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Eigenvalues of Positive Definite Integral Operators on Unbounded Intervals

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Abstract. Let $k(x, y)$ be the positive definite kernel of an integral operator on an unbounded interval of $\mathbb{R}$. If k belongs to class $\mathcal{A}$ defined below, the corresponding operator is compact and trace class. We establish two results relating smoothness of k and its decay rate at infinity along the diagonal with the decay rate of the eigenvalues. The first result deals with the Lipschitz case; the second deals with the uniformly C^1 case. The optimal results known for compact intervals are recovered as special cases, and the relevance of these results for Fourier transforms is pointed out.

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1. Introduction

Given an interval $I \subset \mathbb{R}$, a linear operator $K : L^2(I) \to L^2(I)$ is said to be integral if there exists a measurable function $k(x, y)$ on $I \times I$ such that for all $\phi \in L^2(I)$

$$\phi \longmapsto K(\phi) = \int_I k(x, y)\, \phi(y)\, dy$$

almost everywhere. The function $k(x, y)$ is called the kernel of K. If $k(x, y) = \overline{k(y, x)}$ for almost all $x, y \in I$, then K is self-adjoint. If in addition K satisfies the condition

$$\int_I \int_I k(x, y)\, \phi(y)\overline{\phi(x)}\; dx\, dy \geq 0 \tag{1}$$

for all $\phi \in L^2(I)$, then it is a positive operator. Following standard terminology, we shall call the corresponding kernel $k(x, y)$ an $L^2(I)$-*positive definite kernel*.

This paper shall deal exclusively with positive integral operators and the corresponding positive definite kernels. Its purpose is the study of the asymptotic behavior of eigenvalues of K in the case where I is unbounded.

The case where I is compact, conventionally $I = [0, 1]$, has been thoroughly studied. In the rest of the introduction we describe what is known in this case.

General integral operators with $L^2([0, 1])$ kernels are compact; for self-adjoint operators (i.e. with kernels satisfying $k(x, y) = \overline{k(y, x)}$) the bilinear expansion for the kernel

$$k(x, y) = \sum_{n \geq 1} \lambda_n \, \phi_n(x) \, \overline{\phi_n(y)}, \tag{2}$$

holds, where $\lambda_n \in \mathbb{R}$ are the eigenvalues of K repeated according to (necessarily finite) multiplicity, accumulating only at 0, the $\{\phi_n\}_{n \geq 1}$ are the corresponding $L^2([0, 1])$-orthonormal eigenfunctions spanning the range of K and equality is in the L^2 sense.

If $k(x, y)$ is a positive definite kernel, the eigenvalues $\lambda_n \geq 0$. We may therefore assume without loss of generality that in series (2) the sequence $\{\lambda_n\}_{n \geq 0}$ has been arranged so that eigenvalues are repeated according to multiplicity and the sequence is non-increasing.

The asymptotic behavior of the eigenvalue sequence $\{\lambda_n\}_{n \geq 1}$ is closely related to smoothness properties of the kernel $k(x, y)$. If k is continuous, the classical theorem of Mercer (see e.g. [15]) asserts that eigenfunctions are continuous, convergence of the series (2) is absolute and uniform and the operator K is trace class with

$$\int_0^1 k(x, x) \, dx = \sum_{n \geq 1} \lambda_n,$$

which of course implies $\lambda_n = o(1/n)$. Thus Mercer's theorem may be seen as a first step in this direction.

For general (not necessarily positive definite) kernels it was shown by Weyl [16] that if $k(x, y)$ is C^1 then $\lambda_n = o(1/n^{3/2})$. This estimate may be improved when k is a positive definite kernel, as shown by Reade [9], to $\lambda_n = o(1/n^2)$.

It is natural to ask what happens in the intermediate cases. Reade [10] has shown that if a positive definite kernel k, in addition to continuity, satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, then $\lambda_n = O(1/n^{1+\alpha})$, and that this estimate is best possible as a power of n (but see Remark 4.2 below).

Although higher-order differentiability will not be treated here, we mention that, more generally, positive definite C^p kernels satisfy $\lambda_n = o(1/n^{p+1})$ [11]. In fact the optimal estimates are slightly sharper: $\lambda_n = o(1/n^{p+1})$ for odd p and $\sum_1^\infty n^p \lambda_n < +\infty$ for even p; see Ha [7] and Reade [12]. Cochran and Lukas [5] and Chang and Ha [4] derive the corresponding results for the decay rate of eigenvalues when a suitable higher-order derivative is Lip^α.

2. Preliminaries: The Class $\mathcal{A}$

We shall be concerned in this paper with the study of the asymptotics of eigenvalues of positive integral operators in the case where I is an unbounded (closed) interval in $\mathbb{R}$, which without loss of generality we shall take throughout as $I = [0,+\infty[$. The adaptations to other kinds of unbounded intervals are trivial; we mention however that the case $I = \mathbb{R}$ is particularly significant in view of Fourier transforms, see Remark 6.3 and Corollary 6.4.

In general, integral operators in unbounded domains are not compact. However, we shall restrict attention to positive integral operators whose $L^2[0,+\infty[$ positive definite kernels $k(x,y)$ belong to class $\mathcal{A}$ defined below.

Definition 2.1 (Class $\mathcal{A}$). *An $L^2([0,+\infty[)$ positive definite kernel $k(x,y)$ is said to belong to class $\mathcal{A}$ if:*

1. $k(x,y)$ *is continuous in* $[0,+\infty[\times[0,+\infty[$*;*
2. $k(x,x) \in L^1([0,+\infty[)$*;*
3. $k(x,x) \to 0$ *as* $x \to +\infty$*.*

Note that if $k(x,y)$ is a continuous positive definite kernel (not necessarily in class $\mathcal{A}$), then $\forall x \in I\ k(x,x) \geq 0$ and $\forall x,\, y \in I\ |k(x,y)|^2 \leq k(x,x)k(y,y)$. Note also that a kernel in class $\mathcal{A}$ is uniformly continuous in $[0,+\infty[\times[0,+\infty[$; see [1].

The following description summarizes the results in Buescu [1] relevant for this paper. If $k(x,y)$ is a positive definite kernel in class $\mathcal{A}$, then

$$\|K\|_2 = \|k\|_{L^2} = \left(\int_0^\infty \int_0^\infty |k(x,y)|^2\,dx\,dy\right)^{1/2} \leq \int_0^\infty k(x,x)\,dx < +\infty,$$

where $\|K\|_2$ is the Hilbert-Schmidt norm of K. Thus K is Hilbert-Schmidt and therefore compact. The spectrum of K consists in a sequence $\{\lambda_n\}_{n\geq1}$ of eigenvalues with finite multiplicity converging to zero. Since K is positive the eigenvalues are non-negative; we make the usual standing assumption that in the sequence $\{\lambda_n\}$ they are repeated according to multiplicity and ordered so that $\{\lambda_n\}$ is non-increasing.

