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Brownian Motion on the Hyperbolic Plane and Selberg Trace Formula

Nobuyuki Ikeda

Department of Computer Science, Ritsumeikan University, Kusatsu 525-8577, Japan
E-mail: nobuyuki@theory.cs.ritsumei.ac.jp

and

Hiroyuki Matsumoto

School of Informatics and Sciences, Nagoya University, Chikusa-ku, Nagoya 464-8601, Japan
E-mail: matsu@info.human.nagoya-u.ac.jp

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We will show that the relation of the heat kernels for the Schrödinger operators with uniform magnetic fields on the hyperbolic plane $\mathbf{H}^2$ (the Maass Laplacians) and for the Schrödinger operators with Morse potentials on $\mathbf{R}$ is given by means of a one-dimensional Fourier transform in the framework of stochastic analysis, where the Brownian motion on $\mathbf{H}^2$ plays an important role. By using this relation, we will give the explicit forms of the Green functions. As a typical related problem, we will discuss the Selberg trace formula. The close relation of the trace formula with the corresponding classical mechanics will also be discussed.

Key Words: stochastic analysis; Selberg trace formula; Maass Laplacian; Morse potential.

1. INTRODUCTION

Let $\mathbf{H}^2$ be the upper half plane with rectangular coordinates (x, y), $x \in \mathbf{R}$, $y > 0$, and with the Poincaré metric. $PSL_2(\mathbf{R}) = SL_2(\mathbf{R})/\pm I$, I being the identity element, acts on $\mathbf{H}^2$ as the isometry group by the linear fractional transformation (2.1) below. We consider the Schrödinger operator H_B with a uniform magnetic field on $\mathbf{H}^2$ defined by

$$H_B = \frac{1}{2}\, y^2 \left(\sqrt{-1}\,\frac{\partial}{\partial x} + \frac{B}{y}\right)^2 - \frac{1}{2}\, y^2\, \frac{\partial^2}{\partial y^2}, \qquad B \in \mathbf{R}, \tag{1.1}$$

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and the Schrödinger operator $H^M_{\lambda, B}$ with the Morse potential $V^M_{\lambda, B}$ on $\mathbf{R}$ defined by

$$H^M_{\lambda, B} = -\frac{1}{2}\frac{d^2}{dx^2} + V^M_{\lambda, B}, \qquad V^M_{\lambda, B}(x) = \frac{1}{2}\lambda^2 e^{2x} - \lambda B e^x, \qquad \lambda, B \in \mathbf{R}. \tag{1.2}$$

Since we have two parameters $\lambda, B \in \mathbf{R}$, we will consider the family of all Morse potentials. If we set $D_B = -2H_B + B^2$, we get

$$D_B = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2\sqrt{-1}\, By\frac{\partial}{\partial x}. \tag{1.3}$$

This operator D_B is called the Maass Laplacian and plays an important role in several fields of mathematics. Needless to say, there has been much work on it (see, e.g., [14–16, 24, 25, 41, 42, 44]). The origin of the study of the operator $H^M_{\lambda, B}$ goes back to the work by P. M. Morse [39] in 1929, in which a problem on the molecular vibrations in quantum mechanics has been discussed. See also [30]. For the last twenty years, it has been studied extensively in the physics literature by using the theory of Feynman path integrals (see, e.g., [5, 13, 20]).

The purpose of this paper is to discuss the properties of the operators H_B, $H^M_{\lambda, B}$ and related problems in the framework of stochastic analysis by using the Brownian motion on $\mathbf{H}^2$. This helps us to understand in a unified way some of the results by Comtet [6], Comtet *et al.* [7], Fay [15], Grosche [20], Patterson [41], Yor [50], and so on, in various fields. As a typical related problem, we will study the Selberg trace formula for the semigroup $\exp(-tH_B)$ generated by H_B which acts on automorphic forms with respect to a discrete hyperbolic subgroup of $PSL_2(\mathbf{R})$.

For this purpose we will consider the heat kernels $q_B(t, z_1, z_2)$, $t > 0$, $z_1, z_2 \in \mathbf{H}^2$, of $\exp(-tH_B)$ and $q^M_{\lambda, B}(t, \xi_1, \xi_2)$, $t > 0$, $\xi_1, \xi_2 \in \mathbf{R}$, of $\exp(-tH^M_{\lambda, B})$. Noting that the Brownian motion on $\mathbf{H}^2$ is explicitly given as a Wiener functional by solving the stochastic differential equation (2.3) below, we will show the representations of $q_B(t, z_1, z_2)$ by means of Wiener integrals (see (2.9), (2.10) below) on the one-dimensional Wiener space (W, P). On the other hand the Feynman–Kac formula gives us the integral representation (2.16) below of $q^M_{\lambda, B}(t, \xi_1, \xi_2)$ on W. In both representations we will meet the Wiener functionals of exponential type defined by

$$A_t(w) = \int_0^t \exp(2w_s)\, ds \qquad \text{and} \qquad a_t(w) = \int_0^t \exp(w_s)\, ds. \tag{1.4}$$

One of the representations (2.9) for $q_B(t, z_1, z_2)$ will be useful in the proof of the Selberg trace formula and the other will be used to show the close relation between the two heat kernels. As will be mentioned in Section 8, the first one also helps us to study the Selberg trace formula from the point of view of semi-classical approximation.

By using the representations of the heat kernels, we will show the relation (2.17) below for them. It can be also proved by separation of variables and says that $q_B(t, z_1, z_2)$ and the family $\{q^M_{\lambda, B}(t, \xi_1, \xi_2); \lambda \in \mathbf{R}\}$ are related by means of the one-dimensional Fourier transform with respect to λ. Therefore, if we have some information on one of H_B and $\{H^M_{\lambda, B}\}$, we can derive it on the other one automatically by virtue of the injectivity of the Fourier transform in principle. The relation of the Schrödinger operators with the Morse potentials to the analysis on the hyperbolic spaces has been mentioned in Debiard and Gaveau [10].

In the theory of harmonic analysis on $\mathbf{H}^2$, the Selberg transform has played an important role. We can also calculate it easily for the heat kernel of $\exp(-tH_B)$ by using the Itô formula. In order to obtain the explicit forms of the heat kernel and the Green function for H_B, we have to calculate the inverse of the Selberg transform. As has been pointed out by Selberg himself [46], in the case where $B=0$ the Selberg transform is a composition of a one-dimensional Fourier transform and an Abel-type transform. Therefore the inverse transform is obtained by a repeated use of elementary calculus and, as an easy consequence, the well known explicit form of the heat kernel for H_0, i.e., half of the Laplace–Beltrami operator on $\mathbf{H}^2$, is derived. When $B \neq 0$, the situation is quite different and the calculation of the inverse of the Selberg transform is not easy. In the 1970s, under the influence of Selberg [46], Elstrodt [14], Fay [15], and Patterson [41] have carried it out by using harmonic analysis on $\mathbf{H}^2$ and they have shown the explicit forms of the Green function, the heat kernel, and their spectral decompositions. Comtet [6] has also discussed the various properties of H_B.

For $H^M_{\lambda, B}$ the explicit forms of the Green function and the propagator have been obtained by using the theory of Feynman path integrals in physics literatures (cf. [5, 13, 20]). As will also be shown in this paper, this is an exactly solvable model.

As was mentioned above, the Wiener functionals $A_t(w)$ and $a_t(w)$ given by (1.4) appear in the representations of the heat kernels $q_B(t, z_1, z_2)$ and $q^M_{\lambda, B}(t, \xi_1, \xi_2)$. These Wiener functionals play important roles in the theory of mathematical finance. Letting S be an exponential random variable with parameter $\theta^2/2$ which is independent of $w = \{w_s\}_{s \geqq 0}$, Yor [50] has calculated the Laplace transform of the joint distribution of (A_S, w_S) and has shown the explicit form of the probability density function of (A_t, w_t) in the study of the Asian option (cf. Section 3). Following the idea of [50], Leblanc [31] has calculated the Laplace transform of the joint distribution of (A_S, a_S, w_S) in the study of the Hull–White model. We can obtain the explicit form of the Green function for $H^M_{\lambda, B}$ by a slight modification of their arguments. Moreover the Brownian motion on $\mathbf{H}^2$ and the Wiener functionals of exponential type are closely related to problems of diffusion processes in random media (cf., e.g., [9, 17, 38]).

In the first half of this paper, by using the general theory of the Sturm–Liouville operators, we will first show the explicit form of the Green function for $H^M_{\lambda, B}$ in terms of the Whittaker functions and then, by using the fundamental formula (2.17), we will obtain that form for H_B in terms of the Gauss hypergeometric function. However, what we would like to emphasize is that our arguments can be traced in the reverse way and that we can also show the results of Yor [50] and Leblanc [31] mentioned above. In this sense we will study one object from different points of view. The relation of the Wiener functionals of exponential type to the analysis on the hyperbolic spaces has been mentioned in Yor [50] and, by using his result, Gruet [21] has discussed the explicit form of the heat kernel of the semigroup generated by the Laplace–Beltrami operator on any dimensional hyperbolic space. See also [1, 3] for the related problems.

It should be now mentioned that Grosche [20] has shown a formula of the same type as (2.17) by using the theory of Feynman path integrals. Using this relation, he has obtained the explicit forms of the Green function for H_B and its spectral decomposition from those of $H^M_{\lambda, B}$. In order to obtain the explicit form of the Green function for $H^M_{\lambda, B}$, the physicists have used space–time transform in the calculation of Feynman path integrals (cf. [5, 13, 20]). This method is essentially the same as that of Yor [50] and Leblanc [31], who have used random time change in the theory of diffusion processes. We have learned some methods of calculations from both of them.

The main topic in the latter half is the Selberg trace formula. Letting Γ be a given discrete hyperbolic subgroup of $PSL_2(\mathbf{R})$, we will consider the compact smooth Riemann surface $M = \Gamma \backslash \mathbf{H}^2$ with genus larger than or equal to two and will study the Selberg trace formula for the semigroup generated by H_B acting on automorphic forms with respect to Γ in full generality. By using harmonic analysis on $\mathbf{H}^2$, it has been studied extensively after Selberg [46] by several people (cf. [24, 25, 41, 49], and so on). The Selberg transform mentioned above plays an important role in these works.

On the other hand, McKean [36] has shown the Selberg trace formula for the semigroup generated by the Laplace–Beltrami operator on M regarding the explicit form of the heat kernel on $\mathbf{H}^2$ as known. D'Hoker and Phong [11] have done it when the weight $2B$ of automorphic forms is an integer by combining the explicit form of the heat kernel of $\exp(-tH_B)$ due to Fay [15] with some formulae for the Chebyshev polynomials.

We will prove the trace formula along the same line as that in McKean and d'Hoker–Phong and by using stochastic analysis in the most general framework when Γ is hyperbolic and M is compact. For the calculation corresponding to the null-homotopic classical paths, the explicit form of the heat kernel on the diagonal set itself appears and we will use the result mentioned in the first half of this paper. One of the advantages of our approach consists in the explicit evaluation of the terms corresponding to the not null-homotopic classical paths on M or to the elements in $\Gamma \backslash I$. We will show that we can evaluate them by the equality

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi A_t}} \exp\left(-\frac{x^2}{2A_t}\right) dx = 1$$

if we use the expression (2.9) below of the heat kernel of $\exp(-tH_B)$ by means of the Wiener integral instead of the explicit form. In this step we do not need the precise information of the heat kernel or, equivalently, that of the joint distribution of (A_t, a_t, w_t).

The geodesic flow on a compact Riemann surface with constant negative curvature gives us a typical example of chaotic classical mechanics and some physicists have discussed the Selberg trace formula in relation to problems of quantum chaos by using the theory of Feynman path integrals (cf., e.g., [7, 19, 22]). Some of them have considered the trace formulae as semiclassical asymptotic ones. From the physical point of view it is important to know the roles in quantum mechanics of the quantities of the corresponding classical mechanics, for example, the action integrals along the classical paths and the Jacobi fields along them. Gutzwiller (cf. [22] and his works cited therein) and Comtet *et al.* [7] have discussed the trace formula from this point of view.

We will show that each factor in the term corresponding to the not null-homotopic classical paths is obtained from the meaningful quantities as those of classical mechanics and will see the importance of the second variations of the action integrals along the classical paths. Moreover, in order to show how they come up, we will derive each term by combining the Laplace method with the short time asymptotics of the heat kernel on $\mathbf{H}^2$. It should be mentioned that the integrand in our formula is a Gaussian function and, therefore, we can exactly evaluate the integral without using the semiclassical asymptotics and the Laplace method. This is the reason why we obtain the Selberg trace formula as an exact one from the point of view of semiclassical analysis.

This paper is organized as follows. In Section 2 we will give some fundamental formulae for the heat kernels by using stochastic analysis. Sections 3, 4, and 5 will be devoted to the study of the explicit forms of the Green functions for H_B and $H^M_{\lambda,B}$. The explicit forms of the heat kernels of $\exp(-tH_B)$ and $\exp(-tH^M_{\lambda,0})$ will also be given. In Section 6, introducing the notations, we will recall the Selberg trace formula. Section 7 will be devoted to its proof. The relation of the Selberg trace formula to the corresponding classical mechanics will be discussed in Section 8. We will give a proof of the formula on the moments of the Wiener functional A_t in Section 9.

Before concluding the introduction, we mention a notation which will be used frequently in this paper. We will denote the coupling of a smooth nondegenerate Wiener functional F in the sense of Malliavin and a generalized Wiener functional Φ under the Wiener measure P by

$$\int F(w)\,\Phi(w)\,dP(w)$$

instead of $E[F\Phi]$. For the definitions, see [27, p. 364; 51]. We will use this notation to make the intuitive meaning clear.

2. BASIC FORMULAE FOR HEAT KERNELS

Let $\mathbf{H}^2$ be the upper half plane $\{z=(x, y); x\in\mathbf{R}, y>0\}$ with the Poincaré metric $ds^2=y^{-2}(dx^2+dy^2)$. We identify $\mathbf{H}^2$ as a subset in the complex plane $\mathbf{C}$ as usual. Then $PSL_2(\mathbf{R})=SL_2(\mathbf{R})/\pm I$, I being the identity element, acts on $\mathbf{H}^2$ by the linear fractional transformation

$$\sigma=\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}: z\mapsto \sigma z=\frac{a_\sigma z+b_\sigma}{c_\sigma z+d_\sigma} \tag{2.1}$$

and forms the isometry group on $\mathbf{H}^2$. Throughout, we represent the elements of $PSL_2(\mathbf{R})$ or $SL_2(\mathbf{R})$ in this manner.

Let α be the differential 1-form on $\mathbf{H}^2$ defined by

$$\alpha=\frac{B}{y}\,dx, \qquad B\in\mathbf{R}.$$

Then it is easy to show

$$\delta\alpha=0 \qquad \text{and} \qquad \Box\alpha=-(d\delta+\delta d)\,\alpha=0.$$

Letting Δ be the Laplace–Beltrami operator on $\mathbf{H}^2$, we consider the Schrödinger operator H_B on $\mathbf{H}^2$ with the magnetic vector potential α defined by

$$H_B f=-\tfrac{1}{2}\,\Delta f+\sqrt{-1}\,(df, \alpha)+\tfrac{1}{2}\,\|\alpha\|^2 f, \tag{2.2}$$

where (,) is the inner product in the cotangent bundle $T^*(\mathbf{H}^2)$ determined by the Poincaré metric and $\|\alpha\|^2=(\alpha,\alpha)$. If we rewrite (2.2) by using rectangular coodinates, we can easily show that this definition coincides with (1.1). Since the corresponding magnetic field $d\alpha=By^{-2}\,dx\wedge dy$ is the constant B times the volume form, it is called a Schrödinger operator with a uniform magnetic field on $\mathbf{H}^2$; it has been studied in the literatures of both mathematics and physics (see, e.g., [2, 6, 8, 20, 48]). As is mentioned in the Introduction, H_B is essentially the same operator as the Maass Laplacian given by (1.3). We will consider only H_B in this paper because we use stochastic analysis and the modification is trivial.

