Equivalent Formulations of the Riemann Hypothesis based on Lines of Constant Phase

W.P. Schleich

Institut für Quantenphysik and Center for Integrated Quantum Science and Technology (IQ^{ST}),

Universität Ulm, Albert-Einstein-Allee 11, D-89081, Germany and

Hagler Institute for Advanced Study,

Institute for Quantum Science and Engineering (IQSE),

Texas A&M AgriLife Research,

Texas A&M University,

College Station, TX 77843-4242, USA.

I. Bezděková

Department of Physics, Czech Technical University,

Břehová 7, 115 19 Praha 1 - Czech Republic and

Department of Physics, Faculty of Electrical Engineering and Computer Science,

Technical University of Ostrava 17. Listopadu 15,

Ostrava - Poruba, 70833, Czech Republic

M.B. Kim

Texas A&M University Institute for Advanced Study (TIAS),

Institute for Quantum Science and Engineering (IQSE),

and Department of Physics and Astronomy,

Texas A&M University, College Station, TX 77843-4242, USA.

P.C. Abbott

School of Physics, The University of Western Australia (M013),

35 Stirling Highway, Crawley WA 6009, Australia.

H. Maier

Institut für Zahlentheorie und Wahrscheinlichkeitstheorie,

Universität Ulm, D-89081 Ulm, Germany.

H.L. Montgomery

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA.

J.W. Neuberger

Department of Mathematics, University of North Texas, Denton, TX 76203-5017, USA.

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We prove the equivalence of three formulations of the Riemann Hypothesis for functions f defined by the four assumptions: ($a1$) f satisfies the functional equation $f(1-s) = f(s)$ for the complex argument $s \equiv \sigma + i\tau$, ($a2$) f is free of any pole, ($a3$) for large positive values of σ the phase θ of f increases in a monotonic way without a bound as τ increases, and ($a4$) the zeros of f as well as of the first derivative f' of f are simple zeros. The three equivalent formulations are: ($R1$) All zeros of f are located on the critical line $\sigma = 1/2$, ($R2$) All lines of constant phase of f corresponding to $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ merge with the critical line, and ($R3$) All points where f' vanishes are located on the critical line, and the phases of f at two consecutive zeros of f' differ by π. Our proof relies on the topology of the lines of constant phase of f dictated by complex analysis and the assumptions ($a1$)-($a4$). Moreover, we show that ($R2$) implies ($R1$) even in the absence of ($a4$). In this case ($a4$) is a consequence of ($R2$).

I. INTRODUCTION

The proof of the Riemann Hypothesis is a long-standing problem in mathematics. Whereas numerical techniques and analytic estimates are prevalent in this context geometric approaches are rare. In the present paper we argue in favor of a geometric route towards this conjecture by proving the equivalence of three formulations of the Riemann Hypothesis based on the topology of the lines of constant phase.

A. Riemann Hypothesis and why it might be true

A central problem of Analytic Number Theory [1, 2] is the Riemann Hypothesis [3–9], describing a conjecture about the zeros of the Riemann zeta function ζ [10]. It was first stated by Bernhard Riemann 1859 in his seminal article "On the number of primes below a given number" [11]. Indeed, the Dirichlet series [1]

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}$$

of complex argument $s \equiv \sigma + i\tau$ and defined for $1 < \sigma$ can be meromorphically continued to the entire complex plane, the only singularity being a simple pole at $s = 1$. Since the "trivial" zeros of ζ located at the even negative integers and originating from the poles of the Gamma function Γ are understood the Riemann Hypothesis reads:

All non-trivial zeros of ζ lie on the critical line $\sigma = 1/2$.

Riemann also found a functional equation for ζ which is most conveniently formulated by introducing the function

$$\xi(s) \equiv \pi^{-s/2}(s-1)\Gamma\left(\frac{s}{2}+1\right)\zeta(s) \tag{1}$$

leading to the relation[2]

$$\xi(s) = \xi(1-s). \tag{2}$$

The representation

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \tag{3}$$

of ζ as the Euler product over all primes p valid for $1 < \sigma$ implies that ξ has no zeros for $\sigma > 1$, and thus by virtue of the functional equation Eq. (2) also no zeros for $\sigma < 0$. Since the zeros of ξ are also symmetric with respect to the real axis, they are symmetric with respect to the critical line. For a discussion of the symmetry relations of ξ imposed by the functional equation and complex analysis we refer to the Appendix.

A statement equivalent to the Riemann Hypothesis then reads:

All zeros of ξ lie on the critical line $\sigma = 1/2$.

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[1] It is interesting that ζ is the solution of no algebraic differential equation. However, in Ref. [12] it was shown that ζ satisfies an *infinite* order linear differential equation with analytic coefficients.

[2] Ref. [13] provides a numerical recipe to obtain the Dirichlet series for a given functional equation.

Indeed, the definition of ξ, Eq. (1), implies that the sets of roots of ξ and ζ only differ in the poles of the Gamma function, that is the “trivial” zeros of ζ.

The region $0 < \sigma < 1$ is called the critical strip. A crucial step in the proof of the prime number theorem obtained in 1896 independently by Jacques Hadamard [14] and Charles-Jean de la Vallée Poussin [15] was the proof that there are also no zeros on the boundaries of the critical strip, that is for $\sigma = 0$ and $\sigma = 1$.

There are numerous investigations on the zeros in the critical strip. Indeed, there are infinitely many of such zeros. Moreover, Harald Bohr and Edmund Landau [16] have shown that for an arbitrarily thin parallel strip that contains the critical line, the fraction of zeros lying outside this strip tends to zero as the imaginary part of the roots of ζ approaches $\pm\infty$.

