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APPROXIMATION OF THE ZAKAI EQUATION BY THE SPLITTING UP METHOD*

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Abstract. The objective of this article is to apply an operator splitting method to the time integration of the Zakaï equation. Using this approach the numerical integration can be decomposed into a stochastic step and a deterministic one, both of them much simpler to handle than the original problem. A strong convergence theorem is given, in the spirit of existing results for deterministic problems.

Key words. nonlinear filtering, fractional step methods, Zakaï equation

AMS(MOS) subject classifications. 35K99, 60H15, 65M10

Introduction. We consider in this article an approximation technique for the Zakaï equation of nonlinear filtering. For such a filtering problem, the state and measurement processes are of the form

$$dX_t = g(X_t, t)\,dt + \sigma(X_t, t)\,dV_t,$$
$$dY_t = h(X_t, t)\,dt + dW_t,$$

where X_t denotes the system state at time t, Y_t denotes the measured output of the system at time t, and $\{V_t: t \geqq 0\}$, $\{W_t: t \geqq 0\}$ are independent Brownian motions. The goal of nonlinear filtering is to determine the conditional distribution of the state at time t given the measurements up through time t. The Zakaï equation, given by

$$dy + A^*(t)y\,dt = B(t)y \cdot dW_t, \tag{1}$$

is an evolution equation for the unnormalized conditional density of the state, given the measurements. The operators A and B are defined by

$$A(t)\varphi = \sum_{i,j} a_{ij}\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \sum_i g_i \frac{\partial \varphi}{\partial x_i}$$

and

$$(B(t)\varphi)(x) = \varphi(x) \cdot h(x, t),$$

with A^* denoting the formal adjoint of A, and $a_{ij} = (\frac{1}{2}\sigma\sigma^*)_{ij}$.

We apply the idea of splitting up, considering $A(t)y\,dt - B(t)y \cdot dw$ as the sum of two operators.

Hence we write a sequence of problems of the form

$$d\varphi + A^*(t)\varphi\,dt = 0$$
$$d\psi = B(t)\psi \cdot dw^*$$

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which are considerably simpler than (1). Indeed the φ equation is deterministic and the ψ equation has a closed-form solution.

The technique of splitting up for deterministic partial differential equations has been used extensively by many authors. It might also be noted here that this technique is very much like the Trotter product formula from semigroup theory. We refer here to the work of Temam [8], which is used as the background for our developments, and also to Glowinski [3] and Marchuk [6] for applications in mathematical physics.

We refer to Legland [4] and Elliott and Glowinski [2] for semidiscretization schemes of the Zakaï equation, which, although not related to the splitting up method, bear some analogies with those developed in this article. Legland's newest work, which contains convergence arguments of a probabilistic nature for a fully discrete approximation of the Zakaï equation, also bears some similarities to the splitting-up scheme.

1. Setting of the problem.

1.1. Notation—assumptions. We make the following assumptions on functions of the state and measurement processes:

$$g \in L^\infty(R^n \times (0,\infty); R^n), \sigma \in L^\infty(R^n \times (0,\infty); \mathscr{L}(R^n, R^n)), \tag{1.1}$$

with g and σ Lipschitz in x, uniformly in t, and

$$h \in L^\infty(R^n \times (0,\infty); R^m). \tag{1.2}$$

Let Ω, $\mathscr{A}$, P be a probability space on which exists an m-dimensional standard Wiener process $w(t)$, and let

$$F^t = \sigma(w(s), s \leqq t).$$

Define the second-order differential operator

$$A(t)\varphi = -\sum_{i,j} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \sum_i g_i \frac{\partial \varphi}{\partial x_i} \tag{1.3}$$

where we have set

$$a = \tfrac{1}{2}\sigma\sigma^* \ (a = \text{matrix } a_{ij}). \tag{1.4}$$

We assume that there exists an $\alpha > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geqq \alpha|\xi|^2, \quad \forall \xi \in R^m, \qquad \alpha > 0. \tag{1.5}$$

We shall also define the operator

$$(B(t)\varphi)(x) \equiv \varphi(x)h(x,t). \tag{1.6}$$

Formally the Zakaï equation is written as

$$\begin{aligned} &dy + A^*(t)y\,dt = B(t)y \cdot dw \\ &y(0) = y_0. \end{aligned} \tag{1.7}$$

In the next section, we detail the function space framework necessary to analyze the system (1.7). This type of setting is used in the work of Pardoux [7] and Bensoussan [1].

1.2. Functional set up. Following the variational formulation of partial differential equations (P.D.E.) due to Lions [5], we introduce the Hilbert spaces

$$H = L^2(R^n), \qquad V = H^1(R^n)$$

and identify H with its dual. We denote by V' the dual of V.

