k^2 - a/r)$)

$$\hat{\beta}(r,\rho) = (1/2\pi i r)\int (2ik\rho)^{-\sigma-1}\mathfrak{M}_{\kappa,\sigma+1/2}(2ikr)\Gamma(2\sigma+2)\,d\sigma \tag{6.31}$$

where $\mathfrak{M}_{\kappa,\delta}(z) = M_{\kappa,\delta}(z)/\Gamma(1+2\delta)$. Consequently, $(\hat{\rho} = 2ik\rho,\ \hat{r} = 2ikr)$ formally

$$\hat{\beta}(r,\rho) = (1/2r)\sum_0^\infty \hat{\rho}^{n/2}(-1)^n\mathfrak{M}_{\kappa,1/2(n+1)}(\hat{r})\Gamma(\tfrac{1}{2}(n+2)-\kappa)/n!\Gamma(-\tfrac{1}{2}n-\kappa). \tag{6.32}$$

This procedure can be extended, of course, to obtain new types of generating functions for $\check{\psi}_m^Q(r)$ of type $\sum c_m\check{\psi}_m^P(\rho)\check{\psi}_m^Q(r)$ where $\check{\psi}_m^P(\rho)$ is no longer a power of ρ. Connections of the transmutation approach to the classical developments in [198, 170, 184, 225, 226] will also appear in [53]. Another feature of [53] is to give a complete generalization of the relation (6.30) involving $K(r,\rho)$ of (6.9) $(\check{\beta} = \delta + K(r,\rho)Y(r-\rho))$ and $(\check{\beta}^{-1} \sim \check{\gamma} = \delta + L(r,\rho)Y(r-\rho))L(r,\rho) = (k/2)[1-\rho/r]^{-1/2}I_1[k(\rho/r)^{1/2}(1-\rho/r)^{1/2}]$. Recall $\check{\beta}$ is given by (6.7) and using our general KL method one obtains for the model problem of Examples 6.2 and 6.5

$$\begin{aligned}\check{\gamma}(r,\rho) &= \int_{-i\infty}^{i\infty} r^\sigma(-i\nu/\pi)g_0(\nu,-k,\rho)\,d\nu/f_0(\nu,-k)\\ &= (1/2\rho^{1/2})\int_{-i\infty}^{i\infty}(k/2)^\nu r^\sigma H_\nu^1(k\rho)\,d\nu/\Gamma(\nu)\end{aligned}$$

Writing $\check{\gamma}(r,\rho) = (1/2\pi i)(r\rho)^{-1/2}\int_{-i\infty}^{i\infty}(kr/2)^\nu\Gamma(1-\nu)[J_{-\nu}(k\rho) - e^{-i\pi\nu}J_\nu(k\rho)]\,d\nu$ and evaluating this by residues at $1-\nu = -n$ $(\nu = n+1)$ one obtains $(\bigstar)\check{\gamma}(r,\rho) = (r\rho)^{-1/2}\sum_1^\infty(kr/2)^m J_m(k\rho)/(m-1)! = L(r,\rho)$ for $\rho < r$. Thus (note that $\check{\beta}(\rho,r) + L(r,\rho) = 0$ for $\rho > r$ so $\check{\gamma}(r,\rho)$ is triangular and $\check{\beta}(\rho,r)$ contributes a term $\delta(r-\rho)$) for our model operator of Example 6.5 with $\check{\beta}(r,\rho) = \ker\check{B}$, $\check{B}$: $\tilde{P} \to \tilde{Q}_0$, given as $\delta(r-\rho) + K(r,\rho)Y(r-\rho)$, one has the expansion (6.30) expressing $K(r,\rho)$ as a generating function for $\rho < r$ and the kernel $\check{\gamma}(r,\rho)$ for $\check{\mathscr{B}} = \check{B}^{-1}$ can be constructed via KL theory in the form (6.33) from which

$$\check{\gamma}(r,\rho) = \delta(r-\rho) - [(r\rho)^{-1/2}/2\pi i]\int_{-i\infty}^{i\infty}(kr/2)^\nu\Gamma(1-\nu)\,e^{-i\pi\nu}J_\nu(k\rho)\,d\nu Y(r-\rho).$$

The latter integral represents $\check{\gamma}(r,\rho)$ as a generating function for $\rho < r$ via $(\bigstar)$ above. One can generalize this situation as follows. Let f, φ, etc. refer to a general $\tilde{Q}$ type operator and set $(\bullet\bullet)$ $\check{\beta}(r,\rho) = (1/2\pi i)\int\rho^{-\sigma-1}\varphi(\nu,k,r)\,d\sigma/r$ with $(\blacktriangle\blacktriangle)$ $\check{\gamma}(r,\rho) = -(i/\pi)\int_{-i\infty}^{i\infty}\nu r^\sigma f(\nu,-k,\rho)\,d\nu/\rho f(\nu,-k)$. Note from $\varphi(\nu,k,r) \sim r^{\nu+1/2}$ for large ν we see that $\check{\beta}(r,\rho) = 0$ for $\rho > r$. Now using (6.14) for $-k$ one obtains

$$\check{\gamma}(r,\rho) = (i/2\pi)\int_{-i\infty}^{i\infty} r^\sigma[\{f(-\nu,-k)\varphi(\nu,k,\rho)/\rho f(\nu,-k)\} - \varphi(-\nu,k,\rho)/\rho]\,d\nu. \tag{6.34}$$

For $\rho < r$ (where $\check{\beta}(\rho, r) = 0$) we can write

$$(\blacktriangle)\,\check{\gamma}(r, \rho) = -(1/2\pi i)\int_{-i\infty}^{i\infty} r^{\nu-1/2}[f(-\nu, -k)/f(\nu, -k)]\varphi(\nu, k, \rho)\,d\nu/\rho$$
$$= L(r, \rho).$$

We recall $f(\nu, -k)$ is analytic for Re $\nu > 0$ so formally ($\blacktriangle$) can be evaluated by residues upon closing contours to the right – provided $f(\nu, -k) \neq 0$ for Re $\nu > 0$, which is the requirement for absolutely continuous spectrum. Recall here $f(-\nu, -k) = W(f(-\nu, -k, x), \varphi(-\nu, k, x))$ will have poles determined by those of $\varphi(-\nu, k, x)$ for Re $\nu > 0$ (since $f(-\nu, -k, x)$ is analytic in ν). Various situations are possible (cf. [11, 79, 97, 178]). Typically in physics (simple) poles can occur for $\varphi(-\nu, k, x)$ when $1 - 2\nu = -n$ ($n = 0, 1, \ldots$) and if, e.g., $x\tilde{q}(x)$ is analytic at $x = 0$ then the only poles are at $\nu = \frac{1}{2}, \frac{3}{2}, \ldots$ (there are other situations when, e.g., the only poles are at $\nu = 1, 2, \ldots$). Thus write for such typical situations $\hat{f}_-^n = \lim(\nu \to \nu_n) f(-\nu, -k)(\nu - \nu_n)$. Then

THEOREM 6.21. *Under the hypotheses indicated let $\check{B}$: $\tilde{P} \to \tilde{Q}$ be given formally via the kernel* ($\bullet\bullet$), *$\check{B}$: $\rho^\sigma \to \check{\psi}_\sigma^Q(r)$; then the kernel $\check{\gamma}(r, \rho)$ for $\mathcal{B} = \check{B}^{-1}$ is given by* ($\blacktriangle\blacktriangle$) *and with $L(r, \rho)$ as in* ($\blacktriangle$)$\check{\gamma}(r, \rho) = \delta(r - \rho) + L(r, \rho)Y(r - \rho)$ *so that for $\rho < r$, $\check{\gamma}(r, \rho)$ is expressed via the integral for $L(r, \rho)$ in* ($\blacktriangle$) *as a generating function $L(r, \rho) = \sum_0^\infty r^{n/2}\hat{f}_-^n\varphi(\nu_n, k, \rho)/\rho f(\nu_n, -k)$.*

