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SPECTRAL ANALYSIS OF ABSTRACT FUNCTIONS

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Introduction

In this paper we consider functions $x(t)$ of a real variable t, $-\infty<t<+\infty$, with values in a Hilbert space H, which are in a certain sense a generalization of stationary functions, i.e., which are such that the scalar product

$$(x_{t+\tau}, x_t) = B(\tau) \tag{1.0}$$

does not depend on t. As is well known, a function which satisfies the condition (1.0) can be represented in the form

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi(d\lambda), \tag{2.0}$$

where $\Phi(\Delta)$ is an additive function of the interval Δ with values in H such that

$$(\Phi(\Delta), \Phi(\Delta')) = F(\Delta \cap \Delta'), \tag{3.0}$$

where $F(\Delta)$ is the so-called spectral measure.

In some cases the spectral theory which has been developed for stationary functions (see e.g. [1]) can also be generalized to non-stationary functions. In this paper such a generalization is made for two classes of functions: One such class consists of functions $x(t)$ which can be represented in the form (2.0), where $\Phi(\Delta)$ need not necessarily satisfy the condition (3.0). The other such class consists of functions for which

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (x_{t+\tau}, x_t)\, dt = B(\tau) \tag{4.0}$$

exists for all τ, $-\infty < \tau < +\infty$. Persuing this we study, at the beginning of the paper, additive functions $\Phi(\Delta)$ with values in H. The results which are thereby obtained are valid for functions $\Phi(\Delta)$ defined on a system $\mathfrak{S}$ of subsets of an arbitrary space R; it is only required that the system $\mathfrak{S}$ be a semi-ring (cf. [8]). We also remark that most of the results presented here can be automatically carried over to the case where the space H is an arbitrary Banach space.

1. Abstract Measures and Integration

1°. Let H be a Hilbert space and $\Phi(\Delta)$ an additive function (measure) of the sets of the line $R = \{-\infty < \lambda < \infty\}$ with values in H; $\Phi(\Delta) \in H$ is defined on the class of sets which consists of the empty set $\varnothing$ of the whole line R and all possible finite sums of half intervals $\Delta = [\lambda_1, \lambda_2)$. The additivity property of $\Phi(\Delta)$ consists in the fact that if

$$\Delta = \bigcup_{k=1}^{m} \Delta_k, \qquad \Delta, \Delta_k \in \mathfrak{S},$$

where the Δ_k do not overlap, then

$$\Phi(\Delta) = \sum_{k=1}^{m} \Phi(\Delta_k). \tag{1.1}$$

We set

$$F(\Delta, \Delta') = (\Phi(\Delta), \Phi(\Delta')). \tag{2.1}$$

By (1.1), $F(\Delta, \Delta')$ is an additive function of the sets $\Delta \times \Delta'$ of the plane, defined on the class $\mathfrak{S}^2$. We note that the function $F(\Delta, \Delta')$ has the important property of being positive definite, i.e.

$$F(\Delta, \Delta') = \overline{F(\Delta', \Delta)}, \sum_{k=1}^{m} \sum_{j=1}^{m} \alpha_k \bar{\alpha}_j F(\Delta_k, \Delta_j) \geqq 0 \tag{3.1}$$

for any choice of the complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$ and the sets $\Delta_1, \Delta_2, \cdots, \Delta_m$ of $\mathfrak{S}$.

We call the measure $\Phi(\Delta)$ continuous if, given any monotone decreasing sequence of sets

$$\Delta_n, \ \Delta_n \supseteqq \Delta_{n+1}, \ \bigcap_{n=1}^{\infty} \Delta_n = \varnothing,$$

we have

$$||\Phi(\Delta_n)|| \to 0 \tag{4.1}$$

as $n \to \infty$. (The symbol $||h||$ denotes the norm of the element h, as usual; $||h|| = (h, h)^{1/2}$.) The condition (4.1) is obviously equivalent to the continuity of the function $F(\Delta, \Delta')$, i.e.

$$F(\Delta_m, \Delta_n) \to 0 \tag{5.1}$$

as $m, n \to \infty$.

Theorem 1.1. *For any continuous additive function $F(\Delta, \Delta')$ of the sets $\Delta \times \Delta'$, $\Delta, \Delta' \in \mathfrak{S}$, which has the property* (3.1), *there exists a measure $\Phi(\Delta)$, $\Delta \in \mathfrak{S}$, in the space H such that*

$$(\Phi(\Delta), \Phi(\Delta')) = F(\Delta, \Delta').$$

Proof. Let $\mathfrak{S}_r$ be the class of half intervals $\Delta = [\lambda_1, \lambda_2)$ with rational endpoints λ_1, λ_2. It is well known (see e.g. [1]) that it follows from the positive definiteness property (3.1) that there exist elements $\Phi(\Delta) \in H$, $\Delta \in \mathfrak{S}_r$, such that

$$(\Phi(\Delta), \Phi(\Delta')) = F(\Delta, \Delta').$$

Moreover, let $\Delta_1, \Delta_2, \cdots, \Delta_m$ be non-overlapping and let

$$\Delta = \bigcup_{k=1}^{m} \Delta_k, \qquad \Delta, \Delta_k \in \mathfrak{S}_r, \qquad \Phi'(\Delta) = \sum_{k=1}^{m} \Phi(\Delta_k).$$

We have

$$||\Phi(\varDelta)-\Phi'(\varDelta)||^2 = F(\varDelta, \varDelta) - \sum_{k=1}^{m} F(\varDelta_k, \varDelta) - \sum_{k=1}^{m} F(\varDelta, \varDelta_k) + \sum_{k=1}^{m}\sum_{j=1}^{m} F(\varDelta_k, \varDelta_j)$$

$$= F(\varDelta, \varDelta) - F(\bigcup_{k=1}^{m} \varDelta_k, \varDelta) - F(\varDelta, \bigcup_{k=1}^{m} \varDelta_k) + F(\bigcup_{k=1}^{m} \varDelta_k, \bigcup_{j=1}^{m} \varDelta_j) = 0$$

and thus

$$\Phi(\varDelta) = \sum_{k=1}^{m} \Phi(\varDelta_k).$$

Because of the continuity of F, if

$$\varDelta \in \mathfrak{S} \text{ and } \varDelta = \bigcap_n \varDelta_n, \qquad \varDelta_n \supseteqq \varDelta_{n+1}, \qquad \varDelta_n \in \mathfrak{S}_r,$$

we have

$$||\Phi(\varDelta_m)-\Phi(\varDelta_n)||^2 = ||\Phi(\varDelta_m-\varDelta_n)||^2$$
$$= F(\varDelta_m-\varDelta_n, \varDelta_m-\varDelta_n) \to 0$$

as $n > m \to \infty$. We set

$$\Phi(\varDelta) = \lim_{n\to\infty} \Phi(\varDelta_n),$$

and define $\Phi(\varDelta)$ on the whole class $\mathfrak{S}$ by the property of finite additivity. The function $\Phi(\varDelta)$ so obtained obviously satisfies the conditions of the theorem.