The spectral theorem and standard Hilbert space methods show that, as in the compact case, the kernel $k(x,y)$ satisfies the bilinear series expansion

$$k(x,y) = \sum_{n\geq1} \lambda_n\, \phi_n(x)\, \overline{\phi_n(y)}, \tag{3}$$

where the $\{\phi_n\}_{n\geq1}$ are an $L^2([0,+\infty[)$-orthonormal set of eigenfunctions spanning the range of K and equality is in the sense of convergence in L^2. Note that the eigenvalue 0 does not contribute to the sum (3). Thus if K is a finite rank operator there exists N_0 such that $\lambda_n = 0$ for $n \geq N_0$ and the series (3) is a finite sum. Otherwise $\{\lambda_n\}$ is a positive, non-increasing sequence with $\lambda_n \to 0$ as $n \to +\infty$.

Stronger convergence properties for series (3) are valid for kernels in class $\mathcal{A}$. Eigenfunctions ϕ_n associated with nonzero eigenvalues (which are the ones which contribute nontrivially to the sum (3)) are uniformly continuous and so vanish at infinity. Moreover, the analog of Mercer's theorem holds in this context:

the bilinear series (3) for the kernel is absolutely and uniformly convergent. As a consequence, the operator $K : L^2([0, +\infty[) \to L^2([0, +\infty[)$ is trace class with

$$\operatorname{tr} K = \int_0^\infty k(x, x)\, dx = \sum_{n\geq 1} \lambda_n; \tag{4}$$

from the fact that $k(x, x) \in L^1(\mathbb{R})$ convergence of the series follows, implying that for kernels in class $\mathcal{A}$ the eigenvalues satisfy $\lambda_n = o(1/n)$.

Class $\mathcal{A}$ seems to be the weakest class of positive definite kernels in unbounded domains for which Mercer's theorem, the consequent trace formula (4), and therefore the basic estimate $\lambda_n = o(1/n)$ hold. In fact, it describes rather sharply the minimal conditions on the kernel $k(x, y)$ under which such estimate is at all possible; see counterexamples in [1] as well as more general results in Novitskii [8]. We therefore make class $\mathcal{A}$ the starting point for our study of the asymptotic behavior of eigenvalues of positive definite kernels in unbounded domains. Unless otherwise stated, henceforth all kernels are assumed to belong to class $\mathcal{A}$.

Remark 2.2. In many of the results below, as in Definition 2.1 itself, the diagonal $\{(x, y) \in [0, +\infty[\times [0, +\infty[\colon y = x\}$ plays a prominent role in determining the behavior of $k(x, y)$. Henceforth we abbreviate reference to this set simply as "the diagonal".

3. Preparatory Results

In the sequel we follow closely methods introduced by Reade [9], adapting them to the present context. Critical improvements are made in § 3.3. Those results are crucial for the extension of the known methods to the case of kernels on unbounded domains.

3.1. Best Approximations

Let k be a kernel in class $\mathcal{A}$ and K the associated positive integral operator. It follows from the general theory of compact operators in Hilbert space that, if R is the operator with kernel $\sum_{n=1}^N \lambda_n\, \phi_n(x)\, \overline{\phi_n(y)}$, then R is the best approximation to K in the operator norm by symmetric operators of rank $\leq N$, the minimum distance being $\|K - R\|_{\text{op}} = \lambda_{n+1}$ (see e.g. Gohberg and Krein [6], Theorem III.6.1). Also $\sum_{n=1}^N \lambda_n\, \phi_n(x)\, \overline{\phi_n(y)}$ is the best approximation to $k(x, t)$ by $L^2([0, +\infty[)$ symmetric kernels of rank $\leq N$ which generate compact integral operators, the minimum distance being $(\sum_{n=N+1}^\infty \lambda_n^2)^{1/2}$ (see [15] for a version for integral operators in compact intervals or [6] for a linear operators in Hilbert space).

Lemma 3.1. *If $k(x, y)$ is in class $\mathcal{A}$, then $\sum_{n=1}^N \lambda_n\, \phi_n(x)\, \overline{\phi_n(y)}$ is the best approximation in the trace norm by $L^2([0, +\infty[)$ symmetric kernels of rank $\leq N$.*

Proof. If k is in class $\mathcal{A}$, the operator with kernel k is compact and trace class; we assume its bilinear eigenfunction expansion is given by (3). Defining k_N by $k_N = \sum_{n=1}^N \lambda_n\, \phi_n(x)\, \overline{\phi_n(y)}$, the operator K_N with kernel k_N has rank $\leq N$ and

$$\|K - K_N\|_{\mathrm{tr}} = \sum_{n=N+1}^{\infty} \lambda_n.$$

Let R be any symmetric $L^2([0,+\infty[)$ operator of rank $\leq N$ and denote its kernel by $r(x,y)$. Then $t = k - r$ is the kernel of a compact, symmetric, trace class operator and satisfies the bilinear eigenfunction expansion

$$t(x,y) = \sum_{n=1}^{\infty} \mu_n \psi_n(x)\,\overline{\psi_n(y)}$$

convergent in $L^2([0,+\infty[)$, where the ψ_n are the eigenfunctions of $K - R$ and μ_n are the corresponding eigenvalues ordered in such a way that $\{|\mu_n|\}_{n\geq 0}$ is non-increasing. Then $\|K - R\|_{\mathrm{tr}} = \sum_{n=1}^{\infty} |\mu_n|$. Let

$$s(x,y) = r(x,y) + \sum_{n=1}^{p} \mu_n \psi_n(x)\,\overline{\psi_n(y)}.$$

Then the operator S with kernel s has rank $\leq N + p$ and

$$k(x,y) - s(x,y) = \sum_{n=p+1}^{\infty} \mu_n \psi_n(x)\,\overline{\psi_n(y)}.$$

Then $\|K - S\|_{\mathrm{op}} = |\mu_{p+1}|$ and $\|K - \sum_{n=1}^{N+p} \lambda_n \phi_n(x)\,\overline{\phi_n(y)}\|_{\mathrm{op}} = \lambda_{N+p+1}$. But $\sum_{n=1}^{N+p} \lambda_n \phi_n(x)\,\overline{\phi_n(y)}$ is the best approximation to k by kernels of symmetric operators of rank $\leq N + p$ in the operator norm; therefore $|\mu_{p+1}| \geq \lambda_{N+p+1}$ for all $p \geq 0$. Hence

$$\|K - K_N\|_{\mathrm{tr}} = \sum_{n=N+1}^{\infty} \lambda_n \leq \sum_{n=1}^{\infty} |\mu_n| = \|K - R\|_{\mathrm{tr}}, \tag{5}$$

completing the proof. □

3.2. Square Roots

Any positive operator K in Hilbert space has a unique positive square root S; moreover, S commutes with K, see e.g. [14].

In our case, this fact implies that if K is a positive operator with a class $\mathcal{A}$ kernel satisfying the bilinear eigenfunction expansion (3), the corresponding square root operator S is an $L^2([0,+\infty[)$ positive integral operator. Since K is trace class, standard arguments imply that the positive definite kernel $s(x,y)$ of S satisfies the bilinear expansion

$$s(x,y) = \sum_{n\geq 1} \lambda_n^{1/2}\, \phi_n(x)\,\overline{\phi_n(y)}, \tag{6}$$

where the last equality is in the sense of L^2 convergence.

In general, of course, $s(x,y)$ will not lie in class $\mathcal{A}$, so that the corresponding operator S will not be trace class. However, the following results hold.

Lemma 3.2. *If $k(x,y)$ is in class $\mathcal{A}$ and $s(x,y)$ is the kernel of the corresponding positive square root operator, then for any $f \in L^2([0,+\infty[)$*

$$Sf(x) = \int_0^\infty s(x,y)f(y)\,dy$$

is a continuous function of x.