Another development of GL type arises as follows. Define $(-\nu - \frac{1}{2} = -\sigma - 1)$

$$\begin{aligned}\mathfrak{A}(\xi, \rho) &= (i/2\pi)\int_{-i\infty}^{i\infty} \rho^\sigma[\xi^{\nu-1/2} f(-\nu, -k)/f(\nu, -k) - \xi^{-\nu-1/2}]\,d\nu = \mathfrak{A}(\rho, \xi)\\ &= (i/2\pi)\int_{-i\infty}^{i\infty} \xi^\sigma[\rho^\sigma f(-\nu, -k)/f(\nu, -k) - \rho^{-\nu-1/2}]\,d\nu.\end{aligned} \tag{6.35}$$

THEOREM 6.22. *From $\check{\psi}_\sigma^Q(r) = \langle \beta(r, \rho), \rho^\sigma\rangle$ there is a general GL equation (for $\tilde{P}$ and $\tilde{Q}$) in the form $\check{\gamma}(\xi, r) = \langle \check{\beta}(r, \rho), \mathfrak{A}(\rho, \xi)\rangle$ where $\mathfrak{A}$ is given by* (6.35), *$\check{\beta}$ by* ($\bullet\bullet$), *and $\check{\gamma}$ by* ($\blacktriangle\blacktriangle$). *Further*

$$\begin{aligned}\mathfrak{A}(\eta, r) &= (1/2\pi i)\int_{-i\infty}^{i\infty} \eta^\sigma r^{-\sigma-1}\,d\nu + F(\eta, r) = \delta(\eta - r) + F(\eta, r);\\ F(\eta, r) &= -(1/2\pi i)\int_{-i\infty}^{i\infty} \eta^\sigma r^\sigma[f(-\nu, -k)/f(\nu, -k)]\,d\nu\end{aligned} \tag{6.36}$$

In typical situations as above we have then $F(\eta, r) = \sum_0^\infty (\eta r)^{n/2}\hat{f}_-^n/f(\nu_n, -k)$.

We go now to some new results related to orthogonal polynomials. There are various treatments of orthogonal polynomials of which we mention [5, 72, 96, 100–102, 219]. One way of developing the theory (cf. [72]) is to start with a weight function $\hat{\mu}(x)$ (or suitable $d\mu(x) = \hat{\mu}\,dx$) on say (a, b) and define a moment functional $L(f) = \int_a^b f(x)\,d\mu(x)$. The moments $\mu_n = L(x^n)$ are then known and one looks for polynomials $P_n(x)$ of degree n such that say $\int_a^b P_n(x)P_m(x)\,d\mu(x) = \delta_{mn}$. Various equivalent formulations are possible. For example it is enough to show that (†) $L(\pi(x)P_n(x)) = 0$ for polynomials π of degree $m < n$ while $L(\pi(x)P_n(x)) \neq 0$ for $m = n$. Alternatively, one need only show (††) $L(x^m P_n(x)) = \hat{K}_m \delta_{mn}$ with $\hat{K}_n \neq 0$. Now extend the idea of 'polynomial' and try to produce orthogonal functions relative to a given measure on $[0, \infty)$ for example. The machinery must be different at various places and it turns out that one can magically proceed via transmutation ideas in an intrinsic and canonical manner. This is connected, or course, to the extensions of Krein (continuous analogues of orthogonal polynomials – cf. [5, 67, 100–102, 120, 146, 32]) but our methods have a different flavor and cover more general situations. For the Krein functions let us recall the differential equations in Theorem 4.8; these also arise heuristically from extending certain recursion relations for polynomials orthogonal on the unit circle (cf. [5]). Thus, consider $u_{n+1} = a_n x u_n + b_n v_n$; $v_{n+1} = \bar{b}_n x u_n + a_n v_n$ with say $u_0 = u_0(x) = 1$, $v_0 = v_0(x) = 1$, $a_n = \bar{a}_n$, and $a_n^2 - |b_n|^2 > 0$ ($x = e^{i\theta}$) and extend this to $D_t u = i\lambda u + b(t)v$; $D_t v = \bar{b}(t)u$ as in [5]; this is equivalent to the equations in Theorem 4.8. Generally, we think of $(x, n) \to (\lambda, t)$ in passing from discrete situations to the continuous analogue.

Now the classical techniques for orthogonal polynomials cannot be directly extended since certain procedures simply don't make sense. In order to circumvent this, at the same time give perhaps a more meaningful and canonical procedure, we proceed as follows. Let us work with the analogues of polynomials on the real line or cosine type polynomials (cf. Theorem 4.15) rather than the classical Krein function situation of Theorem 4.8. Thus, in the role of polynomial (of degree t!) we write (this is a model case – the kernel form will be extended later)

$$\pi(\lambda, t) = \cos \lambda t + \int_0^t c(t, s) \cos \lambda s \, ds \tag{6.37}$$

and for moment functional write ($d\mu(\lambda)$ on $[0, \infty)$ being given)

$$L(\cos \lambda t) = \int_0^\infty \cos \lambda t \, d\mu(\lambda) = g(t) \tag{6.38}$$

(one recognizes $g(t)$ as $G'(t)$ in the geophysical problem – cf. (3.21), (6.50), and Theorem 5.4). Note also that a more general $\varphi_\lambda^P(x)$ can be used in the role of $\cos \lambda x$ (see below). Now one expects that the orthogonal functions $f(\lambda, t)$ satisfying $\int_0^\infty f(\lambda, t) f(\lambda, s)\, d\mu(\lambda) = \delta(t-s)$ will arise from a GL kernel $\beta(t, s) = \delta(t-s) + K(t, s)$ for example (in the model case). Thus, one expects

$$f(\lambda, t) = \cos \lambda t + \int_0^t K(t, \tau) \cos \lambda \tau \, d\tau. \tag{6.39}$$

Given that such f exist and are orthogonal one can characterize K, for example by minimization, with a condition analogous to (††) of the form

$$\int_0^\infty \cos \lambda s f(\lambda, t)\, d\mu(\lambda) = 0, \quad \text{for } s < t \tag{6.40}$$

(which says that $\tilde{\beta}(t, s) = \langle f(\lambda, t), \cos \lambda s\rangle_\mu = 0$ for $s < t$). Now using (6.39)–(6.40) one obtains a GL equation; for $s < t$

$$\tilde{\beta}(t, s) = 0 = \langle \cos \lambda t, \cos \lambda s\rangle_\mu + \int_0^t K(t, \tau)\langle \cos \lambda \tau, \cos \lambda s\rangle_\mu\, ds. \tag{6.41}$$

Here one can determine $\mathfrak{A}(t, s) = \langle \cos \lambda t, \cos \lambda s\rangle_\mu$ in terms of the moment functional! Thus, as with the time domain GL equation of the geophysical problem $\mathfrak{A}(t, s) = \delta(t-s) + \Omega(t, s)$ (cf. (3.23)), where we write $d\mu = (2/\pi)\, d\lambda + d\sigma$ (see below for other measures) and (●▲) $\int_0^\infty \cos \lambda t\, d\sigma(\lambda) = g_r(t)$ so that (extending g_r to be even)

$$\Omega(t, s) = \tfrac{1}{2}\int_0^\infty [\cos \lambda(t+s) + \cos \lambda(t-s)]\, d\sigma = \tfrac{1}{2}[g_r(t+s) + g_r(|t-s|)]. \tag{6.42}$$

THEOREM 6.23. *Given* $f(\lambda, t)$ *as in* (6.39) *with* (6.40) *it follows that the kernel* $K(t, s)$ *in* (6.39) *satisfies a GL equation*

$$(s < t), \Omega(t, s) + K(t, s) + \int_0^t K(t, \tau)\Omega(\tau, s)\, d\tau = 0$$

where the symmetric kernel $\Omega(t, s)$ *is given in terms of the moment functional via* (6.42).

