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ON HELIXES IN HILBERT SPACE. I

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§ 1. Introduction

A helix (also called screw curve) in a Hilbert space $\mathfrak{H}$ is a continuous function $x(\cdot)$ on the real number field $\mathfrak{R}$ to $\mathfrak{H}$ for which the inner product $(x(b) - x(a), x(d) - x(c))$ is translation-invariant. Such curves first seem to appear in Wiener's work on the Brownian motion [20], $\mathfrak{H}$ there being the class of L_2 functions over a probability space. The general formulation of the concept for any complex Hilbert space is due to Kolmogorov [6], and for real $\mathfrak{H}$ to von Neumann and Schoenberg [18, 19]. M. Krein [8] extended the notion to include helical arcs defined on subintervals of $\mathfrak{R}$, and Pinsker and Yaglom, cf. [21] and references therein, have extended it to include curves for which the inner products of the n^{th} increments $(n > 1)$ are translation-invariant. As these authors have shown, the concept has interesting ramifications in problems concerning stochastic processes, the embedding of metrics over $\mathfrak{R}$ into Euclidean spaces, infinitely divisible positive definite functions of $\mathfrak{R}$, completely monotone functions and allied subjects [1, 6, 7, 8, 16, 17, 19]. It also has a bearing on some questions in generalized harmonic analysis as Lee [9] has recently indicated.

It is convenient to view the domain $\mathfrak{R}$ of a helix $x(\cdot)$ as representing time, and the helix itself as a path or orbit in $\mathfrak{H}$. One can then speak of $\mathfrak{R}$ as the *time-domain*. This viewpoint is especially poignant when $\mathfrak{H} = L_2(\Omega, \mathfrak{B}, P)$, where $(\Omega, \mathfrak{B}, P)$ is a probability space. Under the temporal interpretation of $\mathfrak{R}$ the helix $x(\cdot)$ then becomes a *stochastic process* (*SP*) *with* (*wide-sense*) *stationary increments*, cf. Doob [1, pp. 99, 551]. Such processes are important, the Brownian movement SP being the first and perhaps most notable example.**

In the study of helixes it has been customary to turn quickly to the spectral representation of the helix, or of certain scalar-valued functions associated with it. In this paper we depart from this tradition by first obtaining a strong time-domain characterization of helixes, which fully reveals their structure. We show that the structure of a helix $x(\cdot)$ is determined completely by (i) the stationary curve $y(\cdot)$ obtained from the action of its unitary shift

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** The Brownian movement SP has of course the very important additional property that *its increments are independent and therefore orthogonal in* $\mathfrak{H}$. We, of course, make n such additional assumption regarding our helixes.

group on an «average vector» α_x, and (ii) a «displacement vector» p_x orthogonal to $y(\cdot)$. We then combine this characterization of the helix with the spectral resolution of its unitary shift group to infer the existence and uniqueness of the spectral representations for it due to Kolmogorov [6] and von Neumann and Schoenberg [19]. But now the measures involved are identified, and to this extent our spectral-domain results are also new. Thus our approach is both more informative and less cumbersome than the traditional one involving the premature use of spectral ideas.

Our work can also be viewed as a furtherance of the use of operator-valued measures in functional analysis. An essential ingredient in our time-domain charasterization of the helix is an $\mathfrak{H}$-to-$\mathfrak{H}$, operator-valued measure $T(\cdot)$ defined on the prering of half-open-closed intervals (a, b) of the time domain $\mathfrak{R}$, cf. (2.17). Previously, we have considered only the measures $T_0(\cdot)$ obtained from this $T(\cdot)$ by restricting the domain of the operators $T(a, b]$ to certain «wandering» subspaces of $\mathfrak{H}$. These restricted measures $T_0(\cdot)$ have a property called *quasi-isometry*, which permits their utilization in the explicit formulation of the Wold decomposition of isometric and unitary flows, and related decompositions in prediction and scattering theory [10, 12, 13, 14]. The present paper shows that *the unrestricted measure $T(\cdot)$ is, itself intrinsic, and its usefulness extends to situations in which it is not quasi-isometric.*

In § 2 we study the helix in the time-domain. After establishing some simple lemmas we prove our main theorems 2.19, 2.20, 2.22 on the time-domain characterization of helixes. In § 3 we turn to the spectral domain, and obtain strong and weak spectral representations of helixes as easy corollaries of our time-domain results and Stone's theorem on unitary groups. We also show the uniqueness of these representations.

The theory of the operator-valued measure $T(\cdot)$ mentioned above logically precedes that of helixes and plays an ancillary role in the latter. To avoid digressions in our development of helix theory, we have accordingly relegated the discussion of $T(\cdot)$ to the Appendix and have omitted certain proofs.

§ 2. The strong time-domain characterization of a helix

Let $\mathfrak{H}$ be a complex Hilbert space and $x(\cdot)$ a function on the real number field $\mathfrak{R}$ to $\mathfrak{H}$, which is continuous with respect to the norm — topology of $\mathfrak{H}$. We shall think of $x(\cdot)$ as being a continuous curve in $\mathfrak{S}$, and of the vector $x(t) = x(s)$ as being a *chord* of the curve $x(\cdot)$. Associated with such a curve $x(\cdot)$ are two subspaces* of $\mathfrak{H}$:

$$\left.\begin{aligned} \mathfrak{M}_x &\underset{d}{=} \mathfrak{S}\{x(t) : t \in \mathfrak{R}\}, \\ \mathfrak{S}_x &\underset{d}{=} \mathfrak{S}\{x(t) - x(s) : s, t \in \mathfrak{R}\} \end{aligned}\right\} \tag{2.1}$$

which we shall call respectively *the subspace* of $x(\cdot)$, and the *chordal-subspace* of $x(\cdot)$. We shall denote by P_x and $P_x^{\perp}$ the orthogonal projections with do-

* The symbol «$\underset{d}{=}$» means «equals by definition». $\forall A \subseteq \mathfrak{H}$, $\mathfrak{S}(A) \underset{d}{=}$ the (least closed linear) subspace spanned by A.

main $\mathfrak{M}_x$ and ranges $\mathfrak{S}_x$ and $\mathfrak{S}_x^{\perp} \cap \mathfrak{M}_x$, respectively. It is easy to see that

$$\left.\begin{aligned} &\forall t \in \mathfrak{R},\ \mathfrak{M}_x = \mathfrak{S}_x + \mathfrak{S}\{x(t)\}, \\ &\forall t \in \mathfrak{R},\ P_x^{\perp}\{x(t)\} = p_x, \text{ where } p_x \underset{d}{=} P_x^{\perp}\{x(0)\}, \\ &\forall t \in \mathfrak{R},\ x(t) = \hat{x}(t) + p_x,\ \hat{x}(t) \underset{d}{=} P_x\{x(t)\}, \\ &\mathfrak{M}_x = \mathfrak{S}_x + \mathfrak{S}(p_x),\ \mathfrak{S}_x \perp \mathfrak{S}(p_x), \\ &\mathfrak{S}_x^{\perp} \cap \mathfrak{M}_x = \mathfrak{S}(p_x) \text{ has dimension 0 or 1.} \end{aligned}\right\} \tag{2.2}$$

Also associated with $x(\cdot)$ are the functions f_x, g_x, γ_x on* $\mathfrak{R}$-to-$\mathfrak{R}_{0+}$, $\mathfrak{R}^2$-to-$\mathfrak{C}$ and $\mathfrak{R}^4$-to-$\mathfrak{C}$, respectively, defined by (writing $x_t = x(t)$, etc.)

$$\left.\begin{aligned} f_x(t) &\underset{d}{=} |x_t - x_0|, && \forall t \in \mathfrak{R}, \\ g_x(s,t) &\underset{d}{=} (x_s, x_t), && \forall s,t \in \mathfrak{R}, \\ \gamma_x(r,s,t,u) &\underset{d}{=} (x_s - x_r, x_u - x_t), && \forall r,s,t,u \in \mathfrak{R}. \end{aligned}\right\} \tag{2.3}$$

We shall call f_x the *chordal length-function* of x, g_x its *covariance kernel* and γ_x its *chordal covariance kernel.* These functions are obviously related by the equations

$$\left.\begin{aligned} &\gamma_x(r,s,t,u) = g_x(r,t) + g_x(s,u) - g_x(r,u) - g_x(s,t), \\ &f_x(t) = \sqrt{\gamma_x(0,t,0,t)} = \{g_x(t,t) + g_x(0,0) - 2\,\mathrm{real}\ g_x(t,0)\}^{1/2}. \end{aligned}\right\} \tag{2.4}$$

The concepts just defined remain meaningful even when $\mathfrak{R}$ is not the real number field but only a topological space. (We have only to reinterpret $x(\cdot)$ as a non-linear manifold in $\mathfrak{H}$, and not necessarily as a curve in $\mathfrak{H}$). We now turn to ideas which depend essentially on the total ordering of $\mathfrak{R}$ and on $\mathfrak{R}$ being a locally compact abelian group.