We denote by

$$(\varphi, \psi) = \int_{R^n} \varphi\psi \, dx$$

the scalar product in H, and by

$$((\varphi, \psi)) = \int_{R^n} (\varphi\psi + D\varphi \cdot D\psi) \, dx,$$

the scalar product in V. We denote the norms on H and V by $|\cdot|$ and $\|\cdot\|$, respectively. The operator D denotes the gradient. The duality between V and V' is referred to as $\langle,\rangle$.

We now write $A(t)$ in divergence form as

$$A(t) = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + a_i \frac{\partial}{\partial x_i}$$

where we have set

$$a_i = \frac{\partial a_{ij}}{\partial x_j} - g_i$$

and

$$A^*(t) = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}(a_i).$$

Note that A^* is the adjoint of A in the Hilbert space H. The operator $A(t)$ belongs to $L^\infty(0, T; \mathscr{L}(V; V'))$ and satisfies the coercivity condition

$$\exists \beta > 0, \lambda \geqq 0 \ni \langle A(t)\varphi, \varphi\rangle + \lambda|\varphi|^2 \geqq \beta\|\varphi\|^2 \quad \forall \varphi \in V, \qquad t \geqq 0. \tag{1.8}$$

This is a consequence of (1.1) and (1.5). Note that A and $A^* \in \mathscr{L}(V; V')$. Now the operator $B(t) \in L^\infty(0, T, \mathscr{L}(H; H^m))$, and we may write more clearly

$$B(t)y \cdot dw = \sum_{j=1}^{m} B^j(t)y \, dw_j,$$

where $B^j(t) \in L^\infty(0, T; \mathscr{L}(H; H))$ corresponds to

$$(B^j(t)\varphi)(x) = \varphi(x)h_j(x, t).$$

We use the notation $L_F^2(0, T; V)$ to denote the Hilbert space of processes $z(t)$ with values in V such that $E\int_0^T \|z(t)\|^2 \, dt < \infty$, and such that, for almost everywhere with respect to time, $z(t)$ is F^t measurable. Naturally, we can replace V by H or any Hilbert space.

We can state the classical result of existence and uniqueness for (1.7) (cf. Pardoux [7], cf. also Bensoussan [1]).

THEOREM 1.1. *Assume* (1.1), (1.2), (1.5). *Then, for each* $y_0 \in H$, *there exists a unique solution of* (1.7) *in the functional space*

$$y(\cdot) \in L_F^2(0, T; V) \cap L^2(\Omega, \mathscr{A}, P; C(0, T; H)).$$

The equation (1.7) can be interpreted as an Ito differential in V', since

$$y(t) = y_0 - \int_0^t A^*(s)y(s) \, ds + \sum_j \int_0^t B^j(s)y(s) \, dw_j.$$

In addition, the following Ito's calculus rule holds (equivalent of an energy equality)

$$d|y(t)|^2 + 2\langle A(t)y(t), y(t)\rangle \, dt = 2\sum_j (y, B^j y) \, dw_j + \sum_j |B^j y|^2 \, dt. \tag{1.9}$$

Note that the integrand in the stochastic integral at the right-hand side of (1.9), $(y, B^j y)$, is almost surely in $L^\infty(0, T)$ but does not belong to $L^2_F(0, T)$. This is the source of technical (although not fundamental) difficulties. To avoid them, we can rely on the following additional result.

PROPOSITION 1.1. *The process* $y(\cdot)$ *satisfies*

$$y(\cdot) \in L^\infty(0, T; L^4(\Omega, \mathcal{A}, P; H)).$$

Proof. We shall derive the a priori estimate without going into the full proof of the results. From (1.9) we deduce

$$d|y(t)|^4 = 2|y(t)|^2\left[-2\langle Ay, y\rangle\, dt + \sum_j |B^j y|^2\, dt + 2\sum_j (y, B^j y)\, dw_j\right] + 4\sum_j (y, B^j y)^2\, dt.$$

From (1.8) we have, among other facts,

$$\langle Ay, y\rangle \geqq -\lambda|y|^2;$$

hence,

$$d|y(t)|^4 \leqq \left[4\lambda|y(t)|^4 + 2|y(t)|^2\sum_j |B^j y|^2 + 4\sum_j (y, B^j y)^2\right] dt + 4|y(t)|^2 \sum_j (y, B^j y)\, dw_j.$$

Taking the mathematical expectation yields

$$\begin{aligned}\frac{d}{dt} E|y(t)|^4 &\leqq 4\lambda E|y(t)|^4 + 2E|y(t)|^2 \sum_j |B^j y|^2 + 4E\sum_j (y, B^j y)^2\\ &\leqq kE|y(t)|^4,\end{aligned}$$

and from Gronwall's inequality, it follows that

$$E|y(t)|^4 \leqq |y_0|^4\, e^{kt},$$

which yields the desired result. □

In the following we shall replace (1.7) by

$$dy + (A^*(t)y + \mu y)\, dt = B(t)y \cdot dw, \qquad y(0) = y_0, \tag{1.10}$$

where μ is a convenient positive constant. Since we derive (1.10) from (1.7) by the transformation $y \to ye^{-\mu t}$ it suffices to consider (1.10).

