LEVEL SETS AND THE DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS

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Abstract. The purpose of this paper is to investigate the connection between the level set structure of a certain class of even, real entire functions f and the distribution of zeros of some functions formed from f. In particular it is shown that for all nonzero real numbers t, the entire functions $f(z+t) \pm f(z-t)$ have infinitely many zeros on the imaginary axis.

1. Introduction. Let $S(A)$ denote the closed strip of width $2A$ in the complex plane $\mathbb{C}$ symmetric about the real axis:

$$S(A) = \{z \in \mathbb{C} : |\operatorname{Im}(z)| \leq A\}, \tag{1.1}$$

where $A \geq 0$.

Definition 1.1. *Let A be such that $0 \leq A < \infty$. We say that a real entire function f belongs to the class $\mathfrak{S}(A)$ if f is of the form*

$$f(z) = ce^{-az^2+bz} z^m \prod_{k=1}^{\infty} (1 - \frac{z}{z_k}) e^{z/z_k}, \tag{1.2}$$

where $a \geq 0$, m is a nonnegative integer, c is a nonzero real number, $z_k \in S(A) \setminus \{0\}$, and $\sum_{k=1}^{\infty} 1/|z_k|^2 < \infty$.

We allow functions in $\mathfrak{S}(A)$ to have only finitely many zeros by letting, as usual, $z_k = \infty$ and $0 = 1/z_k$, $k \geq k_0$, so that the canonical product in (1.2) is a finite product. The

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significance of the class $\mathfrak{S}(A)$ in the theory of entire functions is natural, since $f \in \mathfrak{S}(A)$ if and only if f is the uniform limit on compact sets of a sequence of real polynomials having zeros only in the strip $S(A)$ ([Br, p. 202]). The Gauss-Lucas Theorem [M, p. 22] tells us that this class of polynomials is closed under differentiation, and thus so is $\mathfrak{S}(A)$. The class $\mathfrak{S}(0)$ is also called the *Laguerre-Pólya class*, written $\mathcal{L}$-$\mathcal{P}$, so a function $f \in \mathcal{L}$-$\mathcal{P}$ has only real zeros.

For $f \in \mathfrak{S}(A)$, $\theta \in \mathbb{R}$, and $t \in \mathbb{R}$, we are interested in describing the distribution of zeros of the function

$$g_{t,\theta}(z) = e^{i\theta} f(z+t) + e^{-i\theta} f(z-t). \tag{1.3}$$

To motivate our results and to provide some background information, we recall from the literature (see, for example, [Br]), that if $f \in \mathfrak{S}(A)$, then the zeros of the entire function $g_{it,0}(z) = f(z+it) + f(z-it)$, $(t \in \mathbb{R})$ are "closer" to the real axis than those of f. More precisely, de Bruijn [Br, Theorem 8] showed that the zeros of $g_{it,0}$ all lie in the strip $S(\sqrt{A^2-t^2})$, if $|t| < A$ and $g_{it,0} \in \mathfrak{S}(0)$, if $0 \le A \le |t|$. The relationship between the zero set of $g_{it,0}$ and that of f has been studied by several authors (see, for example, [Br], [K], [M, Sections 17 and 18], [O, p. 271]). In sharp contrast to de Bruijn's result, we will show that for a certain subclass of functions in $\mathfrak{S}(A)$ (see Definition 1.2 below), the entire function $g_{t,\theta}$ has infinitely many zeros on the imaginary axis whenever $t \neq 0$.

We begin by explaining the reason for confining our attention to a subclass of functions, $f \in \mathfrak{S}(A)$. In the first place, the results below show that the growth of f has a major influence on the distribution of zeros of $g_{t,\theta}$. Recall that an entire function f is said to be of *exponential type* [Bo, p. 8] if there are constants C_1 and C_2 such that

$$|f(z)| \leq C_1 \exp(C_2|z|), \quad z \in \mathbb{C}.$$

By way of illustration, consider the real entire function $f(z) = \cos(z)$ and note that $g_{t,0}(z) = 2\cos(z)\cos(t)$ has only real zeros, provided that $\cos(t) \neq 0$. This example shows that it is possible that when $f \in \mathfrak{S}(A)$ is of exponential type, then $g_{t,0} \in \mathfrak{S}(A)$. Now it follows from our main result (Theorem 1.3) that this *always fails* when $f \in \mathfrak{S}(A)$ is an even entire function which is *not* of exponential type. We pause for a moment to introduce a notation for this class of functions.