This theorem is extended and refined below. For now, given (6.39) and the GL equation, we proceed as in Theorem 5.20. First from (6.39) by writing formally (■▲) $K(t, \tau) = (2/\pi)\int_0^\infty [f(\lambda, t) - \cos \lambda t]\cos \lambda\tau\, d\lambda$ one concludes that $K_\tau(t, 0) = 0$ and then for $\blacksquare K = K_{tt} - K_{\tau\tau}$, $\blacksquare K + g(t)\Omega(t, \tau) + \int_0^t \blacksquare K(t, \xi)\Omega(\xi, \tau)\, d\xi = 0$ where $q(t) = 2D_tK(t, t)$. Note here $\Omega_{tt} = \Omega_{\tau\tau}$ and we assume Ω is say twice continuously differentiable. Given unique solutions of the GL equation we obtain

THEOREM 6.24. *Given the hypotheses of Theorem* 6.23 *with unique solutions of the GL equation and* Ω *twice differentiable one has* $Q(D_t)K(t, \tau) = D_\tau^2 K(t, \tau)$ *where* $Q(D_t) = D_t^2 - q(t)$ *and* $q(t) = 2D_tK(t, t)$ *with* $K_\tau(t, 0) = 0$. *Moreover,* (6.39) *determines a transmutation* B: $D^2 \to Q$ *acting on functions with* $g'(0) = 0$.

We remark here that to determine the a_{nm} such that $P_n(x) = \sum_0^n a_{nm}x^m$ represents the orthogonal polynomials relative to $d\mu$, given say (6.34) one writes ($\mu_k = L(x^k))L(x^kP_n(x)) = K_n\delta_{kn} = \sum_0^n a_{nm}\mu_{k+m}$ which is a kind of discrete GL equation (cf. [67] – a somewhat more general situation can also be treated). On the other hand, we note in (■▲) that existence of K is guaranteed, e.g., if $f(\lambda, t) - \cos \lambda t$ has a general Fourier Cosine transform (for triangularity we want say $f(\lambda, t)$ to be entire of exponential type t etc.). In any event, there is a clear analogy between solving for the GL kernel and a coefficient determining procedure in the discrete case. A direct calculation gives

THEOREM 6.25. *Given* $d\mu$ *one solves the GL equation in Theorem* 6.23 *to get* $K(t, \tau)$ *and then defines* $f(\lambda, t)$ *by* (6.39). *The* $f(\lambda, t)$ *are then orthogonal and in fact for f of the form* (6.39) *the GL equation is equivalent to orthogonality. Further, one has* $Q(D_t)f(\lambda, t) = -\lambda^2 f(\lambda, t)$ *with* $f(\lambda, 0) = 1$ *and* $f'(\lambda, 0) = h = -g_r(0)$.

COMMENT 6.26. In earlier sections we had in a sense too much structure available; in the development here and below one can see better what properties are actually being used and how they relate to one another. There are two sets of functions and two measures to play with in writing e.g., $f = \varphi_\lambda^P + \int_0^t K\varphi_\lambda^P\, dx$, $f \sim d\omega$, $\varphi_\lambda^P \sim d\nu$, etc., (cf. [67]).

In the spirit of Theorem 5.11 for example consider now the minimization

$$\Xi_\omega = \int_0^T \int_0^\infty \left[\pi(\lambda, t) - \cos \lambda t - \int_0^t \mathfrak{K}(t, s) \cos \lambda s\, ds\right]^2 d\omega\, dt \tag{6.43}$$

($\mathfrak{K}$ in a suitable class of causal kernels) where $\pi(\lambda, t)$ is some general function with (◆) $\langle \pi(\lambda, t), \cos \lambda s\rangle_\omega = \delta(t-s) + \tilde{\alpha}(t, s)$. The standard minimization procedure for Ξ_ω yields then (assume $\hat{\Xi}_\omega = \int_0^T \int_0^\infty [\pi(\lambda, t) - \cos \lambda t]^2\, d\omega\, dt$ makes sense)

$$\begin{aligned} \Xi_\omega = \hat{\Xi}_\omega &+ 2\int_0^T \int_0^t \mathfrak{K}(t, s)\Omega(t, s)\, ds\, dt - 2\int_0^T \int_0^t \mathfrak{K}(t, s)\tilde{\alpha}(t, s)\, ds\, dt + \\ &+ \int_0^T \int_0^t \mathfrak{K}(t, s)\mathfrak{K}(t, s)\, ds\, dt + \\ &+ \int_0^T \int_0^t \int_0^t \mathfrak{K}(t, s)\Omega(s, \tau)\mathfrak{K}(t, \tau)\, d\tau\, ds\, dt \end{aligned} \tag{6.44}$$

(here again $\mathfrak{A} = \delta + \Omega$ as above). Hence, as in Section 5.

THEOREM 6.27. *Assume* $\pi(\lambda, t)$ *is a suitable general function satisfying* (◆) *with* $\tilde{\alpha}(t, s) = 0$ *for* $s < t$ *and so that* $\hat{\Xi}_\omega$ *makes sense. Then the best least squares approximation for* Ξ_ω *involves a* $\mathfrak{K}_0(t, s)$ *satisfying the GL equation of Theorem* 6.23 *for* $s < t$ *so that* $\min \Xi_\omega = \int_0^T \int_0^\infty [\pi(\lambda, t) - f(\lambda, t)]^2\, d\omega\, dt$.

Let us observe next what happens upon minimizing

$$\Xi_\nu = \int_0^T \int_0^\infty \left[\pi(\lambda, t) - \cos \lambda t - \int_0^t \mathfrak{K}(t, s) \cos \lambda s\, ds\right]^2 d\nu\, dt \tag{6.45}$$

where $d\nu$ is the $P = D^2$ measure relative to which the $\varphi_\lambda^P(x) = \cos \lambda x$ are orthogonal and $\pi(\lambda, t)$ is again some general function, not necessarily represented in the form (6.39) ($\mathfrak{K}$ runs over a suitable class of causal kernels). Let us write (◆◆) $\delta(t-s) + \alpha(t, s) = \langle \pi(\lambda, t), \cos \lambda s\rangle_\nu$ (α need not be triangular) and set $\hat{\Xi}_\nu = \int_0^T \int_0^\infty [\pi(\lambda, t) - \cos \lambda t]^2\, d\nu\, dt$ (assumed to exist). Writing out Ξ_ν now one obtains

$$\begin{aligned}\Xi_\nu &= \hat{\Xi}_\nu + 2\int_0^T\int_0^t \mathfrak{K}(t,s)\delta(t-s)\,ds\,dt - \\ &\quad -2\int_0^T\int_0^t \mathfrak{K}(t,s)[\delta(t-s)+\alpha(t,s)]\,ds\,dt + \\ &\quad +\int_0^T\int_0^t \mathfrak{K}(t,s)\mathfrak{K}(t,s)\,ds\,dt \\ &= \hat{\Xi}_\nu + \int_0^T\int_0^t [\mathfrak{K}(t,s)-\alpha(t,s)]^2\,ds\,dt - \\ &\quad -\int_0^T\int_0^t \alpha^2(t,s)\,ds\,dt.\end{aligned} \tag{6.46}$$

This is clearly analogous to a procedure characterizing the Fourier coefficients in classical trigonometrical expansions and one has

THEOREM 6.28. *Given a general $\pi(\lambda, t)$ with (◆◆) the coefficients $\mathfrak{K}(t, s) = \alpha(t, s)$ for $s \leq t$ provides the minimum value for Ξ_ν and one has a Bessel inequality for suitable $\pi(\lambda, t)$, $\int_0^T \int_0^t \alpha^2(t, s)\,ds\,dt \leq \int_0^T \int_0^\infty [\pi(\lambda, t) - \cos \lambda t]^2\,d\nu\,dt$.*

This result can be thought of as a best least-squares approximation to $\pi(\lambda, t)$ by entire functions of exponential-type t of the form $\cos \lambda t + \int_0^t \mathfrak{K}(t, s) \cos \lambda s\,ds$. Since essentially any such (even) function has this form one obtains equality in the Bessel inequality for $\pi(\lambda, t) = f(\lambda, t) = \cos \lambda t + \int_0^t \alpha(t, s) \cos \lambda s\,ds$. One notes also that in analogy to the conditions (†) and (††) for 'polynomials' of the form (6.37) the condition $\int_0^\infty \pi(\lambda, s) f(\lambda, t)\,d\omega = 0$ $(s < t)$ is equivalent to (6.40); simply invert the formula (6.37). We remark also that the role of the 'kernel polynomial' $K_n(z, x) = \sum \bar{P}_m(z) P_m(x)$ is played here by $\mathfrak{K}_T(\lambda, \mu) = \int_0^T f(\lambda, t) f(\lambda, t)\,dt$ (cf. earlier remarks on the Darboux–Christoffel formula etc. in (4.40)–(4.41) and Theorem 4.8).

