# ASYMPTOTIC ERROR IN THE EIGENFUNCTION EXPANSION FOR THE GREEN'S FUNCTION OF A STURM-LIOUVILLE PROBLEM

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ABSTRACT. We study the asymptotic error arising when approximating the Green's function of a Sturm–Liouville problem through a truncation of its eigenfunction expansion, both for the Green's function of a regular Sturm–Liouville problem and for the Green's function associated with the Hermite polynomials, the associated Laguerre polynomials, and the Jacobi polynomials, respectively. We prove that the asymptotic error obtained on the diagonal can be expressed in terms of the coefficients of the related second-order Sturm–Liouville differential equation, and that the suitable scaling exponent which yields a non-degenerate limit on the diagonal depends on the asymptotic behaviour of the corresponding eigenvalues. We further consider the asymptotic error away from the diagonal and analyse which scaling exponents ensure that it remains at zero. For the Hermite polynomials, the associated Laguerre polynomials, and the Jacobi polynomials, a Christoffel–Darboux type formula, which we establish for all classical orthogonal polynomial systems, allows us to obtain a better control away from the diagonal than what a sole application of known asymptotic formulae gives. As a consequence of our study for regular Sturm–Liouville problems, we identify the fluctuations for the Karhunen–Loève expansion of Brownian motion.

#### 1. Introduction

In a regular Sturm–Liouville problem, we are concerned with a real second-order linear differential operator

$$\mathcal{L} = -\frac{1}{w(x)} \left( \frac{\mathrm{d}}{\mathrm{d}x} \left( p(x) \frac{\mathrm{d}}{\mathrm{d}x} \right) - q(x) \right)$$

for a positive continuously differentiable function  $p: [a,b] \to \mathbb{R}$ , a continuous function  $q: [a,b] \to \mathbb{R}$  as well as a positive continuous function  $w: [a,b] \to \mathbb{R}$  defined on a finite interval I = [a,b], and we are interested in solving the eigenvalue equation, for  $x \in [a,b]$ ,

$$\mathcal{L}\phi(x) = \lambda\phi(x)$$

subject to the separated homogeneous boundary conditions

$$\alpha_1 \phi(a) + \alpha_2 \phi'(a) = 0$$
,  $\alpha_1^2 + \alpha_2^2 > 0$ ,

$$\beta_1 \phi(b) + \beta_2 \phi'(b) = 0$$
,  $\beta_1^2 + \beta_2^2 > 0$ ,

for constants  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ . Non-zero solutions  $\phi$  to the Sturm–Liouville problem are called eigenfunctions with  $\lambda$  being the corresponding eigenvalue.

Since the early works by Sturm and Liouville [18], the theory of Sturm-Liouville problems has seen a vast development with further contributions, among many others, due to Weyl [22], Dixon [6], and Titchmarsh [20, 21]. It is by now well-known that, for a given Sturm-Liouville problem, there exists a family  $\{\phi_n : n \in \mathbb{N}_0\}$  of real eigenfunctions that forms a complete orthogonal set under the weighted inner product in the Hilbert space  $L^2([a,b],w(x)\,\mathrm{d}x)$  which can be ordered such that the corresponding real eigenvalues  $\{\lambda_n : n \in \mathbb{N}_0\}$  form a sequence which strictly increases to infinity.

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The completeness and orthogonality relation of the eigenfunctions can be expressed in the sense of distributions in terms of the Dirac delta function as, for  $x, y \in [a, b]$ ,

$$\sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\int_a^b \left(\phi_n(z)\right)^2 w(z) dz} = \delta(x-y) .$$

Moreover, provided all eigenvalues are non-zero, the Green's function  $G: [a, b] \times [a, b] \to \mathbb{R}$  of the linear differential operator  $\mathcal{L}$  subject to the separated homogeneous boundary conditions stated above has the eigenfunction expansion given by, for  $x, y \in [a, b]$ ,

$$G(x,y) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n \int_a^b (\phi_n(z))^2 w(z) dz}.$$

This expansion is also referred to as the bilinear expansion of the Green's function. Formally, we see that indeed

$$\mathcal{L}G(x,y) = \sum_{n=0}^{\infty} \frac{\mathcal{L}\phi_n(x)\phi_n(y)}{\lambda_n \int_a^b (\phi_n(z))^2 w(z) dz} = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\int_a^b (\phi_n(z))^2 w(z) dz} = \delta(x-y) .$$

The present article is concerned with studying the pointwise limit

(1.1) 
$$\lim_{N \to \infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n \int_a^b (\phi_n(z))^2 w(z) dz} ,$$

provided the eigenvalues are ordered in increasing order, and for a suitable exponent  $\gamma > 0$  which may take a different value on the diagonal  $\{x = y\}$  than away from the diagonal. The resulting limit quantifies the asymptotic error in approximating the Green's function G with its truncation  $G_N$  defined by, for  $x, y \in [a, b]$ ,

$$G_N(x,y) = \sum_{n=0}^{N} \frac{\phi_n(x)\phi_n(y)}{\lambda_n \int_a^b (\phi_n(z))^2 w(z) dz}$$

as  $N \to \infty$ . We further observe that the limit (1.1) of interest can be considered both for regular Sturm–Liouville problems which admit a zero eigenvalue and for singular Sturm–Liouville problems which admit a complete set of orthogonal eigenfunctions.

In a singular Sturm–Liouville problem, the interval I on which the problem is studied may be infinite, in which case suitable singular boundary conditions are imposed as discussed in Littlejohn and Krall [15], the function p may vanish on I and the function w may vanish or be ill-defined at the boundary points of I. A rich class of functions arising as solutions to singular Sturm–Liouville problems are the classical orthogonal polynomials, which are related by linear transformations to the Hermite polynomials, the associated Laguerre polynomials, and the Jacobi polynomials. As conjectured by Aczél [2] and as established by Hahn [12], Feldmann [9], Mikolás [16] and Lesky [14], the classical orthogonal polynomial systems are the only orthogonal polynomial systems which arise as solutions to second-order Sturm–Liouville differential equations.

In our analysis of the asymptotic error in the eigenfunction expansion for the Green's function, we start by determining the limit (1.1) for all regular Sturm–Liouville problems where the coefficients of the linear differential operator  $\mathcal{L}$  satisfy  $p, w \in C^2([a, b])$ . This additional requirement is currently needed as our proof employs the Liouville transformation to reduce the system to a simplified Sturm–Liouville problem for which explicit asymptotic formulae for the eigenfunctions and the eigenvalues are known. It would be of interest to study in future work if this assumption can be relaxed.

**Theorem 1.1.** Fix  $a, b \in \mathbb{R}$ . For a function  $q \in C([a, b])$  and positive functions  $p, w \in C^2([a, b])$ , let  $\{\phi_n : n \in \mathbb{N}_0\}$  be a complete orthogonal set of eigenfunctions for the regular Sturm-Liouville problem

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( p(x) \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \right) - q(x)\phi(x) = -\lambda w(x)\phi(x)$$

on the finite interval [a,b] subject to separated homogeneous boundary conditions. Assume that the eigenfunctions are ordered such that the corresponding eigenvalues  $\{\lambda_n : n \in \mathbb{N}_0\}$  form a strictly increasing sequence. We have, for  $x \in (a,b)$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{\left(\phi_n(x)\right)^2}{\lambda_n \int_a^b \left(\phi_n(z)\right)^2 w(z) dz} = \frac{1}{\pi^2 \sqrt{p(x)w(x)}} \int_a^b \sqrt{\frac{w(z)}{p(z)}} dz$$

and, for  $x, y \in [a, b]$  with  $x \neq y$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n \int_a^b \left(\phi_n(z)\right)^2 w(z) \, \mathrm{d}z} = 0 \; .$$

Moreover, depending on the separated homogeneous boundary conditions, we have

$$\lim_{N\to\infty} N \sum_{n=N+1}^{\infty} \frac{(\phi_n(a))^2}{\lambda_n \int_a^b \left(\phi_n(z)\right)^2 w(z) \,\mathrm{d}z} = \begin{cases} \frac{2}{\pi^2 \sqrt{p(a)w(a)}} \int_a^b \sqrt{\frac{w(z)}{p(z)}} \,\mathrm{d}z & \text{if } \alpha_2 \neq 0 \\ 0 & \text{if } \alpha_2 = 0 \end{cases}$$

as well as

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{(\phi_n(b))^2}{\lambda_n \int_a^b (\phi_n(z))^2 w(z) dz} = \begin{cases} \frac{2}{\pi^2 \sqrt{p(b)w(b)}} \int_a^b \sqrt{\frac{w(z)}{p(z)}} dz & \text{if } \beta_2 \neq 0 \\ 0 & \text{if } \beta_2 = 0 \end{cases}.$$

While [10, Theorem 1.2], which states that, for all  $s, t \in [0, 1]$ ,

(1.2) 
$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{2\sin((n+1)\pi s)\sin((n+1)\pi t)}{(n+1)^2 \pi^2} = \begin{cases} \frac{1}{\pi^2} & \text{if } s = t \text{ and } t \in (0,1) \\ 0 & \text{otherwise} \end{cases},$$

could be considered as a special case of Theorem 1.1, our proof actually depends on this result. Together with Proposition 2.3, it forms one of the base cases which the problem reduces to after applying the Liouville transformation as well as the known asymptotic formulae, and which needs to be proven separately.

As a consequence of Proposition 2.3, we further obtain the fluctuations for the Karhunen–Loève expansion of Brownian motion, just as [10, Theorem 1.2] allowed us to determine the fluctuations for the Karhunen–Loève expansion of a Brownian bridge.

**Theorem 1.2.** Let  $(B_t)_{t\in[0,1]}$  be a Brownian motion in  $\mathbb{R}$  and, for  $N\in\mathbb{N}_0$ , let the fluctuation processes  $(F_t^N)_{t\in[0,1]}$  for the Karhunen–Loève expansion be defined by, for  $t\in[0,1]$ ,

$$F_t^N = \sqrt{N} \left( B_t - \sum_{n=0}^N \frac{2\sin\left(\left(n + \frac{1}{2}\right)\pi t\right)}{\left(n + \frac{1}{2}\right)\pi} \int_0^1 \cos\left(\left(n + \frac{1}{2}\right)\pi r\right) dB_r \right) .$$

The processes  $(F_t^N)_{t\in[0,1]}$  converge in finite dimensional distributions as  $N\to\infty$  to the collection  $(F_t)_{t\in[0,1]}$  of independent Gaussian random variables with mean zero and variance

$$\mathbb{E}\left[\left(F_{t}\right)^{2}\right] = \begin{cases} \frac{1}{\pi^{2}} & \text{if } t \in (0,1) \\ \frac{2}{\pi^{2}} & \text{if } t = 1 \\ 0 & \text{if } t = 0 \end{cases},$$

where  $\mathbb{E}$  denotes expectation with respect to Wiener measure.

After proving Theorem 1.1 and illustrating the result by one example, we turn our attention to singular Sturm–Liouville problems with a focus on classical orthogonal polynomial systems, which arise as solutions to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right) = -\lambda W(x)Y(x)$$

subject to suitable singular boundary conditions. The Hermite polynomial  $H_n : \mathbb{R} \to \mathbb{R}$  of degree  $n \in \mathbb{N}_0$  satisfies the Sturm-Liouville differential equation

$$\left(e^{-x^2} H'_n(x)\right)' = -2n e^{-x^2} H_n(x) .$$

The asymptotic error in approximating the associated Green's function by truncating its bilinear expansion is given by the following result.

**Theorem 1.3.** We have, for  $x \in \mathbb{R}$ ,

$$\lim_{N \to \infty} \sqrt{N} \sum_{n=N+1}^{\infty} \frac{(H_n(x))^2}{2n \, 2^n n! \sqrt{\pi}} = \frac{\exp\left(x^2\right)}{\sqrt{2}\pi}$$

as well as, for  $x, y \in \mathbb{R}$  with  $x \neq y$  and for all  $\gamma < 1$ ,

$$\lim_{N \to \infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{H_n(x)H_n(y)}{2n \, 2^n n! \sqrt{\pi}} = 0 \ .$$

For  $\alpha \in \mathbb{R}$  fixed with  $\alpha > -1$ , the associated Laguerre polynomial  $L_n^{(\alpha)}: [0, \infty) \to \mathbb{R}$  of degree  $n \in \mathbb{N}_0$  solves the Sturm-Liouville differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( x^{\alpha+1} e^{-x} \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} \right) = -nx^{\alpha} e^{-x} L_n^{(\alpha)}(x) .$$

The special case  $\alpha=0$  gives rise to the Laguerre polynomials. As for the Hermite polynomials in Theorem 1.3, and unlike in Theorem 1.1, we need to rescale the error in the approximations for the Green's function by  $\sqrt{N}$  to obtain a non-trivial limit on the diagonal. This is essentially a consequence of both Hermite polynomials and associated Laguerre polynomials having eigenvalues satisfying  $\lambda_n=O(n)$  as  $n\to\infty$  as opposed to  $\lambda_n=O(n^2)$  occurring in all other cases considered.

**Theorem 1.4.** For  $\alpha \in \mathbb{R}$  fixed with  $\alpha > -1$ , we have, for  $x \in (0, \infty)$ ,

$$\lim_{N \to \infty} \sqrt{N} \sum_{n=N+1}^{\infty} \frac{\Gamma(n) \left( L_n^{(\alpha)}(x) \right)^2}{\Gamma(n+\alpha+1)} = \frac{x^{-\alpha-\frac{1}{2}} e^x}{\pi}$$

and, for  $x, y \in (0, \infty)$  with  $x \neq y$  and for all  $\gamma < \frac{1}{2}$ ,

$$\lim_{N\to\infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{\Gamma(n) L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{\Gamma(n+\alpha+1)} = 0 \ .$$

For the associated Laguerre polynomials, we cannot prove the off-diagonal convergence up to the scaling exponent appearing on the diagonal. The result stated is the best we can achieve with our method, discussed later, and using known asymptotic formulae for the polynomials as  $n \to \infty$ . Further notice that both for Hermite polynomials and for associated Laguerre polynomials the limit function formed on the diagonal is still proportional to  $1/\sqrt{PW}$ , as in Theorem 1.1, but with different constants of proportionality.