Definition 1.2. *An even entire function $f \in \mathfrak{S}(A)$ belongs to the class $\mathfrak{S}_\infty(A)$ (written $f \in \mathfrak{S}_\infty(A)$), if f is not of exponential type.*

If in the definition of the class $\mathfrak{S}_\infty(A)$ we omit the assumption that f is even, then, in general, $g_{t,\theta}$ does not vanish on the imaginary axis, as the following example shows. Let $f(x) = (x+1)\exp(-x^2)$. Then with $t = 1/4$ it is easy to check that

$$g_{1/4,0}(iy) = 2\exp(y^2 - 1/16)\{\cos(y/2) + i[y\cos(y/2) - \sin(y/2)/4]\} \neq 0$$

for any $y \in \mathbb{R}$, that is, $g_{1/4,0}$ does not vanish on the imaginary axis. On the other hand, if f is an even, real entire function, then $\overline{f(iy-t)} = f(-iy-t) = f(iy+t)$, whence we have the simple, but important, relation

$$g_{t,\theta}(iy) = 2\,\mathrm{Re}(e^{i\theta} f(iy+t)), \qquad t,\, y \in \mathbb{R}.$$

In particular, the level set $\{z : \mathrm{Re}(e^{i\theta} f(z)) = 0\}$ determines the zeros of $g_{t,\theta}$ on the imaginary axis.

Preliminaries aside, we are now in position to formulate our principal result as follows.

Theorem 1.3. *Let $f \in \mathfrak{S}_\infty(A)$. Then for any $t \in \mathbb{R} \setminus \{0\}$ and for any $\theta \in \mathbb{R}$, the entire function $g_{t,\theta}(z) = e^{i\theta} f(z+t) + e^{-i\theta} f(z-t)$ has infinitely many zeros on the imaginary axis.*

In Section 2, we will establish some of the fundamental properties of functions in $\mathfrak{S}_\infty(A)$ (Lemma 2.1 and Lemma 2.2). Moreover, after a brief review of some facts about level sets, we obtain some quantitative information concerning the level set structure of an entire function f, in terms of the growth of the logarithmic derivative of f (Theorem 2.3). This preliminary investigation enables us to show the existence of at least one purely imaginary zero of $g_{t,\theta}$ (Corollary 2.4). The proof of Theorem 1.3 requires additional analysis of the geometric nature of the level set of a function in $\mathfrak{S}_\infty(A)$ (Lemma 3.1 and Lemma 3.2). Theorem 3.3 gives a lower bound for the number of zeros of $g_{t,\theta}$ on certain segments of the imaginary axis. Since this lower bound is an unbounded function of r, Theorem 1.3 is a consequence of Theorem 3.3. The paper concludes with some remarks and open problems (Section 4).

2. Properties of functions in $\mathfrak{S}_\infty(A)$.

Since $f \in \mathfrak{S}_\infty(A)$ is even, we see from (1.2) that $f(z)$ can be represented in the form

$$f(z) = ce^{-az^2} z^{2m} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right), \tag{2.1}$$

where c is a nonzero real number, $a \geq 0$, m is a nonnegative integer, $z_k \in S(A) \setminus \{0\}$ and $\operatorname{Re} z_k \geq 0$, and $\sum 1/|z_k|^2 < \infty$. The reader will note that $f(z)$ may have some purely imaginary zeros, in which case the representation of $f(z)$ in (2.1) is not unique. The choice of representation is inconsequential. However, to be specific, we assume that if $\operatorname{Re} z_k = 0$, then $\operatorname{Im} z_k > 0$. We use the usual notation of $n(r)$ for the number of zeros of f having modulus at most r. Thus,

$$n(r) = 2\,\mathrm{card}\{k : |z_k| \le r\} + 2m,$$

where the first factor of 2 is required because f is even and the $\{z_k\}$ only account for the zeros of f in the right half-plane. Here $\mathrm{card}(E)$ denotes the cardinality of the set E.

It is clear that if $f \in \mathfrak{S}_\infty(A)$ and if $a = 0$ in (2.1), then f has an infinite number of zeros, since $\mathfrak{S}_\infty(A)$ contains no polynomials. A refinement of this observation is that if $f \in \mathfrak{S}_\infty(A)$ and if $a = 0$ in (2.1), then $n(r)/r$ is unbounded. When the order ρ of f is 1, this follows from a classical theorem of Lindelöf ([Bo, p. 27]) which implies that an even entire function of order 1 is of exponential type if and only if $n(r)/r$ is bounded. Since $f \in \mathfrak{S}_\infty(A)$ is not of exponential type, it remains to consider the case when f has order $\rho > 1$. Recall that the order of a canonical product is equal to the convergence exponent, ρ_1, of its zeros ([Bo, p. 19]), and ([Bo, p. 15])

$$\rho_1 = \varlimsup_{r\to\infty} \frac{\log n(r)}{\log r}.$$

Thus $\rho_1 = \rho > 1$ again implies that $n(r)/r$ is unbounded. The next lemma provides a convenient reformulation of this observation.

