Bull. Malays. Math. Sci. Soc.
https://doi.org/10.1007/s40840-020-01025-3

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Lewis Model Revisited: Option Pricing with Lévy Processes

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Received: 23 August 2019 / Revised: 9 September 2020 / Accepted: 10 September 2020
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Abstract

This paper aims to discuss the mathematical details in Lewis' model by considering the analyticity and integrability conditions of characteristic functions and payoff functions of contingent claims. In his seminal paper, Lewis shows that it is much easier to compute the option value in the Fourier space than computing in terminal security price space. He computes the option value as an integral in the Fourier space, the integrand being some elementary functions and the characteristic functions of a wide range of Lévy processes. The model also illustrates how the residue calculus leads to several variations of option formulas through the contour integrals. In this paper, we provide with, to a reasonable extent, some rigor into the mathematical background of Lewis' model and validate his results for particular Lévy processes. We also simply give the analyticity conditions for the characteristic function of the Carr–Geman–Madan–Yor model and a simple derivation of the characteristic function of Kou's double exponential model.

Keywords Lévy processes · Option pricing · Characteristic function · Fourier transform

Mathematics Subject Classification 62P05 · 60G05

Communicated by Anton Abdulbasah Kamil.

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Published online: 26 September 2020

1 Introduction

In this paper, we aim to provide a more complete mathematical basis for Lewis' model. The proof given in Lewis [35] assumes the existence of a Lebesgue density; however, the results are also applied to pure-jump models where this assumption is not clear at all. A priori one cannot ensure that the unverified claims do not lead to serious mistakes if one were to consider other scenarios not discussed in Lewis' paper. We rework the proof and make precise when it is mathematically valid to use this method.

In a modern[1] sense, the basis of the Theory of Option Pricing goes back to the seminal paper of Robert Merton, "Theory of Rational Option Pricing" [40]. Yet, "The Pricing of Options and Corporate Liabilities" published in the same year by Fischer Black and Myron Scholes [7] opened a huge fertile area in the pricing of options.

During the last five decades, option pricing models have been developed based on the continuous-time stochastic models as a driving force and Brownian motion has been the basic building block in these models. The underlying motive in producing option pricing formulas has always been the "Arbitrage-Free pricing" which eliminates the arbitrage possibilities in the market [26].

Though the models were able to produce arbitrage-free prices, traders and investors have met three important and interrelated empirical facts in the option markets:

- Asymmetric, leptokurtic features; causing left skews for the major index options, higher peaks and heavier tails in return distributions,
- Volatility smile; causing a convex structure for the strike price curve,
- Discontinuous, sudden movements in prices. (See [18,29] for further information.)

Then, once again option pricing models were developed to reflect these empirical facts which may produce a pricing structure complying with the implied volatility determined by the market. These models contain a very wide range of stochastic processes, distributions and techniques, including but not limited to local volatility [23], stochastic volatility [27,28,47,49], jump diffusion [29,41,42] and affine jump diffusion [20,21] models.

Almost in the same period, the option pricing problem was studied in the Fourier space instead of the stock price space and this has provided simple solutions for the unknown stock price distributions by using the characteristic functions of the underlying variables [2,16] where it suffices to be numerically tractable to produce option prices. The risk-neutral probability of finishing in the money is given as

$$P\left(S_t > K\right) = \Delta_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(\frac{\mathrm{e}^{-iu\ln K}\phi_t(u)}{iu}\right)\mathrm{d}u$$

where $\phi_t(u)$, t, S_t and K denote the characteristic function of the stock return distribution, option's maturity, stock price at maturity and the strike price of the option, respectively.

[1] Actually the story goes back to Bachelier's thesis in 1900, and to other precursors such as [8,14,44,48].

Similarly, option delta, Δ_1, is given (the plus term in delta and other probability term being minus for puts) as

$$\Delta_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(\frac{\mathrm{e}^{-iu\ln K}\phi_t(u-i)}{iu\phi_t(-i)}\right)\mathrm{d}u$$

and the option value is determined for calls and puts by the formulae as

$$\begin{aligned} C &= S_0\mathrm{e}^{-qt}\Delta_1 - \mathrm{e}^{-rt}K\Delta_2 \\ P &= \mathrm{e}^{-rt}K\Delta_2 - S_0\mathrm{e}^{-qt}\Delta_1 \end{aligned}$$

where Δ_1, Δ_2, q and r are risk-neutral probabilities of finishing in the money under the stock price measure and riskless bond price measure, continuous dividend rate and risk-free interest rate, respectively, in the Black–Scholes framework [2].

Carr and Madan have first developed alternative methods by using Fourier transform of an option price and the Fourier transform of the time value of an option. By using a factor (α) to obtain a square-integrable function for the transform, they were able to utilize fast Fourier transform to produce the option prices for a numerous amount of strikes simultaneously. (See the details in [16].) Fourier transform makes it possible to provide simple solutions for the underlying instruments when the characteristic functions are known and to help explain some of the deviations from Black–Scholes prices.

