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C-BIMEASURES Λ AND THEIR INTEGRAL EXTENSIONS

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§1. Introduction

Let **C** be the field of complex numbers. **C**-measures on a locally compact topological space E are defined in Ref. 5. [Cf. Ref. 3, Chap. III.] Let E' and E'' be two locally compact topological spaces. Let $\mathcal{K}'_{\mathbf{C}}$ $[\mathcal{K}''_{\mathbf{C}}]$ be the vector space over **C** of the continuous mappings of E' $[E'']$ into **C** with compact support. Let $\mathcal{K}'$ and $\mathcal{K}''$ be similarly defined with **R** replacing **C**. For each pair $(u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$ let $\Lambda\ (u, v)$ be a complex number. For fixed u the mapping $v \to \Lambda(u, v)$ is denoted by $\Lambda\ (u, \cdot)$. The definition of $\Lambda(\cdot, v)$ is similar. With this understood the definition of a **C**-bimeasure is as follows.

The mapping Λ of $\mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$ into **C** *is termed a* **C**-*bimeasure on $E' \times E''$ if $\Lambda(\cdot, v)$ and $\Lambda(u, \cdot)$ are* **C**-*measures on E' and E'', respectively, for fixed $v \in \mathcal{K}''_{\mathbf{C}}$ and $u \in \mathcal{K}'_{\mathbf{C}}$.*

Let K' $[K'']$ be an arbitrary compact subset of E' $[E'']$, and let $\mathcal{K}'_{K'}$ $[\mathcal{K}''_{K''}]$ be the vector subspace of $\mathcal{K}'_{\mathbf{C}}$ $[\mathcal{K}''_{\mathbf{C}}]$ of mappings $u[v]$ with support in K' $[K'']$. Let

$$U(u) = \max_{s \in E'} | u(s) | \qquad U(v) = \max_{t \in E''} | v(t) | . \tag{1.1}$$

A necessary and sufficient condition that a bilinear mapping Λ of $\mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$ into **C** be a **C**-bimeasure is that for each choice of the compact subsets $K' \subset E'$ and $K'' \subset E''$ there exists a constant $M(K', K'')$ such that

$$| \Lambda(u, v) | \leqq M(K', K'')\ U(u)\ U(v) \tag{1.2}$$

for each $(u, v) \in \mathcal{K}'_{K'} \times \mathcal{K}''_{K''}$. [See Ref. 6, §2.]

Bimeasures. If the field **C** is replaced by the field **R** of real numbers, the definition of a **C**-measure reduces to the definition of an **R**-measure or simply a "measure" in the sense of Bourbaki, Ref. 3. In our terminology the definition of a **C**-bimeasure $\Lambda_{\mathbf{R}}$ similarly reduces to the definition of a bimeasure when **R** replaces **C**.

Absolute measures. It should be pointed out that the theory of **C** or **R**-bimeasures is not a theory of **C** or **R**-measures on $E' \times E''$. In this respect bimeasure theory differs from product measure theory. [Cf. Ref. 3, p. 89.] The most striking indication of this is found [Ref. 6, §10] in the fact that a bimeasure $\Lambda_{\mathbf{R}}$ on $E' \times E''$ does not in general admit a decomposition of the form

$$\Lambda_{\mathbf{R}} = F_1 - F_2 \tag{1.3}$$

where F_1 and F_2 are two positive bimeasures on $E' \times E''$. Recall that with each measure μ on E there is associated a unique decomposition $\mu = \mu^+ - \mu^-$ and an absolute measure $| \mu | = \mu^+ + \mu^-$. [See Ref. 3, p. 54.] Since (1.3) is not in general valid one cannot use a relation $| \Lambda_{\mathbf{R}} | = F_2 + F_2$ to define an absolute bimeasure.

It might seem that one could extend the formula

$$|\mu|(f) = \sup_{|u| \leq f} |\mu(u)| \tag{1.4}$$

where (in the notation of Bourbaki) u and f are in $\mathcal{K}(E)$ and $f \geqq 0$. If $\mathcal{K}'_+$ and $\mathcal{K}''_+$ denote the subspaces of $\mathcal{K}'$ and $\mathcal{K}''$ of real positive mappings, one can in fact introduce the number

$$|\Lambda_{\mathbf{R}}|(f, g) = \sup_{|u| \epsilon f, |v| \epsilon g} |\Lambda_{\mathbf{R}}(u, v)| \tag{1.5}$$

for

$$(u, v) \in \mathcal{K}' \times \mathcal{K}'' \qquad (f, g) \in \mathcal{K}'_+ \times \mathcal{K}''_+. \tag{1.6}$$

With $|\Lambda_{\mathbf{R}}|$ so defined $|\Lambda_{\mathbf{R}}|(\cdot, g)$ and $|\Lambda_{\mathbf{R}}|(f, \cdot)$, unlike $|\mu|$, are not in general additive (cf. Ref. 6, §11), so that $|\Lambda_{\mathbf{R}}|$ does not in general admit an extension over $\mathcal{K}' \times \mathcal{K}''$ which is a bimeasure. This means that the theory of the integral extensions of a bimeasure or **C**-bimeasure must depart considerably from the methods used in defining and characterizing integral extensions of a measure.

Let $\mathbf{C}^{E'}$ $[\mathbf{C}^{E''}]$ denote the vector space of mappings of E' $[E'']$ into **C**. Given a pair of mappings

$$(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''} \tag{1.7}$$

and a **C**-bimeasure Λ, our problem is to extend the definition of Λ so as to obtain a unique and natural definition of $\Lambda(x, y)$ as a limit (when possible) of values of Λ on $\mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$. Unlike the corresponding problem for measures this problem involves at least four steps, and these steps may be ordered in different ways.

The Λ-integral of (x, y). Set the given **C**-measure $\Lambda(u, \cdot) = \alpha'_u$. Similarly set $\Lambda(\cdot, v) = \alpha''_v$. The integral $\alpha'_u(y)$ may exist for each $u \in \mathcal{K}'_{\mathbf{C}}$ and the mapping $u \to \alpha'_u(y)$ define a **C**-measure β'_y on E', or this may fail to happen. If the **C**-measure β'_y exists, the integral $\beta'_y(x)$ may or may not exist. The existence of the integral $\alpha''_v(x)$, the **C**-measure $\beta''_x : v \to \alpha''_v(x)$, and the integral $\beta''_x(y)$, are similarly in question. Finally if $\beta'_y(x)$ and $\beta''_x(y)$ exist one may or may not have the equality

$$\beta'_y(x) = \beta''_x(y). \tag{1.8}$$

In case (1.8) is defined and is valid, we set $\Lambda(x, y)$ equal to the common value of the two members of (1.8), call $\Lambda(x, y)$ the Λ-integral of (x, y), and term (x, y) Λ-integrable.

The superior integral Λ^*. In Ref. 6 we have associated a "superior integral" $\Lambda^*(h, k)$ with Λ for each pair (h, k) of positive numerical functions (finite or not), defined on E' and E'' respectively. The functions $\Lambda^*(\cdot, k)$ and $\Lambda^*(h, \cdot)$ are monotone and convex for fixed k and h, respectively. Moreover Λ^* satisfies two fundamental limit theorems established in Ref. 6.

A typical theorem using Λ^* is as follows. If the integrals $\beta'_y(x)$ and $\beta''_x(y)$ exist in (1.8), then (1.8) holds whenever $\Lambda^*(|x|, |y|) < \infty$. This is an analogue of the Fubini theorem for double integrals.

We shall obtain conditions sufficient, and in certain cases necessary, that

β'_y and β''_x exist as **C**-measures and that (x, y) be Λ-integrable. The most satisfactory conditions are in terms of Λ^*.

In this paper a **C**-bimeasure Λ is given and conditions are sought that a pair of mappings (x, y) be Λ-integrable. In a sequel to this paper on "Bilinear functionals on products $A \times B$ of function spaces with duals of integral type" it is the spaces A and B which are given. **C**-bimeasures Λ are sought such that for *each* $(x, y) \in A \times B$, (x, y) is Λ-integrable. The latter study leads to the representation of bounded operators from A to B' the dual of A, or from B to A'.

§2. Λ-integrability

We shall refer to

$$(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}, \qquad (u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}. \tag{2.1}$$

Let Λ be a **C**-bimeasure. By hypothesis $\Lambda(u, \cdot)$ is a **C**-measure on E'' and $\Lambda(\cdot, v)$ a **C**-measure on E'.

DEFINITION (i). *If* (a), *the integral* $\Lambda(\cdot, v)(x)$ *exists for each* $v \in \mathcal{K}''_{\mathbf{C}}$ *and if*

(*b*) *the mapping* $v \longrightarrow \Lambda(\cdot, v)(x)$ *is a* **C**-*measure on* E'' *this measure will be denoted by* $\Lambda(x, \cdot)$ *so that*

$$\Lambda(\cdot, v)(x) = \Lambda(x, \cdot)(v). \tag{2.2}$$

The **C**-*measure* $\Lambda(\cdot, y)$ *is similarly defined.*

DEFINITION (ii). *We shall say that* (x, y) *is* Λ-*integrable if the following three conditions are fulfilled.*

I. *The measures* $\Lambda(x, \cdot)$ *and* $\Lambda(\cdot, y)$ *exist.*

II. *The integrals* $\Lambda(x, \cdot)(y)$ *and* $\Lambda(\cdot, y)(x)$ *exist and*

$$\Lambda(x, \cdot)(y) = \Lambda(\cdot, y)(x). \tag{2.3}$$

DEFINITION (iii). *In case* (x, y) *is* Λ-*integrable we set*

$$\Lambda(x, y) = \Lambda(x, \cdot)(y) = \Lambda(\cdot, y)(x). \tag{2.4}$$

The algebra of Λ-integrals depends upon the following lemma and theorems.

LEMMA 2.1. (α) *If* $\Lambda(\cdot, y_1)$ *and* $\Lambda(\cdot, y_2)$ *exist as* **C**-*measures on* E', *then* $\Lambda(\cdot, y_1 + y_2)$ *is a* **C**-*measure on* E' *and*

$$\Lambda(\cdot, y_1 + y_2) = \Lambda(\cdot, y_1) + \Lambda(\cdot, y_2). \tag{2.5}$$

(β) *If* $\Lambda(x_1, \cdot)$ *and* $\Lambda(x_2, \cdot)$ *exist as* **C**-*measures on* E'', *then* $\Lambda(x_1 + x_2, \cdot)$ *is a* **C**-*measure on* E'' *and*

$$\Lambda(x_1 + x_2, \cdot) = \Lambda(x_1, \cdot) + \Lambda(x_2, \cdot). \tag{2.6}$$

We shall establish (β). For $v \in \mathcal{K}''_{\mathbf{C}}$, $\Lambda(x_1, \cdot)$ and $\Lambda(x_2, \cdot)$ exist as **C**-measures on E'' by hypothesis, so that

$$\Lambda(x_i, \cdot)(v) = \Lambda(\cdot, v)(x_i) \qquad (i = 1, 2) \tag{2.7}$$

in accordance with (2.2). For the **C**-measure $\Lambda(\cdot, v)$

$$\Lambda(\cdot, v)(x_1 + x_2) = \Lambda(\cdot, v)(x_1) + \Lambda(\cdot, v)(x_2). \tag{2.8}$$

The **C**-measures $\Lambda(x_1, \cdot)$ and $\Lambda(x_2, \cdot)$ have a **C**-measure α as a sum and

$$\alpha(v) = \Lambda(x_1, \cdot)(v) + \Lambda(x_2, \cdot)(v).$$

By (2.7) and (2.8) $\alpha(v) = \Lambda(\cdot, v)(x_1 + x_2)$. With $v \to \alpha(v)$, $v \to \Lambda(\cdot, v)(x_1 + x_2)$ is a **C**-measure. In accordance with Definition (i) its value at v is denoted by $\Lambda(x_1 + x_2, \cdot)(v)$. Reference to the preceding equations shows that

$$\Lambda(x_1 + x_2, \cdot)(v) = \Lambda(x_1, \cdot)(v) + \Lambda(x_2, \cdot)(v)$$

so that (2.6) holds.

The proof of (α) is similar.

THEOREM 2.1. (α) *If* (x, y_1) *and* (x, y_2) *are* Λ*-integrable, then* $(x, y_1 + y_2)$ *is* Λ*-integrable and*

$$\Lambda(x, y_1 + y_2) = \Lambda(x, y_1) + \Lambda(x, y_2). \tag{2.9}$$

(β) *Similarly if* (x_1, y) *and* (x_2, y) *are* Λ*-integrable then* $(x_1 + x_2, y)$ *is* Λ*-integrable and*

$$\Lambda(x_1 + x_2, y) = \Lambda(x_1, y) + \Lambda(x_2, y). \tag{2.10}$$

We shall prove (β). To this end we verify the conditions of Definitions (i) and (ii) as follows.