Proof. We have

$$\int_0^\infty s(x,y)f(y)\,dy = \sum_{n\geq 1} \lambda_n^{1/2}\,\langle f,\phi_n\rangle\phi_n(x). \tag{7}$$

We now show the series on the right-hand side to be uniformly and absolutely convergent. In fact

$$\sum_M^N |\lambda_n^{1/2}\langle f,\phi_n\rangle\phi_n(x)| \leq \left(\sum_M^N \lambda_n|\phi_n(x)|^2 \sum_M^N |\langle f,\phi_n\rangle|^2\right)^{1/2}$$

by the Cauchy-Schwartz inequality, and

$$\sum_{n\geq 1} \lambda_n|\phi_n(x)|^2 = k(x,x)$$

uniformly absolutely because k is in class $\mathcal{A}$. On the other hand, Bessel's inequality implies

$$\sum_M^N |\langle f,\phi_n\rangle|^2 \leq \|f\|^2.$$

Thus the series is absolutely and uniformly convergent. Since k is in class $\mathcal{A}$ every $\phi_n(x)$ is continuous (see § 2 or [1]); hence the lemma follows. □

Lemma 3.3. *If K is a positive operator with kernel in class $\mathcal{A}$, S is its positive square root and R is a positive operator of finite rank satisfying*

$$0 \leq R \leq I,$$

where I is the identity operator, then the kernel of SRS lies in class $\mathcal{A}$ and

$$0 \leq SRS \leq K.$$

Proof. Denote the kernels of K, S, R respectively by $k(x,y)$, $s(x,y)$ and $r(x,y)$. The kernel of SRS is then given by

$$g(x,y) = \int_0^\infty \int_0^\infty s(x,u)r(u,v)s(v,y)\,du\,dv. \tag{8}$$

Since $r(u,v)$ is by hypothesis a positive definite kernel and and the corresponding operator has finite rank $N > 0$, it follows from (3) that

$$r(u,v) = \sum_{n=1}^N \mu_n\psi_n(u)\,\overline{\psi_n(v)},$$

where $\mu_n > 0$ are the (strictly positive) eigenvalues of R repeated according to multiplicity and ψ_n are the corresponding orthonormal eigenfunctions. Thus, from (8),

$$g(x,y) = \sum_{n=1}^{N} \mu_n \int_0^\infty s(x,u)\psi_n(u)\,du \,\overline{\int_0^\infty s(y,v)\psi_n(v)\,dv}, \tag{9}$$

which by Lemma 3.2 is continuous, has rank N and is an $L^2([0,+\infty[)$ positive definite kernel (since, for all ϕ, $\langle SRS\phi,\phi\rangle = \langle RS\phi, S\phi\rangle \geq 0$ because R is positive). We now prove that, in addition, it lies in class $\mathcal{A}$.

From (9), using the Cauchy-Schwartz inequality,

$$\begin{aligned} g(x,x) &= \sum_{n=1}^{N} \mu_n \int_0^\infty s(x,u)\psi_n(u)\,du\,\overline{\int_0^\infty s(x,v)\psi_n(v)\,dv} \\ &\leq \sum_{n=1}^{N} \mu_n \left(\int_0^\infty |s(x,u)|^2\,du\right)^{1/2} \|\psi_n\| \left(\int_0^\infty |s(v,x)|^2\,dv\right)^{1/2} \|\psi_n\| \\ &= \sum_{n=1}^{N} \mu_n \left(\int_0^\infty |s(x,u)|^2\,du \int_0^\infty |s(x,v)|^2\,dv\right)^{1/2}. \end{aligned} \tag{10}$$

Taking into account that

$$\int_0^\infty |s(x,t)|^2\,dt = \int_0^\infty s(x,t)\,\overline{s(x,t)}\,dt = \int_0^\infty s(x,t)\,s(t,x)\,dt = k(x,x)$$

and that $k(x,x) \geq 0$, it follows from (10) that

$$g(x,x) \leq \left(\sum_{n=1}^{N} \mu_n\right) k(x,x) = C\,k(x,x)$$

where $C = \sum_{n=1}^{N} \mu_n > 0$. From this inequality and the fact that $k(x,y)$ is in class $\mathcal{A}$ it follows immediately that $g(x,x) \to 0$ as $x \to +\infty$ and $g(x,x) \in L^1(\mathbb{R})$ since it is positive and dominated by $Ck(x,x)$. Thus $g(x,y)$ lies in class $\mathcal{A}$.

The last statement is a simple exercise in linear algebra, and does not depend on R having finite rank. Since $0 \leq R \leq I$, we have for all $\psi \in L^2([0,+\infty[)$

$$0 \leq \langle R\psi,\psi\rangle \leq \langle \psi,\psi\rangle;$$

in particular if $\psi = S\phi$ for some $\phi \in L^2([0,+\infty[)$, we have

$$0 \leq \langle RS\phi, S\phi\rangle = \langle SRS\phi,\phi\rangle \leq \langle S\phi, S\phi\rangle = \langle S^2\phi,\phi\rangle = \langle K\phi,\phi\rangle,$$

thus proving $0 \leq \langle SRS\phi,\phi\rangle \leq \langle K\phi,\phi\rangle$ for all $\phi \in L^2([0,+\infty[)$ or equivalently $0 \leq SRS \leq K$. □

3.3. A Class of Finite Rank Operators

We now define a class of finite rank operators to be used in the approximation of a positive operator K with kernel k in class $\mathcal{A}$.

Let N be a natural number and L be a positive real number. We define $R^{N,L}$ to be the $L^2([0,+\infty[)$ operator with kernel

$$r^{N,L}(u,v) = \frac{N}{L}\sum_{n=1}^{N} \psi_n^{N,L}(u)\,\psi_n^{N,L}(v),$$

where

$$\psi_n^{N,L}(x) = \begin{cases} 1 & \text{if } (n-1)\frac{L}{N} < x \le n\frac{L}{N} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $R^{N,L}$ is an orthogonal projection from $L^2([0,+\infty[)$ into $L^2([(n-1)\frac{L}{N}, n\frac{L}{N}])$. It is thus a positive operator of rank N with $0 \le R^{N,L} \le I$. Its spectrum is $\{0,\, 1\}$, the eigenvalue 1 having multiplicity N and the corresponding orthogonal (unnormalized) eigenfunctions being the ψ_n.