Recently, another kind of stochastic process, namely Lévy processes, have found wide use in option pricing [3,4,45]. Lévy processes can be defined simply as continuous time stochastic processes with independent and stationary increments and may represent the daily stock returns which are characterized by heavy tails and excess kurtosis. However, the densities of Lévy processes are very complicated in stock price (state) space and require many special functions and infinite summations [35].

In [34] and [35], the expected discounted value of the option payoff is computed in Fourier space instead of the state space and this has allowed the following in a very productive way:

1. A single integration for any payoff (path independent),
2. Integrand is typically a compact expression with elementary functions,
3. A suitable integration strip; replacing the integration factor in [16].

Lewis separates the process of state variable which may or may not have a Lebesgue density and the payoff function via the Parseval/Plancherel's theorem and computes the option value as an integral in the Fourier space, the integrand consisting of some elementary functions and the characteristic function of Lévy processes. In [35], it is also illustrated how the residue calculus leads to several variations of option formulas through the contour integrals.

In Lewis' formula, the integration strip simply replaces the damping factor and this makes the formula not so different from Carr and Madan's method in which the option prices are computed for a series of strikes simultaneously. (See also Wu [51].) The key idea in Fourier transform methods in option pricing, as stated in Eberlein et al.

[24], lies in the separation of the underlying process and the payoff function. They show that the valuation formulas hold true under the pointwise convergence and L^2 limit with additional assumptions, in one and several dimensions when the continuity assumption fails for the payoff function and the process has no Lebesgue density.

Raible [43] presents the expectation (integrand) as the bilateral Laplace transform of a convolution and then uses the generalized Fourier transform to obtain the call and put prices.

Lipton [36] uses Fourier and Laplace transforms to solve the pricing equation in the form of a reduced partial differential equation where the inverse Laplace transforms involve the error function in the formula. Also, the approach in Lipton [37] is almost equivalent to Lewis' work and referred to as the Lewis–Lipton approach by some authors. (See the working paper by Schmelzle [46], where the approaches and the methods of [16,34–37] are explained.)

Dufresne et al. [22] show how to use Fourier methods (including the so-called Mellin transforms) and Parseval/Plancherel's theorem to compute prices of puts, calls and stop-loss premiums in insurance applications.

Dampened Fourier transform was followed by further refinements and generalizations. Lee [30] and Crocce et al. [19] present some extensions and unifications particularly producing error bounds for sampling and truncation error in DFT/FFT implementation. Andersen and Lake [1] develop/construct contour deformation to dampen out Fourier oscillations in the integrand by using double-exponential quadrature methods. Boyarchenko and Levendorskiĭ [10,11], using contour deformations with subsequent conformal transformations, increase the efficiency of inverse Fourier transform methods by reducing the number of points in numerical integral significantly.

Binkowski and Kozek [5,6] propose to estimate the unknown characteristic function of the distribution by using the historical data and prove the convergence of empirical characteristic functions in a complex domain employing a version of Lewis' formulas. This approach does not require any particular family of infinitely divisible distributions such as VG, hyperbolic or generalized hyperbolic for the solution of the call and put prices. Boyarchenko and Levendorskiĭ [13,33] provide solutions for non-Gaussian processes by using Fourier techniques.

Boyarchenko and Levendorskiĭ [12], Levendorskiĭ [31], Boyarchenko and Levendorskiĭ [9], Levendorskiĭ and Xie [32] use Lévy processes for the solution of American put, European and some exotic option prices with the implementation of FFT-based methods such as CONV, COS, iFFT and enhanced–refined iFFT. (See also Lord et al. [38].)

The remainder of this paper is organized as follows. Section 2 presents the mathematical background of Lewis' model. In Sect. 3, we rework the critical steps in the derivation of the option price formula, proposing a more rigorous and carefully formulated set of conditions. Section 4 is an application part, and it validates the numerical results of Lewis model for particular benchmark option pricing models and characteristic functions. Section 5 concludes.

2 Background

In this section, we summarize some well-known results related to the topic and introduce the necessary notation. We refer the reader to [17,45] as general references on these topics.

2.1 Analyticity of Characteristic Functions

Throughout this paper, given a function f on a probability space $(E, \mathcal{B}, \mu)$, by "integrable" we mean μ-integrable in the Lebesgue sense, that is, $\int_E |f| d\mu < \infty$. The space of integrable functions will be denoted by $L^1(\mu)$. When $E = \mathbb{R}$ and μ is the Lebesgue measure $\mathcal{L}$, we will simply write $L^1(\mathbb{R})$.