(i) (a) The integral $\Lambda(\cdot, v)(x_1 + x_2)$ exists since $\Lambda(\cdot, v)(x_1)$ and $\Lambda(\cdot, v)(x_2)$ exist by hypothesis. The integral $\Lambda(u, \cdot)(y)$ exists by hypothesis.

(i) (b) The mapping $v \to \Lambda(\cdot, v)(x_1 + x_2)$ is a **C**-measure on E'' by virtue of Lemma 2.1(β). The mapping $u \to \Lambda(u, \cdot)(y)$ is a **C**-measure on E' by hypothesis.

(ii) I. The measures $\Lambda(x_1 + x_2, \cdot)$ and $\Lambda(\cdot, y)$ exist as stated above.

(ii) II. The integral $\Lambda(x_1 + x_2, \cdot)(y)$ exists, since (2.6) holds and the integrals $\Lambda(x_1, \cdot)(y)$ and $\Lambda(x_2, \cdot)(y)$ exist. The integral $\Lambda(\cdot, y)(x_1 + x_2)$ exists since the integrals $\Lambda(\cdot, y)(x_1)$ and $\Lambda(\cdot, y)(x_2)$ exist by hypothesis.

(ii) III. $$\Lambda(x_1 + x_2, \cdot)(y) = \Lambda(\cdot, y)(x_1 + x_2),$$

since

$$\Lambda(x_1, \cdot)(y) = \Lambda(\cdot, y)(x_1) \qquad \Lambda(x_2, \cdot)(y) = \Lambda(\cdot, y)(x_2) \tag{2.11}$$

by hypothesis, while

$$\begin{aligned} \Lambda(x_1 + x_2, \cdot)(y) &= \Lambda(x_1, \cdot)(y) + \Lambda(x_2, \cdot)(y) && \text{(by 2.6)} \\ &= \Lambda(\cdot, y)(x_1) + \Lambda(\cdot, y)(x_2) && \text{by (2.11)} \\ &= \Lambda(\cdot, y)(x_1 + x_2). \end{aligned} \tag{2.12}$$

Thus $(x_1 + x_2, y)$ is Λ-integrable. That (2.10) holds follows from (2.12) and Definition (iii).

The proof of (α) is similar.

For completeness we add the following.

THEOREM 2.2. *If* (x, y) *is* Λ*-integrable and if* a *and* b *are complex constants then* (ax, by) *is* Λ*-integrable and* $\Lambda(ax, by) = ab\Lambda(x, y)$.

Partial maps into $\mathbf{C}$. Let $E_1'[E_1'']$ be a subset of $E'[E'']$. Let $z[w]$ map $E_1'[E_1'']$ into $\mathbf{C}$. For $s \in E' - E_1'[t \in E'' - E_1'']$ suppose $z[w]$ either undefined or such that $\mathcal{R}z(s)$ and $\mathcal{I}z(s)[\mathcal{R}w(t)$ and $\mathcal{I}w(t)]$ are numerical values in $\bar{\mathbf{R}}$, possibly infinite. We term $z[w]$ a *partial map* of $E'[E'']$ into $\mathbf{C}$.

The definitions (i), (ii), (iii) will be understood as extended to the case in which (x, y) is replaced by the pair (z, w). A $\mathbf{C}$-measure $\Lambda(z, \cdot)$, $[\Lambda(\cdot, w)]$ may then exist in the sense of this extended definition; the integrals $\Lambda(z, \cdot)(w)$ and $\Lambda(\cdot, w)(z)$ may also exist. Finally (z, w) is termed Λ-*integrable* if these integrals exist and if

$$\Lambda(z, \cdot)(w) = \Lambda(\cdot, w)(z). \tag{2.13}$$

If (2.13) holds one introduces the Λ-*integral* $\Lambda(z, w) = \Lambda(z, \cdot)(w) = \Lambda(\cdot, w)(z)$.

Lemma 2.1 and Theorems 2.1 and 2.2 then hold with (x, y) replaced by (z, w).

Λ-*negligible* sets. A subset E_0' of $E'[E_0''$ of $E'']$ will be said to be Λ-negligible if $\Lambda(\cdot, v)$-negligible $[\Lambda(u, \cdot)$-negligible] in the sense of Ref. 5, §5 for each $v \in K_{\mathbf{C}}''[u \in K_{\mathbf{C}}']$. If $E_0'[E_0'']$ is Λ-negligible and if the above mapping $z[w]$ restricted to $E' - E_0'[E'' - E_0'']$ maps $E' - E_0'[E'' - E_0'']$ into $\mathbf{C}$, then $z[w]$ will be said to be a mapping $z \in \mathbf{C}^{E'}[w \in \mathbf{C}^{E''}](p, p, \Lambda)$ and finite (p, p, Λ).

Two mappings z' and z'' in $\mathbf{C}^{E'}(p, p, \Lambda)$ will be said to be *equal* (p, p, Λ) if $z'(s) = z''(s)$ except on a Λ-negligible subset of E'. The equality $w'(t) = w''(t)(p, p, \Lambda)$ of two mappings in $\mathbf{C}^{E''}(p, p, \Lambda)$ is similarly defined. Given a partial map z of $E'[w$ of $E'']$ into $\mathbf{C}$ let $\tilde{z} \in \mathbf{C}^{E'}[\tilde{w} \in \mathbf{C}^{E''}]$ be mappings such that $\tilde{z}(s) = z(s)[\tilde{w}(t) = w(t)]$ whenever $z(s) \in \mathbf{C}[w(t) \in \mathbf{C}]$ and such that $\tilde{z}(s) = 0$ $[\tilde{w}(t) = 0]$ for other points $s \in E'[t \in E'']$. We term $\tilde{z}[\tilde{w}]$ the *finite projection* into $\mathbf{C}^{E'}[\mathbf{C}^{E''}]$ of $z[w]$.

If z is a partial map of E' into $\mathbf{C}$ such that $\Lambda(z, \cdot)$ is a $\mathbf{C}$-measure on E' then z is in $\mathbf{C}^{E'}(p, p, \Lambda)$. If z' and z'' are two partial maps of E' into $\mathbf{C}$ which are equal $[p, p, \Lambda]$ then $\Lambda(z', \cdot)$ is a $\mathbf{C}$-measure on E'' if and only if $\Lambda(z'', \cdot)$ is a $\mathbf{C}$-measure on E'' and

$$\Lambda(z', \cdot) = \Lambda(z'', \cdot) = \Lambda(\tilde{z}', \cdot) = \Lambda(\tilde{z}'', \cdot).$$

§3. Approximating and majorizing the integrals $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$

Let (x, y) be given as in (2.1). With the aid of Theorem 8.1 of Ref. 5, the basic approximation Theorem 3.1 can be established.

Notation. We shall refer to the positive real axis $\mathbf{R}_+$ and to its completion $\bar{\mathbf{R}}_+$ by the point $+\infty$. Let $\mathcal{I}_+'[\mathcal{I}_+'']$ be the space of lower semicontinuous mappings of $E'[E'']$ into $\bar{\mathbf{R}}_+$. In addition to the pairs of mappings (x, y) and (u, v) limited as in (2.1) we shall make permanent use of

$$(h, k) \in \bar{\mathbf{R}}_+^{E'} \times \bar{\mathbf{R}}_+^{E''}, \qquad (p, q) \in \mathcal{I}_+' \times \mathcal{I}_+'', \qquad (f, g) \in \mathcal{K}_+' \times \mathcal{K}_+''. \tag{3.0}$$

The filtering sets ${}_-\phi_p'$, ${}_-\phi_q''$, ${}_+\phi_h'$, ${}_+\phi_k''$. In Ref. 6 we have introduced filtering sets of mappings as follows. [Cf. Ref. 2, p. 35.] For $p \in \mathcal{I}_+'$ let ${}_-\phi_p'$ be the ensemble of mappings $f \in \mathcal{K}_+'$ with $f \leqq p$, filtering for the relation $\geqq$. For $q \in \mathcal{I}_+''$ let ${}_-\phi_q''$ be similarly composed of $g \in \mathcal{K}''_+$. For an arbitrary mapping h of E' into $\bar{\mathbf{R}}_+$ let ${}_+\phi'_h$ be the ensemble of mappings $p \in \mathcal{I}'_+$ with $p \geqq h$, filtering for the relation $\leqq$. For an arbitrary mapping k of E'' into $\bar{\mathbf{R}}_+$ let ${}_+\phi''_k$ be similarly composed of $q \in \mathcal{I}''_+$.

We assume throughout this paper that Λ is a **C**-bimeasure.

THEOREM 3.1. *If the integral* $\Lambda(\cdot,\ y)(x)[\Lambda(x,\ \cdot)(y)]$ *exists, then for any* $(p, q) \in {}_+\phi'_{|x|} \times {}_+\phi''_{|y|}$ *and for arbitrary* $\delta > 0$ *there exists* $(u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$ *with* $|u| \leqq p$, $|v| \leqq q$ *such that*

$$(3.1) \qquad |\Lambda(u, v) - \Lambda(\cdot, y)(x)| < \delta\ [\,|\Lambda(u, v) - \Lambda(x, \cdot)(y)| < \delta].$$

Set $\Lambda(\cdot, y) = \alpha$. In our definition in §2 the existence of the integral $\Lambda(\cdot, y)(x)$ presupposes that α is a **C**-measure, and that x is α-integrable. By Theorem 8.1 of Ref. 5 there accordingly exists a $u \in \mathcal{K}'_{\mathbf{C}}$ with $|u| \leqq p$ such that $|\alpha(u) - \alpha(x)| < \frac{1}{2}\delta$, or in terms of Λ

$$(3.2) \qquad |\Lambda(\cdot, y)(u) - \Lambda(\cdot, y)(x)| < \tfrac{1}{2}\delta.$$

By definition of $\Lambda(\cdot, y)$, $\Lambda(u, \cdot)(y) = \Lambda(\cdot, y)(u)$. We now set $\Lambda(u, \cdot) = \beta$ and apply Theorem 8.1 of Ref. 5 to $\beta(y)$, inferring thereby that there exists $v \in \mathcal{K}''_{\mathbf{C}}$ such that $|v| \leqq q$ and

$$(3.3) \qquad |\Lambda(u, \cdot)(v) - \Lambda(u, \cdot)(y)| < \tfrac{1}{2}\delta.$$

Since $\Lambda(u, \cdot)(v) = \Lambda(u, v)$ we conclude from (3.3) and (3.2) that the first inequality in (3.1) is valid.

The proof of the second inequality is similar.

DEFINITION OF Λ^*. We have defined Λ^* in Ref. 6 by setting

$$(3.4) \qquad \Lambda^*(p, q) = \sup_{|u| \leqq p,\ |v| \leqq q} |\Lambda(u, v)|$$

$$(3.5) \qquad \Lambda^*(h, k) = \inf_{p \geqq h,\ q \geqq k} \Lambda^*(p, q)$$

where the notation of (3.0) is employed.

In Ref. 6 we have shown that $\Lambda^*(h, \cdot)$ is monotone and convex on the space of positive numerical functions $k \in \bar{\mathbf{R}}''_+$. Similarly $\Lambda^*(\cdot, k)$ is monotone and convex.

The approximation Theorem 3.1 leads to the following "majorizing" theorem.

THEOREM 3.2. *If the integral* $\Lambda(x, \cdot)(y)$ *exists*

$$(3.6) \qquad |\Lambda(x, \cdot)(y)| \leqq \Lambda^*(|x|, |y|).$$

If the integral $\Lambda(\cdot, y)(x)$ *exists*

$$(3.7) \qquad |\Lambda(\cdot, y)(x)| \leqq \Lambda^*(|x|, |y|).$$

It will be sufficient to establish (3.6) when $\Lambda^*(|x|, |y|) < \infty$. Let (u, v) and (p, q) be chosen as in Theorem 3.1 with $|u| \leqq p$, $|v| \leqq q$. Then $|\Lambda(u, v)| \leqq \Lambda^*(p, q)$ in accordance with the definition of $\Lambda^*(p, q)$. Because of (3.1)

$$|\Lambda(x, \cdot)(y)| \leqq |\Lambda(u, v)| + \delta \leqq \Lambda^*(p, q) + \delta.$$

Since δ is arbitrary $| \Lambda(x, \cdot)(y) | \leqq \Lambda^*(p, q)$. Hence

$$| \Lambda(x, \cdot)(y)| \leqq \inf_{p \geqq |x|,\, q \geqq |y|} \Lambda^*(p, q) = \Lambda^*(| x |, | y |)$$

establishing (3.6). The proof of (3.7) is similar.