2. The splitting up approximation scheme.

2.1. The algorithm. Let N be an integer, which will tend to $+\infty$, and set

$$k = \frac{T}{N+1}.$$

We shall define two processes y_{1k}, y_{2k} depending on k. We split $[0, T]$ in steps 0, $k, \cdots, (N+1)k$. Consider an interval $[rk, (r+1)k[$, $r = 0 \cdots N$, then y_{1k}, y_{2k} are defined on this interval by the relations

$$\begin{aligned} &dy_{1k} + \left(A^*(t)y_{1k} + \frac{\mu}{2} y_{1k}\right) dt = 0\\ &dy_{2k} + \frac{\mu}{2} y_{2k}\, dt = B(t)y_{2k} \cdot dw\\ &y_{1k}(rk) = y_k^r\\ &y_{2k}(rk) = y_k^{r+1/2}\end{aligned} \tag{2.1}$$

and the sequences y_k^r, $y_k^{r+1/2}$ are defined as follows:

$$\begin{aligned} y_k^{r+1/2} &= y_{1k}((r+1)k - 0)\\ y_k^{r+1} &= y_{2k}((r+1)k - 0).\end{aligned} \tag{2.2}$$

Clearly (2.1), (2.2) define completely y_{1k}, y_{2k} in $[rk, (r+1)k[$ once y_k^r is given. As a starting point we set

$$y_k^o = y_0 \tag{2.3}$$

and (2.1), (2.2) define completely y_{1k}, y_{2k} in $[0, T[$. In (2.2) μ is a parameter which will be fixed later. The processes y_{1k}, y_{2k} are right continuous and their discontinuity points are $k, \cdots Nk$ (on $[0, T[$). Since the equation for y_{1k} is deterministic we have

$$\begin{gathered} y_k^r, y_k^{r+1/2} \text{ are } F^{kr} \text{ measurable (with values in } H) \\ y_{1k}(t) \text{ is } F^{kr} \text{ measurable} \quad \forall t \in [kr, (k+1)r[\\ y_{2k}(t) \text{ is } F^t \text{ measurable} \quad \forall t. \end{gathered} \tag{2.4}$$

We can state the following existence result for (2.1).

PROPOSITION 2.1. *The system* (2.1), (2.2) *defines in a unique way* y_{1k}, y_{2k} *in* $L_F^2(0, T, V)$, $L_F^2(0, T; H)$, *respectively.*

Proof. Operating successively in each interval $[rk, (r+1)k[$, the result is clear, since for y_{1k} the deterministic theory applies and for y_{2k} we have an explicit formula for the solution. □

2.2. A priori estimates. We begin by establishing a priori estimates.

PROPOSITION 2.2. *The processes* y_{1k}, y_{2k} *satisfy*

$$E\int_0^T \|y_{1k}\|^2\, dt \leqq C, \quad E\int_0^T |y_{1k}|^2\, dt \leqq C \tag{2.5}$$

$$E|y_{1k}(t)|^4, \quad E|y_{2k}(t)|^4 \leqq C, \quad \forall t \in [0, T[, \tag{2.6}$$

where C does not depend on T or k (for a convenient choice of μ).

Proof. We can write the energy equalities

$$d|y_{1k}|^2 + (\mu|y_{1k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle)\, dt = 0 \tag{2.7}$$

$$d|y_{2k}|^2 + (\mu|y_{2k}|^2 - \sum_j |B^j y_{2k}|^2)\, dt = 2(y_{2k}, B^j y_{2k})\, dw_j \tag{2.8}$$

on $t \in [rk, (r+1)k[$.

We choose μ such that

$$\mu > 2\lambda, \qquad \mu > \sum_j \sup |h_j(x, t)|^2;$$

hence we deduce from (2.7), (2.8) that

$$d(\|y_{1k}(t)\|^2 + |y_{2k}(t)|^2) + \mu(\|y_{1k}\|^2 + |y_{2k}|^2)\, dt \leqq 2(y_{2k}, B^j y_{2k}) dw_j, \quad \text{for each } j.$$

Integrating between $[rk, (r+1)k[$ and taking the mathematical expectation yields,

$$\begin{aligned} &E(|y_{1k}((r+1)k-0)|^2 + |y_{2k}((r+1)k-0)|^2) - E(|y_{1k}(rk)|^2 + |y_{2k}(rk)|^2) \\ &\qquad + \mu E\int_{rk}^{(r+1)k} (\|y_{1k}\|^2 + |y_{2k}|^2)\, dt \leqq 0. \end{aligned} \tag{2.9}$$