In order to study the heat kernel of the semigroup $\exp(-tH_B)$ generated by H_B, we introduce the Brownian motion on $\mathbf{H}^2$. Let $(W^{(2)},\mathscr{B}^{(2)},P^{(2)})$ be the two-dimensional standard Wiener space with the canonical filtration $\{\mathscr{B}_s^{(2)}\}_{s\geqq 0}$: $W^{(2)}$ is the space of all $\mathbf{R}^2$-valued continuous functions $w_\cdot^{(2)}=(w_\cdot^1,w_\cdot^2)$ on $[0,\infty)$ satisfying $w_0^{(2)}=0$ with the topology of uniform convergence on compact sets, $\mathscr{B}^{(2)}$ is the topological σ-field, $\mathscr{B}_s^{(2)}$ is the sub σ-field generated by $\{w_u^{(2)}\}_{0\leqq u\leqq s}$, and $P^{(2)}$ is the two-dimensional Wiener measure. Then the Brownian motion on $\mathbf{H}^2$, that is, the diffusion process generated by $\Delta/2$, is obtained by solving the following stochastic differential equation defined on $(W^{(2)},\mathscr{B}^{(2)},P^{(2)})$:

$$dX(s)=Y(s)\,dw_s^1,\qquad dY(s)=Y(s)\,dw_s^2. \tag{2.3}$$

Denoting by $Z_z(s,w^{(2)})=(X_z(s,w^{(2)}),Y_z(s,w^{(2)}))$ the solution of (2.3) satisfying $(X(0),Y(0))=(x,y)=z$, we have

$$\begin{cases} X_z(s,w^{(2)})=x+\int_0^s y\exp(w_u^2-u/2)\,dw_u^1 \\ Y_z(s,w^{(2)})=y\exp(w_s^2-s/2). \end{cases} \tag{2.4}$$

Therefore, letting $W_{\mathbf{H}^2}$ be the space of all $\mathbf{H}^2$-valued continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact sets and ϕ_z, $z\in\mathbf{H}^2$, be the map from $W^{(2)}$ into the path space $W_{\mathbf{H}^2}$ defined by

$$\phi_z(w^{(2)})=\{Z_z(s,w^{(2)})\}_{s\geqq 0},$$

we see that $\{\phi_z^{-1}P\}_{z\in\mathbf{H}^2}$ is the Brownian motion on $\mathbf{H}^2$. It should be noted that the inverse map ϕ_z^{-1} is also explicitly given by

$$w_s^1=\int_0^s Y_z(u)^{-1}\,dX_z(u),\qquad w_s^2=\log Y_z(s)-\log y+\tfrac{1}{2}s. \tag{2.5}$$

Moreover it is easy to check that $Z_z(s)=Z_z(s, w^{(2)})$ is a smooth non-degenerate Wiener functional in the sense of Malliavin (cf. [27, 33, 34, 51]). Hence, for $z_1, z_2 \in \mathbf{H}^2$, letting $\tilde{\delta}_{z_2}$ be the Dirac δ-function concentrated at z_2 with respect to the measure $dx\,dy$, we obtain the generalized Wiener functional $\tilde{\delta}_{z_2}(Z_{z_1}(s))$ in the sense of Watanabe as the composition of $\tilde{\delta}_{z_2}$ and $Z_{z_1}(s)$ (cf. [27, 51]).

Now let $q_B(t, z_1, z_2)$, $t>0$, $z_i=(x_i, y_i) \in \mathbf{H}^2$, $i=1, 2$, be the heat kernel of $\exp(-tH_B)$ with respect to the volume element $dm(z)=y^{-2}\,dx\,dy$, that is, the fundamental solution to the equation

$$\frac{\partial u}{\partial t}=-H_B u.$$

Then it holds that

$$q_B(t, z_1, z_2)=\int_{W^{(2)}} \exp(-\sqrt{-1}\, I_t(\alpha, w^{(2)}))\, \tilde{\delta}_{z_2}(Z_{z_1}(t, w^{(2)}))\, dP^{(2)} \cdot (y_2)^2, \tag{2.6}$$

where $I_t(\alpha, w^{(2)})$ is the stochastic line integral of the differential 1-form α along the path of $\{Z_{z_1}(s, w^{(2)})\}_{0 \leqq s \leqq t}$ (cf. [26, 27]). The integral in the right hand side of (2.6) is the generalized expectation in the sense of Watanabe [51], that is, the coupling of the smooth Wiener functional $\exp(-\sqrt{-1}\, I_t(\alpha, w^{(2)}))$ and the generalized Wiener functional $\tilde{\delta}_{z_2}(Z_{z_1}(t, w^{(2)}))$. For details, see [27, 51]. Throughout we will use this notation. Since $\delta\alpha=0$, $I_t(\alpha, w^{(2)})$ is a martingale (cf. [26, 27]). Moreover, by the explicit expression (2.4) of the Brownian motion $\{Z_{z_1}(s)\}_{s \geqq 0}$, we have

$$I_t(\alpha, w^{(2)})=\int_0^t \alpha(Z_{z_1}(s)) \circ dZ_{z_1}(s)=\int_0^t \frac{B}{Y_{z_1}(s)} \circ dX_{z_1}(s)=Bw_t^1.$$

Therefore we get a more simple expression of the heat kernel:

$$q_B(t, z_1, z_2)=\int_{W^{(2)}} \exp(-\sqrt{-1}\, Bw_t^1)\, \tilde{\delta}_{z_2}(Z_{z_1}(t, w^{(2)}))\, dP^{(2)} \cdot (y_2)^2. \tag{2.7}$$

We show that the right hand side of (2.7) can be rewritten by using the Wiener functionals of exponential type of one-dimensional standard Brownian motion. The following are the basic formulae in our arguments in some sections. In order to show them, we let $(W, \mathscr{B}, P)$ be the one-dimensional standard Wiener space and $A_t=A_t(w)$, $a_t=a_t(w)$, $t>0$, be the Wiener functionals given by (1.4), i.e.,

$$A_t(w)=\int_0^t \exp(2w_s)\, ds, \qquad a_t(w)=\int_0^t \exp(w_s)\, ds, \tag{2.8}$$

respectively. Note that $a_t(w)^2 < tA_t(w)$ holds for almost all w, which will be used in the following discussions. Moreover we define $w^\xi = \{w_s^\xi\}_{s \geqq 0}$ by $w_s^\xi = w_s + \xi$ for $\xi \in \mathbf{R}$.

THEOREM 2.1. *Let* $z_i = (x_i, y_i) \in \mathbf{H}^2$, $i = 1, 2$, *and* $t > 0$. *Then, setting* $\xi = \log y_1$ *and* $\eta = \log y_2$, *it holds that*

$$q_B(t, z_1, z_2) = e^{-t/8 - B^2 t/2} \left(\frac{y_2}{y_1}\right)^{3/2} \int_W \frac{1}{\sqrt{2\pi A_t}} \times \exp\left[-\frac{1}{2A_t}\left(\frac{x_2 - x_1}{y_1} + \sqrt{-1}\, Ba_t\right)^2\right] \delta_{y_2/y_1}(\exp(w_t))\, dP(w) \tag{2.9}$$

$$= e^{-t/8 - B^2 t/2} \frac{\sqrt{y_1 y_2}}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\,(x_2 - x_1)\lambda}\, d\lambda \times \int_W \exp\left[-\frac{\lambda^2}{2} A_t(w^\xi) + \lambda B a_t(w^\xi)\right] \delta_\eta(w_t^\xi)\, dP(w), \tag{2.10}$$

where δ_ζ *is the Dirac* δ*-function concentrated at* $\zeta \in \mathbf{R}$ *with respect to the Lebesgue measure.*

Proof. We set $(\tilde{w}_s^1, \tilde{w}_s^2) = (w_s^1, w_s^2 - s/2)$, which is a version of Brownian motion under the probability measure $\tilde{P}^{(2)}$ given by

$$d\tilde{P}^{(2)}|_{\mathscr{B}_s^{(2)}} = \exp[\tfrac{1}{2} w_s^2 - \tfrac{1}{8} s]\, dP^{(2)}|_{\mathscr{B}_s^{(2)}}$$

by the Cameron–Martin theorem. Then, by (2.7), we get

$$\begin{aligned} q_B(t, z_1, z_2) &= \int_{W^{(2)}} \exp(-\sqrt{-1}\, B\tilde{w}_t^1) \exp\left[-\frac{1}{2}\tilde{w}_t^2 - \frac{1}{8} t\right] \\ &\quad \times \tilde{\delta}_{z_2}(x_1 + y_1 \beta_t, y_1 \exp(\tilde{w}_t^2))\, d\tilde{P}^{(2)} \cdot (y_2)^2 \\ &= e^{-t/8} (y_2)^2 \left(\frac{y_1}{y_2}\right)^{1/2} \int_{W^{(2)}} \exp(-\sqrt{-1}\, B\tilde{w}_t^1) \\ &\quad \times \tilde{\delta}_{z_2}(x_1 + y_1 \beta_t, y_1 \exp(\tilde{w}_t^2))\, d\tilde{P}^{(2)}, \end{aligned}$$

where the Wiener functional $\beta_t = \beta_t(\tilde{w})$ is given by

$$\beta_t = \int_0^t \exp(\tilde{w}_s^2)\, d\tilde{w}_s^1.$$

Note that the conditional probability distribution of $(\tilde{w}_t^1, \beta_t)$ given $\{\tilde{w}_s^2\}_{s \geqq 0}$ is the Gaussian distribution with mean 0 and covariance matrix V_t given by

$$V_t = \begin{pmatrix} t & a_t(\tilde{w}^2) \\ a_t(\tilde{w}^2) & A_t(\tilde{w}^2) \end{pmatrix}.$$

Then we have

$$\begin{aligned} q_B(t, z_1, z_2) = e^{-t/8}(y_2)^2 \left(\frac{y_1}{y_2}\right)^{1/2} \int_{W^{(2)}} d\tilde{P}^{(2)} \int_{\mathbf{R}^2} \frac{1}{2\pi \sqrt{D_t}} \\ \times \exp\left(-\frac{1}{2}(V_t^{-1}\xi, \xi) - \sqrt{-1}\, B\xi^1\right) \\ \times \tilde{\delta}_{z_2}(x_1 + y_1\xi^2, y_1 \exp(\tilde{w}_t^2))\, d\xi, \end{aligned}$$

where $D_t = \det V_t = tA_t - (a_t)^2$. Moreover

$$\begin{aligned} &\frac{1}{2}(V_t^{-1}\xi, \xi) + \sqrt{-1}\, B\xi^1 \\ &\quad = \frac{A_t}{2D_t}\left(\xi^1 - \frac{a_t\xi^2 - \sqrt{-1}\, BD_t}{A_t}\right)^2 + \frac{1}{2A_t}(\xi^2 + \sqrt{-1}\, Ba_t)^2 + \frac{1}{2}B^2 t. \end{aligned}$$

Carrying out the integration over $\mathbf{R}^2$ and using $\tilde{\delta}_z(c\,\cdot) = c^{-2}\tilde{\delta}_{z/c}(\cdot)$, $c > 0$, we obtain (2.9) because $\{\tilde{w}_s^2\}_{s \geqq 0}$ is a one-dimensional Brownian motion under $\tilde{P}^{(2)}$.

Formula (2.10) is easily derived from (2.9) because the integrand of the right hand side of (2.9) is a Gaussian function and its Fourier transform is of a simple form. ∎

Let $d(z_1, z_2)$ be the hyperbolic distance between z_1 and $z_2 \in \mathbf{H}^2$. Then we have

$$\cosh d(z_1, z_2) = \frac{(x_2 - x_1)^2 + y_1^2 + y_2^2}{2y_1 y_2}, \qquad z_i = (x_i, y_i), \quad i = 1, 2. \tag{2.11}$$

Formula (2.9) says that, if $x_1 = x_2$, $q_B(t, z_1, z_2)$ is non-negative and depends only on y_1/y_2 for every fixed $t > 0$. This means that $q_B(t, z_1, z_2)$ is a function of $d(z_1, z_2)$ in this case. Moreover we show the following proposition on the action of $SL_2(\mathbf{R})$. In the following we denote by $\arg(z)$ the imaginary part of the principal branch of $\log z$ determined by $-\pi < \arg(z) \leqq \pi$ and set

$$z^k = |z|^k \exp(\sqrt{-1}\, k \arg(z)), \qquad z \in \mathbf{C}, \quad k \in \mathbf{R}.$$

PROPOSITION 2.2. (i) *For every* $\mu \in SL_2(\mathbf{R})$, $z_i \in \mathbf{H}^2$, $i=1, 2$, *and* $t>0$, *it holds that*

$$q_B(t, \mu z_1, \mu z_2) = j_\mu^B(z_1)\, q_B(t, z_1, z_2)\, j_\mu^B(z_2)^{-1}, \tag{2.12}$$

where j_μ^B *is the function defined by*

$$j_\mu^B(z) = \left(\frac{c_\mu z + d_\mu}{c_\mu \bar{z} + d_\mu}\right)^B. \tag{2.13}$$

(ii) *There exists a function* g_t *on* $[0, \infty)$ *such that*

$$q_B(t, z_1, z_2) = \left(\frac{z_2 - \bar{z}_1}{z_1 - \bar{z}_2}\right)^B g_t(d(z_1, z_2)). \tag{2.14}$$

Proof. (i) This is an easy consequence of the invariance of H_B under $SL_2(\mathbf{R})$: for any C^2 function f on $\mathbf{H}^2$, $\mu \in SL_2(\mathbf{R})$, $z \in \mathbf{H}^2$,

$$H_B(j_\mu^B(z)^{-1} f(\mu z)) = j_\mu^B(z)^{-1}\, (H_B f)(\mu z)$$

(cf. [15, 16]).

