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Asymptotic analysis for stochastic volatility: martingale expansion

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Abstract A general class of stochastic volatility models with jumps is considered and an asymptotic expansion for European option prices around the Black–Scholes prices is validated in the light of Yoshida's martingale expansion theory. Several known formulas of regular and singular perturbation expansions are obtained as corollaries. An expansion formula for the Black–Scholes implied volatility is given which explains the volatility skew and term structure. The leading term of the expansion is always an affine function of log moneyness, while the term structure of the coefficients depends on the details of the underlying stochastic volatility model. Several specific models which represent various types of term structure are studied.

Keywords Asymptotic expansion · Fast mean reversion · Fractional Brownian motion · Jump-diffusion · Partial Malliavin calculus · Yoshida's formula

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JEL Classification C13 · G13

1 Introduction

The stochastic volatility model is a continuous-time limit of the ARCH model (see Nelson [20]) and a reasonable generalization of the Black–Scholes model. An advantage of this model is the fact that it explains an empirical phenomenon known as the volatility smile or skew. See, e.g., Heston [12], Hull and White [13], Renault and Touzi [23] for details. Unfortunately, no simple option pricing formula has been

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obtained for a general stochastic volatility model (see Heston [12], Nicolato and Venardos [19] for exceptional cases), which apparently causes difficulties in practical use. To overcome this disadvantage, an asymptotic expansion is one of the standard approaches. For example, Yoshida [24] established a small diffusion expansion and his method was utilized by Osajima [22] to validate an asymptotic expansion formula for the SABR model developed by Hagan et al. [11]. Masuda and Yoshida [18] applied the Edgeworth expansion for geometric mixing processes to a specific stochastic volatility model.

The Black–Scholes model seems not so bad when considered as a simple approximation to the true dynamics of the asset price process both theoretically and practically. It is therefore natural to consider an asymptotic expansion around the Black–Scholes prices. Introducing a two time scales model, namely, the so-called fast mean reverting stochastic volatility model, Fouque et al. [3] gave a singular perturbation expansion and its applications. Fouque et al. [5] extended this to a multiscale model. The validity of the singular perturbation expansion was proved by Fouque et al. [4], Conlon and Sullivan [2] for the European call option price for volatility models with certain ergodic volatility factors. The case of a digital option was treated in Fouque et al. [8]. Khasminskii and Yin [15] proved the validity in case that the volatility is an ergodic diffusion on a compact set. Fukasawa [10] succeeded in proving the validity for a quite large class of ergodic diffusions of dimension one by using the Edgeworth expansion for ergodic diffusions developed by Fukasawa [9]. In the formulation of Fukasawa [10], the asset price process $S = \{S_t\}$ is supposed to satisfy

$$\begin{aligned} S_t &= \exp(Z_t), \\ Z_t &= Z_0 + rt - \langle M \rangle_t / 2 + M_t, \\ dM_t &= \varphi(X_t)\, dW_t + \psi(X_t)\, dW'_t, \\ dX_t &= \epsilon^{-2} b_\epsilon(X_t)\, dt + \epsilon^{-1} c(X_t)\, dW_t \end{aligned} \tag{1.1}$$

under a risk-neutral measure P, where (W, W') is a 2-dimensional standard Brownian motion, $r > 0$ is the risk-free rate and $\varphi, \psi, b_\epsilon, c$ are Borel functions on $\mathbb{R}$. Under a suitable condition on the ergodicity of X, the Black–Scholes implied volatility (IV) at time t is expanded as

$$\mathrm{IV} = a \frac{\log(K/S_t)}{T-t} + b + O\big(\epsilon^2\big) \tag{1.2}$$

as $\epsilon \to 0$ for constants a, b which are of $O(\epsilon)$, $O(1)$, respectively, and depend on neither K nor $T - t$, where T and K are the maturity and the strike price of the call/put option respectively. This expansion formula (1.2) explains the volatility skew as a result of nonzero asymptotic covariance between the asset price and the integrated volatility; the constant a in (1.2) is negative if the limit of $\mathrm{Cov}(M_T - M_t, \langle M \rangle_T - \langle M \rangle_t)/\epsilon$ as $\epsilon \to 0$ is negative, which should be related to an empirical observation known as the leverage effect. Note that (1.2) does not incorporate the volatility smile. One has to include higher-order terms in order to capture the smile (see, e.g., Fouque et al. [7]). Nevertheless, Fouque et al. [6] showed that using S&P500 data, the in-sample fit to the volatility surface of (1.2) is fairly good and the variation of the calibrated parameters a, b over time t is relatively small. They reported, however, that the term structure is not like $O((T-t)^{-1})$ as in (1.2), but more like $O((T-t)^{-q})$ with $q \approx 1/2$. See Sects. 1.3 and 2.4 of Fouque et al. [6] for the details. They proposed the use of an inhomogeneous diffusion as X in (1.1).

There are alternative perturbation approaches around the Black–Scholes model. Consider (1.1) with the stochastic differential equation for $X = \{X_t\}$ replaced by

$$dX_t = \big(b_0(X_t) + \epsilon b_1(X_t) + \epsilon^2 b_2(X_t)\big)\,dt + \epsilon c(X_t)\,dW_t.$$

A regular perturbation argument of the corresponding partial differential equation gives an asymptotic expansion of European prices, which results in, e.g., if $b_0 \equiv 0$,

$$\mathrm{IV} = a \log(K/S_t) + b(T-t) + c + O\big(\epsilon^2\big) \tag{1.3}$$

for constants a, b, and c, which are of $O(\epsilon)$, $O(\epsilon)$, and $O(1)$, respectively. See, e.g., Lee [16], Lewis [17], and Fouque et al. [5] for the details.

The purpose of the present paper is to prove the validity of such asymptotic expansions in a unified way for a more general stochastic volatility model with jumps and for a more general option payoff function. We introduce a probabilistic approach which is different from preceding studies that are based on perturbations of partial differential equations, as well as from the other probabilistic approaches of Fukasawa [10] and the martingale method outlined in Fouque et al. [3]. Yoshida's formula for a martingale expansion established by Yoshida [25, 26] plays an essential role, while Fukasawa [10] utilized the Edgeworth theory for ergodic diffusions. The advantage of the martingale expansion compared to the Edgeworth expansion is the fact that it is based only on the convergence of the quadratic variation of a martingale, so that it is not limited to the case of an ergodic diffusion. On the other hand, at the present time, the Edgeworth expansion gives a more accurate estimate of the error under a simpler condition as far as one considers the fast mean reverting model (1.1). The main contribution of this paper is to present a unifying framework for various perturbation expansions around the Black–Scholes price. It also gives an efficient way of identifying the asymptotic approximation.

