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Composition as an integral operator

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Let $\mathbf{S}$ be the unit sphere and $\mathbf{B}$ the unit ball in $\mathbf{C}^n$, and denote by $L^1(\mathbf{S})$ the usual Lebesgue space of integrable functions on $\mathbf{S}$. We define four "composition operators" acting on $L^1(\mathbf{S})$ and associated with a Borel function $\varphi : \mathbf{S} \to \overline{\mathbf{B}}$, by first taking one of four natural extensions of $f \in L^1(\mathbf{S})$ to a function on $\overline{\mathbf{B}}$, then composing with φ and taking radial limits. Classical composition operators acting on Hardy spaces of holomorphic functions correspond to a special case. Our main results provide characterizations of when the operators we introduce are bounded or compact on $L^t(\mathbf{S})$, $1 \leq t < \infty$. Dependence on t and relations between the characterizations for the different operators are also studied.

1. Introduction

Composition operators acting on a space X of functions holomorphic on the unit disk $\mathbf{D}$ in $\mathbf{C}$, or more generally the unit ball $\mathbf{B} = \mathbf{B}_n$ in $\mathbf{C}^n$, have been the subject of a great deal of research. In this setting, a holomorphic self-map φ of $\mathbf{B}$ induces the compo-

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sition operator C_φ, defined for f holomorphic on $\mathbf{B}$ by $C_\varphi f = f \circ \varphi$. The basic problem is to relate function theoretic properties of φ to operator theoretic properties of C_φ. On many of the classical Banach spaces of holomorphic functions on $\mathbf{D}$, including the Hardy spaces $H^p(\mathbf{D})$ and Bergman spaces $L^p_a(\mathbf{D})$, every composition operator is bounded and their study involves other properties, such as when a composition operator is compact. In higher dimensions, when $n \geq 2$, boundedness of a composition operator is not automatic, even on $H^p(\mathbf{B})$ or $L^p_a(\mathbf{B})$. For the definitions of the well-known function spaces mentioned above, one may refer to references [4] and [13], which are good introductions to the extensive literature on composition operators in these settings.

In 1990, D. Sarason [12] introduced the viewpoint of composition operators as integral operators acting on spaces of functions defined on the unit circle $\partial\mathbf{D}$. For φ a holomorphic self-map of $\mathbf{D}$ and $f \in L^1(\partial\mathbf{D})$, $C_\varphi f$ was defined on $\partial\mathbf{D}$ by taking the harmonic extension of f to $\mathbf{D}$, composing with φ, and then taking radial limits. As in the classical setting of composition operators acting on $H^p(\mathbf{D})$, every such operator is bounded, and problems such as characterizing when the operator is compact were studied by Sarason.

In the present paper, we generalize Sarason's approach in two significant ways to define composition operators acting on $L^1(\mathbf{S}) = L^1(\mathbf{S}, d\sigma)$, where $\mathbf{S} = \partial\mathbf{B}$ is the unit sphere in $\mathbf{C}^n$ and $d\sigma$ denotes the normalized surface area measure on $\mathbf{S}$. First, we do not assume that the symbol φ of the operator is holomorphic on $\mathbf{B}$; we only assume that $\varphi : \mathbf{S} \to \overline{\mathbf{B}}$ is Borel measurable. Second, we compose φ with four natural extensions of $f \in L^1(\mathbf{S})$ to a function on $\overline{\mathbf{B}}$, resulting in four different "composition operators". Not surprisingly, not all such operators are bounded, even in dimension one. Our main results provide characterizations of when these operators are bounded or compact. We begin with some background needed to define the operators.

By a reproducing kernel K for the function space X on $\mathbf{B}$ we mean that K is a continuous function on $\mathbf{B} \times \mathbf{S}$ such that

$$f(z) = \int_{\mathbf{S}} f(\zeta) K(z, \zeta)\, d\sigma(\zeta), \quad z \in \mathbf{B}$$

for all $f \in X \cap C(\overline{\mathbf{B}})$. On $\mathbf{B}$ we have several reproducing kernels: the Cauchy kernel K^c, Poisson kernel K^h and Poisson–Szegö kernel K^m given by

$$K^c(z, \zeta) := \frac{1}{(1 - \langle z, \zeta \rangle)^n},$$

$$K^h(z, \zeta) := \frac{1 - |z|^2}{|z - \zeta|^{2n}},$$

$$K^m(z, \zeta) := \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}}$$

for $z \in \mathbf{B}$ and $\zeta \in \mathbf{S}$. Here, and throughout the paper, $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product on $\mathbf{C}^n$, i.e., $\langle z, w \rangle := \sum_{j=1}^n z_j \overline{w}_j$ for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. Also, we define the pluriharmonic Poisson kernel K^p as

$$K^p(z,\zeta) := K^c(z,\zeta) + \overline{K^c(z,\zeta)} - 1.$$

Note that K^c is a reproducing kernel for the holomorphic functions, K^h for the harmonic functions, K^m for the invariant harmonic functions (see [10, Chapter 3]) and K^p for the pluriharmonic functions. We note for later use an easy but useful fact that

$$K^x(r\eta,\zeta) = \overline{K^x(r\zeta,\eta)}, \quad \eta,\zeta \in \mathbf{S},\ 0 < r < 1 \tag{1.1}$$

for each $x \in \{c, h, m, p\}$.

Let $\varphi : \mathbf{S} \to \overline{\mathbf{B}}$ be a Borel function. We say that φ is *holomorphic* if it is σ-almost everywhere given by the boundary function (i.e. the radial limit function) of a holomorphic self-map of $\mathbf{B}$. In case φ is holomorphic, we will identify φ with its holomorphic extension. For each $x \in \{c, h, m, p\}$ we wish to define a "composition operator" C_φ^x on $L^1(\mathbf{S})$, i.e. a linear operator that takes $f \in L^1(\mathbf{S})$ to another function defined on $\mathbf{S}$ that comes from composition of f with φ. Since functions in $L^1(\mathbf{S})$ are only defined on $\mathbf{S}$ modulo sets of σ-measure 0, a problem with the definition of these operators arises if φ takes a subset of $\mathbf{S}$ of positive σ-measure to a set in $\mathbf{S}$ of σ-measure 0. This difficulty does not come up in the classical setting where $n = 1$ and φ is holomorphic (see for example [12, Lemma 2]), but it is present in dimension $n \geq 2$ even if it is assumed that φ is holomorphic. The example below illustrates such difficulty. For the definition of the space $BMOA(\mathbf{B})$ that occurs in the next example, we refer to [15].

Example 1.1 *($n \geq 2$).* There exists $f \in BMOA(\mathbf{B})$ and a holomorphic φ such that $\lim_{r\to 1^-} f(r\varphi(\zeta))$ does not exist at any $\zeta \in \mathbf{S}$.

Proof. Let I be an inner function on $\mathbf{B}$. Namely, let $I : \mathbf{B} \to \mathbf{D}$ be a holomorphic function such that $I(\eta) := \lim_{r\to 1^-} I(r\eta) \in \partial\mathbf{D}$ for almost every $\eta \in \mathbf{S}$; see [11] for the existence of such an inner function. Define $\varphi = (I, 0, \dots, 0)$. It is known that there exists $f \in BMOA(\mathbf{B})$ such that $\lim_{r\to 1^-} f(re^{i\theta}, 0, \dots, 0)$ does not exist for any $\theta \in [0, 2\pi)$; see [14, Theorem 1]. The pair f and φ is the desired example. □

An additional assumption about φ is required to deal with the problem. The pullback measure $\sigma \circ \varphi^{-1}$ is the Borel measure defined for a Borel set $E \subset \overline{\mathbf{B}}$ by $\sigma \circ \varphi^{-1}(E) := \sigma\{\zeta \in \mathbf{S} : \varphi(\zeta) \in E\}$. For the rest of the paper we reserve the letter φ to denote functions satisfying that

$$\varphi : \mathbf{S} \to \overline{\mathbf{B}} \text{ is a Borel function and } \left(\sigma \circ \varphi^{-1}\right)\big|_{\mathbf{S}} \ll \sigma \tag{1.2}$$

where $(\sigma \circ \varphi^{-1})|_{\mathbf{S}}$ is the restriction of the measure $\sigma \circ \varphi^{-1}$ to $\mathbf{S}$. We will see below that this assumption is required for the operators C_φ^x to be well-defined.

Integration against one of the kernels K^x, $x \in \{c, h, m, p\}$, gives an extension of a function $f \in L^1(\mathbf{S})$ to a function f^x on $\mathbf{B}$ that is respectively holomorphic, harmonic, invariant harmonic, or pluriharmonic. That is,

$$f^x(z) = \int_{\mathbf{S}} f(\zeta) K^x(z,\zeta)\, d\sigma(\zeta), \quad z \in \mathbf{B}. \tag{1.3}$$

We then use radial limits (which exist σ-a.e. on $\mathbf{S}$; see for example [1, Theorem 6.39] and [10, Theorems 5.4.8 and 6.2.3]) to extend the definition of f^x from $\mathbf{B}$ to $\overline{\mathbf{B}}$; that is

$$f^x(w) = \lim_{r \to 1^-} f^x(rw) \quad w \in \overline{\mathbf{B}},\ x \in \{c, h, m, p\}. \tag{1.4}$$

This f^x is naturally referred to as the *x-extension* of $f \in L^1(\mathbf{S})$.

It is well known that in some, but not all, settings the function $f^x|_{\mathbf{S}}$ recovers f σ-a.e. as in the next proposition; see for example [1] and [10]. In what follows, $H^t(\mathbf{S})$, $1 \le t < \infty$, denotes the closed subspace of $L^t(\mathbf{S}) = L^t(\mathbf{S}, d\sigma)$, the usual Lebesgue space with norm $\|\cdot\|_t$, consisting of all boundary functions of $H^t(\mathbf{B})$-functions. As is well-known, $H^t(\mathbf{S})$ is isometrically identified with $H^t(\mathbf{B})$; see [10, Theorem 5.6.8].

Proposition 1.2. *The following relations hold:*

(a) *If $x \in \{c, p\}$ and $f \in H^1(\mathbf{S})$, then $f^x|_{\mathbf{S}} = f$ σ-a.e.;*
(b) *If $x \in \{h, m\}$ and $f \in L^1(\mathbf{S})$, then $f^x|_{\mathbf{S}} = f$ σ-a.e.;*
(c) *If $f \in C(\mathbf{S})$ in addition to the hypothesis of (a) or (b), then $f^x \in C(\overline{\mathbf{B}})$;*
(d) *If $x \in \{c, p\}$ and $f \in L^1(\mathbf{S})$, then in general $f^x|_{\mathbf{S}} \ne f$;*
(e) *If $x \in \{c, h, m, p\}$, the transform $f \to f^x|_{\mathbf{S}}$ is L^t-bounded for each $1 < t < \infty$.*

For $f \in L^1(\mathbf{S})$ and $x \in \{c, h, m, p\}$, we define the function $C_\varphi^x f$ on $\mathbf{S}$ by

$$C_\varphi^x f = f^x \circ \varphi.$$

Clearly, this is well defined, because f^x remains the same even if f is altered on a set of σ-measure 0. Also, it should be remarked that this defines $C_\varphi^x f$ off a set of σ-measure 0 on $\mathbf{S}$. To see this, notice that from (1.4) we have

$$C_\varphi^x f(\zeta) = \lim_{r \to 1^-} f^x\big(r\varphi(\zeta)\big), \quad \zeta \in \mathbf{S},$$

and this limit exists precisely when f^x has a radial limit at $\varphi(\zeta)$. Thus $C_\varphi^x f$ has been defined at points $\zeta \in \mathbf{S} \setminus \varphi^{-1}(E)$, where $E \subset \mathbf{S}$ is the set of σ-measure 0 where f^x fails to have a radial limit. Since $\sigma[\varphi^{-1}(E)] = 0$ by the assumption (1.2), $C_\varphi^x f$ has been defined σ-a.e. on $\mathbf{S}$.

In general, C_φ^x is a linear operator from $L^1(\mathbf{S})$ to the vector space of (equivalence classes of) measurable functions on $\mathbf{S}$. From Proposition 1.2(a)–(b), the restriction of C_φ^x for each $x \in \{c, h, m, p\}$ to $H^1(\mathbf{S})$ is the usual composition operator:

$$C_\varphi^x f = f \circ \varphi, \quad f \in H^1(\mathbf{S}) \tag{1.5}$$

where the f in the right-hand side denotes the holomorphic extension of $f \in H^1(\mathbf{S})$. Similarly, Proposition 1.2(b) shows that the restriction of C_φ^x to $L^1(\mathbf{S})$ is the usual composition operator when $x \in \{h, m\}$.

A basic problem in the study of composition operators is to characterize those symbols φ for which the restriction of the composition operator C_φ to a Banach space X is bounded or compact. Before stating our main result, which provides such characterizations for the operators C_φ^x acting on $L^t(S)$, we introduce some notation.

We first introduce the extended kernels. Given $x \in \{c, h, m, p\}$ and $w \in \mathbf{B}$, we denote by $\mathscr{K}^x(\cdot, w)$ the x-extension of $\overline{K^x(w, \cdot)}$, i.e.,

$$\mathscr{K}^x(\cdot, w) = \left[\overline{K^x(w, \cdot)}\right]^x.$$

Note that each $\mathscr{K}^x(\cdot, w)$ is continuous on the whole $\overline{\mathbf{B}}$ by Proposition 1.2(c). More explicitly, we have by (1.3), (1.4) and Proposition 1.2(c)

$$\mathscr{K}^x(z, w) = \begin{cases} \int_{\mathbf{S}} K^x(z, \zeta)\overline{K^x(w, \zeta)}\, d\sigma(\zeta) & \text{if } z \in \mathbf{B} \\ \overline{K^x(w, z)} & \text{if } z \in \mathbf{S}. \end{cases}$$

Except for the Poisson–Szegö kernel, the extended kernels have explicit formulae for $z \in \mathbf{B}$ and $w \in \overline{\mathbf{B}}$:

$$\begin{aligned} \mathscr{K}^c(z, w) &= \frac{1}{(1 - \langle z, w\rangle)^n}, \\ \mathscr{K}^h(z, w) &= \frac{1 - |z|^2|w|^2}{[z, w]^{2n}}, \\ \mathscr{K}^p(z, w) &= \frac{1}{(1 - \langle z, w\rangle)^n} + \frac{1}{(1 - \langle w, z\rangle)^n} - 1 \end{aligned}$$

where $[z, w] = \sqrt{1 - 2\Re(z \cdot \overline{w}) + |z|^2|w|^2}$. The formulae for $\mathscr{K}^c$ and $\mathscr{K}^p$ are easily verified. The formula for $\mathscr{K}^h$ is also well known; see for example [1, p. 122]. Note that the right-hand sides of the formulae above continuously extend to $\overline{\mathbf{B}} \times \overline{\mathbf{B}} \setminus \Delta$ where Δ denotes the diagonal of $\mathbf{S} \times \mathbf{S}$. Such extensions are still denoted by $\mathscr{K}^x$ for $x \in \{c, h, p\}$. Note that

$$\mathscr{K}^x(rz, w) = \mathscr{K}^x(z, rw), \quad x \in \{c, h, p\} \tag{1.6}$$

for $0 < r \le 1$ and $(z, w) \in \overline{\mathbf{B}} \times \overline{\mathbf{B}} \setminus \Delta$.

