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ON THE DIFFERENTIAL EQUATIONS SATISFIED BY CONDITIONAL PROBABILITY DENSITIES OF MARKOV PROCESSES, WITH APPLICATIONS*

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1. Introduction and summary. Consider the vector stochastic differential equation,

$$(1)\qquad dx_i = f_i(x)dt + \sum_k F_{ik}(x)dz_k(t), \qquad i = 1, \cdots, n,$$

where each $z_i(t)$ is an independent Brownian motion process with unit variance parameter. Let x, f and z be vectors with components x_i, f_i and z_i, respectively; let $F(x)$ be the matrix with components $F_{ij}(x)$, and $V(x)$ the matrix with components $v_{ij}(x)$, where $V = FF'$. Let $\hat{P}(a, t)$ be the probability density of $x(t)$ given only the density of $x(t_0)$, $t \geqq t_0$. Under suitable conditions on f and F, it is well-known that (for almost all $z(\cdot)$ functions) there exists a unique solution to (1) which is a Markov process. If $\hat{P}$ is suitably differentiable, then Kolmogorov's forward equation,

$$(2)\qquad \frac{\partial \hat{P}(a, t)}{\partial t} = -\sum_{i=1}^{n} (f_i(a)\hat{P}(a, t))_{a_i} + \frac{1}{2}\sum_{i,j=1}^{n} (v_{ij}(a)\hat{P}(a, t))_{a_i a_j},$$

is satisfied, where the subscript a_i denotes the partial derivative.

A problem of great practical importance arises when noise corrupted observations on x are taken; i.e., the vector‡ $dy = g(x)dt + dw$ is available, where w is a vector Brownian motion process. For example, x may represent a signal stochastic process and dy/dt the (nonlinear function of the) signal plus noise, or x may represent the evolution of a dynamical system driven by a noise process and the interest may be in the estimation of various properties of x or, perhaps, the control of x. In these cases it would be very desirable to have an expression for the probability density of x conditioned upon the observations, as well as upon the initial data. The existence of such an equation is suggested by theorems|| in [3, pp. 287–291]. Here, we derive a partial differential equation satisfied by this conditional density. The equation is of the form (2) with an additional term which contains the ob-

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‡ This could be written without differentials as $b = g(x) + \Psi$, where Ψ is the white Gaussian noise dw/dt.

|| The relation between our results and these theorems is further discussed in the Appendix.

servation in a linear manner, and in many cases, is amenable to convenient analog or digital simulation; hence, the actual conditional density may be obtained as it evolves in time. The equation promises to be of great usefulness in communications and control problems.

The principal result is the following. For any function of time $s(t)$, define $\delta s(t) = s(t + \Delta) - s(t)$ and $ds(t) = s(t + dt) - s(t)$. Let $E\delta w\delta w' = \Sigma\Delta$ and $E\delta z\delta w' = C\Delta$ and assume Σ is nonsingular.† Let $P(a, t \mid t)$ be the conditional density of $x(t)$ given all observations up to t, and let

$$\begin{aligned} d\bar{f}(a, t) &= f(a, t)dt + FC\Sigma^{-1}(dy - g(a)dt), \\ \bar{V} &= V - FC\Sigma^{-1}(FC)', \end{aligned} \tag{3}$$

$$dQ(a, t) = P(a, t \mid t)\cdot(dy - Eg(a)dt)'\Sigma^{-1}(g(a) - Eg(a)), \tag{4}$$

where the expectation E is the conditional expectation using $P(a, t \mid t)$. Then $P(a, t \mid t)$ satisfies

$$\begin{aligned} P(a, t + dt \mid t + dt) &- P(a, t \mid t) \\ &= dP(a, t \mid t) = dQ(a, t) - \sum_1^n (d\bar{f}_i(a, t)\cdot P(a, t \mid t))_{a_i} \\ &\qquad + \tfrac{1}{2} \sum_{i,j=1}^n (\bar{v}_{ij}(a) \cdot P(a, t \mid t))_{a_i a_j} dt. \end{aligned} \tag{5}$$

In certain cases (discussed in §3j) which are reducible to the case where a takes on only values $x^1, \cdots, x^s$, (5) becomes

$$dP(i \mid t) = P(i \mid t)\cdot(dy - Eg(i, t))'\Sigma^{-1}(g(i, t) - Eg(i, t)). \tag{5$'$}$$

Equation (5′) is generally rigorously verifiable.

Although (5) can be rigorously verified in a number of cases, it is, of course, still formal in general (see Appendix), being derived under the assumption that P exists and is suitably differentiable‡. If there is no correlation between the observation noise dw and the noise dz, then $C = 0$ and $d\bar{f}_i = f_i dt$ and $\hat{v}_{ij} = v_{ij}$. In this case the last two terms on the right of (5) are the same as in (2), and (5) differs from (2) only in that the former contains the observation term dQ, where dQ is *linear* in the differential observation dy.