We begin with quoting the following consequence of Morera's theorem (cf. [25], p.121):

Theorem 1 *Suppose $h(x, z)$ is an analytic function on a domain $U \subseteq \mathbb{C}$ for each $x \in \mathbb{R}$ and let ν be a measure on $\mathbb{R}$. If $H(z) := \int_{\mathbb{R}} \nu(dx)h(x, z)$ exists and is a continuous function of $z \in U$ and if $\int \nu(dx)|h(x, z)|$ is locally uniformly bounded (with respect to z), then $H(z)$ is analytic on U.*

Now, suppose ξ is a real-valued random variable with distribution measure Q_ξ. Its characteristic function $\phi(z)$ is the generalized Fourier transform of Q_ξ:

$$\phi(z) = \mathbb{E}\left[\mathrm{e}^{iz\xi}\right] = \int_{\mathbb{R}} Q_\xi(\mathrm{d}x)\mathrm{e}^{izx} = \int_{\mathbb{R}} Q_\xi(\mathrm{d}x)\mathrm{e}^{-\tau x}\mathrm{e}^{iux} \tag{1}$$

where we wrote $z := u + i\tau$. This is only defined for the values of z which make the integral converge, and convergence essentially depends on the parameter $\tau = \Im z$. Now, let us set $a := \inf\left\{\tau : \int_{[0,\infty)} Q_\xi(\mathrm{d}x)\mathrm{e}^{-\tau x} < \infty\right\}$, and similarly let $b := \sup\left\{\tau : \int_{(-\infty,0)} Q_\xi(\mathrm{d}x)\mathrm{e}^{-\tau x} < \infty\right\}$. It is easy to see that $Q_\xi(\mathrm{d}x)\mathrm{e}^{izx}$ is a finite measure on $[0, \infty)$ (resp. $(-\infty, 0]$) for $\Im z > a$ (resp. $\Im z < b$). It follows that (1) exists when $a < \Im z < b$ and is continuous there by the Lebesgue dominated convergence theorem (LDCT). Then, a simple application of Theorem 1 yields that $\phi(z)$ is analytic in this strip. Note that a and b are simply the endpoints of the interval

$$\left\{\tau : \int_{\mathbb{R}} Q_\xi(\mathrm{d}x)\mathrm{e}^{-\tau x} < \infty\right\} = \left\{\tau : \mathbb{E}\left[\mathrm{e}^{-\tau\xi}\right] < \infty\right\} \tag{2}$$

provided it is nonempty. We always have $a \leq 0 \leq b$ if Q_ξ is a finite measure. Moreover, if $a < b$, then by expanding ϕ into a power series at a point on the imaginary axis and using the fact that the disk whose radius is the radius of convergence must contain a singularity point on the boundary, one easily arrives at the result that the points $z = ia$ and $z = ib$ must be singularity points of $\phi(z)$ when a (resp. b) are finite.

2.2 Lévy Processes

Here, the setting is a probability space $(\Omega, \mathcal{B}, Q)$ where Q represents a martingale pricing measure and henceforth all random variables, processes and expectations will be with reference to this probability space. It is assumed that the stock price S_t depends on some Lévy process X_t by the relation

$$S_t = S_0 \exp[(r-q)t + X_t]$$

where r is the risk-free rate and q is the dividend for the stock. The market model requires that $\mathbb{E}[\exp(X_t)] = 1$. We recall the definition of a Lévy process:

Definition 1 A stochastic process $X = (X_t)$, $t \geq 0$ adapted to a filtration $\mathcal{F}$ with $X_0 = 0$ almost surely is called a Lévy process if (i) almost surely $t \to X_t(\omega)$ is a right continuous map with left limits; (ii) for all $t \geq 0$ and $s > 0$, the increment $X_{t+s} - X_t$ is independent of $\mathcal{F}_t$ and has the distribution of X_s.

A Lévy process has a precise form given by the Ito–Lévy decomposition theorem. Denoting by $\mathbb{B}$ the closed unit ball $\{x : |x| \leq 1\}$ and by $\mathbb{B}^c$ its complement, we have:

Theorem 2 *The process X is Lévy if and only if it has the form*

$$X_t = at + \sigma W_t + \int_{[0,t]\times\mathbb{B}} [N(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\,\mu(\mathrm{d}x)]x + \int_{[0,t]\times\mathbb{B}^c} N(\mathrm{d}s, \mathrm{d}x)x \tag{3}$$

where a is some constant, W is a Wiener process and $N = N(\omega)$ is a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}$ independent of W with mean $\mathcal{L} \times \mu$, where μ satisfies $\mu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \mu(\mathrm{d}x) \min\{1, |x|^2\} < \infty. \tag{4}$$

The measure μ above is called the Lévy measure of the process. The first integral in (4) is in fact understood as a limit of integrals on $[0, t] \times (\mathbb{B}\backslash(\varepsilon\mathbb{B}))$ as $\varepsilon \to 0$. Under the stronger assumption that $\int_{\mathbb{R}} \mu(\mathrm{d}x) \min\{|x|, 1\} < \infty$ one can indeed (at the cost of changing the drift term a) replace the two integrals in (3) with the single one $\int_{[0,t]\times\mathbb{R}} N(\mathrm{d}s, \mathrm{d}x)x$, which is now guaranteed to converge.