The leading term of the expansion of the implied volatility is always an affine function of log moneyness, while the term structure of the coefficients depends on the underlying asymptotic model. The expansion formula will be useful in assessing the ability of a specific model to reproduce the volatility term structure in the market. Besides, as proposed in Fouque et al. [3], we can regard the stochastic volatility model as a semi-parametric model with the coefficients of the expansion; they proposed a pricing and hedging method using only the information contained in the observed implied volatility skew via the expansion formula without fully specifying the underlying model. The main result is given in Sect. 2. We analyze specific models in Sect. 3.

2 Martingale expansion

2.1 Stochastic volatility model with jumps

Here, we describe our model for the asset price process. Let $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t\}_{t\geq 0}, P^n)$ be a filtered probability space for each $n \in \mathbb{N}$, where an increasing adapted continuous process Λ^n with $\Lambda^n_0 = 0$, an adapted càg process h^n and a continuous martingale X^n with $X^n_0 = 0$ are defined. Let W be the canonical process $W_t(\omega) = \omega(t)$ of the Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$. Let N be a standard Poisson process and U_j, $j = 1, 2, \dots$ be i.i.d. random variables independent of N defined on a probability space $(\Omega^0, \mathcal{F}^0, P^0)$. Consider the product space $\Omega^0 \times \Omega^n \times \mathcal{W}$ with the product measure $P^0 \otimes P^n \otimes \mu$ and then define a time-changed compound Poisson process C^n as

$$C^n_t = \epsilon_n \sum_{j=1}^{N^n_t} U_j, \qquad N^n_t = N_{\Lambda^n_t},$$

where $\epsilon_n > 0$ is a deterministic sequence with $\epsilon_n \to 0$ as $n \to \infty$. The role of n is explained later. We assume that there exists $\kappa > 0$ such that

$$\int_{\mathbb{R}} e^{\kappa|z|} \nu(dz) < \infty, \tag{2.1}$$

where ν is the distribution of U_j. Then put

$$\nu_k = \int_{\mathbb{R}} z^k \nu(dz), \quad k \in \mathbb{N}.$$

Let $\{\bar{\mathcal{F}}^n_t\}_{t\geq 0}$ be the minimal filtration such that N^n, C^n, and all $\{\mathcal{F}^n_t\}$-adapted processes are $\{\bar{\mathcal{F}}^n_t\}$-adapted and that the usual conditions are satisfied. Let g^n be an $\{\bar{\mathcal{F}}^n_t\}$-adapted process. Let $\{\mathcal{G}^n_t\}_{t\geq 0}$ be the minimal filtration such that W and all $\{\bar{\mathcal{F}}^n_t\}$-adapted processes are $\{\mathcal{G}^n_t\}$-adapted and that the usual conditions are satisfied. Note that every $\{\bar{\mathcal{F}}^n_t\}$-martingale is a $\{\mathcal{G}^n_t\}$-martingale and that W is a $\{\mathcal{G}^n_t\}$-standard Brownian motion. Stochastic integrals, quadratic covariations, and predictable quadratic covariations we consider in the following are all with respect to $\{\mathcal{G}^n_t\}$.

We suppose that the log price process $Z = \log(S/S_0)$ is given by

$$Z = A^n + X^n + \int_0^{\cdot} g^n_s \, dW_s + \int_0^{\cdot} h^n_s \, dC^n_s$$

for a large n, where

$$A^n_t = R(t) - \frac{1}{2}\langle X^n \rangle_t - \frac{1}{2}\int_0^t |g^n_s|^2 \, ds - \int_0^t \int_{\mathbb{R}} \left(e^{h^n_s \epsilon_n z} - 1\right) \nu(dz) \Lambda^n(ds)$$

with a deterministic function R of bounded variation which stands for the cumulative risk-free rate. Note that $e^{-R}S$ is a martingale for each n by Itô's formula due to the definition of A^n under suitable integrability conditions.

Here, we give a brief explanation for the model. If $h^n \equiv 0$, then the price process is continuous and

$$\langle Z \rangle = \langle X^n \rangle + \int_0^{\cdot} |g_s^n|^2 \, ds,$$

which is $\{\bar{\mathcal{F}}_t^n\}$-adapted. This corresponds to the so-called integrated volatility and is deterministic in the Black–Scholes model. A stochastic volatility model refers to a case such that the integrated volatility is not deterministic. The case of $X^n \equiv 0$ is often referred to as the case of no leverage effect because the direction of a price movement is independent of the dynamics of the spot volatility g^n in this case. As is well known, the price movement appears to be negatively correlated to the volatility movement. One of the advantages of stochastic volatility models is the ability to explain this empirical observation called the leverage effect by introducing nonzero X^n. For example, if we set W' being an $\{\mathcal{F}_t^n\}$-standard Brownian motion,

$$\begin{aligned} dX_t^n &= \theta_4 \sqrt{V_t} \, dW_t', \\ dV_t &= -(\theta_1 V_t - \theta_2) \, dt + \theta_3 \sqrt{V_t} \, dW_t', \\ g^n &= \sqrt{\left(1 - \theta_4^2\right) V} \end{aligned}$$

for positive constants $\theta_1, \theta_2, \theta_3$, and $\theta_4 \in (-1, 1)$, then we obtain the Heston model. We assume that g^n is not degenerate in a sense specified later. It implies that the price movement has a component which is not explained by the volatility movement.

In the general case of $h^n \not\equiv 0$, the quadratic variation of the log price process is given by

$$[Z]_t = \langle X^n \rangle_t + \int_0^t |g_s^n|^2 \, ds + \epsilon_n^2 \sum_{j=1}^{N_t^n} \left| h_{\tau_j^n}^n U_j \right|^2,$$

where $\tau_j^n = \inf\{t \geq 0 \mid N_t^n = j\}$, and its predictable version is given by

$$\langle Z \rangle_t = \langle X^n \rangle_t + \int_0^t |g_s^n|^2 \, ds + \epsilon_n^2 \nu_2 \int_0^t |h_s^n|^2 \Lambda^n(ds).$$

The stochastic intensity Λ^n accounts for the clustering of jumps. The integrand h^n plays the role of stochastic volatility for the jump component.