Using (2.2) we deduce

$$E|y_k^{r+1}|^2 - E|y_k^r|^2 + \mu E\int_{rk}^{(r+1)k} (\|y_{1k}\|^2 + |y_{2k}|^2)\, dt \leqq 0. \tag{2.10}$$

Adding up the relations (2.10) for $r=0,\cdots,N$, we easily deduce

$$E\int_0^T \|y_{1k}\|^2\,dt \leqq C,\quad E\int_0^T |y_{2k}|^2\,dt \leqq C,\qquad E|y_k^r|^2 \leqq C, \tag{2.11}$$

where C does not depend on k, nor T, but only on y_0 and μ. Now using (2.7) only, integrated over $[rk,(r+1)k[$, and taking the mathematical expectation yields

$$E|y_k^{r+1/2}|^2 \leqq E|y_k^r|^2 \tag{2.12}$$

and thus also

$$E|y_k^{r+1/2}|^2 \leqq C. \tag{2.13}$$

Similarly

$$E|y_{1k}(t)|^2 \leqq E|y_k^r|^2 \leqq C \qquad \text{for } t\in[rk,(r+1)k[$$

$$E|y_{2k}(t)|^2 \leqq E|y_k^{r+1/2}|^2 \leqq C \qquad \text{for } t\in[rk,(r+1)k[.$$

Therefore we have proven that

$$E|y_{1k}(t)|^2,\ E|y_{2k}(t)|^2 \leqq C,\quad \forall t\in[0,T[. \tag{2.14}$$

We proceed now from (2.7), (2.8) to derive:

$$\begin{aligned} &d|y_{1k}(t)|^4 + 2|y_{1k}(t)|^2(\mu|y_{1k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle)\,dt = 0\\ &d|y_{2k}(t)|^4 + \left[2|y_{2k}(t)|^2\Big(\mu|y_{2k}|^2 - \sum_j |B^j y_{2k}|^2\Big) - 4\sum_j (y_{2k}, B^j y_{2k})^2\right]dt\\ &\qquad = 4|y_{2k}|^2(y_{2k}, B^j y_{2k})\,dw_j, \end{aligned} \tag{2.15}$$

and if μ is slightly larger than before, in particular satisfying

$$\mu \geqq 3\sum_j \sup|h_j|^2.$$

We derive from (2.15) that

$$E|y_{1k}((r+1)k-0)|^4 + E|y_{2k}((r+1)k-0)|^4 \leqq E|y_{1k}(rk)|^4 + E|y_{2k}(rk)|^4.$$

Hence also

$$E|y_k^{r+1}|^4 \leqq E|y_k^r|^4.$$

Therefore,

$$E|y_k^r|^4 \leqq C.$$

From this and (2.15) we easily deduce

$$E|y_{1k}(t)|^4 \leqq C,\qquad E|y_{2k}(t)|^4 \leqq C$$

and the proof of (2.5), (2.6) has been completed.

3. Convergence.

3.1. Statement of the main result. Our main result is the following theorem.

THEOREM 3.1. *Assume* (1.1), (1.2), (1.5). *Then we have*:

$$y_{1k}, y_{2k} \to y \quad \textit{in} \quad L_F^2(0,T;V) \quad \textit{and} \quad L_F^2(0,T;H), \textit{ respectively}; \tag{3.1}$$

$$\begin{aligned} &y_{1k}(t), y_{2k}(t) \to y(t) \quad \textit{in} \quad L^2(\Omega, \mathcal{A}, P; H) \quad \forall t\in[0,T[.\\ &y_{1k}(T-0), y_{2k}(T-0) \to y(T) \quad \textit{in} \quad L^2(\Omega, \mathcal{A}, P; H). \end{aligned} \tag{3.2}$$

3.2. Weak convergence. We can extract subsequences, still denoted y_{1k}, y_{2k} such that

$$y_{1k} \to y_1 \quad \text{in} \quad L^2_F(0, T; V) \text{ weakly,}$$
$$y_{2k} \to y_2 \quad \text{in} \quad L^2_F(0, T; H) \text{ weakly,}$$

and

$$y_{1k}, y_{2k} \to y_1, y_2 \quad \text{in} \quad L^\infty(0, T; L^4(\Omega, \mathcal{A}, P; H)) \text{ weak star.}$$

We first have the following lemma.