The same problem was considered in [1], where x was scalar and $g(x) = x$. A more general problem was discussed in [2] but, as discussed in [1], the results in [2] are incorrect through the omission of certain significant terms. Since the writing of [1], substantial and surprising simplifications (which were initially inapparent) in the form of the scalar equation have been

† Σ and C are assumed to be independent of x; if Σ depended on x, the problem appears to degenerate to one where x can be determined exactly at every t.

‡ When $f = F = 0$, $dP = dQ$ and is simple to verify.

obtained. In this paper, taking advantage of these simplifications, the results for the general vector case with nonlinear observations are derived. These results include, as special cases, many important situations (as will be illustrated) that cannot be represented by the scalar case format.

The derivation is performed in §2. Section 3 discusses several special cases and extensions. The results include as special cases known results [4] for the filtering problem where the signal and noise are Gaussian and finite order Markovian.

Usually, when one has a stochastic differential equation, one seeks properties of the random functions which they define. In this paper, the inverse problem occurs initially: given a random function, what stochastic differential equation does it satisfy? The Appendix contains a discussion of this problem and of the sense in which such an equation is meaningful—as well as of other points which are important in the derivation.

2. The main result. The derivation proceeds by assuming the finite difference model (6) and taking formal limits subsequently.

$$\begin{aligned} \delta x &= f(x)\Delta + F(x)\delta z, \\ \delta y &= g(x)\Delta + \delta w. \end{aligned} \tag{6}$$

Let Y denote the $y(\tau)$, $\tau \leqq t$, the entire set of observations up to t; $\delta y = y(t+\Delta) - y(t)$ is the observation at t given by (6).

The following notation will be used. Let a and α be the generic value of x, and let M and N be any random quantities. Let $P(a, t; M)$ denote the joint density of $x(t)$ and M; $P(a, t \mid M)$ denotes the density of $x(t)$ conditioned upon M; $P(a, t \mid Y)$ will also be written as $P(a, t \mid t)$ or as P; $P(a, t \mid t+\Delta)$ denotes $P(a, t \mid Y, \delta y)$, the density of $x(t)$ conditioned upon the set of past observations Y and also upon the present vector observation δy; $P(M \mid a, t; N)$ denotes the conditional density of M, given $x(t) = a$ and N.

The derivation takes place in two parts. First, let $P(a, t \mid t)$ be given, take the observation δy, and compute $P(a, t \mid t+\Delta) - P(a, t \mid t)$, the change in the conditional density due to the last observation. This change is given by (14). The second part of the derivation assumes the change δx in x, and the Chapman-Kolmogorov equation is applied to include the effects of δx on the conditional density. Formal limits are then taken and the derivation is complete.

Derivation: Part 1. According to the notational convention

$$\begin{aligned} P(a, t \mid t+\Delta) &= P(a, t; Y, \delta y)/P(Y, \delta y) \\ &= \frac{P(\delta y \mid a, t; Y)P(a, t \mid Y)P(Y)}{P(Y)\int P(\delta y \mid a, t; Y)P(a, t \mid Y)\, da}. \end{aligned} \tag{7}$$

Since the distribution of δy is completely specified when a is given, (7) may be written as

$$(8) \qquad \frac{P(a, t \mid t)P(\delta y \mid a, t)}{\int P(\delta y \mid a, t)P(a, t \mid t)\, da}.$$

In fact, as discussed in [1], $P(a, t \mid t)$ is a Markov process in function space. Now, from (5),

$$(9) \qquad P(\delta y \mid a, t) \sim N[g\Delta, \Sigma\Delta],$$

where $N[g\Delta, \Sigma\Delta]$ denotes normal density with mean $g\Delta$ and covariance matrix $\Sigma\Delta$.

Substituting (9) into (8) yields

$$(10) \qquad \frac{P(a, t \mid t) \exp\left[-\frac{1}{2\Delta}(\delta y - g(a)\Delta)'\Sigma^{-1}(\delta y - g(a)\Delta)\right]}{\int P(a, t \mid t) \exp\left[-\frac{1}{2\Delta}(\delta y - g(a)\Delta)'\Sigma^{-1}(\delta y - g(a)\Delta)\right] da},$$

where $da = \prod_1^q da_i$. Equation (10) may be further simplified by deleting the common term $\exp\left[-\frac{1}{2\Delta}\delta y'\Sigma^{-1}\delta y\right]$ from both numerator and denominator†. Thus,

$$(11) \qquad \begin{aligned} R(\Delta, \delta y) &\triangleq \frac{P(a, t \mid t + \Delta)}{P(a, t \mid t)} \\ &= \frac{\exp[\delta y'\Sigma^{-1}g(a) - \frac{1}{2}g'(a)\Sigma^{-1}g(a)\Delta]}{\int P(a, t \mid t) \exp\left[\delta y'\Sigma^{-1}g(a) - \frac{1}{2}g'(a)\Sigma^{-1}g(a)\Delta\right] da}. \end{aligned}$$

