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Stochastic Processes and their Applications 72 (1997) 1–9

stochastic  
processes  
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applications

# Conditions for the completeness of the spectral domain of a harmonizable process

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Received 11 September 1995; received in revised form 23 June 1997

## Abstract

We generalize a theorem of Köthe and Toeplitz on unconditional bases in Hilbert spaces to Hilbert space-valued measures. This leads to a necessary and sufficient condition for the completeness of the spectral domain of a weakly harmonizable process whose shift operator exists and is invertible. A process in this class has a complete spectral domain if and only if it is the image of a stationary process under a topological isomorphism. © 1997 Elsevier Science B.V.

**Keywords:** Weakly harmonizable process; Bimeasure; Completeness of spectral domain; Shift operator

## 1. Introduction

Let  $L_0^2(P)$  denote the zero mean square integrable functions on some probability space. The stochastic processes in this paper are either discrete time ( $T = \mathbb{Z}$ ) or continuous time ( $T = \mathbb{R}$ ) and they are subsets of  $L_0^2(P)$ . Let  $\hat{T} = \Pi := \mathbb{R}/2\pi\mathbb{Z} = (-\pi, \pi]$  if  $T = \mathbb{Z}$  and  $\hat{T} = \mathbb{R}$  if  $T = \mathbb{R}$ , and let  $\mathbb{B}$  denote the Borel  $\sigma$ -field of either  $\Pi$  or  $\mathbb{R}$ .

Recall that a process  $(X_t)_{t \in T}$  is weakly stationary if its covariance function admits the representation  $\text{Cov}(X_s, X_t) = \int_{\hat{T}} e^{i\lambda s} \overline{e^{i\lambda t}} \mu(d\lambda)$ ,  $s, t \in T$ , where  $\mu$  is a finite positive measure on  $\hat{T}$ . The time domain of a stationary process is isometrically isomorphic to its spectral domain  $L^2(\mu)$  and prediction and filtering of the process can thus be carried out in the spectral domain using Fourier methods.

Harmonizable processes are an extension which allow also for these Fourier methods. A stochastic process  $(X_t)_{t \in T}$  is called strongly harmonizable, if its covariance function admits the representation

$$\text{Cov}(X_s, X_t) = \int_{\hat{T}} \int_{\hat{T}} e^{is\lambda} \overline{e^{it\lambda'}} \beta(d\lambda, d\lambda'), \quad \forall s, t \in T \quad (1)$$

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for some measure  $\beta$  on  $\mathbb{B} \otimes \mathbb{B}$ . These processes were introduced by Loève (1948) and Cramér (1951). An important example of strongly harmonizable processes are the periodically correlated discrete time processes (Hurd, 1989, 1991; Dehay, 1994). In this case, the mass of  $\beta$  is concentrated on lines parallel to the diagonal  $\{(\lambda, \lambda), \lambda \in \hat{T}\}$  of  $\hat{T} \times \hat{T}$ . However, the class of strongly harmonizable processes is still not large enough for some applications. For example, it is not closed under linear transformations. A wider extension of stationary processes are the weakly harmonizable processes. These processes have the same covariance representation as strongly harmonizable processes, but the conditions on  $\beta$  are relaxed.  $\beta$  is not necessarily a measure, but can be a bimeasure (i.e.  $\beta$  is a mapping  $\beta: \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{C}$  and  $\beta(\cdot, A)$ ,  $\beta(A, \cdot)$  are measures for all  $A \in \mathbb{B}$ ), which is positive definite (i.e.  $\beta(A, B) = \overline{\beta(B, A)}$  and  $\sum_{i,j} a_i \beta(A_i, A_j) \overline{a_j} \geq 0$  for all  $A_i \in \mathbb{B}$ ,  $a_i \in \mathbb{C}$ ).  $\beta$  is called the spectral bimeasure of the process. The class of weakly harmonizable processes is closed under linear transformations. Indeed, another characterization of weakly harmonizability is that these processes are coordinatewise images of stationary processes under linear mappings (Niemi, 1975, 1977).