2.2 Conditions and results

As is well known, the European option price at time 0 with payoff function f and maturity T is given by

$$e^{-R(T)} E\left[f(S_T)\right] = E\left[F(Z_T)\right], \qquad F(z) = e^{-R(T)} f\left(S_0 \exp(z)\right).$$

Our aim is to prove the validity of asymptotic expansions of such a price around the Black–Scholes price. It suffices to study an asymptotic expansion of the distribution of Z_T for fixed $T > 0$ as $n \to \infty$ around a normal distribution. We can make the model represent various perturbation models by specifying its dependence on n suitably, as we see later in the next section. Since we are considering a sequence of semimartingales, a refinement of the martingale central limit theorem is what we want. Fortunately, it has already given by Yoshida [25, 26]. Here, we give a sufficient condition for Yoshida's formula to be applicable.

Denote by M^n the local martingale part of Z, i.e.,

$$M^n = X^n + \int_0^\cdot g_s^n \, dW_s + \int_0^\cdot h_s^n \big[dC_s^n - \epsilon_n \nu_1 \Lambda^n(ds) \big].$$

Note that

$$\begin{aligned}
\langle M^n \rangle_T &= \langle X^n \rangle_T + \int_0^T |g_s^n|^2 \, ds + \epsilon_n^2 \nu_2 \int_0^T |h_s^n|^2 \Lambda^n(ds), \\
\epsilon_n^{-1} \big([M^n]_T - \langle M^n \rangle_T \big) &= \epsilon_n \sum_{j=1}^{N^n} \big| h^n_{\tau_j^n} U_j \big|^2 - \epsilon_n \nu_2 \int_0^\cdot |h_s^n|^2 \Lambda^n(ds),
\end{aligned}$$

where $\tau_j^n = \inf\{t \geq 0 \mid N_t^n = j\}$, and that

$$\big\langle M^n, \epsilon_n^{-1} \big([M^n] - \langle M^n \rangle \big) \big\rangle_T = \epsilon_n^2 \nu_3 \int_0^T \{h_s^n\}^3 \Lambda^n(ds).$$

Condition 2.1 *There exists a deterministic sequence Σ_n with $\Sigma := \lim_{n\to\infty} \Sigma_n > 0$ such that the sequences*

$$\epsilon_n^{-1} \big(\Sigma_n^{-1} \langle M^n \rangle_T - 1 \big), \quad \epsilon_n^2 \Lambda_T^n, \quad \sup_{0 \leq s \leq T} |h_s^n|, \quad \left\{ \int_0^T |g_s^n|^2 \, ds \right\}^{-1} \quad \text{for } n \in \mathbb{N}$$

are bounded in L^p for any $p > 0$.

Condition 2.2 *The sequence*

$$\begin{aligned}
&\big(\Sigma_n^{-1/2} M_T^n, \Sigma_n^{-1} \epsilon_n^{-1} \big([M^n]_T - \langle M^n \rangle_T \big), \epsilon_n^{-1} \big(\Sigma_n^{-1} \langle M^n \rangle_T - 1 \big), \\
&\Sigma_n^{-3/2} \epsilon_n^{-1} \langle M^n, [M^n] - \langle M^n \rangle \rangle_T \big)
\end{aligned}$$

converges weakly to the distribution of a random variable, say, (N_1, N_2, N_3, N_4).

Remark 2.3 The nondegeneracy condition for g^n in Condition 2.1 is not restrictive in the context of a stochastic volatility model, while it is violated when considering a local volatility model defined as

$$dS_t = S_t \big(r \, dt + \hat{\sigma}(S_t) \, d\hat{W}_t \big)$$

for a standard Brownian motion $\hat{W}$ and a Borel function $\hat{\sigma}$. The reason why we introduce the condition of the nondegeneracy of g^n is that it is quite a convenient condition for the smoothness of the distribution of M^n_T. Yoshida's theory for martingale expansions exploits the smoothness of the distribution of the martingale marginal, which corresponds to the classical Cramér condition in the context of the Edgeworth expansion. Yoshida [25, 26] presented sufficient conditions in terms of the nondegeneracy of the Malliavin covariance of the martingale marginal. Under the nondegeneracy of g^n, we can apply the partial Malliavin calculus with respect to $(\mathcal{W}, \mathcal{H}, \mu)$ to have

$$\int_0^T |g^n_s|^2\, ds$$

as the Malliavin covariance of M^n_T. As a result, we do not need to be involved in the Malliavin calculus on $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t\}, P^n)$ nor $(\Omega^0, \mathcal{F}^0, P^0)$. See the proof of the next theorem for details. The condition on g^n can be removed if the smoothness of the distribution of X^n_T or C^n_T is assured in a suitable manner.

Theorem 2.4 *Let F be a Borel function of polynomial growth. Under Conditions 2.2 and 2.1, it holds*

$$E\big[F(Z_T)\big] = \int_{\mathbb{R}} F\big(R(T) - \Sigma_n/2 + \sqrt{\Sigma_n} z\big)\phi_n(z)\, dz + o(\epsilon_n) \tag{2.2}$$

as $n \to \infty$, where

$$\phi_n(z) = \phi(z) + \epsilon_n \frac{1}{2}\partial_z^2\big(E[\xi|N_1 = z]\phi(z)\big) - \epsilon_n \partial_z\big(E[\eta|N_1 = z]\phi(z)\big),$$

and ϕ is the standard normal density,

$$\xi = \frac{1}{3}N_2 + N_3, \qquad \eta = -\frac{\sqrt{\Sigma}}{2}\left(N_3 + \frac{\sqrt{\Sigma}}{3}N_4\right).$$

Proof Here, we apply Yoshida [26]. Putting

$$\begin{aligned}
\xi_n &= \epsilon_n^{-1}\left\{\frac{1}{3}\big[\Sigma_n^{-1/2}M^n\big]_T + \frac{2}{3}\big\langle\Sigma_n^{-1/2}M^n\big\rangle_T - 1\right\} \\
&= \epsilon_n^{-1}\big(\Sigma_n^{-1}\langle M^n\rangle_T - 1\big) + \frac{1}{3}\Sigma_n^{-1}\epsilon_n^{-1}\big(\big[M^n\big]_T - \langle M^n\rangle_T\big), \\
\xi_n' &= \epsilon_n^{-1}\big(\Sigma_n^{-1}\langle M^n\rangle_T - 1\big), \\
\xi_n'' &= \Sigma_n^{-3/2}\big\langle M^n, \epsilon_n^{-1}\big(\big[M^n\big] - \langle M^n\rangle\big)\big\rangle_T,
\end{aligned}$$

we have

$$\begin{aligned}
Z_T &= R(T) - \frac{1}{2}\Sigma_n - \frac{1}{2}\Sigma_n\big(\Sigma_n^{-1}\langle M^n\rangle_T - 1\big) - \frac{1}{6}\epsilon_n\Sigma_n^{3/2}\xi_n'' + M^n_T + o_p(\epsilon_n) \\
&= R(T) - \frac{1}{2}\Sigma_n + \sqrt{\Sigma_n}\{\mu_n + \epsilon_n\eta_n\},
\end{aligned}$$

where

$$\mu_n = \Sigma_n^{-1/2} M_T^n, \qquad \eta_n = -\frac{\sqrt{\Sigma_n}}{2}\left(\xi_n' + \frac{\sqrt{\Sigma_n}}{3}\xi_n''\right) + o_p(1).$$