COMMENT 6.29. It should be worthwhile to continue the program of extending ideas from orthogonal polynomial theory to more general orthogonal functions via transmutation machinery (cf. [48]).

The above procedures apply very well when $d\omega = (2/\pi)\,d\lambda + d\sigma$ with $d\sigma$ suitably finite. When $d\omega$ grows more rapidly the situation changes and at least two procedures suggest themselves. As a model take the case of Example 5.1 where $d\omega = \hat{\omega}\,d\lambda$ with $\hat{\omega} = c_m^2 \lambda^{2m+1}$ for $Q = Q_m$. For $B: D^2 \to Q_m$ the transmutation kernel is given by (5.5) (with inverse kernel (5.6)) and clearly $\beta = \beta_Q(y, x)$ does not have the form $\delta(x - y) + K(y, x)$ with K a function. We note that in this situation the kernel $A(\xi, x)$ in the general GL equation of Theorem 5.2 has the form

$$\begin{aligned}A(\xi, x) &= \langle \varphi_\lambda^P(\xi), \Omega_\lambda^P(x)\rangle_\omega = c_m^2 \int_0^\infty \cos \lambda\xi \cos \lambda x \lambda^{2m+1}\,d\lambda \\ &= \tfrac{1}{2}\beta_m[|x-\xi|^{-2m-2} + |x+\xi|^{-2m-2}]\end{aligned} \tag{6.47}$$

where $\beta_m = 2\sqrt{\pi}/\Gamma(m+1)\Gamma(-m-\frac{1}{2})$ and $|y|^{-\beta} \sim y_+^{-\beta} + y_-^{-\beta}$ for pseudofunctions $y_\pm^{-\beta}$. One could now try to duplicate some of the previous machinery in a distribution context where objects such as (5.5), (5.6), and (6.47) were prototypical. Another approach, which we prefer here, is to refer a measure $d\omega$ with growth $\sim\lambda^{2m+1}$ to $Q_m = P$ as a point of departure and replace $\cos\lambda x$ by $\varphi_\lambda^P(x) = (1/c_m)J_m(\lambda x)/(\lambda x)^m$ in the preceding machinery (cf. the material on random fields in Section 4). We note first in this connection that for $d\omega = c_m^2\lambda^{2m+1}[d\lambda + d\sigma]$

$$A(\xi, x) = \delta(x-\xi) + x^{2m+1}\int_0^\infty [J_m(\lambda x)/(\lambda x)^m][J_m(\lambda\xi)/(\lambda\xi)^m]\lambda^{2m+1}\,d\sigma \quad (6.48)$$

(note $\langle\varphi_\lambda^P(\xi), \Omega_\lambda^P(x)\rangle_\nu = \delta(x-\xi)$ for $d\nu = c_m^2\lambda^{2m+1}\,d\lambda$). Thus, for say $d\sigma = \hat{\sigma}\,d\lambda$ with suitable $\hat{\sigma}$, the term $\Omega(\xi, x)\Delta_P(x) = A(\xi, x) - \delta(x-\xi)$ will be a function. One knows also (cf. [29, 34]) that if B: $P \to Q$: $\varphi_\lambda^P \to \varphi_\lambda^Q(\Delta_P = x^{2m+1}$ and $\Delta_Q = x^{2m+1}A_Q$ say) then there is a related transmutation $\dot{B}$: $\dot{P} \to \dot{Q}$ where, e.g., if $Qu = (\Delta_Q u')'/\Delta_Q + q(x)u$,

$$\Delta_Q^{1/2}Qu = \dot{Q}[\Delta_q^{1/2}u]; \qquad \dot{Q}w = w'' + \dot{q}w; \quad \dot{q} = q - \Delta_Q^{-1/2}(\Delta_Q^{1/2})''.$$

Then $\dot{B} = \Delta_Q^{1/2}(y)B(x\to y)x^{-m-1/2}$: $\dot{P} \to \dot{Q}$. Now the function $\dot{q}$ will generally have singularities (e.g., for $P = P_m$, $\dot{p} = -(m^2 - \frac{1}{4})/x^2$) but $\dot{\beta}$ = kernel $\dot{B}$ will have the form $\dot{\beta}(y, x) = \delta(x-y) + \dot{K}(y, x)$ with $\dot{K}$ causal. It follows that $\beta(y, x) = A_Q^{-1/2}(y)\,\delta(x-y) + K(y, x)$ where $K(y, x) = \Delta_Q^{-1/2}(y)\dot{K}(y, x)x^{m+1/2}$, which is reminiscent of the geophysical situation (cf. (3.18)). Further, by Volterra inversion $\gamma(x, y) = A_Q^{1/2}(y)\,\delta(x-y) + L(x, y)$ and since $\tilde{\beta}(y, x) = \Delta_P(x)\Delta_Q^{-1}(y)\gamma(x, y)$ (cf. (5.12)) we have $\tilde{\beta}(y, x) = A_Q^{-1/2}(y)\,\delta(x-y) + \tilde{K}(y, x)$. Thus, given the GL equation in Theorem 5.2 one knows $A(\xi, x) = \delta(x-\xi) + \Delta_P(x)\Omega(\xi, x)$ as in (6.48), even though no comparison of measures was made. In any event given measures $d\omega$ with growth as in (6.48) we will try to find orthogonal 'polynomials' $f(\lambda, t)$ in the form $(A \sim A_Q^{-1/2})$

$$f(\lambda, t) = A(t)\varphi_\lambda^P(t) + \int_0^t K(t, \tau)\varphi_\lambda^P(\tau)\,d\tau \quad (6.49)$$

where $A(\xi, x)$ has the form (6.48) $(A(\xi, x) = \delta(x-\xi) + \Omega(\xi, x)\Delta_P(x))$. For moment functional now we specify (cf. (6.38))

$$L(\Omega_\lambda^P(t)) = \int_0^\infty \Omega_\lambda^P(t)\,d\omega(\lambda) = g(t) = \Delta_P(t)\check{W}(t) \quad (6.50)$$

$(W = \hat{\omega}/\hat{\nu}$ with $d\omega = \hat{\omega}\,d\lambda$ and $\check{W}(t) = \int_0^\infty W(\lambda)\varphi_\lambda^P(t)\,d\nu)$. The relation between the moment functional and the kernel $A(\xi, x)$ has an apparently different form now than in (6.42). Thus, recall (5.17) (i.e., $A(\xi, x) = \Delta_P(x)T_\xi^x\check{W}(\xi)$) to write $\hat{\omega} = (1+\hat{\sigma})\hat{\nu}\,d\lambda$ and

$$A(\xi, x) = \delta(x-\xi) + \Delta_P(x)T_\xi^x\Sigma(\xi); \quad \Sigma(t) = \int_0^\infty \varphi_\lambda^P(t)\hat{\sigma}(\lambda)\,d\nu. \quad (6.51)$$

One notes that (6.42) is, of course, a form of generalized translation for $D^2 = P$ and, thus, we have produced the proper 'canonical' formula. Hence, again the moment functional determines the kernel $A(\xi, x)$ via generalized translation. Now (following Theorem 6.21) suppose $f(\lambda, t)$ is given by (6.49) with (cf. (6.40))

$$\int_0^\infty \varphi_\lambda^P(s) f(\lambda, t)\, \mathrm{d}\omega(\lambda) = 0 \tag{6.52}$$

for $s < t$. Then, using $A(\xi, x) = \delta(x-\xi) + \Omega(\xi, x)\Delta_P(x)$ and taking scalar products in (6.49) with $\Omega_\lambda^P(s)$, for $s < t$ one has

$$0 = A(t)\Omega(t, s)\Delta_P(s) + K(t, s) + \int_0^t K(t, \tau)\Omega(\tau, s)\Delta_P(s)\, \mathrm{d}\tau. \tag{6.53}$$

THEOREM 6.30. *Given* (6.49) *with* (6.52) *and the kernel* $A(\xi, x)$ *determined from the moment functional via* (6.51) *it follows that the kernel K in* (6.49) *satisfies a GL type equation* (6.53) *for* $s < t$.

