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ASYMPTOTIC PROPERTIES OF EIGENVALUES OF INTEGRAL EQUATIONS*

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Abstract. The eigenvalue problem is considered for the integral equation

$$\int_0^a K(x,y)\varphi(y)\,dy = \lambda\varphi(x).$$

The eigenvalues and eigenfunctions are studied as functions of the upper limit a. For small values of a with K smooth, expansions of the solutions in powers of a are obtained, and it is shown that $\lambda^{(N)}(a) = O(a^{2N+1})$. For real symmetric kernels it is shown that $|\lambda^{(N)}(a)|$ is an increasing function of a for all values of a. For large values of a with additional conditions on K, expansions of the solutions in powers or fractional powers of a^{-1} are obtained.

Key words. integral equations, asymptotic expansions, Fredholm equations

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1. Introduction. We consider the eigenvalue problem for Fredholm integral equations of the form

$$\int_0^a K(x,y)\varphi(y)\,dy = \lambda\varphi(x). \tag{1.1}$$

Our goal is to determine how the eigenvalues $\lambda(a)$ and the eigenfunctions $\varphi(x, a)$ depend on a, the upper limit of integration. We shall obtain expansions of these quantities for both small and large values of a, and some results for intermediate values, under suitable assumptions on the kernel K.

The reason we have made this study is that, despite the widespread occurrence of integral equations, there are relatively few analytic techniques available for solving them exactly or approximately. This is in sharp contrast to the case of differential equations, for which regular and singular perturbation methods have been developed extensively. Some recent exceptions are the study of (1.1) with limits of integration $-\infty$ and ∞, performed by Sirovich and Knight [1]-[3], in which $K = K[x-y, \varepsilon(x+y)/2]$. They showed how to obtain the high-order eigenvalues asymptotically for ε small. Morrison [4] and Angell and Olmstead [5], [6] investigated nonlinear Volterra equations containing a small parameter and obtained asymptotic expansions of the solutions. Gautesen [7], [8] has studied inhomogeneous linear integral equations and integrodifferential equations with difference kernels involving a large parameter. He obtained asymptotic expansions of the solutions with respect to this parameter for equations in both one and two dimensions. This work was continued by Olmstead and Gautesen [9].

Our analysis begins with the observation that when K and φ are bounded, the integral in (1.1) tends to zero with a. Conversely, as a increases from zero all the eigenvalues emerge or bifurcate from the value $\lambda(0) = 0$. In § 2 we shall show first that for $k = 0, 1, \cdots$ the kth eigenvalue $\lambda^{(k)}(a)$ is $O(a^{2k+1})$ as a tends to zero, provided

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that K is sufficiently smooth at $x=y=0$ and that $K(0,0)\neq 0$. We shall also show how to obtain the expansions of $\lambda^{(k)}(a)$ and $\varphi^{(k)}(x,a)$ in powers of a. This case was treated previously by Slepian [10] who showed how to obtain further terms in the expansions, and also demonstrated that the expansions were useful for computation. By our method we derive the form of expansion which he postulated. As a consequence we can treat the case in which K and any number of its derivatives vanish at $x=y=0$, as we illustrate.

In § 3 we consider the case of large a under the additional assumption that $K=K[a(x-y), x+y]$, i.e., that K is a rapidly varying function of the difference $x-y$. Then (1.1) can often be reduced asymptotically to an eigenvalue problem for a differential equation. In this differential equation a fundamental role is played by the function $\int_{-\infty}^{\infty} K(\xi, 2x)\, d\xi$, which is analogous to the potential function in the one-dimensional Schrödinger equation. We also consider the special case in which $K[a(x-y)]$ does not depend on $x+y$. Some of the results for this case were obtained before by Olmstead and Gautesen [9].

In § 4 we show how to obtain asymptotic expansions of λ and φ for large a in terms of the corresponding quantities for $a=\infty$, assuming that the latter exist.

We conclude this Introduction by proving a theorem based on the following result.

LEMMA. *Suppose that the eigenvalue $\lambda(a)$ and corresponding eigenfunction $\varphi(x,a)$ of* (1.1) *are differentiable with respect to a, and that $K(x,y)=K(y,x)$. Then*

$$\frac{d\lambda(a)}{da}=\lambda(a)\varphi^2(a,a)\Big/\int_0^a \varphi^2(x,a)\,dx. \tag{1.2}$$

Proof. Differentiation of (1.1) with respect to a yields

$$\int_0^a K(x,y)\partial_a\varphi(y,a)\,dy+K(x,a)\varphi(a,a)=\frac{d\lambda}{da}\varphi(x,a)+\lambda(a)\partial_a\varphi(x,a). \tag{1.3}$$

We multiply (1.3) by $\varphi(x,a)$, integrate with respect to x from zero to a, and use (1.1) and the symmetry of K. Then the first term in (1.3) cancels the last term, and the two remaining terms yield (1.2). An analogous result can be obtained when K is not symmetric.

With $d\lambda/da$ given by (1.2), (1.3) can be solved for $\partial_a\varphi(x,a)$. The solution can be made unique by imposing a normalization condition on φ, and orthogonality conditions if λ is not simple. Then (1.2) and (1.3) can be viewed as differential equations for λ and φ as functions of a. Initial conditions for these equations are provided by the results of the next section.

When K is real and symmetric, λ is real and φ can be chosen to be real. Then (1.2) yields the following result.

THEOREM. *When K is real and symmetric and λ and φ are differentiable with respect to a, each positive eigenvalue of* (1.1) *increases monotonically with a and each negative eigenvalue decreases monotonically.*

2. Expansions for $a \ll 1$.

To solve (1.1) for $a\ll 1$ we first introduce the new variables $\xi=x/a$, $\eta=y/a$, and $\phi(\xi,a)=\varphi(a\xi,a)$, and rewrite (1.1) in the form

$$\int_0^1 K(a\xi, a\eta)\phi(\eta)\,d\eta=a^{-1}\lambda\phi(\xi). \tag{2.1}$$

Next we assume that $K(x,y)$ has as many derivatives at the origin as we use in our calculation. Then we seek expansions of an eigenvalue $\lambda(a)$ and corresponding normalized eigenfunction $\phi(\xi,a)$ of (2.1) in the forms

$$\begin{aligned}\lambda(a)&=\lambda_1 a+\lambda_2 a^2+\lambda_3 a^3+\cdots,\\ \phi(\xi,a)&=\phi_0(\xi)+a\phi_1(\xi)+a^2\phi_2(\xi)+\cdots.\end{aligned} \tag{2.2}$$

Now we substitute (2.2) into (2.1), expand K in a finite Taylor series about $x=y=0$, and equate coefficients of each power of a. This yields a sequence of equations of which we shall write the first three. In these and in the rest of this section, K and its derivatives are evaluated at $x=y=0$:

$$K\int_0^1 \phi_0(\eta)\,d\eta=\lambda_1\phi_0(\xi), \tag{2.3}$$

$$K\int_0^1 \phi_1\,d\eta+\xi K_x\int_0^1 \phi_0\,d\eta+K_y\int_0^1 \eta\phi_0\,d\eta=\lambda_2\phi_0(\xi)+\lambda_1\phi_1(\xi), \tag{2.4}$$

$$\begin{aligned}&K\int_0^1 \phi_2\,d\eta+\xi K_x\int_0^1 \phi_1\,d\eta+K_y\int_0^1 \eta\phi_1\,d\eta+\frac{1}{2}\xi^2K_{xx}\int_0^1 \phi_0\,d\eta+\xi K_{xy}\int_0^1 \eta\phi_0\,d\eta\\&\quad+\frac{1}{2}K_{yy}\int_0^1 \eta^2\phi_0\,d\eta=\lambda_3\phi_0(\xi)+\lambda_2\phi_1(\xi)+\lambda_1\phi_2(\xi).\end{aligned} \tag{2.5}$$

Next we use (2.2) for ϕ in the normalization condition $\int_0^1\phi^2(\xi,a)\,d\xi=1$ and equate coefficients of the first three powers of a to get

$$\int_0^1 \phi_0^2\,d\xi=1,\qquad \int_0^1 \phi_0\phi_1\,d\xi=0,\qquad \int_0^1 (\phi_1^2+2\phi_0\phi_2)\,d\xi=0. \tag{2.6}$$