Given an operator K with kernel k in class $\mathcal{A}$ and square root S, it follows from Lemma 3.3 that $0 \le SR^{N,L}S \le K$ and that $SR^{N,L}S$ is in class $\mathcal{A}$; in particular it is trace class and has a continuous kernel. Therefore

$$\begin{aligned}
&\|K - SR^{N,L}S\|_{\mathrm{tr}} \\
&= \int_0^\infty k(x,x)\,dx - \int_0^\infty\int_0^\infty\int_0^\infty s(x,u)r^{N,L}(u,v)s(v,x)\,du\,dv\,dx \\
&= \int_0^\infty k(x,x)\,dx - \int_0^\infty\int_0^\infty r^{N,L}(u,v)\left(\int_0^\infty s(x,u)s(v,x)\,dx\right)du\,dv \\
&= \int_0^\infty k(x,x)\,dx - \int_0^\infty\int_0^\infty r^{N,L}(u,v)k(v,u)\,du\,dv.
\end{aligned}$$

Since $\int_0^L r^{N,L}(u,v)\,du = \int_0^L r^{N,L}(u,v)\,dv = 1$ for all $u, v \in]0, L]$, we have

$$\begin{aligned}
\int_0^L k(x,x)\,dx &= \int_0^L\int_0^L r^{N,L}(u,v)k(u,u)\,du\,dv \\
&= \int_0^L\int_0^L r^{N,L}(u,v)k(v,v)\,du\,dv.
\end{aligned}$$

Noting that $\operatorname{supp} r^{N,L} \subset [0,L]^2$, we have

$$\int_0^\infty\int_0^\infty r^{N,L}(u,v)k(v,u)\,du\,dv = \int_0^L\int_0^L r^{N,L}(u,v)k(v,u)\,du\,dv.$$

Hence we may either write

$$\begin{aligned}
&\|K - SR^{N,L}S\|_{\mathrm{tr}} \\
&= \int_0^L k(x,x)\,dx - \int_0^L \int_0^L r^{N,L}(u,v)k(v,u)\,du\,dv + \int_L^\infty k(x,x)\,dx \\
&= \int_0^L \int_0^L r^{N,L}(u,v)\left(k(u,u) - k(v,u)\right)\,du\,dv + \int_L^\infty k(x,x)\,dx \\
&= \sum_{n=1}^N \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \left(k(u,u) - k(v,u)\right)\,du\,dv + \int_L^\infty k(x,x)\,dx
\end{aligned} \tag{11}$$

or

$$\begin{aligned}
&\|K - SR^{N,L}S\|_{\mathrm{tr}} \\
&= \int_0^L \int_0^L r^{N,L}(u,v)\left(\frac{k(u,u)+k(v,v)}{2} - k(v,u)\right)\,du\,dv + \int_L^\infty k(x,x)\,dx \\
&= \sum_{n=1}^N \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \left(\frac{k(u,u)+k(v,v)}{2} - k(v,u)\right) dudv + \int_L^\infty k(x,x)dx
\end{aligned} \tag{12}$$

Equations (11) and (12) will be used in § 4 and § 5 in the proof of our main results.

Remark 3.4. Recalling that symmetry of the operator K implies $k(u,v) = \overline{k(v,u)}$ for all u, v, it is immediate to conclude that the contribution of the imaginary part of k to the integral $\int_0^L \int_0^L r^{N,L}(u,v)k(v,u)\,du\,dv$ is zero. The same observation obviously applies to integration in any square symmetric with respect to the diagonal. Consequently, in equations (11) and (12) we may regard $k(u,v)$ as being a real-valued function without any loss of generality.

4. Asymptotics of Eigenvalues: The Lipschitz Case

Our first result concerns the asymptotic behavior of the eigenvalues of any kernel in class $\mathcal{A}$ satisfying the following Lipschitz condition.

Definition 4.1. *We say that a kernel $k(x,y)$ in class $\mathcal{A}$ is* $\mathrm{Lip}^{\alpha,s}$ with respect to x on the diagonal *if*

$$|k(x,x) - k(x,y)| \le A(x)|x-y|^\alpha \qquad \textit{for all } x,y \in \mathbb{R}^+,$$

where $0 < \alpha \le 1$ and $A(x)$ is a positive locally integrable function satisfying

$$\int_0^L A(x)\,dx \le AL^s$$

for all $L > 0$ and some $A > 0$, $s \ge 0$.

Remark 4.2. Stronger conditions than the one in Definition 4.1 are commonly used in the related literature (see e.g. [5], [4]). A function $k(x,y) : I^2 \to \mathbb{R}$ is said to be Lip^α with respect to the first variable if

$$|k(x,z) - k(y,z)| \le A(z)|x-y|^\alpha \quad \text{for all } x,y,z \in I, \tag{13}$$

where $0 < \alpha \le 1$ and $A(z)$ is taken to be a positive constant or a positive function verifying some appropriate integrability requirement.

The condition in Definition 4.1, being sufficient for our purposes, corresponds to the restriction of (13) to the points (x,z) such that $x = z$ (hence the term *Lipschitz with respect to the first variable on the diagonal*). Our choice therefore emphasizes the key role played by the behavior of the kernel k along the diagonal in the study of the properties of the related operator K.

Observe also that, if we take I to be $[0,+\infty[$, in contrast to the case where I is a compact interval, the conditions imposed on $A(z)$ are not only more general but also affect directly the results obtained for the issues under study, namely the decay rate of the eigenvalues of K. Therefore, an appropriate condition on the growth of $A(z)$ is required in this case. We have taken it to be that $A(z)$ be locally integrable and $\int_0^L A(z)\,dz \le AL^s$ for some real numbers $A > 0$ and $s \ge 0$. Notice, for instance, that the cases $A(z) =$ constant and $A(z)$ integrable in $[0,+\infty[$ are covered, respectively, by taking $s = 1$ and $s = 0$.

Finally, note that Lipschitz conditions on $k(x,y)$ are sometimes required on both variables (see e.g. Reade [10]). For symmetric kernels, however, it follows from $k(x,y) = \overline{k(y,x)}$ that if k is Lipschitz with respect to one of the variables it is automatically Lipschitz with respect to the other. Incidentally, our proofs do not depend on the use of this property.

The next result describes how, for a kernel in class $\mathcal{A}$, Lipschitz continuity on the diagonal and the rate of decay at infinity on the diagonal allow us to control the decay rate of eigenvalues.

Theorem 4.3. *Suppose $k(x,y)$ is a positive definite kernel in class $\mathcal{A}$ and is $\mathrm{Lip}^{\alpha,s}$ with respect to x on the diagonal. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the integral operator with kernel k. Then the following statements hold.*

(i) *If $\beta > 1$, $k(x,x) = O(1/x^\beta)$ as $x \to +\infty$ and $\gamma = \dfrac{(\alpha+1)\beta + s - 1}{\alpha + \beta + s - 1}$, then $\lambda_n = O(1/n^\gamma)$.*

(ii) *If $k(x,x) = O(1/x^\beta)$ as $x \to +\infty$ for all $\beta > 1$, then $\lambda_n = o(1/n^\gamma)$ for all $\gamma \in]1, 1+\alpha[$.*

(iii) *If $k(x,x)$ has compact support, then $\lambda_n = O(1/n^{1+\alpha})$.*

Proof. Let K be the operator with kernel k and S be its square root. Choose L in $\mathbb{R}^+$ and N in $\mathbb{N}$. Then $|u - v| < L/N$ for all u, v in $[(n-1)L/N, nL/N]$, $n = 1,\dots,N$. Defining $R^{N,L}$ as in § 3.3 and recalling equation (11), we have

$$\|K - SR^{N,L}S\|_{\mathrm{tr}} = \sum_{n=1}^{N} \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} (k(u,u) - k(v,u))\, du\, dv + \int_{L}^{\infty} k(x,x)\, dx. \quad (14)$$

Using the fact that k is $\mathrm{Lip}^{\alpha,s}$, the first term on the right hand side of (14) satisfies

