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Trading strategy with stochastic volatility in a limit order book market

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Abstract
In this paper, we employ the *Heston stochastic volatility model* to describe the stock's volatility and apply the model to derive and analyze trading strategies for dealers in a security market with price discovery. The problem is formulated as a stochastic optimal control problem, and the controlled state process is the dealer's mark-to-market wealth. Dealers in the security market can optimally determine their ask and bid quotes on the underlying stocks continuously over time. Their objective is to maximize an expected profit from transactions with a penalty proportional to the variance of cumulative inventory cost. We provide an approximate, analytically tractable solution to the stochastic control problem. Numerical experiments are given to illustrate the effects of various parameters on the performances of trading strategies.

Keywords Limit order book (LOB) · Dynamic programming (DP) · Hamilton–Jacobi–Bellman (HJB) equation · Market impact · Stochastic volatility (SV) model

JEL Classification C5 · C6 · D8 · D9 · G4

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1 Introduction

The optimal trading strategy of dealers in a limit order book (LOB) market has been widely studied in early 1990s, see, for example, Cebiroglu and Horst (2015), Chiarella et al. (2015), Platania et al. (2018), Saito and Takahashi (2017) and Gould et al. (2013) for a detailed survey. Ho and Stoll (1981) provided one of the early studies on the behavior of a monopolistic dealer in a single stock situation. Avellaneda and Stoikov (2008) proposed a quantitative model for LOB by making use of its statistical properties coupled with the utility framework of Ho and Stoll. They approximately derived simple expressions for the bid and ask prices in terms of the model parameters. This approximate solution greatly simplified the simulations that the authors performed in their paper. Guéant et al. (2012a, b) transformed the original stochastic control problem arising from Avellaneda–Stoikov's model into a system of linear ordinary differential equations (ODEs) and provided simple and easy-to-compute expressions for optimal quotes when the trader is willing to liquidate a portfolio. Due to the tractable, theoretical and empirical appeal, most of the studies are based on the assumption that the volatility of the underlying security is constant over time or is independent of the changes in the price level of the underlying security. Nevertheless, certain empirical characteristics cast doubts on the constancy of volatility in the context of market microstructure and other aspects of financial applications such as option pricing and hedging. Figure 1 depicts the implied volatilities for European call options written on the VSTOXX index on March 31, 2014, over various maturities. The closing value of the index on March 31, 2014, was $V_0 = 17.6639$. As in stock or foreign exchange markets, one may notice that the *implied volatility smile* seems to be more pronounced for the shortest maturity and becomes a bit less pronounced for longer maturities.

There are three categories of nonconstant volatility models (Bouchaud et al. 2004; Gould et al. 2013; Hendershott et al. 2011; Mitra 2009; Wyart et al. 2008), and they include (i) time-dependent deterministic volatility $\sigma(t)$, (ii) local volatility $\sigma(t, S(t))$: volatility dependent on the current stock price S_t and time t, (iii) stochastic volatility: volatility driven by an additional random process. Fodra and Labadie (2012) extended the market-making models with inventory constraints of Avellaneda and Stoikov (2008) and Guéant et al. (2012a, b) to the case of a rather general class of mid-price processes,

$$\mathrm{d}S(t) = b(t, S(t))\mathrm{d}t + \sigma(t, S(t))\mathrm{d}W(t),$$

under either exponential or linear PnL (profit and loss) utility functions with (without) an inventory-risk-aversion parameter. This diffusion modeling framework allows a market maker to make directional bets on market trends while keeping his inventory risk under control. Subsequently, Fodra and Labadie (2013) completed and extended their work in Fodra and Labadie (2012). Their new model admitted several state variables (e.g., market spread, stochastic volatility and intensities of market orders) provided that the whole system is Markovian.

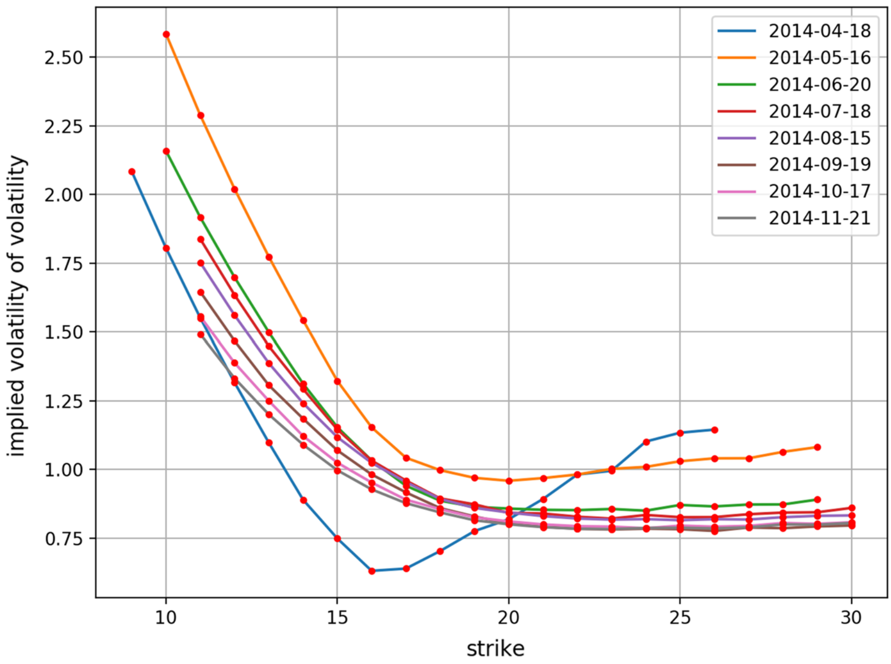


Fig. 1 Implied volatilities for European call options on the VSTOXX index on March 31, 2014, over all maturities

In this paper, we adopt Heston's mean-reverting stochastic volatility model[1] coupled with an arithmetic Brownian motion to set up our model

$$\begin{cases} \mathrm{d}S_t &= \sqrt{\nu_t} dW_t \\ \mathrm{d}\nu_t &= \theta(\alpha - \nu_t)\mathrm{d}t + \xi\sqrt{\nu_t}\mathrm{d}B_t. \end{cases}$$

Here $\{W_t\}_{t\geq 0}$ and $\{B_t\}_{\geq 0}$ are correlated standard Brownian motions with correlation coefficient ρ. In this paper, we mainly study an optimal quoting strategy that is chosen by a stock market maker market maker in a LOB with stochastic volatility. In our model, the market maker's objective is to maximize the expected revenues from transactions with a penalty proportional to the variance of cumulative inventory cost rather than the traditional expected utility setup. It appears to be straightforward and familiar to practitioners. Furthermore, this paper puts together the stock market mechanisms and the price discovering processions, to discuss how the market maker updates his/her quotes in the stock market. Similar methods with Avellaneda and Stoikov (2008) and Fodra and Labadie (2012, 2013), such as the ansatz about the form of the value function and a linear approximation of the order arrival terms, are used to derive an approximate, analytically tractable solution to the stochastic control problem. This approximate solution in turn greatly simplifies the simulations we perform in our numerical experiments.

[1] Heston's model stands out from other stochastic volatility models here because there exists an analytical solution for European options that take the correlation between stock price and volatility into consideration (Heston 1993).

The remainder of the paper is structured as follows. In Sect. 2, we introduce a fundamental model with stochastic volatility, under which we study optimal trading strategies in a setting of price discovery. Concluding remarks are given in Sect. 3.

2 Stock market making in a limit order book

Technological innovation has completely changed the role of a dealer, especially with the growth of electronic exchanges such as Nasdaq's Inet. Orders are placed in an automatic and electronic order-driven platform and wait in the *limit order book (LOB)* to be executed. Typically, there are two types of orders for market participants to post: market orders (MOs) and limit orders (LOs).

- A *limit* order is an order to trade a certain amount of a security at a given specified price [supplies liquidity].
- A *market* order is an order to buy/sell a certain quantity of the security at the best available price in the LOB [demands liquidity].

Market makers (MMs) play a crucial role in markets. They contribute to market quality by providing improved liquidity and price efficiency, enabling investors to trade in a timely manner and at a lower transaction cost. MMs facilitate trade and profit from making the spread and from their execution skills, and must be quick to adapt to changing market conditions. Liquidity takers, on the other hand, take liquidity in the market and are usually subject to fees, see, for example, Cartea et al. (2015), for a detailed discussion. In the following sections, we shall focus on MMs' market-making strategy in a market with stochastic volatility.