If $K(0,0)\neq 0$ and $\lambda_1\neq 0$, (2.3) implies that ϕ_0 is a constant. Then (2.6) shows that $\phi_0(\xi)=1$ and (2.3) yields $\lambda_1=K(0,0)$. By using these results in (2.4) we obtain

$$\xi K_x+\frac{1}{2}K_y+K\int_0^1 \phi_1\,d\eta=\lambda_2+K\phi_1(\xi). \tag{2.7}$$

From (2.7) we see that ϕ_1 is a linear function of ξ. Upon setting $\phi_1=A+B\xi$ in (2.7) and comparing coefficients of 1 and ξ, we find that $B=K_x/K$ and $\lambda_2=\frac{1}{2}[K_x(0,0)+K_y(0,0)]$. Then we compute A using (2.6) and we finally get

$$\phi_1(\xi)=\frac{K_x(0,0)}{K(0,0)}\left(\xi-\frac{1}{2}\right).$$

The problem (2.5) can be solved similarly. The function ϕ_2 must be of the form $a+b\xi+c\xi^2$. When this is substituted into (2.5), along with our previous formulae for λ_1, λ_2, ϕ_0, and ϕ_1, it yields equations for b, c, and λ_3. Their solutions are

$$c=\frac{1}{2}\frac{K_{xx}}{K},\qquad b=\frac{1}{K}\left[\frac{1}{2}K_{yx}-\lambda_2\frac{K_x}{K}\right]$$

and

$$\begin{aligned}\lambda_3&=\frac{1}{6}K_{yy}+\frac{1}{12}\frac{K_xK_y}{K}+K\left(\frac{b}{2}+\frac{c}{3}\right)+\frac{\lambda_2}{2}\frac{K_x}{K}\\&=\frac{1}{6}[K_{xx}+K_{yy}]+\frac{1}{4}K_{yx}+\frac{1}{12}\frac{K_xK_y}{K}.\end{aligned}$$

The constant a can be computed from (2.6).

We have now computed the first three terms in the expansion of an eigenvalue and eigenfunction of the integral equation (2.1). We shall denote them $\lambda^{(0)}$ and $\varphi^{(0)}$. To compute other eigenvalues we must change the assumption that $\lambda_1\neq 0$, which we made in solving (2.3). Therefore, we now assume that $\lambda_1 = 0$. Then (2.3) and the fact that $K(0,0) \neq 0$ show that $\int_0^1 \phi_0 \, d\eta = 0$. With $\lambda_1 = 0$, (2.4) reduces to

$$K_y \int_0^1 \eta\phi_0 \, d\eta + K \int_0^1 \phi_1 \, d\eta = \lambda_2 \phi_0(\xi). \tag{2.8}$$

If $\lambda_2 \neq 0$, (2.8) implies that ϕ_0 is a constant. This contradicts the fact that $\int_0^1 \phi_0(\eta) \, d\eta = 0$, unless $\phi_0(\eta) \equiv 0$. But this cannot be since ϕ_0 is an eigenfunction, so we must conclude that $\lambda_2 = 0$. Then (2.8) gives a relation between the first moment of ϕ_0 and the zeroth moment of ϕ_1, though these functions remain undetermined at this stage. To determine λ_3 and ϕ_0, we must consider (2.5) which becomes, with $\lambda_1 = \lambda_2 = 0$,

$$\begin{aligned} &K \int_0^1 \phi_2 \, d\eta + \xi K_x \int_0^1 \phi_1 \, d\eta + K_y \int_0^1 \eta\phi_1 \, d\eta \\ &\quad + \xi K_{xy} \int_0^1 \eta\phi_0 \, d\eta + \frac{1}{2} K_{yy} \int_0^1 \eta^2 \phi_0 \, d\eta = \lambda_3 \phi_0(\xi). \end{aligned} \tag{2.9}$$

When $\lambda_3 \neq 0$, it follows from (2.9) that ϕ_0 is a linear function of ξ. Since $\int_0^1 \phi_0 \, d\xi = 0$, we have $\phi_0(\xi) = C(\xi - \frac{1}{2})$, and C is determined by normalization as $C = \sqrt{12}$. By integrating (2.9) over (0, 1) we obtain an expression relating the three integrals $\int_0^1 \phi_2 \, d\xi$, $\int_0^1 \xi\phi_1 \, d\xi$, and $\int_0^1 \xi^2 \phi_0 \, d\xi$; this is the solvability condition for (2.9). By using this relation back in (2.9) we obtain

$$\lambda_3 \phi_0(\xi) = \left(\xi - \frac{1}{2}\right)\left[K_{xy} \int_0^1 \eta\phi_0 \, d\eta + K_x \int_0^1 \phi_1 \, d\eta\right]. \tag{2.10}$$

In view of (2.8) with $\lambda_2 = 0$, we can rewrite (2.10) as

$$\lambda_3 \phi_0(\xi) = \left(\xi - \frac{1}{2}\right)\left[K_{xy} - \frac{K_x K_y}{K}\right] \int_0^1 \eta\phi_0 \, d\eta,$$

which yields

$$\lambda_3 = \frac{1}{12}\left[K_{xy} - \frac{K_x K_y}{K}\right]. \tag{2.11}$$

If the kernel is a product, $K(x, y) = A(x)B(y)$, then $K(x, y)K_{xy}(x, y) = K_x(x, y)K_y(x, y)$ and (2.11) shows that $\lambda_3 = 0$. This is in agreement with the fact that an integral equation with a product kernel has only one nonzero eigenvalue.

The leading terms in the expansions of two eigenvalues and corresponding eigenfunctions have now been computed. Next, we compute the leading term in the expansion of the third eigenvalue and eigenfunction. From this calculation it will become transparent how to determine the kth eigenvalue and eigenfunction. We have already assumed that $\lambda_1 = 0$, and concluded that $\lambda_2 = 0$. We now assume that $\lambda_3 = 0$, and then (2.9) yields the two relations

$$K_x \int_0^1 \phi_1 \, d\eta + K_{xy} \int_0^1 \eta\phi_0 \, d\eta = 0, \tag{2.12}$$

$$K \int_0^1 \phi_2 \, d\eta + K_y \int_0^1 \eta\phi_1 \, d\eta + \frac{1}{2} K_{yy} \int_0^1 \eta^2 \phi_0 \, d\eta = 0. \tag{2.13}$$

From (2.12) and (2.8) with $\lambda_2 = 0$ we obtain

$$\int_0^1 \eta\phi_0\, d\eta = \int_0^1 \phi_1\, d\eta = 0,$$

provided that $K_{xy}K \neq K_x K_y$.

We now proceed to $O(a^3)$ in the expansion of (1.2) to obtain

$$\begin{aligned}\lambda_4\phi_0(\xi) = K\int_0^1 \phi_3\, d\eta + \xi K_x \int_0^1 \phi_2\, d\eta + K_y \int_0^1 \eta\phi_2\, d\eta \\ + \frac{1}{2!}\sum_{i=0}^{2}\binom{2}{i}(D_x^i D_y^{2-i}K)\xi^i \int_0^1 \eta^{2-i}\phi_1\, d\eta \\ + \frac{1}{3!}\sum_{i=0}^{3}\binom{3}{i}(D_x^i D_y^{3-i}K)\xi^i \int_0^1 \eta^{3-i}\phi_0\, d\eta.\end{aligned} \tag{2.14}$$

Here $D_x^i D_y^j K$ is the mixed ith x, jth y derivative of K at $x = y = 0$ and $\binom{n}{i}$ is the binomial coefficient. Since ϕ_0 is orthogonal to both one and ξ, and ϕ_1 is orthogonal to one, only the terms with $i = 0$ and $i = 1$ in the sums in (2.14) are nonzero. Hence the right side of (2.14) is linear in ξ. If $\lambda_4 \neq 0$, (2.14) implies that ϕ_0 is linear in ξ. Then the conditions $\int_0^1 \phi_0\, d\eta = \int_0^1 \eta\phi_0\, d\eta = 0$ force $\phi_0 = 0$, which is impossible. Therefore we must have $\lambda_4 = 0$.