2.5. **Def.** A curve $x(\cdot)$ in $\mathfrak{H}$ is called *an indefinite integral* of a continuous function $y(\cdot)$ in $\mathfrak{H}$, iff**

$$\forall t \in \mathfrak{R},\ x(t) = x(0) + \int_0^t y(t)\,dt,$$

the last being a Lebesgue — Bochner (in fact Riemann) integral in $\mathfrak{H}$.

When $x(\cdot)$ and $y(\cdot)$ are related as in 2.5. we obviously have

$$\left.\begin{aligned} &\forall a,b \in \mathfrak{R},\ x(b) - x(a) = \int_a^b y(t)\,dt \in \mathfrak{M}_y, \\ &\forall t \in \mathfrak{R},\ y(t) = \lim_{h \to 0} \frac{1}{h}\{x(t+h) - x(t)\} \in \mathfrak{S}_x. \end{aligned}\right\} \tag{2.6}$$

The first of these statements shows that $\mathfrak{S}_x \subseteq \mathfrak{M}_y$, and the second that $\mathfrak{M}_y \subseteq \mathfrak{S}_x$; thus

$$\mathfrak{S}_x = \mathfrak{M}_y. \tag{2.7}$$

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* $\mathfrak{R}_{0+} \underset{d}{=} \{x : x \in \mathfrak{R}\ \&\ x \geqslant 0\}$, $\mathfrak{R}_{+} \underset{d}{=} \{x : x \in \mathfrak{R}\ \&\ x > 0\}$.
$\mathfrak{C}$ denotes the complex number field.

** «iff» abbreviates «if and only if».

2.8. **Def.** (a) By a *stationary curve* in $\mathfrak{H}$ we mean a continuous function $x(\cdot)$ on $\mathfrak{R}$ to $\mathfrak{H}$ for which the covariance kernel g_x is translation invariant i. e.

$$\forall a, b, t \in \mathfrak{R}, \quad g_x(a+t, b+t) = g_x(a, b).$$

(b) By a *helix* (or *screw curve*) in $\mathfrak{H}$ we mean a continuous function $x(\cdot)$ on $\mathfrak{R}$ to $\mathfrak{H}$ for which the chordal covariance kernel $\gamma_x(\cdot\cdot\cdot\cdot)$ is translation invariant, i. e.

$$\forall a, b, c, d, t \varepsilon \mathfrak{R}, \quad \gamma_x(a+t, b+t, c+t, d+t) = \gamma_x(a, b, c, d).$$

For a helix $x(\cdot)$ we obviously have

$$|x(b) - x(a)| = \sqrt{\gamma_x(a, b, a, b)} = \sqrt{\gamma_x(0, b-a, 0, b-a)} = f_x(b-a). \quad (2.9)$$

Schoenberg and von Neumann [19] call ϕ a *screw function*, when it is the chordal length-function f_x of a helix $x(\cdot)$. The growth of such a function is given by the following Lemma:

2.10. **Lemma** (Growth of a screw function). *Let $x(\cdot)$ be a helix. Then*

(a) *$f_x(\cdot)$ is an even continuous function on $\mathfrak{R}$ to $\mathfrak{R}_{0+}$ such that $f(0) = 0$;*

(b) $\forall t \in \mathfrak{R}, \; 0 \leqslant f_x(t) \leqslant \max\limits_{0 \leqslant t \leqslant 1} f_x(t) + |t| f_x(1).$

P r o o f. (a) is utterly obvious.

(b) We first note that $\forall n \geqslant 1$,

$$f_x(n) = |x(n) - x(0)| \leqslant \sum_{k=0}^{n-1} |x(k+1) - x(k)| \leqslant \sum_{k=0}^{n-1} |x(1) - x(0)| \leqslant n f_x(1).$$

It follows that $\forall t > 0$,

$$f_x(t) = |x(t) - x(0)| \leqslant |x(t) - x([t])| + |x([t]) - x(0)| \leqslant$$
$$\leqslant f_x(t - [t]) + f_x([t]) \leqslant M + [t] f_x(1) \leqslant M + t f_x(1), \; M \underset{d}{=} \max_{0 \leqslant t \leqslant 1} f_x(t).$$

Also, since $f_x(\cdot)$ is even, therefore $\forall t < 0$,

$$f_x(t) = f_x(|t|) \leqslant M + |t| f_x(1). \; \blacksquare^*$$

Let $x(\cdot)$ be a helix in $\mathfrak{H}$. Then $x(0) - x(\cdot)$ and $x(\cdot)$ being continuous, on $\mathfrak{R}$, are strongly (Bochner-) measurable on $\mathfrak{R}$, cf. [3, Thm. 3.5.3]. Also by 2.10 (b),

$$0 \leqslant \int_{\mathfrak{R}} e^{-|t|} |x(0) - x(t)| \, dt = \int_{\mathfrak{R}} e^{-|t|} f_x(t) \, dt \leqslant$$
$$\leqslant M \int_{\mathfrak{R}} e^{-|t|} dt + f_x(1) \int_{\mathfrak{R}} |t| e^{-|t|} dt < \infty$$

where $M \underset{d}{=} \max\limits_{0 \leqslant t \leqslant 1} f_x(t)$; and so

$$0 \leqslant \int_{\mathfrak{R}} e^{-|t|} |x(t)| \, dt \leqslant \int_{\mathfrak{R}} e^{-|t|} |x(0) - x(t)| \, dt + |x(0)| \int_{\mathfrak{R}} e^{-|t|} dt < \infty.$$

Hence, cf. [3, Thm. 3.7.4], we get the following result concerning the chords of a helix:

* The symbol $\blacksquare$ signifies the completion of a proof.

2.11. **Lemma.** *Let* $x(\cdot)$ *be a helix. Then*

$$x(0) - x(\cdot) \,\&\, x(\cdot) \in L_1(\mathfrak{R}, Bl.(\mathfrak{R}), p; \mathfrak{H})^*,$$

where the last is the class of functions on $\mathfrak{R}$ *to* $\mathfrak{H}$ *which are (Lebesgue — Bochner) integrable with respect to the exponential distribution* $p(\cdot)$ *over* $\mathfrak{R}$ *such that* $p'(t) = e^{-t}\chi_{[0,\infty]}(t)$; *also*

$$\int_0^\infty e^{-t}\{x(0) - x(t)\}\,dt \in \mathfrak{S}_x \,\&\, \int_0^\infty e^{-t}x(t)\,dt \in \mathfrak{M}_p.$$

212. Def. Let $x(\cdot)$ be a helix in $\mathfrak{H}$. Then the vector

$$\alpha_x \underset{d}{=} \sqrt{2}\int_0^\infty e^{-t}\{x(0) - x(t)\}\,dt \in \mathfrak{S}_x$$

is called the *average vector* of $x(\cdot)$.

