# On Singularities in the Heston Model

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**Abstract**

In this note we provide characterization of the singularities of the Heston characteristic function. In particular, we show that all the singularities are pure imaginary.

## 1 Problem Formulation

Consider the Heston stochastic volatility model, which under risk-neutral measure and with zero drift has the following dynamics

$$\begin{aligned} dS_t &= S_t\sqrt{v_t}\,dW_t^{(1)}, \\ dv_t &= \lambda(\bar{v}-v_t)\,dt + \eta\sqrt{v_t}\,(\rho\,dW_t^{(1)} + \sqrt{1-\rho^2}\,dW_t^{(2)}), \end{aligned}$$

where the parameters $\lambda, \eta$, and $\bar{v}$ are nonnegative, $\rho \in [-1,1]$, and the initial values $S_0$ and $v_0$ are positive.

The Heston characteristic function is defined as

$$\phi_H(u,\tau) = \mathbb{E}\big[e^{iu\log(S_\tau/S_0)}\big], \quad \alpha < \Im(u) < \beta.$$

Results of Heston [2] and Lewis [3] show that on the strip of convergence $\alpha < \Im(u) < \beta$ the Heston characteristic function coincides with

$$\phi(u,\tau) = e^{C(u,\tau)\bar{v} + D(u,\tau)v_0}, \quad u \in \mathbb{Z},$$

where

$$\begin{aligned} D(u,\tau) &= r_- \frac{1-e^{-d\tau}}{1-ge^{-d\tau}}, \quad C(u,\tau) = \lambda\left[r_-\tau - \frac{2}{\eta^2}\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right], \\ r_\pm &= \frac{\beta \pm d}{\eta^2}, \quad d = \sqrt{\beta^2 + 2\alpha\eta^2}, \quad g = \frac{r_-}{r_+}, \\ \alpha &= \frac{u^2}{2} - \frac{iu}{2}, \quad \beta = \lambda + \rho\eta i u. \end{aligned}$$

It is common, with slight abuse of notation, to refer to $\phi(u,\tau)$ as the Heston characteristic function.

Using a result[1] of Lukacs [4], Lewis [3] points out that $\phi(u,\tau)$ has singularities on the imaginary axis at the boundaries of the strip of convergence. Whether there are any other singularities (necessarily complex-conjugate) on that boundary could not be readily established. Furthermore, no conclusions can be made about singularities outside of the strip of convergence. The purpose of this note is to provide full characterization of the singularities of $\phi(u,\tau)$.

# 2 Main Result

The following theorem, although presented as an existence result, allows for construction of the singularities of $\phi(u,\tau)$ via standard numerical methods.

**Theorem 1.** *All singularities of $\phi(u,\tau)$ are pure imaginary.*

*Proof.* Assume $\eta > 0$, as for $\eta = 0$ we have the Black-Scholes model whose characteristic function is free of singularities (see, e.g., Lewis [3]).

To simplify notation we put $is = u$ and show that the (essential) singularities of $\phi(is,\tau)$ are real. To this end, we show that the transcendental equation

$$\frac{r_+}{r_-} = e^{-d\tau}, \tag{1}$$

[1] As noted in Lukacs [4], this is a corollary of a more general result on Laplace transforms, e.g. Theorem II.5b of Widder [5].

where

$$\begin{aligned} \beta &= \lambda - \rho\eta s && \text{(2a)} \\ d &= \sqrt{\beta^2 - \eta^2 s(s-1)} && \text{(2b)} \\ r_\pm &= \frac{\beta \pm d}{\eta^2} && \text{(2c)} \end{aligned}$$

has only real roots.