Lemma 2.1. *Let $f \in \mathfrak{S}_\infty(A)$ and suppose that f is given by* (2.1) *with $a = 0$. Set*

$$\sigma(r) = \sum_{0<\operatorname{Re} z_k \le r} \frac{1}{|z_k|}. \tag{2.2}$$

Then

$$\varlimsup_{r\to\infty} \left[\sigma(r) - \sigma(r/2)\right] = \infty. \tag{2.3}$$

Proof. Let $f \in \mathfrak{S}_\infty(A)$ be given by (2.1) with $a = 0$, and assume that $r \geq 4A$. If $3r/4 < |z_k|$ and $\operatorname{Re} z_k > 0$, then

$$\frac{3}{4}r < |z_k| \leq \operatorname{Re} z_k + A \leq \operatorname{Re} z_k + \frac{r}{4},$$

and so $r/2 < \operatorname{Re} z_k$. Thus

$$\sigma(r) - \sigma(r/2) = \sum_{\frac{r}{2} < \operatorname{Re} z_k \leq r} \frac{1}{|z_k|} \geq \sum_{\frac{3}{4}r < |z_k| \leq r} \frac{1}{|z_k|} \geq \frac{1}{2}\frac{n(r) - n(3r/4)}{r}. \tag{2.4}$$

Proceeding with a proof by contradiction, we assume that (2.3) does not hold. Then there is a positive number M such that $0 \leq \sigma(r) - \sigma(r/2) \leq M$, and by (2.4)

$$n(r) - n(3r/4) \leq 2Mr, \qquad r \geq 4A.$$

Recalling that $r \geq 4A$, let K be the largest integer such that $r(3/4)^K \geq 4A$. Then

$$n(r) = \sum_{j=0}^{K} \left[n(\frac{r3^j}{4^j}) - n(\frac{r3^{j+1}}{4^{j+1}}) \right] + n(\frac{r3^{K+1}}{4^{K+1}}) \leq 2M \sum_{j=0}^{\infty} \frac{r3^j}{4^j} + n(4A) = 8Mr + n(4A),$$

$r \geq 4A$, and so $n(r)/r$ is bounded. This is the desired contradiction, since it was observed before the statement of the lemma that $n(r)/r$ is unbounded, and the proof is complete. $\square$

In the sequel, we will require the following elementary preparatory result concerning the behavior of the logarithmic derivative of f in $\mathfrak{S}_\infty(A)$ in a vertical strip of the form

$$V_t(r) = \{z = x + iy \,:\, 0 \leq x \leq t,\ y \geq r\}, \qquad (r > 0,\, t > 0).$$

Lemma 2.2. *Let $f(z) \in \mathfrak{S}_\infty(A)$ and let $r \geq 2A + 1$. Then*

$$\operatorname{Im}\left(\frac{f'}{f}(z)\right) < 0, \qquad \operatorname{Im} z > A. \tag{2.5}$$

Furthermore, there is a positive absolute constant B such that

$$\left|\operatorname{Im}\left(\frac{f'}{f}(z)\right)\right| \geq 2a(r-1) + B[\sigma(r) - \sigma(r/2)], \quad z \in V_t(r-1) \setminus V_t(r+1), \tag{2.6}$$

where $a \geq 0$ comes from (2.1) and $\sigma(r)$ is defined by (2.2).

Proof. With $f(z)$ expressed in the form (2.1), let $z_k = x_k + iy_k$ denote the zeros of $f(z)$ with $x_k > 0$. Then, with $z = x + iy$, logarithmic differentiation yields

$$\operatorname{Im}\left(\frac{f'}{f}(z)\right) = -2ay - \frac{2my}{x^2+y^2} - \sum_{k=1}^{\infty}\left[\frac{y-y_k}{(x-x_k)^2+(y-y_k)^2} + \frac{y+y_k}{(x+x_k)^2+(y+y_k)^2}\right],$$

whence (2.5) is clear, since $a \geq 0$ and $A \geq y_k$. Dropping the middle term on the right side of the display above gives

$$\left|\operatorname{Im}\left(\frac{f'}{f}(z)\right)\right| \geq 2ay + \sum_{k=1}^{\infty}\left[\frac{y-y_k}{(x-x_k)^2+(y-y_k)^2} + \frac{y+y_k}{(x+x_k)^2+(y+y_k)^2}\right].$$

Using the hypothesis that $r - 1 \geq 2A$, we have $y \geq r - 1 \geq 2A \geq 2y_k$ for all $z \in V_t(r-1) \setminus V_t(r+1)$. Hence, for such z we get the estimate that

$$\begin{aligned}
\left|\operatorname{Im}\left(\frac{f'}{f}(z)\right)\right| - 2a(r-1) &\geq \frac{1}{2}\sum_{k=1}^{\infty}\frac{y}{(x-x_k)^2+(y-y_k)^2} \\
&\geq c_1 \sum_{r/2 \leq x_k \leq r} \frac{y}{x_k^2+y^2} \\
&\geq c_2 \sum_{r/2 \leq x_k \leq r} \frac{r}{x_k^2+r^2} \\
&\geq c_3 \sum_{r/2 \leq x_k \leq r} \frac{1}{x_k} \geq c_4 \sum_{r/2 < x_k \leq r} \frac{1}{|z_k|},
\end{aligned}$$

where c_j, $1 \leq j \leq 4$, is a positive absolute constant. Since the last term in the display above is equal to $c_4[\sigma(r) - \sigma(r/2)]$, this is equivalent to (2.6), and completes the proof. □