We also note that

$$\alpha_x + \beta_x = \sqrt{2}x(0), \text{ where } \beta_x \underset{d}{=} \sqrt{2}\int_0^\infty e^{-t}x(t)\,dt \in \mathfrak{M}_x. \tag{2.13}$$

It follows from (2.2) that

$$\mathfrak{S}_x = \mathfrak{M}_x \Leftrightarrow x(0) \in \mathfrak{S}_x \Leftrightarrow \beta_x = \sqrt{2}x(0) - \alpha_x \in \mathfrak{S}_x.$$

Karhunen [5, p. 55] showed that a continuous curve $y(\cdot)$ in $\mathfrak{H}$ is *stationary* if there exists a strongly continuous *shift-group* $(V_t, t \in \mathfrak{R})$ of unitary operators on $\mathfrak{M}_y$ onto $\mathfrak{M}_y$ such that

$$\forall a, t \in \mathfrak{R},\ V_t\{y(a)\} = y(a+t). \tag{2.14}$$

We owe to von Neumann and Schoenberg [19, p. 238] the much harder analogous result for a helix, viz.

2.15. **Theorem.** *Let* $x(\cdot)$ *be a continuous function on* $\mathfrak{R}$ *to* $\mathfrak{H}$. *Then* $x(\cdot)$ *is a helix if there exists a strongly continuous group* $(U_t, t \in \mathfrak{R})$ *of unitary operators on the chordal subspace* $\mathfrak{S}_x$ *onto* $\mathfrak{S}_x$ *such that*

$$\forall a, b, t \in \mathfrak{R},\ U_t\{x(b) - x(a)\} = x(b+t) - x(a+t). \tag{1}$$

It is easy to see that if $x(\cdot)$ is a helix, then the group $(U_t, t \in \mathfrak{R})$ satisfying the conditions of the last theorem is unique. We shall call it *the shift group of the helix*. The operators U_t of the shift group of a helix have unitary extensions $\tilde{U}_t$ on $\mathfrak{H}$ onto $\mathfrak{H}$, which can be chosen so as to make $(\tilde{U}_t, t \in \mathfrak{R})$ a strongly continuous group of unitary operators on $\mathfrak{H}$ onto $\mathfrak{H}$. Such an extention is not of course unique.

Now in our work on stationary curves in $\mathfrak{H}$ and on isometric and unitary flows on $\mathfrak{H}$ [14, 10, 12] we have associated with each strongly continuous group $(U_t, t \in \mathfrak{R})$ of unitary operators on $\mathfrak{H}$ onto $\mathfrak{H}$ the operator-valued measure $T_U(\cdot)$ defined on the pre-ring

$$\mathfrak{P} \underset{d}{=} \{(a, b] : a, b \in \mathfrak{R} \,\&\, a \leqslant b\} \tag{2.16}$$

* Bl.($\mathfrak{R}$) denotes the family of all Borel subsets of $\mathfrak{R}$.

of bounded half-open-closed intervals $(a, b]$ of $\mathfrak{R}$ by the formula

$$T_U(a, b] \underset{d}{=} \frac{1}{\sqrt{2}} \left\{ U_b - U_a - \int_a^b U_t dt \right\}. \tag{2.17}$$

But in the past we have been interested primarily in* $\mathrm{Rstr.}_W T_U(\cdot)$, where W is a *wandering subspace* of $\mathfrak{H}$ with respect to T_U, i. e. is such that $\forall w$, $w' \in W$ and $\forall J$, $K \in \mathfrak{P}$,

$$(T_U(J) w, T_U(K) w') = \mathrm{Leb.}\,(J \cap K) \cdot (w, w').$$

It follows from the last equation that $T_U(J)/\sqrt{\mathrm{Leb.}\,(J)}$ is an isometry on W onto $\mathfrak{H}$. We have therefore called the restriction of $T_U(\cdot)$ to W a *quasi-isometric measure.*** In this paper we have to consider the unrestricted measure $T_U(\cdot)$ itself. It has several nice properties, cf. [13, § 15]; the ones we need are given in the Appendix.

The nexus between a helix $x(\cdot)$ and the measure $T_U(\cdot)$ stemming from its shift group $(U_t, t \in \mathfrak{R})$ is revealed by the following crucial lemma:

2.18. **Main Lemma** (Switching property). *Let $x(\cdot)$ be a helix in $\mathfrak{H}$ with shift group $(U_t, t \in \mathfrak{R})$. Then for all $(a, b]$, $(c, d] \in \mathfrak{P}$,*

$$T_U(a, b]\{x(d) - x(c)\} = T_U(c, d]\{x(b) - x(a)\}.$$

P r o o f. The proof is straightforward. Abbreviating $x(t)$ by x_t, we have

$$\sqrt{2} T_U(a, b](x_d - x_c) = \left(U_b - U_a - \int_a^b U_t dt\right)(x_d - x_c). \tag{1}$$

Now from the linearity of the U_t and eqn. 2.15 (1) it follows that

$$(U_b - U_a)(x_d - x_c) = x_{b+d} - x_{a+d} - (x_{b+c} - x_{a+c}) = (U_d - U_c)(x_b - x_a). \tag{2}$$

The linearity of U_t, the eqn. 2.15 (1), along with [3, Thm. 3.3.4] and some simple substitutions also show that

$$\left(\int_a^b U_s ds\right)(x_d - x_c) = \int_a^b x_{s+d} ds - \int_a^b x_{s+c} ds =$$

$$= \int_{a+d}^{b+d} x_t dt - \int_{a+c}^{b+c} x_u du, \text{ where } t = s + d,\ u = s + c,$$

$$= \int_{c+b}^{d+b} x_u du - \int_{c+a}^{d+a} x_u du =$$

$$= \int_c^d x_{s+b} ds - \int_c^d x_{t+a} dt, \text{ where } s = u - b,\ t = u - a,$$

$$= \left(\int_c^d U_s ds\right)(x_b - x_a). \tag{3}$$

On combining (2), (3) and (1), we get the desired equality. ■

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* $\mathrm{Rstr.}_A F$ denotes the restriction of the function F to the set $A \cap \mathrm{dom.}\, F$.

** Such measures provide an elegant concept of integration, which is useful in making explicit several representation theorems for operators on a Hilbert space [13, § 10—15].

The switching property given in the last lemma along with the simple properties of the measure $T_U(\cdot)$ recorded in Thm. A. 2 (Appendix) yield our main theorems on the (strong) time—domain characterization and structure of helixes:

2.19. **Main Theorem I.** *Let $x(\cdot)$ be a helix in $\mathfrak{H}$ with shift group $(U_t, t \in \mathfrak{R})$ and average vector α_x. Then*

(a) $$\forall (a, b] \in \mathfrak{P},\ x(b) - x(a) = T_U(a, b](\alpha_x);$$

(b) $\mathcal{S}_x$ *is the cyclic subspace* $\mathfrak{S}\{U_t(\alpha_x) : t \in \mathfrak{R}\} \underset{d}{=} \mathfrak{M}_{\mathfrak{R}}\{\alpha_x\}$,

cf. (A. 1) (iii)

Proof. (a) From the definition of the average vector, the well-known properties of the Lebesgue — Bochner integral, the last lemma and the inversion formula A.2 (d), we get

$$T_U(a, b](\alpha_x) = T_U(a, b]\left[\sqrt{2}\int_0^\infty e^{-t}\{x(0) - x(t)\}\,dt\right], \quad \text{by 2.12,}$$

$$= -\sqrt{2}\int_0^\infty e^{-t}\,T(a, b]\{x(t) - x(0)\}\,dt, \quad \text{cf. [3, Thm. 3.7.12],}$$

$$= -\sqrt{2}\int_0^\infty e^{-t}\,T(0, t]\{x(b) - x(a)\}\,dt, \quad \text{by 2.18,}$$

$$= \left\{-\sqrt{2}\int_0^\infty e^{-t}\,T(0, t]\,dt\right\}\{x(b) - x(a)\}, \quad \text{cf. [3, Thm. 3.3.4],}$$

$$= x(b) - x(a), \quad \text{by A.2 (d).}$$

(b) Since cf. 2.12, $\alpha_x \in \mathcal{S}_x$ and each U_t carries $\mathcal{S}_x$ onto itself, therefore $U_t(\alpha_x) \in \mathcal{S}_x$, and so

$$\mathfrak{M}_{\mathfrak{R}}\{\alpha_x\} \underset{d}{=} \mathfrak{S}\{U_t(\alpha_x) : t \in \mathfrak{R}\} \subseteq \mathcal{S}_x. \tag{1}$$