Theorems 3.1 and 3.2 can be extended as follows.

THEOREM 3.3. *Let z and w be partial maps of E' and E'' respectively into* $\mathbf{C}$ *such that the integral $\Lambda(\cdot, w)(z)[\Lambda(z, \cdot)(w)]$ exists. Then for $| \tilde{z} | = h$ and $| \tilde{w} | = | k |$, for any $(p, q) \in {}_+\phi_h' \times {}_+\phi_k''$, and for arbitrary $\delta > 0$, there exists $(u, v) \in \mathcal{K}_{\mathbf{C}}' \times \mathcal{K}_{\mathbf{C}}''$ with $| u | \leqq p$, $| v | \leqq q$ such that*

$$(3.8) \qquad | \Lambda(u, v) - \Lambda(\cdot, w)(z) | < \delta \qquad [| \Lambda(u, v) - \Lambda(z, \cdot)(w) | < \delta].$$

If the integral $\Lambda(\cdot, w)(z)$ exists the integral $\Lambda(\cdot, \tilde{w})(\tilde{z})$ exists and equals $\Lambda(\cdot, w)(z)$. Similarly the existence of $\Lambda(z, \cdot)(w)$ implies that $\Lambda(\tilde{z}, \cdot)(\tilde{w})$ exists and equals $\Lambda(z, \cdot)(w)$. Theorem 3.1 applies to the pair $(\tilde{z}, \tilde{w})$ and Theorem 3.3 is inferred.

THEOREM 3.4. *If the integral $\Lambda(z, \cdot)(w)$ exists*

$$(3.9) \qquad | \Lambda(z, \cdot)(w) | \leqq \Lambda^*(| \tilde{z} |, | \tilde{w} |).$$

If the integral $\Lambda(\cdot, w)(z)$ exists

$$(3.10) \qquad | \Lambda(\cdot, w)(z) | \leqq \Lambda^*(| \tilde{z} |, | \tilde{w} |).$$

The proof of Theorem 3.4 differs from that of Theorem 3.2 only in that the choice of (u, v) and (p, q) in the present proof are in accordance with Theorem 3.3 rather than Theorem 3.1.

§4. Λ-integrability by the method of decomposition

Given a mapping $x \in \mathbf{C}^E$, the mappings $\mathcal{R}x$ and $\mathcal{I}x$ are well defined, as well as their positive and negative components,

$$(4.0) \qquad [\mathcal{R}x]^+, \qquad [\mathcal{R}x]^-, \qquad [\mathcal{I}x]^+, \qquad [\mathcal{I}x]^-.$$

The respective mappings (x_1, x_2, x_3, x_4) appearing in (4.0) will be termed the *Riesz components* of x. The Riesz components (y_1, y_2, y_3, y_4) of a mapping $y \in \mathbf{C}^{E''}$ are similarly defined.

We shall seek conditions for the Λ-integrability of a pair of mappings $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$. Let

$$(4.1) \qquad (x_1, x_2, x_3, x_4) \qquad (y_1, y_2, y_3, y_4)$$

be respectively the Riesz components of x and y. It would seem a priori likely that conditions for the Λ-integrability of (x, y) could be obtained by the *method of decomposition*, that is by a study of the Λ-integrability of the pairs (x_i, y_j) $i, j = 1, \cdots, 4$. This method is useful for some purposes. It is limited by the following fact. The hypothesis that the integral $\Lambda(x, \cdot)(y)[\Lambda(\cdot, y)(x)]$ exists does not imply that the respective integrals $\Lambda(x_i, \cdot)(y_j)[\Lambda(\cdot, y_j)(x_i)]$ exist, as we shall show by counter example. Thus the fundamental theorem that the existence of $\Lambda(x, \cdot)(y)$

and $\Lambda(\cdot, y)(x)$ and the finiteness of $\Lambda^*(|x|, |y|)$ implies the Λ-integrability of (x, y) does not appear capable of proof by the direct method of decomposition.

In this section we shall establish a lemma and theorem on Λ-integrability, using the method of decomposition. In the next section we present the basic limit theorems on Λ^* in vector form using methods which are apparently more powerful than the method of decomposition.

The first "decomposition" lemma follows.

LEMMA 4.1. *Let the Riesz components of the mappings x and y be given as in* (4.1).

(i) *If each $\Lambda(x_i, \cdot)$ is a* **C**-*measure on E'', then $\Lambda(x, \cdot)$ is a* **C**-*measure on E'' and*

$$\Lambda(x, \cdot) = \Lambda(x_1, \cdot) - \Lambda(x_2, \cdot) + i\Lambda(x_3, \cdot) - i\Lambda(x_4, \cdot). \tag{4.2}$$

(ii) *If each $\Lambda(\cdot, y_i)$ is a* **C**-*measure on E', $\Lambda(\cdot, y)$ is a* **C**-*measure on E' and*

$$\Lambda(\cdot, y) = \Lambda(\cdot, y_1) - \Lambda(\cdot, y_2) + i\Lambda(\cdot, y_3) - i\Lambda(\cdot, y_4). \tag{4.3}$$

This lemma follows at once from Lemma 2.1.

We are led to the following theorem.

THEOREM 4.1. *If the Riesz components of mappings $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$ are given in* (4.1) *and if each pair (x_i, y_j), $i, j = 1, \cdots, 4$, is Λ-integrable, then (x, y) is Λ-integrable.*

If each pair (x_i, y_j) is Λ-integrable it follows from Lemma 4.1 that the measures $\Lambda(x, \cdot)$ and $\Lambda(\cdot, y)$ exist and are given by (4.2) and (4.3) respectively. By hypothesis the integrals

$$\Lambda(x_i, \cdot)(y_j) = \Lambda(\cdot, y_j)(x_i) \qquad (i, j = 1, \cdots, 4) \tag{4.4}$$

exist and are equal as indicated. With the aid of (4.2) one sees that $\Lambda(x, \cdot)(y)$ exists and is representable in the form

$$\Lambda(x, \cdot)(y) = \textstyle\sum_{ij} a_{ij}\Lambda(x_i, \cdot)(y_j) \qquad (i, j = 1, \cdots, 4) \tag{4.5}$$

where a_{ij} is in **C** and $a_{ij} = a_{ji}$. With the aid of (4.3) one similarly sees that the integral $\Lambda(\cdot, y)(x)$ exists and that

$$\Lambda(\cdot, y)(x) = \textstyle\sum_{ij} a_{ij}\Lambda(\cdot, y_i)(x_j) \qquad (i, j = 1, \cdots, 4). \tag{4.6}$$

It follows then from (4.4), (4.5) and (4.6) and the relation $a_{ij} = a_{ji}$ that $\Lambda(x, \cdot)(y) = \Lambda(\cdot, y)(x)$.

Counter example. This example will show that the existence of the integral $\Lambda(x, \cdot)(y)$ does not in general imply the existence of each integral $\Lambda(x_i, \cdot)(y_j)$ even when $\Lambda^*(|x|, |y|) < \infty$.

Let E' be the pair of points $\{1, 2\}$ with discrete topology and E'' the interval $[0, 1]$. Let μ denote Lebesgue measure on $[0, 1]$, and let k be a positive function on $[0, 1]$ which is not Lebesgue integrable. Suppose further that $k(s)$ is bounded by a constant B for $s \in E''$. For $(u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$ let a **C**-bimeasure Λ on $E' \times E''$ be defined by its values

$$\Lambda(u, v) = [u(1) + u(2)]\mu_e(v), \tag{4.7}$$

where $\mu_e(v) = \mu(\mathcal{R}v) + i\mu(\mathcal{I}v)$, and defines a $\mathbf{C}$-measure μ_e on E'' which extends the measure μ. Let $u_0 \in \mathcal{K}'_{\mathbf{C}}$ have the values $u_0(1) = 1$, $u_0(2) = -1$. Then $u_0^+(1) = 1$, $u_0^+(2) = 0$, $u_0^-(1) = 0$, $u_0^-(2) = 1$ so that $| u_0 | (1) = 1$ and $| u_0 | (2) = 1$. Using the definition of Λ^* we see that with $i = 1, 2$

$$(4.8)' \qquad \Lambda^*(| u_0 |, q) = \sup_{|u(i)| \leq 1} | u(1) + u(2) | \, | \mu |^*(q) = 2 | \mu |^*(q)$$

$$(4.8)'' \qquad \Lambda^*(| u_0 |, k) = 2 \inf_{q \geqq k} \mu^*(q) \leqq 2B$$

noting that an admissible choice of q in (4.8)″ is obtained by setting $q(t) = B$ for $t \in E''$.

Since $u_0(1) + u_0(2) = 0$, it follows from (4.7) that the $\mathbf{C}$-measure $\Lambda(u_0, \cdot) \equiv 0$. The integral $\Lambda(u_0, \cdot)(k)$ accordingly exists. Moreover (4.7) implies that the $\mathbf{C}$-measure $\Lambda(u_0^+, \cdot) = \mu_e$. Since k is not Lebesgue integrable the integral $\mu(k)$ does not exist. Thus $\Lambda(u_0^+, \cdot)(k)$ does not exist even though $\Lambda^*(| u_0 |, k) < \infty$ in accordance with (4.8) and the integral $\Lambda(u_0, \cdot)(k)$ exists.

Because of the limitations of the method of decomposition in the study of Λ-integrability we turn in the next section to new vectorial methods.

§5. The superior integral Λ^*

With the aid of the first and second limit theorems for Λ^* as established in Ref. 6, and as here extended in vector form, and with the aid of the majorizing Theorem 3.2, conditions sufficient for Λ-integrability can be established. The extent to which these conditions are necessary will also be studied.

Use will be made of the pairs of positive mappings (h, k), (p, q) and (f, g), in (3.0).

The vectors $\mathbf{z}$ *and finite images* $\dot{\mathbf{z}}$. Functions h, k, p, q, f, g are numerical valued and positive. They are respectively limited to six well-defined function spaces. Let Z be any one of these six spaces. Let the vector $\mathbf{z}$ represent a point (z_1, z_2, z_3, z_4) in the product space

$$(5.1) \qquad Z \times Z \times Z \times Z = [Z]^4.$$

Referring to the definition of $\tilde{z}_i$ in §2 set

$$(5.2) \qquad \tilde{z}_1 - \tilde{z}_2 + i(\tilde{z}_3 - \tilde{z}_4) = \dot{\mathbf{z}}.$$

Note that this *finite image* $\dot{\mathbf{z}}$ of $\mathbf{z}$ is well-defined while $z_1 - z_2$ is not defined if $z_1(s) = z_2(s) = +\infty$ for some s. Similarly $z_3 - z_4$ might not be defined.

Vector pairs

$$(5.3) \quad (\mathbf{h}, \mathbf{k}) \in [\bar{\mathbf{R}}_+^{E'}]^4 \times [\bar{\mathbf{R}}_+^{E''}]^4; \qquad (\mathbf{p}, \mathbf{q}) \in [\mathcal{I}'_+]^4 \times [\mathcal{I}''_+]^4; \qquad (\mathbf{f}, \mathbf{g}) \in [\mathcal{K}'_+]^4 \times [\mathcal{K}''_+]^4$$

have well-defined finite images $(\dot{\mathbf{h}}, \dot{\mathbf{k}})$, $(\dot{\mathbf{p}}, \dot{\mathbf{q}})$, $(\dot{\mathbf{f}}, \dot{\mathbf{g}})$ in $\mathbf{C}^{E'} \times \mathbf{C}^{E''}$. Given $x \in \mathbf{C}^{E'}$ [$y \in \mathbf{C}^{E''}$], we say that a vector $\mathbf{x}[\mathbf{y}]$ *canonically represents* $x[y]$ if the components of $\mathbf{x}[\mathbf{y}]$ are the Riesz components of x [of y].

