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Factorization theorems and the structure of operators on Hilbert space

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1. Introduction

The purpose of this paper is to settle Conjecture 2.14 of [2]. We begin by recalling some definitions necessary for the statement of our result.

Let $\mathscr{H}$ be a complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators acting on $\mathscr{H}$. The scalar product of two vectors $x, y \in \mathscr{H}$ will be denoted $\langle x, y \rangle$. We recall that the ultraweak topology on $\mathscr{L}(\mathscr{H})$ is the weak* topology generated by the duality between $\mathscr{L}(\mathscr{H})$ and the space of trace-class operators. A linear subspace $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is said to have property $(\mathbb{A}_1)$ (or D_σ in the terminology of [22]) provided that for every weak*-continuous functional φ on $\mathscr{M}$ there exist $x, y \in \mathscr{H}$ such that

$$\varphi(A) = \langle Ax, y \rangle \tag{1.1}$$

for all A in $\mathscr{M}$; relation (1.1) will be indicated symbolically as $\varphi = [x \otimes y]$. The subspace $\mathscr{M}$ has property $(\mathbb{A}_1(r))$, where $r \geq 1$, if given φ weak* continuous and $s > r$, there exist x, y satisfying (1.1) and the inequality $\|x\|\,\|y\| \leq s\|\varphi\|$.

Assume now that $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a subalgebra which is isometrically isomorphic and weak* homeomorphic with the algebra H^∞ of bounded analytic functions in the unit disc; in other words, assume that there exists an isometric isomorphism $\phi\colon H^\infty \to \mathscr{A}$ which is also a homeomorphism in the corresponding weak* topologies. It was conjectured in [2] that $\mathscr{A}$ must have property $(\mathbb{A}_1)$. The main result of this paper is as follows.

1.2 Theorem. *$\mathscr{A}$ has property* $(\mathbb{A}_1(1))$.

Property $(\mathbb{A}_1)$ is known to have consequences related to invariant subspaces and dilation theory; [8] contains an account of recent related developments. In particular the above theorem implies (and provides a new proof of) the Brown, Chevreau, Pearcy theorem from [14]: Every contraction T on a Hilbert space, whose spectrum contains the unit circle, has nontrivial invariant subspaces.

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In order to see how remarkable property $(\mathbb{A}_1)$ is, let us note that every weak*-continuous functional φ on $\mathscr{A}$ can be extended to a weak*-continuous functional on $\mathscr{L}(\mathscr{H})$. Therefore one can find sequences $\{x_n\}$ and $\{y_n\}$ in $\mathscr{H}$ such that $\sum_{n=1}^{\infty}\|x_n\|\,\|y_n\| < \infty$ and

$$\varphi = \sum_{n=1}^{\infty} [x_n \otimes y_n]. \tag{1.3}$$

Property $(\mathbb{A}_1)$ says that one can collapse this sum to a single term. This is not possible for very large algebras such as $\mathscr{L}(\mathscr{H})$, and in fact it is not possible for certain singly generated weak*-closed algebras (cf. [22] and [30]). Particular cases of Theorem 1.2 have been known for some time. In order to discuss the history of the subject, we denote by T the operator in $\mathscr{A}$ that corresponds (via the isomorphism with H^∞) to the identity function of the unit disc. Brown, Chevreau and Pearcy [13] proved the conjecture in case $\{T^n\}$ converges strongly to zero and the left essential spectrum of T is dominating for the unit disc. Apostol [1] considered the case of operators such that $\{T^n\}$ and $\{T^{*n}\}$ converge strongly to zero and the essential resolvent of T grows fast enough inside the unit circle. Bercovici, Foias and Pearcy [6] settled the case in which the essential resolvent grows fast enough, without any assumptions on the powers of T; see also [9], [10], and Robel [24] for the case of a dominating essential spectrum. An important development is due to Sheung [26], who imported certain techniques from subnormal operators to treat some operators with dominating spectrum (rather than essential spectrum). Exner [20] considered normal operators with dominating spectrum. Westwood [29] settled the case of operators with dominating spectrum such that $\{T^n\}$ and $\{T^{*n}\}$ converge strongly to zero. Some finer spectral conditions on the line of dominating spectra were obtained by Chevreau and Pearcy [18] and [19]. An important new idea was given in the work of Brown [12], who settled the case of operators such that $\{T^n\}$ and $\{T^{*n}\}$ converge strongly to zero and the resolvent of T grows fast enough. The condition on the powers of T was removed by Brown, Chevreau and Pearcy [14]. See also Prunaru [23] for the case of dominating spectra. Brown's technique was adapted by the author [4] to prove the conjecture in case either $\{T^n\}$ or $\{T^{*n}\}$ converges strongly to zero, with no additional spectral conditions on T. Chevreau [16] used the methods of [4] to show that $\mathscr{A}$ has a weaker property $(\mathbb{A}_{1/2})$, and hence $\mathscr{A}$ is weakly closed. Finally, in this paper we remove the condition on the powers of T.

While this paper was being written we learned from Professor Carl Pearcy that Chevreau proved that $\mathscr{A}$ has property $(\mathbb{A}_1(r))$ for some universal constant r.

I wish to thank the referee for his thorough work, which helped improve the exposition of this paper.

2. Preliminaries

We begin by reformulating the main result in function theoretic terms. Let $\mathbb{T} = \{\zeta \in \mathbf{C}: |\zeta| = 1\} = \{e^{it}: 0 \leq t < 2\pi\}$ denote the unit circle in the complex plane. On $\mathbb{T}$ we consider normalized arclength measure $dt/2\pi$. The spaces L^p are to be understood as $L^p(\mathbb{T}, dt/2\pi)$. We recall that a function $f \in L^1$ is uniquely determined by its Fourier coefficients

$$\hat{f}(n) = \int_{\mathbb{T}} f e_{-n}, \qquad n \in \mathbf{Z},$$

where $e_n(\zeta) = \zeta^n$, $\zeta \in \mathbb{T}$. The space $H_0^1 \subset L^1$ consists of those functions $f \in L^1$ for which $\hat{f}(n) = 0$ when $n \leq 0$. The class of a function $f \in L^1$ in the quotient L^1/H_0^1 is denoted $[f]$. Consider next a separable, complex Hilbert space $\mathscr{D}$, and the Hilbert space $L^2(\mathscr{D})$ of all (classes of) measurable, square integrable functions $x: \mathbb{T} \to \mathscr{D}$. If $x, y \in L^2(\mathscr{D})$ we can define a function $x \cdot y \in L^1$ by setting

$$(x \cdot y)(\zeta) = \langle x(\zeta), y(\zeta) \rangle$$

for almost every $\zeta \in \mathbb{T}$.