We shall say that the function $\Phi(\varDelta)$ is of bounded variation if for any element $h \in H$, the additive numerical function $F_h(\varDelta)$,

(6.1) $$F_h(\varDelta) = (\Phi(\varDelta), h),$$

is of bounded variation, i.e.

(7.1) $$\sup_{\varDelta_k} \sum_{k=1}^{m} |F_h(\varDelta_k)| = \operatorname{Var} F_h < \infty,$$

where the sup is taken over all finite families $\varDelta_1, \varDelta_2, \cdots, \varDelta_m$ of non-overlapping sets of $\mathfrak{S}$ (cf. [4], [5]). It is not hard to show (see e.g. [4]) that if $\Phi(\varDelta)$ is of bounded variation, then there exists a constant M which does not depend on h such that

(8.1) $$\operatorname{Var} F_h \leqq M \cdot ||h||.$$

It follows from (8.1) that

(9.1) $$\sup_{\varDelta_k, \alpha_k} \left\| \sum_{k=1}^{m} \alpha_k \Phi(\varDelta_k) \right\| \leqq M,$$

where the sup is taken over all finite families $\varDelta_1, \varDelta_2, \cdots, \varDelta_m$ of non-overlapping sets of $\mathfrak{S}$ and over arbitrary complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$, $|\alpha_k| \leqq 1$, since

$$\left| \left(\sum_{k=1}^{m} \alpha_k \Phi(\varDelta_k), h \right) \right| = \left| \sum_{k=1}^{m} a_k (\Phi(\varDelta_k), h) \right| \leqq \sum_{k=1}^{n} |F_h(\varDelta)| \leqq \operatorname{Var} F_h \leqq M\,||h||$$

for any h, and in particular for $h = \sum_{k=1}^{m} \alpha_k \Phi(\varDelta_k)$.

We define the variation of the measure Φ on the set $\varDelta \in \mathfrak{S}$ by the formula

(10.1) $$\operatorname{Var}_\varDelta \Phi = \sup_{\varDelta_k, \alpha_k} \left\| \sum_{k=1}^{m} \alpha_k \Phi(\varDelta_k) \right\|,$$

where the sup is taken over all finite families $\Delta_1, \Delta_2, \cdots, \Delta_m, \Delta_k \subseteqq \Delta$, of non-overlapping sets of $\mathfrak{S}$ and over arbitrary complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$, $|\alpha_k| \leqq 1$. We note that $\mathrm{Var}_\Delta\, \Phi$ as a set function is monotone and subadditive, i.e. if $\Delta_1 \cap \Delta_2 = \varnothing$, $\Delta = \Delta_1 \cup \Delta_2$, then

$$\mathrm{Var}_{\Delta_1} \Phi \leqq \mathrm{Var}_\Delta\, \Phi \leqq \mathrm{Var}_{\Delta_1} \Phi + \mathrm{Var}_{\Delta_2} \Phi. \tag{11.1}$$

Obviously

$$\mathrm{Var}_\Delta\, F_h \leqq \mathrm{Var}_\Delta\, \Phi ||h|| \tag{12.1}$$

holds for any set $\Delta \in \mathfrak{S}$ and any element $h \in H$.

We call the abstract measure $\Phi(\Delta)$ weakly continuous if the numerical measure $F_h(\Delta) = (\Phi(\Delta), h)$ is continuous for every $h \in H$, i.e. if the sequence $\Delta_n \in \mathfrak{S}$ is such that if $\Delta_n \supseteqq \Delta_{n+1}$ and $\bigcap_{n=1}^{\infty} \Delta_n = \varnothing$, then

$$F_h(\Delta_n) \to 0 \tag{13.1}$$

as $n \to \infty$. We denote by $H(\Phi)$ the linear closure of the elements $\Phi(\Delta)$, $\Delta \in \mathfrak{S}$.

Theorem 2.1. *Every weakly continuous abstract measure $\Phi(\Delta)$ of bounded variation, $\Phi(\Delta) \in H$, defined on $\mathfrak{S}$, can be uniquely extended to a weakly continuous measure $\tilde{\Phi}(\Delta)$, $\Phi(\Delta) \in H(\Phi)$, which is now defined on the algebra $\tilde{\mathfrak{S}}$ of all Lebesgue measurable sets of the line R, where* $\mathrm{Var}_R\, \tilde{\Phi} = \mathrm{Var}_R\, \Phi$.

Proof. We take any element $h \in H(\Phi)$ and consider the numerical measure $F_h(\Delta)$. By (13.1) it is σ-additive on $\mathfrak{S}$ and therefore can be uniquely extended to the algebra $\tilde{\mathfrak{S}}$ of all Lebesgue measurable sets of the line R (see e.g. [8]). The value of this measure $F_h(\Delta)$ for any fixed $\Delta \in \mathfrak{S}$ defines a functional of $h \in H(\Phi)$ where because of the obvious relations

$$\begin{gathered} F_{h_1+h_2}(\Delta) = F_{h_1}(\Delta) + F_{h_2}(\Delta), \qquad F_{\alpha h}(\Delta) = \bar{\alpha} F_h(\Delta), \\ F_h(\Delta) \leqq \mathrm{Var}_R\, F_h \leqq \mathrm{Var}_R\, \Phi \cdot ||h|| \end{gathered} \tag{14.1}$$

this functional is linear; therefore, by Riesz' theorem on the general form of a linear functional, there exists an element $\tilde{\Phi}(\Delta) \in H(\Phi)$ such that

$$F_h(\Delta) = (\Phi(\Delta), h) \tag{15.1}$$

for all $h \in H(\Phi)$. If $\Delta \in \mathfrak{S}$, then $\tilde{\Phi}(\Delta) = \Phi(\Delta)$. Obviously, the function $\tilde{\Phi}(\Delta)$, $\Delta \in \tilde{\mathfrak{S}}$, defined by the relation (15.1) is additive. Moreover

$$\mathrm{Var}_\Delta\, F_h \leqq \mathrm{Var}_{\Delta'}\, F_h \leqq \mathrm{Var}_{\Delta'}\, \Phi \cdot ||h|| \tag{16.1}$$

for any $\Delta \in \tilde{\mathfrak{S}}$ and $\Delta' \in \tilde{\mathfrak{S}}$, $\Delta \subseteqq \Delta'$, and $h \in H(\Phi)$, and therefore

$$\mathrm{Var}_\Delta\, \tilde{\Phi} \leqq \sup_{||h||=1} \mathrm{Var}_\Delta\, F_h \leqq \mathrm{Var}_{\Delta'}\, \Phi. \tag{17.1}$$

Thus, the measure $\tilde{\Phi}(\Delta)$ defined by the relation (14.1) satisfies the conditions of the theorem.