In general, bimeasures cannot be extended to measures on  $\hat{T} \otimes \hat{T}$ , so the integral in (1) is not a Lebesgue integral, but a strict integral in the sense of Morse and Transue (Chang and Rao, 1986).

Let  $f, g: \hat{T} \rightarrow \mathbb{C}$  be measurable and  $\beta$  a bimeasure on  $\mathbb{B} \times \mathbb{B}$ .  $(f, g)$  is strictly  $\beta$ -integrable, if the following holds:

(1)  $f$  is  $\beta(\cdot, A)$ -integrable,  $g$  is  $\beta(A, \cdot)$ -integrable for all  $A \in \mathbb{B}$ . If this holds, then  $\beta_1^E: A \rightarrow \int_E f(x) \beta(dx, A)$  and  $\beta_2^F: A \rightarrow \int_F g(y) \beta(A, dy)$  are measures for all  $E, F \in \Sigma$  (Houdré, 1987, Lemma 2).

(2)  $f$  is  $\beta_2^E$ -integrable and  $g$  is  $\beta_1^F$ -integrable for all  $E, F \in \mathbb{B}$ . One defines now

$$\int_E \int_F^* (f, g) d\beta := \int_E f(x) \beta_2^F(dx) = \int_F g(x) \beta_1^E(dx).$$

The equality of the integrals holds by Houdré (1987, Theorem 4).

If  $\beta$  is a measure on  $\mathbb{B} \otimes \mathbb{B}$  and  $f, g$  are bounded then this integral is equal to the usual Lebesgue integral, i.e. to  $\int_E \int_F f(x) g(y) \beta(dx, dy)$ .

For a positive-definite bimeasure  $\beta$  define

$$\mathcal{L}_*^2(\beta) := \{f : (f, \bar{f}) \text{ is strict } \beta\text{-integrable}\}$$

the spectral domain of  $\beta$ . This is a pre-Hilbert space with the inner product  $(f, g)_\beta := \int \int^* (f, \bar{g}) d\beta$  (cf. Chang and Rao, 1986).

Now let  $(X_t)_{t \in T}$  be a weakly harmonizable process with spectral bimeasure  $\beta$ . As for a stationary process,  $(X_t)_{t \in T}$  has an integral representation  $X_t = \int_{\hat{T}} e^{i\lambda t} Z(d\lambda)$  where  $Z$  is a stochastic measure (i.e.  $L_0^2(P)$ -valued) and the integral is in the sense of Dunford–Schwartz (1960, IV.10).  $Z$  is called the stochastic spectral measure of  $(X_t)_{t \in T}$ . It is related to the spectral bimeasure by  $\beta(A, B) = EZ(A) \overline{Z(B)}$ ,  $\mathcal{L}_*^2(\beta) = L^1(Z)$  and

$$\text{Cov} \left( \int f dZ, \int g dZ \right) = \int \int^* (f, \bar{g}) d\beta \quad (2)$$

(Chang and Rao, 1986). We then say that  $\beta$  is induced by  $Z$ . Note that  $(X_t)_{t \in T}$  is stationary if  $Z$  is orthogonally scattered (i.e.  $A \cap B = \emptyset$  then  $Z(A) \perp Z(B)$ ) or equivalently the mass of  $\beta$  is concentrated on the diagonal, i.e.  $\beta(A, B) = \beta(A \cap B)$ .

It is well known that  $H_Z := \text{sp}\{Z(A), A \in \mathbb{B}\} = \text{sp}\{\int_{\hat{T}} e^{it} dZ, t \in T\}$ . Indeed, stochastic measures on  $\hat{T}$  are regular (this follows from Dunford and Schwartz, 1960, IV.10.5) and hence the set of integrals of continuous functions is dense in  $H_Z$ . The assertion now follows from the fact that continuous functions can be approximated uniformly on compact sets by trigonometric polynomials (Naimark, 1959, p. 406, Corollary 4).