2.1 Model setup

In this section, we adopt Heston's stochastic volatility model to study the impact of changing volatility on MMs' market-making strategy. To start with, we consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ endowed with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. All random componenets of our model will be defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. To highlight the impact of stochastic volatility on MMs' trading strategy, we simply follow the setting in Avellaneda and Stoikov (2008) and assume that the stock mid-price S_t evolves over time according to an arithmetic Brownian motion with stochastic volatility. More specifically, the Heston mean-reverting stochastic volatility model is adopted as follows:

$$\begin{cases} \mathrm{d}S_t = \sqrt{\nu_t}\mathrm{d}W_t \\ \mathrm{d}\nu_t = \theta(\alpha - \nu_t)\mathrm{d}t + \xi\sqrt{\nu_t}\mathrm{d}B_t. \end{cases} \tag{2.1}$$

The process $\{\nu_t\}_{t\geq 0}$ originates from the *Cox–Ingersoll–Ross (CIR)* interest rate process (Cox et al. 1985). In the stochastic differential equations, θ, α and ξ are positive constants and $\{B_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ are two standard Brownian motions with constant correlation coefficient ρ, such that

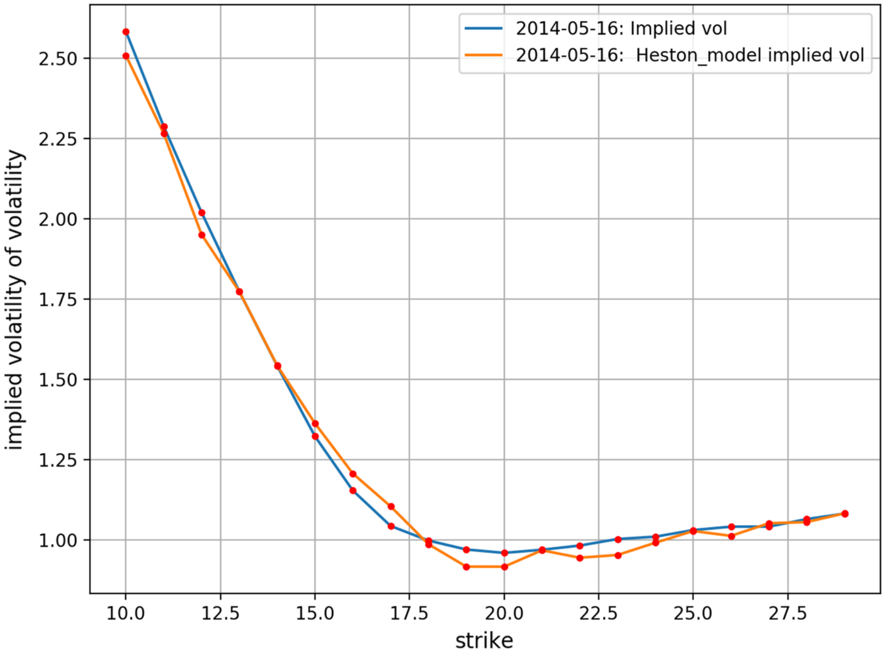


Fig. 2 Implied volatilities for European call options on the VSTOXX on March 31, 2014, with $(\hat{\theta}, \hat{\alpha}, \hat{\xi}, \hat{\nu}_0, \hat{\rho}) = (4.72, 0.43, 5.30, 0.50, 0.40)$ and MSE $= 0.002$

$$W_t = \rho B_t + \sqrt{1 - \rho^2}\,\tilde{B}_t$$

where $\{B_t\}_{\geq 0}$ and $\{\tilde{B}_t\}_{t \geq 0}$ are two independent Brownian motions. One advantage of the CIR model is that ν_t in (2.1) does not become negative. If ν_t reaches zero, the term multiplying dB_t vanishes and the positive drift term $\theta\alpha dt$ drives the variance process back into positive territory. If the parameters obey the Feller condition (Albrecher et al. 2007): $2\theta\alpha > \xi^2$, then the variance process ν_t is always positive and cannot reach zero.

Note that the stochastic volatility is not observable in the financial markets. In order to apply Model (2.1), it is necessary to calibrate it. The data sets used to calibrate the model usually are observations of the asset prices and/or of the prices of derivatives on the asset at some known times. Historical option prices are adopted, here, to calibrate our model. Figure 2 shows the calibration results for European call options on the VSTOXX index on March 31, 2014, over the maturity 2014-05-16 using the Monte Carlo method.

2.2 Price discovery and state feedback control problem

Prices move in response to new public information that cause traders to simultaneously revise their belief. The process of trading itself may also generate price movements. If market makers believe that some traders may posses private information about fundamental asset value, a buy (sell) order is associated with an upward (downward) revision of beliefs. We assume, following Almgren (2001), Almgren (2005), Moro (2009), Tóth (2011), that the revision in beliefs is positively correlated with the net order flow:

$$\mathrm{d}S_t = \sqrt{\nu_t}\mathrm{d}W_t - \eta(t)\Big(\mathrm{d}N_t^b - \mathrm{d}N_t^a\Big) \tag{2.2}$$

where N_t^b is the amount of stocks bought and N_t^a is the amount of stocks sold by the market maker, at time t and $\eta(t)$ is a function of t, representing the price impact incurred by the trades, and may be in general related to the states S_t and ν_t.

Remark 1 Price impact models are nowadays well studied under several viewpoints, and the literature on the topic is quite rich, see, for instance, Almgren (2001), Almgren (2012), Bayraktar and Ludkovshi (2014), Colaneri et al. (2019), Gatheral and Schied (2013), Lehalle et al. (2018), Schneider and Lillo (2019), Zabaljauregui and Campi (2019). Three categories of price impact models are generally used to analyze the price impact phenomenon. They are the constant model, the coordinated variation model and the two-variable model:

1. **[Constant model]** $\eta(t) \equiv c$, where $c > 0$ is a constant. With this form, each trade conveys the same amount of information regardless of the trading period;
2. **[Coordinated variation model]** $\nu_t\eta(t) \equiv c$, where $c > 0$ is a constant. That is, ν_t and $\eta(t)$ vary perfectly inversely, supposing that the arrival rate of trade events is the single source of uncertainty. If each trade event brings both a fixed amount of price variance and the opportunity to trade a fixed number of shares for a particular cost, then we obtain the above relation. This assumption is reasonable for markets during "normal" periods, though it can be severely violated during exceptional events when volatility sharply increases, but liquidity is also withdrawn;
3. **[Two-variable model]**

$$\begin{cases} \eta(t) = \bar{\eta}e^{\zeta(t)} \\ \mathrm{d}\zeta(t) = a(t)\mathrm{d}t + b(t)\mathrm{d}B_L(t), \end{cases}$$

   where $a(t)$ and $b(t)$ are coefficients whose values may depend on $\eta(t)$ and ν_t, and $B_L(t)$ and $B(t)$ are correlated standard Brownian motions, with a constant coefficient of correlation $\rho_0(0 \le \rho_0 \le 1)$. The two-variable model is an extension of the coordinated variation model. The coordinated case can be recovered by making the following assumptions:

   (a) $\rho_0 = 1$;
   (b)

$$\begin{cases} a(t) = \frac{\xi^2 - 2\theta(\alpha-\nu)}{2\nu} \\ b(t) = -\frac{\xi}{\sqrt{\nu}}. \end{cases}$$

One simple way to study the market impact is to consider the constant model. This considerably simplifies the problem via dimension reduction and allows us to exhibit the essential features of price discovery process in an optimal market-making problem without introducing much complexity. We now set $\eta(t) \equiv \eta$ and analyze MMs' market-making strategy in a market with stochastic volatility.[2] To begin with, we present the following notations that will be used in this section:

- p_t^a / p_t^b: ask/bid quote at time t.
- δ_t^a / δ_t^b: ask/bid premium at time t, denoted as

$$\delta_t^a = p_t^a - S_t \quad \text{and} \quad \delta_t^b = S_t - p_t^b.$$

- N_t^a and N_t^b: two independent *Poisson processes* with rates λ_t^a and λ_t^b, respectively, which are assumed to be independent of the Brownian motions $\{W_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$. The arrival rates of buy and sell orders that will reach the dealer also depend on the ask and bid premiums, i.e.,

$$\lambda_t^a = \lambda^a(\delta_t^a) \quad \text{and} \quad \lambda_t^b = \lambda^b(\delta_t^b).$$

Following Avellaneda and Stoikov (2008), we assume, in this paper, that

$$\lambda^a(\delta) = \lambda^b(\delta) = A \exp(-k\delta). \tag{2.3}$$

- X_t: wealth in cash at time t.
- q_t: number of stocks held at time t.
- $\mathrm{E}_t[\cdot]$: conditional expectation $\mathrm{E}[\cdot|\mathcal{F}_t]$.
- $\mathcal{C}^{\cdot,2,1}(\mathbb{Z} \times \Re_+ \times [0, T])$: a space of functions $f(q, \nu, t)$ which is continuously differentiable in t and twice continuously differentiable in ν.

Consider an active dealer in a LOB market, quoting a bid price p_t^b and an ask price p_t^a at no cost at time t. The dealer is committed to, respectively, buy and sell *one* share of stock at these prices. The wealth in cash, X_t, jumps whenever there is a buy or sell order,

$$\mathrm{d}X_t = p_t^a \mathrm{d}N_t^a - p_t^b \mathrm{d}N_t^b. \tag{2.4}$$

The number of stocks held at time t is then $q_t = q_0 + N_t^b - N_t^a$. The mark-to-market wealth, $X_t + q_t S_t$, then follows

$$\mathrm{d}\Big(X_t + q_t S_t\Big) = \underbrace{\delta_t^a \mathrm{d}N_t^a + \delta_t^b \mathrm{d}N_t^b}_{\text{(revenues)}} + \underbrace{q_t \mathrm{d}S_t}_{\text{(inventory value)}}.$$

[2] The value of η is relatively small when compared with the stock price S_t. The parameter for NASDAQ stock FARO in 2013 is 1.41×10^{-4}, SMH, 5.45×10^{-6} and INTC, 6.15×10^{-7} (see Cartea et al. 2015, Chap. 4). However, such a small number will have a great influence on the profitability of a high-frequency trading (HFT) strategy, e.g., market-making strategies in a LOB (see, for instance Rishi Narang 2013, for more on this).