For the proof of (ii), we set

$$\mu = \frac{1}{\alpha - \beta}\begin{pmatrix} 1 & -\alpha \\ 1 & -\beta \end{pmatrix},$$

where α, β ($\alpha < \beta$) are the endpoints of the Euclidean semi-circle in the upper half plane whose center is real and which passes through z_1 and z_2. Then

$$\mu z_i = \frac{1}{\alpha - \beta}\,\frac{z_i - \alpha}{z_i - \beta} \in \sqrt{-1}\,\mathbf{R}, \qquad i = 1, 2.$$

By (2.12), we get

$$q_B(t, z_1, z_2) = j_\mu^B(z_1)^{-1}\, q_B(t, \mu z_1, \mu z_2)\, j_\mu^B(z_2).$$

As has been remarked above, $q_B(t, \mu z_1, \mu z_2)$ is a function of $d(\mu z_1, \mu z_2) = d(z_1, z_2)$ for any fixed $t > 0$. The rest of the proof is an easy consequence of elementary geometry and we omit it. ▮

We now mention the relation between $q_B(t, z_1, z_2)$ and the heat kernel of the semigroup generated by the Schrödinger operator with the Morse potential. We consider the potential function $V^M_{\lambda, B}$ on $\mathbf{R}$ given by (1.2), i.e.,

$$V^M_{\lambda, B}(x) = \tfrac{1}{2}\lambda^2 e^{2x} - \lambda B e^x, \qquad \lambda, B \in \mathbf{R},$$

and the Schrödinger operator $H^M_{\lambda, B}$ defined by

$$H^M_{\lambda, B} = -\frac{1}{2}\frac{d^2}{dx^2} + V^M_{\lambda, B}. \tag{2.15}$$

We denote by the same notation $H^M_{\lambda, B}$ the unique self-adjoint realization on $L^2(\mathbf{R})$. Moreover we let $q^M_{\lambda, B}(t, \xi, \eta)$, $\xi, \eta \in \mathbf{R}$, be the heat kernel of the semigroup $\exp(-tH^M_{\lambda, B})$, that is, the fundamental solution to the equation

$$\frac{\partial u}{\partial t} = -H^M_{\lambda, B} u.$$

Then, by the Feynman–Kac formula, we have

$$\begin{aligned} q^M_{\lambda, B}(t, \xi, \eta) &= \int_W \exp[-\tfrac{1}{2}\lambda^2 A_t(w^\xi) + \lambda B a_t(w^\xi)]\, \delta_\eta(w^\xi_t)\, dP(w) \\ &= \int_W \exp[-\tfrac{1}{2}(\lambda e^\xi)^2 A_t(w) + \lambda e^\xi B a_t(w)]\, \delta_{\eta - \xi}(w_t)\, dP(w) \end{aligned} \tag{2.16}$$

for $t > 0$ and $\xi, \eta \in \mathbf{R}$. Then (2.10) implies the following.

THEOREM 2.3. *Let $z_i = (x_i, y_i) \in \mathbf{H}^2$, $i = 1, 2$, and $t > 0$. Then it holds that*

$$\begin{aligned} q_B(t, z_1, z_2) = e^{-t/8 - B^2 t/2} \frac{\sqrt{y_1 y_2}}{2\pi} \int_{-\infty}^{\infty} & e^{-\sqrt{-1}\,(x_1 - x_2)\,\lambda} \\ & \times q^M_{\lambda, B}(t, \log y_1, \log y_2)\, d\lambda. \end{aligned} \tag{2.17}$$

It should be mentioned that Grosche has shown a formula of the same type for the propagators by using the theory of Feynman path integrals (cf. (15) in [20, p. 115]). As was mentioned in the Introduction, the formula (2.17) above can also be found by separation of variables in rectangular coordinates. The study of the operator H_B in these coordinates leads us to use the Bessel functions and the Whittaker functions (cf. Terras [49, Sect. 3.2]). For details, see Sections 3 and 4.

Remark 2.1. Formula (2.17) is an analogue of the following one for the heat kernel (Gauss kernel) on $\mathbf{R}^2$: identifying $z_i \in \mathbf{R}^2$ with $|z_i| \exp(\sqrt{-1}\,\theta_i) \in \mathbf{C}$, $0 \leqq \theta_i < 2\pi$, $i = 1, 2$,

$$g(t, z_1, z_2) \equiv \frac{1}{2\pi t} e^{-|z_2 - z_1|^2/2t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{\sqrt{-1}\, n(\theta_1 - \theta_2)} p_n(t, |z_1|, |z_2|), \tag{2.18}$$

where $p_n(t, \xi, \eta)$ is the fundamental solution to the equation

$$\frac{\partial u}{\partial t} = L_n u, \qquad L_n = \frac{1}{2}\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - \frac{n^2}{\xi^2}\right).$$

By combining the skew product representation of the two-dimensional standard Brownian motion with the eigenfunction expansion of the transition density of the Brownian motion on S^1, we can show (2.18) in the framework of stochastic analysis (cf. [28]). Formula (2.18) is also shown in the following way. We first remember the equality

$$g(t, z_1, z_2) = \frac{1}{2\pi} \int_0^{\infty} e^{-\lambda^2 t/2} J_0(|z_2 - z_1|\, \lambda)\, \lambda\, d\lambda$$

(cf. [32, p. 133]). Throughout, we denote by J_n the Bessel function of the first kind of order n. Then, by using the addition theorem

$$J_0(\lambda\, |z_2 - z_1|) = \sum_{n=-\infty}^{\infty} e^{\sqrt{-1}\, n(\theta_1 - \theta_2)} J_n(\lambda\, |z_1|)\, J_n(\lambda\, |z_2|)$$

[18, p. 979; 32, p. 124], we get (2.18).

3. EXPLICIT FORMULAE, I

In this and the following two sections we study the explicit forms of the Green functions for the operators H_B and $H^M_{\lambda, B}$ by using the fundamental formula (2.17). The explicit forms of the heat kernels of the semigroups $\exp(-tH_B)$ and $\exp(-tH^M_{\lambda, 0})$ will also be given. We will consider the case where $B = 0$ in the latter half of this section.

The explicit form of the Green function for H_B has been studied by Elstrodt [14], Fay [15], Patterson [41], and so on, by using harmonic analysis on $\mathbf{H}^2$. Their arguments are based on the idea of Selberg [46], which is illustrated as follows. Let $q(z_1, z_2)$ be an integral kernel of the form

$$q(z_1, z_2) = \left(\frac{z_2 - \bar{z}_1}{z_1 - \bar{z}_2}\right)^B g(d(z_1, z_2)). \tag{3.1}$$

If there exists a function h satisfying

$$h(\lambda)\, f(z_1) = \int_{\mathbf{H}^2} q(z_1, z_2)\, f(z_2)\, dm(z_2)$$

for every eigenfunction f of H_B corresponding to the eigenvalue λ, it is called the Selberg transform of the integral kernel $q(z_1, z_2)$. Since y^s, $s \in \mathbf{C}$, is an eigenfunction of H_B corresponding to the eigenvalue $\lambda = (-s(s-1) + B^2)/2$, h is a function on the whole $\mathbf{C}$.

The heat kernel $q_B(t, z_1, z_2)$ is one of the typical examples of the integral kernels satisfying (3.1). We first show that the Selberg transform of $q_B(t, z_1, z_1)$ is an exponential function. Let $\{Z_z(s, w^{(2)})\}_{s \geqq 0}$ be the Brownian motion on $\mathbf{H}^2$ starting from z given by (2.4) and f be an eigenfunction of H_B corresponding to the eigenvalue λ. Then, by the Itô formula, it is easy to show

$$\begin{aligned}
&\exp(-\sqrt{-1}\, Bw_t^1)\, f(Z_z(t, w^{(2)})) - f(z) \\
&\quad = \text{a martingale} + \int_0^t \exp(-\sqrt{-1}\, Bw_s^1)(-H_B f)(Z_z(s, w^{(2)}))\, ds \\
&\quad = \text{a martingale} - \lambda \int_0^t \exp(-\sqrt{-1}\, Bw_s^1)\, f(Z_z(s, w^{(2)}))\, ds.
\end{aligned} \tag{3.2}$$

Hence, setting

$$\begin{aligned}
v(t, z; f) &= \int_{W^{(2)}} \exp(-\sqrt{-1}\, Bw_t^1)\, f(Z_z(t, w^{(2)}))\, dP^{(2)}(w) \\
&= \int_{\mathbf{H}^2} q_B(t, z, z')\, f(z')\, dm(z')
\end{aligned}$$

and integrating all terms in (3.2) with respect to $P^{(2)}$, we get

$$v(t, z; f) = f(z) - \lambda \int_0^t v(s, z; f)\, ds$$

and

$$v(t, z; f) = e^{-\lambda t} f(z).$$

Therefore the Selberg transform $h_t(\lambda)$ of $q_B(t, z_1, z_2)$ is given by

$$h_t(\lambda) = e^{-\lambda t}. \tag{3.3}$$

This calculation is essentially the same as that in harmonic analysis on $\mathbf{H}^2$ (cf., e.g., Terras [49]). We have used the Itô formula instead of the Green formula.

Remark 3.1. If we use (2.9), we can easily show

$$\int_{\mathbf{H}^2} q_B(t, z_1, z_2)\, y_2^s\, dm(z_2) = e^{-\lambda t} y_1^s, \qquad \lambda = (-s(s-1)/2 + B^2)/2,$$

which also proves (3.3).

If we can explicitly calculate the inverse of the Selberg transform, we obtain the explicit form of the heat kernel $q_B(t, z_1, z_2)$. While this is a hard problem in general, it can be reduced to elementary calculations when $B = 0$ and H_B is half of the Laplace–Beltrami operator. In this case the Selberg transform is essentially a composition of the one-dimensional Fourier transform and an Abel-type transform. Therefore, as Selberg himself [46] has pointed out, the inverse transform is explicitly written down as follows. Let h be the image by the Selberg transform of an integral kernel $q(z_1, z_2)$, which depends only on $r = d(z_1, z_2)$. Noting that y^s, $s \in \mathbf{C}$, is an eigenfunction of H_0 corresponding to the eigenvalue $\lambda = s(s-1)/2$ and setting $s = 1/2 + \sqrt{-1}\,\tau$, we consider h as a function in τ and denote it by the same notation h. We now set

$$g(b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\, b\tau} h(\tau)\, d\tau$$

and $w = e^b + e^{-b} - 2$. Moreover we define the function Q by

$$Q(w) = g(b).$$

Then the inverse Selberg transform is given by

$$q(z_1, z_2) = -\frac{1}{\pi} \int_r^{\infty} \frac{dQ(w)}{\sqrt{w - \alpha(r)}}, \qquad \alpha(r) = 2(\cosh r - 1).$$

See also Terras [49]. Applying this for the Selberg transform h_t of the heat kernel $q_0(t, z_1, z_2)$ given by

$$h_t(\tau) = \exp(-t/8 - t\tau^2/2),$$

we get the following well known formula:

$$q_0(t, z_1, z_2) = \frac{\sqrt{2}\, e^{-t/8}}{(2\pi t)^{3/2}} \int_r^\infty \frac{b e^{-b^2/2t}}{\sqrt{\cosh b - \cosh r}}\, db, \qquad r = d(z_1, z_2). \tag{3.4}$$

By using (3.4), we show the explicit form of the heat kernel $q^M_{\lambda,0}(t, \xi, \eta)$.

PROPOSITION 3.1. *For $t > 0$, $\xi, \eta \in \mathbf{R}$, it holds that*

$$\begin{aligned} q^M_{\lambda,0}(t, \xi, \eta) = \frac{1}{\sqrt{2\pi t^3}} \int_{|\xi-\eta|}^\infty & b e^{-b^2/2t} \\ & \times J_0(\sqrt{2}\, e^{(\xi+\eta)/2}\, |\lambda| \sqrt{\cosh b - \cosh(\xi - \eta)})\, db. \end{aligned} \tag{3.5}$$

Proof. Let $\tilde{q}_\lambda(t, \xi, \eta)$ be the function defined by the right hand side of (3.5). By virtue of the injectivity of the Fourier transform and Theorem 2.3, (3.5) follows if we show

$$\begin{aligned} & e^{-t/8} \frac{\sqrt{y_1 y_2}}{2\pi} \int_{-\infty}^\infty \cos[(x_2 - x_1)\lambda]\, \tilde{q}_\lambda(t, \xi, \eta)\, d\lambda \\ & \quad = \frac{\sqrt{2}\, e^{-t/8}}{(2\pi t)^{3/2}} \int_r^\infty \frac{b e^{-b^2/2t}}{\sqrt{\cosh b - \cosh r}}\, db, \end{aligned}$$

where $r = d(z_1, z_2)$, $z_i = (x_i, y_i)$, $i = 1, 2$, $y_1 = \exp(\xi)$, $y_2 = \exp(\eta)$.
For this we note

$$\int_0^\infty \cos \beta\lambda\, J_0(\alpha\lambda)\, d\lambda = \begin{cases} \dfrac{1}{\sqrt{\alpha^2 - \beta^2}}, & 0 \leqq \beta < \alpha \\ 0, & \beta > \alpha \end{cases}$$

(cf. [18, p. 731]). Moreover it is easy to show

$$(\sqrt{2}\, e^{(\xi+\eta)/2} \sqrt{\cosh b - \cosh(\xi - \eta)})^2 - (x_2 - x_1)^2 = 2 y_1 y_2 (\cosh b - \cosh r)$$

and

$$|\xi - \eta| = |\log(y_1/y_2)| \leqq r.$$

Then we get

$$\frac{\sqrt{y_1 y_2}}{2\pi} \int_{-\infty}^{\infty} \cos[(x_2 - x_1)\lambda] \, \tilde{q}_\lambda(t, \xi, \eta) \, d\lambda$$

$$= \frac{\sqrt{y_1 y_2}}{\sqrt{2\pi^3 t^3}} \int_{|\log(y_1/y_2)|}^{\infty} b e^{-b^2/2t} \, db$$

$$\times \int_0^{\infty} \cos[(x_2 - x_1)\lambda] \, J_0(\sqrt{2 y_1 y_2} \, \lambda \sqrt{\cosh b - \cosh(\xi - \eta)}) \, d\lambda$$

$$= \frac{\sqrt{2}}{(2\pi t)^{3/2}} \int_r^{\infty} \frac{b e^{-b^2/2t}}{\sqrt{\cosh b - \cosh r}} \, db.$$

Rigorously speaking, we must justify the exchange of the order of the integrations at the second line. It is not difficult and we omit it. ▮

Remark 3.3. Alili and Gruet [1] have also obtained (3.5) by a different method. See also [10].

The explicit expression (3.5) is naturally guessed by combining the Feynman–Kac formula (2.16) with the following explicit expression of the moments of the Wiener functional $A_t = A_t(w)$. The result is essentially due to Yor [50].

LEMMA 3.2. *For every $t > 0$ and $n = 0, 1, 2, ...,$ it holds that*

$$\int_W (A_t(w^\xi))^n \, \delta_\eta(w_t^\xi) \, dP(w)$$

$$= \frac{e^{n(\xi + \eta)}}{n!(2\pi t^3)^{1/2}} \int_{|\xi - \eta|}^{\infty} b e^{-b^2/2t} (\cosh b - \cosh(\xi - \eta))^n \, db. \tag{3.6}$$

The proof will be given in Section 9.

Next we give the explicit form of the Green function for the Liouville Hamiltonian $H^M_{\lambda, 0}$. For this we consider the conditional distribution $P(A_t \in D \mid w_t = \xi)$, $D \subset (0, \infty)$, of A_t given w_t. We have

$$p(t, 0, \xi) \, E[f(A_t) \mid w_t = \xi] = \int_W f(A_t) \, \delta_\xi(w_t) \, dP(w)$$

for every bounded smooth function f on $\mathbf{R}$ and the conditional distribution has a density $a_t(\xi, x)$,

$$P(A_t \in dx \mid w_t = \xi) = a_t(\xi, x) \, dx,$$

where $E[\,\cdot \mid w_t = \xi]$ denotes the conditional expectation under the condition $w_t = \xi$ and $p(t, \xi, \eta)$ is the Gauss kernel on $\mathbf{R}$,

$$p(t, \xi, \eta) = \frac{1}{\sqrt{2\pi t}} \exp(-|\xi - \eta|^2/2t).$$

Yor [50] showed, first,

$$\int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2}\left(\mu^2 t + \frac{\xi^2}{t}\right)\right] a_t(\xi, x)\, dt = e^{-(\mu+1)\xi} p_x^\mu(1, e^\xi), \quad (3.7)$$

and, second, the explicit expression of $a_t(\xi, x)$ from (3.7), where $p_u^\mu(a, r)$ is the probability density of the Bessel process with index μ:

$$p_u^\mu(a, r) = \left(\frac{r}{a}\right)^\mu \frac{r}{u} \exp(-(a^2 + r^2)/2u)\, I_\mu(u^{-1} ar) \quad (3.8)$$

(cf., e.g., [27, p. 239; 43, p. 422]).