The characteristic function $\phi_t(z) = \mathbb{E}[\exp(izX_t)]$ is given by

$$\phi_t(z) = \exp\left\{t\left(iaz - \frac{1}{2}z^2\sigma^2 + \int_{\mathbb{B}} \mu(\mathrm{d}x)\left[\mathrm{e}^{izx} - 1 - izx\right] + \int_{\mathbb{B}^c} \mu(\mathrm{d}x)\left[\mathrm{e}^{izx} - 1\right]\right)\right\}$$

which is known as the Lévy–Khintchine formula. Recalling (2), we see that the strip of analyticity of $\phi_t(z)$ is determined by the parameters $\tau = \Im z$ such that $\mathbb{E}[\exp(-\tau X_t)] < \infty$, or, equivalently,

$$\int_{\mathbb{B}} \mu(\mathrm{d}x)(\mathrm{e}^{-\tau x} - 1 + \tau x) + \int_{\mathbb{B}^c} \mu(\mathrm{d}x)(\mathrm{e}^{-\tau x} - 1) < \infty.$$

The Lévy condition (4) implies that the first term above is finite and $\mu(\mathbb{B}^c) < \infty$. Therefore, this last inequality is equivalent to $\int_{\mathbb{B}^c} \mu(\mathrm{d}x)\mathrm{e}^{-\tau x} < \infty$. (Heuristically, it is the distribution of big jumps that determine the convergence strip of ϕ_t.)

Now, we discuss some extra information we gain in the "diffusion" models, that is when $\sigma \neq 0$ in (3). Recall that a convolution of measures is absolutely continuous if one of the factors is. The distribution of σW_t is absolutely continuous (with respect to the Lebesgue measure), and its density function is a Schwarz function. Independence implies that Q_{X_t} is the convolution of the four terms corresponding to (3); thus, by what we noted above, it is absolutely continuous with respect to the Lebesgue measure when $\sigma \neq 0$.

3 Computing the Value of the Derivative

In what follows below, Q_{X_t} is the distribution measure of the Lévy process X_t in (3) and ϕ_t is its (generalized) Fourier transform, namely

$$\phi_t(z) = \mathbb{E}\left[\mathrm{e}^{izX_t}\right] = \int_{\mathbb{R}} Q_{X_t}(\mathrm{d}x)\mathrm{e}^{izx}.$$

Suppose the payoff of a derivative, as a function of the stock price S_t, is given by $\tilde{w}$. Then, the value of this derivative with maturity t is

$$V_0 = \mathrm{e}^{-rt}\int_{\mathbb{R}^+} Q_{S_t}(\mathrm{d}x)\tilde{w}(x) = \mathrm{e}^{-rt}\int_{\mathbb{R}} Q_{\ln S_t}(\mathrm{d}x)w(x), \tag{5}$$

where Q_{S_t} is the distribution measure of S_t and $w(x) := \tilde{w}(\mathrm{e}^x)$. Let us write $\ln S_t = \ln S_0 + (r-q)t + X_t =: Y_t + X_t$. Note that Y_t is constant for a fixed t; therefore, $Q_{\ln S_t}$ is a translation of Q_{X_t}. The main idea in [35] was to use Theorem 39 from [50] which is similar in spirit to Parseval/Plancherel's theorem. Unfortunately, (a) the proof given in Titchmarsh has a gap[2], (b) Lewis assumes the existence of a (Lebesgue) density function of Q_{X_t} and uses it as one of the functions in Titchmarsh's theorem, but such a density function need not exist in pure-jump models, and (c) Lewis does not verify one of the conditions of the theorem, namely the continuity of a certain convolution at 0.

A simple inspection of the proof in Titchmarsh reveals that we can formulate the following modified version which covers all the cases one would encounter in practice, including those considered by Lewis:

[2] For Titchmarsh's proof to be complete, using the notation there, one has to show that

$$\lim_{\lambda\to+\infty}\int_{-\lambda}^{\lambda} F(t)G(t)\mathrm{d}t = \lim_{\lambda\to+\infty}\int_{-\infty}^{\infty} F(t)G(t)\mathrm{e}^{-\frac{t^2}{4\lambda}}\mathrm{d}t.$$

The limits need not even exist unless FG is integrable, but this is not guaranteed by the assumptions in the theorem.