For $x = m$ when $n \ge 2$, no explicit formula of closed form is available; the main difficulty is the fact that the invariant-harmonicity is not dilation-invariant (see [10, Theorem 4.4.10]). Nevertheless, we have natural growth estimate

$$\mathscr{K}^m(z, w) \approx \frac{(1 - |z|^2|w|^2)^n}{|1 - \langle z, w\rangle|^{2n}} \tag{1.7}$$

for $z, w \in \mathbf{B}$; see [2, Lemma 4.1] for the lower estimate and the remark after [2, Corollary 4.6] for the upper estimate. Also, we can still naturally extend $\mathscr{K}^m$ to $\overline{\mathbf{B}} \times \overline{\mathbf{B}} \setminus \Delta$ as follows. First, noting that $\mathscr{K}^m$ is symmetric on $\mathbf{B} \times \mathbf{B}$, we extend $\mathscr{K}^m$ to $\overline{\mathbf{B}} \times \mathbf{B}$ by symmetry. So, $\mathscr{K}^m(\zeta, w) = K^m(w, \zeta)$ for $\zeta \in \mathbf{S}$ and $w \in \mathbf{B}$. Next, noting that $K^m(w, \eta)$ continuously extends to the zero function on $\mathbf{S} \setminus \{\eta\}$, we simply define $\mathscr{K}^m(\zeta, \eta) = 0$ for $\zeta, \eta \in \mathbf{S}$ with $\zeta \neq \eta$. Now, one can check that such an extension, still denoted by $\mathscr{K}^m$, is also symmetric on $\overline{\mathbf{B}} \times \overline{\mathbf{B}} \setminus \Delta$ and continuous in each variable separately. Although not needed in this paper, we remark that $\mathscr{K}^m$ is actually continuous on $\overline{\mathbf{B}} \times \overline{\mathbf{B}} \setminus \Delta$. We remark that the dilation-commuting property as in (1.6) is no longer true for $\mathscr{K}^m$.

Given φ as in (1.2) and $x \in \{c, h, m, p\}$, using the extended kernels introduced above, we now define the functions

$$\Lambda^x_{\varphi,1}(z) := \left\| \mathscr{K}^x\big(\varphi(\cdot), z\big) \right\|_1, \quad z \in \overline{\mathbf{B}},$$

and, for $1 < t < \infty$,

$$\Lambda^x_{\varphi,t}(z) := \begin{cases} \frac{\|\mathscr{K}^x(\varphi(\cdot), z)\|_t}{\|K^x(z, \cdot)\|_t} & \text{if } z \in \mathbf{B} \\ 0 & \text{if } z \in \mathbf{S}. \end{cases}$$

Note that these functions are well defined, because each $\mathscr{K}^x(\varphi(\cdot), z)$ with $z \in \mathbf{S}$ is a Borel function defined on $\mathbf{S}$ off the set $\varphi^{-1}\{z\}$ of σ-measure 0. The definition of $\Lambda^x_{\varphi,t}$ for $1 < t < \infty$ requiring to be 0 on the boundary may seem peculiar. We define it in this way only for the purpose of stating the next theorem in a unified way.

Theorem 1.3. *Let $x \in \{c, h, m, p\}$, $1 \le t < \infty$, and assume φ satisfies (1.2). Then the following statements hold:*

(a) *C^x_φ is bounded on $L^t(\mathbf{S})$ if and only if $\Lambda^x_{\varphi,t}$ is bounded on $\mathbf{B}$;*
(b) *C^x_φ is compact on $L^t(\mathbf{S})$ if and only if $\Lambda^x_{\varphi,t} \in C(\overline{\mathbf{B}})$.*

Note that the restriction of $\Lambda^x_{\varphi,t}$ to $\mathbf{B}$ is always continuous and hence, when $1 < t < \infty$, the statement that $\Lambda^x_{\varphi,t} \in C(\overline{\mathbf{B}})$ is equivalent to $\Lambda^x_{\varphi,t}(z) \to 0$ as $|z| \to 1^-$. Thus Condition (b) in Theorem 1.3 can be viewed as a "little-oh" version of Condition (a). Our proof for the boundedness characterization in the theorem above actually yields norm estimates; see Propositions 3.1 and 3.5. Also, one can easily recover Sarason's one-variable characterization for L^1-compactness; see Corollary 4.9.

When $1 < t < \infty$, Carleson measure methods are available and provide alternate characterizations of when C^x_φ is bounded or compact on $L^t(\mathbf{S})$; see Section 2.3. Carleson measure methods do not provide a characterization of when C^x_φ to be bounded or compact on $L^1(\mathbf{S})$, but can be used to establish relationships with the operators on $L^t(\mathbf{S})$.

Theorem 1.4. *Let $x \in \{c, h, m, p\}$ and φ be as in (1.2). Then the following statements hold:*

(a) *If C_φ^x is bounded (respectively compact) on $L^1(\mathbf{S})$, then C_φ^x is bounded (compact) on $L^t(\mathbf{S})$ for all $t \in (1, \infty)$;*
(b) *For each $x \in \{c, h, m, p\}$ there exists φ such that C_φ^x is compact on $L^t(\mathbf{S})$, $1 < t < \infty$, but C_φ^x is not bounded on $L^1(\mathbf{S})$.*

Finally, we remark that the Poisson kernel K^h for the unit ball of the Euclidean space of real dimension d is given by

$$K^h(\xi, \eta) = \frac{1 - |\xi|^2}{|\xi - \eta|^d}$$

for ξ and η in $\mathbf{R}^d$ with $|\xi| < 1$ and $|\eta| = 1$. Our results for the operator C_φ^h have natural formulations in this setting, and remain valid with the same proofs. This comment does not extend to the operators C_φ^x for $x \in \{c, m, p\}$, due to the appearance of the Hermitian inner product $\langle \cdot, \cdot \rangle$ in the corresponding kernels.

The rest of the paper is organized as follows: Section 2 is a collection of some general background material that we need, along with some immediate consequences for the operators we consider. In particular, the Carleson measure methods available when $1 < t < \infty$ are used to establish relations between boundedness or compactness of C_φ^x and that of C_φ^y on $L^t(\mathbf{S})$; see Theorem 2.3. Section 3 has proofs of our characterizations using $\Lambda_{\varphi,t}^x$ of when the operators are bounded, and applications including the proof of Theorem 1.4(a). Results concerning compactness of the operators are then established in Section 4. The focus of Section 5 is the action of the operators on $L^1(\mathbf{S})$. We show that most, but not all, of the relations between when C_φ^x or C_φ^y is bounded or compact on $L^t(\mathbf{S})$, $1 < t < \infty$, fail when $t = 1$ and Carleson measure methods are not available.

Constants. Throughout the paper we use the same letter C to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be sometimes specified in parentheses. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \leq CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Background

In this section we collect some basic notions and related facts to be used in our proofs.

2.1. Norm estimates for the reproducing kernels

We first recall the well-known integral estimates related to the reproducing kernels under consideration. Given α real, put

$$I_\alpha(z) = \int_{\mathbf{S}} \frac{d\sigma(\zeta)}{|1 - \langle z, \zeta \rangle|^{n+\alpha}} \quad \text{and} \quad J_\alpha(z) = \int_{\mathbf{S}} \frac{d\sigma(\zeta)}{|z - \zeta|^{2n-1+\alpha}}$$

for $z \in \mathbf{B}$. The growth estimates for these integrals are well known:

$$I_\alpha(z) \approx J_\alpha(z) \approx \begin{cases} (1-|z|^2)^{-\alpha} & \text{if } \alpha > 0 \\ 1 + \log(1-|z|^2)^{-1} & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha < 0 \end{cases} \tag{2.1}$$

for $z \in \mathbf{B}$. Proofs can be found, for example, in [15, Theorem 1.12] and [3, Lemma 2.4] for I_α and J_α, respectively.

As an immediate consequence of (2.1), we have the following norm estimates for reproducing kernels for $1 < t < \infty$:

$$\left\|K^x(z,\cdot)\right\|_t^t \approx \begin{cases} (1-|z|^2)^{n(1-t)} & \text{if } x = c, m \\ (1-|z|^2)^{(2n-1)(1-t)} & \text{if } x = h \end{cases} \tag{2.2}$$

for $z \in \mathbf{B}$. Also, we have

$$\left\|K^c(z,\cdot)\right\|_1 \approx 1 + \log\bigl(1-|z|^2\bigr)^{-1},$$

but $\|K^x(z,\cdot)\|_1 = 1$ for $x = m, h$. For $x = p$, since $|K^p(z,\cdot)| \lesssim |K^c(z,\cdot)|$, we have, for each $1 \le t < \infty$,

$$\left\|K^p(z,\cdot)\right\|_t \lesssim \left\|K^c(z,\cdot)\right\|_t. \tag{2.3}$$

When $1 < t < \infty$, by the Korányi–Vagi Theorem (see [10, Theorem 6.3.1]) asserting that the Cauchy transform followed by the Korányi maximal function is L^t-bounded, there is a constant $C = C(t,n) > 0$ such that

$$\|f\|_t \le C\|\operatorname{Re} f\|_t \tag{2.4}$$

for functions $f \in H^t(\mathbf{S})$ with $\operatorname{Im} f(0) = 0$; see [10, Section 7.1.7]. So, the estimate in (2.3) can be reversed for $1 < t < \infty$; see also Lemma 3.4 below. We remark in passing that the reverse estimate of (2.3) is also valid when $t = 1$ and $n \ge 2$, as can be seen by using (3.2) to convert integration over the sphere to a weighted integral over the unit disk, and then that harmonic conjugation is L^1-bounded on the standard weighted Bergman spaces of the unit disk.

2.2. Normal family argument

The term *normal family* refers to a family of functions with the property that every sequence in the family contains a subsequence converging uniformly on compact subsets of the domain. As is well known, a family of holomorphic functions that is uniformly bounded on each compact subset of the domain is a normal family. An argument using that result is often called a normal family argument. Such a normal family argument extends to harmonic functions and hence to pluriharmonic functions; see [1, Theorem 2.6]. We failed to locate a reference for the invariant harmonic case, which is not clear to us. However, the following is enough for our purpose. The cases $x = c, p, h$ are also included in the statement for easier reference later.

Lemma 2.1. *Let $x \in \{c, h, m, p\}$. Given a bounded set $\mathcal{F}$ in $L^1(\mathbf{S})$, let $\mathcal{F}^x := \{f^x : f \in \mathcal{F}\}$. Then $\mathcal{F}^x$ is a normal family on $\mathbf{B}$.*

Proof. The cases $x = c, p, h$ are easily seen from the remark above. To treat the case $x = m$, we first introduce some notation. Given $z \in \mathbf{B}$, let τ_z be the involutive automorphism of $\mathbf{B}$ that exchanges 0 and z. It is known that

$$f^m \circ \tau_z = (f \circ \tau_z)^m, \quad f \in L^1(\mathbf{S}); \tag{2.5}$$

see [10, Theorem 3.3.8]. Also, $\rho(z, w) := |\tau_a(b)|$ is known to be a metric, called the pseudohyperbolic metric, on $\mathbf{B}$; see [15, Corollary 1.22].

Let $E \subset \mathbf{B}$ be a compact set. We claim there is a constant $C = C(E) > 0$ such that

$$\left|f^m(a) - f^m(b)\right| \le C\rho(a, b)\|f\|_1, \quad a, b \in E \tag{2.6}$$

for all $f \in L^1(\mathbf{S})$. With this granted, we see that $\mathcal{F}^m$ is equicontinuous on each compact subset, which is the key to the proof; the lemma follows then from the standard argument using the Arzela–Ascoli Theorem and the diagonal process.

Let $f \in L^1(\mathbf{S})$. Since

$$\left|f^m(0) - f^m(z)\right| \le \|f\|_1 \sup_{\eta\in\mathbf{S}} \left|1 - \left(\frac{1 - |z|^2}{|1 - \langle z, \eta\rangle|^2}\right)^n\right| \lesssim \frac{|z|}{(1 - |z|)^n}\|f\|_1$$

for $z \in \mathbf{B}$, we see that

$$\left|f^m(0) - f^m(z)\right| \le C_1|z|\|f\|_1, \quad z \in E$$

for some constant $C_1 = C_1(E) > 0$. Now, given $a, b \in E$, we have by (2.5)

$$\left|f^m(a) - f^m(b)\right| = \left|(f \circ \tau_a)^m(0) - (f \circ \tau_a)^m\big(\tau_a(b)\big)\right| \le C_1\rho(a, b)\|f \circ \tau_a\|_1$$

Meanwhile, we have again by (2.5)

$$\|f \circ \tau_a\|_1 = |f \circ \tau_a|^m(0) = |f|^m(a) \le \left(\frac{1 + |a|}{1 - |a|}\right)^n \|f\|_1 \le C_2\|f\|_1$$

for some constant $C_2 = C_2(E) > 0$. Combining these observations, we conclude (2.6), as claimed. The proof is complete. $\square$

2.3. Carleson measures

We recall the notions of Carleson measures that are needed in our work. Let $1 < t < \infty$ and $x \in \{c, h, m, p\}$. Let μ be a positive finite Borel measure on $\overline{\mathbf{B}}$. We say that μ is an *x-Carleson measure for $L^t(\mathbf{S})$* if there exists some constant $C > 0$ such that

$$\int_{\overline{\mathbf{B}}} \left|f^x\right|^t d\mu \le C \int_{\mathbf{S}} \left|f^x\right|^t d\sigma, \quad f \in L^t(\mathbf{S}). \tag{2.7}$$

That is, μ is an x-Carleson measure for $L^t(\mathbf{S})$ if and only if the mapping $f^x|_{\mathbf{S}} \mapsto f^x$ is continuous from $L^t(\mathbf{S})$ to $L^t(\mu)$. We write $N^x(\mu)$ for the infimum of the constants C for which inequality (2.7) holds, so $[N^x(\mu)]^{1/t}$ is the norm of this mapping. If, in addition, this mapping is compact, then μ is said to be a *compact* x-Carleson measure for $L^t(\mathbf{S})$. Characterizations for (compact) x-Carleson measures for $L^t(\mathbf{S})$ are given in terms of Carleson sets that are balls defined using a metric appropriate for the kernel K^x.

For $\zeta \in \mathbf{S}$ and $0 < \delta < 1$, let

$$S^x(\zeta, \delta) = \left\{z \in \overline{\mathbf{B}} : \left|1 - \langle z, \zeta\rangle\right| < \delta\right\}, \quad x \in \{c, m, p\},$$

and

$$S^h(\zeta, \delta) = \left\{z \in \overline{\mathbf{B}} : |z - \zeta| < \delta\right\}.$$

Now, we put

$$M^x_\delta(\mu) := \sup_\zeta \frac{\mu[S^x(\zeta, \delta)]}{\sigma[S^x(\zeta, \delta) \cap \mathbf{S}]}.$$

Note that c-Carleson measures for $L^t(\mathbf{S})$ are precisely the well-known Carleson measures for $H^t(\mathbf{B})$ and that the two notions of x-Carleson measures for $x \in \{c, p\}$ coincide by (2.4). We have the following characterizations for each $1 < t < \infty$:

- μ is an x-Carleson measure for $L^t(\mathbf{S})$ $\iff$ $\sup_\delta M^x_\delta(\mu) < \infty$;
- μ is a compact x-Carleson measure for $L^t(\mathbf{S})$ $\iff$ $M^x_\delta(\mu) \to 0$ as $\delta \to 0^+$.

A reference for the case $x \in \{c, p\}$, i.e. for Carleson measures for $H^t(\mathbf{B})$, is [4, Theorem 2.37]. While we have not been able to find a reference for the characterization of m-Carleson measures, it should be well known that they also coincide with the Hardy space Carleson measures. Indeed, in all cases the necessity of the characterizing condition is established using natural test functions and simple estimates of the kernel. The proof of sufficiency in the Hardy space case given in [8] goes through for $x = m$ with almost no change. A comment is needed regarding just one part of the proof: the pointwise estimate of the Korányi maximal function of $f \in H^t(\mathbf{B})$ by the Hardy–Littlewood maximal function of f associated with non-isotropic balls. That this estimate remains valid when $x = m$ is the content of [10, Theorem 5.4.5]. The characterization when $x = h$ can be found in [7] for measures supported on $\mathbf{B}$; the extension to measure supported on $\overline{\mathbf{B}}$ is standard. Alternatively, it can be observed that Euclidean (rather than non-isotropic) versions of the key ingredients of the proof in the Hardy space case are well known.

