Holomorphic flows on simply connected regions have no limit cycles

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The dynamical system or flow $\dot{z} = f(z)$, where f is holomorphic on $\mathbb{C}$, is considered. The behaviour of the flow at critical points coincides with the behaviour of the linearization when the critical points are non-degenerate: there is no center-focus dichotomy. Periodic orbits about a center have the same period and form an open subset. The flow has no limit cycles in simply connected regions. The advance mapping is holomorphic where the flow is complete. The structure of the separatrices bounding the orbits surrounding a center is determined. Some examples are given including the following: if a quartic polynomial system has 4 distinct centers, then they are collinear.

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1. INTRODUCTION

There have been a number of studies of dynamical systems $\dot{z} = f(z)$ or $\dot{z} = f(z,t)$where f is holomorphic in z and $z \in \mathbb{C}^n$ or some subset. These are called holomorphic or conformal flows [1, 6, 7]. Some closely related work has been done on Newton flows, that is dynamical systems of the form

$$\dot{z} = -\frac{f(z)}{f'(z)}.$$

See [2, 3, 4, 5]. For example this type of flow is used by Benzinger in [2] to prove that holomorphic flows with rational function right hand sides do not have limit cycles.

In this article the primary interest is to explore holomorphic flows on $\mathbb{C}$ (i.e. $n = 1$) to better understand complex, even entire, functions. Applications to functions such as the Riemann Ξ function will be developed in a later article.

First it is shown that the flow characterizes the function. Section 2 is a compilation of local properties. These are straight forward and can be found in the literature, for example [8]. They are included here for ease of reference. Section 3 is the main part of this paper. There it is proved that, under suitable restrictions, the advance mapping is holomorphic. This is then used to prove that holomorphic flows have no limit cycle. In addition a description of the global neighbourhood of a center for every entire flow it derived.

Section 4 contains some examples, again indicating the restricted behaviour of holomorphic flows. We are especially interested in flows for entire functions which have only centers in the finite plane. It is shown that if the flow is polynomial of degree less than 5, with centers only, then the centers must be on a line.

THEOREM 1.1. *If the complex functions f and g are holomorphic on an open connected subset $\Omega \subset \mathbb{C}$ and not identically zero, and $\dot{z} = f(z)$ and $\dot{z} = g(z)$, have the same critical points and the same integral paths, then there exists an $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ such that, for all $z \in \Omega$, $f(z) = \alpha g(z)$.*

Proof. If $\alpha : \Omega \to \mathbb{C}$ is real valued and holomorphic, and Ω open and connected, then, by the Cauchy-Riemann equations, α is constant.

Let Z be the (isolated) set of critical points. If z is not a critical point, then $f(z) = \alpha(z) \cdot g(z)$ where $\alpha(z) \in \mathbb{R}$ is non-zero. But then $\alpha(z) = f(z)/g(z)$, so $\alpha(z)$ is holomorphic and hence constant on $\Omega \setminus Z$, hence on Ω. ∎

It follows that the integral paths of $\dot{z} = f(z)$, for holomorphic f, determine f determine f up to multiplication by a real constant.

EXAMPLE 1.1. Let

$$f(z) = (1 + \frac{z}{3i})(1 - \frac{z}{3i})^3.$$

The phase portrait of f it plotted below. It is a quartic polynomial with one simple zero, which is a centre, and one zero of order 3.

THEOREM 1.2. *If the complex functions f and g are holomorphic and not identically zero on an open connected subset $\Omega \subset \mathbb{C}$, and $|f(z)| = |g(z)|$ for all $z \in \Omega$, then there exists an $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that for all $z \in \Omega$, $f(z) = \alpha g(z)$.*

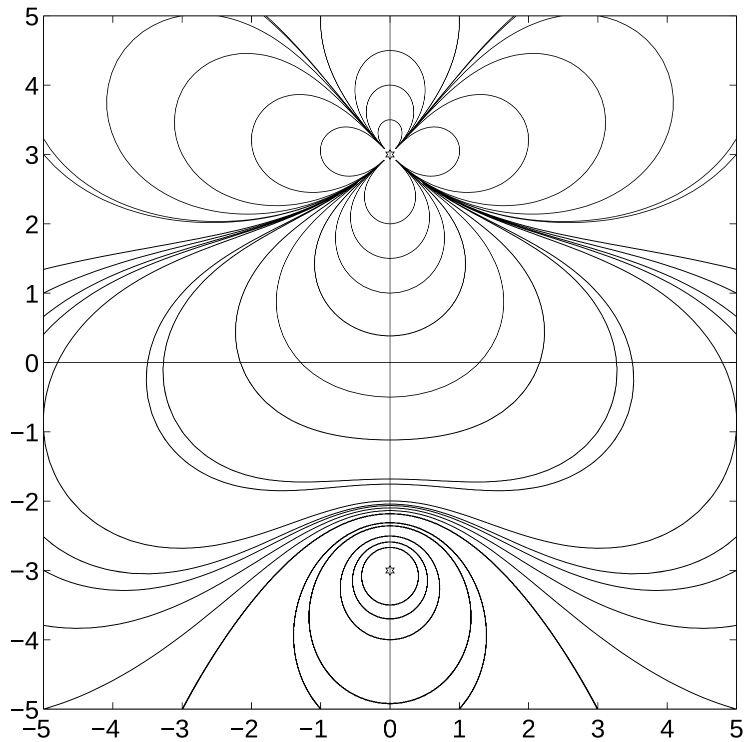

FIG. 1. Holomorphic flow for $f(z)$ a polynomial of degree 4.