In our next result, we will make use of several fundamental, albeit elementary, properties of level curves which we briefly review here for the reader's convenience (cf. [CSV]). We first recall that if f is a nonconstant real entire function, a component, γ, of the level set

$$\{z \in \mathbb{C} : \operatorname{Re}(e^{i\theta} f(z)) = 0\} \tag{2.7}$$

is a piecewise analytic curve. Such a curve γ is called a *level curve* of f. Note that by the local mapping properties of an analytic function, every zero of f is on some level curve of f. If z_0 is a critical point of f, that is, if $f'(z_0) = 0$, and if z_0 is on the level curve γ, then z_0 is said to be a *branch point* of γ. We observed earlier (cf. Section 1) that if $f \in \mathfrak{S}(A)$, then $f' \in \mathfrak{S}(A)$ also and hence no branch points lie outside the strip $S(A)$.

Now, if γ has no branch points, then again it follows from the local mapping properties of f that the restriction of f to γ is locally a homeomorphism. Thus, if there are no branch points on $\gamma(z_0, z_1)$, then f is one–to–one on $\gamma(z_0, z_1)$, and the restriction of f to $\gamma(z_0, z_1)$ is a homeomorphism. Hence in this case f can have at most one zero on γ. (Simple examples show that f need not have a zero on γ.) Since f is nonconstant, by the maximum principle γ cannot be a closed bounded curve. Moreover, if γ has no branch points, then γ is an analytic curve which separates the plane. Finally, we also note that if f is a nonconstant, even, real entire function, then its level curves are symmetric about the coordinate axes. By virtue of this twofold symmetry of the level curves, we will confine (for the most part) our analysis to the first quadrant.

The next theorem gives quantitative information on the structure of the level set (2.7) of an entire function f, in terms of the growth of f'/f. In the next section this will be used, together with Lemma 2.2, to show that, in a certain sense, the level curves of $f \in \mathfrak{S}_\infty(A)$

in the first quadrant approach the imaginary axis.

Theorem 2.3. *Let f be an entire function and let $\theta \in \mathbb{R}$. Set $P = \{z = x + iy : \operatorname{Re}(e^{i\theta} f(z)) > 0\}$ and $N = \{z = x + iy : \operatorname{Re}(e^{i\theta} f(z)) < 0\}$. Suppose that the disk*

$$D\big(z_0, \delta(z_0)\big) = \{z = x + iy \ : \ |z - z_0| < \delta(z_0)\} \subset P \cup N.$$

Then

$$\delta(z_0) \leq 2 \left| \frac{f}{f'}(z_0) \right|.$$

Before the proof, consider the simple example where $f(z) = \exp(-z^2/2)$ and $\theta = 0$. Then the level set $\{z : \operatorname{Re} f(z) = 0\} = \{x + iy : \cos(xy) = 0\}$ is a family of hyperbolas. Theorem 2.3 says that if the disk $D\big(z_0, \delta(z_0)\big)$ is disjoint from these hyperbolas, then $\delta(z_0) \leq 2/|z_0|$. It is easy to see that this is the correct order of magnitude for our estimate of $\delta(z_0)$, in the sense that $\overline{\lim}_{|z_0| \to \infty} |z_0| \delta(z_0) > 0$.

Proof of Theorem 2.3. We prove the theorem under the assumption that the disk $D\big(z_0, \delta(z_0)\big)$ is contained in the set P. (The proof is the same, *mutatis mutandis*, when this disk is contained in N.) For notational convenience, let $\mathbb{D}$ denote the unit disk centered at the origin and let $\mathbb{H}$ denote the open right half-plane. Consider the composition, $h = \varphi \circ (e^{i\theta} f) \circ \psi$, of the maps

$$\mathbb{D} \xrightarrow{\psi} D\big(z_0, \delta(z_0)\big) \xrightarrow{(e^{i\theta} f)} \mathbb{H} \xrightarrow{\varphi} \mathbb{D},$$

where $\psi(z) = z\delta(z_0) + z_0$ and $\varphi(w) = \dfrac{w - e^{i\theta} f(z_0)}{w + e^{-i\theta} \overline{f(z_0)}}$. Then, $h(z) = \varphi(e^{i\theta} f(z\delta(z_0) + z_0))$ and a computation yields

$$|h'(0)| = |\varphi'(e^{i\theta} f(z_0))||f'(z_0)|\delta(z_0) = \frac{\delta(z_0)}{2} \left| \frac{f'(z_0)}{\operatorname{Re}(e^{i\theta} f(z_0))} \right|.$$