With $\lambda_4 = 0$, (2.14) gives two additional relations between the moments of ϕ_j ($j = 0$, 1, 2, 3):

$$K\int_0^1 \phi_3\, d\eta + K_y\int_0^1 \eta\phi_2\, d\eta + \frac{1}{2!}K_{yy}\int_0^1 \eta^2\phi_1\, d\eta + \frac{1}{3!}K_{yyy}\int_0^1 \eta^3\phi_0\, d\eta = 0, \tag{2.15}$$

$$K_x\int_0^1 \phi_2\, d\eta + \frac{1}{2!}\binom{2}{1}K_{xy}\int_0^1 \eta\phi_1\, d\eta + \frac{1}{3!}\binom{3}{1}K_{xyy}\int_0^1 \eta^2\phi_0\, d\eta = 0. \tag{2.16}$$

At next order in the expansion of (1.2) we get

$$\begin{aligned}\lambda_5\phi_0(\xi) = K\int_0^1 \phi_4\, d\eta + \xi K_x\int_0^1 \phi_3\, d\eta \\ + K_y\int_0^1 \eta\phi_3\, d\eta + \frac{1}{2!}\sum_{i=0}^{2}\binom{2}{i}(D_x^i D_y^{2-i}K)\xi^i\int_0^1 \eta^{2-i}\phi_2\, d\eta \\ + \frac{1}{3!}\sum_{i=0}^{2}\binom{3}{i}(D_x^i D_y^{3-i}K)\xi^i\int_0^1 \eta^{3-i}\phi_1\, d\eta \\ + \frac{1}{4!}\sum_{i=0}^{2}\binom{4}{i}(D_x^i D_y^{4-i}K)\xi^i\int_0^1 \eta^{4-i}\phi_0\, d\eta.\end{aligned} \tag{2.17}$$

The orthogonality relations for ϕ_0 and ϕ_1 previously obtained have allowed us to reduce the upper limits on the last two sums.

The right side of (2.17) is quadratic in ξ, so we write (2.17) as $\lambda_5\phi_0(\xi) = A + B\xi + C\xi^2$. Then since ϕ_0 is orthogonal to one and ξ, we can determine A and B in terms of C and obtain

$$\begin{aligned}\lambda_5\phi_0(\xi) = C\left(\xi^2 - \xi + \frac{1}{6}\right), \\ C = \frac{1}{4!}\binom{4}{2}(D_x^2 D_y^2 K)\int_0^1 \eta^2\phi_0\, d\eta + \frac{1}{3!}\binom{3}{2}(D_x^2 D_y K)\int_0^1 \eta\phi_1\, d\eta \\ + \frac{1}{2!}\binom{2}{2}(D_x^2 K)\int_0^1 \phi_2\, d\eta.\end{aligned} \tag{2.18}$$

We can determine the moments of ϕ_1 and ϕ_2 which occur in the expression for C by solving the following linear system, which comes from (2.13) and (2.16):

$$\begin{pmatrix} K & K_y \\ K_x & \frac{1}{2!}\binom{2}{1} K_{xy} \end{pmatrix} \begin{pmatrix} \int_0^1 \phi_2\, d\eta \\ \int_0^1 \eta\phi_1\, d\eta \end{pmatrix} = -\begin{pmatrix} \frac{1}{2!} K_{yy} \\ \frac{1}{3!}\binom{3}{1} K_{xyy} \end{pmatrix} \int_0^1 \eta^2\phi_0\, d\eta.$$

Once C is expressed in terms of $\int_0^1 \eta^2\phi_0\, d\eta$, we multiply (2.18) by ξ^2 and integrate to compute λ_5 as

$$\begin{aligned} \lambda_5 &= \int_0^1 \xi^2\left(\xi^2-\xi+\frac{1}{6}\right) d\xi \cdot \frac{C}{\int_0^1 \xi^2\phi_0\, d\xi} \\ &= \frac{C}{180}\left(\int_0^1 \xi^2\phi_0\, d\xi\right)^{-1}. \end{aligned} \tag{2.19}$$

Equations (2.18) and (2.19) give the leading terms in the third eigenvalue and eigenfunction, $\lambda^{(2)}$ and $\phi^{(2)}$.

It is now clear how to compute the higher eigenvalues and eigenfunctions. For each integer $N \geqq 0$ we seek an eigenfunction and eigenvalue with expansions of the forms

$$\phi = \phi_0 + a\phi_1 + \cdots + a^N\phi_N + O(a^{N+1}),$$

$$\lambda = \lambda_{2N+1}a^{2N+1} + O(a^{2N+2}).$$

We find that the ϕ_i satisfy the orthogonality conditions

$$\int_0^1 \phi_i(\eta)\eta^j\, d\eta = 0, \qquad i \geqq 0, \quad j>0, \quad i+j<N. \tag{2.20}$$

The equation for λ_{2N+1} is

$$\lambda_{2N+1}\phi_0(\xi) = \sum_{l=0}^{2N} \frac{1}{l!} \sum_{i=0}^{l} \binom{l}{i} (D_x^i D_y^{l-i} K)\xi^i \int_0^1 \eta^{l-i}\phi_{2N-l}\, d\eta \tag{2.21}$$

and the lower order equations yield the conditions

$$\sum_{l=0}^{M} \frac{1}{l!} \sum_{i=0}^{l} \binom{l}{i} (D_x^i D_y^{l-i} K)\xi^i \int_0^1 \eta^{l-i}\phi_{M-l}\, d\eta = 0 \tag{2.22}$$

for all ξ and for $M=0, 1, 2, \cdots, 2N-1$. Rearranging the double sum in (2.22), we obtain the equivalent conditions

$$\sum_{l=i}^{M} \frac{1}{l!}\binom{l}{i} (D_x^i D_y^{l-i} K) \int_0^1 \eta^{l-i}\phi_{M-l}\, d\eta = 0, \qquad i=0, 1, \cdots, M, \tag{2.23}$$

$$M = 0, 1, \cdots, 2N-1.$$

If $l>N$ then from (2.20) we obtain $\int_0^1 \eta^{l-i}\phi_{2N-l}\, d\eta = 0$ for $l \geqq i > N$ and hence (2.21) simplifies to

$$\lambda_{2N+1}\phi_0(\xi)=CP_N(\xi),$$

(2.24)
$$C=\sum_{l=N}^{2N}\frac{1}{l!}\binom{l}{N}(D_x^N D_y^{l-N}K)\int_0^1 \eta^{l-N}\phi_{2N-l}\,d\eta$$
$$=\sum_{j=0}^{N}\frac{1}{(j+N)!}\binom{j+N}{N}(D_x^N D_y^j K)\int_0^1 \eta^j\phi_{N-j}\,d\eta.$$

Here $P_N(\xi)=\xi^N+\cdots$ is the unique polynomial of degree N with leading coefficient one that is orthogonal to the set of functions $\{1, \xi, \cdots, \xi^{N-1}\}$ on the unit interval.

To complete the determination of λ_{2N+1}, we must express C in terms of $\int_0^1 \eta^N\phi_0\,d\eta$. To this end, we return to (2.23) and isolate only those relations that involve $\phi_0, \cdots, \phi_N$. The other relations would be used in computing the corrections to the leading term (i.e., λ_l for $l>2N+1$). When $M=N$, (2.23) is identically satisfied in view of (2.20) except when $i=0$. When $i=0$, we obtain the new relation

(2.25)
$$\sum_{l=0}^{N}\frac{1}{l!}(D_y^l K)\int_0^1 \eta^l\phi_{N-l}\,d\eta=0.$$

For $N+1\leqq M\leqq 2N-1$, only the relation with $i=M-N$ does not contain the functions ϕ_k, $k>N$ so that, after a shift of summation index, we obtain from (2.23) the $N-1$ additional relations

(2.26)
$$\sum_{i=0}^{N}\frac{1}{(i+j)!}\binom{i+j}{j}(D_x^j D_y^i K)\int_0^1 \eta^i\phi_{N-i}\,d\eta=0$$

for $j=1, 2, \cdots, N-1$. Equations (2.25)-(2.26) give N homogeneous equations for the $N+1$ unknowns $\int_0^1 \phi_{N-i}\eta^i\,d\eta$, $i=0, 1, \cdots, N$. These equations can be used to express the moments $\int_0^1 \phi_{N-i}\eta^i\,d\eta$ for $i<N$, which appear in the expression for C, in terms of $\int_0^1 \phi_0\eta^N\,d\eta$. Upon using these expressions in (2.24) we can write

$$C=C^*\int_0^1 \phi_0\eta^N\,d\eta,$$

and then we obtain

$$\lambda_{2N+1}=C^*\int_0^1 P_N(\xi)\xi^N\,d\xi.$$

This completes the analysis of the case $K(0,0)\neq 0$ and we now summarize our main results as follows.