Theorem 3 *Suppose that (i) ν is a finite measure on $\mathbb{R}$, (ii) f is bounded and integrable, (iii) $\hat{f}$ is integrable, and (iv) f continuous ν-almost everywhere. Then,*

$$\int_{\mathbb{R}} \nu(\mathrm{d}x) f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} (\mathrm{d}u) \hat{f}(-u) \hat{\nu}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} (\mathrm{d}u) \hat{f}(u) \hat{\nu}(-u).$$

Proof The second equality is clear, so we prove the first one. For any $\lambda > 0$ and $t \in \mathbb{R}$

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\mathrm{d}u) \hat{f}(-u) \mathrm{e}^{-\frac{u^2}{4\lambda}+iut} = \lim_{r\to\infty} \int_{-\infty}^{\infty} (\mathrm{d}u) \mathrm{e}^{-\frac{u^2}{4\lambda}+iut} \int_{-r}^{r} (\mathrm{d}x) f(x) \mathrm{e}^{-iux} \\
&= \lim_{r\to\infty} \int_{-r}^{r} (\mathrm{d}x) f(x) \int_{-\infty}^{\infty} (\mathrm{d}u) \mathrm{e}^{-\frac{u^2}{4\lambda}+iu(t-x)} = 2\sqrt{\pi\lambda} \lim_{r\to\infty} \int_{-r}^{r} (\mathrm{d}x) f(x) \mathrm{e}^{-\lambda(t-x)^2}
\end{aligned}$$

where the interchanges of limits and integrals are justified by conditions (ii) and (iii). Furthermore, the convergence of the limit is uniform (in t) when t is restricted to a compact interval. Therefore, for any $s > 0$

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\mathrm{d}u) \hat{f}(-u) \mathrm{e}^{-\frac{u^2}{4\lambda}} \int_{-s}^{s} \nu(\mathrm{d}t) \mathrm{e}^{iut} = \int_{-s}^{s} \nu(\mathrm{d}t) \int_{-\infty}^{\infty} (\mathrm{d}u) \hat{f}(-u) \mathrm{e}^{-\frac{u^2}{4\lambda}+iut} \\
&= 2\sqrt{\pi\lambda} \lim_{r\to\infty} \int_{-s}^{s} \nu(\mathrm{d}t) \int_{-r}^{r} (\mathrm{d}x) f(x) \mathrm{e}^{-\lambda(t-x)^2} \\
&= 2\sqrt{\pi\lambda} \lim_{r\to\infty} \int_{-s}^{s} \nu(\mathrm{d}t) \int_{-r+t}^{r+t} (\mathrm{d}x) f(t-x) \mathrm{e}^{-\lambda x^2} \\
&= 2\sqrt{\pi\lambda} \lim_{r\to\infty} \int_{-r-s}^{r+s} (\mathrm{d}x) \mathrm{e}^{-\lambda x^2} \int_{\max(-s,x-r)}^{\min(s,x+r)} \nu(\mathrm{d}t) f(t-x) \\
&= 2\sqrt{\pi\lambda} \int_{-\infty}^{\infty} (\mathrm{d}x) \mathrm{e}^{-\lambda x^2} \int_{-s}^{s} \nu(\mathrm{d}t) f(t-x)
\end{aligned}$$

where we used Fubini and LDCT as well as conditions (i)–(iii). Now, we can let $s \to \infty$ and note that we can do the necessary interchanges of integration and limit to get

$$\int_{-\infty}^{\infty} (\mathrm{d}u) \mathrm{e}^{-\frac{u^2}{4\lambda}} \hat{f}(-u) \hat{\nu}(u) = 2\pi \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (\mathrm{d}x) \mathrm{e}^{-\lambda x^2} \int_{-\infty}^{\infty} \nu(\mathrm{d}t) f(t-x).$$

Our assumptions imply that the innermost integral on the right is continuous at $x = 0$ and bounded as a function of x. As we let $\lambda \to 0$, we can use the LDCT on the left and Theorem 16 in [50] on the right to obtain

$$\int_{\mathbb{R}} (\mathrm{d}u) \hat{f}(-u) \hat{\nu}(u) = 2\pi \int_{\mathbb{R}} \nu(\mathrm{d}t) f(t).$$

□

Returning to (5), we may not be able to apply this directly to $Q_{\ln S_t}$ and w because the latter need not be integrable in the first place. (For example, for a call option with strike price K we have $w(x) = (\mathrm{e}^x - K)^+$.) But we can rewrite the integrand as $Q_{\ln S_t}(\mathrm{d}x)w(x) = Q_{\ln S_t}(\mathrm{d}x)\mathrm{e}^{\tau x}\mathrm{e}^{-\tau x}w(x)$ for any $\tau \in \mathbb{R}$. Suppose τ is such that the following hold:

(C1) $\int_{\mathbb{R}} Q_{X_t}(\mathrm{d}x)\mathrm{e}^{\tau x} < \infty$,
(C2) $w(x)\mathrm{e}^{-\tau x}$ belongs to $L^1(\mathbb{R})$ and is bounded,
(C3) the Fourier transform of $w(x)\mathrm{e}^{-\tau x}$ belongs to $L^1(\mathbb{R})$,
(C4) $w(x)$ is continuous $Q_{\ln S_t}$-almost everywhere.

