Counting distinct zeros of the Riemann zeta-function

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Abstract. Bounds on the number of simple zeros of the derivatives of a function are used to give bounds on the number of distinct zeros of the function.

The Riemann ξ-function is defined by $\xi(s) = H(s)\zeta(s)$, where $H(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)$ and $\zeta(s)$ is the Riemann ζ-function. The zeros of $\xi(s)$ and its derivatives are all located in the critical strip $0 < \sigma < 1$, where $s = \sigma + it$. Since $H(s)$ is regular and nonzero for $\sigma > 0$, the nontrivial zeros of $\zeta(s)$ exactly correspond to those of $\xi(s)$. Let $\rho^{(j)} = \beta + i\gamma$ denote a zero of the j^{th} derivative $\xi^{(j)}(s)$, and denote its multiplicity by $m(\gamma)$. Define the following counting functions:

$$N^{(j)}(T) = \sum_{\rho^{(j)}=\beta+i\gamma} 1 \qquad \text{zeros of } \xi^{(j)}(\sigma+it) \text{ with } 0<t<T$$

$$N(T) = N^{(0)}(T) \qquad \text{zeros of } \xi(\sigma+it) \text{ with } 0<t<T$$

$$N_s^{(j)}(T) = \sum_{\substack{\rho^{(j)}=\beta+i\gamma \\ m(\gamma)=1}} 1 \qquad \text{simple zeros of } \xi^{(j)}(\sigma+it) \text{ with } 0<t<T$$

$$N_{s,\frac{1}{2}}^{(j)}(T) = \sum_{\substack{\rho^{(j)}=\frac{1}{2}+i\gamma \\ m(\gamma)=1}} 1 \qquad \text{simple zeros of } \xi^{(j)}(\tfrac{1}{2}+it) \text{ with } 0<t<T$$

$$M_r(T) = \sum_{\substack{\rho^{(0)}=\beta+i\gamma \\ m(\gamma)=r}} 1 \qquad \text{zeros of } \xi(\sigma+it) \text{ of multiplicity } r \text{ with } 0<t<T$$

$$M_{\leq r}(T) = \sum_{\substack{\rho^{(0)}=\beta+i\gamma \\ m(\gamma)\;\;r}} 1 \qquad \text{zeros of } \xi(\sigma+it) \text{ of multiplicity } \leq r \text{ with } 0<t<T$$

where all sums are over $0 < \gamma < T$, and zeros are counted according to their multiplicity. It is well known that $N^{(j)}(T) \sim \frac{1}{2\pi}T\log T$. Let

$$\alpha_j = \liminf_{T\to\infty} \frac{N_{s,\frac{1}{2}}^{(j)}(T)}{N^{(j)}(T)}. \qquad\qquad \beta_j = \liminf_{T\to\infty} \frac{N_s^{(j)}(T)}{N^{(j)}(T)}.$$

Thus, β_j is the proportion of zeros of $\xi^{(j)}(s)$ which are simple, and α_j is the proportion which are simple and on the critical line. The best currently available bounds are $\alpha_0 > 0.40219$, $\alpha_1 > 0.79874$, $\alpha_2 > 0.93469$, $\alpha_3 > 0.9673$, $\alpha_4 > 0.98006$, and $\alpha_5 > 0.9863$. These bounds were obtained by combining Theorem 2 of [C2] with the methods of [C1]. Trivially, $\beta_j \geq \alpha_j$.

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Let $N_d(T)$ be the number of distinct zeros of $\xi(s)$ in the region $0 < t < T$. That is,

$$N_d(T) = \sum_{n=1}^{\infty} \frac{M_n(T)}{n}. \tag{1}$$

It is conjectured that all of the zeros of $\xi(s)$ are distinct: $N_d(T) = N(T)$, or equivalently, all of the zeros are simple: $N_s^{(0)}(T) = N(T)$. From the bound on α_0 we have $N_s^{(0)}(T) > \kappa\, N(T)$, with $\kappa = 0.40219$. We will use the bounds on β_j to obtain the following

Theorem. *For T sufficiently large,*

$$N_d(T) > k\, N(T),$$

with $k = 0.63952\ldots$. Furthermore, given the bounds on β_j, this result is best possible.

