SEMI-NORMED VECTOR SPACES WITH DUALS OF INTEGRAL TYPE

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## 1. Introduction.

A classical example of a vector space with a dual space of integral type is the Banach space $C$ of functions $u$ mapping the interval $[0,1]$ of the $t$-axis continuously into the field $\mathbf{R}$ of real numbers with a norm $\max |u(t)|$. Each element $\mu$ in the dual $C'$ of $C$ is represented by a Stieltjes integral

$$\mu(u) = \int_0^1 u(t)\, dg(t)$$

where $g$ is of bounded Jordan variation on $[0,1]$. The classical spaces [1] $L_p$, $l_p$, $c$, $c_0$ as defined in Banach [2], as well as the Orlicz spaces, have duals of integral type.

The earlier papers of the authors were concerned with bilinear functions $\Lambda$ on the product $A \times B$ of two function spaces $A$ and $B$ with duals $A'$ and $B'$ of integral type. The theory of such spaces and of the integral representations of $\Lambda$ had applications to variational theory [3], to the problem of Pringsheim convergence of double Fourier series [4], and to the theory of representation of operators from $A$ to $B'$ and from $B$ to $A'$.

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1. M. Morse and W. Transue. Functionals $F$ bilinear over the product $A \times B$ of two $p$-normed vector spaces II. *Annals of Math.* 51 (1950) 576—614.

2. S. Banach. Théorie des opérations linéaires. Warsaw, 1932.

3. M. Morse and W. Transue. A characterization of the bilinear sums associated with the classical second variation. *Annali di Matematica*, 28 (1949) 25—68.

4. M. Morse and W. Transue. The Fréchet variation and the Pringsheim convergence of double Fourier series. Contributions to Fourier Analysis. *Annals of Math. Studies* No. 25, Princeton University Press, 1950.

The generality of this earlier theory was limited by a restriction of the distribution functions to subsets of euclidean spaces, as well as by the special methods used to extend these distribution functions over the euclidean spaces involved. We shall give the earlier theory a more general setting.

To this end we shall take $E$ as a locally compact topological space, and make use of the Bourbaki theory of integration on $E$. The Bourbaki real measure theory extended to $\mathbf{C}$-measures, i. e., complex valued measures, forms a natural basis for the theory of bilinear functions. We shall introduce products $E' \times E''$ of locally compact topological spaces $E'$, $E''$, and define $\mathbf{C}$-bimeasures $\Lambda$ on such products. We are then led to $\Lambda$-integrals as continuous extensions of $\mathbf{C}$-bimeasures in a way partially analogous to the Bourbaki definition of an integral on $E$ as an extension of a measure on $E$.

Bimeasures (real-valued) are to be sharply distinguished from product measures on $E' \times E''$. Product measures can be partially ordered so as to define a space of Riesz-type (B 17)[5]. There can be no analogous ordering of our spaces of bimeasures as we shall show. As a consequence the decomposition of a measure on $E$ into the difference of two positive measures on $E$ can have no counterpart for bimeasures. Moreover there is no analogue for $\Lambda$ of the absolute measure $|\mu|$ associated with a measure, at least no strict analogue which is a bimeasure. However the superior integral $|\mu|^*$ of $|\mu|$ has an analogue $\Lambda^*$ for $\Lambda$ which serves not only to generalize the Fréchet variation, but also plays an essential role in the definition of $\Lambda$-integrals.

The present paper serves two purposes. It begins with a systematic study of $\mathbf{C}$-measures. It continues with a characterization of a class of vector spaces $A$ whose duals are of integral type. We shall term such spaces $A$, $MT$-spaces. If $A$ is an $MT$-space the dual $A'$ of $A$ is representable as a vector space of $\mathbf{C}$-measures on $E$. The semi-norm $\mathfrak{N}^A(f)$ of $f \in A$ is defined in terms of elements in $A'$ and admits a natural extension over all of $\mathbf{C}^E$. These $MT$-spaces specialize into the classical Banach function spaces.

5. N. Bourbaki. Éléments de mathématique. Vol. II, deuxième edition, also Vols XIII and XVI. Hermann and Cie., Paris.

A typical reference, Bourbaki, Vol. XVI, page 100, will be abbreviated as B XVI 100. Most of the references will be to Volume XIII, *Intégration*, and in this case XIII will be omitted.

Spaces of $MT$-type have many of the properties of the spaces $\mathfrak{L}_P$ (B 125—142). In particular the vector subspace $\mathfrak{F}^A$ of $\mathbf{C}^E$ on which $\mathfrak{N}^A(f) < \infty$ is complete, although $A$ itself may not be complete. In case $A$ is complete many other theorems on $\mathfrak{L}_P$ have valid analogues for $A$.

The results obtained in Part I of this paper are needed in the forthcoming papers on "**C**-bimeasures $\Lambda$ and their superior integrals $\Lambda^*$" and on "**C**-bimeasures and their integral extensions". In Part II a basis is laid for a theory of integral representations of bounded operators from $A$ to $B'$ or $B$ to $A'$ when $A$ and $B$ are of $MT$-type.

## Part I. C-measures and their integral extensions

### 2. C-linear forms.

The field of real numbers is denoted by $\mathbf{R}$. One may also regard $\mathbf{R}$ as a vector space (B VI 3) "over" the field $\mathbf{R}$. Similarly $\mathbf{C}$ denotes the field of complex numbers or the corresponding vector space over the field $\mathbf{C}$.

Let $E$ be a locally compact topological space, and let $X$ be a vector subspace of $\mathbf{R}^E$ over $\mathbf{R}$. Let $X_\mathbf{C}$ be the vector subspace of $\mathbf{C}^E$ over $\mathbf{C}$ generated by $X$. Let $\alpha$ be a $\mathbf{C}$-linear form on $X_\mathbf{C}$, that is a mapping of $X_\mathbf{C}$ into $\mathbf{C}$, additive, and homogeneous with respect to operators in $\mathbf{C}$. We refer also to $\mathbf{R}$-linear forms $\beta$ on $X$ ($X_\mathbf{C}$). Such forms map $X$ ($X_\mathbf{C}$) into $\mathbf{R}$ and are additive, and homogeneous with respect to operators in $\mathbf{R}$.

Real **C**-linear forms. Let $\beta$ be an $\mathbf{R}$-linear form on $X$. For $x_1$ and $x_2$ in $X$, $x_1 + ix_2$ is in $X_\mathbf{C}$. Let $\beta_e$ be an extension of $\beta$ over $X_\mathbf{C}$ defined by setting

$$\beta_e(x_1 + ix_2) = \beta(x_1) + i\beta(x_2)\,. \tag{2.1}$$

One verifies that $\beta_e$ is a $\mathbf{C}$-linear form on $X_\mathbf{C}$. A $\mathbf{C}$-linear form $\gamma$ on $X_\mathbf{C}$ will be termed real if its values on $X$ are real. It may, of course, have complex values on $X_\mathbf{C}$. The form $\beta_e$ defined in (2.1) is a real $\mathbf{C}$-linear form on $X_\mathbf{C}$.

Conversely every real $\mathbf{C}$-linear form $\gamma$ on $X_\mathbf{C}$ may be represented in the form

$$\gamma(x_1 + ix_2) = \beta_e(x_1 + ix_2) = \beta(x_1) + i\beta(x_2)\,, \tag{2.2}$$

where $\beta$ is an $\mathbf{R}$-linear form on $X$ given by the restriction $\gamma|X$ of $\gamma$ to $X$, and $\beta_e$ is defined by (2.2). We state the following lemma.

**Lemma** 2.1. *An arbitrary* **C**-*linear form* $\eta$ *on* $X_{\mathbf{C}}$ *has a representation* $\eta = \mu_e + i\nu_e$, *where* $\mu$ *and* $\nu$ *are* **R**-*linear forms on* $X$, *uniquely determined by* $\eta$.

If $\eta$ has the above representation, then for any $x \in X$

$$\mu(x) = \operatorname{Re} \eta(x) \qquad \nu(x) = \operatorname{Im} \eta(x) \tag{2.3}$$

so that $\mu$ and $\nu$ are uniquely determined by $\eta$.

If $\eta$ is given as **C**-linear on $X_{\mathbf{C}}$, one can define $\mu$ and $\nu$ as **R**-linear forms on $X$ by (2.3) and then extend $\mu$ and $\nu$ over $X_{\mathbf{C}}$ as real **C**-linear forms $\mu_e$ and $\nu_e$. Then $\eta(x) = \mu_e(x) + i\nu_e(x)$ for $x \in X$. Finally $\eta(y) = \mu_e(y) + i\nu_e(y)$ for $y \in X_{\mathbf{C}}$, by virtue of the **C**-linearity of $\eta$ and of $\mu_e$ and $\nu_e$.

## 3. The C-measure $\alpha$ and associated absolute measure $|\alpha|$.

As in B 48 let $\mathfrak{C}_{\mathbf{C}}$ denote the vector space over **C** of continuous mappings of $E$ into **C**, and let $\mathfrak{K}_{\mathbf{C}}$ be the subspace of $\mathfrak{C}_{\mathbf{C}}$ of mappings with compact support. For each mapping $x \in \mathfrak{C}_{\mathbf{C}}$ which is bounded in $E$ set

$$\sup_{t \in E} |x(t)| = U(x). \tag{3.1}$$

We use $U(x)$ rather than $\|x\|$ to avoid ambiguity later. The number $U(x)$ defines a norm on $\mathfrak{K}_{\mathbf{C}}$. The corresponding topology is that of "uniform convergence".

By a **C**-measure on $E$ is meant any **C**-linear form $\beta$ on $\mathfrak{K}_{\mathbf{C}}$ such that, for any compact subset $K$ of $E$, the restriction of $\beta$ to the subspace of functions in $\mathfrak{K}_{\mathbf{C}}$ with support in $K$, is continuous for the topology of uniform convergence. (Cf. B 50). For a **C**-linear form on $\mathfrak{K}_{\mathbf{C}}$ to be a **C**-measure it is necessary and sufficient that for each compact subset $K$ of $E$ there exist a constant $M_K$ such that for each $u \in \mathfrak{K}_{\mathbf{C}}$ with support in $K$

$$|\beta(u)| \leqq M_K U(u). \tag{3.2}$$

We write $\beta(u)$ in the form $\int u d\beta$ and term $\int u d\beta$ the *integral* of $u$ with respect to $\beta$.

The term "measure" as distinguished from "**C**-measure" will be used exclusively in the sense of B XIII. A measure $\mu$ on $E$ is defined on $\mathfrak{K}$ and not on $\mathfrak{K}_{\mathbf{C}}$, is **R**-linear and not **C**-linear, and has real values. A measure $\mu$ on $E$ is termed positive if $\mu(f) \geqq 0$ for each $f \in \mathfrak{K}_+$, that is for each non-negative $f \in \mathfrak{K}$. One then writes $\mu \geqq 0$. We come to a fundamental definition and theorem.

**Theorem 3.1.** *Corresponding to each* $\mathbf{C}$-*measure* $\eta$ *on* $E$ *there exists a unique positive measure* $|\eta|$, *termed the absolute measure defined by* $\eta$, *and such that for* $f \in \mathfrak{K}_+$

$$(3.3) \qquad |\eta|(f) = \sup_{|u| \leqq f} |\eta(u)| \qquad (u \in \mathfrak{K}_{\mathbf{C}}).$$

It is understood that $|u|$ denotes the mapping of $E$ into $\mathbf{R}$ such that $|u|(t) = |u(t)|$ for each $t \in E$.

To show that $|\eta|$ is a measure we refer to the theorem (B 54) that each positive $\mathbf{R}$-linear form on $\mathfrak{K}$ is a measure on $E$. It remains to show that (3.3) defines a unique positive $\mathbf{R}$-linear form $|\eta|$ on $\mathfrak{K}$. For this it is sufficient (B 34) to show that for $f', f'' \in \mathfrak{K}_+$

$$(3.4) \qquad |\eta|(f' + f'') = |\eta|(f') + |\eta|(f'').$$

To establish (3.4) we first show that

$$(3.5) \qquad |\eta|(f' + f'') \leqq |\eta|(f') + |\eta|(f'').$$

**Proof of 3.5.** For each $u \in \mathfrak{K}_{\mathbf{C}}$ choose an angle $\Theta$ such that $e^{i\Theta}\eta(u)$ is real and non-negative. One has $\eta(e^{i\Theta}u) = |\eta(u)|$. Let $Z_\eta$ be the subset of $u \in \mathfrak{K}_{\mathbf{C}}$ such that $\eta(u)$ is real and non-negative. It follows from the definition of $|\eta|$ that for $f \in \mathfrak{K}_+$

$$(3.6) \qquad |\eta|(f) = \sup_{|u| \leqq f} \eta(u) \qquad (u \in Z_\eta).$$

For $t \in E$, $u \in Z_\eta$ and $|u| \leqq f' + f''$ set

$$(3.7) \qquad f'_u(t) = \frac{f'(t)}{f'(t) + f''(t)} u(t), \quad f''_u(t) = \frac{f''(t)}{f'(t) + f''(t)} u(t),$$
$$(f'(t) + f''(t) > 0)$$

and set $f'_u(t) = f''_u(t) = 0$ when $f'(t) = f''(t) = 0$. So defined $f'_u$ and $f''_u$ are in $\mathfrak{K}_{\mathbf{C}} \cap Z_\eta$. Moreover $u = f'_u + f''_u$, and $|f'_u| \leqq f'$, $|f''_u| \leqq f''$. From (3.6) and the definition of $f'_u$, $f''_u$, for $u \in Z_\eta$

$$|\eta|(f' + f'') = \sup_{|u| \leqq f' + f''} \eta(u) = \sup_{|u| \leqq f' + f''} [\eta(f'_u) + \eta(f''_u)]$$
$$= \sup_{|u| \leqq f' + f''} [|\eta(f'_u)| + |\eta(f''_u)|] \leqq \sup_{|u'| \leqq f'} |\eta(u')| + \sup_{|u''| \leqq f''} |\eta(u'')|$$

with $u', u'' \in \mathfrak{K}_{\mathbf{C}}$. Relation (3.5) follows.

**Proof of 3.4.** With $u'$ and $u''$ in $Z_\eta$, (3.6) gives

$$|\eta|(f') = \sup_{|u'| \leqq f'} \eta(u'), \quad |\eta|(f'') = \sup_{|u''| \leqq f''} \eta(u'')$$

so that

$$|\eta|(f') + |\eta|(f'') \leq \sup_{|u'+u''|\leq f'+f''} \eta(u'+u'') \leq |\eta|(f'+f'') .$$

Thus the inequality is excluded in (3.5), thereby establishing the theorem.