On $L^2(\mathscr{D})$ consider the unitary operator U of multiplication by the independent variable: $(Ux)(\zeta) = \zeta x(\zeta)$, $x \in L^2(\mathscr{D})$, almost everywhere. Finally, let $\mathscr{H} \subset L^2(\mathscr{D})$ be a semi-invariant subspace for U, and let T denote the compression of U to $\mathscr{H}$. The fact that $\mathscr{H}$ is semi-invariant means (by definition) that

$$T^n = P_{\mathscr{H}} U^n | \mathscr{H}, \qquad n = 0, 1, 2, \ldots .$$

Assume that T generates a weak*-closed algebra $\mathscr{A}$ that is isometrically isomorphic and weak* homeomorphic to H^∞. Then (as proved in [4]; cf. Corollary 10) $\mathscr{H}$ has the following property:

2.1 *Property.* For every subset $\sigma \subset \mathbb{T}$ with positive measure $|\sigma|$, every $\varepsilon > 0$, and every finite set $\{\xi_1, \xi_2, \ldots, \xi_p\} \subset \mathscr{H}$, there exists $x \in \mathscr{H}$, $x \neq 0$, such that

(i) x is essentially bounded;
(ii) $\langle x, \xi_j \rangle = 0$, $1 \leq j \leq p$; and
(iii) $\|\chi_{\mathbb{T}\setminus\sigma} x\| < \varepsilon \|\chi_\sigma x\|$.

Moreover, every algebra $\mathscr{A}$ which is isometrically isomorphic and weak* homeomorphic to H^∞ can be realized, up to a unitary equivalence, as the algebra generated by an operator T obtained as a compression of the operator U on $L^2(\mathscr{D})$. In addition, there is a bijection between weak* continuous functionals φ on $\mathscr{A}$ and elements ψ in L^1/H_0^1 such that $\|\varphi\| = \|\psi\|$ and, for $x, y \in \mathscr{H}$, we have $[x \otimes y] = \varphi$ if and only if $[x \cdot y] = \psi$. (See [8], Proposition 8.3, for details of this correspondence.) Thus Theorem 1.2 can be reformulated as follows:

2.2 Theorem. *Suppose that $\mathscr{H} \subset L^2(\mathscr{D})$ is semi-invariant and has Property 2.1. Then for every $\varepsilon > 0$ and every $\psi \in L^1/H_0^1$ there exist vectors $x, y \in \mathscr{H}$ such that $[x \cdot y] = \psi$ and $\|x\|\,\|y\| \leq (1 + \varepsilon)\|\psi\|$.*

Note that once the equation $[x \cdot y] = \psi$ is solved, one can easily obtain a solution such that $\|x\| = \|y\|$. In this case the last estimate can be written as $\|x\|, \|y\| \leq (1 + \varepsilon)^{1/2}\|\psi\|^{1/2}$.

We will prove the main result under this function theoretical form. A main ingredient is the following result from [5] (cf. Theorem 10).

2.3 THEOREM. *Suppose that $\mathscr{H} \subset L^2(\mathscr{D})$ has Property 2.1. Given $f \in L^1$, $\varepsilon > 0$, and vectors $\xi_1, \xi_2, \ldots, \xi_p \in L^2(\mathscr{D})$ there exist vectors $x, y \in \mathscr{H}$ such that*

(i) $\|x\| \leq \|f\|_1^{1/2}$, $\|y\| \leq \|f\|_1^{1/2}$;

(ii) $\langle x, \xi_j \rangle = \langle y, \xi_j \rangle = 0$, $1 \leq j \leq p$; *and*

(iii) $\|f - x \cdot y\|_1 < \varepsilon$.

3. Dilations and vanishing lemmas

In this section $\mathscr{H} \subset L^2(\mathscr{D})$ is a fixed semi-invariant subspace for U, and T denotes the compression of U to $\mathscr{H}$. Of course U is not, generally, the minimal unitary dilation of T. Therefore we set $\mathscr{K}_+ = \bigvee_{n=0}^{\infty} U^n\mathscr{H}$ and $\mathscr{K}_- = \bigvee_{n=-\infty}^{0} U^n\mathscr{H}$. Then $\mathscr{K}_+$ is invariant for U, and $\mathscr{K}_-$ is invariant for U^*. We set $U_+ = U|\mathscr{K}_+$ and $U_-^* = U^*|\mathscr{K}_-$; thus U_+ and U_-^* are isometries. An important fact (see Chapter II of [27]) is that $\mathscr{K}_+ \ominus \mathscr{H}$ is invariant for U^*, so that $T = U_-|\mathscr{H}$ and $T^* = U_+^*|\mathscr{H}$. In addition, the spaces $\mathscr{K}_+ \ominus \mathscr{H}$ and $\mathscr{K}_- \ominus \mathscr{H}$ are orthogonal.

3.1 LEMMA. *If $x \in \mathscr{K}_+$ and $y \in \mathscr{K}_-$, then $[x \cdot y] = [P_{\mathscr{H}}x \cdot P_{\mathscr{H}}y]$.*

Proof. We only need to show that $(x \cdot y)\hat{}(k) = (P_{\mathscr{H}}x \cdot P_{\mathscr{H}}y)\hat{}(k)$ for $k \leq 0$ or, equivalently, that

$$\langle U^n x, y \rangle = \langle U^n P_{\mathscr{H}}x, P_{\mathscr{H}}y \rangle \tag{3.2}$$

for $n \geq 0$. Let us write $x = x_1 + x_2$, $y = y_1 + y_2$, with $x_1, y_1 \in \mathscr{H}$, $x_2 \in \mathscr{K}_+ \ominus \mathscr{H}$, and $y_2 \in \mathscr{K}_- \ominus \mathscr{H}$. Then $U^n x_2 \in \mathscr{K}_+ \ominus \mathscr{H}$ for $n \geq 0$, and $\mathscr{K}_+ \ominus \mathscr{H}$ is orthogonal onto $(\mathscr{K}_- \ominus \mathscr{H}) \oplus \mathscr{H}$. Thus $\langle U^n x_2, y_1 + y_2 \rangle = 0$, $n \geq 0$. Also, $U^n x_1 \in \mathscr{K}_+ = \mathscr{H} \oplus (\mathscr{K}_+ \ominus \mathscr{H})$ and $y_2 \in \mathscr{K}_- \ominus \mathscr{H}$ is orthogonal onto $\mathscr{K}_+$, whence $\langle U^n x_1, y_2 \rangle = 0$, $n \geq 0$. The equality (3.2), and hence the lemma, follow at once.