Here, the remainder term of $o_p(1)$ is $\mathcal{F}_T^n$-measurable and bounded in L^p uniformly in sufficiently large n for any $p > 0$ by Condition 2.1 and (2.1). By Conditions 2.1 and 2.2 with the aid of the Burkholder–Davis–Gundy inequality, (μ_n, ξ_n, η_n) is uniformly bounded in L^p for any $p > 0$ and converges weakly to

$$\left(N_1, \left(\frac{1}{3}N_2 + N_3\right), -\frac{\sqrt{\Sigma}}{2}\left(N_3 + \frac{\sqrt{\Sigma}}{3}N_4\right)\right).$$

Now, we apply Theorem 1 of Yoshida [26] to $\mu_n + \epsilon_n \eta_n$. To verify the conditions of the theorem, as mentioned in Remark 2.3, we apply the partial Malliavin calculus with the Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$. In other words, we define a Malliavin operator $\mathcal{L}^n$ as the Ornstein–Uhlenbeck operator associated with W only. Then

$$\Sigma_n^{-1} \int_0^T |g_s^n|^2 \, ds$$

is the Malliavin covariance of μ_n. In the notation of Yoshida [26], we take $Y_n = \mu_n$, $s_n = \sigma_{Y_n}$ and

$$\sigma_{Y_n} = \Sigma_n^{-1} \int_0^T |g_s^n|^2 \, ds,$$

$$\kappa_n = \Sigma_n^{-2} \epsilon_n^2 \sum_{j=1}^{N_T^n} \left|h_{\tau_j^n} U_j\right|^4,$$

$$\lambda_n = \Sigma_n^{-2} \epsilon_n^2 \nu_4 \int_0^T |h_s|^4 \Lambda^n(ds),$$

all of which are $\bar{\mathcal{F}}_T^n$-measurable as well as ξ_n, η_n. The (partial) Malliavin derivative of an $\bar{\mathcal{F}}_T^n$-measurable functional is always 0, so that

$$\mathcal{S}_0^n = \{\sigma_{Y_n}, \mu_n, \xi_n, \eta_n, \kappa_n, \lambda_n\},$$

$$\mathcal{S}_1^n = \left\{\sigma_{Y_n}, \mu_n, \xi_n, \eta_n, \kappa_n, \lambda_n, -\Sigma_n^{-1/2} \int_0^\cdot g_s^n \, dW_s\right\},$$

and $\mathcal{S}_\ell^n = \mathcal{S}_1^n$ for all $\ell \geq 2$. Notice that $\mathcal{S}_\ell^n$ is uniformly in n and ℓ bounded in L^p for any $p > 0$ by Condition 2.1 and (2.1). Thus, all the assumptions of Theorem 1 of Yoshida [26] are verified. □

Notice that

$$P_f(\sigma) := e^{-R(T)} \int f\big(S_0 \exp\big(R(T) - \sigma^2 T/2 + \sigma\sqrt{T} z\big)\big)\phi(z)\, dz$$

is the Black–Scholes price for the payoff function f with volatility σ, so that considering $F(z) = e^{-R(T)} f(S_0 \exp(z))$, the leading term of the expansion (2.2) corresponds to the Black–Scholes price with σ_n such that $\sigma_n^2 T = \Sigma_n$.

Theorem 2.5 *Let F be a Borel function of polynomial growth. Suppose that Conditions 2.1 and 2.2 hold with (N_1, N_2, N_3) being normal and N_4 being a constant. Then (2.2) holds with*

$$\phi_n(z) = \phi(z)\left\{1 + \frac{\epsilon_n}{2}\left\{\delta_3\left(z^2 - 1\right) + \left(\rho_{13} + \frac{1}{3}\rho_{12}\right)\left(z^3 - 3z\right) - \sqrt{\Sigma}\left\{\left(\delta_3 + \frac{\sqrt{\Sigma}}{3}\rho_{12}\right)z + \rho_{13}\left(z^2 - 1\right)\right\}\right\}\right\},$$

where

$$\delta_3 = E[N_3], \qquad \rho_{12} = E[N_1 N_2], \qquad \rho_{13} = E[N_1 N_3].$$

In particular, the put option price

$$e^{-R(T)} E\left[(K - S_T)_+ \middle| S_0 = S\right]$$

with $K > 0$ is expanded as

$$P_f(\sigma_n) + \frac{1}{2}\epsilon_n \sigma_n \sqrt{T} S \phi(d_1)\left\{\delta_3 - \rho_{13} d_2 - \frac{1}{3}\rho_{12}(d_2 - \sigma_n\sqrt{T})\right\} + o(\epsilon_n),$$

where $\sigma_n^2 T = \Sigma_n$ and P_f is the Black–Scholes price with $f(s) = (K - s)_+$, namely

$$P_f(\sigma_n) = K e^{-R(T)} \Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + R(T) + \Sigma_n/2}{\sqrt{\Sigma_n}}, \qquad d_2 = d_1 - \sqrt{\Sigma_n}.$$

Proof The formula for ϕ_n is obtained from a simple calculation by noting that

$$E[N_1] = E[N_2] = 0, \qquad E\left[N_1^2\right] = 1, \qquad N_4 = \rho_{12},$$

which follow from the uniform integrability. For the put option price, using the fact that $K\phi(d_2) = Se^R \phi(d_1)$, observe that

$$\int \left(K - S\exp\left(R(T) - \Sigma_n/2 + \sqrt{\Sigma_n} z\right)\right)_+ \phi(z)\left(z^3 - 3z\right) dz$$
$$= -\sqrt{\Sigma_n} S e^{R(T)}\left\{\phi(d_1)(d_2 - \sqrt{\Sigma_n}) + \Phi(-d_1)\Sigma_n\right\},$$
$$\int \left(K - S\exp\left(R(T) - \Sigma_n/2 + \sqrt{\Sigma_n} z\right)\right)_+ \phi(z)\left(z^2 - 1\right) dz$$
$$= \sqrt{\Sigma_n} S e^{R(T)}\left\{\phi(d_1) - \Phi(-d_1)\sqrt{\Sigma_n}\right\},$$

and that

$$\int \big(K - S\exp\big(R(T) - \Sigma_n/2 + \sqrt{\Sigma_n}z\big)\big)_+ \phi(z)z\,dz = -Se^{R(T)}\Phi(-d_1)\sqrt{\Sigma_n}.$$