Moreover, setting $M^x(\mu) := \sup_\delta M^x_\delta(\mu)$, we have

$$M^x(\mu) \approx N^x(\mu). \tag{2.8}$$

Of particular importance is that the characterization of (compact) x-Carleson measures for $L^t(\mathbf{S})$ is independent of $t > 1$, and that the characterization is the same for $x \in \{c, m, p\}$. But the characterization differs for $x = h$ when $n > 1$, since

$$\sigma\big[S^x(\zeta, \delta) \cap \mathbf{S}\big] \approx \begin{cases} \delta^n & \text{if } x \in \{c, m, p\} \\ \delta^{2n-1} & \text{if } x = h. \end{cases} \tag{2.9}$$

When $x = h$ this is elementary. When $x \in \{c, m, p\}$ see, for example, [10, Proposition 5.1.4]. Finally, we note that the restriction $t > 1$ for $x \in \{h, m, p\}$ comes from the same restriction in the L^t-boundedness of the Hardy–Littlewood maximal function as well as in the Korányi–Vagi Theorem mentioned after (2.3), when proving the sufficiency of the characterizing conditions. On the other hand, one may remove the restriction $t > 1$ when $x = c$, considering $f \in H^t(\mathbf{B})$ in (2.7) instead of f^x, and the characterization remains the same for $0 < t \le 1$; see [7, Theorem 4.3].

2.4. Pullback measures

The relevance of Carleson measures to composition operators comes from the idea of *pullback measure*. Associated with φ as in (1.2) is the pullback measure $\sigma \circ \varphi^{-1}$, which is the Borel measure defined for a Borel set $E \subset \overline{\mathbf{B}}$ by $\sigma \circ \varphi^{-1}(E) = \sigma\{\zeta \in S : \varphi(\zeta) \in E\}$. Use of a change-of-variable formula from measure theory ([5, p. 163]) shows that

$$\|C^x_\varphi f\|^t_t = \int_{\mathbf{S}} |f^x \circ \varphi|^t \, d\sigma = \int_{\overline{\mathbf{B}}} |f^x|^t \, d(\sigma \circ \varphi^{-1}) \tag{2.10}$$

for any $f \in L^t(\mathbf{S})$, $1 \le t < \infty$ and $x \in \{c, h, m, p\}$. This gives the following proposition. In what follows, $\|C^x_\varphi\|_{L^t(\mathbf{S})}$ denotes the operator norm of C^x_φ acting on $L^t(\mathbf{S})$.

Lemma 2.2. *Let $x \in \{c, h, m, p\}$, $1 < t < \infty$, and φ be as in (1.2). Then C^x_φ is bounded (respectively compact) on $L^t(\mathbf{S})$ if and only if $\sigma \circ \varphi^{-1}$ is a (compact) x-Carleson measure for $L^t(\mathbf{S})$. Moreover, the operator norm satisfies*

$$\|C^x_\varphi\|^t_{L^t(\mathbf{S})} \approx N^x\big(\sigma \circ \varphi^{-1}\big) \approx M^x\big(\sigma \circ \varphi^{-1}\big);$$

the constants suppressed above depend on x and t, but are independent of φ.

Proof. If $\sigma \circ \varphi^{-1}$ is a Carleson measure, then use of (2.10), (2.7) and Proposition 1.2(e) shows that C_φ^x is bounded on $L^t(\mathbf{S})$ with $\|C_\varphi^x\|_{L^t(\mathbf{S})}^t \lesssim N^x(\sigma \circ \varphi^{-1})$. Conversely, if C_φ^x is bounded on $L^t(\mathbf{S})$ and $f \in L^t(\mathbf{S})$ then

$$\int_{\overline{\mathbf{B}}} |f^x|^t \, d(\sigma \circ \varphi^{-1}) = \|C_\varphi^x f^x\|_t^t \le \|C_\varphi^x\|^t \|f^x\|_t^t,$$

and so $\sigma \circ \varphi^{-1}$ is an x-Carleson measure for $L^t(\mathbf{S})$ and $N^x(\sigma \circ \varphi^{-1}) \le \|C_\varphi^x\|_{L^t(\mathbf{S})}^t$. This, together with (2.8), completes the proof for C_φ^x bounded. We note that the dependence of the constants on x and t comes from the application of Proposition 1.2(e). The proof for C_φ^x compact is similar and so is omitted. $\square$

We mention some immediate consequences of Lemma 2.2.

Theorem 2.3. *Let φ be as in (1.2). Then the following statements hold.*

(a) *If $x \in \{c, h, m, p\}$ and $1 < t_1, t_2 < \infty$, then C_φ^x is bounded (respectively compact) on $L^{t_1}(\mathbf{S})$ if and only if C_φ^x is bounded (compact) on $L^{t_2}(\mathbf{S})$.*
(b) *If $x, y \in \{c, m, p\}$ and $1 < t < \infty$, then C_φ^x is bounded (respectively compact) on $L^t(\mathbf{S})$ if and only if C_φ^y is bounded (compact) on $L^t(\mathbf{S})$.*
(c) *If $x \in \{c, m, p\}$ and $1 < t < \infty$, then $C_\varphi^x : H^1(\mathbf{S}) \to L^1(\mathbf{S})$ is bounded (respectively compact) if and only if C_φ^x is bounded (compact) on $L^t(\mathbf{S})$.*
(d) *If $x \in \{c, m, p\}$, $1 < t < \infty$, and C_φ^h is bounded (respectively compact) on $L^t(\mathbf{S})$, then C_φ^x is bounded (compact) on $L^t(\mathbf{S})$.*

For (a) note that the characterizations of (compact) x-Carleson measures are independent of $1 < t < \infty$. For (b) note that (compact) x-Carleson measures for $x \in \{c, m, p\}$, precisely being the same as those for the Hardy spaces, coincide. For (c) note from (1.5) and (2.10) that $C_\varphi^x : H^1(\mathbf{S}) \to L^1(\mathbf{S})$ is bounded (respectively compact) if and only if $\sigma \circ \varphi^{-1}$ is a (compact) Carleson measure for $H^1(\mathbf{S})$. In case of (c) note

$$\|C_\varphi^x\|_{H^1(\mathbf{S}) \to L^1(\mathbf{S})} \approx N^x(\sigma \circ \varphi^{-1}) \approx \|C_\varphi^x\|_{L^t(\mathbf{S})}^t$$

where $\|C_\varphi^x\|_{H^1(\mathbf{S}) \to L^1(\mathbf{S})}$ denotes the operator norm of $C_\varphi^x : H^1(\mathbf{S}) \to L^1(\mathbf{S})$. For (d) note that C_φ^h is bounded (compact) on $L^t(\mathbf{S})$ if and only if $\sigma \circ \varphi^{-1}$ is a (compact) Carleson measure for the harmonic Hardy space $h^t(\mathbf{B})$, which is isometrically isomorphic to $L^t(\mathbf{S})$ when $t > 1$. Since $H^t(\mathbf{B})$ is isometrically isomorphic to $H^t(\mathbf{S}) \subset L^t(\mathbf{S})$, we see that $\sigma \circ \varphi^{-1}$ is a (compact) Carleson measure for $H^t(\mathbf{S})$, and the result follows as in (b). An example will be presented in Section 5 that shows the converse to (d) fails badly: for $t \ge 1$, there exists φ such that C_φ^x is compact on $L^t(\mathbf{S})$ for $x \in \{c, m, p\}$, but C_φ^h is not bounded.

2.5. Miscellany

We first mention some remarks for holomorphic symbols. So, assume φ is holomorphic in the following three remarks.

(1) In conjunction with (1.5) we note that if the standard composition operator C_φ maps $H^1(\mathbf{B})$ into $H^t(\mathbf{B})$ for some $0 < t < \infty$, then each C_φ^x, when restricted to $H^1(\mathbf{S})$, is precisely the same as C_φ if a Hardy function is identified with its boundary function. To see this, let $f \in H^1(\mathbf{S})$ (or $H^1(\mathbf{B})$) and put $f_r(z) := f(rz)$ for $0 < r < 1$. Then, as $r \to 1^-$, we have $f_r \to f$ in $H^1(\mathbf{B})$ and thus $C_\varphi f_r \to C_\varphi f$ in $H^t(\mathbf{B})$. Also, note that the boundary function of $C_\varphi f_r$ is $f_r \circ \varphi$, which is obvious by the continuity of f_r on $\mathbf{S}$. Thus, by Fatou's Lemma and (1.5), we obtain

$$0 = \lim_{r\to 1^-} \int\limits_{\mathbf{S}} |C_\varphi f_r - C_\varphi f|^t \, d\sigma \geq \int\limits_{\mathbf{S}} \left|C_\varphi^x f - C_\varphi f\right|^t d\sigma,$$

which shows that $C_\varphi^x f$ is the boundary function of $C_\varphi f$, as asserted.

(2) The absolute continuity hypothesis in (1.2) is satisfied if C_φ is bounded on $H^t(\mathbf{B})$ for some/all $0 < t < \infty$; see [4, Corollary 3.38]. Thus (1.2) is satisfied for all holomorphic self-maps of $\mathbf{D}$, which is not the case on multi-dimensional balls (see the function exhibited in the proof of Example 1.1).

(3) We see from (1.5), Lemma 2.2 and [4, Theorem 3.35] that when $x \in \{c, m, p\}$, C_φ^x is bounded (respectively compact) on $L^t(\mathbf{S})$ for some/all $1 < t < \infty$ if and only if C_φ is bounded (compact) on $H^t(\mathbf{B})$ for some/all $0 < t < \infty$.

Finally, we mention an elementary result (see, for example, [9, p. 90] or [6, Lemma 3.17]) from real analysis that will be used repeatedly, following the approach of Sarason in [12].

Lemma 2.4. *Let $f \in L^1(\mathbf{S})$ and $\{f_j\}$ be a sequence of functions in $L^1(\mathbf{S})$ such that $f_j \to f$ σ-a.e. on $\mathbf{S}$. Then $\|f_j\|_1 \to \|f\|_1$ if and only if $\|f_j - f\|_1 \to 0$.*

3. Boundedness

In this section we prove the boundedness parts of our results stated in the Introduction. Proof for the boundedness part of Theorem 1.3 is split in the next two propositions, since they differ when $t = 1$ or $1 < t < \infty$. We first characterize boundedness for the case $t = 1$.

Proposition 3.1. *Let $x \in \{c, h, m, p\}$ and φ be as in (1.2). Then C_φ^x is bounded on $L^1(\mathbf{S})$ if and only if $\Lambda_{\varphi,1}^x$ is bounded on $\mathbf{B}$. Moreover, the operator norm satisfies $\|C_\varphi^x\|_{L^1(\mathbf{S})} = \sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^x(z)$.*

Proof. Let $z \in \mathbf{B}$. For $x \in \{c, h, p\}$ we choose $k_z := K^h(z, \cdot)$ as a test function. Note that $\mathscr{K}^x$ is harmonic on $\mathbf{B}$ in each variable separately. So, from the reproducing property of k_z, we see that

$$\begin{aligned}(k_z)^x(w) &= \int_S K^h(z,\eta)K^x(w,\eta)\,d\sigma(\eta), \quad w \in \mathbf{B} \\ &= \int_S K^h(z,\eta)\mathscr{K}^x(w,\eta)\,d\sigma(\eta) \\ &= \mathscr{K}^x(w,z),\end{aligned}$$

and so

$$C_\varphi^x k_z = (k_z)^x \circ \varphi = \mathscr{K}^x\big(\varphi(\cdot), z\big).$$

Since $\|k_z\|_1 = 1$, integration on $\mathbf{S}$ shows that $\|C_\varphi^x\|_{L^1(\mathbf{S})} \geq \|\mathscr{K}^x(\varphi(\cdot), z)\|_1$, $z \in \mathbf{B}$. Thus we conclude

$$\big\|C_\varphi^x\big\|_{L^1(\mathbf{S})} \geq \sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^x(z).$$

This inequality also holds for $x = m$, with the same proof except for choosing $k_z := K^m(z, \cdot)$ as a test function in this case.

We now prove the reverse inequality. Let $f \in L^1(\mathbf{S})$ and assume f^x is defined at $\varphi(\zeta)$, $\zeta \in \mathbf{S}$. For $x \neq m$ note from (1.4) and (1.6)

$$\big(f^x \circ \varphi\big)(\zeta) = \lim_{r\to 1^-} \int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(\varphi(\zeta), r\eta\big)\,d\sigma(\eta). \tag{3.1}$$

This remains valid for $x = m$, even though (1.6) is no longer true in that case. In fact, when $\varphi(\zeta) \in \mathbf{S}$, the above is certainly true by (1.1). On the other hand, when $\varphi(\zeta) \in \mathbf{B}$, we have

$$\begin{aligned}\big(f^x \circ \varphi\big)(\zeta) &= \lim_{r\to 1^-} \int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(r\varphi(\zeta), \eta\big)\,d\sigma(\eta) \\ &= \int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(\varphi(\zeta), \eta\big)\,d\sigma(\eta) \\ &= \int_{\mathbf{S}} f(\eta) \lim_{r\to 1^-} \mathscr{K}^x\big(\varphi(\zeta), r\eta\big)\,d\sigma(\eta) \\ &= \lim_{r\to 1^-} \int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(\varphi(\zeta), r\eta\big)\,d\sigma(\eta);\end{aligned}$$

the second and the last equalities hold by the Dominated Convergence Theorem and the third equality holds by the continuity of $\mathscr{K}^x(\varphi(\zeta), \cdot)$ on $\overline{\mathbf{B}}$.

So, for any $x \in \{c, h, m, p\}$, we have by (3.1), Fatou's Lemma and Fubini's Theorem

$$\begin{aligned} \left\|C_\varphi^x f\right\|_1 &= \int_{\mathbf{S}} \left|\left(f^x \circ \varphi\right)(\zeta)\right| d\sigma(\zeta) \\ &\le \liminf_{r \to 1^-} \int_{\mathbf{S}} \int_{\mathbf{S}} \left|f(\eta)\right| \left|\mathscr{K}^x\left(\varphi(\zeta), r\eta\right)\right| d\sigma(\eta)\, d\sigma(\zeta) \\ &\le \|f\|_1 \sup_{z \in \mathbf{B}} \Lambda_{\varphi,1}^x(z) \end{aligned}$$

and thus conclude

$$\left\|C_\varphi^x\right\|_{L^1(\mathbf{S})} \le \sup_{z \in \mathbf{B}} \Lambda_{\varphi,1}^x(z),$$

which completes the proof. □

For the proof below (and later use), we recall the following slice integration formula for $n > 1$ (see [10, Section 1.4.5] or [15, Lemma 1.10]):

$$\int_{\mathbf{S}} \psi\left(\langle \eta, \xi \rangle\right) d\sigma(\eta) = \frac{n-1}{\pi} \int_{\mathbf{D}} \psi(\lambda)\left(1 - |\lambda|^2\right)^{n-2} dA(\lambda) \tag{3.2}$$

for any positive measurable function ψ on $\mathbf{D}$ and $\xi \in \mathbf{S}$. Here, A denotes the area measure on $\mathbf{D}$.

Corollary 3.2. *If $n \ge 1$ and C_φ^c is bounded on $L^1(\mathbf{S})$, then $\sigma[\varphi^{-1}(\mathbf{S})] = 0$. If $n \ge 2$ and C_φ^p is bounded on $L^1(\mathbf{S})$, then $\sigma[\varphi^{-1}(\mathbf{S})] = 0$.*

We remark that the statement for C_φ^p does not extend to $n = 1$. For example, with id denoting the identity map of $\mathbf{S}$, note that $C_{id}^p = C_{id}^h$ is the identity operator on $L^1(\mathbf{S})$ in the one-dimensional case.

Proof. It is easily seen from Fatou's Lemma that $\Lambda_{\varphi,1}^c(\eta) \le \sup_{z \in \mathbf{B}} \Lambda_{\varphi,1}^c(z)$ for all $\eta \in \mathbf{S}$. Thus we have

$$\begin{aligned} \sup_{z \in \mathbf{B}} \Lambda_{\varphi,1}^c(z) &\ge \int_{\mathbf{S}} \Lambda_{\varphi,1}^c(\eta)\, d\sigma(\eta) \\ &= \int_{\mathbf{S}} \int_{\mathbf{S}} \frac{d\sigma(\eta)}{|1 - \langle \varphi(\zeta), \eta \rangle|^n}\, d\sigma(\zeta) \\ &\ge \int_{\varphi^{-1}(\mathbf{S})} \int_{\mathbf{S}} \frac{d\sigma(\eta)}{|1 - \langle \varphi(\zeta), \eta \rangle|^n}\, d\sigma(\zeta). \end{aligned}$$

Note that the inner integral of the above diverges for each $\zeta \in \varphi^{-1}(\mathbf{S})$. This is elementary when $n = 1$; when $n \geq 2$ it is easily seen using (3.2). So, the result for $x = c$ is a consequence of Proposition 3.1.