It is also known that there are infinitely many zeros on the critical line. Improving on earlier work by Godfrey Harold Hardy and John Edensor Littlewood [17], Atle Selberg [18], and Norman Levinson [19], John Brian Conrey [20] has established that the zeros on the critical line comprise at least 40% of all zeros.

There are also several hints that the Riemann Hypothesis is true. Apart from numerical work [21] one indication emerges from the Mertens function

$$M(x) \equiv \sum_{n \leq x} \mu(n)$$

involving the Möbius function

$$\mu(n) \equiv \begin{cases} (-1)^{\nu(n)} & \text{if } n \text{ is square free} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We call n square free if it has no repeated prime factors, and $\nu(n)$ is the number of primes contained in n.

According to Littlewood [22], we have the estimate

$$M(x) = \mathscr{O}(x^a) \quad \text{for all} \quad a > 1/2, \tag{5}$$

if and only if the Riemann Hypothesis is true.

It has been shown by Gábor Halász [23] that estimates analogous to Eq. (5) are true with probability one, if μ is replaced by a random multiplicative function [24].

B. Riemann Hypothesis and why it is important

From the product representation, Eq. (3), of ζ Euler could deduce some elementary properties of the sequence of prime numbers p such as the divergence of the infinite series

$$S \equiv \sum_p \frac{1}{p}. \tag{6}$$

Riemann employed the powerful technique of complex integration together with the Euler product, Eq. (3), and established the basic connection between the distribution of prime numbers and the zeros of the Riemann zeta function. Indeed, the function

$$\pi_0(x) \equiv \lim_{\varepsilon\to 0} \frac{1}{2}[\pi(x-\varepsilon)+\pi(x+\varepsilon)] \tag{7}$$

composed of the function $\pi = \pi(x)$ which counts the number of primes below x, provides us with the relation

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \pi_0(x^{1/n}) \tag{8}$$

where

$$\pi_0(x) = \frac{1}{2}\left(\sum_{p^n<x} \frac{1}{n} + \sum_{p^n\leq x} \frac{1}{n}\right). \tag{9}$$

The connection to the zeros then reads

$$\pi_0(x) = \mathrm{li}(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \ln(2\pi) - \frac{1}{2}\ln(1-x^{-2}), \tag{10}$$

where ρ and li denote the zeros of ζ and the integral logarithm

$$\mathrm{li}(x) \equiv \lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} \mathrm{d}t\,\frac{1}{\ln t} + \int_{1+\varepsilon}^{x} \mathrm{d}t\,\frac{1}{\ln t}\right), \tag{11}$$

respectively.

The central question about the distribution $\pi = \pi(x)$ of prime numbers is the error term

$$E(x) \equiv \pi(x) - \mathrm{li}(x). \tag{12}$$

From Eq. (10), it is apparent that E depends in a crucial manner on the real part of the zeros of ζ. It can be shown that $E(x)$ satisfies the estimate

$$E(x) = \mathscr{O}(x^{1/2+\varepsilon}), \tag{13}$$

if and only if the Riemann Hypothesis is true.

Although the Riemann Hypothesis is mainly known for its connection to the distribution of prime numbers and arithmetic functions related to them, there are relations to physics. Indeed, David Hilbert and George Pólya [5] found that the Riemann Hypothesis would follow if the zeros would be eigenvalues of an operator $1/2 + i\hat{T}$, where $\hat{T}$ is Hermitian. In this context the recent constructive proof [25] of the Hilbert-Pólya conjecture is most interesting. Moreover, the statistics of the nearest neighbor separations [26] of the nontrivial zeros of the Riemann zeta function is closely related to the energy eigenvalue distributions of random matrices [27] which play a central role in nuclear physics and quantum chaos [28].

C. Geometrical approach

The results summarized in Sec. I A rely heavily on analytic estimates [29]. So far there have been relatively few geometric considerations. For example, lines of constant phase and height of ζ have played a crucial role in the work of Albert Utzinger [30] and Andreas Speiser [31]. In particular, Speiser [31] found a statement equivalent to the Riemann Hypothesis:

All non-trivial roots of ζ' must be to the right of the critical line.

Since the pioneering work of Utzinger and Speiser, lines of constant phase and height of ζ have been studied [32] especially from the perspectives of flows [33–35], such as the holomorphic flow [36, 37], or the Newton flow [38–41]. Most interesting are in this connection the X rays [42] of ζ.

Unfortunately, due to the presence of the pole at $s = 1$ and the trivial zeros of ζ along the negative real axis the resulting flow is rather complicated. Fortunately, the function ξ defined by Eq. (1) is free of these complications and the corresponding flow lines are highly symmetric [43, 44].

D. Central Result

In the spirit of a geometrical approach towards the Riemann Hypothesis, we have studied the lines of constant phase of ζ from the point of view of the Newton flow [38, 41, 45] in the complex plane [39] and on the Riemann sphere [40]. In our analysis we have identified the crucial role of a subclass of phase lines, that is the separatrices which divide the flow of the phase lines into the different domains of the complex plane representing the basins of attraction for the individual zeros. We have also shown for an elementary example [40] that information about the separatrices in infinity provides us with insight into the location of the roots at the center of the complex plane.

In the present article we study the distribution of zeros of a rather general class of appropriately differentiable functions $f = f(s)$ by using the arrangements of their lines of constant phase and making the following assumptions:

(*a1*) f satisfies the functional equation

$$f(1-s) = f(s), \tag{14}$$

(*a2*) f is free of any pole,

(*a3*) for large positive values of σ the phase θ of f increases in a monotonic way without a bound as τ increases, and

(*a4*) the zeros of f as well as of the first derivative f' of f are simple zeros.

The set of functions f specified by these assumptions is not empty. For example,

$$c(s) \equiv \cosh\left(s - \frac{1}{2}\right) \tag{15}$$

is an element of this class. Indeed, it is straight forward to verify that c satisfies the assumptions (*a1*)-(*a4*).