This lemma shows that in proving Theorem 2.2 we may as well prove that we can write $\psi = [x \cdot y]$ with $x \in \mathscr{K}_+$ and $y \in \mathscr{K}_-$. This simple observation greatly simplifies our calculations.

Let us note now that U_+ is an isometry on $\mathscr{K}_+$ and hence it has a Wold decomposition. Thus we can write $\mathscr{K}_+ = \mathscr{M} \oplus \mathscr{R}$, where $\mathscr{M}$ and $\mathscr{R}$ are reducing subspaces for U_+, $U_+|\mathscr{M}$ is a unilateral shift, and $U_+|\mathscr{R}$ is a unitary operator (called the residual part of U_+; see [27], Chapter II, §2). Analogously, we can write $\mathscr{K}_- = \mathscr{M}_* \oplus \mathscr{R}_*$, where $\mathscr{M}_*$ and $\mathscr{R}_*$ are reducing for U_-, $U_-^*|\mathscr{M}_*$ is a unilateral shift and $U_-|\mathscr{R}_*$ is a unitary operator (the *-residual part).

3.3 LEMMA. (i) *If* $\{x_n\} \subset \mathscr{M}$ *is a sequence weakly convergent to zero, and* $y \in \mathscr{K}_-$, *then* $\lim_{n\to\infty}\|[x_n \cdot y]\| = 0$.

(ii) *If* $x \in \mathscr{K}_+$ *and* $\{y_n\} \subset \mathscr{M}_*$ *is a sequence weakly convergent to zero then* $\lim_{n\to\infty}\|[x \cdot y_n]\| = 0$.

Proof. For reasons of symmetry it suffices to prove (i). Thus, let $\{x_n\} \subset \mathscr{M}$ be weakly convergent to zero, and $y \in \mathscr{K}_-$. By Lemma 3.1 it suffices to consider the case in which $y \in \mathscr{H}$. Assume therefore that $y = m + r$ with $m \in \mathscr{M}$ and $r \in \mathscr{R}$. Since clearly

$$[x_n \cdot y] = [x_n \cdot m] + [x_n \cdot r] = [x_n \cdot m],$$

it suffices to prove the lemma for $y = m \in \mathscr{M}$. Furthermore, since $\|[x_n \cdot m]\| \le \|x_n\|\,\|m\|$ and $\{\|x_n\|\}$ is bounded, it suffices to consider a total set of elements $m \in \mathscr{M}$. Such a set is $\bigcup_{N=1}^{\infty}\ker(U_+^{*N})$ (recall that $U_+|\mathscr{M}$ is a unilateral shift). Now, if $U_+^{*N}m = 0$, then

$$(x_n \cdot m)\hat{\ }(-k) = \langle U^k x_n, m\rangle = \langle U_+^k x_n, m\rangle = \langle x_n, U_+^{*k}m\rangle = 0$$

for $k \ge N$, and hence

$$\|[x_n \cdot m]\| \le \sum_{k=0}^{N-1} |\langle U^k x_n, m\rangle|.$$

The relation $\lim_{n\to\infty}\|[x_n \cdot m]\| = 0$ follows at once for $m \in \ker(U_+^{*N})$, and the lemma is proved.

4. Some approximation results

The ideas in the following approximation arguments are contained essentially in [19]. Unfortunately the modifications needed are fairly substantial and therefore we include complete proofs. In the following result $L^1(\omega)$ is identified with the set of those functions in L^1 such that $\chi_{\mathbb{T}\setminus\omega}f = 0$ almost everywhere.

4.1 LEMMA. *Let V be an absolutely continuous unitary operator on a space* $\mathscr{N}$, *and let* $\omega \subset \mathbb{T}$ *be a Borel set such that Lebesgue measure on* ω *is a scalar spectral measure for V. Assume furthermore that* $\mathscr{L} \subset \mathscr{N}$ *is invariant for V and* $\bigvee_{n=-\infty}^{0} V^n\mathscr{L} = \mathscr{N}$. *Then for every* $\delta > 0$ *and every function* $f \in L^1(\omega)$, *such that* $f \ge 0$ *almost everywhere, there exists* $x \in \mathscr{L}$ *such that* $\|f - x \cdot x\|_1 < \delta$.

Proof. There are two cases: either $\mathscr{L} = \mathscr{N}$ or $\mathscr{L} \ne \mathscr{N}$. If $\mathscr{L} = \mathscr{N}$ then for every $f \in L^1(\omega)$, $f \ge 0$, we can find $x \in \mathscr{L}$ such that $f = x \cdot x$. If $\mathscr{L} \ne \mathscr{N}$ then $V|\mathscr{L}$ has a unilateral shift as a direct summand, and in this case we must have

$|\mathbb{T} \setminus \omega| = 0$. The function $(\delta/2 + f)^{1/2}$ is the absolute value of an outer function in H^2, and it follows immediately that there exists $x \in \mathscr{L}$ such that $\delta/2 + f = x \cdot x$. Clearly $\|f - x \cdot x\|_1 < \delta$, as desired.

We will also need the following lemma, whose proof is dual to the one above.

4.2 LEMMA. *Let V be an absolutely continuous unitary operator on a space $\mathscr{N}$, and let $\omega \subset \mathbb{T}$ be a Borel set such that Lebesgue measure on ω is a scalar spectral measure for V. Assume furthermore that $\mathscr{L} \subset \mathscr{N}$ is invariant for V^*, and $\bigvee_{n=0}^{\infty} V^n \mathscr{L} = \mathscr{N}$. Then for every $\delta > 0$ and every function $f \in L^1(\omega)$, such that $f \geq 0$ almost everywhere, there exists $x \in \mathscr{L}$ such that $\|f - x \cdot x\|_1 < \delta$.*

Let U, $\mathscr{H}$, $\mathscr{K}_+$, $\mathscr{K}_-$, $\mathscr{M}$, $\mathscr{M}_*$, $\mathscr{R}$, and $\mathscr{R}_*$ be as in Section 3. There are Borel subsets Δ and Δ_* of $\mathbb{T}$ (possibly empty) such that the spectral measures of $U|\mathscr{R}$ and $U|\mathscr{R}_*$ are equivalent to Lebesgue measure on Δ and Δ_*, respectively. Note that functions in $\mathscr{R}$ [resp., $\mathscr{R}_*$] are zero almost everywhere on $\mathbb{T} \setminus \Delta$ (resp., $\mathbb{T} \setminus \Delta_*$).