The results then follow from a straightforward calculation. □

Corollary 2.6 *Under the same conditions as Theorem 2.5, the Black–Scholes implied volatility*

$$P_f^{-1}\big(e^{-R(T)}E\big[f(S_T)\big|S_0 = S\big]\big), \qquad f(s) = (K - s)_+$$

is expanded as

$$\sigma_n\left\{1 + \frac{\epsilon_n}{2}\left\{\delta_3 - \rho_{13}d_2 - \frac{1}{3}\rho_{12}(d_2 - \sigma_n\sqrt{T})\right\}\right\} + o(\epsilon_n).$$

In particular, the implied volatility is an affine function of log moneyness $\log(K/S_0)$ *up to* $o(\epsilon_n)$, *since* d_2 *is so.*

Proof Use the fact that the Black–Scholes put price P_f satisfies

$$\frac{\partial P_f}{\partial \sigma} = \sqrt{T}S\phi(d_1).$$

□

Remark 2.7 The error estimate $o(\epsilon_n)$ is a weaker result than the estimates obtained in the preceding studies. For example, Fouque et al. [4], Fouque et al. [8] estimated the order of the error which depends on the smoothness of the payoff function. Fukasawa [10] proved the stronger result that the error is $O(\epsilon_n^2)$ independently of the smoothness of the payoff. Our present framework is too general to give such a precise estimate. In fact we are assuming only the convergence of semimartingales. To go further, we have to specify the rate of the convergence of, at least, $\epsilon_n^{-1}(\Sigma_n^{-1}\langle M^n\rangle_T - 1)$. An extension of Yoshida's formula to a higher order expansion is left for further research.

Proposition 2.8 *Suppose that Condition 2.1 holds. Assume that there exists a sequence of square-integrable martingales* Q^n *such that*

$$\epsilon_n^{-1}\big(\Sigma_n^{-1}\langle M^n\rangle_T - 1\big) = \int_0^T Q_s^n \mu(ds) + o_p(1)$$

for a deterministic finite measure μ *on* $[0, T]$ *and that*

$$|x|^2 1_{\{|x|>a\}} * \nu_t^n \to 0$$

in probability for all $t \in [0, T]$, $a > 0$, *where* ν^n *is the predictable compensator of the random measure associated to the jumps of* Q^n. *If the predictable covariation of the sequence of martingales*

$$\big(M^n,\ \epsilon_n^{-1}\big([M^n] - \langle M^n\rangle\big),\ Q^n\big)$$

converges in probability to a deterministic continuous process, then Condition 2.2 holds with (N_1, N_2, N_3) *being normal and* $N_4 = E[N_1 N_2]$.

Proof Use Jacod and Shiryaev [14], VIII.3.24 and VI.1.17. □

3 More specific models

3.1 Pure jump effect

Let us analyze the effect of jumps by considering a simple model of deterministic volatility. Suppose that

$$R(T) = rT, \qquad X^n \equiv 0, \qquad g_s^n \equiv g > 0, \qquad h_s^n \equiv 1, \qquad \Lambda_s^n = \epsilon_n^{-2}\lambda s$$

for constants $r, g, \lambda > 0$. Then we have a simple jump-diffusion model

$$Z_T = rT - \frac{g^2 T}{2} - \epsilon_n^{-2}\lambda T \int \left(e^{\epsilon_n z} - 1\right)\nu(dz) + gW_T + \epsilon_n \sum_{j=1}^{N_T^n} U_j,$$

where N^n is a Poisson process with $E[N_t^n] = \epsilon_n^{-2}\lambda t$. As before, ν is the law of the i.i.d. random variables U_j. The larger n, the more and the smaller the jumps. We are considering the case that Z_T converges weakly to a normal distribution by the central limit theorem. In other words, we are trying to approximate the compound Poisson process by a Brownian motion. Putting

$$\Sigma_n = \Sigma = g^2 T + \nu_2 \lambda T, \qquad \alpha = \nu_3 \lambda T \Sigma^{-3/2},$$

we have

$$\Sigma_n^{-1} \langle M^n \rangle_T = \Sigma^{-1}\left(g^2 T + \nu_2 \lambda T\right) = 1$$

and

$$\Sigma_n^{-3/2} \langle M^n, \epsilon_n^{-1}\left([M^n] - \langle M^n \rangle\right)\rangle_T = \Sigma^{-3/2}\nu_3 \lambda T = \alpha.$$

In particular, Condition 2.1 is satisfied. It is also easy to see that the predictable quadratic covariation of $(M^n, \epsilon_n^{-1}([M^n] - \langle M^n \rangle))$ is deterministic and independent of n. Hence we can apply Theorem 2.5 and Proposition 2.8 with $Q^n \equiv 0$ to have (2.2) with

$$\phi_n(z) = \phi(z)\left\{1 + \frac{\epsilon_n \alpha}{6}\left\{\left(z^3 - 3z\right) - \sigma^2 T z\right\}\right\}$$

with $\sigma = \sqrt{g^2 + \nu_2 \lambda}$. By Corollary 2.6, the Black–Scholes implied volatility IV at time 0 is expanded as

$$\mathrm{IV} = \sigma\left\{1 - \frac{\epsilon_n \alpha}{6}(d_2 - \sigma\sqrt{T})\right\} + o(\epsilon_n) = a\frac{\log(K/S)}{T} + b + o(\epsilon_n),$$

where

$$a = \frac{\nu_3 \lambda}{6\sigma^3}\epsilon_n, \qquad b = \sigma - \frac{\nu_3\lambda}{6\sigma^3}\left(r - \frac{3}{2}\sigma^2\right)\epsilon_n.$$

Notice that the last form of the expansion is exactly the same as in the fast mean reverting model (1.2). Here, the volatility skew results not from the stochastic volatility, but the non-zero skewness $\nu_3\lambda$ of the marginal distribution of the compound Poisson process. The slope of the volatility skew is negative if $\nu_3 < 0$, which represents a distribution with longer left tail.