On the other hand, by (a), $\forall (a, b] \in \mathfrak{P}$

$$x(b) - x(a) = T_U(a, b](\alpha_x) =$$

$$= \frac{1}{\sqrt{2}}\left\{U_b - U_a - \int_a^b U_t\,dt\right\}(\alpha_x) \in \mathfrak{M}_{\mathfrak{R}}\{\alpha_x\},$$

and so

$$\mathcal{S}_x \underset{d}{=} \mathfrak{S}\{x(b) - x(a) : (a, b] \in \mathfrak{P}\} \subseteq \mathfrak{M}_{\mathfrak{R}}\{\alpha_x\}. \tag{2}$$

(b) follows from (1) and (2). ∎

2.20. **Main Theorem II** (Converse of the last). *Let $x(\cdot)$ be a continuous function on $\mathfrak{R}$ to $\mathfrak{H}$ such that*

$$\forall (a, b] \in \mathfrak{P},\ x(b) - x(a) = T_U(a, b](\alpha),$$

where $(U_t, t \in \mathfrak{R})$ is a strongly continuous group of unitary operators on $\mathfrak{H}$ onto $\mathfrak{H}$ and $\alpha \in \mathfrak{H}$. Then $x(\cdot)$ is a helix in $\mathfrak{H}$ for which the average vector α_x is α, the chordal subspace $\mathcal{S}_x$ is the cyclic subspace $\mathfrak{M}_{\mathfrak{R}}\{\alpha\}$, cf. (A. 1) (iii), *and the shift group is* (Rstr. $_{\mathfrak{M}_{\mathfrak{R}}\{\alpha\}}\, U_t$, $t \in \mathfrak{R}$).

P r o o f. Let $(a, b] \in \mathfrak{P}$ and $t \in \mathfrak{R}$. Then from the hypothesis and the stationarity property A.2 (c), it follows that

$$x(b+t) - x(a+t) = T_U(a+t,\ b+t](\alpha) = \\ = U_t T_U(a, b](\alpha) = U_t\{x(b) - x(a)\}.$$

Hence by 2.15

$$x(\cdot) \text{ is a helix,} \tag{1}$$

and obviously

$$\text{the shift group of } x(\cdot) = (\text{Rstr. }_{\mathfrak{S}_x}U_t,\ t \in \mathfrak{R}). \tag{2}$$

Next, from 2.12 and our hypothesis,

$$\alpha_x \underset{d}{=} -\sqrt{2}\int_0^\infty e^{-t}\{x(t) - x(0)\}\,dt = -\sqrt{2}\int_0^\infty e^{-t}\,T_U(0, t](\alpha)\,dt =$$

$$= \left\{-\sqrt{2}\int_0^\infty e^{-t}\,T_U(0, t]\,dt\right\}(\alpha), \qquad \text{cf. [3, Thm. 3.3.4],}$$

$$= \alpha \qquad \text{by. A.2 (d)).} \tag{3}$$

Also, by 2.19 (b) and (3),

$$\mathfrak{S}_x = \mathfrak{M}_{\mathfrak{R}}\{\alpha_x\} = \mathfrak{M}_{\mathfrak{R}}\{\alpha\}. \tag{4}$$

Hence (2) can be written

$$\text{the shift group of } x(\cdot) = (\text{Rstr. }_{\mathfrak{M}_{\mathfrak{R}}\{\alpha\}}U_t,\ t \in \mathfrak{R}). \tag{5}$$

The results (1), (3) and (5) constitute the theorem.

N o t e. There are plenty of continuous functions $x(\cdot)$ on $\mathfrak{R}$ to $\mathfrak{H}$ satisfying the hypothesis of the last theorem. For let

$$x(t) = \begin{cases} x_0 + T_U(0, t]\,\alpha, & t \geqslant 0, \\ x_0 - T_U(t, 0]\,\alpha, & t < 0, \end{cases} \tag{2.21}$$

where x_0 is any vector in $\mathfrak{H}$. The fact that the hypothesis of the last theorem then holds is easily verified by considering the separate cases $a < b < 0$, $a < 0 \leqslant b$, and $0 \leqslant a < b$.

The last two theorems reveal the full structure of all helixes in $\mathfrak{H}$. The following reformulation of their contents links the concept of helix with the concepts of stationary curve, and the indefinite integral of such a curve, cf. 2.8, 2.5:

2.22. **Main Theorem III** (Underlying stationary curve).

(a) *Let $x(\cdot)$ be a continuous function on $\mathfrak{R}$ to $\mathfrak{H}$. Then the following conditions are equivalent*:

(α) *$x(\cdot)$ is a helix in $\mathfrak{H}$*;

(β) *$\exists$ a stationary curve $y(\cdot)$ in $\mathfrak{H}$ such that $x(\cdot)$ is the difference between $y(\cdot)$ and an indefinite integral of $y(\cdot)$, i. e.*

$$\forall t \in \mathfrak{R},\ x(t) = y(t) - \left\{y(0) - x(0) + \int_0^t y(s)\,ds\right\}.$$

(b) *For a helix $x(\cdot)$ in $\mathfrak{H}$, the curve $y(\cdot)$ in* (β) *is unique, and given by*

$$y(t) = U_t(\alpha_x/\sqrt{2}), \quad \forall t \in \mathfrak{R},$$

where $(U_t,\ t \in \mathfrak{R})$ is the shift group of $x(\cdot)$ and α_x is its average vector. Thus $\mathfrak{M}_y = \mathcal{S}_x$, and $y(\cdot)$ has exactly the same shift group as $x(\cdot)$.

Proof. (a) Let (α) hold, $(U_t,\ t \in \mathfrak{R})$ be the shift group of the helix $x(\cdot)$, and α_x be its average vector. Then by Thm. 2.19 (a), $\forall t \geqslant 0$

$$x(t) - x(0) = \frac{1}{\sqrt{2}}\left\{U_t - 1 - \int_0^t U_s ds\right\}(\alpha_x) = y(t) - y(0) - \int_0^t y(t)\, dt, \tag{1}$$

where

$$y(t) \underset{a}{=} U_t(\alpha_x/\sqrt{2}), \quad \forall t \in \mathfrak{R}.$$

Since this $y(\cdot)$ is obviously a stationary curve, (1) is indeed the equation given in (β). For $t < 0$, a similar attack on $x(0) - x(t)$ yields the equation in (β). This establishes (β).

Next let (β) hold. Then $\forall (a, b] \in \mathfrak{P}$,

$$x(b) - x(a) = y(b) - y(a) - \int_a^b y(s)\, ds = T_U(a, b]\{\sqrt{2}y(0)\}, \tag{2}$$

where $(U_t,\ t \in \mathfrak{R})$ is the shift group of the stationary curve $y(\cdot)$. By (2) and Thm. 2.20, $x(\cdot)$ is a helix, i. e. we have (α).

(b) Let $x(\cdot)$ be a helix, and suppose that $y(\cdot)$ and $z(\cdot)$ are stationary curves both satisfying the condition (β). Then, cf. (2), we have

$$\forall (a, b] \in \mathfrak{P}, \quad x(b) - x(a) = T_U(a, b]\{\sqrt{2}y(0)\}, \tag{3}$$

$$\forall (a, b] \in \mathfrak{P}, \quad x(b) - x(a) = T_V(a, b]\{\sqrt{2}z(0)\}, \tag{3'}$$

where $(U_t,\ t \in \mathfrak{R})$ and $(V_t,\ t \in \mathfrak{R})$ are the shift groups of the stationary curves $y(\cdot)$ and $z(\cdot)$. By (3) and Thm. 2.20,

$$\alpha_x = \sqrt{2}y(0), \tag{4}$$

$$\mathcal{S}_x = \mathfrak{S}\{U_t\{\sqrt{2}y(0)\} : t \in \mathfrak{R}\} = \mathfrak{M}_y, \tag{5}$$

$$\text{the shift group of } x(\cdot) = (\text{Rstr.}_{\mathcal{S}_x} U_t,\ t \in \mathfrak{R}) = (U_t,\ t \in \mathfrak{R}), \tag{6}$$

Similarly from (3′) and Thm. 2.20,

$$\alpha_x = \sqrt{2}z(0), \tag{4'}$$

$$\mathcal{S}_x = \mathfrak{M}_z, \tag{5'}$$

$$\text{the shift group of } x(\cdot) = (V_t,\ t \in \mathfrak{R}). \tag{6'}$$

It follows at once from (6) & (6′) and (4) & (4′) that

$$\forall t \in \mathfrak{R}, \quad y(t) = U_t\{y(0)\} = V_t\{z(0)\} = z(t). \tag{7}$$

Thus, at most one stationary curve $y(\cdot)$ can satisfy the condition (β).