The proof for $x = p$ is similar:

$$\begin{aligned}\sup_{z\in\mathbf{B}} \Lambda^p_{\varphi,1}(z) + 1 &\geq \int_{\mathbf{S}} \big(\Lambda^p_{\varphi,1}(\eta) + 1\big)\, d\sigma(\eta)\\ &\geq \int_{\varphi^{-1}(\mathbf{S})} \int_{\mathbf{S}} \frac{|2\operatorname{Re}(1-\langle\varphi(\zeta),\eta\rangle)^n|}{|1-\langle\varphi(\zeta),\eta\rangle|^{2n}}\, d\sigma(\eta)\, d\sigma(\zeta).\end{aligned}$$

Since $n \geq 2$ and $\varphi(\zeta) \in \mathbf{S}$, (3.2) is available to compute the inner integral to be

$$\int_{\mathbf{S}} \frac{|2\operatorname{Re}(1-\langle\varphi(\zeta),\eta\rangle)^n|}{|1-\langle\varphi(\zeta),\eta\rangle|^{2n}}\, d\sigma(\eta) = \frac{2(n-1)}{\pi} \int_{\mathbf{D}} \frac{|\operatorname{Re}(1-\lambda)^n|}{|1-\lambda|^{2n}} \big(1-|\lambda|^2\big)^{n-2}\, dA(\lambda).$$

This integral can be seen to diverge by using polar coordinates centered at $\lambda = 1$ and integrating over a small sector, and the result again holds by Proposition 3.1. □

Now, we turn to the case $1 < t < \infty$, where some auxiliary estimates are needed. First, we need the following estimate as to how the kernels grow on certain Carleson sets.

Lemma 3.3. *Let $\zeta_0 \in \mathbf{S}$, $\delta \in (0,1)$ and put $z = (1-\delta)\zeta_0$. Then there are constants $C_1 = C_1(n) > 0$ and $c_2 = c_2(n) > 0$ such that*

$$\big|\mathscr{K}^x(z,w)\big| \geq C_1 \times \begin{cases} \delta^{-n} & \text{if } x = c, m, p\\ \delta^{-(2n-1)} & \text{if } x = h\end{cases}$$

for $w \in S^x(\zeta_0, c_2\delta)$.

Before the proof, we remark that we can take $c_2 = 1$ when $x \in \{c, h, m\}$. It is when $x = p$ and $n > 1$ that $0 < c_2 < 1$ is necessary.

Proof. For $x \neq p$ the proof is a straightforward estimate using the explicit formula for $\mathscr{K}^x(w,z)$ or (1.7), and will be omitted.

For $x = p$, assume first that $\delta \in (0, 1/16)$. Choose $c_2 \in (0,1)$, depending only on n, so small that $\operatorname{Re}(a^n) \geq 1/2$ for all a lying in the disk with center at 1 and radius c_2. Let $w \in S^p(\zeta_0, c_2\delta)$ and put $\lambda = \langle\zeta_0, w\rangle$. Note

$$\left|1 - \frac{1-(1-\delta)\lambda}{\delta}\right| = \frac{(1-\delta)|1-\lambda|}{\delta} < c_2,$$

which means that $\frac{1-(1-\delta)\lambda}{\delta}$ lies in the disk with center at 1 and radius c_2. Now, since $|1-(1-\delta)\lambda| \le 2\delta$ and $\delta < 1/16$, we obtain

$$\mathscr{K}^p(z,w) = \frac{2\,\mathrm{Re}[1-(1-\delta)\lambda]^n}{|1-(1-\delta)\lambda|^{2n}} - 1 \ge \frac{1}{(2\delta)^{2n}} \cdot \frac{2\delta^n}{2} - 1 \ge \frac{\delta^{-n}}{4^{n+1}},$$

which completes the proof when $\delta \in (0, 1/16)$. The extension to $\delta \in (0,1)$ can be accomplished by replacing c_2 by $c_2/16$. The proof is complete. $\square$

We have the optimal norm estimate (2.2) for the reproducing kernels except for the pluriharmonic case. In the pluriharmonic case, we have an upper estimate (2.3) for $x = p$. What is needed here is the lower estimate for $1 < t < \infty$. We do not know a reference and thus a proof is provided below. Other cases are restated for easier reference.

Lemma 3.4. *Given $1 < t < \infty$, the estimate*

$$\left\|K^x(z,\cdot)\right\|_t^t \approx \begin{cases} (1-|z|^2)^{n(1-t)} & \text{if } x = c,p,m \\ (1-|z|^2)^{(2n-1)(1-t)} & \text{if } x = h \end{cases}$$

holds for $z \in \mathbf{B}$.

Proof. We only need to establish the lower estimate for $x = p$. Let $z \in \mathbf{B}$, $z \ne 0$, put $\zeta_0 = z/|z|$ and set $E_z := S^p(z/|z|, c_2(1-|z|)) \cap \mathbf{S}$, where c_2 is the constant provided by Lemma 3.3. Note that $z = (1-\delta)\zeta_0$ where $\delta = 1-|z|$. Thus by Lemma 3.3 we have

$$\left|K^p(z,\zeta)\right| \gtrsim \left(1-|z|^2\right)^{-n}, \quad \zeta \in E_z$$

so that

$$\left\|K^p(z,\cdot)\right\|_t^t \ge \int_{E_z} \left|K^p(z,\zeta)\right|^t d\sigma(\zeta) \gtrsim \frac{\sigma(E_z)}{(1-|z|^2)^{nt}} \approx \left(1-|z|^2\right)^{n(1-t)},$$

which completes the proof. $\square$

We are now ready to characterize boundedness for the case $1 < t < \infty$.

Proposition 3.5. *Let $x \in \{c,h,m,p\}$, $1 < t < \infty$ and φ be as in (1.2). Then C_φ^x is bounded on $L^t(\mathbf{S})$ if and only if $\varLambda_{\varphi,t}^x$ is bounded on $\mathbf{B}$. Moreover, the operator norm satisfies*

$$\left\|C_\varphi^x\right\|_{L^t(\mathbf{S})} \approx \sup_{z\in\mathbf{B}} \varLambda_{\varphi,t}^x(z);$$

the constants suppressed above depend on x and t, but are independent of φ.

Proof. Fix any $x \in \{c, h, m, p\}$, let $z \in \mathbf{B}$ and choose $k_z = \overline{K^x(z, \cdot)}$ as a test function. Then

$$(k_z)^x(w) = \int_S \overline{K^x(z,\eta)} K^x(w,\eta)\, d\sigma(\eta) = \mathscr{K}^x(w,z), \quad w \in \mathbf{B}$$

and so

$$C_\varphi^x k_z = (k_z)^x \circ \varphi = \mathscr{K}^x\big(\varphi(\cdot), z\big).$$

Hence

$$\big\|C_\varphi^x\big\|_{L^t(\mathbf{S})} \geq \frac{\|\mathscr{K}^x(\varphi(\cdot), z)\|_t}{\|K^x(z, \cdot)\|_t} = \Lambda_{\varphi,t}^x(z)$$

and this is true for any $z \in \mathbf{B}$. Taking the supremum over $z \in \mathbf{B}$, we obtain

$$\big\|C_\varphi^x\big\|_{L^t(\mathbf{S})} \geq \sup_{z \in \mathbf{B}} \Lambda_{\varphi,t}^x(z).$$

For the reverse inequality, let $\zeta_0 \in \mathbf{S}$, $\delta \in (0, 1)$ and put $w = (1 - \delta)\zeta_0$. First, consider the case $x \in \{c, m, p\}$. Using Lemmas 3.3 and 3.4, we see that

$$\begin{aligned}
\big[\Lambda_{\varphi,t}^x(w)\big]^t &\approx \big(1 - |w|\big)^{n(t-1)} \int_{\mathbf{S}} \big|\mathscr{K}^x\big(\varphi(\cdot), w\big)\big|^t\, d\sigma(\zeta) \\
&\gtrsim \delta^{n(t-1)} \int_{\varphi^{-1}[S^x(\zeta_0, c_2\delta)]} \delta^{-nt}\, d\sigma(\zeta) \\
&= \frac{(\sigma \circ \varphi^{-1})[S^x(\zeta_0, c_2\delta)]}{\delta^n}.
\end{aligned}$$

This estimate is independent of ζ_0 and δ, so taking the supremum yields

$$\sup_{z \in \mathbf{B}} \big[\Lambda_{\varphi,t}^x(z)\big]^t \gtrsim M^x\big(\sigma \circ \varphi^{-1}\big).$$

Hence $\sigma \circ \varphi^{-1}$ is an x-Carleson measure with the norm estimate $N^x(\sigma \circ \varphi^{-1}) \approx M^x(\sigma \circ \varphi^{-1}) \lesssim \|\Lambda_{\varphi,t}^x\|_\infty^t$. Since $\|C_\varphi^x\|_{L^t(\mathbf{S})}^t \approx N(\sigma \circ \varphi^{-1})$ from Lemma 2.2, we conclude $\|C_\varphi^x\|_{L^t(\mathbf{S})} \lesssim \sup_{z \in \mathbf{B}} \Lambda_{\varphi,t}^x(z)$, which completes the proof for $x \in \{c, m, p\}$.

The proof when $x = h$ is similar, using the norm estimate for $\|K^h(z, \cdot)\|_t^t$ from (2.2) and the lower bound for $K^h(w, z)$, $w \in S^h(\zeta_0, c_2\delta)$, from Lemma 3.3. □

As an application we now show that L^1-boundedness implies L^t-boundedness for each $1 < t < \infty$, which is the content of the boundedness part of Theorem 1.4(a). In view of this result, one may wonder whether its converse would hold. The answer turns out to be *no*; see Proposition 4.8 in the next section.

Proposition 3.6. *Let $x \in \{c, h, m, p\}$, $1 < t < \infty$ and φ be as in (1.2). If C_φ^x is bounded on $L^1(\mathbf{S})$, then C_φ^x is bounded on $L^t(\mathbf{S})$. Moreover, the operator norms satisfy $\|C_\varphi^x\|_{L^t(\mathbf{S})}^t \le C\|C_\varphi^x\|_{L^1(\mathbf{S})}$ for some constant $C = C(x, t) > 0$.*

Proof. Since L^1-boundedness of C_φ^x implies the boundedness of $C_\varphi^x : H^1(\mathbf{S}) \to L^1(\mathbf{S})$, the case $x \neq h$ is contained in Theorem 2.3(c). So, let $x = h$. Suppose C_φ^h is bounded on $L^1(\mathbf{S})$. By Lemma 2.2, to show C_φ^h is bounded on $L^t(\mathbf{S})$, it suffices to show that $\sigma \circ \varphi^{-1}$ is an h-Carleson measure. Given $\zeta_0 \in \mathbf{S}$ and $\delta \in (0, 1)$, put $w = (1 - \delta)\zeta_0$. Then we see from Lemma 3.3 that

$$\Lambda_{\varphi,1}^h(w) \ge \int\limits_{\varphi^{-1}[S^h(\zeta_0, c_2\delta)]} \left|\mathscr{K}^h\big(\varphi(\zeta), w\big)\right| d\sigma(\zeta) \gtrsim \frac{(\sigma \circ \varphi^{-1})[S^h(\zeta_0, c_2\delta)]}{\delta^{2n-1}},$$

and this estimate is independent of ζ_0 and $\delta \in (0, 1)$. Taking the supremum over all ζ_0 and δ yields $M^h(\sigma \circ \varphi^{-1}) \lesssim \sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^h(z)$. Now, since $\|C_\varphi^h\|_{L^t(\mathbf{S})}^t \approx M^h(\sigma \circ \varphi^{-1})$ from Lemma 2.2, and $\sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^h(z) = \|C_\varphi^h\|_{L^1(\mathbf{S})}$ from Proposition 3.1, the proof is complete. □

We now close the section with the following remarks.

(1) When $x \in \{h, m\}$, as in the proof of Proposition 4.7 in the next section, the Riesz–Thorin Interpolation Theorem (see [16, Theorem 2.5]) could also be used to prove Proposition 3.6 with norm estimate

$$\left\|C_\varphi^x\right\|_{L^t(\mathbf{S})}^t \le \left\|C_\varphi^x\right\|_{L^1(\mathbf{S})}, \tag{3.3}$$

since in these cases C_φ^x is bounded on $L^\infty(\mathbf{S})$ with operator norm 1. When $x \in \{c, p\}$ this method does not work, as C_φ^x is not bounded on $L^\infty(\mathbf{S})$ in general. To see examples of C_φ^c and C_φ^p which are not bounded on $L^\infty(\mathbf{S})$, simply consider $\varphi = id$. Note that C_{id}^c is the Cauchy transform. As is well known, the Cauchy transform (and hence C_{id}^p as well) is not bounded on $L^\infty(\mathbf{S})$. In fact the Cauchy transform takes $L^\infty(\mathbf{S})$ into the space of functions of bounded mean oscillation with respect to nonisotropic balls; see [15, Theorem 5.15].

(2) One may also derive (3.3) by an elementary argument. In fact, when $x \in \{h, m\}$, note that $\mathscr{K}^x(\varphi(\zeta), r\eta)\, d\sigma(\eta)$ is a probability measure for each $0 < r < 1$. So, given $f \in L^t(\mathbf{S})$, applications of (3.1), Fatou's Lemma and Jensen's Inequality yield

$$\left\|C_\varphi^x f\right\|_t^t \le \liminf_{r\to 1^-} \int\limits_{\mathbf{S}} \int\limits_{\mathbf{S}} \left|f(\eta)\right|^t \mathscr{K}^x\big(\varphi(\zeta), r\eta\big)\, d\sigma(\eta)\, d\sigma(\zeta).$$

Now, computing the ζ-integration first, we obtain

$$\left\|C_\varphi^x f\right\|_t^t \le \|f\|_t^t \sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^x(z) = \|f\|_t^t \left\|C_\varphi^x\right\|_{L^1(\mathbf{S})},$$

which yields (3.3).

4. Compactness

In this section we prove the compactness parts of our results stated in the Introduction. Recall that a linear operator on a Banach space X is said to be compact if any bounded sequence $\{x_j\}$ in X contains a subsequence $\{x_{j_k}\}$ for which Tx_{n_k} converges in X.

As in the case of boundedness, proof for the compactness part of Theorem 1.3 is split in the two Propositions 4.1 and 4.6 below. This time the case $1 < t < \infty$ is easier to handle and so we first characterize compactness for that case.

Proposition 4.1. *Let $x \in \{c, h, m, p\}$, $1 < t < \infty$ and φ be as in (1.2). Then C^x_φ is compact on $L^t(\mathbf{S})$ if and only if $\Lambda^x_{\varphi,t} \in C(\overline{\mathbf{B}})$.*

Proof. We first prove the necessity. Fix any $x \in \{c, h, m, p\}$ and suppose C^x_φ is compact on $L^t(\mathbf{S})$. Note that $\Lambda^x_{\varphi,t}$ is clearly continuous on $\mathbf{B}$ and was defined to be 0 on $\mathbf{S}$. So, in order to see $\Lambda^x_{\varphi,t} \in C(\overline{\mathbf{B}})$, it suffices to show that

$$\Lambda^x_{\varphi,t}(z) \to 0 \quad \text{as } |z| \to 1^-. \tag{4.1}$$

Given $z \in \mathbf{B}$, put $f_z := \mathscr{K}^x(\cdot, z)/\|K^x(z,\cdot)\|_t$ so that $\|f_z\|_t = 1$ and $\Lambda^x_{\varphi,t}(z) = \|C^x_\varphi f_z\|_t$. Now, suppose that (4.1) fails to hold. Then one can find an $\epsilon > 0$ and a sequence $\{z^j\} \subset \mathbf{B}$ such that z^j convergent to a boundary point, say $\eta_0 \in \mathbf{S}$, and

$$\left\|C^x_\varphi f_{z^j}\right\|_t \ge \epsilon > 0 \tag{4.2}$$

for all j. Since C^x_φ is compact, we may assume, by passing to a subsequence if necessary, that $\{C^x_\varphi f_{z^j}\}$ is norm convergent in $L^t(\mathbf{S})$. On the other hand, note from Lemma 3.4 that $C^x_\varphi f_{z^j} = \mathscr{K}^x(\varphi(\cdot), z^j)/\|K^x(z^j,\cdot)\|_t \to 0$ pointwise as $j \to +\infty$ on $\mathbf{S} \setminus \varphi^{-1}(\eta_0)$, and hence σ-a.e. on $\mathbf{S}$ by (1.2). Hence $C^x_\varphi f_{z^j} \to 0$ in norm, which contradicts (4.2). Hence (4.1) holds, and the proof of the necessity is complete.

