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A remark on the spectral domain of nonstationary processes

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Abstract

The question of completeness of the spectral domain generated by a bimeasure is discussed. The bimeasure is derived from a spectral representation of a given process of weak class (C). In particular by an example it is shown that the spectral domain of a weak class (C) process is not necessarily complete. A sufficient condition for completeness is established.

Keywords: Spectral domain; Weak harmonizability; Bimeasure; Stochastic integral; Karhunen class; Cramér class

1. Introduction

The problem of completeness of the spectral domain was firstly mentioned in Cramér (1951), where the spectral domain is defined for a special class of nonstationary processes introduced by Cramér and denoted as the C-class processes now. Cramér shows that the spectral domain has all the properties of a Hilbert space except completeness. Even in the case of multivariable stationary processes one cannot simply prove the completeness of the spectral domain. In the multidimensional case this was established by Rozanov (1967) and independently by Rosenberg (1964). For nonstationary processes like strongly or weakly harmonizable processes (for definitions see Rao (1989a)) the question of spectral domain completeness has been open for many years. But, in a recent paper of Chang and Rao (1986) a general completeness result is stated. One can also find there on page 76 a remark concerning a possible generalization of their approach to prove the completeness of the spectral domain in the case of weak class (C) (cf. also Rao 1989a, p. 607). The aim of this paper is to construct a simple example of a C-class process whose spectral domain is not complete and, furthermore, to provide a sufficient condition for completeness.

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A function F on $S \times S$ (S being a locally compact set) into **C** is defined to be of locally finite (Fréchet) variation if the restriction of F to each bounded Borel set $B \times B$ of $S \times S$ has finite (Fréchet) variation.

Definition 1 (see Hannan, Krishanaiah, Rao (1985), p. 291). A second order process $X = \{X_t, t \in T\} \subset \mathscr{L}_0^2(P)$ is of Cramér class (or class (C)) if its covariance function r is representable as

$$r(s,t) = \int_S \int_S g_s(\lambda) \bar{g}_t(\lambda') \beta(\mathrm{d}\lambda, \mathrm{d}\lambda'), \quad s, t \in T$$

relative to a family $\{g_t(\cdot), t \in T\}$ of Borel functions and a positive definite function β of locally bounded variation on $S \times S$ and each g_t satisfying the (Lebesgue) integrability condition

$$0 \leq \int_S \int_S g_t(\lambda) \bar{g}_t(\lambda') \beta(\mathrm{d}\lambda, \mathrm{d}\lambda') < \infty, \quad t \in T.$$

If β has locally finite Fréchet variation then the integrals are in Morse–Transue sense, and X is called to be of weak class (C).

Why is the question of completeness so important? For the explicit construction of optimal estimators or predictors, which are based on minimal quadratic mean error, the completeness of the spectral domain is a necessary condition. If the completeness of the spectral domain is not valid, then the construction of the best linear estimates cannot be based on the solution of the integral equations arising from the projection equations and leading to explicit results in several cases. For easy reference we repeat the definition of the spectral function and spectral domain.

Definition 2 (see Chang and Rao (1986), p. 69). Let $(S, \mathscr{A})$ be a measurable space and β be a bimeasure on $\mathscr{A} \times \mathscr{A}$ positive definite. The set $\mathscr{L}^2(\beta)$ defined by

$$\mathscr{L}^2(\beta) = \left\{ f: S \to C,\ \mathscr{A}\text{-measurable},\ \int_S \int_S f(\lambda) \overline{f(\mu)} \beta(\mathrm{d}\lambda, \mathrm{d}\mu) < \infty \right\}$$

is called the spectral domain of the process for which β is the spectral function.