We consider (1) and (2) as a system in $d$ and $s$. Equation (1) can be written as

$$d = (-\lambda + \rho\eta s)\tanh(\tau d/2). \tag{3}$$

From (2a) and (2b) we get

$$-(1-\rho^2)\eta^2 s^2 + s(\eta^2 - 2\rho\eta\lambda) + \lambda^2 - d^2 = 0, \tag{4}$$

so, with $q := \sqrt{1-\rho^2}$, we can express $s$ in terms of $d$: for $q \neq 0$

$$s_{1/2} = \frac{\eta - 2\rho\lambda \pm \sqrt{(\eta - 2\rho\lambda)^2 + 4q^2(\lambda^2 - d^2)}}{2q^2\eta}, \tag{5}$$

and for $q = 0$ and $2\rho\lambda - \eta \neq 0$

$$s = \frac{d^2 - \lambda^2}{\eta^2 - 2\lambda\rho\eta}. \tag{6}$$

If $q = 0$ and $2\rho\lambda - \eta = 0$ from from (1), (2), and (4) we obtain $d = \lambda$, $\rho = 1$, $\eta = 2\lambda$, which implies that the only singularity is

$$s = \frac{1}{1 - e^{-\lambda\tau}}.$$

If $d = 0$ we have equality in (3), while from (5) and (6) it follows that the roots in $s$ are real.

For $d \neq 0$ substituting (5) in (3) yields

$$d = \left(-\lambda + \rho\frac{\eta - 2\rho\lambda \pm \sqrt{(\eta - 2\rho\lambda)^2 + 4q^2(\lambda^2 - d^2)}}{2q^2}\right)\tanh(\tau d/2),$$

while substituting (6) in (3) gives

$$d = \left(-\lambda + \rho\frac{d^2 - \lambda^2}{\eta - 2\rho\lambda}\right)\tanh(\tau d/2).$$

which imply, respectively,

$$\big(2dq^2\coth(\tau d/2)+2\lambda-\rho\eta\big)^2=\rho^2((\eta-2\rho\lambda)^2+4q^2(\lambda^2-d^2)), \tag{7}$$

and

$$d\coth(\tau d/2)+\lambda=\frac{\rho}{\eta-2\rho\lambda}(\lambda^2-d^2). \tag{8}$$

With

$$\frac{\tau d}{2}=iz, \quad a=\frac{\tau(\eta-2\lambda\rho)}{4q}\operatorname{sgn}(\rho), \quad b=\frac{\tau\lambda}{2}, \quad c=\frac{|\rho|}{q}$$

Lemma 1 implies that the roots of (7) are either real or pure imaginary. For the special case (8), Lemma 2 with

$$\frac{\tau d}{2}=iz, \quad b=\frac{\tau\lambda}{2}, \quad c=\frac{2\rho}{\tau(\eta-2\rho\lambda)}$$

implies that the corresponding roots are also either real or pure imaginary.

Therefore, it follows that for $d\neq 0$ the expression in the brackets in (3) is real (being ratio of either real or imaginary numbers), which in turn implies that the solutions of the transcendental equation (1) are real in $s$. $\square$

**Lemma 1.** *For real $a$ and real nonnegative $b, c$ the roots of the equation*

$$(z\cot(z)+b-ac)^2=c^2(a^2+b^2+z^2), \quad z\in\mathbb{Z} \tag{9}$$

*are real or pure imaginary.*

*Proof.* For $c=0$ the result follows from Lemma 9. If $c>0$ from Lemma 3 we have that for sufficiently large $N$ equation (9) has $4N+2$ roots inside the square with vertices $(N+1/2)(\pm\pi,\pm i\pi)$. On the other hand, from Lemma 4 and Lemma 6 it follows that there are $4N+2$ real or pure imaginary roots inside the same square, so the result follows. $\square$

**Lemma 2.** *For real nonnegative $b$ and real $c$ the roots of the equation*

$$z\cot(z)+b=c(b^2+z^2), \quad z\in\mathbb{Z} \tag{10}$$

*are real or pure imaginary.*

*Proof.* For $c = 0$ the result follows from Lemma 9. Putting $a = 0$ in Lemma 3 we conclude that for every $c \neq 0$ and sufficiently large $N$ equation

$$(z\cot(z) + b)^2 = c^2(b^2 + z^2)^2, \quad z \in \mathbb{Z}$$

has $4N+4$ roots inside the square with vertices $(N+1/2)(\pm\pi, \pm i\pi)$. On the other hand, from Lemma 5 and Lemma 7 it follows that both equations

$$z\cot(z) + b = \pm c(b^2 + z^2)$$

have $2N+2$ real or pure imaginary roots inside the same square, whence the result follows. $\square$

In the next lemma we make repeated use of the Rouché's theorem[2]. The version we use is given, for example, in Theorem 9.2.3 of Hille [1].