The expression for $y(t)$ claimed in (b) is clear from (7) and (4). The remaining claims are covered by (5) and (6). ∎

It is worth noting the relation between the chordal covariance kernel of a helix $x(\cdot)$ and the covariance kernel of the underlying stationary curve $y(\cdot)$:

2.23. **Corollary.** *Let $x(\cdot)$ be a helix in $\mathfrak{H}$ and $y(\cdot)$ be the (unique) stationary curve in $\mathfrak{H}$ related to $x(\cdot)$ as in Thm.* 2.22 (β). *Then*

$$\forall s, t \in \mathfrak{R},\ g_y(s, t) = \int_0^\infty \int_0^\infty e^{-(u+v)} \gamma_x(s, s+u, t, t+v)\, du dv.$$

P r o o f. By Thm. 2. 22 (b)

$$g_y(s, t) = (U_s(\alpha_x/\sqrt{2}),\ U_t(\alpha_x/\sqrt{2})), \tag{1}$$

and by 2.12 and 2.15 (1)

$$U_s(\alpha_x/\sqrt{2}) = \int_0^\infty e^{-u} \{x(s) - x(s+u)\}\, du. \tag{2}$$

On substituting in the RHS of (1) the Bochner integral occuring in (2) and the corresponding integral with t replacing s, and apllying to the resulting inner product the Lma. 4.2 of [11], we get the desired expression. ∎

We end this section by referring to a question that is readily amenable to the purely time-domain methods developed here, viz. the equivalence between our Def. 2.8 (b) and Kolmogorov's definition of a helix [6, p. 7]. He defines the latter as any orbit $(K_t x,\ t \in \mathfrak{R})$, where $x \in \mathfrak{H}$ and $(K_t,\ t \in \mathfrak{R})$ is a strongly continuous group of *affine motions* of the type

$$K_t(\cdot) = r(t) + V_t(\cdot)$$

where $r(t) \in \mathfrak{H}$ and V_t is a unitary operator on $\mathfrak{H}$ onto $\mathfrak{H}$. Letting $\mathfrak{K}$ be the class of such orbits, the equivalence in question, viz.

$$x(\cdot) \in \mathfrak{K} \Leftrightarrow x(\cdot) \text{ is a helix}$$

can be proved easily without using spectral ideas by appeal to Thm. 2.15.

§ 3. Spectral representations of a helix

The spectral representations of a helix can be easily obtained from its time-domain charasterization on taking the spectral resolution of its shift group. This becomes evident from the following triviality:

3.1. **Triviality** (Spectral representation of $T_U(\cdot)$). *Let $E(\cdot)$ be the spectral (projection-valued) measure on the σ-algebra Bl. $(\mathfrak{R})$ of the strongly continuous group $(U_t,\ t \in \mathfrak{R})$ of unitary operators U_t on $\mathfrak{H}$ onto $\mathfrak{H}$, so that*

$$\forall t \in \mathfrak{R},\ U_t = \int_{\mathfrak{R}} e^{it\lambda} E(d\lambda).$$

Then $\forall a, b \in \mathfrak{R}$ such that $a \leqslant b$,

$$T_U(a, b] = \frac{1}{\sqrt{2}} \int_{\mathfrak{R}} (e^{ib\lambda} - e^{ia\lambda}) \frac{\lambda + i}{\lambda} E(d\lambda).$$

P r o o f. Using the expression for U_t, we have from (2.17)

$$T_U(a, b] \underset{d}{=} \frac{1}{\sqrt{2}}\left\{U_b - U_a - \int_a^b U_t dt\right\} = \frac{1}{\sqrt{2}}\int_{\mathfrak{R}}\left\{e^{ib\lambda} - e^{ia\lambda} - \int_a^b e^{it\lambda}dt\right\}E(d\lambda),$$

where the change in the order of integration involved in the last step is easily justified, cf. [3, p. 62]. Simplification of the integrand on the RHS yields the desired expression. ∎

This triviality enables us to translate quickly our structure theorems on helixes (2.19, 2.20, 2.22) into spectral terms:

3.2. **Theorem** (Spectral representations of a helix). (a) *Let $x(\cdot)$ be a continuous function on $\mathfrak{R}$ to $\mathfrak{H}$. Then the following conditions are equivalent:*

(α) *$x(\cdot)$ is a helix in $\mathfrak{H}$,*

(β) $\exists$ *a bounded, $\mathfrak{H}$-valued, c.a.o.s. measure $\rho(\cdot)$ on $Bl.(\mathfrak{R})$,* cf. Def. A. 8, *such that $\forall a, b \in \mathfrak{R}$,*

$$x(b) - x(a) = \frac{1}{\sqrt{2}}\int_{\mathfrak{R}}(e^{ib\lambda} - e^{ia\lambda})\frac{\lambda + i}{\lambda}\rho(d\lambda),$$

(γ) $\exists$ *a bounded, non-negative, c. a. measure $\mu(\cdot)$ on $Bl.(\mathfrak{R})$ such that $\forall a, b, c, d \in \mathfrak{R}$,*

$$\gamma_x(a, b, c, d) = \frac{1}{2}\int_{\mathfrak{R}}(e^{ib\lambda} - e^{ia\lambda})(e^{-id\lambda} - e^{-ic\lambda})\frac{1+\lambda^2}{\lambda^2}\mu(d\lambda).$$

(b) *For a helix $x(\cdot)$ in $\mathfrak{H}$ the measures $\rho(\cdot)$ and $\mu(\cdot)$ described in* (β), (γ) *may be chosen to be*

$$\rho(\cdot) = E(\cdot)\alpha_x, \quad \mu(\cdot) = |E(\cdot)\alpha_x|^2,$$

where $E(\cdot)$ is the spectral measure of the shift-group $(U_t, t \in \mathfrak{R})$ of $x(\cdot)$ and α_x is its average vector *.

P r o o f. (a) We shall show that (α) $\Rightarrow$ (β) $\Rightarrow$ (γ) $\Rightarrow$ (α).

Let (α) hold and $a, b \in \mathfrak{R}$, say $a < b$ for definiteness. Then, U_t, $E(\cdot)$, α_x being defined as in (b), it follows from Thm. 2.19 and Triv. 3.1 that

$$x(b) - x(a) = T_U(a, b](\alpha_x) = \frac{1}{\sqrt{2}}\int_{\mathfrak{R}}(e^{ib\lambda} - e^{ia\lambda})\frac{\lambda + i}{\lambda}E(d\lambda)(\alpha_x). \tag{1}$$

Since $\rho(\cdot) \underset{d}{=} E(\cdot)\alpha_x$ is a bounded, $\mathfrak{H}$-valued, c.a.o.s. measure on Bl. $(\mathfrak{R})$, we have established (β).