For  $\alpha, \beta \in \mathbb{R}$  fixed with  $\alpha, \beta > -1$ , the Jacobi polynomial  $P_n^{(\alpha,\beta)} : [-1,1] \to \mathbb{R}$  of degree  $n \in \mathbb{N}_0$  satisfies the Sturm-Liouville differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( (1-x)^{\alpha+1} (1+x)^{\beta+1} \frac{\mathrm{d}P_n^{(\alpha,\beta)}(x)}{\mathrm{d}x} \right) = -n(n+\alpha+\beta+1)(1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) ,$$

and the asymptotic error in the eigenfunction expansion for the associated Green's function is given in the theorem below. We see that the limit function formed on the diagonal is exactly the same one which appears in Theorem 1.1 for regular Sturm—Liouville problems as

$$\int_{-1}^{1} \sqrt{\frac{W(z)}{P(z)}} dz = \int_{-1}^{1} \frac{1}{\sqrt{1-z^2}} dz = \pi.$$

The analysis we employ for the Jacobi polynomials in fact allows us to determine the asymptotic error even for those families of Jacobi polynomials which do not arise as classical orthogonal polynomial systems but, with  $\alpha, \beta \in \mathbb{R}$ , are defined by, for  $n \in \mathbb{N}_0$  and  $x \in [-1, 1]$ ,

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+\beta+1)_k (\alpha+k+1)_{n-k} \left(\frac{x-1}{2}\right)^k.$$

**Theorem 1.5.** For  $\alpha, \beta \in \mathbb{R}$  fixed, we have, for  $x \in (-1, 1)$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \frac{\left(P_n^{(\alpha,\beta)}(x)\right)^2}{n(n+\alpha+\beta+1)} = \frac{(1-x)^{-\alpha-\frac{1}{2}}(1+x)^{-\beta-\frac{1}{2}}}{\pi}$$

and, for  $x, y \in (-1, 1)$  with  $x \neq y$  and for all  $\gamma < 2$ ,

$$\lim_{N\to\infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \frac{P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)}{n(n+\alpha+\beta+1)} = 0.$$

We show later that [11, Theorem 1.5] which quantifies an integrated version of the completeness and orthogonality property for Legendre polynomials can be deduced from Theorem 1.5. Moreover, we have the following convergence results for the Legendre polynomials  $\{P_n : n \in \mathbb{N}_0\}$ , the Chebyshev polynomials of the first kind  $\{T_n : n \in \mathbb{N}_0\}$  and the Chebyshev polynomials of the second kind  $\{U_n : n \in \mathbb{N}_0\}$ , which all are scalar multiplies of Jacobi polynomials.

Corollary 1.6. We have, for  $x, y \in (-1, 1)$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{(2n+1)P_n(x)P_n(y)}{2n(n+1)} = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{if } x = y\\ 0 & \text{if } x \neq y \end{cases}$$

as well as

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{T_n(x)T_n(y)}{n^2} = \begin{cases} \frac{1}{2} & \text{if } x = y\\ 0 & \text{if } x \neq y \end{cases}$$

and

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{U_n(x)U_n(y)}{n(n+2)} = \begin{cases} \frac{1}{2(1-x^2)} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

When proving the convergence results for the classical orthogonal polynomials, we use, in principle, the same method which was employed to establish [11, Theorem 1.5] and we split the analysis into an on-diagonal and an off-diagonal part. The pointwise convergence on the diagonal is deduced from a convergence of moments, and two local uniform bounds which allow us to apply the Arzelà–Ascoli theorem. For the convergence away from the diagonal, we find a Christoffel–Darboux type formula, see Proposition 3.4, which together with known asymptotic formulae for the polynomials yields the desired result. By using the Christoffel–Darboux type formula, we obtain a better control away from the diagonal than a sole application of the asymptotic formulae would give.

The main advantage compared to [11] is that we have a streamlined way to deal with the moment convergence. Proposition 3.1 and Proposition 3.2 imply that for a family of classical orthogonal polynomials which is subject to suitable boundary conditions the limit moments on the diagonal satisfy the same recurrence relation as the moments of the desired limit function. The one subtlety which appears to remain, and which forces us to employ a workaround for the Jacobi polynomials, is checking that the limit moments satisfy the right initial condition. In doing so, we establish a number of interesting identities for Hermite polynomials and associated Laguerre polynomials.

The paper is organised as follows. After starting by reviewing the Liouville transformation for Sturm-Liouville problems, we give an account, in Section 2.1, of the asymptotic formulae for eigenvalues and eigenfunctions of systems in Liouville normal form. We then prove Theorem 1.1 and Proposition 2.3, which implies Theorem 1.2, in Section 2.2, and we illustrate Theorem 1.1 by one example given in Section 2.3. Our considerations for classical orthogonal polynomials follow in Section 3. We first prove Proposition 3.1 and Proposition 3.2, which allow us to streamline the moment analysis on the diagonal, and we further derive the Christoffel-Darboux type formula stated in Proposition 3.4, applicable to any family of classical orthogonal polynomials, which is used for the off-diagonal analysis. We apply these results to prove Theorem 1.3 in Section 3.1, Theorem 1.4 in Section 3.2, and finally, Theorem 1.5 in Section 3.3. Our proof of Theorem 1.5 relies on Proposition 3.18, which demonstrates that Theorem 1.1 extends to cases beyond separated homogeneous boundary conditions. Throughout, we use the convention that  $\mathbb{N}$  denotes the positive integers and  $\mathbb{N}_0$  the non-negative integers.

# 2. Green's function of a regular Sturm-Liouville problem

We prove Theorem 1.1 by applying the Liouville transformation to reduce the analysis to systems in Liouville normal form, for which the asymptotic error in the eigenfunction expansion for the Green's function is deduced from the asymptotics for the eigenfunctions as well as the eigenvalues and from four base cases which we establish directly.

The reason for assuming  $p, w \in C^2([a, b])$  in Theorem 1.1 is that a given Sturm–Liouville differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( p(x) \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \right) - q(x)\phi(x) = -\lambda w(x)\phi(x)$$

is transformed by the Liouville transformation

$$t = \int_a^x \sqrt{rac{w(z)}{p(z)}} \, \mathrm{d}z$$
 and  $u(t) = \sqrt[4]{p(x)w(x)}\phi(x)$ 

and with

$$\tilde{q}(t) = \frac{q(x)}{w(x)} + \frac{1}{\sqrt[4]{p(x)w(x)}} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\sqrt[4]{p(x)w(x)}\right)$$

into the Liouville normal form

$$\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} - \tilde{q}(t)u(t) = -\lambda u(t)$$

on the finite interval [0, c] where

$$c = \int_a^b \sqrt{\frac{w(z)}{p(z)}} \, \mathrm{d}z \;,$$

see Birkhoff and Rota [3, Theorem 10.6]. In particular, the additional assumption  $p, w \in C^2([a, b])$  together with the standard assumptions of p and w being positive and  $q \in C([a, b])$  ensures that  $\tilde{q} \in C([0, c])$ . According to [3, Corollary 10.1 and Corollary 10.2], the Liouville transformation transforms the original regular Sturm-Liouville problem with separated homogeneous boundary conditions into another regular Sturm-Liouville problem with separated homogeneous boundary conditions which has the same eigenvalues as the original system and where eigenfunctions are mapped to eigenfunctions, except that the new eigenfunctions are now orthogonal with respect to a constant weight function.

Hence, the Liouville transformation allows us to prove Theorem 1.1 by analysing Sturm–Liouville problems in Liouville normal form subject to separated homogeneous boundary conditions. For these systems asymptotic formulae for the eigenfunctions and eigenvalues are known as presented in the subsequent section.

2.1. Asymptotics for eigenfunctions and eigenvalues. In our discussion of the asymptotic formulae for eigenfunctions and eigenvalues of systems in Liouville normal form, we follow the analysis in Birkhoff and Rota [3, Section 10.10 to Section 10.13] with explicitly completing the extension to all possible separated homogeneous boundary conditions.

Fix  $c \in \mathbb{R}$  with c > 0. We consider a regular Sturm–Liouville problem in Liouville normal form on the interval [0, c], for  $q \in C([0, c])$ ,

$$\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} - q(t)u(t) = -\lambda u(t)$$

subject to the separated homogeneous boundary conditions

$$\alpha_1 u(0) + \alpha_2 u'(0) = 0$$
,  $\alpha_1^2 + \alpha_2^2 > 0$ ,

$$\beta_1 u(c) + \beta_2 u'(c) = 0$$
,  $\beta_1^2 + \beta_2^2 > 0$ ,

for constants  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ . By Sturm-Liouville theory, there exists a family  $\{u_n : n \in \mathbb{N}_0\}$  of real eigenfunctions that forms a complete orthonormal set and such that the corresponding real eigenvalues  $\{\lambda_n : n \in \mathbb{N}_0\}$  form a sequence strictly increasing to infinity. In particular, as we are interested in asymptotic formulae for  $u_n$  and  $\lambda_n$  as  $n \to \infty$ , we may and do assume throughout our analysis that the function  $Q \in C([0,c])$  defined by

$$Q(t) = \lambda - q(t)$$

is strictly positive. For a given solution u to the Sturm-Liouville problem with eigenvalue  $\lambda$ , the modified amplitude R and the modified phase  $\theta$  are defined by

(2.1) 
$$u = \frac{R}{\sqrt[4]{Q}}\sin(\theta) \quad \text{and} \quad u' = R\sqrt[4]{Q}\cos(\theta) ,$$

which is called the modified Prüfer substitution. We see that the modified amplitude, chosen to be non-negative, and the modified phase, subject to shifts by  $2\pi$ , are determined by the equations

$$(R(t))^{2} = \sqrt{Q(t)} (u(t))^{2} + \frac{1}{\sqrt{Q(t)}} (u'(t))^{2}$$

as well as

$$\cot(\theta(t)) = \frac{1}{\sqrt{Q(t)}} \frac{u'(t)}{u(t)}$$
 provided  $u(t) \neq 0$ 

and

$$\tan(\theta(t)) = \sqrt{Q(t)} \frac{u(t)}{u'(t)}$$
 provided  $u'(t) \neq 0$ .

Differentiating these equations and using u'' = -Qu, we obtain the modified Prüfer system

$$\theta'(t) = \sqrt{\lambda - q(t)} - \frac{q'(t)}{4(\lambda - q(t))} \sin(2\theta(t)) ,$$
  
$$\frac{R'(t)}{R(t)} = \frac{q'(t)}{4(\lambda - q(t))} \cos(2\theta(t)) .$$

By comparing solutions to the modified Prüfer system with solutions to

$$\tilde{\theta}'(t) = \sqrt{\lambda}$$
 and  $\left(\log \tilde{R}(t)\right)' = 0$ ,

and by exploiting the property that, for given initial values, solutions to an ordinary differential equation depend continuously on the ordinary differential equation, we deduce that, as  $\lambda \to \infty$ ,

(2.2) 
$$\theta(t) = \theta(0) + \sqrt{\lambda}t + O\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{and} \quad R(t) = R(0) + O\left(\frac{1}{\lambda}\right) ,$$

see proof of [3, Theorem 10.7] for details. For the remaining analysis it is necessary to distinguish according to the type of separated homogeneous boundary conditions imposed, where we essentially differentiate if the Dirichlet part or the Neumann part of the boundary conditions dominates for large  $\lambda$ . This gives rise to four different cases.

**Proposition 2.1.** The real eigenvalues  $\{\lambda_n : n \in \mathbb{N}_0\}$  ordered in increasing order of a regular Sturm-Liouville problem in Liouville normal form satisfy, as  $n \to \infty$ ,

$$c\sqrt{\lambda_n} + O\left(\frac{1}{n}\right) = \begin{cases} n\pi & \text{if } \alpha_2, \beta_2 \neq 0\\ \left(n + \frac{1}{2}\right)\pi & \text{if } \alpha_2 \neq 0 \text{ and } \beta_2 = 0\\ \left(n + \frac{1}{2}\right)\pi & \text{if } \alpha_2 = 0 \text{ and } \beta_2 \neq 0\\ \left(n + 1\right)\pi & \text{if } \alpha_2 = \beta_2 = 0 \end{cases}.$$

*Proof.* Let us first consider the case  $\alpha_2, \beta_2 \neq 0$ . For an eigenfunction corresponding to some large eigenvalue  $\lambda_n$ , the associated modified phase  $\theta_n$  then satisfies

$$\cot\left(\theta_n(0)\right) = -\frac{\alpha_1}{\alpha_2} \frac{1}{\sqrt{\lambda_n - q(0)}} \quad \text{and} \quad \cot\left(\theta_n(c)\right) = -\frac{\beta_1}{\beta_2} \frac{1}{\sqrt{\lambda_n - q(c)}} \ .$$

It follows that we can choose  $\theta_n$  such that, as  $n \to \infty$ ,

$$\theta_n(0) = \frac{\pi}{2} + \frac{\alpha_1}{\alpha_2 \sqrt{\lambda_n}} + O\left(\frac{1}{\sqrt{\lambda_n^3}}\right).$$

Since the eigenfunction corresponding to the eigenvalue  $\lambda_n$  has exactly n zeros in the interval (0, c), see [3, Theorem 10.5], we further obtain that

$$\theta_n(c) = \frac{\pi}{2} + n\pi + O\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

This implies that

$$\theta_n(c) - \theta_n(0) = n\pi + O\left(\frac{1}{\sqrt{\lambda_n}}\right)$$

and a comparison with (2.2) shows that we indeed have, as  $n \to \infty$ .

$$c\sqrt{\lambda_n} = n\pi + O\left(\frac{1}{n}\right) .$$

We argue similarly in the three remaining cases, except we need to take care of phase shifts by  $\frac{\pi}{2}$ . If  $\alpha_2 \neq 0$  and  $\beta_2 = 0$  then, as before, we can choose the modified phase such that, as  $n \to \infty$ ,

$$\theta_n(0) = \frac{\pi}{2} + O\left(\frac{1}{\sqrt{\lambda_n}}\right) ,$$

whereas  $\tan(\theta_n(c)) = 0$  due to  $\beta_2 = 0$  and the fact that the eigenfunction corresponding to the eigenvalue  $\lambda_n$  has exactly n zeros in the open interval (0,c) imply that

$$\theta_n(c) = \pi + n\pi$$
.

A comparison with (2.2) again yields the claimed result. Likewise, if  $\alpha_2 = 0$  and  $\beta_2 \neq 0$ , we can choose the modified phase such that, as  $n \to \infty$ ,

$$\theta_n(0) = 0$$
 and  $\theta_n(c) = \frac{\pi}{2} + n\pi + O\left(\frac{1}{\sqrt{\lambda_n}}\right)$ ,

and, if  $\alpha_2 = \beta_2 = 0$ , such that

$$\theta_n(0) = 0$$
 and  $\theta_n(c) = (n+1)\pi$ ,

which, as above, imply the stated asymptotic formulae.

We are now in a position to deduce the asymptotic formula for the normalised eigenfunction  $u_n$  as  $n \to \infty$ . We see that the normalised eigenfunctions of a Sturm-Liouville problem in Liouville normal form are asymptotically close to the normalised eigenfunctions of the Sturm-Liouville problem  $u'' = -\lambda u$  subject to suitable boundary conditions.