We shall refer to *product filtering sets*

$$(5.4) \qquad \Phi'(\mathbf{p}),\ {}_-\Phi''(\mathbf{q}),\ {}_+\Phi'(\mathbf{h}),\ {}_+\Phi''(\mathbf{k}),$$

defined as follows. The product of a finite number of filtering sets has been defined in §7 of Ref. 5. Given $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{h}, \mathbf{k})$ as in (5.3) we introduce the product filtering sets,

$$\prod_{i=1}^4 {}_-\phi'_{p_i} = {}_-\Phi'(\mathbf{p}), \qquad \prod_{i=1}^4 {}_-\phi''_{q_i} = {}_-\Phi''(\mathbf{q}) \tag{5.5}$$

in the product spaces $[\mathcal{K}'_+]^4$ and $[\mathcal{K}''_+]^4$ respectively, as well as the product filtering sets,

$$\prod_{i=1}^4 {}_+\phi'_{h_i} = {}_+\Phi'(\mathbf{h}), \qquad \prod_{i=1}^4 {}_+\phi''_{k_i} = {}_+\Phi''(\mathbf{k}). \tag{5.6}$$

Following notational conventions introduced in §7 of Ref. 5, the limits with respect to the filters of sections of the respective filtering sets (5.4) will be indicated by symbols

$$\lim_{\mathbf{f}\uparrow\mathbf{p}}, \qquad \lim_{\mathbf{g}\uparrow\mathbf{q}}, \qquad \lim_{\mathbf{p}\downarrow\mathbf{h}}, \qquad \lim_{\mathbf{q}\downarrow\mathbf{k}} \tag{5.7}$$

placed before the function whose limit is desired. Given one of the above vectors $\mathbf{z}$ let

$$|\mathbf{z}| = z_1 + z_2 + z_3 + z_4 .$$

With this understood the first limit theorem on Λ^* in Ref. 6, §6 has the following extension.

THEOREM 5.1. *If* $\Lambda^*(|\mathbf{p}|, |\mathbf{q}|) < \infty$ *then*

$$\lim_{\mathbf{f}\uparrow\mathbf{p}} \Lambda^*(|\dot{\mathbf{p}} - \dot{\mathbf{f}}|, |\mathbf{q}|) = 0; \qquad \lim_{\mathbf{g}\uparrow\mathbf{q}} \Lambda^*(|\mathbf{p}|, |\dot{\mathbf{q}} - \dot{\mathbf{g}}| = 0. \tag{5.8}$$

Note first that the hypothesis $\Lambda^*(|\mathbf{p}|, |\mathbf{q}|) < \infty$ implies that

$$\Lambda^*(p_i, |\mathbf{q}|) < \infty \qquad (i = 1, 2, 3, 4). \tag{5.9}$$

To establish the first relation in (5.8) observe further that the convexity of $\Lambda^*(\cdot, |\mathbf{q}|)$ and the relation

$$|\dot{\mathbf{p}} - \dot{\mathbf{f}}| \leqq \sum_{i=1}^4 |p_i - f_i| \tag{5.10}$$

together imply that

$$\Lambda^*(|\dot{\mathbf{p}} - \dot{\mathbf{f}}|, |\mathbf{q}|) \leqq \sum_{i=1}^4 \Lambda^*(p_i - f_i, |\mathbf{q}|).$$

Moreover $|\mathbf{q}|$ is clearly in $\mathcal{I}''_+$ with the mappings q_i, $i = 1, 2, 3, 4$. It follows from Theorem 6.1 of Ref. 6. that

$$\lim_{f_i \uparrow p_i} \Lambda^*(p_i - f_i, |\mathbf{q}|) = 0 \qquad (i = 1, \cdots 4).$$

The first relation in (5.8) is a consequence.

The proof of the second relation in (5.8) is similar.

The second limit theorem on Λ^* in §7 of Ref. 6 similarly leads to a theorem on $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$.

THEOREM 5.2. *Suppose that* $\Lambda^*(|x|, |y|) < \infty$ *and that vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *canoni-*

cally represent x and y, then

$$\lim_{\mathbf{p} \downarrow \mathbf{x}} \Lambda^*(|\dot{\mathbf{p}} - \dot{\mathbf{p}}'|, |y|) = 0, \qquad \lim_{\mathbf{q} \downarrow \mathbf{y}} \Lambda^*(|x|, |\dot{\mathbf{q}} - \dot{\mathbf{q}}'|) = 0 \tag{5.11}$$

where $\mathbf{p}'$ is in the section $S'(\mathbf{p})$ of ${}_+\Phi'(\mathbf{x})$ and $\mathbf{q}'$ is in the section $S''(\mathbf{q})$ of ${}_+\Phi''(\mathbf{y})$.

We shall make use of the following corollary of Theorem 5.1.

COROLLARY 5.1. (i). *If $\Lambda^*(|\mathbf{p}|, |y|) < \infty$ then*

$$\lim_{\mathbf{f} \uparrow \mathbf{p}} \Lambda^*(|\dot{\mathbf{p}} - \dot{\mathbf{f}}|, |y|) = 0. \tag{5.12}$$

(ii). *If $\Lambda^*(|x|, |\mathbf{q}|) < \infty$ then*

$$\lim_{\mathbf{g} \uparrow \mathbf{q}} \Lambda^*(|x|, |\dot{\mathbf{q}} - \dot{\mathbf{g}}|) = 0. \tag{5.13}$$

Let (y_1, y_2, y_3, y_4) be the Riesz components of y. To establish (i) one first notes that the hypothesis $\Lambda^*(|\mathbf{p}|, |y|) < \infty$ implies that for some $\mathbf{q} \in [\mathcal{I}''_+]^4$ with $q_i \geqq y_i$, $\Lambda^*(|\mathbf{p}|, |\mathbf{q}|) < \infty$. Relation (5.12) now follows from the first relation in (5.8). The relation (5.13) is similarly established.

Theorem 5.3 extends Cor. 7.1 of Ref. 6. In stating this theorem we adopt the convention that when vectors $\mathbf{a}$ and $\mathbf{b}$ are in $[\bar{\mathbf{R}}^E]^4$ and $a_i \leqq b_i$, $i = 1, \cdots, 4$, we write $\mathbf{a} \leqq \mathbf{b}$.

THEOREM 5.3. *Suppose that $\Lambda^*(|x|, |y|) < \infty$ and that $\mathbf{y}$ canonically represents y. If $\mathbf{q}'$ varies in the section $S(\mathbf{q})$ of ${}_+\Phi''(\mathbf{y})$ and if $\mathbf{k}'$ and $\mathbf{k}''$ are in $[\bar{\mathbf{R}}_+^{E''}]^4$ and vary so that*

$$\mathbf{q}' \leqq \mathbf{k}' \leqq \mathbf{q}, \qquad \mathbf{q}' \leqq \mathbf{k}'' \leqq \mathbf{q} \tag{5.14}$$

then

$$\lim_{\mathbf{q} \downarrow \mathbf{y}} \Lambda^*(|x|, |\dot{\mathbf{k}}' - \dot{\mathbf{k}}''|) = 0. \tag{5.15}$$

Relation (5.15) follows from Cor. 7.1 of Ref. 6 on noting that

$$\Lambda^*(|x|, |\dot{\mathbf{k}}' - \dot{\mathbf{k}}''|) \leqq \sum_{i=1}^4 \Lambda^*(|x|, |k_i' - k_i''|).$$

Theorem 5.3 has a *dual* in which the roles of the first and second arguments are interchanged.

We make use of the following in the next sections. If p is in $\mathcal{I}'_+$ and if $v \to \Lambda(\cdot, v)(p)$ defines a measure on E'', p is finite (p, p, Λ). For the integral $\Lambda(\cdot, v)(p)$ then exists for each $v \in \mathcal{K}''_{\mathbf{C}}$, implying that p is finite $(p, p, \Lambda(\cdot, v))$. If $\Lambda(p, \cdot)$ is a $\mathbf{C}$-measure on E'', $\Lambda(\tilde{p}, \cdot)$ is a $\mathbf{C}$-measure on E'' and

$$\Lambda(p, \cdot) = \Lambda(\tilde{p}, \cdot). \tag{5.16}$$

§6. C-measures $\Lambda(\dot{\mathbf{p}}, \cdot)$ and $\Lambda(\cdot, \dot{\mathbf{q}})$

We are here concerned with pairs of vectors $(\mathbf{p}, \mathbf{q})$ limited as in (5.3). The results obtained in this section are essential in the study of the Λ-integrability of a general pair of mappings $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$. A major lemma follows.

LEMMA 6.1. *If p is in $\mathcal{G}'_+$, a necessary and sufficient condition that $\Lambda(p, \cdot)$ exist as a* **C**-*measure on E'' is that*

$$\Lambda^*(p, g) < \infty \tag{6.1}$$

for each $g \in \mathcal{K}''_+$.

We first prove the condition sufficient.

To show that $\Lambda(p, \cdot)$ is a **C**-measure one must verify (a) and (b) of Definition (i) of §2.

(a) The integral $\Lambda(\cdot, v)(p)$ exists for $v \in \mathcal{K}''_{\mathbf{C}}$ if the superior integral $| \Lambda(\cdot, v) |^*(p) < \infty$. [Lemma 5.2, Ref. 5.] Now $| v |$ is in $\mathcal{K}''_+$, and by Lemma 3.1 of Ref. 6, and (6.1),

$$| \Lambda(\cdot, v) |^*(p) \leqq \Lambda^*(p, | v |) < \infty. \tag{6.2}$$

Hence $\Lambda(\cdot, v)(p)$ exists.

(b) That the mapping $v \rightarrow \Lambda(\cdot, v)(p)$ of $\mathcal{K}''_{\mathbf{C}}$ into **C** is a **C**-measure on E'' is seen as follows. Let K be a compact subset of E'', and $g \in \mathcal{K}''_+$ such that $g \geqq \phi_K$, where ϕ_K is the characteristic function of K. Since $\Lambda(\cdot, v)(p)$ exists

$$| \Lambda(\cdot, v)(p) | \leqq \Lambda^*(p, | v |) \qquad \text{[by Th. 3.4].}$$

In case K contains the support of v, $U(v)g \geqq | v |$ so that

$$U(v)\Lambda^*(p, g) \geqq \Lambda^*(p, | v |) \geqq | \Lambda(\cdot, v)(p) |.$$

Thus $\Lambda(p, \cdot)$ is a **C**-measure on E''.

We now prove the condition (6.1) necessary.

Given $g \in \mathcal{K}''_+$ let K be a compact subset of E'' which includes the support of g. Let H_K be the subset of $v \in \mathcal{K}''_{\mathbf{C}}$ with support K. With the norm $v \rightarrow U(v)$, H_K is a Banach space. Note that $g \in H_K$. For each $u \in \mathcal{K}'_{\mathbf{C}}$ there exists a positive constant $M(u)$ such that

$$\sup_{v \in H_K} \frac{| \Lambda(u, v) |}{U(v)} = M(u) < \infty \qquad [v \neq 0] \tag{6.3}$$

since $\Lambda(u, \cdot)$ is a **C**-measure. Concerning $M(u)$ we shall prove the following.

(β). *For fixed K and variable $u \in \mathcal{K}'_{\mathbf{C}}$ with $| u | \leqq p \in \mathcal{G}''_+$, $M(u)$ admits a bound B.*

If (β) were false there would exist a sequence of mappings $u_n \in \mathcal{K}'_{\mathbf{C}}$ with $| u_n | \leqq p$ such that $M(u_n) \uparrow \infty$ as $n \uparrow \infty$. Consider then the sequence of linear forms V_n on H_K with values $V_n(v) = \Lambda(u_n, v)$. For fixed n, V_n is continuous on H_K.

For $v \in H_K$ set $\Lambda(\cdot, v) = \alpha$. By hypothesis on Λ, α is a **C**-measure on E''. Since $\alpha(p)$ exists by hypothesis $| \alpha | (p)$ exists by Lemma 4.1 of Ref. 5. Thus for fixed $v \in H_K$ and for $n = 1, 2, \cdots$,

$$| V_n(v) | = | \Lambda(\cdot, v)(u_n) | \leqq | \Lambda(\cdot, v) |(| u_n |) \leqq | \Lambda(\cdot, v) | (p).$$

Now $\Lambda(\cdot, v)(p)$ exists by hypothesis so that $| \Lambda(\cdot, v) | (p) < \infty$. Thus $| V_n(v) |$ is bounded for $n = 1, 2, \cdots$, and fixed $v \in H_K$. It follows from Theorem 5 of

Banach, p. 80, that for some constant B_0, $\| V_n \| \leqq B_0$ for $n = 1, 2, \cdots$. But $\| V_n \| = M(u_n)$ by (6.3). Thus $M(u_n) \leqq B_0$ for $n = 1, 2, \cdots$. From this contradiction we infer the truth of (β).

To prove that $\Lambda^*(p, g) < \infty$ we start with the definition

$$\Lambda^*(p, g) = \sup_{|u| \leqq p,\ |v| \leqq g} | \Lambda(u, v) | \qquad [(u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}]. \tag{6.4}$$

Since v is in H_K this becomes

$$\leqq M(u)U(g) \leqq BU(g) < \infty$$

in accordance, respectively, with (6.3) and (β). This establishes the necessity of the condition (6.1).

To Lemma 6.1 corresponds a dual lemma giving a necessary and sufficient condition that $\Lambda(\cdot, q)$ be a $\mathbf{C}$-measure on E'' for $q \in \mathcal{I}''_+$.