2.2.1 The objective and stochastic control approach

Suppose the dealer is to liquidate $q_0 = Q$ shares of the orders before time T. He/she decides to place limit orders to liquidate his/her shares. Compare with market orders that would at least save the cost of crossing the spread, and perhaps even achieve better performance by posting deeper in the LOB. The risk in such a decision is that he/she may not execute his/her shares. Assume that the unexecuted shares will be executed through a market order and the fee incurred is β/share. In addition, the dealer may also have a certain sense of urgency to get rid of his/her shares, represented by penalizing holding inventories different from zero throughout the strategy. Denote

$$\mathrm{d}Z_t = \delta_t^a \mathrm{d}N_t^a + \delta_t^b \mathrm{d}N_t^b \quad \text{and} \quad \mathrm{d}I_t = q_t \mathrm{d}S_t$$

with Z_t and I_t representing, respectively, the cumulative revenues from transactions and the inventory value at time t. At any time t, we aim to find an optimal strategy by solving the following optimization problem:

$$\max_{\left(\delta_u^a, \delta_u^b\right)_{u\in[t,T]} \in \mathcal{A}} \left\{ E_t[Z_T - \beta q_T] - \frac{\gamma}{2} E_t \left[\int_t^T (\mathrm{d}I_u)^2 \right] \right\}, \tag{2.5}$$

where $\mathcal{A}$ is a set of admissible processes on $[0, T]$, bounded from below. Note that N_t^a and N_t^b are two independent *Poisson processes*.

$\cdot$	$\mathrm{d}t$	$\mathrm{d}W_t$	$\mathrm{d}N_t^a$	$\mathrm{d}N_t^b$
$\mathrm{d}t$	0	0	0	0
$\mathrm{d}W_t$	0	$\mathrm{d}t$	0	0
$\mathrm{d}N_t^a$	0	0	$\mathrm{d}N_t^a$	0
$\mathrm{d}N_t^b$	0	0	0	$\mathrm{d}N_t^b$

Thus, $\left(\mathrm{d}I_t\right)^2 = q_t^2 \nu_t \mathrm{d}t + q_t^2 \eta^2 \left(\mathrm{d}N_t^a + \mathrm{d}N_t^b\right)$. The optimization problem can then be written as,

$$\max_{\left(\delta_u^a, \delta_u^b\right)_{u\in[t,T]} \in \mathcal{A}} E_t \left[\int_t^T \left[\left(\delta_u^a + \beta - \frac{\gamma}{2} q_u^2 \eta^2\right) \mathrm{d}N_u^a + \left(\delta_u^b - \beta - \frac{\gamma}{2} q_u^2 \eta^2\right) \mathrm{d}N_u^b \right] \right.$$
$$\left. - \frac{\gamma}{2} \int_t^T q_u^2 \nu_u \mathrm{d}u \right].$$

Remark 2 For a market maker to remain in business, the first condition that needs to be satisfied is $p_t^a \geq p_t^b$, $p_t^a < \infty$ and $p_t^b > -\infty$. In other words,

$$\mathcal{A} = \left\{ \left(\delta_t^a, \delta_t^b\right)_{t \geq 0} | \delta_t^a + \delta_t^b \geq 0, \delta_t^a, \delta_t^b < \infty \right\}.$$

Let $V(q, \nu, t)$ denote the *value function* conditional on initial values $q_t = q$ and $\nu_t = \nu$. Then we have the following proposition for the value function V.

Proposition 1 *Suppose the value function V is sufficiently smooth—$V \in \mathcal{C}^{\cdot,2,1}(\mathbb{Z} \times \Re_+ \times [0, T])$. Then, it satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:*

$$\begin{aligned} & V_t + \theta(\alpha - \nu)V_\nu + \frac{1}{2}\xi^2 \nu V_{\nu\nu} - \frac{\gamma}{2} q^2 \nu \\ & + \max_{\delta_t^a \in \mathcal{A}} \lambda^a\left(\delta_t^a\right)\left[\delta_t^a + \beta - \frac{\gamma\eta^2}{2}(q-1)^2 + V(q-1, \nu, t) - V(q, \nu, t)\right] \\ & + \max_{\delta_t^b \in \mathcal{A}} \lambda^b\left(\delta_t^b\right)\left[\delta_t^b - \beta - \frac{\gamma\eta^2}{2}(q+1)^2 + V(q+1, \nu, t) - V(q, \nu, t)\right] = 0 \end{aligned} \tag{2.6}$$

with the boundary condition $V(q, \nu, T) = 0$.

HJB equation (2.6) provides a necessary condition for the value function $V(q, \nu, t)$. The verification theorem, on the other hand, guarantees that if a classical solution to the associated HJB equation is found, it is the value function and the resulting control is an optimal one. Before we prove the verification theorem, we need the following proposition and its proof can be found in "Appendix A." To avoid confusion, in the following sections, we denote

- $\{\nu_u^{t,\nu}\}_{u \geq t}$: a trajectory of $\{\nu_u\}_{u \geq t}$ given $\nu_t = \nu$.
- $\{q_u^{t,q,\left(\delta_t^a, \delta_t^b\right)_{t \geq 0}}\}_{u \geq t}$: a trajectory of $\{q_u\}_{u \geq t}$ given $q_t = q$ and a trading strategy $\left(\delta_t^a, \delta_t^b\right)_{t \geq 0}$.
- $\mathbb{I}_{\{A\}}$: the indicator function of an event A.

Proposition 2 *Consider an arbitrary control process $\left(\delta_t^a, \delta_t^b\right)_{t \geq 0} \in \mathcal{A}$ and the processes*

$$\begin{cases} \mathrm{d}\nu_u^{t,\nu} = \theta\left(\alpha - \nu_u^{t,\nu}\right)\mathrm{d}u + \xi\sqrt{\nu_u^{t,\nu}}\mathrm{d}B_u \\ \mathrm{d}q_u^{t,q,\left(\delta_t^a, \delta_t^b\right)_{t \geq 0}} = \mathrm{d}N_u^b - \mathrm{d}N_u^a. \end{cases}$$

The point processes N_t^a and N_t^b have stochastic intensities

$$\lambda^a\left(\delta_t^a\right) = A\exp\left(-k\delta_t^a\right)\mathbb{I}_{\{\mathrm{d}N_t^a > 0\}} \quad \textit{and} \quad \lambda^b\left(\delta_t^b\right) = A\exp\left(-k\delta_t^b\right)\mathbb{I}_{\{\mathrm{d}N_t^b > 0\}},$$

respectively.

(a) *Let* $\phi(q,\nu,t)$ *be a function in* $\mathcal{C}^{\cdot,2,1}(\mathbb{Z}\times\Re_+\times[0,T])$ *satisfying the following conditions:*

(i) *For all* $\left(\delta_t^a,\delta_t^b\right)_{t\geq 0}\in\mathcal{A}$,

$$\begin{aligned}\phi_t &+ \theta(\alpha-\nu)\phi_\nu + \frac{1}{2}\xi^2\nu\phi_{\nu\nu} - \frac{\gamma}{2}q^2\nu\\ &+ \lambda^a\left(\delta_t^a\right)\left[\delta_t^a + \beta - \frac{\gamma\eta^2}{2}(q-1)^2 + \phi(q-1,\nu,t) - \phi(q,\nu,t)\right]\\ &+ \lambda^b\left(\delta_t^b\right)\left[\delta_t^b - \beta - \frac{\gamma\eta^2}{2}(q+1)^2 + \phi(q+1,\nu,t) - \phi(q,\nu,t)\right] \leq 0\end{aligned}$$

(ii) $\phi(q,\nu,T)\geq 0$, *for all* $(q,\nu)\in\mathbb{Z}\times\Re_+$

(iii) $\mathbb{E}\left[\int_0^T\left[\sqrt{\nu_u^{t,\nu}}\cdot\phi_\nu\left(q_u^{t,q,\left(\delta_t^a,\delta_t^b\right)_{t\geq 0}},\nu_u^{t,\nu},u\right)\right]^2\mathrm{d}u\right]<\infty$, *for all* $\left(\delta_t^a,\delta_t^b\right)_{t\geq 0}\in\mathcal{A}$

(iv) $\phi(q,\nu,t)$ *satisfies a polynomial growth condition, i.e., there exist a constant* C *and integers* m,n *such that* $|\phi(q,\nu,t)|\leq C\left(1+|q|^n+|\nu|^m\right)$.