**Proposition 2.2.** The normalised eigenfunctions  $\{u_n : n \in \mathbb{N}_0\}$  of a regular Sturm-Liouville problem in Liouville normal form whose corresponding eigenvalues  $\{\lambda_n : n \in \mathbb{N}_0\}$  form a strictly increasing sequence satisfy, for  $t \in [0,c]$  and as  $n \to \infty$ ,

$$u_n(t) + O\left(\frac{1}{n}\right) = \begin{cases} \sqrt{\frac{2}{c}}\cos\left(\frac{n\pi t}{c}\right) & \text{if } \alpha_2, \beta_2 \neq 0\\ \sqrt{\frac{2}{c}}\cos\left(\left(n + \frac{1}{2}\right)\frac{\pi t}{c}\right) & \text{if } \alpha_2 \neq 0 \text{ and } \beta_2 = 0\\ \sqrt{\frac{2}{c}}\sin\left(\left(n + \frac{1}{2}\right)\frac{\pi t}{c}\right) & \text{if } \alpha_2 = 0 \text{ and } \beta_2 \neq 0\\ \sqrt{\frac{2}{c}}\sin\left(\frac{(n+1)\pi t}{c}\right) & \text{if } \alpha_2 = \beta_2 = 0 \end{cases}.$$

*Proof.* From the modified Prüfer system and (2.2), we obtain, as  $\lambda \to \infty$ ,

$$\theta'(t) = \sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{and} \quad \int_{\theta(0)}^{\theta(c)} (\sin(\theta))^2 d\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_{\theta(0)}^{\theta(c)} = \frac{\sqrt{\lambda}c}{2} + O(1) .$$

It follows that

$$\int_0^c \left(\sin\left(\theta(t)\right)\right)^2 dt = \left(\frac{1}{\sqrt{\lambda}} + O\left(\frac{1}{\sqrt{\lambda^3}}\right)\right) \int_{\theta(0)}^{\theta(c)} \left(\sin(\theta)\right)^2 d\theta = \frac{c}{2} + O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Using this together with (2.1) and (2.2) yields

$$\int_0^c \left(u(t)\right)^2 dt = \left(R(0) + O\left(\frac{1}{\lambda}\right)\right)^2 \left(\frac{c}{2\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right) .$$

Thus, the modified amplitude of a normalised eigenfunction satisfies, as  $\lambda \to \infty$ ,

(2.3) 
$$R(0) = \sqrt{\frac{2\sqrt{\lambda}}{c}} \left( 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right) .$$

For the modified phase  $\theta_n$  which corresponds to the normalised eigenfunction with eigenvalue  $\lambda_n$  as in the proof of Proposition 2.1, by using (2.2), we find that, as  $n \to \infty$ , if  $\alpha_2 \neq 0$ ,

$$\sin\left(\theta_n(t)\right) = \sin\left(\frac{\pi}{2} + \sqrt{\lambda_n}t + O\left(\frac{1}{\sqrt{\lambda_n}}\right)\right) = \cos\left(\sqrt{\lambda_n}t + O\left(\frac{1}{\sqrt{\lambda_n}}\right)\right) ,$$

whereas, if  $\alpha_2 = 0$ ,

$$\sin(\theta_n(t)) = \sin\left(\sqrt{\lambda_n}t + O\left(\frac{1}{\sqrt{\lambda_n}}\right)\right).$$

The claimed result follows from (2.1), (2.2), (2.3) and after applying the asymptotic formula for the eigenvalues from Proposition 2.1 as well as the mean value theorem.

2.2. Asymptotic error in the eigenfunction expansion. Using the Liouville transformation and the asymptotic formulae for the eigenfunctions and eigenvalues of Sturm–Liouville problems in Liouville normal form, we prove Theorem 1.1 by reducing the analysis to four base cases, which take care of different types of separated homogeneous boundary conditions. One base case is covered by [10, Theorem 1.2], another one by Proposition 2.3 below, and the remaining two cases can be deduced from these results.

When proving Proposition 2.3, we employ a similar proof strategy as was used for [10, Theorem 1.2] and [11, Theorem 1.5], that is, we split the analysis into an on-diagonal and an off-diagonal part, with the pointwise convergence on the diagonal being a consequence of a convergence of moments and a locally uniform convergence, which allows for an application of the Arzelà–Ascoli theorem. The main difference is that we do not compute the moments on the diagonal explicitly, and instead illustrate a powerful approach exploiting the Sturm–Liouville differential equation which we apply in our analysis for classical orthogonal polynomial systems in Section 3.

**Proposition 2.3.** We have, for  $s, t \in [0, 1]$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{2 \sin\left(\left(n + \frac{1}{2}\right) \pi s\right) \sin\left(\left(n + \frac{1}{2}\right) \pi t\right)}{\left(n + \frac{1}{2}\right)^{2} \pi^{2}} = \begin{cases} \frac{1}{\pi^{2}} & \text{if } s = t \text{ and } t \in (0, 1) \\ \frac{2}{\pi^{2}} & \text{if } s = t = 1 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* We start by proving the claimed convergence on the diagonal  $\{s = t\}$  and then deduce the off-diagonal convergence from the on-diagonal convergence. For  $n \in \mathbb{N}_0$ , the function  $u_n : [0,1] \to \mathbb{R}$  given by

$$u_n(t) = \sqrt{2}\sin\left(\left(n + \frac{1}{2}\right)\pi t\right)$$

is a normalised eigenfunction of the Sturm-Liouville problem

(2.4) 
$$\frac{d^2 u(t)}{dt^2} = -\lambda u(t) \text{ subject to } u(0) = 0, \quad u'(1) = 0,$$

with corresponding eigenvalue

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 \pi^2 \ .$$

Hence, we are interested in determining the pointwise limit of the functions  $S_N : [0,1] \to \mathbb{R}$  defined by, for  $N \in \mathbb{N}_0$ ,

$$S_N(t) = N \sum_{n=N+1}^{\infty} \frac{(u_n(t))^2}{\lambda_n}$$

as  $N \to \infty$ . Due to  $u_n(0) = 0$  for all  $n \in \mathbb{N}_0$  we have  $\lim_{N \to \infty} S_N(0) = 0$ , as claimed. For t = 1, we need to compute

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{2}{\left(n + \frac{1}{2}\right)^2 \pi^2} ,$$

which is indeed  $\frac{2}{\pi^2}$  because

(2.5) 
$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^2} \le \lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \left(\frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}}\right) = \lim_{N \to \infty} \frac{N}{N + \frac{1}{2}} = 1$$

and

$$(2.6) \qquad \lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2} \geq \lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{3}{2}}\right) = \lim_{N \to \infty} \frac{N}{N+\frac{3}{2}} = 1 \; .$$

We further observe that the bound which was used to deduce (2.5) shows that, for all  $N \in \mathbb{N}_0$  and for all  $t \in [0, 1]$ ,

$$(2.7) |S_N(t)| \le \frac{2}{\pi^2} .$$

Since the Green's function  $G: [0,1] \times [0,1] \to \mathbb{R}$  of the Sturm–Liouville problem (2.4) is given by  $G(s,t) = \min(s,t)$ ,

$$S_N(t) = N\left(t - \sum_{n=0}^{N} \frac{(u_n(t))^2}{\lambda_n}\right).$$

It follows that

$$S_N'(t) = N\left(1 - \sum_{n=0}^N \frac{2u_n(t)u_n'(t)}{\lambda_n}\right) = N\left(1 - \sum_{n=0}^N \frac{4\sin\left((2n+1)\pi t\right)}{(2n+1)\pi}\right).$$

Similar to the proof of [10, Lemma 4.2], we use

$$\sum_{n=0}^{N} \cos((2n+1)\pi t) = \frac{\sin(2(N+1)\pi t)}{2\sin(\pi t)}$$

to rewrite, for  $t \in (0,1)$ ,

$$\sum_{n=0}^{N} \left( \frac{(-1)^n}{(2n+1)\pi} - \frac{\sin((2n+1)\pi t)}{(2n+1)\pi} \right) = -\sum_{n=0}^{N} \int_{\frac{1}{2}}^{t} \cos((2n+1)\pi r) dr = -\int_{\frac{1}{2}}^{t} \frac{\sin(2(N+1)\pi r)}{2\sin(\pi r)} dr.$$

As in the proof of [10, Lemma 4.2], integration by parts shows that, for all  $\varepsilon > 0$ , the family

$$\left\{ N \int_{\frac{1}{2}}^{t} \frac{\sin(2(N+1)\pi r)}{2\sin(\pi r)} \, \mathrm{d}r : N \in \mathbb{N}_{0} \text{ and } t \in [\varepsilon, 1-\varepsilon] \right\}$$

is uniformly bounded. Since the Leibniz series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$  takes the value  $\frac{\pi}{4}$ , we further have

$$N\left(\frac{1}{4} - \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)\pi}\right) = N \sum_{n=N+1}^{\infty} \frac{(-1)^n}{(2n+1)\pi} ,$$

which, by a similar telescoping argument as above, is bounded uniformly in  $N \in \mathbb{N}_0$ . It follows that the derivative  $S_N'$  is locally uniformly bounded on the open interval (0,1). This together with (2.7) implies that, by the Arzelà-Ascoli theorem, the functions  $S_N$  converge locally uniformly on (0,1) as  $N \to \infty$ . In particular, the pointwise limit function is continuous on (0,1) and we can use a moment argument to identify it. This is where we exploit the Sturm-Liouville differential equation. Integration by parts and (2.4) imply that, for all  $n, k \in \mathbb{N}_0$ ,

$$\lambda_n \int_0^1 t^k (u_n(t))^2 dt = -\int_0^1 t^k u_n(t) u_n''(t) dt = \int_0^1 t^k (u_n'(t))^2 dt + \int_0^1 k t^{k-1} u_n(t) u_n'(t) dt.$$

On the other hand, we obtain from (2.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{k+1}\left(u_n'(t)\right)^2 + \lambda_n t^{k+1}\left(u_n(t)\right)^2\right) = (k+1)t^k\left(u_n'(t)\right)^2 + (k+1)\lambda_n t^k\left(u_n(t)\right)^2 \ ,$$

which implies

$$\int_0^1 t^k (u'_n(t))^2 dt + \lambda_n \int_0^1 t^k (u_n(t))^2 dt = \frac{\lambda_n (u_n(1))^2}{k+1} = \frac{2\lambda_n}{k+1}.$$

It suffices to observe that, for  $k \in \mathbb{N}_0$  fixed and as  $n \to \infty$ ,

$$\int_0^1 kt^{k-1} u_n(t) u_n'(t) dt = \frac{1}{2} \left( 2k - \int_0^1 k(k-1)t^{k-2} (u_n(t))^2 dt \right) = O(1) ,$$

which can be seen, for instance, from

$$\int_0^1 k(k-1)t^{k-2} (u_n(t))^2 dt \le k(k-1) \int_0^1 (u_n(t))^2 dt = k(k-1),$$

to deduce that, for  $k \in \mathbb{N}_0$  fixed and as  $n \to \infty$ ,

$$2\lambda_n \int_0^1 t^k (u_n(t))^2 dt = \frac{2\lambda_n}{k+1} + O(1) .$$

Since  $\lambda_n = O(n^2)$  as  $n \to \infty$ , this yields

$$\int_{0}^{1} \frac{t^{k} (u_{n}(t))^{2}}{\lambda_{n}} dt = \frac{1}{(k+1)\lambda_{n}} + O(n^{-4}).$$

Using Fubini's theorem and the bound, for  $N \in \mathbb{N}$ ,

$$\int_{N+1}^{\infty} \frac{1}{r^4} \, \mathrm{d}r \le \sum_{n=N+1}^{\infty} \frac{1}{n^4} \le \frac{1}{N^4} + \int_{N+1}^{\infty} \frac{1}{r^4} \, \mathrm{d}r$$

as well as (2.5) and (2.6), we conclude, for all  $k \in \mathbb{N}_0$ ,

$$\lim_{N \to \infty} \int_0^1 t^k S_N(t) dt = \frac{1}{(k+1)\pi^2} = \int_0^1 \frac{t^k}{\pi^2} dt.$$

As the limit function was shown to be continuous on (0,1), this implies that, for  $t \in (0,1)$ ,

$$\lim_{N \to \infty} S_N(t) = \frac{1}{\pi^2} \ .$$

It remains to prove the pointwise convergence away from the diagonal. As for [10, Theorem 1.2] this follows from the on-diagonal convergence. For  $t \in (0,1)$ , we see that

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{\cos\left(2\left(n + \frac{1}{2}\right)\pi t\right)}{\left(n + \frac{1}{2}\right)^2 \pi^2} = \lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{1 - 2\left(\sin\left(\left(n + \frac{1}{2}\right)\pi t\right)\right)^2}{\left(n + \frac{1}{2}\right)^2 \pi^2} = \frac{1}{\pi^2} - \frac{1}{\pi^2} = 0.$$

Applying the identity, for  $n \in \mathbb{N}_0$ ,

$$2\sin\left(\left(n+\frac{1}{2}\right)\pi s\right)\sin\left(\left(n+\frac{1}{2}\right)\pi t\right) = \cos\left(\left(n+\frac{1}{2}\right)\pi (t-s)\right) - \cos\left(\left(n+\frac{1}{2}\right)\pi (t+s)\right) ,$$

we deduce the claimed convergence to zero for  $s \neq t$ .

Using characteristic functions as in the proof of [11, Theorem 1.6], Theorem 1.2 is a consequence of Proposition 2.3 since the fluctuation processes  $(F_t^N)_{t\in[0,1]}$  are zero-mean Gaussian processes whose covariance functions, due to (2.8), are exactly given by

$$N \sum_{n=N+1}^{\infty} \frac{2 \sin \left( \left( n + \frac{1}{2} \right) \pi s \right) \sin \left( \left( n + \frac{1}{2} \right) \pi t \right)}{\left( n + \frac{1}{2} \right)^2 \pi^2} .$$

Before we turn to the proof of Theorem 1.1, we show how Proposition 2.3 and [10, Theorem 1.2], whose result is (1.2), allow us to analyse the remaining two base cases. From [10, Theorem 1.2], it follows that, for  $t \in (0, 1)$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{\cos(2n\pi t)}{n^2 \pi^2} = 0 ,$$

see [10, Corollary 1.3]. This together with [10, Theorem 1.2] and

$$2\cos(n\pi s)\cos(n\pi t) = \cos(n\pi(t-s)) + \cos(n\pi(t+s))$$

as well as

$$N\sum_{n=N+1}^{\infty} \frac{2\left(\cos(n\pi t)\right)^2}{n^2\pi^2} = N\left(\sum_{n=N+1}^{\infty} \frac{2}{n^2\pi^2} - \sum_{n=N+1}^{\infty} \frac{2\left(\sin(n\pi t)\right)^2}{n^2\pi^2}\right)$$

implies that, for  $s, t \in [0, 1]$ ,

(2.9) 
$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{2\cos(n\pi s)\cos(n\pi t)}{n^2 \pi^2} = \begin{cases} \frac{1}{\pi^2} & \text{if } s = t \text{ and } t \in (0,1) \\ \frac{2}{\pi^2} & \text{if } s = t = 0 \text{ or } s = t = 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we conclude, for  $s, t \in [0, 1]$ ,

(2.10) 
$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{2\cos\left(\left(n + \frac{1}{2}\right)\pi s\right)\cos\left(\left(n + \frac{1}{2}\right)\pi t\right)}{\left(n + \frac{1}{2}\right)^{2}\pi^{2}} = \begin{cases} \frac{1}{\pi^{2}} & \text{if } s = t \text{ and } t \in (0, 1) \\ \frac{2}{\pi^{2}} & \text{if } s = t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof of Theorem 1.1. Since  $p, w \in C^2([a, b])$ , we can apply the Liouville transformation to the given regular Sturm-Liouville problem. Under this transformation, the family  $\{\phi_n : n \in \mathbb{N}_0\}$  of eigenfunctions on [a, b] is transformed to the family  $\{u_n : n \in \mathbb{N}_0\}$  of eigenfunctions on [0, c] defined by

$$u_n(t) = \sqrt[4]{p(x)w(x)}\phi_n(x) .$$

where

$$t = \int_a^x \sqrt{\frac{w(z)}{p(z)}} dz$$
 and  $c = \int_a^b \sqrt{\frac{w(z)}{p(z)}} dz$ .