Now, to prove the sufficiency, let $x \in \{c, p, m\}$ and assume $\Lambda^x_{\varphi,t} \in C(\overline{\mathbf{B}})$. Given $\zeta_0 \in \mathbf{S}$ and $\delta \in (0, 1)$, put $z := (1 - \delta)\zeta_0$ so that $1 - |z| = \delta$. Then

$$\begin{aligned}\frac{\|\mathscr{K}^x(\varphi(\cdot), z)\|_t^t}{\|K^x(z,\cdot)\|_t^t} &\ge \frac{1}{\|K^x(z,\cdot)\|_t^t} \int\limits_{\varphi^{-1}(S^x(\zeta_0,c_2\delta))} \left|\mathscr{K}^x\big(\varphi(\zeta), z\big)\right|^t d\sigma(\zeta) \\ &\gtrsim \frac{\delta^{-nt}\sigma[\varphi^{-1}(S^x(\zeta_0, c_2\delta))]}{\delta^{-nt+n}}\end{aligned}$$

where the last inequality holds by Lemma 3.3 and (2.2). Since

$$\frac{\|\mathscr{K}^x(\varphi(\cdot), z)\|_t}{\|K^x(z,\cdot)\|_t} = \Lambda^x_{\varphi,t}(z) \to 0$$

as $|z| \to 1^-$ by continuity of $\Lambda^x_{\varphi,t}$, we conclude by Lemma 2.2 that C^x_φ is compact on $L^t(\mathbf{S})$. The argument for $x = h$, using the alternate lower bounds provided by Lemmas 3.3 and 3.4, is similar. This completes the proof of the sufficiency and thus of the proposition. □

We now turn to the compactness characterization for the case $t = 1$. We need some preliminary lemmas.

Lemma 4.2. *Let $x \in \{c, h, m, p\}$ and φ be as in (1.2). Assume, in addition, φ takes $\mathbf{S}$ into a compact subset of $\mathbf{B}$. Then C^x_φ is compact on $L^t(\mathbf{S})$ for each $1 \le t < \infty$.*

Proof. Fix $x \in \{c, h, m, p\}$ and $1 \le t < \infty$. Since $\varphi(\mathbf{S})$ is contained in a compact subset of $\mathbf{B}$, it is easily seen from Lemma 3.4 that $\Lambda^x_{\varphi,t}$ is bounded on $\overline{\mathbf{B}}$. So, C^x_φ is bounded on $L^t(\mathbf{S})$ by Propositions 3.1 and 3.5. Now, using Lemma 2.1, the rest of the proof is a standard normal family argument. □

Lemma 4.3. *Let $x \in \{c, h, m, p\}$ and φ be as in (1.2), and define the function G^x on $\overline{\mathbf{B}} \setminus \{0\}$ by*

$$G^x_\varphi(\nu\eta) := \int_{\mathbf{S}} \left| \mathscr{K}^x\big(\nu\varphi(\zeta), \eta\big) - \mathscr{K}^x\big(\varphi(\zeta), \eta\big) \right| d\sigma(\zeta) \tag{4.3}$$

for $0 < \nu \le 1$ and $\eta \in \mathbf{S}$. Then, for $0 < s < 1$,

$$\left\| C^x_{s\varphi} - C^x_\varphi \right\|_{L^1(\mathbf{S})} \le \limsup_{r \to 1^-} \Big(\sup_{\eta \in \mathbf{S}} \big[G^x_\varphi(rs\eta) + G^x_\varphi(r\eta) \big] \Big).$$

Proof. Fix $0 < s < 1$. Note that $s\varphi$ satisfies (1.2), because $\sigma \circ (s\varphi)^{-1}|_{\mathbf{S}}$ is the zero measure. Given $f \in L^1(\mathbf{S})$, using Fatou's Lemma, we have

$$\begin{aligned} \left\| \big(C^x_{s\varphi} - C^x_\varphi\big) f \right\|_1 &\le \liminf_{r \to 1^-} \int_{\mathbf{S}} \left| \int_{\mathbf{S}} f(\eta) \big[K^x\big(rs\varphi(\zeta), \eta\big) - \mathscr{K}^x\big(r\varphi(\zeta), \eta\big) \big] d\sigma(\eta) \right| d\sigma(\zeta) \\ &\le \|f\|_1 \limsup_{r \to 1^-} \Big(\sup_{\eta \in \mathbf{S}} \big[G^x_\varphi(rs\eta) + G^x_\varphi(r\eta) \big] \Big), \end{aligned}$$

as required. □

We remark that if $x \in \{c, h, p\}$, then (1.6) is available to give

$$G^x_\varphi(\nu\eta) = \int_{\mathbf{S}} \left| \mathscr{K}^x\big(\varphi(\zeta), \nu\eta\big) - \mathscr{K}^x\big(\varphi(\zeta), \eta\big) \right| d\sigma(\zeta).$$

Now use of Lemma 2.4 shows that if $\Lambda^x_{\varphi,1} \in C(\overline{\mathbf{B}})$, then $G^x_\varphi \in C(\overline{\mathbf{B}} \setminus \{0\})$ and vanishes on $\mathbf{S}$. Hence, for $x \in \{c, h, p\}$, $\Lambda^x_{\varphi,1} \in C(\overline{\mathbf{B}})$ implies $\|C^x_{s\varphi} - C^x_\varphi\|_{L^1(\mathbf{S})} \to 0$ as $s \to 1^-$, from Lemma 4.3. Since each $C^x_{s\varphi}$ is compact on $L^1(\mathbf{S})$ by Lemma 4.2, it follows that C^x_φ is as well.

These remarks do not extend to $x = m$, since (1.6) is not available in that case. We will see in Proposition 4.6 below that the extension to $x = m$ is valid, though the proof is much more involved. The next two lemmas will be used in that proof.

Lemma 4.4. *Let φ be as in (1.2). If $\Lambda^m_{\varphi,1} \in C(\overline{\mathbf{B}})$, then $C^m_\varphi : H^1(\mathbf{S}) \to L^1(\mathbf{S})$ is compact.*

Proof. As in the proof of Lemma 4.3, each $s\varphi$, $0 < s < 1$, satisfies (1.2) and $C^m_{s\varphi} : H^1(\mathbf{S}) \to L^1(\mathbf{S})$ is compact by Lemma 4.2. So it suffices to show

$$\left\| C^m_{s\varphi} - C^m_\varphi \right\| \to 0 \quad \text{as } s \to 1^- \tag{4.4}$$

where $\|C^m_{s\varphi} - C^m_\varphi\|$ denote the operator norm acting from $H^1(\mathbf{S})$ into $L^1(\mathbf{S})$.

Let Q^m be the function defined on $\overline{\mathbf{B}} \setminus \{0\}$ by

$$Q^m(\nu\eta) := \int_{\mathbf{S}} \left| \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \eta\big) \right| d\sigma(\zeta),$$

for $0 < \nu \le 1$ and $\eta \in \mathbf{S}$. From the hypothesis that $\Lambda^m_{\varphi,1} \in C(\overline{\mathbf{B}})$ and Lemma 2.4, we see that Q^m is a continuous function vanishing on $\mathbf{S}$. So, given $\epsilon > 0$, we can fix a $\nu \in (0, 1)$ such that

$$\sup_{\eta \in \mathbf{S}} Q^m(\nu\eta) < \epsilon. \tag{4.5}$$

Let $f \in H^1(\mathbf{S})$ and identify it with $f \in H^1(\mathbf{B})$. Let $0 < s < 1$ and put $f_s(z) := f(sz)$.

Note $C^m_{s\varphi} f = f(s\varphi) = f_s(\varphi) = C^m_\varphi f_s$ for each $0 < s < 1$, because f is holomorphic. It follows from (3.1) that

$$\begin{aligned}
\big(C^m_{s\varphi} - C^m_\varphi\big) f(\zeta) &= C^m_\varphi (f_s - f)(\zeta) \\
&= \lim_{r \to 1^-} \int_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta), r\eta\big) \big[f_s(\eta) - f(\eta)\big] \, d\sigma(\eta)
\end{aligned}$$

for σ-almost every $\zeta \in \mathbf{S}$. Thus, by Fatou's Lemma, we have

$$\left\| \big(C^m_{s\varphi} - C^m_\varphi\big) f \right\|_1 \le \liminf_{r \to 1^-} \int_{\mathbf{S}} \left| \int_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta), r\eta\big) \big[f_s(\eta) - f(\eta)\big] \, d\sigma(\eta) \right| d\sigma(\zeta). \tag{4.6}$$

Meanwhile, note from Fubini's Theorem

$$\int_{\mathbf{S}} \int_{\mathbf{S}} \left| \mathscr{K}^m\big(\varphi(\zeta), r\eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \eta\big) \right| \left| f_s(\eta) - f(\eta) \right| d\sigma(\eta) \, d\sigma(\zeta)$$

$$\leq \|f_s - f\|_1 \Big[\sup_{\eta\in\mathbf{S}} Q^m(r\eta)\Big]$$

$$\leq 2\|f\|_1 \Big[\sup_{\eta\in\mathbf{S}} Q^m(r\eta)\Big] \to 0 \quad \text{as } r \to 1^-.$$

Therefore, by the triangle inequality and Fubini's Theorem, for the ν fixed above, we obtain from (4.6)

$$\begin{aligned}\big\|\big(C^m_{s\varphi} - C^m_{\varphi}\big)f\big\|_1 &\leq \int_{\mathbf{S}}\left|\int_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\big[f_s(\eta) - f(\eta)\big]\,d\sigma(\eta)\right| d\sigma(\zeta)\\ &\leq I + II,\end{aligned}$$

where

$$I := \int_{\mathbf{S}} Q^m(\nu\eta)\big|f_s(\eta) - f(\eta)\big|\,d\sigma(\eta) < 2\epsilon\|f\|_1, \tag{4.7}$$

by (4.5), and

$$II := \int_{\mathbf{S}}\left|\int_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta),\nu\eta\big)\big[f_s(\eta) - f(\eta)\big]\,d\sigma(\eta)\right| d\sigma(\zeta).$$

To estimate II, we first note from the reproducing property

$$f(\nu s z) = \int_{\mathbf{S}} \mathscr{K}^m(z,\nu\eta)f_s(\eta)\,d\sigma(\eta), \quad z \in \mathbf{S},$$

because $\mathscr{K}^m(z,\nu\eta) = \overline{K^m(\nu\eta,z)} = K^m(\nu z,\eta)$ by (1.1). This also remains valid for $z \in \mathbf{B}$. To see it, note from Fubini's Theorem, (1.1), and the reproducing property of the kernel that

$$\begin{aligned}\int_{\mathbf{S}} \mathscr{K}^m(z,\nu\eta)f_s(\eta)\,d\sigma(\eta) &= \int_{\mathbf{S}} f_s(\eta)\left\{\int_{\mathbf{S}} K^m(z,\xi)\overline{K^m(\nu\eta,\xi)}\,d\sigma(\xi)\right\} d\sigma(\eta)\\ &= \int_{\mathbf{S}}\left\{\int_{\mathbf{S}} K^m(\nu\xi,\eta)f_s(\eta)\,d\sigma(\eta)\right\}K^m(z,\xi)\,d\sigma(\xi)\\ &= \int_{\mathbf{S}} f_{\nu s}(\xi)K^m(z,\xi)\,d\sigma(\xi)\\ &= f(\nu s z).\end{aligned}$$

A similar argument yields

$$f(\nu s z) = \int_{\mathbf{S}} \mathscr{K}^m(sz, \nu\eta) f(\eta)\, d\sigma(\eta), \quad z \in \overline{\mathbf{B}}.$$

Therefore, we have

$$\int_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big) f_s(\eta)\, d\sigma(\eta) = \int_{\mathbf{S}} \mathscr{K}^m\big(s\varphi(\zeta), \nu\eta\big) f(\eta)\, d\sigma(\eta)$$

at σ-almost every $\zeta \in \mathbf{S}$ and thus by Fubini's Theorem

$$\begin{aligned} II &\le \int_{\mathbf{S}} \int_{\mathbf{S}} \big|\mathscr{K}^m\big(s\varphi(\zeta), \nu\eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big)\big| \cdot \big|f(\eta)\big|\, d\sigma(\eta)\, d\sigma(\zeta) \\ &\le \|f\|_1 \sup_{\eta \in \mathbf{S}} \int_{\mathbf{S}} \big|\mathscr{K}^m\big(s\varphi(\zeta), \nu\eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big)\big|\, d\sigma(\zeta). \end{aligned}$$

Now, since $0 < \nu < 1$, the Dominated Convergence Theorem can be used to see the (uniform) continuity of the mapping $(s, \eta) \mapsto \mathscr{K}^m(s\varphi(\cdot), \nu\eta)$ from $[0,1] \times \mathbf{S}$ to $L^1(\mathbf{S})$, yielding

$$II < \epsilon \|f\|_1,$$

provided s is sufficiently close to 1. Along with (4.7), this shows that $C^m_{s\varphi} - C^m_{\varphi}$ boundedly takes $H^1(\mathbf{S})$ into $L^1(\mathbf{S})$ and, moreover, that (4.4) holds. The proof is complete. □

Lemma 4.5. *Let φ be as in (1.2) and put*

$$\widetilde{G}^m(r\eta) := \int_{\mathbf{S}} \left| \frac{(1 - |\varphi(\zeta)|^2)^n}{|1 - \langle \varphi(\zeta), r\eta\rangle|^{2n}} - \mathscr{K}^m\big(\varphi(\zeta), \eta\big) \right| d\sigma(\zeta)$$

for $0 < r \le 1$ and $\eta \in \mathbf{S}$. If $\Lambda^m_{\varphi,1} \in C(\overline{\mathbf{B}})$, then $\widetilde{G}^m \in C(\overline{\mathbf{B}} \setminus \{0\})$.

Proof. Assume $\Lambda^m_{\varphi,1} \in C(\overline{\mathbf{B}})$. Since $\widetilde{G}^m$ is clearly continuous on $\mathbf{B}\setminus\{0\}$ by the Dominated Convergence Theorem, it suffices to prove $\widetilde{G}^m$ is continuous at every point in $\mathbf{S}$, where $\widetilde{G}^m$ vanishes. Given $s \in (0,1)$ and $\eta \in \mathbf{S}$, we use temporary notation

$$E_s(\eta) := \varphi^{-1}\big[S^m(\eta, s)\big] \cap \mathbf{S}$$

for short.