With a measure $\mu$ on $E$ there is associated in B 54 a positive measure $|\mu|$ such that for $f \in \mathfrak{K}_+$

$$(3.8) \qquad |\mu|(f) \sup_{|u|\leq f} \mu(u) \qquad (u \in \mathfrak{K}) .$$

**Lemma** 3.1. *If $\mu$ is a measure on $E$ and $\mu_e$ is its extension as a real* **C***-measure, then $|\mu| = |\mu_e|$.*

For $f \in \mathfrak{K}_+$ it follows from (3.6) that

$$(3.9) \qquad |\mu_e|(f) = \sup_{|u|\leq f} \mu_e(u) \qquad (u \in Z_{\mu_e}) .$$

Now $\mu_e(u)$ is real, so that if one sets $u = u_1 + iu_2$ with $u_1$ and $u_2$ real, $\mu_e(u) = \mu(u_1)$. Since $|u_1| \leq |u|$, (3.9) and (3.8) respectively imply that

$$(3.10) \qquad |\mu_e|(f) \leq \sup_{|u_1|\leq f} \mu(u_1) = |\mu|(f) \qquad (u_1 \in \mathfrak{K}) .$$

A reference to the definition of $|\mu_e|(f)$ in (3.3) shows that the inequality is to be excluded in (3.10). Thus $|\mu_e|(f) = |\mu|(f)$, and hence $|\mu| = |\mu_e|$ (cf. B 34).

**Lemma** 3.2. *If $\alpha$ and $\beta$ are* **C***-measures on $E$ then $|\alpha| + |\beta| \geqq |\alpha + \beta|$ .*

For $u \in \mathfrak{K}_{\mathbf{C}}$, $\alpha(u) + \beta(u) = (\alpha + \beta)(u)$ by definition of $\alpha + \beta$, so that $|\alpha(u)| + |\beta(u)| \geqq |(\alpha + \beta)(u)|$ . It follows from (3.3) that

$$|\alpha|(f) + |\beta|(f) \geqq |\alpha + \beta|(f) \quad \text{for } f \in \mathfrak{K}_+ .$$

**Lemma** 3.3. *If $\mu$ and $\nu$ are measures, and if $\alpha$ a* **C***-measure such that $\alpha = \mu_e + i\nu_e$, then*

$$(3.11) \qquad |\mu| + |\nu| \geqq |\alpha| , \quad |\mu| \leqq |\alpha| , \quad |\nu| \leqq |\alpha| .$$

Since $|i\nu_e| = |\nu_e|$ the first relation in (3.11) follows from Lemmas 3.1 and 3.2.

To show that $|\mu| \leq |\alpha|$ recall that for $f \in \mathfrak{K}_+$

$$(3.12) \qquad |\mu|(f) = \sup_{|u|\leq f} \mu(u) \qquad (u \in \mathfrak{K})$$

in accordance with (3.8). Since $\mu(u) + i\nu(u) = \alpha(u)$ , $|\mu(u)| \leq |\alpha(u)|$ . From (3.12) then, for $u \in \mathfrak{K}$, $f \in \mathfrak{K}_+$ , and $u' \in \mathfrak{K}_{\mathbf{C}}$ ,

$$|\mu|(f) \leqq \sup_{|u| \leqq f} |\alpha(u)| \leqq \sup_{|u'| \leqq f} |\alpha(u')| = |\alpha|(f) \qquad \text{(Cf. 3.3)}.$$

Thus $|\mu| \leqq |\alpha|$. Similarly $|\nu| \leqq |\alpha|$.

**Lemma** 3.4. *If $\mu$ and $\nu$ are measures on $E$, $\mu_e + i\nu_e$ is a $\mathbf{C}$-measure on $E$. If $\eta$ is a $\mathbf{C}$-measure there exist unique measures $\mu$ and $\nu$ such that $\eta = \mu_e + i\nu_e$.*

Let $\mu$ be a measure on $E$ and $K$ a compact subset of $E$. If $u_1 + iu_2$ is in $\mathfrak{K}_{\mathbf{C}}$ with support in $K$ and $u_i$ real, then $u_i$ is in $\mathfrak{K}$ with support in $K$. Now

$$\begin{aligned} |\mu_e(u_1 + iu_2)| &= |\mu(u_1) + i\mu(u_2)| \leqq |\mu(u_1)| + |\mu(u_2)| \\ &\leqq M_K[U(u_1) + U(u_2)] \leqq 2M_K\, U(u_1 + iu_2) \end{aligned}$$

where $M_K$ is a constant associated with $K$ and $\mu$. Thus $\mu_e$ is a $\mathbf{C}$-measure. Similarly $\nu_e$ is a $\mathbf{C}$-measure. Hence $\mu_e + i\nu_e$ is a $\mathbf{C}$-measure.

If $\eta$ is a $\mathbf{C}$-measure it follows from Lemma 2.1 that $\eta = \mu_e + i\nu_e$ where $\mu$ and $\nu$ are $\mathbf{R}$-linear forms on $\mathfrak{K}$ such that for each $u \in \mathfrak{K}$, $\eta(u) = \mu(u) + i\nu(u)$. Since $|\mu(u)| \leqq |\eta(u)|$ and $|\nu(u)| \leqq |\eta(u)|$ one infers that $\mu$ and $\nu$ are measures. Moreover $\mu$ and $\nu$ are uniquely determined by the condition $\eta(u) = \mu(u) + i\nu(u)$, valid for $u \in \mathfrak{K}$.

*If $\mu$ and $\nu$ are measures we shall say that $\mu_e + i\nu_e$ is the canonical form of the $\mathbf{C}$-measure $\alpha = \mu_e + i\nu_e$.*

## 4. The spaces $\mathfrak{F}^1_{\mathbf{C}}(\alpha)$ and $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$.

Let $\alpha$ be a $\mathbf{C}$-measure on $E$. For each $x \in \mathbf{C}^E$ the number $N_1[|x|, |\alpha|]$ is well defined (B 127). It may be infinite. We here set

$$(4.1) \qquad N_1[|x|, |\alpha|] = N^1_{\mathbf{C}}[x, \alpha].$$

Let the vector subspace of $\mathbf{C}^E$ on which $N^1_{\mathbf{C}}[x, \alpha] < \infty$ be denoted by $\mathfrak{F}^1_{\mathbf{C}}(\alpha)$, and topologized by $N^1_{\mathbf{C}}$ as a semi-norm. When $\alpha$ is fixed we usually write $N^1_{\mathbf{C}}[x]$ and $\mathfrak{F}^1_{\mathbf{C}}$ in place of $N^1_{\mathbf{C}}[x, \alpha]$ and $\mathfrak{F}^1_{\mathbf{C}}(\alpha)$, respectively.

Note that $\mathfrak{K}_{\mathbf{C}}$ is a vector subspace of $\mathfrak{F}^1_{\mathbf{C}}$. Let $\mathfrak{L}^1_{\mathbf{C}}$ denote the closure of $\mathfrak{K}_{\mathbf{C}}$ in $\mathfrak{F}^1_{\mathbf{C}}$. The space $\mathfrak{L}^1_{\mathbf{C}}$ is supposed topologized by $N^1_{\mathbf{C}}$.

For each $u \in \mathfrak{K}_{\mathbf{C}}$ the relations (4.4) below hold with $x = u$, by virtue of the definition (3.3) of $|\alpha|$, the definition of the superior integral (B 104), and of $N_1$ (B 127) respectively. It follows that the $\mathbf{C}$-linear form $\alpha$ is uniformly continuous on $\mathfrak{K}_{\mathbf{C}}$ (topologized as a subspace of $\mathfrak{L}^1_{\mathbf{C}}$). We infer from B II 151 that $\alpha$ admits a unique extension over $\mathfrak{L}^1_{\mathbf{C}}$ as a uniformly continuous $\mathbf{C}$-linear form. This extended form is again denoted by $\alpha$. For $x \in \mathfrak{L}^1_{\mathbf{C}}(\alpha)$ one writes

$$\alpha(x) = \int x \, d\alpha$$

and terms $\alpha(x)$ the $\alpha$-integral of the $\alpha$-integrable function $x$.

For each positive measure $\mu$ the space $\mathfrak{L}^1(\mu)$ is defined as in B 131.

**Lemma 4.1.** *The application $x \to |x|$ maps the space $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ uniformly continuously into $\mathfrak{L}^1(|\alpha|)$. The space $\mathfrak{L}^1(|\alpha|)$ is a subspace of $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$.*

Suppose that $x$ and $y$ are in $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$, and that for some $e > 0$

$$N^1_{\mathbf{C}}[x - y, \alpha] < e. \tag{4.2}$$

On recalling that $||x| - |y|| \leqq |x - y|$ we infer that

$$N_1[|x| - |y|, |\alpha|] \leqq N_1[|x - y|, |\alpha|] = N^1_{\mathbf{C}}[x - y, \alpha] < e. \tag{4.3}$$

Given $e > 0$ and $x \in \mathfrak{L}^1_{\mathbf{C}}(\alpha)$ there exists a mapping in $\mathfrak{K}_{\mathbf{C}}$, say $y_e$, such that $N^1_{\mathbf{C}}[x - y_e, \alpha] < e$ since $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ is the closure of $\mathfrak{K}_{\mathbf{C}}$ in $\mathfrak{F}^1_{\mathbf{C}}(\alpha)$. Since $|y_e|$ is in $\mathfrak{K}$ and since (4.3) holds for arbitrary $e > 0$ with $y$ replaced by $y_e$, $|x|$ is $|\alpha|$-integrable. Since (4.2) implies (4.3) the mapping $x \to |x|$ is uniformly continuous. Finally the fact that $\mathfrak{L}^1(|\alpha|)$ is a subspace of $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ with a topology derived from that of $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ is readily verified.

**Lemma 4.2.** *If $\alpha$ is a $\mathbf{C}$-measure and if $x \in \mathfrak{L}^1_{\mathbf{C}}(\alpha)$ then*

$$\left|\int x \, d\alpha\right| \leqq \int |x| \, d|\alpha| = \int^{\bullet} |x| \, d|\alpha| = N^1_{\mathbf{C}}[x, \alpha]. \tag{4.4}$$

We have already seen that (4.4) holds for $x \in \mathfrak{K}_{\mathbf{C}}$. Moreover $|x|$ is $|\alpha|$-integrable by Lemma 4.1, and the two equalities in (4.4) follow from B 146.

The first relation in (4.4) will be established by a limiting process, making use of a sequence of elements $x_n \in \mathfrak{K}_{\mathbf{C}}$ such that $x_n$ tends to $x$ in $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ as $n \uparrow \infty$. The mapping $x \to \int x \, d\alpha$ of $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ into $\mathbf{C}$ is continuous as we have noted in defining the integral $\alpha$ as a continuous extension of the measure $\alpha$. The application $x \to \int |x| \, d|\alpha|$ can be written in the form $x \to |x| \to \int |x| \, d|\alpha|$ so that its continuity follows with the aid of Lemma 4.1. The validity of the first relation in (4.4) for each $x_n$ thus implies its validity for $x$ as a limit in $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ of the sequence $(x_n)$. This establishes the lemma.

Making use of the fact that for $x \in \mathbf{C}^E$

$$N^1_{\mathbf{C}}[x, \alpha] = N^1_{\mathbf{C}}[\bar{x}, \alpha] \qquad (\alpha \text{ a } \mathbf{C}\text{-measure})$$

where $\bar{x}$ is the conjugate of $x$, one readily establishes the following lemma.

Lemma 4.3. Let $x \in \mathbf{C}^E$ have the form $x = x_1 + ix_2$, where $x_i$ is real, $i = 1, 2$. A necessary and sufficient condition that $x$ be $\alpha$-integrable is that $\bar{x}$ be $\alpha$-integrable, or equivalently that $x_i$, $i = 1, 2$, be $\alpha$-integrable.

Lemma 4.4. Let $x \in \mathbf{R}^E$ have the form $x = x^+ - x^-$ (B 18). A necessary and sufficient condition that $x$ be $\alpha$-integrable is that $x^+$ and $x^-$ be $\alpha$-integrable.

It is clearly sufficient that $x^+$ and $x^-$ be $\alpha$-integrable. The necessity of the condition follows from the relations

$$2x^+ = |x| + x, \qquad 2x^- = |x| - x$$

and the fact that $|x|$ is $\alpha$-integrable with $x$.

Riesz components of $x \in \mathbf{C}^E$. Each $x \in \mathbf{C}^E$ admits a cononical representation of the form

$$x = x_1 + ix_2 = x_1{}^+ - x_1{}^- + i\,(x_2{}^+ - x_2{}^-) \tag{4.5}$$

in which $x_i$ is real. The mappings $x_1{}^+$, $x_1{}^-$, $x_2{}^+$, $x_2{}^-$ will be termed Riesz components of $x$.

Lemmas 4.3 and 4.4 combine as follows.

Lemma 4.5. A necessary and sufficient condition that an $x \in \mathbf{C}^E$ be $\alpha$-integrable is that the Riesz-components of $x$ be $\alpha$-integrable.

Notation. For fixed measure $\mu$ let the semi-norm whose value at $x \in \mathbf{R}^E$ is $N_1[x, \mu]$ be denoted by $N_1[\,.\,, \mu]$. For fixed $\mathbf{C}$-measure $\alpha$ let $N^1_{\mathbf{C}}[\,.\,, \alpha]$ be similarly defined.

## 5. $\alpha$-Negligible mappings and sets in $E$.

If $\alpha$ is a $\mathbf{C}$-measure and $x$ is in $\mathbf{C}^E$ we term $x$ $\alpha$-negligible if $|x|$ is $|\alpha|$-negligible in the sense of B p. 118. A subset $A$ of $E$ is termed $\alpha$-negligible if $A$ is $|\alpha|$-negligible. The phrase "presque partout dans $E$ par rapport à $\alpha$" will be symbolized by (p, p, $\alpha$) and shall be equivalent to (p, p, $|\alpha|$). We note the following consequence of Theorem 1 of B 119.

(i) A necessary and sufficient condition that $x \in \mathbf{C}^E$ be $\alpha$-negligible is that $x$ vanish (p, p, $\alpha$) in $E$.

When $\mu$ is a measure the extension of $\mu$ as a real $\mathbf{C}$-measure has been denoted by $\mu_e$. According to Lemma 3.1, $|\mu| = |\mu_e|$. Hence subsets $A$ of $E$ which are $|\mu|$-negligible are $|\mu_e|$-negligible and conversely. Bourbaki defines a $\mu$-negligible set in fascicule XIII only when $\mu \geqq 0$. If $\mu$ is a non-positive measure a subset $A$ of $E$ might be called $\mu$-negligible if $|\mu|$-negligible. With this convention $A$ is $\mu$-negligible if and only if $\mu_e$-negligible.

The classes $\tilde{x}^\alpha$. Let $\alpha$ be a $\mathbf{C}$-measure. Let $E_1$ be a subset of $E$ such that $E - E_1$ is $\alpha$-negligible. A mapping $y \in \mathbf{C}^{E_1}$ will be said to be a mapping in $\mathbf{C}^E$ (p, p, $\alpha$). If $x$ defines a mapping in $\mathbf{C}^E$ (p, p, $\alpha$) the class of all such mappings $y$ for which $x(t) = y(t)$ (p, p, $\alpha$) will be denoted $\tilde{x}^\alpha$. If $\mu$ is a positive measure and $x$ is a mapping in $\mathbf{R}^E$ (p, p, $\mu$) the equivalence class $\tilde{x}$ of B 120 will here be more precisely denoted by $\tilde{x}^\mu$ whenever clarity is thereby served.