Result 1. Let K be smooth at the origin with $K(0,0)\neq 0$ and $a\ll 1$. Then the first eigenvalue of the Fredholm equation (1.1) has the asymptotic expansion

$$\lambda^{(0)}=K(0,0)a+\frac{1}{2}[K_x(0,0)+K_y(0,0)]a^2$$
$$+\left\{\frac{1}{6}[K_{xx}(0,0)+K_{yy}(0,0)]+\frac{1}{4}K_{xy}(0,0)+\frac{1}{12}\frac{K_x(0,0)K_y(0,0)}{K(0,0)}\right\}a^3+O(a^4),$$

and the corresponding eigenfunction has the expansion

$$\phi^{(0)}(\xi)=1+\frac{K_x(0,0)}{K(0,0)}\left(\xi-\frac{1}{2}\right)a+O(a^2).$$

For each $N \geqq 0$ the leading terms in the expansions of the Nth eigenvalue and eigenfunction are

$$\lambda^{(N)} = C^* \left[\int_0^1 P_N(\xi)\xi^N \, d\xi \right] a^{2N+1} + O(a^{2N+2}),$$

$$\phi^{(N)}(\xi) = P_N(\xi) \Big/ \sqrt{\int_0^1 P_N^2(\xi)\, d\xi} + O(a).$$

Here $P_N(\xi)$ is the polynomial of degree N with leading coefficient one which is orthogonal to $1, \xi, \cdots, \xi^{N-1}$ on the unit interval. The constant C^* is computed by first solving the linear system

$$A\mathbf{v} = -\mathbf{b}$$

where $\mathbf{b}$ is the N-vector with ith component

$$b_i = \frac{1}{N!\,i!}(D_x^i D_y^N K)(0,0), \qquad i = 0, 1, \cdots, N-1$$

and A is an $N \times N$ matrix with components

$$A_{ij} = \frac{1}{(i+j)!}\binom{i+j}{j}(D_x^i D_y^j K)(0,0), \qquad i, j = 0, 1, \cdots, N-1.$$

In terms of the components v_i of the solution vector $\mathbf{v}$, C^* is given by

$$C^* = \sum_{i=0}^{N} \frac{1}{(i+N)!}\binom{i+N}{N}(D_x^N D_y^i K)(0,0)v_i, \qquad v_N \equiv 1.$$

Slepian [10] considered the slightly more general eigenvalue problem

$$\int_a^b K(c\xi, c\eta) w(\eta)\phi(\eta)\, d\eta = \lambda\phi(\xi)$$

where $w(\cdot)$ is a weight function. He assumed that $K(0,0) = 1$ and that $K(\cdot,\cdot)$ can be expanded as a double series involving $P_n(\xi)$ and $P_n(\eta)$, which are polynomials orthogonal on $[a, b]$ with weight $w(\cdot)$. When $w = 1$, as we have assumed, these polynomials are closely related to Legendre polynomials. Using this expansion of the kernel, Slepian obtained the expansions

$$\lambda_n = c^{2n} \sum_{i=0}^{\infty} \chi_i(n) c^i,$$

$$\psi_n(\xi) = \sum_{j=0}^{\infty} c^j \phi_j^{(n)}(\xi) = \sum_{j=0}^{\infty} c^j \sum_{k=-j}^{J} A_k^j(n) P_{n+k}(\xi)$$

for the eigenvalues and eigenfunctions. He developed a recursion scheme for computing the coefficients in these series. It involves inverting an $N \times N$ matrix closely related to A in Result 1.

Now we briefly discuss the case $K(0,0) = 0$, with K linear to leading order for small x and y, i.e., $(\nabla K)(0,0) \neq \mathbf{0}$. Then by setting $\lambda = \lambda_2 a^2 + \lambda_3 a^3 + O(a^4)$, we obtain from (1.2) at leading order in a

$$\lambda_2 \phi_0(\xi) = \xi K_x \int_0^1 \phi_0 \, d\eta + K_y \int_0^1 \eta\phi_0 \, d\eta. \tag{2.27}$$

It follows that ϕ_0 is linear in ξ. Setting $\phi_0(\xi)=A+B\xi$ in (2.27), we obtain the matrix eigenvalue problem

$$\begin{pmatrix} \frac{1}{2}K_y & \frac{1}{3}K_y \\ K_x & \frac{1}{2}K_x \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\lambda_2\begin{pmatrix} A \\ B \end{pmatrix}. \tag{2.28}$$

Thus there are two possibilities for λ_2,

$$\lambda_2=\frac{1}{2}\left[\frac{K_x(0,0)+K_y(0,0)}{2}\pm\sqrt{\left(\frac{K_x(0,0)+K_y(0,0)}{2}\right)^2+\frac{K_x(0,0)K_y(0,0)}{3}}\right].$$

This yields the leading terms in the first two eigenvalues $\lambda^{(0)}$ and $\lambda^{(1)}$. The eigenvalues need not be real since the kernel is not symmetric to leading order, unless $K_x(0,0)=K_y(0,0)$.

Finding the correction term λ_3 involves analyzing the problem

$$\begin{aligned} &\lambda_2\phi_1(\xi)-\xi K_x\int_0^1\phi_1\,d\eta-K_y\int_0^1\eta\phi_1\,d\eta \\ &\quad=-\lambda_3\phi_0(\xi)+\frac{1}{2}\xi^2K_{xx}\int_0^1\phi_0\,d\eta+\xi K_{xy}\int_0^1\eta\phi_0\,d\eta+\frac{1}{2}K_{yy}\int_0^1\eta^2\phi_0\,d\eta. \end{aligned} \tag{2.29}$$

We now set $\phi_0=A_0+B_0\xi$ and $\phi_1=A_1+B_1\xi+C_1\xi^2$, with A_0 and B_0 computed from (2.28). Then we obtain from (2.29) the equations

$$\lambda_2C_1=\tfrac{1}{2}K_{xx}(A_0+\tfrac{1}{2}B_0)$$

and

$$\lambda_2\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}-\begin{pmatrix} \frac{1}{2}K_y & \frac{1}{3}K_y \\ K_x & \frac{1}{2}K_x \end{pmatrix}\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$
$$=\begin{pmatrix} \dfrac{C_1}{4}K_y+\lambda_3A_0+\dfrac{1}{2}K_{yy}\left(\dfrac{A_0}{3}+\dfrac{B_0}{4}\right) \\ \dfrac{C_1}{3}K_x+\lambda_3B_0+K_{xy}\left(\dfrac{A_0}{2}+\dfrac{B_0}{3}\right) \end{pmatrix}.$$

Since λ_2 is an eigenvalue of (2.28), this inhomogeneous problem will have a solution only if the right side is orthogonal to all solutions of the matrix problem adjoint to (2.28). This condition determines λ_3 in terms of the previously computed λ_2, A_0, B_0, C_1. The correction terms λ_k for $k\geqq 4$ can be obtained similarly.

We have now determined expansions of the first two eigenvalues $\lambda^{(0)}$ and $\lambda^{(1)}$. To compute the others, we proceed as in the case $K(0,0)\neq 0$. We omit the details, but it is not hard to show that if $\lambda_2=0$, then $\lambda_3=\lambda_4=0$. Thus the third eigenvalue $\lambda^{(2)}$ is $O(a^5)$ with the coefficient being computed just as it was in the case $K(0,0)\neq 0$. The fourth eigenvalue $\lambda^{(3)}$ is $O(a^7)$ and the calculation of all further eigenvalues is identical to that in the previous case.

These results show that the main difference between the cases where $K(x,y)$ is constant and where K is linear for small x and y lies in the fact that in the former case $\lambda^{(0)}=O(a)$ and $\lambda^{(1)}=O(a^3)$, while in the latter case $\lambda^{(0)}$ and $\lambda^{(1)}$ are both $O(a^2)$. If $K(x,y)$ is quadratic to leading order for small x and y, the first three eigenvalues $\lambda^{(0)}$, $\lambda^{(1)}$, and $\lambda^{(2)}$ are $O(a^3)$ and the remaining ones are $O(a^{2N+1})$, $N\geqq 3$. The expansions of the first three eigenvalues can be computed by analyzing a 3×3 matrix problem analogous to (2.28), while the remaining ones can be computed by using the procedure outlined in Result 1.