If β is the spectral function of a weak class (C) process relative to a family $\{g(t,\cdot), t \in T\}$ on $(S, \mathscr{A})$, then clearly $g(t,\cdot) \in \mathscr{L}^2(\beta)$ for each $t \in T$. Since β is positive definite, the β-integral

$$(f,h) \to \int_S \int_S f(\lambda) \overline{h(\mu)} \beta(\mathrm{d}\lambda, \mathrm{d}\mu) \tag{1}$$

for $f, h \in \mathscr{L}^2(\beta)$, defines a semi-inner product on $\mathscr{L}^2(\beta)$. The crucial question is now: is the space $\mathscr{L}^2(\beta)$ complete? Theorem 4 in Chang and Rao, p. 76 asserts that in the case of a weakly harmonizable process the corresponding spectral domain $\mathscr{L}^2(\beta)$ is complete.

2. A counterexample

We next shall construct a random process, which belongs to the class (C), but its spectral domain is not complete with respect to the topology induced by the semi-inner product defined in (1). Let $T = \mathbf{R}_1$ be the real numbers, let $S = \mathcal{N}$ be the natural numbers with $\mathscr{A}$ equal to the class of all subsets of S with the discrete topology. Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of mutually independent random variables with distribution $N(0, \sigma_n^2)$ such that

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty .$$

Let us consider $X_n = \sum_{k=1}^{n} Y_k$ and let $g(t,j): T \times S \to \mathbf{C}$ be a sequence of Borel measurable functions such that for each $t \in \mathbf{R}_1$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g(t,j)\overline{g(t,k)}R(j,k) \tag{2}$$

is finite, where $R(j,k) = E\{X_j X_k\} = \sum_{l=1}^{j \wedge k} \sigma_l^2$. Consider the process

$$x(t) = \sum_{j=1}^{\infty} g(t,j) X_j, \quad t \in \mathbf{R}_1,$$

where the summation is understood in the quadratic mean sense. The functions $g(t,\cdot)$ can be chosen, e.g.

$$g(t,j) = \mathrm{e}^{-j|t|}.$$

Then condition (2) is fulfilled because $R(\cdot,\cdot)$ is bounded. Let us prove that the spectral domain of $\{x(t), t \in \mathbf{R}_1\}$ is not complete.

The considered process belongs to the weak class (C) because its spectral function $\beta(\cdot,\cdot)$ is defined and finite on the δ-ring $\mathscr{B}_0 \times \mathscr{B}_0$ of all compact (bounded) subsets in $\mathcal{N}$. Surely

$$\beta(A,B) = E\left\{\sum_{j \in A} X_j \cdot \sum_{k \in B} X_k\right\} = \sum_{j \in A} \sum_{k \in B} R(j,k) \quad (A, B \in \mathscr{B}_0).$$

$\beta(\cdot,\cdot)$ possesses even locally bounded Vitali variation thanks to additivity of the random set function $\mu(A) = \sum_{j \in A} X_j$. Hence the process $\{x(t), t \in \mathbf{R}_1\}$ belongs to the class (C). The corresponding spectral domain is formed by all sequences $a = \{a_j\}_{j=1}^{\infty}$ with finite

$$\|a\|_\beta^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j \overline{a_k} R(j,k).$$

Let us consider the sequence $\{\delta_n\}_{n=1}^{\infty}$ of elements belonging to $\mathscr{L}^2(\beta)$

$$\delta_n = \{\delta_{nj}\}_{j=1}^{\infty},$$

where δ_{nj} is the Kronecker delta. Let us prove that $\{\delta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathscr{L}^2(\beta)$. Evidently

$$\|\delta_n\|_\beta^2 = R(n, n)$$

and

$$\|\delta_n - \delta_m\|_\beta^2 = R(n, n) - R(n, m) - R(m, n) + R(m, m)$$
$$= E\{(X_n - X_m)^2\}.$$