As we have already seen, the mapping  $V: \mathcal{L}_*^2(\beta) \rightarrow H_Z (= \text{sp}\{X_t, t \in T\})$ ,  $f \rightarrow \int f dZ$  is an isometry. It is easy to see that  $V$  is surjective if and only if  $\mathcal{L}_*^2(\beta)$  is complete. Thus, the spectral domain and the time domain of the process are isometrically isomorphic if and only if the spectral domain is complete.

One of the reasons to introduce weakly harmonizable processes was to retain the powerful Fourier methods of the stationary processes, so that, for example, extrapolation and interpolation can be carried out in the spectral domain of the process, using again the methods of Fourier analysis. But the transition from the spectral domain to the time domain of a process can only be made in full generality if the spectral domain is complete. For example, Chang and Rao (1986) need this completeness to solve filter equations, and Houdré (1991) for obtaining autoregressive predictors. A solution to this problem might be to take the algebraic completion of the spectral domain. But the members of this space need not to be functions in the usual sense. Also the isometry to the time domain can no longer be described by the elegant integral representation of the process.

Cramér, who defined the spectral domain, raised the problem of the completeness of this spectral domain: “the set  $\mathcal{A}_2(\rho)$  will have all the properties of Hilbert space, except possibly the completeness property” (Cramér, 1951). In the 1980s, Rao et al. (Chang and Rao, 1986; Rao, 1989a, b; Mehlmann, 1991) came up with proofs for the completeness of the spectral domain, but these proofs worked only in special cases. Then in 1991, Miamee and Salehi were able to show that, in general,  $\mathcal{L}_*^2(\beta)$  is not complete (cf. Miamee and Salehi, 1991; Michálek and Rüschendorf, 1994). In 1995, Miamee and Schröder (1995) gave conditions for the completeness of the spectral domain in certain cases (see also Michálek and Rüschendorf, 1994).

In this work, we represent a necessary and sufficient condition for the completeness of the spectral domain for a broad class of weakly harmonizable processes. This result is not only of interest for harmonizable processes, but also in functional analysis, in fact we prove:

*Let  $Z$  be a Hilbert space-valued measure such that  $\int f dZ = 0$  implies  $f = 0$   $Z$ -almost everywhere. Then  $\{\int f dZ, f \in L^1(Z)\}$  is complete if and only if there exists an orthogonally scattered measure  $\hat{Z}$  and an isomorphism  $V$  with  $V(\hat{Z}) = Z$ .*

The following result due to Rao (Rao, 1989, p. 605, Proposition 3.2) gives a sufficient condition for the completeness of the spectral domain of a weakly harmonizable process (with  $T = \mathbb{R}$  or  $\mathbb{Z}$ ).

*Let  $\beta$  be a positive-definite bimeasure on  $\mathbb{B} \times \mathbb{B}$ . Then  $\mathcal{L}_*^2(\beta)$  is complete if there is a positive finite measure  $\mu$  on  $\mathbb{B}$  and a continuous linear mapping  $W: L^2(\mu) \rightarrow L^2(\mu)$*

with closed range such that  $L^2(\mu) \subset \mathcal{L}_*^2(\beta)$  and

$$\int \int^* (f, \bar{g}) d\beta = \int W(f) \overline{W(g)} d\mu, \quad \forall f, g \in L^2(\mu). \quad (3)$$

There is always a finite measure  $\mu$  and a linear continuous mapping from  $L^2(\mu)$  to  $L^2(\mu)$  such that (3) holds, but its range is not necessarily closed. This follows from the Grothendieck inequality (Rao, 1989, Theorem 2). If  $L^2(\mu) = \mathcal{L}_*^2(\beta)$ , the closed range condition is necessary. The closed range condition is clearly satisfied if there are constants  $C, c > 0$  such that

$$C \|f\|_{\mu, 2} \geq \|f\|_{\beta} \geq c \|f\|_{\mu, 2}, \quad \forall f \in L^2(\mu)$$

holds. This is the sufficient condition for the completeness of the spectral domain obtained by Truong-Van (1981) (Theorem 6, cf. also Michalék and Rüschendorf, 1994). In the next section we shall see that this condition is close to being necessary for completeness.