Next, let (β) hold, and for abbreviation write

$$\theta_{a,b}(\lambda) \underset{d}{=} \frac{1}{\sqrt{2}}(e^{ib\lambda} - e^{ia\lambda})(\lambda + i)/\lambda. \tag{2}$$

Then by (β)

$$x(b) - x(a) = \int_{\mathfrak{R}}\theta_{a,b}(\lambda)\rho(d\lambda), \quad x(d) - x(c) = \int_{\mathfrak{R}}\theta_{c,d}(\lambda)\rho(d\lambda).$$

Hence by the well-known properties of such integrals, cf. [11, (5.2) (b)],

$$\gamma_x(a, b, c, d) = \left(\int_{\mathfrak{R}}\theta_{a,b}(\lambda)\rho(d\lambda), \int_{\mathfrak{R}}\theta_{c,d}(\lambda)\rho(d\lambda)\right) = \int_{\mathfrak{R}}\theta_{a,b}(\lambda)\overline{\theta_{c,d}(\lambda)}\mu(d\lambda), \tag{3}$$

* These are in fact the only possible choices, as we shall see in Thm. 3.4 below.

where $\mu(\cdot) \underset{d}{=} |\rho(\cdot)|^2$. The last integrand is precisely the one occurring in (γ) and $\mu(\cdot)$ is a bounded, non-negative, c. a. measure on Bl. ($\mathfrak{R}$). Thus (γ).

Finally let (γ) hold. Since obviously

$$\theta_{a+t,\, b+t}(\lambda) = e^{it\lambda}\theta_{a,\, b}(\lambda),$$

it follows from (γ) that $\forall\ a, b, c, d, t \in \mathfrak{R}$,

$$\gamma_x(a+t,\ b+t,\ c+t,\ d+t) = \int_{\mathfrak{R}} e^{it\lambda}\theta_{a,\, b}(\lambda)\, \overline{e^{-it\lambda}\theta_{c,\, d}(\lambda)}\, \mu(d\lambda) =$$

$$= \int_{\mathfrak{R}} \theta_{a,\, b}(\lambda)\, \overline{\theta_{c,\, d}(\lambda)}\, \mu(d\lambda) = \gamma_x(a, b, c, d).$$

Hence by Def. 2.8 (b), $x(\cdot)$ is a helix. Thus (α).

This not only completes the proof of (a) but also that of (b), for the desired expressions for the measures $\rho(\cdot)$ and $\mu(\cdot)$ follow from (1) and (3) above. ∎

We proceed to prove the uniqueness of the spectral representations for a helix given in 3.2 (β), (γ). For this we need the following theorem on stationary curves:

3.3. **Theorem.** (a) *Let $y(\cdot)$ be a continuous function on $\mathfrak{R}$ to $\mathfrak{H}$. Then the following conditions are equivalent*

(α) *$y(\cdot)$ is a stationary curve in $\mathfrak{H}$,*

(β) *$y(\cdot)$ is the Fourier-Stieltjes transform of a bounded, $\mathfrak{H}$-valued, c. a. o. s. measure $\rho(\cdot)$ over ($\mathfrak{R}$, $Bl.(\mathfrak{R})$, μ), i. e.*

$$\forall t \in \mathfrak{R},\ y(t) = \int_{\mathfrak{R}} e^{it\lambda}\rho(d\lambda)$$

where $\mu(\cdot) \underset{d}{=} |\rho(\cdot)|^2$ is a bounded measure, cf. Def. A. 3.

(b) *For a stationary curve $y(\cdot)$ in $\mathfrak{H}$ the measure $\rho(\cdot)$ described in (β) is unique*; *in fact* $\forall\ a, b \in \mathfrak{R}$ such that $a < b$,

$$\rho(a, b) + \frac{1}{2}(\rho\{a\} + \rho\{b\}) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \overline{\xi_{(a,\, b]}(t)}\, y(t)\, dt,$$

where the last is a Bochner integral in $\mathfrak{H}$ involving the $L_2(\mathfrak{R})$-valued measure $\xi_{(\cdot)}$ defined by *

$$\xi_B(t) = \frac{1}{\sqrt{2\pi}} \int_B e^{it\lambda} d\lambda,\quad B \in Bl.(\mathfrak{R})\ \&\ Leb.\ (B) < \infty.$$

P r o o f r e m a r k s. 3.3 is just a sharpened version of a known result, cf. e. g. [1, p. 527, 4.1]. We shall therefore omit the proof. The inversion formula in (b) is proved by using the definition of the FS transform and the Fubini-type theorem [11, 5.20] for the product measure $\rho \times L$, where L is Lebesgue measure over $[-T, T]$ **. ∎

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* In our previous work we have referred to $\xi_{(\cdot)}$ as the *Fourier — Plancherel measure* over $\mathfrak{R}$, because of its central role in the harmonic analysis of $\mathfrak{R}$, cf. [13, § 6].

** This formula extends P. Lévy's classical inversion formula for the FS transform to bounded, $\mathfrak{H}$-valued, c.a.o.s. measures. An even wider extension seems possible, in which $\rho(\cdot)$ is merely a bounded, $\mathfrak{H}$-valued, c. a. measure and $y(\cdot)$ is just a *harmonizable curve* in $\mathfrak{H}$, cf. [15].

3.4. **Theorem** (Uniqueness of spectral representation). *For a helix $x(\cdot)$ in $\mathfrak{H}$, the measures $\rho(\cdot)$ and $\mu(\cdot)$ described in Thm.* 3.2 (β), (γ) *are unique.*

P r o o f. Let $E(\cdot)$ be the spectral measure of the shift group $(U_t, t \in \mathfrak{R})$ of the helix $x(\cdot)$, and suppose that $\forall\, a, b \in \mathfrak{R}$

$$x(b) - x(a) = \frac{1}{\sqrt{2}} \int_{\mathfrak{R}} \theta_{a,b}(\lambda)\, \rho(d\lambda), \tag{1}$$

where ρ is a bounded, $\mathfrak{H}$-valued, c. a. o. s. measure on Bl.($\mathfrak{R}$), and $\theta_{a,b}$ is defined as in (2) in the proof of 3.2. Since

$$\sqrt{2}\theta_{a,b}(\lambda) = e^{ib\lambda} - e^{ia\lambda} - \int_a^b e^{it\lambda} dt,$$

an application of the Fubini Thm. [11, 5.20 (c)] to the product measure $\rho \times L$, where L is Lebesgue measure over $[a, b]$, takes us from (1) to

$$x(b) - x(a) = y(b) - y(a) - \int_a^b y(t)\, dt, \tag{2}$$

where

$$y(t) = \frac{1}{\sqrt{2}} \int_{\mathfrak{R}} e^{it\lambda} \rho(d\lambda). \tag{3}$$

By Thm. 3.3 (a), $y(\cdot)$ is a stationary curve in $\mathfrak{H}$, which by (2) satisfies the equation in 2.22 (β). Hence by Thm. 2.22 (b) $y(\cdot)$ is unique and given by

$$y(t) = U_t(\alpha_x/\sqrt{2}) = \frac{1}{\sqrt{2}} \int_{\mathfrak{R}} e^{it\lambda} E(d\lambda)\, \alpha_x, \quad \forall t \in \mathfrak{R}. \tag{4}$$

By (3) and (4) the bounded, c. a. o. s. measures $\rho(\cdot)$ and $E(\cdot)\alpha_x$ have the same Fourier — Stieltjes transform. Hence by Thm. 3.3 (b) $\rho(\cdot) = E(\cdot)\alpha_x$. Thus the measure $\rho(\cdot)$ appearing in 3.2 (β) is unique.

To turn to the uniqueness of $\mu(\cdot)$, suppose that $\forall\, a, b, c, d \in \mathfrak{R}$,

$$\gamma_x(a, b, c, d) = \int_{\mathfrak{R}} \theta_{a,b}(\lambda)\, \overline{\theta_{c,d}(\lambda)}\, \mu(d\lambda), \tag{5}$$

where $\mu(\cdot)$ is a bounded, non-negative, c. a. measure on Bl. ($\mathfrak{R}$). We shall show that*

$$\forall s \in \mathfrak{R}, \int_0^\infty \int_0^\infty e^{-(u+v)} \gamma_x(s, s+u, 0, v)\, du dv = \frac{1}{2} \tilde{\mu}(s), \tag{I}$$

where $\tilde{\mu}(\cdot)$ is the Fourier Stieltjes transform of μ. (I) shows that the kernel $\gamma_x(\cdot\cdot\cdot\cdot)$ determines the FS transform $\tilde{\mu}$, and hence (by Lévy's Thm.) the measure μ itself. Since $x(\cdot)$ determines $\gamma_x(\cdot\cdot\cdot\cdot)$, it follows that the measure $\mu(\cdot)$, appearing in 3.2 (γ) is uniquely determined by $x(\cdot)$.