Given $\eta \in \mathbf{S}$ and $0 < r < 1$, note

$$\frac{(1 - |z|^2)^n}{|1 - \langle z, r\eta\rangle|^{2n}} \le \frac{2^n}{(1 - r)^n}, \quad \text{for } z \in \overline{\mathbf{B}},$$

and

$$\left|\frac{1-\langle z,r\eta\rangle}{1-\langle z,\eta\rangle}\right| \geq 1-\left|\frac{(1-r)\langle z,\eta\rangle}{1-\langle z,\eta\rangle}\right| \geq \frac{1}{2}, \quad \text{for } z \notin S^m\big(\eta, 2(1-r)\big).$$

Therefore, given arbitrary $\eta_0 \in \mathbf{S}$ and $0 < \delta < 1$, we have

$$\begin{aligned}\int\limits_{E_\delta(\eta_0)} \frac{(1-|\varphi(\zeta)|^2)^n}{|1-\langle\varphi(\zeta), r\eta\rangle|^{2n}} d\sigma(\zeta) &= \int\limits_{E_\delta(\eta_0)\setminus E_{2(1-r)(\eta)}} + \int\limits_{E_\delta(\eta_0)\cap E_{2(1-r)(\eta)}} \\ &\lesssim \int\limits_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\, d\sigma(\zeta) + \frac{\sigma[E_{2(1-r)}(\eta)]}{(1-r)^n}.\end{aligned}$$

Thus, setting

$$I(\eta) := \int\limits_{\mathbf{S}} \big|\mathscr{K}^m\big(\varphi(\zeta),\eta\big) - \mathscr{K}^m\big(\varphi(\zeta),\eta_0\big)\big|\, d\sigma(\zeta)$$

we obtain

$$\begin{aligned}&\int\limits_{E_\delta(\eta_0)} \left|\frac{(1-|\varphi(\zeta)|^2)^n}{|1-\langle\varphi(\zeta), r\eta\rangle|^{2n}} - \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\right| d\sigma(\zeta) \\ &\quad\lesssim \int\limits_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\, d\sigma(\zeta) + \frac{\sigma[E_{2(1-r)}(\eta)]}{(1-r)^n} \\ &\quad\lesssim \int\limits_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta),\eta_0\big)\, d\sigma(\zeta) + I(\eta) + M^m_{2(1-r)}\big(\sigma\circ\varphi^{-1}\big).\end{aligned}$$

Note that $\sigma \circ \varphi^{-1}$ is a compact m-Carleson measure, since $C^m_\varphi : H^1(\mathbf{S}) \to L^1(\mathbf{S})$ is compact by Lemma 4.4. Thus the last term of the above tends to 0 uniformly in η as $r \to 1^-$. Meanwhile, since

$$\lim_{\eta\to\eta_0} \int\limits_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\, d\sigma(\zeta) = \int\limits_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta),\eta_0\big)\, d\sigma(\zeta)$$

by the continuity of $\Lambda^m_{\varphi,1}$ on $\mathbf{S}$, we have $I(\eta) \to 0$ as $\eta \to \eta_0$ by Lemma 2.4. Also, note

$$\lim_{r\eta\to\eta_0} \int\limits_{\mathbf{S}\setminus E_\delta(\eta_0)} \left|\frac{(1-|\varphi(\zeta)|^2)^n}{|1-\langle\varphi(\zeta), r\eta\rangle|^{2n}} - \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\right| d\sigma(\zeta) = 0$$

by the Dominated Convergence Theorem. It follows from these observations that

$$\limsup_{r\eta\to\eta_0, 0<r<1} \widetilde{G}^m(r\eta) \lesssim \int_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta), \eta_0\big)\, d\sigma(\zeta).$$

In the display above note that the right-hand tends to 0 as $\delta \to 0^+$, because $\|\mathscr{K}^m(\varphi(\cdot), \eta_0)\|_1 = \Lambda^m_{\varphi,1}(\eta_0) < \infty$. Since the left-hand side is independent of δ, we conclude that

$$\limsup_{r\eta\to\eta_0, 0<r<1} \widetilde{G}^m(r\eta) = 0 = \widetilde{G}^m(\eta_0).$$

Hence $\widetilde{G}^m$ is continuous at every point in $\mathbf{S}$ as required, and the proof is complete. $\square$

We are now ready to characterize the compactness for the case $t = 1$.

Proposition 4.6. *Let $x \in \{c, h, m, p\}$ and φ be as in (1.2). Then C^x_φ is compact on $L^1(\mathbf{S})$ if and only if $\Lambda^x_{\varphi,1} \in C(\overline{\mathbf{B}})$.*

Proof. We first prove the necessity. So, suppose C^x_φ is compact on $L^1(\mathbf{S})$ and let $\{w^j\}$ be a sequence of points in $\overline{\mathbf{B}}$ with $w^j \to w_0$. To show $\Lambda^x_{\varphi,1} \in C(\overline{\mathbf{B}})$, it suffices to show that there is a subsequence $\{w^{j_k}\}$ such that $\Lambda^x_{\varphi,1}(w^{j_k}) \to \Lambda^x_{\varphi,1}(w_0)$. Let $k_w := \mathscr{K}^h(w, \cdot)$ for $x \in \{c, p, h\}$ and $k_w := \mathscr{K}^m(w, \cdot)$ for $x = m$. Then $\|k_w\|_1 = 1$ for $w \in \mathbf{B}$ and $\|k_w\|_1 = 0$ for $w \in \mathbf{S}$. Thus $\{k_{w^j}\}$ is a bounded sequence in $L^1(\mathbf{S})$, and since C^x_φ is compact it follows that $\{C^x_\varphi k_{w^j}\} = \{\mathscr{K}^x(\varphi(\cdot), w^j)\}$ has a subsequence $\{\mathscr{K}^x(\varphi(\cdot), w^{j_k})\}$ that converges in norm. Since $\mathscr{K}^x(\varphi(\cdot), w^j) \to \mathscr{K}^x(\varphi(\cdot), w_0)$ σ-a.e., it follows that $\mathscr{K}^x(\varphi(\cdot), w^{j_k}) \to \mathscr{K}^x(\varphi(\cdot), w_0)$ in norm. Hence,

$$\Lambda^x_{\varphi,1}\big(w^{j_k}\big) = \big\|\mathscr{K}^x\big(\varphi(\cdot), w^{j_k}\big)\big\|_1 \to \big\|\mathscr{K}^x\big(\varphi(\cdot), w_0\big)\big\|_1 = \Lambda^x_{\varphi,1}(w_0)$$

as required, and this completes the proof that $\Lambda^x_{\varphi,1} \in C(\overline{\mathbf{B}})$.

Sufficiency has already been proved for $x \in \{c, h, p\}$ in the remarks following the proof of Lemma 4.3. So for the rest of the proof we put $x = m$, and assume $\Lambda^m_{\varphi,1} \in C(\overline{\mathbf{B}})$. As in those remarks, it suffices to show that

$$\big\|C^m_{s\varphi} - C^m_\varphi\big\|_{L^1(\mathbf{S})} \to 0 \quad \text{as } s \to 1^-.$$

From Lemma 4.3, it suffices to show

$$\lim_{s\to 1^-}\Big(\sup_{\eta\in\mathbf{S}} G^m_\varphi(s\eta)\Big) = 0, \tag{4.8}$$

where G^m_φ was defined in (4.3). Let $z \in \overline{\mathbf{B}}$ and $\eta \in \mathbf{S}$. When $s < 1$, $\mathscr{K}^m(sz, \eta) = K^m(sz, \eta)$, so the explicit formula of the kernel can be used. For $s \in [0, 1)$ we have

$$\begin{aligned}K^m(sz,\eta) &= \frac{(1-|sz|^2)^n}{|1-\langle sz,\eta\rangle|^{2n}} \\ &= \frac{[(1-|z|^2)+|z|^2(1-s^2)]^n}{|1-\langle z,s\eta\rangle|^{2n}} \\ &= \frac{(1-|z|^2)^n}{|1-\langle z,s\eta\rangle|^{2n}} + \mathcal{O}(1)\cdot\sum_{k=1}^{n}\frac{(1-s)^k(1-|z|^2)^{n-k}}{|1-\langle z,s\eta\rangle|^{2n}}\end{aligned}$$

where $\mathcal{O}(1)$ is uniform in z and s. Thus

$$\left|K^m(sz,\eta) - \frac{(1-|z|^2)^n}{|1-\langle z,s\eta\rangle|^{2n}}\right| \lesssim \sum_{k=1}^{n}\frac{(1-s)^k}{|1-\langle z,s\eta\rangle|^{n+k}}, \qquad z\in\overline{\mathbf{B}}.$$

Hence, setting

$$g_{a,k} := \frac{(1-|a|)^k}{(1-\langle\,\cdot\,,a\rangle)^{n+k}}, \qquad a\in\mathbf{B},\ k=1,2,\dots,n$$

we obtain

$$G^m_\varphi(s\eta) \lesssim \widetilde{G}^m_\varphi(s\eta) + \sum_{k=1}^{n}\left\|C^m_\varphi g_{s\eta,k}\right\|_1.$$

The function $\widetilde{G}^m_\varphi$ was introduced in Lemma 4.5, where it was shown that $\widetilde{G}^m_\varphi(s\eta)\to 0$ as $s\to 1^-$ uniformly in η. So, to complete the proof, it suffices to show the sum in the right-hand side of the display above converges to 0 uniformly in η as $s\to 1^-$.

Note that $\{g_{s\eta,k}\}$ is a bounded set in $H^1(\mathbf{S})$ by (2.1). Also, note that, given $a^j\to\eta_0\in\mathbf{S}$, $\{C^m_\varphi g_{a^j,k}\} = \{g_{a^j,k}\circ\varphi\}$ (with k fixed) converges pointwise to 0 on $\mathbf{S}\setminus\varphi^{-1}\{\eta_0\}$ and hence σ-a.e. by (1.2). Since $C^m_\varphi : H^1(\mathbf{S})\to L^1(\mathbf{S})$ is compact by Lemma 4.4, a subsequence converges to 0 in norm. It follows that

$$\lim_{s\to 1^-}\sup_{\eta\in\mathbf{S}}\sum_{k=1}^{n}\left\|C^m_\varphi g_{s\eta,k}\right\|_1 = 0,$$

as desired. This completes the proof. □

Having proved the compactness characterizations, we now prove the following, which is the content of the compactness part of Theorem 1.4(a).

Proposition 4.7. *Let $x\in\{c,h,m,p\}$, $1<t<\infty$ and φ be as in (1.2). If C^x_φ is compact on $L^1(\mathbf{S})$, then C^x_φ is compact on $L^t(\mathbf{S})$.*

Proof. As in the proof of Proposition 3.6, the case $x\neq h$ is contained in Theorem 2.3(c). For $x=h$, from Proposition 4.6 and the hypothesis that C^h_φ is compact on $L^1(\mathbf{S})$ we have that $\Lambda^h_{\varphi,1} \in C(\overline{\mathbf{B}})$. So by Lemma 2.4 it follows that G^h_φ, defined in (4.3), is continuous and vanishes on $\mathbf{S}$. Thus Lemma 4.3 shows that given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left\|C^h_{s\varphi} - C^h_\varphi\right\|_{L^1(\mathbf{S})} \le \epsilon, \quad \text{for all } s \in (1-\delta, 1).$$

Also, the operator $C^h_{s\varphi} - C^h_\varphi$ clearly acts boundedly on $L^\infty(\mathbf{S})$ with

$$\left\|C^h_{s\varphi} - C^h_\varphi\right\|_{L^\infty(\mathbf{S})} \le 2.$$

It now follows from the Riesz–Thorin Interpolation Theorem that

$$\left\|C^h_{s\varphi} - C^h_\varphi\right\|_{L^t(\mathbf{S})} \le 2^{1-1/t}\epsilon^{1/t}, \quad \text{for all } s \in (1-\delta, 1);$$

the slightly different norm estimate implicit in the proof of the Marcinkiewicz Interpolation Theorem in [16, p. 43] could also be used. (An alternate approach, available since C^h_φ is an integral operator, is to use Schur's test; see [12, Lemma 1].) Hence

$$\lim_{s\to 1^-} \left\|C^h_{s\varphi} - C^h_\varphi\right\|_{L^t(\mathbf{S})} = 0.$$

Since each $C^h_{s\varphi}$, $0 < s < 1$ is compact on $L^t(\mathbf{S})$ from Lemma 4.2, C^h_φ is also. The proof is complete. □

Applying our compactness characterization, we can show by explicit examples that L^t-compactness for each $1 < t < \infty$ may not imply L^1-boundedness. This, in particular, shows that the converse of Proposition 3.6 does not hold.

Proposition 4.8. *For each $x \in \{c, h, m, p\}$ there exists φ_x such that $C^x_{\varphi_x}$ is compact on $L^t(\mathbf{S})$, $1 < t < \infty$, but $C^x_{\varphi_x}$ is not bounded on $L^1(\mathbf{S})$.*

We fix as a standard reference point

$$\mathbf{e} := (1, 0, \ldots, 0) \in \mathbf{S}. \tag{4.9}$$

Proof. Put

$$d_x(\zeta) := \begin{cases} |1 - \langle \mathbf{e}, \zeta\rangle| & \text{if } x = c, p, m \\ |\mathbf{e} - \zeta| & \text{if } x = h, \end{cases} \qquad \zeta \in \mathbf{S}$$

and

$$h_x(s) := \begin{cases} s|\log s|^{1/n} & \text{if } x = c, p, m \\ s|\log s|^{1/(2n-1)} & \text{if } x = h, \end{cases} \qquad 0 \le s < \epsilon$$

where $\epsilon > 0$ is a sufficiently small number chosen so that $0 \le h_x(s) \le 1$. Given $x \in \{c, h, m, p\}$, put $V_x = S^x(\mathbf{e}, \epsilon) \cap \mathbf{S}$ and define

$$\varphi_x(\zeta) := \big(1 - (h_x \circ d_x)(\zeta), 0, \ldots, 0\big)\chi_{V_x}(\zeta)$$

where χ_{V_x} denotes the characteristic function of V_x. Clearly, this function satisfies (1.2).

We show that $C^x_{\varphi_x}$ is not bounded on $L^1(\mathbf{S})$, only for the case $x = c$; the proofs for other cases are similar and thus omitted. By Fatou's Lemma $\Lambda^c_{\varphi_c,1}(\mathbf{e}) \le \liminf_{r\to 1^-} \Lambda^c_{\varphi_c,1}(r\mathbf{e})$, and so by Proposition 3.1 it suffices to prove that $\Lambda^c_{\varphi_c,1}(\mathbf{e}) = \infty$. Note

$$\big|\mathscr{K}^c\big(\varphi_c(\zeta), \mathbf{e}\big)\big| = \frac{1}{|1 - \langle \mathbf{e}, \zeta\rangle|^n}\left(\log \frac{1}{|1 - \langle \mathbf{e}, \zeta\rangle|}\right)^{-1}, \quad \zeta \in V_c$$

and so

$$\Lambda^c_{\varphi_c,1}(\mathbf{e}) = \int\limits_{V_c} \frac{1}{|1 - \langle \mathbf{e}, \zeta\rangle|^n}\left(\log \frac{1}{|1 - \langle \mathbf{e}, \zeta\rangle|}\right)^{-1} d\sigma(\zeta).$$

It is elementary to see that this integral diverges when $n = 1$. When $n > 1$, one can use (3.2) to see that the integral also diverges. So, $C^c_{\varphi_c}$ is not bounded on $L^1(\mathbf{S})$.

Now, we let $1 < t < \infty$ and show that each $C^x_{\varphi_x}$ is compact on $L^t(\mathbf{S})$. By Lemma 2.2 it suffices to show that $\sigma \circ \varphi_x^{-1}$ is a compact x-Carleson measure. It is easy to see

$$\big|1 - \big\langle \mathbf{e}, \varphi_x(\zeta)\big\rangle\big| \le \big|1 - \big\langle \eta, \varphi_x(\zeta)\big\rangle\big|$$

for any $\zeta, \eta \in \mathbf{S}$. Thus we have

$$\varphi_x^{-1}\big(S^x(\eta, \delta)\big) \subset \varphi_x^{-1}\big(S^x(\mathbf{e}, \delta)\big), \quad \eta \in \mathbf{S},\ 0 < \delta < 1$$

and consequently it suffices to consider the Carleson sets $S^x(\mathbf{e}, \delta)$.