2⁰. Because of the preceding theorem, we can assume without loss of generality that the weakly continuous abstract measure $\Phi(\Delta)$ of bounded variation is defined on all the Lebesgue measurable sets of the line R. Let the measurable function $\varphi(\lambda)$ take only a finite number of values:

$$\varphi(\lambda) = a_k \quad \text{for } \lambda \in \Delta_k, \qquad k = 1, 2, \cdots, m. \tag{18.1}$$

We define the integral $I(\varphi) = \int_{-\infty}^{\infty} \varphi(\lambda)\Phi(d\lambda)$ by the formula

$$I(\varphi) = \sum_{k=1}^{m} a_k \Phi(\Delta_k). \tag{19.1}$$

The integral of a bounded measurable function $\varphi(\lambda)$ is defined as (cf. [4])

$$I(\varphi) = \int_{-\infty}^{\infty} \varphi(\lambda)\Phi(d\lambda) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(\lambda)\Phi(d\lambda), \tag{20.1}$$

where $\varphi_n(\lambda)$ is a sequence of functions of the form (19.1) which converges uniformly to $\varphi(\lambda)$ as $n \to \infty$, and the limit is understood in the sense of convergence in the norm. The definition (20.1) is correct, since, if $\psi_m(\lambda)$ is a sequence of functions of the form (18.1) which also converges to $\varphi(\lambda)$ uniformly as $m \to \infty$, then

$$||I(\varphi_n) - I(\psi_m)|| \leqq \sup_{\lambda} |\varphi_n(\lambda) - \psi_m(\lambda)| \operatorname{Var}_R \Phi \to 0 \tag{21.1}$$

as $n, m \to \infty$. We define the integral $I_\Delta(\varphi)$ over the set Δ by the relation

$$I_\Delta(\varphi) = \int_\Delta \varphi(\lambda)\Phi(d\lambda) = \int_{-\infty}^{\infty} \varphi(\lambda)\chi_\Delta(\lambda)\Phi(d\lambda), \tag{22.1}$$

where $\chi_\Delta(\lambda)$ is the characteristic function of the set Δ.

Moreover, we say that the unbounded function $\varphi(\lambda)$ is integrable, if there exists a sequence $\varphi_n(\lambda)$ of functions of the form (18.1) which converges to $\varphi(\lambda)$ uniformly in λ on every set $A_N = \{\lambda: |\varphi(\lambda)| \leqq N\}$ for any fixed N, and if in addition

$$||I_{R/A_N}(\varphi_n)|| \to 0 \tag{23.1}$$

uniformly in n as $N \to \infty$. We denote this convergence by $\varphi_n(\lambda) \Rightarrow \varphi(\lambda)$. In this case we set

$$I(\varphi) = \int_{-\infty}^{\infty} \varphi(\lambda)\Phi(d\lambda) = \lim_{n\to\infty} I(\varphi_n). \tag{24.1}$$

(The limit in (24.1) exists and does not depend on the sequence $\varphi_n(\lambda)$, $\varphi_n(\lambda) \Rightarrow \varphi(\lambda)$.)

We note the following basic properties of the integral $I(\varphi)$:

(1) $I_\Delta(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 I_\Delta(\varphi_1) + \alpha_2 I_\Delta(\varphi_2)$.

(2) The inequality

$$||I_\Delta(\varphi)|| \leqq \sup_{\lambda\in\Delta} |\varphi(\lambda)| \operatorname{Var}_\Delta \Phi \tag{25.1}$$

holds.

(3) If $\varphi_n(\lambda)$ are integrable and $\varphi_n(\lambda) \Rightarrow \varphi(\lambda)$ as $n \to \infty$, then $\varphi(\lambda)$ is also integrable and

$$I(\varphi) = \lim_{n\to\infty} I(\varphi_n).$$

(4) If $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_1 \cap \Delta_2 = \varnothing$, then $I_\Delta(\varphi) = I_{\Delta_1}(\varphi) + I_{\Delta_2}(\varphi)$; if moreover $\varphi(\lambda)$ is bounded, then the integral $I_\Delta(\varphi)$ is a weakly continuous measure of bounded variation and

$$\operatorname{Var}_\Delta I(\varphi) = \sup_{|\alpha_k|\leqq 1,\, \Delta_k \subseteqq \Delta} \left\| \sum_k \alpha_k I_{\Delta_k}(\varphi) \right\| \leqq \sup_{\lambda\in\Delta} |\varphi(\lambda)| \operatorname{Var}_\Delta \Phi.$$

3⁰. The continuous abstract measure $\Phi(\Delta)$ is related in a one-to-one way to the continuous additive function $F(\Delta, \Delta')$,

$$F(\Delta, \Delta') = (\Phi(\Delta), \Phi(\Delta'))$$

(cf. Theorem 1.1), and

$$\begin{gathered} F(\Delta, \Delta') = \overline{F(\Delta', \Delta)}, \\ \sum_{k=1}^{m} \sum_{j=1}^{m} \alpha_k \bar{\alpha}_j F(\Delta_k, \Delta_j) \geqq 0 \end{gathered} \tag{26.1}$$

for any complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$ and measurable sets $\Delta_1, \Delta_2, \cdots, \Delta_m$. We define the variation of the function $F(\Delta, \Delta')$ on the set $\Delta \times \Delta'$ by the relation

$$\operatorname{Var}_{\Delta, \Delta'} F = \sup \left| \sum_{k=1}^{m} \sum_{j=1}^{m} \alpha_k \bar{\beta}_j F(\Delta_k, \Delta'_j) \right|, \tag{27.1}$$

where the sup is taken over all finite families of non-intersecting measurable sets $\Delta_1, \Delta_2, \cdots, \Delta_m, \Delta_k \subseteqq \Delta$ and $\Delta'_1, \Delta'_2, \cdots, \Delta'_n, \Delta'_j \subseteqq \Delta'$, and complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m, |\alpha_n| \leqq 1$, and $\beta_1, \beta_2, \cdots, \beta_n, |\beta_j| \leqq 1$. It follows from (26.1) that

$$\begin{gathered} \operatorname{Var}_{\Delta, \Delta'} F \leqq (\operatorname{Var}_{\Delta, \Delta} F)^{1/2} \cdot (\operatorname{Var}_{\Delta', \Delta'} F)^{1/2}, \\ \operatorname{Var}_{\Delta_1 \cup \Delta_2, \Delta'} F \leqq \operatorname{Var}_{\Delta_1, \Delta'} F + \operatorname{Var}_{\Delta_2, \Delta'} F. \end{gathered} \tag{28.1}$$

If $F(\Delta, \Delta')$ is of bounded variation, i.e.