While the following is a special case of Proposition 4.1 below and can be proved by the general theory of the Sturm–Liouville operators, we discuss it separately in this section because we can show it on the basis of the well known formula (3.4).

PROPOSITION 3.3. (i) *For $\alpha > 0$ and $\xi < \eta$, it holds that*

$$\int_0^\infty e^{-\alpha^2 t/2} q_{\lambda,0}^M(t, \xi, \eta)\, dt = 2I_\alpha(|\lambda|\, e^\xi)\, K_\alpha(|\lambda|\, e^\eta), \quad (3.9)$$

where I_α is the modified Bessel function of the first kind and K_α is the Macdonald function.

(ii) *If $\xi < \eta$, it holds that*

$$\int_0^\infty e^{-\alpha^2 t/2} p(t, \xi, \eta)\, dt \int_0^\infty \exp(-\tfrac{1}{2}\lambda^2 e^{2\xi} x)\, a_t(\eta - \xi, x)\, dx$$

$$= 2I_\alpha(|\lambda|\, e^\xi)\, K_\alpha(|\lambda|\, e^\eta). \quad (3.10)$$

Proof. We first give a proof of (3.9) based on the formula (3.5), which was proved from (3.4). We have

$$\int_0^\infty e^{-\alpha^2 t/2} q^M_{\lambda,0}(t,\xi,\eta)\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\eta-\xi}^\infty J_0(\sqrt{2\lambda^2 e^{\xi+\eta}\cosh b-\lambda^2(e^{2\xi}+e^{2\eta})})\,b\,db$$

$$\times\int_0^\infty t^{-3/2} e^{-(\alpha^2 t+b^2/t)/2}\,dt.$$

Remember the integral representation

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-z^2/4t} t^{-\nu-1}\,dt$$

of K_ν and $K_{1/2}(z)=(\pi/2z)^{1/2} e^{-z}$ (cf. [32, pp. 119, 112]). Then we get

$$\int_0^\infty e^{-\alpha^2 t/2} q^M_{\lambda,0}(t,\xi,\eta)\,dt=\int_{\eta-\xi}^\infty e^{-\alpha b} J_0(\sqrt{2\lambda^2 e^{\xi+\eta}\cosh b-\lambda^2(e^{2\xi}+e^{2\eta})})\,db.$$

Finally, by using the integral representation of the product of the Bessel functions

$$I_\alpha(u)\,K_\alpha(v)=\tfrac{1}{2}\int_{\log(v/u)}^\infty e^{-\alpha b} J_0(\sqrt{2uv\cosh b-u^2-v^2})\,db,\qquad u>0,\quad v>0$$

(cf. [32, p. 140]), we obtain (3.9).

While (3.10) follows from (3.9), we give a proof based on the result of Yor [50]. By (3.7) and (3.8), we get

$$\int_0^\infty e^{-\alpha^2 t/2} p(t,\xi,\eta)\,dt \int_0^\infty \exp(-\lambda^2 e^{2\xi} x/2)\, a_t(\eta-\xi,x)\,dx$$

$$=\int_0^\infty \exp(-\lambda^2 e^{2\xi}x/2)\, e^{-(\alpha+1)(\eta-\xi)} p^\alpha_x(1,e^{\eta-\xi})\,dx$$

$$=\int_0^\infty \exp\left[-\frac{1}{2}\left(x+\frac{(|\lambda|\,e^\xi)^2+(|\lambda|\,e^\eta)^2}{2x}\right)\right] I_\alpha\left(\frac{(|\lambda|\,e^\xi)(|\lambda|\,e^\eta)}{x}\right)dx.$$

By using another integral representation of the product of I_α and K_α,

$$I_\alpha(u)\,K_\alpha(v)=\frac{1}{2}\int_0^\infty \exp\left[-\frac{1}{2}x-\frac{1}{2x}(u^2+v^2)\right] I_\alpha(x^{-1}uv)\,\frac{dx}{x},\qquad 0<u<v \tag{3.11}$$

(cf. [18, p. 725]), we get (3.10). ∎

Remark 3.3. Tracing the proof of (3.10) in the reverse way, we can prove Yor's result (3.7) from the explicit form of the Green function for $H^M_{\lambda,0}$ given by (3.9).

4. EXPLICIT FORMULAE, II

Assuming that Re E is sufficiently large, we let $G^M_{\lambda,B}(\xi,\eta;E)$ be the Green function for $H^M_{\lambda,B}$:

$$G^M_{\lambda,B}(\xi,\eta;E)=\int_0^\infty e^{-Et}q^M_{\lambda,B}(t,\xi,\eta)\,dt. \tag{4.1}$$

Moreover we denote the Green function for H_B by G_B. By straightforward calculations, we give the explicit form of $G^M_{\lambda,B}$ in terms of the Whittaker functions $W_{\mu,\nu}$, $M_{\mu,\nu}$. For details of the Whittaker functions, we refer to [4, 18]. It should be noted that the definition of the function $M_{\mu,\nu}$ in [4] is different from that in [18] by a multiplicative constant $\Gamma(1+2\nu)$. We follow the convention in [18]. Furthermore, by improving Comtet's method [6], we derive the explicit form (4.3), below, of G_B due to Elstrodt [14] from (4.2), below. See also Fay [15] and Patterson [41] for the approach from harmonic analysis on $\mathbf{H}^2$.

PROPOSITION 4.1. (i) *For $\alpha>0$ and $\eta>\xi$, it holds that*

$$G^M_{\lambda,B}(\xi,\eta;\alpha^2/2)$$
$$=\begin{cases}\dfrac{\Gamma(\alpha-|B|+1/2)}{2\,|\lambda|\,\Gamma(1+2\alpha)}\,e^{-(\xi+\eta)/2}W_{|B|,\alpha}(2\,|\lambda|\,e^{\eta})\,M_{|B|,\alpha}(2\,|\lambda|\,e^{\xi})\\[2ex] \dfrac{\Gamma(\alpha+|B|+1/2)}{2\,|\lambda|\,\Gamma(1+2\alpha)}\,e^{-(\xi+\eta)/2}W_{-|B|,\alpha}(2\,|\lambda|\,e^{\eta})\,M_{-|B|,\alpha}(2\,|\lambda|\,e^{\xi})\end{cases} \tag{4.2}$$

according to $\lambda B>0$ and $\lambda B<0$.

(ii) ([14]) *It holds that*

$$G_B(z_1,z_2;\alpha^2/2)=\frac{1}{4\pi}\left(\frac{z_2-\bar z_1}{z_1-\bar z_2}\right)^B\frac{\Gamma(s+B)\,\Gamma(s-B)}{\Gamma(2s)}\,\sigma^{-s}$$
$$\times F(s+B,s-B;2s;\sigma^{-1}), \tag{4.3}$$

where $\sigma=(1+\cosh r)/2$, $r=d(z_1,z_2)$, F is the Gauss hypergeometric function, and $s=s(\alpha,B)$ is the positive root of $x(x-1)=\alpha^2+B^2$ given by

$$s=\tfrac{1}{2}+\sqrt{\alpha^2+B^2+\tfrac{1}{4}}.$$

Proof. For the proof of (4.2), we first show the case where $\lambda > 0$. The following argument is not dependent on the signature of B. By the Whittaker differential equation [4, p. 10; 18, p. 1059]), it is easy to show that

$$\phi_1(x) = e^{-x/2} W_{B,\alpha}(2\lambda e^x) \qquad \text{and} \qquad \phi_2(x) = e^{-x/2} M_{B,\alpha}(2\lambda e^x)$$

are the linearly independent solutions of the equation

$$(\tfrac{1}{2}\alpha^2 + H^M_{\lambda,B})\, u = 0.$$

Moreover, by the integral representation of the Whittaker functions,

$$W_{B,\alpha}(z) = \frac{z^B e^{-z/2}}{\Gamma(\alpha - B + 1/2)} \int_0^\infty e^{-t} t^{\alpha - B - 1/2} (1 + t/z)^{\alpha + B - 1/2}\, dt$$

$$M_{B,\alpha}(z) = \frac{\Gamma(1+2\alpha)(z/4)^{\alpha+1/2}}{\Gamma(\alpha - B + 1/2)\, \Gamma(\alpha + B + 1/2)} \int_0^1 t^{\alpha - 1/2}$$
$$\times \left\{ \left(\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}} \right)^B e^{-z\sqrt{1-t}/2} + \left(\frac{1-\sqrt{1-t}}{1+\sqrt{1-t}} \right)^B e^{z\sqrt{1-t}/2} \right\} \frac{dt}{\sqrt{1-t}}$$

($\mathrm{Re}(\alpha \pm B + 1/2) > 0$; cf. [4, pp. 63, 93]), we get

$$\lim_{x \to \infty} \phi_1(x) = \lim_{x \to -\infty} \phi_2(x) = 0$$

and

$$\lim_{x \to -\infty} \phi_1(x) = \lim_{x \to \infty} \phi_2(x) = +\infty.$$

Finally, noting that the Wronskian is given by

$$\phi_1' \phi_2 - \phi_1 \phi_2' = \frac{2\lambda \Gamma(1+2\alpha)}{\Gamma(\alpha - B + 1/2)}$$

(cf. [4, p. 25]), we get

$$G^M_{\lambda,B}(\xi, \eta; \alpha^2/2) = \frac{\Gamma(\alpha - B + 1/2)}{2\lambda \Gamma(1+2\alpha)} e^{-(\xi+\eta)/2} W_{B,\alpha}(2\lambda e^{\eta})\, M_{B,\alpha}(2\lambda e^{\xi})$$

by the general theory of the Sturm–Liouville operators (cf. [28, 35]). Noting that

$$G^M_{\lambda,B}(\xi, \eta; \alpha^2/2)$$
$$= \frac{\Gamma(\alpha + |B| + 1/2)}{2\lambda \Gamma(1+2\alpha)} e^{-(\xi+\eta)/2} W_{-|B|,\alpha}(2\lambda e^{\eta})\, M_{-|B|,\alpha}(2\lambda e^{\xi}) \quad \text{for } B < 0,$$

we obtain (4.2) in the case where $\lambda > 0$. When $\lambda < 0$, the proof of (4.2) can be reduced to that in the case where $\lambda > 0$ if we change the signature of B.

For the proof of (4.3), we consider only the case where $B > 0$ for simplicity. The other case can be proved in the same way. Set $z_i = (x_i, y_i)$, $i = 1, 2$. Then, by Theorem 2.3, we get

$$\begin{aligned} G_B(z_1, z_2; \alpha^2/2) &= \frac{\sqrt{y_1 y_2}}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\,(x_2 - x_1)\,\lambda}\, d\lambda \\ &\quad \times \int_0^\infty e^{-(\alpha^2 + B^2 + 1/4)\,t/2} q^M_{\lambda, B}(t, \log y_1, \log y_2)\, dt \\ &= \frac{\sqrt{y_1 y_2}}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\,(x_2 - x_1)\,\lambda} G^M_{\lambda, B}(\log y_1, \log y_2; (s - 1/2)^2/2)\, d\lambda. \end{aligned}$$

Moreover we remember Proposition 2.2, which implies that (4.3) follows in general if we show it when $x_1 = x_2$ and $y_2 > y_1$. Therefore we assume these in the following. Then, by applying (4.2), we obtain

$$\begin{aligned} G_B(z_1, z_2; \alpha^2/2) &= \frac{\Gamma(s - B)}{4\pi\Gamma(2s)} \int_0^\infty W_{B, s-1/2}(2\lambda y_2)\, M_{B, s-1/2}(2\lambda y_1)\, \frac{d\lambda}{\lambda} \\ &\quad + \frac{\Gamma(s + B)}{4\pi\Gamma(2s)} \int_0^\infty W_{-B, s-1/2}(2\lambda y_2)\, M_{-B, s-1/2}(2\lambda y_1)\, \frac{d\lambda}{\lambda}. \end{aligned}$$

Now we note that

$$\begin{aligned} &\int_0^\infty e^{-\alpha x} I_\nu(\beta x)\, dx \\ &\quad = \frac{\beta^\nu}{\sqrt{\alpha^2 - \beta^2}\,(\alpha + \sqrt{\alpha^2 - \beta^2})^\nu}, \qquad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > |\operatorname{Re} \beta| \end{aligned}$$

(cf. [18, p. 708]), and the integral representation of the product of the Whittaker functions: if $\operatorname{Re}(\mu - \nu + 1/2) > 0$, $\operatorname{Re} \mu > 0$, $a_1 > a_2$,

$$\begin{aligned} W_{\nu, \mu}(a_1 t)\, M_{\nu, \mu}(a_2 t) &= \frac{t\sqrt{a_1 a_2}\,\Gamma(1 + 2\mu)}{\Gamma(\mu - \nu + 1/2)} \int_0^\infty \exp\left(-\frac{1}{2}(a_1 + a_2)\, t \cosh x\right) \\ &\quad \times \left[\coth\left(\frac{x}{2}\right)\right]^{2\nu} I_{2\mu}(t\sqrt{a_1 a_2} \sinh x)\, dx \end{aligned} \tag{4.4}$$

(cf. [4, p. 85; 18, p. 729]). Then we get

$$
\begin{aligned}
&G_B(z_1, z_2; \alpha^2/2)\\
&\quad = \frac{\sqrt{y_1 y_2}}{2\pi} \int_0^\infty \varphi_B(x)\, dx \int_0^\infty e^{-\lambda(y_1+y_2)\cosh x} I_{2s-1}(2\lambda \sqrt{y_1 y_2}\sinh x)\, d\lambda\\
&\quad = \frac{\sqrt{y_1 y_2}}{2\pi} \int_0^\infty \frac{\varphi_B(x)}{\sqrt{(y_1+y_2)^2\cosh^2 x - 4y_1 y_2 \sinh^2 x}}\\
&\qquad \times \frac{(2\sqrt{y_1 y_2}\sinh x)^{2s-1}}{\{(y_1+y_2)\cosh x + \sqrt{(y_1+y_2)^2\cosh^2 x - 4y_1 y_2\sinh^2 x}\}^{2s-1}}\, dx,
\end{aligned}
$$

where

$$
\varphi_B(x) = \left[\left(\tanh\frac{x}{2}\right)^{2B} + \left(\tanh\frac{x}{2}\right)^{-2B}\right].
$$

Since we have assumed that $x_1 = x_2$ and $y_2 > y_1$, it holds that $\log(y_2/y_1) = d(z_1, z_2) = r$. Therefore we easily obtain

$$
G_B(z_1, z_2; \alpha^2/2) = \frac{1}{4\pi}\int_0^\infty \varphi_B(x)\, \frac{(\sinh x)^{2s-1}}{\phi(r, x)\{\cosh r/2\cosh x + \phi(r, x)\}^{2s-1}}\, dx, \tag{4.5}
$$

where

$$
\phi(r, x) = \sqrt{\sinh^2 r/2\cosh^2 x + 1}.
$$

In order to evaluate the right hand side of (4.5), we change the variable from x into θ by

$$
\left(\tanh\frac{x}{2}\right)^2 = \frac{e^{-\theta}\sinh r/2 + \cosh r/2}{e^{\theta}\sinh r/2 + \cosh r/2}.
$$

The following are easily verified:

$$
e^{\theta} = \frac{\cosh r/2 + \phi(r, x)}{\sinh r/2(\cosh x - 1)}
$$

and

$$
e^{\pm\theta}\sinh r/2 + \cosh r/2 = \frac{\cosh r/2\cosh x + \phi(r, x)}{\cosh x \mp 1}.
$$

Then we get

$$\int_0^\infty \left(\tanh\frac{x}{2}\right)^{2B} \frac{(\sinh x)^{2s-1}}{\phi(r,x)\{\cosh r/2 \cosh x + \phi(r,x)\}^{2s-1}}\,dx = F_{B,s}(r),$$

where

$$F_{B,s}(r) = \int_0^\infty (e^{-\theta}\sinh r/2 + \cosh r/2)^{-(s-B)}\,(e^{\theta}\sinh r/2 + \coth r/2)^{-(s+B)}\,d\theta.$$

By virtue of the equality

$$F_{B,s}(r) + F_{-B,s}(r) = \frac{\Gamma(s-B)\,\Gamma(s+B)}{\Gamma(2s)}\,\sigma^{-s}F(s+B, s-B; 2s; \sigma^{-1})$$

(cf. Comtet [6]), we obtain 4.3 when $x_2 = x_1$ and $y_2 > y_1$. ▮

Remark 4.1. If we combine (4.3) with Theorem 2.3, we can prove (4.2) by tracing the argument in its proof in the reverse way. By using a method based on Feynman path integrals, Grosche [20] has also obtained (4.2) when $\lambda B > 0$.