We present two methods for determining lower bounds for $N_d(T)$. These methods employ combinatorial arguments involving the β_j. We note that the added information that α_j detects zeros on the critical line is of no use in improving our result. Everything below is phrased in terms of the Riemann ξ-function, but the manipulations work equally well for any function such that it and all of its derivatives have the same number of zeros. We write $f(T) \gtrsim g(T)$ for $f(T) \geq g(T) + o(N(T))$ as $T \to \infty$. For example, $N_s^{(j)}(T) \gtrsim \beta_j N(T)$ means $N_s^{(j)}(T) \geq (\beta_j + o(1))\, N(T)$ as $T \to \infty$.

The first method starts with the following inequality of Conrey, Ghosh, and Gonek [CGG]. A simple counting argument yields

$$N_d(T) \geq \sum_{r=1}^{R} \frac{M_{\leq r}(T)}{r(r+1)} + \frac{M_{\leq R+1}(T)}{R+1}. \tag{2}$$

To obtain lower bounds for $M_{\leq r}(T)$ we note that if ρ is a zero of $\xi(s)$ of order $m \geq n+2$ then ρ is a zero of order $m - n \geq 2m/(n+2) \geq 2$ for $\xi^{(n)}(s)$. Thus,

$$N_s^{(n)}(T) \leq N(T) - \frac{2}{n+2}\big(N(T) - M_{\leq n+1}(T)\big),$$

which gives

$$M_{\leq n}(T) \gtrsim \left(\frac{\beta_{n-1}(n+1) - n + 1}{2}\right) N(T). \tag{3}$$

The bounds for α_j now give: $M_{\leq 1}(T) \gtrsim 0.40219N(T)$, $M_{\leq 2}(T) \gtrsim 0.69812N(T)$, $M_{\leq 3}(T) \gtrsim 0.86938N(T)$, $M_{\leq 4}(T) \gtrsim 0.91825N(T)$, $M_{\leq 5}(T) \gtrsim 0.94019N(T)$, and $M_{\leq 6}(T) \gtrsim 0.9520N(T)$. Inserting these bounds into inequality (2) with $R = 5$ gives $N_d(T) \gtrsim 0.62583N(T)$. We note that the lower bounds for $M_{\leq n}(T)$ are best possible in the sense that, for each n separately, equality could hold in (3). However, inequality (3) is not simultaneously sharp for all n, and this possibility imparts some weakness to the result. A lower bound for $N_d(T)$ was calculated in [CGG] in a spirit similar to the above computation, but it was mistakenly assumed that $M_{\leq n}(T) \gtrsim \beta_{n-1}N(T)$, rendering their bound invalid.

Our second method eliminates the loss inherent in the first method. We start with this

Lemma. *In the notation above,*

$$N_s^{(n)}(T) \le \sum_{j=1}^{n+1} M_j(T) + n \sum_{j=n+2}^{\infty} \frac{M_j(T)}{j}.$$

Proof. Suppose ρ is a zero of order j for $\xi(s)$. If $j \ge n+2$ then ρ is a zero of order $j-n$ for $\xi^{(n)}(s)$, so $\xi^{(n)}(s)$ has at least $\displaystyle\sum_{j=n+2}^{\infty} \frac{(j-n)M_j(T)}{j}$ zeros of order ≥ 2. Thus,

$$\begin{aligned} N_s^{(n)}(T) &\le N^{(n)}(T) - \sum_{j=n+2}^{\infty} \frac{(j-n)M_j(T)}{j} \\ &= \sum_{j=0}^{\infty} M_j(T) - \sum_{j=n+2}^{\infty} \frac{(j-n)M_j(T)}{j} \\ &= \sum_{j=1}^{n+1} M_j(T) + n \sum_{j=n+2}^{\infty} \frac{M_j(T)}{j}, \end{aligned}$$

as claimed.