$$\operatorname{Var}_{R,R} F < \infty, \tag{29.1}$$

then in analogy to Section 2⁰ we can define the integral

$$I(\varphi, \psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\lambda) \bar{\psi}(\mu) F(d\lambda, d\mu)$$

first for "step functions" $\varphi(\lambda), \psi(\lambda)$ of the form (18.1) as

$$I(\varphi, \psi) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k \bar{b}_j F(\Delta_k, \Delta'_j), \tag{30.1}$$

and then for integrable functions $\varphi(\lambda)$ and $\psi(\lambda)$ by passing to the limit

$$I(\varphi, \psi) = \lim_{n, m \to \infty} I(\varphi_n, \psi_m), \tag{31.1}$$

where $\varphi_n(\lambda)$ and $\psi_m(\lambda)$ are sequences of step functions $\varphi_n(\lambda) \Rightarrow \varphi(\lambda)$, $\psi_m(\lambda) \Rightarrow \psi(\lambda)$. The convergence $\varphi_n(\lambda) \Rightarrow \varphi(\lambda)$ means that for any N the sequence $\varphi_n(\lambda) \to \varphi(\lambda)$ converges uniformly on the set $A_N = \{\lambda : |\varphi(\lambda)| \leqq N\}$ and

$$\int_{R-A_N} \int_{R-A_N} \varphi_n(\lambda) \overline{\varphi_n(\mu)} F(d\lambda, d\mu) \to 0$$

uniformly in n as $N \to \infty$. The integral defined in this way has all the properties analogous to (25.1). We note that if $\Phi(\Delta)$ is the abstract measure corresponding to the function $F(\Delta, \Delta')$ (see Theorem 1.1) then

$$(I(\varphi), I(\psi)) = I(\varphi, \psi). \tag{32.1}$$

In the case where the ordinary variation $V(F)$ of the function $F(\Delta, \Delta')$

$$V(F) = \sup_{\Delta_k, \Delta'_j} \sum_k \sum_j |F(\Delta_k, \Delta'_j)| < \infty,$$

our integral $I(\varphi, \psi)$ agrees with the ordinary Lebesgue integral.

We call the additive function $F(\Delta, \Delta')$ strongly continuous if, given a decreasing sequence of sets Δ_n,

$$\Delta_n \supseteqq \Delta_{n+1}, \qquad \bigcap_{n=1}^{\infty} \Delta_n = \varnothing,$$

we have

$$\mathrm{Var}_{\Delta_n, \Delta_n} F \to 0$$

as $n \to \infty$.

4⁰. Consider a function $x(t)$ of the real variable t, $-\infty < t < +\infty$, with values in H, $x(t) \in H$, which is such that the linear closure $H(x)$ of the elements $x(t)$, $-\infty < t < +\infty$, is separable. The function $x(t)$ is called measurable if the numerical function $||x(t)||$ is measurable (see e.g. [5]). The measurable function $x(t)$ is called integrable on the set T, if (cf. [4], [5])

$$\int_T ||x(t)||\, dt < \infty. \tag{33.1}$$

As shown in [5], if $x(t)$ is integrable, then there exists a sequence of "step functions" $x_n(t)$, $x_n(t) \in H(x)$, $x_n(t) = x_k$ for $t \in T_{kn} \subseteqq T$, which converges to $x(t)$ almost uniformly (i.e. for any $\varepsilon > 0$ one can find a set T_0 of Lebesgue measure $l(T_0) \leqq \varepsilon$ such that $||x_n(t) - x(t)|| \to 0$ uniformly in t, $t \in T - T_0$, as $n \to \infty$) and such that

$$\int_T ||x_n(t) - x_m(t)||\, dt \to 0 \tag{34.1}$$

as $n, m \to \infty$. If we set

$$\sum_k x_{kn}\, l(T_{kn}) = \int_T x_n(t)\, dt \tag{35.1}$$

the series in (35.1) converges in norm; then the limit (in the norm)

$$\lim_{n\to\infty} \int_T x_n(t) dt = \int_T x(t)\, dt \tag{36.1}$$

is called the integral of the function $x(t)$ (cf. [4], [5]).

Moreover, let the function $\varphi(\lambda, t)$ be measurable in the pair of variables (λ, t) and let it be integrable with respect to the abstract measure $\Phi(\Delta)$, $\Phi(\Delta) \in H$, for almost all t. Consider the function $x(t)$, where

$$x(t) = \int_{-\infty}^{\infty} \varphi(\lambda, t)\, \Phi(d\lambda). \tag{37.1}$$

The function $x(t)$ is measurable, since the numerical function $(x(t), h)$,

$$(x(t), h) = \int_{-\infty}^{\infty} \varphi(\lambda, t) F_h(d\lambda), \tag{38.1}$$

is measurable for any $h \in H$, and it is integrable on the set T if the numerical function $||x(t)||$,

$$||x(t)||^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\lambda, t)\overline{\varphi(\mu, t)}\, F(d\lambda, d\mu),$$

is integrable on T and

$$h_1 = \int_T x(t)\,dt = \int_T \int_{-\infty}^{\infty} \varphi(\lambda, t)\Phi(d\lambda)dt. \tag{39.1}$$

Now let $\varphi(\lambda, t)$ be such that the function $\psi(\lambda)$, $\psi(\lambda) = \int_T \varphi(\lambda, t)\,dt$, is integrable with respect to the measure $\Phi(\Delta)$:

$$h_2 = \int_{-\infty}^{\infty} \psi(\lambda)\Phi(d\lambda) = \int_{-\infty}^{\infty}\int_T \varphi(\lambda, t)\,dt\,\Phi(d\lambda). \tag{40.1}$$

For any $h \in H$ we have

$$\begin{aligned}(h_1, h) &= \int_T (x(t), h)dt = \int_T \int_{-\infty}^{\infty} \varphi(\lambda, t)\, F_h(d\lambda)dt \\ &= \int_{-\infty}^{\infty}\int_T \varphi(\lambda, t)\,dt\,F_h(d\lambda) = \int_{-\infty}^{\infty} \psi(\lambda)\, F_h(d\lambda) = (h_2, h),\end{aligned}$$

and therefore $h_1 = h_2$, i.e.

$$\int_T \int_{-\infty}^{\infty} \varphi(\lambda, t)\Phi(d\lambda)\,dt = \int_{-\infty}^{\infty}\int_T \varphi(\lambda, t)\,dt\,\Phi(d\lambda). \tag{41.1}$$

2. Harmonisable Abstract Functions

1°. Consider an abstract function $x(t)$ of a real variable with values in the Hilbert space H. We shall call the function $x(t)$ harmonisable if it can be represented in the form

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda), \tag{1.2}$$

where $\Phi(\Delta)$ is an abstract measure of bounded variation with values in H; $\Phi(\Delta)$ is defined on all measurable sets and is strongly continuous (strong continuity of $\Phi(\Delta)$ means that for any sequence of measurable sets Δ_n, $\Delta_n \supseteqq \Delta_{n+1}$, $\bigcap_{n=1}^{\infty} \Delta_n = \varnothing$, we have $\mathrm{Var}_{\Delta_n} \Phi \to 0$ as $n \to \infty$). We now examine some examples of harmonisable functions.