Further setting

$$s = \int_{a}^{y} \sqrt{\frac{w(z)}{p(z)}} \, \mathrm{d}z$$

and observing that

$$\int_{a}^{b} (\phi_{n}(z))^{2} w(z) dz = \int_{0}^{c} (u_{n}(r))^{2} dr,$$

we can write

$$(2.11) N \sum_{n=N+1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n \int_a^b (\phi_n(z))^2 w(z) dz} = N \sum_{n=N+1}^{\infty} \frac{1}{\sqrt[4]{p(x)w(x)}} \frac{u_n(s)u_n(t)}{\sqrt[4]{p(y)w(y)}} \frac{u_n(s)u_n(t)}{\lambda_n \int_0^c (u_n(r))^2 dr}.$$

Moreover, since p and w are positive functions by assumption, the four cases distinguished by  $\alpha_2 = 0$  or  $\alpha_2 \neq 0$ , and  $\beta_2 = 0$  or  $\beta_2 \neq 0$  remain invariant under the Liouville transformation, whilst the values of the non-zero constants might change. In particular, if we have  $\alpha_2, \beta_2 \neq 0$  in the original Sturm-Liouville problem then, by Proposition 2.2, we obtain, as  $N \to \infty$ ,

$$N \sum_{n=N+1}^{\infty} \frac{u_n(s)u_n(t)}{\lambda_n \int_0^c (u_n(r))^2 dr} = N \sum_{n=N+1}^{\infty} \frac{2\cos\left(\frac{n\pi s}{c}\right)\cos\left(\frac{n\pi t}{c}\right) + O\left(\frac{1}{n}\right)}{c\lambda_n}.$$

Applying Proposition 2.1, which for  $\alpha_2, \beta_2 \neq 0$  gives that, as  $n \to \infty$ ,

$$c^2 \lambda_n = n^2 \pi^2 + O(1)$$
.

we conclude

$$(2.12) N \sum_{n=N+1}^{\infty} \frac{u_n(s)u_n(t)}{\lambda_n \int_0^c (u_n(r))^2 dr} = N \sum_{n=N+1}^{\infty} \left( \frac{2c \cos\left(\frac{n\pi s}{c}\right) \cos\left(\frac{n\pi t}{c}\right)}{n^2 \pi^2} + O\left(\frac{1}{n^3}\right) \right) .$$

It suffices to note that

$$0 \le \lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{1}{n^3} \le \lim_{N \to \infty} \sum_{n=N+1}^{\infty} \frac{1}{n^2} = 0$$

to deduce the claimed result from (2.11), (2.12) and (2.9). The remaining three cases given in terms of the separated homogeneous boundary conditions follow similarly from Proposition 2.1 and Proposition 2.2 as well as from (2.10), Proposition 2.3 and [10, Theorem 1.2], respectively.  $\square$ 

2.3. Illustrating example. We present one example to illustrate Theorem 1.1 which goes beyond the four base cases encountered above but where the eigenfunctions and eigenvalues can still be computed explicitly, allowing for the inclusion of a plot.

# **Example 2.4.** We consider the eigenvalue problem

$$\phi''(x) + 3\phi'(x) + 2\phi(x) = -\lambda\phi(x)$$
 subject to  $\phi(0) = 0$ ,  $\phi(1) = 0$ ,

which can be rewritten as the Sturm-Liouville problem

(2.13) 
$$\left(e^{3x}\phi'(x)\right)' + 2e^{3x}\phi(x) = -\lambda e^{3x}\phi(x)$$
 subject to  $\phi(0) = 0$ ,  $\phi(1) = 0$ .

Thus, we are in the setting of Theorem 1.1 with a = 0, b = 1 and, for  $x \in [0, 1]$ ,

$$p(x) = w(x) = e^{3x} .$$

A direct computation shows that the normalised eigenfunctions are given by, for  $n \in \mathbb{N}_0$ ,

$$\phi_n(x) = \sqrt{2} e^{-3x/2} \sin((n+1)\pi x)$$
,

with corresponding eigenvalues

$$\lambda_n = \frac{1 + 4(n+1)^2 \pi^2}{4} \ .$$

To construct the Green's function  $G: [0,1] \times [0,1] \to \mathbb{R}$  of the Sturm-Liouville problem (2.13), we use solutions to the homogeneous differential equation

$$v''(x) + 3v'(x) + 2v(x) = 0$$

one subject to v(0) = 0 and another one subject to v(1) = 0. Taking

$$v_1(x) = e^{-x} - e^{-2x}$$
 as well as  $v_2(x) = e^{-2x} - e^{-1-x}$ 

and computing

$$p(x) (v_1'(x)v_2(x) - v_1(x)v_2'(x)) = 1 - e^{-1},$$

we obtain, for  $x, y \in [0, 1]$ ,

$$G(x,y) = \begin{cases} \frac{\left(e^{-x} - e^{-2x}\right)\left(e^{-2y} - e^{-1-y}\right)}{1 - e^{-1}} & \text{if } 0 \le x \le y\\ \frac{\left(e^{-y} - e^{-2y}\right)\left(e^{-2x} - e^{-1-x}\right)}{1 - e^{-1}} & \text{if } y < x \le 1 \end{cases}.$$

In particular, we have, for  $N \in \mathbb{N}_0$ ,

$$N\sum_{n=N+1}^{\infty}\frac{\left(\phi_n(x)\right)^2}{\lambda_n}=N\left(G(x,x)-\sum_{n=0}^{N}\frac{\left(\phi_n(x)\right)^2}{\lambda_n}\right)\;.$$

A plot of this function for N=100 is shown in Figure 1, which nicely illustrates that, on the diagonal away from its endpoints, the rescaled error in approximating the Green's function is close to be given by  $x\mapsto \mathrm{e}^{-3x}/\pi^2$ , as asserted by Theorem 1.1.

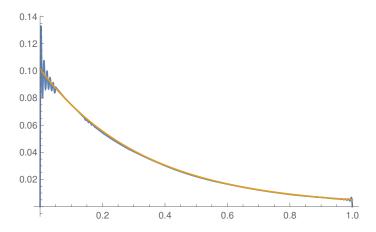


FIGURE 1. Rescaled error in approximating Green's function for N=100 (blue) and  $x\mapsto {\rm e}^{-3x}/\pi^2$  (yellow) on (0,1).

## 3. Green's function formed by classical orthogonal polynomials

We prove Theorem 1.3, Theorem 1.4 and Theorem 1.5, that is, we derive the asymptotic error in the eigenfunction expansion for the Green's function associated with the Hermite polynomials, the associated Laguerre polynomials and the Jacobi polynomials. After we first show that the limit moments on the diagonal satisfy the desired recurrence relation and find a Christoffel-Darboux type formula for any family of classical orthogonal polynomials, we then need to consider the Hermite polynomials, the associated Laguerre polynomials and the Jacobi polynomials separately to conclude the proofs. This involves using asymptotic formulae for the orthogonal polynomials to obtain sufficiently strong bounds and, for the Hermite polynomials and the associated Laguerre polynomials, showing that the moments on the diagonal satisfy the necessary initial condition. For the Jacobi polynomials, the latter seems challenging which is why we employ a workaround. For a family  $\{Y_n : n \in \mathbb{N}_0\}$  of classical orthogonal polynomials on the interval I of orthogonality, there exists a linear function L and a polynomial Q of degree at most two as well as a family  $\{\lambda_n : n \in \mathbb{N}_0\}$  such that, for all  $n \in \mathbb{N}_0$  and for all  $x \in I$ ,

(3.1) 
$$Q(x)Y_n''(x) + L(x)Y_n'(x) + \lambda_n Y_n(x) = 0.$$

They arise by means of linear transformation from the Jacobi polynomials if Q is of degree two and has two distinct zeros, from the associated Laguerre polynomials if Q is linear, and from the Hermite polynomials if Q is constant. In particular, the three main theorems proved in this section allow us to deduce the considered asymptotic error for other classical orthogonal polynomial systems.

Throughout, we assume that the degree of a polynomial is given by its index. Using the integrating factor

$$R(x) = \exp\left(\int_{-\infty}^{x} \frac{L(z)}{Q(z)} dz\right) ,$$

we can write (3.1) in its standard Sturm-Liouville form

$$\left(P(x)Y_n'(x)\right)' = -\lambda_n W(x)Y_n(x) \quad \text{with} \quad P(x) = R(x) \quad \text{and} \quad W(x) = \frac{R(x)}{Q(x)} \; ,$$

where  $\{\lambda_n : n \in \mathbb{N}_0\}$  is the family of eigenvalues corresponding to  $\{Y_n : n \in \mathbb{N}_0\}$ . As discussed in Lesky [14, Satz 3], the eigenvalues are mutually distinct and given by

(3.2) 
$$\lambda_n = -n\left(\frac{n-1}{2}Q'' + L'\right) ,$$

which is indeed a constant. Moreover, it follows that  $\lambda_n \neq 0$  for all  $n \in \mathbb{N}$  since  $\lambda_0 = 0$ . It is further shown, see Lesky [14, Satz 4], that at a boundary point  $c \in \partial I$  of the interval of orthogonality with respect to the weight function  $W : I \to \mathbb{R}$ , we have

$$P(c) = R(c) = 0$$
 if  $c \in \mathbb{R}$ 

and

$$\lim_{x \to c} x^k W(x) = \lim_{x \to c} \frac{x^k R(x)}{Q(x)} = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad \text{if } c = \pm \infty \ .$$

We remark that for a given interval I of orthogonality and a suitable weight function W, we can recover the associated family  $\{Y_n : n \in \mathbb{N}_0\}$  of classical orthogonal polynomials by applying the Gram–Schmidt process to the monomials  $\{x^n : n \in \mathbb{N}_0\}$  on I with respect to the  $L^2(I, W(x) dx)$  inner product and by imposing a normalisation condition.

The one system of polynomials arising from Sturm–Liouville differential equations which is notably missing in our discussion above compared to the classification given in Bochner [4] are the Bessel polynomials. As discussed by Littlejohn and Krall in [15], no sufficiently satisfying weight function has been identified for the Bessel polynomials, apart from the orthogonality relation obtained by integrating over the unit circle in the complex plane with respect to  $e^{-2/x}$  given in Krall and Frink [13]. Recalling that the weight function features in our expression for the asymptotic error, one may wonder if one could use an error analysis of the eigenfunction expansion for the Green's function associated with the Bessel polynomials to find a good candidate for the long sought-after weight function. However, initial plots suggest that this is not feasible.

For a given family  $\{Y_n : n \in \mathbb{N}_0\}$  of classical orthogonal polynomials on the interval I, let  $M_n$  denote the square of the  $L^2(I, W(x) dx)$  norm of  $Y_n$ , that is,

$$M_n = \int_I W(x) \left( Y_n(x) \right)^2 dx .$$

Choosing  $\tau \in \{\frac{1}{2}, 1\}$  such that  $\sqrt{\lambda_n} = O(n^{\tau})$  as  $n \to \infty$ , whose existence is guaranteed by (3.2) and the result that the eigenvalues  $\lambda_n$  are mutually distinct, we are then interested in studying, for  $x, y \in I \setminus \partial I$ ,

(3.3) 
$$\lim_{N \to \infty} N^{\tau} \sum_{n=N+1}^{\infty} \frac{Y_n(x)Y_n(y)}{M_n \lambda_n} ,$$

with the one caveat that in the off-diagonal convergence for the associated Laguerre polynomials we can only deal with scaling exponents strictly smaller than  $\frac{1}{2}$ .

In our analysis, we use the same approach which already proved powerful in [10, 11] and which was demonstrated in the previous section, that is, we split our considerations into an on-diagonal part, consisting of a moment argument and local uniform bounds feeding into the Arzelà–Ascoli theorem, and an off-diagonal part. While we are able to establish in general that, under suitable boundary conditions, in the limit as  $N \to \infty$  the moments on the diagonal satisfy the same recurrence relation as the desired limit function and to derive a general Christoffel–Darboux type formula used for the off-diagonal convergence, it seems difficult to develop a full general analysis. In particular, to show that the moments satisfy the desired initial condition, to obtain the local uniform bounds and to complete the analysis away from the diagonal, we need to distinguish between the Hermite

polynomials, the associated Laguerre polynomials and the Jacobi polynomials. We further rely on existing asymptotic expansions for these polynomials as given in Szegő [19, Chapter 8].

We start by determining the recurrence relation satisfied, in the limit as  $N \to \infty$ , by the moments of the function (3.3) restricted to the diagonal  $\{x=y\}$ . As the final part of the moment analysis requires us to consider the three main families of classical orthogonal polynomials separately, we delay the observation that the integrals and series needed in the following are well-defined and finite until then. Moreover, while the boundary conditions and the asymptotic assumptions in the propositions below are satisfied whenever we want to apply the propositions, we still include the assumptions explicitly to make it clear what is needed.

**Proposition 3.1.** Let  $\{Y_n : n \in \mathbb{N}_0\}$  be a family of classical orthogonal polynomials which are orthogonal on the interval I = (a,b) with respect to the weight function  $W: I \to \mathbb{R}$  and which together with the family  $\{\lambda_n : n \in \mathbb{N}_0\}$  of eigenvalues solve the Sturm-Liouville differential equation

$$(P(x)Y'_n(x))' = -\lambda_n W(x)Y_n(x) ,$$

or equivalently

$$Q(x)Y_n''(x) + L(x)Y_n'(x) + \lambda_n Y_n(x) = 0$$
.

Assume that, for all  $l, n \in \mathbb{N}_0$  and for  $c \in \{a, b\}$ ,

(3.4) 
$$\lim_{x \to c} x^{l+1} P(x) W(x) (Y_n(x))^2 = \lim_{x \to c} x^l P(x) Y_n(x) = \lim_{x \to c} x^l P(x) Y_n'(x) = 0$$

and that, for  $l \in \mathbb{N}_0$  fixed and as  $n \to \infty$ ,

(3.5) 
$$\int_{I} (x^{l} P(x))' P(x) Y_{n}(x) Y_{n}'(x) dx = O(M_{n}).$$

Then the limit moments, for  $k \in \mathbb{N}_0$ ,

$$m_k = \lim_{N \to \infty} N^{\tau} \sum_{n=N+1}^{\infty} \int_I \frac{x^k \left(W(x)\right)^2 \left(Y_n(x)\right)^2}{M_n \lambda_n} \, \mathrm{d}x$$

satisfy the recurrence relation

$$(k+1)Q(0)m_k + \left(L(0) + \left(k + \frac{1}{2}\right)Q'(0)\right)m_{k+1} + \left(L'(0) + \frac{k}{2}Q''(0)\right)m_{k+2} = 0.$$

*Proof.* Using the Sturm-Liouville differential equation, we obtain that, for all  $k, n \in \mathbb{N}_0$ ,

$$\lambda_n \int_I x^k P(x) W(x) (Y_n(x))^2 dx = -\int_I x^k (P(x) Y_n'(x))' P(x) Y_n(x) dx.$$

After integrating by parts, we have

$$\lambda_n \int_I x^k P(x) W(x) (Y_n(x))^2 dx$$

$$= -x^k (P(x))^2 Y_n(x) Y_n'(x) \Big|_a^b + \int_I x^k (P(x) Y_n'(x))^2 dx + \int_I (x^k P(x))' P(x) Y_n(x) Y_n'(x) dx.$$

Due to the boundary conditions (3.4), which give

$$x^{k} (P(x))^{2} Y_{n}(x) Y'_{n}(x) \Big|_{a}^{b} = 0$$
,

and the asymptotic assumption (3.5), the above reduces to, for  $k \in \mathbb{N}_0$  fixed and as  $n \to \infty$ ,

$$\lambda_n \int_I x^k P(x) W(x) (Y_n(x))^2 dx = \int_I x^k (P(x) Y_n'(x))^2 dx + O(M_n).$$

Integrating by parts yet again and using (3.4), we further see that

$$\int_{I} x^{k} (P(x)Y'_{n}(x))^{2} dx = -\frac{2}{k+1} \int_{I} x^{k+1} P(x)Y'_{n}(x) (P(x)Y'_{n}(x))' dx$$
$$= \frac{2\lambda_{n}}{k+1} \int_{I} x^{k+1} P(x)W(x)Y_{n}(x)Y'_{n}(x) dx$$

as well as

$$\int_{I} x^{k} P(x) W(x) (Y_{n}(x))^{2} dx$$

$$= -\frac{2}{k+1} \int_{I} x^{k+1} P(x) W(x) Y_{n}(x) Y_{n}'(x) dx - \frac{1}{k+1} \int_{I} x^{k+1} (P(x) W(x))' (Y_{n}(x))^{2} dx$$

Combining these identities yields, for  $k \in \mathbb{N}_0$  fixed and as  $n \to \infty$ ,

$$(3.6) 2\lambda_n \int_I x^k P(x) W(x) (Y_n(x))^2 dx + \frac{\lambda_n}{k+1} \int_I x^{k+1} (P(x)W(x))' (Y_n(x))^2 dx = O(M_n).$$

Observing that

$$P(x)W(x) = Q(x) \left(\frac{R(x)}{Q(x)}\right)^2 \quad \text{and} \quad \left(P(x)W(x)\right)' = 2\left(L(x) - \frac{1}{2}Q'(x)\right) \left(\frac{R(x)}{Q(x)}\right)^2 \ ,$$

we can rewrite (3.6) as

$$2\lambda_n \int_I x^k Q(x) (W(x))^2 (Y_n(x))^2 dx + \frac{2\lambda_n}{k+1} \int_I x^{k+1} \left( L(x) - \frac{1}{2} Q'(x) \right) (W(x))^2 (Y_n(x))^2 dx$$
  
=  $O(M_n)$ .