**Lemma 3.** *Let $C_N$, $N \in \mathbb{N}$ denote the square in complex plane with vertices at $(N + \frac{1}{2})(\pm\pi, \pm i\pi)$. Then for real $a$, nonnegative $b, c$, and $d = 1, 2$ there exists $N_0 \in \mathbb{N}$ such that for every integer $N > N_0$ the equation*

$$(z\cot(z) + b - ac)^2 = c^2(a^2 + b^2 + z^2)^d, \quad z \in \mathbb{Z} \tag{11}$$

*has $4N + 2d$ roots inside $C_N$.*

*Proof.* Consider the case $d = 1$, $c > 1$ and the case $d = 2$, $c > 0$ together. On the right vertical side of $C_N$ we have

$$|\cot(z)| = \left|\cot\left(\frac{\pi}{2} + N\pi + iy\right)\right| = |\tan(iy)| = \left|\frac{e^y - e^{-y}}{e^y + e^{-y}}\right| < 1, \tag{12}$$

while on the upper horizontal side we have

$$|\cot(z)| = \left|\frac{e^{2iz} + 1}{e^{2iz} - 1}\right| = \left|\frac{1 + e^{-(2N+1)\pi}e^{2ix}}{1 - e^{-(2N+1)\pi}e^{2ix}}\right| \leq \frac{1 + e^{-(2N+1)\pi}}{1 - e^{-(2N+1)\pi}}.$$

Together with (12) and the fact that $|\cot(z)| = |\cot(-z)|$ this implies

$$|\cot(z)| \leq \frac{1 + e^{-(2N+1)\pi}}{1 - e^{-(2N+1)\pi}} =: k_N, \quad z \in C_N.$$

[2] A different proof of Lemma 2 (dealing with the case of real roots only) appears as solution to Problem E1295 in *American Mathematical Monthly*, Vol. 65., No. 6, p. 450.

For $z \in C_N$ have

$$\begin{aligned}\frac{|(z\cot(z)+b-ac)^2|}{|c^2(a^2+b^2+z^2)|^d} &\leq \frac{(|z\cot(z)|+|b-ac|)^2}{|c^2(a^2+b^2+z^2)|^d} \\ &\leq \left(\frac{k_N}{c}+\frac{|b-ac|}{|cz|}\right)^2\left|\frac{z^2}{(a^2+b^2+z^2)^d}\right|.\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$, the last expression tends to $(2-d)/c^2 < 1$ uniformly in $z$ as $N \to \infty$, so for sufficiently large $N$ we have

$$|(z\cot(z)+b-ac)^2| < |c^2(a^2+b^2+z^2)|^d, \quad z \in C_N.$$

Therefore, by Rouché's theorem the number of roots of (11) inside $C_N$ is equal to the number of poles of $z \mapsto (z\cot(z)+b-ac)^2 - c^2(a^2+b^2+z^2)^d$ inside $C_N$ plus the number of zeros of $z \mapsto c^2(a^2+b^2+z^2)^d$ inside $C_N$ (considering their multiplicities). For sufficiently large $N$ those two numbers are $4N$ and $2d$ respectively, whence the equation (11) has $4N+2d$ roots inside $C_N$.

Consider now $d=1$, $0<c<1$. Let $D_N$ be the square vertices at $(\pm N\pi, \pm Ni\pi)$, and let $D_N^\epsilon$ denote $D_N$ extended with semicircles of radius $\epsilon$ so that the poles of $\cot(z)$ at $\pm N\pi$ are inside $D_N^\epsilon$, but the real zeros of (11) in $(N\pi,(N+1/2)\pi)$ and $(-(N+1/2)\pi,-N\pi)$ described in Lemma 4 remain outside. For ease of exposition in what follows we make $\epsilon$ smaller if necessary, which can be done without invalidating previously established statements.