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* The eqn. (I) is of course suggested by Cor. 2.23, since

$$g_y(t, 0) = \{ | E(\cdot)(\alpha_x / \sqrt{2}) |^2 \}(t) = \frac{1}{2} \tilde{\mu}(t).$$

The proof of (I) rests on the observation that

$$\theta_{a,b}(\lambda) = \xi_{(a,b]}(\lambda)\sqrt{\pi}(i\lambda - 1), \quad \xi_{(a,b]}(\lambda) \underset{d}{=} \frac{e^{ib\lambda} - e^{ia\lambda}}{\sqrt{2\pi} i\lambda},$$

and that therefore from (5)

$$\gamma_x(s, s+u, 0, v) = \pi \int_{\mathfrak{R}} e^{is\lambda} \xi_{(0,u]}(\lambda) \overline{\xi_{(0,v]}(\lambda)} (1 + \lambda^2) \mu(d\lambda).$$

It follows from Fubini's Thm. that

$$\text{LHS}(I) = \pi \int_{\mathfrak{R}} e^{is\lambda} \left| \int_0^\infty e^{-u} \xi_{(0,u]}(\lambda)\, du \right|^2 (1 + \lambda^2) \mu(d\lambda). \tag{6}$$

But a simple computation shows that

$$\int_0^\infty e^{-u} \xi_{(0,u]}(\lambda)\, du = \frac{1}{\sqrt{2\pi}} \frac{1}{1 - i\lambda},$$

and so the RHS of (6) does indeed reduce to $\tilde{\mu}(s)/2$. This establishes (I), and completes the proof. ∎

The representations obtained in Thm. 3.2 (a) are different from but equivalent to those given by Kolmogorov [6, Thms. 2—5]. Indeed, his original expressions may be had from ours, which are much closer to Doob's [1, p. 552], by splitting $\int_{\mathfrak{R}}$ into $\int_{\mathfrak{R}-\{0\}}$ and $\int_{\{0\}}$. Thus from 3.2 (β) with $b = t$, $a = 0$ we get

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{2}} \int_{\mathfrak{R}-\{0\}} (e^{it\lambda} - 1) \frac{\lambda + i}{\lambda} \rho(d\lambda) - \frac{1}{\sqrt{2}} t\rho\{0\} + x(0) = \\ &= \frac{1}{\sqrt{2}} \int_{\mathfrak{R}-\{0\}} (e^{it\lambda} - 1)\Phi(d\lambda) + tu + v, \end{aligned} \tag{3.5}$$

where $u, v \in \mathfrak{H}$ and $\Phi(\cdot)$ is an $\mathfrak{H}$-valued, c.a.o.s. measure over $\mathfrak{R} - \{0\}$ of a certain kind; in fact $u = -\rho\{0\}/\sqrt{2} = -E\{0\}(\alpha_x/\sqrt{2})$, $v = x(0)$ and $\Phi(B) = \int_B \{(\lambda + i)/\sqrt{2}\lambda\}\rho(d\lambda)$, $B \in \text{Bl.}(\mathfrak{R} - \{0\})$. (3.5) is Kolmogorov's spectral representation for $x(\cdot)$, [6, Thm. 4]. His spectral representation for the covariance $\gamma(0, \tau_1, 0, \tau_2)$, [6, Thm. 2], may be similarly derived from 3.2 (γ), and we can show that his constant θ is in fact $|E\{0\}(\alpha_x/\sqrt{2})|^2$.

Kolmogorov's results differ from ours in another inessential respect. He asserts for instance, [6, Thm. 2], that given an infinite-dimensional $\mathfrak{H}$ and a function $\gamma(\cdot\cdot\cdot\cdot)$ satisfying his spectral conditions, there exists a helix $x(\cdot)$ in $\mathfrak{H}$ having $\gamma_x(\cdot\cdot\cdot\cdot) = \gamma(\cdot\cdot\cdot\cdot)$. This assertion may be deduced from our results as follows:

As shown in the last proof the given $\gamma(\ldots.)$ satisfying 3.2 (γ) determines via eqn. (I) the FS transform $\tilde{\mu}(\cdot)$ of the measure $\mu(\cdot)$ occurring in 3.2 (γ). By Khinchine's Thm. there exists a stationary curve $y(\cdot)$ in $\mathfrak{H}$ for which the covariance function is $\tilde{\mu}(\cdot)$. Now let

$$\forall t \in \mathfrak{R}, \; x(t) \underset{d}{=} y(t) - y(0) - \int_0^t y(s)\, ds.$$

Then by Thm. 2.22, $x(\cdot)$ is a helix in $\mathfrak{H}$, and a routine calculation shows that $\gamma_x(\cdot\cdot\cdot\cdot) = \gamma(\cdot\cdot\cdot\cdot)$.

When the Hilbert space $\mathfrak{H}$ is over $\mathfrak{R}$ the representations given in Thm. 3.2 do not make sense, since they involve nonreal numbers. For such spaces we have the following result, part (a) of which is due essentially to Schoenberg and von Neumann [18; 19, Thm. 1]:

3.6. **Corollary.** *Let $\mathfrak{H}$ be a real Hilbert space. Then*

(a) *for a continuous function $x(\cdot)$ on $\mathfrak{R}$ to $\mathfrak{H}$ the following conditions are equivalent:*

(α) *$x(\cdot)$ is a helix in $\mathfrak{H}$,*

(β) *$\exists$ a bounded, non-negative, c.a. measure μ on $Bl.(\mathfrak{R})$ such that*, cf. (2.3),

$$\forall t \in \mathfrak{R},\ f_x(t)^2 = \frac{1}{2}\int_{\mathfrak{R}} \left(\frac{\sin t\lambda/2}{\lambda/2}\right)^2 (1+\lambda^2)\,\mu(d\lambda);$$

(b) *when $x(\cdot)$ is a helix, the measure $\mu(\cdot)$ in (β) is given by $\mu(\cdot) = |E(\cdot)\alpha_x|^2$, where α_x is the average vector of $x(\cdot)$, and $E(\cdot)$ is the spectral measure of the shift group of $x(\cdot)$ in the complexification $\mathscr{H}$ of $\mathfrak{H}$.*

P r o o f. (a) We first note that if $\mathscr{H}$ is a *complex* Hilbert space and $x(\cdot)$ a helix in $\mathscr{H}$, then indeed by (2.4) and 3.2 (γ)

$$f_x(t)^2 = \gamma(0, t, 0, t) = \frac{1}{2}\int_{\mathfrak{R}} |e^{it\lambda} - 1|^2 \frac{1+\lambda^2}{\lambda^2}\mu(d\lambda) =$$

$$= \frac{1}{2}\int_{\mathfrak{R}} \left(\frac{\sin t\lambda/2}{\lambda/2}\right)^2 (1+\lambda^2)\,\mu(d\lambda), \tag{1}$$

as desired.

Next, we complexify the given (real) Hilbert space $\mathfrak{H}$ à la Schoenberg and von Neumann [19]. We then complete their proof of (a), but utilize (1) in the process. The resulting proof of the implication (β) $\Rightarrow$ (α) parallels theirs, but that of the implication (α) $\Rightarrow$ (β) is much shorter than theirs, since the existence of $\mu(\cdot)$ is ensured by (1) and does not have to be proved ab initio.

(b) follows at once from (1), 3.2 (b) and 3.4.