Proof. With the same notation as in Theorem 1.1 above, if $z \in \Omega \setminus Z$, let $h(z) = f(z)/g(z)$. Then $|h(z)| = 1$ so $h : \Omega \to \mathbb{C}$ is constant, $h(z) = e^{i\theta}$ say. It follows that the level curves of $|f|$ determine f up to multiplication by a constant direction. ▌

2. LOCAL PROPERTIES

THEOREM 2.1. *Let f be a meromorphic function on $\Omega \subset \mathbb{C}$ and let $a \in \Omega$ be such that $f(a) = 0$ and $f'(a) \neq 0$. Let $\dot{z} = f(z)$ be the corresponding flow. Then*

(a) the critical point $z = a$ is non-degenerate,
(b) $\lambda_{\pm} = u_x \pm iv_x = \{f'(a), \overline{f'(a)}\}$,
(c) $\lambda_{\pm}$ is real if and only if $\lambda_1 = \lambda_2 = \Re f'(a)$,

(d) at the critical point the linearization of f has either a center, focus or node.

Proof. Let $f(z) = u+iv$. Since the equation for the system is equivalent to $\dot{x} = u, \dot{y} = v$ the linearization is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Hence the characteristic polynomial

$$\begin{aligned} p(\lambda) &= u_x v_y - \lambda(u_x + v_y) + \lambda^2 - v_x u_y \\ &= \lambda^2 - 2u_x\lambda + |f'(a)|^2 \end{aligned}$$

Therefore the sum of the roots $\lambda_+ + \lambda_- = 2u_x$ and product is $\lambda_+\lambda_- = f'(a)\overline{f'(a)}$. But the equation $p(\lambda) = 0$ is real, so the roots are complex conjugate. Therefore

$$\begin{aligned} \lambda_+ &= u_x + iv_x \\ \lambda_- &= u_x - iv_x, \end{aligned}$$

hence (a).

Since $f'(a) \neq 0$, $\lambda_\pm \neq 0$, so the critical point is non-degenerate.

(b) Follows since if $\lambda_\pm$ is real then $v_x = 0$ and so $\lambda_+ = \lambda_- = \Re f'(a)$.

(c) Since $\lambda_+ = \lambda_-$ when the eigenvalues are real, saddles do not exist. ∎

Theorem 2.2. *If $\dot{z} = f(z)$ has a simple pole at $z = a$ then, for some $\epsilon > 0$, the flow has the same orbits as a saddle on $B(a,\epsilon) \setminus \{a\}$*

Proof. The flow has integral curves which coincide with those of

$$\dot{z} = \frac{f(z)}{|f(z)|^2}$$

provided $f(z) \neq 0$. Assume, without loss of generality, the simple pole is at $z = 0$. Near $z = 0$,

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots$$

where $c_{-1} \neq 0$. Therefore

$$\begin{aligned}\frac{f(z)}{|f(z)|^2} &= \frac{1}{\overline{f(z)}} = \frac{1}{\overline{\frac{c_{-1}}{z} + c_0 + c_1 z + \cdots}} \\ &= \frac{\bar{z}}{\overline{c_{-1}}} \frac{1}{\overline{1 + \frac{c_0 z}{c_{-1}} + \cdots}}\end{aligned}$$

so the linearization near $z = 0$ is of the form

$$\dot{x} + \dot{y} = (a + ib)(x - iy)$$

where a and b are real with $a^2 + b^2 \neq 0$. It follows that the characteristic polynomial is $\lambda^2 - (a^2 + b^2)$, so the singular point behaves as a saddle. ∎

Theorem 2.3. *Let $\dot{z} = f(z)$ be a holomorphic flow on Ω with a center at z_o. Then all closed orbits in Ω with interior in Ω and z_o in the interior, have the same period, namely $2\pi i / f'(z_o)$.*

Proof. Let Γ be a closed orbit. Then, because there are no saddle points, Γ has one zero of f in its interior, say z_o, and that zero is simple. (check) If T is the period:

$$T = \int_0^T dt = \int_\Gamma \frac{dt}{dz} dz = \int_\Gamma \frac{dz}{f(z)} = \frac{2\pi i}{f'(z_o)}.$$

∎

Theorem 2.4. *Let $\dot{z} = f(z)$ have a critical point at $z = z_o$. Then the order m of the zero at z_o is the index of the critical point at z_o.*

Proof. Let Γ be a simple closed curve with interior containing z_o and no other zero of f. Then if I is the index of z_o:

$$I = \frac{\Delta \arg f(z)}{2\pi} = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = m.$$

∎

Theorem 2.5. *If $\dot{z} = f(z)$ is a holomorphic flow and z_o a zero of f of order $m \geq 2$, $f(z) = (z - z_o)^m g(z)$, where $g(z_o) \neq 0$, then there are $2(m-1)$ sectors for the flow at z_o and all sectors are elliptic.*

Proof. Assume, without loss of generality, that $z_o = 0$ and that, in a neighbourhood of 0, f has the representation $f(z) = az^m(1+z\phi(z))$ where ϕ is holomorphic on the neighbourhood.