Let pairs $(\mathbf{p}, \mathbf{q})$ be conditioned as in (5.3).

THEOREM 6.1. (i) *A necessary and sufficient condition that $\Lambda(p_j, \cdot)$ be a $\mathbf{C}$-measure on E'' for $j = 1, 2, 3, 4$ is that for each $g \in \mathcal{K}''_+$*

$$\Lambda^*(| \mathbf{p} |, g) < \infty. \tag{6.5}$$

(ii) *A necessary and sufficient condition that $\Lambda(\cdot, q_j)$ be a $\mathbf{C}$-measure on E' for $j = 1, 2, 3, 4$ is that for each $f \in \mathcal{K}''_+$*

$$\Lambda^*(f, | \mathbf{q} |) < \infty. \tag{6.6}$$

We consider (i) and prove the condition (6.5) sufficient. Since $\Lambda^*(| \mathbf{p} |, g) < \infty$ we infer that $\Lambda^*(p_j, g) < \infty$, $j = 1, \cdots, 4$ for each $g \in \mathcal{K}''_+$, so that by Lemma 6.1 $\Lambda(p_j, \cdot)$ is a $\mathbf{C}$-measure on E''.

We now prove the condition (6.5) of (i) necessary. It follows from Lemma 6.1 that $\Lambda^*(p_j, g) < \infty$ for $j = 1, 2, 3, 4$. Since $| \mathbf{p} | = p_1 + p_2 + p_3 + p_4$ the convexity of Λ^* in its first argument implies that (6.5) holds as stated.

The proof of (ii) is similar.

COROLLARY 6.1. *If* (6.5) *holds* $\Lambda(\dot{\mathbf{p}}, \cdot)$ *is a* $\mathbf{C}$*-measure on* E''. *If* (6.6) *holds* $\Lambda(\cdot, \dot{\mathbf{q}})$ *is a* $\mathbf{C}$*-measure on* E'.

If (6.5) holds $\Lambda(p_j, \cdot)$, $j = 1, 2, 3, 4$, is a $\mathbf{C}$-measure on E'', $p_j < \infty\ (p, p, \Lambda)$ and $\Lambda(p_j, \cdot) = \Lambda(\tilde{p}_j, \cdot)$ in accordance with (5.16). It follows from Lemma 2.1 that $\Lambda(\dot{\mathbf{p}}, \cdot)$ is a $\mathbf{C}$-measure on E''. The case of $\Lambda(\cdot, \dot{\mathbf{q}})$ is similar.

COROLLARY 6.2. *If* $E'[E'']$ *is compact* $\Lambda(p, \cdot)[\Lambda(\cdot, q)]$ *is a* $\mathbf{C}$*-measure on* $E''[E']$ *whenever* $p[q]$ *is positive, lower semi-continuous and bounded.*

Recall that $\Lambda^*(f, g) < \infty$ for each pair $(f, g) \in \mathcal{K}'_+ \times \mathcal{K}''_+$ by virtue of Theorem 2.1 of Ref. 6. This applies in particular to the case in which f is a constant mapping f_0 with the value $\sup_s p(s) \mid (s \in E')$. With this choice of f_0

$$\Lambda^*(p, g) \leqq \Lambda^*(f_0, g) < \infty$$

so that $\Lambda(p, \cdot)$ is a $\mathbf{C}$-measure on E'' by Lemma 6.1. The case of q is similar.

§7. The relation $\Lambda(x, \cdot)(y) = \Lambda(\cdot, y)(x)$

In this section we shall be concerned with mappings $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$ such that the integrals $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$ exist in the sense of §2, and shall set

$$\Lambda(x, \cdot)(y) - \Lambda(\cdot, y)(x) = D(x, y). \tag{7.1}$$

When these integrals exist a necessary and sufficient condition that (x, y) be Λ-integrable is that $D(x, y) = 0$.

Let $\mathbf{y}$ be a vector canonically representing y. [See §5.] We shall make repeated use of the fact that when the integral $\Lambda(x, \cdot)(y)$ exists,

$$\lim_{\mathbf{q}\downarrow\mathbf{y}} \Lambda(x, \cdot)(\dot{\mathbf{q}}) = \Lambda(x, \cdot)(y) \qquad [\mathbf{q} \in {}_{+}\Phi(\mathbf{y})] \tag{7.2}$$

in accordance with Lemma 7.1 of Ref. 5.

We shall also refer to vectors $(\mathbf{p}, \mathbf{q})$ conditioned as in (5.3). When the integral $\Lambda(x, \cdot)(\dot{\mathbf{q}})$ exists and $\Lambda(|x|, |\mathbf{q}|) < \infty$

$$\lim_{\mathbf{g}\uparrow\mathbf{q}} \Lambda(x, \cdot)(\dot{\mathbf{g}}) = \Lambda(x, \cdot)(\dot{\mathbf{q}}) \tag{7.3}$$

where $\mathbf{g}$ is in ${}_{-}\Phi(\mathbf{q})$. The formula (7.3) will result from Lemma 7.2 of Ref. 5 once the implication

$$\Lambda^*(|x|, |\mathbf{q}|) < \infty \Rightarrow |\Lambda(x, \cdot)|^*(q_i) < \infty \qquad (i = 1, \cdots, 4) \tag{7.4}$$

has been established. Since $q_i \leqq |\mathbf{q}|$

$$|\Lambda(x, \cdot)|^*(q_i) \leqq |\Lambda(x, \cdot)|^*(|\mathbf{q}|) = \sup_{|v| \leqq |\mathbf{q}|} |\Lambda(x, \cdot)(v)|$$

where v is in $\mathcal{K}''_{\mathbf{C}}$. Taken with Theorem 3.2 this becomes

$$\leqq \sup_{|v| \leqq |\mathbf{q}|} \Lambda^*(|x|, |v|) = \Lambda^*(|x|, |\mathbf{q}|) < \infty \tag{7.5}$$

thus establishing (7.4). Set $\Lambda(x, \cdot) = \alpha$ and note that

$$N_1(q_i, |\alpha|) = |\Lambda(x, \cdot)|^*(q_i) < \infty \qquad (i = 1, \cdots, 4)$$

by virtue of (7.4). Relation (7.3) follows from Lemma 7.2 of Ref. 5.

Formulas for $\Lambda(x, \cdot)(y)$ *and* $\Lambda(\cdot, y)(x)$. If the integrals $\Lambda(\cdot, y)(x)$ and $\Lambda(x, \cdot)(y)$ exist, if $\Lambda^*(|x|, |y|) < \infty$, and if $\mathbf{x}$ and $\mathbf{y}$ canonically represent x and y, then

$$\Lambda(x, \cdot)(y) = \lim_{\mathbf{q}\downarrow\mathbf{y}} \lim_{\mathbf{g}\uparrow\mathbf{q}} \lim_{\mathbf{p}\downarrow\mathbf{x}} \lim_{\mathbf{f}\uparrow\mathbf{p}} \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}}) \tag{7.6}$$

$$\Lambda(\cdot, y)(x) = \lim_{\mathbf{p}\downarrow\mathbf{x}} \lim_{\mathbf{f}\uparrow\mathbf{p}} \lim_{\mathbf{r}\downarrow\mathbf{y}} \lim_{\mathbf{e}\uparrow\mathbf{r}} \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{e}}) \tag{7.7}$$

where $\mathbf{p} \in {}_{+}\Phi'(\mathbf{x})$; $\mathbf{r}, \mathbf{q} \in {}_{+}\Phi''(\mathbf{y})$; $\mathbf{f} \in {}_{-}\Phi'(\mathbf{p})$; $\mathbf{g} \in {}_{-}\Phi''(\mathbf{q})$; $\mathbf{e} \in {}_{-}\Phi''(\mathbf{r})$.

To establish these formulas let $(\mathbf{p}_0, \mathbf{q}_0)$ be chosen (as is possible since $\Lambda^*(|x|, |y|) < \infty$) so that

$$(\mathbf{p}_0, \mathbf{q}_0) \in {}_{+}\Phi'(\mathbf{x}) \times {}_{+}\Phi''(\mathbf{y}), \qquad \Lambda^*(|\mathbf{p}_0|, |\mathbf{q}_0|) < \infty. \tag{7.8}$$

Let $S'(\mathbf{p}_0)$ and $S''(\mathbf{q}_0)$ be the sections of ${}_{+}\Phi'(\mathbf{x})$ and ${}_{+}\Phi''(\mathbf{y})$ respectively in which $\mathbf{p} \leqq \mathbf{p}_0$, $\mathbf{q} \leqq \mathbf{q}_0$. Restrict $\mathbf{p}$ to $S'(\mathbf{p}_0)$ and $\mathbf{r}, \mathbf{q}$ to $S''(\mathbf{q}_0)$.

PROOF OF (7.6). We start with (7.2). To $\Lambda(x, \cdot)(\dot{\mathbf{q}})$ in (7.2) one can apply (7.3) provided $\mathbf{q}$ is in $S''(\mathbf{q}_0)$, since this condition on $\mathbf{q}$ implies that

$$\Lambda^*(|x|, |\mathbf{q}|) \leqq \Lambda^*(|\mathbf{p}_0|, |\mathbf{q}_0|) < \infty.$$

Thus (7.2) and (7.3) yield the formula

$$\Lambda(x, \cdot)(y) = \lim_{\mathbf{q}\downarrow y} \lim_{\mathbf{g}\uparrow \mathbf{q}} \Lambda(x, \cdot)(\dot{\mathbf{g}}). \tag{7.9}$$

The existence of $\Lambda(x, \cdot)$ as a $\mathbf{C}$-measure implies that $\Lambda(x, \cdot)(\dot{\mathbf{g}}) = \Lambda(\cdot, \dot{\mathbf{g}})(x)$ in accordance with (2.2). To (7.2) there corresponds a dual formula with the roles of x and y interchanged. Using this dual formula, the procedure used in establishing (7.9) applied to $\Lambda(\cdot, \dot{\mathbf{g}})(x)$ shows that

$$\Lambda(x, \cdot)(\dot{\mathbf{g}}) = \lim_{\mathbf{p}\downarrow x} \lim_{\mathbf{f}\uparrow \mathbf{p}} \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}}). \tag{7.10}$$

Together (7.9) and (7.10) establish (7.6). The proof of (7.7) is similar.

A formula for $D(x, y)$. If the integrals $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$ exist and if $\Lambda^*(|x|, |y|) < \infty$, then (7.6) and (7.7) hold and with trivial modifications yield the formulas

$$\Lambda(x, \cdot)(y) = \lim_{\mathbf{q}\downarrow y} \lim_{\mathbf{g}\uparrow \mathbf{q}} \lim_{\mathbf{p}\downarrow x} \lim_{\mathbf{f}\uparrow \mathbf{p}} \lim_{\mathbf{r}\downarrow y} \lim_{\mathbf{e}\uparrow \mathbf{r}} \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}}) \tag{7.11}$$

$$\Lambda(\cdot, y)(x) = \lim_{\mathbf{q}\downarrow y} \lim_{\mathbf{g}\uparrow \mathbf{q}} \lim_{\mathbf{p}\downarrow x} \lim_{\mathbf{f}\uparrow \mathbf{p}} \lim_{\mathbf{r}\downarrow y} \lim_{\mathbf{e}\uparrow \mathbf{r}} \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{e}}) \tag{7.12}$$

$$D(x, y) = \lim_{\mathbf{q}\downarrow y} \lim_{\mathbf{g}\uparrow \mathbf{q}} \lim_{\mathbf{p}\downarrow x} \lim_{\mathbf{f}\uparrow \mathbf{p}} \lim_{\mathbf{r}\downarrow y} \lim_{\mathbf{e}\uparrow \mathbf{r}} \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}} - \dot{\mathbf{e}}). \tag{7.13}$$

We come to a fundamental theorem.