Let $\mathbf{L}^1_{\mathbf{C}}(\alpha)$ denote the normed Hausdorff space "associated" with $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$. (Cf. B 131) The elements of $\mathbf{L}^1_{\mathbf{C}}(\alpha)$ are the equivalence classes $\tilde{x}^\alpha$, where $x$ is an element in $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$. The norm of $\tilde{x}^\alpha$ is taken as $N^1_{\mathbf{C}}[x, \alpha]$ and may be denoted by $N^1_{\mathbf{C}}[\tilde{x}, \alpha]$. That $N^1_{\mathbf{C}}[x, \alpha]$ is independent of the choice of $x \in \mathbf{C}^E$ in the class $\tilde{x}^\alpha$ is affirmed as follows.

(ii) *If $x$ and $y$ are in $\mathbf{C}^E$ and if $x(t) = y(t)$ (p, p, $\alpha$) then $N^1_{\mathbf{C}}[x, \alpha] = N^1_{\mathbf{C}}[y, \alpha]$. If $x$ is $\alpha$-integrable, $y$ is $\alpha$-integrable, and $\alpha(x) = \alpha(y)$.*

If $z \in \mathbf{C}^E$ and $z = 0$ (p, p, $\alpha$) then $|z| = 0$ (p, p, $|\alpha|$). Hence $|z|$ is $|\alpha|$-negligible so that

$$0 = \int^* |z| \, d\, |\alpha| = N_1[|z|, |\alpha|] = N^1_{\mathbf{C}}(z, \alpha).$$

If then $x = y$, (p, p, $\alpha$), $N^1_{\mathbf{C}}[x - y, \alpha] = 0$ and

$$N^1_{\mathbf{C}}[x, \alpha] \leqq N^1_{\mathbf{C}}[x - y, \alpha] + N^1_{\mathbf{C}}[y, \alpha] = N^1_{\mathbf{C}}[y, \alpha].$$

A reversal of the roles of $x$ and $y$ shows that $N^1_{\mathbf{C}}[x, \alpha] = N^1_{\mathbf{C}}[y, \alpha]$ as stated in (ii).

If $x$ is $\alpha$-integrable, $y$ is $\alpha$-integrable, since the condition $N^1_{\mathbf{C}}[x-y, \alpha] = 0$ implies that $y$ is in the closure of $x$ in $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$. From the continuity of $\alpha$ over $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ we conclude that $\alpha(x) = \alpha(y)$. Thus (ii) is established.

**Definition.** *A mapping $x \in \mathbf{C}^E$ (p, p, $\alpha$) will be said to be $\alpha$-integrable if $\tilde{x}^\alpha$ contains an $\alpha$-integrable mapping $y \in \mathbf{C}^E$. One then sets $\alpha(x) = \alpha(y)$ and terms $\alpha(x)$ the $\alpha$-integral* of $x$. A mapping is admitted to $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ if and only if $x$ is $\alpha$-integrable and $x$ is in $\mathbf{C}^E$.

We make an application of these conventions.

The space $\mathfrak{J}_+$. Recall that $\overline{\mathbf{R}}_+$ is the space $\mathbf{R}_+$ completed by the point $+\infty$, and that $\mathfrak{J}_+$ is the vector space of all positive, lower semi-continuous mappings $p$ of $E$ into $\overline{\mathbf{R}}_+$.

Lemma 5.1. If $\alpha$ is a $\mathbf{C}$-measure on $E$ and if $x$ maps $E$ into $\overline{\mathbf{R}}_+$, a necessary and sufficient condition that $x$ be $\alpha$-integrable is that $x$ be $|\alpha|$-integrable.

Let $E_1$ be the subset of $E$ on which $x(t) = \infty$. Let $y \in \mathbf{R}_+^E$ be such that $y(t) = x(t)$ for $t \in E - E_1$ and $y(t) = 0$ for $t \in E_1$. It follows from our conventions and from (ii) that $x$ is $\alpha$-integrable if and only if $E_1$ is $\alpha$-negligible and $y$ is $\alpha$-integrable, or equivalently if $E_1$ is $|\alpha|$-negligible and $y$ is $|\alpha|$-integrable (Lemma 4.1), or equivalently again if $x$ is $|\alpha|$-integrable in the sense of Bourbaki.

Lemma 5.2. If $\alpha$ is a $\mathbf{C}$-measure on $E$ and $p \in \mathfrak{J}_+$, a necessary and sufficient condition that $p$ be $\alpha$-integrable is that $|\alpha|^*(p) < \infty$.

By the preceding lemma $p$ is $\alpha$-integrable if and only if $|\alpha|$-integrable, or equivalently by Prop. 5 B 150, if and only if $|\alpha|^*(p) < \infty$.

## 6. Filtering sets $_+\varphi_h$ and $_-\varphi_p$.

Let $h$ be an arbitrary mapping of $E$ into $\overline{\mathbf{R}}_+$. Let $_+\varphi_h$ denote the ensemble of mappings $p \in \mathfrak{J}_+$, $p \geqq h$, filtering (i. e. filtrant) for the relation $\geqq$ (B II 35). If $\mu$ is a positive measure on $E$ a mapping $p \in {_+\varphi_h}$, finite or not, is $\mu$-integrable if and only if $N_1[p, u] < \infty$ (B 150). However $p$ is not admitted as an element of $\mathfrak{L}^1(\mu)$ unless finite-valued (B 132).

A convention as to limits. In classical analysis a sequence $(x_n)$ of functions is frequently given formally, in such a way that the convergence of $(x_n)$ to $x$ is defined a priori only for some subsequence $x_m, x_{m+1}, \ldots$, of $(x_n)$. It is then convenient to say that the original sequence $(x_n)$ converges to $x$. There is here an analogous convention involving a filtering set. Let $A$ be an arbitrary space and $B$ a subspace of $A$. Let $H$ be a filtering set in $A$ such that some "section" $S(a)$ of $H$ is in $B$, but $H$, as a whole, is not necessarily in $B$. If $F$ maps $B$ into a topological Hausdorff space $F(B)$, we shall say that $F$ has the limit $b \in F(B)$ with respect to $H$, if $F$ has the limit $b$ with respect to $S(a)$ regarded as a filtering set on $B$.

A limit of $F$ with respect to $H$ is independent of the section $S(a)$ of $H$ in $B$ used to define a filtering set in $B$. In accordance with Bourbaki (B XVI, p. 13, no. 19) we write

$$b = \lim_{x \in H} F(x) \tag{6.0}$$

when the limit exists and equals $b$. With this notation we state the following lemma.

Lemma 6.1. *If $\mu$ is a positive measure on $E$ and if $h$ is a positive, numerical function on $E$, finite or not, but $\mu$-integrable, then*

$$\lim_{p \in {}_{+}\varphi_h} N_1[p-h, \mu] = 0 \tag{6.1}$$

*and*

$$\lim_{p \in {}_{+}\varphi_h} \mu(p) = \mu(h). \tag{6.2}$$

Let $e > 0$ be given. By virtue of Theorem 3 of B 151, there exists a $\mu$-integrable $p \in {}_{+}\varphi_h$ such that

$$\int (p-h)\, d\mu < e.$$

Since $p-h \geqq 0$ and $p-h$ is $\mu$-integrable

$$N_1[p-h, \mu] = \int (p-h)\, d\mu < e. \tag{6.3}$$

From (6.3) and the monotonicity of the norm $N_1$, relation (6.1) follows. Thus the image in $\mathbf{L}_1(\mu)$ of the "filter of sections" of ${}_{+}\varphi_h$ converges in $\mathbf{L}_1(\mu)$ to $\tilde{h}$. Relation (6.2) follows, since the integral $\mu$ is continuous on $\mathbf{L}_1(\mu)$.

Notation. We shall find it both simplifies and clarifies our notation, particularly when we come to $\mathbf{C}$-bimeasures $\Lambda$ and their integrals, if we regard the relation

$$\lim_{p \in {}_{+}\varphi_h} \mu(p) = \lim_{p \downarrow h} \mu(p) \tag{6.4}$$

as a definition of the right member, and prefer the right member as notation whenever the limit on the left exists. A similar notational change will be made in (6.1).

Given $p \in \mathfrak{J}_+$, let ${}_-\varphi_p$ denote the set of mappings $f \in \mathfrak{K}_+$, with $f \leqq p$, filtering for the relation $\leqq$. If a mapping $F$ of some section of ${}_-\varphi_p$ into $\mathbf{R}$ is defined; then we shall write

$$(6.5) \qquad \lim_{f \in {}_-\varphi_p} F(f) = \lim_{f \uparrow p} F(f)$$

and prefer the notation on the right whenever the limit on the left exists. With this understood we have the following lemma.

**Lemma** 6.2. *If $\mu$ is a positive measure on $E$ and if $p \in \mathfrak{J}_+$ is $\mu$-integrable, then*

$$(6.6) \qquad \lim_{f \uparrow p} N_1[p - f, \mu] = 0$$

*and*

$$(6.7) \qquad \lim_{f \uparrow p} \mu(f) = \mu(p).$$

The proof of Lemma 6.2 is similar to that of Lemma 6.1.

We use these lemmas to prove the following theorem.

**Theorem** 6.1. *If $\mu$ and $\nu$ are positive measures on $E$, and if $x \in \mathbf{R}^E$ is $\mu$ and $\nu$-integrable, then $x$ is $(\mu + \nu)$-integrable and*

$$(6.8) \qquad (\mu + \nu)(x) = \mu(x) + \nu(x).$$

Write $x = x^+ - x^-$. If $x$ is $\mu$ and $\nu$-integrable $x^+$ and $x^-$ are $\mu$ and $\nu$-integrable (B 136). It is accordingly sufficient to establish the theorem in case $x \geqq 0$. Suppose then $x \geqq 0$.

By Lemma 6.1

$$(6.9) \qquad 0 = \lim_{p \downarrow x} \{N_1[p - x, \mu] + N_1[p - x, \nu]\}.$$

By virtue of the relation $(\mu + \nu)^* = \mu^* + \nu^*$ of B 113 we infer from (6.9) that

$$0 = \lim_{p \downarrow x} N_1[p - x, \mu + \nu].$$

Thus $p$ converges to $x$ according to ${}_+\varphi_x$ in the topology of $\mathfrak{F}^1(\mu + \nu)$. Hence $x$ is in $\mathfrak{L}^1(\mu + \nu)$.

Since $x \geqq 0$ it follows from (4.4) that $(\mu + \nu)(x) = (\mu + \nu)^*(x)$, $\mu(x) = \mu^*(x)$, $\nu(x) = \nu^*(x)$, so that the relation $(\mu + \nu)^* = \mu^* + \nu^*$ implies (6.8). The theorem follows.

The filtering set $H_p$. Let $H_p$ be an arbitrary ensemble of mappings $f \in \mathfrak{K}_+$, with $f \leqq p$ and $\sup f = p$, filtering for the relation $\leqq$. The filtering set ${}_-\varphi_p$ is a set $H_p$.

Lemma 6.2 admits the following extension.

Lemma 6.3. *If $\alpha$ is a $\mathbf{C}$-measure and if $p \in \mathfrak{J}'_+$ is $\alpha$-integrable, then*

$$\lim_{f \in H_p} N_1[p - f, |\alpha|] = 0 \qquad \lim_{f \in H_p} \alpha(f) = \alpha(p).$$

The first limit relation follows from (2) of B 105 on replacing $\mu^*$ of Bourbaki by $|\alpha|^*$. The second relation then follows from the continuity of $\alpha$ on $\mathbf{L}^1_{\mathbf{C}}(\alpha)$.

## 7. The filtering sets ${}_-\Phi_{\mathbf{p}}$, ${}_+\Phi_{\mathbf{x}}$.

Let $p_i \in \mathfrak{J}_+$ $(i = 1, 2, 3, 4)$ be given. It will be convenient to introduce the vector

$$\mathbf{p} = (p_1, p_2, p_3, p_4) \tag{7.1}$$

representing $\mathbf{p}$ as a point in the product space

$$\mathfrak{J}_+ \times \mathfrak{J}_+ \times \mathfrak{J}_+ \times \mathfrak{J}_+ = [\mathfrak{J}_+]^4. \tag{7.2}$$

Let $\dot{\mathbf{p}}$ be an image of $\mathbf{p}$ in $\mathbf{C}^E$ with value at $s \in E$ given by the equation

$$\dot{\mathbf{p}}(s) = (p_1(s) - p_2(s)) + i(p_3(s) - p_4(s)) \tag{7.3}$$

when each $p_i(s)$ is finite, otherwise by (7.3) with each value $p_i(s)$ which is infinite replaced by 0. We shall have occasion to make these replacements only on subsets of $E$ which are negligible with respect to some given $\mathbf{C}$-measure.

For the vector $\mathbf{h} = (h_1, h_2, h_3, h_4)$ represented by a point in the product space $[\mathbf{R}^E_+]^4$ an image $\dot{\mathbf{h}}$ in $\mathbf{C}^E$ is defined as was the image $\dot{\mathbf{p}}$ of $\mathbf{p}$. For a vector $\mathbf{f} = (f_1, f_2, f_3, f_4)$ represeneed by a point in $[\mathfrak{K}_+]^4$ the image $\dot{\mathbf{f}}$ reduces to $f_1 - f_2 + i(f_3 - f_4)$.

Since each $x \in \mathbf{C}^E$ has a *canonical* representation $x = \dot{\mathbf{h}}$ in which $(h_1, h_2, h_3, h_4)$ are Riesz components of $x$ it follows that $\mathbf{h} \to \dot{\mathbf{h}}$ maps $[\mathbf{R}^E_+]^4$ *onto* $\mathbf{C}^E$.

The product of filtering sets. Let $H_i$ be a filtering set in a space $A_i$, $i = 1, \dots, n$ with elements in $H_i$ ordered by a relation $\sigma_i$. The product $\prod_{i=1}^{n} H_i$ is a subset of $\prod_{i=1}^{n} A_i$. With $(\sigma_i)$ we associate a partial order $\sigma$ of points $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\prod_{i=1}^{n} H_i$. The points $\mathbf{x}$ and $\mathbf{y}$ will be said to satisfy the relation $\mathbf{x}(\sigma)\mathbf{y}$ if and only if $x_i(\sigma_i)y_i$ for $i = 1, \dots, n$. It is clear that if $\mathbf{a}$ and $\mathbf{b}$ are given in $\prod_{i=1}^{n} H_i$, there always exists $\mathbf{c}$ in this product such that $\mathbf{a}(\sigma)\mathbf{c}$ and $\mathbf{b}(\sigma)\mathbf{c}$. Ordered by the relation $\sigma$, $\prod_{i=1}^{n} H_i$ is thus a filtering set in $\prod_{i=1}^{n} A_i$, termed the product of the filtering sets $H_i$.

Given $\mathbf{p} \in [\mathfrak{Z}_+]^4$ the filtering set ${}_-\varphi_{p_i}$, $i = 1, \dots, 4$, is defined in $\mathfrak{R}_+$. The product filtering set

$$\prod_{i=1}^{4} {}_-\varphi_{p_i} = {}_-\Phi_{\mathbf{p}} \tag{7.4}$$

is here introduced, and is well defined in $[\mathfrak{R}_+]^4$. Given $\mathbf{h} \in [\mathbf{R}_+^E]^4$ the product filtering set

$$\prod_{i=1}^{h} {}_+\varphi_{h_i} = {}_+\Phi_{\mathbf{h}} \tag{7.5}$$

is defined in $[\mathfrak{Z}_+]^4$. We shall employ ${}_+\Phi_{\mathbf{h}}$ only in case $(h_1, h_2, h_3, h_4)$ are Riesz components of $\dot{\mathbf{h}}$.