Now (6.53) contains two unknowns $A(t)$ and $K(t, \tau)$. When, in fact, there is an underlying A operator as above we know $A = A_Q^{-1/2}$. In practice one uses an integrated form of such a GL equation as in the geophysical problem to obtain an equation involving K alone, which can then be solved for K (with A later determined via K by a separate argument). One way around this here is to write (★) $A(t)\hat{K}(t, s) = K(t, s)$; $\Omega(t, s)\Delta_P(s) = \hat{\Omega}(t, s)$. Now $\hat{\Omega}_t(0, s) = 0$ and from (6.49) ($\hat{f} = f/A$)

$$\begin{aligned} &\hat{f}(\lambda, t) - \varphi_\lambda^P(t) = \int_0^t \hat{K}(t, \tau)\varphi_\lambda^P(\tau)\, \mathrm{d}\tau; \\ &\int_0^\infty [\hat{f}(\lambda, t) - \varphi_\lambda^P(t)]\varphi_\lambda^P(\tau)\, \mathrm{d}\nu(\lambda) = \hat{K}(t, \tau)\Delta_P^{-1}(\tau). \end{aligned} \tag{6.54}$$

Hence, $(\hat{K}/\Delta_P)_\tau(t, 0) = 0$ and $\hat{K}(t, 0) = 0$ (for $m > -\frac{1}{2}$). There results (rewriting (6.53) in terms of $\hat{K}$ and $\hat{\Omega}$)

THEOREM 6.31. *Given* (6.49) *with* (★) *and* (6.53) *where* (6.53) *has unique solutions, and* $\hat{\Omega}$ *is twice continuously differentiable, one has* (*for* $q = 2\hat{K}'(t, t)$)

$$[P(D_t) - q(t)]\hat{K}(t, x) = P^*(D_s)\hat{K}(t, s)$$

(*also* $(\hat{K}/\Delta_P(\tau))_\tau(t, 0) = 0$ *and* $\hat{K}(t, 0) = 0 - m > -\frac{1}{2}$). *Further, from* (6.54), $P(D_t)\hat{f} = -\lambda^2\hat{f} + q(t)\hat{f}$.

Now from [32], if we are dealing with the situation above where $\beta = A_Q^{-1/2}\delta + K$ and $\varphi_\lambda^Q = A_Q^{-1/2}\varphi_\lambda^P + \int K\varphi_\lambda^P\, \mathrm{d}x$ (with suitable K) then for $Qu = (\Delta_Q u')'/\Delta_Q + \hat{q}u$ ($\Delta_Q = A_Q\Delta_P = A_Q x^{2m+1}$) one has in fact

$$\begin{aligned} q(y) &= 2[A_Q^{1/2}(y)K(y, y)]' = 2\hat{K}'(y, y) \\ &= -[\hat{q} + \tfrac{1}{4}(A_Q'/A_Q)^2 - \tfrac{1}{2}(A_Q''/A_Q) - (A_Q'/A_Q)((m+\tfrac{1}{2})/y)] \end{aligned} \tag{6.55}$$

$(Q_m^*(D_x)K(y,x) = Q(D_y)K(y,x)$ for $x<y$ and from (6.54), for $m>0$ at least, our K will be suitable).

Now write $Q_0 f = (\Delta_P A_Q f')'/\Delta_P A_Q = [(\Delta_P'/\Delta_P) + (A_Q'/A_Q)]f' + f''$ and one obtains $(\hat{f} = A_Q^{1/2} f) A_Q^{-1/2} P\hat{f} = Q_0 f + q_0 f$;

$$q_0 = [(A_Q'/A_Q)((m+\tfrac{1}{2})/t) + \tfrac{1}{2}(A_Q''/A_Q) - \tfrac{1}{4}(A_Q'/A_Q)^2]f.$$

We note that (6.55) represents $2K'(y,y)$ as $q = -\hat{q} + q_0$. Now use Theorem 6.31 and the above to obtain $Q_0 f + \hat{q} f = A_Q^{-1/2} P\hat{f} + (\hat{q} - q_0) f = -\lambda^2 f + (q + \hat{q} - q_0) f = -\lambda^2 f$. For simplicity take $\hat{q} = 0$ to obtain a typical theorem

THEOREM 6.32. *The connection* (6.49) (*with* $A = A_Q^{-1/2}$) *will arise from an underlying differential operator* $Q = Q_0$ $(\Delta_Q = A_Q \Delta_P)$ *with* B: $\varphi_\lambda^P \to f$ *in* (6.49) *representing a transmutation* B: $P = Q_m \to Q$ *provided* A_Q *is determined from the equation* $C'' + ((2m+1)/t)C' - qC = 0$ *where* $C = A_Q^{1/2} = 1/A$ *and* q *is known by first solving* (*uniquely*) *the GL equation* (6.53) *in the form* $0 = \hat{\Omega}(t,s) + \hat{K}(t,s) + \int_0^t \hat{K}(t,\tau)\hat{\Omega}(\tau,s)\,d\tau$ $(s<t)$ *for* $\hat{K}$ *and then using* (6.55) *to determine* $q = 2\hat{K}'(t,t)$. *The resulting* $f(\lambda,t)$ *constructed via* (6.49) *are then orthogonal relative to* $d\omega$.

COMMENT 6.33. The equation for C in Theorem 6.31 is of a standard form and will have solutions for quite general (reasonable) q (cf. [57]). We have omitted any details concerning hypotheses on $\hat{\sigma}$ involving the differentiability of $\hat{\Omega}$ which would guarantee the existence of an underlying differential problem. The role of Goursat problems and techniques from linear estimation theory is apparent. We leave open also various questions about other choices of q above.

7. Systems of Equations

We will sketch some developments for systems following [6, 7, 87, 88, 154, 163, 164]. The paper [88], which we follow in part, is especially illuminating and it also treats various other topics in the system context such as scattering theory via the M equation, band extensions, entropy, Lax–Phillips scattering, characteristic operator functions, etc. We go first for background to [6, 163, 164], for example, for canonical equations (cf. also [78, 108, 154] – let us mention explicitly the enormous contributions of M. Krein to this area as well as many other areas touched upon in this article). Thus, let H be a separable Hilbert space and J an operator in H such that (*) $J^* = -J$ and $J^2 = -I$. Set $P_\pm = [I \mp iJ]/2$ with $P_+ + P_- = I$ and $J = [iP_+ - iP_-]$; the $P_\pm$ are projections and we write $H_\pm = P_\pm H$. Consider (**) $JD_r x - Vx = f$ where $x(r) \in H$, $f(r) \in H$, and $V(r) \in B(H) =$ bounded linear operators in $H(V \in L^\infty(0,\infty; B(H))$ and eventually also $V \in L^1$ is used). Assume also $V(r)$ to be self-adjoint and consider the operator (◆) $A_0 = JD_r - V$ in $L^2(0,\infty; H)$ defined on $D(A_0) = \{$absolutely continuous vector functions x with compact support such that $A_0 x \in L^2(0,\infty; H)$ and $x(0) = 0\}$. One knows, e.g., there exists a J unitary operator $U(r) \in B(H)$ satisfying $JD_r U -$