Combining the Lemma with (1) we get

$$N_s^{(n)}(T) \le nN_d(T) + n \sum_{j=1}^{n+1} \left(\frac{1}{n} - \frac{1}{j}\right) M_j(T). \tag{4}$$

Let I_n denote the inequality (4). Then, in the obvious notation, a straightforward calculation finds that the inequality

$$I_J \ + \ \sum_{n=1}^{J-1} 2^{J-n-1} I_n$$

is equivalent to

$$\left(2^J - 1\right) N_d(T) + \sum_{n=1}^{J+1} \frac{M_n(T)}{n} \ \ge \ 2^{J-1} M_1(T) + N_s^{(J)}(T) + \sum_{n=1}^{J-1} 2^{J-n-1} N_s^{(n)}(T). \tag{5}$$

This implies

$$\begin{aligned} N_d(T) &\ge 2^{-J} \left(2^{J-1} N_s^{(0)}(T) + N_s^{(J)}(T) + \sum_{n=1}^{J-1} 2^{J-n-1} N_s^{(n)}(T) \right) \\ &\gtrsim 2^{-J} \left(2^{J-1} \beta_0 + \beta_J + \sum_{n=1}^{J-1} 2^{J-n-1} \beta_n \right) N(T). \end{aligned} \tag{6}$$

Choose $J = 5$ and use the trivial inequality $\beta_j \ge \alpha_j$ and the bounds for α_j to obtain the Theorem.

Finally, we show that our result is best possible. In other words, if our lower bounds for the β_j were actually equalities, then the lower bound given by (6) is sharp. We will accomplish this by showing that the $M_n(T)$, the number of zeros of $\xi(s)$ with multiplicity exactly n, can be assigned values which achieve the bounds on β_j, and which yield a value of $N_d(T)$ which is arbitrarily close to the lower bound given by (6).

Suppose we have lower bounds for β_j, for $0 \le j \le J$, and let $K \ge J+2$. Suppose we had the following four equalities:

$$M_1(T) = \beta_0 N(T),$$

$$M_K(T) = \tfrac{K}{K-J}(1-\beta_J)N(T),$$

$$M_{J+1}(T) = \frac{J+1}{2}\left(\beta_J - \beta_{J-1} - \frac{1-\beta_J}{K-J}\right)N(T),$$

and for $2 \le n \le J$,

$$M_n(T) = \frac{n}{2}\left(\frac{3\beta_{n-1}}{2} - \beta_{n-2} - 2^{n-J-1}\beta_J - \frac{1-\beta_J}{2^{J-n+1}(K-J)} - \sum_{j=n}^{J-1} 2^{n-j-2}\beta_j\right)N(T)$$

and $M_j(T) = 0$ otherwise. Then $\sum_{j=1}^{\infty} M_j(T) = N(T)$ and for $0 \le n \le J$ we have

$$\sum_{j=1}^{n+1} M_j(T) + n\sum_{j=n+2}^{\infty}\frac{M_j(T)}{j} = \beta_n N(T), \tag{7}$$

and

$$\sum_{n=1}^{\infty}\frac{M_n(T)}{n} = 2^{-J}\left(2^{J-1}\beta_0 + \beta_J + \sum_{n=1}^{J-1} 2^{J-n-1}\beta_n\right)N(T) + \frac{(1-\beta_J)2^{-J}}{K-J}N(T). \tag{8}$$

Since the left side of (8) is $N_d(T)$ and the second term on the right side can be made arbitrarily small by choosing K large, we conclude that (6) is sharp. There are two things left to check. The given values of $M_n(T)$ must be positive when K is large. It is easy to check this for $J=5$ and our lower bounds for β_j. And since we supposed that our bounds for β_j are sharp, we must show that $N_s^{(j)}(T) = \beta_j N(T)$. To see this, note that, generically, the left side of (7) equals $N_s^{(j)}(T)$. In other words, the zeros of the derivatives of a generic function are all simple, except for those which are "tied up" in high-order zeros of the original function.

By computing further values of α_j, enabling us to take a larger value of J in (6), we could improve the result slightly: this is due to a decrease in the loss in passing from (5) to (6). The bound $M_{\le 6}(T) \gtrsim 0.952N(T)$ implies that this improvement could increase the lower bound we obtained by at most $0.00021N(T)$.

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