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Modeling chaotic quantum systems by tridiagonal random matrices

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The spectral properties of "chaotic quantum systems" are modeled using tridiagonal random matrices. Unlike the Gaussian-orthogonal-ensemble (GOE) method, both the smooth and fluctuating parts of the spectral properties can be modeled simultaneously. We model the recent experiment by Gräf *et al.* [Phys. Rev. Lett. **69**, 1296 (1992)] as an example. This approach also provides an approximate construction of a GOE-type spectrum without any need for unfolding. By changing one control variable, one can bring the model from obeying δ-function level-spacing statistics, to being GOE-like or to being Poisson-like.

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Random-matrix theory has been very successful in describing the spectral properties of complicated systems [1, 2]. Perhaps the best known example of this theory is the Gaussian orthogonal ensemble (GOE) which is used to model complicated systems with time-reversal invariance. Due to the intense interest in studying quantum integrability [3–7], GOE has attracted much attention in recent years. It is well known that the smooth part of the level density from GOE is a Wigner semicircle, which in general is not the same as for the system one is modeling. In order to study spectral fluctuations, the energy levels must be unfolded so that the local average density of states of the resulting spectrum is unity.

The primary objective of this work is to generate a GOE-type spectrum with a smooth level density of unity. Our intent is to format the problem as simply as possible, hence the choice of tridiagonal random matrices. Deviation from full size random matrices can be traced back to the 1950s when Dyson [8], Wigner [9], and Engleman [10] studied special cases of bordered random matrices. By generalizing the approach to produce a smooth uniform density spectrum, a spectrum of known smooth density can be generated. This allows us to model not only quantum level fluctuations, but also the smooth level density for certain given systems. Another advantage is that tridiagonal matrices require much less computation than the usual GOE approach with full matrices.

Consider a real symmetric Hamiltonian represented by an $N \times N$ matrix

$$H = H_0 + \lambda H_1 \tag{1}$$

where H_0 is diagonal, and

$$(H_0)_{n,n} = E_{\rm sm}(n), \quad n = 1, \ldots, N \tag{2}$$

is a monotonic increasing smooth function. H_1 is a tridiagonal real symmetric matrix whose elements are defined as

$$(H_1)_{n,n} \equiv d_n = \begin{cases} g_1[E_{\rm sm}(2) - E_{\rm sm}(1)], & n = 1 \\ g_n[E_{\rm sm}(n+1) - E_{\rm sm}(n-1)]/2, & 1 < n < N \\ g_N[E_{\rm sm}(N) - E_{\rm sm}(N-1)], & n = N \end{cases}$$

$$(H_1)_{n,n+1} = (H_1)_{n+1,n} \equiv c_n = g_{N+n}[E_{\rm sm}(n+1) - E_{\rm sm}(n)], \quad n = 1, \ldots, N-1, \tag{3}$$

$$(H_1)_{i,j} = 0, \quad |i-j| > 1,$$

where $(g_i, i = 1, \ldots, 2N-1)$ is a sequence of random numbers of normal Gaussian distribution. This definition ensures that the matrix elements of H_1 are of the order of $\Delta E_{\rm sm}$. Without the H_1 term, the eigenvalues of H are just $E_{\rm sm}(n)$. When we turn H_1 on, keeping λ small, λH_1 can be viewed as a perturbation. Up to the second order,

$$E_n = E_{\rm sm}(n) + d_n + p_n - p_{n+1},$$
$$p_n = \frac{c_{n-1}^2}{E_{\rm sm}(n) - E_{\rm sm}(n-1) + d_n - d_{n-1}} \tag{4}$$

This perturbation introduces random fluctuations into

the spectrum, but retains the global spectral properties as determined by H_0. To see this, let us take a local average of E_n around n. It is not difficult to see that $\overline{d_n} = 0$ and $\overline{p_n} \approx \overline{p_{n+1}}$, thus

$$\overline{E_n} = \overline{E_{\rm sm}(n)} \quad . \tag{5}$$

In other words, the smooth level density is determined by $E_{\rm sm}(n)$. Since the off-diagonal matrix elements act as a source for level repulsion, one is unlikely to find degenerate levels. This is the underlying reason for finding a GOE-like spectrum.