Since the results of the next section do not depend on the structure of  $\mathbb{R}$  or  $\mathbb{I}$ , we will formulate and prove them for arbitrary measurable spaces. So let  $(\Omega, \Sigma)$  be a measurable space and  $\beta$  a positive-definite bimeasure defined on  $\Sigma \times \Sigma$ .  $\beta$  is induced by the stochastic measure  $Z$ . Since the strict integral is not handy, we shall make use of (2) and use instead the Dunford–Schwartz integral for vector measures. By  $\|Z\|(\cdot)$ , we will denote the semivariation of the stochastic measure  $Z$  (Dunford and Schwartz, 1960, IV.10.3). In this paper, we will use only the Hilbert space properties of  $L_0^2(P)$ . By  $\|\cdot\|$  we refer to the norm of this space.

The next theorem is crucial for the next section and due to Drewsowski (1974a, p. 217, 3.10; 1974b, p. 799, Example 2.3.3).

**Theorem.** (a)  $\|\cdot\|_Z$  defined via  $\|f\|_Z := \sup_{|g| \leq |f|} \|\int g dZ\|$ ,  $f \in L^1(Z)$ , is a norm on  $L^1(Z)$ . This norm is equivalent to the norm

$$\|f\| := \sup_{A \in \Sigma} \left\| \int_A f dZ \right\| \left( \geq \frac{1}{4} \|f\|_Z \right).$$

(b)  $(L^1(Z), \|\cdot\|_Z)$  is a Banach space.

(c) If  $f_n, f \in L^1(Z)$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \|f - f_n\|_Z = 0$ , then  $(f_n)_{n \in \mathbb{N}}$  converges to  $f$  in  $Z$ -measure (i.e.,  $\forall \varepsilon > 0 \lim_{n \rightarrow \infty} \|Z\|(\{|f_n - f| > \varepsilon\}) = 0$ ). Hence any subsequence of  $(f_n)_{n \in \mathbb{N}}$  contains a subsequence converging  $Z$ -a.e. to  $f$ .

Note that  $\|f\|_Z = \sup_{|g| \leq |f|} \|g\|_{\beta}$ ! Finally, note that the simple functions are dense in  $(L^1(Z), \|\cdot\|_Z)$ . This follows from the results in Kwapien and Woyczyński (1992, Proposition 7.1.1, Definition 7.3.1).

## 2. A condition for the completeness of the spectral domain

Miamee and Schröder (1995) obtained necessary conditions for the completeness of the spectral domain of a weakly harmonizable process. In particular, for the class of

spectral bimeasures  $\beta$  such that for some positive finite measure  $\mu$ ,  $\mu(\{f \neq 0\}) = 0$  is equivalent to  $\|f\|_\beta = 0$ . In this section we investigate the completeness of the spectral domain of a bimeasure  $\beta$  such that  $\|\cdot\|_Z = 0$  is equivalent to  $\|\cdot\|_\beta = 0$ , which is a weaker condition than the one studied by Miamee and Schröder.

**Theorem 1.** *Let  $\|\cdot\|_\beta = 0$  be equivalent to  $\|\cdot\|_Z = 0$ . Then the following statements are equivalent:*

- (1)  $\mathcal{L}_*^2(\beta)$  is complete.
- (2) There exists a  $K > 0$  such that for all  $f \in L^1(Z)$

$$K\|f\|_\beta \geq \|f\|_Z.$$

- (3) There exists a positive finite measure  $\mu$  and constants  $c, C > 0$  such that  $L^2(\mu) = L^1(Z) (= \mathcal{L}_*^2(\beta))$  and

$$C\|\cdot\|_{\mu,2} \geq \|\cdot\|_\beta \geq c\|\cdot\|_{\mu,2}.$$

**Proof.** (1)  $\Rightarrow$  (2): Note that  $T: L^1(Z) \rightarrow \mathcal{L}_*^2(\beta)$ ,  $f \mapsto f$ , is a linear, one to one mapping and that  $(L^1(Z), \|\cdot\|_Z)$  and  $\mathcal{L}_*^2(\beta)$  are Banach spaces. This implies that  $T$  has a continuous linear inverse (Dunford and Schwartz, 1960, II.2.2). Consequently, there exists a  $K > 0$  satisfying  $K\|f\|_\beta \geq \|f\|_Z$ , for all  $f \in L^1(Z)$ .