As $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$ there exists $\lim_{n\to\infty} X_n = X_\infty$ in L^2 and we see that $\{\delta_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Further, $\lim_{n\to\infty} \delta_n = (0, 0, 0, \ldots)$ coordinatewise. Completeness of the spectral domain would imply the existence of limit sequence $\Delta = \{\Delta_n\}_{n=1}^{\infty} \in \mathscr{L}^2(\beta)$ such that

$$\|\delta_n - \Delta\|_\beta^2 \to 0, \quad n \to \infty.$$

Simultaneously, we would have

$$X_\infty = \sum_{n=1}^{\infty} \Delta_n X_n.$$

As consequence we then obtain

$$E\{X_\infty Y_k\} = \lim_{N\to\infty} E\left\{\sum_{n=1}^{N} \Delta_n X_n Y_k\right\}$$
$$= \lim_{N\to\infty} \sum_{n=k}^{N} \Delta_n \sigma_k^2 = \sigma_k^2 \sum_{n=k}^{\infty} \Delta_n.$$

On the other hand,

$$E\{X_\infty Y_k\} = \lim_{n\to\infty} E\{X_n Y_k\} = \lim_{n\to\infty} \sum_{j=1}^{n} E\{Y_j Y_k\} = \sigma_k^2.$$

Comparing these two results we obtain for every $k \in \mathscr{N}$

$$\sum_{n=k}^{\infty} \Delta_n = 1,$$

a contradiction. Hence X_∞ does not admit a representation of the form $\sum_{n=1}^{\infty} \alpha_n X_n$. This fact shows that the spectral domain $\mathscr{L}^2(\beta)$ of the process $\{x(t), t \in \mathbf{R}_1\}$ is not complete.

Further, in this example the covariance function $R(\cdot,\cdot)$ defines a scalar product on the linear set of all sequences $a = \{a_j\}_{j=1}^{\infty}$ satisfying

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j \overline{a_k} R(j,k) < \infty .$$

But this linear set is not complete and, therefore, does not form a Hilbert space.

If we consider $Z_n = X_n/2^n$ instead of X_n and $r_n = 2^n \delta_n$ instead of δ_n, then $\| r_n - r_m \|_{\beta_1} = \| \delta_n - \delta_m \|_{\beta}$, where $\beta_1(A,B) = \mathsf{E}(\sum_{j\in A} Z_j \cdot \sum_{k\in B} Z_k)$. Then β_1 is even of finite Vitali variation and the random process

$$y(t) = \sum_{j=1}^{\infty} \mathrm{e}^{itj} Z_j$$

is strongly harmonizable but the corresponding spectral domain is not complete.

For the elements of $\mathscr{L}^2(\beta)$ generated by the process $\{x(t), t \in \mathbf{R}_1\}$ we can prove the following property.

Lemma 1. *For any* $a = \{a_j\}_{j=1}^{\infty} \in \mathscr{L}^2(\beta)$ *from the previous example there exists* $\sum_{j=1}^{\infty} a_j$, *but is nonabsolutely convergent, in general.*

Proof. When $a \in \mathscr{L}^2(\beta)$ then the sum $\sum_{j=1}^{\infty} a_j X_j$ is convergent in the quadratic mean sense. Hence, the weak convergence is true also, i.e. for every random variable $Y \in \mathscr{L}_2(P)$ there exists the limit

$$\lim_{N\to\infty} E\left\{ \sum_{j=1}^{N} a_j X_j Y \right\} = \sum_{j=1}^{\infty} a_j E\{X_j Y\}.$$

If we put $Y = Y_k$ we find that $E\{X_j Y_k\} = \delta_{jk}\sigma_k^2$. From this fact we obtain that

$$\lim_{n\to\infty} \sum_{j=k}^{n} a_j$$

exists for every $k \in \mathcal{N}$. □

Remark 1. (a) After submission of the first version of this manuscript we were informed about the paper due to Miamee and Salehi (1991) where a counterexample is also presented. The example constructed in this paper seems to be more elementary and allows to construct further related counterexamples.

(b) We should like to note that the proof of completeness in a more special situation given in Rao (1989a) fails at two places. On one hand, on p. 604, line 6 from above, the consequence of weak convergence needs the continuity of the limiting measure (which is by no means to assure). On the other hand, the author uses the implicit assumption that $(A^{1/2})^{-1}$, A is the covariance operator, is bounded (cf. p. 603) which is not valid, in general.