Lemmas 4.1 and 4.2 hold even if $\mathscr{L}$ is merely assumed to be a linear manifold. The reason is that the set $\{x \cdot x : x \in \mathscr{L}\}$ is dense in $\{x \cdot x : x \in \mathscr{L}^-\}$. This justifies the following proof since $P_{\mathscr{R}}\mathscr{H}$ might not be closed.

4.3 COROLLARY. (i) *Let $\delta > 0$ and $f \in L^1(\Delta)$ be such that $f \geq 0$ almost everywhere. There exists $z \in \mathscr{H}$ such that $\|f - P_{\mathscr{R}}z \cdot P_{\mathscr{R}}z\|_1 < \delta$.*

(ii) *Let $\delta > 0$ and $f \in L^1(\Delta_*)$ be such that $f \geq 0$ almost everywhere. There exists $z \in \mathscr{H}$ such that $\|f - P_{\mathscr{R}_*}z \cdot P_{\mathscr{R}_*}z\|_1 < \delta$.*

Proof. For reasons of symmetry we only prove (i). Lemma 4.2, with ω, $\mathscr{N}$, $\mathscr{L}$ and V replaced by Δ, $\mathscr{R}$, $P_{\mathscr{R}}\mathscr{H}$ and $U|\mathscr{R}$, respectively, implies the result immediately. One only has to verify that $P_{\mathscr{R}}\mathscr{H}$ is invariant for $(U|\mathscr{R})^*$ and that $\bigvee_{n=0}^{\infty} U^n P_{\mathscr{R}}\mathscr{H} = \mathscr{R}$, and this is an easy exercise.

The following lemma also has a version for $\mathscr{R}_*$. We leave the statement and proof of this version to the interested reader.

4.4 LEMMA. *Let $z \in \mathscr{H}$, $\delta > 0$, and let $u \in L^\infty$ be a positive function, bounded away from zero. There exists a vector $w \in \mathscr{H}$ such that*

(i) $\|P_{\mathscr{M}}w\| < \delta$; *and*

(ii) $\|(P_{\mathscr{R}}w)(\zeta)\| = u(\zeta)\|(P_{\mathscr{R}}z)(\zeta)\|$ *almost everywhere.*

Proof. There is an outer function $\psi \in H^\infty$ such that $|\psi(\zeta)| = u(\zeta)$ almost everywhere. Set

$$y = \psi(U_+)^* z = \psi(T)^* z,$$

and note that

$$(P_{\mathscr{R}}y)(\zeta) = (\psi(U|\mathscr{R})^* P_{\mathscr{R}}z)(\zeta) = \overline{\psi(\zeta)} P_{\mathscr{R}}z(\zeta)$$

almost everywhere. It suffices then to set $w = T^{*n}y = U_+^{*n}y$, where n is chosen such that $\|U_+^{*n}P_{\mathscr{M}}y\| = \|P_{\mathscr{M}}w\| < \delta$.

We are now ready for an approximation procedure very similar to Theorem 3.11 in [19]. We note for further use that for $x, y \in \mathscr{K}$ we have $P_{\mathscr{R}}x \cdot y = x \cdot P_{\mathscr{R}}y = P_{\mathscr{R}}x \cdot P_{\mathscr{R}}y$. This follows from the fact that $\mathscr{R}$ is a reducing space for U, and hence $\langle U^n P_{\mathscr{R}}x, y\rangle = \langle U^n x, P_{\mathscr{R}}y\rangle = \langle U^n P_{\mathscr{R}}x, P_{\mathscr{R}}y\rangle$ for all integers n.

4.5 Proposition. (i) *Let* $\varepsilon > 0$, $x \in \mathscr{K}_+$, $y \in \mathscr{K}_-$, *a measurable set* $\sigma \subset \Delta$, *and* $f \in L^1(\sigma)$ *be given. There exist* $x_1 \in \mathscr{K}_+$ *and* $y_1 \in \mathscr{K}_-$ *with the following properties*:

(a) $\|x \cdot y + f - x_1 \cdot y_1\| < \varepsilon$;
(b) $\|x_1\| \le (1 + \varepsilon)(\|x\| + \|f\|_1^{1/2})$;
(c) $\chi_{\mathbb{T}\setminus\sigma}(x - x_1) = 0$; *and*
(d) $\|y - y_1\| \le 3\|f\|_1^{1/2}$.

(ii) *Let* $\varepsilon > 0$, $x_1 \in \mathscr{K}_+$, $y_1 \in \mathscr{K}_-$, *a measurable set* $\sigma_* \subset \Delta_*$, *and* $g \in L^1(\sigma_*)$ *be given. There exist* $x' \in \mathscr{K}_+$ *and* $y' \in \mathscr{K}_-$ *with the following properties*:

(a_*) $\|x_1 \cdot y_1 + g - x' \cdot y'\|_1 < \varepsilon$;
(b_*) $\|x_1 - x'\| \le 3\|g\|_1^{1/2}$;
(d_*) $\|y'\| \le (1 + \varepsilon)(\|y_1\| + \|g\|_1^{1/2})$; *and*
(e_*) $\chi_{\mathbb{T}\setminus\sigma_*}(y_1 - y') = 0$.