Assuming that the appropriate moments of $P(a, t \mid t)$ exist, (11) may be differentiated any number of times with respect to the infinitesimals Δ and δy_i. We wish to obtain an expansion of (11) which contains all terms of order Δ or less. Since $E\delta y\delta y' = \Sigma\Delta$, the expansion must be carried to the second degree in the components of δy, and to the first degree in Δ. It is easily shown that the remainder in the expansion has a mean value of smaller order than Δ and a mean square value of smaller order than Δ^2.

The differentiation of (11) is straightforward. Recalling that E refers to the expectation using $P(a, t \mid t)$, we have

$$\begin{aligned} R_\Delta(0, 0) &= -\tfrac{1}{2}[g'(a)\Sigma^{-1}g(a) - E(g'(a)\Sigma^{-1}g(a))], \\ R(0, 0) &= 1, \\ (12) \qquad R_{\delta y}(0, 0) &= \Sigma^{-1}g(a) - \Sigma^{-1}Eg(a), \end{aligned}$$

† If Σ depended upon x, this could not be done.

$$R_{\delta y,\delta y}(0, 0) = (\Sigma^{-1}g(a))(\Sigma^{-1}g(a))' - 2\Sigma^{-1}g(a)(\Sigma^{-1}Eg(a))' + 2(\Sigma^{-1}Eg(a))(\Sigma^{-1}Eg(a))' - E(\Sigma^{-1}g(a)(\Sigma^{-1}g(a))'),$$

where $R_{\delta y}$ and $R_{\delta y,\delta y}$ are the gradient and Jacobian, respectively, of R with respect to δy. Thus,

$$\begin{aligned} &P(a, t \mid t + \Delta) = \\ (13)\quad &P(a, t \mid t)[1 + R_\Delta(0, 0)\Delta + R'_{\delta y}(0, 0)\delta y + \tfrac{1}{2}\delta y' R_{\delta y,\delta y}(0, 0)\delta y] + r, \end{aligned}$$

where $Er \sim o(\Delta)$, $Er^2 \sim o(\Delta^2)$.

Although there is frequent occurrence of terms such as $E\delta y_i \delta y_j$ in probability theory, (13) is unusual in that these random terms are included without expectations. It would appear that these terms substantially complicate the result. It is quite remarkable that the term $\delta y_i \delta y_j$ may be replaced everywhere by its expectation without altering the result at all. The arguments for this are given in the Appendix: the replacement will be used hereafter in the text.† The simplification was not apparent in the earlier work. We have $E\delta y \delta y' = E[g(a)\Delta + \delta w][g(a)\Delta + \delta w]' = \Sigma\Delta + o(\Delta)$. Various terms in (12) may now be rewritten; e.g., replace $\delta y'\Sigma^{-1}g(a) \cdot (\Sigma^{-1}g(a))'\delta y = g(a)'\Sigma^{-1}\delta y \delta y' \Sigma^{-1} g(a)$ by $g'(a)\Sigma^{-1}g(a)\Delta + o(\Delta)$.

Now, adding and subtracting the terms $P(a, t \mid t)\ (Eg)'\Sigma^{-1}g\Delta$ and $P(a, t \mid t)\ (Eg)'\Sigma^{-1}(Eg)$, using the expectation substitutions for the second order terms, and rearranging terms yields

$$\begin{aligned} (14)\quad P(a, t \mid t + \Delta) - P(a, t \mid t) &\overset{\Delta}{=} \delta Q(a, t) \\ &= P(a, t \mid t)\cdot(\delta y - Eg\Delta)'\Sigma^{-1}(g - Eg) + r, \end{aligned}$$

where, again, $Er^2 \sim o(\Delta^2)$.

Completion of derivation. We are now prepared to use a modification of the Chapman-Kolmogorov [3] equation to complete our derivation by including the effects of $\delta x(t)$ on the conditional distribution. The method is a modification of the usual formal approach to the derivation of (2).

In general, by the definition of conditional probability,

$$P(a, t + \Delta \mid t + \Delta) = \int P(\alpha, t \mid t + \Delta)P(a, t + \Delta \mid \alpha, t;\ Y, \delta y)\, d\alpha.$$

If no observations are taken, it reduces to

$$\hat{P}(a, t + \Delta) = \int \hat{P}(\alpha, t)P(a, t + \Delta \mid \alpha, t)\, d\alpha.$$

† Although the second order random terms cannot be neglected since their expectation is of the order of Δ, their contribution is essentially deterministic. See Appendix for more details.