Let $a = Re^{i\beta}$ and $z = re^{i\theta}$. Then since the index $m \geq 2$, the critical point 0 is not a focus or a center. Hence, ([9], Theorem 1.10.2), there are explicit directions θ at which the flow approaches or leaves 0. Each of these directions satisfies

$$\begin{aligned}
\tan(\theta) &= \frac{\dot{y}}{\dot{x}} \\
&= \lim_{r\to 0+} \frac{\Im(f(z))}{\Re(f(z))} \\
&= \lim_{r\to 0+} \frac{\Im[Rr^m e^{i(\alpha+m\theta)}(1+re^{i\theta}\phi(re^{i\theta}))]}{\Re[Rr^m e^{i(\alpha+m\theta)}(1+re^{i\theta}\phi(re^{i\theta}))]} \\
&= \lim_{r\to 0+} \frac{\sin(\alpha+m\theta)[1+\Re re^{i\theta}\phi(re^{i\theta})]+\cos(\alpha+m\theta)\Im[re^{i\theta}\phi(re^{i\theta})]}{\cos(\alpha+m\theta)(1+\Re[re^{i\theta}\phi(re^{i\theta})])-\sin(\alpha+m\theta)\Im[re^{i\theta}\phi(re^{i\theta})]} \\
&= \frac{\sin(\beta+m\theta)}{\cos(\beta+m\theta)}.
\end{aligned}$$

Therefore $\sin((m-1)\theta+\beta) = 0$ so

$$\theta = \frac{n\pi}{m-1} + \frac{\beta}{m-1}, n \in \mathbb{Z}$$

leading to $2(m-1)$ distinct directions.

But the index $I = 1 + \frac{e-h}{2}$, where e is the number of elliptic sectors and h the number of hyperbolic sectors, so $e - h = 2(m-1)$. Therefore $e = 2(m-1)$, since that is the total number of sectors, and therefore $h = 0$. ∎

Theorem 2.6. *If $\dot{z} = f(z)$ is a holomorphic flow and z_o a pole of f of order $m \geq 2$, $f(z) = (z-z_o)^{-m}g(z)$, where $g(z_o) \neq 0$, then there are $2(m+1)$ sectors for the flow at z_o and all sectors are hyperbolic.*

Proof. The proof is similar to that of the previous theorem, replacing m by $-m$. ∎

Theorem 2.7. *Let $\dot{z} = f(z) = u + iv = (\alpha+i\beta)z^m$ where $m \geq 2$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha^2+\beta^2 \neq 0$ are such that $u(x,y) = u(-x,y)$ and $v(x,y) = -v(-x,y)$. Then if m is even $\beta = 0$ and if m is odd $\alpha = 0$.*

Proof. If m is even, $(x+iy)^m = A(x,y) + iB(x,y)$ where $A(x,y) = A(-x,y)$ and $B(x,y) = -B(-x,y)$. Hence

$$u + iv = f(z) = (\alpha + i\beta)(A + iB)$$

implies $u(x,y) = \alpha A(x,y) - \beta B(x,y)$ and so $\alpha A(x,y) - \beta B(x,y) = \alpha A(-x,y) - \beta B(-x,y)$ and therefore $2\beta B(x,y) = 0$, so $\beta = 0$.

The proof for m odd is similar. ▮

THEOREM 2.8. *Let f be homomorphic on $\mathbb{C}$, $\dot{z} = f(z)$, and $\overline{f(z)} = f(\overline{z}) = f(-\overline{z})$. Then the integral paths are symmetric with respect to reflection in the x and y axes. If any point on an axis is a center for the linearized system, then it is a center for the original system $\dot{z} = f(z)$.*

Proof. If $f(z) = u + iv$ then for all x, y

$$\begin{aligned} &(1) \quad u(-x,y) = u(x,y) \\ &(2) \quad v(-x,y) = -v(x,y) \\ &(3) \quad u(x,-y) = u(x,y) \\ &(4) \quad v(x,-y) = -v(x,y) \end{aligned}$$