Thus, it seems that regardless of the order of vanishing of $K(x, y)$ near $x=y=0$, the following rule applies, which may be called "conservation of asymptotic orders." If the Taylor series expansion of K at $(0,0)$ starts with a homogeneous polynomial of degree M then

$$\lambda^{(N)}=O(a^{M+1}), \quad 0\leqq N\leqq M, \qquad \lambda^{(N)}=O(a^{2N+1}), \quad N\geqq M+1.$$

Thus, regardless of M, we have

$$\lambda^{(0)}\lambda^{(1)}\cdots\lambda^{(N-1)}=O(a^{N^2}), \qquad N\geqq M+1.$$

Although we have considered only one-dimensional Fredholm equations in this section, the basic ideas should carry over to higher-dimensional problems.

3. Expansions for $a\gg 1$. To study (1.1) for $a\gg 1$ we restrict K to be either (i) a symmetric difference kernel $K(|x-y|)$, (ii) a difference kernel $K(x-y)$, or (iii) a symmetric kernel $K[|x-y|, (x+y)/a]$ with slow dependence on $x+y$ for a large. First we will obtain results for case (i) valid in any number of dimensions, and then show that the analysis of case (ii) can be reduced to that of case (i). For case (iii), however, the structure of the problem is changed drastically by the apparently small alteration of the kernel from the form in case (i) and it requires a separate analysis. In all cases we will assume the finiteness of whatever moments of K with respect to $\xi=x-y$ we need,

$$\int_{-\infty}^{\infty}|\xi|^n K(\xi, z)\,d\xi<\infty, \qquad n=1,2,\cdots,n_0.$$

We also assume that for $x=y$, $K>0$. With these assumptions $aK(a|x-y|, x+y)$ tends to a multiple of the delta function $\delta(x-y)$ as $a\to\infty$.

Let us begin with the case (i), in wnich (2.1) can be written as follows:

$$\int_0^1 aK(a|x-y|)\phi(y)\,dy=\lambda\phi(x). \tag{3.1}$$

Upon letting a tend to infinity in (3.1), and assuming that $\phi(x)$ and λ tend to limits $\phi_0(x)$ and λ_0, we obtain $\int_{-\infty}^{\infty}K(|\xi|)\,d\xi\,\phi_0(x)=\lambda_0\phi_0(x)$. This yields

$$\lambda_0=\int_{-\infty}^{\infty}K(|\xi|)\,d\xi, \tag{3.2}$$

which shows that all eigenvalues of (3.1) approach the same limit as $a\to\infty$. To determine $\phi_0(x)$ and the splitting apart of the eigenvalues for finite values of a, we set $y=x+a^{-1}\xi$ in (3.1) and expand $\phi(x+a^{-1}\xi)$ in a Taylor series to obtain

$$\begin{aligned}\lambda\phi(x)&=\int_{-\infty}^{\infty}K(|\xi|)\,d\xi\,\phi(x)+\frac{1}{2}\int_{-\infty}^{\infty}\xi^2K(|\xi|)\,d\xi\,\phi''(x)a^{-2}\\&\quad+\frac{1}{4!}\int_{-\infty}^{\infty}\xi^4K(|\xi|)\,d\xi\,\phi^{(iv)}(x)a^{-4}+O(a^{-6}).\end{aligned} \tag{3.3}$$

Now we seek λ and ϕ in the forms

$$\begin{aligned}\lambda(a)&=\int_{-\infty}^{\infty}K(|\xi|)\,d\xi+a^{-1}\lambda_1+a^{-2}\lambda_2+\cdots,\\ \phi(x,a)&=\phi_0(x)+a^{-1}\phi_1(x)+\cdots.\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and equating coefficients of the first three powers of a^{-1} yields $\lambda_1=0$ and

$$\lambda_2\phi_0(x)=\gamma\phi_0''(x), \qquad \gamma=\frac{1}{2}\int_{-\infty}^{\infty}\xi^2K(|\xi|)\,d\xi, \tag{3.5}$$

$$\lambda_3\phi_0(x)+\lambda_2\phi_1(x)=\gamma\phi_1''(x). \tag{3.6}$$

Thus ϕ_0 satisfies the homogeneous second-order differential equation (3.5), while ϕ_1 satisfies its inhomogeneous form (3.6).

The general solution of (3.5) is

$$\phi_0(x)=a_0\cos(-\lambda_2/\gamma)^{1/2}x+b_0\sin(-\lambda_2/\gamma)^{1/2}x. \tag{3.7}$$

To determine the constants in (3.7) we require additional conditions, such as boundary conditions at $x=0$ and $x=1$. Therefore we set $x=0$ in (3.1) and let $a\to\infty$ to get $\int_0^\infty K(|\xi|)\,d\xi\,\phi_0(0)=\lambda_0\phi_0(0)$ which is consistent with (3.2) only if $\phi_0(0)=0$ assuming that $\int_0^\infty K\,d\xi\neq 0$. Proceeding similarly at $x=1$ yields $\phi_0(1)=0$. With these boundary conditions we find that λ_2 and the normalized eigenfunction $\phi_0(x)$ are given by

$$\begin{aligned}\lambda_2&=-\gamma(N\pi)^2, \qquad N=1,2,\cdots,\\ \phi_0(x)&=2^{1/2}\sin N\pi x.\end{aligned} \tag{3.8}$$

The preceding derivation of the boundary conditions is not satisfactory because the convergence of the eigenfunction $\phi(x,a)$ to $\phi_0(x)$ as $a\to\infty$ is not uniform near the endpoints, as we shall now show. Therefore to derive the boundary conditions correctly we will construct boundary layer expansions of ϕ near the endpoints and match them to ϕ_0. Since the boundary layers are of width a^{-1}, near $x=0$ we introduce the local variable $\xi=ax$. Then we write ϕ in the form

$$\phi(x,a)=\psi(\xi,a)=\psi_0(\xi)+a^{-1}\psi_1(\xi)+\cdots.$$

Substituting this expansion into (3.1), together with (3.4) for λ, and equating coefficients of powers of a^{-1} yields

$$\int_0^\infty K(|\xi-\eta|)\psi_0(\eta)\,d\eta=\lambda_0\psi_0(\xi), \tag{3.9}$$

$$\begin{gathered}\int_0^\infty K(|\xi-\eta|)\psi_1(\eta)\,d\eta=\lambda_0\psi_1(\xi),\\ \int_0^\infty K(|\xi-\eta|)\psi_2(\eta)\,d\eta=\lambda_0\psi_2(\xi)+\lambda_2\psi_0(\xi).\end{gathered} \tag{3.10}$$

Equation (3.9) is of Wiener–Hopf type, so it can be solved by Fourier transformation followed by a suitable factorization. First we define the functions $F(\alpha)$ and $G(\alpha)$ as the one-sided Fourier transforms

$$F(\alpha)=\int_0^\infty e^{i\alpha\xi}\psi_0(\xi)\,d\xi,$$

$$G(\alpha)=\int_{-\infty}^0 e^{i\alpha\xi}\int_0^\infty K(|\xi-\eta|)\psi_0(\eta)\,d\eta\,d\xi.$$

Then after Fourier transformation of (3.9) we can write it as

$$F(\alpha)=G(\alpha)[\lambda_0-\hat{K}(\alpha)]^{-1}. \tag{3.11}$$

Here $\hat{K}$ is the two-sided Fourier transform of K. From (3.2) we see that $\lambda_0 = \hat{K}(0)$, and the symmetry of K implies that $\hat{K}'(0) = 0$. Therefore the denominator in (3.11) vanishes to second order at $\alpha = 0$. If $G(0) \neq 0$, as we assume, then $F(\alpha)$ has a double pole at $\alpha = 0$. This implies that as $\xi \to \infty$, $\psi_0(\xi)$ has the form

$$\psi_0(\xi) \sim A_0 + B_0 \xi \quad \text{as } \xi \to \infty.$$

We can now employ the asymptotic matching condition

$$\phi(x, a)|_{x \ll 1} \sim \psi(\xi, a)|_{\xi \gg 1}. \tag{3.12}$$

First we expand $\phi_0(x)$, for x small and use the form of $\psi_0(\xi)$ for ξ large given above. Then matching yields

$$a_0 + b_0(-\lambda_2/\gamma)^{1/2} x + a^{-1}\phi_1(0) \sim A_0 + B_0 \xi = axB_0 + A_0.$$

Since the right side is $O(a)$ and the left side $O(1)$ we must have $B_0 = 0$. But then $\psi_0(\xi) \equiv 0$ and therefore $A_0 = 0$. Then the matching requires $a_0 = 0$, which shows that $\phi_0(0) = 0$. In the same way, by boundary layer analysis and matching near $x = 1$, we find that $\phi_0(1) = 0$. This determines the boundary conditions satisfied by ϕ_0, and shows that they are the same as those given by the argument which ignored the boundary layers. Therefore the results (3.8) for $\phi_0(x)$ and λ_2 are correct.