LEMMA 3.1. *The functions y_1 and y_2 are equal to a common function η.*

Proof. Consider (2.1). We can integrate the first equation backward (since it is deterministic), to obtain

$$y_k^{r+1/2} - y_{1k}(t) + \int_t^{(r+1)k} \left(\frac{\mu}{2} y_{1k} + A^* y_{1k}\right) ds = 0,$$

and integrating the second equation forward, we have also

$$y_{2k}(t) - y_k^{r+1/2} + \int_{rk}^t \frac{\mu}{2} y_{2k}\, ds = \int_{rk}^t \sum_j B^j y_{2k}\, dw_j.$$

Adding up, we get (recall $t \in [rk, (r+1)k[$)

$$(y_{2k}(t) - y_{1k}(t)) + \int_t^{(r+1)k} \left(\frac{\mu}{2} y_{1k} + A^* y_{1k}\right) ds + \int_{rk}^t \frac{\mu}{2} y_{2k}\, ds = \int_{rk}^t \sum_j B^j y_{2k}\, dw_j$$

from which we deduce:[1]

$$\begin{aligned} &E\|y_{2k}(t) - y_{1k}(t)\|^2_{V'} \\ &\leqq 3E\left(\int_t^{(r+1)k} \left\|\frac{\mu}{2} y_{1k} + A^* y_{1k}\right\|_{V'} ds\right)^2 \\ &\qquad + 3E\left(\int_{rk}^t \frac{\mu}{2} \|y_{2k}\|_{V'}\, ds\right)^2 + 3E\left(\left\|\int_{rk}^t \sum_j B^j y_{2k}\, dw_j\right\|_{V'}\right)^2 \\ &\leqq CkE \int_{rk}^{(r+1)k} (\|y_{1k}\|^2 + |y_{2k}|^2)\, ds + CE\int_{rk}^{(r+1)k} |y_{2k}|^2\, ds \\ &\leqq CkE\int_0^t (\|y_{1k}\|^2 + |y_{2k}|^2)\, ds + Ck^{1/2}\left(E\int_0^t |y_{2k}|^4\, ds\right)^{1/2} \\ &\leqq Ck^{1/2}. \end{aligned}$$

Since $y_{2k} - y_{1k} \to y_2 - y_1$ in $L^2_F(0, T; V')$ weakly, we deduce from Fatou's Lemma that

$$\int_0^t E\|y_1 - y_2\|^2_{V'}\, dt = 0;$$

hence $y_1 = y_2 = \eta$.

[1] We use successively $\|X + Y + Z\|^2 \leqq 3(\|X\|^2 + \|Y\|^2 + \|Z\|^2)$ for all X, Y, Z in a Hilbert space; the fact that $A^* \in \mathcal{L}(V; V')$; Cauchy-Schwartz inequality $|\int_a^b f\,dx|^2 \leqq (b-a)\int_a^b |f|^2\, dx$; $\|\phi\|_{V'} \leqq K_1|\phi|_H \leqq K_2\|\phi\|_V$.

Naturally $\eta \in L_F^2(0, T; V) \cap L^\infty(0, T; L^4(\Omega, \mathscr{A}, P; H))$. $\square$

Our objective now is to check that η satisfies (1.7) and thus $\eta = y$, by the uniqueness.

LEMMA 3.2. $\eta = y$.

Proof. We write from (2.1)

$$y_k^{r+1/2} - y_k^r + \int_{rk}^{(r+1)k} \left(\frac{\mu}{2} y_{1k} + A^* y_{1k}\right) ds = 0$$

$$y_k^{r+1} - y_k^{r+1/2} + \int_{rk}^{(r+1)k} \frac{\mu}{2} y_{1k}\, ds = \int_{rk}^{(r+1)k} \sum_j B^j y_{2k}\, dw_j,$$

hence adding up

$$y_k^{r+1/2} - y_k^r + \int_{rk}^{(r+1)k} \left(\frac{\mu}{2}(y_{1k} + y_{2k}) + A^* y_{1k}\right) ds = \int_{rk}^{(r+1)k} \sum_j B^j y_{2k}\, dw_j. \tag{3.3}$$

Adding up these relations for $r = 0, \cdots, q-1$, yields

$$y_k^q - y_0 + \int_0^{qk} \left(\frac{\mu}{2}(y_{1k} + y_{2k}) + A^* y_{1k}\right) ds = \int_0^{qk} \sum_j B^j y_{2k}\, dw_j. \tag{3.4}$$

Also from (2.1) we have for $t \in [rk, (r+1)k[$,

$$y_{1k}(t) - y_k^r + \int_{rk}^t \left(A^* y_{1k} + \frac{\mu}{2} y_{1k}\right) ds = 0. \tag{3.5}$$