Since $h : \mathbb{D} \longmapsto \mathbb{D}$ is analytic and $h(0) = 0$, $|h'(0)| \leq 1$, by Schwarz's lemma. Therefore, we have the estimate

$$\delta(z_0) \leq 2 \left| \frac{\operatorname{Re}(e^{i\theta} f(z_0))}{f'}(z_0) \right| \leq 2 \left| \frac{f}{f'}(z_0) \right|,$$

as asserted. □

Additional properties of the level curves of functions in $\mathfrak{S}_\infty(A)$ will needed to prove Theorem 1.3. These properties will be established in the next section. At this juncture, we are able to conclude only the following much weaker result. We include it here to indicate how Theorem 2.3, in conjunction with the preceding lemmas, can be used to establish the existence of a purely imaginary zero of the function $g_{t_0,\theta}$ defined below by (2.8).

Corollary 2.4. *Let $f \in \mathfrak{S}_\infty(A)$, let $\theta \in \mathbb{R}$, and let $t > 0$. Then there is a t_0, $0 < t_0 \leq t$, such that the real entire function*

$$g_{t_0,\theta}(z) = e^{i\theta} f(z + t_0) + e^{-i\theta} f(z - t_0) \tag{2.8}$$

has at least one purely imaginary zero in the upper half-plane.

Proof. Fix $t > 0$ and $f \in \mathfrak{S}_\infty(A)$. We will show that $\operatorname{Re}(e^{i\theta} f)$ vanishes at some point in $V_t(0)$. If not, then without loss of generality assume that $\operatorname{Re}(e^{i\theta} f) > 0$ in $V_t(0)$. This means that for all $y \geq t/2$ the disk $D\big(\frac{t}{2} + iy, \frac{t}{2}\big)$ is contained in the strip $V_t(0)$, where $e^{i\theta} f$ has positive real part. Thus, Theorem 2.3 gives

$$\left| \frac{f'}{f}\left(\frac{t}{2} + iy\right) \right| \leq \frac{4}{t}, \quad y \geq \frac{t}{2},$$

which contradicts (2.6). Indeed, when $a > 0$ in the representation (2.1) of f, the contradiction is immediate. When $a = 0$, we get the contradiction by using the assertion of Lemma 2.1 that $\overline{\lim}_{r\to\infty}[\sigma(r) - \sigma(r/2)] = \infty$. Hence there exists $z_0 = t_0 + iy_0 \in V_t(0)$ such that $\mathrm{Re}(e^{i\theta} f(z_0)) = 0 = g_{t_0,\theta}(iy_0)$, as required. $\square$

3. Proof of Theorem 1.3.

The proof of Theorem 1.3 requires a deeper analysis of the relationship between the geometric nature of the level set structure of a real entire function $f \in \mathfrak{S}_\infty(A)$ and the distribution of its zeros. We use Arg to denote the principal branch of the argument and, as in Section 2, we denote by $V_t(r)$ the vertical strip

$$V_t(r) = \{z = x + iy \,:\, 0 \le x \le t,\ y \ge r\}, \qquad (r > 0,\, t > 0).$$

Lemma 3.1. *Let $t > 0$ and $A > 0$ be fixed. For all $\varepsilon > 0$, there exists $M < \infty$ such that if $z \in V_t(M)$ and $\pm iw \in V_A(M)$, then*

$$\left|\mathrm{Arg}\left(\frac{z}{z^2 - w^2}\right) + \frac{\pi}{2}\right| < \varepsilon.$$

Proof. Since $|\,\mathrm{Arg}(z) - \pi/2| < \varepsilon/2$ when $z \in V_t(M)$ and M is sufficiently large, and

$$-\,\mathrm{Arg}\left(\frac{z}{z^2 - w^2}\right) = \mathrm{Arg}\left(z - \frac{w^2}{z}\right) = \mathrm{Arg}(z) + \mathrm{Arg}\left(1 - \frac{w^2}{z^2}\right),$$

it suffices to show that $|\,\mathrm{Arg}\left(1 - w^2/z^2\right)| < \varepsilon/2$ for large M. It is clear from the hypotheses on z and w that M can be chosen so that

$$|\,\mathrm{Arg}(-w^2/z^2)| = |\,\mathrm{Arg}(w^2) + \mathrm{Arg}(-1/z^2)| \le |\,\mathrm{Arg}(w^2)| + |\,\mathrm{Arg}(-1/z^2)| < \varepsilon/2.$$

The result now follows from the inequality $|\,\mathrm{Arg}\left(1 - w^2/z^2\right)| < |\,\mathrm{Arg}(-w^2/z^2)|$. $\square$

We associate with $f \in \mathfrak{S}_\infty(A)$ a sequence of positive numbers $\{r_n\}$ as follows: When $a = 0$ in the representation (2.1) of f, we use Lemma 2.1 to choose $\{r_n\}$ so that

$$\lim_{n\to\infty} [\sigma(r_n) - \sigma(r_n/2)] = \infty. \tag{3.1}$$

When $a > 0$ in (2.1), we simply set $r_n = n$. This definition of $\{r_n\}$ assures that

$$\lim_{n\to\infty} [a(r_n - 1) + \sigma(r_n) - \sigma(r_n/2)] = \infty, \tag{3.2}$$

for all $f \in \mathfrak{S}_\infty(A)$.