Let the point $a = iy$ on the y-axis be a zero of $f(z)$. By (2), $v(0,y) = 0$. By Theorem 1.1, $\lambda_{\pm} = u_x \pm iv_x$. By (1) $-u_x(-x,y) = u_x(x,y)$ so $-u_x(0,y) = u_x(0,y)$ and thus $u_x(0,y) = 0$. Hence $\lambda_{\pm} = \pm iv_x$, and a is a center for the linearized system. A system of two first order differential equations, where time t is the independent variable, is said to be invariant with respect to the y-axis if it is invariant under the transformation $(t,x) \to (-t,-x)$. But

$$\begin{aligned} \dot{x} &= u(x,y) \ \to \dot{x} = u(-x,y) = u(x,y) \text{ by (1)} \\ \dot{y} &= v(x,y) \ \to -\dot{y} = v(-x,y) = -v(x,y) \text{ by (2)} \end{aligned}$$

Hence ([9], Theorem 2.10.6), $z = a$ is a center for the system $\dot{z} = f(z)$. ▮

For holomorphic flows we have the following local-global principle at each simple zero:

THEOREM 2.9. *Let $\dot{z} = f(z) = (z - z_o)g(z)$ where f is holomorphic on a neighbourhood of z_o and $g(z_o) = a = \alpha + i\beta \neq 0$ (so z_o is a simple zero of f). Then the flow has at z_o:*

(a) a focus if $\alpha \neq 0$ and $\beta \neq 0$,

(b) a node if $\beta = 0$,
(c) a center if $\alpha = 0$,
that is to say the critical point z_o *has the same type as its linearization* $f(z) = (z - z_o)g(z_o)$.

Proof. In cases (a) and (b), $\alpha \neq 0$ so, by Theorem 1.1, the linearization has no eigenvalue with zero real part. The result then follows by the Hartman-Grobman theorem [9]. Case (c) is the theorem of Benzinger [2]. ▌

THEOREM 2.10. *If* $f(z) \neq 0$, *the curvature of the orbit passing through* z *is given by*

$$\kappa = \frac{|v_x|}{|f(z)|}.$$

Proof. Let $\dot{z} = f(z) = u + iv$. If $\mathbf{r} = (x(t), y(t))$ is an integral path, the curvature

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

where $\dot{x} = u$, $\dot{y} = v$, $\ddot{x} = uu_x + vu_y$, and $\ddot{y} = uv_x + vv_y$. The given expression for κ follows after substituting and simplifying. ▌

COROLLARY 2.1. $\kappa = 0$ *at* $z = a$ *if and only if* $\Im f'(a) = 0$.

COROLLARY 2.2. *If* $f(a) = 0$ *and* $\Im f'(a) \neq 0$ *then* κ *is unbounded in a neighbourhood of* a. *If* $f(a) \neq 0$ *then* κ *is bounded in a neighbourhood of* a. *Hence the curvature is unbounded in the neighbourhood of a center or focus. At a node it is zero.*

Conjecture: If f is non-constant, every periodic orbit has at most a finite number of points of zero curvature.

3. NO LIMIT CYCLES FOR HOLOMORPHIC FLOWS

THEOREM 3.1. *Let* $\dot{z} = f(z)$ *where* f *is holomorphic on the open region* $\Omega \subset \mathbb{C}$ *on which the vector field is complete. Then the solution* $\gamma(z,t)$ *satisfying* $\gamma(z,0) = z$ *for all* $z \in \Omega$ *and*

$$\frac{d\gamma(z,t)}{dt} = f(\gamma(z,t)$$

for all $t \in \mathbb{R}$ is holomorphic on Ω in that, for fixed T the mapping $z \to \gamma(z,T)$ is holomorphic.

Proof. Fix $T \in \mathbb{R}$. Then

$$(1)\ \gamma(z,T) = z + \int_0^T f(\gamma(z,s))ds.$$

Let $f(z) = u(x,y) + iv(x,y)$ and $\gamma(z,t) = A(x,y,t) + iB(x,y,t)$. Then, A and B are real analytic in (x,y). Partially differentiate (1) with respect to x and y to obtain:

$$(2) \quad \frac{\partial A}{\partial x} + i\frac{\partial B}{\partial x} = 1 + \int_0^T [(u_x\frac{\partial A}{\partial x} + u_y\frac{\partial B}{\partial x}) + i(v_x\frac{\partial A}{\partial x} + v_y\frac{\partial B}{\partial x})]ds$$

$$(3) \quad \frac{\partial A}{\partial y} + i\frac{\partial B}{\partial y} = i + \int_0^T [(u_x\frac{\partial A}{\partial y} + u_y\frac{\partial B}{\partial y}) + i(v_x\frac{\partial A}{\partial y} + v_y\frac{\partial B}{\partial y})]ds$$

where the functions on the left are evaluated at (x,y,T) and those on the right at $(A(x,y,s), B(x,y,s))$ in the case of the derivatives of u and v or (x,y,s) in the case of the derivatives of A and B.