We show that the following three statements are equivalent:

(*R1*) All zeros of f are located on the critical line.

(*R2*) All lines of constant phase of f corresponding to $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ merge with the critical line.

(*R3*) All points where f' vanishes are located on the critical line, and the phases of f at two consecutive zeros of f' differ by π.

Our proof outlined in the next section brings out most clearly how the four assumptions (*a1*)-(*a4*) conspire together to produce the equivalence of the three formulations of the Riemann Hypothesis. In this process (*a4*) plays a prominent role but we emphasize that in the case of ξ this assumption has not been proven yet, although it is generally accepted.

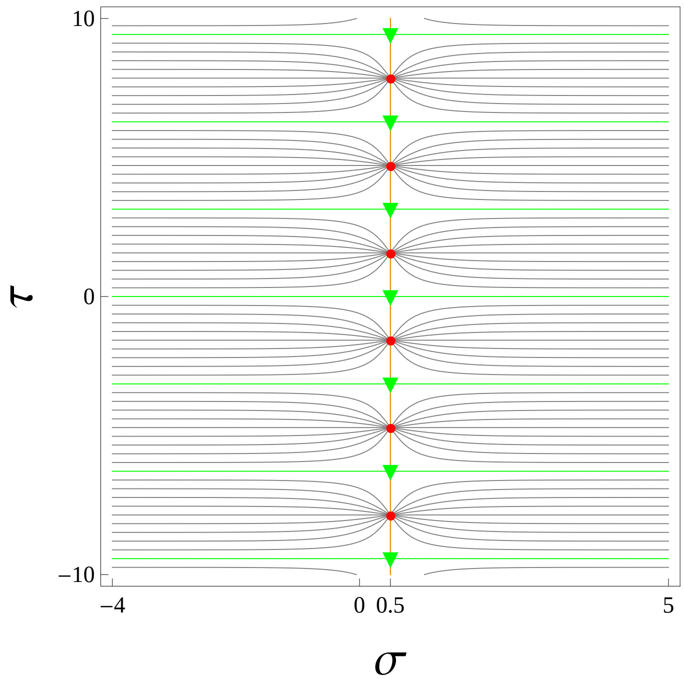

Figure 1: Lines of constant phase (black curves) of the elementary example $c(s) \equiv \cosh(s - 1/2)$ satisfying the assumptions $(a1)$-$(a4)$. We identify the horizontal lines (green curves) merging with the critical line (orange curve) at a zero (green triangles) of the derivative c' of c as separatrices channeling the flow of phase lines into the individual zeros (red dots) of c. Points where c' vanishes are surrounded by white domains since zeros of c' strongly repel the phase lines. In this and the following figures phase lines enjoy a phase difference of $\pi/10$.

E. Overview

Our article is organized as follows: In order to lay the ground work for our proof of the equivalence we analyze in Section II the restrictions on the lines of constant phase imposed by the assumptions $(a1)$-$(a4)$. In particular, we emphasize the degrees of freedom *not* determined by them. They are specified by the individual formulations $(R1)$-$(R3)$. Throughout this section we heavily rely as a guiding principle on the Newton flow [38] of phase lines as discussed in Ref. [39].

Section III which constitutes the main part of our article contains the proof of the equivalence of our three formulations of the Riemann Hypothesis for the class of functions defined by the assumptions $(a1)$-$(a4)$. Our argument is solely based on the topology of the lines of constant phase enforced through these assumptions by complex analysis [46] and illustrated