Then we have $\phi(q,\nu,t)\geq V(q,\nu,t)$ *for all* $(q,\nu,t)\in\mathbb{Z}\times\Re_+\times[0,T]$.

(b) *If for each* $(q,\nu,t)\in\mathbb{Z}\times\Re_+\times[0,T]$, *there exists a pair* $\left(\delta_t^{*,a},\delta_t^{*,b}\right)$ *such that*

(v)

$$\begin{aligned}\phi_t &+ \theta(\alpha-\nu)\phi_\nu + \frac{1}{2}\xi^2\nu\phi_{\nu\nu} - \frac{\gamma}{2}q^2\nu\\ &+ \lambda^a\left(\delta_t^{*,a}\right)\left[\delta_t^{*,a} + \beta - \frac{\gamma\eta^2}{2}(q-1)^2 + \phi(q-1,\nu,t) - \phi(q,\nu,t)\right]\\ &+ \lambda^b\left(\delta_t^{*,b}\right)\left[\delta_t^{*,b} - \beta - \frac{\gamma\eta^2}{2}(q+1)^2 + \phi(q+1,\nu,t) - \phi(q,\nu,t)\right] = 0\end{aligned}$$

(vi) $\phi(q,\nu,T)=0$, *for all* $(q,\nu)\in\mathbb{Z}\times\Re_+$

(vii) $\phi(q,\nu,t)$ *satisfies a polynomial growth condition.*

Then $\phi(q,\nu,t)=V(q,\nu,t)$ *and* $(\delta_t^{*,a},\delta_t^{*,b})_{t\geq 0}\in\mathcal{A}$ *is an optimal control process.*

We note that the problem in Eq. (2.5) is *finite*, i.e., the value function $V(q,\nu,t)$ has a finite upper bound, and that the objective function is concave. As discussed in Yong and Zhou (1999), Ch. 2, the problem admits an optimal control.

Theorem 1 (Verification theorem and optimal quotes) *Let* $V(q,\nu,t)$ *be a function in* $\mathcal{C}^{\cdot,2,1}(\mathbb{Z}\times\Re_+\times[0,T])$ *satisfying HJB equation* (2.6)*, i.e., the classical solution of the HJB equation exists. If* $V(q,\nu,t)$ *satisfies a polynomial growth condition and*

$$\mathbb{E}\left[\int_0^T\left[\sqrt{\nu_u}V_\nu(q_u,\nu_u,u)\right]^2\mathrm{d}u\right]<\infty$$

holds, then $V(q,\nu,t)$ is the value function of control problem (2.5). Assume further that the derivatives $\frac{\partial\lambda^a\left(\delta^a\right)}{\partial\delta^a}$ and $\frac{\partial\lambda^b\left(\delta^b\right)}{\partial\delta^b}$ are nonzero. The optimal ask and bid quotes at time t, $\left(\delta_t^{*,a},\delta_t^{*,b}\right)(q,\nu,t)$, are given by

$$\Bigg(-\frac{\lambda^a\left(\delta_t^{*,a}\right)}{\left(\partial\lambda^a/\partial\delta^a\right)\left(\delta_t^{*,a}\right)}-\beta+\frac{\gamma\eta^2}{2}(q-1)^2+V(q,\nu,t)-V(q-1,\nu,t),$$
$$-\frac{\lambda^b\left(\delta_t^{*,b}\right)}{\left(\partial\lambda^b/\partial\delta^b\right)\left(\delta_t^{*,b}\right)}+\beta+\frac{\gamma\eta^2}{2}(q+1)^2+V(q,\nu,t)-V(q+1,\nu,t)\Bigg), \tag{2.7}$$

respectively. Furthermore, the value function $V(q,\nu,t)$ satisfies the following nonlinear partial differential equation (PDE):

$$V_t+\theta(\alpha-\nu)V_\nu+\frac{1}{2}\xi^2\nu V_{\nu\nu}-\frac{\gamma}{2}q_t^2\nu-\frac{(\lambda^a(\delta_t^{*,a}))^2}{(\partial\lambda^a/\partial\delta^a)(\delta_t^{*,a})}$$
$$-\frac{(\lambda^b(\delta_t^{*,b}))^2}{(\partial\lambda^b/\partial\delta^b)(\delta_t^{*,b})}=0 \tag{2.8}$$

with boundary condition $V(q,\nu,T)=0$.

Proof Conclusions can be directly deduced from Propositions 1 and 2. □

2.3 The quote: a linear approximation approach

The optimal control in Eq. (2.7) can be derived through an intuitive, two-step procedure: first, solve Eq. (2.8) for V, then solve Eq. (2.7) for $\left(\delta_t^{*,a},\delta_t^{*,b}\right)$. The main computational difficulty lies in solving Eq. (2.8), since it contains not only continuous variables t and ν, but also a discrete variable q.

Assume that the arrival rates of buy and sell orders that will reach the dealer take exponential form (2.3). It is not difficult to check that the derivatives $\frac{\partial\lambda^a(\delta^a)}{\partial\delta^a}$ and $\frac{\partial\lambda^b(\delta^b)}{\partial\delta^b}$ are nonzero. From Theorem 1, V solves the following nonlinear PDE:

$$V_t+\theta(\alpha-\nu)V_\nu+\frac{1}{2}\xi^2\nu V_{\nu\nu}-\frac{\gamma}{2}q^2\nu+\frac{A}{k}\left(e^{-k\delta_t^{*,a}}+e^{-k\delta_t^{*,b}}\right)=0. \tag{2.9}$$

We follow Fodra and Labadie (2012, 2013) and Avellaneda and Stoikov (2008) and adopt a linear approximation of the order arrival terms

$$\frac{A}{k}\left(e^{-k\delta_t^{*,a}}+e^{-k\delta_t^{*,b}}\right)\approx\frac{A}{k}\left(2-k(\delta_t^{*,a}+\delta_t^{*,b})\right) \tag{2.10}$$

in Eq. (2.9).

Theorem 2 *Assume the arrival rates of buy and sell orders that will reach the dealer take exponential form* (2.3)*. For an active dealer, with the linear approximation of order arrival terms* (2.10)*, an approximation of the value function*

$$\tilde{V}(q,\nu,t) = -\frac{\gamma A}{\theta}(\nu-\alpha)\left[\frac{1}{\theta}\left(1-e^{-\theta(T-t)}\right)-e^{-\theta(T-t)}(T-t)\right]-\frac{\gamma\alpha A}{2}(T-t)^2$$
$$-\frac{\gamma q^2}{2\theta}(\nu-\alpha)\left[1-e^{-\theta(T-t)}\right]-\frac{\gamma q^2}{2}\alpha(T-t)$$

is obtained. The optimal ask and bid quotes $\left(\delta_t^{*,a},\delta_t^{*,b}\right)$ *can then be approximated by* $\left(\hat{\delta}_t^{*,a},\hat{\delta}_t^{*,b}\right)$*:*

$$\begin{cases}\hat{\delta}_t^{*,a} = \frac{1}{k}-\beta+\frac{\gamma\eta^2}{2}(q-1)^2-\left(\frac{\gamma}{2\theta}(\nu-\alpha)[1-e^{-\theta(T-t)}]+\frac{\gamma}{2}\alpha(T-t)\right)(2q-1),\\ \hat{\delta}_t^{*,b} = \frac{1}{k}+\beta+\frac{\gamma\eta^2}{2}(q+1)^2+\left(\frac{\gamma}{2\theta}(\nu-\alpha)[1-e^{-\theta(T-t)}]+\frac{\gamma}{2}\alpha(T-t)\right)(2q+1).\end{cases}$$

Moreover, we have
$|\delta_t^{*,a}-\hat{\delta}_t^{*,a}| = O\left((\delta_t^{*,a})^2\right)$ *and* $|\delta_t^{*,b}-\hat{\delta}_t^{*,b}| = O\left((\delta_t^{*,b})^2\right)$.

Proof Please refer "Appendix B" for more details. □

It is not difficult to verify that $\left(\hat{\delta}_t^{*,a},\hat{\delta}_t^{*,b}\right)_{t\geq 0}\in\mathcal{A}$. For more information, see the following remarks.

Remark 3 – Let $\left(\tilde{\delta}_t^{*,a},\tilde{\delta}_t^{*,b}\right)_{t\in[0,T]}$ denote the optimal quotes without price impact ($\eta_t\equiv 0$), then

$$\begin{cases}\hat{\delta}_t^{*,a} = \tilde{\delta}_t^{*,a}+\frac{\gamma\eta^2}{2}(q-1)^2\\ \hat{\delta}_t^{*,b} = \tilde{\delta}_t^{*,b}+\frac{\gamma\eta^2}{2}(q+1)^2,\\ \hat{\delta}_t^{*,a}+\hat{\delta}_t^{*,b} = \tilde{\delta}_t^{*,a}+\tilde{\delta}_t^{*,b}+\gamma\eta^2(q^2+1),\end{cases}$$

the price impact cost $\frac{\gamma\eta^2}{2}(q\pm 1)^2$ widens the ask and bid premiums, so is the dealer's spread, and the extent of this influence depends on dealer's position in the stock market. This feature has the following immediate consequences:

(i) No matter what position the dealer holds in the stock market, long $q>0$ or short $q<0$, the price impact always moves the stock price in a "wrong direction." A market maker taking the other side of one order can hope she/he is able to take the other side of another order immediately, and at a profit that at least equals the bid–ask spread. This is the embodiment of "buy low, sell high."

