AN INTEGRAL EQUATION RELATED TO BESSEL FUNCTIONS

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1. **Introduction.** A study of the integral equation,

$$2 \int_0^\infty \exp(-\alpha \mid s - t \mid - 2t) f(t)\, dt = \lambda f(s) \qquad (\alpha > 0) \tag{1.1}$$

provides a rapid way of obtaining results concerning Bessel functions. This equation arose in the researches of Professor Mark Kac on random noise in radio receivers.

The general procedure will be to examine the kernel and, after solving the equation, use the properties of its corresponding eigenvalues and eigenfunctions. It will be shown that the kernel is positive definite and that the solutions of the equation are Bessel functions. This information and a few theorems from the theory of positive definite kernels, notably Mercer's theorem, lead with great simplicity and rapidity to the proof that the sum of the reciprocals of the squares of the roots of $J_p(z)$ is equal to $[4(p + 1)]^{-1}$ and to the proof that $J_p(z)$ has no complex roots, where in each case $p > -1$. With equal rapidity the positive definiteness of the kernel leads to the completeness in L_2 over (0, 1) of the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0; J_{\alpha-1}(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$, and to the completeness in C over (0, 1) of the set $\{J_p(r_n z)\}$ $[p > -1; J_p(r_n)' = 0; r_n > 0, n = 1, 2, 3, \cdots]$. It is also possible to prove that the set $\{1, J_0(r_n z)\}$ $[J_1(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$ is complete in C over (0, 1). The author thanks Professor W. A. Hurwitz for suggesting this latter proof. The use of the completeness of a kernel to prove the completeness of a set (in C) was first employed by Kneser [3; §32] but, in our opinion, in a more laborious manner due to a less convenient kernel. It may be worth mentioning that we found it rather curious that our integral equation, which arose in a perfectly natural way in a physical problem, led directly to the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0; J_{\alpha-1}(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$. In Kneser's treatment one is led to the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0; J_\alpha(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$.

2. **The kernel and its properties.** We may transform the integral equation into one with a symmetrical kernel. If we multiply the integral equation (1.1) by e^{-s} and call

$$e^{-t} f(t) = \phi(t),$$

we get

$$2 \int_0^\infty \exp[-\alpha \mid s - t \mid - (s + t)] \phi(t)\, dt = \lambda \phi(s) \qquad (\alpha > 0) \tag{2.1}$$

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as the new equation with the symmetric kernel

$$K(s, t) = 2 \exp [-\alpha \mid s - t \mid - (s + t)] \qquad (\alpha > 0).$$

Since

$$\int_0^\infty \int_0^\infty \mid K(s, t) \mid^2 dt\, ds < 1,$$

we note that the kernel belongs to L_2 over $(0, \infty)$. An important fact is that the kernel is *positive definite*. We may write

$$\begin{aligned} K(s, t) &= \exp [-\alpha \mid s - t \mid - (s + t)] \\ &= (\alpha/\pi) \exp (-s - t) \int_{-\infty}^\infty (\alpha^2 + u^2)^{-1} \exp [iu(s - t)]\, dt, \end{aligned}$$

the Fourier transform of a positive function. Hence, by a well-known theorem [1; 74], it is positive definite.

Since the kernel is symmetric, belongs to L_2 over $(0, \infty)$, and is positive definite, we list some of its properties [2; Part I, 6].

(i) The kernel is complete in L_2 .

(ii) The eigenvalues are all positive.

(iii) The set of eigenfunctions belonging to the kernel is complete in L_2 over $(0, \infty)$.

(iv) The kernel can be expanded in an absolutely and uniformly convergent series of its eigenfunctions,

$$K(s, t) = \sum_{n=1}^\infty \lambda_n \phi_n(s) \phi_n(t),$$

and furthermore

$$\int_0^\infty K(s, s)\, ds = \sum_{n=1}^\infty \lambda_n ,$$

which also converges absolutely (Mercer's theorem). Note that our λ_n $(n = 1, 2, 3, \cdots)$ are the reciprocals of the eigenvalues as usually defined.

3. **The solution of the equation.** The form of the solution of the equation (1.1) may be obtained by reducing the integral equation to a differential equation of a known type.