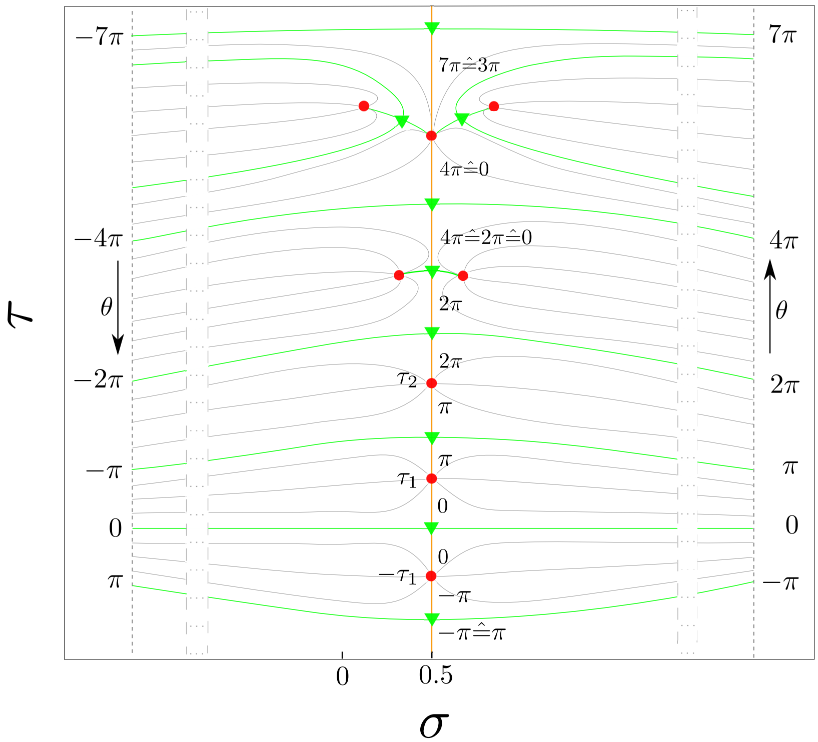

Figure 2: Key elements of our proof of the equivalence of the three formulations (*R1*), (*R2*), and (*R3*) of the Riemann Hypothesis, and consequences of the assumptions (*a1*)-(*a4*) on the separatrices directing the flow of lines of constant phase of $f = f(s)$ in the complex plane, and thus on the distribution of zeros. Due to the functional equation, Eq. (14) of f, its phase θ is anti-symmetric with respect to the critical line $\sigma = 1/2$ depicted by the orange line. As a result, f is real on it and can only assume the phases $k\pi$ where k is integer. Moreover, complex analysis enforces [46] an anti-symmetry of θ with respect to the real axis which is a line of constant phase with $\theta = 0$ provided f does not have a zero there. At $s = 1/2$, the first derivative of f vanishes as indicated by the green triangle. A simple zero on the critical line with imaginary part τ_1, denoted by a red dot and no zeros off the line requires phase lines of a π-interval to approach from the right *and* from the left. If f enjoys more zeros their domains of attraction need to be fenced off by separatrices shown by the green curves which start at infinity and merge with the critical line at right angles where again f' vanishes. Since f is free of poles infinity is indeed the only source of phase lines. For a zero located off the critical line the functional equation automatically enforces an additional zero which is its mirror image. When *three* consecutive points on the critical line where f' vanishes are caught between two zeros of f, *two* off-axis zeros emerge, and the phase difference between two consecutive separatrices is 2π rather than π. Similarly, when we have one zero on the critical line, and two off the line the phase difference between two consecutive separatrices can be a less than π. The value of k in the phase $k\pi$ on the critical line indicated in the figure is dictated by the distribution of zeros. Due to the 2π-periodicity of $\exp(i\theta)$ values of θ such as $-\pi$ and π, or 4π, 2π and 0 are equivalent.

for the elementary example of c defined by Eq. (15) in Fig. 1, and in more general terms in Fig. 2. In particular, we identify the crucial role of the separatrices, that is lines of constant phase that cross each other in points where f' vanishes, as *the* factor determining the distribution of zeros of f.

A crucial ingredient of our proof of the equivalence is the assumption (*a4*) stating that the zeros of f and f' are simple ones. As mentioned earlier this property is not known for ξ yet. Nevertheless, we can profit from the knowledge on the topology of the lines of constant phase acquired in the proof and verify the formulation (*R1*) starting from (*R2*) without ever using the fact that we deal with simple zeros of f or f'. We dedicate Sec. IV to prove this statement and emphasize that it might open up a new approach towards the Riemann Hypothesis.

Another consequence of our equivalence is a necessary but not sufficient condition for the validity of the Riemann Hypothesis. We illustrate this condition in Sec. V by a function [47] which is very similar to ξ, but is known to have zeros off the critical line. In Sec. VI we summarize our results and provide an outlook.

For the sake of completeness we derive in the Appendix the symmetry and anti-symmetry relations of the absolute value $|f|$ and the phase θ of f with respect to the critical line, and the real axis. Indeed, throughout our analysis these arguments play a crucial role.

II. PROPERTIES OF PHASE LINES

Before we turn to the proof of the equivalence, we analyze in this section the consequences of the assumptions (*a1*)-(*a4*) on the topology of the lines of constant phase. In particular, we show that these assumptions enormously restrict the possible locations of the zeros of f and of f'.

A. Phase Lines approach from Infinity

In order to bring out this fact most clearly we first note that with the help of the Newton flow [38] we can imprint a direction onto the lines of constant phase of a function $f = f(s)$. Indeed, they are born either in a pole or in infinity, and terminate at a zero. Since f is free of any poles as expressed by assumption (*a2*) all phase lines must approach from infinity.

It is in this context that the functional equation, Eq. (14), that is the assumption (*a1*) plays a central role. As shown in the Appendix it enforces together with the symmetry relation

$$f^*(s) = f(s^*) \tag{16}$$

of complex analysis [46], where the symbol $*$ indicates the complex conjugate, the anti-symmetry of the phase θ of f with respect to the critical line $\sigma = 1/2$.

Since according to (*a3*) θ *increases* for increasing τ at $\sigma = +\infty$ in a monotonic way without a bound, the anti-symmetry of θ implies that θ *decreases* in the same way at $\sigma = -\infty$. Hence, *all* phase lines initiate either from $\sigma = +\infty$ or $\sigma = -\infty$, and approach the center of the complex plane where they terminate in zeros as exemplified in Fig. 2. Moreover, due to the monotonic increase of θ without a bound there must be an infinite amount of zeros.

According to (*a4*) f enjoys only simple zeros which requires a complete 2π-interval of phase lines to end there. When a zero is located on the critical line phase lines from a π-interval emerging from $\sigma = +\infty$, and an equivalent set of the negative phase variety approaching from $\sigma = -\infty$ suffice. Here the phase lines corresponding to $\theta = \pi$ and $\theta = -\pi$ are provided by the critical line.

Indeed, as shown in the Appendix the functional equation, Eq. (14), that is assumption (*a1*) together with the symmetry relation Eq. (16) ensures that f is real along the critical line. Hence, θ can assume the values $k\pi$ where k is an integer. When we traverse a simple zero located on the critical axis θ jumps by π.

B. Bridges to Infinity

However, the critical line cannot be isolated in the complex plane, it must be connected to infinity by at least two phase lines. The real axis provides one such connection.

Indeed, as shown in the Appendix the symmetry relation, Eq. (16) ensures that f is real also along the real axis. Thus here the phase of f can assume the values $k\pi$ as well.

At the intersection of the critical line with the real axis four lines of equivalent phase meet and the two incoming lines on the real axis approaching from infinity are orthogonal on the two outgoing ones on the critical line. Moreover, at the crossing the phase of f is maintained.

On first sight this feature is surprising since in general two lines of constant phase cannot cross. This feature is a consequence of the uniqueness of $\theta \bmod 2\pi$. However, an exception of this rule occurs at points in the complex plane where the first derivative f' of f vanishes. Since according to the assumption *(a4)* f' has only simple zeros, the two crossing phase lines are indeed orthogonal on each other and the roots of f and f' are distinct. Hence, there cannot be a zero of f at the crossing of the critical line with the real axis, only a zero of f' is allowed.