Similarly as before, on the right vertical side of $D_N$ we have

$$|\tan(z)| = |\tan(N\pi+iy)| = |\tan(iy)| = \left|\frac{e^y-e^{-y}}{e^y+e^{-y}}\right| < 1, \tag{13}$$

while on the upper horizontal side we have

$$|\tan(z)| = \left|\frac{e^{2iz}-1}{e^{2iz}+1}\right| = \left|\frac{1-e^{-2N\pi}e^{2ix}}{1+e^{-2N\pi}e^{2ix}}\right| \leq \frac{1+e^{-2N\pi}}{1-e^{-2N\pi}}. \tag{14}$$

Together with (13) and the fact that $|\cot(z)| = |\cot(-z)|$ this implies that for sufficiently small $\epsilon > 0$

$$|\cot(z)| \geq \frac{1-e^{-2N\pi}}{1+e^{-2N\pi}} =: k_n, \quad z \in D_N^\epsilon. \tag{15}$$

On $D_N^\epsilon$ we have

$$\frac{|c^2(a^2+b^2+z^2)|}{|(z\cot(z)+b-ac)^2|} \leq \frac{c^2}{\left(|\cot(z)|-\left|\frac{b-ac}{z}\right|\right)^2} + \frac{|c^2(a^2+b^2)|}{\left(|z||\cot(z)|-|b-ac|\right)^2},$$

so for $N$ large enough

$$\frac{|c^2(a^2+b^2+z^2)|}{|(z\cot(z)+b-ac)^2|} \le \frac{c^2}{\left(k_N - \left|\frac{b-ac}{z}\right|\right)^2} + \frac{|c^2(a^2+b^2)|}{\left(|z|\,k_N - |b-ac|\right)^2}.$$

Since $\lim_{n\to\infty} k_n = 1$, the last expression tends to $c^2 < 1$ uniformly in $z$ as $N \to \infty$, so for sufficiently large $N$ we have

$$|c^2(a^2+b^2+z^2)| < |(z\cot(z)+b-ac)^2|, \quad z \in D_N^\epsilon.$$

Therefore, by Rouché's theorem the number of roots of (11) inside $D_N^\epsilon$ is equal to the number of poles of $z \mapsto (z\cot(z)+b-ac)^2 - c^2(a^2+b^2+z^2)$ inside $D_N^\epsilon$ plus the number of zeros minus the number of poles of $z \mapsto (z\cot(z)+b-ac)^2$ inside $D_N^\epsilon$ (considering their multiplicities). The two mappings have common poles, so we are left with number of zeros of the second mapping, which for sufficiently small $\epsilon$, according to Lemma 9, is $4N$. Therefore, from Lemma 8, and taking into account two real zeros in $(-(N+1/2)\pi, -N\pi) \cup (N\pi, (N+1/2)\pi)$ whose existence is established in Lemma 4, we conclude that for $d = 1$, $0 < c < 1$ and sufficiently large $N$ there are $4N+2$ zeros inside $C_N$.

Finally, consider the case $c = 1$, $d = 1$. Put $\alpha = b - ac$, $\beta^2 = a^2 + b^2$, so that we get

$$\frac{\cos(2z)}{\sin^2(z)}z^2 + \alpha z\frac{\sin(2z)}{\sin^2(z)} - (\beta^2 - \alpha^2) = 0, \tag{16}$$

or, equivalently,

$$2\cot(2z)\cot(z)z^2 + 2\alpha z\cot(z) - (\beta^2 - \alpha^2) = 0.$$

On $D_N^\epsilon$ we have

$$\frac{|2\alpha z\cot(z) - (\beta^2-\alpha^2)|}{|2\cot(2z)\cot(z)z^2|} \le \frac{\left|2\alpha - \frac{\beta^2-\alpha^2}{z\cot(z)}\right|}{2|z||\cot(2z)|} \le \frac{|2\alpha| + \frac{|\beta^2-\alpha^2|}{|z|k_N}}{2|z|k_{2N}}.$$