Since  $\tau$  is chosen such that  $\sqrt{\lambda_n} = O(n^{\tau})$  as  $n \to \infty$ , it follows that

$$\int_{I} \frac{x^{k} Q(x) (W(x))^{2} (Y_{n}(x))^{2}}{M_{n} \lambda_{n}} dx + \frac{1}{k+1} \int_{I} \frac{x^{k+1} (L(x) - \frac{1}{2} Q'(x)) (W(x))^{2} (Y_{n}(x))^{2}}{M_{n} \lambda_{n}} dx 
= O\left(\frac{1}{n^{4\tau}}\right).$$

Using

$$\lim_{N\to\infty}\sqrt{N}\sum_{n=N+1}^{\infty}\frac{1}{n^2}=0\quad\text{as well as}\quad\lim_{N\to\infty}N\sum_{n=N+1}^{\infty}\frac{1}{n^4}=0$$

together with the property that both Q and  $L - \frac{1}{2}Q'$  are polynomials, we deduce the claimed recurrence relation for the limit moments  $\{m_k : k \in \mathbb{N}_0\}$ .

According to the following proposition, under suitable boundary conditions, weighted moments of the function  $x\mapsto 1/\sqrt{P(x)W(x)}$  defined on the interval I satisfy the same recurrence relation as the limit moments considered above.

**Proposition 3.2.** Provided that, for all  $l \in \mathbb{N}_0$  and for  $c \in \{a, b\}$ ,

(3.7) 
$$\lim_{x \to b} \frac{x^{l+1}R(x)}{\sqrt{Q(x)}} - \lim_{x \to a} \frac{x^{l+1}R(x)}{\sqrt{Q(x)}} = 0,$$

the weighted moments defined by, for  $k \in \mathbb{N}_0$  and some constant  $C \in \mathbb{R}$ ,

$$\widetilde{m}_k = C \int_I \frac{x^k (W(x))^2}{\sqrt{P(x)W(x)}} dx$$

satisfy the recurrence relation

$$(k+1)Q(0)\widetilde{m}_k + \left(L(0) + \left(k + \frac{1}{2}\right)Q'(0)\right)\widetilde{m}_{k+1} + \left(L'(0) + \frac{k}{2}Q''(0)\right)\widetilde{m}_{k+2} = 0.$$

Proof. Since

$$\frac{(W(x))^2}{\sqrt{P(x)W(x)}} = \frac{R(x)}{(Q(x))^{\frac{3}{2}}} \quad \text{as well as} \quad \left(\frac{R(x)}{\sqrt{Q(x)}}\right)' = \frac{R(x)}{(Q(x))^{\frac{3}{2}}} \left(L(x) - \frac{1}{2}Q'(x)\right) ,$$

integration by parts and the boundary condition (3.7) give that, for all  $k \in \mathbb{N}_0$ ,

$$\int_{I} \frac{x^{k} Q(x) (W(x))^{2}}{\sqrt{P(x)W(x)}} dx = \int_{I} \frac{x^{k} R(x)}{\sqrt{Q(x)}} dx = -\frac{1}{k+1} \int_{I} \frac{x^{k+1} R(x)}{(Q(x))^{\frac{3}{2}}} \left( L(x) - \frac{1}{2} Q'(x) \right) dx$$

$$= -\frac{1}{k+1} \int_{I} \frac{x^{k+1} \left( L(x) - \frac{1}{2} Q'(x) \right) (W(x))^{2}}{\sqrt{P(x)W(x)}} dx.$$

As in the proof of Proposition 3.1, this yields the claimed recurrence relation.

The second main ingredient for our subsequent analysis which we establish for general classical orthogonal polynomial systems is the Christoffel–Darboux type formula given in Proposition 3.4. For its derivation, we need an extension of Szegő [19, Theorem 3.2.1] to orthogonal polynomials which are not assumed to be orthonormal. It characterises two of the three coefficients appearing in the three-term recurrence relation satisfied by the orthogonal polynomials  $\{Y_n : n \in \mathbb{N}_0\}$  in terms of the square  $M_n$  of the  $L^2(I, W(x) dx)$  norm of  $Y_n$  and the leading coefficient  $K_n$  of  $Y_n$ .

**Lemma 3.3.** Let  $\{Y_n : n \in \mathbb{N}_0\}$  be a family of classical orthogonal polynomials on the interval I with respect to the weight function  $W : I \to \mathbb{R}$ . We have the three-term recurrence relation, for  $n \in \mathbb{N}$  and  $x \in I$ ,

$$Y_{n+1}(x) = (A_n x + B_n) Y_n(x) - C_n Y_{n-1}(x)$$

with constants  $A_n, B_n, C_n \in \mathbb{R}$ , where

$$A_n = \frac{K_{n+1}}{K_n} \quad and \quad C_n = \frac{K_{n-1}K_{n+1}}{K_n^2} \frac{M_n}{M_{n-1}} \; .$$

*Proof.* Adapting Szegő [19, Proof of Theorem 3.2.1], we observe that  $Y_{n+1}(x) - A_n x Y_n(x)$  defines a polynomial on I of degree n. Hence, this polynomial can be written as a linear combination of  $Y_0, Y_1, \ldots, Y_n$ . Similarly, for all  $m \in \mathbb{N}_0$ , the polynomial defined by  $x Y_m(x)$  can be expressed as a linear combination of  $Y_0, Y_1, \ldots, Y_{m+1}$ . By orthogonality, it follows that, as long as  $0 \le m < n-1$ ,

$$\int_{I} (Y_{n+1}(x) - A_n x Y_n(x)) Y_m(x) W(x) dx = 0.$$

Thus, the polynomial defined by  $Y_{n+1}(x) - A_n x Y_n(x)$  is a linear combination of  $Y_{n-1}$  and  $Y_n$  only. It remains to identify the coefficient in front of  $Y_{n-1}$ . Using the three-term recurrence relation and orthogonality, we deduce that

(3.8) 
$$A_n \int_I x Y_n(x) Y_{n-1}(x) W(x) dx - C_n M_{n-1} = \int_I Y_{n+1}(x) Y_{n-1}(x) W(x) dx = 0.$$

When expressing the polynomial defined by  $xY_{n-1}(x)$  as a linear combination of  $Y_0, Y_1, \ldots, Y_n$ , the coefficient in front of  $Y_n$  is  $K_{n-1}/K_n$ . We conclude

$$\int_{I} x Y_n(x) Y_{n-1}(x) W(x) \, \mathrm{d}x = \frac{K_{n-1} M_n}{K_n} \;,$$

which together with (3.8) yields

$$C_n = \frac{K_{n-1}K_{n+1}}{K_n^2} \frac{M_n}{M_{n-1}} \, .$$

as required.

The standard Christoffel–Darboux formula, see Szegő [19, Theorem 3.2.2], asserts that, for  $N \in \mathbb{N}$  and  $x, y \in I$ ,

$$(x-y)\sum_{n=0}^{N} \frac{Y_n(x)Y_n(y)}{M_n} = \frac{K_N}{K_{N+1}M_N} \left( Y_{N+1}(x)Y_N(y) - Y_N(x)Y_{N+1}(y) \right) .$$

The Christoffel–Darboux type formula stated below extends the Christoffel–Darboux type formula, see [11, Proposition 5.1], derived for integrals of Legendre polynomials. It enters the analysis in exactly the same way as [11, Proposition 5.1] did in establishing the convergence away from the diagonal for [11, Theorem 1.5].

**Proposition 3.4.** Let  $\{Y_n : n \in \mathbb{N}_0\}$  be a family of classical orthogonal polynomials and let  $\{\lambda_n : n \in \mathbb{N}_0\}$  be the corresponding family of eigenvalues. Setting, for  $n \in \mathbb{N}_0$  and  $x, y \in I$ ,

$$D_{n+1}(x,y) = Y_{n+1}(x)Y_n(y) - Y_n(x)Y_{n+1}(y) ,$$

we have, for  $N \in \mathbb{N}$ ,

$$(x-y)\sum_{n=1}^{N} \frac{Y_n(x)Y_n(y)}{M_n \lambda_n}$$

$$= \frac{K_N}{K_{N+1}M_N} \frac{D_{N+1}(x,y)}{\lambda_N} - \frac{K_0}{K_1M_0} \frac{D_1(x,y)}{\lambda_1} + \sum_{n=1}^{N-1} \frac{K_n}{K_{n+1}M_n} D_{n+1}(x,y) \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right).$$

*Proof.* Since the eigenvalues are known to be mutually distinct and as  $\lambda_0 = 0$ , the expressions we consider are all well-defined. From the three-term recurrence relation given in Lemma 3.3 for the family  $\{Y_n : n \in \mathbb{N}_0\}$  of classical orthogonal polynomials, it follows that, for  $n \in \mathbb{N}$  and for  $x, y \in I$ ,

$$D_{n+1}(x,y) = A_n(x-y)Y_n(x)Y_n(y) + C_nD_n(x,y) ,$$

which implies that, for  $N \in \mathbb{N}$ ,

$$(x-y)\sum_{n=1}^{N} \frac{Y_n(x)Y_n(y)}{M_n\lambda_n} = \sum_{n=1}^{N} \frac{D_{n+1}(x,y)}{A_nM_n\lambda_n} - \sum_{n=1}^{N} \frac{C_nD_n(x,y)}{A_nM_n\lambda_n} .$$

Using the expressions for  $A_n$  and  $C_n$  from Lemma 3.3, we deduce

$$\begin{split} &(x-y)\sum_{n=1}^{N}\frac{Y_{n}(x)Y_{n}(y)}{M_{n}\lambda_{n}}\\ &=\sum_{n=1}^{N}\frac{K_{n}}{K_{n+1}M_{n}}\frac{D_{n+1}(x,y)}{\lambda_{n}}-\sum_{n=0}^{N-1}\frac{K_{n}}{K_{n+1}M_{n}}\frac{D_{n+1}(x,y)}{\lambda_{n+1}}\\ &=\frac{K_{N}}{K_{N+1}M_{N}}\frac{D_{N+1}(x,y)}{\lambda_{N}}-\frac{K_{0}}{K_{1}M_{0}}\frac{D_{1}(x,y)}{\lambda_{1}}+\sum_{n=1}^{N-1}\frac{K_{n}}{K_{n+1}M_{n}}D_{n+1}(x,y)\left(\frac{1}{\lambda_{n}}-\frac{1}{\lambda_{n+1}}\right)\;, \end{split}$$

as claimed.  $\Box$ 

In particular, note that if, for  $x, y \in I$ ,

$$\lim_{n\to\infty} \frac{K_n}{K_{n+1}M_n} \frac{D_{n+1}(x,y)}{\lambda_n} = 0 ,$$

the Christoffel–Darboux type formula yields

$$(x-y)\sum_{n=N+1}^{\infty} \frac{Y_n(x)Y_n(y)}{M_n\lambda_n} = \sum_{n=N}^{\infty} \frac{K_n}{K_{n+1}M_n} D_{n+1}(x,y) \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) - \frac{K_N}{K_{N+1}M_N} \frac{D_{N+1}(x,y)}{\lambda_N} .$$

This implication is so powerful in our subsequent analysis because, as  $n \to \infty$ ,

$$\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} = O\left(\frac{1}{n^{2\tau+1}}\right)$$

while  $\lambda_n^{-1} = O\left(n^{-2\tau}\right)$ , which gives rise to better estimates for the terms in the second series above than for the terms in the first series. Essentially, the Christoffel–Darboux type formula encompasses a cancellation which is otherwise missed when bounding summands separately.

Having established two main results for general classical orthogonal polynomial systems, we now consider the Hermite polynomials, the associated Laguerre polynomials and the Jacobi polynomials separately, in that order. The reason for choosing this order is that the proofs for the Hermite polynomials are the cleanest and already convey the used strategy well, whilst the proofs for the Jacobi polynomials contain the most involved expressions.

Throughout the study in the following three subsections, we frequently use Stirling's formula in the form, for  $z \in \mathbb{R}$  as  $z \to \infty$ ,

(3.9) 
$$\Gamma(z+1) \sim \sqrt{2\pi z} \left(\frac{z}{e}\right)^z$$

as well as the asymptotics for the Gamma function that, for  $\alpha \in \mathbb{R}$  and as  $z \to \infty$ ,

(3.10) 
$$\Gamma(z+\alpha) \sim \Gamma(z)z^{\alpha} ,$$

see [1, 6.1.39]. We further need the Legendre duplication formula [1, 6.1.18] asserting, for  $z \in \mathbb{R}$ ,

(3.11) 
$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \ .$$

3.1. **Hermite polynomials.** The Hermite polynomials  $\{H_n : n \in \mathbb{N}_0\}$  are the classical orthogonal polynomials on the interval  $I = \mathbb{R}$  with respect to the weight function  $W : \mathbb{R} \to \mathbb{R}$  given by

$$W(x) = \exp\left(-x^2\right)$$

subject to the normalisations, for  $n \in \mathbb{N}_0$ ,

$$M_n = \int_{-\infty}^{\infty} e^{-x^2} (H_n(x))^2 dx = 2^n n! \sqrt{\pi}$$
 and  $K_n = 2^n$ ,

see Szegő [19, Chapter 5.5]. These Hermite polynomials are also referred to as the physicist Hermite polynomials to distinguish them from the probabilist Hermite polynomials which are orthogonal on  $\mathbb{R}$  with respect to the weight function defined by  $\exp\left(-\frac{1}{2}x^2\right)$ .