These correspond to the four conditions of Theorem 3. We will later return to the question of finding a suitable τ. Note that the finiteness of the measure $Q_{X_t}(\mathrm{d}x)\mathrm{e}^{\tau x}$ is equivalent to that of $Q_{\ln S_t}(\mathrm{d}x)\mathrm{e}^{\tau x}$ as the two distribution measures are translations of each other. We also remark that as we observed at the end of Sect. 2.2, when $\sigma \neq 0$ (the diffusion case) Q_{X_t} is absolutely continuous with respect to the Lebesgue measure, so w is allowed to have a Lebesgue-null discontinuity set.

Now, we apply Theorem 3 to the measure $\nu(\mathrm{d}x) = Q_{\ln S_t}(\mathrm{d}x)\mathrm{e}^{\tau x}$ and $f(x) := w(x)\mathrm{e}^{-\tau x}$. We first compute the transforms $\hat{\nu}(-u)$ and $\hat{f}(u)$:

$$\begin{aligned}
\hat{\nu}(-u) &= \int_{-\infty}^{\infty} Q_{\ln S_t}(\mathrm{d}x)\mathrm{e}^{\tau x}\mathrm{e}^{-iux} = \int_{-\infty}^{\infty} Q_{X_t+Y_t}(\mathrm{d}x)\mathrm{e}^{i(-u-i\tau)x} \\
&= \int_{-\infty}^{\infty} Q_{X_t}(\mathrm{d}x)\mathrm{e}^{i(-u-i\tau)(x+Y_t)} = \mathrm{e}^{i(-u-i\tau)Y_t}\int_{-\infty}^{\infty} Q_{X_t}(\mathrm{d}x)\mathrm{e}^{i(-u-i\tau)x} \\
&= \mathrm{e}^{i(-u-i\tau)Y_t}\phi_t(-u-i\tau).
\end{aligned}$$

The transform of $f(x) = w(x)\mathrm{e}^{-\tau x}$ at u yields

$$\hat{f}(u) = \int_{-\infty}^{\infty}(\mathrm{d}x)w(x)\mathrm{e}^{-\tau x}\mathrm{e}^{iux} = \int(\mathrm{d}x)w(x)\mathrm{e}^{i(u+i\tau)x} = \hat{w}(u+i\tau)$$

which is the (generalized) Fourier transform of $w(x)$. Thus, by Theorem 3, Eq. (5) becomes

$$V_0 = \frac{\mathrm{e}^{-rt}}{2\pi}\int_{-\infty}^{\infty}(\mathrm{d}u)\left[\mathrm{e}^{i(-u-i\tau)Y_t}\phi_t(-u-i\tau)\right]\hat{w}(u+i\tau)$$

provided $\hat{f}(u) \in L^1(\mathbb{R})$. Now, putting $z = u + i\tau$ and observing $\mathrm{d}z = \mathrm{d}u$, we get

$$V_0 = \frac{\mathrm{e}^{-rt}}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty}(\mathrm{d}z)\mathrm{e}^{-izY_t}\phi_t(-z)\hat{w}(z), \tag{6}$$

the integral being taken on the horizontal line $\Im z = \tau$. This is the option formula in [35]. The main innovation there was to compute this integral using residue calculus methods.

It remains to find a τ for which conditions (C1)–(C4) hold. The results from Sect. 2.1 indicate that $\phi_t(-z)$ is analytic on a certain horizontal strip and any value of τ from this strip makes $Q_{X_t}(\mathrm{d}x)\mathrm{e}^{\tau x}$ a finite measure. Using very similar techniques, one can prove the following:

Proposition 1 *Suppose there are numbers $a < b$ such that $w(x)e^{-\tau x}, x \in \mathbb{R}$, is bounded for $\tau = a$ and $\tau = b$. Then, for any $\tau \in (a, b)$, the function $w(x)e^{-\tau x}$ belongs to $L^p(\mathbb{R})$ for any $p \geq 1$ and is bounded and the generalized Fourier transform*

$$\hat{w}(z) = \int_{\mathbb{R}} (\mathrm{d}x)\mathrm{e}^{izx} w(x)$$

is analytic on $a < \Im z < b$.

Therefore, if the strip of Prop. 1 overlaps with the strip of analyticity of $\phi_t(-z)$ which is the mirror reflection of that of $\phi_t(z)$, then a value of τ, such that $\Im z = \tau$ is in this overlap, satisfies (C1) and (C2).