Proof. For reasons of symmetry it suffices to prove (i). We may, and shall, assume without loss of generality that $f \ne 0$. Fix $\delta > 0$ and choose, by virtue of Corollary 4.3(i) a vector $z \in \mathscr{H}$ such that $\||f| - P_{\mathscr{R}}z \cdot P_{\mathscr{R}}z\|_1 < \delta$. Choose also a unimodular function $u \in L^\infty$ such that

$$\|f - uP_{\mathscr{R}}z \cdot P_{\mathscr{R}}z\|_1 < \delta.$$

Define next a subset $\sigma' \subset \sigma$ by

$$\sigma' = \left\{\zeta \in \sigma: \left\|(P_{\mathscr{R}}z)(\zeta)\right\| \ge \left\|(P_{\mathscr{R}}y)(\zeta)\right\|\right\}.$$

Lemma 4.4 implies the existence of $w \in \mathscr{H}$ such that $\|P_{\mathscr{M}}w\| < \delta$, and

$$\begin{aligned}\left\|(P_{\mathscr{R}}w)(\zeta)\right\| &= (2 - \delta)\left\|(R_{\mathscr{R}}z)(\zeta)\right\| \text{ a.e. on } \sigma',\\ &= \delta\left\|(P_{\mathscr{R}}z)(\zeta)\right\| \text{ a.e. on } \mathbb{T} \setminus \sigma'.\end{aligned}$$

Let us estimate for further use $\|w\|$ and $\|\chi_{\pi\setminus\sigma}w\|$. We have

$$\begin{aligned}\|w\| &\leq \|P_{\mathscr{M}}w\| + \|P_{\mathscr{R}}w\| \leq \delta + (2-\delta)\|P_{\mathscr{R}}z\|\\ &\leq \delta + (2-\delta)(\|f\|_1 + \delta)^{1/2},\end{aligned}$$

and since $\sigma' \subset \sigma$,

$$\begin{aligned}\|\chi_{\mathbb{T}\setminus\sigma}w\| &\leq \|P_{\mathscr{M}}w\| + \|\chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}w\| = \|P_{\mathscr{M}}w\| + \delta\|\chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}z\|\\ &< \delta + \delta\big(\|\chi_{\mathbb{T}\setminus\sigma}f\|_1 + \delta\big)^{1/2} = \delta + \delta^{3/2} < 2\delta.\end{aligned}$$

We define $y_1 = y + w$. Before defining x_1 we note that for almost every $\zeta \in \mathbb{T}\setminus\sigma'$ we have

$$\begin{aligned}\|(P_{\mathscr{R}}y_1)(\zeta)\| &= \|(P_{\mathscr{R}}y)(\zeta) + (P_{\mathscr{R}}w)(\zeta)\|\\ &\geq \|(P_{\mathscr{R}}y)(\zeta)\| - \|(P_{\mathscr{R}}w)(\zeta)\|\\ &= \|(P_{\mathscr{R}}y)(\zeta)\| - \delta\|(P_{\mathscr{R}}z)(\zeta)\|\\ &\geq (1-\delta)\|(P_{\mathscr{R}}y)(\zeta)\|,\end{aligned}$$

while for almost every $\zeta \in \sigma'$,

$$\begin{aligned}\|(P_{\mathscr{R}}y_1)(\zeta)\| &\geq \|(P_{\mathscr{R}}w)(\zeta)\| - \|(P_{\mathscr{R}}y)(\zeta)\|\\ &= (2-\delta)\|(P_{\mathscr{R}}z)(\zeta)\| - \|(P_{\mathscr{R}}y)(\zeta)\|\\ &\geq (1-\delta)\|(P_{\mathscr{R}}z)(\zeta)\|.\end{aligned}$$

Therefore

$$\|(P_{\mathscr{R}}y_1)(\zeta)\| \geq (1-\delta)\max\{\|(P_{\mathscr{R}}y)(\zeta)\|, \|(P_{\mathscr{R}}z)(\zeta)\|\}$$

almost everywhere. It follows that we can choose a measurable function $g(\zeta)$ such that

$$g(\zeta)\|(P_{\mathscr{R}}y_1)(\zeta)\|^2 = (P_{\mathscr{R}}x \cdot P_{\mathscr{R}}y)(\zeta) + u(\zeta)(P_{\mathscr{R}}z \cdot P_{\mathscr{R}}z)(\zeta)$$

almost everywhere on σ, and $g(\zeta) = 0$ when $\zeta \notin \sigma$ or $(P_{\mathscr{R}}y_1)(\zeta) = 0$. Moreover, we have

$$\begin{aligned}&|g(\zeta)|\,\|(P_{\mathscr{R}}y_1)(\zeta)\|\\ &\quad\leq \frac{\|(P_{\mathscr{R}}x)(\zeta)\|\|(P_{\mathscr{R}}y)(\zeta)\| + \|(P_{\mathscr{R}}z)(\zeta)\|^2}{\|(P_{\mathscr{R}}y_1)(\zeta)\|}\\ &\quad\leq \frac{\big(\|(P_{\mathscr{R}}x)(\zeta)\| + \|(P_{\mathscr{R}}z)(\zeta)\|\big)\max\{\|(P_{\mathscr{R}}y)(\zeta)\|, \|(P_{\mathscr{R}}z)(\zeta)\|\}}{\|(P_{\mathscr{R}}y_1)(\zeta)\|}\\ &\quad\leq \frac{1}{1-\delta}\big(\|(P_{\mathscr{R}}x)(\zeta)\| + \|(P_{\mathscr{R}}z)(\zeta)\|\big)\end{aligned}$$

almost everywhere. Therefore the function $\xi \in \mathscr{R}$ defined by $\xi(\zeta) = g(\zeta)(P_{\mathscr{R}}y_1)(\zeta)$ satisfies the inequality

$$\|\xi\| \leq \frac{1}{1-\delta}(\|\chi_\sigma P_{\mathscr{R}}x\| + \|P_{\mathscr{R}}z\|),$$

and the identity

$$\xi \cdot y_1 = \chi_\sigma(x \cdot P_{\mathscr{R}}y + uP_{\mathscr{R}}z \cdot P_{\mathscr{R}}z).$$

Define now $x_1 = x - \chi_\sigma P_{\mathscr{R}}x + \xi = P_{\mathscr{M}}x + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x + \xi$. To conclude the proof we have to verify that x_1 and y_1 satisfy the conditions of the proposition for sufficiently small δ. Condition (c) is clearly satisfied since both $\chi_\sigma P_{\mathscr{R}}x$ and ξ are supported on σ. Next we have $\|y - y_1\| = \|w\|$, and by the estimate of w obtained above, (d) is satisfied if