As discussed by Littlejohn and Krall in [15], the Hermite polynomials solve the differential equation, for  $n \in \mathbb{N}_0$  and  $x \in \mathbb{R}$ ,

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 ,$$

whose Sturm-Liouville form is

(3.12) 
$$\left(e^{-x^2} H'_n(x)\right)' = -2n e^{-x^2} H_n(x) .$$

In particular, using the notations introduced previously, we have

$$Q(x) = 1$$
,  $L(x) = -2x$ ,  $P(x) = W(x) = \exp(-x^2)$  and  $\lambda_n = 2n$ .

For our analysis, we need the two identities, for  $n \in \mathbb{N}$ ,

$$(3.13) H'_n(x) = 2nH_{n-1}(x)$$

and

(3.14) 
$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x) ,$$

see [19, 5.5.10], giving rise to the three-term recurrence relation

(3.15) 
$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) ,$$

which is consistent with Lemma 3.3. We further rely on the asymptotic formula that, for  $x \in \mathbb{R}$  and as  $n \to \infty$ ,

(3.16) 
$$H_n(x) = e^{\frac{1}{2}x^2} \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \cos\left(x\sqrt{2n} - \frac{n\pi}{2}\right) \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) ,$$

a consequence of [1, 13.5.16 and 13.6.38], where the bound on the error term is uniform on every finite real interval. By the Legendre duplication formula (3.11), we have

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}+1\right) = 2^{-n}\sqrt{\pi}\Gamma(n+1)$$
,

which together with Stirling's formula (3.9) implies that, as  $n \to \infty$ ,

$$\frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right) \sim \sqrt{2}\left(\frac{2n}{e}\right)^{\frac{n}{2}}.$$

Thus, for all  $K \in \mathbb{R}$  with K > 0, there exists a positive constant  $C \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}_0$  and for all  $x \in [-K, K]$ ,

$$(3.17) |H_n(x)| \le C \left(\frac{2n}{e}\right)^{\frac{n}{2}}.$$

Moreover, we observe that due to the exponential decay in  $\exp(-x^2)$  as  $x \to \pm \infty$ , for all  $l, n \in \mathbb{N}_0$ ,

(3.18) 
$$\lim_{x \to \pm \infty} x^l e^{-x^2} H_n(x) = \lim_{x \to \pm \infty} x^l e^{-x^2} H'_n(x) = 0.$$

The first lemma in this subsection is later needed to verify that the limit moments on the diagonal satisfy the correct initial condition.

**Lemma 3.5.** For all  $n \in \mathbb{N}_0$ , we have

$$\int_{-\infty}^{\infty} e^{-2x^2} (H_n(x))^2 dx = 2^{n-\frac{1}{2}} \Gamma\left(n + \frac{1}{2}\right).$$

*Proof.* The Sturm-Liouville differential equation (3.12) yields, for  $n \in \mathbb{N}_0$  and  $x \in \mathbb{R}$ ,

$$\left(e^{-2x^2} \left(H'_n(x)\right)^2\right)' = -4n e^{-2x^2} H_n(x) H'_n(x) .$$

Integrating by parts as well as using (3.18), the identity (3.13) and the recurrence relation (3.15), we obtain, for  $n \in \mathbb{N}$ ,

$$\int_{-\infty}^{\infty} e^{-2x^2} (H'_n(x))^2 dx = 2n \int_{-\infty}^{\infty} e^{-2x^2} 2x H_n(x) H'_n(x) dx$$
$$= 4n^2 \int_{-\infty}^{\infty} e^{-2x^2} (H_{n+1}(x) + 2n H_{n-1}(x)) H_{n-1}(x) dx.$$

Applying (3.13) to the left hand side above implies that

$$(3.19) (1-2n) \int_{-\infty}^{\infty} e^{-2x^2} (H_{n-1}(x))^2 dx = \int_{-\infty}^{\infty} e^{-2x^2} H_{n+1}(x) H_{n-1}(x) dx.$$

On the other hand, by using (3.13), integrating by parts and noting (3.18), and concluding with (3.13) as well as (3.14), we can argue, for  $n \in \mathbb{N}$ ,

$$2(n+1) \int_{-\infty}^{\infty} e^{-2x^2} (H_n(x))^2 dx = \int_{-\infty}^{\infty} e^{-2x^2} H_n(x) H'_{n+1}(x) dx$$
$$= \int_{-\infty}^{\infty} e^{-2x^2} (4x H_n(x) - H'_n(x)) H_{n+1}(x) dx$$
$$= \int_{-\infty}^{\infty} e^{-2x^2} (2H_{n+1}(x) + 2nH_{n-1}(x)) H_{n+1}(x) dx .$$

Putting this together with (3.19) gives, for  $n \in \mathbb{N}$ ,

$$\int_{-\infty}^{\infty} e^{-2x^2} (H_{n+1}(x))^2 dx$$

$$= (n+1) \int_{-\infty}^{\infty} e^{-2x^2} (H_n(x))^2 dx + n(2n-1) \int_{-\infty}^{\infty} e^{-2x^2} (H_{n-1}(x))^2 dx ,$$

which is a recurrence relation for the integrals we wish to evaluate. Since

$$\int_{-\infty}^{\infty} e^{-2x^2} (H_0(x))^2 dx = \int_{-\infty}^{\infty} e^{-2x^2} dx = \sqrt{\frac{\pi}{2}} = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}},$$

$$\int_{-\infty}^{\infty} e^{-2x^2} (H_1(x))^2 dx = \int_{-\infty}^{\infty} e^{-2x^2} 4x^2 dx = \sqrt{\frac{\pi}{2}} = \sqrt{2} \Gamma(\frac{3}{2}),$$

and, for  $n \in \mathbb{N}$ ,

$$2(n+1)\,\Gamma\left(n+\frac{1}{2}\right)+n(2n-1)\,\Gamma\left(n-\frac{1}{2}\right)=2\left(2n+1\right)\Gamma\left(n+\frac{1}{2}\right)=4\,\Gamma\left(n+\frac{3}{2}\right)\;,$$

the claimed result follows by induction.

The next two lemmas feed into the Arzelà–Ascoli theorem to deduce a locally uniform convergence on the diagonal, which allows us to identify the pointwise limit from the moment analysis.

**Lemma 3.6.** Fix  $K \in \mathbb{R}$  with K > 0. The family

$$\left\{\sqrt{N}\sum_{n=N+1}^{\infty}\frac{H_n(x)H_n(y)}{2n\,2^nn!}:N\in\mathbb{N}\ and\ x,y\in[-K,K]\right\}$$

is uniformly bounded.

*Proof.* According to the bound (3.17), there exists a constant  $C \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$  and for all  $x, y \in [-K, K]$ ,

$$\left| \frac{H_n(x)H_n(y)}{2n \, 2^n n!} \right| \le \frac{C^2}{2n \, 2^n n!} \left( \frac{2n}{e} \right)^n.$$

By Stirling's formula (3.9), we have, as  $n \to \infty$ ,

$$\frac{1}{2^n n!} \left(\frac{2n}{e}\right)^n = \frac{1}{n!} \left(\frac{n}{e}\right)^n \sim \frac{1}{\sqrt{2\pi n}}.$$

Hence, there exists a positive constant  $D \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$  and for all  $x, y \in [-K, K]$ ,

(3.20) 
$$\left| \frac{H_n(x)H_n(y)}{2n \, 2^n n!} \right| \le D n^{-\frac{3}{2}} \,,$$

which implies that, for all  $N \in \mathbb{N}$ ,

$$\left| \sqrt{N} \sum_{n=N+1}^{\infty} \frac{H_n(x) H_n(y)}{2n \, 2^n \, n!} \right| \le \sqrt{N} \sum_{n=N+1}^{\infty} D n^{-\frac{3}{2}} \le D \sqrt{N} \left( N^{-\frac{3}{2}} + \int_N^{\infty} z^{-\frac{3}{2}} \, \mathrm{d}z \right) \le 3D \;,$$

as required.

**Lemma 3.7.** Fix  $K \in \mathbb{R}$  with K > 0. The family

$$\left\{\sqrt{N}\sum_{n=N+1}^{\infty}\frac{H_n(x)H_{n-1}(x)}{2^n n!}: N \in \mathbb{N} \text{ and } x \in [-K, K]\right\}$$

is uniformly bounded.

*Proof.* Using the recurrence relation (3.15), we can rewrite

$$\sum_{n=N+1}^{\infty} \frac{H_n(x)H_{n-1}(x)}{2^n n!} = \sum_{n=N+1}^{\infty} \frac{2x \left(H_n(x)\right)^2 - H_n(x)H_{n+1}(x)}{2n \, 2^n n!} ,$$

which implies that

(3.21) 
$$\sum_{n=N+1}^{\infty} \frac{H_n(x)H_{n-1}(x)}{2^n(n-1)!} \left(\frac{1}{n} + \frac{1}{n-1}\right) = \sum_{n=N+1}^{\infty} \frac{2x\left(H_n(x)\right)^2}{2n \, 2^n n!} + \frac{H_N(x)H_{N+1}(x)}{2N \, 2^N N!} .$$

Since, for  $n \in \mathbb{N}$  with  $n \geq 2$ ,

$$\frac{1}{n} + \frac{1}{n-1} = \frac{1}{n} \left( 2 + \frac{1}{n-1} \right) ,$$

we deduce that

$$2\sum_{n=N+1}^{\infty} \frac{H_n(x)H_{n-1}(x)}{2^n n!} = \sum_{n=N+1}^{\infty} \frac{2x \left(H_n(x)\right)^2}{2n \, 2^n n!} - \sum_{n=N+1}^{\infty} \frac{H_n(x)H_{n-1}(x)}{(n-1) \, 2^n n!} + \frac{H_N(x)H_{N+1}(x)}{2N \, 2^N N!} \ .$$

Crucially, thanks to the telescoping-like rearrangement in (3.21), the terms of the third series in the above identity have an additional factor of n-1 in the denominator compared to the terms of the series we are interested in. As a result, we can use the bound (3.17) to conclude. For all  $n \in \mathbb{N}$  with  $n \geq 2$  and for all  $x \in [-K, K]$ , we have

$$\left| \frac{H_n(x)H_{n-1}(x)}{(n-1)2^n n!} \right| \le \frac{C^2}{(n-1)2^n n!} \left( \frac{2n}{e} \right)^{\frac{n}{2}} \left( \frac{2(n-1)}{e} \right)^{\frac{n-1}{2}}$$

From Stirling's formula (3.9), it follows that there exists  $D \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$  with  $n \ge 2$  and for all  $x \in [-K, K]$ ,

$$\left| \frac{H_n(x)H_{n-1}(x)}{(n-1)2^n n!} \right| \le Dn^{-2} .$$

Moreover, we can choose  $D \in \mathbb{R}$  such that, for all  $N \in \mathbb{N}$  and for all  $x \in [-K, K]$ ,

$$\left| \frac{H_N(x)H_{N+1}(x)}{2N \, 2^N N!} \right| \le DN^{-1} \ .$$

This together with the bound (3.20) obtained in the proof of Lemma 3.6 implies that

$$\left| \sqrt{N} \sum_{n=N+1}^{\infty} \frac{H_n(x)H_{n-1}(x)}{2^n n!} \right| \le \sqrt{N} \left( \sum_{n=N+1}^{\infty} DK n^{-\frac{3}{2}} + \sum_{n=N+1}^{\infty} Dn^{-2} + DN^{-1} \right) \le 3D(1+K) ,$$

which establishes the claimed result.

Putting everything together, we can determine the asymptotic error in approximating the Green's function associated with the Hermite polynomials by truncating the bilinear expansion.

Proof of Theorem 1.3. We start by proving the claimed convergence on the diagonal. For  $N \in \mathbb{N}_0$ , let  $S_N : \mathbb{R} \to \mathbb{R}$  be defined by

$$S_N(x) = \sqrt{N} \sum_{n=N+1}^{\infty} \frac{(H_n(x))^2}{2n \, 2^n n! \sqrt{\pi}}.$$

We have, for all  $x \in \mathbb{R}$ ,

$$S'_N(x) = \sqrt{N} \sum_{n=N+1}^{\infty} \frac{H_n(x)H_{n-1}(x)}{2^{n-1}n!\sqrt{\pi}}$$

due to (3.13) and since the series converges uniformly on finite intervals by the estimates given in the proof of Lemma 3.7. From Lemma 3.6, Lemma 3.7 and the Arzelà–Ascoli theorem, it follows that the functions  $S_N$  converge locally uniformly on  $\mathbb{R}$  as  $N \to \infty$ . Thus, the pointwise limit function is continuous on  $\mathbb{R}$  and we can identify it by considering its moments. The symmetry of the Hermite polynomials, implied by the analogue [19, 5.5.3] of the Rodrigues formula, shows that, for all  $k \in \mathbb{N}_0$  and for all  $N \in \mathbb{N}_0$ ,

$$\int_{-\infty}^{\infty} x^{2k+1} e^{-2x^2} S_N(x) dx = 0 ,$$

which is consistent with the claimed limit function being even. Moreover, for the even weighted moments and in terms of the limit moments considered in Proposition 3.1, we have

$$m_{2k} = \lim_{N \to \infty} \int_{-\infty}^{\infty} x^{2k} e^{-2x^2} S_N(x) dx$$
,

where we can interchange summation and integration because all terms are non-negative. Since, for  $l \in \mathbb{N}_0$  fixed, the function

$$x \mapsto \left( \left( x^l e^{-x^2} \right)' e^{-x^2} \right)' e^{x^2}$$

is bounded on  $\mathbb{R}$  and using (3.18), we see that, as  $n \to \infty$ ,

$$\int_{-\infty}^{\infty} \left( x^l e^{-x^2} \right)' e^{-x^2} H_n(x) H_n'(x) dx = -\frac{1}{2} \int_{-\infty}^{\infty} \left( \left( x^l e^{-x^2} \right)' e^{-x^2} \right)' (H_n(x))^2 dx = O(M_n).$$

This together with (3.18) shows that we can employ Proposition 3.1 to deduce that the even limit moments satisfy the recurrence relation

$$m_{2k+2} = \frac{k+1}{2} m_{2k} \, ,$$

which, according to Proposition 3.2 and since  $R(x) = \exp(-x^2)$  as well as Q(x) = 1, is the same recurrence relation as satisfied by the even weighted moments

$$\widetilde{m}_{2k} = \frac{1}{\sqrt{2}\pi} \int_{-\infty}^{\infty} x^{2k} e^{-x^2} dx$$
.