However, there could be more bridges from infinity to the critical line than just the real axis. Indeed, when we have several simple zeros on the critical line the consecutive pieces connecting them differ in their respective phases by π. Since according to *(a2)* f is free of a pole these phase line segments must really be born in infinity. Hence, lines of constant phase corresponding to $k\pi$ with k integer must start from $\sigma = +\infty$ as well as $\sigma = -\infty$, and merge with the critical line.

Moreover, the anti-symmetry of θ with respect to the critical line requires that always a line corresponding to $k\pi$ and one corresponding to $-k\pi$ merge with the critical axis at the same point, where f' vanishes. Since according to *(a4)* we deal with simple zeros of f they must be different from the ones of f'.

In the case of simple zeros of f' as guaranteed by *(a4)* *two* phase lines enter the root of f' and emerge from it. These lines are orthogonal on each other as is the case at the intersection of the critical line and the real axis.

Hence, consecutive zeros on the critical line are separated by phase lines corresponding to $k\pi$ and $-k\pi$ with $k = 0, \pm 1, \pm 2, \cdots$, as dictated by the requirements of continuity of phase and phase jumps of π when traversing consecutive zeros. These phase lines initiate at $\sigma = +\infty$ and $\sigma = -\infty$, traverse the complete complex plane, and merge with the critical line. They divide the complex plane into individual horizontal stripes as shown in Fig. 2.

C. Zeros off the Critical Line

Finally we address the situation of a zero *not* being on the critical line as illustrated in the two top domains of Fig. 2. In this case the non-crossing rule of phase lines does not allow the phase lines from across the critical line help make up the 2π-interval. Thus all phase line have to approach the zero either from $\sigma = +\infty$, provided the zero is to the right of the critical line, or from $\sigma = -\infty$, if the zero is to the left of it. In this process some phase lines have to first pass the zero then turn around and return to it.

It is exactly this inversion of the flow of phase lines which is at the very heart of the Lagarias criterion [48, 49], as discussed in a future publication. Moreover, it is the deviation of the amount of enclosed phase lines from π which will allow us in Sec. III to develop a new geometric approach towards the Riemann Hypothesis.

However, we emphasize that the assumptions (*a1*)-(*a4*) do not enforce the locations of the zeros of f of f'. This task is only achieved by the additional information contained in the formulations (*R2*) and (*R3*).

III. PROOF OF EQUIVALENCE

We are now in the position to prove the equivalence of our three formulations of the Riemann Hypothesis introduced in Sec. I D. Our proof proceeds as follows: We start from the statement (*R1*) and verify (*R2*). Then we prove (*R3*) assuming (*R2*) is correct, and finally close the cycle by demonstrating that (*R1*) is a consequence of (*R3*). Since we have already discussed in great detail the restrictions imposed by the assumptions (*a1*)-(*a4*) on the topology of phase lines we can be rather brief.

A. From Riemann Hypothesis to separatrices

We begin by assuming that the formulation (*R1*) is correct and verify that it implies (*R2*). Hence, we assume that all zeros of f are on the critical line.

For this purpose we recall from Sec. II that at its intersection with the real axis, that is at $s = 1/2$, is a zero of f' but not of f. Hence, the phase of f along the real axis is constant and can be chosen to be zero, that is

$$\theta(\sigma, \tau = 0) = 0. \tag{17}$$

Due to the phase difference of π between two consecutive pieces of the critical line also two consecutive phase lines merging from the left or from the right with it must have the phase difference π. Moreover, since *all* zeros of f are on the critical line, *all* phase lines corresponding to $k\pi$ merge with it at points which are not zeros of f which represents (*R2*).

B. From separatrices to their crossing points

Next we assume that the statement (*R2*) is correct and deduce from it (*R3*). For this purpose it suffices to assume that at least one zero of f' is not on the critical line as exemplified by the top part of Fig. 2.

The two lines entering this simple root of f' from infinity must both come either from $\sigma = +\infty$, or from $\sigma = -\infty$ depending on the root of f' being located either to the right, or to the left of the critical line. Moreover, one of the emerging lines must lead to a zero enclosed by the two incoming lines. Since they surround a complete 2π-interval of phase lines this set must also contain a phase line corresponding to $k_o\pi$ with k_0 inter. Obviously this line terminates in the zero and does not merge with the critical line in contrast to the formulation (*R2*) we started from, claiming that *all* phase lines with $k\pi$ merge with the critical line. Hence, *all* roots of f' are on the critical line, and they must be located between the zeros of f endowing them with a phase difference of π. Both features constitute the formulation (*R3*).

C. From crossing points to Riemann Hypothesis

Finally, we close the circle of logic and derive (*R1*) by starting from (*R3*). For this purpose we assume that there exists a root of f which is *not* located on the critical line as exemplified by the two top zones of Fig. 2.

In this case phase lines from a complete 2π-interval terminating in this zero must either come from $\sigma = +\infty$, or $\sigma = -\infty$ depending on this zero of f being located either to the right, or to the left of the critical line. This bunch is surrounded by two phase lines entering a root of f' which according to the formulation (*R3*) is on the critical line.

Since the incoming and outgoing lines are orthogonal to and aligned with the critical line, respectively, it is impossible for *two* phase lines to enter a single root of f' from the right or the left side of the critical line only. Hence, the two phase lines must merge with the critical axis in *two* roots of f'.

Still these phase lines have to terminate in a zero of f which cannot be on the critical line between these roots of f' because their phase difference is 2π rather than π. However, this requires an additional root of f' trapped between the other two. As a result, the phase difference of f at these *three* consecutive roots of f' is an integer multiple of 2π, in contrast to our initial formulation ($R3$). As a result, all zeros of f must be on the critical line which is the statement ($R1$).