In normal times, in the absence of a very short-term trend, this appears to be at least somewhat achievable. But frequently, large moves happen because there is real information in the marketplace that leads to a prolonged or extended trend. Market makers in these situations are characterized as "picking up nickels in front of a steamroller," which is also known as *adverse selection risk*.[3]

(ii) Recall that each time a dealer trades, the spread he pays for a roundtrip trade, due to the costs incurred by price impact, is exactly $\gamma\eta^2(q^2+1)$: downside risk for buy trades + upside risk for sell trades. We therefore conclude that the dealer's net gain per roundtrip trade is actually $\left(\tilde{\delta}_t^{*,a}+\tilde{\delta}_t^{*,b}\right)$, the same as that in the model without price impact.

– **[Dependence on** ν**]**:

$$\begin{cases} \frac{\partial\hat{\delta}_t^{*,a}}{\partial\nu}<0, & \frac{\partial\hat{\delta}_t^{*,b}}{\partial\nu}>0, \quad \text{if } q_t>0 \\ \frac{\partial\hat{\delta}_t^{*,a}}{\partial\nu}>0, & \frac{\partial\hat{\delta}_t^{*,b}}{\partial\nu}>0, \quad \text{if } q_t=0 \\ \frac{\partial\hat{\delta}_t^{*,a}}{\partial\nu}>0, & \frac{\partial\hat{\delta}_t^{*,b}}{\partial\nu}<0, \quad \text{if } q_t<0 \end{cases}$$

and $\frac{\partial\left(\hat{\delta}_t^{*,a}+\hat{\delta}_t^{*,b}\right)}{\partial\nu}>0$. The rationale behind this is that an increase in the variance ν will lead to an increase in the inventory risk. Hence, to reduce this risk, dealers having a long position will try to lower their ask and bid prices so as to encourage selling and discourage purchasing. Similarly, dealers with a short position will try to raise prices to encourage purchasing and to discourage selling. As a conclusion, due to the increase in price risk, the bid–ask spread, which reflects the risk a market maker is facing, widens.

– Note that

$$\hat{\delta}_t^{*,a}+\hat{\delta}_t^{*,b}=\frac{2}{k}+\frac{\gamma}{\theta}(\nu-\alpha)\left[1-e^{-\theta(T-t)}\right]+\gamma\alpha(T-t)+\gamma\eta^2(q^2+1).$$

As $\eta_t\equiv 0$ and $\theta\to 0$,

$$\hat{\delta}_t^{*,a}+\hat{\delta}_t^{*,b}=\frac{2}{k}+\gamma\nu(T-t).$$

This corresponds to the results in Avellaneda and Stoikov (2008) where the variance ν is a constant.

– As in Avellaneda and Stoikov (2008), define the price adjustment variable m_t:

$$m_t=\hat{\delta}_t^{*,a}-\hat{\delta}_t^{*,b}=-2\beta-2\left(\frac{\gamma}{\theta}(\nu-\alpha)\left[1-e^{-\theta(T-t)}\right]+\gamma\alpha(T-t)+\gamma\eta^2\right)q.$$

[3] The concept of the adverse selection risk was first introduced by Bagehot (1971) and formalized by Copeland and Galai (1983), Glosten and Milgrom (1985) and others. Adverse selection in the sense that applies to capital markets is defined as a situation in which there is a tendency for bad outcomes to occur, due to asymmetric information between a buyer and a seller.

Table 1 Model parameters

Initial price (S_0)	17.6639
Time horizon (T)	1 h
Initial volatility (ν_0)	$\frac{0.50}{252}$
Time step (dt)	0.005
Initial inventory (Q)	0
Risk aversion (γ)	0.10
Reversion rate (θ)	$\frac{4.72}{252}$
Long-term volatility (α)	$\frac{0.43}{252}$
Volatility of volatility (ξ)	$\frac{5.30}{252}$
Correlation coefficient (ρ)	0.40
Elasticity (κ)	1.2
Intensity (A)	360
Clearing fee (β)	0.001

As $\theta \to 0$, we have $m_t \to -2\beta - 2\gamma[\nu(T-t) + \eta^2]q$. The principal function of m_t is to promote adjustment of the dealer's inventory when it is temporarily thrown out of balance by the random arrival of transactions. Consistently with the results in Avellaneda and Stoikov (2008), the price adjustment variable is negative (positive) when the inventory is greater (less) than a certain number: when $m_t < 0$, both the bid price and ask price are "low" and the dealer has an incentive to sell rather than to purchase, and as a result, it reduces the dealer's inventory level; when $m_t > 0$ the dealer has an incentive to purchase rather than to sell, and as a result, it will raise her/her inventory level.

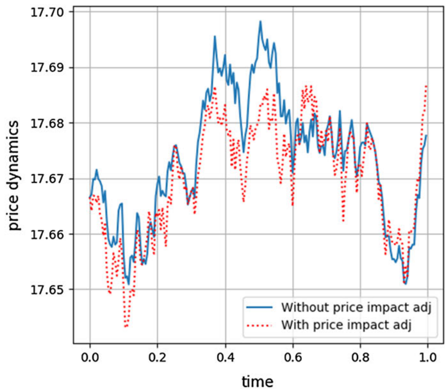
2.4 Numerical experiments

In this section, we illustrate the results of our analysis with some numerical examples. For the purpose of illustration, we use the set of baseline parameters given in Table 1.[4]

Simulations are obtained through the following procedure:

Step 1:	Compute the agent's quotes $\hat{\delta}_t^a$, $\hat{\delta}_t^b$ and other state variables at time t.
Step 2:	With probability $\lambda^a\left(\delta_t^a\right)\mathrm{d}t$, $\mathrm{d}N_t^a = 1$, $\mathrm{d}X_t = S_t + \delta_t^a$, $\mathrm{d}S_t = \eta_t$;
	With probability $\lambda^b\left(\delta_t^b\right)\mathrm{d}t$, $\mathrm{d}N_t^b = 1$, $\mathrm{d}X_t = -S_t + \delta_t^b$, $\mathrm{d}S_t = -\eta_t$;
	The mid-price is updated by a random increment $\pm\sqrt{v_t}\sqrt{dt}$;
	The volatility is updated accordingly by a random increment:
	$\theta(\alpha - v_t)\mathrm{d}t + \xi\sqrt{v_t}(\rho\sqrt{t} \pm \sqrt{1-\rho^2}\sqrt{t})$ or $\theta(\alpha - v_t)\mathrm{d}t + \xi\sqrt{v_t}(-\rho\sqrt{t} \pm \sqrt{1-\rho^2}\sqrt{t})$.
Step 3:	Set $t := t + \mathrm{d}t$, and return to Step 1.

4 The set of baseline parameters is the same with that for European call options with maturity 2014-05-16, but on daily basis.



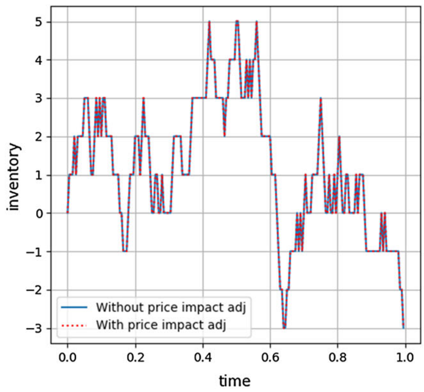


Fig. 3 Path simulation of the stock price and the associated evolution of the inventory w.r.t market-making strategies with/without price impact adjustment ($\eta = 0.003$)

2.4.1 The effect of price impact

To test the effect of price impact, we simply assume that there is only one market maker present in the security market. The market maker places his/her limit orders around the "fundamental price" that he/she believes is fair. The number of stocks held by the market maker jumps whenever there is a buy or sell order. We first plot the path simulation of the stock price and the associated evolution of the inventory with respect to market-making strategies with/without *price impact adjustment*[5] in Fig. 3.

We then plot the order placement policies (ask/bid quotes) of the market-making strategies with/without price impact adjustment in Fig. 4. The associated PnLs are depicted in Fig. 5. As we can see from Fig. 5, market-making strategies, whether with or without price impact adjustment, would accumulate revenues during "normal times" (no prolonged trend and auto-correlated order flows). Price impact adjustment would further enhance the profitability of this strategy.

To better understand the details of the impact-adjusted market-making strategy, we focus on the response process adopted by the dealer. For example, around time $t = 0.55$, there is a sequence of auto-correlated buy orders, which may be caused by real information in the marketplace. Dealers who had detected this information in their models would lower their bid premiums and raise their ask premiums to avoid risk arising from information asymmetry. Meanwhile, they would widen their ask–bid spreads to further compensate themselves for the increasing liquidity risk.

To emphasize the effect of price impact on dealer's market-making strategy, we set $\eta = 0.003$ and run 10,000 simulations of the model. The performances of the trading strategies with/without price impact adjustment are displayed in Table 2 and Fig. 6.