We shall adopt notational conventions similar to those indicated in (6.4) and (6.5), preferring the right member of the notational identity

$$\lim_{\mathbf{p} \in {}_+\Phi_{\mathbf{h}}} F(\mathbf{p}) = \lim_{\mathbf{p} \downarrow \mathbf{h}} F(\mathbf{p}) \qquad [\mathbf{p} \in [\mathfrak{Z}_+]^4] \tag{7.6}$$

to the left. We similarly write

$$\lim_{\mathbf{f} \in {}_-\Phi_{\mathbf{p}}} M(\mathbf{f}) = \lim_{\mathbf{f} \uparrow \mathbf{p}} M(\mathbf{f}) \qquad [\mathbf{f} \in [\mathfrak{R}_+]^4]. \tag{7.7}$$

Lemma 6.1 has the following extension. Lemma 7.1 will play a vital role in our treatment of the integral extensions of $\mathbf{C}$-bimeasures.

**Lemma 7.1.** *If $\alpha$ is a $\mathbf{C}$-measure on $E$, if $x \in \mathbf{C}^E$ is $\alpha$-integrable and if $(h_1, h_2, h_3, h_4)$ are the Riesz components of $x$, then*

$$\lim_{\mathbf{p} \downarrow \mathbf{h}} N_{\mathbf{C}}^1 [\dot{\mathbf{p}} - x, \alpha] = 0, \qquad \lim_{\mathbf{p} \downarrow \mathbf{h}} \alpha(\dot{\mathbf{p}}) = \alpha(x). \tag{7.8}$$

The $\alpha$-integrability of $x$ implies the $\alpha$-integrability of $h_i$, $i = 1, 2, 3, 4$, by Lemma 4.5. By Lemma 4.1 each $h_i$ is $|\alpha|$-integrable. It follows from

Lemma 6.1 that with $p_i \in {}_+\varphi_{h_i}$

$$\lim_{p_i \downarrow h_i} N_1[p_i - h_i, |\alpha|] = 0 \qquad [i = 1, \ldots, 4]. \tag{7.9}$$

It is clear that

$$|\dot{\mathbf{p}} - x| \leqq \sum_{i=1}^{4} (p_i - h_i) \tag{7.10}$$

so that the first relation in (7.8) follows from (7.9).

The second relation in (7.8) is then implied by the first relation in (7.8) and the continuity of the integral $\alpha$ on $\mathbf{L}^1_{\mathbf{C}}(\alpha)$.

Lemma 7.2. *If $\alpha$ is a $\mathbf{C}$-measure on $E$ and $\mathbf{p} = (p_1, p_2, p_3, p_4)$ is a vector such that $p_i$ is in $\mathfrak{J}_+$, and $N_1[p_i, |\alpha|] < \infty$ for $i = 1, \ldots, 4$, then*

$$\lim_{\mathbf{f} \uparrow \mathbf{p}} N^1_C[\dot{\mathbf{p}} - \dot{\mathbf{f}}, \alpha] = 0, \quad \lim_{\mathbf{f} \uparrow \mathbf{p}} \alpha(\dot{\mathbf{f}}) = \alpha(\dot{\mathbf{p}}). \tag{7.11}$$

It follows from the conditions on $p_i$ that $p_i$ is $|\alpha|$-integrable (Lemma 5.2), so that one can apply Lemma 6.2 to $p_i$, with $\mu$ replaced by $|\alpha|$. Thus

$$\lim_{f_i \uparrow p_i} N_1[p_i - f_i, |\alpha|] = 0 \qquad [i = 1, \ldots, 4]. \tag{7.12}$$

The first relation in (7.11) follows from (7.12), and the second relation follows from the continuity of the integral $\alpha$ on $\mathbf{L}^1_{\mathbf{C}}(\alpha)$.

## 8. Complex truncation.

If $\alpha$ is a $\mathbf{C}$-measure on $E$ and if $x \in \mathbf{C}^E$ is $\alpha$-integrable, it follows from the definition in § 4 of $\alpha(x)$ as an integral, and from (4.4), that, corresponding to a prescribed $e > 0$, there exists a $u \in \mathfrak{K}_{\mathbf{C}}(E)$ such that

$$e > N^1_C[u - x, \alpha] \geqq |\alpha(u - x)|. \tag{8.0}$$

Thus $u$ approximates $x$ in $\mathfrak{L}^1_{\mathbf{C}}$, and $\alpha(u)$ approximates $\alpha(x)$. For later purposes this approximation needs to be supplemented as in Theorem 8.1. To this end a basic definition is required.

*The truncated image $s_* x$ of $x$.* For $x \in \mathbf{C}^E$ and $s \in \mathbf{R}^E_+$ set

$$\rho^x_s(t) = \inf\left[\frac{s(t)}{|x(t)|}, 1\right] \qquad [t \in E] \tag{8.1}$$

when $x(t) \neq 0$, and set $\rho^x_s(t) = 1$ when $x(t) = 0$. The truncated image $s_* x$ of $x$ by $s$ is defined by setting

$$s_* x = \rho^x_s x. \tag{8.2}$$

It is clear that

$$(8.3)\qquad |s * x| \leqq \inf[s, |x|].$$

Moreover

$$(8.4)\qquad |x-(s * x)| = (1-\rho_s^x)|x| \leqq [|x|-s]^+ \leqq ||x|-s|.$$

It is to be noted that $s * x$ is continuous on $E$ if $x$ and $s$ are continuous on $E$.

Theorem 8.1. *If $\alpha$ is a $\mathbf{C}$-measure on $E$, and if $x \in \mathbf{C}^E$ is $\alpha$-integrable, then corresponding to a prescribed $e>0$, and to an arbitrary $p \in {}_+\varphi_{|x|}$, there exists a $u \in \mathfrak{K}_{\mathbf{C}}(E)$ such that (8.0) holds with $|u| \leqq p$.*

Since $x$ is $\alpha$-integrable, $|x|$ is $|\alpha|$-integrable, by Lemma 4.1. By Lemma 6.1 there exists an $r \in {}_+\varphi_{|x|}$ with $r \leqq p$, such that if $N_1$ is the norm with respect to the measure $|\alpha|$,

$$(8.5)\qquad N_1[r-|x|] < e/4.$$

Since $r$ is $|\alpha|$-integrable (B 150 Prop. 5] it follows from Lemma 6.2 that there exists an $s \in {}_-\varphi_r$ such that $N_1[r-s] < e/4$, so that

$$(8.6)\qquad N_1[s-|x|] < e/2.$$

Since $x$ is $\alpha$-integrable there exists a $u_0 \in \mathfrak{K}_{\mathbf{C}}(E)$ such that

$$(8.7)\qquad N_{\mathbf{C}}^1[u_0-x, \alpha] < e/4 \rightleftarrows N_1[u_0-x] < e/4.$$

We shall show that the lemma is satisfied by $u = s * u_0$. Such a $u$ is in $\mathfrak{K}_{\mathbf{C}}(E)$ and

$$(8.8)\qquad |u| \leqq s \leqq r \leqq p, \qquad |u-u_0| \leqq |s-|u_0||$$

using (8.3) and (8.4). Hence

$$(8.9)\qquad |u-x| \leqq |u-u_0| + |u_0-x| \leqq |s-|u_0|| + |u_0-x|$$

$$(8.10)\qquad \leqq |s-|x|| + ||x|-|u_0|| + |u_0-x| \leqq |s-|x|| + 2|u_0-x|.$$

From (8.10), (8.7) and (8.6) one infers that

$$N_1[u-x] \leqq N_1[s-|x|] + 2N_1[u_0-x] < e.$$

Relation (8.0) thus holds, with $|u| \leqq p$ in accordance with (8.8).

## 9. The decomposition of C-measures.

In § 3 we have introduced the canonical form

$$(9.1)\qquad \alpha = \mu_e + i\nu_e$$

of a $\mathbf{C}$-measure $\alpha$. We term $\mu^+$, $\mu^-$, $\nu^+$, $\nu^-$ *the Riesz components* of $\alpha$. In § 4 we have introduced the canonical form

$$(9.2) \qquad x = x_1 + ix_2 = x_1^+ - x_1^- + i(x_2^+ - x_2^-)$$

of an arbitrary mapping $x \in \mathbf{C}^E$ and termed $x_1^+$, $x_1^-$, $x_2^+$, $x_2^-$ the Riesz components of $x$.

In proving the theorems of this section statement (a) will be usefull.

(a) If $\mu_1$ and $\mu_2$ are two positive measures such that $\mu_1 \leqq \mu_2$ then any $x \in \mathbf{R}^E$ which is $\mu_1$-integrable is $\mu_2$-integrable.

This follows with the aid of the relation $N_1[y, \mu_1] \leqq N_1[y, \mu_2]$, valid for each $y \in \mathbf{R}^E$.

We begin with Theorem 9.1.

Theorem 9.1. If $\alpha$ and $\beta$ are $\mathbf{C}$-measures on $E$, and if $x \in \mathbf{C}^E$ is $\alpha$-and $\beta$-integrable, then $x$ is $(\alpha+\beta)$-integrable and

$$(9.3) \qquad (\alpha + \beta)(x) = \alpha(x) + \beta(x) .$$

We begin by proving (i).

(i) A mapping $p \in \mathfrak{J}_+$ which is $\alpha$-and $\beta$-integrable is $(\alpha+\beta)$-integrable, and

$$(9.4) \qquad (\alpha + \beta)(p) = \alpha(p) + \beta(p) .$$

Since $p$ is $|\alpha|$-and $|\beta|$-integrable it follows from Theorem 6.1 that $p$ is $(|\alpha| + |\beta|)$-integrable. Set $|\alpha| + |\beta| = \mu$. By Prop. 5 B 150, $\mu^*(p) < \infty$. By Lemma 3.2 $|\alpha + \beta| \leqq |\alpha| + |\beta|$, implying $|\alpha + \beta|^*(p) \leqq \mu^*(p) < \infty$, so that $p$ is $|\alpha + \beta|$-integrable. It follows from Lemma 7.2 that the limits with respect to $u \in {}_-\varphi_p$ of $(\alpha+\beta)(u)$, $\alpha(u)$ and $\beta(u)$ are respectively $(\alpha + \beta)(p)$, $\alpha(p)$ and $\beta(p)$. Since $(\alpha+\beta)(u) = \alpha(u) + \beta(u)$, relation (9.4) holds.

Proof of theorem. Because of Lemma 4.5, it will be sufficient to prove the theorem for $x \in \mathbf{R}^E_+$. Suppose then that $x = h \in \mathbf{R}^E_+$.

Note that $h$ is $|\alpha|$-and $|\beta|$-integrable by Lemma 5.1, hence $(|\alpha| + |\beta|)$-integrable by Theorem 6.1. Since $|\alpha + \beta| \leqq |\alpha| + |\beta|$, $h$ is $|\alpha + \beta|$-integrable by (a), and hence $\alpha + \beta$-integrable by Lemma 5.1. It follows from Lemma 7.1 that the limits with respect to $p \in {}_+\varphi_h$ of $(\alpha+\beta)(p)$, $\alpha(p)$ and $\beta(p)$ are respectively $(\alpha + \beta)(h)$, $\alpha(h)$ and $\beta(h)$. For some section $S(p_0)$ of ${}_+\varphi_h$, and for all $p \in S(p_0)$, $p$ will be $\alpha, \beta$, and $(\alpha + \beta)$-integrable, so that (9.4) will hold. Passing to the limit with respect to $p \in {}_+\varphi_h$ we conclude that

$$(\alpha + \beta)(h) = \alpha(h) + \beta(h) .$$

The theorem is thus established when $x \geqq 0$ and follows in the general case.

Corollary 9.1. Under the hypotheses of the theorem $x$ is $(c_1\alpha+c_2\beta)$-integrable and

$$(c_1\alpha + c_2\beta)(x) = c_1\alpha(x) + c_2\beta(x) \tag{9.5}$$

for arbitrary $c_1$ and $c_2$ in $\mathbf{C}$.

Theorem 9.2. Let a $\mathbf{C}$-measure $\alpha$ be written in the canonical form $\mu_e+i\nu_e$. A necessary and sufficient condition that an $x\in\mathbf{C}^E$ be $\alpha$-integrable is that $x$ be $\mu_e$-and $\nu_e$-integrable. If $x$ is $\alpha$-integrable

$$\alpha(x) = \mu_e(x) + i\nu_e(x)\,. \tag{9.6}$$

If $x$ is $\mu_e$-and $\nu_e$-integrable, it is $(\mu_e+i\nu_e)$-integrable, and (9.6) holds, by virtue of Corollary 9.1.

To show that an $\alpha$-integrable $x$ is $\mu_e$- and $\nu_e$-integrable it is sufficient, in accordance with Lemma 4.5, to suppose that $x=h\in\mathbf{R}_+^E$. Now $h$ is $|\alpha|$-integrable. Since $|\alpha|\geqq|\mu|$ and $|\alpha|\geqq|\nu|$ (Lemma 3.3), it follows from (a) that $h$ is $|\mu|$-and $|\nu|$-integrable, or equivalently $|\mu_e|$- and $|\nu_e|$-integrable. Hence $h$ is $\mu_e$- and $\nu_e$-integrable by Lemma 5.1.

If $x$ is $\alpha$-integrable, it is thus $\mu_e$- and $\nu_e$-integrable, so that (9.6) holds by Corollary 9.1.

Theorem 9.3. If $\mu$ is a measure, a necessary and sufficient condition that an $x\in\mathbf{C}^E$ be $\mu_e$-integrable is that the Riesz components of $x$ be $\mu^+$-and $\mu^-$-integrable. If $x$ is $\mu_e$-integrable

$$\mu_e(x) = (\mu^+)_e(x) - (\mu^-)_e(x)\,. \tag{9.7}$$

By virtue of Lemma 4.5 it is sufficient to establish the theorem for $x=h\in\mathbf{R}_+^E$. If $h$ is $\mu_e$-integrable it is $|\mu_e|=|\mu|$-integrable by Lemma 5.1. Since $\mu^+\leqq|\mu|$ and $\mu^-\leqq|\mu|$, $h$ is $\mu^+$-and $\mu^-$-integrable by (a). Conversely a $h\in\mathbf{R}_+^E$ which is $\mu^+$-and $\mu^-$-integrable is $\mu^++\mu^-=|\mu|$-integrable by Theorem 6.1, and hence $\mu_e$-integrable by Lemma 5.1.

To establish (9.7) for a $\mu_e$-integrable $x$, set $x=x_1+ix_2$, with $x_i\in\mathbf{R}^E$, $i=1, 2$. Then $x_i$ is $\mu_e$-integrable by Lemma 4.3. By virtue of the definition of $\mu_e$

$$\mu_e(x_i) = \mu(x_i) = \mu^+(x_i) - \mu^-(x_i) = (\mu^+)_e(x_i) - (\mu^-)_e(x_i) \qquad (i=1, 2)$$

and (9.7) follows.

Theorems 9.2 and 9.3 combine as follows.