3.2 The leverage effect with jumps

Extending the preceding example, consider

$$R(T) = rT, \qquad X^n \equiv 0, \qquad h^n_s \equiv 1, \qquad \Lambda^n_s = \epsilon_n^{-2}\lambda s, \qquad g^n_s = g\left(Y^n_s\right)$$

with

$$Y^n_s = y + \epsilon_n c_1 W'_s + c_2\left\{\epsilon_n^2 \sum_{j=1}^{N^n_s} U_j - \nu_1 \lambda s\right\}$$

for a Borel function g and constants y, c_1, c_2, where W' is a standard Brownian motion independent of W. Note that Y^n is recurrent and has stationary increments. If ν_1, c_2 are negative and g is increasing, the spot volatility g^n jumps up when the asset price drops. This behavior is consistent with observations from recent stock markets. Assume that the function g has bounded derivatives up to order 2 and is positive, bounded away from 0. Condition 2.1 is then satisfied. Putting

$$\Sigma_n = \Sigma = g(y)^2 T + \nu_2 \lambda T, \quad \alpha = \nu_3 \lambda T \Sigma^{-3/2},$$

we have

$$\begin{aligned}
\Sigma_n^{-1}\langle M^n\rangle_T - 1 &= \Sigma^{-1}\int_0^T \left(g\left(Y^n_s\right)^2 - g(y)^2\right) ds \\
&= \Sigma^{-1} 2 g(y) g'(y) \int_0^T \left(Y^n_s - y\right) ds + o_p(\epsilon_n),
\end{aligned}$$

$$\Sigma_n^{-3/2}\left\langle M^n, \epsilon_n^{-1}\left(\left[M^n\right] - \langle M^n\rangle\right)\right\rangle_T = \Sigma^{-3/2}\nu_3\lambda T = \alpha.$$

Note that the predictable covariation of the martingale

$$\left(M^n, \epsilon_n^{-1}\left(\left[M^n\right] - \langle M^n\rangle\right), \epsilon_n^{-1}\left(Y^n - y\right)\right)$$

converges in probability to a deterministic continuous process. Hence we can apply Theorem 2.5 and Proposition 2.8 with $Q^n = \epsilon_n^{-1}(Y^n - y)$. The asymptotic distribution (N_1, N_2, N_3) is normal with

$$\delta_3 = E[N_3] = 0, \qquad \rho_{12} = E[N_1 N_2] = \alpha,$$

$$\rho_{13} = E[N_1 N_3] = g(y) g'(y) c_2 \nu_2 \lambda T^2 \Sigma^{-3/2}.$$

By Corollary 2.6, we have

$$\begin{aligned}\mathrm{IV} &= \sigma\left\{1-\frac{\epsilon_n}{2}\left\{\rho_{13}d_2+\frac{1}{3}\alpha(d_2-\sigma\sqrt{T})\right\}\right\}+o(\epsilon_n)\\ &= \left(\gamma_1+\frac{\gamma_2}{T}\right)\log(K/S)+\sigma-r(T\gamma_1+\gamma_2)+\frac{\sigma^2}{2}(T\gamma_1+3\gamma_2)+o(\epsilon_n),\end{aligned}$$

where

$$\sigma=\sqrt{g(y)^2+\nu_2\lambda},\qquad \gamma_1=\frac{g(y)g'(y)c_2\nu_2\lambda}{2\sigma^3}\epsilon_n,\qquad \gamma_2=\frac{\nu_3\lambda}{6\sigma^3}\epsilon_n.$$

The above expansion formula with $\sigma, \gamma_1, \gamma_2$ is almost of the same form as what was obtained in Fouque et al. [5], while they derived it from a combination of regular and singular perturbation techniques for a continuous stochastic volatility model with multiscale parameters. They also showed that the formula reproduces well the term structure of the implied volatility skew in market. Here, the roles of fast and slow scales are played respectively by the pure jump martingale with high intensity and the correlated factor of the volatility.

3.3 Fractional Brownian motion

This example is motivated by a result of Alós et al. [1]. Consider a simple stochastic volatility model

$$Z_t=rt-\frac{1}{2}\int_0^t g\big(Y^n_s\big)^2\,ds+\int_0^t g\big(Y^n_s\big)\big[\theta\,dW'_s+\sqrt{1-\theta^2}\,dW_s\big] \tag{3.1}$$

with

$$Y^n_s=y+\epsilon_n W^H_s,\qquad W^H_t=\int_0^t K_H(t,s)\,dW'_s,$$

where $\theta\in(-1,1)$, $y\in\mathbb{R}$ are constants, (W,W') is a 2-dimensional Brownian motion and

$$\begin{aligned}K_H(t,s)=c_H\bigg[\left(\frac{t}{s}\right)^{H-1/2}&(t-s)^{H-1/2}-(H-1/2)s^{1/2-H}\\ &\times\int_s^t u^{H-3/2}(u-s)^{H-1/2}\,du\bigg]\end{aligned}$$

with $H\in(0,1/2)$,

$$c_H=\sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+1/2)}}.$$

Note that W^H is a fractional Brownian motion, so that it has stationary increments. See Nualart [21], Sect. 5.1, for details. Notice that

$$\left(\int_0^T W_t^H \, dt, W_T'\right)$$

has a normal distribution with covariance $c_H' T^{H+3/2}$, where

$$c_H' = \frac{c_H \beta(3/2 - H, H + 1/2)}{(H + 3/2)(H + 1/2)}.$$

We assume that g has bounded derivatives up to order 2 and is positive, bounded away from 0. Conditions 2.1, 2.2 are then easily verified with

$$\Sigma_n = \Sigma = g(y)^2 T \tag{3.2}$$

and the asymptotic distribution (N_1, N_2, N_3, N_4) is normal with $N_2 = N_4 = 0$,

$$\delta_3 = E[N_3] = 0, \qquad \rho_{13} = E[N_1 N_3] = 2\theta g'(y) c_H' T^H / g(y). \tag{3.3}$$

By Corollary 2.6, we have

$$\mathrm{IV} = \sigma \left\{1 - \frac{\epsilon_n}{2} \rho_{13} d_2\right\} + o(\epsilon_n) = a T^{H-1/2} \log(K/S) + \sigma + b T^{H+1/2} + o(\epsilon_n), \tag{3.4}$$

where

$$\sigma = g(y), \qquad a = \frac{\theta g'(y) c_H'}{\sigma} \epsilon_n, \qquad b = -a\left(r - \frac{\sigma^2}{2}\right).$$

Thus, we obtain a stochastic volatility model with term structure of $O((T-t)^{-q})$, $0 < q < 1/2$ for the slope of the volatility skew. See Fouque et al. [6] for an empirical study reporting that $q \approx 1/2$ is suitable.