In our case, since dw is allowed to depend on dz but not on x, the distribution of $x(t + \Delta)$ is completely determined when δy and $x(t)$ are given. Thus

$$P(a, t + \Delta \mid t + \Delta) = \int P(\alpha, t \mid t + \Delta) P(a, t + \Delta \mid \alpha, t; \delta y)\, d\alpha. \tag{15}$$

To complete the procedure, multiply (15) by an arbitrary triply differentiable function $h(a)$, such that (19) holds and the integrals (16) exist. Thus from (15),

$$\begin{aligned}
&\int h(a) P(a, t + \Delta \mid t + \Delta)\, da \\
&\quad = \iint h(\alpha + (a - \alpha)) P(\alpha, t \mid t + \Delta) P(a, t + \Delta \mid \alpha, t; \delta y)\, d\alpha\, da \\
&\quad = \iint \Big[h(\alpha) + h_a'(\alpha)(a - \alpha) \\
&\qquad + \frac{1}{2}(a - \alpha)' h_{a,a}(\alpha)(a - \alpha) + o((a - \alpha)'(a - \alpha)) \Big] \\
&\qquad \times \{ P(\alpha, t \mid t + \Delta) P(a, t + \Delta \mid \alpha, t; \delta y)\, d\alpha\, da \}.
\end{aligned} \tag{16}$$

Now, the density $P(a, t + \Delta \mid \alpha, t; \delta y)$ is normal. Since it is conditioned upon δy and $x(t)$, it is also conditioned upon δw. From standard theorems† on conditional normal variables [10],

$$\begin{aligned}
E[a - \alpha \mid \delta y, x(t) = \alpha] &= E[f(\alpha)\Delta + F(\alpha)\delta z \mid \delta y = g(\alpha)\Delta + \delta w] \\
&\overset{\Delta}{=} \delta\hat{f} = f(\alpha)\Delta + (FC)\Sigma^{-1}(\delta y - g(\alpha)\Delta), \\
E[(a - \alpha)^2 \mid \delta y, x(t) = \alpha] &= E[(f(\alpha)\Delta + F(\alpha)\delta z)^2 \mid g(\alpha)\Delta + \delta w] \\
&\overset{\Delta}{=} \bar{V}\Delta = V\Delta - FC\Sigma^{-1}(FC)'\Delta + o(\Delta).
\end{aligned}$$

Substituting these results into the last line of (16) yields

$$\int \Big[h(\alpha) + h_a'(\alpha)(\delta\hat{f}(\alpha)) + \frac{1}{2}\sum_{i,j} h_{a_i a_j}(\alpha)\bar{v}_{ij}(\alpha)\Delta \Big] P(\alpha, t \mid t + \Delta)\, d\alpha, \tag{17}$$

where $\bar{v}_{ij}$ is the (i, j)th entry of the matrix $\bar{V}$. Recall that $P(\alpha, t \mid t + \Delta) = P(\alpha, t) + \delta Q(\alpha, t)$.

† Given two normal vectors s, t, with $Es = \mu_s$, $Et = \mu_t$, $Est' = \Sigma_{12}$, $Ess' = \Sigma_{11}$, $Ett' = \Sigma_{22}$, we have $E[s \mid t] = \mu_s + \Sigma_{12}\Sigma_{22}^{-1}(t - \mu_t)$ and $\mathrm{Cov}[s \mid t] = \Sigma_{11} - \Sigma_{12}'\Sigma_{22}^{-1}\Sigma_{12}$. (See [10].)

Consider the term

$$\delta\hat{f}(\alpha)[P(\alpha, t \mid t) + \delta Q(\alpha, t)] = \\ [f(\alpha)\Delta + FC\Sigma^{-1}(\delta y - g(\alpha)\Delta)] \\ \cdot [1 + (\delta y - Eg(\alpha)\Delta)'\Sigma^{-1}(g(\alpha) - Eg(\alpha))]P(a, t \mid t).$$

Upon replacing $\delta y \delta y'$ by its expectation $\Sigma\Delta + o(\Delta)$ as discussed earlier, and rearranging, the term becomes* $\delta\bar{f}P \overset{\Delta}{=} [f\Delta + FC\Sigma^{-1}(\delta y - Eg\Delta) + o(\Delta)]P$. Upon replacing this in (17) and assuming (19), (17) may be partially integrated to yield (18).

$$\text{(18)} \qquad \int h(a)\left[P + \delta Q - \sum_i (\delta\bar{f}_i P)_{a_i} + \frac{\Delta}{2}\sum_{i,j} (\bar{v}_{ij} P)_{a_i a_j} + o(\Delta)\right].$$

$$\text{(19)} \qquad 0 = (\delta\bar{f}_i P)h \,\Big|_{a_i=-\infty}^{a_i=\infty} = (\bar{v}_{ij} P)h_{a_j} \,\Big|_{a_i=-\infty}^{a_i=\infty} = (\bar{v}_{ij} P)_{a_i} h \,\Big|_{a_i=-\infty}^{a_i=\infty}.$$