3. A sufficient condition for the completeness of stochastic integrals

The problem of completeness can be set up quite generally for each random set function $Z(\cdot,\cdot)$ defined on $\mathscr{B}\times\Omega$, where $\mathscr{B}$ is a set system in S and (Ω,σ,P) is an underlying probability space. For simplicity we put

$$E\{Z(A,\omega)\}=0$$

for each $A\in\mathscr{B}$ and demand the existence of

$$E\{Z(A,\omega)\overline{Z(B,\omega)}\}=\beta(A,B) \tag{3}$$

for each A, $B\in\mathscr{B}$. Let $\mathscr{L}_2(P)$ be the Hilbert space of all square integrable random variables on (Ω,σ). Let $\mathscr{L}_2(Z)$ be the closure in $\mathscr{L}_2(P)$ of the linear set generated by all the random variables

$$\xi(\omega)=\sum_{j=1}^{p}\lambda_j Z(A_j,\omega),\quad \lambda_j\in\mathbf{C},\quad A_j\in\mathscr{B}.$$

The subspace $\mathscr{L}_2(Z)$ can be understood as the set of all the linear estimates derived from random set function $Z(\cdot,\cdot)$. In case $\mathscr{B}$ is a σ-algebra in S we can define the stochastic integral in the sense of Bartle–Dunford–Schwartz

$$\int_E f(s)\,Z(\mathrm{d}s,\omega),\quad E\in\mathscr{B}.$$

Here, the notion of a stochastic integral is understood as a special case of the Bartle–Dunford–Schwartz integral as presented, e.g. in Dunford–Schwartz (1958). A function f defined on $(S,\mathscr{B})$ is integrable with respect to a stochastic measure $Z(\cdot,\cdot):(\Omega,\mathscr{B})\to\mathbf{C}$ if

(1) there exists a sequence of $\mathscr{B}$-measurable simple functions $\{f_n\}_{n=1}^{\infty}$ such that

$$f_n\to f\quad \text{a.e. } [Z]$$

(2) for each $E\in\mathscr{B}$ $\{\int_E f_n\,\mathrm{d}Z\}_{n=1}^{\infty}$ is convergent in the quadratic mean sense.

Then, we define $\int_E f\,\mathrm{d}Z=\lim_{n\to\infty}\int_E f_n\,\mathrm{d}Z$, $E\in\mathscr{B}$. It is worth noting that the existence of $\lim_{n\to\infty}\int_S f_n\,\mathrm{d}Z$ does not imply in general the existence of $\lim_{n\to\infty}\int_E f_n\,\mathrm{d}Z$. We can use the notation $\int_S \psi_E f\,\mathrm{d}Z$ also for $\int_E f\,\mathrm{d}Z$ a special case because for each simple function g on S(ψ_E is the indicator of $E\in\mathscr{B}$)

$$\int_S \psi_E g\,\mathrm{d}Z=\int_E g\,\mathrm{d}Z.$$

In this sense the stochastic integral $\int f\,\mathrm{d}Z$ is a linear estimate.

The following question arises naturally: Under which conditions can each element of $\mathscr{L}_2(Z)$ be expressed in the form of a stochastic integral? The previous example shows that in general there is no equality between $\mathscr{L}_2(Z)$ and the class of stochastic integrals. This question is closely connected with the completeness of a spectral domain because the random set function $Z(\cdot,\cdot)$ induces a bimeasure $\beta(\cdot,\cdot)$ on $\mathscr{B}\times\mathscr{B}$ by the relation (3). This problem was studied, e.g. in the paper due to Grandell (1976) in the case where $Z(\cdot,\cdot)$ is a double stochastic Poisson process. Grandell proved that in this case the completeness of the spectral domain is valid under certain conditions. The correspondence between the stochastic integral and the element of $\mathscr{L}^2(\beta)$ is given by the one-to-one isometric mapping

$$\int_S f\,\mathrm{d}Z \leftrightarrow f$$

which can be unambiguously enlarged to $\mathscr{L}^2(Z)$ and $\overline{\mathscr{L}_2(\beta)}$, the closure in the topology defined by the semi-inner product

$$\langle f, g\rangle_\beta = \int_S\int_S f(s)\overline{g(t)}\,\beta(\mathrm{d}s,\mathrm{d}t).$$

Obviously the stochastic integrals form an everywhere dense subset in $\mathscr{L}_2(Z)$.