**Corollary** 9.2. *If $\alpha$ is a* $\mathbf{C}$*-measure, a necessary and sufficient condition that an $x \in \mathbf{C}^E$ be $\alpha$-integrable is that each of the four Riesz components of $x$ be integrable with respect to each of the four Riesz components of $\alpha$.*

$\alpha$-Measurability. An $x \in \mathbf{C}^E$ will be said to be $\alpha$-measurable if $|\alpha|$-measurable in the sense defined in B 180. With the aid of Cor. 2 of B 185 it follows that $x$ is $\alpha$-measurable if and only if its Riesz components are $|\alpha|$-measurable. But by Lemma 4.5 $x$ is $\alpha$-integrable if and only if its Riesz components are $|\alpha|$-integrable. Theorem 5 of B 194 accordingly yields the following

**Theorem** 9.4. *An $x \in \mathbf{C}^E$ is $\alpha$-integrable with respect to a* $\mathbf{C}$*-measure $\alpha$ if and only if $x$ is $\alpha$-measurable, and if $N^1_{\mathbf{C}}[x, \alpha] < \infty$.*

A subset $H$ of $E$ is termed $\alpha$-measurable if the characteristic function of $H$ is $\alpha$-measurable.

## Part II. Vector spaces $A$ with duals $A'$ of integral type

### 10. Conditions I and II on $A$.

We shall be concerned with a vector subspace $A$ of $\mathbf{C}^E$ provided with a semi-norm $\mathfrak{N}^A$. The value of $\mathfrak{N}^A$ at $x \in A$ is denoted by $\mathfrak{N}^A(x)$. The subspace $A$ is a $\mathbf{C}$-module (B VI 5).

Monotone semi-norms. Given $x$ and $y \in A$ we term $\mathfrak{N}^A$ monotone if the condition $|x| \leq |y|$ implies $\mathfrak{N}^A(x) \leq \mathfrak{N}^A(y)$. When $\mathfrak{N}^A$ is monotone the relation $|x| = |y|$ implies $\mathfrak{N}^A(x) = \mathfrak{N}^A(y)$. The latter relation does not however imply $|x| = |y|$ in general. A semi-norm is termed trivial if null for each element in $A$.

$MT$-Spaces. We shall admit vector spaces $A$ which satisfy Conditions I and II, terming such spaces $MT$-spaces.

**Condition** I. *$A$ shall be a vector subspace of $\mathbf{C}^E$ with a non-trivial, monotone, semi-norm $\mathfrak{N}^A$, and shall contain $\mathfrak{K}_{\mathbf{C}}$ as an everywhere dense subspace. When $x \in A$, $|x|$ and the conjugate $\bar{x}$ of $x$ shall be in $A$.*

**Lemma** 10.1. *If a vector space $A$ satisfies Condition I, then for each compact subset $K$ of $E$ there exists a cons-*

tant $M_K \geqq 0$ such that for each $u \in \mathfrak{K}_{\mathbf{C}}$ whose support is in $\mathfrak{K}$

$$\mathcal{N}^A(u) \leqq M_K\, U(u)\,. \tag{10.0}$$

Let $f \in \mathfrak{K}$ map $E$ into $[0\,,\,1]$ with $f(t) = 1$ for $t \in K$. It is clear that (10.0) holds with $M_K = 0$ in case $u = 0$. In any other case set $U(u) = \rho^{-1}$. Then $|\rho u| \leqq f$, and the monotonicity of $\mathcal{N}^A$ implies that $\mathcal{N}^A(\rho u) \leqq \mathcal{N}^A(f)$. Thus (10.0) holds with $M_K = \mathcal{N}^A(f)$.

The form $\widehat{\eta}$. If $\eta$ is an element in the dual $A'$ of $A$ let $\widehat{\eta}$ denote the restriction $\eta\,|\mathfrak{K}_{\mathbf{C}}$ of $\eta$ to $\mathfrak{K}_{\mathbf{C}}$. Lemma 10.1 has the following corollary.

**Corollary** 10.1. *If $\eta$ is an element in the dual $A'$ of a vector space $A$ satisfying Condition I, then $\widehat{\eta} = \eta\,|\mathfrak{K}_{\mathbf{C}}$ is a $\mathbf{C}$-measure on $E$.*

If $\eta$ is in $A'$ set

$$\|\eta\|_{A'} = \sup_{x \in A} \frac{|\eta(x)|}{\mathcal{N}^A(x)} \qquad [\text{for } \mathcal{N}^A(x) \neq 0]\,. \tag{10.1}$$

Then

$$|\eta(x)| \leqq \|\eta\|_{A'}\, \mathcal{N}^A(x) \qquad [x \in A] \tag{10.2}$$

and $\|\eta\|_{A'}$ gives the values of a norm on $A'$. It follows from (10.2) and (10.0) that for each $u \in \mathfrak{K}_{\mathbf{C}}$ whose support is contained in $K$

$$|\widehat{\eta}(u)| \leqq \|\eta\|_{A'}\, M_K\, U(u) \tag{10.3}$$

so that $\widehat{\eta}$ is a $\mathbf{C}$-measure on $E$.

**Condition** II. *If $A$ satisfies Condition I and if $\eta$ is in $A'$, Condition II requires that each $x \in A$ shall be $\widehat{\eta}$-integrable and that*

$$\eta(x) = \int x\, d\widehat{\eta} \qquad [x \in A]\,. \tag{10.4}$$

It is because of Condition II that we refer to $MT$-spaces $A$ as having duals $A'$ *of integral type*.

We shall introduce an isomorph $\mathcal{A}'$ of $A'$.

*The measure dual* $\mathcal{A}'$ *of* $A$. Let $\mathfrak{M}_{\mathbf{C}}$ denote the vector space of all $\mathbf{C}$-measures on $E$. The preceding mapping $\eta \to \widehat{\eta}$ of $A'$ into $\mathfrak{M}_C$ is clearly an algebraic homomorphism. It is an isomorphism since the antecedent of a null $\widehat{\eta}$ under the mapping $\eta \to \widehat{\eta}$ is a null $\eta$ by (10.4). Let $\mathcal{A}'$ be the isomorphic image of $A'$ under this mapping. We shall term $\mathcal{A}'$ the *measure dual of* $A$, distinguishing $\mathcal{A}'$ from $A'$ the *dual of* $A$.

If $\alpha \in \mathcal{A}'$, each $x \in A$ is $\alpha$-integrable by Condition II. For each $\alpha \in \mathcal{A}'$

and for a range of $x \in A$ and $u \in \mathfrak{R}_{\mathbf{C}}$ such that $\mathcal{N}^A(x) \neq 0$ and $\mathcal{N}^A(u) \neq 0$ set

$$\|\alpha\|_{\mathcal{M}'} = \sup_{x \in A} \frac{|\int x\, d\alpha|}{\mathcal{N}^A(x)} = \sup_{u \in \mathfrak{R}_{\mathbf{C}}} \frac{|\int u\, d\alpha|}{\mathcal{N}^A(u)} . \tag{10.5}$$

The equality of the two suprema in (10.5) is a consequence of the fact that $\mathfrak{R}_{\mathbf{C}}$ is everywhere dense in $A$, and $\int x\, d\alpha$ and $\mathcal{N}^A(x)$ vary continuously with $x \in A$.

By definition of $\mathcal{M}'$ a $\mathbf{C}$-measure $\alpha \in \mathcal{M}'$ equals $\widehat{\eta}$ for a suitable $\eta \in A'$. From (10.1), (10.4) and (10.5) we infer that

$$\|\alpha\|_{\mathcal{M}'} = \|\eta\|_{A'} \qquad [\text{for } \eta \in A' \text{ and } \widehat{\eta} = \alpha] . \tag{10.6}$$

Recall that $\|\eta\|_{A'}$ defines a norm in $A'$ and $\|\alpha\|_{\mathcal{M}'}$ a norm in $\mathcal{M}'$.

*The space* $\Omega^A$. If $A$ is an $MT$-space

$$A \subset \bigcap_{\alpha \in \mathcal{M}'} \mathfrak{L}^1_{\mathbf{C}}(\alpha) = \Omega^A \tag{10.7}$$

introducing $\Omega^A$. We shall see that this inclusion becomes an equality for some but not all spaces $A$.

The following theorem characterizes a $\mathbf{C}$-measure in the measure dual $\mathcal{M}'$ of $A$. It has an important generalization for bimeasures.

**Theorem** 10.1. (i) *If $\alpha$ is a $\mathbf{C}$-linear form on $\mathfrak{R}_{\mathbf{C}}$ such that*

$$\sup_{u \in \mathfrak{R}_{\mathbf{C}}} |\alpha(u)| = G < \infty \qquad [\text{for } \mathcal{N}^A[u] \leqq 1] \tag{10.8}$$

*then $\alpha$ is a $\mathbf{C}$-measure in $\mathcal{M}'$ and $G = \|\alpha\|_{\mathcal{M}'}$.*

(ii) *Conversely each $\mathbf{C}$-measure $\alpha \in \mathcal{M}'$ is a $\mathbf{C}$-linear form on $\mathfrak{R}_{\mathbf{C}}$ such that (10.8) holds with $G = \|\alpha\|_{\mathcal{M}'}$.*

**Proof** of (i). Recall that $\mathfrak{R}_{\mathbf{C}}$ is everywhere dense in $A$. When (10.8) holds $\alpha$ maps $\mathfrak{R}_{\mathbf{C}}$ uniformly continuously into the complete Hausdorff space $\mathbf{C}$. We infer from Theorem 1 of B II 151 that the form $\alpha$ can be continuously extended over $A$ so as to define a $\mathbf{C}$-linear form $\eta \in A'$. Since $\widehat{\eta} = \alpha$ it follows from the definition of $\mathcal{M}'$ that $\alpha \in \mathcal{M}'$. From (10.5) we conclude that $G = \|\alpha\|_{\mathcal{M}'}$.

**Proof** of (ii). For a $\mathbf{C}$-measure $\alpha \in \mathcal{M}'$, $\alpha = \widehat{\eta}$ for some $\eta \in A'$. Then $\|\alpha\|_{\mathcal{M}'}$, as defined in (10.5), is finite, since (10.6) holds and $\|\eta\|_{A'}$ is finite by hypothesis. On setting $G = \|\alpha\|_{\mathcal{M}'}$, (10.8) holds in accordance with (10.5).

## 11. The extension of $\mathfrak{N}^A$ over $\mathbf{C}^E$.

Our extension of the domain of definition of $\mathfrak{N}^A$ from $A$ to all of $\mathbf{C}^E$ is made possible by a representation (11.2) of $\mathfrak{N}^A$ in terms of the measure dual $\mathcal{A}'$ of $A$. We begin with a known formula (11.0). We shall refer to *subunit* $\mathbf{C}$*-measures* in $\mathcal{A}'$, that is to $\mathbf{C}$-measures $\alpha \in \mathcal{A}'$ such that $\|\alpha\|_{\mathcal{A}'} \leqq 1$.

**Lemma** 11.1. *For fixed* $x \in A$

$$\mathfrak{N}^A(x) = \sup_{\alpha \in \mathcal{A}'_u} \left| \int x \, d\alpha \right| \tag{11.0}$$

*where* $\mathcal{A}'_u$ *is the set of subunit* $\mathbf{C}$*-measures in* $\mathcal{A}'$.

Let $m$ be the sup in (11.0). Form (10.5), it appears that for $x \in A$

$$\left| \int x \, d\alpha \right| \leqq \|\alpha\|_{\mathcal{A}'} \mathfrak{N}^A(x) \qquad [\alpha \in \mathcal{A}'] .$$

That $\mathfrak{N}^A(x) \geq m$ follows. According to an extension of the Hahn-Banach Theorem (B XV 111 Cor. 3) there exists a form $\eta \in A'$ such that for the given fixed $x \in A$, $\eta(x) = \mathfrak{N}^A(x)$ and $\|\eta\|_{A'} = 1$. By (10.6) $\|\widehat{\eta}\|_{\mathcal{A}'} = 1$. For this $\eta$, by Condition II,

$$\int x \, d\widehat{\eta} = \eta(x) = \mathfrak{N}^A(x) . \tag{11.1}$$

This $\eta$ gives a subunit $\widehat{\eta}$, admissible as an $\alpha$ in (11.0). We conclude that $\mathfrak{N}^A(x) = m$.

We shall represent $\mathfrak{N}^A(x)$ as follows.

**Theorem** 11.1. *For each* $x \in A$

$$\mathfrak{N}^A(x) = \sup_{\alpha \in \mathcal{A}'_u} \int |x| \, d|\alpha| \tag{11.2}$$

*where* $\mathcal{A}'_u$ *is the set of subunit* $\alpha \in \mathcal{A}'$.

To establish this theorem we shall need the following lemma.

**Lemma** 11.2. *If* $\alpha \in \mathcal{A}'$, *then* $|\alpha|_e \in \mathcal{A}'$, *and*

$$\|\alpha\|_{\mathcal{A}'} = \| |\alpha|_e \|_{\mathcal{A}'} . \tag{11.3}$$

The lemma will follow from Theorem 10.1 provided one can show that for $\mathfrak{N}^A(u) \leqq 1$

$$\sup_{u \in \mathfrak{K}_\mathbf{C}} \left| \int u \, d|\alpha|_e \right| = \|\alpha\|_{\mathcal{A}'} . \tag{11.4}$$

By Lemma 3.1, $|\alpha| = | |\alpha|_e |$. From (4.4) and (3.3), respectively, for $u$ and $v \in \mathfrak{K}_\mathbf{C}$

$$\left| \int u \, d|\alpha|_e \right| \leqq \int |u| \, d|\alpha| = \sup_{|v| \leqq |u|} \left| \int v \, d\alpha \right| \leqq \sup_{|v| \leqq |u|} \|\alpha\|_{\mathcal{A}'} \mathfrak{N}^A(v) = \|\alpha\|_{\mathcal{A}'} \mathfrak{N}^A(u)$$

using (10.5) and the monotonicity of $\mathfrak{N}^A$. If then $m$ denotes the sup in (11.4), $m \leqq \|\alpha\|_{\mathfrak{A}'}$. To exclude the inequality we use (4.4) to infer that for $u \in \mathfrak{K}_{\mathbf{C}}$ $|\int u\, d\alpha| \leqq \int |u|\, d\,|\alpha| \leqq |\int |u|\, d\,|\alpha|_e| \leqq m\, \mathfrak{N}^A(|u|) = m\, \mathfrak{N}^A(u)$, and conclude from (10.5) that $\|\alpha\|_{\mathfrak{A}'} \leqq m$. Hence (11.4) holds and the lemma follows.

Proof of Theorem 11.1. Let $s$ denote the right member of (11.2). We first show that $\mathfrak{N}^A(x)$, as given by (11.0), is at most $s$. If $\alpha$ is a $\mathbf{C}$-measure and $x$ is in $A$,

$$|\int x\, d\alpha| \leqq \int |x|\, d\,|\alpha| \qquad \text{[by (4.4)]}\,. \tag{11.5}$$

Taking the sup as in (11.0), it follows from (11.5) that $\mathfrak{N}^A(x) \leqq s$.