Let t be fixed. Apply (3.4) with $q = [t/k]$ and (3.5) with $r = [t/k]$. Adding up, we obtain:

$$y_{1k}(t) - y_0 + \int_0^t \left(A^* y_{1k} + \frac{\mu}{2} y_{1k}\right) ds + \int_0^{k[t/k]} \frac{\mu}{2} y_{2k}\, ds = \int_0^{k[t/k]} \sum_j B^j y_{2k}\, dw_j. \tag{3.6}$$

Note that

$$\begin{aligned} E\left|\int_{k[t/k]}^t y_{2k}\, ds\right|^2 &\leqq \left(t - k\left[\frac{t}{k}\right]\right)\left(\int_{k[t/k]}^t E|y_{2k}|^2\, ds\right) \\ &\leqq C(k - [t/k]) \to 0 \end{aligned} \tag{3.7}$$

$$\begin{aligned} E\left|\int_{k[t/k]}^t \sum_j B^j y_{2k}\, dw_j\right|^2 &= E\int_{k[t/k]}^t \sum_j |B^j y_{2k}|^2\, ds \\ &\leqq CE\int_{k[t/k]}^t |y_{2k}|^2\, ds \\ &\leqq C\left(t - k\left[\frac{t}{k}\right]\right)^{1/2}\left(E\int_{k[t/k]}^t |y_{2k}|^4\, ds\right)^{1/2} \\ &\leqq C\left(t - k\left[\frac{t}{k}\right]\right)^{1/2}. \end{aligned} \tag{3.8}$$

Also

$$\int_0^t \left(A^* y_{1k} + \frac{\mu}{2} y_{1k} + \frac{\mu}{2} y_{2k}\right) ds \to \int_0^t (A^*\eta + \mu\eta)\, ds \quad \text{in } L^2(\Omega, \mathscr{A}, P; V') \text{ weakly}. \tag{3.9}$$

Let us check that

$$\int_0^t \sum_j B^j y_{2k}\, dw_j \to \int_0^t \sum_j B^j \eta\, dw_j \quad \text{in } L^2(\Omega, \mathcal{A}, P; H) \text{ weakly.} \tag{3.10}$$

Since

$$E\left|\int_0^t \sum_j B^j y_{2k}\, dw_j\right|^2 \leqq C,$$

we can assert that at least for subsequences $\int_0^t \sum_j B^j y_{2k}\, dw_j \to \chi$ in $L^2(\Omega, \mathcal{A}, P; H)$ weakly. To check that χ coincides with the right-hand side of (3.10), it is sufficient to prove that

$$E\left[\left(v, \int_0^t \sum_j B^j \eta\, dw_j\right)\xi\right] = E[(v, \chi)\xi] \tag{3.11}$$

for all $v \in H$, and $\xi \in L^2(\Omega, \mathcal{A}, P)$. This follows from the separability of H. Now since $\chi \in L^2(\Omega, F^t, P; H)$ (because $L^2(\Omega, F^t, P; H)$ is a closed subspace of $L^2(\Omega, \mathcal{A}, P; H)$ in which $\int_0^t \sum_j B^j y_{2k}\, dw_j$ stands), and since $\int_0^t \sum_j B^j \eta\, dw_j$ belongs to $L^2(\Omega, F^t, P; H)$, it is sufficient to check (3.11) with $\xi \in L^2(\Omega, F^t, P)$. Now a dense subspace of $L^2(\Omega, F^t, P)$ is made of linear combinations of random variables of the form

$$\theta(t) = \exp\left(\int_0^t \beta(s) \cdot dw(s) - \frac{1}{2}\int_0^t |\beta(s)|^2\, ds\right)$$

where β is in $L^\infty(0, t; R^m)$ (and deterministic). This follows from the fact that F^t is generated by $w(s)$, $s \leqq t$. Therefore, it is sufficient to check (3.11) with $\xi = \theta(t)$ for any β fixed. But then, what we have to prove is that

$$E\left[\left(v, \int_0^t \sum_j B^j y_{2k}\, dw_j\right)\theta(t)\right] \to E\left[\left(v, \int_0^t \sum_j B^j \eta\, dw_j\right)\theta(t)\right]. \tag{3.12}$$

However we can calculate both sizes of (3.12) by Ito's calculus, and (3.12) amounts to

$$E\int_0^t \sum_j \beta_j(s)(v, B^j y_{2k}(s))\, ds \to E\int_0^t \sum_j \beta_j(s)(v, \beta^j \eta(s))\, ds,$$

which immediately follows from the weak convergence of y_{2k} to η in $L_F^2(0, T; H)$.