To calculate λ_3 we multiply (3.6) by $\phi_0(x)$ and integrate from $x = 0$ to $x = 1$, which yields the solvability condition

$$\lambda_3 = 2^{1/2} N\pi\gamma[\phi_1(0) - (-1)^N \phi_1(1)]. \tag{3.13}$$

To determine the boundary values of ϕ_1, which occur in (3.13), we again use (3.12). Now since $\psi_0(\xi) \equiv 0$ we have $\psi(\xi, a) \sim a^{-1}\psi_1(\xi)$ where ψ_1 satisfies (3.10). Since this problem is the same as (3.9) we have

$$\psi_1(\xi) \sim A_1 + B_1 \xi \quad \text{as } \xi \to \infty. \tag{3.14}$$

Then the matching condition (3.12) becomes

$$x\phi_0'(0) + a^{-1}\phi_1(0) \sim a^{-1}A_1 + B_1 x. \tag{3.15}$$

Thus $B_1 = \phi_0'(0) = 2^{1/2} N\pi$. Once B_1 is fixed, the solution $\psi_1(\xi)$ of (3.10) is uniquely determined and then A_1 is determined by (3.14). The matching condition (3.15) finally gives

$$\phi_1(0) = A_1. \tag{3.16}$$

The construction of the boundary layer expansion near $x = 1$ and the matching procedure are similar to those at $x = 0$. We introduce η defined by $\eta = a(1 - x)$ with $\phi(x) = \tilde{\psi}(\eta) \sim a^{-1}\tilde{\psi}_1(\eta)$. For large η we have

$$\tilde{\psi}_1(\eta) \sim \tilde{A}_1 + \tilde{B}_1 \eta$$

so that

$$(x - 1)\phi_0'(1) + a^{-1}\phi_1(1) \sim a^{-1}(\tilde{A}_1 + \tilde{B}_1 \eta).$$

Hence $\tilde{B}_1 = -\phi_0'(1) = (-1)^{N+1} 2^{1/2} N\pi$ and $\phi_1(1) = \tilde{A}_1$. The symmetry of the kernel and the conditions above give $\psi_1(t) = (-1)^{N+1}\tilde{\psi}_1(t)$ and hence $\tilde{A}_1 = (-1)^{N+1} A_1$. Then from (3.13) we get

$$\lambda_3 = 2^{1/2} N\pi[A_1 + (-1)^{N+1}\tilde{A}_1]\gamma = 2^{3/2} N\pi\gamma A_1.$$

The constant A_1 is determined by the solution of (3.10), but it can be given explicitly only for some special forms of the kernel.

Now we summarize our results for case (i).

Result 2. When $a \gg 1$ and the kernel has the form $K(a|x-y|)$, the eigenvalue $\lambda^{(N)}$, $N \geqq 1$ of (1.1) has the asymptotic expansion

$$\lambda^{(N)} = \int_{-\infty}^{\infty} K(|\xi|)\, d\xi - \frac{(N\pi)^2}{2a^2} \int_{-\infty}^{\infty} \xi^2 K(|\xi|)\, d\xi + \frac{2(N\pi)^2}{a^3}\left(\int_{-\infty}^{\infty} \xi^2 K(|\xi|)\, d\xi\right) A + O(a^{-4}).$$

The expansion of the corresponding eigenfunction is

$$\phi^{(N)}(x) \sim \begin{cases} 2^{1/2} N\pi a^{-1}\psi(ax), & x = O(a^{-1}), \\ 2^{1/2}\sin(N\pi x), & x = O(1), \quad 1-x = O(1), \\ 2^{1/2} N\pi a^{-1}(-1)^{N+1}\psi(a(1-x)), & 1-x = O(a^{-1}). \end{cases}$$

Here $\psi(t)$ is the solution of the Wiener-Hopf equation

$$\int_0^{\infty} K(|t-s|)\psi(s)\, ds = \psi(t), \qquad t > 0;$$

normalized by the condition $\psi(t) \sim t$, $t \to \infty$, and A is defined by $A = \lim_{t\to\infty}[\psi(t) - t]$.

Although only the first two moments of K appear in our expansion of $\lambda^{(N)}$, higher order moments appear at higher orders in a^{-1}. If some moment is infinite, the expansions of $\lambda^{(N)}$ and $\phi^{(N)}$ do not consist of powers of a^{-1} beyond the corresponding order. We also note that $\lambda_2 a^{-2} = -(N\pi/a)^2\gamma$ becomes comparable to λ_0 when $N = O(a)$. Therefore our expansion is not valid for such large values of N. A different analysis, probably of WKB type, would then be needed.

Wiener-Hopf boundary layers of the type encountered here also occurred in the work of Olmstead and Gautesen [9]. They mostly considered the inhomogeneous version of (3.1). Various special forms of the kernel led to applications in heat conduction, diffraction theory, and molecular dissociation.

Let us now consider a bounded domain Ω in R^n and replace (3.1) by the integral equation

$$a^n \int_{\Omega} K(a|x-y|)\phi(y)\, dy = \lambda\phi(x), \qquad x \in \Omega. \tag{3.17}$$

This corresponds to the original equation (1.1) over the magnified domain $a\Omega$. Suppose that the first two moments of K are finite and that $\phi(x) \to \phi_0(x)$ and $\lambda \to \lambda_0$ as $a \to \infty$. Then letting $a \to \infty$ in (3.17) yields

$$\lambda_0 = \int_{R^n} K(|\xi|)\, d\xi.$$

Thus as before, all eigenvalues tend to λ_0 as $a \to \infty$.

To obtain the eigenfunction ϕ and a correction to the eigenvalue we write $\lambda = \lambda_0 - \mu a^{-2} + \cdots$, $\phi(x, a) = \phi_0(x) + \cdots$. Then by using these expansions in (3.17) and proceeding as before, we obtain

$$\left(\frac{S_n}{2n}\int_0^{\infty} r^{n+1} K(r)\, dr\right)\Delta\phi_0(x) + \mu\phi_0(x) = 0, \qquad x \in \Omega. \tag{3.18}$$

Here $S_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in R^n. As before we find that ϕ_0 satisfies the boundary condition $\phi_0(x) = 0$, $x \in \partial\Omega$. Then (3.18) with this boundary condition is an eigenvalue problem which has eigenfunctions $\phi_0^{(N)}(x)$ and eigenvalues $\mu^{(N)}$.

We have thus obtained the following result.

Result 3. When $a \gg 1$, the eigenvalues and eigenfunctions of (3.17) are

$$\lambda^{(N)} = S_n \int_0^\infty r^{n-1} K(r)\, dr - a^{-2}\mu^{(N)} + O(a^{-3}),$$

$$\phi^{(N)}(x) = \phi_0^{(N)}(x) + O(a^{-1}).$$

Here $(\mu^{(N)}, \phi_0^{(N)}(x))$, are eigenvalues and eigenfunctions of the Dirichlet problem

$$\frac{1}{2n} S_n \left(\int_0^\infty r^{n+1} K(r)\, dr \right) \Delta\phi_0^{(N)}(\mathbf{x}) + \mu^{(N)} \phi_0^{(N)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega,$$

$$\phi_0^{(N)}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega$$

and $S_n = 2\pi^{n/2}/\Gamma(n/2)$.

Obtaining the $O(a^{-3})$ term in the eigenvalue expansion requires a more careful boundary layer analysis near $\partial\Omega$. By employing suitable local coordinates, we can reduce the boundary layer problem to a one-dimensional Wiener–Hopf equation. (See Gautesen [8] for an example of this type in two dimensions.)