In (19), when $a_i = \pm\infty$, the a_j, $j \neq i$, are arbitrary. Equating (18) to the left hand side of (16) and recalling the arbitrariness of h yields, in the limit,†

$$\begin{aligned} dP &\overset{\Delta}{=} P(a, t + dt \mid t + dt) - P(a, t \mid t) \\ \text{(20)} \qquad &= dQ - \textstyle\sum_i (d\bar{f}_i P)_{a_i} + \frac{1}{2} \sum_{i,j} (\bar{v}_{ij} P)_{a_i a_j} dt, \\ dQ &= P(dy - Egdt)' \,\Sigma^{-1}(g - Eg). \end{aligned}$$

The equation (20) is the culmination of all our efforts. Observe that, as all the components of Σ tend to ∞ (as the value of the observations decreases), (20) tends to Kolmogorov's forward diffusion equation (in differential form). From a formal point of view, (20) may be divided through by dt and viewed as a differential equation with the observation dy/dt as a driving term or input.

It is easy to obtain a set of ordinary differential equations for the conditional moments of P. The method is given below.

3. Discussion of special cases and extensions.

3a. No dynamics. The simplest case is where $f = dz = 0$. Here x is an unknown vector. If some initial distribution $P(a,t_0)$ is assigned to x, then

$$\text{(21)} \qquad dP = P(dy - Egdt)'\Sigma^{-1}(g - Eg)$$

represents the conditional distribution.

* For brevity, $P = P(a, t)$, $\delta Q = \delta Q(a, t)$, $g(a) = g$ and $f(a) = f$ are used when no confusion will arise.

† As $\Delta \to 0$, the expectation of the $o(\Delta)$ in (17) is $o(\Delta)$ and its mean square value is $o(\Delta^2)$. Thus, we have (20) valid in the mean square sense, as discussed in the Appendix.

A special case of importance is where x may take on only finitely many values, $x^1, \cdots, x^s$. Since (21) must hold for each x^i, it reduces to a set of s ordinary differential equations with a simple analog computer representation, even for fairly general observation forms g.

$$Eg_i = \sum_j g_i(a^j)P(a^j, t \mid t).$$

3b. Linear dynamics. The case where $f(x) = Ax$, F = constant, $g(x) = Gx$, and $P(a, 0)$ is Gaussian, where A and G are matrices, has been discussed in [4], where the ordinary differential equation for the conditional expectation of x was obtained. With our form, it is possible to compute all the moments of P in any case; in the linear case, with linear observations, it may be verified that our results specialize to those in [4]. This is, of course, the optimum filter for finite order Gaussian Markov processes.

3c. Filtering. The general problem here may be viewed as an optimum filtering problem, where $dx = fdt + Fdz$ represents the process, and dy is the nonlinear noisy observation. Then (20), or the equations for the moments, represent the form of the optimum filter, i.e., the simulation of (20) yields a running estimate of the conditional probability.

3d. Dependent observation noise. Up to now, the observation noise dw/dt has been white Gaussian. Assume $\beta = dw/dt$ is a correlated process and let it be represented as $d\beta = k(\beta)dt + d\epsilon$, where ϵ is a vector Brownian motion process with $E\delta\epsilon = 0$ and covariance $(\delta\epsilon) = \Sigma\Delta$. The observation is $b \overset{\Delta}{=} dy/dt = g(x) + \beta$. If the observation is considered to be

$$dy = dg + d\beta = (\dot{g} + k(\beta))dt + d\epsilon, \tag{22}$$

the previous theory may be applied: put $\dot{g} + k(\epsilon)$ whenever g appeared. Now, the distribution of β must also be estimated, and the x included the components of β. It appears to be typical of the estimation or filtering problem that, whenever observation noise is correlated, the noise as well as the quantity of interest must be estimated. The differentiation is not easy to simulate. If the observation is assumed to be $\beta + x + d\Psi/dt$, where Ψ is the Brownian motion, then by expanding the state vector x by adjoining β, the theory of the last section may be applied.

3e. Unknown system parameters or system order. Let γ be a constant parameter which either simply parametrizes f or determines the order of the system $dx = fdt + Fdz; f = f(x, \gamma)$. Let γ be given some initial distribution $P(\gamma, 0)$. Then, our results apply to the augmented system

$$dx = f(x, \gamma)dt + Fdz,$$

$$d\gamma = 0,$$

and we merely replace x by the vector $[x, \gamma]$ in the results. $P(a, \gamma, 0) = P(a, 0)P(\gamma, 0)$. This is a general solution to what has been called partial observability by some authors [5].