(2)  $\Rightarrow$  (1): Since  $\|f\|_Z \geq \|f\|_\beta$ , both norms are equivalent.

(3)  $\Rightarrow$  (2): It follows directly from the definition of  $\|\cdot\|_Z$  that  $C\|\cdot\|_{\mu,2} \geq \|\cdot\|_Z$ , hence  $C/c\|\cdot\|_\beta \geq \|\cdot\|_Z$ .

(2)  $\Rightarrow$  (3): We define  $|f|_Z := \inf_{|h|=|f|} \|\int h dZ\| = \inf_{|h|=|f|} \|h\|_\beta$ , and prove that  $\|\cdot\|_Z^2$  is superadditive while  $|\cdot|_Z^2$  is subadditive:

Indeed, let  $f, g \in L^1(Z)$  and  $fg = 0$ . Then

$$\begin{aligned} \|f + g\|_Z^2 &= \sup_{|f'| \leq |f|, |g'| \leq |g|} \sup_{\alpha \in \mathbb{C}, |\alpha|=1} (\alpha f', \alpha f')_\beta + (g', g')_\beta + 2 \operatorname{Re}(\alpha f', g')_\beta \\ &\geq \sup_{|f'| \leq |f|, |g'| \leq |g|} (f', f')_\beta + (g', g')_\beta \\ &= \|f\|_Z^2 + \|g\|_Z^2. \end{aligned}$$

In a similar way one shows  $|f + g|_Z^2 \leq |f|_Z^2 + |g|_Z^2$ .

Next, we prove that

$$\mu(A) := \sup \left\{ \sum_{i \in I} |1_{A_i}|_Z^2; (A_i)_{i \in I} \text{-finite measurable partitions of } A \right\}$$

is a positive finite measure.

The properties  $\mu(\emptyset) = 0$ ,  $\mu \geq 0$ , and the additivity follow readily from the definition of  $\mu$ .

Since  $\|Z\|(\cdot)$  is continuous at the empty set (Dunford and Schwartz, 1960, IV.10.5) and  $\mu(A) \leq \|1_A\|_Z^2 = \|Z\|(A)^2$ , we conclude that  $\mu$  is also continuous. Thus  $\mu$  is a measure.

It is clear that  $|1_A|_Z^2 \leq \mu(A) \leq \|1_A\|_Z^2$  for all  $A \in \Sigma$ . Let  $f$  be a simple function. From the last inequality, the subadditivity of  $|\cdot|_Z^2$ , and the superadditivity of  $\|\cdot\|_Z^2$ , it follows that

$$|f|_Z^2 \leq \|f\|_{\mu,2}^2 \leq \|f\|_Z^2.$$

By the assumptions there exists a  $K > 0$  such that  $K\|\cdot\|_\beta \geq \|\cdot\|_Z$ . Now let  $f \in L^1(Z)$ ; then

$$K|f|_Z = K \inf_{|g|=|f|} \|g\|_\beta \geq \inf_{|g|=|f|} \|g\|_Z = \|f\|_Z,$$

where the last equality holds since  $\|\cdot\|_Z$  depends only on the modulus of a function. Thus  $K|\cdot|_Z \geq \|\cdot\|_Z$ .

Hence, the norms  $\|\cdot\|_{\mu,2}$  and  $\|\cdot\|_Z$  are equivalent for the simple functions. Since  $(L^1(Z), \|\cdot\|_Z)$  and  $L^2(\mu)$  are Banach spaces, since the simple functions are dense in both spaces and since the convergence in both spaces implies the almost everywhere convergence of some subsequence, the norms are equivalent for  $L^1(Z) = L^2(\mu)$ .