We rewrite (1.1) in the form

$$\begin{aligned} e^{-\alpha s} \int_0^s \exp [(\alpha - 2)t] f(t)\, dt + e^{\alpha s} \int_s^\infty \exp [-(\alpha + 2)t] f(t)\, dt \\ = (\lambda/2) f(s) \qquad (\alpha > 0). \end{aligned} \tag{3.1}$$

Differentiating twice with respect to s, and substituting (3.1) in the result we obtain

$$f''(s) + [(4\alpha/\lambda) \exp(-2s) - \alpha^2]f(s) = 0 \qquad (\lambda \neq 0).$$

This reduces to Bessel's differential equation of order α by the following substitutions,

$$f(s) = g[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] \tag{3.2}$$

and

$$x = 2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}. \tag{3.3}$$

Thus, we get

$$x^2g''(x) + xg'(x) + (x^2 - \alpha^2)g(x) = 0,$$

whose general solution is

$$g(x) = \begin{cases} AJ_\alpha(x) + BJ_{-\alpha}(x) & (\alpha \neq \text{an integer}) \\ AJ_\alpha(x) + BY_\alpha(x) & (\alpha = \text{an integer};\ x \neq 0). \end{cases}$$

Therefore, we have, by virtue of (3.2) and (3.3), that the general form of the solution of (1.1) is

$$f(s) = \begin{cases} AJ_\alpha[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] + BJ_{-\alpha}[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] & (\alpha \neq \text{an integer}), \\ AJ_\alpha[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] + BY_\alpha[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] & (\alpha = \text{an integer}). \end{cases}$$

Now, consider the original equation (1.1). If we let $s \to \infty$, the left side approaches zero. Hence, there is a boundary condition on $f(s)$ that it approach zero as $s \to \infty$. From which we conclude that B must be zero. Now, since A is arbitrary, we may assume it to be 1. Hence, the solution of (1.1) must be of the form,

$$f(s) = J_\alpha[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] \qquad (\alpha, \lambda > 0), \tag{3.4}$$

with no restrictions on whether α is an integer or not.

As is to be expected from the theory of integral equations, this form of the solution does not hold for all values of λ. In order to determine the eigenvalues consider

$$I = 2\int_0^\infty \exp[-\alpha \mid s - t \mid - 2t]J_\alpha[2(\alpha/\lambda)^{\frac{1}{2}}e^{-t}]\, dt \qquad (\alpha, \lambda > 0).$$

It can be written

$$I = 2\int_0^\infty \exp[-\alpha \mid s - t \mid - 2t] \sum_{n=0}^\infty \frac{(-1)^n[(\alpha/\lambda)e^{-2t}]^{n+\alpha/2}}{n!\Gamma(n + \alpha + 1)}\, dt \qquad (\alpha, \lambda > 0).$$

If we reverse the order of integration and summation, after some simplification we find that

$$I = \lambda J_\alpha[2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}] - (\lambda/\alpha)^{\frac{1}{2}}e^{-\alpha s}J_{\alpha-1}[2(\alpha/\lambda)^{\frac{1}{2}}].$$

We have, therefore, that the necessary and sufficient conditions for (3.4) to be the non-trivial solution of the integral equation (1.1) is that

$$J_{\alpha-1}[2(\alpha/\lambda)^{\frac{1}{2}}] = 0 \qquad (\alpha, \lambda > 0). \tag{3.5}$$

Thus, the eigenvalues are

$$\lambda_n = 4\alpha r_n^{-2} \qquad [\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]. \tag{3.6}$$

4. **Results involving roots of Bessel functions.** (i) *The sum of the reciprocals of the squares of* $J_p(z)$ $(p > -1)$ *is equal to* $[4(p+1)]^{-1}$. Applying Mercer's theorem, we have

$$\sum_{n=1}^{\infty} \lambda_n = \int_0^\infty K(s, s)\, ds = 1. \tag{4.1}$$

Letting $\alpha = p + 1$ in (3.6) and putting the result in (4.1), we obtain

$$\sum_{n=1}^{\infty} r_n^{-2} = [4(p+1)]^{-1} \qquad [p > -1;\ J_p(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots].$$

Note that the completeness of the set of Bessel functions is unnecessary for this proof.

(ii) $J_p(z)$ *has no complex roots* $(p > -1)$. Since $K(s, t)$ is positive definite, its eigenvalues, λ_n, are positive. Since α is positive also, $(\alpha/\lambda)^{\frac{1}{2}}$ is real. Letting $\alpha = p + 1$ in (3.5) and (3.6), we have proved that $J_p(z)$ has no complex roots $(p > -1)$.

(iii) *An identity.* If we substitute the solution (3.4) and the value of λ (3.6) in (1.1) and let $x = e^{-s}$ and $z = e^{-t}$, we obtain

$$\int_0^1 z \exp[-\alpha \mid \log z/x \mid] J_\alpha(r_n z)\, dz = 2\alpha r_n^{-2} J_\alpha(r_n x) \qquad (\alpha > 0).$$

Allowing x to approach 1, we have

$$\int_0^1 z^{\alpha+1} J_\alpha(r_n z)\, dz = 2\alpha r_n^{-2} J_\alpha(r_n) \qquad (\alpha > 0).$$

Repeated integration by parts gives the identity

$$J_\alpha(r_n) = \frac{r_n^2 \Gamma(n+2)}{4(\alpha^2 + \alpha) - r_n^2} \sum_{p=1}^{\infty} \frac{(r_n/2)^p J_{\alpha+p}(r_n)}{\Gamma(\alpha + p + 2)}$$
$$[\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots].$$

The details of this and the other items where the work is briefly outlined appear in the author's M. S. thesis of the same title.