In the simplest case, we can take $E_{\rm sm} = n$. With this particular choice, the smooth level density is unity, and there is no need for unfolding the spectrum. When $\lambda = 0$, we have $E_n = n$, and the nearest-level spacing x obeys δ-function statistics: $P(x) = \delta(x-1)$. This result holds for any spectrum in which E_n is a smooth function of n, as is often seen in one-dimensional systems. At the other extreme, one may set λ to be infinite, effectively setting $H = H_1$ up to a scale factor. The disappearance of H_0 eliminates the dominant term which defined the smooth level density, thus the resulting spectrum no longer has unit density of states on average. In this case, unfolding is needed. Our numerical result shows that the eigenvalues of H_1 obey Poisson level statistics. This conclusion is supported by both the nearest-level spacing and the spectral rigidity Δ_3 statistics. In Figs. 1 and 2 we plot these two quantities for 1060 levels, and compare with the analytical result for Poisson level statistics. These two limits are both generic characteristics of integrable systems, so we can say that by changing λ, we have brought our model from one type of integrable system to another. This has some similarity to changing a system from one dynamical symmetry to another [11, 12]. By a similar argument, we expect to see "chaotic" level statistics between the two limits. Indeed, we found that at $\lambda = 0.7$,

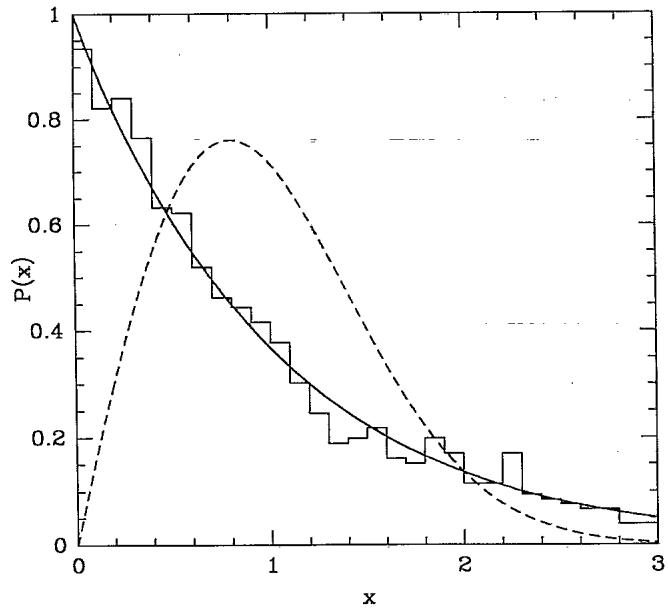

FIG. 1. Nearest-level spacing distribution for $H = H_1$ as compared to Poisson level statistics $P(x) = \exp(-x)$. $N = 1060$.

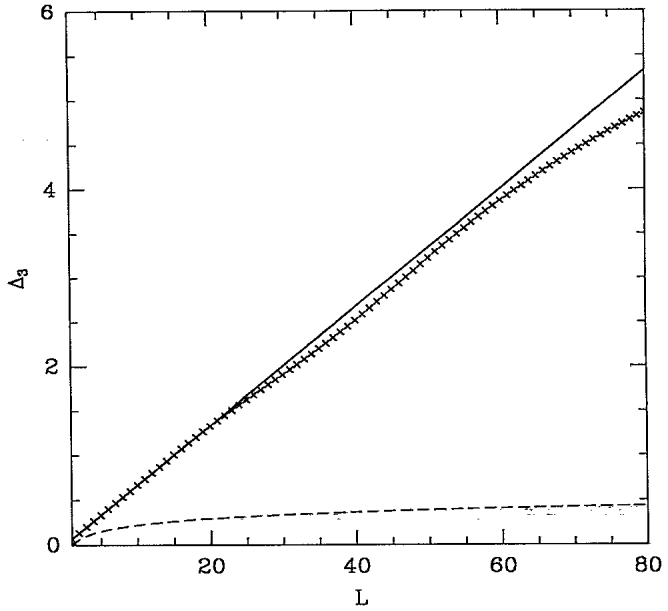

FIG. 2. The Δ_3 level statistics for $H = H_1$, $N = 1060$. Soild line: Result for Poisson level statistics; dashed line: GOE result.

the spectrum of H is very close to GOE. The nearest-level spacing fits better while the Δ_3 statistic staturates below the GOE value for large L, a phenomenon predicted by Berry [4] for systems with few degrees of freedom. We will not provide a separate set of plots for the GOE-like statistics obtained for $E_{\rm sm} = n$, because the result is very similar to the example which we will show for the stadium billiard problem.