Collecting results we can assert from (3.6) that

$$\forall t \quad y_{1k}(t) \to y_0 - \int_0^t (A^*\eta + \mu\eta)\, ds + \int_0^t \sum_j B^j \eta\, dw_j$$

in $L^2(\Omega, \mathcal{A}, P; H)$ weakly. Since $y_{1k}(t)$ is bounded in $L^\infty(\Omega, \mathcal{A}, P; H)$ and $y_{1k}(\cdot)$ converges weakly to $\eta(\cdot)$ in $L_F^2(0, T; H)$, we necessarily have

$$\eta(t) = y_0 - \int_0^t (A^*\eta + \mu\eta)\, ds + \int_0^t \sum_j B^j_\eta\, dw_j$$

and thus $\eta = y$.

From the uniqueness of the limit, we can assert that

$$y_{1k} \to y \quad \text{in } L_F^2(0, T; V) \text{ weakly}, \qquad y_{2k} \to y \quad \text{in } L_F^2(0, T; H) \text{ weakly},$$

and both sequences also converge in $L^\infty(0, T; L^4(\Omega, \mathcal{A}, P; H))$ weak star. $\square$

3.3. Strong convergence. Consider (2.7), (2.8) which yield, integrating between rk and $(r+1)k$ and taking the mathematical expectation,

$$E|y_k^{r+1/2}|^2 - E|y_k^r|^2 + E\int_{rk}^{(r+1)k} (\mu|y_{1k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle)\,ds = 0$$

$$E|y_k^{r+1}|^2 - E|y_k^{r+1/2}|^2 + E\int_{rk}^{(r+1)k} \left(\mu|y_{2k}|^2 - \sum_j |B^j y_{2k}|^2\right) ds = 0.$$

Adding up we get

$$E|y_k^{r+1/2}|^2 - E|y_k^r|^2 + E\int_{rk}^{(r+1)k} \left(\mu|y_{1k}|^2 + \mu|y_{2k}|^2 - \sum_j |B^j y_{2k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle\right) ds = 0. \tag{3.13}$$

Adding up these relations for $r = 0, \cdots, q-1$, yields

$$E|y_k^q|^2 - |y_0|^2 + E\int_0^{qk} \left(\mu|y_{1k}|^2 + \mu|y_{2k}|^2 - \sum_j |B^j y_{2k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle\right) ds = 0. \tag{3.14}$$

Now from (2.7) we have for $t \in [rk, (r+1)k[$,

$$E|y_{1k}(t)|^2 - E|y_k^r|^2 + E\int_{rk}^{t} (\mu|y_{1k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle)\,ds = 0. \tag{3.15}$$

Let t be fixed. Apply (3.14) with $q = [t/k]$ and (3.15) with $r = [t/k]$. Adding up, we obtain

$$\begin{aligned} &E|y_{1k}(t)|^2 - |y_0|^2 + E\int_0^t (\mu|y_{1k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle)\,ds \\ &\quad + E\int_0^{[t/k]k} \left(\mu|y_{1k}|^2 - \sum_j |B^j y_{1k}|^2\right) ds = 0. \end{aligned} \tag{3.16}$$

Now consider the expression[2]

$$\begin{aligned} X_k(t) &= E|y(t) - y_{1k}(t)|^2 + 2E\int_0^t \langle A(y - y_{1k}), y - y_{1k}\rangle\,ds \\ &\quad + E\int_0^t \mu|y - y_{1k}|^2\,ds + E\int_0^{[t/k]k} \left(\mu|y - y_{2k}|^2 - \sum_j |B^j(y - y_{2k})|^2\right) ds \\ &= X_k^1(t) + X_k^2(t) + X_k^3(t) \end{aligned}$$

with

$$\begin{aligned} X_k^1(t) &= E|y(t)|^2 + 2E\int_0^t \langle Ay, y\rangle\,ds \\ &\quad + E\int_0^t \mu|y|^2\,ds + E\int_0^{[t/k]k} \left(\mu|y|^2 - \sum_j |B^j y|^2\right) ds \\ &\to E|y(t)|^2 + 2E\int_0^t \langle Ay, y\rangle\,ds + 2\mu E\int_0^t |y|^2\,ds - E\int_0^t \sum_j |B^j y|^2\,ds \\ &= |y_0|^2; \end{aligned}$$

[2] In $X_k(t)$ the terms do not go to 0 individually, at least a priori.