Lemma 3.2. *Let $f \in \mathfrak{S}_\infty(A)$ and let $\varepsilon > 0$. Then, for all n sufficiently large,*

$$\left|\operatorname{Arg} \frac{f'}{f}(z) + \frac{\pi}{2}\right| < 2\varepsilon, \qquad z \in V_t(r_n - 1) \setminus V_t(r_n + 1).$$

Proof. Let $\varepsilon > 0$ and choose M as in Lemma 3.1. From the representation (2.1) for $f \in \mathfrak{S}_\infty(A)$, we have

$$\frac{f'}{f}(z) = \frac{2m}{z} + \sum_{k=1}^{N} \frac{2z}{z^2 - z_k^2} - 2az + \sum_{k=N+1}^{\infty} \frac{2z}{z^2 - z_k^2},$$

where N is chosen so that $\operatorname{Re} z_k \geq M$ when $k \geq N$. With N now fixed, the first two terms on the right in the display above tend to 0 as $|z| \to \infty$. We also have from (2.6) and (3.2) that

$$\lim_{n\to\infty} \inf\{|f'(z)/f(z)| : z \in V_t(r_n - 1) \setminus V_t(r_n + 1)\} = \infty,$$

and so

$$\left|\operatorname{Arg}\left(\frac{f'}{f}(z)\right) - \operatorname{Arg}\left(-2az + \sum_{k=N+1}^{\infty} \frac{2z}{z^2 - z_k^2}\right)\right| < \varepsilon,$$

for n sufficiently large and $z \in V_t(r_n - 1) \setminus V_t(r_n + 1)$. Lemma 3.1 asserts that each term in the series above lies in the sector $\{w : |\operatorname{Arg} w + \pi/2| < \varepsilon\}$. Since $-2az$ also lies in this sector when n is sufficiently large, and the sector is closed under addition, the result follows. $\square$

Next we examine what these lemmas say about the geometry of the level curves in $V_t(r)$ of $f \in \mathfrak{S}_\infty(A)$. A level curve in the set (2.7) can be parameterized by $z(s)$ such that

$$e^{i\theta} f(z(s)) = is, \qquad s \in \mathbb{R}.$$

Throughout this section, $z(s)$ will always represent such a parameterization. It is clear that $z(s)$ extends to be analytic in a neighborhood of any point s_0 such that $z(s_0)$ is not a branch point of the level set, and in particular when $|\operatorname{Im} z(s_0)| > A$.

Differentiation of the equation in the display above leads to the relation

$$z'(s)\frac{f'}{f}(z(s)) = \frac{1}{s}. \tag{3.3}$$

Thus we see from Lemma 3.2 that if $t > 0$ and $z(s) \in V_t(r_n - 1) \setminus V_t(r_n + 1)$, then $\operatorname{Arg} z'(s) \to \pi/2$ as $n \to \infty$. Geometrically, this says that the tangents to the level curves in this set approach the vertical for n large. We also observe from (3.3) and (2.5), that

$$\operatorname{Arg} z'(s) = -\operatorname{Arg} \frac{f'}{f}(z(s)) > 0, \quad \operatorname{Im} z(s) > A, \tag{3.4}$$

and so all level curves in $V_t(A)$ are monotone, in the sense that their tangents are never horizontal. These observations are key to the proof of the next theorem.

For $f \in \mathfrak{S}_\infty(A)$ and $t > 0$, we introduce the following notation:

(1) $\ell(r; g_{t,\theta}) = \operatorname{card}(\{z : g_{t,\theta}(z) = 0\} \cap [0, ir])$ denotes the number of zeros $g_{t,\theta}$ on the imaginary axis between 0 and ir;

(2) $k_{t,\theta}(r) = \text{card}(\{z : \text{Re}(e^{i\theta} f(z)) = 0\} \cap [ir, t+ir])$ denotes the number of intersections the line segment $[ir, t+ir]$ has with the level set (2.7);

(3) $j_{t,\theta}(y_1, y_2) = \text{card}(\{z : \text{Re}(e^{i\theta} f(z)) = 0\} \cap [t+iy_1, t+iy_2])$ denotes the number of intersections the line segment $[t+iy_1, t+iy_2]$ has with the level set (2.7).

We note that each of these quantities is finite, since distinct analytic curves intersect in a discrete set.