Example 1.2. The function $x(t)$ is stationary, i.e., the scalar product $(x(t+\tau), x(t)) = B(\tau)$ does not depend on t. In this case

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda); \tag{2.2}$$

here the measure $\Phi(\Delta)$ is such that

$$(\Phi(\Delta), \Phi(\Delta')) = F(\Delta \cap \Delta'), \tag{3.2}$$

where $F(\Delta)$ is a positive bounded measure on the line.

Example 2.2. Let $x(t)$ be a stationary function

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda),$$

and let

$$p(t) = \int_{-\infty}^{\infty} e^{i\lambda t} P(d\lambda), \tag{4.2}$$

where $P(\Delta)$ is a complex measure on the line. Then the abstract function $y(t) = x(t)p(t)$ can be represented in the form

$$y(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \psi(d\lambda), \tag{5.2}$$

where the abstract measure $\psi(\Delta)$ has the form

$$\psi(\Delta) = \int_{-\infty}^{\infty} P(\Delta-\mu)\Phi(d\mu). \tag{6.2}$$

EXAMPLE 3.2. Let $x(t)$ be a stationary function as before, of the form (2.2), let H_x be the linear closure of the elements $x(t)$, $-\infty < t < +\infty$, and let A be a linear operator from H_x to some subspace $H_y \subseteqq H$. Consider the function $y(t) = Ax(t)$. We have

$$y(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \psi(d\lambda), \tag{7.2}$$

where the measure $\psi(\Delta)$ is defined by the relation

$$\begin{gathered} \psi(\Delta) = A\Phi(\Delta), \\ \operatorname{Var}_\Delta \psi \leqq \|A\| \operatorname{Var}_\Delta \Phi = \|A\| F(\Delta), \end{gathered} \tag{8.2}$$

and $F(\Delta)$ is a positive measure on the line, defined by the relation (3.2).

Theorem 1.2.[1] *In order that the abstract function $x(t)$, $x(t) \in H$, be harmonisable, i.e., in order that it can be represented in the form* (1.2), *it is necessary and sufficient that the numerical function $B(u, v) = (x(u), x(v))$ can be represented in the form*

$$B(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda u-\mu v)} F(d\lambda, d\mu), \tag{9.2}$$

where $(\Phi(\Delta), \Phi(\Delta')) = F(\Delta, \Delta')$, $\Phi(\Delta) \in H(x)$ ($F(\Delta, \Delta')$ is the strongly continuous additive function described in Section 1, 3^0).

PROOF. The necessity is obvious. To prove the sufficiency consider the space L of functions $\varphi(\lambda)$ for which

$$I(\varphi, \varphi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\lambda)\overline{\varphi(\mu)}\, F(d\lambda, d\mu) \tag{10.2}$$

exists. We define the scalar product $(\varphi, \psi) = I(\varphi, \psi)$ on L and we identify all functions φ, ψ for which $\|\varphi-\psi\| = 0$, $\|\varphi\|^2 = (\varphi, \varphi)$. We define the mapping T from the space L to $H(x)$ by the relation $Te^{i\lambda t} = x(t)$. It is isometric, i.e.

$$(Te^{i\lambda u}, Te^{i\lambda v}) = B(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda u-\mu v)} F(d\lambda, d\mu) = (e^{i\lambda u}, e^{i\lambda v}).$$

Because of the strong continuity of $F(\Delta, \Delta')$ (see Section 1, 3^0), the characteristic function $\chi_\Delta(\lambda)$ of the half intervals $\Delta = [\lambda_1, \lambda_2)$ can be approximated by linear combinations $\sum_k a_k e^{i\lambda t_k}$. Therefore the mapping T can be extended, pre-

[1] See [10].

serving the isometry, to characteristic functions $\chi_\Delta(\lambda)$, $\Delta \in \mathfrak{S}$. Set $\Phi(\Delta) = T\chi_\Delta(\lambda)$. The measure $\Phi(\Delta)$ is a continuous measure of bounded variation, $(\Phi(\Delta), \Phi(\Delta')) = F(\Delta, \Delta')$, defined on $\mathfrak{S}$. By Theorem 1.1 it can be extended to all measurable sets with

$$(\Phi(\Delta), \Phi(\Delta')) = F(\Delta, \Delta') \tag{11.2}$$

for any measurable sets Δ and Δ'. It follows from the strong continuity of $F(\Delta, \Delta')$ and from relation (11.2) that the measure $\Phi(\Delta)$ itself is strongly continuous. This proves the theorem.

Theorem 2.2. *Let $x(t)$ be a harmonisable function*

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda).$$

Then

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,e^{-i\lambda_0 t}\,dt = \Phi(\lambda_0), \\ &\lim_{T_1, T_2\to\infty} \frac{1}{T_1 T_2}\int_0^{T_1}\int_0^{T_2} B(u, v)\,e^{-i(\lambda_0 u-\mu_0 v)}\,dudv = F(\lambda_0, \mu_0). \end{aligned} \tag{12.2}$$

Proof. It follows from Section 1, 4^0 that the function $x(t)e^{-i\lambda_0 t}$ is integrable on every finite interval and that

$$\frac{1}{T}\int_0^T x(t)e^{-i\lambda_0 t}\,dt = \int_{-\infty}^{\infty}\frac{1}{T}\int_0^T e^{i(\lambda-\lambda_0)t}\,dt\Phi(d\lambda).$$

Let $\chi_{\lambda_0}(\lambda) = 1$ if $\lambda = \lambda_0$ and $\chi_{\lambda_0}(\lambda) = 0$ if $\lambda \neq \lambda_0$. We have

$$\begin{aligned} &\left\|\frac{1}{T}\int_0^T x(t)\,e^{-i\lambda_0 t}\,dt - \Phi(\lambda_0)\right\| \\ &= \left\|\int_{-\infty}^{\infty}\left[\frac{1}{T}\int_0^T e^{i(\lambda-\lambda_0)t}\,dt - \chi_{\lambda_0}(\lambda)\right]\Phi(d\lambda)\right\| \leqq \left\|\int_{|\lambda-\lambda_0|\geqq\varepsilon}\left[\frac{1}{T}\int_0^T e^{i(\lambda-\lambda_0)t}\,dt\right]\Phi(d\lambda)\right\| \\ &+ \left\|\int_{0<|\lambda-\lambda_0|<\varepsilon}\left[\frac{1}{T}\int_0^T e^{i(\lambda-\lambda_0)t}\,dt\right]\Phi(d\nu)\right\| \leqq \frac{2}{T\varepsilon}\operatorname{Var}\Phi + \operatorname{Var}_{\Delta_\varepsilon}\Phi \end{aligned}$$

for any T and $\varepsilon > 0$, $\Delta_\varepsilon = \{\lambda\colon 0 < |\lambda-\lambda_0| < \varepsilon\}$. Since $\bigcap_\varepsilon \Delta_\varepsilon = \varnothing$, then $\operatorname{Var}_{\Delta_\varepsilon}\Phi \to 0$ as $\varepsilon \to 0$, and choosing $\varepsilon = T^{-1/2}$, we obtain

$$\left\|\frac{1}{T}\int_0^T x(t)\,e^{-i\lambda_0 t}\,dt - \Phi(\lambda_0)\right\| \leqq \frac{2}{\sqrt{T}}\operatorname{Var}\Phi + \operatorname{Var}_{\Delta_{T^{-\frac{1}{2}}}}\Phi \to 0$$

as $T \to \infty$, and moreover

$$\begin{aligned} &\frac{1}{T_1 T_2}\int_0^{T_1}\int_0^{T_2} B(u, v)\,e^{-i(\lambda_0 u-\mu_0 v)}\,dudv \\ &= \left(\frac{1}{T_1}\int_0^{T_1} x(u)e^{-i\lambda_0 u}\,du,\ \frac{1}{T_2}\int_0^{T_2} x(v)\,e^{-i\mu_0 v}\,dv\right) \to (\Phi(\lambda_0), \Phi(\mu_0)) = F(\lambda_0, \mu_0), \end{aligned}$$

as was to be proved.