It is clear that dealers, in awareness of the impact incurred by the auto-correlated order flows, would adjust their trading strategies accordingly and this leads to not only a relatively higher return but also a relatively lower standard deviation. It is also worth

[5] Price impact adjustment here refers to whether the dealer's model takes the price impact phenomena into consideration.

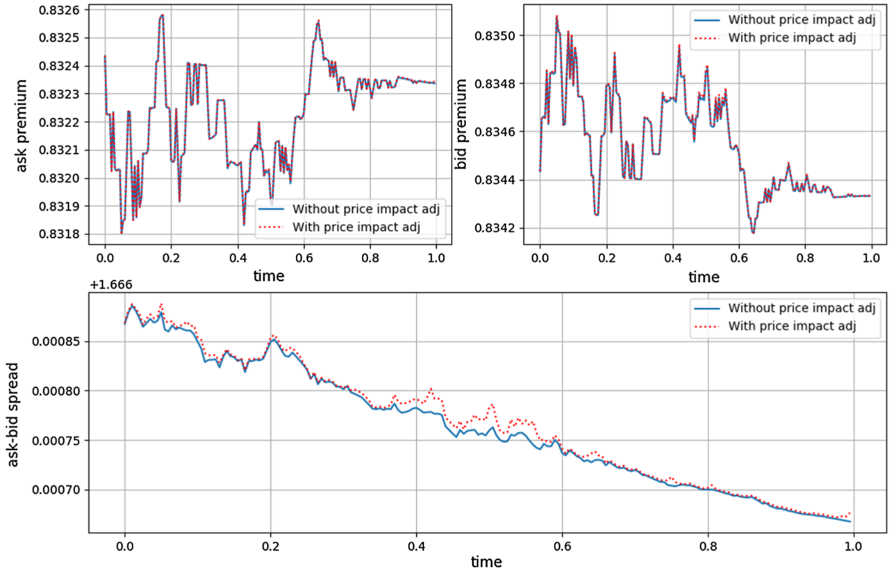


Fig. 4 Order placement policies of the market-making strategies with/without price impact adjustment

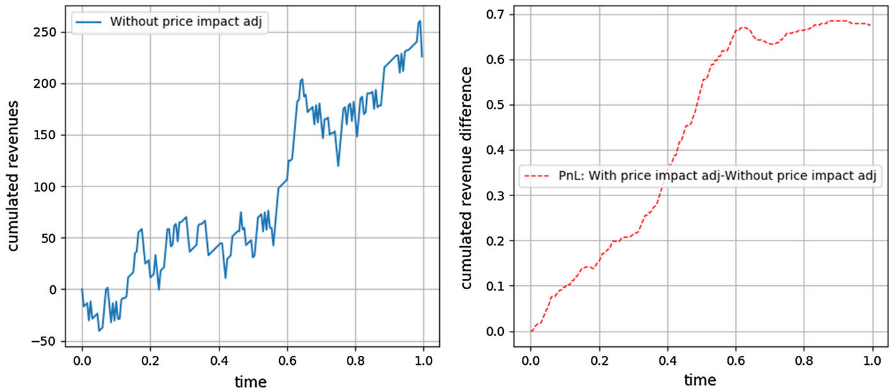


Fig. 5 PnLs of the market-making strategies with/without price impact adjustment

Table 2 A comparison of trading strategies

Strategy	Profit	Std (profit)	q_T	Std (q_T)
Without price impact adjustment	246.20	32.27	-0.11	9.33
With price impact adjustment	**246.43**	**30.25**	$\mathbf{-0.00}$	**2.07**

noticing that an impact-adjusted market-making strategy that trades 100,000 shares[6] per trade would make extra \$23,000 per day, which comes to \$5.8 million per year. It

6 In our experiment, dealers are committed to, respectively, buy/sell *one* share of stock at their quoting prices.

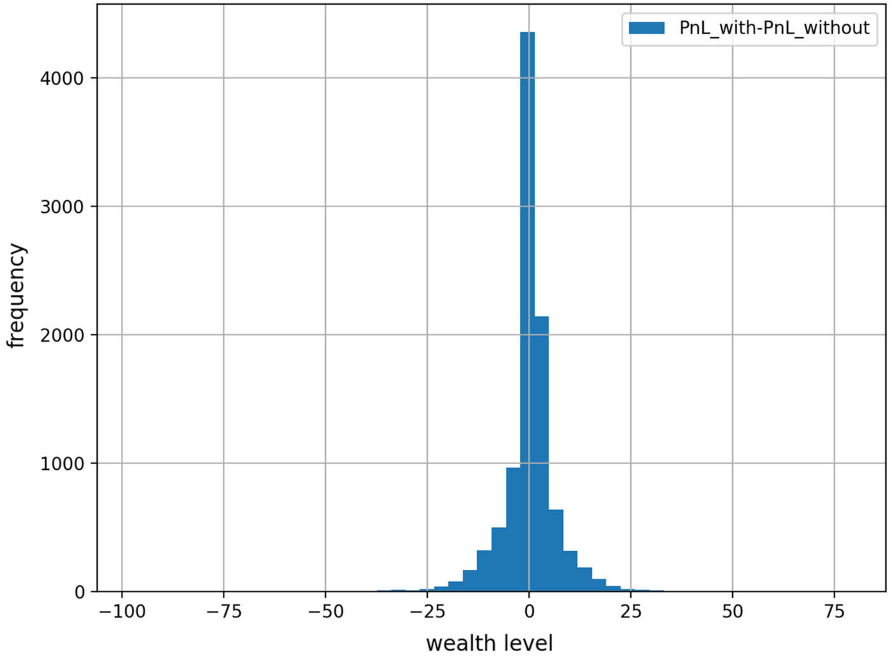


Fig. 6 A comparison of trading strategies: $\mathrm{PnL}_{\mathrm{with}} - \mathrm{PnL}_{\mathrm{without}}$

is a great number when compared with the average profitability of an high-frequency trading strategy in US equities (in relatively good times), which is \$0.001 per share (one-tenth of a penny) (Narang 2013).

2.4.2 The effect of risk aversion and price impact on trading curves

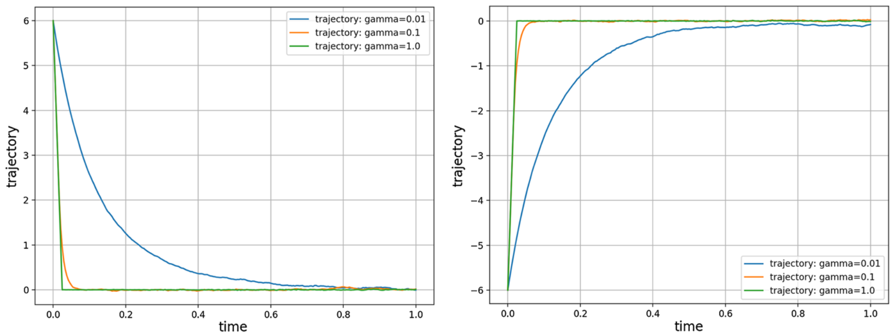
To test the effect of risk aversion, we first plot in Fig. 7a, b the trading curves with initial inventory $Q = 6$ and $Q = -6$, respectively, subject to Case (1) $\gamma = 0.01$, Case (2) $\gamma = 0.1$ and Case (3) $\gamma = 1$, under the setting of $\eta = 0$, when the optimal order placement policy is adopted.

We then plot in Fig. 8a, b the trading curves subject to Case (1) under different settings of η:

- without price impact: $\eta = 0$;
- with price impact: $\eta = 0.003$,

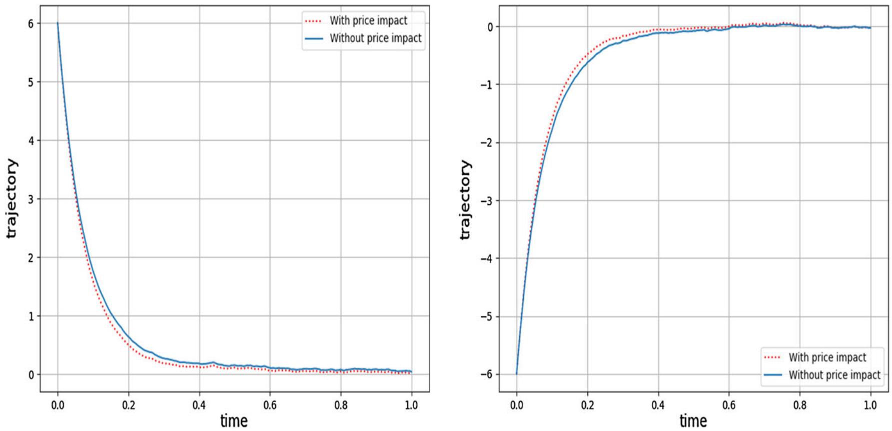
to test the effect of price impact.