$VU=0$; $U(0)=I$ (J unitary means $U^*JU=UJU^*=J$) and $g\in L^2(0,\infty;H)$ with compact support is in $R(A_0)$ if and only if $\int_0^\infty U^*(r)g(r)\,\mathrm{d}r=0$. One can also assume that $JV=-VJ$ without loss of generality. It is easy to see that A_0 in (♦) is symmetric in $\mathscr{H}=L^2(0,\infty;H)$ and $D(A_0^*)=\{x\in\mathscr{H}$ which are locally absolutely continuous with $A_0x\in\mathscr{H}\}$. By standard procedures one finds symmetric extensions A_p of A_0 as restrictions of A_0^* to domains $D(A_p)=\{x\in D(A_0^*);\ P_{\mathfrak{H}}x(0)=x(0)$ or $x(0)\in P_{\mathfrak{H}}H=\mathfrak{H}\}$ where $\mathfrak{H}\subset H$ is a $(-iJ)$-neutral subspace characterized by $[x,y]$ $(=i[(A_0x,y)-(x,A_0^*y)])=((-iJ)x(0),\ y(0))=0$ for $x(0),y(0)\in\mathfrak{H}$. In order to have a self-adjoint extension of A_0 it is necessary and sufficient that $\dim H_+=\dim H_-$ and then $P_{\mathfrak{H}}=\frac{1}{2}[I+K+K^*]=P$ where $K\colon H_+\to H_-$ is an isometry onto ($P^*=P$). Note also $JP+PJ=J$ (i.e., $JP=QJ=(I-P)J$) in the self-adjoint case with $K^*K=P_+$ and $KK^*=P_-$. In general, $\mathfrak{H}=PH$ is characterized by $PJP=0$. Further, one notes that $D(A_P^*)$ involves $y(0)\in H\text{–}JPH$ and self-adjointness requires $PH=H\text{–}JPH$ ($(J\hat{h},\hat{h})=0$ for $\hat{h}=Ph$). Consider now (■) $[A_0-\lambda]U(x,\lambda)=0$; $U(0,\lambda)=I$ with $U(x,\lambda)\in B(H)$ and λ real momentarily; thus $U'+\lambda JU+JVU=0$. Evidently $U^*JU=UJU^*=J$ (as for $U=U(x,0)$ above). Now write $\Phi(x,\lambda)=U(x,\lambda)P$ (so Φ satisfies (■) with $\Phi(0,\lambda)=P$) and set

$$\tilde{f}(\lambda)=\Phi(f,\lambda)=P\int_0^\infty U^*(x,\lambda)f(x)\,\mathrm{d}x=\int_0^\infty \Phi^*(x,\lambda)f(x)\,\mathrm{d}x \tag{7.1}$$

for H valued $f\in K^2=\{L^2$ with compact support$\}$. This function $\Phi(f,\lambda)$ is a directing functional in the sense of Krein satisfying in particular the condition that $(A_0-\lambda)g=f$ if and only if $\Phi(f,\lambda)=0$. The spectral theory can be built up from these considerations but we proceed differently following [88].

Now set $U_0(x,\lambda)=\mathrm{e}^{-\lambda Jx}$ and $\Omega(x,\lambda)=U_0^{-1}U=\mathrm{e}^{\lambda Jx}U(x,\lambda)$ (this is called a normalized matrizant in the finite-dimensional situation). One obtains by iterative procedures (★)$\Omega(t,\lambda)=I+\int_0^t \mathrm{e}^{2\lambda sJ}w(t,s)\,\mathrm{d}s$ where

$$w(t,s)+JV(s)=\int_s^t V(u)\,Jw(u,u-s)\,\mathrm{d}s \tag{7.2}$$

and with hypotheses as indicated one can write $\|U(t,\lambda)\|\le\nu(t)\,\mathrm{e}^{|\mathrm{Im}\,\lambda|t}$ where $\nu(t)=\mathrm{e}^{\int_0^t\|V(s)\|\,\mathrm{d}s}$. Now note that $\mathrm{e}^{\mu J}=P_+\mathrm{e}^{i\mu}+P_-\mathrm{e}^{-i\mu}$ and set

$$\begin{aligned} w_+(t,(t-u)/2)&=P_+w(t,(t-u)/2)P_++P_-w(t,(t-u)/2)P_-;\\ w_-(t,(t+u)/2)&=P_+w(t,(t+u)/2)P_-+P_-w(t,(t+u)/2)P_+ \end{aligned} \tag{7.3}$$

THEOREM 7.1. *Under the hypotheses indicated one can write* $U(t,\lambda)=U_0(t,\lambda)+\int_{-t}^t K(t,s)U_0(s,\lambda)\,\mathrm{d}s$ *where* $K(t,s)=\frac{1}{2}[w_+(t,(t-s)/2)+w_-(t,(t+s)/2)]$.

This is, of course, a transmutation type formula and it can also be expressed somewhat differently as in [163, 88]. Let us indicate a model situation.

EXAMPLE 7.2. For illustrative purposes one can take (1 could also be I_n)

$$P=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad Q=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad J=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

$$P_+=\tfrac{1}{2}\begin{pmatrix}1&-i\\i&1\end{pmatrix}\quad\text{and}\quad P_-=\tfrac{1}{2}\begin{pmatrix}1&i\\-i&1\end{pmatrix}$$

so

$$I=P_++P_-,\qquad J=iP_+-iP_-$$

and

$$e^{-Jx\lambda}=U_0=P_+e^{-i\lambda x}+P_-e^{i\lambda x}=\begin{pmatrix}\cos\lambda x&-\sin\lambda x\\\sin\lambda x&\cos\lambda x\end{pmatrix}.$$

Also e.g.,

$$V=\begin{pmatrix}s&r\\r&-s\end{pmatrix}\quad\text{with}\quad r^*=r,\ s^*=s,$$

and thus $JV=-VJ$. Further, one checks easily that $PJP=0$ knd $JP=QJ$ with $P^*=P$ and $PH\perp JPH$. In addition (●) $(P-Q)U_0(-x,\lambda)=U_0(x,\lambda)(P-Q)$. We will assume (●) holds now in the remaining discussion. It clearly extends to $2n$ dimensions and thence to any separable H essentially as a choice of basis vectors. Note also that $H_+=\frac{1}{2}[H-JHJ]=P_+HP_++P_-HP_-$ with $H_-=\frac{1}{2}[H+JHJ]=P_+HP_-+P_-HP_+$ gives a decomposition into J commuting and J anticommuting parts (e.g., $JH_+=\frac{1}{2}[JH+HJ]=\frac{1}{2}[H-JHJ]J=H_+J$).

Now assume the property (●) and write (◆◆)

$$\begin{aligned}\hat K(t,s)&=K(t,s)+K(t,-s)(P-Q)\\&=\tfrac{1}{2}[w(t,(t+s)/2)+w(t,(t-s)/2)]P\\&\quad+\tfrac{1}{2}J[w(t,(t+s)/2)-w(t,(t-s)/2)]JQ.\end{aligned}$$

Then an elementary calculation yields

$$U(t,\lambda)=U_0(t,\lambda)+\int_0^t\hat K(t,s)U_0(s,\lambda)\,ds. \tag{7.4}$$

THEOREM 7.3. *Under the hypotheses indicated plus* (●) *and smoothness on V as needed one has* (7.4) *with* (◆◆) *and this represents a transmutation* $JD_s\to JD_t-V(t)$ *with* $V(t)=J\hat K(t,t)-\hat K(t,t)J$. *Thus,* $\hat K$ *appears as the solution of a 'Goursat' problem* $JD_t\hat K-V(t)\hat K=-D_s\hat KJ$ *with* $\hat K(t,0)=0$.

The spectral theory can now be presented in various ways. In [6] for example (cf. also [88, 163, 164]) one takes $\Phi=UP$ with $\Omega=U_0^{-1}U$ and sets $G(\lambda)=\lim\Omega(x,\lambda)$ as $x\to\infty$ so by (★)

$$G(\lambda) = I + \int_0^\infty e^{2\lambda s J} w(\infty, s)\, \mathrm{d}s = \Omega_\infty(\lambda) = A(\lambda) \tag{7.5}$$

where $w(\infty, s)$ can be assumed to make sense (assume here $V \in L^1(H)$ also). One can define on PH (▲■) $\Delta(\lambda) = [PG^*(\lambda) G(\lambda) P]^{-1}$ and note also that $\Omega_\infty^* J \Omega_\infty = I$. Let $L_2^\Delta(\mathfrak{H})$, $\mathfrak{H} = PH$, be determined by the norm

$$\|f\|_\Delta = [(1/\pi) \int_{-\infty}^\infty (\Delta(\lambda) f(\lambda), f(\lambda))_{\mathfrak{H}}\, \mathrm{d}\lambda]$$

(in fact $L^2(\mathfrak{H}) = L_2^\Delta(\mathfrak{H})$ since $\|\Delta\|$ is bounded above and below independently of λ). One has then (see (7.1) for $\tilde{f}(\lambda) = \Phi(f, \lambda)$)

THEOREM 7.4. *With hypotheses as indicated* $f(x) = (1/\pi) \int_{-\infty}^\infty \Phi(x, \lambda) \Delta(\lambda) \tilde{f}(\lambda)\, \mathrm{d}\lambda$ *for* $f \in L^2(\mathfrak{H})$ *and* $\int_0^\infty \|f\|^2\, \mathrm{d}x = (1/\pi) \int_{-\infty}^\infty (\Delta(\lambda) \tilde{f}, \tilde{f})\, \mathrm{d}\lambda$.