The equivalence of  $\|\cdot\|_\beta$  and  $\|\cdot\|_Z$  implies the equivalence of  $\|\cdot\|_\beta$  and  $\|\cdot\|_{\mu,2}$ .  $\square$

Let  $\beta$ ,  $Z$  and  $\mu$  be as in the second condition of Theorem 1. Then there is (up to isometric isomorphism) a unique orthogonally scattered Hilbert space valued measure  $\hat{Z}$  satisfying  $\|\hat{Z}(A)\|^2 = \mu(A)$ ,  $L^1(\hat{Z}) = L^2(\mu)$  and

$$\left( \int f d\hat{Z}, \int g d\hat{Z} \right) = \int f \bar{g} d\mu, \quad \forall f, g \in L^2(\mu).$$

Take, for example,  $\hat{Z}(A) := 1_A \in L^2(\mu)$ . Since  $\|\cdot\|_{\mu,2}$  and  $\|\cdot\|_\beta$  are equivalent, there is a continuous and continuously invertible linear map  $V: H_{\hat{Z}} \rightarrow H_Z$  with  $V \circ \hat{Z} = Z$ .

Now, let  $(x_i)_{i \in \mathbb{N}}$  be a normed sequence in some Hilbert space, and let  $Z$  be the measure on  $\mathbb{N}$  with  $Z(\{i\}) = x_i/i^2$ ,  $i \in \mathbb{N}$ . It is easy to show that  $(a_i)_{i \in \mathbb{N}} \in L^1(Z)$  if and only if  $\sum_{i=1}^\infty a_i x_i (= \int_{\mathbb{N}} a dZ)$  is unconditionally convergent. Hence, it is easy to see that Theorem 1 is a generalization of the theorem of Köthe and Toeplitz (Nikolskii, 1986, p. 137), which states that if  $(x_n)_{n \in \mathbb{N}}$  is a normed unconditional base of a Hilbert space, then there exist constants  $c, C > 0$  such that

$$c \sum_{i=1}^n |a_i|^2 \leq \left\| \sum_{i=1}^n a_i x_i \right\|^2 \leq C \sum_{i=1}^n |a_i|^2, \quad a_i \in \mathbb{C}, \quad i = 1, \dots, n \in \mathbb{N}.$$

In other words, any normed unconditional base of a separable Hilbert space is topologically isomorphic to some orthonormal base.

### 3. The shift operator condition

We now turn to the question of finding under which conditions  $\|\cdot\|_\beta = 0$  and  $\|\cdot\|_Z = 0$  are equivalent, where  $\beta$  is the spectral bimeasure and  $Z$  the stochastic spectral measure of some weakly harmonizable process.

The next theorem gives a necessary and sufficient condition for this equivalence.

**Theorem 2.** Let  $(X_t)_{t \in \mathbb{Z}}$  be a weakly harmonizable process with spectral bimeasure  $\beta$  and stochastic spectral measure  $Z$ .

Then the following two statements are equivalent:

- (1)  $\|\cdot\|_\beta = 0$  is equivalent to  $\|\cdot\|_Z = 0$ .
- (2) The shift operator  $S: \mathcal{L}_*^2(\beta) \rightarrow \mathcal{L}_*^2(\beta)$ ,  $S(f) := \exp(i \cdot) f$  is a well-defined invertible linear mapping, i.e.  $\|f\|_\beta = 0 \Leftrightarrow \|\exp(i \cdot) f\|_\beta = 0$ .