3f. Determination of the conditional moments. The moments

$$
\begin{aligned}
m_i &= \int a_i P(a, t \mid t)\, da, \\
m(j_1, \cdots, j_r, t) &= \int \prod_{i=1}^{r} (a_i - m_i)^{j_i} P(a, t \mid t)\, da, \qquad (23)\\
c(j_1, \cdots, j_r, t) &= \int \prod_{i=1}^{r} a_i^{j_i} P(a, t \mid t)\, da,
\end{aligned}
$$

satisfy ordinary differential equations, on the right hand side of which the observations appear linearly. The procedure is simple and we merely indicate it here.

We have

$$
dc(j_1, \cdots, j_r, t) = \int \prod_{i=1}^{r} a_i^{j_i} [P(a, t + dt \mid t + dt) - P(a, t \mid t)]\, da. \tag{24}
$$

Let $h(a) = \prod_{i=1}^{r} a_i^{j_i}$. Equating the left hand side of (16) and (17) yields

$$
\begin{aligned}
&\int h(a)[P(a, t + dt \mid t + dt) - P(a, t \mid t)]\, da \\
&\quad = \int h(a) dQ\, da + \int \left[h_a'(a)\, d\bar{f}(a) + \frac{dt}{2} \sum_{i,j} h_{a_i a_j}(a) \bar{v}_{ij} \right] P(a, t \mid t)\, da.
\end{aligned} \tag{25}
$$

Upon performing the integration in (25), dc is obtained. This question is also discussed in [1].

3g. Applications to optimal stochastic control theory. The function $P(0, t)$ is a Markov process, and appears to be the most natural quantity which one may consider as the state variable of the differential system (1). To extend the form (1) to the optimal control formulation, write $dx = f(x, u)dt + F(x, u)dz$, where u is a control function which is to be determined so as to minimize some error criterion, say

$$
E \int_{t_0}^{T} k(x, u, t)\, dt = E \int_{t_0}^{T} \int P(a, t \mid t) k(a, u, t)\, da\, dt, \tag{26}
$$

where E is the expectation over all random variables. (See [1], [6], [7], [8].) Here the optimal control u^0 will be a functional of P.

It is possible to write a second order partial differential equation whose dependent variable is the minimum of (26) and whose independent variables are $P(a, t_0 \mid t_0)$ and t_0, and which yields many properties of u^0. This will not be done here. The equation is analogous to those appearing in [6], [7], [8].† The method of derivation is exactly that used in [1] for the scalar x and linear g case.

3h. Poisson z. The results may be extended to all dz for which the Chapman-Kolmogorov equation is valid; in particular, an equation for P may also be obtained when z is a Poisson process.

3i. The results have numerous applications to special problems in statistical communication theory; these will be considered elsewhere.

3j. Previous cases extended to case where $P(a,t_0)$ is concentrated at only finitely many points, and $x(t)$ is not necessarily generated by a differential equation. For the most general case, $P(a,t_0)$ is a sufficient statistic for control purposes; that is, the minimum of (26) can be written as a functional of $P(a, t_0 \mid t_0)$ for any t_0. When $F = 0$ and $P(a, t_0)$ is concentrated at only finitely many points, it is not usually convenient to take the point of view of §3g. Here, P is not differentiable with respect to a and (15) is a sum; $P(a, t + \Delta \mid \alpha, t)$ is either zero or is concentrated at only one point for any given α.

Although the formerly derived results are not valid for this case, an extremely simple extension is available—in fact, the extension is rigorously verifiable (it is essentially the case discussed in Appendix 2).

To view the results in a fairly general form, let us have a choice of n possible curves $x^i(t)$, $i = 1, \cdots, n$; the ith having conditional probability $p(i \mid t)$ at t. Each x^i could be the solution to the equation $\dot{x} = f(x)$ with a different initial condition, or with a different value of some parameter; or it could be an arbitrary signal function. The observation is $dy = g(a)dt + dw$, where a takes one of the values $x^i(t)$, $i = 1, \cdots, n$. We will write $g(i, t) = g(x^i(t))$.

The method is the following. Instead of keeping track of the arguments at which P is concentrated, as part of the procedure of generating P, we keep track of these arguments separately—and assume that the values of each $x^i(t)$ are available; thus, P is applied to the state i, $i = 1, \cdots, n$, which is not subject to dynamical changes. Carrying previous arguments over, we obtain

$$dP(i \mid t) = P(i \mid t)\cdot(dy - Eg)'\Sigma^{-1}(g - Eg). \tag{27}$$

For this problem, (26) is rewritten as

$$E \int_{t_0}^{T} \sum_i p(i \mid t) k(x^i, u, t)\, dt. \tag{28}$$

† Due to the presence of P, the equation contains functional derivatives as well as ordinary derivatives.