REMARK 7.5. We will indicate here some of the ideas which can go into a proof of this theorem; all details are not spelled out but the essential is present. It is profitable here to use the technique of [7, 88] which refines the approach of [6, 163, 164] although dealing only with finite-dimensional situations. This is achieved in particular by splitting off the space PH in a $2n \times 2n$ situation to deal with 2 blocks of $n \times n$ matrices. Thus, in Example 7.2 think of P with $I_n \sim 1$ and $U = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ so that $UP \sim \left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right) = X$. Then one can work with X in H or U in PH equivalently. First, however, let us record a few formulas. We recall $JD_x U - VU = \lambda U$ (and $\Omega' = \mathrm{e}^{2\lambda x J} V J \Omega$). Set now $F^\#(\lambda) = F(\lambda^*)^*$ and one obtains

$$J - U_\lambda^\# J U_{\mu^*} = (\lambda - \mu^*) \int_0^t U_\lambda^\# U_{\mu^*}\, \mathrm{d}s. \tag{7.6}$$

In particular, for $\mu^* = \lambda$ one has (★) $J - U^\#(t, \lambda) J U(t, \lambda) = 0$ and $U^{-1} = -JU^\# J$. Setting $\lambda = \omega^*$ and $\mu^* = \omega$ one has also

$$[J - U_\omega^* J U_\omega]/(\omega^* - \omega) = \int_0^t U^*(s, \omega) U(s, \omega)\, \mathrm{d}s \geqslant 0.$$

Consider now

$$\Lambda_\mu^t(\lambda) = [J - U_\lambda^\# J U_{\mu^*}]/2\pi(\lambda - \mu^*). \tag{7.7}$$

One can calculate with such objects as reproducing kernels and relate the matter (following [7, 88] directly to the idea of deBranges spaces, etc. (cf. [84, 88]). We will do this differently however (as in [88]) where we use $n \times n$ blocks $A, B, F = A + iB$, $E = A - iB$, and $X = \left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$. Define for suitable $f \in K^2$ ($K^2 = L^2(H)$ with compact support $- R^{2n} \sim H$ here and $f \sim (f_1, f_2)$ as a column vector)

$$f^\blacktriangle(\lambda) = \int_0^\infty X^\#(s, \lambda) f(s)\, \mathrm{d}s; \qquad \tilde{f}(\lambda) = \int_0^\infty X_0^\#(s, \lambda) f(s)\, \mathrm{d}s \tag{7.8}$$

where $X_0 = U_0 P = (\cos \lambda x I_n, \sin \lambda x I_n)$ as a column vector ($U_0 = \mathrm{e}^{-\lambda J x}$ with $J =$

$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ – cf. Example 7.2). Note also $V \sim \begin{pmatrix} S & R \\ R & -S \end{pmatrix}$ with $R^* = R$, $S^* = S$ and from (7.4) with $X = UP$, $X_0 = U_0P$ one has, e.g.,

$$X = X_0 + \int_0^t \hat{K}(t, s)X_0(s, \lambda)\, ds. \tag{7.9}$$

The inversion formula we will obtain below, following [88] is then

$$f(x) = (1/\pi)\int_{-\infty}^{\infty} X(s, \lambda)\Delta_\infty(\lambda)f^\blacktriangle(\lambda)\, d\lambda \tag{7.10}$$

with

$$\Delta_t(\lambda) = [E^\#(t, \lambda)E(t, \lambda)]^{-1}\ (=[F^\#(t, \lambda)F(t, \lambda)]^{-1})$$

and $\Delta_\infty = \lim \Delta_t$ as $t \to \infty$. Let us note here in the model situation described in Example 7.2.

$$UP = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \quad PU^* = \begin{pmatrix} A^* & B^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad PU^*UP = E^*E = A^*A + B^*B$$

(since $B^*A = A^*B$ from $J = U^*JU$). Thus, $\Delta_\infty \sim \Delta$ in (▲■) (note $G \sim \Omega$ and $\Omega^*\Omega = U^*U$). We remark also that notation (X, X) or X^*X is used interchangably at times. The Parseval formula at level t in the form

$$\int_0^t (f(s), g(s))\, ds = \int_0^t f^*(s)g(s)\, ds = \frac{1}{\pi}\int_{-\infty}^{\infty} (f^\blacktriangle(\lambda), \Delta_t(\lambda)g^\blacktriangle(\lambda))\, d\lambda \tag{7.11}$$

extends then to the corresponding formula with $t = \infty$. One constructs U etc., as before and the entries in $U(t, \lambda)$ are entire functions of λ of exponential type t with the $n \times n$ blocks A, B, C, D invertible. One refers to W_m (Wiener algebra) as the set of $m \times m$ matrix functions $F(\lambda) = c_F + \hat{k}_F(\lambda)$ where c_F is constant and $k_F \in L^1_{m\times m}(R)$ ($\hat{k}_F = \int_{-\infty}^{\infty} k_F\, e^{i\lambda x}\, dx = \mathcal{F}k_F$). Thus, $c_F = F(\infty)$ and $k_F = \mathcal{F}^{-1}[F - c_F]$. If $c_F = I$, one refers to W^I and W_+ (resp. W_-) refers to F such that $\mathcal{F}^{-1}F(x) = 0$ for $x < 0$ (resp. $x > 0$). One also writes $W^I_\pm = W^I \cap W_\pm$. This, one is dealing with functions $F \in W_+$ (resp. W_-) analytic for $\mathrm{Im}\,\lambda > 0$ (resp. $\mathrm{Im}\,\lambda < 0$). One shows easily that $\Omega(\lambda, t) = U_0^{-1}U \in W^I$ for example. One says now (C_+ denotes $\mathrm{Im}\,\lambda > 0$, $C_- \sim \mathrm{Im}\,\lambda < 0$) that a pair of $n \times n$ matrix valued entire functions (E_+, E_-) is a deBranges pair if (●●●) $E_+E_+^\# = E_-E_-^\#$ on C; $\det E_+ \neq 0$ on C_+; $\det E_- \neq 0$ on C_-; and $\Sigma = E_+^{-1}E_-$ is inner on C_+. Here Σ is $\tilde{J}$ inner if it is meromorphic with $\Sigma\tilde{J}\Sigma^* \leq \tilde{J}$ at points of analyticity while $\Sigma\tilde{J}\Sigma^* = \tilde{J}$ on R – for $\tilde{J} = I$ one speaks of inner. Let us denote by $\mathfrak{H}^2$ (resp. $\mathfrak{K}^2$) the Hardy spaces over C_+ (resp. C_-) of n-vector functions $(f_i(\lambda)) = f(\lambda)$ (i.e., the entries, say for $\mathfrak{H}^2$, $f_i(\lambda)$ are analytic in C_+ and $\sup_{\eta>0}\int |f_i(\xi + i\eta)|^2\, d\xi < \infty$ – equivalently, $f_i(\lambda) = \int_0^\infty F(x)\, e^{i\lambda x}\, dx$ with $F \in L^2$). Now with a deBranges pair (E_+, E_-) as indicated one associates a deBranges space $B(E_+, E_-)$ of $n \times 1$ vector valued entire functions f such that $E_+^{-1}f \in \mathfrak{H}^2 - \Sigma\mathfrak{H}^2$. One has then (cf. [88])

THEOREM 7.6. *Let* (E_+, E_-) *be a deBranges pair. Then* $B = B(E_+, E_-)$ *is a reproducing kernel space relative to the inner product* $\langle f, g\rangle = \int_{-\infty}^{\infty} (E_+^{-1}f, E_+^{-1}g)\,d\lambda$ *with reproducing kernel* $\Lambda_\omega(\lambda) = [E_+(\lambda)E_+^*(\omega) - E_-(\lambda)E_-^*(\omega)]/[-2\pi i(\lambda - \omega^*)]$ (*i.e., for* $h \in \mathfrak{H} \sim C^n \sim PH$, $\Lambda_\omega h \in B$ *and for* $f \in B$, $\langle f, \Lambda_\omega h\rangle_B = (f(\omega), h)$).