In a completely analogous way we establish

Theorem 3.2. *Let $x(t)$ be a harmonisable function*

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi(d\lambda),$$

and let the interval $\Delta = (\lambda_1, \lambda_2)$ be such that $\Phi(\lambda_1) = \Phi(\lambda_2) = 0$. Then

$$\lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-i\lambda_2 t} - e^{-i\lambda_1 t}}{-it} x(t)\,dt = \Phi(\Delta). \tag{13.2}$$

If $\Delta' = (\lambda_1', \lambda_2')$, $\Phi(\lambda_1') = \Phi(\lambda_2') = 0$, then

$$\lim_{T_1, T_2\to\infty} \int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \frac{e^{-i\lambda_2 u} - e^{-i\lambda_1 u}}{-iu} \cdot \frac{e^{+i\lambda_2' v} - e^{+i\lambda_1' v}}{iv} B(u, v)\,du\,dv = F(\Delta, \Delta').$$

2⁰. Let $x(t)$ be a harmonisable function

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \Phi(d\lambda).$$

We denote by A_φ the linear operator which establishes a correspondence between the function $x(t)$ and the function

$$y(t) = A_\varphi x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \varphi(\lambda) \Phi(d\lambda). \tag{14.2}$$

Example 4.2. The linear "sliding average" operation

$$y(t) = \sum_{k=1}^{m} c_k x(t + t_k) = \int_{-\infty}^{\infty} e^{i\lambda t} \left(\sum_{k=0}^{m} c_k e^{i\lambda t_k}\right) \Phi(d\lambda)$$

is an operator of this type.

Example 5.2. Let

$$\varphi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} \tilde{\varphi}(t)\,dt,$$

where

$$\int_{-\infty}^{\infty} |\tilde{\varphi}(t)|\,dt < \infty.$$

Since

$$||x(t)||^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda-\mu)t} F(d\lambda, d\mu) \leqq \mathrm{Var}_{R,R}\, F, \tag{15.2}$$

the function $x(t+t_0)\tilde{\varphi}(t)$ is integrable and

$$\int_{-\infty}^{\infty} e^{i\lambda t}\varphi(\lambda)\Phi(d\lambda) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\lambda(t+\tau)}\tilde{\varphi}(\tau)\,d\tau\right]\Phi(d\lambda)$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\lambda(t+\tau)}\Phi(d\lambda)\right]\tilde{\varphi}(\tau)\,d\tau = \int_{-\infty}^{\infty} \varphi(\tau)\, x(t+\tau)\,d\tau.$$

The operator A_φ has the form

$$y(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\varphi(\lambda)\Phi(d\lambda) = \int_{-\infty}^{\infty} \tilde{\varphi}(\tau - t)\, x(\tau)\,d\tau. \tag{16.2}$$

EXAMPLE 6.2. Let the function $x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda)$ have the derivative $x'(t)$, i.e.

$$\left\| x'(t) - \frac{x(t+\varepsilon) - x(t)}{\varepsilon} \right\| \to 0 \tag{17.2}$$

as $\varepsilon \to 0$. Set

$$y_\varepsilon(t) = \frac{x(t+\varepsilon) - x(t)}{\varepsilon}, \qquad \varphi_\varepsilon(\lambda) = \frac{e^{i\lambda\varepsilon} - 1}{\varepsilon}.$$

It follows from (17.2) that

$$||y_{\varepsilon_1} - y_{\varepsilon_2}|| = \left\| \int_{-\infty}^{\infty} e^{i\lambda t}[\varphi_{\varepsilon_1}(\lambda) - \varphi_{\varepsilon_2}(\lambda)]\Phi(d\lambda) \right\| \to 0,$$

and since for any Λ

$$\left\| \int_{-\Lambda}^{\Lambda} e^{i\lambda t}[\varphi_{\varepsilon_1}(t) - \varphi_{\varepsilon_2}(\lambda)]\Phi(d\lambda) \right\| \leqq \sup_{|\lambda| \leqq \Lambda} |\varphi_{\varepsilon_1}(t) - \varphi_{\varepsilon_2}(\lambda)| \operatorname{Var} \Phi \to 0$$

as $\varepsilon_1, \varepsilon_2 \to 0$, we have

$$\left\| \int_{|\lambda| > \Lambda} e^{i\lambda t}[\varphi_{\varepsilon_1}(\lambda) - \varphi_{\varepsilon_2}(\lambda)]\Phi(d\lambda) \right\| \to 0$$

uniformly in ε as $\Lambda \to \infty$, and thus

$$e^{i\lambda t}\varphi_\varepsilon(\lambda) = \frac{e^{i\lambda(t+\varepsilon)} - e^{i\lambda t}}{\varepsilon} \Rightarrow i\lambda e^{i\lambda t},$$

whence it follows that the function $i\lambda e^{i\lambda t}$ is integrable and

$$x'(t) = \int_{-\infty}^{\infty} i\lambda e^{i\lambda t}\Phi(d\lambda). \tag{18.2}$$

Finally, the converse is also true: if

$$\frac{e^{i\lambda(t+\varepsilon)} - e^{i\lambda t}}{\varepsilon} \to i\lambda e^{i\lambda t},$$

then the function

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda)$$

has the strong derivative

$$x'(t) = \int_{-\infty}^{\infty} i\lambda e^{i\lambda t}\Phi(d\lambda).$$

Thus the linear operation A_φ corresponding to the function $\varphi = i\lambda$ is the differentiation operator.