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} (k(u,u) - k(v,u))\, du\, dv \\
&\quad \le \sum_{n=1}^{N} \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} A(u)\, du\, dv \left(\frac{L}{N}\right)^{\alpha} \\
&\quad = \int_0^L A(u)\, du \left(\frac{L}{N}\right)^{\alpha} \\
&\quad \le A \frac{L^{\alpha+s}}{N^{\alpha}}.
\end{aligned}$$

Therefore

$$\|K - SR^{N,L}S\|_{\mathrm{tr}} \le A \frac{L^{\alpha+s}}{N^{\alpha}} + \int_{L}^{\infty} k(x,x)\, dx. \quad (15)$$

We now use the hypotheses on the decay rate of $k(x,x)$. To prove statement (i), suppose there exist $\beta > 1$, $B > 0$ and x_0 such that $k(x,x) \le B/x^{\beta}$ for $x > x_0$. Set $\gamma = \dfrac{(\alpha+1)\beta + s - 1}{\alpha + \beta + s - 1}$. For each N we take L as a function of N, defining $L = L(N) = N^{\frac{\alpha+1-\gamma}{\alpha+s}}$; observe that in the range of parameters under consideration we have $1 < \gamma < 1+\alpha$ and consequently $L(N) \to +\infty$ as $N \to +\infty$.

We now consider the sequence of finite rank operators K_N defined by $K_N = SR^{N,L(N)}S$. Inequality (15) then becomes

$$\|K - K_N\|_{\mathrm{tr}} \le \frac{A}{N^{\gamma-1}} + \int_{L(N)}^{\infty} k(x,x)\, dx.$$

Taking N_0 large enough that $L(N) > x_0$ for all $N \ge N_0$ and noting that $(\frac{\alpha+1-\gamma}{\alpha+s})(1-\beta) = 1-\gamma$, the second term on the right hand side may be estimated as

$$\begin{aligned}
\int_{L(N)}^{\infty} k(x,x)\, dx &\le \int_{L(N)}^{\infty} \frac{B}{x^{\beta}}\, dx \\
&\le \frac{B}{\beta-1} \frac{1}{L(N)^{\beta-1}} \\
&= \frac{B}{\beta-1} N^{\left(\frac{\alpha+1-\gamma}{\alpha+s}\right)(1-\beta)} \\
&= \frac{B}{\beta-1} \frac{1}{N^{\gamma-1}}.
\end{aligned}$$

Hence

$$\|K - K_N\|_{\mathrm{tr}} \leq \left(A + \frac{B}{\beta - 1}\right) \frac{1}{N^{\gamma-1}}.$$

Recalling the results in §3, namely (5), we conclude that

$$\sum_{n=N+1}^{\infty} \lambda_n \leq \frac{C}{N^{\gamma-1}}$$

as $N \to +\infty$, with $C = A + B/(\beta - 1)$. This condition implies

$$\lambda_n = O\left(\frac{1}{n^\gamma}\right)$$

as $n \to +\infty$, completing the proof of statement (i).

To prove statement (ii), suppose now that $k(x,x) = O(1/x^\beta)$ for every $\beta > 1$. Then, by the above result, $\lambda_n = O(1/n^\gamma)$ for every γ in $]1, 1+\alpha[$. This fact actually implies the stronger statement $\lambda_n = o(1/n^\gamma)$ for every γ in $]1, 1+\alpha[$. In fact, if there were $\gamma_0 \in]1, 1+\alpha[$ such that $n^{\gamma_0} \lambda_n \to C$ for some $C > 0$, then λ_n would not be $O(1/n^\gamma)$ for $\gamma_0 < \gamma < 1+\alpha$, contradicting the previous result. So under this hypothesis on $k(x,x)$

$$\lambda_n = o\left(\frac{1}{n^\gamma}\right)$$

for every γ in $]1, 1+\alpha[$, proving statement (ii).

We now prove statement (iii). Suppose that $k(x,x)$ has compact support and let L be chosen such that $\operatorname{supp} k(x,x) \subset [0, L]$. Then we obviously have $\int_L^\infty k(x,x)\,dx = 0$. Fix L and for each $N \in \mathbb{N}$ define $K_N = SR^{N,L}S$. Proceeding as in the proof of statement (i), inequality (15) now yields

$$\|K - K_N\|_{\mathrm{tr}} \leq A \frac{L^{\alpha+s}}{N^\alpha}.$$

Since in this case L and A are fixed, it follows again from (5) that

$$\sum_{n=N+1}^{\infty} \lambda_n = O\left(\frac{1}{N^\alpha}\right)$$

as $N \to +\infty$, which implies

$$\lambda_n = O\left(\frac{1}{n^{1+\alpha}}\right),$$

as $n \to +\infty$. This completes the proof of the theorem. □

Remark 4.4. With respect to statement (ii) of Theorem 4.3, it is relevant to observe that the condition $\lambda_n = o(1/n^\gamma)$ for all $\gamma \in]1, \gamma_m[$ does not carry through to $\gamma = \gamma_m$. In fact, not even the weaker condition $\lambda_n = O(1/n^{\gamma_m})$ is in general implied, as is readily shown by the counterexample $\lambda_n = \dfrac{\log n}{n^{\gamma_m}}$, so that no statement for decay rate with exponent γ_m is in general valid.

Remark 4.5. A few interesting observations can be made from the study of the limiting cases in

$$\gamma = \frac{(\alpha+1)\beta + s - 1}{\alpha + \beta + s - 1} \tag{16}$$

and corresponding results on the decay rate of the eigenvalues λ_n of the operator K. Suppose we fix $\alpha \in]0,1]$ and $s > 0$. Then if k lies in class $\mathcal{A}$, is $\mathrm{Lip}^{\alpha,s}$ on the diagonal and $k(x,x) = O(1/x^{\beta})$ for some $\beta > 1$, Theorem 4.3 asserts that $\lambda_n = O(1/n^{\gamma})$ with γ given by (16). This implies, in particular, that $\lambda_n = o(1/n)$, a fact already known for any kernel in class $\mathcal{A}$, see § 2. This is the strongest decay rate if β is only known to be greater than 1. Accordingly, observe that $\gamma \to 1$ as $\beta \to 1$ irrespective of the values of α and s.

In a similar way, (16) yields $\gamma \to 1 + \alpha$ as $\beta \to +\infty$, while assertions (ii) and (iii) in Theorem 4.3 establish, respectively, that $\lambda_n = o(1/n^{\gamma})$ for all $\gamma \in]1, 1+\alpha[$ if $k(x,x) = O(1/x^{\beta})$ for all $\beta > 1$ and that $\lambda_n = O(1/n^{1+\alpha})$ if $k(x,x)$ has compact support. Again we find agreement between results in Theorem 4.3 and limiting values of γ given by (16). Note that in both cases $\beta \to 1$ and $\beta \to +\infty$ these limiting values are independent of s. The effect of this parameter can only be noticed for intermediate range values of β. Expectably, the strongest decay rate for the eigenvalues of K is attained at $s = 0$.

The limiting case $\alpha = 0$ may be interpreted as simply requiring that k lies in class $\mathcal{A}$, hence implying that $\lambda_n = o(1/n)$. Accordingly, we have $\gamma \to 1$ when $\alpha \to 0$ independently of β and s in (16). In fact, a slight modification of the proof of Theorem 4.3 yields the known estimate of the decay rate of the eigenvalues in the case where the kernel is only known to be in class $\mathcal{A}$.