In the present situation one will have a deBranges pair $(F^\#, E^\#) = (A^\# - iB^\#, A^\# + iB^\#) \sim (E_+, E_-)$. To see how this works one can show first that ($\Sigma = E_+^{-1}E_-$), $E_+^{-1}\Lambda_\omega(\lambda)h \in \mathfrak{H}^2$ and is perpendicular to $\Sigma\mathfrak{H}^2$ for $\omega \in C_+$ (and R). Next, $E_+^{-1}f \in \mathfrak{H}^2 - \Sigma\mathfrak{H}^2$ if and only if $f \in E_+\mathfrak{H}^2 \cap E_-\mathfrak{K}^2$ and arguing as above for $E_-^{-1}\Lambda_\omega(\lambda)h$ with $\omega \in C_-$ one obtains $\Lambda_\omega(\lambda)h \in B$ for $\omega \in C$. Finally, using $E_+E_+^\# = E_-E_-^\#$, one obtains

$$\begin{aligned}\langle f, \Lambda_\omega h\rangle_B &= \left(\int_{-\infty}^{\infty} [E_+^{-1}(\lambda)f(\lambda)]/[2\pi i(\lambda - \omega)]\,d\lambda,\ E_+^*(\omega)h\right) - \\ &\quad - \left(\int_{-\infty}^{\infty} [E_-^{-1}(\lambda)f(\lambda)]/[2\pi i(\lambda - \omega)]\,d\lambda,\ E_-^*(\omega)h\right) \\ &= (f(\omega), h).\end{aligned} \tag{7.12}$$

Now one can use $(F^\#, E^\#)$ as (E_+, E_-) to define a deBranges space B_t so that (recall $F_\mu^{\#*} = F_{\mu*}$ and cf. (7.6))

$$\begin{aligned}\Lambda_\mu^t(\lambda) &= [F^\#(t,\lambda)F^{\#*}(t,\mu) - E^\#(t,\lambda)E^{\#*}(t,\mu)]/(-2\pi i(\lambda - \mu^*)) \\ &= [X^\#(t,\lambda)JX^{\#*}(t,\mu)]/[-\pi(\lambda - \mu^*)] \\ &= (1/\pi)\int_0^t X^\#(s,\lambda)X(s,\mu^*)\,ds\end{aligned} \tag{7.13}$$

(recall $PJP = 0$). Define now $f^\blacktriangle$ as in (7.8) and one obtains

THEOREM 7.7. *The map* $f \to f^\blacktriangle$ *determined by* (7.8) *maps* $L^2(0, T) \to B_T$, $1-1$ *onto with* $\|f^\blacktriangle\|_B^2 = \pi\|f\|_{L^2}^2$ *and the inverse map is given by* (7.10) *with Parseval formula* (7.11). *One has also* $\Delta_\infty(\lambda) = I - \hat{h}(\lambda)$ (>0 *for* $\lambda \in R$) *with* $h \in L^1$.

To show that (7.10) is the correct inverse one writes heuristically, using (7.13) and Theorem 7.6

$$\begin{aligned}(f^\blacktriangle(\mu), h) &= \int (F^{\#-1}f^\blacktriangle, F^{\#-1}\Lambda_\mu^t(\lambda)h)\,d\lambda \\ &= \int ([F^{-1}F^{*-1}]f^\blacktriangle, \Lambda_\mu^t(\lambda)h)\,d\lambda \\ &= \int (\Lambda_\mu^{t*}\Delta_t f^\blacktriangle, h)\,d\lambda \\ &= \int \left(\left[(1/\pi)\int_0^t X^\#(s,\lambda)X(s,\mu^*)\,ds\right]^* \Delta_t f^\blacktriangle, h\right) d\lambda \\ &= \left(\int_0^t X^\#(s,\mu)\left[(1/\pi)\int_{-\infty}^{\infty} X(s,\lambda)\Delta_t f^\blacktriangle\,d\lambda\right] ds,\ h\right)\end{aligned} \tag{7.14}$$

This shows that (7.10) is formally the correct formula for $f^{\blacktriangle} \in B_t$ and the extension for $t \to \infty$ is immediate. The GL equation for the present context goes as follows (following [88]) for the $\hat{K}$ of (7.4) (cf. also Remark 5.16). Thus, $\Delta_\infty(\lambda) = 1 - \hat{h}(\lambda)$ from Theorem 7.7 and $\hat{h} = \int_{-\infty}^{\infty} h(s)\, e^{i\lambda s}\, ds = \mathscr{F}h$. Set $Hg = h * g$ (and ($\bullet\bullet$) $Nf(s)$ is defined by $(f = (f_1, f_2))$ $\sqrt{2}Nf(s) = f_1(s) - if_2(s)$ for $s > 0$ with $\sqrt{2}Nf(s) = f_1(-s) + if_2(-s)$ for $s < 0$. Then (cf. (7.8)) $\tilde{f}(\lambda) = \mathscr{F}(Nf)/\sqrt{2}$. We write $\mathscr{H}(t, s) = \ker N^*HN = \ker \mathscr{H}$. Then, recalling (7.4) in the transmutational form $U = (I + \hat{K})U_0$ so that $X = (I + \hat{K})X_0$ as in (7.9) one has

$$\begin{aligned} f^{\blacktriangle}(\lambda) &= \langle X, f\rangle = \int X^{\#} f = \langle (I + \hat{K})X_0, f\rangle \\ &= \langle X_0, (I + \hat{K}^*)f\rangle = [(I + \hat{K}^*)f]^{\sim} = [N(I + \hat{K}^*)f]^{\wedge}/\sqrt{2}. \end{aligned}$$

We write the Parseval formula determined by Theorem 7.7 in the form

$$\begin{aligned} \int_0^\infty (f, g)\, ds &= (1/\pi) \int_{-\infty}^{\infty} (f^{\blacktriangle}, \Delta_\infty g^{\blacktriangle})\, d\lambda = (1/2\pi) \int_{-\infty}^{\infty} ([N(I + \hat{K}^*)f]^{\wedge}, \\ &\quad (I - \hat{h})[N(I + \hat{K}^*)g]^{\wedge})\, d\lambda = \int [N(I + \hat{K}^*)f, (I - H)[N(I + \hat{K}^*)]g)\, ds \qquad (7.15) \\ &= \int (f, (I + \hat{K})N^*(I - H)[N(I + \hat{K}^*)]g)\, ds. \end{aligned}$$

It follows that $(I + \hat{K})[N^*(I - H)N](I + \hat{K}^*) = I$ which is the GL equation in factorized form. Writing $N^*(I - H)N = I - N^*HN = I - \mathscr{H}$ one has

THEOREM 7.8. *The GL equation for* $\hat{K}$ *of* (7.4) $(I + \hat{L} = (I + \hat{K})^{-1})$ *is given by* $(I + \hat{K})\mathscr{H}(I + \hat{K}^*) = I$ *or* $(I + \hat{L})(I + \hat{L}^*) = \mathscr{H}$ *which exhibits* $I + \hat{L}$ (*resp.* $I + \hat{L}^*$) *as lower* (*resp. upper*) *triangular factors of the positive operator* $I - \mathscr{H}$ (*relative to a chain of projections* $P_T : f \to f$ $(0 \leq s \leq T)$ *and* $f \to 0$ $(s \geq T)$.

One can connect this framework to readout impulse responses and the spectral representation of kernels as in the geophysical situation but we omit the details (cf. [32, 63]). There are also applications to Dirac equations etc. but we will not pursue this (the unified GL, M, etc. equations of [156] are spectralized in [63]).

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