THEOREM 7.1. *If the integrals* $\Lambda(\cdot, y)(x)$ *and* $\Lambda(x, \cdot)(y)$ *exist and if* $\Lambda^*(|x|, |y|) < \infty$ *then*

$$\Lambda(\cdot, y)(x) = \Lambda(x, \cdot)(y). \tag{7.14}$$

Use will be made of (7.13). Let the ordered sequence of the six limits in (7.13) be denoted by L so that (7.13) takes the form $D(x, y) = L\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}} - \dot{\mathbf{e}})$. For $\mathbf{r}, \mathbf{q}$ in $S''(\mathbf{q}_0)$ write

$$\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}} - \dot{\mathbf{e}}) = \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{g}} - \dot{\mathbf{q}}) + \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{r}} - \dot{\mathbf{e}}) + \Lambda(\dot{\mathbf{f}}, \dot{\mathbf{q}} - \dot{\mathbf{r}}). \tag{7.15}$$

The derivation of (7.9) and (7.10) shows that the integrals $\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{q}})$ and $\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{r}})$ exist. Making use of $(\mathbf{p}_0, \mathbf{q}_0)$ as chosen in (7.8) and $\mathbf{p} \leqq \mathbf{p}_0$, $\mathbf{q} \leqq \mathbf{q}_0$, Theorem 3.2 implies that for $\mathbf{f} \in {}_{-}\Phi'(\mathbf{p})$

$$|\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{q}} - \dot{\mathbf{g}})| \leqq \Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{q}} - \dot{\mathbf{g}}|) \tag{7.16}$$

$$|\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{r}} - \dot{\mathbf{e}})| \leqq \Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{r}} - \dot{\mathbf{e}}|) \tag{7.17}$$

$$|\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{q}} - \dot{\mathbf{r}})| \leqq \Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{q}} - \dot{\mathbf{r}}|). \tag{7.18}$$

Taking account of the constancy of $\mathbf{p}_0$ relative to the limit operation L we find that

$$L\Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{q}} - \dot{\mathbf{g}}|) = \lim_{\mathbf{q}\downarrow\mathbf{y}} \lim_{\mathbf{g}\uparrow\mathbf{q}} \Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{q}} - \dot{\mathbf{g}}|) = 0 \tag{7.19}$$

where the inner limit is zero by virtue of Theorem 5.1. Similarly

$$L\Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{r}} - \dot{\mathbf{e}}|) = \lim_{\mathbf{r}\downarrow\mathbf{y}} \lim_{\mathbf{e}\uparrow\mathbf{r}} \Lambda^*(|p_0|, |\dot{\mathbf{r}} - \dot{\mathbf{e}}|) = 0. \tag{7.20}$$

It follows from (7.13), (7.15), (7.16), (7.17), (7.19) and (7.20) that

$$D(x, y) = L\Lambda(\dot{\mathbf{f}}, \dot{\mathbf{q}} - \dot{\mathbf{r}}). \tag{7.21}$$

From (7.21), (7.18) and the nature of L, for $\mathbf{q}$ and $\mathbf{r}$ in $S''(\mathbf{q}_0)$

$$|D(x, y)| \leqq \sup_{\mathbf{q},\mathbf{r}} \Lambda^*(|\mathbf{p}_0|, |\dot{\mathbf{q}} - \dot{\mathbf{r}}|). \tag{7.22}$$

Set $q_i' = \inf(q_i, r_i)$ for $i = 1, 2, 3, 4$. Then $\mathbf{q}'$ is in $S''(q_0)$ and $\mathbf{q}' \leqq \mathbf{q} \leqq \mathbf{q}_0$, $\mathbf{q}' \leqq \mathbf{r} \leqq \mathbf{q}_0$. By Theorem 5.3 the right member of (7.22) is at most a prescribed $e > 0$ if the section $S''(\mathbf{q}_0)$ is sufficiently advanced in ${}_+\Phi''(\mathbf{y})$. It follows that $D(x, y) = 0$, and the proof of the theorem is complete.

COROLLARY 7.1. *If z and w are partial maps of E' and E'' respectively into* **C** *such that the integrals $\Lambda(z, \cdot)(w)$ and $\Lambda(\cdot, w)(z)$ exist and $\Lambda^*(|z|, |w|) < \infty$, then*

$$\Lambda(\cdot, w)(z) = \Lambda(z, \cdot)(w). \tag{7.23}$$

If the integral $\Lambda(z, \cdot)(w)$ exists the integral $\Lambda(\tilde{z}, \cdot)(\tilde{w})$ exists and equals $\Lambda(z, \cdot)(w)$. Similarly with $\Lambda(\cdot, w)(z)$. Moreover

$$\Lambda^*(|\tilde{z}|, |\tilde{w}|) \leqq \Lambda^*(|z|, |w|) < \infty.$$

Corollary 7.1 follows then from Theorem 7.1.

§8. Counter examples

Each example is associated with Theorem 7.1.

EXAMPLE 1. *This example shows that the integrals $\Lambda(p, \cdot)(q)$ and $\Lambda(\cdot, q)(p)$ may exist while $\Lambda(p, \cdot)(q) \neq \Lambda(\cdot, q)(p)$.*

Sequential spaces $\mathcal{E}'$ and $\mathcal{E}''$. Let $\mathcal{E}$ be a space of the points $1, 2, \cdots$ with discrete topology. The subset K_n of points $1, 2, \cdots, n$ of $\mathcal{E}$ is compact, and each compact subset of $\mathcal{E}$ is contained in some K_n for proper choice of n. Since any mapping of $\mathcal{E}$ into **C** is continuous, the space $K_{\mathbf{C}}(\mathcal{E})$ is the space of functions u with values $u(i)$, $1, 2, \cdots$, such that $u(i) = 0$ for i exceeding some integer. The characteristic function of $\mathcal{E}$ is continuous but not in $\mathcal{K}_{\mathbf{C}}$. It is however the supremum of the functions $\omega_n \in \mathcal{K}_{\mathbf{C}}$ such that

$$\omega_n(i) = 1, i = 1, \cdots, n; \qquad \omega_n(i) = 0, i > n. \tag{8.1}$$

Let the spaces $\mathcal{E}'$ and $\mathcal{E}''$ be homeomorphic images of $\mathcal{E}$ with points again denoted by $1, 2, \cdots$.

The **C**-*bimeasures* Λ_A. Let A be a matrix with complex elements a_{ij}, $(i, j = 1, 2, \cdots)$. For each $(u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}$ set

$$(8.2) \qquad \Lambda_A(u, v) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u(i)v(j)a_{ij}.$$

There exist integers m_u and n_v such that $u(i) = 0$ for $i > m_u$, and $v(j) = 0$ for $j > n_v$. Thus at most a finite number of terms in (8.2) differ from zero. Moreover

$$(8.3) \qquad \Lambda_A(u, v) \leqq \sum_{i=1}^{m_u} \sum_{j=1}^{n_v} | a_{ij} | U(u)U(v).$$

One infers that Λ_A is a **C**-bimeasure on $\mathcal{E}' \times \mathcal{E}''$.

Essentially finite matrices A. If at most a finite number of the constants a_i, in any row or column of A differ from zero, the matrix A will be termed essentially finite. We shall prove the following.

(α) *If A is essentially finite then corresponding to mappings $(h, k) \in \mathbf{R}_+^{E'} \times \mathbf{R}_+^{E''}$ $\Lambda_A(\cdot, k)$ is a* **C**-*measure on $\mathcal{E}'$, and $\Lambda_A(h, \cdot)$ a* **C**-*measure on $\mathcal{E}''$.*

The mappings (h, k) are continuous so that in accordance with the definition of Λ^*

$$(8.4) \qquad \Lambda_A^*(h, k) = \sup_{|u| \leqq h, |v| \leqq k} | \Lambda(u, v) | \qquad [(u, v) \in \mathcal{K}'_{\mathbf{C}} \times \mathcal{K}''_{\mathbf{C}}].$$

In particular let k be replaced by $g \in \mathcal{K}''_+$. For some integer n, $g(j) = 0$ for $j > n$. Since A is essentially finite, for fixed n there exists an integer $m(n)$ such that $a_{ij} = 0$ if $j \leqq n$ and $i \geqq m(n)$. Hence

$$| \Lambda_A^*(h, g) | \leqq \sum_{i=1}^{m(n)} \sum_{j=1}^{n} | h(i)\, g(j)\, a_{ij} | < \infty.$$

It follows from Lemma 6.1 that $\Lambda_A(h, \cdot)$ is a **C**-measure on $\mathcal{E}''$. Similarly $\Lambda_A(\cdot, k)$ is a **C**-measure on $\mathcal{E}'$.

EXAMPLE 1. *The choice of* A. In this example we shall set $a_{ij} = 1$ for $i = j - 1$ and $j = 2, 3, \cdots$ and set $a_{ij} = -1$ for $i = j + 1$ and $j = 1, 2, \cdots$. Otherwise set $a_{ij} = 0$. The constants a_{ij} are the elements of the matrix

$$(8.5) \qquad A = \left\| \begin{array}{ccccc} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \cdots \\ 0 & -1 & 0 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{array} \right\|$$

Let Λ_A now be defined by (8.2) using the constants of the matrix (8.5).

The **C**-*measures $\Lambda_A(\cdot, J)$ and $\Lambda_A(I, \cdot)$ in Ex.* 1. Let I and J be the characteristic functions of $\mathcal{E}'$ and $\mathcal{E}''$ respectively. Since the matrix (8.5) is essentially finite $\Lambda_A(\cdot, J)$ and $\Lambda_A(I, \cdot)$ are **C**-measures on $\mathcal{E}'$ and $\mathcal{E}''$, respectively, in accordance with (α) above. We continue by proving (β).

(β) *In Ex.* 1 *the* **C**-*measure $\Lambda_A(\cdot, J)$ is a discrete* **C**-*measure of mass* 1 *at the point $t = 1$ of $\mathcal{E}'$.*

Recall that for $u \in \mathcal{K}'_{\mathbf{C}}$ and $g \in {}_{-}\phi'_J$

$$\Lambda_A(u, \cdot)(J) = \lim_{g \uparrow J} \Lambda_A(u, \cdot)(g) \tag{8.6}$$

by Lemma 6.2 of Ref. 5. The mappings ω_n defined in (8.1), ordered according to increasing n form a filtering set in $\mathcal{K}''_+$ and define a Fréchet filter of sections [Ref. 2, p. 33] equivalent to the filter of sections of $_-\phi_J$. Hence (8.6) implies that

$$\Lambda_A(u, \cdot)(J) = \lim_{m \uparrow \infty} \Lambda_A(u, \cdot)(\omega_m) = \lim_{m \uparrow \infty} \Lambda_A(u, \omega_m). \tag{8.7}$$

If $u(i) = v(i) = 0$ for $i \geqq n$, it is readily seen that when A is the matrix (8.5)

$$\Lambda_A(u, \omega_{n+1}) = \sum_{i=1}^{n} \sum_{j=1}^{n+1} u(i)a_{ij} = u(1)$$

$$\Lambda_A(\omega_{n+1}, v) = \sum_{i=1}^{n+1} \sum_{j=1}^{n} v(j)a_{ij} = -v(1).$$

Hence by (8.7) $u(1) = \Lambda_A(u, \cdot)(J)$. But $\Lambda_A(\cdot, J)$ is a **C**-measure by (α), so that by (2.2)

$$\Lambda_A(u, \cdot)(J) = \Lambda_A(\cdot, J)(u) = u(1). \tag{8.8}$$

Thus $\Lambda_A(\cdot, J)$ is a mapping $u \to u(1)$ of $\mathcal{K}'_{\mathbf{C}}$ into **C**, and by definition [Ref. 3, p. 51] is a **C**-measure of mass 1 at the point $s = 1$ of $\mathcal{E}'$.

Example 1 *as counter example.* Set $\Lambda_A(\cdot, J) = \alpha$. Using ($\beta$) we shall see that I, the characteristic function of $\mathcal{E}'$ is α-integrable and that $\alpha(I) = 1$. In fact the relation $\alpha(u) = u(1)$ implies that $|\alpha|^*(|u|) = |u(1)|$ and $|\alpha|^*(I) = 1$. Let $e \in \mathcal{K}''_{\mathbf{C}}$ be the characteristic function of the point $t = 1$. Since $|\alpha|^*(I) < \infty$, I is α-integrable. Hence $|\alpha|^*(I - e) = |\alpha|(I) - |\alpha|(e) = |\alpha|^*(I) - |\alpha|^*(e) = 0$. We infer that $\alpha(I) = \alpha(e)$ so that $\Lambda_A(\cdot, J)(I) = \alpha(e) = 1$.

One similarly shows that $\Lambda_A(I, \cdot)$ is a discrete **C**-measure of mass -1 at the point $t = 1$ of $\mathcal{E}''$, and that $\Lambda_A(I, \cdot)(J) = -1$. Hence

$$\Lambda_A(I, \cdot)(J) \neq \Lambda_A(\cdot, J)(I). \tag{8.9}$$

Thus the integrals $\Lambda_A(I, \cdot)(J)$ and $\Lambda_A(\cdot, J)(I)$ exist and are unequal.