Theorem 3.3. *Let $f \in \mathfrak{S}_\infty(A)$, let $t > 0$ and let $\{r_n\}$ be as in (3.1). Then*

$$\ell(r_n; g_{t,\theta}) \geq \frac{t}{10} \left(2a(r_n - 1) + B[\sigma(r_n) - \sigma(r_n/2)]\right),$$

for all n sufficiently large, where a comes from the representation (2.1) of f and B is the positive absolute constant from (2.6).

Remark 3.4. As noted in (3.2), this lower bound for $\ell(r_n; g_{t,\theta})$ is unbounded. Hence it is an immediate consequence of Theorem 3.3 that if $f \in \mathfrak{S}_\infty(A)$ and $t > 0$, then $g_{t,\theta}$ has infinitely many zeros on the imaginary axis. Thus, since $g_{-t,\theta} = g_{t,\theta}$, Theorem 1.3 is a consequence of Theorem 3.3.

Proof of Theorem 3.3. For the proof, we analyze how a level curve may enter and leave a rectangle of the form $\{x + iy : 0 < x \leq t, \quad y_0 \leq y \leq r\}$. As was observed above, it follows from (3.4) that all level curves in $V_t(A)$ are monotone in the sense that their tangents are never horizontal. Also, when $e^{i\theta} \neq \pm i$ no level curve in the set (2.7) can intersect the positive imaginary axis, since f is real and nonzero there. Thus the level curves entering the top of the rectangle must all exit either on the right side or the bottom of the rectangle, provided $y_0 \geq A$. On the other hand, when $e^{i\theta} = \pm i$ the imaginary axis is a level curve in the set (2.7). It again follows that a level curve entering the top of the rectangle must exit on the right or bottom of the rectangle, provided $y_0 \geq A$, since f has no branch points outside the strip $S(A)$. Hence, in both cases,

$$k_{t,\theta}(r) \leq k_{t,\theta}(y_0) + j_{t,\theta}(y_0, r), \quad A \leq y_0 < r. \tag{3.5}$$

We note that strict inequality is possible since, for example, a curve may both enter and exit on the right side.

Next, using first Theorem 2.3 and then (2.6), we have that

$$\delta(x + ir_n) \leq 2\left|\frac{f}{f'}(x + ir_n)\right| \leq \frac{2}{2a(r_n - 1) + B[\sigma(r_n) - \sigma(r_n/2)]} = \Delta(r_n), \tag{3.6}$$

for all n sufficiently large, if $\mathrm{Re}(e^{i\theta} f) \neq 0$ in the disk $D(x + ir_n, \delta(x + ir_n))$. Also, using (3.2), we see that $\Delta(r_n) \to 0$ as $n \to \infty$.

Suppose for a moment that the portion of the level curves

$$(V_t(r_n - \Delta(r_n)) \setminus V_t(r_n + \Delta(r_n))) \cap \{z : \mathrm{Re}(e^{i\theta} f(z)) = 0\} \tag{3.7}$$

are actually vertical lines. Since they divide $[ir_n, t + ir_n]$ into $k_{t,\theta}(r_n) + 1$ pieces, and (3.6) implies that the length of each piece is at most $2\Delta(r_n)$, we get the estimate $k_{t,\theta}(r_n) \geq t/(2\Delta(r_n)) - 1$.

For the actual level curves, it follows from (3.4) and Lemma 3.2 that

$$|\operatorname{Arg}(z'(s)) - \pi/2| < 1/100, \quad z(s) \in V_t(r_n - 1) \setminus V_t(r_n + 1), \tag{3.8}$$

for n large. Thus the vertical projection of the set (3.7) onto the line segment $[ir_n, t + ir_n]$ has length less than $t/2$, provided n is large enough so that $\Delta(r_n) \leq t$ and (3.8) holds.

Hence

$$k_{t,\theta}(r_n) \geq \frac{t}{4\Delta(r_n)} - 1, \qquad n \geq n_0. \tag{3.9}$$

Combining this estimate with (3.5), and recalling the definition of $\Delta(r_n)$ given in (3.6) and that $2a(r_n - 1) + B[\sigma(r_n) - \sigma(r_n/2)] \to \infty$ from (3.2), we have

$$j_{t,\theta}(y_0, r_n) \geq \frac{t}{10}\left(2a(r_n - 1) + B[\sigma(r_n) - \sigma(r_n/2)]\right),$$

for all n sufficiently large. Since $g_{t,\theta}(iy) = 0$ if and only if $t + iy$ is in the level set (2.7), $j_{t,\theta}(y_0, r_n)$ represents the number of zeros $g_{t,\theta}$ has in the interval $[iy_0, ir_n]$ on the imaginary axis, and thus the proof is complete. $\square$

4. Concluding remarks and open problems.

We note that the lower bound for the number of zeros of $g_{t,\theta}$ on the positive imaginary axis, given in Theorem 3.3, is independent of θ. The explanation for this is that the lower bound was established by way of estimates involving $k_{t,\theta}(r)$ in (3.5) and (3.9), where we recall that $k_{t,\theta}(r)$ denotes the number of intersections the line segment $[ir, t + ir]$ has with the level set $\{z : \operatorname{Re}(e^{i\theta} f(z)) = 0\}$. The next proposition, which is of independent interest, shows that outside the strip $S(A)$ the curves in $\{z : \operatorname{Re}(e^{i\theta} f(z)) = 0\}$ separate the curves in the level set $\{z : \operatorname{Re} f(z) = 0\}$. An immediate consequence is that $|k_{t,\theta}(r) - k_{t,0}(r)| \leq 1$, that is, $k_{t,\theta}(r)$ is essentially independent of θ.