Since f is holomorphic, u and v satisfy the Cauchy-Riemann equations. Using these and equating the real and imaginary parts of (2) and (3) gives:

$$(4) \quad \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = \int_0^T [u_x(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}) + u_y(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x})]ds$$

$$(5) \quad \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = \int_0^T [-u_y(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}) + u_x(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x})]ds$$

and so

$$\mathbf{C}(x,y,T) = \int_0^T \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} \mathbf{C}(x,y,s)ds$$

where

$$\mathbf{C}(x,y,s) = \begin{pmatrix} \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \\ \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \end{pmatrix}$$

and where the elements of the 2×2 matrix appearing in the integral are evaluated at $(A(x,y,s), B(x,y,s))$.

Differentiate the integral equation with respect to T to obtain the system of two ODE's

$$\dot{\mathbf{C}}(x,y,t) = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} \mathbf{C}(x,y,t).$$

Since $\gamma(z,0) = z = A(x,y,0) + iB(x,y,0)$ it follows immediately that $\mathbf{C}(x,y,0) = 0$. By existence and uniqueness of solutions of the system, the map $z \to \gamma(z,T)$ satisfies the Cauchy-Riemann equations for each T and all z. Hence, for each T, it represents a holomorphic function. ▮

Lemma 3.1. *Let Γ be a periodic solution to $\dot{z} = f(z)$, where f is holomorphic on the simply connected open subset Ω of $\mathbb{C}$. Then there is an open set G containing all points in the interior of Γ and on the graph of Γ on which the flow is complete.*

Proof. At each point z_o in the interior of Γ the orbit starting at that point is bounded, hence ([9],Corollary 2.4.2) $\gamma(z_o,t)$ exists for all $t \in \mathbb{R}$. The same is true for each point on Γ and therefore [[9], Theorem 2.4.4) on a neighbourhood of each point. Finally the union of these neighbourhoods and the interior of Γ is an open subset of $\mathbb{C}$ on which the flow is complete. ▮

Theorem 3.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected region and let f be holomorphic on Ω. Then the flow $\dot{z} = f(z)$ has no limit cycles in Ω.*

Proof. Assume there is at least one limit cycle. Since the real and imaginary parts of f are analytic, by the theorem of Poincaré, the flow has at most a finite number of limit cycles in any bounded subregion of Ω. Therefore there exists a limit cycle Γ with no other limit cycles in its interior. Inside this cycle there must be a single simple zero z_o, by the principle of the argument, which must therefore by Theorem 2.1 be a center, a node or a focus.

(a) The point z_o cannot be a center, since in that case B would be filled entirely with periodic orbits, each of which has the same period by Theorem 2.3 below, leading to a return map for Γ which is the identity, contradicting Γ being a limit cycle. If it were not filled with periodic orbits than, by the Poincare-Bendixson Theorem, there would be a limit cycle interior to Γ, which is impossible.

(b) Let z_o be a focus or node (stable or unstable): Let $z \to \gamma(z,t)$ be the mapping describing the flow. Then, by Lemma 3.1, there is a neigbourhood V of Γ and its interior region B, such that the mapping is holomorphic on V for each fixed $t \in \mathbb{R}$.

Let $0 < \delta < T$ where T is the period of the flow on Γ and let $g(z) = \gamma(z,\delta)$ if z_o is stable and let $g(z) = \gamma(z,-\delta)$ if z_o is unstable.

Then g is holomorphic on $\overline{B}$. Let $U = \{z : |z| < 1\}$ be the open unit disk. Then by the Riemann Mapping Theorem, there is a conformal map $\theta : B \to U$, which is injective and surjective and has an extension to a homeomorphism (also called θ) from $\overline{B}$ to $\overline{U}$.

This follows because boundary of B is the graph of Γ, a Jordan curve, being a homeomorphic image of the unit circle, so each boundary point is simple.

Let $h(z) = \theta \circ g \circ \theta^{-1}(z)$. Then $h : U \to U$, $h(0) = 0$ and h is holomorphic on U. It follows from the Schwartz Lemma that h is either a rotation or satisfies $|h(z)| < |z|$ for all $z \in U$.

If h is a rotation, $h(z) = e^{i\alpha}z$ for some α with $0 \leq \alpha \leq 2\pi$. Let $z_1 = \frac{1}{2}$. Then there exists a subsequence of $\mathbb{N}$ and point $z_2 \in U$ such that $h^{n_j}(z_1) \to z_2$. If $z_1 = \theta(z_3)$ with $z_3 \in B \setminus \{z_o\}$, then

$$g^{n_{j_k}}(z_3) \to \theta^{-1}(z_2) \in B \setminus \{z_o\}.$$

But $g^{n_{j_k}}(z_3) = \gamma(z_3, n_{j_k}\delta)$ which converges to z_o or a point on Γ. This contradiction shows that z_o is not a focus if h is a rotation.