3. The Spectrum of an Abstract Function

1°. Let $x(t)$ be an abstract function of the real variable t, $-\infty < t < +\infty$, with values in a Hilbert space H. We shall say that the function $x(t)$ has a spectrum (cf. [7], [8]) if the numerical function $||x(t)||^2$ is integrable on every finite interval and if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (x(t+\tau), x(t))\,dt = B(\tau) \tag{1.3}$$

exists for all τ, where the function $B(\tau)$ is continuous. Let the function $x(t)$ have a spectrum. The function $B(\tau)$ is positive definite, i.e.

$$\sum_{k,j=1}^{m} B(\tau_k-\tau_j)\alpha_k\bar{\alpha}_j \geqq 0 \tag{2.3}$$

for any $\tau_1, \tau_2, \cdots, \tau_m$ and complex $\alpha_1, \alpha_2, \cdots, \alpha_m$, since

$$\begin{aligned}\sum_{k,j=1}^{m} B(\tau_k-\tau_j)\alpha_k\bar{\alpha}_j &= \lim_{T\to\infty}\sum_{k,j=1}^{m}\alpha_k\bar{\alpha}_j\frac{1}{T}\int_0^T (x(t+\tau_k-\tau_j), x(t))\,dt\\ &= \lim_{T\to\infty}\sum_{k,j=1}^{m}\alpha_k\bar{\alpha}_j\frac{1}{T}\int_{-\tau_j}^{T-\tau_j}(x(t+\tau_k), x(t+\tau_j))\,dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\left\|\sum_{k=1}^{m}\alpha_k x(t+\tau_k)\right\|^2 dt \geqq 0.\end{aligned}$$

Because of its positive definiteness, the function $B(\tau)$ can be represented in the form

$$B(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau}F(d\lambda), \tag{3.3}$$

where $F(\Delta)$ is a positive definite measure; we shall call $F(\Delta)$ the spectral measure (the spectrum) of the function $x(t)$.

EXAMPLE 1.3. Let $x(t)$ be a stationary function

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda).$$

In this case $B(\tau) = (x(t+\tau), x(t))$ and the spectral measure $F(\Delta)$ is

$$F(\Delta) = \|\Phi(\Delta)\|^2. \tag{4.3}$$

EXAMPLE 2.3. Let $x(t)$ be a harmonisable function

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda),$$

where [2] the ordinary variation $V(F)$ of the function $F(\Delta, \Delta')$:

$$\begin{aligned} V(F) &= \sup_{\Delta_k, \Delta'_j}\sum_{k,j}|F(\Delta_k, \Delta'_j)| < \infty,\\ F(\Delta, \Delta') &= (\Phi(\Delta), \Phi(\Delta')).\end{aligned} \tag{5.3}$$

In this case $F(\Delta, \Delta')$ can be extended to a measure on the plane. We have (cf. [9])

$$\begin{aligned}\frac{1}{T}\int_0^T (x(t+\tau), x(t))dt &= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda\tau}e^{i(\lambda-\mu)t}F(d\lambda, d\mu)\,dt\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda\tau}\frac{1}{T}\int_0^T e^{i(\lambda-\mu)t}dtF(d\lambda, d\mu) \to \int_{-\infty}^{\infty} e^{i\lambda\tau}F(d\lambda, d\lambda),\end{aligned}$$

[2] If the condition (5.3) is not met, then the harmonisable function $x(t)$ may not have a spectrum in the sense of the definition (1.3).

i.e., the function $x(t)$ has a spectrum, where the spectral measure $F(\Delta)$ is

$$F(\Delta) = \int_\Delta F(d\lambda, d\lambda). \tag{6.3}$$

2⁰. Let the function $x(t)$ have the spectrum $F(\Delta)$, i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (x(t+\tau), x(t))\,dt = \int_{-\infty}^{\infty} e^{i\lambda\tau} F(d\lambda). \tag{7.3}$$

Consider the function $y(t)$

$$y(t) = \sum_{k=1}^{m} c_k x(t+\tau_k). \tag{8.3}$$

It is easy to see that the function $y(t)$ also has a spectrum $G(\Delta)$,

$$\begin{gathered} \lim_{T\to\infty} \frac{1}{T}\int_0^T (y(t+\tau), y(t))\,dt = \int_{-\infty}^{\infty} e^{i\lambda\tau} \left| \sum_{k=1}^{m} c_k e^{i\lambda\tau_k}\right|^2 F(d\lambda), \\ G(\Delta) = \int_\Delta \left| \sum_{k=1}^{m} c_k e^{i\lambda\tau_k}\right|^2 F(d\lambda). \end{gathered} \tag{9.3}$$

Let $\|x(t)\|$ be uniformly bounded. Consider the function

$$y(t) = \int_{-\infty}^{\infty} \tilde\varphi(\tau - t)\, x(\tau)\, d\tau, \tag{10.3}$$

where the numerical function $\tilde\varphi(t)$ is such that $\int_{-\infty}^{\infty} |\tilde\varphi(t)|dt < \infty$ and

$$\varphi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t}\tilde\varphi(t)\,dt$$

is square integrable with respect to $F(\Delta)$, i.e.

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 F(d\lambda) < \infty.$$

We have

$$\begin{aligned} \frac{1}{T}\int_0^T (y(t+\tau), y(t))\,dt &= \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \tilde\varphi(u) \int_{-\infty}^{\infty} \overline{\tilde\varphi(v)}\, (x(t+\tau+u), x(t+v))\, du\,dv\,dt \\ &= \int_{-\infty}^{\infty} \tilde\varphi(u) \int_{-\infty}^{\infty} \overline{\tilde\varphi(v)}\, \frac{1}{T}\int_0^T (x(t+\tau+u), x(t+v))\, dt\,du\,dv, \end{aligned}$$

and, since as $T \to \infty$ the functions

$$\psi_T(u, v) = \frac{1}{T}\int_0^T (x(t+\tau+u), x(t+v))\,dt \to \int_{-\infty}^{\infty} e^{i\lambda(\tau+u-v)} F(d\lambda)$$

for every (u, v) and moreover are uniformly bounded, then

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde\varphi(u)\overline{\tilde\varphi(v)}\,\psi_T(u, v)\,du\,dv &\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde\varphi(u)\overline{\tilde\varphi(v)} \int_{-\infty}^{\infty} e^{i\lambda(\tau+u-v)} F(d\lambda) \\ &= \int_{-\infty}^{\infty} e^{i\lambda\tau} \left| \int_{-\infty}^{\infty} e^{i\lambda u}\tilde\varphi(u)\,du \right|^2 F(d\lambda), \end{aligned}$$

and thus

$$\lim_{T\to\infty}\int_0^T (y(t+\tau),\, y(t))\,dt = \int_{-\infty}^{\infty} e^{i\lambda\tau}\,|\varphi(\lambda)|^2\,F(d\lambda), \tag{11.3}$$

i.e., the function

$$y(t) = \int_{-\infty}^{\infty} \tilde{\varphi}(\tau-t)\,x(\tau)\,d\tau$$

has a spectrum $G(\varDelta)$, where

$$G(\varDelta) = \int_{\varDelta} |\varphi(\lambda)|^2\,F(d\lambda). \tag{12.3}$$

3⁰. In the spectral theory of stationary functions the question of linear extrapolation is very important (see e.g. [1]), i.e. the question of whether it is possible to linearly approximate the value of the function $x(t)$ at the time $t > \tau$ in terms of the values of $x(t)$ for $t \leqq \tau$. More precisely this means the following. Let H_τ^- be the linear closure of the elements $x(t)$, $-\infty < t \leqq \tau$, and let $\hat{x}_\tau(t) = P_\tau x(t)$, where P_τ is the operator of projection on H_τ^-. The element $\hat{x}_\tau(t)$ gives the best linear approximation to the value of the function $x(t)$ at the time $t > \tau$ in terms of the values of $x(t)$ for $t \leqq \tau$, and

$$||x(t)-\hat{x}_\tau(t)|| = \min_{h\in H_\tau} ||x(t)-h|| = \sigma_{t-\tau}^2. \tag{13.3}$$

The extrapolation problem consists of finding the elements $\hat{x}_\tau(t)$ (see e.g. [1], [2]).