$$\delta + (2-\delta)(\|f\|_1 + \delta)^{1/2} \leq 3\|f\|_1^{1/2}. \tag{4.6}$$

To estimate x_1 we write

$$\begin{aligned}
\|x_1\| &= \left(\|P_{\mathscr{M}}x\|^2 + \|\chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x\|^2 + \|\xi\|^2\right)^{1/2} \\
&\leq \left[\|P_{\mathscr{M}}x\|^2 + \|\chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x\|^2 + \frac{1}{(1-\delta)^2}(\|\chi_\sigma P_{\mathscr{R}}x\| + \|P_{\mathscr{R}}z\|)^2\right]^{1/2} \\
&\leq \frac{1}{1-\delta}\left[\|P_{\mathscr{M}}x\|^2 + \|\chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x\|^2 + (\|\chi_\sigma P_{\mathscr{R}}x\| + \|P_{\mathscr{R}}z\|)^2\right]^{1/2} \\
&= \frac{1}{1-\delta}\left[\|x\|^2 + 2\|\chi_\sigma P_{\mathscr{R}}x\|\,\|P_{\mathscr{R}}z\| + \|P_{\mathscr{R}}z\|^2\right]^{1/2} \\
&\leq \frac{1}{1-\delta}\left(\|x\|^2 + 2\|x\|\,\|P_{\mathscr{R}}z\| + \|P_{\mathscr{R}}z\|^2\right)^{1/2} \\
&= \frac{1}{1-\delta}(\|x\| + \|P_{\mathscr{R}}z\|) \\
&\leq \frac{1}{1-\delta}\left(\|x\| + (\|f\|_1 + \delta)^{1/2}\right),
\end{aligned}$$

and we see that (b) is satisfied if

$$\frac{1}{1-\delta}\left(\|x\| + (\|f\|_1 + \delta)^{1/2}\right) \leq (1+\varepsilon)\left(\|x\| + \|f\|_1^{1/2}\right). \tag{4.7}$$

Finally we note that

$$\begin{aligned}
x \cdot y &= P_{\mathscr{M}}x \cdot y + P_{\mathscr{R}}x \cdot y \\
&= P_{\mathscr{M}}x \cdot y + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x \cdot y + \chi_\sigma P_{\mathscr{R}}x \cdot y \\
&= P_{\mathscr{M}}x \cdot y + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x \cdot y + \chi_\sigma x \cdot P_{\mathscr{R}}y,
\end{aligned}$$

and hence

$$\begin{aligned} x_1 \cdot y_1 &= \left(P_{\mathscr{M}}x + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x + \xi\right) \cdot (y + w) \\ &= P_{\mathscr{M}}x \cdot y + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x \cdot y + \xi \cdot y_1 + P_{\mathscr{M}}x \cdot w + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x \cdot w \\ &= P_{\mathscr{M}}x \cdot y + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x \cdot y + \chi_\sigma x \cdot P_{\mathscr{R}}y + \chi_\sigma uP_{\mathscr{R}}z \cdot P_{\mathscr{R}}z + P_{\mathscr{M}}x \cdot w \\ &\quad + \chi_{\mathbb{T}\setminus\sigma}P_{\mathscr{R}}x \cdot w \\ &= x \cdot y + f + r, \end{aligned}$$

where the remainder is

$$r = \left(\chi_\sigma uP_{\mathscr{R}}z \cdot P_{\mathscr{R}}z - f\right) + x \cdot P_{\mathscr{M}}w + P_{\mathscr{R}}x \cdot \left(\chi_{\mathbb{T}\setminus\sigma}w\right).$$

Thus we have

$$\begin{aligned} \|r\| &\le \delta + \|x\|\,\|P_{\mathscr{M}}w\| + \|x\|\,\|\chi_{\mathbb{T}\setminus\sigma}w\| \\ &\le \delta + \delta\|x\| + 2\delta\|x\| = \delta(1 + 3\|x\|), \end{aligned}$$

and we see that (a) is satisfied if

$$\delta(1 + 3\|x\|) < \varepsilon. \tag{4.8}$$

The proposition follows because (4.6), (4.7) and (4.8) are satisfied for sufficiently small δ.

The two parts of Proposition 4.5 can now be combined to yield the following result.

4.9 THEOREM. *Fix measurable sets $\sigma \subset \Delta$ and $\sigma_* \subset \Delta_*$ such that $\sigma \cap \sigma_* = \varnothing$ and $\sigma \cup \sigma_* = \Delta \cup \Delta_*$. Let $\alpha > 0$, $x \in \mathscr{K}_+$, $y \in \mathscr{K}_-$ and $h \in L^1(\Delta \cup \Delta_*)$ be given. There exist $x' \in \mathscr{K}_+$ and $y' \in \mathscr{K}_-$ with the following properties*:

(A) $\|x \cdot y + h - x' \cdot y'\|_1 < \alpha$;
(B) $\|x'\| \le (1 + \alpha)(\|x\| + 4\|h\|_1^{1/2})$;
(C) $\|\chi_{\mathbb{T}\setminus\sigma}(x - x')\| \le 3\|h\|_1^{1/2}$;
(D) $\|y'\| \le (1 + \alpha)(\|y\| + 4\|h\|_1^{1/2})$; *and*
(E) $\|\chi_{\mathbb{T}\setminus\sigma_*}(y - y')\| \le 3\|h\|_1^{1/2}$.

Proof. Set $\varepsilon = \alpha/2$ and apply Proposition 4.5.(i) with $f = \chi_\sigma h$ to obtain vectors x_1 and y_1 satisfying (a), (b), (c) and (d). Then apply part (ii) of the same proposition with $g = \chi_{\sigma_*}h$ to obtain x' and y' that satisfy (a_*), (b_*), (d_*) and (e_*). We must show that x' and y' satisfy the conditions of the theorem. We have $h = f + g$ so that

$$x \cdot y + h - x' \cdot y' = \left(x \cdot y + f - x_1 \cdot y_1\right) + \left(x_1 \cdot y_1 + g - x' \cdot y'\right),$$

and hence $\|x \cdot y + h - x' \cdot y'\| < 2\varepsilon = \alpha$. Thus (A) is satisfied. Next we verify

(B):

$$\begin{aligned}\|x'\| &\le \|x_1 - x'\| + \|x_1\| \le 3\|g\|_1^{1/2} + (1+\varepsilon)\big(\|x\| + \|f\|_1^{1/2}\big)\\ &\le 3\|h\|_1^{1/2} + (1+\varepsilon)\big(\|x\| + \|h\|_1^{1/2}\big) \le (1+\varepsilon)\big(\|x\| + 4\|h\|_1^{1/2}\big)\\ &\le (1+\alpha)\big(\|x\| + 4\|h\|_1^{1/2}\big).\end{aligned}$$