It suffices to prove  $m_0 = \widetilde{m}_0$  to conclude that the functions  $S_N$  converge pointwise as  $N \to \infty$  to the claimed limit function. Applying Lemma 3.5 and the asymptotics (3.10), we obtain

$$m_0 = \lim_{N \to \infty} \sqrt{N} \sum_{n=N+1}^{\infty} \frac{2^{n-\frac{1}{2}}}{2n \, 2^n n! \sqrt{\pi}} \, \Gamma\left(n + \frac{1}{2}\right) = \lim_{N \to \infty} \sqrt{N} \sum_{n=N+1}^{\infty} \frac{1}{2n\sqrt{2n\pi}} = \frac{1}{\sqrt{2\pi}} \,,$$

and a direct computation gives

$$\widetilde{m}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{\sqrt{2\pi}}.$$

It remains to establish the off-diagonal convergence, which is a consequence of Proposition 3.4 and the bound (3.17). Due to the estimate (3.17) and Stirling's formula (3.9), for  $x, y \in \mathbb{R}$  fixed, there exists a positive constant  $D \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$ ,

(3.22) 
$$\left| \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{2n \, 2^{n+1} n!} \right| \le Dn^{-1} .$$

Thus, applying Proposition 3.4 to the Hermite polynomials and using

$$\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} = \frac{1}{2n} - \frac{1}{2(n+1)} = \frac{1}{2n(n+1)} \;,$$

we have that, for all  $N \in \mathbb{N}$ ,

$$(x-y)\sum_{n=N+1}^{\infty} \frac{H_n(x)H_n(y)}{2n \, 2^n n! \sqrt{\pi}} = \sum_{n=N}^{\infty} \frac{1}{2n(n+1)} \frac{D_{n+1}(x,y)}{2^{n+1} n! \sqrt{\pi}} - \frac{1}{2^{N+1} N! \sqrt{\pi}} \frac{D_{N+1}(x,y)}{2N} .$$

It follows that

$$\left| (x-y) \sum_{n=N+1}^{\infty} \frac{H_n(x)H_n(y)}{2n \, 2^n n! \sqrt{\pi}} \right| \le D \left( \sum_{n=N}^{\infty} \frac{1}{n(n+1)} + \frac{1}{N} \right) = \frac{2D}{N} ,$$

which implies that provided  $x \neq y$ , for all  $\gamma < 1$ ,

$$\lim_{N\to\infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{H_n(x)H_n(y)}{2n \, 2^n n! \sqrt{\pi}} = 0 \; ,$$

as claimed.  $\Box$ 

The proof of Theorem 1.3 illustrates two aspects very nicely. Firstly, it demonstrates why we weigh the moments considered in Proposition 3.1 and Proposition 3.2 by  $W^2$  instead of just by W as this is needed to ensure that the moments remain finite. Secondly, by using the Christoffel–Darboux type formula given in Proposition 3.4 we improve the off-diagonal convergence in Theorem 1.3 by one factor of N compared to what we could deduce from the bound (3.17) alone.

Remark 3.8. By employing a similar workaround to the one we use when studying the Jacobi polynomials, it is possible to deduce the convergence on the diagonal in Theorem 1.3 from the asymptotic formula (3.16) and Lemma 3.6 as well as Lemma 3.7. The approach presented above still has its own benefits as it nicely demonstrates Proposition 3.1, it only uses (3.17) as opposed to the full asymptotic formula (3.16), and it led us to Lemma 3.5.

3.2. Associated Laguerre polynomials. For a parameter  $\alpha \in \mathbb{R}$  with  $\alpha > -1$ , the associated Laguerre polynomials  $\{L_n^{(\alpha)} : n \in \mathbb{N}_0\}$  are the classical orthogonal polynomials on  $I = [0, \infty)$  with respect to the weight function  $W : (0, \infty) \to \mathbb{R}$  defined by

$$W(x) = x^{\alpha} e^{-x}$$

which are normalised such that, for  $n \in \mathbb{N}_0$ ,

$$M_n = \int_0^\infty x^{\alpha} e^{-x} \left( L_n^{(\alpha)}(x) \right)^2 dx = \frac{\Gamma(n+\alpha+1)}{n!} \text{ and } K_n = \frac{(-1)^n}{n!},$$

see [19, Chapter 5.1]. We adapt the convention that the superscript for the associated Laguerre polynomials is reserved for the parameter  $\alpha > -1$ , and not to indicate any number of derivatives. According to Littlejohn and Krall [15], the associated Laguerre polynomials solve the differential equation, for  $n \in \mathbb{N}_0$  and  $x \in [0, \infty)$ ,

(3.23) 
$$x \frac{\mathrm{d}^2 L_n^{(\alpha)}(x)}{\mathrm{d}x^2} + (\alpha + 1 - x) \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} + nL_n^{(\alpha)}(x) = 0 ,$$

which rewrites as

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( x^{\alpha+1} e^{-x} \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} \right) = -nx^{\alpha} e^{-x} L_n^{(\alpha)}(x) .$$

In particular, we have

$$Q(x) = x$$
,  $L(x) = \alpha + 1 - x$ ,  $P(x) = x^{\alpha+1} e^{-x}$  and  $\lambda_n = n$ .

Furthermore, the associated Laguerre polynomials satisfy the recurrence relation, for  $n \in \mathbb{N}$  and  $x \in [0, \infty)$ ,

$$(3.24) (n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)$$

as well as the identity, for  $n \in \mathbb{N}$  and  $x \in [0, \infty)$ ,

(3.25) 
$$x \frac{dL_n^{(\alpha)}(x)}{dx} = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) ,$$

see [19, 5.1.10 and 5.1.14]. The bounds we use in our subsequent analysis for the associated Laguerre polynomials are consequences of the asymptotic formula [19, Theorem 8.22.1], which is due to Fejér [7, 8].

**Theorem 3.9** (Fejér's formula). For all  $\alpha \in \mathbb{R}$  with  $\alpha > -1$ , we have, as  $n \to \infty$ ,

$$L_n^{(\alpha)}(x) = \frac{e^{\frac{x}{2}} n^{\frac{\alpha}{2} - \frac{1}{4}}}{\sqrt{\pi} r^{\frac{\alpha}{2} + \frac{1}{4}}} \cos\left(2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(n^{\frac{\alpha}{2} - \frac{3}{4}}\right) ,$$

where the bound on the error term is uniform in  $x \in [\varepsilon, K]$  for  $\varepsilon, K \in \mathbb{R}$  with  $0 < \varepsilon < K$ .

Due to the exponential decay in  $\exp(-x)$  as  $x \to \infty$ , we further obtain that, for all  $l, n \in \mathbb{N}_0$ ,

(3.26) 
$$\lim_{x \to \infty} x^{l} x^{\alpha} e^{-x} L_{n}^{(\alpha)}(x) = \lim_{x \to \infty} x^{l} x^{\alpha} e^{-x} \frac{dL_{n}^{(\alpha)}(x)}{dx} = 0,$$

and since  $\alpha > -1$ , we have, for all  $l, n \in \mathbb{N}_0$ ,

(3.27) 
$$\lim_{x \to 0} x^l x^{\alpha+1} e^{-x} L_n^{(\alpha)}(x) = \lim_{x \to 0} x^l x^{\alpha+1} e^{-x} \frac{dL_n^{(\alpha)}(x)}{dx} = 0,$$

which yield the boundary conditions needed for Proposition 3.1 to deduce the recurrence relation satisfied by the limit moments on the diagonal. As for the Hermite polynomials, we compute the zeroth limit moment explicitly, which makes use of the following two results. The first lemma feeds

directly into the second one, but we include it as a separate statement because the result is used a second time in the proof of Theorem 1.4.

**Lemma 3.10.** For all  $\alpha \in \mathbb{R}$  with  $\alpha > -1$  and for all  $n \in \mathbb{N}_0$ , we have

$$\int_0^\infty x^{2\alpha+1} e^{-2x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx = 0.$$

*Proof.* Note that the integrals we consider are well-defined since  $2\alpha + 1 > -1$ . Integration by parts and (3.26) yield, for  $n \in \mathbb{N}_0$ ,

$$\int_0^\infty x^{2\alpha+1} e^{-2x} \left( \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} \right)^2 \mathrm{d}x = -\int_0^\infty x \left( x^{2\alpha+1} e^{-2x} \left( \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} \right)^2 \right)' \mathrm{d}x.$$

Using the differential equation (3.23) satisfied by the associated Laguerre polynomials, we deduce

$$\int_0^\infty x^{2\alpha+1} e^{-2x} \left( \frac{dL_n^{(\alpha)}(x)}{dx} \right)^2 dx = \int_0^\infty x^{2\alpha+1} e^{-2x} \left( 2nL_n^{(\alpha)}(x) + \frac{dL_n^{(\alpha)}(x)}{dx} \right) \frac{dL_n^{(\alpha)}(x)}{dx} dx ,$$

which implies the claimed result for  $n \neq 0$ . For the case n = 0, it suffices to note that  $L_0^{(\alpha)}(x) = 1$  for all  $x \in [0, \infty)$ .

Due to the identity (3.25), Lemma 3.10 shows that, if  $\alpha > -\frac{1}{2}$  then, for all  $n \in \mathbb{N}$ ,

$$n \int_0^\infty x^{2\alpha} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx = (n+\alpha) \int_0^\infty x^{2\alpha} e^{-2x} L_n^{(\alpha)}(x) L_{n-1}^{(\alpha)}(x) dx.$$

While the integrals are not well-defined when  $\alpha \in (-1, -\frac{1}{2}]$ , we still obtain, for  $\alpha > -1$  and  $n \in \mathbb{N}$  fixed, as  $\varepsilon \to 0$ ,

$$(3.28) \quad n \int_{0}^{\infty} x^{2\alpha} e^{-2x} \left( L_{n}^{(\alpha)}(x) \right)^{2} dx - (n+\alpha) \int_{0}^{\infty} x^{2\alpha} e^{-2x} L_{n}^{(\alpha)}(x) L_{n-1}^{(\alpha)}(x) dx = O\left(\varepsilon^{2\alpha+2}\right) .$$

This is used in the proof of the next lemma, where extra care is needed to ensure that it also works for  $\alpha \in (-1, -\frac{1}{2}]$ .

**Lemma 3.11.** For  $\alpha \in \mathbb{R}$  with  $\alpha > -1$  and for all  $n \in \mathbb{N}_0$ , we have

$$\int_0^\infty x^{2\alpha+1} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx = \frac{\Gamma(n+\alpha+1)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{3}{2}\right)}{2\pi \left(\Gamma(n+1)\right)^2}.$$

*Proof.* For  $\varepsilon > 0$  and  $n \in \mathbb{N}_0$ , set

$$e_{n,\varepsilon}^{(\alpha)} = -\frac{1}{2}x^{2\alpha+1} e^{-2x} \left(L_n^{(\alpha)}(x)\right)^2 \Big|_{x=\varepsilon}$$
.

Integration by parts and (3.26) give

$$\int_{\varepsilon}^{\infty} x^{2\alpha+1} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx = e_{n,\varepsilon}^{(\alpha)} + \frac{1}{2} \int_{\varepsilon}^{\infty} e^{-2x} \left( x^{2\alpha+1} \left( L_n^{(\alpha)}(x) \right)^2 \right)' dx$$

$$= e_{n,\varepsilon}^{(\alpha)} + \left( \alpha + \frac{1}{2} \right) \int_{\varepsilon}^{\infty} x^{2\alpha} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx + \int_{\varepsilon}^{\infty} x^{2\alpha+1} e^{-2x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx ,$$

which by Lemma 3.10 implies that, as  $\varepsilon \to 0$ ,

(3.29) 
$$\int_{\varepsilon}^{\infty} x^{2\alpha+1} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx \\ = e_{n,\varepsilon}^{(\alpha)} + \left( \alpha + \frac{1}{2} \right) \int_{\varepsilon}^{\infty} x^{2\alpha} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx + O\left( \varepsilon^{2\alpha+2} \right) .$$

On the other hand, using the recurrence relation (3.24), we obtain

$$\int_{\varepsilon}^{\infty} x^{2\alpha+1} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx$$

$$= \int_{\varepsilon}^{\infty} x^{2\alpha} e^{-2x} \left( (2n + \alpha + 1) L_n^{(\alpha)}(x) - (n+1) L_{n+1}^{(\alpha)}(x) - (n+\alpha) L_{n-1}^{(\alpha)}(x) \right) L_n^{(\alpha)}(x) dx.$$

From (3.28), it follows that

(3.30) 
$$\int_{\varepsilon}^{\infty} x^{2\alpha+1} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx$$

$$= \int_{\varepsilon}^{\infty} x^{2\alpha} e^{-2x} \left( (n+\alpha+1) L_n^{(\alpha)}(x) - (n+1) L_{n+1}^{(\alpha)}(x) \right) L_n^{(\alpha)}(x) dx + O\left(\varepsilon^{2\alpha+2}\right) .$$

Subtracting (3.29) from (3.30), we conclude

$$\left(n + \frac{1}{2}\right) \int_{\varepsilon}^{\infty} x^{2\alpha} e^{-2x} \left(L_n^{(\alpha)}(x)\right)^2 dx$$

$$= e_{n,\varepsilon}^{(\alpha)} + (n+1) \int_{\varepsilon}^{\infty} x^{2\alpha} e^{-2x} L_n^{(\alpha)}(x) L_{n+1}^{(\alpha)}(x) dx + O\left(\varepsilon^{2\alpha+2}\right) ,$$

which together with (3.28) and (3.29) yields, as  $\varepsilon \to \infty$ ,

$$\begin{split} &(n+\alpha+1)\left(n+\frac{1}{2}\right)\int_{\varepsilon}^{\infty}x^{2\alpha+1}\operatorname{e}^{-2x}\left(L_{n}^{(\alpha)}(x)\right)^{2}\,\mathrm{d}x\\ &=(n+1)^{2}\int_{\varepsilon}^{\infty}x^{2\alpha+1}\operatorname{e}^{-2x}\left(L_{n+1}^{(\alpha)}(x)\right)^{2}\,\mathrm{d}x+(n+\alpha+1)^{2}e_{n,\varepsilon}^{(\alpha)}-(n+1)^{2}e_{n+1,\varepsilon}^{(\alpha)}+O\left(\varepsilon^{2\alpha+2}\right)\;. \end{split}$$

According to [19, 5.1.7], the associated Laguerre polynomials satisfy

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} ,$$

which implies that, as  $\varepsilon \to 0$ ,

$$(n+\alpha+1)^2 e_{n,\varepsilon}^{(\alpha)} - (n+1)^2 e_{n+1,\varepsilon}^{(\alpha)} = O\left(\varepsilon^{2\alpha+2}\right) .$$

Thus, we can take the limit as  $\varepsilon \to \infty$  to deduce, for  $n \in \mathbb{N}_0$ ,

(3.31) 
$$\int_0^\infty x^{2\alpha+1} e^{-2x} \left( L_{n+1}^{(\alpha)}(x) \right)^2 dx = \frac{(n+\alpha+1)\left(n+\frac{1}{2}\right)}{(n+1)^2} \int_0^\infty x^{2\alpha+1} e^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 dx.$$

We further compute that

(3.32) 
$$\int_0^\infty x^{2\alpha+1} e^{-2x} \left( L_0^{(\alpha)}(x) \right)^2 dx = \int_0^\infty x^{2\alpha+1} e^{-2x} dx = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+2}}.$$

Using the Legendre duplication formula (3.11) and  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ , this can be rewritten as

$$\frac{\Gamma(2\alpha+2)}{2^{2\alpha+2}} = \frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{3}{2}\right)}{2\pi} \,.$$

The claimed identity then follows by induction from (3.31) and (3.32).