Equation (28) yields that the sufficient state variables for control purposes are all the $p(i \mid t)$ and their (effective) arguments $x^i(t)$ (occasionally some of the x^i can be derived from the others—and may be eliminated as state variables).

APPENDIX

The appendices contain several interesting facts and demonstrations relevant to our method of deriving the differential equations satisfied by certain stochastic processes, such as conditional probabilities. Appendix 1 contains some general remarks and in Appendix 2, the results are verified for some simple cases.

Appendix 1. We first discuss the meaning of the obtained stochastic equations by means of an example. Consider the scalar function $x = e^z$ where $z(t)$ is Brownian motion; $z(t) \sim N(0,\sigma^2 t)$. We are interested in a differential equation which represents x: since $z(t)$ is nowhere differentiable [3], the equation *cannot* be obtained in the usual formal manner. Consider

$$\delta x = e^{z+\delta z} - e^z = x(e^{\delta z} - 1) = x\left(\delta z + \frac{\delta z^2}{2} + \cdots\right). \tag{A.1}$$

Truncate the power series expansion and note that

$$E\left[\delta x - x\left(\delta z + \frac{\delta z^2}{2}\right)\right] = o(\Delta), \tag{A.2}$$

$$E\left[\delta x - x\left(\delta z + \frac{\delta z^2}{2}\right)\right]^2 = o(\Delta^2). \tag{A.3}$$

Thus, in the mean square sense, we have the differential equation

$$dx = x\left(dz + \frac{dz^2}{2}\right). \tag{A.4}$$

Note that, if the $dz^2/2$ term were omitted, (A.2) would be $O(\Delta)$ and (A.3) would be $O(\Delta^2)$; the errors would be of the order of dt, and the resulting solution would be meaningless.

Now, divide the time interval t into n equal sections and let $\Delta = t/n$. Let $\delta z_i = z((i+1)\Delta) - z(i\Delta)$. Thus a discrete approximation to (A.4) is

$$\delta x_i = x_i\left(\delta z_i + \frac{\delta z_i^2}{2}\right)$$

or

$$x_n = x_0 \prod_1^n \left(1 + \delta z_i + \frac{\delta z_i^2}{2}\right).$$

Now it is easily shown that

$$E[x_n - e^z] \to 0,\ E[x_n - e^z]^2 \to 0, \text{ as } \Delta \to 0,\ n \to \infty. \tag{A.5}$$

Thus, again in the mean square sense, (A.4) represents $x = e^z$. If the $dz^2/2$ terms were omitted, (A.5) would tend to some nonzero quantity. This holds in the general case also, since the truncation errors add linearly. Thus, the presence of the second order term dz^2 or $dy_i dy_j$ is justified.

There are some theorems in [3, pp. 286–291] which prove that, given a suitably regular continuous Markov process such as $x = e^z$, x has a representation of the form

$$\text{(A.6)} \quad \begin{aligned} dx = E[x(t + dt) - x(t) \mid x(t)]dt \\ + E^{1/2}[(x(t + dt) - x(t))^2 \mid x(t)]du' \end{aligned}$$

where u is a Brownian motion process. The major problem appears to be the identification of the process u. Let $Edz^2 = \sigma^2 dt$. It may be shown here that $\sigma du = dz$, or

$$\text{(A.7)} \qquad dx = x\sigma^2 dt/2 + xdz.$$

In finite difference form $x_n = x_0 \prod_1^n (1 + \delta z_i + \sigma^2\Delta/2)$. It is verifiable, by direct computation, that

$$\text{(A.8)} \quad E\left[\prod_1^n \left(1 + \delta z_i + \frac{\delta z_i^2}{2}\right) - \prod_1^n (1 + \delta z_i + \sigma^2\Delta/2)\right]^2 \to 0,$$

as $\Delta \to 0$, thus proving the validity of the replacement.

Now, we briefly discuss the nature and interpretation of $(dz)^i$, $i \geqq 2$. According to the derivation,

$$\text{(A.9)} \quad x(t) = \sum_1^n \delta x_i = \sum_1^n x_i \delta z_i + \sum_{i=1}^n x_i \frac{(\delta z_i)^2}{2} + \sum_{i=1}^n x_i \left(\frac{\delta z_i^3}{3!} + \cdots\right)$$

is an exact expression for $x(t)$. This suggests the integral

$$\text{(A.10)} \qquad \int (dz)^2,$$

which may be interpreted as the limit of Riemann sums. With this interpretation,

$$\int_0^t (dz)^2 = \sum (\delta z_i)^2.$$

In the limit as $\Delta \to 0$, $n\Delta = t$, $\sum(\delta z_i)^2$ tends to a constant, $\sigma^2 t$, with probability one and in mean square. With this definition of (A.10), the integral over any measurable t set may be defined and stochastic integrals of the form $\int x(dz)^2 = \int x(dt\, \sigma^2)$ considered. Similarly, for the higher terms $(dz)^i$, $i > 2$, whose integrals degenerate to zero with probability one and in mean square. Thus, the replacement of dz^2 by $\sigma^2 dt$ is again justified.