To show that $\mathfrak{N}^A(x) \geqq s$ recall that for $\alpha \in \mathfrak{A}'_u$, $|\alpha|_e$ is in $\mathfrak{A}'_u$ (Lemma 11.2). For $x \in A$

$$\int |x|\, d\,|\alpha| = \int |x|\, d\,|\alpha|_e \leqq \mathfrak{N}^A(|x|) = \mathfrak{N}^A(x)$$

where the middle inequality is implied by (11.0). On taking the sup of $\int |x|\, d\,|\alpha|$ as in (11.2) we conclude that $s \leqq \mathfrak{N}^A(x)$. Hence $s = \mathfrak{N}^A(x)$ and the theorem follows.

The extension of $\mathfrak{N}^A$ defined. The representation (11.2) of $\mathfrak{N}^A(x)$ is a priori valid for $x \in A$. It can be written in the equivalent form

$$\mathfrak{N}^A(x) = \sup_{\alpha \in \mathfrak{A}'_u} \int^* |x|\, d\,|\alpha| \tag{11.6}$$

using the superior integral defined in B 109. The right member of (11.6) is however defined over all of $\mathbf{C}^E$, and will serve to define $\mathfrak{N}^A(x)$ for arbitrary $x \in \mathbf{C}^E$. The value of $\mathfrak{N}^A(x)$ when $x$ is not in $A$ may be $+\infty$. However it is not necessarily $+\infty$ when $x$ is not in $A$ as will be seen in § 14 when $A = \mathfrak{L}^1_{\mathbf{C}}(\beta)$.

For $x$ and $y \in \mathbf{C}^E$, and $\lambda \in \mathbf{C}$,

$$\mathfrak{N}^A(\lambda x) = |\lambda|\, \mathfrak{N}^A(x) \qquad [\text{for } \lambda \neq 0]$$
$$\mathfrak{N}^A(x+y) \leqq \mathfrak{N}^A(x) + \mathfrak{N}^A(y)\,.$$

These relations are a consequence of the positive homogeneity and convexity of the superior integral. We note that the extension of $\mathfrak{N}^A$ has the monotone character over $\mathbf{C}^E$ which it possessed originally over $A$.

**12. The countable convexity of $\mathfrak{N}^A$ and the completeness of $\mathfrak{F}^A$.**

*MT*-spaces $A$ possess some of the main characteristics of the spaces $\mathfrak{L}^p_{\mathbf{C}}(\alpha)$ as we shall see in this section. In particular the space $\mathfrak{F}^A$ generalizing $\mathfrak{F}^p_{\mathbf{C}}(\alpha)$ is complete.

N e g l i g i b i l i t y. A mapping $x \in \mathbf{C}^E$ will be termed $A'$-n e g l i g i b l e if $\mathfrak{N}^A(x) = 0$. Thus $x$ is $A'$-negligible if and only if $|x|$ is $A'$-negligible. A subset $H$ of $E$ will be termed $A'$-negligible if the characteristic function of $H$ is $A'$-negligible. It follows from (11.6) that a necessary and sufficient condition that a function $x$ (set $H$) be $A'$-negligible is that it be $|\alpha|$-negligible or equivalently $\alpha$-negligible, for each $\mathbf{C}$-measure in $\mathcal{A}'$. An $x \in A$ is thus $A'$-negligible if and only if (in the notation of Bourbaki) $N_1[|x|, |\alpha|] = 0$ for each $\alpha \in \mathcal{A}'$.

Two mappings $x$ and $y \in \mathbf{C}^E$ such that $\mathfrak{N}^A(x-y) = 0$ will be said to be $A'$-e q u i v a l e n t. If $x$ and $y$ are $A'$-equivalent $\mathfrak{N}^A(x) = \mathfrak{N}^A(y)$. A property of points in $E$ which is valid or invalid for each point $t \in E$ will be said to hold (p , p , $A'$) if it is valid for each point of $E$, excepting at most points $t$ in an $A'$-negligible subset of $E$.

$A'$-m e a s u r a b i l i t y. An $x \in \mathbf{C}^E$ will be said to be $A'$-measurable if $\alpha$-measurable for each $\mathbf{C}$-measure $\alpha \in \mathcal{A}'$.

From Theorem 1 of B 119 and the above criteria for $A'$-negligibility one obtains the following.

(i) A m a p p i n g $x \in \mathbf{C}^E$ i s $A'$-n e g l i g i b l e i f a n d o n l y i f $x(t) = 0$, (p , p , $A'$) .

That $\mathfrak{N}^A$ is c o u n t a b l y c o n v e x is affirmed in (12.2) of the following theorem.

**Theorem** 12.1. G i v e n $f_r \in \mathbf{C}^E$, $r = 1, 2, \ldots$, s e t $g_n = f_1 + f_2 + \ldots + f_n$. I f $\liminf |g_n|$ e q u a l s a f u n c t i o n $g \in \mathbf{R}^E$ (p , p , $A'$) t h e n

$$\mathfrak{N}^A(g) \leq \liminf_{n \uparrow \infty} \mathfrak{N}^A(g_n) \tag{12.1}$$

$$\mathfrak{N}^A(g) \leq \mathfrak{N}^A(f_1) + \mathfrak{N}^A(f_2) + \ldots . \tag{12.2}$$

Since $\liminf\limits_{n \uparrow \infty} |g_n| = g$ (p , p , $|\alpha|$) for each $\mathbf{C}$-measure $\alpha \in \mathcal{A}'$, it follows from the lemma of Fatou (Prop. 14 B 112) that

$$\int^* g \, d|\alpha| \leq \liminf_{n \uparrow \infty} \int^* |g_n| \, d|\alpha| . \tag{12.3}$$

Let $e > 0$ be given. By virtue of the definition (11.6) of $\mathfrak{N}^A(g)$ there exists an $\alpha \in \mathcal{A}'_u$ such that

$$\mathcal{M}^A(g) \leqq \int^* g d|\alpha| + e. \tag{12.4}$$

Hence

$$\mathcal{M}^A(g) \leqq \liminf_{n\uparrow\infty} \int^* |g_n| d|\alpha| + e \leqq \liminf_{n\uparrow\infty} \mathcal{M}^A(g_n) + e \tag{12.5}$$

applying (11.6). Since $e > 0$ is arbitrary, (12.1) holds. Finally (12.1) implies (12.2), since

$$\mathcal{M}^A(g_n) \leqq \mathcal{M}^A(f_1) + \mathcal{M}^A(f_2) + \dots + \mathcal{M}^A(f_n).$$

**Corollary** (12.1). *Relations (12.1) and (12.2) hold in particular if $|g_n|$ converges* (p , p , $A'$) *to* $g$.

*The space* $\mathfrak{F}^A$. Let $\mathfrak{F}^A$ be the normed subspace of $\mathbf{C}^E$ of mappings $f$ such that $\mathcal{M}^A(f) < \infty$. We suppose $\mathfrak{F}^A$ assigned a uniform topology defined by $\mathcal{M}^A$.

*Convergence in* $\mathfrak{F}^A$. *Let* $f_1 + f_2 + \dots$ be a series with $f_n \in \mathfrak{F}^A$ and set $s_n = f_1 + f_2 + \dots + f_n$. We say that the series converges in $\mathfrak{F}^A$ to a mapping $s \in \mathbf{C}^E$ if $\mathcal{M}^A(s_n - s) \to 0$ as $n \uparrow \infty$. A limit $s$ is necessarily in $\mathfrak{F}^A$.

**Theorem** 12.2. *Let $f_1 + f_2 + \dots$ be a series of mappings $f_n \in \mathfrak{F}^A$ such that*

$$\mathcal{M}^A(f_1) + \mathcal{M}^A(f_2) + \dots < \infty. \tag{12.6}$$

*Then $|f_1| + |f_2| + \dots$ converges* (p , p , $A'$). *Further any finite function $s$ to which $f_1 + f_2 + \dots$ converges* (p , p , $A'$) *is in $\mathfrak{F}^A$ and such that for each $n$*

$$\mathcal{M}^A(s - s_n) \leqq \mathcal{M}^A(f_{n+1}) + \mathcal{M}^A(f_{n+2}) + \dots. \tag{12.7}$$

*Thus $s_n$ converges to $s$ in the topology of $\mathfrak{F}^A$.*

It follows from the definition of the semi-norm $N_1$ in B 127, and of $\mathcal{M}^A$ in (11.6), that for each $\alpha \in \mathcal{A}'_u$ and $f \in \mathbf{C}^E$

$$N_1(|f|, |\alpha|) \leqq \mathcal{M}^A(f).$$

Under the hypothesis (12.6)

$$\sum_{n=1}^{\infty} N_1(|f_n|, |\alpha|) < \infty \qquad [\alpha \in \mathcal{A}'_u].$$

Set $k(t) = |f_1(t)| + |f_2(t)| + \dots$. According to Prop. 6 (B 130), $k(t)$ is finite (p , p , $|\alpha|$) for each $\alpha \in \mathcal{A}'_u$, and hence for each $\alpha \in \mathcal{A}'$. Thus $k(t)$

is finite (p , p , $A'$). Set $h(t)=k(t)$ when $k(t)<\infty$, and set $h(t)=0$ at other points $t\in E$. Since $|f_1|+|f_2|+\dots$ converges (p , p , $A'$) to $h$, $\mathfrak{N}^A(h)<\infty$ by Cor. 12.1. Thus $h$ is in $\mathfrak{F}^A$.

Hence $f_1+f_2+\dots$ converges (p , p , $A'$) to a mapping $s\in \mathbf{C}^E$. The series $f_{n+1}+f_{n+2}+\dots$ converges (p , p , $A'$) to $s-s_n$. One has

$$|s(t)-s_n(t)|\leqq |f_{n+1}(t)|+|f_{n+2}(t)|+\dots \qquad [\text{p , p , } A'] \,. \tag{12.8}$$

The right member of (12.8) converges (p , p , $A'$) to a value $S_n(t)$. Moreover $S_n(t)$ can be taken finite and such that $|s(t)-s_n(t)|\leqq S_n(t)$. Hence

$$\mathfrak{N}^A(s-s_n)\leqq \mathfrak{N}^A(S_n),$$

and (12.7) follows on applying (12.2) with $S_n$ in place of $g$, and the right member of (12.8) in place of $f_1+f_2+\dots$ .

Corollary 12.2. *The space $\mathfrak{F}^A$ is complete.*
This corollary is an obvious consequence of Theorem 12.2. For details see the analogous proof of Prop. 5 (B 130).

Corollary 12.3. *If $A$ is closed in $\mathfrak{F}^A$, $A$ is complete.*

*Example.* That not all $MT$-spaces are complete may be seen as follows. It will be shown in § 14 that for a fixed non-null $\mathbf{C}$-measure $\alpha$ the space $A=\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ is an $MT$-space, and that the subspace $\mathfrak{K}_{\mathbf{C}}$ of $A$ with a semi-norm induced by that of $A$ is also an $MT$-space. The space $A$ is complete while the subspace $\mathfrak{K}_{\mathbf{C}}$ is not in general closed in $A$, and hence not complete.

Theorem 3 of B 133 has the following generalization.

Theorem 12.3. *Let $(g_k)$, $k=1, 2, \dots$, be a Cauchy sequence in an $MT$-space $A$. There then exists a subsequence $(g_{k_n})$ of $(g_k)$ defining differences $f_n=g_{k_{n+1}}-g_{k_n}$, $n=1, 2, \dots$, such that*

$$\sum_{n=1}^{\infty}\mathfrak{N}^A(f_n)<\infty\,, \tag{a}$$

(b) *the series $|f_1|+|f_2|+\dots$ converges* (p , p , $A'$),
(c) *there exists a function $s\in\mathfrak{F}^A$, equal* (p , p , $A'$), *to the limit of the sequence $(g_{k_n})$,*
(d) *if $A$ is complete, $s$ is $A'$-equivalent to a mapping $s_0\in A$, and the sequence $(g_k)$ converges to $s_0$ in the topology of $A$.*

That the differences $f_n$ can be chosen so that (a) is true, follows readily from the hypothesis that $(g_k)$ is a Cauchy sequence. Statements (b) and (c) now follow from Theorem 12.2.

To establish (d) set $\mathfrak{F}^A = G$. Let $R$ be the relation between pairs of elements in $G$ defined by $A'$-equivalence. Introduce the space $\tilde{G} = G \mid R$ of equivalence classes of $G$ with a "separated structure associated" with the uniform structure of $G$ as defined by the semi-norm $\mathfrak{N}^A$ (B II 127). The space $\tilde{G}$ is complete (B II 149) since $G$ is complete (Cor. 12.2). Let $\tilde{x}$ denote the equivalence class of $x \in G$. Since $(g_{k_n})$ converges (p , p , $A'$) in $\mathfrak{F}^A$ to $s$, the sequence $(\tilde{g}_{k_n})$ converges in $\tilde{G}$ to $\tilde{s}$. Since $\tilde{G}$ is "separated" and complete the Cauchy sequence $(\tilde{g}_k)$ converges in $\tilde{G}$ to $\tilde{s}$ and only to $\tilde{s}$. All this is true whether $A$ is complete or not.

Let $\tilde{A}$ be the image of $A$ under the canonical mapping $x \to \tilde{x}$ of $G$ onto $\tilde{G}$. If now $A$ is complete, $\tilde{A}$ is complete (B II 149). Hence $\tilde{A}$ is closed in the separated space $\tilde{G}$ (B 149 Prop. 6). Thus $\tilde{s}$ is in $\tilde{A}$, and (d) follows.

## 13. Compatible MT-spaces.

Observe first that

$$A \subset [\Omega^A \cap \mathfrak{F}^A] \tag{13.1}$$

and that $\Omega^A \cap \mathfrak{F}^A$ is the largest vector subspace of $\mathfrak{F}^A$ composed of mappings $f \in \mathbf{C}^E$ which are integrable with respect to each $\alpha \in \mathcal{M}'$. Moreover for each $\alpha \in \mathcal{M}'$ and $f \in \Omega^A \cap \mathfrak{F}^A$, in accordance with (11.6)

$$| \int f d\alpha | \leqq \int |f| \, d\, |\alpha| \leqq \|\alpha\|_{\mathcal{M}'} \mathfrak{N}^A (f) \,. \tag{13.2}$$

The definition (11.6) implies more broadly that

$$N^1_{\mathbf{C}} [f \,, \alpha] \leqq \|\alpha\|_{\mathcal{M}'} \mathfrak{N}^A (f) \qquad [f \in \mathfrak{F}^A] \,. \tag{13.3}$$

From (13.3) we infer that $N^1_{\mathbf{C}} (f \,, \alpha) < \infty$ on $\mathfrak{F}^A$, so that a secondary topology $T_\alpha$ is defined by $N^1_{\mathbf{C}} [\,.\,, \alpha]$ on $\mathfrak{F}^A$. From (13.3) we can then infer that the primary topology on $\mathfrak{F}^A$, namely the topology $T_A$ induced by $\mathfrak{N}^A$, is at least as fine as $T_\alpha$. With the aid of this fact we can establish the following lemma.