In [35, p. 10], it was remarked (without any justification) that in the strip where $\hat{w}(z)$ exists, one has $w(x)\mathrm{e}^{-\tau x} \to 0$ as $|x| \to \infty$. This may not be the case indeed, since an integrable function need not converge to 0 as $|x| \to \infty$. We prefer to state the proposition above by directly requiring that $w(x)\mathrm{e}^{-\tau x}$ be bounded for the extreme values of τ, since this is precisely what is needed to make the proof work.

The usual kind of payoff functions has Fourier transforms with good decay along horizontal lines. The borderline values of τ for the "strip of boundedness" in Prop. 1 usually coincide with those of the maximal strip of analyticity as given in Sect. 2.1. In the call option example $w(x) = (\mathrm{e}^x - K)^+$, both strips would be $1 < \Im z < \infty$. Moreover,

$$\hat{w}(z) = \frac{K^{iz+1}}{iz - z^2}, \quad \Im z > 1.$$

This function is $O(1/|z|^2)$ as $|z| \to \infty$, thus integrable along $\Im z = \tau$. Covered call or put option cases are similar. So the integrability requirement (C3) is always satisfied for put and call options. See [35] for some other examples.

Finally, (C4) is also easily satisfied in virtually all cases one uses in practice. A typical payoff function is either monotone or piecewise monotone which means there is at most a countable set of discontinuity points. Functions with countably many discontinuity points include functions which have bounded variation on finite intervals, and this class practically includes all payoff functions. As we remarked above, if the model assumes diffusion ($\sigma \neq 0$), then $Q_{\ln S_t}$ is absolutely continuous with respect to the Lebesgue measure; therefore, for any such payoff function, (C4) holds. In pure-jump models, it is not possible to make generic statements about the distribution of X_t without extra assumptions on the Lévy measure. It could be argued that a sensible pure-jump model should allow infinitely many jumps in any time interval (the Type II processes in Lewis). But when this is the case we have

$$\int \mu(\mathrm{d}x) \min\{1, |x|\} = +\infty,$$

Table 1 Alan Lewis' first formula

K	Log-normal (BS)		Jump diffusion (Merton) $\lambda = 0.1$ $\mu = 0$ $\delta = 0.50$		VG $v = 0.20$ $\theta = -0.14$		CGMY $C = 1.5$, $G = 8$, $M = 12$, $Y = 0.5$		Double exponential (Kou) $\eta_2 = 40$, $\eta_1 = 50$, $p = 0.4$, $\lambda = 1$	
	Call	Put	Call	Put	Call	Put	Call	Put	Call	Put
45	6.5598	0.4487	6.6969	0.5859	6.6808	0.5697	6.8936	0.7826	6.5721	0.4611
50	3.1272	1.8927	3.3257	2.0912	3.004	1.7695	3.4293	2.1948	3.1471	1.9126
55	1.1589	4.8009	1.3966	5.0387	0.9664	4.6084	1.3726	5.0147	1.1762	4.8182

Some numerical values for various strike levels. Case $S_0 = 50, r = 0.1, t = 0.25, \sigma = 0.25, q = 0$. All values are exactly the same with the benchmark values (Black–Scholes, Merton jump diffusion, variance gamma, CGMY and Kou models). $\Im z = 1.1$ for call options and $\Im z = -0.5$ for put options. Each case is valid for $z \in \mathbb{S}_V = \mathbb{S}_w \cap \mathbb{S}_X$, the intersection of the strips of analyticity

Table 2 Drift parameter for martingale pricing and strip of analyticity in Lewis' model

Model	Drift term a	Strip of analyticity $\mathbb{S}_X$
Lognormal	$-\sigma^2/2$	Entire z plane
Jump with LN $\left(\mu, \delta^2\right)$	$-\frac{\sigma^2}{2} - \lambda\left(\exp\left(\mu + \delta^2/2\right) - 1\right)$	Entire z plane
Double exponential	$-\frac{\sigma^2}{2} - \lambda\left(\frac{(q\eta_1 - p\eta_2) - 1}{(1-\eta_1)(1+\eta_2)}\right)$	$-\eta_2 < \Im z < \eta_1, 1 < \eta_1, 0 < \eta_2$
VG	$\frac{1}{v}\ln\left(1 - v\theta - \frac{1}{2}\sigma^2 v\right)$	$\beta - \alpha < \Im z < \beta + \alpha$
CGMY	$-C\Gamma(-Y)\times\left[(M-1)^Y - M^Y + (G+1)^Y - G^Y\right]$	$-M < \Im z < G$
Normal inverse Gaussian	$-\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}\right)$	$\beta - \alpha < \Im z < \beta + \alpha$
Finite moment log-stable	$\sigma^\alpha \sec\frac{\pi\alpha}{2}$	$\Im z < 0$

which implies that $\mu(\mathbb{R}) = +\infty$. But this last condition is equivalent to $Q_{\ln S_t}$ not having atoms for all $t > 0$ [45, Theorem 27.4]. Therefore, for such processes, all countable sets will be $Q_{\ln S_t}$-null and condition (C4) will hold.