Appendix 2. Now, using the *limit of the Riemann sum* definition of the integrals we prove, in an indirect although instructive way, that the replacement of dy^2 by $dt\ \sigma^2$ is justified. We limit ourselves, only for simplicity, to a scalar case with no dynamics where x takes the values 0 or 1.†‡

Let $P(1,t) = P_1$. No generality is lost in letting $g(a) = a$. Here, $dP = dQ$ and, by rearranging (13), we obtain

$$\begin{aligned} dP = \frac{P}{2\sigma^2}\Big[&2(dy - m\,dt)(a - m) \\ &+ \left(\frac{dy^2}{\sigma^2} - dt\right)((a - m)^2 - m_2)\Big] + r, \end{aligned} \tag{A.11}$$

where $Er \sim o(dt)$, $Er^2 \sim o(dt^2)$. Also $r(t) - Er(t)$ is an orthogonal process and orthogonal to any function of $P(a, t)$. Upon replacing dy^2 by its average value and neglecting r, we have

$$d\tilde{P} = \frac{\tilde{P}}{\sigma^2}[(dy - \tilde{m}\,dt)(a - \tilde{m})]. \tag{A.12}$$

Also

$$m = Ea = P_1, \quad m_2 = E(a - Ea)^2 = (1 - P_1)P_1. \tag{A.13}$$

Thus, letting $\tilde{P}_1(1 - \tilde{P}_1) = k(\tilde{P}_1) = \tilde{k}$,

$$\begin{aligned} \sigma^2 dP_1 &= k[(dy - P_1 dt) + (dy^2/\sigma^2 - dt)(1 - 2P_1)/2] + r, \\ \sigma^2 d\tilde{P}_1 &= \tilde{k}(dy - \tilde{P}_1 dt) = \tilde{k}[(dy - P_1 dt) + (P_1 - \tilde{P}_1)dt], \end{aligned} \tag{A.14}$$

where $Er \sim o(dt)$, $Er^2 \sim o(dt^2)$.

Now, P_1 and $\tilde{P}_1$ are always in the interval [0, 1]. Letting $e_\tau = P(\tau) - \tilde{P}_1(\tau)$, we have the error

$$\begin{aligned} \sigma^2 e_t = \int_0^t (k - \tilde{k})(dy - P_1\,d\tau) + \int_0^t \tilde{k} e_\tau\,d\tau + \sigma^2 \int_0^t r \\ + \int_0^t k(dy^2/\sigma^2 - d\tau)(1 - 2P_1)/2. \end{aligned} \tag{A.15}$$

Note that $(dy - P_1 dt)$ and $(dy^2/\sigma^2 - dt)$ are orthogonal processes, and are orthogonal to any function of $P_1(\tau)$ or $\tilde{P}_1(\tau)$, $\tau \leqq t$. Using these facts,

$$\begin{aligned} \sigma^4 E e_t^2 &= E\int_0^t (k - \tilde{k})^2 \sigma^2\,d\tau + \int_0^t o(d\tau) \\ &+ E\int_0^t \tilde{k} e_\tau\,d\tau \left\{\int_0^t (k - \tilde{k})(dy - P_1\,d\tau) + \int_0^t k(dy^2/\sigma^2 - d\tau)(1 - 2P_1)/2\right\}. \end{aligned}$$

† Again, the reference quoted above implies some sort of replacement, but the Brownian motion in the differential equation is not identified. The technique here identifies all terms in terms of the observations and properties of the conditional densities.

‡ Generally, the replacement of second order terms by their average values is the easiest part to verify; it is more difficult to prove that our other limiting operations are valid.

Now, observing that k satisfies a Lipschitz condition for P in [0, 1] and using Schwartz's inequality on the last product of integrals yields

$$Ee_t^2 \leqq K_1 \int_0^t Ee_\tau^2 \, d\tau + K_2 \left\{ \int_0^t Ee_\tau^2 \, d\tau \right\}^{1/2} + \int_0^\tau o(d\tau) \tag{A.16}$$

for some positive and finite K_1 and K_2. Thus $Ee_t^2 = 0$, since $P_1(0) = \tilde{P}_1(0)$, and the validity of (A.12) is proved.

In all cases checked, Doob's representation theorems (referred to in the text) yield equations of our form, where the observation noise process w is identified with the Brownian motion. For the problem of this Appendix, there are two families of stochastic processes. The first are the *family of actual sample functions* P_1, when $a = 1$ ($dy = dt + dw$); the second when $a = 0$ ($dy = dw$). Applying Doob's theorems to each of these yields the representation (A.12).

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