Remark 4.6. Reade [10] shows that, in the case where $\mathrm{supp}\, k \subset [0,L]$ and k is regarded as the kernel of an $L^2([0,L])$ operator, then if k is Lip^{α} for $0 < \alpha \le 1$ the estimate $\lambda_n = O(1/n^{1+\alpha})$ is optimal. This is done by explicitly constructing kernels $k(x,y)$ which are Lip^{α} and whose eigenvalues are $\lambda_n = 1/n^{1+\alpha}$. Significantly, our results in Theorem 4.3 (iii) reproduce this optimal estimate. This is of course no coincidence; for a general discussion of the connection between these results see Remark 6.1 below.

First note that the condition that $k(x,x)$ has compact support actually implies that $k(x,y)$ has compact support. In fact, if $\mathrm{supp}\, k(x,x) = [0,L]$ then $\mathrm{supp}\, k(x,y) \subset [0,L]^2$, as implied by the inequality $|k(x,y)|^2 \le k(x,x)\, k(y,y)$ valid for k in class $\mathcal{A}$. Thus statement (iii) in Theorem 4.3 is a result about compactly supported $\mathrm{Lip}^{\alpha,s}$ $L^2([0,+\infty)[)$ kernels (by definition s is in this case irrelevant).

Reade's result implies that statement (iii) in Theorem 4.3 is also optimal. To see this, consider positive definite Lip^{α} kernels with support $S \subset [0,L]^2$ such that $k_{|\partial S} = 0$ and which admit a Lip^{α} extension $\tilde{k}$ to the zero function outside S. k is the kernel of an $L^2([0,L])$ operator to which Reade's results apply; $\tilde{k}$ is the kernel of an $L^2([0,+\infty])$ operator to which Theorem 4.3 applies. It is trivial to show, e.g. from expansion (3), that the eigenvalues of these operators are the same (as well as the eigenfunctions, with the obvious interpretation of the eigenfunctions of k being the restriction to $[0, L]$ of the eigenfunctions of $\tilde{k}$, which are 0 outside $[0, L]$). Thus optimality of the estimate in the $L^2([0, L])$ case implies optimality of statement (iii) in Theorem 4.3.

This fact together with Remark 4.5 above strongly suggests that statements (i) and (ii) in Theorem 4.3 are also optimal.

5. Asymptotics of Eigenvalues: The Differentiable Case

Our next result describes how, for a kernel in class $\mathcal{A}$, uniform continuity of a single partial derivative on the diagonal and the rate of decay at infinity on the diagonal allow us to control the decay rate of eigenvalues.

Theorem 5.1. *Let $k(x, y)$ be a differentiable kernel in class $\mathcal{A}$ and suppose that $\dfrac{\partial k}{\partial x}$ is uniformly continuous on the diagonal. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the integral operator with kernel k. Then the following statements hold.*

(i) *If $\beta > 1$, $k(x, x) = O(1/x^\beta)$ as $x \to +\infty$ (resp. $k(x, x) = o(1/x^\beta)$ as $x \to +\infty$) and $\gamma = \dfrac{2\beta}{1+\beta}$ then $\lambda_n = O(1/n^\gamma)$ (resp. $\lambda_n = o(1/n^\gamma)$).*

(ii) *If $k(x, x) = O(1/x^\beta)$ for all $\beta > 1$, then $\lambda_n = o(1/n^\gamma)$ for all γ in $]1, 2[$.*

(iii) *If $k(x, x)$ has compact support, then $\lambda_n = o(1/n^2)$.*

Proof. Let K be the operator with kernel k and S its positive square root. According to the hypothesis of uniform continuity of $\dfrac{\partial k}{\partial x}$ on the diagonal, for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\left|\frac{\partial k}{\partial u}(u, v) - \frac{\partial k}{\partial u}(u, u)\right| < \epsilon$$

for all (u, v) satisfying $|v - u| < \delta$. For every $L \in \mathbb{R}^+$ and every $N \in \mathbb{N}$ such that $L/N < \delta$, define $R^{N,L}$ as in § 3.3. Recalling equation (12), we have

$$\begin{aligned}
&\|K - SR^{N,L}S\|_{\mathrm{tr}} \\
&= \sum_{n=1}^{N} \frac{N}{L} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}} \left(\frac{k(u, u) + k(v, v)}{2} - k(v, u)\right) du dv + \int_L^\infty k(x, x) dx.
\end{aligned}$$

For all u, $v \in [0, +\infty[$ define $f_c(t)$ by

$$f_c(t) = \frac{k(c - t, c - t) + k(c + t, c + t)}{2} - k(c + t, c - t)$$

where $c = \frac{u+v}{2}$. Then f_c is differentiable for every $t \in \mathbb{R}$; furthermore, $f_c(0) = 0$ and $f_c(\frac{v-u}{2}) = \frac{k(u,u)+k(v,v)}{2} - k(v, u)$. By the Lagrange theorem there exists θ between 0 and $\frac{v-u}{2}$ such that

$$\frac{k(u,u)+k(v,v)}{2}-k(v,u)=f_c'(\theta)\left(\frac{v-u}{2}\right).$$

Recalling Remark 3.4, since in equation (12) $k(u,v)$ only appears upon integration in squares symmetric with respect to the diagonal, the imaginary part cancels out, so we may discard it. The real part, on the other hand, satisfies $k(u,v)=k(v,u)$ and $\frac{\partial k}{\partial v}(u,v)=\frac{\partial k}{\partial u}(v,u)$ by symmetry of k. Differentiating f and reordering terms, we obtain

$$f_c'(\theta)=\frac{\partial k}{\partial u}(c+\theta,c+\theta)-\frac{\partial k}{\partial u}(c+\theta,c-\theta)+\frac{\partial k}{\partial u}(c-\theta,c+\theta)-\frac{\partial k}{\partial u}(c-\theta,c-\theta).$$

Now, if $|v-u|\le L/N$, then $|v-u|<\delta$ and

$$\begin{aligned}&\frac{k(u,u)+k(v,v)}{2}-k(v,u)\ \le\ |f_c'(\theta)|\frac{L}{2N}\\ &\le\frac{L}{2N}\left\{\left|\frac{\partial k}{\partial u}(c+\theta,c+\theta)-\frac{\partial k}{\partial u}(c+\theta,c-\theta)\right|+\left|\frac{\partial k}{\partial u}(c-\theta,c+\theta)-\frac{\partial k}{\partial u}(c-\theta,c-\theta)\right|\right\}\\ &\le\frac{L}{N}\,\epsilon.\end{aligned}$$

We therefore have

$$\begin{aligned}\|K-SR^{N,L}S\|_{\mathrm{tr}}\ &\le\ \sum_{n=1}^{N}\frac{N}{L}\int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}}\int_{(n-1)\frac{L}{N}}^{n\frac{L}{N}}\frac{L}{N}\epsilon\,du\,dv+\int_L^\infty k(x,x)\,dx\\ &=\ \frac{L^2}{N}\,\epsilon+\int_L^\infty k(x,x)\,dx.\end{aligned}\tag{17}$$