Now we consider case (ii) where the kernel is a difference kernel that is not necessarily symmetric. We begin by denoting the solution of (2.1) $\hat{\phi}(x)$ and writing it in the form

$$\hat{\phi}(x) = e^{acx}\phi(x).$$

Then (2.1) becomes

$$\int_0^1 aK[a(x-y)]\, e^{ac(y-x)} \phi(y)\, dy = \lambda\phi(x).$$

We choose the constant c to eliminate the first odd power in the expansion of the preceding integral:

$$\int_{-\infty}^\infty \xi K(\xi)\, e^{-c\xi}\, d\xi = 0. \tag{3.19}$$

Then the kernel is symmetric through terms of second degree, so we can expand the integral equation for $a \gg 1$ as before to get

$$\lambda = \int_{-\infty}^\infty e^{-c\xi} K(\xi)\, d\xi + \frac{\mu}{a^2} + O(a^{-3}),$$

$$\frac{1}{2}\left[\int_{-\infty}^\infty \xi^2\, e^{-c\xi} K(\xi)\, d\xi \right] \phi_0''(x) - \mu\phi_0(x) = 0,$$

$$\phi_0(0) = \phi_0(1) = 0.$$

By using the previous results we see that the eigensolution to our original problem is

$$\lambda^{(N)} = \int_{-\infty}^\infty e^{-c\xi} K(\xi)\, d\xi - \frac{(N\pi)^2}{2a^2} \int_{-\infty}^\infty \xi^2\, e^{-c\xi} K(\xi)\, d\xi + O(a^{-3}), \tag{3.20}$$

$$\hat{\phi}^{(N)}(x) = e^{acx}[\phi_0^{(N)}(x) + O(a^{-1})],$$

where $\phi_0^{(N)}(x)$ is given by Result 2. The eigenfunctions $\hat{\phi}^{(N)}$ in (3.20) are unnormalized.

The existence of a solution to (3.19) follows from the assumption that the integral exists for all real c. In contrast to case (i), the eigenfunctions $\hat{\phi}^{(N)}$ in (3.20) will be concentrated in the region $x=O(a^{-1})$ if $c<0$ and in the region $1-x=O(a^{-1})$ if $c>0$. Thus, it is important to have the boundary layer corrections to $\phi_0^{(N)}(x)$, as given in Result 2.

Next we analyze case (iii) for which (2.1) is

$$\int_0^1 K(a|x-y|, x+y)\phi(y)\, dy = \lambda a^{-1}\phi(x) \tag{3.21}$$

and we define

$$V(x)=\int_0^1 K(|\xi|, 2x)\, d\xi. \tag{3.22}$$

The analysis of the previous cases does not apply since letting $a\to\infty$ in (3.21) yields $V(x)\phi(x)\sim\lambda\phi(x)$ which makes sense only if $V(x)$ is a constant. We assume that the maximum of $V(x)$ on the interval $[0, 1]$ occurs at a finite number of points, some of which may be the boundary points $x=0$, 1.

First we consider the case where $V(x)$ is strictly increasing on the interval $[0, 1]$ so that $V'(x)>0$. The key to analyzing (3.21) lies in the observation that the function $-V(x)$ is similar to the potential function $U(x)$ in the Schrödinger equation $\psi''(x)+a^2[E-U(x)]\psi(x)=0$. The eigenvalues of the Schrödinger equation for $E=O(1)$ and $a\to\infty$ satisfy $E\to U(x^*)$ where x^* is the minimum of $U(x)$ on $[0, 1]$. Motivated by this analogy, we expand (3.21) for $a\gg 1$ to obtain

$$\begin{aligned}&\frac{1}{a}\phi(x)\int_{-\infty}^{\infty} K(|\xi|, 2x)\, d\xi+\frac{\phi''(x)}{2a^3}\int_{-\infty}^{\infty}\xi^2 K(|\xi|, 2x)\, d\xi\\ &+\frac{\phi(x)}{2a^3}\int_{-\infty}^{\infty}\xi^2 K_{22}(|\xi|, 2x)\, d\xi\\ &+\frac{\phi'(x)}{a^3}\int_{-\infty}^{\infty}\xi^2 K_2(|\xi|, 2x)\, d\xi+\cdots=\lambda a^{-1}\phi(x).\end{aligned} \tag{3.23}$$

Here K_2 stands for the derivative of the kernel with respect to its second argument.

Now we set

$$\lambda(a)=\int_{-\infty}^{\infty} K(|t|, 2)\, dt+\mu a^{-2/3}+o(a^{-2/3}),$$

$$\xi=(1-x)a^{2/3},$$

$$\phi(x, a)=\psi(\xi, a)=\psi_0(\xi)+o(1).$$

Then we obtain from (3.23), to leading order, the problem

$$\psi_0(\xi)\left[\mu+2\xi\int_{-\infty}^{\infty} K_2(|t|, 2)\, dt\right]=\frac{1}{2}\left[\int_{-\infty}^{\infty} t^2 K(|t|, 2)\, dt\right]\psi_0''(\xi). \tag{3.24}$$

An argument identical to that used in case (i) yields the boundary condition

$$\psi_0(0)=0. \tag{3.25}$$

In order that $\psi(\xi)$ be bounded, we reject the exponentially growing solutions to (3.24), which is equivalent to requiring

$$\psi_0(\infty)=0. \tag{3.26}$$

Equation (3.24) can be transformed easily into the standard Airy equation. Its solution satisfying (3.26) is

$$\psi_0(\xi) = \mathrm{Ai}\,[\beta(\mu + \xi V'(1))],$$

(3.27)
$$\beta = \left[\frac{2}{(V'(1))^2 \int_{-\infty}^{\infty} t^2 K(|t|, 2)\, dt}\right]^{1/3}.$$

The condition (3.25) then implies that

$$\mu^{(N)} = -\frac{1}{\beta}\,\delta_N, \qquad N = 1, 2, \cdots,$$

where the roots of Ai $(z) = 0$ are denoted $z = -\delta_1, -\delta_2, \cdots$ with $\delta_k > 0$ and $\delta_{k+1} > \delta_k$.

From (3.27) we see that the eigenfunctions for this case have all their zeros in the region $x - 1 = O(a^{-2/3})$. They are analogous to "whispering gallery" modes since they correspond to waves localized near a boundary. The case in which $V(x)$ is strictly decreasing on $[0, 1]$ is completely analogous with the modes localized near $x = 0$ rather than $x = 1$. To obtain the terms in ϕ beyond ψ_0 we would again have to analyze the Wiener–Hopf equation for $x - 1 = O(a^{-1})$. We note that the thickness of the boundary layer in which (3.24) holds is $O(a^{-2/3})$, which is thicker than the $O(a^{-1})$ layer corresponding to the Wiener–Hopf problem.

Next we consider the case in which the maximum of $V(x)$ occurs at an interior point x_0 with $V''(x_0) < 0$. We introduce into (3.21) the scaling

$$\lambda = \int_{-\infty}^{\infty} K(|t|, 2x_0)\, dt + \mu a^{-1} + o(a^{-1}),$$

$$\eta = (x - x_0)a^{1/2},$$

$$\phi(x, a) = \psi(\eta, a) = \psi_0(\eta) + o(1).$$

Expanding the resulting equation for $a \gg 1$ yields

(3.28)
$$\psi_0(\eta)\left[\mu - \frac{1}{2} V''(x_0)\eta^2\right] = \frac{1}{2}\left[\int_{-\infty}^{\infty} t^2 K(|t|, 2x_0)\, dt\right]\psi_0''(\eta).$$

The solution of (3.28) that decays as $\eta \to \pm\infty$ is given by

(3.29)
$$\psi_0^{(N)}(\eta) = e^{-\eta^2/4\alpha^2}\,\mathrm{He}_N\left(\frac{\eta}{\alpha}\right)$$

with He_N the Nth Hermite polynomial and

$$\alpha = 2^{-1/2}\left[\frac{\int_{-\infty}^{\infty} t^2 K(|t|, 2x_0)\, dt}{|V''(x_0)|}\right]^{1/4}.$$

The corresponding eigenvalue is

$$\mu^{(N)} = -\left(N + \frac{1}{2}\right)|V''(x_0)|^{1/2}\left[\int_{-\infty}^{\infty} t^2 K(|t|, 2x_0)\, dt\right]^{1/2}$$

for $N = 0, 1, 2, \cdots$.

Sirovich and Knight [1], [2] derived a quantization rule to calculate the large eigenvalues of an equation similar to our (3.21). Their rule breaks down when the area enclosed by a certain curve becomes small. For this case, they introduced a new scaling and obtained eigenfunctions of the type (3.29).

