Criteria equivalent to the Riemann Hypothesis

J. Cisło, M. Wolf

Instytut Fizyki Teoretycznej Uniwersytet Wrocławski PL-50-204 Wrocław Pl. Maxa Borna 9

Abstract

We give a brief overview of a few criteria equivalent to the Riemann Hypothesis. Next we concentrate on the Riesz and Báez-Duarte criteria. We proof that they are equivalent and we provide some computer data to support them. It is not compressed to six pages version of the talk delivered by M.W. during the XXVII Workshop on Geometrical Methods in Physics, 28 June – 6 July, 2008, Białowieża, Poland.

1. Introduction

Euler investigated the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

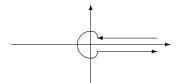
for real s > 1. In particular, he found expression giving values of the zeta function for even arguments:

$$\zeta(2m) = \frac{|B_{2m}|\pi^{2m}}{2(2m)!} \tag{2}$$

where B_n are the Bernoulli numbers. Riemann showed [23] that the integral $(s \neq 1)$:

$$\zeta(s) = \frac{\Gamma(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$
 (3)

where the integration is performed on the contour



is well defined on the whole complex plane without s=1, where $\zeta(s)$ has the simple pole, and is equal to (1) on the right of the line $\Re[s]=1$. The $\zeta(s)$ function has trivial zeros: $s=-2,-4,-6,\ldots$ i.e. $\zeta(-2n)=0$. Besides that there exists the infinity of zeros $\rho=\sigma+it$ in the critical strip, $0\leq \Re[\rho]=\sigma\leq 1$. If ρ is zero, then also $\overline{\rho}$ and $1-\rho$ are zeros, thus zeros are located symmetrically around the critical line $\Re[s]=\frac{1}{2}$, see e.g. [27]. The Riemann Hypothesis (RH) states that all non-trivial zeros of $\zeta(s)$ lie on the critical line $s=\frac{1}{2}+it$. Presently the requirement that they are simple is often added. It is one of the best known open problems in mathematics, see e.g. [8],[9]. In the last years, after the Clay Mathematics Institute granted 1 million US\$ award for solving the dilemma of the RH, there have appeared on the arxiv many preprints claiming to have proved (e.g.,[25], [14]) or disproved (e.g., [1], [21]) the RH, but so far all of them have been withdrawn as errors were found in them.

Already Riemann calculated numerically a few first nontrivial zeros of $\zeta(s)$ [9]. Below there is a short list of numerical determinations of nontrivial zeros of $\zeta(s)$:

J.P. Gram(1903): 15 zeros are on the critical line [11]

:

A. Turing (1953): 1104 zeros are on the critical line [28]

:

D.H. Lehmer (1956): 25000 zeros are on the critical line.

:

A few years ago S. Wedeniwski (2005) was leading the internet project Zetagrid [31] which during four years determined that 250×10^{12} zeros are on the critical line: $s = \frac{1}{2} + it$, |t| < 29,538,618,432.236. The present record belongs to K. Gourdon(2004) [10]: the first 10^{13} zeros are on the critical line.

2. Some Criteria equivalent to the RH

There are probably well over one hundred statements equivalent to the RH, see eg. [27], [12], [30]. Riemann's original aim was to prove the guess made by 15-years-old Gauss, namely that the number $\pi(x)$ of primes < x is well approximated by the logarithmic integral:

$$\pi(x) \approx \operatorname{Li}(x) = \int_2^\infty \frac{du}{\log(u)}.$$
 (4)

In this spirit in 1901, Koch proved [29] that the Riemann Hypothesis is equivalent to the following error term for the expression for the prime counting function: $\pi(x)$ is given by:

$$\pi(x) = \operatorname{Li}(x) + \mathcal{O}(\sqrt{x}\ln(x)). \tag{5}$$

Another similar criterion is

RH
$$\Leftrightarrow \pi(x) = \text{Li}(x) + \mathcal{O}(x^{1/2+\epsilon})$$
 for each $\epsilon > 0$. (6)

The following criterion is of interest to mathematical physicists. In 1955 Arne Buerling [3] proved that the Riemann Hypothesis is equivalent to the assertion that $\mathcal{N}_{(0,1)}$ is dense in $L^2(0,1)$. Here $\mathcal{N}_{(0,1)}$ is the space of functions

$$N_{(0,1)} = \left\{ \sum_{k=1}^{n} c_k \rho\left(\frac{\theta_k}{x}\right), \quad 0 < \theta_k < 1, \sum_{k=1}^{n} c_k = 0, \quad n = 1, 2, 3, \dots \right\}$$
 (7)

where $\rho(u) = u - \lfloor u \rfloor$ is a fractional part of u. The function $\zeta(s)$ does not have zeros in the half-plane $\sigma > \frac{1}{q}$, $1 < q < \infty$ iff the set $N_{(0,1)}$ is dense in $L^q(0,1)$.