Since $\lim_{n\to\infty} k_n = 1$, the last expression tends to zero uniformly in $z$ as $N \to \infty$, so for sufficiently large $N$ we obtain

$$|2\alpha z\cot(z) - (\beta^2-\alpha^2)| < |2\cot(2z)\cot(z)z^2|, \quad z \in D_N^\epsilon,$$

that is,

$$\left|\frac{\cos(2z)}{\sin^2(z)}z^2\right| > \left|\alpha z\frac{\sin(2z)}{\sin^2(z)} - (\beta^2-\alpha^2)\right|, \quad z \in D_N^\epsilon.$$

Thus, by Rouché's theorem this implies that the number of roots of (16) inside $D_N^\epsilon$ equals the number of zeros of $z \mapsto \frac{\cos(2z)}{\sin^2(z)} z^2$ inside $D_N^\epsilon$, which is $4N$. Therefore, reasoning as in the previous part of the proof we conclude that for $c = 1$ we have $4N + 2$ zeros of (11) inside $C_N$ for $N$ large enough. □

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# A Appendix

**Lemma 4.** *For $N$ sufficiently large, equation* (9) *has* $4N - 2$ *real roots in* $(-(N + 1/2)\pi, -\pi) \cup (\pi, (N + 1/2)\pi)$.

*Proof.* By Lemma 8 for every $N > 1$ equation (9) as two real roots in each of the intervals $(-(k + 1)\pi, -k\pi)$ and $(k\pi, (k + 1)\pi)$, $k = 1, 2, \dots N - 1$.

Rewrite (9) as

$$z \cot(z) = -(b - ac) \pm c\sqrt{a^2 + b^2 + z^2}. \tag{17}$$

For $N > 0$

$$\lim_{z \to N\pi+} z \cot(N\pi) = +\infty, \quad z \cot(N\pi + \pi/2) = 0, \tag{18}$$

so we conclude that for sufficiently large $N$ the equation with plus sign has one real root in $(N\pi, (N + 1/2)\pi)$, hence by symmetry in $(-(N + 1/2)\pi, -N\pi)$. □

**Lemma 5.** *For $N$ sufficiently large equation* (10) *has* $2N$ *real roots in* $(-(N + 1/2)\pi, -\pi) \cup (\pi, (N + 1/2)\pi)$ *if* $c > 0$, *and* $2N - 2$ *real roots if* $c < 0$.

*Proof.* By Lemma 8 for every $N > 1$ equation (10) has one real root in each of the intervals $(-(k + 1)\pi, -k\pi)$ and $(k\pi, (k + 1)\pi)$, $k = 1, 2, \dots N - 1$.

Rewrite (10) as

$$z \cot(z) = -b + c(b^2 + z^2). \tag{19}$$

From (18) and (19) we conclude that if $c > 0$ for sufficiently large $N$ equation (10) has one real root in $(N\pi, (N + 1/2)\pi)$, hence by symmetry in $(-(N + 1/2)\pi, -N\pi)$. □

**Lemma 6.** *For real $a$, nonnegative $b$, and $c > 0$ equation* (9) *has either four real roots in $(-\pi, \pi)$, or two real roots in $(-\pi, \pi)$ and two imaginary roots.*

*Proof.* The proof follows by simple geometrical considerations. For $z = 0$ the right-hand side of (17) assumes two values

$$\alpha_1 := -(b - ac) + c\sqrt{a^2 + b^2}, \quad \alpha_1 := -(b - ac) - c\sqrt{a^2 + b^2}.$$

Since $ac - c\sqrt{b^2 + a^2} \leq 0$ we have $\alpha_2 \leq 0$. On the other hand, the function $x \mapsto x\cot(x)$ is zero at the origin and strictly decreases on $[0, \pi)$, with a discontinuity of the second kind at $\pi$. Thus, (17) has one real root corresponding to the intersection of $x \mapsto x\cot(x)$ and $x \mapsto -(b - ac) - c\sqrt{a^2 + b^2 + z^2}$ on $(0, \pi)$.

If $\alpha_1 < 1$ following the same argument we conclude that there is another real root in $(0, \pi)$ corresponding to the intersection of $x \mapsto x\cot(x)$ and $x \mapsto -(b - ac) + c\sqrt{a^2 + b^2 + z^2}$. If $\alpha_1 = 1$ we have a double root at zero.