**Proof.** (1)  $\Rightarrow$  (2): This follows immediately from  $\|f e^{i \cdot}\|_Z = \|f\|_Z$ .

(2)  $\Rightarrow$  (1): Let  $\|f\|_\beta = 0$ . It follows from the theorem stated at the end of Section 1, that there is a  $A \in \mathbb{B}$  such that  $\|f\|_Z \leq 5 \|f 1_A\|_\beta$ . Since  $\|Z\|$  is regular, it is easy to see that there exist trigonometric polynomials  $(p_n)_{n \in \mathbb{N}}$  with  $\|p_n\|_\infty \leq 2$  for all  $n \in \mathbb{N}$ , such that  $\lim_{n \rightarrow \infty} \|p_n - 1_A\|_Z = 0$  (the trigonometric polynomials are dense in the continuous functions with respect to the uniform convergence (Naimark, 1959, p. 406, Corollary 4)). By the theorem previously cited there is a subsequence  $(p_{n_k})_{k \in \mathbb{N}}$  converging  $Z$ -almost everywhere to  $1_A$ . Applying the theorem of dominated convergence (Dunford and Schwartz, 1960, IV.10.10) we have  $\lim_{k \rightarrow \infty} \int p_{n_k} f dZ = \int_A f dZ$ . But if  $p = \sum_{j=-m}^m a_j e^{ij \cdot}$ , then

$$\left\| \int p f dZ \right\| = \|p f\|_\beta \leq \sum_{j=-m}^m |a_j| \|(e^{i \cdot})^j f\|_\beta = \sum_{j=-m}^m |a_j| \|S^j(f)\|_\beta = 0.$$

Hence  $0 = 5 \left\| \int_A f dZ \right\| \geq \|f\|_Z$ .  $\square$

**Remark.** A sufficient condition for the existence of an invertible shift operator  $S$  of the spectral domain is the existence of an invertible shift operator  $\hat{S}$  of the time domain  $H_X := \overline{\text{sp}\{X_t, t \in \mathbb{Z}\}}$ , i.e., there is a continuous linear and continuously invertible mapping  $\hat{S}: H_X \rightarrow H_X$  with  $\hat{S}(X_t) = X_{t+1}$  for all  $t \in \mathbb{Z}$ .

Indeed, it is sufficient to show that

$$\hat{S} \left( \int f dZ \right) = \int S(f) dZ, \quad \forall f \in L^1(Z). \quad (4)$$

$U: f \rightarrow e^{i \cdot} f$  and  $V: f \rightarrow \int f dZ$  are continuous linear mappings from  $(L^1(Z), \|\cdot\|_Z)$  to  $(L^1(Z), \|\cdot\|_Z)$ , respectively  $H_Z$ . If  $p$  is a trigonometric polynomial, then  $\hat{S}(V(p)) = V(U(p))$  holds. By an argument similar to the one given in the proof of Theorem 2 it follows that the trigonometric polynomials are dense in  $(L^1(Z), \|\cdot\|_Z)$ . Thus, (4) is true for all  $f \in L^1(Z)$ .

A similar shift operator criterion holds for weakly harmonizable continuous-time processes.

Let  $(X_t)_{t \in \mathbb{Z}}$  be a weakly harmonizable process whose spectral domain is complete and assume that the shift operator  $S$  of the process exists and is invertible. Then by the last two theorems the operators  $S^k$ ,  $k \in \mathbb{Z}$ , are uniformly bounded. But is the uniform boundedness of the operators  $S^k$ ,  $k \in \mathbb{Z}$ , sufficient for the completeness of the spectral domain?

The following theorem, due to Nagy (1948), gives an affirmative answer (see also Truong-Van, 1981).

**Theorem.** *Let  $(X_t)_{t \in \mathbb{Z}}$  be a stochastic process, and let  $S$  be the shift operator of the process (i.e.,  $S$  is a linear continuous mapping on the time domain of  $(X_t)_{t \in \mathbb{Z}}$ ,  $S(X_t) = X_{t+1}$  for all  $t \in \mathbb{Z}$ ). If  $S$  is invertible and moreover the mappings  $S^k$ ,  $k \in \mathbb{Z}$ , are uniformly bounded (in the operator norm), then  $(X_t)_{t \in \mathbb{Z}}$  is weakly harmonizable and there exists a finite positive measure  $\mu$ , such that  $\|\cdot\|_{\mu,2}$  and  $\|\cdot\|_\beta$  are equivalent, where  $\beta$  is the spectral bimeasure of  $(X_t)_{t \in \mathbb{Z}}$ . Hence  $\mathcal{L}_*^2(\beta)$  is complete.*

## Acknowledgements

The author would like to thank C. Houdré, J. Kallsen and the anonymous referee for their valuable advice on the composition of this paper.

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