So far we have demonstrated a very simple way of constructing a GOE-like spectrum with the local average level-spacing one; this type of spectrum is useful because it contains pure quantum fluctuation properties. It is independent of any "unfolding procedure" used to separate the smooth part of the spectrum from the fluctuating part. This type of spectrum has been used as input to one-dimensional problems [13, 14].

Very recently, Gräf *et al.* studied experimentally a two-dimensional quantum stadium billard problem by means of a superconducting microwave resonator of high Q value [15]. They measured 1060 energy levels and analyzed both the fluctuating part and the smooth part of the spectrum. Here we are going to use our random tridiagonal matrix to model both parts of the spectrum. Specifically, we wish to find $E_{\rm sm}$ and λ such that both aspects of the observed spectrum can be reproduced without actual knowledge of the experimental data. According to Eqs. (1) and (3) and Ref. [15], the number of cumulative levels up to energy E is (taking $\frac{\hbar^2}{2m} = 1$)

$$\begin{aligned} n(E) = {} & \frac{A}{4\pi}E - \frac{P}{4\pi}\sqrt{E} \\ & + \frac{a\sqrt{2}}{(2\pi)^{3/2}\sqrt{r}}E^{1/4} \\ & \times \sum_{m=1}^{\infty} m^{-3/2}\cos(2mr\sqrt{E} - 3\pi/4), \end{aligned} \tag{6}$$

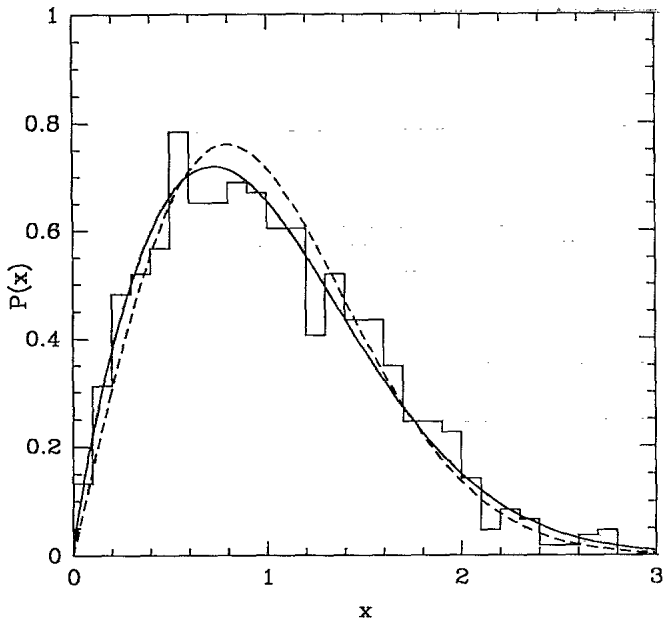

FIG. 3. Nearest-level spacing distribution of 1060 levels for $\lambda = 0.7$. Histogram: our result; solid smooth curve: Brody distribution best fit to experiment data; dashed line: GOE result.

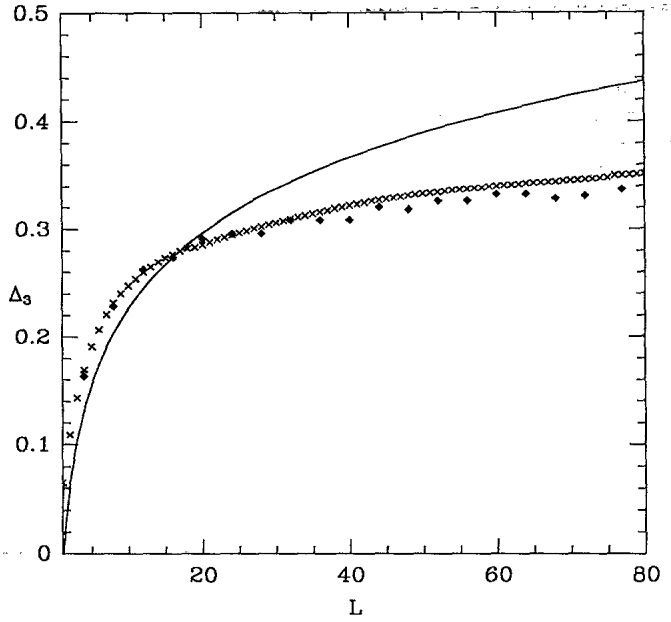

FIG. 4. The Δ_3 level statistics for 1060 levels when $\lambda = 0.7$. Solid curve: GOE result; crosses: our results; diamonds: a few experimental points taken from Fig. 4 of Ref. [15].