If $|h(z)| < |z|$ for all $z \in U$, $h^n(z_1) \to 0 \in U$. Hence $g^n(z_3) \to z_o$. But $\gamma^{n_j}(z_3) = \gamma(z_3, -n_j\delta)$ converges to point on Γ. This contradiction completes the proof that z_o cannot be a focus or a node, so therefore the limit cycle Γ does not exist. ∎

THEOREM 3.3. *Let $\dot{z} = f(z)$ be an entire flow with center at x_o. Let P be the set consisting of x_o together with the union of all of the closed orbits of the flow which contain x_o in their interior. Then P is an open subset of $\mathbb{C}$ and ∂P consists of the at most countable union of a set of separatrices $\{\gamma(x_\lambda, t) : \lambda \in \Lambda\}$ where each $\gamma(x_\lambda, t)$ is unbounded as $t \to \pm\infty$.*

Proof. 1. Let x_o be a center. Let

$$P = \{y \mid y \text{ is on a periodic orbit about } x_o\} \cup \{x_o\}.$$

Then, since $\dot{z} = f(z)$ has no limit cycle (Theorem 3.2), P is connected. If $y \in P$ and $y \neq x_o$, then by the continuous dependence of solutions on initial conditions, any trajectory starting sufficiently close to y will circle x_o, so must be a periodic orbit, since there are no limit cycles. Hence P is open.

2. $\partial P = B$ is closed in $\mathbb{C}$. Then if $B = \emptyset$ the proof is complete. Otherwise proceed as follows:

3. If $x_o \in B$, let

$$C = \{t \in [0, \infty) \mid \text{ for all } s \text{ with } 0 \leq s \leq t,\ \gamma(x_o, s) \in b\}.$$

Let $t_o = \sup C$. If $t_o < \infty$, let $x_1 = \gamma(x_o, t_o)$. Then $x_1 \in B$ and there is a neighbourbood N of x_1 free of critical points, for which the flow is locally uniform. Therefore there is a value $t > t_o$ with $\gamma(x_o, s) \in B$ for all s with $t_o \le s \le t$, contradicting the definition of t_o. Hence $\gamma(x_o, t) \in B$ for all $t \in [0, \infty)$. Similarly $\gamma(x_o, t) \in B$ for all $t \in (-\infty, 0]$. Therefore

$$\{\gamma(x_o, t) \mid t \in \mathbb{R}\} \subset B.$$

4. By 3. we can write

$$B = \sqcup_{\lambda \in \Lambda} \{\gamma(x_\lambda, t) \mid t \in \mathbb{R}\} = \sqcup_{\lambda \in \Lambda} B_\lambda$$

where the index set Λ is non-empty and the union disjoint.

5. No B_λ can be periodic: If so let $y \in B_\lambda$ and let T be an open transversal at y. Let T have a parametrization so that points in the interior of B_λ are smaller than points in P and such that $T \subset P \cup P_\lambda$. Then the pair of sets (P, P_λ) disconnects T, which is not possible.

6. Each B_λ is closed: If not there is an $x \in \omega(B_\lambda)$ or $x \in \alpha(B_\lambda)$. Since the flow has no limit cycle, x must be a critical point, so must be center, focus, node or point with only elliptic sectors. Since B is closed, $x \in B$, so it must have at least one hyperbolic sector, which is a contradiction.

7. Each B_λ is unbounded, since otherwise it would not be closed.

8. $|\Lambda| \le \aleph_o$: By 7. each B_λ divides $\mathbb{C}$ into three subsets: $\mathbb{C} = Q_\lambda \cup B_\lambda \cup P_\lambda$ where P_λ and Q_λ are open and $P \subset Q_\lambda$. Then for $\alpha \ne \beta$, $P_\alpha \cap Q_\beta = \emptyset$ so $|\Lambda| \le \aleph_o$, since $\mathbb{C}$ is separable. ∎

An example with $B = \emptyset$ is $\dot z = iz$. The flow $\dot z = iz(z^n - 1)$ with center $x_o = 0$ has $|\Lambda| = n$. Constructing an example with $|\Lambda| = \aleph_o$ is an unsolved problem.

This theorem and the theorem of Benzinger [2], showing that rational function flows on $\mathbb{C}$ do not have limit cycles, lead to the following natural conjecture.

Conjecture: Let $f : \Omega \to \mathbb{C}$ be meromorphic, where Ω is an open. Then the flow $\dot z = f(z)$ does not have a limit cycle.

4. EXAMPLES

Example 4.1. If $\{z_j : 1 \le j \le n\}$ are distinct points on a line L in $\mathbb{C}$ then there is a θ such that

$$f(z) = e^{i\theta} \prod_{j=1}^{n} (z - z_j)$$

has a center for $\dot{z} = f(z)$ at each z_j: If L cuts the real axis at $x = \eta$ and at an angle β set $w = e^{i\beta}(z - \eta)$ so each $w_j = e^{i\beta}(z_j - \eta)$ is real and thus $\dot{w} = i\prod_{j=1}^{n}(w - w_j)$ has a center at each w_j. Changing variables

$$\dot{z} = ie^{in\beta}\prod_{j=1}^{n}(z - z_j)$$

gives $\theta = \frac{\pi}{2} + n\beta$. If L is parallel to OX and cuts OY at γ, set $w = z - i\gamma$ and derive $\theta = \frac{\pi}{2}$.