In Case (1), we have $\gamma = 0.01$, which presents a trader who is risk-neutral. In Case (3), $\gamma = 1$, which describes a risk-averse trader who wishes to sell quickly to reduce exposure to volatility risk. In Case (2), we have $\gamma = 0.1$, which lies between the above two extremes. We can see from Fig. 7 that traders with higher risk aversion prefer to close their positions quickly to avoid risk arising from stock's volatility. Similar conclusions, except for the liquidation speed, can be drawn for the case of



(a) Trading curve with $Q = 6$. **(b)** Trading curve with $Q = -6$.

Fig. 7 Trading curves for dealers with different risk aversion under the setting of $\eta = 0$



(a) Trading curve with $Q = 6$. **(b)** Trading curve with $Q = -6$.

Fig. 8 Trading curves subject to Case (1) under different settings of price impact parameter

$\eta > 0$. Compared with the case of $\eta = 0$, dealers who had detected signals implied by the order flow would like to close their positions as soon as possible to avoid loss arising from price impact.

2.4.3 Strategy comparison

We now test the performance of our strategies with/without price impact adjustment under the Heston stochastic volatility model. The performances of our strategies with $\eta = 0.003$ are displayed in Table 3.

Comparing with the strategy without price impact adjustment, traders with price impact adjustment obtain a slightly higher return and lower volatility. The more cautious the trader is, the lower the PnL and risk. It is not difficult to understand why the impact-adjusted strategy results in a slightly higher return than the other strategy since it is adapted to the market sentiment.

Table 3 A comparison of market-making strategies

Strategy	PnL	Std (PnL)	Final q_T	Std (final q_T)
Risk aversion: $\gamma = 0.01$				
Without price impact adjustment	264.67	41.60	−0.10	9.36
With price impact adjustment	264.99	39.19	−0.02	5.44
Risk aversion: $\gamma = 0.1$				
Without price impact adjustment	246.20	32.27	−0.11	9.33
With price impact adjustment	246.43	30.25	−0.00	2.07
Risk aversion: $\gamma = 1$				
Without price impact adjustment	194.91	19.29	0.03	8.62
With price impact adjustment	195.24	18.42	−0.02	0.82

3 Concluding remarks

In this paper, we adopted a stochastic volatility model to describe the dynamics of the underlying stock's volatility and derived mean-quadratic-variation optimal trading strategies for market making in the stock markets. A stochastic control approach was adopted to solve the associated optimization problems. An important topic for future research is to develop accurate and efficient methods to solve the resulting HJB equation. This is particularly important because the optimal trading strategy cannot be obtained without solving the resulting HJB equation. Here we employed the Heston's mean-reverting stochastic volatility model to describe the dynamics of the mid-price process. It may not be unreasonable in a short-term trading environment. However, at that short timescale, intra-day seasonality and stock–vol correlation will also play important roles in the market-making process. In future research, we will focus our work on this issue. Generally, volatility tends to be correlated with high trading volume and company specific news (e.g., earning announcements), and other important further research issues may include taking into account these empirical characteristics and extending the model to more general cases, for example, the case of a trend, which is predictable, in the price dynamics, the effect of news events on securities markets, the application of HMM in the LOBs and the case of a multiple-dealer competitive market (Yang et al. 2019).

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4 Appendix

A. Proof of Proposition 2

Proof (a) Let $\left(\delta_t^a, \delta_t^b\right)_{t\geq 0} \in \mathcal{A}$. Then by Dynkin's formula we have

$$\begin{aligned}
&\mathbb{E}_t\left[\phi\left(q_{T_R}^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_{T_R}^{t,\nu}, T_R\right)\right]\\
&\quad= \phi(q,\nu,t) + \mathbb{E}_t\Bigg[\int_t^{T_R}\Bigg(\phi_t + \theta(\alpha - \nu_u^{t,\nu})\phi_\nu + \frac{1}{2}\xi^2\nu_u^{t,\nu}\phi_{\nu\nu}\\
&\qquad+\left[\phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}} - 1, \nu_u^{t,\nu}, u\right) - \phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u\right)\right]\lambda^a(\delta_u^a)\\
&\qquad+\left[\phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}} + 1, \nu_u^{t,\nu}, u\right) - \phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u\right)\right]\lambda^b(\delta_u^b)\Bigg)\mathrm{d}u\Bigg]\\
&\qquad+\mathbb{E}_t\left[\int_t^{T_R}\xi\sqrt{\nu_u^{t,\nu}}\phi_\nu(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u)\mathrm{d}B_u\right],
\end{aligned}$$

where $T_R = \min\left\{R, T, \inf\{s > t; \mathbb{E}_t\left[\int_t^s \nu_u^{t,\nu}\phi_\nu^2\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u\right)\mathrm{d}u\right] \geq R\}\right\}$. Note that $T_R \to T$, as $R \to \infty$ (by condition (iii)). The stopped process

$$\left\{\int_t^{T_R}\xi\sqrt{\nu_u^{t,\nu}}\phi_\nu\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u\right)\mathrm{d}B_u\right\}_{R\geq 0}$$

is then a martingale under the probability measure $\mathcal{P}$.
From Condition (i), we have

$$\begin{aligned}
&\mathbb{E}_t\left[\phi\left(q_{T_R}^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_{T_R}^{t,\nu}, T_R\right)\right]\\
&\quad= \phi(q,\nu,t) + \mathbb{E}_t\Bigg[\int_t^{T_R}\Bigg(\phi_t + \theta(\alpha - \nu_u^{t,\nu})\phi_\nu + \frac{1}{2}\xi^2\nu_u^{t,\nu}\phi_{\nu\nu}\\
&\qquad+\left[\phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}} - 1, \nu_u^{t,\nu}, u\right) - \phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u\right)\right]\lambda^a(\delta_u^a)\\
&\qquad+\left[\phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}} + 1, \nu_u^{t,\nu}, u\right) - \phi\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}, \nu_u^{t,\nu}, u\right)\right]\lambda^b(\delta_u^b)\Bigg)\mathrm{d}u\Bigg]\\
&\quad\leq \phi(q,\nu,t) - \mathbb{E}_t\Bigg[\int_t^{T_R}\Bigg(-\frac{\gamma}{2}\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}\right)^2\nu_u^{t,\nu}\\
&\qquad+\left(\delta_u^a + \beta - \frac{\gamma\eta^2}{2}\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}} - 1\right)^2\right)\lambda^a(\delta_u^a)\\
&\qquad+\left(\delta_u^b - \beta - \frac{\gamma\eta^2}{2}\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}} + 1\right)^2\right)\lambda^b(\delta_u^b)\Bigg)\mathrm{d}u\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \phi(q,\nu,t) - \Bigg\{\mathbb{E}_t\Bigg[\int_t^{T_R}\left(\delta_u^a+\beta-\frac{\gamma\eta^2}{2}\Big(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}-1\Big)^2\right)\mathrm{d}N_u^a \\
&\quad +\left(\delta_u^b-\beta-\frac{\gamma\eta^2}{2}\Big(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}+1\Big)^2\right)\mathrm{d}N_u^b \\
&\quad -\frac{\gamma}{2}\int_t^{T_R}\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}\right)^2\nu_u^{t,\nu}\mathrm{d}u\Bigg]\Bigg\}.
\end{aligned}
\tag{4.1}
$$

Meanwhile, we have

$$
\begin{aligned}
\mathbb{E}_t\left[\int_t^{T_R}\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}\right)^2\nu_u^{t,\nu}\mathrm{d}u\right] &\leq \mathbb{E}_t\left[\int_t^{T}\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}\right)^2\nu_u^{t,\nu}\mathrm{d}u\right] \\
&= \int_t^T \mathbb{E}_t\left[\left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}\right)^2\nu_u^{t,\nu}\right]\mathrm{d}u \\
&< \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigg|\mathbb{E}_t\Bigg[\int_t^{T_R}\left(\delta_u^a+\beta-\frac{\gamma\eta^2}{2}\Big(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}-1\Big)^2\right)\mathrm{d}N_u^a \\
&\qquad +\left(\delta_u^b-\beta-\frac{\gamma\eta^2}{2}\Big(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}+1\Big)^2\right)\mathrm{d}N_u^b\Bigg]\Bigg| \\
&\leq \mathbb{E}_t\Bigg[\int_t^{T}|\delta_u^a+\beta-\frac{\gamma\eta^2}{2}\Big(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}-1\Big)^2|\mathrm{d}N_u^a \\
&\qquad +|\delta_u^b-\beta-\frac{\gamma\eta^2}{2}\Big(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}+1\Big)^2|\mathrm{d}N_u^b\Bigg] \\
&< \infty.
\end{aligned}
$$

Since ϕ satisfies a polynomial growth condition, we have[7]

$$
\left|\phi\Big(q_{T_R}^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}},\nu_{T_R}^{t,\nu},T_R\Big)\right| \leq C\left(1+\max_{u\in[t,T]}|q_u^{t,q,(\delta_t^{*,a},\delta_t^{*,b})_{t\geq 0}}|^n+|\nu_{T_R}^{t,\nu}|^m\right).
\tag{4.2}
$$