**Lemma** 13.1. *The closure $\bar{A}$, in the topology $T_A$ of $A$ in $\mathfrak{F}^A$, is in $\Omega^A \cap \mathfrak{F}^A$.*

The closure $A_\alpha$, in the topology $T_\alpha$, of $A$ in $\mathfrak{F}^A$ is in $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$. For $A \subset \mathfrak{L}^1_{\mathbf{C}}(\alpha)$, and $\mathfrak{L}^1_{\mathbf{C}}(\alpha)$ is closed in $\mathfrak{F}^1_{\mathbf{C}}(\alpha) \supset \mathfrak{F}^A$ in the topology $T_\alpha$. Since the topology $T_A$ is as fine as $T_\alpha$ on $\mathfrak{F}^A$, $\bar{A} \subset A_\alpha$. Hence $\bar{A} \subset \mathfrak{L}^1_{\mathbf{C}}(\alpha)$ so that $\bar{A} \subset \Omega^A \cap \mathfrak{F}^A$.

We introduce a definition.

**Definition.** Two $MT$-spaces $A$ and $B$ will be said to be compatible if

$$(13.4) \qquad \mathcal{A}' = \mathcal{B}', \quad \mathfrak{N}^A(f) = \mathfrak{N}^B(f) \qquad [\text{for } f \in \mathfrak{K}_{\mathbf{C}}].$$

If $A$ and $B$ are compatible $MT$-spaces it follows from the definitions that

$$(13.5) \qquad \Omega^A = \Omega^B, \ \mathfrak{F}^A = \mathfrak{F}^B, \ \mathfrak{N}^A(x) = \mathfrak{N}^B(x) \qquad [\text{for } x \in \mathbf{C}^E].$$

The $MT$-spaces compatible with a given $MT$-space form an equivalence class. Concerning this equivalence class we have the following theorem.

**Theorem** 13.1. *If $A$ is an $MT$-space, the closure $\bar{A}$ of $A$ in $\mathfrak{F}^A$ is an $MT$-space compatible with $A$, and any $MT$-space compatible with $A$ is a vector subspace of $\bar{A}$. The smallest $MT$-space compatible with $A$ is $\mathfrak{K}_{\mathbf{C}}$.*

It is understood that $\bar{A}$ and $\mathfrak{K}_{\mathbf{C}}$ are here normed by the restrictions of $\mathfrak{N}^A$ to $\bar{A}$ and $\mathfrak{K}_{\mathbf{C}}$ respectively.

That $\mathfrak{K}_{\mathbf{C}}$ with a topology induced by $\mathfrak{N}^A$ is an $MT$-space compatible with $A$ follows with the aid of Theorem 10.1. That $\mathfrak{K}_{\mathbf{C}}$ is the smallest such space follows from the fact that $\mathfrak{K}_{\mathbf{C}}$ is a subspace of each $MT$-space $A$.

We continue with a proof of (a), (b), (c), (d).

(a). *$\bar{A}$ satisfies Conditions* I. Under (1) to (5) we consider the sub-conditions on $\bar{A}$ which must be satisfied. (1) That $\bar{A}$ is non-trivial and contains $\mathfrak{K}_{\mathbf{C}}$ follows from the fact that $\bar{A} \supset A$. (2) Moreover $\mathfrak{K}_{\mathbf{C}}$ is everywhere dense in $A$ and hence in $\bar{A}$. (3) If $f$ is in $\bar{A}$ we must show that $|f|$ is in $\bar{A}$. If $f$ is in $\bar{A}$ there exists a sequence $(f_n)$ of elements in $A$ such that $\mathfrak{N}^A(f_n - f) \to 0$ as $n \uparrow \infty$. Recall that

$$(13.6) \qquad \big| |f_n| - |f| \big| \leq |f_n - f|.$$

Now $|f_n|$ and $|f|$ are in $\mathfrak{F}^A$ with $f_n$ and $f$, since $\mathfrak{N}^A(f_n) = \mathfrak{N}^A(|f_n|)$ and $\mathfrak{N}^A(f) = \mathfrak{N}^A(|f|)$. Hence $|f_n| - |f|$ is in $\mathfrak{F}^A$. It follows from (13.6) that

$$(13.7) \qquad \mathfrak{N}^A(|f_n| - |f|) \leq \mathfrak{N}^A(f_n - f) \to 0 \qquad [\text{as } n \uparrow \infty].$$

Since $|f_n|$ is in $A$ with $f_n$, it follows from (13.7) that $|f|$ is in $\bar{A}$. (4) That $\bar{f}$ is in $\bar{A}$ with $f$ follows similarly from the relation

$$|\bar{f}_n - \bar{f}| = |f_n - f| .$$

(5) On $\bar{A}$, $\mathfrak{N}^A$ satisfies the condition of monotonicity, since it satisfies this condition on $\mathfrak{F}^A$.

Thus $\bar{A}$ satisfies Conditions I.

(b). $\bar{A}$ satisfies Conditions II. We must show that if $\eta$ is in $\bar{A}'$, and $f \in \bar{A}$, then

$$\eta(f) = \int f d\hat{\eta}, \tag{13.8}$$

where $\hat{\eta}$ is the restriction of $\eta$ to $\mathfrak{K}_{\mathbf{C}}$. Since $\bar{A} \supset A$, $\hat{\eta}$ is in $\mathcal{A}'$. The relation (13.8) holds for $f \in A$ and hence for $f \in \bar{A}$; for $\eta$ is continuous in $\bar{A}$ since $\eta$ is in the dual of $\bar{A}$, and $\int f d\hat{\eta}$ varies continuously with $f \in \bar{A}$ by virtue of the relation (13.2) (with $\hat{\eta} = \alpha$ therein).

Thus $\bar{A}$ is an $MT$-space.

(c). $\bar{A}$ is compatible with $A$. Setting $\bar{A} = H$ it is sufficient to show that $\mathfrak{N}^A = \mathfrak{N}^H$ on $\mathfrak{K}_{\mathbf{C}}$ and that $\mathcal{A}' = \mathcal{H}'$. (1) The norm $\mathfrak{N}^H$, by hypothesis, is defined on $H$ by restriction of $\mathfrak{N}^A$ to $\bar{A} \supset \mathfrak{K}_{\mathbf{C}}$. Hence $\mathfrak{N}^H$ and $\mathfrak{N}^A$ have the same values on $\mathfrak{K}_{\mathbf{C}}$. (2) Since $H \supset A$, and since $\mathfrak{N}^H | A = \mathfrak{N}^A | A$, $\mathcal{H}' \subset \mathcal{A}'$. Now $H \subset \Omega^A \cap \mathfrak{F}^A$ dy Lemma 13.1, so that (13.2) is satisfied by $\alpha \in \mathcal{A}'$ and $f \in H$. It follows that $\alpha$ is in $\mathcal{H}'$ so that $\mathcal{A}' \subset \mathcal{H}'$. Hence $\mathcal{A}' = \mathcal{H}'$.

(d). If $B$ is an $MT$-space compatible with $A$, then $\bar{A} \supset B$. To see this we use the fact that $\bar{A}$ and $\bar{B}$ are closures of $\mathfrak{K}_{\mathbf{C}}$ in $\mathfrak{F}^A = \mathfrak{F}^B$ (Lemma 13.1). Hence $A \subset \bar{A} = \bar{B} \supset B$.

This establishes the theorem.

## 14. The space $\mathfrak{L}^1_{\mathbf{C}}(\beta)$.

Let $\beta \neq 0$ be a fixed $\mathbf{C}$-measure, and set $\mathfrak{L}^1_{\mathbf{C}}(\beta) = B$. We understand that for $f \in B$

$$\mathfrak{N}^B(f) = N^1_{\mathbf{C}}[f , \beta] \tag{14.1}$$

and shall show (1) that $B$ is an $MT$-space, (2) that $\Omega^B = B$, and (3) that when $\mathfrak{N}^B$ has been extended with the aid of (11.6), (14.1) holds for each $f \in \mathbf{C}^E$. We begin with a characterization of the measure dual $\mathcal{B}'$ of $B$.

**Definition.** A **C**-measure $\alpha$ on $E$ will be said to be dominated by a measure $\mu \geqq 0$ on $E$ if there exists a constant $M$ such that

$$|\alpha(f)| \leqq M\mu(|f|) \qquad [\text{for } f \in \mathfrak{K}_{\mathbf{C}}]. \tag{14.2}$$

When (14.2) holds we denote the minimum constant $M$ such that (14.2) holds by $M_\mu^\alpha$.

Among the **C**-measures $\alpha$ dominated by $\mu$ is $\mu_c$. If $\alpha_1$ and $\alpha_2$ are **C**-measures dominated by $\mu$, $\alpha_1 + \alpha_2$ is a **C**-measure dominated by $\mu$. The relation

$$(\alpha_1 + \alpha_2)(f) = \alpha_1(f) + \alpha_2(f)$$

implies that

$$|(\alpha_1 + \alpha_2)(f)| \leqq [M_\mu^{\alpha_1} + M_\mu^{\alpha_2}]\, \mu(|f|)$$

so that

$$M_\mu^{\alpha_1+\alpha_2} \leqq M_\mu^{\alpha_1} + M_\mu^{\alpha_2}. \tag{14.3}$$

If $c \in \mathbf{C}$, and if $\alpha$ is dominated by $\mu$, $c\alpha$ is dominated by $|c|\mu$ and

$$M_\mu^{c\alpha} = |c|\, M_\mu^\alpha. \tag{14.4}$$

**Lemma** 14.1. *The set of* **C***-measures $\alpha$ dominated by a positive measure $\mu$ forms a vector space $G_\mu$ over the field* **C***, with a norm whose values are $M_\mu^\alpha$.*

Since $M_\mu^\alpha$ satisfies (14.3) and (14.4), it remains only to note that the condition $M_\mu^\alpha = 0$ implies that $\alpha = 0$.

**Lemma** 14.2. *If a* **C***-measure $\alpha$ is dominated by a positive measure $\mu$, then*

$$|\alpha|^* \leqq M_\mu^\alpha \mu^*. \tag{14.5}$$

Introduce the measure $\nu = \mu M_\mu^\alpha$. In accordance with Prop. 15 (B 113), the inequality (14.5) will follow if $|\alpha| \leqq \nu$. Using the definition of $|a|(g)$ for $g \in \mathfrak{K}_+$,

$$|\alpha|(g) = \sup_{|f| \leqq g} |\alpha(f)| \leqq M_\mu^\alpha \mu(g) \qquad [f \in \mathfrak{K}_{\mathbf{C}}].$$

Thus $|\alpha| \leqq \nu$ and (14.5) follows.

**Definition.** A **C**-measure $\alpha$ will be said to be dominated by the $\mathbf{C}$-measure $\beta$ if $\alpha$ is dominated by the measure $|\beta|$.

Lemma 14.3. *If the $\mathbf{C}$-measure $\alpha$ is dominated by $\beta$ then for each $f \in \mathbf{C}^E$*

$$N_{\mathbf{C}}^1[f, \alpha] \leqq M_{|\beta|}^{\alpha} N_{\mathbf{C}}^1[f, \beta] \tag{14.6}$$

*and*

$$\mathfrak{F}_{\mathbf{C}}^1(\beta) \subset \mathfrak{F}_{\mathbf{C}}^1(\alpha), \qquad \mathfrak{L}_{\mathbf{C}}^1(\beta) \subset \mathfrak{L}_{\mathbf{C}}^1(\alpha). \tag{14.7}$$

Relation (14.6) is an immediate consequence of (14.5). The first inclusion in (14.7) is implied by (14.6). We infer further from (14.6) that the topology of $\mathfrak{F}_{\mathbf{C}}^1(\beta)$, as defined by $N_{\mathbf{C}}^1(\,.\,, \beta)$, is as fine on $\mathfrak{F}_{\mathbf{C}}^1(\beta)$ as the topology of $\mathfrak{F}_{\mathbf{C}}^1(\alpha)$, defined by $N_{\mathbf{C}}^1[\,.\,, \alpha]$. The second inclusion in (14.7) follows.

We draw the following inference. When $\alpha$ is dominated by $\beta$ the integral $\alpha(f)$ is defined over $\mathfrak{L}_{\mathbf{C}}^1(\alpha)$ and hence over $\mathfrak{L}_{\mathbf{C}}^1(\beta)$, and satisfies the relation

$$\left| \int f d\alpha \right| \leqq M_{|\beta|}^{\alpha} N_{\mathbf{C}}^1[f, \beta] \qquad [f \in \mathfrak{L}_{\mathbf{C}}^1(\beta)] \tag{14.8}$$

as a consequence of (14.6).

Lemma 14.4. *The space $B = \mathfrak{L}_{\mathbf{C}}^1(\beta)$ satisfies Conditions I of* § 10.

The semi-norm $N_{\mathbf{C}}^1[\,.\,, \beta]$ on $B$ is not trivial since $\beta \neq 0$ by hypothesis. Morever $\mathfrak{K}_{\mathbf{C}}$ is everywhere dense in $B$. The monotonicity of semi-norm follows from the relation

$$N_{\mathbf{C}}^1[f, \beta] = \int^{*} |f| \, d\,|\beta|, \tag{14.9}$$

and the monotonicity of the superior integral. If $f$ is in $B$, $|f|$ is in $B$ by Lemma 4.1, and $\overline{f}$ is in $B$ by Lemma 4.3.

Theorem 14.1. *Let $\beta \neq 0$ be a $\mathbf{C}$-measure on $E$, and set $B = \mathfrak{L}_{\mathbf{C}}^1(\beta)$. The dual $B'$ of $B$ is isomorphic to the vector space $G_\beta$ of $\mathbf{C}$-measures $\alpha$ dominated by $\beta$, under the mapping $\eta \to \widehat{\eta}$ in which $\eta \in B'$ corresponds to its restriction $\widehat{\eta} = \eta \,|\mathfrak{K}_{\mathbf{C}}$.*

Since $B$ satisfies Conditions I the image $\widehat{\eta}$ of $\eta \in B'$ is a $\mathbf{C}$-measure $\alpha$. (Cor. 10.1). The image of $B'$ under the mapping $\eta \to \widehat{\eta}$ is a vector space over $\mathbf{C}$. Since $\eta$ is in $B'$

$$|\eta(f)| \leqq M \, N_{\mathbf{C}}^1[f, \beta] \qquad [f \in B], \tag{14.10}$$

where $M$ is a posititve constant. In particular

$$|\eta(g)| \leqq M\,|\beta|(|g|) \qquad [g \in \mathfrak{K}_{\mathbf{C}}], \tag{14.11}$$

so that $\widehat{\eta}$ is dominated by $\beta$.

Moreover $\eta \to \widehat{\eta}$ maps $B'$ onto $G_\beta$, since the integral $\alpha(f)$ in (14.8) not only gives the values of an element in $B'$, but is also an extension of the arbitrary measure $\alpha$ (in Lemma 14.3) dominated by $\beta$. The mapping $\eta \to \widehat{\eta}$ is also $1-1$. For a null $\widehat{\eta}$ implies a null $\eta$, since $\mathfrak{K}_{\mathbf{C}}$ is everywhere dense in $B$, and $\eta$ is a continuous extension of $\widehat{\eta}$ over $B$.

Theorem 14.2. *If $\beta$ is a non-null $\mathbf{C}$-measure, the vector space $B = \mathfrak{L}^1_{\mathbf{C}}(\beta)$ is an $MT$-space.*

Lemma 14.4 affirms that $B$ satisfies Conditions I. If $\eta \in B'$, then $\alpha = \widehat{\eta}$ is a $\mathbf{C}$-measure dominated by $\beta$. (Theorem 14.1) Each $f \in B$ is then $\alpha$-integrable by (14.7). The relation

$$\eta(f) = \int f\,d\alpha \tag{14.12}$$

is valid for $f \in \mathfrak{K}_{\mathbf{C}}$ because $\alpha = \widehat{\eta}$. But the two members of (14.12) are continuous on $B$, the right member by virtue of (14.8), and the left member because $\eta$ is in $B'$. Hence (14.2) holds for $f \in B$. Thus $B$ satisfies Conditions II as well as I.