EXAMPLE 2. *This example shows that it is not necessary that $\Lambda^*(p, q) < \infty$ in order that (p, q) be Λ-integrable.*

Let a_{ij} be defined as in Ex. 1. We introduce $b_{ij} = (-1)^{(i+1)} a_{ij}$, defining a matrix B of the form

$$B = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & -1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{Vmatrix}$$

Let Λ_B be defined in terms of B by (8.2), b_{ij} replacing a_{ij}. The analysis of Ex. 1 gives $\Lambda_B(I, \cdot)(J) = \Lambda_B(\cdot, J)(I) = 1$. In Ex. 1 $\Lambda_A^*(I, J) = \infty$ in accordance with Theorem 7.1. Directly from the definition of Λ^* one sees that $\Lambda_B^* = \Lambda_A^*$ so that $\Lambda_B^*(I, J) = \infty$. Thus (I, J) is Λ_B-integrable while $\Lambda_B^*(I, J) = \infty$.

§9. Necessary and sufficient conditions that $\Lambda(\cdot, q)(p) = \Lambda(p, \cdot)(q)$

We shall refer to pairs of mappings (p, q) and (f, g) conditioned as in (3.0). We shall assume that the integrals $\Lambda(\cdot, q)(p)$ and $\Lambda(p, \cdot)(q)$ exist in the sense of §2, and seek necessary and sufficient conditions that

$$\Lambda(\cdot, q)(p) = \Lambda(p, \cdot)(q) \tag{9.1}$$

or equivalently, necessary and sufficient conditions that (p, q) be Λ-integrable when the integrals $\Lambda(\cdot, q)(p)$ and $\Lambda(p, \cdot)(q)$ exist. We have already seen in Theorem 7.1 that a sufficient condition that (9.1) hold is that $\Lambda^*(p, q) < \infty$. But this condition is not necessary as Ex. 2 in §8 shows.

Filtering sets H_p', H_q'', H_{pq}. Let H_p', $[H_q'']$ be an arbitrary ensemble of mappings $f \in \mathcal{K}_+'$ with $\sup f = p$, $[g \in \mathcal{K}_+''$ with $\sup g = q]$ filtering for the relation $\leqq$. Let H_{pq} be the product on $\mathcal{K}_+' \times \mathcal{K}_+''$ of the filtering sets H_p' and H_q''.

If the integral $\Lambda(\cdot, q)(p)$ exists then

$$\Lambda(\cdot, q)(p) = \lim_{f \in H_p'} \lim_{g \in H_q''} \Lambda(f, g) \tag{9.2}$$

as we shall see. Similarly, if the integral $\Lambda(p, \cdot)(q)$ exists,

$$\Lambda(p, \cdot)(q) = \lim_{g \in H_q''} \lim_{f \in H_p'} \Lambda(f, g). \tag{9.3}$$

To establish (9.2) we infer from Lemma 6.3 of Ref. 5 that

$$\Lambda(\cdot, q)(p) = \lim_{f \in H_p'} \Lambda(\cdot, q)(f). \tag{9.4}$$

The hypothesis that $\Lambda(\cdot, q)(p)$ exists implies, by definition of this integral in §2, that the integral $\Lambda(f, \cdot)(q)$ exists and equals $\Lambda(\cdot, q)(f)$. By a second application of Lemma 6.3 of Ref. 5.

$$\Lambda(f, \cdot)(q) = \lim_{g \in H_q''} \Lambda(f, \cdot)(g). \tag{9.5}$$

Now $\Lambda(f, \cdot)(g)$ is by definition $\Lambda(f, g)$ so that (9.2) follows from (9.4) and (9.5). The proof of (9.3) is similar.

We infer the following.

THEOREM 9.1. *If the integrals $\Lambda(p, \cdot)(q)$ and $\Lambda(\cdot, q)(p)$ exist a necessary and sufficient condition that they have a common value is that*

$$\lim_{f \in H_p'} \lim_{g \in H_q''} \Lambda(f, g) = \lim_{g \in H_q''} \lim_{f \in H_p'} \Lambda(f, g). \tag{9.6}$$

When (9.6) *holds these limits give the value*

$$\Lambda(p, q) = \Lambda(p, \cdot)(q) = \Lambda(\cdot, q)(p).$$

To obtain conditions of a different sort that (9.1) hold set

$$D(p, q) = \Lambda(p, \cdot)(q) - \Lambda(\cdot, q)(p) \tag{9.7}$$

as in (7.1).

LEMMA 9.1. *If the integrals* $\Lambda(p, \cdot)(q)$ *and* $\Lambda(\cdot, q)(p)$ *exist*

$$D(p, q) = \lim_{f \in H'_p} \Lambda(p - f, \cdot)(q) = -\lim_{g \in H''_q} \Lambda(\cdot, q - g)(p). \tag{9.8}$$

We shall establish the first relation in (9.8). The relation (9.4) can be written in the form

$$\Lambda(\cdot, q)(p) = \lim_{f \in H'_p} \Lambda(f, \cdot)(q) \tag{9.9}$$

since $\Lambda(\cdot, q)(f) = \Lambda(f, \cdot)(q)$ when the integral $\Lambda(\cdot, q)(p)$ exists. Moreover when $\Lambda(p, \cdot)(q)$ exists $\Lambda(p, \cdot)$ is a **C**-measure on E'', so that $\Lambda(p, \cdot) - \Lambda(f, \cdot) = \Lambda(p - f, \cdot)$ in accordance with Lemma 2.1 (β). Using this relation and (9.9), one obtains the first relation in (9.8).

The second relation is similarly proved.

We infer the following.

THEOREM 9.2. *If the integrals* $\Lambda(p, \cdot)(q)$ *and* $\Lambda(\cdot, q)(p)$ *exist two equivalent necessary and sufficient conditions that these integrals be equal are that*

$$\lim_{f \in H'_p} \Lambda(p - f, \cdot)(q) = 0 \qquad \lim_{g \in H''_q} \Lambda(\cdot, q - g)(p) = 0.$$

As a corollary one obtains a simple confirmation of Theorem 7.1 when $(x, y) = (p, q)$.

COROLLARY 9.1. *If the integrals* $\Lambda(p, \cdot)(q)$ *and* $\Lambda(\cdot, q)(p)$ *exist, a sufficient condition that they be equal is that* $\Lambda^*(p, q) < \infty$.

In fact it follows from Theorem 3.2 that

$$| \Lambda(p - f, \cdot)(q) | \leqq \Lambda^*(p - f, q)$$

so that in accordance with the limit Theorem 6.1 of Ref. 6

$$\lim_{f \in H'_p} | \Lambda(p - f, \cdot)(q) | \leqq \lim_{f \in H'_p} \Lambda^*(p - f, q) = 0.$$

Thus the first condition of Theorem 9.2 is satisfied, and hence $\Lambda(p, \cdot)(q) = \Lambda(\cdot, q)(p)$.

Further sufficient conditions. The preceding analysis suggests other sufficient conditions that (9.1) hold.

Uniform H''_q convergence. Granting that the integral $\Lambda(\cdot, q)(p)$ exists, $\Lambda(f, g)$ converges to $\Lambda(f, \cdot)(q)$ according to $g \in H''_q$, as (9.5) shows. We say that this convergence is uniform with respect to f in some section Σ' of H'_p, if for arbitrary $e > 0$ there exists a section S''_e of H''_q such that

$$| \Lambda(f, g) - \Lambda(f, \cdot)(q) | < e \qquad [\text{for } g \in S''_e, f \in \Sigma']. \tag{9.10}$$

We term such convergence uniform H''_q-convergence.

Uniform H'_p-convergence. Uniform H'_p convergence of $\Lambda(f, g)$ to $\Lambda(\cdot, g)(p)$ for g in some section Σ'' of H''_q is similarly defined.

Convergence according to H_{pq}. If F maps $\mathcal{K}'_+ \times \mathcal{K}''_+$ into $\mathbf{C}$ one says that $F(f, g)$ converges to A according to H_{pq} if the limit of F with respect to $(f, g) \in H_{pq}$ is A.

We shall establish Theorem 9.2.

THEOREM 9.2. *If the integrals $\Lambda(p, \cdot)(q)$ and $\Lambda(\cdot, q)(p)$ exist, each of the following conditions are sufficient that*

$$\Lambda(p, \cdot)(q) = \Lambda(\cdot, q)(p). \tag{9.11}$$

(a) *The convergence of $\Lambda(f, g)$ to $\Lambda(f, \cdot)(q)$ according to $g \in H''_q$ shall be uniform for f in some section Σ' of H'_p.*

(b) *The convergence of $\Lambda(f, g)$ to $\Lambda(\cdot, g)(p)$ according to $f \in H'_p$ shall be uniform for g in some section Σ'' of H''_q.*

(c) *$\Lambda(f, g)$ shall converge according to H_{pq}.*

In case (a), (b), *or* (c) *is satisfied, $\Lambda(p, q)$ equals*

$$\lim_{(f,g) \in H_{pq}} A(f, g) = \lim_{f \in H'_p} \lim_{g \in H''_q} \Lambda(f, g) = \lim_{g \in H''_p} \lim_{f \in H'_q} \Lambda(f, g). \tag{9.12}$$

We establish the following.

(1) *The condition* (c) *is sufficient that* (9.1) *hold.* This follows from the fact that the H_{pq}-convergence of Λ to a limit Λ_0 implies that the two limits appearing in (9.6) exist and are equal. Moreover, the limit Λ_0 equals the limit $\Lambda(p, q)$ of the two members of (9.6), so that (9.12) holds.

(2) *The condition* (a) *implies* (c). Let $e > 0$ be given. Since the integral $\Lambda(\cdot, q)(p)$ exists, $\Lambda(\cdot, q)(f)$ converges to $\Lambda(\cdot, q)(p)$ according to $f \in H'_p$. There accordingly exists a section S'_e of H'_p such that

$$| \Lambda(\cdot, q)(p) - \Lambda(\cdot, q)(f) | < e \qquad [f \in S'_e]. \tag{9.13}$$

Condition (a) presupposes that

$$| \Lambda(f, g) - \Lambda(f, \cdot)(q) | < e \tag{9.14}$$

for g in some section S''_e of H''_q and for f in some section Σ' of H'_p independent of e. From (9.13) and (9.14) and the relation $\Lambda(f, \cdot)(q) = \Lambda(\cdot, q)(f)$ we infer that

$$| \Lambda(f, g) - \Lambda(\cdot, q)(p) | < 2e \tag{9.15}$$

for $g \in S''_e$, and $f \in S'_e \cap \Sigma'$. Hence $\Lambda(f, g)$ converges according to H_{pq}. Thus (a) implies (c), so that (a) is sufficient that (9.1) hold.

(3) *The condition* (b) *similarly implies* (c). Hence (b) is sufficient that (9.1) hold.

§10. The Λ-integrability of mappings $(\dot{\mathbf{p}}, \dot{\mathbf{q}})$

We return to pairs of vectors $(\mathbf{p}, \mathbf{q})$ limited as in (5.3). We are in a position to give sufficient conditions not only that $\Lambda(\dot{\mathbf{p}}, \cdot)$ and $\Lambda(\cdot, \dot{\mathbf{q}})$ exist as $\mathbf{C}$-measures, but also that the integrals $\Lambda(\dot{\mathbf{p}}, \cdot)(\dot{\mathbf{q}})$ and $\Lambda(\cdot, \dot{\mathbf{q}})(\dot{\mathbf{p}})$ exist and are equal. The result here obtained will serve as a model for the case of Borel mappings (x, y) in §11.

THEOREM 10.1. *Sufficient conditions that a pair of mappings* $(\dot{\mathbf{p}}, \dot{\mathbf{q}})$, *conditioned as in* (5.3), *be Λ-integrable, are that* $\Lambda^*(\,|\mathbf{p}|, |\mathbf{q}|\,) < \infty$, *and that*

(1) $\Lambda^*(\,|\mathbf{p}|, g) < \infty$ *for each* $g \in \mathcal{K}''_+$

(2) $\Lambda^*(f, |\mathbf{q}|\,) < \infty$ *for each* $f \in \mathcal{K}'_+$.

We have seen in Corollary 6.1 that condition (1) is sufficient that $\Lambda(\dot{\mathbf{p}}, \cdot)$ be a $\mathbf{C}$-measure on E'', and (2) sufficient that $\Lambda(\cdot, \dot{\mathbf{q}})$ be a $\mathbf{C}$-measure on E'. Let $\tilde{p}_i$ $[\tilde{q}_i]$ be the finite projection of p_i $[q_i]$, $(i = 1, 2, 3, 4)$. Recall that $\dot{\mathbf{p}} = \tilde{p}_1 - \tilde{p}_2 + i(\tilde{p}_3 - \tilde{p}_4)$, $\dot{\mathbf{q}} = \tilde{q}_1 - \tilde{q}_2 + i(\tilde{q}_3 - \tilde{q}_4)$.