IV. A GEOMETRICAL APPROACH TOWARDS THE RIEMANN HYPOTHESIS

In our proof of the equivalence of the three formulations of the Riemann Hypothesis presented in the preceding section the assumption ($a4$) that f and f' enjoy simple roots has played a central role. We now show that ($R2$) implies ($R1$) even in the absence of ($a4$) and outline a geometrical approach towards the Riemann Hypothesis based on the function ξ defined by Eq. (1).

A. From separatrices to Riemann Hypothesis

Due to the functional equation, Eq. (14), that is ($a1$) the phase of f is anti-symmetric with respect to $\sigma = 1/2$ and the critical line must have the phases $k\pi$. Since $k\pi$ is equivalent to $-k\pi$, the continuity of phase enforces the corresponding phase lines to merge at the same point on the critical line. As a result, two incoming and two outgoing phase lines of equivalent phases are orthogonal on each other representing a simple root of f' located on the critical line.

The assumption ($a2$), that is f is free of a pole, ensures that all field lines approach from infinity. Moreover, the assumption ($a3$) that θ increases in a monotonic way without a bound guarantees that we have a continuous set of phase lines starting from $\sigma = +\infty$. We now invoke ($R2$) which implies that the phase difference between consecutive phase lines merging with the critical line is π. Hence, the zero of f must be located on the critical line, with a π-interval of phase lines approaching from $\sigma = +\infty$, and due to the anti-symmetry of θ a π-interval coming in from $\sigma = -\infty$. Moreover, this fact also guarantees that f must have simple zeros since a complete 2π-interval of phase lines terminates there.

Finally, all zeros of f' must be on the critical line since any zero of f' not located on it would lead to a zero of f that is also off which would require consecutive separatrices to have a phase difference different from π.

B. Application to ξ

We conclude this section by outlining a geometrical approach towards the Riemann Hypothesis based on this result. Needless to say, we do not pursue it at this point.

From the definition, Eq. (1), of ξ we recognize that ξ is free of any poles. Moreover, it satisfies [1] the functional equation, Eq. (14). In addition, the phase θ increases [1] at $\sigma = +\infty$ in a monotonic way without a bound. Hence, ξ satisfies the assumptions (*a1*), (*a2*) and (*a3*). As a result, we can immediately formulate the following lemma:

All zeros of ξ are on the critical line if all lines of constant phase with $k\pi$ and k integer merge with the critical line.

In other words we have to show that all lines of ξ with $\theta = k\pi$ are separatrices as shown in Figs. 3 and 4 for selected domains of the complex plane. Hence, the question of the Riemann Hypothesis has been transferred to the identification of separatrices from the sea of phase lines.

V. A NECESSARY BUT NOT SUFFICIENT CONDITION

A crucial ingredient of our argument leading from (*R3*) to (*R1*) is the assumption that the phases of f at two consecutive roots of f' on the critical line differ by π. Moreover, the formulation (*R3*) also assumes that *all* zeros of f' are on the critical line. We now use a function [1, 47] which is in many ways similar to ξ to illustrate that a breakdown of one of these two properties leads immediately to zeros that are located off the critical line as indicated in Fig. 2.

For this purpose we display in Fig. 5 the lines of constant phase of

$$\xi_g(s) \equiv \left(\frac{\pi}{5}\right)^{-s/2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) g(s), \tag{18}$$

with

$$g(s) \equiv \frac{1}{5^s}\left[\zeta\left(s, \frac{1}{5}\right) + \tan\theta_g\,\zeta\left(s, \frac{2}{5}\right) - \tan\theta_g\,\zeta\left(s, \frac{3}{5}\right) - \zeta\left(s, \frac{4}{5}\right)\right] \tag{19}$$

containing linear combinations of Hurwitz zeta functions

$$\zeta(s, a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{20}$$

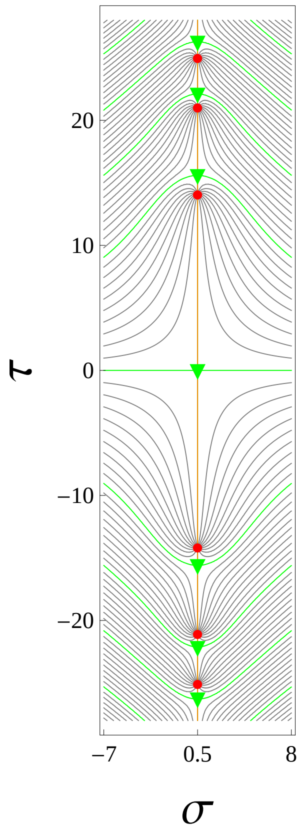

Figure 3: Lines of constant phase (black curves) of the function ξ defined by Eq. (1) in the neighborhood of the real axis and along the critical line (orange curve). Here we show the first three zeros (red dots) of ξ with positive imaginary parts, and the first three roots with negative imaginary parts. The picture is symmetric with respect to the real axis as well as the critical line. The flows of phase lines approaching the individual zeros are separated from each other by separatrices (green curves) which initiate at infinity and terminate at zeros after they have merged with the critical line at a zero of ξ' indicated by a green triangle. Provided the phase difference between two consecutive separatrices is π, as is the case in this part of the complex plane, the enclosed zeros must be located on the critical line and they have to be simple zeros. Moreover, the zeros of ξ' must be simple ones as well.

of different parameters a with

$$\tan\theta_g \equiv \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}. \tag{21}$$

Although ξ_g has similar properties as ξ and, in particular, satisfies the functional equation, Eq. (14), it is known [47] to have zeros off the critical line. For example, in the domain of