Theorem 4.1. *Let $\dot{z} = f(z) = \alpha(z - z_1)\cdots(z - z_n)$ be a flow where $\alpha \in \mathbb{C} \setminus \{0\}$, the z_i are distinct, and each is a center (for the linearized flow). If $n \leq 4$ then the z_i are collinear.*

Proof. If $n = 1$ or 2 there is nothing to prove. If $n = 3$ linearize about each of the points z_i to obtain the equations:

$$\begin{aligned} \alpha(z_1 - z_2)(z_1 - z_3) &= i\alpha_1 \\ \alpha(z_2 - z_1)(z_2 - z_3) &= i\alpha_2 \\ \alpha(z_3 - z_1)(z_3 - z_2) &= i\alpha_3 \end{aligned}$$

where the α_i are non-zero real numbers. Dividing the first two of these equations leads to

$$\frac{z_1 - z_3}{z_2 - z_3} = \beta \in \mathbb{R} \setminus \{0\}$$

so $z_1 = z_3 + \beta(z_2 - z_3)$ and therefore $\{z_1, z_2, z_3\}$ are collinear.

If $n = 4$ proceed as above and derive the equations:

$$\alpha(z_1 - z_2)(z_1 - z_3)(z_1 - z_4) = i\alpha_1 \tag{1}$$
$$\alpha(z_2 - z_1)(z_2 - z_3)(z_2 - z_4) = i\alpha_2 \tag{2}$$
$$\alpha(z_3 - z_1)(z_3 - z_2)(z_3 - z_4) = i\alpha_3 \tag{3}$$
$$\alpha(z_4 - z_1)(z_4 - z_2)(z_4 - z_3) = i\alpha_4 \tag{4}$$

Divide (1) by (2), (3) by (4) and (2) by (3) to obtain:

$$\frac{(z_1 - z_3)(z_1 - z_4)}{(z_2 - z_3)(z_2 - z_4)} = -\frac{\alpha_1}{\alpha_2} \tag{5}$$

$$\frac{(z_3 - z_1)(z_3 - z_2)}{(z_4 - z_1)(z_4 - z_2)} = -\frac{\alpha_3}{\alpha_4} \tag{6}$$

$$\frac{(z_2 - z_1)(z_2 - z_4)}{(z_3 - z_1)(z_3 - z_4)} = -\frac{\alpha_2}{\alpha_3} \tag{7}$$

Multiplying (5) and (6):

$$\big(\frac{z_1 - z_3}{z_4 - z_2}\big)^2 = \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4} \in \mathbb{R} \setminus \{0\}$$

Also, by symmetry,

$$\big(\frac{z_1 - z_2}{z_3 - z_4}\big)^2 = \in \mathbb{R} \setminus \{0\} \text{ and } \big(\frac{z_1 - z_4}{z_3 - z_2}\big)^2 = \in \mathbb{R} \setminus \{0\}.$$

If (Case I) $(z_1 - z_3)/(z_4 - z_2) \in \mathbb{R}$ then $z_1 - z_3 \parallel z_4 - z_2$. By (5), (6) and (7), $z_1 - z_4 \parallel z_2 - z_3$ and $z_2 - z_1 \parallel z_3 - z_4$. The only configuration of distinct points for which this is possible is when $\{z_1, z_2, z_3, z_4\}$ are collinear.

If (Case II) $(z_1 - z_3)/(z_4 - z_2) = i\beta_1$ for some non-zero real number β_1, then by symmetry, we may assume also that $(z_1 - z_2)/(z_3 - z_4) = i\beta_2$ and $(z_1 - z_4)/(z_3 - z_2) = i\beta_3$ for non-zero β_i, else the result would follow as in Case I. But then $z_1 - z_3 \perp z_4 - z_2$, $z_1 - z_2 \perp z_3 - z_4$ and $z_1 - z_4 \perp z_3 - z_2$, an impossible configuration for four distinct points in $\mathbb{R}^2$. ▮

Note that the same type of proof would enable the conditions to be relaxed so that ($n - 1$ of) the $f'(z_j)$ are parallel and lead to conclusions such as if $f(z)$ is a cubic with two nodes or two centers then the zeros are collinear. The same conclusion applies to a quartic with three nodes or three centers. The next example shows that the theorem cannot be extended to quintics.

Example 4.2. Define a polynomial flow of degree $n = 5$ by

$$\dot{z} = f(z) = iz(z-1)(z+1)(z-i)(z+i).$$

Then the flow has centers at $\{0, \pm 1, \pm i\}$ and, more generally, if $\{r_1, \cdots, r_N\}$ are distinct strictly positive real numbers, then

$$\dot{z} = f(z) = iz\prod_{j=1}^{N}(z^4 - r_j^4)$$

has centers at $\{\pm r_j, \pm i r_j : 1 \leq j \leq N\} \cup \{0\}$.