We state a theorem which justifies in a remarkable way the definition (11.6) of $\mathfrak{N}^A$ over $\mathbf{C}^E$.

Theorem 14.3. *If $\beta \neq 0$ is a $\mathbf{C}$-measure on $E$, the semi-norm $\mathfrak{N}^B$ of the vector space $B = \mathfrak{L}^1_{\mathbf{C}}(\beta)$, as extended in definition over $\mathbf{C}^E$ by (11.6), is identical with $N^1_{\mathbf{C}}[\cdot\,,\beta]$, as defined in (4.1) over $\mathbf{C}^E$.*

We begin by proving (a).

(a) If $\alpha$ is a subunit $\mathbf{C}$-measure in $\mathfrak{B}'$

$$N^1_{\mathbf{C}}[f\,,\alpha] \leqq N^1_{\mathbf{C}}[f\,,\beta] \qquad [f \in \mathbf{C}^E]. \tag{14.13}$$

For a subunit $\alpha \in \mathfrak{B}'$, $\|\alpha\|_{\mathfrak{B}'} \leqq 1$, by definition. According to the definition (10.5) of this norm, and (14.1)

$$\|\alpha\|_{\mathfrak{B}'} = \sup_{f \in B} \left|\int f\,d\alpha\right| \qquad [N^1_{\mathbf{C}}[f\,,\beta] \leqq 1]. \tag{14.14}$$

On the other hand $M^{\alpha}_{|\beta|}$ is the least constant $M$ such that for $f \in \mathfrak{K}_{\mathbf{C}}$ (or equivalently for $f \in B$)

$$(14.15) \qquad |\int f d\alpha| \leqq M |\beta| (|f|) = MN^1_{\mathbf{C}}[f, \beta].$$

From (14.14) and (14.15) we infer that

$$(14.16) \qquad M^{\alpha}_{|\beta|} = \|\alpha\|_{\mathfrak{B}'}$$

for each $\alpha \in \mathfrak{B}'$. Hence for a subunit $\alpha \in \mathfrak{B}'$, (14.13) holds as a consequence of (14.6).

To establish the theorem write (11.6) in the form

$$(14.17) \qquad \mathfrak{N}^B(f) = \sup_{\alpha \in \mathfrak{B}'_u} N^1_{\mathbf{C}}[f, \alpha] \qquad [f \in \mathbf{C}^E].$$

Now $\beta$ is a subunit $\alpha \in \mathfrak{B}'$ since

$$\|\beta\|_{\mathfrak{B}'} = \sup_x \frac{|\int x d\beta|}{N^1_{\mathbf{C}}(x, \beta)} \leqq 1 \qquad [x \in \mathfrak{L}^1_{\mathbf{C}}(\beta)].$$

Taking account of (14.13), the right member of (14.17) must then equal $N^1_{\mathbf{C}}[f, \beta]$, as affirmed in the theorem.

Theorem 14.4. *If $B$ is the $MT$-space $\mathfrak{L}^1_{\mathbf{C}}(\beta)$, then $\Omega^B = B$.*

According to Theorem 14.1, $\mathfrak{B}'$ is the vector space of $\mathbf{C}$-measures dominated by $\beta$, including $\beta$ itself. It follows from (14.7) that for each $\alpha \in \mathfrak{B}'$, $\mathfrak{L}^1_{\mathbf{C}}(\beta) \subset \mathfrak{L}^1_{\mathbf{C}}(\alpha)$. That $\Omega^B = B$ now follows from the definition of $\Omega^B$ in (10.7).

## **15.** The Banach space $H = \mathfrak{K}_{\mathbf{C}}$.

For $f \in H$ set $\mathfrak{N}^H(f) = U(f)$. We shall characterize $\Omega^H$ and $\mathfrak{F}^H$. The measure dual $\mathcal{H}'$ of $H$ is composed of $\mathbf{C}$-measures $\alpha$, bounded in the sense that

$$(15.1) \qquad |\alpha(f)| = \|\alpha\|_{\mathcal{H}'} U(f) \qquad [f \in H]$$

where

$$(15.2) \qquad \infty > \|\alpha\|_{\mathcal{H}'} = \sup_{U(f) \leqq 1} |\alpha(f)| \qquad [f \in H].$$

Following the proof of (8) B 59, with $f \in \mathfrak{K}_{\mathbf{C}}$ and $g \in \mathfrak{K}_+$,

$$\sup_{U(f) \leqq 1} |\alpha(f)| = \sup_{0 \leqq g \leqq 1} [\sup_{|f| \leqq g} |\alpha(f)|] = \sup_{0 \leqq g \leqq 1} |\alpha|(g)$$

so that we have the relation,

$$(15.3) \qquad \|\alpha\|_{\mathcal{H}'} = \int^* d|\alpha| \qquad [\alpha \in \mathcal{H}'].$$

We continue with a proof of (i).

(i) The space $H=\mathfrak{K}_{\mathbf{C}}$, normed by the values $U(f)$ is an $MT$-space.

One sees that $H$ satisfies Conditions I. If $\eta$ is in $H'$ its restriction $\widehat{\eta}$ to $\mathfrak{K}_{\mathbf{C}}$ is identical with $\eta$. Hence $H'$ is isomorphic to the measure dual $\mathcal{H}'$ of $H$ under the identity $\eta=\widehat{\eta}$. Condition II is thus satisfied so that $H$ is an $MT$-space.

$A'$-measurability. If $A$ is a space of $MT$-type, a mapping $f\in\mathbf{C}^E$ will be termed $\mathcal{A}'$-measurable if $\alpha$-measurable for each $\alpha\in\mathcal{A}'$.

Theorem 15.1. For the $MT$-space $H=\mathfrak{K}_{\mathbf{C}}$ normed by the values $U(f)$, $\Omega^H$ is the subspace of $x\in\mathbf{C}^E$, bounded, and measurable with respect to each bounded $\mathbf{C}$-measure on $E$. Moreover $\Omega^H\subset\mathfrak{F}^H$.

It is clear from the definition (10.8) of $\Omega^H$ and from Theorem 9.4 that each $f\in\Omega^H$ is $\alpha$-measurable for each $\alpha\in\mathcal{H}'$. If $|f|$ were not bounded there would exist a sequence of points $x_n\in E$ such that $|f(x_n)|\to\infty$ as $n\to\infty$. There would then exist a sequence $(m_n)$ of real positive numbers $m_n$ such that

$$(15.4)\qquad \sum_{n=1}^{\infty} m_n|f(x_n)|=\infty\,,\qquad \sum_{n=1}^{\infty} m_n<\infty\,.$$

Let $\mu$ be a discrete measure on $E$ defined by masses $m_n$ placed at the respective points $x_n$. Then $\mu$ is a bounded measure, and hence (cf. (15.3)) $\mu_c$ a bounded $\mathbf{C}$-measure. Now $|f|$ is in $\Omega^H$. Hence $|f|$ is in $\mathfrak{L}^1_{\mathbf{C}}(\mu_e)$, so that $\mu_e(|f|)$ is finite. But the value of $\mu_e(|f|)$ is given by the left sum in (15.4) and so is infinite. We infer that $|f|$ is bounded.

If then $f\in\Omega^H$, $|f|\leqq M_f$ for some constant $M_f$. By (11.6) and (15.3), respectively,

$$\mathfrak{N}^H(f)\leqq M_f\sup_{\alpha\in\mathcal{H}'_u}\int^* d\,|\alpha|=M_f\sup_{\alpha\in\mathcal{H}'_u}\|\alpha\|_{\mathcal{H}'}\leqq M_f\,.$$

It follows that $\Omega^H\subset F^H$.

The following lemma shows the meaning of $H'$-negligibility.

Lemma 15.1. A mapping $f\in\mathbf{C}^E$ is $H'$-negligible if and only if $f(t)=0$ on $E$.

Given $f\in\mathbf{C}^E$ let $\mu$ be the discrete measure defined by a single mass $m=1$ placed at a point $t\in E$. Then $\mu_e$ is a $\mathbf{C}$-measure in $\mathcal{H}'$. If $f$ is

$H'$-negligible, then by definition of $H'$-negligibility

$$0 = \int^* |f| \, d\, |\mu_e| = \int^* |f| \, d\, |\mu| = |f(t_j)| = 0 .$$

Thus $f(t) = 0$ for $t \in E$ if $f$ is $H'$-negligible. The converse is trivial.

It has been seen in § 12 that $\Omega^B = \mathfrak{L}^1_{\mathbf{C}}(\beta)$ in case $B = \mathfrak{L}^1_{\mathbf{C}}(\beta)$. In contrast we here establish the following

**Lemma** 15.2. *If $E$ is not a finite set then a representation of $\Omega^H$ of the form*

$$\Omega^H = \bigcap_{i=1}^{n} \mathfrak{L}^1_{\mathbf{C}}(\alpha_i) \tag{15.5}$$

*in which $\alpha_1, \ldots, \alpha_n$ is a finite set of $\mathbf{C}$-measures in $\mathcal{H}'$ is impossible.*

We simplify the proof by first establishing (b).

(b) *If $\Omega^H$ admits the representation (15.5) then $\Omega^H$ also admits a representation of the form*

$$\Omega^H = \mathfrak{L}^1_{\mathbf{C}}(\mu_e) \tag{15.6}$$

*where $\mu = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|$, and $\mu_e$ is in $\mathcal{H}'$.*

By virtue of Lemma 11.2 $|\alpha_i|_e$ is in $\mathcal{H}'$ for $i = 1, \ldots, n$. Now $\mu$ is a measure, and

$$\mu_e = |\alpha_1|_e + |\alpha_2|_e + \ldots + |\alpha_n|_e .$$

Since $\mathcal{H}'$ is an additive group, $\mu_e$ is in $\mathcal{H}'$. From the relation $|\alpha_i| \leqq \mu$ it follows that

$$\mathfrak{L}^1_{\mathbf{C}}(\alpha_i) \supset \mathfrak{L}^1_{\mathbf{C}}(\mu_e) \qquad [i = 1, \ldots, n] .$$

Hence $\Omega^H \supset \mathfrak{L}^1_{\mathbf{C}}(\mu_e)$ if (15.5) holds. Since $\mu_e \in \mathcal{H}'$, (15.6) follows.

It is therefore sufficient to show that $\Omega^H$ admits no representation of the form (15.6).

Let $t_i$ be a sequence of distinct points in $E$. For each $i$ let $\mu_e(t_i)$ denote the $\mu_e$-measure of the subset of $E$ containing the single point $t_i$. Then the series $\mu_e(t_1) + \mu_e(t_2) + \ldots$ must converge. Otherwise an integer $m$ exists such that

$$\sum_{i=1}^{m} \mu_e(t_i) > \|\mu_e\|_{\mathcal{H}'} ,$$

so that if $g \in \mathfrak{K}_{\mathbf{C}}$ maps $E$ into $[0, 1]$ with $g_m(t_i) = 1$ for $i = 1, \ldots, m$, then

$$\mu_e(g) \geqq \sum_{i=1}^{m} \mu_e(t_i) > \|\mu_e\|_{\mathcal{H}'}$$

contrary to the relation

$$\mu_e(g) \leqq \|\mu_e\|_{\mathcal{H}'} U(g) = \|\mu_e\|_{\mathcal{H}'} .$$

Granting the convergence of the series $\mu_e(t_1) + \mu_e(t_2) + \ldots$ we infer the existence of an $f \in R_{+}^{E}$ such that $f(t) = 0$, except at most for $t = t_i$, $i = 1, 2, \ldots$, while

$$\limsup_{i \uparrow \infty} f(t_i) = +\infty , \qquad \sum_{i=1}^{\infty} f(t_i)\, \mu_e(t_i) = \sum_{i=1}^{\infty} f(t_i)\, \mu(t_i) < \infty .$$

It follows that $f \in \mathfrak{L}^1(\mu)$. Cf. Ex. in B p. 181. Hence $f \in \mathfrak{L}^1_{\mathbf{C}}(\mu_e) \subset \Omega^H$.

Let $\varepsilon_t$ be the discrete measure with a mass $m = 1$ at $t$. If $k_1 + k_2 + \ldots$ is a convergent series of positive constants $\nu = k_1 \varepsilon_{t_1} + k_2 \varepsilon_{t_2} + \ldots$ is a bounded measure. Since the sequence $f(t_i)$ is unbounded, one can further condition the sequence $(k_i)$ so that the series

$$k_1 f(t_1) + k_2 f(t_2) + \ldots$$

diverges. This implies that $f$ is not in $\mathfrak{L}^1(\nu)$. Hence $f$ is not in $\mathfrak{L}^1_{\mathbf{C}}(\nu_e)$, and so not in $\Omega^H$. From this contradiction we infer that (15.5) cannot hold and the theorem follows.

We conclude with a characterization of $\Omega^H$, a priori more inclusive than that given in Theorem 15.1.

**Theorem** 15.2. *The space $\Omega^H$ is the vector subspace $G$ of $\mathbf{C}^E$ of all $\mathcal{H}'$-measurable functions $f$ for which*

$$\sup_{\alpha} N^1_{\mathbf{C}}[f, \alpha] < \infty \tag{15.7}$$

*taking the sup over all subunit $\alpha \in \mathcal{H}'$.*

An $f \in G$ is $\alpha$-integrable for each $\alpha \in \mathcal{H}'$ by virtue of Theorem 9.4, so that $G \subset \Omega^H$. But $\Omega^H \subset \mathfrak{F}^H$ by Theorem 15.1, so that an $f \in \Omega^H$ satisfies the conditions of the theorem on an element in $G$. Hence $G \supset \Omega^H$. We conclude that $G = \Omega^H$.

Theorems 15.1 and 15.2 give the following

**Corollary** 15.2. If $f \in \mathbf{C}^E$ is $\alpha$-measurable for each bounded $\mathbf{C}$-measure $\alpha$ on $E$, and if (15.7) holds, then $|f|$ is bounded.

**Theorem** 15.3. For arbitrary $f \in \mathbf{C}^E$ and for $A = H$

$$\mathfrak{N}^A(f) = \sup_{x \in E} |f(x)| . \tag{15.8}$$

Let $M$ be the sup in (15.8). Given $a \in E$ let $\mu$ be the subunit measure on $E$ with mass 1 at $a$ and total mass 1. Then $\mu_e(|f|) = |f(a)|$. Hence $\mathfrak{N}^A(f) \geqq |f(a)|$ from (11.6), and so $\mathfrak{N}^A(f) \geqq M$. But for an arbitrary subunit $\mathbf{C}$-measure $\alpha$

$$\int^* |f| \, d\,|\alpha| \leqq M \int^* d\,|\alpha| = M \, ||\alpha||_{\mathcal{H}'} = M \tag{15.9}$$

using (15.3). Hence from (11.6) $\mathfrak{N}^A(f) \leqq M$. Thus (15.8) is valid.

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