The space $\mathscr{L}^2(\beta)$ is also closely connected with the reproducing kernel Hilbert space (RKHS) derived from a given random process. If the underlying process can be expressed in the form of a stochastic integral in the sense of Bartle–Dunford–Schwartz

$$x(t) = \int_S f(t,\lambda)\,Z(\mathrm{d}\lambda), \quad t\in T,$$

where $f(t,\cdot)\in\mathscr{L}^2(\beta)$ then the corresponding RKHS is described in Lemma 2.

Lemma 2. *Let*

$$R(s,t) = E\{x(s)\overline{x(t)}\} = \int_S\int_S f(s,\lambda)\overline{f(t,\mu)}\,\beta(\mathrm{d}\lambda,\mathrm{d}\mu).$$

Then

$$H(R) = \left\{g^* : g^*(t) = \int_S\int_S g(\lambda)\overline{f(t,\mu)}\,\beta(\mathrm{d}\lambda,\mathrm{d}\mu),\ g\in\mathscr{L}^2(\beta,f_t)\right\}$$

is the RKHS of $\{x(t), t\in T\}$ where $\mathscr{L}^2(\beta,f_t)$ is the subspace of $L^2(\beta)$ generated by the functions $\{f(t,\cdot), t\in T\}$.

Proof. As $R(s,t)=\int_S\int_S f(s,\lambda)\overline{f(t,\mu)}\,\beta(\mathrm{d}\lambda,\mathrm{d}\mu)$, $R(s,\cdot)$ belongs to $H(R)$ for each $s\in T$. On the other hand, for every $g\in\mathscr{L}^2(\beta,f_t)$ we have

$$(g^*(\cdot), R(s,\cdot))_{H(R)} = \int_S\int_S g(\lambda)\overline{f(t,\mu)}\,\beta(\mathrm{d}\lambda,\mathrm{d}\mu) = g^*(t)$$

for each $t\in T$. These two properties define the RKHS of the process $\{x(t), t\in T\}$. $\square$

The following Proposition gives some sufficient conditions for the completeness of the class of stochastic integrals.

Proposition 1. *Let $\nu(\cdot)$ be a σ-finite measure on $(S, \mathscr{B})$ such that each $f \in \mathscr{L}^2(\beta)$ belongs to $\mathscr{L}_2(\nu)$ and for some constants $0 < c \le C < \infty$ the following conditions*

$$\text{(A1)} \qquad \int_S \int_S f(\lambda)\overline{f(\mu)}\,\beta(\mathrm{d}\lambda, \mathrm{d}\mu) \ge c \int_S |f(\lambda)|^2\,\nu(\mathrm{d}\lambda),$$

$$\text{(A2)} \qquad \int_S \int_S f(\lambda)\overline{f(\mu)}\,\beta(\mathrm{d}\lambda, \mathrm{d}\mu) \le C \int_S |f(\lambda)|^2\,\nu(\mathrm{d}\lambda),$$

hold. Then each element $\xi \in \mathscr{L}_2(Z)$ can be expressed as a stochastic integral (in the sense of Bartle–Dunford–Schwartz)

$$\xi = \int_S f(\lambda)\,Z(\mathrm{d}\lambda).$$

Proof. Let $\int_S f_n(\lambda) Z(\mathrm{d}\lambda) \to \eta$ in the quadratic mean sense. We have to prove the existence of $f: S \to C$ ($\mathscr{A}$-measurable) such that