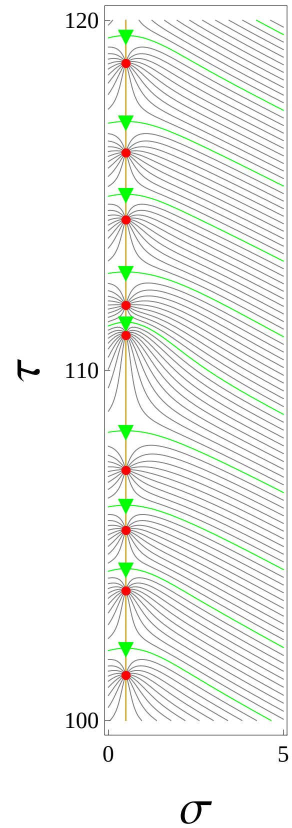

Figure 4: Lines of constant phase (black curves) of the function ξ, for larger positive imaginary parts than in Fig. 3, involving nine zeros (red dots) of ξ along the critical line (orange curve) and nine roots of ξ' (green triangles). Due to the symmetry of the flows with respect to the critical line we only depict the domain to the right of it. Again two consecutive separatrices (green lines) always enclose a phase interval of π indicating that the zeros of ξ are located on the critical line. If this behavior continues along the critical line to infinity, the Riemann Hypothesis is correct. Although this figure seems to suggest it, the example of the function ξ_g defined by Eq. (18) and shown in Figs. 5 and 6 demonstrates that this feature is not guaranteed.

the complex plane around $\tau \cong 86$ we recognize from Fig. 5 three consecutive zeros of ξ_g' on it. Here the phase difference between two consecutive separatrices enclosing together with the critical line the off-axis zero is 2π rather π. Consequently, the phase difference between ξ_g at two consecutive zeros of ξ_g' cannot be π. Two roots of ξ_g symmetrically located off the critical line are the consequence of this violation of part of the formulation $(R3)$.

This observation gives rise to the following lemma:

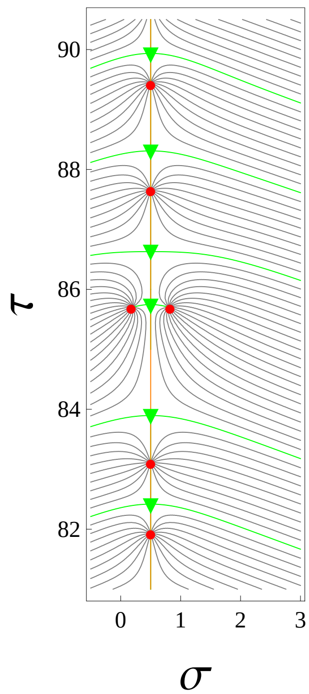

Figure 5: Violation of the formulation ($R3$) of the Riemann Hypothesis and appearance of two zeros (red dots) off the critical line (orange line) illustrated by the function $\xi_g = \xi_g(s)$ defined by Eq. (18). In the domain of the complex plane shown here *three* consecutive points (green triangles) on the critical line exist where ξ_g' vanishes. Since these three zeros of ξ_g' are caught between two consecutive zeros of ξ_g on the critical line, their mutual phase difference cannot be π in violation of ($R3$). As a result, two zeros of ξ_g off the critical line appear demonstrating a violation of the formulation ($R1$). Moreover, the phase difference between the two consecutive separatrices merging with the critical line and enclosing together with it one of the two off-axis zeros is 2π rather than π, which is also in contradiction to the formulation ($R2$).

A necessary condition for the validity of the Riemann Hypothesis is that between two consecutive zeros of the function ξ on the critical line there is only a single root of ξ'.

We emphasize that this condition is not sufficient as illustrated by Fig. 6. Here we depict again the function ξ_g, however now for values of τ larger than in Fig. 5. In this domain the two lines of constant phase approaching the critical line from the right differ by 3π rather than π, allowing a root of ξ_g' to be located off the critical line. This additional phase difference of 2π feeds a zero off the critical line as indicated in Fig. 7 which amplifies the relevant neighborhood of Fig. 6. It is interesting that in this situation there is a *single* point between two consecutive zeros of ξ_g on the critical line where ξ_g' vanishes, and still the zeros are off-axis.

The examples depicted in Figs. 5 and 6 demonstrate in a striking way the importance of the phase difference of π between consecutive phase lines merging with the critical line in a point different from a zero of the function.

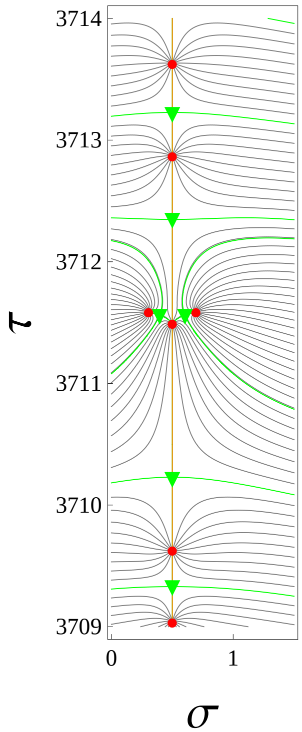

Figure 6: In contrast to Fig. 5 where *three* consecutive roots of ξ_g' are located *on* the critical line we now focus on a domain of the complex plane where two zeros of ξ_g' are located *off* the critical line. This feature which represents another violation of the formulation ($R3$) of the Riemann Hypothesis leads immediately to zeros off the critical line, and thus to a violation of ($R1$). Moreover, the phase difference between two consecutive separatrices merging with the critical line and enclosing with it the off-axis zero is 3π rather than π, which is also in contradiction to ($R2$). The two separatrices leading to the off-axis zero of ξ_g' correspond to the phase 0.7885π.