§ 4. Further analysis of helixes

It is easy to see that every displacement $x(\cdot) = y(\cdot) + r$, $r \in \mathfrak{H}$, of a stationary curve $y(\cdot)$ in $\mathfrak{H}$ is a helix in $\mathfrak{H}$, and so also is any indefinite integral $x(\cdot)$ of such a $y(\cdot)$. The converses of these results are of course false: the Brownian movement SP $x(\cdot)$ is a helix in $\mathscr{L} \underset{d}{=} L_2(\Omega, \mathfrak{B}, P)$, which is neither the displacement of a stationary curve $y(\cdot)$ in $\mathscr{L}$ nor an indefinite integral of such a $y(\cdot)$*.

The further analysis of helixes concerns the extra conditions required in order that these curves may possess special properties such as that of being the indefinite integral of a stationary curve, or of being the displacement of such a curve, or of having orthogonal increments. The results in §§ 2, 3

* The latter $y(\cdot)$, the so-called *white noise SP*, exists only as a *generalized SP*, i. e. as a continuous linear operator on $\mathfrak{D}$ to $\mathscr{L}$, where $\mathfrak{D}$ is the vector space of functions in $C^\infty(\mathfrak{R})$ with pre-compact support endowed with the Schwartz topology, cf. [4].

provide a firm foundation on which to base such special analysis. To put it more fully, we find that the occurrence of these specific characteristics depends on *the position of the average vector* α_x *of our helix* $x(\cdot)$ *with respect to the domain and range of the infinitesimal generator* iH *of the shift group* $(U_t, t \in \mathfrak{R})$ *of* $x(\cdot)$ *and with respect to the set of «wandering vectors» of the associated measure* $T_U(\cdot)$. These concepts, so far dispensable, first have to be studied. As this paper is already rather long, we shall pursue these more specialized aspects of helix theory in a second paper.

Appendix. Properties of the measure $T_U(\cdot)$

In this appendix it will be understood that

(A.1)

(i)
$$\mathfrak{P} \underset{d}{=} \{(a, b] : a, b \in \mathfrak{R} \,\&\, a \leqslant b\},$$

(ii) $(U_t, t \in \mathfrak{R})$ is a strongly continuous group of unitary operators U_t on $\mathfrak{H}$ onto $\mathfrak{H}$,

(iii) $\forall A \subseteq \mathfrak{R} \,\&\, \forall X \subseteq \mathfrak{H}, \ \mathfrak{M}_A(X) \underset{d}{=} \mathfrak{S}\{U_t(X) : t \in A\}$,

(iv)* $\forall (a, b] \in \mathfrak{P}, T_U(a, b] \underset{d}{=} \frac{1}{\sqrt{2}}\left\{U_b - U_a - \int_a^b U_t dt\right\}$,

(v) iH is the infinitesimal generator of $(U_t, t \in \mathfrak{R})$,

(vi) $V \underset{d}{=} (H - iI)(H + iI)^{-1} \underset{d}{=}$ the Cayley transform of H,

(vii) $E(\cdot)$ is the spectral measure on Bl. $(\mathfrak{R})$ of $(U_t, t \in \mathfrak{R})$, so that $\forall t \in \mathfrak{R}, \ U_t = \int_{\mathfrak{R}} e^{it\lambda} E(d\lambda)$.

The following theorem is fundamental:

A. 2. **Theorem.** (a) $T_U(\cdot)$ *is a f. a.*, $\mathfrak{H}$*-to-*$\mathfrak{H}$, *continuous operator-valued measure on the pre-ring* $\mathfrak{P}$.

(b) $T_U(\cdot)$ *is strongly regular, i. e.* $\forall (a, b] \in \mathfrak{P}$,
$$\operatorname*{slim}_{h \to 0+} T_U(a - h, b] = T_U(a, b] = \operatorname*{slim}_{h \to 0+} T_U(a, b + h].$$

(c) $T_U(\cdot)$ *is stationary with shift group* $(U_t, t \in \mathfrak{R})$, *i. e.*
$$\forall t \in \mathfrak{R} \,\&\, \forall J \in \mathfrak{P}, \ U_t \cdot T_U(J) = T_U(J + t) = T_U(J) \cdot U_t.$$

(d)
$$\forall t \in \mathfrak{R}, -\sqrt{2} \int_t^\infty e^{t-s} T_U(t, s] \, ds = U_t,$$
$$-\sqrt{2} \int_0^\infty e^{-s} T_U(0, s] \, ds = I.$$

(e) $\forall X \subseteq \mathfrak{H}, \ \mathfrak{M}_{\mathfrak{R}}(X) = \mathfrak{S}\{T_U(J)(X) : J \in \mathfrak{P}\}$,

$\forall X \subseteq \mathfrak{H} \,\&\, \forall a \in \mathfrak{R}, \ \mathfrak{M}_{[a, \infty)}(X) = \mathfrak{S}\{T_U(J)(X) : J \in \mathfrak{P} \,\&\, J \subset [a, \infty)\}$.

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* The operator-valued integrals appearing in (A.1) (iv), A.2 (d) and elsewhere in his Appendix are all Riemann or improper Riemann, and defined by using the strong operator-topology cf. [3, Thm. 3.3.4].

P r o o f. The proof is the same as that of the corresponding results [13, (15.7) — (15.8′)] in which an isometric semigroup $(S_t,\ t \geqslant 0)$ replaces the group $(U_t,\ t \in \mathfrak{R})$.

This is the only result on the measure $T_U(\cdot)$ needed in this paper. But in this appendix it is convenient to recall another concept also used above:

A. 3. **Def.** By an *$\mathfrak{H}$-valued, countably additive, orthogonally scattered (c. a. o. s.) measure over* $(\mathfrak{R},\ \mathfrak{B},\ \mu)$, where $\mathfrak{B}$ is any pre-ring over $\mathfrak{R}$, is meant a function ξ on $\mathfrak{B}$ to $\mathfrak{H}$ such that

$$\forall J,\ K \in \mathfrak{B}, \qquad (\xi(J),\ \xi(K)) = \mu(J \cap K),$$

where μ is a c. a., non-negative ($< \infty$) measure on $\mathfrak{B}$.

The theory of such measures is more fully expounded in [11].

N o t e a d d e d i n P r o o f. The writer has found recently that the time-domain analysis of helixes given above extends to Banach spaces $\mathfrak{X}$, cf. Abstract in Amer. Math. Sos. Notices 17(1970), 1062. A helix in $\mathfrak{X}$ has to be defined in terms of a group of isometries U_t, $t \in \mathfrak{R}$ on $\mathfrak{X}$ onto $\mathfrak{X}$ as in 2.15(1). The concepts of average vector, T_U measure and the main theorems 2.19, 2.20, 2.22 then survive. An obvious application is to stochastic processes having wide-sense stationary increments in $L_p(\Omega, \mathfrak{B}, P)$, $1 \leqslant p \leqslant \infty$.

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О СПИРАЛЯХ В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ

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(Резюме)

Пусть $x(\cdot)$ — спираль в гильбертовом пространстве $\mathfrak{H}$, т. е. непрерывная функция на поле действительных чисел $\mathfrak{R}$ со значениями в $\mathfrak{H}$, такая, что для всех a, b, c, d, $t \in \mathfrak{R}$ $(x(b+t) - x(a+t),\ x(d+t) - x(c+t)) = (x(b) - x(a),\ x(d) - x(c))$. Мы показываем, что $x(b) - x(a) = T_U(a, b](\alpha_x)$, где $T_U(\cdot)$ — операторно-значная мера, построенная по группе сдвигов $(U_t,\ t \in \mathfrak{R})$ функции $x(\cdot)$ и $\alpha_x \in \mathfrak{H}$ — «средний вектор», связанный с $x(\cdot)$ (§ 2). Этот результат полностью описывает все спирали в рассматриваемой временно́й области: структура каждой спирали определяется соответствующей стационарной кривой $y(\cdot) = (U_t(\alpha_x),\ t \in \mathfrak{R})$ и «вектором перемещения» $p_x \perp y(\cdot)$. С помощью спектрального представления группы сдвигов $x(\cdot)$ мы из этих результатов легко получаем сильное и слабое спектральное представление спирали в смысле Колмогорова [6], [7] и фон Неймана и Шенберга [18], [19] и, кроме того, полностью описываем участвующие в них меры (§ 3).