In fact Beurling proved that the following three statements regarding a number $q \in (1, \infty)$ are equivalent:

- (1) $\zeta(s)$ has no zeros in $\sigma > 1/q$
- (2) $\mathcal{N}_{(0,1)}$ is dense in $L^{q}(0,1)$
- 3) The characteristic function $\chi_{(0,1)}$ is in the closure of $\mathcal{N}_{(0,1)}$ in $L^q(0,1)$

The following ideas show that the validity of the RH is very delicate and subtle. Let us introduce the function

$$\xi(iz) = \frac{1}{2} \left(z^2 - \frac{1}{4} \right) \pi^{-\frac{z}{2} - \frac{1}{4}} \Gamma\left(\frac{z}{2} + \frac{1}{4} \right) \zeta\left(z + \frac{1}{2} \right). \tag{8}$$

We can see from the above formula that the RH \Leftrightarrow all zeros of $\xi(iz)$ are real. The point is that $\xi(z)$ can be expressed as the following Fourier transform:

$$\frac{1}{8}\xi\left(\frac{z}{2}\right) = \int_0^\infty \Phi(t)\cos(zt)dt,\tag{9}$$

where

$$\Phi(t) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) e^{-\pi n^2 e^{4t}}.$$
 (10)

And now we follow the rule of Polya [22]: if one can not solve a particular problem, maybe it is possible to solve more general problem. So, we introduce the family of functions $H(z, \lambda)$ parameterized by λ as the following Fourier transform:

$$H(z,\lambda) = \int_0^\infty \Phi(t)e^{\lambda t^2} \cos(zt)dt. \tag{11}$$

Thus we have $H(z,0) = \frac{1}{8}\xi(\frac{1}{2}z)$ N. G. De Bruijn [7] proved that (1950):

- 1. $H(z,\lambda)$ has only real zeros for $\lambda \geq \frac{1}{2}$
- 2. If $H(z, \lambda)$ has only real zeros for some λ' , then $H(z, \lambda)$ has only real zeros for each $\lambda > \lambda'$.

And here comes bad news: in 1976 Ch. Newman [19] has proved that there exists parameter λ_1 such that $H(z, \lambda_1)$ has at least one non-real zero. Thus, there exists

such constant Λ in the interval $-\infty < \Lambda < \frac{1}{2}$ that $H(z,\lambda)$ has real zeros $\Leftrightarrow \lambda > \Lambda$. The Riemann Hypothesis is equivalent to $\Lambda \leq 0$. This constant Λ is now called the de_Bruijn-Newman constant. Newman believes that $\Lambda \geq 0$. The computer determination has provided the numerical values of de_Bruijn-Newman constant, here is a sample of results:

Csordas et al (accuracy of calculations 360 digits) (1988) $-50 < \Lambda$

:

te Riele (accuracy of calculations 250 digits) (1991) $-5 < \Lambda$

:

Odlyzko (2000)
$$-2.7 \cdot 10^{-9} < \Lambda$$

This last result [20] was obtained using the following pair of zeros of the Riemann $\zeta(s)$ function near zero number 10^{20} that are unusually close together (thus they almost violate the simplicity of zeros):

$$k = 10^{20} + 71810732, \quad \gamma_{k+1} - \gamma_k < 0.000145$$
 (12)

Because the gap in which Λ catching the RH is so squeezed, Odlyzko noted in [20], that "...the Riemann Hypohesis, if true, is just barely true".

Li Criterium (1997) [15]: Riemann Hypothesis is true iff the sequence:

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} (s^{n-1} \log \xi(s))|_{s=1}$$

where

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

fulfills:

$$\lambda_n \ge 0 \quad \text{for } n = 1, 2, \dots \tag{13}$$

Explicit expression: $\lambda_n = \sum_{\rho} (1 - (1 - 1/\rho)^n)$. K. Maślanka [18], [17] gave explicit expression for λ_i and performed extensive computer calculations of these constants confirming (13).

A sensation was stirred in 2000, when the elementary criterion for the RH was invented by Lagarias [13]: the Riemann Hypothesis is equivalent to the inequalities:

$$\sigma(n) \equiv \sum_{d|n} d \le H_n + e^{H_n} \log(H_n)$$
 (14)

for each n = 1, 2, ..., where H_n is the *n*-th harmonic number $H_n = \sum_{j=1}^n \frac{1}{j}$. Here $\sigma(n)$ is the sum of all divisors of n. The illustration of (14) is shown on Fig.1. In the paper [4] the maxima of $\sigma(n)$ were studied.