$$X_k^2(t) = -2E(y(t), y_{1k}(t)) - 2E\int_0^t (\langle Ay, y_{1k}\rangle + \langle Ay_{1k}, y\rangle)\,ds - 2\mu E\int_0^t (y, y_{1k})\,ds - 2\mu E\int_0^{[t/k]k} (y, y_{2k})\,ds + 2E\int_0^{[t/k]k} \sum_j (B^j y, B^j y_{2k})\,ds$$
$$\to -2E|y(t)|^2 - 4E\int_0^t \langle Ay, y\rangle\,ds - 4\mu E\int_0^t |y|^2\,ds + 2E\int_0^t \sum_j |B^j y|^2\,ds = -2|y_0|^2$$

and from (3.16)

$$X_k^3(t) = |y_0|^2.$$

Therefore $X_k(t)\to 0$, for all t. Remark that μ was chosen so that

$$\mu > \sum_j \sup |h_j|$$

which implies

$$\mu |y - y_{2k}|^2 \geqq \sum |B^i(y - y_{2k})|^2.$$

Moreover, we have $E\int_0^t |y - y_{1k}|^2 \to 0$ which yields $y_{1k}\to y$ strongly in $L_F^2(0, T; H)$ and $E|y_{1k}(t) - y(t)|^2 \to 0$, so that $y_{1k}\to y$ in $L^2(\Omega, A, P, H)$. Next the coercivity condition $\langle A(y - y_{1k}), y - y_{1k}\rangle + \lambda |y - y_{1k}|^2 \geqq \|y - y_{1k}\|^2$ gives the $L_F^2(0, T; V)$ convergence, because the left side of the inequality is controlled by $X_k(t)\to 0$.

This implies

$$\begin{aligned} &y_{1k}\to y \quad \text{in } L_F^2(0, T; V) \text{ strongly},\\ &y_{1k}(t)\to y(t) \quad \text{in } L^2(\Omega, \mathcal{A}, P; H) \quad \forall t\in [0, T[, \text{ strongly},\\ &y_{1k}(T-0)\to y(T-0) \quad \text{in } L^2(\Omega, \mathcal{A}, P; H), \text{ strongly}. \end{aligned} \tag{3.17}$$

Similarly we can check that the following expression analogous to (3.16) holds:

$$\begin{aligned} &E|y_{2k}(t)|^2 - E|y_k^{1/2}|^2 + E\int_0^t \Big(\mu |y_{2k}|^2 - \sum_k |B^j y_{2k}|^2\Big)\,ds\\ &\quad + E\int_k^{([t_k]+1)k} (\mu |y_{1k}|^2 + 2\langle Ay_{1k}, y_{1k}\rangle)\,ds = 0. \end{aligned} \tag{3.18}$$

Using this identity, and constructing an expression similar to $X_k(t)$, which is easily guessed from the structure of (3.18), we can prove the remainder of the results ($y_{2k}\to y$ in $L_F^2(0, T, H)$ strongly).

The proof of Theorem 3.1 has been completed. □

4. Remarks.

4.1. Explicit solution. The equation for y_{2k} can be explicitly solved, namely,

$$y_{2k}(x, t) = y_k^{r+1/2}(x)\, e^{-\mu/2k} \exp \int_{rk}^t \sum_j h_j(x, s)\, dw_j(s) - \frac{1}{2}\int_{rk}^t |h(x, s)|^2\,ds$$

whereas y_{1k} is solution of the classical Fokker Plank equation. therefore we can write for y_{1k} the following equation:

$$
\begin{aligned}
&\frac{\partial y_{1k}}{\partial t}+\left(A^*(t)y_{1k}+\frac{\mu}{2}y_{1k}\right)\\
&\quad=\sum_{r=1\cdots N}\delta_{rk}(t)y_{1k}(x,t-0)\\
&\quad\cdot\left\{\exp\left[-\frac{\mu k}{2}+\int_{t-k}^{t}\sum_j h_j\,dw_j-\frac{1}{2}\int_{t-k}^{t}|h|^2\,ds\right]-1\right\},\qquad y_{1k}(0)=y_0
\end{aligned}
\tag{4.1}
$$

which can be considered as the approximation of the original Zakaï equation.

We can understand the right-hand side as follows. For k small and assuming h continuous in time, the term within brackets is equivalent to $-(\mu k/2)+\sum_j h_j(x,t)\times(w_j(t)-w_j(t-k))$, up to second-order terms. Comparing with the original Zakaï equation, it means that we have replaced the term

$$y(x,t)\left(-\frac{\mu}{2}\,dt+\sum_j h_j(x,t)\,dw_j\right)$$

with the sum of impulses

$$\sum_{r=1\cdots N}\delta_{rk}(t)y(x,t-0)\left(-\frac{\mu k}{2}+\sum_j h_j(x,t)(w_j(t)-w_j(t-k))\right).$$

4.2. Fully numerical scheme. It remains, of course, to discretize completely (4.1) both in time and space. This can be done using classical tools of numerical analysis. Numerical results will be reported elsewhere.

4.3. Extension. Our variational techniques directly inspired from the deterministic case bear two serious limitations. First, g, h must be bounded, which leaves out of the framework of the linear case. More importantly, the case when there is correlation between the system noise and the observation noise, which leads to an operator B involving the gradient of y, seems to be out of the scope of our theory.

The first limitation is purely technical, and can be overcome using Sobolev spaces with weights.

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