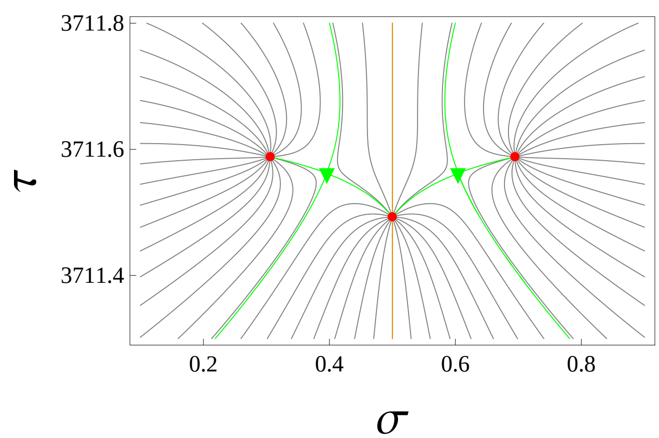

Figure 7: Lines of constant phase in the neighborhood of the three zeros of ξ_g shown in Fig. 6. The two roots of ξ_g' located symmetrically off the critical line are clearly visible as green triangles in the white spots. Locally we can approximate the behaviour of ξ_g by a polynomial of degree three giving rise to separatrices that are not associated with the phase $k\pi$. For a more detailed discussion we refer to Appendix C in Ref. [39].

VI. CONCLUSIONS AND OUTLOOK

We are now in a position to summarize our main results and provide an outlook. Throughout our article we have considered a class of functions defined by specific assumptions. Based on the behavior of the lines of constant phase we could determine possible topologies for the distributions of zeros, and prove an equivalence of three formulations of the Riemann Hypothesis involving the zeros of the function and its first derivative along the critical line, and the separatrices.

As a biproduct we have found a geometric approach towards the Riemann Hypothesis. Indeed, all zeros of ξ are located on the critical line provided every phase line corresponding to $k\pi$ is a separatrix. In this case the zeros of ξ as well as of ξ' are simple. Moreover, all roots of ξ' are located on the critical line. Finally, we have obtained a necessary but not sufficient condition for the validity of the Riemann Hypothesis.

It is interesting that this criterion is closely related to the one proposed by Ernest Oliver Tuck [50–52] and based on a ratio involving the first and second derivative of ξ. Likewise, we could only but allude to a connection to the Lagarias criterion [48, 49]. Insight into both topics springs from the Newton flow [38] of ξ and will be illuminated in a future publication.

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APPENDIX : SYMMETRY RELATIONS

In this appendix we recall the symmetry relations imposed by the functional equation, Eq. (14), as well as complex analysis [46] on the amplitude and the phase of f. In particular, we show that f is real along the critical line as well as along the real axis.

We start our discussion with the functional equation, Eq. (14) which identifies the critical line $\sigma = 1/2$ as a symmetry axis. Indeed, when we use the decomposition $s \equiv \sigma + i\tau$ we find

$$f(1-\sigma-i\tau) = f\left[\frac{1}{2} - \left(\sigma - \frac{1}{2}\right) - i\tau\right] = f(\sigma+i\tau) = f\left[\frac{1}{2} + \left(\sigma - \frac{1}{2}\right) + i\tau\right], \qquad (22)$$

which with the help of the relation

$$f^*(s) = f(s^*), \qquad (23)$$

leads us to the symmetry relation

$$f^*\left[\frac{1}{2} - \left(\sigma - \frac{1}{2}\right) + i\tau\right] = f\left[\frac{1}{2} + \left(\sigma - \frac{1}{2}\right) + i\tau\right]. \qquad (24)$$

With the representation

$$f \equiv |f|e^{i\theta} \qquad (25)$$

of f by its absolute value $|f| = |f(s)|$ and its phase $\theta = \theta(s)$, we obtain from Eq. (24) the identities

$$\left|f\left[\frac{1}{2} - \left(\sigma - \frac{1}{2}\right) + i\tau\right]\right| = \left|f\left[\frac{1}{2} + \left(\sigma - \frac{1}{2}\right) + i\tau\right]\right| \qquad (26)$$

and

$$\theta\left[\frac{1}{2} - \left(\sigma - \frac{1}{2}\right) + i\tau\right] = -\theta\left[\frac{1}{2} + \left(\sigma - \frac{1}{2}\right) + i\tau\right]. \qquad (27)$$

Hence, $|f|$ and θ are symmetric and anti-symmetric with respect to the critical line.

This anti-symmetry of θ restricts the values of θ along this line. Indeed, for $\sigma = 1/2$ we find from Eq. (27) the relation

$$\theta\left(\frac{1}{2} + i\tau\right) = -\theta\left(\frac{1}{2} + i\tau\right), \qquad (28)$$

or

$$\theta\left(\frac{1}{2} + i\tau\right) = k\pi, \qquad (29)$$

where k is an integer. In the last step we have used the fact that the phase is defined only mod 2π and $\theta = +\pi$ is equivalent to $\theta = -\pi$.

Hence, on the critical line θ can only assume the values $k\pi$ as indicated in Fig. 2. As a consequence, f is real along the critical line.

We conclude this appendix by turning to the real axis and note that, Eq. (23) predicts a symmetry of $|f|$ and an anti-symmetry of θ with respect to $\tau = 0$, that is

$$|f(\sigma + i\tau)| = |f(\sigma - i\tau)| \tag{30}$$

and

$$\theta(\sigma + i\tau) = -\theta(\sigma - i\tau). \tag{31}$$

The anti-symmetry property of θ, Eq. (31), which for $\tau = 0$ reads $\theta(\sigma) = -\theta(\sigma)$ enforces again the phase condition

$$\theta(\sigma) = k\pi, \tag{32}$$

where k is an integer and f is also real along the real axis.

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