Again, for all $n \in \mathbb{N}$ let

$$\dot{z} = f(z) = iz(z^n - 1)$$

has centers at 0 and each of the n'th roots of unity. To see this note that at each such root of unity, $f'(z_o) = i((n+1)z_o^n - 1) = in$.

Example 4.3. Phase portrait for $\dot{z} = f(z) = iz(z-1)(z+1)(z-i)(z+i)$.

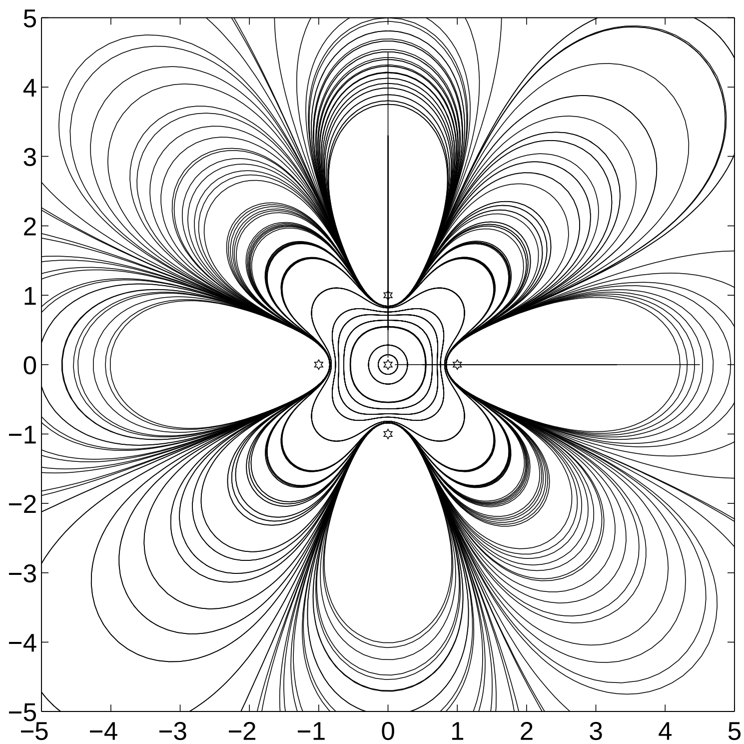

FIG. 2. Flow with five centers.

Theorem 4.2. *Let $f(z)$ be a polynomial of degree $n \geq 2$ with simple zeros $\{z_1, \cdots, z_n\}$. Then*

$$\sum_{j=1}^{n} \frac{1}{f'(z_j)} = 0.$$

Proof. Integrate $1/f(z)$ over a circle of radius R sufficiently large to contain all of the zeros of f, and let $R \to \infty$. ▮

COROLLARY 4.1. *With respect to the flow $\dot{z} = f(z)$:*
(a) If $z_1, \cdots, z_{n-1}$ are nodes, so is z_n.
(b) If $z_1, \cdots, z_{n-1}$ are centers, so is z_n.
(c) If z_1 is a focus, then the remaining zeros cannot be all nodes or all centers.
(d) If there exists only centers and nodes then there is more than one center and more than one node.
(e) No cubic system has only centers and nodes.
(f) Each cubic system has at least one focus.

Conjecture: Let $\dot{z} = f(z)$ be a polynomial system with five simple zeros, each of which is a node. Assume that the system is normalized so that $f(0) = f(1) = 0$ and $f'(0) = -1$. Then

$$f(z) = z(z-1)(z+1)(z-i)(z+i) = z(z^4 - 1).$$

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REFERENCES

1. Anosov, D.V., Aranson, S.Kh., Arnold, V.I., Bronshtein, I.U., Grines, V.Z., Il'yashenko, Yu.S. *Ordinary Differential Equations and Smooth Dynamical Systems*, Springer, 1998.

2. Benzinger, H. *Plane automomous systems with rational vector fields*, Trans. Amer. Math. Soc. bf 326, (1991), 465-484.

3. Benzinger, H. *Julia sets and differential equations*, Proc. Amer. Math. Soc. **117**, (1993), 939-946.

4. Jongen, H.Th., Jonker,P., and Twilt,F. *The continuous Newton-method for meromorphic functions*, Springer-Verlag, Lecture Notes in Mathematics **810**(1980) p181-239, Proceedings of the conference Geometrical Approaches to Differential Equations, Schevanlingen, 1979.

5. Jongen, H.Th., Jonker,P., and Twilt,F. *The continuous desingularized Newton method for meromorphic functions*, Acta Applic. Math. **13**, (1988), 81-121.

6. Lloyd, N. G. *On the number of periodic solutions of the equation* $\dot{z} = z^N + p_1(t)z^{N-1} + \cdots + p_N(t)$, Proc. Lond. Math. Soc (3) **27** (1973), 667-700.

7. Lloyd, N. G. *An analytic differential equations*, Proc. Lond. Math. Soc (3) **30** (1975), 430-444.

8. Needham, T. *Visual Complex Analysis*, Oxford, 1997.

9. Perko, L. *Differential Equations and Dynamical Systems, Second Edition*, Springer, 1996.