Eigenfunctions of Positive Definite Integral Covariance Operators Via The Bubnov-Galerkin Method And The The Christoffel-Darboux kernel

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March 18, 2024

1 Compact Integral Covariance Operators

Consider a compact integral operator K acting on functions in $L^2(D)$, the space of square-integrable functions over a domain D, where we take to be the noncompact domain of semi-infinite positive half-line $D=[0,\infty)$. This operator is associated with a reproducing kernel Hilbert space (RKHS) as a consequence of the positive definiteness of the kernel function $k(x,y)$ which has an RKHS whose existence is assured by Bocherns theorem which says that function is positive definite if its Fourier transform is non-negative and non-decreasing and these will be the conditions for which this article applies. The goal is to solve the eigenvalue problem for K:

$$(K\phi)(x)=\lambda\phi(x) \tag{1}$$

where ϕ represents an eigenfunction of K, and λ is the corresponding eigenvalue. The action of K on ϕ is defined by:

$$(K\phi)(x)=\int_D k(x,y)\,\phi(y)\,dy \tag{2}$$

An orthonormal basis $\{\phi_i\}_{i=1}^{\infty}$ in $L^2(D)$, consisting of orthogonal polynomials, serves as the basis functions for the RKHS.

Right now the orthonormal basis will assume to be known but it will be shown how one can always be found or constructed if the sequence for the kernel does not correspond to any of the off-the-shelf known orthognal polynomial sequences, classical or otherwise.

1.1 The Bubnov-Galerkin method

The Bubnov-Galerkin method is applied to find the eigenfunctions of K by expressing a solution ϕ_n as a linear combination of the basis functions:

$$\phi_n(x)=\sum_{i=1}^{n} c_i\,\phi_i(x) \tag{3}$$

with c_i being coefficients to determine. The orthonormality of these basis functions implies that the mass matrix is the identity matrix I, leading to:

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} \tag{4}$$

where δ_{ij} is the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise.

The eigenvalue problem then transforms into:

$$\int_D k(x, y) \sum_{i=1}^{n} c_i \phi_i(y) \, dy = \lambda \sum_{j=1}^{n} c_j \phi_j(x) \tag{5}$$

By multiplying both sides by $\phi_k(x)$ and integrating over D, and utilizing the orthonormality of the basis functions, we simplify the equation to:

$$\sum_{i=1}^{n} c_i \int_D \int_D \phi_k(x) \, k(x, y) \, \phi_i(y) \, dy \, dx = \lambda c_k \tag{6}$$

for each k. The system of equations derived from the above formulation can be solved as a standard eigenvalue problem:

$$A \vec{c} = \lambda \vec{c} \tag{7}$$

where A is the matrix with elements

$$A_{ki} = \int_D \int_D \phi_k(x) \, k(x, y) \, \phi_i(y) \, dy \, dx \tag{8}$$

, and $\vec{c}$ is the vector of coefficients c_i.

2 The Christoffel-Darboux kernel

Leveraging the Christoffel-Darboux kernel we can construct a sequence of optimal (in the sense that their truncating at finite N minimzes the error of the approximation for any finite N) finite rank approximations, we can express the N-th partial sum as a ratio of the N and N-1 polynomials . This allows us to compute the integrals involved in A_{ki} exactly, without the need for direct integration over the kernel, possibly taking the limit via

Let's denote the N-th partial sum as $K_N(x, y)$, and let $\phi_i(x)$ and $\phi_k(x)$ be the N-th order polynomials. Then, we have:

$$K_N(x, y) = \sum_{j=1}^{N} \frac{\phi_j(x) \, \phi_j(y)}{\lambda_j} \tag{9}$$

Using this approximation, we can express A_{ki} as:

$$A_{ki} = \int_D \int_D \phi_k(x) K_N(x, y) \phi_i(y)\, dy\, dx \tag{10}$$

Substituting the expression for $K_N(x, y)$, we get:

$$A_{ki} = \sum_{j=1}^{N} \frac{1}{\lambda_j} \int_D \int_D \phi_k(x)\ \phi_j(x)\ \phi_i(y)\ dy\ dx \tag{11}$$

This can be simplified further using the orthonormality condition of the basis functions. Let C_{ijk} represent the triple product integral:

$$C_{ijk} = \int_D \phi_k(x)\ \phi_j(x)\ \phi_i(y)\ dx \tag{12}$$

Then, A_{ki} can be expressed as:

$$A_{ki} = \sum_{j=1}^{N} \frac{C_{ijk}}{\lambda_j} = \sum_{j=1}^{N} \frac{\int_D \phi_k(x)\ \phi_j(x)\ \phi_i(y)\ dx}{\lambda_j} \tag{13}$$

Now, to compute C_{ijk} exactly, we need to solve the triple product integral in closed form, which can be done analytically based on the properties of the basis functions and the domain D. This approach provides an exact solution for A_{ki}, avoiding the need for numerical integration and ensuring accuracy, stability, and computational efficiency.

2.1 The Eigenfunctions

Given the matrix elements A_{ki} as computed using the Christoffel-Darboux kernel approximation(which is exact in the limit), we can express the eigenfunctions $\phi_k(x)$ in terms of the coefficients c_i as follows:

$$\phi_k(x) = \sum_{i=1}^{n} c_i\, \psi_{ki}(x)$$

where $\psi_{ki}(x)$ are the basis functions used to construct the RKHS. The specific form of $\psi_{ki}(x)$ depends on the choice of basis functions. If orthogonal polynomials are used as the basis functions, then $\psi_{ki}(x)$ would correspond to the i-th polynomial in the expansion of the k-th eigenfunction.

In terms of the matrix elements A_{ki}, we can express the eigenfunctions as:

$$\phi_k(x) = \sum_{i=1}^{n} \psi_{ki}(x) \frac{c_i}{\lambda_k} \sum_{j=1}^{N} \frac{C_{ijk}}{\lambda_j}$$

where C_{ijk} are the triple product integrals and λ_k are the eigenvalues. This expression provides the eigenfunctions in terms of the coefficients c_i and the basis functions used to construct the RKHS.