Below we summarize the main formulas.

Result 4. Consider the integral equation (2.1) with the kernel $K(a|x-y|, x+y)$. The eigensolutions for $a \gg 1$ depend on the function $V(x)=\int_{-\infty}^{\infty} K(|t|, 2x)\,dt$.

(i) If $V'(x)>0$ for $x\in[0,1]$, then

$$\lambda^{(N)}=\int_{-\infty}^{\infty} K(|t|,2)\,dt-\frac{1}{\beta}\,\delta_N a^{-2/3}+o(a^{-2/3}),$$

$$\phi^{(N)}(x,a)=\mathrm{Ai}\,[\beta\xi V'(1)-\delta_N]+o(1), \qquad \xi=(1-x)a^{2/3},$$

$$\beta=\left[\frac{1}{2}(V'(1))^2\int_{-\infty}^{\infty} t^2K(|t|,2)\,dt\right]^{-1/3}.$$

Here $z=-\delta_N$ is the Nth root of $\mathrm{Ai}\,(z)=0$.

(ii) If $V(x_0)\geqq V(x)$ for $x\in[0,1]$ and $V''(x_0)<0$, then

$$\lambda^{(N)}=V(x_0)-\frac{(N+\frac{1}{2})}{a}|V''(x_0)|^{1/2}\left[\int_{-\infty}^{\infty} t^2K(|t|,2x_0)\,dt\right]^{1/2}+o(a^{-1}),$$

$$\phi^{(N)}(x)=e^{-\eta^2/4\alpha^2}\,\mathrm{He}_N\left(\frac{\eta}{\alpha}\right)+o(1), \qquad \eta=(x-x_0)a^{1/2},$$

$$\alpha=2^{-1/2}\left[\int_{-\infty}^{\infty} t^2K(|t|,2x_0)\,dt\right]^{1/4}|V''(x_0)|^{-1/4}.$$

Here He_N is the Nth Hermite polynomial.

(iii) If $V'(x)<0$ for $x\in[0,1]$, then

$$\lambda^{(N)}=\int_{-\infty}^{\infty} K(|t|,0)\,dt-\gamma^{-1}\delta_N a^{-2/3}+o(a^{-2/3}),$$

$$\phi^{(N)}(x,a)=\mathrm{Ai}\,[\gamma\xi|V'(0)|-\delta_N]+o(1), \qquad \xi=xa^{2/3},$$

$$\gamma=\left[\frac{1}{2}[V'(0)]^2\int_{-\infty}^{\infty} t^2K(|t|,0)\,dt\right]^{-1/3}.$$

We close this section by discussing briefly the case when the second moment of the kernel is infinite. For an example we take

$$K(a|x-y|)=\frac{1}{1+a^2(x-y)^2}.$$

As in the analysis of case (i) we can conclude that

$$\lambda\sim\lambda_0=\int_{-\infty}^{\infty} K(|\xi|)\,d\xi=\pi.$$

To proceed further for large a, with x bounded away from zero and one, we approximate the integral equation by the infinite order ordinary differential equation (ODE)

$$\left(\frac{\lambda-\pi}{a}\right)\phi(x)=\frac{1}{a^2}\left[\log\left(\frac{1-x}{x}\right)\right]\phi'(x)-\frac{1}{a^2}\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi(x)$$
$$+\frac{1}{a^2}\sum_{n=2}^{\infty}\frac{\phi^{(n)}(x)}{n!(n-1)}[(1-x)^{n-1}+(-1)^n x^{n-1}]+o(a^{-2}). \tag{3.30}$$

Then after setting

$$\phi(x)=\phi_0(x)+o(1) \quad \text{and} \quad \lambda=\pi+\mu a^{-1}+\cdots,$$

we can simplify (3.30) to

$$\mu\phi_0(x)+\frac{\phi_0(1)}{1-x}+\frac{\phi_0(0)}{x}={}^*\!\int_0^1 \frac{\phi_0'(t)}{t-x}\,dt, \qquad 0<x<1. \tag{3.31}$$

Here $*$ is used to denote a principal value integral. As before, we conclude that $\phi_0(0)=\phi_0(1)=0$. Thus μ and ϕ_0 are determined by (3.31).

If the interval of integration were $(-\infty,\infty)$ rather than $(0,1)$, then (3.31) could be reduced to a differential equation of the form (3.5) by differentiating it on the one hand, applying a Hilbert transform on the other, and then eliminating the Hilbert transform of ϕ_0' between the two resulting equations. The eigenvalue condition on μ could then be given explicitly. However, we have not been able to obtain the eigenvalue μ explicitly for the finite interval problem.

If the kernel has the form

$$K(z)=1/[1+|z|^\beta], \qquad 1<\beta\leqq 2,$$

we have

$$\lambda=\int_{-\infty}^{\infty}[1+|t|^\beta]^{-1}\,dt+\mu a^{1-\beta}+\cdots.$$

The problem for (μ,ϕ_0) is given by

$$\begin{aligned}\mu\phi_0(x)&=\frac{(1-x)^{1-\beta}}{1-\beta}\phi_0(1)+\frac{x^{1-\beta}}{1-\beta}\phi_0(0)\\ &\quad+\frac{1}{\beta-1}\int_0^1\frac{\operatorname{sgn}(z-x)}{|z-x|^{\beta-1}}\phi_0'(z)\,dz, \qquad 0<x<1,\end{aligned} \tag{3.32}$$

which appears too difficult to solve explicitly.

4. Further properties of the eigenvalues. Let us now suppose that the kernel is symmetric and that

$$\int_0^\infty\int_0^\infty |K(x,y)|^2\,dx\,dy<\infty.$$

We consider the integral equation

$$\int_0^a K(x,y)\phi(y)\,dy=\lambda\phi(x) \tag{4.1}$$

and ask how the eigenvalues for the finite interval problem approach those of the infinite interval problem. The latter are guaranteed to exist by the general theory of Fredholm equations with symmetric L_2 kernels. We denote by λ_∞ and $\phi_\infty(x)$ the eigenvalues and eigenfunctions for the problem with $a=\infty$. Then we assume that the corresponding quantities for $a<\infty$ can be expressed as

$$\begin{aligned}\lambda&=\lambda_\infty+f(a)\mu+\cdots,\\ \phi(x)&=\phi_\infty(x)+g(a)\psi(x)+\cdots,\end{aligned} \tag{4.2}$$

where the order functions f and g are to be determined. Clearly, $f(\infty)=g(\infty)=0$.

Substituting (4.2) into (4.1) yields, to leading order, the balance

$$f(a)\mu\phi_\infty(x)+\lambda_\infty g(a)\psi(x) \sim g(a)\int_0^a K(x,y)\psi(y)\,dy-\int_a^\infty K(x,y)\phi_\infty(y)\,dy. \tag{4.3}$$

Multiplying (4.3) by $\phi_\infty(x)$ and integrating from $x=0$ to $x=\infty$ yields

$$f(a)\mu \sim -\lambda_\infty \int_a^\infty \phi_\infty^2(y)\,dy \Big/ \int_0^\infty \phi_\infty^2(y)\,dy.$$

Thus, the correction to the eigenvalue λ_∞ involves the tail of the infinite interval eigenfunction. The function $\psi(x)$ can be computed by setting $f(a)=g(a)=\int_a^\infty \phi_\infty^2(y)\,dy$ and solving the inhomogeneous problem (4.3) after replacing $\sim$ by $=$. Thus we have obtained Result 5.

Result 5. If $K(x,y)$ is a symmetric L_2 function over $(0,\infty)\times(0,\infty)$, then the eigenvalues for (4.1) have the expansions

$$\lambda^{(N)}=\lambda_\infty^{(N)}\left[1-\int_a^\infty [\phi_\infty^{(N)}(y)]^2\,dy(1+o(1))\right],$$

where $\lambda_\infty^{(N)}$ and $\phi_\infty^{(N)}$ are the eigenvalues and normalized eigenfunctions for (4.1) with $a=\infty$.

This result shows that the *magnitude* of the eigenvalue increases as a increases in agreement with the theorem in the Introduction. This behavior is exactly opposite that of second-order ordinary differential equations for which the eigenvalues decrease as the length of the interval increases. This opposite behavior is related to the fact that in some cases integral operators, such as that in (4.1), are inverses of differential operators.

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