where A is the area, P the perimeter, r the radius of the curved wall of the stadium, and a the length of the rectangular part of the stadium. This function determines the smooth part of the spectrum and that coming from the periodic classical orbits. By inverting it at integer values of n, we have the function $E_{\rm sm}(n)$ which determines H_0. From the previous discussion, we see that upon diagonalizing H, this smooth part of the spectrum will be reproduced. We indeed see this in our resulting spectrum.

Once we know the smooth part of the spectrum, the unfolding procedure is a trivial task. With the unfolded spectrum, we can study the level fluctuation and compare with the analysis provided in Ref. [15]. In that paper, the nearest-level spacing distribution is fitted using a Brody distribution [1] $P(x) = c_1 x^{\omega} \exp(-c_2 x^{\omega+1})$ with $\omega = 0.82$. Normalization and average level spacing require that $c_1 = 1.47$, $c_2 = 0.81$. We found that at $\lambda = 0.7$, our spectrum reproduces their fit quite well. Even more remarkable is the result for Δ_3. With the same λ value, our model reproduces the experimental data very well. Comparing with GOE, we reproduce saturation of Δ_3 naturally [4]. These results are plotted in Figs. 3 and 4. Thus with our model, we were able to reproduce three aspects of the experimental data: the smooth level density, the nearest-level spacing, and the Δ_3 spectrum rigidity. The ability to reproduce Δ_3 saturation lies in the fact that the effect of λH_1 on the spectrum is local in the perturbation approximation; thus it is less effective in altering the long-range correlation of the spectrum determined by H_0.

It is worthwhile to discuss the physical insight brought by Eq. (1). The separation of the whole model Hamiltonian into a major part and a fluctuation term is analogous to a similar separation of the spectrum. H_0 gives an approximate semiclassical solution, while λH_1 introduces fluctuations. The model suggests that there might exist a basis in which the semiclassical approximation to a system is diagonal, and in this basis, the fluctuation aspect can be taken into account in a simple way. Research is under way to study the current model in a semiclassical approach and a possible relationship with Gutzwiller periodic orbital theory [16].

We have demonstrated a tridiagonal random matrix approach to modeling "chaotic quantum systems." By comparison with experiment, our approach is better than the traditional GOE approach, at least for the two-dimensional billiard problem. This approach has also been tested for the case where $E_{\rm sm}(n)$ is not primarily linear, and yields similar results. It was also noticed that matrices with additional nonzero off-diagonal bands do not change the result significantly as long as the $E_{\rm sm}$ remains dominant.

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[1] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[2] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, San Diego, 1991).
[3] M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).
[4] M. V. Berry, Proc. R. Soc. London, Ser. A **400**, 229 (1985).
[5] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[6] O. Bohigas, M. J. Giannoni, and C. Schmit, J. Phys. (Paris) Lett. **45**, L1015 (1984).
[7] O. Bohigas, R. U. Haq, and A. Pandey, Phys. Rev. Lett. **54**, 1645 (1985).
[8] F. J. Dyson, Phys. Rev. **92**, 1331 (1953).
[9] E. P. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957).
[10] R. Engleman, Nuovo Cimento **10**, 615 (1958).
[11] H. Wu, D. H. Feng, and M. Vallières, J. Phys. G **16**, L149 (1990).
[12] Y. Alhassid, A. Novoselsky, and N. Whelan, Phys. Rev. Lett. **65**, 2971 (1990).
[13] Hua Wu, M. Vallières, D. H. Feng, and D. W. L. Sprung, Phys. Rev. A **42**, 1027 (1990).
[14] Hua Wu, D. W. L. Sprung, M. Vallières, and D. H. Feng, Phys. Rev. E **47**, 2444 (1993).
[15] H. D. Gräf, H. L. Harney, H. Lengler, A. Richter, P. Schardt, and H. A. Weidenmüller, Phys. Rev. Lett. **69**, 1296 (1992).
[16] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Berlin, 1990).