Continuing under the assumption that $x = c$, the other cases being similar, note that h_c is invertible (when ϵ is sufficiently small). Thus, for $\zeta \in V_c$, we see that $\varphi_c(\zeta) \in S^c(\mathbf{e}, \delta)$ if and only if $d_c(\zeta) < h_c^{-1}(\delta)$. Thus $\varphi_c^{-1}[S^c(\mathbf{e}, \delta)] = S^c(\mathbf{e}, h_c^{-1}(\delta))$ for all δ sufficiently small. Hence

$$\frac{(\sigma \circ \varphi_c^{-1})[S^c(\mathbf{e}, \delta)]}{\sigma[S^c(\mathbf{e}, \delta)]} \approx \left(\frac{h_c^{-1}(\delta)}{\delta}\right)^n \to 0 \quad \text{as } \delta \to 0^+,$$

and so $\sigma \circ \varphi^{-1}$ is a compact c-Carleson measure. This completes the proof. $\square$

As another consequence of our compactness characterization, we can easily recover Sarason's result. Recall that (1.2) is satisfied by all holomorphic self-maps of $\mathbf{D}$; see Section 2.5.

Corollary 4.9. *(See Sarason [12].) Let φ be a holomorphic self-map of $\mathbf{D}$ such that $\varphi(0) = 0$. Then C^h_φ is compact on $L^1(\partial\mathbf{D})$ if and only if $\Lambda^h_{\varphi,1}(\eta) = 1$ for all $\eta \in \partial\mathbf{D}$.*

Proof. Note that $\mathscr{K}^h(\varphi(\cdot), z)$ is a bounded harmonic function on $\mathbf{D}$ for each $z \in \mathbf{D}$. We thus have

$$\Lambda^h_{\varphi,1}(z) = \mathscr{K}^h\big(\varphi(0), z\big) = \mathscr{K}^h(0, z) = 1$$

for all $z \in \mathbf{D}$. So, the corollary is immediate from Proposition 4.6. $\square$

5. Examples for $t = 1$

From Theorem 2.3, if $x, y \in \{c, p, m\}$, $t > 1$, and C^x_φ is bounded (respectively compact) on $L^t(\mathbf{S})$, then C^y_φ is bounded (compact) on $L^t(\mathbf{S})$. Also, if C^h_φ is bounded (respectively compact) on $L^t(\mathbf{S})$, $1 < t < \infty$, then C^x_φ is bounded (compact) on $L^t(\mathbf{S})$ for $x \in \{c, p, m\}$. Example 5.2 below shows that the converse fails badly: C^x_φ, $x \in \{c, p, m\}$, can be compact while C^h_φ is not bounded.

Carleson measure methods were used to get these results for $t > 1$. When $t = 1$, Carleson measure methods are not available and the situation is quite different. In this section we consider the 12 implications of the type

$$\text{If } C^x_\varphi \text{ is bounded on } L^1(\mathbf{S}), \text{ then } C^y_\varphi \text{ is bounded on } L^1(\mathbf{S}),$$

where $x \neq y$. We show that 2 of these implications hold (see Proposition 5.1 below), while 9 of the remaining 10 fail. In fact we show that 7 of these fail badly, in that C^x_φ can be compact while C^y_φ is not bounded. The last cases that we were not able to resolve will be stated as questions at the end of this section. We begin with the two implications that do hold:

Proposition 5.1. *Let φ be as in (1.2). If C^c_φ is bounded (respectively compact) on $L^1(\mathbf{S})$, then C^x_φ is bounded (compact) on $L^1(\mathbf{S})$ for $x = p, m$.*

Proof. We first consider $x = p$. We have $\mathscr{K}^p = 2\operatorname{Re}\mathscr{K}^c - 1$ and so the boundedness part of the statement is clear from Proposition 3.1. For compactness, if $z, w \in \overline{\mathbf{B}}$, then

$$\big\|\mathscr{K}^p\big(\varphi(\cdot), z\big) - \mathscr{K}^p\big(\varphi(\cdot), w\big)\big\|_1 \le 2\big\|\mathscr{K}^c\big(\varphi(\cdot), z\big) - \mathscr{K}^c\big(\varphi(\cdot), w\big)\big\|_1.$$

Proposition 2.4 now shows that $\Lambda^c_{\varphi,1} \in C(\overline{\mathbf{B}})$ implies $\Lambda^p_{\varphi,1} \in C(\overline{\mathbf{B}})$, and so the statement for compactness follows from Proposition 4.6.

We now consider the boundedness for $x = m$. Let $z \in \mathbf{B}$. Since $K^m(\varphi(\zeta), \eta) \le 2^n|K^c(\varphi(\zeta), \eta)|$, we have by Fubini's Theorem and Fatou's Lemma

$$\begin{aligned}\left\|\mathscr{K}^m(\varphi(\cdot),z)\right\|_1 &= \int\limits_{\mathbf{S}}\left\{\int\limits_{\mathbf{S}} K^m(\varphi(\zeta),\eta)\,d\sigma(\zeta)\right\}K^m(z,\eta)\,d\sigma(\eta)\\ &\le 2^n \liminf_{r\to 1}\int\limits_{\mathbf{S}}\left\{\int\limits_{\mathbf{S}}\left|\mathscr{K}^c(\varphi(\zeta),r\eta)\right|d\sigma(\zeta)\right\}K^m(z,\eta)\,d\sigma(\eta)\\ &\le 2^n \sup_{w\in\mathbf{B}}\left\|\mathscr{K}^c(\varphi(\cdot),w)\right\|_1 \int\limits_{\mathbf{S}} K^m(z,\eta)\,d\sigma(\eta).\end{aligned}$$

Since the last integral above is equal to 1, this shows that $\Lambda^m_{\varphi,1}$ is bounded by $2^n\Lambda^c_{\varphi,1}$. Hence the result for boundedness follows from Proposition 3.1.

Finally, we consider the compactness for $x = m$. Assume that C^c_φ is compact on $L^1(\mathbf{S})$, or equivalently by Proposition 4.6 that $\Lambda^c_{\varphi,1}$ is continuous on $\overline{\mathbf{B}}$. Now by Proposition 4.6 again, to see that C^m_φ is compact, it suffices to show that $\Lambda^m_{\varphi,1}$ is continuous on $\overline{\mathbf{B}}$. Since $\Lambda^m_{\varphi,1}$ is continuous on $\mathbf{B}$, it suffices to show that it is radially uniformly continuous. Let $\epsilon > 0$ and $\eta \in \mathbf{S}$. By Lemma 2.4, there is $\delta > 0$ independent of $\eta \in \mathbf{S}$ such that for all $\xi \in S^m(\eta,\delta)$

$$\int\limits_{\mathbf{S}}\left|\left|K^c(\varphi(\zeta),\eta)\right| - \left|K^c(\varphi(\zeta),\xi)\right|\right|d\sigma(\zeta) < \epsilon, \tag{5.1}$$

since $\Lambda^c_{\varphi,1}$ is (uniformly) continuous on $\overline{\mathbf{B}}$.

Note

$$\begin{aligned}\left|K^m(z,\eta) - K^m(z,\xi)\right| &= \left(1-|z|^2\right)^n\left|\left|K^c(z,\eta)\right|^2 - \left|K^c(z,\xi)\right|^2\right|\\ &\le 2^{n+1}\left|\left|K^c(z,\eta)\right| - \left|K^c(z,\xi)\right|\right|\end{aligned}$$

for $\zeta \in \mathbf{S}$ and $z \in \mathbf{B}$. With K^m replaced by $\mathscr{K}^m$ this estimate extends to all $z \in \mathbf{S}$, since $\mathscr{K}^m(z,\eta) = 0$ when $z \in \mathbf{S}$. Thus we see from (5.1)

$$\int\limits_{\mathbf{S}}\left|\mathscr{K}^m(\varphi(\zeta),\eta) - \mathscr{K}^m(\varphi(\zeta),\xi)\right|d\sigma(\zeta) < 2^{n+1}\epsilon \tag{5.2}$$

for $\xi \in S^m(\eta,\delta)$. Let $0 < r < 1$. Since $\int_{\mathbf{S}} K^m(r\eta,\xi)\,d\sigma(\xi) = 1$, using Corollary 3.2 we have

$$\begin{aligned}&\Lambda^m_{\varphi,1}(r\eta) - \Lambda^m_{\varphi,1}(\eta)\\ &\quad= \int\limits_{\mathbf{S}}\left\{\int\limits_{\mathbf{S}}\left[\mathscr{K}^m(\varphi(\zeta),\xi) - \mathscr{K}^m(\varphi(\zeta),\eta)\right]K^m(r\eta,\xi)\,d\sigma(\xi)\right\}d\sigma(\zeta).\end{aligned}$$

Thus, setting $M := \sup_{z\in\mathbf{B}}\Lambda^m_{\varphi,1}(z)$, we obtain by Fubini's Theorem and (5.2)

$$\begin{aligned}
&\left|\Lambda^m_{\varphi,1}(r\eta) - \Lambda^m_{\varphi,1}(\eta)\right| \\
&\quad \le \int_{\mathbf{S}} \left\{ \int_{\mathbf{S}} \left|\mathscr{K}^m\big(\varphi(\zeta),\xi\big) - \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\right| d\sigma(\zeta) \right\} K^m(r\eta,\xi)\, d\sigma(\xi) \\
&\quad \le 2^{n+1}\epsilon \int_{S^m(\eta,\delta)} K^m(r\eta,\xi)\, d\sigma(\xi) + 2M \int_{\mathbf{S}\setminus S^m(\eta,\delta)} K^m(r\eta,\xi)\, d\sigma(\xi) \\
&\quad \le 2^{n+1}\epsilon + 2M \frac{(1-r^2)^n}{\delta^{2n}}.
\end{aligned}$$

Therefore we have

$$\left|\Lambda^m_{\varphi,1}(r\eta) - \Lambda^m_{\varphi,1}(\eta)\right| < \big(2^{n+1}+1\big)\epsilon, \quad \eta \in \mathbf{S}$$

for all r sufficiently close to 1. The proof is complete. $\square$

We now turn to the examples that demonstrate the failures of the implications discussed at the beginning of this section.

Recall $C^p_\varphi = C^m_\varphi = C^h_\varphi$ in the one-dimensional case. So, the restriction $n \ge 2$ in the next example is required, by Theorem 2.3(b) for $1 < t < \infty$, and by Proposition 5.1 for $t = 1$. In what follows, $\mathbf{e} \in \mathbf{S}$ denotes the point specified in (4.9).

Example 5.2. Let $1 \le t < \infty$, $x \in \{c, p, m\}$, and $n \ge 2$. Then there exists φ such that C^x_φ is compact on $L^t(\mathbf{S})$, but C^h_φ is not bounded on $L^t(\mathbf{S})$.

Proof. First, consider the case $1 < t < \infty$. In this case we use the Carleson measure characterizations. Fix $\frac{n}{2n-1} < a < 1$ and let

$$\varphi(\zeta) := \big(L_a(\zeta_1), 0, \dots, 0\big)$$

where $L_a(\lambda) = 1 - (1-\lambda)^a$, $\lambda \in \mathbf{D}$. We remark that L_a is a conformal map of $\mathbf{D}$ onto a teardrop-shaped region in $\mathbf{D}$ with vertex at 1. Clearly, φ satisfies (1.2). Since $|\varphi(\zeta) - \mathbf{e}| = |1-\zeta_1|^a = |1 - \langle\varphi(\zeta),\mathbf{e}\rangle|$, we have

$$\begin{aligned}
\big(\sigma\circ\varphi^{-1}\big)\big[S^x(\mathbf{e},\delta)\big] &= \big(\sigma\circ\varphi^{-1}\big)\big[S^h(\mathbf{e},\delta)\big] \\
&= \sigma\big\{\zeta \in \mathbf{S} : |1-\zeta_1| < \delta^{1/a}\big\} \\
&\approx \delta^{n/a} \quad \text{by (3.2)}.
\end{aligned}$$

Accordingly, using (2.9), we obtain

$$M^h_\delta\big(\sigma\circ\varphi^{-1}\big) \gtrsim \frac{\delta^{n/a}}{\delta^{2n-1}} \to \infty \quad \text{as } \delta \to 0^+,$$

which, together with Lemma 2.2, shows that C_φ^h is not bounded on $L^t(\mathbf{S})$. On the other hand, using (2.9), we see

$$\frac{(\sigma\circ\varphi^{-1})[S^x(\mathbf{e},\delta)]}{\sigma[S^x(\mathbf{e},\delta)]} \approx \frac{\delta^{n/a}}{\delta^n} \to 0 \quad \text{as } \delta\to 0^+.$$

As in the proof of Proposition 4.8, this shows that C_φ^x is compact on $L^t(\mathbf{S})$. This completes the proof for $t > 1$.

Now, we consider the case $t = 1$. This time we let

$$\varphi(\zeta) := \bigl(1 - h(\operatorname{Re}\zeta_1), 0, \dots, 0\bigr)$$

where $h(s) = (1-s)^{1/2}$ for $0 \le s \le 1$ and $h(s) = 1$ otherwise. Again, φ clearly satisfies (1.2). A little manipulation using (3.2) yields

$$\Lambda_{\varphi,1}^h(\mathbf{e}) \approx \int_{-1}^{1} \frac{(1-s^2)^{n-3/2}}{h^{2n-1}(s)}\,ds \approx 1 + \int_0^1 \frac{ds}{1-s} = \infty.$$

Thus we see from Proposition 3.1 and Fatou's Lemma that C_φ^h is not bounded on $L^1(\mathbf{S})$.

To prove that C_φ^x is compact on $L^1(\mathbf{S})$, we only need to consider for $x = c$ by Proposition 5.1. Since $\varphi(\mathbf{S})$ touches $\mathbf{S}$ only at $\mathbf{e}$, it suffices to show that $\Lambda_{\varphi,1}^c$ is continuous at $\mathbf{e}$ by Proposition 4.6. Let $w \in \overline{\mathbf{B}}$. Since $\operatorname{Re}(1-\overline{w}_1) \ge 0$, we have

$$\bigl|1 - \langle\varphi(\zeta), w\rangle\bigr| = \bigl|h(\operatorname{Re}\zeta_1) + (1-\overline{w}_1)\bigl(1 - h(\operatorname{Re}\zeta_1)\bigr)\bigr| \ge h(\operatorname{Re}\zeta_1)$$

so that

$$\bigl|\mathscr{K}^c\bigl(\varphi(\zeta), w\bigr)\bigr| \le \frac{1}{|1-\operatorname{Re}\zeta_1|^{n/2}}, \qquad \zeta\in\mathbf{S},\ w\in\overline{\mathbf{B}}.$$

Note by (3.2)

$$\int_{\mathbf{S}} \frac{d\sigma(\zeta)}{|1-\operatorname{Re}\zeta_1|^{n/2}} \approx 1 + \int_0^1 \frac{(1-s)^{n-3/2}}{(1-s)^{n/2}}\,ds < \infty;$$

this is where the restriction $n \ge 2$ comes into play. Thus we conclude via the Dominated Convergence Theorem that $\Lambda_{\varphi,1}^c$ is continuous at $\mathbf{e}$, as required. The proof is complete. $\square$

Next we give a simple example that shows C_φ^h and C_φ^m may be bounded on $L^1(\mathbf{S})$, while C_φ^p is not. Recall that id denotes the identity map of $\mathbf{S}$.

Example 5.3. For $n \geq 2$, C_{id}^h and C_{id}^m are bounded on $L^1(\mathbf{S})$, but C_{id}^c and C_{id}^p are not bounded on $L^1(\mathbf{S})$.

Proof. C_{id}^h and C_{id}^m are simply the identity operator on $L^1(\mathbf{S})$ by Proposition 1.2(b), and so bounded. That C_{id}^p is not bounded is a consequence of Corollary 3.2, and it follows from Proposition 5.1 that C_{id}^c is not bounded. □

The restriction $n \geq 2$ in the next two examples is required, since $C_\varphi^p = C_\varphi^m$ when $n = 1$. Also, when $n = 1$, C_{id}^p is bounded on $L^1(\mathbf{S})$ but C_{id}^c is not, as discussed above. But a different example is required to differentiate between the behavior of C_φ^p and C_φ^c on $L^1(\mathbf{S})$ when $n \geq 2$, since in that case C_{id}^p is not bounded.

Example 5.4. Let $n \geq 2$. Then there is φ satisfying (1.2) such that C_φ^m is compact on $L^1(\mathbf{S})$, but C_φ^c and C_φ^p are not bounded on $L^1(\mathbf{S})$.