Remark 4.2. We can prove (4.2) by combining a result in Leblanc [31] with the methods in Grosche [20] in the following way. We consider only the case where $B > 0$ for simplicity. Let $(X, Q_a^{(2\alpha)})$ be the Bessel process with index 2α satisfying $X_0 = a$ and $\tilde{W}$ be the space of all **R**-valued continuous functions on $[0, \infty)$. Then Lemma 1.4 in [31] shows

$$\begin{aligned}
&G^M_{\lambda,B}(\xi,\eta;\alpha^2/2)\\
&\quad= \int_0^\infty e^{-\alpha^2 t/2}\,dt \int_W \exp(-\tfrac{1}{2}\lambda^2 e^{2\xi}A_t + \lambda B e^{\xi} a_t)\,\delta_{\eta-\xi}(w_t)\,dP(w)\\
&\quad= e^{\eta-\xi}\int_0^\infty 4^{\alpha+1}\exp(\lambda B e^{\xi} t)\,dt\\
&\qquad\times \int_{\tilde{W}} X_t^{-2(1+\alpha)}\exp(-\tfrac{1}{8}\lambda^2 e^{2\xi}\int_0^t X_s^2\,ds)\,\delta_{\exp(\eta-\xi)}(X_t^2/4)\,dQ_2^{(2\alpha)}(X)\\
&\quad= e^{-(\alpha+1/2)(\eta-\xi)}\int_0^\infty \exp(\lambda B e^{\xi} t)\,dt\\
&\qquad\times \int_{\tilde{W}} \exp(-\tfrac{1}{8}\lambda^2 e^{2\xi}\int_0^t X_s^2\,ds)\,\delta_{2\exp((\eta-\xi)/2)}(X_t)\,dQ_2^{(2\alpha)}(X).
\end{aligned}$$

The integral with respect to $Q_2^{(2\alpha)}$ above is evaluated by the formula

$$\int_{\tilde{W}} \exp\left(-\frac{b^2}{2}\int_0^t X_s^2\, ds\right) \delta_r(X_t)\, dQ_a^{(2\alpha)}$$
$$= \frac{br^{1+2\alpha}a^{-2\alpha}}{\sinh bt} \exp\left[-\frac{b(r^2+a^2)\cosh bt}{2\sinh bt}\right] I_{2\alpha}\left(\frac{bar}{\sinh bt}\right)$$

(cf. [43, p. 443]). Therefore we obtain

$$G^M_{\lambda, B}(\xi, \eta; \alpha^2/2) = \int_0^\infty \frac{\lambda e^\xi \exp(\lambda B e^\xi t)}{\sinh \lambda e^\xi t/2} \exp\left(-\lambda(e^\xi + e^\eta) \coth \frac{\lambda e^\xi t}{2}\right)$$
$$\times I_{2\alpha}\left(\frac{2\lambda e^{(\xi+\eta)/2}}{\sinh \lambda e^\xi t/2}\right) dt.$$

Moreover, changing the variable by $u = \pm\lambda e^\xi t/2$ according to the signature of λ, we get

$$G^M_{\lambda, B}(\xi, \eta; \alpha^2/2) = 2\int_0^\infty \frac{e^{2Bu}}{\sinh u} \exp(-\lambda(e^\xi + e^\eta)\coth u)\, I_{2\alpha}\left(\frac{2\lambda e^{(\xi+\eta)/2}}{\sinh u}\right) du$$

and

$$G^M_{\lambda, B}(\xi, \eta; \alpha^2/2) = 2\int_0^\infty \frac{e^{-2Bu}}{\sinh u} \exp(\lambda(e^\xi + e^\eta)\coth u)\, I_{2\alpha}\left(\frac{-2\lambda e^{(\xi+\eta)/2}}{\sinh u}\right) du,$$

for $\lambda > 0$ and $\lambda < 0$, respectively. Then, using (4.4) and changing the variable by $1/\sinh u = \sinh v$ according to Grosche [20], we get (4.2).

Before closing this section, we mention the spectra of the operators $H^M_{\lambda, B}$ and H_B. By the analytic continuation of $G^M_{\lambda, B}(\xi, \eta; E)$ in E, we can show that the discrete spectrum of $H^M_{\lambda, B}$ appears as a pole of $\Gamma(\alpha - |B| + 1/2)$ when $\lambda B > 0$ and is given by

$$-\tfrac{1}{2}(|B| - 1/2 - m)^2, \qquad m = 0, 1, ..., [|B| - 1/2]_-,$$

where $[x]_-$ denotes the largest integer smaller than x. Note that the eigenvalues are independent of λ. From (4.3) we see that, when $|B| > 1/2$, H_B has the point spectrum

$$c_m = c_m(B) = (m + 1/2)\,|B| - \tfrac{1}{2}m(m+1), \qquad m = 0, 1, ..., [|B| - 1/2]_-. \tag{4.6}$$

For details on the spectral properties of H_B, see [2, 6, 8, 41].

5. SPECTRAL DECOMPOSITIONS

In the previous sections, we have shown the explicit form of the Green function G_B for H_B by calculating directly the Fourier transform of $G^M_{\lambda,B}$ for $H^M_{\lambda,B}$ and by using various properties of special functions. We have followed the lines of Grosche [20] and the important point consists in the use of the close relation between the operators H_B and $H^M_{\lambda,B}$. Grosche [20] has, first, shown the explicit form of the Green function $G^M_{\lambda,B}$ for $H^M_{\lambda,B}$ and, second, its spectral decomposition, which itself is interesting. Finally, on the basis of the correspondence of the generalized eigenfunctions, which can also be shown by using our fundamental formula (2.17), he has derived the spectral decomposition of G_B by calculations similar to those in Comtet [6].

The purpose of this section is to complete Grosche's story of obtaining the characteristic quantities of the operator H_B from those of $H^M_{\lambda,B}$ by discussing the spectral decompositions of the Green functions and the heat kernels. In order to follow the lines of Grosche's arguments and calculations, we must note the following two points. (1) As has been stated in Proposition 4.1, it is necessary to consider $G^M_{\lambda,B}$ separately according to the signature of λB and make a minor change of Grosche's arguments when $\lambda B<0$. (2) Although Comtet [6] has omitted some of the details of the proof, Proposition 2.2 shows that his proof has not lost generality (see Remark 5.1, below).

First of all we consider the Green function $G^M_{\lambda,B}$ for the Morse Hamiltonian $H^M_{\lambda,B}$. In the same way as that in the proof of Proposition 4.1, we can show easily that, if $\lambda B>0$,

$$\begin{aligned}
&H^M_{\lambda,B}(e^{-x/2}W_{|B|,\,|B|-m-1/2}(2\,|\lambda|\,e^x))\\
&\qquad=-\tfrac{1}{2}(|B|-m-1/2)^2\,e^{-x/2}W_{|B|,\,|B|-m-1/2}(2\,|\lambda|\,e^x),\\
&H^M_{\lambda,B}(e^{-x/2}W_{|B|,\,\sqrt{-1}\,p}(2\,|\lambda|\,e^x))\\
&\qquad=\tfrac{1}{2}p^2e^{-x/2}W_{|B|,\,\sqrt{-1}\,p}(2\,|\lambda|\,e^x),\qquad p>0,
\end{aligned}$$

for $m=0, 1, ..., [\,|B|-1/2\,]_-$ and, if $\lambda B<0$,

$$H^M_{\lambda,B}(e^{-x/2}W_{-|B|,\,\sqrt{-1}\,p}(2\,|\lambda|\,e^x))=\tfrac{1}{2}p^2e^{-x/2}W_{-|B|,\,\sqrt{-1}\,p}(2\,|\lambda|\,e^x),\qquad p>0.$$

Remember the equality

$$W_{m+(\lambda+1)/2,\,\lambda/2}(x)=(-1)^m\,x^{(\lambda+1)/2}e^{-x/2}L^{(\lambda)}_m(x),$$

where $L_m^{(\lambda)}$ is the Laguerre polynomial (cf. [4, p. 13; 18, p. 1063]). Then, normalizing the eigenfunctions and using the general theory of the Sturm–Liouville operators [28, 35], we arrive at the following result of Grosche [20] on the spectral decomposition of the Green function $G^M_{\lambda, B}(\xi, \eta; E)$.

PROPOSITION 5.1 [20]. *If* $\lambda B > 0$, *it holds that*

$$G^M_{\lambda, B}(\xi, \eta; E) = \sum_{m=0}^{[|B|-1/2]_-} \frac{(2\,|B|-2m-1)\, m!(2\lambda)^{2\,|B|-2m-1}}{E-(|B|-m-1/2)^2/2}$$
$$\times \exp[-\lambda(e^{\xi}+e^{\eta})+(|B|-n-1/2)(\xi+\eta)]$$
$$\times L_m^{(2\,|B|-2m-1)}(2\,|\lambda|\, e^{\xi})\, L_m^{(2\,|B|-2m-1)}(2\,|\lambda|\, e^{\eta})$$
$$+\frac{1}{2\pi^2\,|\lambda|}\int_0^{\infty} \frac{dp}{E+p^2/2}\, p \sinh 2\pi p\, |\Gamma(\sqrt{-1}\, p-|B|+1/2)|^2$$
$$\times e^{(\xi+\eta)/2}\, W_{|B|, \sqrt{-1}\, p}(2\,|\lambda|\, e^{\xi})\, W_{|B|, \sqrt{-1}\, p}(2\,|\lambda|\, e^{\eta}) \qquad (5.1)$$

when $\mathrm{Re}\, E > (|B|-1/2)^2/2$ *and, if* $\lambda B < 0$, *it holds that*

$$G^M_{\lambda, B}(\xi, \eta; E) = \frac{1}{2\pi^2\,|\lambda|}\int_0^{\infty} \frac{dp}{E+p^2/2}\, p \sinh 2\pi p\, |\Gamma(\sqrt{-1}+|B|+1/2)|^2$$
$$\times e^{-(\xi+\eta)/2}\, W_{-|B|, \sqrt{-1}\, p}(2\,|\lambda|\, e^{\xi})\, W_{-|B|, \sqrt{-1}\, p}(2\,|\lambda|\, e^{\eta}) \qquad (5.2)$$

when $\mathrm{Re}\, E > 0$.

For details of the proof, see Grosche [20].

By straightforward calculations of the Fourier transforms of the right hand sides of both (5.1) and (5.2), we can arrive at the spectral decomposition of the Green function $G_B(z_1, z_2; E)$, which has been obtained by Fay [15] and Patterson [41]. In fact, by using the same calculations as those in Comtet [6], we get the following: Setting

$$\sigma = \frac{1+\cosh d(z_1, z_2)}{2} \qquad \text{and} \qquad E = \frac{1}{2}\,(s(s-1)-B^2),$$

it holds that

$$
\begin{aligned}
&G_B(z_1, z_2; E) \\
&\quad = \left(\frac{z_2 - \bar{z}_1}{z_1 - \bar{z}_2}\right)^B \sum_{m=0}^{[|B| - 1/2]_-} \frac{(-1)^m}{\pi m!} \frac{(2|B| - 2m - 1)\,\Gamma(2|B| - m)}{\Gamma(2|B| - 2m)} \sigma^{m - |B|} \\
&\qquad \times \frac{F(2|B| - m, -m; 2|B| - 2m, \sigma^{-1})}{(|B| - m)(1 - |B| + m) - s(1 - s)} \\
&\qquad + \left(\frac{z_2 - \bar{z}_1}{z_1 - \bar{z}_2}\right)^B \frac{1}{4\pi^2 \sqrt{-1}} \int_{1/2 - \sqrt{-1}\,\infty}^{1/2 + \sqrt{-1}\,\infty} d\lambda \frac{(2\lambda - 1)\sin 2\pi\lambda}{\sin \pi(\lambda - B)\sin \pi(\lambda + B)} \\
&\qquad \times \frac{1}{\lambda(1 - \lambda) - s(1 - s)} \sigma^{\lambda - 1} F(1 - \lambda + B, 1 - \lambda - B; 1; \tau^{-1})
\end{aligned}
$$

if $\mathrm{Re}\, E > -|B|/2$, where $\tau = (\cosh d(z_1, z_2) + 1)/(\cosh d(z_1, z_2) - 1)$.

Remark 5.1. Comtet [6] has considered only the case where $x_2 = x_1$ "for simplicity." However, by virtue of Proposition 2.2, it is in fact sufficient to consider only this case.

Remark 5.2. Patterson [41] has shown that the right hand side of (5.3) is equal to that of (4.3).

Using (5.3) above, Fay [15] has shown the explicit form of the heat kernel $q_B(t, z_1, z_2)$. We recall the form to derive the formula (5.6), below, which will be necessary in the proof of the Selberg trace formula in Section 7. In order to discuss it, we prepare some notations. We define the function $\theta = \theta(u, r)$, $0 < r \leqq u$, by

$$e^{\pm\theta} \sinh r = e^u - \cosh r \pm e^{u/2} \sqrt{2(\cosh u - \cosh r)}$$

and set

$$f_\pm^B(u, r) = e^{\pm 2B\theta} (\sqrt{2} \sinh u/2 \mp \sqrt{\cosh u - \cosh r})^{2B},$$

respectively. Then Fay [15] has shown that the function $g_t(r)$ in Proposition 2.2 is given by

$$
\begin{aligned}
g_t(r) = &\frac{e^{-B^2 t/2 - t/8} (\cosh r - 1)^{-B}}{(2\pi t)^{3/2}} \\
&\times \int_r^\infty \frac{u e^{-u^2/2t}}{\sqrt{2(\cosh u - \cosh r)}} (f_+^B(u, r) + f_-^B(u, r))\, du. \qquad (5.4)
\end{aligned}
$$

It should be noted that, as is reported by d'Hoker and Phong [12], Fay himself has mentioned to them that the explicit form of $g_t(r)$ in [15] is not correct. In fact Oshima [40] has shown (5.4) when B is a half integer. Furthermore we can show (5.4) for general B by a slight modification of his arguments.