Analogously,

$$\begin{aligned}\|y'\| &\le (1+\varepsilon)\big(\|y_1\| + \|g\|_1^{1/2}\big) \le (1+\varepsilon)\big(\|y\| + \|y - y_1\| + \|g\|_1^{1/2}\big)\\ &\le (1+\varepsilon)\big(\|y\| + 3\|f\|_1^{1/2} + \|g\|_1^{1/2}\big) \le (1+\varepsilon)\big(\|y\| + 4\|h\|_1^{1/2}\big)\\ &\le (1+\alpha)\big(\|y\| + 4\|h\|_1^{1/2}\big),\end{aligned}$$

which proves (D). Condition (E) is verified because

$$\begin{aligned}\big\|\chi_{\mathbb{T}\setminus\sigma_*}(y - y')\big\| &\le \big\|\chi_{\mathbb{T}\setminus\sigma_*}(y - y_1)\big\| + \big\|\chi_{\mathbb{T}\setminus\sigma_*}(y_1 - y')\big\|\\ &= \big\|\chi_{\mathbb{T}\setminus\sigma_*}(y - y_1)\big\| \le \|y - y_1\|\\ &\le 3\|f\|_1^{1/2} \le 3\|h\|_1^{1/2},\end{aligned}$$

and (C) is proved in an analogous manner. The theorem follows.

5. Proof of the main result

The main ingredient in the proof is the following approximation result which is a combination of Theorems 2.3 and 4.9, together with the vanishing lemmas of Section 3. The notation is that established in Sections 3 and 4. In particular, $\sigma \subset \Delta$ and $\sigma_* \subset \Delta_*$ are two disjoint Borel sets such that $\sigma \cup \sigma_* = \Delta \cup \Delta_*$.

5.1 THEOREM. *Assume that $\mathscr{H}$ satisfies Property 2.1, and let $\delta > 0$, $x_0 \in \mathscr{K}_+$, $y_0 \in \mathscr{K}_-$, and $k \in L^1$ be given. There exist $x' \in \mathscr{K}_+$ and $y' \in \mathscr{K}_-$, with the following properties*:

(i) $\|[x_0 \cdot y_0 + k - x' \cdot y']\| < \delta$;

(ii) $\|x'\| \le (1+\delta)[\|x_0\| + \|k\|_1^{1/2} + 4\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}]$;

(iii) $\|\chi_{\mathbb{T}\setminus\sigma}(x_0 - x')\| \le \|k\|_1^{1/2} + 3\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}$;

(iv) $\|y'\| \le (1+\delta)[\|y_0\| + \|k\|_1^{1/2} + 4\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}]$; *and*

(v) $\|\chi_{\mathbb{T}\setminus\sigma_*}(y_0 - y')\| \le \|k\|_1^{1/2} + 3\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}$.

Proof. Theorem 2.3 implies the existence of orthogonal sequences $\{x^{(n)}\}, \{y^{(n)}\}$ in $\mathscr{H}$ such that $\|x^{(n)}\| \le \|k\|_1^{1/2}$, $\|y^{(n)}\| \le \|k\|_1^{1/2}$ and $\lim_{n\to\infty}\|k - x^{(n)} \cdot y^{(n)}\|_1 = 0$. Set $\alpha = \delta/4$ and note that Lemma 3.3 implies the existence of n such that, upon setting $\xi = x^{(n)}$ and $\eta = y^{(n)}$, we have

$$\|\xi\| \leq \|k\|_1^{1/2}, \qquad \|\eta\| \leq \|k\|_1^{1/2},$$

$$\big\|[P_{\mathscr{M}}\xi \cdot y_0]\big\| < \alpha, \qquad \big\|[x_0 \cdot P_{\mathscr{M}_*}\eta]\big\| < \alpha, \quad \text{and}$$

$$\|k - \xi \cdot \eta\|_1 < \alpha.$$

We apply next Theorem 4.9 to

$$x = x_0 + \xi,\ y = y_0 + \eta \text{ and}$$

$$h = -\big(x_0 \cdot P_{\mathscr{R}_*}\eta + P_{\mathscr{R}}\xi \cdot y_0\big) \in L^1(\Delta \cup \Delta_*).$$

Note that $\|h\|_1 \leq \|x_0\|\,\|\eta\| + \|\xi\|\,\|y_0\| \leq \|k\|_1^{1/2}(\|x_0\| + \|y_0\|)$, and hence Theorem 5.1 yields x' and y' such that

(A) $\|(x_0 + \xi) \cdot (y_0 + \eta) - x_0 \cdot P_{\mathscr{R}_*}\eta - P_{\mathscr{R}}\xi \cdot y_0 - x' \cdot y'\|_1 < \alpha$;
(B) $\|x'\| \leq (1 + \alpha)[\|x_0 + \xi\| + 4\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}]$;
(C) $\|\chi_{\mathbb{T}\setminus\sigma}(x_0 + \xi - x')\| \leq 3\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}$;
(D) $\|y'\| \leq (1 + \alpha)[\|y_0 + \eta\| + 4\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}]$; and
(E) $\|\chi_{\mathbb{T}\setminus\sigma_*}(y_0 + \eta - x')\| \leq 3\|k\|_1^{1/4}(\|x_0\| + \|y_0\|)^{1/2}$.

We will show now that x' and y' satisfy conditions (i)–(v). Conditions (ii)–(v) follow immediately from conditions (B)–(E) because $\alpha < \delta$ and

$$\|x_0 + \xi\| \leq \|x_0\| + \|\xi\| \leq \|x_0\| + \|k\|_1^{1/2},$$

$$\|y_0 + \eta\| \leq \|y_0\| + \|\eta\| \leq \|y_0\| + \|k\|_1^{1/2}.$$

To verify (i) we write

$$\begin{aligned}
&x_0 \cdot y_0 + k - x' \cdot y' \\
&\quad = (x_0 + \xi) \cdot (y_0 + \eta) - \xi \cdot \eta - x_0 \cdot \eta - \xi \cdot y_0 + k - x' \cdot y' \\
&\quad = (x_0 + \xi) \cdot (y_0 + \eta) + h - x' \cdot y' - x_0 \cdot P_{\mathscr{M}_*}\eta - P_{\mathscr{M}}\xi \cdot y_0 + k - \xi \cdot \eta
\end{aligned}$$

so that

$$\begin{aligned}
&\big\|[x_0 \cdot y_0 + k - x' \cdot y']\big\| \\
&\quad \leq \big\|(x_0 + \xi) \cdot (y_0 + \eta) + h - x' \cdot y'\big\|_1 + \big\|[x_0 \cdot P_{\mathscr{M}_*}\eta]\big\| + \big\|[P_{\mathscr{M}}\xi \cdot y_0]\big\| \\
&\qquad + \|k - \xi \cdot \eta\|_1 \\
&\quad < \alpha + \alpha + \alpha + \alpha = \delta.
\end{aligned}$$

The theorem is proved.