We now use the hypotheses on the decay rate of $k(x,x)$. To prove statement (i), suppose there exist $\beta>1$, $B>0$ and $x_0>0$ such that $k(x,x)\le B/x^\beta$ for every $x>x_0$. Set $\gamma=\frac{2\beta}{1+\beta}$, define $L=L(N)$ by $L(N)=N^{1-\gamma/2}$ and the operator K_N by $K_N=SR^{N,L(N)}S$. Observe that, since $\beta>1$, it follows that $1<\gamma<2$, whence $L(N)\to+\infty$ and $L(N)/N=N^{-\gamma/2}\to 0$ as $N\to+\infty$. Note also that, by the results in § 2, the operators K_N have rank N and are in class $\mathcal{A}$. Then there is a natural number N_0 such that $L(N)\ge x_0$ and $L(N)/N<\delta$ for every $N\ge N_0$. For such values of N inequality (17) becomes

$$\|K-K_N\|_{\mathrm{tr}}\le\frac{\epsilon}{N^{\gamma-1}}+\int_{L(N)}^\infty k(x,x)\,dx.$$

The second term on the right hand side can be estimated as follows:

$$\begin{aligned}\int_{L(N)}^{\infty} k(x,x)\,dx &\le \int_{L(N)}^{\infty} \frac{B}{x^{\beta}}\,dx \\ &= \frac{B}{\beta-1}\,\frac{1}{L(N)^{\beta-1}} \\ &= \frac{B}{\beta-1}\,\frac{1}{N^{(1-\gamma/2)(1-\beta)}} \\ &= \frac{B}{\beta-1}\,\frac{1}{N^{\gamma-1}}.\end{aligned}$$

Hence

$$\|K-K_N\|_{\mathrm{tr}} \le \left(\epsilon + \frac{B}{\beta-1}\right)\frac{1}{N^{\gamma-1}}.$$

According to (5), it follows that $\displaystyle\sum_{n=N+1}^{\infty} \lambda_n \le \frac{C}{N^{\gamma-1}}$, with $C = \epsilon + B(\beta-1)$. If $k(x,x) = O(1/x^{\beta})$ as $x \to +\infty$ the above condition is verified for some fixed $C > 0$, which implies that $\lambda_n = O(1/n^{\gamma})$ as $n \to +\infty$. If $k(x,x) = o(1/x^{\beta})$, the same condition is established for every $C > 0$; we therefore derive in this case the stronger conclusion that $\lambda_n = o(1/n^{\gamma})$. This completes the proof of statement (i).

Suppose now that $k(x,x) = O(1/x^{\beta})$ for every $\beta > 1$. Then, by the above result, it follows that $\lambda_n = O(1/n^{\gamma})$ for every $\gamma \in]1,2[$. Reasoning as in the proof of statement (ii) of Theorem 4.3, it follows that $\lambda_n = o(1/n^{\gamma})$ for every $\gamma \in]1,2[$. This proves statement (ii).

Finally, suppose that $k(x,x)$ has compact support and let L be such that $\operatorname{supp} k(x,x) \subset [0,L]$. Then we obviously have $\int_L^{\infty} k(x,x)\,dx = 0$. For every $\epsilon > 0$ choose N_0 such that

$$\left|\frac{\partial k}{\partial u}(u,v) - \frac{\partial k}{\partial u}(u,u)\right| < \epsilon$$

for all (u,v) satisfying $|v-u| < L/N_0$. For each $N > N_0$ define the rank N, class $\mathcal{A}$ operator $K_N = SR^{N,L}S$, where L and N are now independent. Performing the same procedure as in the proof of statement (i), inequality (17) now yields

$$\|K-K_N\| \le \epsilon\frac{L^2}{N}.$$

Since L is fixed and $\epsilon > 0$ is arbitrary, this implies that

$$\sum_{n=N+1}^{\infty} \lambda_n = o\left(\frac{1}{N}\right)$$

as $N \to +\infty$, from which it follows that

$$\lambda_n = o\left(\frac{1}{n^2}\right)$$

as $n \to +\infty$. This completes the proof of the theorem. □

Remark 5.2. As pointed out in Remark 4.4, we observe that statement (ii) of Theorem 5.1 cannot be strengthened to a statement about decay rate with exponent 2, even in the weaker form $\lambda_n = O(1/n^2)$.

Remark 5.3. In the hypotheses of Theorem 5.1 we have assumed $k(x,y)$ to have a uniformly continuous first partial derivative with respect to x on the diagonal. Some observations are relevant to the discussion of this choice.

Firstly, we note that the same condition could have been imposed to the partial derivative with respect to the variable y instead. In fact, for symmetric kernels the two conditions are easily shown to be equivalent. Each of them implies, therefore, the apparently stronger condition that k is uniformly C^1 on the diagonal. Secondly, it is interesting to observe that, as in the case of Theorem 4.3, or indeed of Definition 2.1 of class $\mathcal{A}$, the essential requirements on the behavior of k may be restricted to the diagonal with no consequence on the proofs, a fact which is not apparent in the previous literature. Finally we note that uniform continuity of the derivative, which trivially derives from the C^1 condition in the compact domain case (see Remark 6.1 below), must be explicitly imposed in the case of unbounded domains. Significantly, this is true for the partial derivatives of k along the diagonal in the hypotheses of Theorem 5.1 as it is for k itself as a consequence of lying in class $\mathcal{A}$ (see [1]).

Remark 5.4. Reade [10, 13] shows that, in the case where supp $k \subset [0,L]$ and k is regarded as the kernel of an $L^2([0,L])$ operator, then if k is C^1 the estimate $\lambda_n = O(1/n^2)$ is optimal. This is done by explicitly constructing kernels $k(x,y)$ which are C^1 and whose eigenvalues are $\lambda_n = 1/n^2$. Theorem 5.1 (iii) reproduces this optimal estimate. In view of the similarity with Remark 4.6 we abbreviate the discussion.

If $k(x,x)$ has compact support then $k(x,y)$ has compact support. Thus statement (iii) in Theorem 5.1 is a result about compactly supported C^1, $L^2([0,+\infty)[)$ kernels. Consider positive definite C^1 kernels with support $S \subset [0,L]^2$ such that $k_{|\partial S} = 0$ and which admit a C^1 extension $\tilde{k}$ to the zero function outside S. k is the kernel of an $L^2([0,L])$ operator to which Reade's results apply; $\tilde{k}$ is the kernel of an $L^2([0,+\infty])$ operator to which Theorem 5.1 applies. From expansion (3) the eigenvalues of these operators are the same. Thus optimality of the estimate in the $L^2([0,L])$ case implies optimality of statement (iii) in Theorem 5.1.

This fact strongly suggests that statements (i) and (ii) in Theorem 5.1 are also optimal.

6. Final Remarks

Remark 6.1. Clearly, the last assertions of Theorems 4.3 and 5.1 bear a direct connection to the formally identical results known for positive trace class operators defined on a compact interval, namely those which apply to kernels verifying a Lipschitz condition of order α or a condition of continuous differentiability described in § 1.

In fact, Theorems 4.3 and 5.1 require somewhat weaker (yet sufficient) versions of the above referred conditions if the domains of the kernels are assumed, in particular, to be compact (see Remarks 4.6 and 5.4). Moreover, the steps taken in our proofs of the third statements of both Theorems 4.3 and 5.1 collapse, upon assumption of compactness of support of k, precisely into Reade's proofs of his results.