We end this section by giving a derivation of an explicit evaluation of $g_t(0) = q_B(t, z, z)$ from (5.4). Formula (5.6), below, is a special case of the explicit formulae given in Hejhal [25] and Patterson [41]. By (5.4), it is easy to show

$$q_B(t, z, z) = \frac{e^{-B^2 t/2 - t/8}}{(2\pi t)^{3/2}} \int_0^\infty \frac{u e^{-u^2/2t} \cosh Bu}{\sinh u/2}\, du. \tag{5.5}$$

The Plancherel identity gives us

$$q_B(t, z, z) = \frac{e^{-B^2 t/2 - t/8}}{8t} \int_{-\infty}^\infty \frac{1}{\cosh^2 \pi\xi} \left(e^{-(\xi + \sqrt{-1}\, B)^2\, t/2} + e^{-(\xi - \sqrt{-1}\, B)^2\, t/2}\right) d\xi.$$

We first consider the case where B is not a half integer. Then, noting that $\cosh \pi\xi$, $\xi \in \mathbf{C}$, has zeros at $(m + 1/2)\sqrt{-1}$, $m \in \mathbf{Z}$, and changing the path of integration into $\mathbf{R} \pm \sqrt{-1}\, B$, we obtain by the calculus of residues

$$\begin{aligned} q_B(t, z, z) = &\frac{1}{4\pi} \sum_{0 \leqq m < |B| - 1/2} (2\,|B| - 2m - 1)\, e^{-c_m t} \\ &+ \frac{1}{2\pi}\, e^{-(1/8 + B^2/2)\, t} \int_0^\infty \frac{b e^{-b^2 t/2} \sinh 2\pi b}{\cosh 2\pi b + \cos 2\pi B}\, db, \end{aligned} \tag{5.6}$$

for every $t > 0$ and $z \in \mathbf{H}^2$, where the c_m's are the eigenvalues of H_B on $L^2(\mathbf{H}^2)$ given by (4.6). Finally, noting that the right hand sides of both (5.5) and (5.6) are continuous in B, we get (5.6) for every $B \in \mathbf{R}$.

6. SELBERG TRACE FORMULA

In this section we recall the statement of the Selberg trace formula (cf. [25, 41]) for the heat kernel of $\exp(-tH_B)$ acting on automorphic forms of arbitrary rational weight satisfying (6.4), below, in order to discuss the term corresponding to the not null-homotopic classical paths from the point of view of stochastic analysis and to fix the notations.

We assume that a hyperbolic discrete subgroup Γ of the isometry group $PSL_2(\mathbf{R})$ on $\mathbf{H}^2$ is given. Γ is hyperbolic means that each element $\gamma \in \Gamma$ is conjugate in $PSL_2(\mathbf{R})$ to an element γ_a of the form

$$\gamma_a = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \qquad a > 1. \tag{6.1}$$

Then the quotient space $M = \Gamma \backslash \mathbf{H}^2$ is a compact smooth Riemann surface with constant negative curvature -1 and with genus $g \geqq 2$.

We introduce the automorphic forms in order to consider H_B as an operator on M in a sense. For this purpose we prepare some notions and notations (cf. [16, 25, 41]). Denoting by $\arg(z)$ the imaginary part of the principal branch of $\log z$ determined by $-\pi < \arg(z) \leqq \pi$, we set

$$w(\mu, \nu) = \arg(c_\mu \nu z + d_\mu) + \arg(c_\nu z + d_\nu) - \arg(c_{\mu\nu} z + d_{\mu\nu}) \tag{6.2}$$

for $\mu, \nu \in SL_2(\mathbf{R})$. $w(\mu, \nu)$ takes the value in $\{0, \pm 2\pi\}$ and is independent of the choice of z because

$$(c_\mu \nu z + d_\mu)(c_\nu z + d_\nu) = c_{\mu\nu} z + d_{\mu\nu}$$

and the right hand side of (6.2) is a continuous function in z. Moreover we set

$$\sigma_B(\mu, \nu) = \exp(2\sqrt{-1}\, B w(\mu, \nu))$$

and call it a factor system of weight $2B$, following Fischer [16].

Next we let $\hat{\Gamma}$ be the subgroup of $SL_2(\mathbf{R})$ which covers Γ under the canonical projection $SL_2(\mathbf{R}) \to PSL_2(\mathbf{R})$. Then a function χ on $\hat{\Gamma}$, if it exists, is called a multiplier system of weight $2B \in \mathbf{R}$ with respect to Γ if

$$|\chi(\mu)| = 1, \quad \chi(-I) = \exp(-2\sqrt{-1}\,\pi B), \quad \text{and} \quad \chi(\mu\nu) = \sigma_B(\mu, \nu)\, \chi(\mu)\, \chi(\nu) \tag{6.3}$$

hold for every $\mu, \nu \in \hat{\Gamma}$. It is known (cf. [16, 25, 42]) that, if Γ is hyperbolic, the multiplier system with respect to Γ exists (uniquely) if and only if B is a rational number of the form

$$B \in \frac{\mathbf{Z}}{2(g-1)} = \frac{2\pi \mathbf{Z}}{\mathrm{vol}(M)}. \tag{6.4}$$

The equality comes from the Gauss–Bonnet theorem. The condition (6.4) is called the Dirac quantization condition in the physics literature (cf. [7]).

We will assume (6.4) in this and the next sections and denote by χ_B the multiplier system with weight $2B$. We now introduce the automorphic forms. A function f on $\mathbf{H}^2$ is called an automorphic form of weight $2B$ with respect to Γ if

$$f(\mu z) = \chi_B(\mu)\, j_\mu^B(z)\, f(z)$$

holds for every $\mu \in \Gamma$ and $z \in \mathbf{H}^2$, where j_μ^B is the function given by (2.13). We denote by $\mathscr{H}_B$ the space of all measurable automorphic forms of weight $2B$ with respect to Γ which are square integrable on a fundamental domain of Γ with respect to the volume element dm on $\mathbf{H}^2$. When $B=0$, $\mathscr{H}_0$ is the space of all measurable Γ-periodic functions which are square integrable on the fundamental domain.

Then the operator H_B first defined on

$$\mathscr{D}_B = \{ f \in \mathscr{H}_B ;\ f \text{ is of } C^2 \text{ class} \}$$

is essentially self-adjoint on this space and its unique self-adjoint realization H_B^Γ on $\mathscr{H}_B$ has compact resolvents (cf. [14, 16, 44]). Moreover the semigroup $\exp(-tH_B^\Gamma)$ generated by H_B^Γ is of trace class.

In order to mention the trace formula, we need more notations. We put

$$l(\gamma) = \inf_{z \in \mathbf{H}^2} d(z, \gamma z)$$

for $\gamma \in \Gamma$. Since γ is hyperbolic, $l(\gamma) > 0$ and is equal to $\log a^2$ if γ is conjugate to γ_a given by (6.1). Note that $l(\gamma^n) = nl(\gamma)$ for every $n \in \mathbf{N}$. Moreover an element of Γ is called primitive if it cannot be expressed as a positive power of another element. The primitive elements in Γ correspond to the prime periodic geodesics on M and it is known that any element in Γ is expressed uniquely as a positive power of a primitive one. For details, see [36]. We denote by Γ_0 the subset of Γ consisting of all primitive elements which are inconjugate each other. Then the following explicit form of the Selberg trace formula is a special case of the trace formulae shown in Hejhal [25] and Patterson [41]. See also the footnote in Selberg [46, p. 83].

THEOREM 6.1 [25, 41]. *Assume* (6.4) *and let* $\{\lambda_k^\Gamma\}_{k=1}^\infty$ *be the eigenvalues of* H_B^Γ. *Then it holds that*

$$
\begin{aligned}
&\operatorname{Tr}(\exp(-tH_B^{\Gamma}))\\
&\quad=\sum_{k=1}^{\infty}\exp(-t\lambda_k^{\Gamma})\\
&\quad=\frac{\operatorname{vol}(M)}{4\pi}\sum_{0\leqq m<|B|-1/2}(2\,|B|-2m-1)\exp(-c_m t)\\
&\qquad+\frac{\operatorname{vol}(M)}{2\pi}\,e^{-(1/8+B^2/2)\,t}\int_0^{\infty}\frac{be^{-b^2t/2}\sinh 2\pi b}{\cosh 2\pi b+\cos 2\pi B}\,db\\
&\qquad+\sum_{\gamma\in\Gamma_0}\sum_{n=1}^{\infty}\chi_B(\gamma^n)\,\frac{e^{-t/8}}{2\sqrt{2\pi t}}\,\frac{l(\gamma)}{\sinh(nl(\gamma)/2)}\,e^{-(n^2l(\gamma)^2/2t+B^2t/2)} \qquad (6.5)
\end{aligned}
$$

for any $t>0$, *where* c_m, $m=0, 1, ..., [\,|B|-1/2\,]_-$, *are given by* (4.6).

One of the purposes of this paper is to consider the third term of the right hand side of the trace formula (6.5) by using the basic formula (2.9). It will be given in the next section.

We end this section by giving an application of the trace formula (6.5) to the study of the least eigenvalue of H_B^{Γ}. It is important in the calculation of the determinant of H_B^{Γ} (see [11, 45]).

COROLLARY 6.2. *When* $B=0$ *or* $|B|\geqq 1$, *the limit*

$$N_B=\lim_{t\to\infty} e^{|B|\,t/2}\operatorname{Tr}(\exp(-tH_B^{\Gamma}))$$

exists and

$$
N_B=\begin{cases}
1, & \text{if}\quad B=0,\\[4pt]
1+\dfrac{\operatorname{vol}(M)}{4\pi}, & \text{if}\quad |B|=1,\\[8pt]
\dfrac{\operatorname{vol}(M)(2\,|B|-1)}{4\pi}, & \text{if}\quad |B|>1.
\end{cases}
$$

We can prove the corollary in the same way as d'Hoker and Phong [11] and we omit the details.

7. PROOF OF THE TRACE FORMULA

In this section we give a proof of the trace formula (6.5) by using the results in the previous sections. The explicit expression (5.6) of $q_B(t, z, z)$ gives us the first and the second terms of the right hand side of (6.5) and the third term is derived by using the fundamental formula (2.9).

First of all we give the integral kernel of the semigroup $\exp(-tH_B^\Gamma)$. We set

$$q_B^\Gamma(t, z_1, z_2) = \sum_{\sigma \in \Gamma} \chi_B(\sigma)\, j_\sigma^B(z_2)\, q_B(t, z_1, \sigma z_2)$$

for $t > 0$ and $z_i \in \mathbf{H}^2$, $i = 1, 2$. Then the following proposition shows that q_B^Γ is the heat kernel of $\exp(-tH_B^\Gamma)$ acting on the space $\mathscr{H}_B$ of automorphic forms.

Proposition 7.1 [25, 41]. *$q_B^\Gamma(t, z_1, z_2)$ is an automorphic form in z_1 of weight $2B$ with respect to Γ.*

Proof. By using (2.12), we get

$$\begin{aligned} &q_B^\Gamma(t, \mu z_1, z_2) \\ &\quad = \sum_{\sigma \in \Gamma} \chi_B(\sigma)\, j_\sigma^B(z_2)\, q_B(t, \mu z_1, \sigma z_2) \\ &\quad = \sum_{\sigma \in \Gamma} \chi_B(\sigma)\, j_\sigma^B(z_2)\, j_\mu^B(z_1)\, q_B(t, z_1, \mu^{-1}\sigma z_2)\, j_\mu^B(\mu^{-1}\sigma z_2)^{-1} \\ &\quad = j_\mu^B(z_1) \sum_{\sigma \in \Gamma} \chi_B(\mu\sigma)\, j_{\mu\sigma}^B(z_2)\, q_B(t, z_1, \sigma z_2)\, j_\mu^B(\sigma z_2)^{-1}. \end{aligned}$$

Note that the function $j_\sigma^B(z)$ satisfies

$$j_\mu^B(\sigma z)\, j_\sigma^B(z) = \sigma_{B(\mu, \sigma)}\, j_{\mu\sigma}^B(z), \qquad \mu, \sigma \in SL_2(\mathbf{R}), \quad z \in \mathbf{H}^2, \tag{7.1}$$

which is the characterization of the factor system $\sigma_B(\mu, \sigma)$. Then it is easy to see

$$\chi_B(\mu\sigma)\, j_{\mu\sigma}^B(z_2)\, j_\mu^B(\sigma z_2)^{-1} = \chi_B(\mu)\, \chi_B(\sigma)\, j_\sigma^B(z_2)$$

by the definition of the multiplier system χ_B. Therefore we obtain

$$q_B^\Gamma(t, \mu z_1, z_2) = \chi_B(\mu)\, j_\mu^B(z_1)\, q_B^\Gamma(t, z_1, z_2). \quad \blacksquare$$

Remark 7.1. D'Hoker and Phong [11] have used another expression for $q_B^\Gamma(t, z_1, z_2)$,

$$q_B^\Gamma(t, z_1, z_2) = \sum_{\sigma \in \Gamma} \overline{\chi_B(\sigma)\, j_\sigma^B(z_1)}\, q_B(t, \sigma z_1, z_2),$$

which has been also given in Fay [15]. We can prove that these two expressions coincide by an argument similar to that in the proof of Proposition 7.1.

Now we proceed to the proof of Theorem 6.1. We remember that every $\sigma \in \Gamma$ is written in the form

$$\sigma = \mu^{-1}\gamma^n\mu$$

for some $n \in \mathbf{N}$, $\mu \in \Gamma$, and $\gamma \in \Gamma_0$. We can choose γ, n, and the conjugacy class $[\mu] \in \Gamma/\Gamma_\gamma$ uniquely, where Γ_γ is the centralizer of γ. Furthermore the expression is independent of the choice of the representative μ (cf. [36]). Then, letting $\mathscr{F}$ be a fundamental domain of Γ, we have

$$\begin{aligned}
&\operatorname{Tr}(\exp(-tH_B^\Gamma)) \\
&\quad = \int_{\mathscr{F}} q_B(t, z, z)\, dm(z) + \sum_{\sigma \neq I} \int_{\mathscr{F}} \chi_B(\sigma)\, j_\sigma^B(z)\, q_B(t, z, \sigma z)\, dm(z) \\
&\quad = \operatorname{vol}(\mathscr{F})\, g_t(0) + \sum_{\gamma \in \Gamma_0} \sum_{n=1}^{\infty} \sum_{[\mu] \in \Gamma/\Gamma_\gamma} I(\gamma, n, [\mu]),
\end{aligned} \tag{7.2}$$

where

$$I(\gamma, n, [\mu]) = \int_{\mu\mathscr{F}} \chi_B(\mu^{-1}\gamma^n\mu)\, j_{\mu^{-1}\gamma^n\mu}^B(\mu^{-1}z)\, q_B(t, \mu^{-1}z, \mu^{-1}\gamma^n z)\, dm(z). \tag{7.3}$$

Therefore the explicit expression (5.6) of $g_t(0)$ gives us the first and the second terms of the right hand side of (6.5).

The rest of this section is devoted to the evaluation of the second term of the right hand side of (7.2). For this we first show that the integrand in the right hand side of (7.3) is independent of μ.