$$\eta = \int_S f(\lambda) Z(\mathrm{d}\lambda).$$

According to (A1), $\int_S |(f_n - f_m)(\lambda)|^2\,\nu(\mathrm{d}\lambda) \to 0$ as $n, m \to \infty$. Due to the completeness of $\mathscr{L}_2(\nu)$ there exists $f \in \mathscr{L}_2(\nu)$ such that

$$\int_S |(f_n - f)(\lambda)|^2\,\nu(\mathrm{d}\lambda) \to 0$$

as $n \to \infty$. Let us prove that $\int_S f(\lambda)\,Z(\mathrm{d}\lambda)$ does exist. We see that for some subsequence $f_n \to f$ a.s. $[\nu]$. This together with (A2) prove that $f_n \to f$ a.s. $[Z]$. Further, $\{\int_E f_n(\lambda) Z(\mathrm{d}\lambda)\}_{n=1}^{\infty}$ is convergent for each $E \in \mathscr{B}$. These two conditions are sufficient for existence of a Bartle–Dunford–Schwartz integral, see Bartle–Dunford–Schwartz (1958). There is no problem to prove that the semivariation of $Z(\cdot,\cdot)$ (which is finite because $Z(\cdot,\cdot)$ is defined on a σ-algebra) is vanishing if and only if the measure $\nu(\cdot)$ is vanishing. This follows from the properties of semivariation and from (A2) because for each $A \in \mathscr{B}$

$$c\nu(A) \le \beta(A, A) \le C\nu(A).$$

Using these facts we can immediately prove that ($f_n \to f$ a.e. $[\nu]$) $\Rightarrow$ ($f_n \to f$ a.e. $[Z]$) as necessary for the construction of the Bartle–Dunford–Schwartz integral. It remains to prove that $\{\int_E f_n \mathrm{d}Z\}$ is convergent for each $E \in \mathscr{B}$. This can be proved by use of (A1) and (A2) in a very simple way, namely

$$E\left\{\left|\int_E f_n \mathrm{d}Z - \int_E f_m \mathrm{d}Z\right|^2\right\} = \int_E \int_E (f_n - f_m)(\lambda)\overline{(f_n - f_m)}(\mu)\,\beta(\mathrm{d}\lambda, \mathrm{d}\mu)$$

$$\le C \int_E |f_n - f_m|^2\,\mathrm{d}\nu \le C \int_S |f_n - f_m|^2\,\mathrm{d}\nu \xrightarrow[n,m\to\infty]{} 0$$

because of the validity of (A2). This completes the proof. □

Remark 2. (a) By Theorem 1, p. 264 in Rao (1989b) there exists generally a measure ν satisfying (A2). ν is called Grothendieck measure, so a sufficient condition for completeness is the existence of a Grothendieck measure satisfying condition (A1).

(b) As was indicated by a referee some arguments related to Proposition 1 can be found in Truong–Van (1981).

(c) Let us show that the conditions A1 and A2 hold for an important case, namely for the Karhunen class. When $Z(\cdot,\cdot)$ is an orthogonal measure then

$$\int_S \int_S f(\lambda)\overline{f(\mu)}\,\beta(\mathrm{d}\lambda,\mathrm{d}\mu) = \int_S |f(\lambda)|^2\,\nu(\mathrm{d}\lambda),$$

where $\mu(A \cap B) = \beta(A, B)$. Then, of course $\nu = \mu$ and $c = C = 1$.

Another example where the conditions (A1) and (A2) are valid is a Cox process $Z(\cdot,\cdot)$ with a mixing measure $\lambda(\cdot,\cdot)$. Then

$$\beta(A, B) = K(A \times B) + \mu(A \cap B),$$

where $K(A \times B) = E\{\lambda(A,\cdot)\lambda(B,\cdot)\}$, $E\{Z(A,\cdot)\} = \mu(A)$. If, e.g.

$$K(A \times B) \le C\mu(A)\mu(B)$$

for each $A, B \in \mathscr{B}$ then $\int_S\int_S f(\lambda)\overline{f(\mu)}\,\beta(\mathrm{d}\lambda,\mathrm{d}\mu) \le (C+1)\int_S |f|^2\,\mathrm{d}\mu$ and the completeness of the class of stochastic integrals is proved. Grandell (1976) considered this sufficient condition for a Cox process.