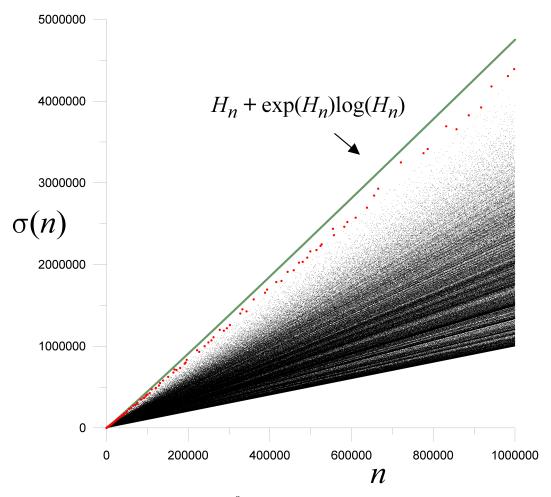


Fig.1 The plot of $\sigma(n)$ for $1 < n < 10^6$. In red are plotted values of $\sigma(n)$ which approach the threshold values closer than 5%. Data for this plot was obtained with the free package PARI/GP [26]

3. Criteria of Riesz and Báez-Duarte

In 1916 Riesz [24] introduced the function:

$$R(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k! \zeta(2k+2)}$$
 (15)

and next he proved that:

$$RH \Leftrightarrow R(x) = \mathcal{O}\left(x^{1/4+\epsilon}\right).$$
 (16)

It involves uncountably many values of x and L. Báez-Duarte [2] found a discrete version of (16). In turn, his derivation was based on the representation for $\zeta(s)$ found by K. Maślanka [16]:

$$\zeta(s) = \frac{1}{(s-1)\Gamma(1-s/2)} \times \sum_{k=0}^{\infty} \frac{\Gamma(k-\frac{s}{2})}{k!} \sum_{j=0}^{k} (-1)^{j} {k \choose j} (2j+1)\zeta(2j+2).$$
 (17)

Using (17) L. Báez-Duarte proved that the Riemann Hypothesis is equivalent to the fact that the sequence

$$c_k = \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}$$
 (18)

decreases to zero like

$$c_k = \mathcal{O}(k^{-\frac{3}{4}+\epsilon})$$
 for each ϵ . (19)

In 2005 one of us [32] started the computer calculation of c_k . The plot of c_k is presented in Fig.2. The envelops are given by the equations

$$y(k) = \pm Ak^{-\frac{3}{4}}, \quad A = 0.777506... \times 10^{-5}.$$
 (20)

In [5] and [6] we have shown that c_k and R(x) are "entangled" by means of the relation:

$$\sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = \frac{e^x}{x} R(x) \tag{21}$$

Next we have proved that

Theorem: For any real number $\delta > -3/2$ we have

$$R(x) = O(x^{\delta+1}) \Leftrightarrow c_k = O(k^{\delta}). \tag{22}$$

The proof of this theorem is based on the fact that $R(k)/k \approx c_k$ for integer k. In more detail, in [6] we have shown that:

$$\left| \frac{R(k)}{k} - c_k \right| \le \frac{3\sqrt{\pi}}{16} k^{-3/2} + \mathcal{O}(k^{-2}). \tag{23}$$

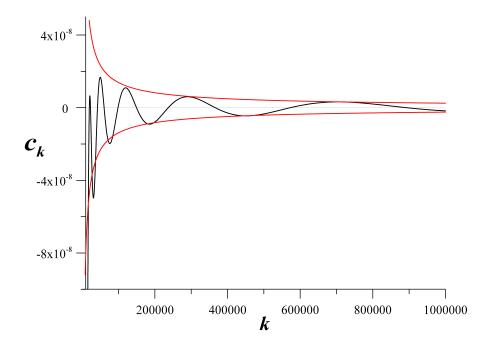


Fig.2 The plot of c_k given by (18)

We have also obtained the value of the sum:

$$\sum_{k=0}^{\infty} (-1)^k c_k = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\zeta(2k)} = 0.7825279853253842\dots$$
 (24)

Finally, we mention the approximate relation which follows from (23) entangling in a strange manner values of $\zeta(2j+2)$:

$$\sum_{j=0}^{\infty} \frac{(-1)^j k^j}{j! \zeta(2j+2)} \approx \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}.$$
 (25)

We have checked numerically that the difference between these two sums very quickly tends to zero with increasing k.

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