Lemma 7.2. *For every $z \in \mu\mathscr{F}$, it holds that*

$$\chi_B(\mu^{-1}\gamma^n\mu)\, j_{\mu^{-1}\gamma^n\mu}^B(\mu^{-1}z)\, q_B(t, \mu^{-1}z, \mu^{-1}\gamma^n z) = \chi_B(\gamma^n)\, j_{\gamma^n}^B(z)\, q_B(t, z, \gamma^n z). \tag{7.4}$$

Proof. By (6.3), (7.1), and (2.12), we have

$$\begin{aligned}
\chi_B(\mu^{-1}\gamma^n\mu) &= \sigma_B(\mu^{-1}\gamma^n, \mu)\, \sigma_B(\mu^{-1}, \gamma^n)\, \chi_B(\mu^{-1})\, \chi_B(\gamma^n)\, \chi_B(\mu), \\
j_{\mu^{-1}\gamma^n\mu}^B(\mu^{-1}z) &= \sigma_B(\mu^{-1}\gamma^n, \mu)^{-1}\, \sigma_B(\mu^{-1}, \gamma^n)^{-1}\, j_{\mu^{-1}}^B(\gamma^n z)\, j_{\gamma^n}^B(z)\, j_\mu^B(\mu^{-1}z)
\end{aligned}$$

and

$$q_B(t, \mu^{-1}z, \mu^{-1}\gamma^n z) = j^B_{\mu^{-1}}(z)\, q_B(t, z, \gamma^n z)\, j^B_{\mu^{-1}}(\gamma^n z)^{-1}.$$

Therefore we obtain

$$\begin{aligned}\chi_B(\mu^{-1}\gamma^n\mu)\, j^B_{\mu^{-1}\gamma^n\mu}(\mu^{-1}z)\, q_B(t, \mu^{-1}z, \mu^{-1}\gamma^n z)\\ = \chi_B(\mu^{-1})\,\chi_B(\gamma^n)\,\chi_B(\mu)\, j^B_{\gamma^n}(z)\, j^B_\mu(\mu^{-1}z)\, j^B_{\mu^{-1}}(z)\, q_B(t, z, \gamma^n z).\end{aligned}$$

Moreover, by using (6.3) and (7.1) again, we get

$$\chi_B(\mu^{-1})\,\chi_B(\mu)\, j^B_\mu(\mu^{-1}z)\, j^B_{\mu^{-1}}(z) = 1$$

and (7.4) because $\chi_B(I) = 1$ and $j^B_I(z) = 1$ for the identity I in $SL_2(\mathbf{R})$. ∎

We take $a = a(\gamma) > 1$ and $\tau \in PSL_2(\mathbf{R})$ for $\gamma \in \Gamma_0$ such that

$$\gamma = \tau^{-1}\gamma_a\tau, \tag{7.5}$$

where γ_a is the magnification given by (6.1). Moreover we remember that

$$\mathscr{F}_\gamma = \bigcup_{[\mu] \in \Gamma/\Gamma_\gamma} \mu\mathscr{F}$$

is a fundamental domain of Γ_γ and we may assume that the fundamental domain $\tau\mathscr{F}_\gamma$ of the cyclic group $\{\gamma_a^n\}_{n=-\infty}^{\infty}$ is given by

$$\mathscr{F}_{(a)} = \{(x, y) \in \mathbf{H}^2;\ 1 < y \leqq a^2\}. \tag{7.6}$$

Then we show the following.

LEMMA 7.3. *For $\gamma \in \Gamma_0$ it holds that*

$$\int_{\mathscr{F}_\gamma} j^B_{\gamma^n}(z)\, q_B(t, z, \gamma^n z)\, dm(z) = \int_{\mathscr{F}_{(a)}} q_B(t, z, a^{2n}z)\, dm(z), \tag{7.7}$$

where $a = a(\gamma)$ is given by (7.5).

Proof. Choose $\tau \in SL_2(\mathbf{R})$ so that (7.5) is satisfied. Then, since $\tau\mathscr{F}_\gamma = \mathscr{F}_{(a)}$, we get

$$\int_{\mathscr{F}_\gamma} j^B_{\gamma^n}(z)\, q_B(t, z, \gamma^n z)\, dm(z) = \int_{\mathscr{F}_{(a)}} j^B_{\tau^{-1}\gamma_a^n\tau}(\tau^{-1}z)\, q_B(t, \tau^{-1}z, \tau^{-1}\gamma_a^n z)\, dm(z).$$

By (7.1) and (2.12), we have

$$j^B_{\tau^{-1}\gamma^n_a\tau}(\tau^{-1}z)=\sigma_B(\tau^{-1}\gamma^n_a,\tau)^{-1}\,\sigma_B(\tau^{-1},\gamma^n_a)^{-1}\,j^B_{\tau^{-1}}(\gamma^n_a z)\,j^B_{\gamma^n_a}(z)\,j_\tau{}^B(\tau^{-1}z)$$

and

$$q_B(t,\tau^{-1}z,\tau^{-1}\gamma^n_a z)=j^B_{\tau^{-1}}(z)\,q_B(t,z,\gamma^n_a z)\,j^B_{\tau^{-1}}(\gamma^n_a z)^{-1}.$$

Moreover, by the definitions of j^B_τ and σ_B, it is easy to show

$$j^B_{\gamma^n_a}(z)=1,\qquad \sigma_B(\tau^{-1},\gamma^n_a)=1,\qquad\text{and}\qquad \sigma_B(\tau^{-1}\gamma^n_a,\tau)=\sigma_B(\tau^{-1},\tau)$$

because γ^n_a is diagonal (cf. [16]). Therefore, by using (7.1) again, we get

$$j^B_{\tau^{-1}\gamma^n_a\tau}(\tau^{-1}z)\,q_B(t,\tau^{-1}z,\tau^{-1}\gamma^n_a z)=q_B(t,z,\gamma^n_a z)$$

and the proof of (7.7) is completed. ∎

We now evaluate the second term of the right hand side of (7.2). We have proved

$$\begin{aligned}\sum_{\sigma\neq I}\chi_B(\sigma)&\int_{\mathscr{F}} j^B_\sigma(z)\,q_B(t,z,\sigma z)\,dm(z)\\&=\sum_{\gamma\in\Gamma_0}\sum_{n=1}^{\infty}\chi_B(\gamma^n)\int_1^{a^2}\frac{dy}{y^2}\int_{\mathbf{R}} q_B(t,z,a^{2n}z)\,dx,\end{aligned}$$

where $a=a(\gamma)>1$ is given in (7.5). Now we use (2.9). Using the same notations as those in Section 2, we have

$$\begin{aligned}\int_1^{a^2}&\frac{dy}{y^2}\int_{\mathbf{R}} q_B(t,z,a^{2n}z)\,dx\\&=e^{-t/8-B^2t/2}a^{3n}\int_1^{a^2}\frac{dy}{y^2}\int_{\mathbf{R}}dx\int_W\frac{1}{\sqrt{2\pi A_t}}\\&\quad\times\exp\left[-\frac{1}{2A_t}\left(\frac{(a^{2n}-1)\,x}{y}+\sqrt{-1}\,Ba_t\right)^2\right]\delta_{a^{2n}}(\exp(w_t))\,dP.\end{aligned}\tag{7.8}$$

Then, noting that

$$\int_{\mathbf{R}}\frac{1}{\sqrt{2\pi A_t}}\exp\left[-\frac{1}{2A_t}\left(\frac{(a^{2n}-1)\,x}{y}+\sqrt{-1}\,Ba_t\right)^2\right]dx=\frac{y}{a^{2n}-1}\tag{7.9}$$

holds for all $y>0$ and $a>1$, we can rewrite the right hand side of (7.8) into a form which is independent of A_t and a_t by integrating in x first. That is, we obtain by (7.9)

$$\int_1^{a^2} \frac{dy}{y^2} \int_{\mathbf{R}} q_B(t, z, a^{2n}z)\, dx = e^{-t/8 - B^2t/2} \frac{a^{3n}}{a^{2n}-1} \int_1^{a^2} \frac{dy}{y} \int_W \delta_{a^{2n}}(\exp(w_t))\, dP$$
$$= e^{-t/8 - B^2t/2} \frac{\log a^2}{\sqrt{2\pi t}\,(a^n - a^{-n})} \exp\left[-\frac{(\log a^{2n})^2}{2t} \right]. \tag{7.10}$$

Therefore we get

$$\int_1^{a^2} \frac{dy}{y^2} \int_{\mathbf{R}} q_B(t, z, a^{2n}z)\, dx = \frac{e^{-t/8}}{2\sqrt{2\pi t}} \frac{l(\gamma)}{\sinh(nl(\gamma)/2)} \exp(-n^2 l(\gamma)^2/2t - B^2t/2)$$

because $l(\gamma) = \log a^2$ (see Section 6). The derivation of the third term of the right hand side of (6.5) is completed.

8. CLASSICAL MECHANICS AND TRACE FORMULA

In this section we study the relation between the Selberg trace formula and the corresponding classical mechanics. We mainly consider the case where $B=0$. The purpose is to show that every factor of each element in the infinite sum in the right hand side of the trace formula (6.5) is meaningful as quantities of classical mechanics and appears naturally. To do this we derive each term in the infinite sum by using the short time asymptotics of the heat kernel.

The Lagrangian and the Hamiltonian of the classical mechanics on $\mathbf{H}^2$ corresponding to the Schrödinger operator H_B are given by

$$L(z, \dot{z}) = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2) + \frac{B\dot{x}}{y} \quad \text{and} \quad H(p, z) = \frac{1}{2} y^2 (p_x - By^{-1})^2 + \frac{1}{2} y^2 p_y^2,$$

respectively. The classical path is obtained by solving the equation

$$\ddot{x} - \frac{2\dot{x}\dot{y}}{y} - B\dot{y} = 0, \qquad \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} + B\dot{x} = 0. \tag{8.1}$$

This equation is explicitly solved (see, [2, 6, 48]). The solution curve describes an arc of a Euclidean circle or a half line and, in particular, the circle or the half line is contained in the upper half plane if $|B| > 1$.

Moreover, for every hyperbolic element γ in $SL_2(\mathbf{R})$, there exist γ-invariant classical paths. It is easy to show that

$$z_\pm^B(s) = p(\pm 1 + \sqrt{-1}\, k) \exp(\mp kBs) \qquad (p, k > 0)$$

are the classical paths invariant under the action of the magnifications γ_a^n, $n=1, 2, \ldots$, where γ_a is given by (6.1). Note that $z_\pm^B$ move on the Euclidean lines. When $B=0$, the classical path invariant under γ_a^n moves on the imaginary axis and is given by

$$z_\pm^0(s) = \sqrt{-1}\, p \exp(\pm ks) \qquad (p, k > 0).$$

If $\mu^{-1}\gamma\mu = \gamma_a$, $\mu(z_\pm^B)$ are γ-invariant classical paths. For the invariant classical path $z_\pm^B(s)$ satisfying $z_\pm^B(t) = \gamma_a^n(z_\pm^B(0))$, we can compute the action integral as

$$S(t, z_\pm^B) = \int_0^t L(z_\pm^B, \dot{z}_\pm^B)(s)\, ds = \frac{n^2(\log a^2)^2}{2t} - \frac{B^2 t}{2} = \frac{n^2 l(\gamma)^2}{2t} - \frac{B^2 t}{2}.$$

In the following, restricting ourselves to the typical case where $B=0$ and to each term

$$\frac{e^{-t/8}}{\sqrt{2\pi t}} \frac{l(\gamma)}{2\sinh(nl(\gamma)/2)}\, e^{-n^2 l(\gamma)^2/2t}$$

in the infinite sum in the right hand side of (6.5), we show that each factor is meaningful as a quantity of the classical path and that it naturally appears. The following consideration might help us to understand the Gutzwiller semiclassical trace formula (cf. [22, 23]). As was seen above, the exponent is determined by the action integral itself.

Let γ be a primitive element in Γ which is conjugate to the magnification γ_a, $a>1$, given by (6.1) and let us rewrite the factor in front of the exponential factor as

$$\frac{l(\gamma)}{2\sinh(nl(\gamma)/2)} = l(\gamma)\{J_1(nl(\gamma))^{-1}\, J_2(nl(\gamma))\}^{1/2}, \tag{8.2}$$

where the functions J_1 and J_2 are given by

$$J_1(\xi) = \frac{4\xi(\sinh \xi/2)^2}{\sinh \xi} \qquad \text{and} \qquad J_2(\xi) = \frac{\xi}{\sinh \xi},$$

respectively. The first factor $l(\gamma)$ of the right hand side of (8.2) comes from the hyperbolic width of the fundamental domain $\mathscr{F}_{(a)}$ given by (7.6). Note that $l(\gamma)$ is the length of the geodesic connecting $\sqrt{-1}$ and $\gamma_a(\sqrt{-1}) = a^2\sqrt{-1}$.

In order to see the origin of J_1 and J_2, we derive each term in the infinite sum by using the short time asymptotics of the heat kernel on $\mathbf{H}^2$. It is well known (cf. Molchanov [37]) that

$$q_0(t, z_1, z_2) = \frac{e^{-t/8}}{2\pi t} J_2(d(z_1, z_2))^{1/2} \exp(-d(z_1, z_2)^2/2t) \cdot (1 + o(1)) \tag{8.3}$$

holds as $t \downarrow 0$. Note that, for every $\rho > 0$, (8.3) holds uniformly in z_1, z_2, satisfying $d(z_1, z_2) < \rho$. The factor $\exp(-t/8)$ in the right hand side of (8.3), which we put for later convenience, comes from the curvature of $\mathbf{H}^2$. The exponent $d(z_1, z_2)^2/2t$ is the action integral $S(t, z_1, z_2)$ along the geodesic $c_{z_1, z_2} = \{c_{z_1, z_2}(s)\}_{0 \leqq s \leqq t}$ satisfying $c_{z_1, z_2}(0) = z_1$ and $c_{z_1, z_2}(t) = z_2$. As is mentioned in [37], the factor $J_2(d(z_1, z_2))$ can be written as a determinant expressed in terms of the Jacobi fields along c_{z_1, z_2}, which are obtained from the second variations of the action integral along c_{z_1, z_2}. Furthermore we can show

$$\det \begin{pmatrix} \dfrac{\partial^2 S(t, z_1, z_2)}{\partial x_1 \partial x_2} & \dfrac{\partial^2 S(t, z_1, z_2)}{\partial x_1 \partial y_2} \\ \dfrac{\partial^2 S(t, z_1, z_2)}{\partial y_1 \partial x_2} & \dfrac{\partial^2 S(t, z_1, z_2)}{\partial y_1 \partial y_2} \end{pmatrix} = \sqrt{\det g(z_1)}\, \frac{J_2(d(z_1, z_2))}{t} \sqrt{\det g(z_2)}, \tag{8.4}$$

where $g(z) = (g_{ij}(z)) = (y^{-2}\delta_{ij})$ is the component of the metric tensor. Therefore the main term of the right hand side of (8.3) can be considered as an analogue of the semiclassical (Van Vleck) propagator on $\mathbf{H}^2$ which some physicists use (cf. [7]).