If $x(t)$ is a stationary function, then the function $\hat{x}_{t-\tau}(t)$ of t ($\tau = \text{const.}$) satisfies the condition

$$\inf_{y(t)}\lim_{T\to\infty}\frac{1}{T}\int_0^T ||x(t)-y(t)||^2\,dt = \lim_{T\to\infty}\int_0^T ||x(t)-\hat{x}_{t-\tau}(t)||^2\,dt, \tag{14.3}$$

where the inf is taken over all functions $y(t)$ of the form

$$y(t) = \sum_{\tau_k \geqq \tau} c_k x(t-\tau_k). \tag{15.3}$$

Now let $x(t)$ be an arbitrary function with a spectrum, i.e.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T (x(t+\tau),\, x(t))\,dt = \int_{-\infty}^{\infty} e^{i\lambda\tau}F(d\lambda). \tag{16.3}$$

We shall call the function $y(t)$ a stationary approximation for $x(t)$ with lag τ if $y(t)$ has the form

$$y(t) = \int_{-\infty}^{t-\tau} \tilde{\varphi}(u-t)\,x(u)\,du + \sum_{\tau_k\geqq\tau} c_k x(t-\tau_k), \tag{17.3}$$

where the numerical function $\tilde{\varphi}(t)$, $\tilde{\varphi}(t) = 0$ for $t > 0$, satisfies the conditions stated after equation (10.3). Consider the mean deviation $\sigma_\tau^2(y)$:

$$\sigma_\tau^2(y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T ||x(t)-y(t)||^2\,dt = \int_{-\infty}^{\infty} |e^{i\lambda\tau}-\varphi(\lambda)|^2\,F(d\lambda), \tag{18.3}$$

where

$$\varphi(\lambda) = \int_{-\infty}^{-\tau} e^{i\lambda t}\tilde{\varphi}(t)\,dt + \sum_{\tau_k\geqq\tau} c_k e^{-i\lambda\tau_k}.$$

By equation (18.3), the mean deviation $\sigma_\tau^2(y)$ lies in the range

$$0 \leqq \sigma_\tau^2(y) \leqq \int_{-\infty}^{\infty} F(d\lambda). \tag{19.3}$$

We set

$$\sigma_\tau^2 = \inf_{y(t)} \sigma_\tau^2(y) = \inf_{\varphi(\lambda)} \int_{-\infty}^{\infty} |e^{i\lambda\tau} - \varphi(\lambda)|^2 F(d\lambda). \tag{20.3}$$

We note that if $\sigma_\tau^2 = 0$ for some $\tau = \tau_0 > 0$, then $\sigma_\tau^2 \equiv 0$ for all τ. In fact, because of the monotoneness, $\sigma_\tau^2 = 0$ for $\tau \leqq \tau_0$; if $\varepsilon > 0$ is arbitrarily small and

$$\varphi(\lambda) = \sum_{\tau_k \geqq \tau_0} c_k e^{-i\lambda\tau_k}$$

is such that

$$\int_{-\infty}^{\infty} |e^{i\lambda\tau_0} - \varphi(\lambda)|^2 F(d\lambda) \leqq \varepsilon,$$

then for $\tau_1 \leqq 2\tau_0$ we have

$$\int_{-\infty}^{\infty} \Big| e^{i\lambda\tau_1} - e^{i\lambda(\tau_0 - \tau_1)} \sum_{\tau_k \geqq \tau_0} c_k e^{-i\lambda\tau_k} \Big|^2 F(d\lambda) = \int_{-\infty}^{\infty} |e^{i\lambda\tau_0} - \varphi(\lambda)|^2 F(d\lambda) \leqq \varepsilon$$

and consequently $\sigma_{\varepsilon_1}^2 \leqq \varepsilon$, $\sigma_{\varepsilon_1}^2 = 0$, etc.

We shall call the function $x(t)$ singular if $\sigma_\tau^2 = 0$ for $\tau > 0$ (cf. [1]) and regular if $\sigma_\tau^2 \to \int_{-\infty}^{\infty} F(d\lambda)$ as $\tau \to \infty$. It follows from [2] that the function $x(t)$ is singular if and only if

$$\int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1+\lambda^2} d\lambda = -\infty. \tag{21.3}$$

(Here $f(\lambda) = F(d\lambda)/d\lambda$ and the integral in (21.3) is by definition equal to $-\infty$ in the case where $f(\lambda) = 0$ on a set of positive measure.) Similarly for the regularity of the function $x(t)$ it is necessary and sufficient that the spectral measure $F(\Delta)$ be absolutely continuous and that

$$\int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1+\lambda^2} d\lambda > -\infty. \tag{22.3}$$

The least mean square deviation σ_τ^2 has the form

$$\sigma_\tau^2 = \int_{-\infty}^{\infty} |e^{i\lambda\tau} - \varphi_0(\lambda)|^2 F(d\lambda), \tag{23.3}$$

where the function $\varphi_0(\lambda)$ is the boundary value of the analytic function $\Gamma(z)$, $z = \lambda + i\nu$, defined by

$$\Gamma(z) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log f(\mu)}{1+\mu^2} \frac{1-\mu z}{z-\mu} d\mu \right\} \tag{24.3}$$

as $\nu \to 0$. In the case where (22.3) holds, but the spectral measure $F(\Delta)$ is not absolutely continuous, we have

$$\lim_{\tau\to\infty} \sigma_\tau^2 = \int_{-\infty}^{\infty} f(\lambda)\, d\lambda. \tag{25.3}$$

We note that $\inf_y \sigma_\tau^2(y)$ may not be attained, but that it is easy to find a minimizing sequence for the function $y(t)$ of the form (17.3) by using (23.3) and (24.3).

EXAMPLE 3.3. Let the function $x(t)$ have the absolutely continuous spectrum $F(\Delta)$, $f(\lambda) = 1/(a^2+\lambda^2)$. Then $\inf_y \sigma_\tau^2(y)$ is attained for functions of the form

$$y(t) = e^{-a\tau} x(t-\tau). \tag{26.3}$$

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SPECTRAL ANALYSIS OF ABSTRACT FUNCTIONS

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(*Summary*)

In this paper a spectral theory is developed for abstract functions similar to the well-known spectral theory for stationary random processes.