4 Applying Lewis' First Formula

As presented in Tables 1 and 2, option prices are obtained for five models and for the parameters referred in the tables. A sample MATLAB code is provided in Appendix A for the variance gamma model [39]. All the calculations are valid for the region consisting of an analyticity strip for characteristic function $\phi(-z)$ which is given by $\mathbb{S}_X$ and the strips for the call and put options which are given as $\Im z > 1$ and $\Im z < 0$, respectively. In order to calculate the correct call (resp. put) option prices, the $\mathbb{S}_X$ strip must intersect the strip of $\Im z > 1$ (resp. $\Im z < 0$). The numerical results fit the exact model values in these regions as presented in Table 1. A derivation of the characteristic function of the double exponential jump process [29] is provided in Appendix B, for a different format is referred in [35]. We have also added the drift component of the Carr–Geman–Madan–Yor (CGMY) process [15] and its analyticity strip to Table 2.1 of [35] which was not available at that time.

Table 2 gives the martingale drift correction term a for various option pricing models with exponential Lévy processes. As this is computed with the model parameters mentioned in Table 2, the option values can be obtained very easily with integration in a suitable analyticity strip. The α and β in the column of strip of analyticity for the VG model are given as follows (see the Notes in [35], page 17, Table 2.1):

$$\alpha = \left[\frac{2}{v\sigma^2} + \frac{\theta^2}{\sigma^4}\right]^{1/2}, \qquad \beta = \frac{\theta}{\sigma^2}.$$

5 Conclusion

In this paper, we have discussed the analyticity of characteristic functions in Fourier spaces. Then, we discussed the Lévy processes, its characteristic functions and Lévy–Khintchine decomposition particularly in the context of option pricing. We refer to [35] all along the paper which deserves a special attention, for Lewis' method uses residue calculus to calculate the option prices very simply and practically in a Fourier space, instead of deriving the PDE and dealing with complicated density functions. This facilitates the computation of option prices enormously. We try to contribute to the Lewis' model by well-structuring some mathematical parts. Finally, we provide some numerical results validating the option prices obtained by five important option pricing models.

Acknowledgements We would like to express our special thanks to the referees for their very valuable and constructive suggestions during the publication process of our paper. We are grateful for their careful guidance and critiques which helped us improve the content significantly.

Appendix A

A MATLAB code for variance gamma call option:

```
function callvg= integ_vgcall(S,rf,sg,t,K,q,v,theta)
a = (1/v)*log(1-v*theta-0.5*sg^2*v)
k = log(S/K)+(rf+a-q)*t;
fun = @(z)exp(-i*k*z).*(1+i*z*v*theta+0.5*sg^2*v*z.^2)
              .^(-t/v)./(z.^2-z.*i)
% Value of integral truncated at 10,000 Imz = 1.1
alint1 = quad(fun,0+1.1*i,10000+1.1*i)
callvg = -(1/pi)*K*real(alint1)*exp(-rf*t)
```

Appendix B

The characteristic function of the double exponential pure-jump process can be written as

$$\phi_t(z) = \exp\left(\lambda t \int Q_{\ln V}(\mathrm{d}y)(\mathrm{e}^{izy} - 1)\right)$$

where $Q_{\ln V}(\mathrm{d}y)$ is the Lévy measure of the process and y is an (double) exponential variable with means $1/\eta_1$ (positive part) and $1/\eta_2$ (negative part).

Applying the integral for $y > 0$ and $y < 0$ parts separately

$$\phi_t(z) = \exp\left(\lambda t \left[p \int_0^{\infty} \left(\mathrm{e}^{izy} - 1\right) \eta_1 \mathrm{e}^{-\eta_1 y} \mathrm{d}y + q \int_{-\infty}^{0} \left(\mathrm{e}^{izy} - 1\right) \eta_2 \mathrm{e}^{\eta_2 y} \mathrm{d}y\right]\right)$$

and taking the integrals, we obtain

$$\begin{aligned}
&\exp\left(\lambda t \left[-p\eta_1 \left(\frac{1}{iz - \eta_1} + \frac{1}{\eta_1}\right) + q\eta_2 \left(\frac{1}{iz + \eta_2} - \frac{1}{\eta_2}\right)\right]\right) \\
&\quad = \exp\left(\lambda t \left[\frac{-piz}{iz - \eta_1} + \frac{-qiz}{iz + \eta_2}\right]\right).
\end{aligned}$$

With simple algebra, this will yield

$$\phi_t(z) = \exp\left(\lambda t \left[\frac{z^2 + iz(q\eta_1 - p\eta_2)}{(iz + \eta_2)(iz - \eta_1)}\right]\right).$$

Recall that here $p, q > 0$ and $p + q = 1$ represent the probabilities of upward and downward jumps as in the Kou [29] model.

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