If $N = \sum_{i=1}^{\tau} \varepsilon_{X_i}$ is a random sampling process, i.e. τ, $\{X_i\}_{i=1}^{\infty}$ are independent, $\{X_i\}_{i=1}^{\infty}$ an i.i.d. sequence,

$$E\{\tau^2\} < \infty, \qquad p_n = P\{\tau = n\}.$$

Then $\int f\,\mathrm{d}N \in \mathscr{L}_2(P)$ if and only if $f \in \mathscr{L}_2(\nu)$ where $\nu = P^{X_1}$ and we obtain

$$E\left\{\left(\int f\,\mathrm{d}N\right)^2\right\} = \sum_{n=1}^{\infty} p_n E\left\{\left(\sum_{i=1}^{n} f(X_i)\right)^2\right\}$$

$$= E\{\tau\}E\{(f(X_1))^2\} + 2\sum_{n=1}^{\infty} p_n n(n-1)(E\{f(X_1)\})^2.$$

Therefore, $\beta(A, B) = E\{\tau\}\nu(A \cap B) + (E\{\tau^2\} - E\{\tau\})\nu(A)\nu(B)$.

From Proposition 1 we have completeness. If more generally, the sequence $\{X_i\}_{i=1}^{\infty}$ is exchangeable and $\alpha = P^{(X_i, X_j)}$ then

$$\beta(A, B) = E\{\tau\}\nu(A \cap B) + (E\{\tau^2\} - E\{\tau\})\alpha(A \times B).$$

If $\mathrm{d}\alpha/(\mathrm{d}\nu \otimes \nu) = h$, $h \ge c > 0$, then again completeness is obtained.

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References

H. Cramér, A contribution to the theory of stochastic processes, Proc. Second Berkeley Symp. Math. Statist. Probab. (1951) 329–339.

H. Cramér, On some classes of nonstationary stochastic processes, Proc. Fourth Berkeley Symp. Math. Statist. Proab. 2 (1961) 55–77.

D.K. Chang and M.M. Rao, Bimeasures and nonstationary processes, in: M.M. Rao ed., Real and Stochastic Analysis (Wiley, New York, 1986) pp. 7–118.

J. Grandell, Doubly Stochastic Poisson Processes, LNM 529 (Springer, Berlin, 1976).

E.J. Hannan, P.R. Krishnaiah and M.M. Rao, eds., Handbook of Statistics, Vol. 5 (Elsevier, Amsterdam, 1985) pp. 279–310.

A.G. Miamee and H. Salehi, An example of a harmonizable process whose spectral domain is not complete, Scand. J. Statist. 18 (1991) 249–254.

M.M. Rao, A view of harmonizable processes, in: Y. Dodge, ed., Statistical Data Analysis and Inference, (1989a) pp. 597–615.

M.M. Rao, Bimeasures and harmonizable processes, in: H. Heyer, ed., Probability Measures on Groups IX, Lecture Notes in Mathematics, Vol. 1379 (Springer, Berlin, 1989b) pp. 254–298.

M. Rosenberg, The square integrability of matrix-valued functions with respect to a nonnegative hermitian measure, Duke Math. J. 31 (1964) 291–298.

Yu. A. Rozanov, Stationary Random Processes (Holden-Day, San Francisco, CA, 1967).

L. Rüschendorf, Inference for random sampling processes, Stochastic Process. Appl. 32 (1989) 129–140.

B. Truong-Van, Une généralisation du théorème de Kolmogorov–Aronszajn, Processus V-bornés q-dimensionnels: domaine spectral § dilatations stationnaires. Ann. Inst. Henri Poincaré 17 (1) (1981) 31–49.