We are now ready for the proof of our main result, Theorem 2.2. Assume that Property 2.1 holds, $\varepsilon > 0$ and $f \in L^1$. By Lemma 3.1 it suffices to prove the existence of $x \in \mathscr{K}_+$ and $y \in \mathscr{K}_-$ such that $[x \cdot y] = [f]$ and $\|x\|\,\|y\| \leq (1 + \varepsilon)\|f\|_1$. Let $\{\delta_n\}$ be a sequences of positive numbers. We claim that there exist sequences $\{x_n\} \subset \mathscr{K}_+$ and $\{y_n\} \subset \mathscr{K}_-$ with the following properties:

(0) $\|x_0\| \leq \|f\|_1^{1/2}$, $\|y_0\| \leq \|f\|_1^{1/2}$,

(i) $\|[f] - [x_n \cdot y_n]\| < \delta_{n+1}^2$,

(ii) $\|x_{n+1}\| \leq (1 + \delta_{n+1}^2)[\|x_n\| + \delta_n + 4\delta_n^{1/2}(\|x_n\| + \|y_n\|)^{1/2}]$,

(iii) $\|\chi_{\mathbb{T}\setminus\sigma}(x_{n+1} - x_n)\| \leq \delta_n + 3\delta_n^{1/2}(\|x_n\| + \|y_n\|)^{1/2}$,

(iv) $\|y_{n+1}\| \leq (1 + \delta_{n+1}^2)[\|y_n\| + \delta_n + 4\delta_n^{1/2}(\|x_n\| + \|y_n\|)^{1/2}]$, and

(v) $\|\chi_{\mathbb{T}\setminus\sigma_*}(y_{n+1} - y_n)\| \leq \delta_n + 3\delta_n^{1/2}(\|x_n\| + \|y_n\|)^{1/2}$

for all $n \geq 0$. Indeed, the existence of x_0 and y_0 follows from Theorem 2.3. If x_n and y_n have already been chosen, let $k_n \in L^1$ be such that $[k_n] = [f] - [x_n \cdot y_n]$ and $\|k_n\|_1 < \delta_n^2$. Then an application of Theorem 5.1 with δ, x_0, y_0 and k of that theorem replaced by δ_{n+1}^2, x_n, y_n and k_n, respectively, yields vectors x_{n+1} and y_{n+1} with the desired properties. The vectors x and y will be obtained as weak limits of $\{x_n\}$ and $\{y_n\}$. First, let us impose some restriction on $\{\delta_n\}$. Fix a strictly increasing sequence $\{\varepsilon_n\}$ of positive numbers such that $(1 + \varepsilon_n)^2 < 1 + \varepsilon$, and choose $\{\delta_n\}$ such that $\sum_{n=0}^{\infty}\delta_n^{1/2} < \infty$, and

$$\begin{aligned}\left(1 + \delta_{n+1}^2\right)\left[(1 + \varepsilon_n)\|f\|_1^{1/2} + \delta_n + 4\delta_n^{1/2}\left(2(1 + \delta_n)\|f\|_1^{1/2}\right)\right]\\ \leq (1 + \varepsilon_{n+1})\|f\|_1^{1/2}.\end{aligned}$$

With this choice, (ii) and (iv) imply that $\|x_n\| \leq (1 + \varepsilon_n)\|f\|_1^{1/2}$ and $\|y_n\| \leq (1 + \varepsilon_n)\|f\|_1^{1/2}$. Thus there exists a sequence of integers $n_1 < n_2 < \ldots$ such that $\{x_{n_j}\}_j$ and $\{y_{n_j}\}_j$ converge weakly to vectors x and y, respectively, satisfying $\|x\| \leq (1 + \varepsilon)^{1/2}\|f\|_1^{1/2}$ and $\|y\| \leq (1 + \varepsilon)^{1/2}\|f\|_1^{1/2}$. To conclude the proof we only have to show that $[x \cdot y] = [f]$ or, equivalently, that $\langle U^k x, y\rangle = \hat{f}(-k)$ for $k \geq 0$. Since

$$\lim_{j\to\infty}\left|\langle U^k x_{n_j}, y_{n_j}\rangle - \hat{f}(-k)\right| \leq \lim_{j\to\infty}\left\|\left[x_{n_j} \cdot y_{n_j}\right] - [f]\right\| = 0,$$

we only have to prove that $\lim_{j\to\infty}\langle U^k x_{n_j}, y_{n_j}\rangle = \langle U^k x, y\rangle$ for $k \geq 0$. To see this we note that

$$\begin{aligned}\langle U^k x_{n_j}, y_{n_j}\rangle &= \langle U^k(\chi_\sigma + \chi_{\mathbb{T}\setminus\sigma})x_{n_j}, y_{n_j}\rangle\\ &= \langle U^k\chi_{\mathbb{T}\setminus\sigma}x_{n_j}, y_{n_j}\rangle + \langle U^k x_{n_j}, \chi_\sigma y_{n_j}\rangle.\end{aligned}$$

Because $\sum_{n=0}^{\infty}\delta_n^{1/2} < \infty$, the sequence $\{\chi_{\mathbb{T}\setminus\sigma}x_{n_j}\}$ is Cauchy in norm by (iii) and its norm limit must certainly be $\chi_{\mathbb{T}\setminus\sigma}x$. Thus

$$\lim_{j\to\infty}\langle U^k\chi_{\mathbb{T}\setminus\sigma}x_{n_j}, y_{n_j}\rangle = \langle U^k\chi_{\mathbb{T}\setminus\sigma}x, y\rangle.$$

Analogously, since $\sigma \subset \mathbb{T} \setminus \sigma_*$, the sequence $\{\chi_\sigma y_{n_j}\}_j$ converges in norm to $\chi_\sigma y$ and hence

$$\lim_{j\to\infty}\langle U^k x_{n_j}, \chi_\sigma y_{n_j}\rangle = \langle U^k x, \chi_\sigma y\rangle.$$

Thus

$$\lim_{j\to\infty} \langle U^k x_{n_j}, y_{n_j}\rangle = \langle U^k \chi_{\mathbb{T}\setminus\sigma} x, y\rangle + \langle U^k x, \chi_\sigma y\rangle = \langle U^k x, y\rangle,$$

as desired. Theorem 2.2 is proved.

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