Thus, based on the above considerations and the symmetry around the origin it follows that in $(-\pi, \pi)$ equation (17) has four real roots if $\alpha_1 \leq 1$, and two real roots if $\alpha_1 > 1$. Therefore, to complete the proof we show that (17) has two imaginary roots if $\alpha_1 > 1$.

Put $z = iy, y \in \mathbb{R}$ in (17) to get

$$y\coth(y) = -(b - ac) \pm c\sqrt{a^2 + b^2 - y^2}. \tag{20}$$

On the left-hand side we have a continuous function equal to one at the origin that tends to infinity as $y$ increases. Note that $\alpha_1 > 1$ implies $a^2 + b^2 > 0$ Thus, on the right-hand side we have a semi-circle starting at $(0, \alpha_1)$ on the ordinate, entering into the right half-plane, and ending at $(0, \alpha_2)$ on the ordinate, half-encircling the point $(0, 1)$ (as $\alpha_1 > 1$ and $\alpha_2 \leq 0$). Therefore, there must exist $y_0 > 0$ for which the equality holds in (20). Since $-y_0$ also solves (20), we have two imaginary solutions. □

**Lemma 7.** *Assume $b \geq 0$. For $c > 0$ equation* (10) *has either two real roots in $(-\pi, \pi)$ or two imaginary roots. If $c < 0$ equation* (10) *has two real roots in $(-\pi, \pi)$ and two imaginary roots.*

*Proof.* At $z = 0$ the right-hand side of (17) equals $-b + cb^2$. The function $x \mapsto x\cot(x)$ is zero at the origin and strictly decreases on $[0, \pi)$, with a discontinuity of the second kind at $\pi$. Thus, if $c < 0$ or $c > 0$ and $-b + cb^2 < 1$ there is one real root in $(0, \pi)$, hence by symmetry in $(-\pi, 0)$. If $-b + cb^2 = 1$

we have a double root at the origin. Next, with $z = iy, y \in \mathbb{R}$ equation (10) becomes

$$y \coth(y) = -b + c(b^2 - y^2). \tag{21}$$

Threfore, if $c > 0$ and $-b + cb^2 > 1$ the right-hand side dominates the left-hand side at the origin, while the opposite is true for sufficiently large $y$. From the continuity of the two functions it then follows that (21) has one positive root, hence by symmetry one negative root. Finally, if $c < 0$ the left-hand side dominates the right-hand side at the origin, while the opposite is true for sufficiently large $y$, giving a pair of imaginary roots. □

**Lemma 8.** *For every positive integer $k$ equation* (11) *has two real roots in each of the intervals* $(-(k+1)\pi, -k\pi)$ *and* $(k\pi, (k+1)\pi)$.

*Proof.* The result follows from the fact that on each of those intervals the range of the map $x \mapsto x \cot(x)$ is the whole real line, while the maps $x \mapsto -b + ac \pm c(a^2 + b^2 + x^2)^{d/2}$ are bounded. □

**Lemma 9.** *For $a \in \mathbb{R}$ the equation*

$$z \cot(z) = a, \quad z \in \mathbb{Z} \tag{22}$$

*has $2N$ roots inside the square with vertices $(\pm N\pi, \pm Ni\pi)$. The roots are real or pure imaginary.*

*Proof.* For $a = 0$ the roots are the zeros of $\cos(z)$. If $a \neq 0$ from (13) and (14) we conclude that for sufficiently large $N$

$$|a \tan(z)| < |z|, \quad z \in D_N.$$

Thus, by Rouché's theorem

$$z = a \tan(z)$$

has $2N + 1$ roots inside the square with vertices $(\pm N\pi, \pm Ni\pi)$. If $k > 0$ it has two real roots in $(-(k+1)\pi/2, k\pi/2) \cup (k\pi/2, (k+1)\pi/2)$ if either $a > 0$ and $k$ is even, or $a < 0$ and $k$ is odd.

On the other hand, in $(-\pi, \pi)$ there are three roots (counting their multiplicities) if $a \geq 1$ and one root if $0 < a < 1$. In the latter case there are two imaginary roots (c.f. example on page 255 of Hille [1]). Since (22) has one root less at the origin, the result follows. □

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