To see this, we just need to observe that the equality $\int_L^\infty k(x,x)\,dx = 0$, which is ensured by compactness of the support of the kernel k on the diagonal when the domain of k is $[0,+\infty[^2$, is trivially implicit in the assumption that the domain of k is $[0,L]^2$.

Remark 6.2. As remarked in §2, our results are stated and proved in the case where $I = [0,+\infty[$ for convenience only. They are valid, with minimal rephrasing, for $L^2(I)$ integral operators K and respective positive definite kernels $k(x,y)$ for other kinds of unbounded intervals.

Of particular significance (see Remark 6.3 below) is the case where $I = \mathbb{R}$. In this case we are dealing with $L^2(\mathbb{R})$ positive definite kernels; class $\mathcal{A}(\mathbb{R})$ (corresponding to class $\mathcal{A}$ of Definition 2.1) kernels are now required to be continuous in $\mathbb{R}^2$, with $k(x,x) \in L^1(\mathbb{R})$ and $k(x,x) \to 0$ as $|x| \to +\infty$. All the results and proofs, in particular Theorems 4.3 and 5.1, are valid simply by replacing the required asymptotic behavior of $k(x,x)$ as $x \to +\infty$ by the corresponding requirement as $|x| \to +\infty$.

Remark 6.3. The case $I = \mathbb{R}$ is particularly relevant because there is a close connection between positive definiteness of a continuous $L^2(\mathbb{R})$ kernel $k(x,y)$ and that of its Fourier transform $\hat{k}(\nu_1,\nu_2)$. More specifically, it is possible to show that if k is in class $\mathcal{A}(\mathbb{R})$ as defined in Remark 6.2, then its "rotated" Fourier transform $\tilde{k}(\nu_1,\nu_2) = \hat{k}(\nu_1,-\nu_2)$ is a positive definite kernel with the same eigenvalues λ_n as k and whose associated eigenfunctions are the Fourier transforms of the corresponding eigenfunctions of k. Moreover, if $k^{1/2}(x,x) \in L^1(\mathbb{R})$ then the $L^2(\mathbb{R})$ integral operator $\tilde{K}$ with kernel $\tilde{k}$ is trace class with the same trace as K; see Buescu *et al.* [2] for details.

If k is in class $\mathcal{A}(\mathbb{R})$, a sufficient condition for $k^{1/2}(x,x) \in L^1(\mathbb{R})$ may be formulated in terms of the asymptotic behavior of $k(x,x)$ as $k(x,x) = O(1/x^\beta)$ for some $\beta > 2$. In view of Theorems 4.3 and 5.1 we may then derive the following

Corollary 6.4. *Suppose $k(x,y)$ is a positive definite kernel in class $\mathcal{A}(\mathbb{R})$ with $k(x,x) = O(1/x^\beta)$ as $|x| \to +\infty$ for some $\beta > 2$ and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the integral operator with kernel k with associated eigenfunctions ϕ_n. Let $\hat{k}(\nu_1,\nu_2)$ be the double Fourier transform of $k(x,y)$, $\tilde{k}(\nu_1,\nu_2) = \hat{k}(\nu_1,-\nu_2)$ and $\hat{\phi}_n$ be the Fourier transform of ϕ_n. Then the following statements hold.*

(i) *$\tilde{k}$ is a positive definite kernel in class $\mathcal{A}(\mathbb{R})$ with absolutely and uniformly convergent eigenfunction expansion*

$$\tilde{k}(\nu_1,\nu_2) = \sum_{n\geq 1} \lambda_n\, \hat{\phi}_n(\nu_1)\, \overline{\hat{\phi}_n(\nu_2)} \tag{18}$$

and the operator $\tilde{K}$ with kernel $\tilde{k}$ is trace class with

$$tr\ \tilde{K} = tr\ K = \int_{-\infty}^{+\infty} k(x,x)\,dx = \int_{-\infty}^{+\infty} \tilde{k}(\nu,\nu)\,d\nu = \sum_{n\geq 1} \lambda_n.$$

(ii) *If $k(x,y)$ is $Lip^{\alpha,s}$ with respect to x on the diagonal, then $\lambda_n = O(1/n^\gamma)$, where $\gamma = \dfrac{(\alpha+1)\beta+s-1}{\alpha+\beta+s-1}$.*

(iii) *If $k(x,y)$ is differentiable and $\dfrac{\partial k}{\partial x}$ is uniformly continuous on the diagonal then $\lambda_n = O(1/n^\gamma)$, where $\gamma = \dfrac{2\beta}{1+\beta}$.*

Note that, since the eigenvalues of the operators with kernels k and $\tilde{k}$ coincide, assertions (ii) and (iii) in Corollary 6.4 may be interpreted directly as statements about decay rates of eigenvalues of the operator with kernel $\tilde{k}$, that is, on the Fourier transform side. These results may be relevant for applications to engineering, namely in the field of signal processing [3].

References

[1] J. Buescu, *Positive integral operators in unbounded domains.* Jour. Math. Anal. Appl., (2004), 244–255.

[2] J. Buescu, F. Garcia, I. Lourtie, A. Paixão, *Positive definiteness, integral equations and Fourier transforms.* Jour. Int. Eq. Appl., **16** 1 (2004), 33–52.

[3] J. Buescu, F. Garcia, I. Lourtie, *$L^2(\mathbb{R})$ nonstationary processes and the sampling theorem.* IEEE Sign. Proc. Lett. **8**, 4 (2001), 117–119.

[4] C. Chang, C. Ha, *On eigenvalues of differentiable positive definite kernels.* Integr. Equ. Oper. Theory **33** (1999), 1–7.

[5] J. Cochran, M. Lukas, *Differentiable positive definite kernels and Lipschitz continuity.* Math. Proc. Camb. Phil. Soc. **104** (1988), 361–369.

[6] I. Gohberg, M. Krein, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space.* A.M.S., Providence, 1969.

[7] C. Ha, *Eigenvalues of differentiable positive definite kernels.* SIAM J. Math. Anal. **17** (1986), 2, 415–419.

[8] I. M. Novitskii, *Representation of kernels of integral operators by bilinear series.* Siberian Math. J. **25** (1984), 3, 774–778. Translated form the Russian: Sibirsk. Mat. Zh. **25** (1984), 5, 114–118.

[9] J. Reade, *Eigenvalues of positive definite kernels.* SIAM J. Math. Anal. **14** (1983), 1, 152–157.

[10] J. Reade, *Eigenvalues of Lipschitz kernels.* Math. Proc. Camb. Phil. Soc. **93** (1983), 1, 135–140.

[11] J. Reade, *Eigenvalues of positive definite kernels II.* SIAM J. Math. Anal. **15** (1984), 1, 137–142.

[12] J. Reade, *Positive definite C^p kernels.* SIAM J. Math. Anal. **17** (1986), 2, 420–421.

[13] J. Reade, *Eigenvalues of smooth positive definite kernels.* Proc. Edimburgh Math. Soc. **35** (1990), 41–45.

[14] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis (revised and enlarged edition).* Academic Press, San Diego, 1980.

[15] F. Riesz, B. Nagy, *Functional Analysis.* Ungar, New York, 1952.

[16] H. Weyl, *Das Asymptotische Verteilunggesetz der Eigenwerte linearer partieller Differentialgleichungen.* Math. Ann. **71** (1912), 2, 441–479.

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