5. **Results involving the completeness of sets of Bessel functions.** (i) *Completeness in* L_2 *of* $\{J_\alpha(r_n z)\}$ $[\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$. Let $f(z) \in L_2$ over $(0, 1)$ and suppose that

$$\int_0^1 J_\alpha(r_n z) f(z)\, dz = 0 \qquad [\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots].$$

Letting $z = e^{-s}$ and

$$f(z) = f(e^{-s}) = h(s),$$

we get

$$\int_0^\infty e^{-s} J_\alpha(r_n e^{-s}) h(s)\, ds = 0 \qquad [\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots].$$

However, from property (iii), §2, of positive definite kernels, the set $\{e^{-s} J_\alpha(r_n e^{-s})\}$ $[\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$ is complete in L_2 over $(0, \infty)$ since it belongs to the positive definite kernel

$$K(s, t) = 2 \exp\left[-\alpha \mid s - t \mid - (s + t)\right] \qquad (\alpha > 0).$$

Hence, $h(s) = 0$ almost everywhere. Hence, $f(z) = 0$ almost everywhere. Therefore, we have the main result of this investigation that the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$ is complete in L_2 over $(0, 1)$.

(ii) *Completeness in* C *of* $\{J_p(r_n z)\}$ $[p > -1;\ J_p(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$. Let $h(z) \in C$ over $(0, 1)$ and let us define

$$f(z) = z^{\alpha-1} h(z) \qquad (\alpha > 0) \tag{5.1}$$

and

$$F(z) = \int_0^z f(t)\, dt. \tag{5.2}$$

Suppose

$$\int_0^1 J_{\alpha-1}(r_n z) h(z)\, dz = 0 \qquad [\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots].$$

Using the relation

$$\frac{d}{dz}\left[z^{-p} J_p(z)\right] = -z^{-p} J_{p+1}(z)$$

in the above, we have

$$\int_0^1 J_{\alpha-1}(r_n z) h(z)\, dz = \left[J_{\alpha-1}(r_n z) z^{-(\alpha-1)} F(z)\right]_0^1 + r_n \int_0^1 J_\alpha(r_n z) z^{-(\alpha-1)} F(z)\, dz = 0$$

$$(\alpha > 0).$$

The first member of the right side vanishes at both the upper and lower limits since $J_{\alpha-1}(r_n) = 0$ and $z^{-\alpha+1}J_\alpha(r_n z)$ $(\alpha > 0)$ is bounded as $z \to 0$, while $F(0) = 0$. Thus, we have

$$\int_0^1 J_{\alpha-1}(r_n z)h(z)\, dz = r_n \int_0^1 J_\alpha(r_n z) z^{-(\alpha-1)}F(z)\, dz = 0$$

$$[\alpha > 0;\ J_{\alpha-1}(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots].$$

Now, from (5.1) and (5.2),

$$z^{-(\alpha-1)}F(z) = z^{-(\alpha-1)} \int_0^z t^{\alpha-1}h(t)\, dt \qquad (\alpha > 0).$$

Since $h(z)$ is bounded over $(0, 1)$, say by M, we have

$$| z^{-(\alpha-1)}F(z) | \leq \frac{Mz}{\alpha} \qquad (0 \leq z \leq 1;\ \alpha > 0),$$

and, hence, $z^{-\alpha+1}F(z)$ is certainly in L_2 over $(0, 1)$. Applying the results of (i) of this section we conclude that $F(z) = 0$ and, hence, $f(z) = 0$. Thus, by virtue of (5.1), we have $h(z) = 0$. Therefore, if we let $\alpha = p + 1$, we have the result that the set of functions $\{J_p(r_n z)\}$ $[p > -1;\ J_p(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$ is complete in C over $(0, 1)$.

(iii) *Completeness in C of* $\{1,\ J_0(r_n z)\}$ $[J_1(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$. Let $f(z) \in C$ and define as in (5.2)

$$F(z) = \int_0^z f(t)\, dt.$$

Suppose

$$\int_0^1 f(z)\, dz = 0 \tag{5.3}$$

and

$$\int_0^1 J_0(r_n z) f(z)\, dz = 0.$$

Integrating by parts, we obtain

$$\int_0^1 J_0(r_n z) f(z)\, dz = [J_0(r_n z)F(z)]_0^1 - r_n \int_0^1 J_0'(r_n z)F(z)\, dz$$

$$= \int_0^1 J_1(r_n z)F(z)\, dz = 0,$$

the first member of the right side vanishing by virtue of (5.2) and (5.3). Applying the results of (ii) of this section to the case when $p = 1$, we conclude that the set $\{1,\ J_0(r_n z)\}$ $[J_1(r_n) = 0;\ r_n > 0,\ n = 1, 2, 3, \cdots]$ is complete in C over $(0, 1)$.

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