Directly applying the dominated convergence theorem and sending R in Eq. (4.1) to infinity yields

$$
\begin{aligned}
\phi(q,\nu,t) \geq \mathbb{E}_t\Bigg[\int_t^T&\left(\delta_u^a+\beta-\frac{\gamma\eta^2}{2}(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}-1)^2\right)\mathrm{d}N_u^a \\
&+\left(\delta_u^b-\beta-\frac{\gamma\eta^2}{2}(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}+1)^2\right)\mathrm{d}N_u^b
\end{aligned}
$$

[7] The right-hand-side term of Eq. (4.2) is integrable.

$$-\frac{\gamma}{2}\int_t^T \left(q_u^{t,q,(\delta_t^a,\delta_t^b)_{t\geq 0}}\right)^2 \nu_u^{t,\nu} du\Bigg]$$
$$\geq V^{(\delta_t^a,\delta_t^b)_{t\geq 0}}(q,\nu,t).$$

Since $\left(\delta_t^a, \delta_t^b\right)_{t\geq 0} \in \mathcal{A}$ is arbitrary, we conclude that

$$\phi(q,\nu,t) \geq V(q,\nu,t) \quad \text{for all } (q,\nu,t) \in \mathbb{Z}\times\Re_+\times[0,T]. \tag{4.3}$$

(b) Apply the above argument to $\left(\delta_t^a, \delta_t^b\right)_{t\geq 0} = \left(\delta_t^{*,a}, \delta_t^{*,b}\right)_{t\geq 0}$. The calculations above give equality and hence

$$\phi(q,\nu,t) = V^{\left(\delta_t^{*,a},\delta_t^{*,b}\right)_{t\geq 0}}(q,\nu,t) \leq V(q,\nu,t) \tag{4.4}$$

for all $(q,\nu,t) \in \mathbb{Z}\times\Re_+\times[0,T]$. Combining (4.3) and (4.4), we get the results in (b). □

B. Proof of Theorem 2

Proof Due to our choice of "*mean-variance*" objective function, we are able to simplify the problem with an ansatz on the form of the value function:

$$V(q,\nu,t) = f(\nu,t) + g(\nu,t)q + h(\nu,t)q^2. \tag{4.5}$$

Then, we have

$$\begin{cases} \delta_t^{*,b} = 1/k + b - g(\nu,t) - h(\nu,t)(2q+1) \\ \delta_t^{*,a} = 1/k - b + g(\nu,t) + h(\nu,t)(2q-1). \end{cases}$$

The ask–bid spread $\delta_t^{*,a} + \delta_t^{*,b} = \frac{2}{k} - 2h(\nu,t)$ is independent of the inventory. Substituting Eq. (4.5) and the linear approximation of the order arrival terms into Eq. (2.9) and grouping terms of q yields

$$\begin{cases} g_t + \theta(\alpha-\nu)g_\nu + \frac{1}{2}\xi^2\nu g_{\nu\nu} = 0 \\ g(\nu,T) = 0. \end{cases} \tag{4.6}$$

By the Feynman–Kac formula, $g(\nu,t) = 0$. Grouping terms in the coefficients of q^2 yields

$$\begin{cases} h_t + \theta(\alpha-\nu)h_\nu + \frac{1}{2}\xi^2\nu h_{\nu\nu} - \frac{\gamma}{2}\nu = 0 \\ h(\nu,T) = 0, \end{cases}$$

whose solution can be directly obtained using the Feynman–Kac formula

$$
\begin{aligned}
h(\nu, t) &= E_t\left[\int_t^T -\frac{1}{2}\gamma\nu_u \mathrm{d}u\right] \\
&= -\frac{1}{2}\gamma\int_t^T E_t[\nu_u]\mathrm{d}u \\
&= -\frac{\gamma}{2\theta}(\nu-\alpha)\left[1-e^{-\theta(T-t)}\right]-\frac{\gamma}{2}\alpha(T-t). \qquad (4.7)
\end{aligned}
$$

Grouping terms in the coefficients of q^0 yields

$$
\begin{cases} f_t+\theta(\alpha-\nu)f_\nu+\frac{1}{2}\xi^2\nu f_{\nu\nu}+2Ah(\nu,t)=0 \\ f(\nu,T)=0. \end{cases}
$$

Thus,

$$
\begin{aligned}
f(\nu,t) &= \mathbb{E}_t\left[\int_t^T 2Ah(\nu_r, r)\mathrm{d}r\right] \\
&= -\frac{\gamma A}{\theta}(\nu-\alpha)\left[\frac{1}{\theta}\left(1-e^{-\theta(T-t)}\right)-e^{-\theta(T-t)}(T-t)\right]-\frac{\gamma\alpha A}{2}(T-t)^2.
\end{aligned}
$$

By now, we have got an approximation to the solution of the HJB equation, which is given by

$$
\begin{aligned}
\tilde{V}(q,\nu,t) = &-\frac{\gamma A}{\theta}(\nu-\alpha)\left[\frac{1}{\theta}\left(1-e^{-\theta(T-t)}\right)-e^{-\theta(T-t)}(T-t)\right]-\frac{\gamma\alpha A}{2}(T-t)^2 \\
&-\frac{\gamma q^2}{2\theta}(\nu-\alpha)\left[1-e^{-\theta(T-t)}\right]-\frac{\gamma q^2}{2}\alpha(T-t).
\end{aligned}
$$

We now analyze the difference between the approximate and the exact solutions under the Heston stochastic volatility model. Let

$$
V^i(q,\nu,t) = -\frac{\gamma q^2}{2\theta}(\nu-\alpha)\left[1-e^{-\theta(T-t)}\right]-\frac{\gamma q^2}{2}\alpha(T-t).
$$

Suppose that $V(q,\nu,t) = w^q(t,\nu)+V^i(q,\nu,t)$, where $(w^q)_{q\in\mathbb{N}}$ is a family of functions in $\mathcal{C}^{1,2}(t,\nu)$, and $\mathbb{N}$ is the set of natural numbers. Substituting the above expression into Eq. (2.9) yields

$$
\begin{cases} w_t^q+\theta(\alpha-\nu)w_\nu^q+\dfrac{1}{2}\xi^2\nu w_{\nu\nu}^q+\epsilon^q(\nu,t)=0 \\ w^q(\nu,T)=0, \end{cases} \qquad (4.8)
$$

where

$$\begin{aligned}\epsilon^q(\nu,t) &= \frac{A}{k}\left(e^{-k\delta_t^{*,a}} + e^{-k\delta_t^{*,b}}\right)\\ &= \frac{A}{k}\Big[e^{\left(-k(w^q-w^{q-1}+\frac{1}{k}-b+\left(\frac{\gamma}{2\theta}(\nu-a)(1-e^{-\theta(T-t)})+\frac{\gamma}{2}\alpha(T-t)\right)(-2q+1)\right)}\\ &\quad +e^{\left(-k(w^q-w^{q+1}+\frac{1}{k}+b+\left(\frac{\gamma}{2\theta}(\nu-a)(1-e^{-\theta(T-t)})+\frac{\gamma}{2}\alpha(T-t)\right)(2q+1)\right)}\Big]\end{aligned} \tag{4.9}$$

We first note that

$$\begin{cases}\underbrace{\delta_t^{*,a}}_{\text{exact}} - \underbrace{\hat{\delta}_t^{*,a}}_{\text{approx}} = w^q - w^{q-1} = E_t\left[\int_t^T(\epsilon^q - \epsilon^{q-1})(\nu_r, r)\mathrm{d}r\right]\\ \underbrace{\delta_t^{*,b}}_{\text{exact}} - \underbrace{\hat{\delta}_t^{*,b}}_{\text{approx}} = w^q - w^{q+1} = E_t\left[\int_t^T(\epsilon^q - \epsilon^{q+1})(\nu_r, r)\mathrm{d}r\right],\end{cases}$$

and that

$$\epsilon^q(\nu,t) = \frac{A}{k}\left(2 - k(\delta_t^{*,a}+\delta_t^{*,b}) + \frac{k^2}{2}\left((\delta_t^{*,a})^2 + (\delta_t^{*,b})^2\right) + o((\delta_t^{*,a})^2 + (\delta_t^{*,b})^2)\right).$$

Since $\delta_t^{*,a}$ and $\delta_t^{*,b}$ are relatively small and $\delta_t^{*,a}+\delta_t^{*,b}$ is independent of q, we have

$$\begin{aligned}|\delta_t^{*,a} - \hat{\delta}_t^{*,a}| &= \left|\mathbb{E}_t\left[\int_t^T\left(\epsilon^q - \epsilon^{q-1}\right)(\nu_r, r)\mathrm{d}r\right]\right|\\ &\le \mathbb{E}_t\left[\int_t^T\left|\left(\epsilon^q - \epsilon^{q-1}\right)(\nu_r, r)\right|\mathrm{d}r\right]\\ &= O\Big(\left|\delta_r^{*,a}(q)\delta_r^{*,b}(q) - \delta_r^{*,a}(q-1)\delta_r^{*,b}(q-1)\right|\Big].\end{aligned}$$

That is, the differences between the exact and the approximate ask quotes can be very small. (Similar arguments can be directly applied to the bid side.) □

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