For $H > 1/2$, we replace K_H, c_H with

$$c_H = \left\{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}\right\}^{1/2},$$

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} \, du.$$

Then W_H is a fractional Brownian motion with long memory. See Nualart [21], Sect. 5.1, for details. Conditions 2.1, 2.2 are again easily verified with (3.2) and the asymptotic distribution (N_1, N_2, N_3) is normal with $N_2 = 0$ and (3.3), where c_H' is replaced with

$$c_H' = \frac{c_H \beta(3/2 - H, H - 1/2)}{(H + 3/2)(H + 1/2)}.$$

Thus, (3.4) remains valid for $H \in (1/2, 1)$.

3.4 Regular perturbation expansion: slowly varying and small vol-of-vol models

Here, we prove the validity of regular perturbation expansion formulas, which were derived from a regular perturbation argument of partial differential equations. Consider (3.1) with Y^n satisfying the stochastic differential equation

$$dY_s^n = \big(b_0\big(Y_s^n\big) + \epsilon_n b_1\big(Y_s^n\big) + \epsilon_n^2 b_2\big(Y_s^n\big)\big)\,ds + \epsilon_n c\big(Y_s^n\big)\,dW_s', \qquad Y_0^n = y.$$

For brevity, we assume again that g has bounded derivatives up to order 2 and is positive, bounded away from 0. Let y_t be the solution of the ordinary differential equation

$$\frac{dy_t}{dt} = b_0(y_t), \qquad y_0 = y.$$

We also assume that the coefficients b_0, b_1, b_2, c are so smooth that for the continuous processes

$$D_t^n = \epsilon_n^{-1}\big(Y_t^n - y_t\big),$$

the sequence of random variables

$$\int_0^T \big|D_t^n\big|^2\,dt$$

is uniformly bounded in L^p for any $p > 0$ and D^n converges in probability to the strong solution D of the stochastic differential equation

$$dD_t = \big(b_0'(y_t)D_t + b_1(y_t)\big)\,dt + c(y_t)\,dW_t', \qquad D_0 = 0.$$

It is easy to see that

$$D_t = \int_0^t \exp\left\{\int_s^t b_0'(y_u)\,du\right\}\big[b_1(y_s)\,ds + c(y_s)\,dW_s'\big],$$

so that D is a Gaussian process. Conditions 2.1, 2.2 are then satisfied with

$$\Sigma_n = \Sigma = \int_0^T g(y_t)^2\,dt.$$

We have $N_2 = N_4 = 0$ and

$$N_1 = \Sigma^{-1/2}\int_0^T g(y_t)\big[\theta\,dW_t' + \sqrt{1-\theta^2}\,dW_t\big],$$

$$N_3 = 2\Sigma^{-1}\int_0^T g(y_t)g'(y_t)D_t\,dt,$$

which is normal with

$$\delta_3 = E[N_3] = 2\Sigma_n^{-1} \int_0^T g(y_t) g'(y_t) \int_0^t \exp\left\{\int_s^t b_0'(y_u)\,du\right\} b_1(y_s)\,ds\,dt,$$

$$\begin{aligned}\rho_{13} &= E[N_1 N_3]\\ &= 2\Sigma_n^{-3/2}\theta \int_0^T g(y_t) g'(y_t) \int_0^t \exp\left\{\int_s^t b_0'(y_u)\,du\right\} c(y_s) g(y_s)\,ds\,dt.\end{aligned}$$

If $b_0 \equiv 0$, then $y_t \equiv y$, so that

$$\delta_3 = \frac{g'(y) b_1(y)}{g(y)} T, \qquad \rho_{13} = \frac{\theta g'(y) c(y)}{g(y)} \sqrt{T}.$$

In this case, by Corollary 2.6, the implied volatility is expanded as

$$\mathrm{IV} = a \log(K/S) + \sigma + bT + o(\epsilon_n),$$

where

$$\sigma = g(y), \qquad a = \frac{\theta g'(y) c(y)}{2\sigma} \epsilon_n, \qquad b = -a\left(r - \frac{\sigma^2}{2}\right) + \frac{g'(y) b_1(y)}{2\sigma} \epsilon_n.$$

See Lee [16] and Lewis [17] for earlier studies on this expansion.

3.5 Singular perturbation expansion: fast mean reverting model

Here, we prove the validity of a singular perturbation expansion. We treat a general multi-dimensional ergodic diffusion. Let $\bar{W} = (\bar{W}^1, \dots, \bar{W}^d)$ be a d-dimensional standard Brownian motion and consider the stochastic differential equation

$$dY_t^n = \left(\epsilon_n^{-2} b\left(Y_t^n\right) + \epsilon_n^{-1} b_1\left(Y_t^n\right)\right) dt + \epsilon_n^{-1} c\left(Y_t^n\right) d\bar{W}_t, \qquad Y_0^n = y,$$

where $b = (b^j)$, $b_1 = (b_1^j) : \mathbb{R}^m \to \mathbb{R}^m$, $c = (c_i^j) : \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^d$. We assume that the stochastic differential equation

$$d\hat{Y}_t^n = \left(b\left(\hat{Y}_t^n\right) + \epsilon_n b_1\left(\hat{Y}_t^n\right)\right) dt + c\left(\hat{Y}_t^n\right) d\hat{W}_t, \qquad \hat{Y}_0^n = y \tag{3.5}$$

has a unique weak solution $\hat{Y}^n$ and that $\hat{Y}^n$ is ergodic for each n. Denote by π^n the ergodic distribution. The case where $\hat{Y}^n$ is a perturbed OU process was treated by Fouque et al. [3]. Let $\psi = (\psi_1, \dots, \psi_m) : \mathbb{R}^m \to \mathbb{R}^m$ be a Borel function and put

$$\psi_0 = \sum_{i,j=1}^m \psi_i \psi_j a^{ij}, \quad a^{ij} = \sum_{k=1}^d c_k^i c_k^j.$$

Let us set

$$R(T) = rT, \qquad dX_s^n = \theta \psi\left(Y_s^n\right) c\left(Y_s^n\right) d\bar{W}_s = \theta \sum_{k,j} \psi_k\left(Y_s^n\right) c_j^k\left(Y_s^n\right) d\bar{W}_s^j,$$

$$g_s^n = \sqrt{\left(1-\theta^2\right)\psi_0\left(Y_s^n\right)}, \qquad h_s^n \equiv 0,$$

where $\theta \in (-1, 1)$. Here, we define g^n so that

$$\langle M^n\rangle_T = \int_0^T \psi_0(Y_s^n)\,ds, \quad M^n = X^n + \int_0^\cdot g_s^n\,dW_s.$$