Proposition 4.1. *Let $f \in \mathfrak{S}_\infty(A)$ and assume that $\operatorname{Re} f(x_1 + iy_0) = \operatorname{Re} f(x_2 + iy_0) = 0$, where $x_1 < x_2$ are real and $y_0 > A$. Then, for all $\theta \in \mathbb{R}$, there exists $x_0 \in [x_1, x_2]$ such that $\operatorname{Re}(e^{i\theta} f(x_0 + iy_0)) = 0$.*

Proof. Let $f \in \mathfrak{S}_\infty(A)$ and let $\theta \in \mathbb{R}$. Since all the zeros of f lie in the strip $S(A)$, there is an analytic branch of $\log f$ defined on the set $\{z : \operatorname{Im} z > A\}$. With $\arg f$ denoting the imaginary part of $\log f$, we get

$$\frac{\partial}{\partial x} \arg f(z) = \operatorname{Im}(\log f)'(z) = \operatorname{Im} \frac{f'}{f}(z) < 0, \quad \operatorname{Im} z > A,$$

from (2.5). Thus, the function $\theta + \pi/2 + \arg f$ is monotone on the line segment $[x_1 + iy_0, x_2 + iy_0]$ and its range contains an interval of length π. It therefore assumes on this set a value that is an integral multiple of π, which is equivalent to the assertion of the proposition. $\square$

Open Problems 4.2. Special cases of Theorem 1.3 can also be expressed in terms of some simple infinite order differential operators. Thus, if $D_z = d/dz$ denotes differentiation with respect to z, then for any $f \in \mathfrak{S}(A)$, $e^{tD_z} f(z) = f(z+t)$. Therefore, by Theorem 1.3, for $t \in \mathbb{R} \setminus \{0\}$ and $f \in \mathfrak{S}_\infty(A)$, the entire functions

$$g_{t,0}(z) = 2\cosh(tD_z) f(z) = f(z+t) + f(z-t)$$

and

$$g_{t,\pi/2}(z) = 2i \sinh(tD_z) f(z) = i\big(f(z+t) - f(z-t)\big)$$

have infinitely many zeros on the imaginary axis. Repeated applications of these operators to a function $f \in \mathfrak{S}_\infty(A)$ lead to several questions. Does, for example, $(\cosh(tD_z))^2 f(z) = f(z+2t) + 2f(z) + f(z-2t))$ $(t \in \mathbb{R} \setminus \{0\})$ have infinitely many zeros on the imaginary axis? The problem of characterizing the functions in $\mathfrak{S}_\infty(A)$ such that $\exp(t^2 D_z^2/2) f(z) = \lim_{n\to\infty} (\cosh(tD_z/\sqrt{n}))^n f(z)$, $t \in \mathbb{R} \setminus \{0\}$, is an entire function having some nonreal zeros appears to be very difficult.

Remark 4.3. Let $f \in \mathfrak{S}_\infty(A)$ and set $w(x,t) = f(x+t)+f(x-t)$, where $x,\ t \in \mathbb{R}$. Then w satisfies the wave equation $w_{tt} = w_{xx}$. This suggests the physical interpretation that $w(x,t)$ (that is, $g_{t,0}$ restricted to the real axis) represents two waves traveling in opposite directions with unit speed.

Open Problems 4.4. (a) Characterize the functions $f \in \mathfrak{S}_\infty(A)$ such that all the zeros of $g_{t,0}(z) = f(z+t)+f(z-t)$ lie on the coordinate axes. (b) Suppose that $f \in \mathfrak{S}_\infty(A) \cap \mathcal{L}\text{-}\mathcal{P}$, so that $f \in \mathfrak{S}_\infty(A)$ has only real zeros (cf. Section 1). Then easy examples show that $g_{t,0}$ need not have any nonreal zeros off the imaginary axis. (Consider, for example, $f(z) = \exp(-z^2) \in \mathfrak{S}_\infty(A) \cap \mathcal{L}\text{-}\mathcal{P}$.) For what functions $f \in \mathfrak{S}_\infty(A) \cap \mathcal{L}\text{-}\mathcal{P}$ is it true that $g_{t,0}(z) = f(z+t) + f(z-t)$ has nonreal zeros other than those on the imaginary axis?

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