We consider the family $\Phi_{n, a}$ of all geodesics $c_x = \{c_x(s)\}_{0 \leqq s \leqq t}$, $x \in \mathbf{R}$, connecting $x + \sqrt{-1}$ and $a^{2n}(x + \sqrt{-1})$. As was seen above, the action integral $S_{n, a}(x) = S(t, x + \sqrt{-1}, a^{2n}(x + \sqrt{-1}))$ along $c_x \in \Phi_{n, a}$ is given by

$$S_{n, a}(x) = \frac{d_a(x)^2}{2t}$$

where $d_a(x)$ is the length of c_x. The asymptotic formula (8.3) implies

$$\begin{aligned} &\int_1^{a^2} \frac{dy}{y^2} \int_{-\infty}^{\infty} q_0(t, z, a^{2n}z)\, dx \\ &\quad = \int_1^{a^2} \frac{dy}{y^2} \int_{-\infty}^{\infty} \frac{e^{-t/8}}{2\pi t} \left(\frac{d(z, a^{2n}z)}{\sinh d(z, a^{2n}z)} \right)^{1/2} \\ &\qquad \times \exp(-d(z, a^{2n}z)^2/2t)\, dx (1 + o(1)) \\ &\quad = \int_1^{a^2} \frac{dy}{y} \int_{-\infty}^{\infty} \frac{e^{-t/8}}{2\pi t} \left(\frac{d_a(x)}{\sinh d_a(x)} \right)^{1/2} \exp(-d_a(x)^2/2t)\, dx (1 + o(1)) \end{aligned} \tag{8.5}$$

as $t \downarrow 0$, because

$$\begin{aligned} d_a(x) &= d(x+\sqrt{-1},\, a^{2n}(x+\sqrt{-1})) \\ &= d(y(x+\sqrt{-1}),\, ya^{2n}(x+\sqrt{-1})), \qquad y>0. \end{aligned}$$

It is easy to show that

$$d_a(0) = \log a^{2n} = nl(\gamma), \quad d'_a(0) = 0, \quad d''_a(0) = \frac{4(\sinh d_a(0)/2)^2}{\sinh d_a(0)} = \frac{J_1(nl(\gamma))}{nl(\gamma)},$$

and that $d_a(x)$ attains the minimum uniquely at $x=0$. This implies that the action integral $S_{n,a}(x)$ along c_x restricted to $\Phi_{n,a}$ attains the minimum at c_0. It is also easy to show that

$$S''_{n,a}(0) = J_1(nl(\gamma))/t,$$

which means that J_1 comes from the second variation at c_0 of the action integral restricted to $\Phi_{n,a}$. Therefore, by the Laplace method, we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1}{2\pi t}\left(\frac{d_a(x)}{\sinh d_a(x)}\right)^{1/2} \exp(-d_a(x)^2/2t)\, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi t}\left(\frac{d_a(0)}{\sinh d_a(0)}\right)^{1/2} \\ &\quad \times \exp\left[-\frac{d_a(0)^2 + d_a(0)\, d''_a(0)\, x^2}{2t}\right] dx(1+o(1)) \\ &= \left(\frac{d_a(0)}{\sinh d_a(0)}\right)^{1/2} e^{-d_a(0)^2/2t} \frac{1}{\sqrt{2\pi t}} \left(\frac{1}{d_a(0)\, d''_a(0)}\right)^{1/2} \\ &\quad \times \left(\frac{d_a(0)\, d''_a(0)}{2\pi t}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[\frac{-d_a(0)\, d''_a(0)}{2t} x^2\right] dx(1+o(1)) \\ &= \frac{1}{\sqrt{2\pi t}} \{J_1(nl(\gamma))^{-1} J_2(nl(\gamma))\}^{1/2} \exp(-n^2 l(\gamma)^2/2t) \cdot (1+o(1)) \end{aligned}$$

as $t \downarrow 0$. Combining this with (8.5), we obtain

$$\int_1^{a^2} \frac{dy}{y^2} \int_{-\infty}^{\infty} q_0(t, z, a^{2n}z)\, dx$$
$$= \frac{e^{-t/8}}{\sqrt{2\pi t}}\, l(\gamma) \{J_1(nl(\gamma))^{-1} J_2(nl(\gamma))\}^{1/2} \exp(-n^2 l(\gamma)^2/2t) \cdot (1+o(1))$$

and arrive at the term in the infinite sum in the right hand side of the trace formula (6.5).

Now set

$$V_t(c_0) = \int_0^t |c_0(s)|^2\, ds = \int_0^t |\exp(nl(\gamma)\, s/t)|^2\, ds.$$

It is worth our while to note that

$$\frac{d_a(0)\, d_a''(0)}{t} = \frac{(e^{nl(\gamma)}-1)^2}{V_t(c_0)}$$

holds and that $V_t(c_0)$ can be obtained by replacing $\{\exp(w_s)\}_{0 \leqq s \leqq t}$ in the Wiener functional

$$A_t = \int_0^t |\exp(w_s)|^2\, ds$$

with the geodesic $c_0 = \{c_0(s)\}_{0 \leqq s \leqq t}$ connecting $\sqrt{-1}$ and $a^{2n}\sqrt{-1}$. Furthermore, as was mentioned in Section 2, the stochastic process $(\{\exp(w_s)\}_{0 \leqq s \leqq t}, P)$ is obtained by the Cameron–Martin transform of the second component of the Brownian motion $(\{(X_z(s), Y_z(s))\}_{0 \leqq s \leqq t}, P^{(2)})$ with $z=(0, 1)$ on $\mathbf{H}^2$ given by (2.4).

Combining (7.4) and (7.10) with the consideration above, we obtain

$$\sum_{\gamma \in \Gamma_0} \sum_{n=1}^{\infty} \int_1^{a^2} \frac{dy}{y^2} \int_{-\infty}^{\infty} q_0(t, z, a^{2n}z)\, dx$$
$$= \sum_{\gamma \in \Gamma_0} \sum_{n=1}^{\infty} e^{-t/8} a^n \int_1^{a^2} \frac{dy}{y} \int_W dP \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi A_t}}$$
$$\times \exp\left[-\frac{(a^{2n}-1)^2}{2A_t} x^2 \right] \delta_{\log a^{2n}}(w_t)\, dx$$

$$= \sum_{\gamma \in \Gamma_0} \sum_{n=1}^{\infty} e^{-t/8} \int_1^{a^2} \frac{dy}{y} \int_W dP \int_{-\infty}^{\infty} \frac{a^n}{\sqrt{2\pi V_t(c_0)}}$$

$$\times \exp\left[-\frac{(a^{2n}-1)^2}{2V_t(c_0)} x^2 \right] \delta_{\log a^{2n}}(w_t)\, dx$$

$$= \sum_{\gamma \in \Gamma_0} \sum_{n=1}^{\infty} e^{-t/8} l(\gamma) \int_{-\infty}^{\infty} \frac{1}{2\pi t} \left(\frac{d_a(0)}{\sinh d_a(0)} \right)^{1/2}$$

$$\times \exp\left[-\frac{d_a(0)^2 + d_a(0)\, d_a''(0)\, x^2}{2t} \right] dx$$

$$= \sum_{\gamma \in \Gamma_0} \sum_{n=1}^{\infty} \Big\{ \text{the semiclassical approximation of}$$

$$\int_1^{a^2} \frac{dy}{y^2} \int_{-\infty}^{\infty} q_0(t, z, a^{2n} z)\, dx \Big\},$$

where $a = a(\gamma) > 1$ is given by (7.5). Therefore, studying from Gutzwiller's point of view (cf. [22, 23]), we obtain the Selberg trace formula as an exact one.

9. PROOF OF LEMMA 3.4

Lemma 3.4 is essentially due to Yor [50], while the result itself has not been mentioned. We begin by introducing some results in [50] because we give a proof following Yor's method.

LEMMA 9.1 (Theorem 1 in Yor [50]). (i) *Setting* $\phi(z) = z^2/2$,

$$\int_0^{\infty} e^{-\lambda t} E[(A_t)^n \exp(\sqrt{-1}\, \alpha w_t)]\, dt = n! \prod_{j=0}^{n} \{\lambda - \phi(\sqrt{-1}\, \alpha + 2j)\}^{-1}$$

holds for every $\alpha \in \mathbf{R}$ *and non-negative integer* n *if* $\operatorname{Re} \lambda$ *is sufficiently large, where* $E[\cdot]$ *means the expectation with respect to the one-dimensional Wiener measure* P.

(ii) *For every* $t > 0$, $\alpha \in \mathbf{R}$ *and non-negative integer* n, *it holds that*

$$E[(A_t)^n \exp(\sqrt{-1}\, \alpha w_t)] = \sum_{j=0}^{n} C(n, j; \alpha) \exp(t\phi(\sqrt{-1}\, \alpha + 2j)), \quad (9.1)$$

where the constant $C(n, j; \alpha)$ *is given by*

$$C(n, j; \alpha) = 2^{-n}(-1)^{n-j}\ {}_nC_j \prod_{\substack{k \neq j \\ 0 \leqq k \leqq n}} (\sqrt{-1}\,\alpha + (j+k))^{-1}$$

for $j = 0, 1, ..., n$ *when* $n \geqq 1$ *and* $C(0, 0; \alpha) = 1$.

For the proof of Lemma 3.2, we express the right hand side of (3.6) by means of elementary functions. Define the functions $f_{j,k}$, $F_{j,n}$, and G_j^t, $j, k = 0, 1, ..., n$, by

$$f_{j,k}(x) = \begin{cases} e^{-(j+k)x}, & x \geqq 0, \\ 0, & x < 0, \end{cases}$$

$$F_{j,n}(x) = (f_{j,0} * \cdots * f_{j,j-1} * f_{j,j+1} * \cdots * f_{j,n})(x),$$

$$G_j^t(x) = e^{-2jx} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t},$$

respectively, where $*$ is the usual convolution of functions on $\mathbf{R}$ given by

$$(f * g)(x) = \int_{\mathbf{R}} f(x-y)\, g(y)\, dy.$$

Then it is easy to see

$$\hat{F}_{j,n}(\alpha) = \prod_{\substack{k \neq j \\ 0 \leqq k \leqq n}} (\sqrt{-1}\,\alpha + j + k)^{-1} \quad \text{and} \quad \hat{G}_j^t(\alpha) = \exp(t\phi(\sqrt{-1}\,\alpha + 2j)), \tag{9.2}$$

where $\hat{f}$ is the Fourier transform defined by

$$\hat{f}(\alpha) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\,\alpha x} f(x)\, dx.$$

Moreover we set $F_{0,0}(x) = \delta_0(x)$, the Dirac δ-function concentrated at 0 with respect to the Lebesgue measure.

LEMMA 9.2. *For every* $t > 0$, *non-negative integer* n, $j = 0, 1, ..., n$, $\xi \in \mathbf{R}$, *it holds that*

$$(F_{j,n} * G_j^t)(\xi) = \frac{1}{n!} \exp(-2j\xi) \int_0^{\infty} \frac{x - \xi}{t\sqrt{2\pi t}} \exp(jx - (x-\xi)^2/2t)(1 - e^{-x})^n\, dx \tag{9.3}$$

and

$$\sum_{j=0}^{n} (-1)^{n-j} {}_nC_j \, (F_{j,n} * G_j^t)(-\xi)$$
$$= \frac{2^n e^{n\xi}}{n!} \int_{\xi}^{\infty} \frac{x}{t\sqrt{2\pi t}} \exp(-x^2/2t)(\cosh x - \cosh \xi)^n \, dx. \tag{9.4}$$

Proof. When $j=n=0$, the proof of (9.3) is easy. When $n \geqq 1$, we first note that the elementary equality

$$\prod_{k=0}^{n} (\xi - c_k)^{-1} = \sum_{k=0}^{n} (\xi - c_k)^{-1} \prod_{\substack{m \neq k \\ 0 \leqq m \leqq n}} (c_k - c_m)^{-1}$$

holds for every sequence $\{c_k\}_{k=0}^{n}$ consisting of different numbers. Then, by (9.2), the injectivity of the Fourier transform implies

$$\hat{F}_{j,n}(\alpha) = \sum_{\substack{l \neq j \\ 0 \leqq l \leqq n}} (\sqrt{-1}\,\alpha + j + l)^{-1} \prod_{\substack{m \neq j, l \\ 0 \leqq m \leqq n}} (m-l)^{-1}$$
$$= \sum_{\substack{l \neq j \\ 0 \leqq l \leqq n}} (j-l) \frac{(-1)^l}{(n-l)!\,l!} (\sqrt{-1}\,\alpha + j + l)^{-1}$$

and

$$F_{j,n}(x) = \begin{cases} \dfrac{1}{n!} \displaystyle\sum_{l=0}^{n} (-1)^l (j-l) \, {}_nC_l \exp(-(j+l)\,x), & x \geqq 0, \\ 0, & x < 0. \end{cases}$$

Therefore we get

$$(F_{j,n} * G_j^t)(\xi) = (G_j^t * F_{j,n})(\xi)$$
$$= \frac{1}{n!} \sum_{l=0}^{n} (-1)^l (j-l) \, {}_nC_l \int_0^{\infty} \exp(-2j(\xi - x))$$
$$\times \frac{1}{\sqrt{2\pi t}} \exp((j+l)\,x - (x-\xi)^2/2t) \, dx$$
$$= -\frac{1}{n!} \sum_{l=0}^{n} (-1)^l \, {}_nC_l \exp(-2j\xi) \frac{1}{\sqrt{2\pi t}} \exp(-\xi^2/2t)$$
$$+ \frac{1}{n!} \sum_{l=0}^{n} (-1)^l \, {}_nC_l \exp(-2j\xi)$$
$$\times \int_0^{\infty} \frac{x-\xi}{t\sqrt{2\pi t}} \exp((j-l)\,x - (x-\xi)^2/2t) \, dx,$$

which is equal to the right hand side of (9.3) by virtue of the binomial theorem.

Equality (9.4) is an easy consequence of (9.3). ∎

Proof of Lemma 3.2. We note that

$$\int_W (A_t(w^\xi))^n \, \delta_\eta(w_t^\xi) \, P(dw) = e^{2n\xi} \int_W (A_t(w))^n \, \delta_{\eta-\xi}(w_t) \, P(dw).$$

Therefore we obtain (3.6) if we show that

$$\int_W (A_t(w))^n \, \delta_\zeta(w_t) \, P(dw) = \frac{e^{n\zeta}}{n!} \int_\zeta^\infty \frac{x}{t\sqrt{2\pi t}} \exp(-x^2/2t)(\cosh x - \cosh \zeta)^n \, dx \tag{9.5}$$

holds for every $\zeta \in \mathbf{R}$. By (9.1) and (9.2), we have

$$E[(A_t)^n \, e^{\sqrt{-1}\,\alpha w_t}] = 2^{-n} \sum_{j=0}^n (-1)^{n-j} \, {}_nC_j (F_{j,n} * G_j^t)^\wedge(\alpha)$$

and

$$E[(A_t)^n; \, w_t \in d\zeta] = 2^{-n} \sum_{j=0}^n (-1)^{n-j} \, {}_nC_j (F_{j,n} * G_j^t)(-\zeta) \, d\zeta.$$

Therefore, by using (9.4), we obtain

$$\begin{aligned} &\int_W (A_t(w))^n \, \delta_\zeta(w_t) \, P(dw) \\ &\quad = 2^{-n} \sum_{j=0}^n (-1)^{n-j} \, {}_nC_j (F_{j,n} * G_j^t)(-\zeta) \\ &\quad = \frac{e^{n\zeta}}{n!} \int_\zeta^\infty \frac{x}{t\sqrt{2\pi t}} e^{-x^2/2t} (\cosh x - \cosh \zeta)^n \, dx. \quad ∎ \end{aligned}$$

Remark 9.1. Formula (9.5) can also be shown by induction in n if we follow the method in Kac [29].

Remark 9.2. Denoting the right hand side of (9.5) by μ_n, it is easy to show

$$\sum_{n=1}^\infty \mu_n^{-1/2n} < \infty.$$

Therefore, as has been pointed out by Yor [50], Carleman's sufficient condition for the unique solvability of the Stiltjes moment problem does not hold (cf. [47]). Therefore it is not easy to know the explicit form of the joint distribution of (A_t, w_t) from the sequence of the moments.

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