In this model, Y^n determines the volatility of the price and the price movement has a component which is driven by Y^n. Observe that $\hat{Y}_s^n = Y^n_{\epsilon_n^2 s}$ is a strong solution of (3.5) with respect to the standard Brownian motion $\hat{W}_s = \epsilon_n^{-1}\bar{W}_{\epsilon_n^2 s}$. Then we have

$$M_T^n = \epsilon_n\theta \int_0^{T/\epsilon_n^2} \psi(\hat{Y}_t^n)c(\hat{Y}_t^n)\,d\hat{W}_t + \epsilon_n \int_0^{T/\epsilon_n^2} \sqrt{(1-\theta^2)\psi_0(\hat{Y}_t^n)}\,d\hat{W}_t',$$

where $\hat{W}_s' = \epsilon_n^{-1} W_{\epsilon_n^2 s}$. Let $\mathcal{L}^n$ be the generator of (3.5), i.e.,

$$\mathcal{L}^n = \sum_{j=1}^m (b^j + \epsilon_n b_1^j)\partial_j + \frac{1}{2}\sum_{i,j=1}^m a^{ij}\partial_i\partial_j,$$

and suppose that the Poisson equation

$$\mathcal{L}^n F^n = \psi_0 - \pi^n[\psi_0]$$

has a smooth solution F^n on $\mathbb{R}^m$ for each $n \in \mathbb{N}$, where $\pi^n[\psi_0]$ is the integral of ψ_0 with respect to π^n. Putting $\Sigma_n = T\pi^n[\psi_0]$ and $f_k^n = \partial_k F^n$, we have

$$\begin{aligned}\epsilon_n^{-1}\left(\Sigma_n^{-1}\langle M^n\rangle_T - 1\right) &= \frac{\epsilon_n}{T\pi^n[\psi_0]}\left(F^n(Y_T^n) - F^n(Y_0)\right)\\ &\quad - \frac{\epsilon_n}{T\pi^n[\psi_0]}\sum_{k,j}\int_0^{T/\epsilon_n^2} f_k^n(\hat{Y}_s^n)c_j^k(\hat{Y}_s^n)\,d\hat{W}_s^j\end{aligned}$$

by Itô's formula. Under Condition 2.1 and suitable conditions on convergence and integrability of

$$\epsilon_n^2 \int_0^{T/\epsilon_n^2} f_i^n(\hat{Y}_s^n) f_j^n(\hat{Y}_s^n) a^{ij}(\hat{Y}_s^n)\,ds,$$

we apply Proposition 2.8 with

$$Q^n = -\frac{\epsilon_n}{T\pi^n[\psi_0]}\sum_{k,j}\int_0^{\cdot/\epsilon_n^2} f_k^n(\hat{Y}_s^n)c_j^k(\hat{Y}_s^n)\,d\hat{W}_s^j$$

and μ being the delta measure at T to have Condition 2.2. Then we apply Theorem 2.5 with

$$\begin{aligned}&N_2 = N_4 = 0, \qquad \delta_3 = E[N_3] = 0,\\ &\rho_{13} = E[N_1N_3] = -\frac{\theta}{\sqrt{T}\pi[\psi_0]^{3/2}}\sum_{i,j}\pi\left[f_i\psi_j a^{ij}\right],\end{aligned}$$

where π is the ergodic distribution of a diffusion Y which satisfies (3.5) with ϵ_n replaced by 0. Similarly, f is the derivative of the solution F of the Poisson equation $\mathcal{L}F = \psi_0 - \pi[\psi_0]$ with respect to the generator $\mathcal{L}$ of Y. Here, we have assumed that

$$\pi^n[\psi_0] = \pi[\psi_0] + o(1), \qquad \pi^n\left[f_i^n \psi_j a^{ij}\right] = \pi\left[f_i \psi_j a^{ij}\right] + o(1).$$

Hence the asymptotic expansion of European option prices does not depend on the initial value y of Y^n, that is, the spot volatility up to $o(\epsilon_n)$. The implied volatility is expanded as (1.2). Notice that f appears in the coefficients of the expansion, which might cause a problem in practice because no analytic formula is available in general for the solution of the Poisson equation except in the one-dimensional case. However, if Y is a symmetric diffusion and there exists a function $\Psi : \mathbb{R}^m \to \mathbb{R}$ such that $\partial_j \Psi = \psi_j$, we have

$$\sum_{i,j} \pi\left[f_i \psi_j a^{ij}\right] = \sum_{i,j} \pi\left[\partial_i F \partial_j \Psi a^{ij}\right] = -2\pi[\mathcal{L}F\Psi] = 2\pi\left[\left(\pi[\psi_0] - \psi_0\right)\Psi\right]$$

by the integration-by-parts formula or, equivalently, a well-known identity for the Dirichlet form. See Fukasawa [10] for a more detailed analysis based on the Edgeworth expansion in the one-dimensional case.

4 Concluding remarks

We have seen that Yoshida's formula allows us to prove the validity of various kinds of asymptotic expansions around the Black–Scholes model. We have not needed any smoothness assumption for the payoff function h. The leading term of the asymptotic expansion of the Black–Scholes implied volatility is always an affine function of log moneyness, while the term structure of the coefficients depends on the underlying asymptotic model. We have studied several specific models which represent various kinds of term structures. Although we concentrate on the case where the asset price process is one-dimensional, an extension to the multidimensional case is straightforward since Yoshida's formula is itself for multidimensional martingales. Besides, the multidimensional formula allows us to incorporate a stochastic interest rate into the stochastic volatility model. For example, we can treat a stochastic cumulative risk-free rate process R^n such that

$$R^n_T - R(T) = \epsilon_n M^R_T$$

with a martingale M^R. By Yoshida's formula, we can see that the stochastic interest rate induces a volatility level correction which does not depend on the strike price K, but the maturity T of the put options up to $o(\epsilon_n)$.

As mentioned in Remark 2.3, we assumed the nondegeneracy of g^n as a sufficient condition for the smoothness of the distribution. This simple condition does not serve any more when considering local volatility models. Nevertheless, by estimating the Malliavin covariance of the log price in a suitable manner, it will be possible to validate by Yoshida's formula an asymptotic expansion for, e.g., the CEV model

$$dS_t = rS_t\,dt + \sigma S_t^{1-\epsilon_n}\,dW_t$$

with an artificial parameter ϵ_n. Letting $\epsilon_n \to 0$, this can also be seen as a perturbation model around the Black–Scholes model. A formal calculation gives that the corresponding expansion of the implied volatility should be of the same form as in the regular perturbation case (1.3). It remains, however, for further research to validate an expansion for path-dependent option prices as well as for American option prices.

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