In the proof below we will use the non-isotropic triangle inequality (see, for example, [10, Proposition 5.1.2])

$$\left|1 - \langle z^1, z^2 \rangle\right|^{1/2} \leq \left|1 - \langle z^1, z^3 \rangle\right|^{1/2} + \left|1 - \langle z^3, z^2 \rangle\right|^{1/2} \tag{5.3}$$

valid for all $z^1, z^2, z^3 \in \overline{\mathbf{B}}$.

Proof. Define a sequence $\{q_k\} \subset \mathbf{B}$ by

$$q_k = \left(1 - \frac{1}{2^k}, \sqrt{\frac{2}{2^k} - \frac{1}{2^{2k}} - \frac{1}{2^k(\log k)^2}}, 0, \dots, 0\right), \quad k = 2, 3, \dots$$

so that

$$\left|1 - \langle q_k, \mathbf{e} \rangle\right| = \frac{1}{2^k} \quad \text{and} \quad 1 - |q_k|^2 = \frac{1}{2^k(\log k)^2}. \tag{5.4}$$

Let $\{E_k\}_{k=2}^\infty$ be a partition of $\mathbf{S}$ into Borel sets E_k such that $\sigma(E_k) = \frac{c_n}{k 2^{nk}}$, where $(c_n)^{-1} = \sum_{k=2}^\infty \frac{1}{k 2^{nk}}$. Denote by χ_k the characteristic function of E_k and define $\varphi : \mathbf{S} \to \mathbf{B}$ by $\varphi = \sum_{k=2}^\infty q_k \chi_k$. Then φ is a Borel function and

$$\sigma \circ \varphi^{-1} = c_n \sum_{k=2}^\infty \frac{\delta_k}{k 2^{nk}},$$

where δ_k is the point mass at q_k. Clearly φ satisfies (1.2) since $\varphi^{-1}(\mathbf{S}) = \emptyset$. Also, note

$$\Lambda_{\varphi,1}^x(w) = c_n \sum_{k=2}^\infty \frac{1}{k 2^{nk}} \left|\mathscr{K}^x(q_k, w)\right|, \quad w \in \overline{\mathbf{B}} \tag{5.5}$$

for any x.

Note

$$\Lambda^p_{\varphi,1}(\mathbf{e}) + 1 \geq c_n \sum_{k=2}^{\infty} \frac{1}{k2^{nk}} \frac{2}{|1-\langle q_k, \mathbf{e}\rangle|^n} = c_n \sum_{k=2}^{\infty} \frac{2}{k} = \infty. \tag{5.6}$$

Thus, from Fatou's Lemma and Proposition 3.1, C^p_φ is not bounded on $L^1(\mathbf{S})$ which in turn implies C^c_φ is not bounded on $L^1(\mathbf{S})$ by Proposition 5.1.

We now turn to the proof that C^m_φ is compact on $L^1(\mathbf{S})$. Since $q_k \to \mathbf{e}$, it follows from (1.7) that the sequence $\{\mathscr{K}^m(q_k, \cdot)\}_k$ is uniformly bounded on each compact subset of $\overline{\mathbf{B}} \setminus \{\mathbf{e}\}$. Accordingly, the series in (5.5) for $x = m$ converges uniformly on each compact subset of $\overline{\mathbf{B}} \setminus \{\mathbf{e}\}$. So, $\Lambda^m_{\varphi,1}$ is continuous on $\overline{\mathbf{B}} \setminus \{\mathbf{e}\}$, because each $\mathscr{K}^m(q_k, \cdot)$ is continuous on $\overline{\mathbf{B}}$. Thus, by Proposition 4.6 it suffices to show that $\Lambda^p_{\varphi,1}$ is continuous at $\mathbf{e}$. For this it is enough to show that there is a constant $C = C(n) > 0$ such that

$$C\big|\Lambda^m_{\varphi,1}(w) - \Lambda^m_{\varphi,1}(\mathbf{e})\big| \leq \frac{1}{\log M} + \sum_{k=2}^{M-1} \frac{|\mathscr{K}^m(q_k, w) - \mathscr{K}^m(q_k, \mathbf{e})|}{k2^{nk}} \tag{5.7}$$

for any integers M, N with $N - 2 \geq M \geq 3$ and $w \in S^m(\mathbf{e}, 2^{-N})$.

Let $M \geq 3$ be a given positive integer. As a preliminary step towards (5.7), we need certain estimate for the series $\sum_{k=M}^{\infty} \frac{\mathscr{K}^m(q_k,w)}{k2^{nk}}$. First, we show that there is a constant $C = C(n) > 0$ such that

$$\sum_{k=M}^{\ell-2} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} \leq \frac{C}{\log M}, \qquad w \in S^m\big(\mathbf{e}, 2^{-\ell}\big) \tag{5.8}$$

for $\ell \geq M + 2$. To see this, for $M \leq k \leq \ell - 2$ and $w \in S^m(\mathbf{e}, 2^{-\ell})$, note by (5.3)

$$\big|1 - \langle q_k, w\rangle\big| \geq \frac{1}{2}\big|1 - \langle q_k, \mathbf{e}\rangle\big| - \big|1 - \langle w, \mathbf{e}\rangle\big| \geq \frac{1}{2^{k+1}} - \frac{1}{2^\ell} \geq \frac{1}{2^{k+2}}$$

so that using (1.7) we get that

$$\mathscr{K}^m(q_k, w) \lesssim \frac{(1-|q_k|^2)^n + (1-|w|^2)^n}{|1-\langle q_k, w\rangle|^{2n}} \lesssim 2^{2kn}\left[\frac{1}{2^{kn}(\log k)^{2n}} + \frac{1}{2^{\ell n}}\right].$$

Since $\sum_{k=M}^{\ell-2} [k(\log k)^{2n}]^{-1} \lesssim (\log M)^{1-2n} \leq (\log M)^{-1}$ and $\sum_{k=M}^{\ell-2} 2^{kn}/(k2^{\ell n}) \lesssim M^{-1}$, this yields (5.8). Next, since

$$\mathscr{K}^m(q_k, w) \lesssim \frac{1}{(1-|q_k|^2)^n} = 2^{nk}(\log k)^{2n},$$

there is a constant $C = C(n) > 0$ such that

$$\sum_{k=\ell-1}^{\ell+2} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} \le C\frac{(\log \ell)^{2n}}{\ell}, \quad w \in \overline{\mathbf{B}} \tag{5.9}$$

for all integers $\ell \ge 2$. Finally, we show that there is a constant $C = C(n) > 0$ such that

$$\sum_{k=\ell+3}^{\infty} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} \le \frac{C}{\ell}, \quad w \notin S^m\left(\mathbf{e}, 2^{-\ell-1}\right) \tag{5.10}$$

for positive integers ℓ. To see this, for $k \ge \ell + 3$ and $w \notin S^m(\mathbf{e}, 2^{-\ell-1})$, note by (5.3)

$$\left|1 - \langle q_k, w\rangle\right| \ge \frac{1}{2}\left|1 - \langle w, \mathbf{e}\rangle\right| - \left|1 - \langle \mathbf{e}, q_k\rangle\right| \ge \frac{1}{2^{\ell+2}} - \frac{1}{2^k} \ge \frac{1}{2^{\ell+3}}$$

so that, again using (1.7),

$$\mathscr{K}^m(q_k, w) \lesssim \frac{2^n}{|1 - \langle q_k, w\rangle|^n} \le 2^{4n+\ell n}.$$

Since $\sum_{k=\ell+3}^{\infty} 2^{n\ell}/(k2^{nk}) \lesssim 1/\ell$, this yields (5.10).

Now, we proceed to the proof of (5.7). Since

$$\sum_{k=M}^{\infty} \frac{\mathscr{K}^m(q_k, \mathbf{e})}{k2^{nk}} = \sum_{k=M}^{\infty} \frac{1}{k(\log k)^{2n}} \lesssim \frac{1}{\log M},$$

we have

$$\begin{aligned}
&\left|\Lambda^m_{\varphi,1}(w) - \Lambda^m_{\varphi,1}(\mathbf{e})\right| \\
&\quad \lesssim \sum_{k=2}^{\infty} \frac{|\mathscr{K}^m(q_k, w) - \mathscr{K}^m(q_k, \mathbf{e})|}{k2^{nk}} \\
&\quad = \sum_{k=2}^{M-1} \frac{|\mathscr{K}^m(q_k, w) - \mathscr{K}^m(q_k, \mathbf{e})|}{k2^{nk}} + \sum_{k=M}^{\infty} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} + \frac{1}{\log M}
\end{aligned} \tag{5.11}$$

for $w \in \overline{\mathbf{B}}$. Let N be a given positive integer with $N \ge M + 2$ and fix $w \in S^m(\mathbf{e}, 2^{-N})$. Choose $\ell \ge N$ such that $w \in S^m(\mathbf{e}, 2^{-\ell}) \setminus S^m(\mathbf{e}, 2^{-\ell-1})$. Then we see from (5.8), (5.9) and (5.10) that the second term of the above is dominated by some constant (depending only on n) times

$$\frac{1}{\log M} + \frac{(\log \ell)^{2n}}{\ell} + \frac{1}{\ell} \lesssim \frac{1}{\log M} + \frac{(\log M)^{2n}}{M} + \frac{1}{M} \lesssim \frac{1}{\log M};$$

the constants suppressed in these estimates are independent of M and N. From this and (5.11) we conclude (5.7), as asserted. The proof is complete. □

Remark. (1) Closely looking at the proof of Example 5.4, one may check that the proof goes through whenever (5.4) is satisfied and the first component of each q_k is real; the latter condition is needed for the first inequality in (5.6). Note that construction of such sequence $\{q_k\}$ is possible only when $n \geq 2$.

(2) In conjunction with previous remark, note that there is a sequence $\{q_k\} \subset \mathbf{D}$ satisfying (5.4). So, by the same proof, one may see that there is φ satisfying (1.2) such that $C_\varphi^h = C_\varphi^m = C_\varphi^p$ is compact on $L^1(\partial\mathbf{D})$, but C_φ^c is not bounded on $L^1(\partial\mathbf{D})$.

Example 5.5. Let $n \geq 2$. Then there is φ such that C_φ^p is compact on $L^1(\mathbf{S})$ but C_φ^c and C_φ^m are not bounded on $L^1(\mathbf{S})$.

Proof. Define a sequence $\{a_k\}$ of complex numbers by

$$a_k = 1 - \frac{e^{i\pi/2n}}{2^k}, \quad k = 1, 2, \ldots$$

so that

$$|1 - a_k| = \frac{1}{2^k} \quad \text{and} \quad \operatorname{Re}(1 - a_k)^n = 0.$$

Also, since

$$\frac{1 - |a_k|^2}{2} = \frac{1}{2^k}\left(\cos\frac{\pi}{2n} - \frac{1}{2^{k+1}}\right),$$

we have

$$1 - |a_k| \approx \frac{1}{2^k}, \quad k = 1, 2, \ldots; \tag{5.12}$$

it is this step where the restriction $n \geq 2$ comes into play. This in particular shows $\{a_k\} \subset \mathbf{D}$.

Now, as in the proof of Example 5.4, take a Borel function $\varphi : \mathbf{S} \to \mathbf{B}$ such that

$$\Lambda_{\varphi,1}^x(w) = c_n \sum_{k=1}^{\infty} \frac{1}{k2^{nk}} \left|\mathscr{K}^x(a_k\mathbf{e}, w)\right|, \quad w \in \overline{\mathbf{B}} \tag{5.13}$$

for any x; this time we take $c_n = (\sum_{k=1}^{\infty} \frac{1}{k2^{nk}})^{-1}$. It is easily checked that $\Lambda_{\varphi,1}^m(\mathbf{e}) = \infty$. So, as in the proof of Example 5.4, C_φ^m is not bounded on $L^1(\mathbf{S})$, which in turn implies C_φ^c is not bounded on $L^1(\mathbf{S})$ by Proposition 5.1.

Turning to the proof that C_φ^p is compact on $L^1(\mathbf{S})$, first note that an argument similar to the one used in the previous example will show that $\Lambda_{\varphi,1}^p$ is continuous on $\overline{\mathbf{B}} \setminus \{\mathbf{e}\}$. Thus, by Proposition 4.6 it suffices to show that $\Lambda_{\varphi,1}^p$ is continuous at $\mathbf{e}$. To this end we will prove

$$\left|\Lambda^p_{\varphi,1}(w) - \Lambda^p_{\varphi,1}(\mathbf{e})\right| \le \frac{C}{N} \tag{5.14}$$

for $w \in S^c(\mathbf{e}, 2^{-N}) \setminus S^c(\mathbf{e}, 2^{-N-1})$ and for some constant $C > 0$ independent of N and w. Here, and in the rest of the proof, N denotes an arbitrary positive integer.

To begin with, let $w \in \overline{\mathbf{B}}$. Note

$$\mathscr{K}^p(a_k\mathbf{e}, w) = 2\frac{\mathrm{Re}(1 - a_k\overline{w_1})^n}{|1 - a_k\overline{w_1}|^{2n}} - 1$$

and, in particular,

$$\mathscr{K}^p(a_k\mathbf{e}, \mathbf{e}) = 2\frac{\mathrm{Re}(1 - a_k)^n}{|1 - a_k|^{2n}} - 1 = -1$$

for each k. Hence we have by (5.13)

$$\begin{aligned}
\left|\Lambda^p_{\varphi,1}(w) - \Lambda^p_{\varphi,1}(\mathbf{e})\right| &\le c_n \sum_{k=1}^{\infty} \frac{1}{k2^{nk}} \left|\left|\mathscr{K}^p(a_k\mathbf{e}, w)\right| - 1\right| \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{k2^{nk}} \frac{|\,\mathrm{Re}(1 - a_k\overline{w_1})^n|}{|1 - a_k\overline{w_1}|^{2n}} \\
&= \sum_{k \le N+1} + \sum_{k > N+1}.
\end{aligned}$$

We now restrict $w \in S^c(\mathbf{e}, 2^{-N}) \setminus S^c(\mathbf{e}, 2^{-N-1})$ and estimate each sum of the above separately. For the second term, since

$$|1 - a_k\overline{w_1}| \ge |1 - \overline{w_1}| - |1 - a_k| \ge \frac{1}{2^{N+1}} - \frac{1}{2^k}$$

for each k, we have, as in the proof of Example 5.4,

$$\sum_{k > N+1} \lesssim \frac{1}{N}. \tag{5.15}$$

To estimate the first sum, note

$$|1 - w_1| \approx 2^{-N} \le 2|1 - a_k|, \quad k \le N + 1.$$

Hence, using that $\mathrm{Re}(1 - a_k)^n = 0$, we have

$$\begin{aligned}
\left|\mathrm{Re}(1 - a_k\overline{w_1})^n\right| &= \left|\mathrm{Re}\left[(1 - a_k) + a_k(1 - \overline{w_1})\right]^n\right| \\
&\lesssim |1 - a_k|^{n-1}|1 - w_1| \\
&\approx \frac{1}{2^{k(n-1)}2^N}, \quad k \le N + 1.
\end{aligned}$$

This, together with (5.12), yields

$$\sum_{k\leq N+1} \lesssim \sum_{k=1}^{N+1} \frac{1}{k2^{nk}} \frac{2^{2nk}}{2^{k(n-1)}2^N} = \frac{1}{2^N} \sum_{k=1}^{N+1} \frac{2^k}{k} \lesssim \frac{1}{N}. \tag{5.16}$$

Now, we conclude (5.14) by (5.15) and (5.16). The proof is complete. □

Proposition 5.1 and the examples in this section leave unresolved one (respectively three) implication(s) of the type "If C_φ^x is bounded (compact) on $L^1(\mathbf{S})$, does it follow that C_φ^y is bounded on $L^1(\mathbf{S})$?". We end with statements of these remaining cases as questions.

Question 5.6. *For $n \geq 2$, if C_φ^h is bounded on $L^1(\mathbf{S})$, does it follow that C_φ^m is bounded on $L^1(\mathbf{S})$?*

Question 5.7. *For $n \geq 2$ and $x \in \{c, m, p\}$, if C_φ^h is compact on $L^1(\mathbf{S})$, does it follow that C_φ^x is bounded on $L^1(\mathbf{S})$?*

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