PROOF THAT $\Lambda(\tilde{p}_i, \cdot)(\tilde{q}_j)$ EXISTS. It follows from condition (1) and Lemma 6.1 that for $i = 1, 2, 3, 4$, that $p_i < \infty$ (p, p, Λ) and that $\Lambda(\tilde{p}_i, \cdot) = \Lambda(p_i, \cdot)$ is a $\mathbf{C}$-measure on E''. Set $\Lambda(p_i, \cdot) = \alpha_i$. If $|\alpha_i|^*(q_j) < \infty$, then the integral $\alpha_i(q_j)$ exists by Lemma 5.2 of Ref. 5, and hence the integral $\alpha_i(\tilde{q}_j)$ exists. By virtue of the Bourbaki definition of the superior integral

$$|\alpha_i|^*(\tilde{q}_j) \leqq |\alpha_i|^*(q_j) = \sup_{g \leqq q_j} |\alpha_i|(g) \qquad [g \in \mathcal{K}''_+].$$

Hence by our definition of $|\alpha_i|(g)$ in §3 of Ref. 5

$$|\alpha_i|^*(q_j) = \sup_{|u| \leqq q_j} |\Lambda(p_i, \cdot)(u)| \leqq \Lambda^*(p_i, q_j) \leqq \Lambda^*(\,|\mathbf{p}|, |\mathbf{q}|\,)$$

where u is in $\mathcal{K}'_{\mathbf{C}}$ and where the middle inequality follows from the relation $|\Lambda(p_i, \cdot)(u)| \leqq \Lambda^*(p_i, |u|\,)$ of Theorem 3.4. Hence $|\alpha_i|^*(q_j) < \infty$ so that the integral $\alpha_i(\tilde{q}_j) = \Lambda(\tilde{p}_i, \cdot)(\tilde{q}_j)$ exists. Similarly $\Lambda(\cdot, \tilde{q}_j)(\tilde{p}_i)$ exists.

Thus the integrals $\Lambda(\tilde{p}_i, \cdot)(\tilde{q}_j)$ exist. Similarly the integrals $\Lambda(\cdot, \tilde{q}_i)(\tilde{p}_j)$ exist. The integrals $\Lambda(\dot{\mathbf{p}}, \cdot)(\dot{\mathbf{q}})$ and $\Lambda(\cdot, \dot{\mathbf{q}})(\dot{\mathbf{p}})$ accordingly exist. These integrals are equal by virtue of Theorem 7.1, since

$$\Lambda^*(\,|\dot{\mathbf{p}}|, |\dot{\mathbf{q}}|\,) \leqq \Lambda^*(\,|\mathbf{p}|, |\mathbf{q}|\,) < \infty.$$

Hence $(\dot{\mathbf{p}}, \dot{\mathbf{q}})$ is Λ-integrable in accordance with Definition (iii) of §2.

§11. The Λ-integrability of Borel mappings (x, y)

Theorems 10.1 and 7.1 give conditions that a pair of mappings $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$ be Λ-integrable. In Theorem 7.1 it is assumed that the integrals $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$ exist. A general approach to the problem of the existence of these integrals can be made when x and y are Borel measurable mappings (written B-measurable mappings).

Borel measurable sets $\mathcal{B}$. A class $\mathcal{B}$ of subsets of E will be termed the class of B-measurable sets of E [cf. Ref. 4] if $\mathcal{B}$ is the smallest class containing all compact sets of E, and such that

(1) if A_1 and A_2 are in $\mathcal{B}$, then $A_1 \cap CA_2 \in \mathcal{B}$;

(2) if $A_i \in \mathcal{B}$ $(i = 1, 2, \cdots)$ then $\text{Union}_i\, A_i \in \mathcal{B}$.

B-measurable mappings. Given $f \in \mathbf{R}^E$ let E_f denote the set $E \mid (f(s) \neq 0)$. The mapping f is termed *B-measurable* if and only if for each $a \in \mathbf{R}$ the set $f_a =$

$E_f \mid (f(s) \geqq a)$ is B-measurable. A mapping $x \in \mathbf{C}^E$ is termed B-measurable if $\mathbf{R}x$ and $\mathcal{I}x$ are B-measurable. If f is B-measurable E_f is the union of the set f_n for $n = -1, -2, \cdots$ and so is B-measurable. By the usual argument one shows that $-f$ is B-measurable with f, and $-x$ with x.

If α is a $\mathbf{C}$-measure on E a subset A of E [mapping $x \in \mathbf{C}^E$] is termed *α-measurable* if the set A [mapping x] is $\mid \alpha \mid$-measurable on E in the sense of Bourbaki [Ref. 3, p. 180]. The set E is always α-measurable.

LEMMA 11.1. *If a subset A of E [a mapping $x \in \mathbf{C}^E$] is B-measurable, it is α-measurable for each $\mathbf{C}$-measure α on E.*

We first consider A. The class H of $\mid \alpha \mid$-measurable sets on E is a clan [Ref. 3, p. 185, Cor. 4] which contains each compact subset of E [Ref. 3, p. 156, Cor. to Prop. 10 and p. 195, Cor. 1] and each denumerable union of sets in H [Ref. 3, p. 189, Cor. 2]. The class H must then contain $\mathcal{B}$ by definition of $\mathcal{B}$. In particular $A \in H$.

Consider a B-measurable $x \in \mathbf{C}^E$. Set $\mathcal{R}x = f$. By hypothesis f is B-measurable so that for each $a \in \mathbf{R}$ the set f_a is B-measurable. As seen in the preceding paragraph the set f_a is α-measurable. So also then is E_f. Since E is α-measurable we infer that $E - E_f$ is α-measurable. To show that f is α-measurable note that the set $S_a = E \mid (f(s) \geqq a)$ is the set f_a when $a > 0$, and the set $f_a \cup (E - E_f)$ when $a \leqq 0$. Thus S_a is α-measurable for each $a \in \mathbf{R}$. It follows [Prop. 9, Ref. 3, p. 192] that f is α-measurable.

The mapping $\mathcal{I}x$ is similarly proved α-measurable. This completes the proof of Lemma 11.1.

The Riesz components of $x \in \mathbf{C}^E$. If $x \in \mathbf{C}^E$ is B-measurable the Riesz components of x are B-measurable as we now recall. The definition of the B-measurability of x implies that $\mathcal{R}x$ and $\mathcal{I}x$ are B-measurable. Set $\mathcal{R}x = f$. If f is B-measurable f^+ is B-measurable. For the set $f_a^+ = f_a$ when $a > 0$, and $f_a^+ = f_0$ when $a \leqq 0$. Hence f^+ is B-measurable. To show that f^- is B-measurable recall that $-f$ is B-measurable with f. Since $f^- = (-f)^+$ it is clear that f^- is B-measurable. Thus the Riesz components of x are B-measurable.

We continue with the following lemma.

LEMMA 11.2. *If h is positive and B-measurable, and if $\Lambda^*(h, g) < \infty$ for each $g \in \mathcal{K}_+''$, then $\Lambda(h, \cdot)$ is a $\mathbf{C}$-measure on E''.*

Given $g \in \mathcal{K}_+''$, we show first that the integral $\Lambda(\cdot, g)(h)$ exists. Set $\Lambda(\cdot, g) = \alpha$. Then α is a $\mathbf{C}$-measure on E'. Now h is $\mid \alpha \mid$-measurable by Lemma 11.1. For the integral $\alpha(h)$, or equivalently $\mid \alpha \mid (h)$, to exist, it is sufficient [Ref. 3, p. 194, Th. 5] that $\mid \alpha \mid^*(h)$ be finite. Since $\Lambda^*(h, g) < \infty$ there exists $p_1 \in {}_+\phi_h$ such that $\Lambda^*(p_1, g) < \infty$. With p in the section $S'(p_1)$ of ${}_+\phi_h'$ on which $p \leqq p_1$,

$$(11.1) \qquad \mid \alpha \mid^*(h) = \inf_p \mid \alpha \mid^*(p) = \inf_p \mid \Lambda(\cdot, g) \mid^*(p) \leqq \Lambda^*(p_1, g)$$

in accordance with L. 3.1 of Ref. 6. Thus $\mid \alpha \mid^*(h)$ is finite so that $\Lambda(\cdot, g)(h)$ exists.

That the integral $\Lambda(\cdot, v)(h)$ exists for each $v \in \mathcal{K}_{\mathbf{C}}''$ follows now from the fact that if g is any one of the Riesz components of v, $\Lambda(\cdot, g)(h)$ exists. Knowing that

$\Lambda(\cdot, v)(h)$ exists, it follows as in (b) of the proof of Lemma 6.1 that the mapping $v \to \Lambda(\cdot, v)(h)$ is a **C**-measure on E''.

Granting that x and y are B-measurable mappings of E' and E'' respectively into **C** we shall verify the following.

(a) *The condition* $\Lambda^*(|x|, g) < \infty$ *for each* $g \in \mathcal{K}''_+$ *implies that* $\Lambda(x, \cdot)$ *is a* **C**-*measure on* E''.

(b) *The condition* $\Lambda^*(f, |y|) < \infty$ *for each* $f \in \mathcal{K}'_+$ *implies that* $\Lambda(\cdot, y)$ *is a* **C**-*measure on* E'.

(c) *If the conditions of* (a) [(b)] *are satisfied, the condition* $\Lambda^*(|x|, |y|) < \infty$ *implies that the integral* $\Lambda(x, \cdot)(y)$ $[\Lambda\cdot, y)(x)]$ *exists.*

Proof of (a). Let (x_1, x_2, x_3, x_4) be the Riesz components of x. The condition $\Lambda^*(|x|, g) < \infty$ implies that $\Lambda^*(x_i, g) < \infty$ $(i = 1, \cdots, 4)$ and hence by Lemma 11.2 that $\Lambda(x_i, \cdot)$ is a **C**-measure on E''. It follows from Lemma 4.1 that $\Lambda(x, \cdot)$ is the **C**-measure,

$$\Lambda(x, \cdot) = \Lambda(x_1, \cdot) - \Lambda(x_2, \cdot) + i\Lambda(x_3, \cdot) - i\Lambda(x_4, \cdot).$$

Proof of (b). The proof of (b) is similar to that of (a).

Proof of (c). Let h and k respectively be Riesz components of x and y. To show that the integral $\Lambda(x, \cdot)(y)$ exists if the conditions of (a) are satisfied, and if $\Lambda^*(|x|, |y|) < \infty$, it will be sufficient to show that under these hypotheses each integral $\Lambda(h, \cdot)(k)$ exists. The condition $\Lambda^*(|x|, |y|) < \infty$ implies that $\Lambda^*(h, k) < \infty$, and this in turn implies that for some $q_1 \in {}_+\phi''_k$, $\Lambda^*(h, q_1) < \infty$.

Set $\Lambda(h, \cdot) = \alpha$. Then α is a **C**-measure on E'' under the conditions of (a). Moreover $\alpha(k)$, or equivalently $|\alpha|(k)$, exists if $|\alpha|^*(k) < \infty$ [Ref. 3, p. 194, Th. 5]. To show that $|\alpha|^*(k) < \infty$ recall that

$$|\alpha|^*(k) = \inf_{q \in {}_+\phi''_k} |\alpha|^*(q).$$

It will be sufficient to show that $|\alpha|^*(q_1) < \infty$. Now with $v \in \mathcal{K}''_{\mathbf{C}}$ and $|v| \leqq q_1$

$$|\alpha|^*(q_1) = \sup_v |\alpha(v)| = \sup_v |\Lambda(h, \cdot)(v)| \leqq \Lambda^*(h, q_1)$$

by virtue of Theorem 3.2. Thus $|\alpha|^*(k)$ is finite so that $\alpha(k)$ exists. Thus the integral $\Lambda(x, \cdot)(y)$ exists.

The integral $\Lambda(\cdot, y)(x)$ similarly exists if the conditions of (b) are satisfied and if $\Lambda^*(|x|, |y|) < \infty$.

The results (a), (b), and (c) taken with Theorem 7.1 give a fundamental theorem.

Theorem 11.1. *If the mappings* $(x, y) \in \mathbf{C}^{E'} \times \mathbf{C}^{E''}$ *are Borel measurable, then* $\Lambda(x, \cdot)$ *and* $\Lambda(\cdot, y)$ *are* **C**-*measures on* E' *and* E'' *respectively whenever*

(α) $\Lambda^*(|x|, g) < \infty$ *for each* $g \in \mathcal{K}''_+$,

(β) $\Lambda^*(f, |y|) < \infty$ *for each* $f \in \mathcal{K}'_+$.

If in addition $\Lambda^*(|x|, |y|) < \infty$, *the integrals* $\Lambda(x, \cdot)(y)$ *and* $\Lambda(\cdot, y)(x)$ *exist and are equal.*

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