Note that the above proof simplifies significantly for  $\alpha > -\frac{1}{2}$  because in that regime all the integrals considered are well-defined on  $[0,\infty)$  and the error terms  $e_{n,\varepsilon}^{(\alpha)}$  vanish as  $\varepsilon \to 0$ . For the Laguerre polynomials  $\{L_n : n \in \mathbb{N}_0\}$ , which are the associated Laguerre polynomials with

$$\int_0^\infty e^{-2x} (L_n(x))^2 dx = \frac{1}{2^{2n+1}} {2n \choose n} \text{ and } \int_0^\infty x e^{-2x} (L_n(x))^2 dx = \frac{1}{4^{n+1}} {2n \choose n}.$$

The next two results are the local uniform bounds later needed to apply the Arzelà-Ascoli theorem.

**Lemma 3.12.** Fix  $\varepsilon, K \in \mathbb{R}$  with  $0 < \varepsilon < K$  and  $\alpha \in \mathbb{R}$  with  $\alpha > -1$ . The family

parameter  $\alpha = 0$ , Lemma 3.11 as well as the identity (3.29) show that

$$\left\{\sqrt{N}\sum_{n=N+1}^{\infty}\frac{\Gamma(n)L_{n}^{(\alpha)}(x)L_{n}^{(\alpha)}(y)}{\Gamma(n+\alpha+1)}:N\in\mathbb{N}\ and\ x,y\in[\varepsilon,K]\right\}$$

is uniformly bounded.

*Proof.* As a consequence of Fejér's formula, see Theorem 3.9, there exists a positive constant  $C \in \mathbb{R}$ , depending on  $\varepsilon$  and K, such that, for all  $n \in \mathbb{N}$  and for all  $x, y \in [\varepsilon, K]$ ,

$$\left| L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) \right| \le C n^{\alpha - \frac{1}{2}} .$$

Using the asymptotics (3.10) for the Gamma function, we deduce that  $C \in \mathbb{R}$  can further be chosen such that

$$\left| \frac{\Gamma(n)L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)}{\Gamma(n+\alpha+1)} \right| \le Cn^{-\alpha-1}n^{\alpha-\frac{1}{2}} = Cn^{-\frac{3}{2}}.$$

It follows that, for all  $N \in \mathbb{N}$  and for all  $x, y \in [\varepsilon, K]$ ,

$$\left| \sqrt{N} \sum_{n=N+1}^{\infty} \frac{\Gamma(n) L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{\Gamma(n+\alpha+1)} \right| \le \sqrt{N} \sum_{n=N+1}^{\infty} C n^{-\frac{3}{2}} \le 3C ,$$

which proves the claimed uniform boundedness.

**Lemma 3.13.** Fix  $\varepsilon, K \in \mathbb{R}$  with  $0 < \varepsilon < K$  and  $\alpha \in \mathbb{R}$  with  $\alpha > -1$ . The family

$$\left\{\sqrt{N}\sum_{n=N+1}^{\infty}\frac{\Gamma(n)}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(x)\frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x}:N\in\mathbb{N}\ and\ x\in[\varepsilon,K]\right\}$$

is uniformly bounded.

*Proof.* According to (3.25), we have, for all  $n \in \mathbb{N}$ ,

(3.33) 
$$\frac{\Gamma(n)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{L_n^{(\alpha)}(x) \left(nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)\right)}{nx}$$

We proceed by using a telescoping-like series to find an alternative expression for the series we are interested in such that Fejér's formula provides sufficient control over the terms of the new series

to imply the claimed uniform boundedness. We observe that, for  $N \in \mathbb{N}$ ,

$$\begin{split} \sum_{n=N+1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{(n+\alpha)L_n^{(\alpha)}(x)L_{n-1}^{(\alpha)}(x)}{n} + \sum_{n=N+1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \frac{(n+1)L_n^{(\alpha)}(x)L_{n+1}^{(\alpha)}(x)}{n} \\ &= \sum_{n=N+1}^{\infty} \frac{\Gamma(n+1)L_n^{(\alpha)}(x)L_{n-1}^{(\alpha)}(x)}{\Gamma(n+\alpha)} \left(\frac{1}{n} + \frac{1}{n-1}\right) - \frac{\Gamma(N+2)}{\Gamma(N+\alpha+1)} \frac{L_N^{(\alpha)}(x)L_{N+1}^{(\alpha)}(x)}{N} \\ &= \sum_{n=N+1}^{\infty} \frac{\Gamma(n)L_n^{(\alpha)}(x)L_{n-1}^{(\alpha)}(x)}{\Gamma(n+\alpha)} \left(2 + \frac{1}{n-1}\right) - \frac{\Gamma(N+2)}{\Gamma(N+\alpha+1)} \frac{L_N^{(\alpha)}(x)L_{N+1}^{(\alpha)}(x)}{N} \; . \end{split}$$

Applying the recurrence relation (3.24), we deduce

$$(3.34) \sum_{n=N+1}^{\infty} \frac{\Gamma(n+1) \left(L_n^{(\alpha)}(x)\right)^2}{\Gamma(n+\alpha+1)} \left(2 + \frac{\alpha+1}{n}\right) - \sum_{n=N+1}^{\infty} \frac{\Gamma(n) L_n^{(\alpha)}(x) L_{n-1}^{(\alpha)}(x)}{\Gamma(n+\alpha)} \left(2 + \frac{1}{n-1}\right)$$

$$= \sum_{n=N+1}^{\infty} \frac{\Gamma(n) x \left(L_n^{(\alpha)}(x)\right)^2}{\Gamma(n+\alpha+1)} - \frac{\Gamma(N+2)}{\Gamma(N+\alpha+1)} \frac{L_N^{(\alpha)}(x) L_{N+1}^{(\alpha)}(x)}{N}.$$

By Fejér's formula, see Theorem 3.9, and due to the asymptotics (3.10), we can find  $C \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$  with  $n \geq 2$  and for all  $x \in [\varepsilon, K]$ ,

$$\left| \frac{\Gamma(n) \left( L_n^{(\alpha)}(x) \right)^2}{\Gamma(n+\alpha+1)} \right| \le C n^{-\frac{3}{2}} \quad \text{as well as} \quad \left| \frac{\Gamma(n-1) L_n^{(\alpha)}(x) L_{n-1}^{(\alpha)}(x)}{\Gamma(n+\alpha)} \right| \le C n^{-\frac{3}{2}}$$

and, for all  $N \in \mathbb{N}$ ,

$$\left| \frac{\Gamma(N+2)}{\Gamma(N+\alpha+1)} \frac{L_N^{(\alpha)}(x) L_{N+1}^{(\alpha)}(x)}{N} \right| \le C N^{-\frac{1}{2}}.$$

From these estimates, by combining (3.33) as well as (3.34), and since  $x \in [\varepsilon, K]$ , it follows that

$$\left| \sqrt{N} \sum_{n=N+1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} \right|$$

$$\leq \frac{\sqrt{N}C}{\varepsilon} \left( (\alpha+1) \sum_{n=N+1}^{\infty} n^{-\frac{3}{2}} + \sum_{n=N+1}^{\infty} n^{-\frac{3}{2}} + K \sum_{n=N+1}^{\infty} n^{-\frac{3}{2}} + N^{-\frac{1}{2}} \right)$$

$$\leq \frac{3(\alpha+3+K)C}{\varepsilon} ,$$

as required.

As in the proof of Lemma 3.7, we note that simply applying the known asymptotic formula for the polynomials does not give us sufficient control to obtain the desired uniform bound in Lemma 3.13. Instead, we first need to perform a telescoping-like rearrangement. This approach is motivated by a true telescoping series which appears when establishing the corresponding uniform bound needed in the proof of [11, Theorem 1.5]. We stress that exactly the same idea is employed and works in our subsequent analysis for the Jacobi polynomials, which might not be immediately evident due to the lengthiness of the expressions involved.

We are finally in a position to derive the asymptotic error in the eigenfunction expansion for the Green's function corresponding to the associated Laguerre polynomials.

Proof of Theorem 1.4. We start by considering the convergence on the diagonal for which we study the functions  $S_N: (0, \infty) \to \mathbb{R}$  given by, for  $N \in \mathbb{N}_0$ ,

$$S_N(x) = \sqrt{N} \sum_{n=N+1}^{\infty} \frac{\Gamma(n) \left( L_n^{(\alpha)}(x) \right)^2}{\Gamma(n+\alpha+1)}$$

in the limit  $N \to \infty$ . As a consequence of Lemma 3.13, we have, for all  $x \in (0, \infty)$ ,

$$S_N'(x) = \sqrt{N} \sum_{n=N+1}^{\infty} \frac{2\Gamma(n)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} \ .$$

Therefore, Lemma 3.12, Lemma 3.13 and the Arzelà–Ascoli theorem imply that the functions  $S_N$  converge as  $N \to \infty$  to a continuous limit function on  $(0, \infty)$ . To identify this limit function, we consider the moments defined by, for  $k \in \mathbb{N}_0$ ,

$$p_k = \lim_{N \to \infty} \int_0^\infty x^k x^{2\alpha + 1} e^{-2x} S_N(x) dx$$

where the integrals are well-defined since  $2\alpha + 1 > -1$ . As positivity of the terms in the series allows us to interchange integration and summation, the moments defined above are related to the limit moments considered in Proposition 3.1 by

$$p_k = m_{k+1} .$$

For all  $l \in \mathbb{N}$ , we observe that the function

$$x \mapsto \left( \left( x^l x^{\alpha+1} e^{-x} \right)' x^{\alpha+1} e^{-x} \right)' x^{-\alpha} e^x$$

is bounded on  $(0, \infty)$  since  $l + \alpha > 0$ . Using integration by parts as well as (3.26) and (3.27), we can then argue that, as  $n \to \infty$ ,

$$\int_0^\infty (x^l x^{\alpha+1} e^{-x})' x^{\alpha+1} e^{-x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx$$

$$= -\frac{1}{2} \int_0^\infty ((x^l x^{\alpha+1} e^{-x})' x^{\alpha+1} e^{-x})' (L_n^{(\alpha)}(x))^2 dx = O(M_n) .$$

While we do not consider the limit moment  $m_0$ , we still have to check that the condition (3.5) in Proposition 3.1 is satisfied for l = 0 since, as we see below, this relates the limit moments  $m_1$  and  $m_2$ . Applying Lemma 3.10 prior to integrating by parts, we obtain, for  $n \in \mathbb{N}_0$ ,

$$\int_0^\infty (x^{\alpha+1} e^{-x})' x^{\alpha+1} e^{-x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx$$

$$= (\alpha+1) \int_0^\infty x^{2\alpha+1} e^{-2x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx - \int_0^\infty x^{2\alpha+2} e^{-2x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx$$

$$= \frac{1}{2} \int_0^\infty (x^{2\alpha+2} e^{-2x})' \left( L_n^{(\alpha)}(x) \right)^2 dx .$$

Due to the function  $x \mapsto (x^{2\alpha+2} e^{-2x})' x^{-\alpha} e^x$  being bounded on  $(0, \infty)$ , it follows that, as  $n \to \infty$ ,

$$\int_0^\infty (x^{\alpha+1} e^{-x})' x^{\alpha+1} e^{-x} L_n^{(\alpha)}(x) \frac{dL_n^{(\alpha)}(x)}{dx} dx = O(M_n) ,$$

as needed. Together with (3.26) and (3.27), this means that Proposition 3.1 yields, for all  $k \in \mathbb{N}_0$ ,

$$m_{k+2} = \left(\alpha + k + \frac{3}{2}\right) m_{k+1}$$
 and  $p_{k+1} = \left(\alpha + k + \frac{3}{2}\right) p_k$ .

Even though we could show that Proposition 3.2 gives rise to the same recurrence relation for suitably weighted moments of the claimed limit function for the full parameter range  $\alpha > -1$ , it is easier to establish this directly. We have, for all  $k \in \mathbb{N}_0$ ,

$$\widetilde{p}_{k+1} = \frac{1}{\pi} \int_0^\infty x^{k+1} x^{\alpha + \frac{1}{2}} e^{-x} dx = \frac{\alpha + k + \frac{3}{2}}{\pi} \int_0^\infty x^k x^{\alpha + \frac{1}{2}} e^{-x} dx = \left(\alpha + k + \frac{3}{2}\right) \widetilde{p}_k.$$

Thus, the desired pointwise convergence on the diagonal follows once we know that  $p_0 = \tilde{p}_0$ . From Lemma 3.11 and the asymptotics (3.10) for the Gamma function, we see that, as  $n \to \infty$ ,

$$\frac{\Gamma(n)}{\Gamma(n+\alpha+1)} \int_0^\infty x^{2\alpha+1} \, \mathrm{e}^{-2x} \left( L_n^{(\alpha)}(x) \right)^2 \, \mathrm{d}x = \frac{\Gamma(n) \Gamma\left(n+\frac{1}{2}\right) \Gamma\left(\alpha+\frac{3}{2}\right)}{2\pi \left(\Gamma(n+1)\right)^2} \sim \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{2\pi n^{\frac{3}{2}}} \; .$$

It follows that

$$p_0 = \lim_{N \to \infty} \sqrt{N} \sum_{n=N+1}^{\infty} \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{2\pi n^{\frac{3}{2}}} = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{\pi} ,$$

as needed because of

$$\widetilde{p}_0 = \frac{1}{\pi} \int_0^\infty x^{\alpha + \frac{1}{2}} e^{-x} dx = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{\pi}.$$

We are left with proving the convergence away from the diagonal. Fix  $x, y \in (0, \infty)$  with  $x \neq y$ . By Fejér's formula and due to the asymptotics (3.10), there exists a positive constant  $C \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$ ,

(3.35) 
$$\left| \frac{\Gamma(n+2)}{\Gamma(n+\alpha+1)} \frac{L_{n+1}^{(\alpha)}(x) L_n^{(\alpha)}(y) - L_n^{(\alpha)}(x) L_{n+1}^{(\alpha)}(y)}{n} \right| \le C n^{-\frac{1}{2}}.$$

Thus, applying Proposition 3.4 to the associated Laguerre polynomials and using

$$\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} ,$$

we obtain, for  $N \in \mathbb{N}$ ,

$$(x-y)\sum_{n=N+1}^{\infty} \frac{\Gamma(n)L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)}{\Gamma(n+\alpha+1)} = \frac{\Gamma(N+2)}{\Gamma(N+\alpha+1)} \frac{D_{N+1}(x,y)}{N} - \sum_{n=N}^{\infty} \frac{\Gamma(n)D_{n+1}(x,y)}{\Gamma(n+\alpha+1)} .$$

Since (3.35) gives

$$\left| \frac{\Gamma(n)D_{n+1}(x,y)}{\Gamma(n+\alpha+1)} \right| \le Cn^{-\frac{3}{2}} ,$$

we deduce

$$\left| (x - y) \sum_{n = N + 1}^{\infty} \frac{\Gamma(n) L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{\Gamma(n + \alpha + 1)} \right| \le C \left( N^{-\frac{1}{2}} + \sum_{n = N}^{\infty} n^{-\frac{3}{2}} \right) \le \frac{4C}{\sqrt{N}}.$$

This shows that with  $x \neq y$ , for all  $\gamma < \frac{1}{2}$ ,

(3.36) 
$$\lim_{N \to \infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{\Gamma(n) L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{\Gamma(n+\alpha+1)} = 0 ,$$

as claimed.  $\Box$ 

The reason why we obtain a weaker off-diagonal convergence result for the associated Laguerre polynomials compared to the Hermite polynomials is that Fejér's formula only allows us to obtain the bound (3.35) whereas for the Hermite polynomials we have (3.22). It would be of interest to investigate if (3.36) remains true for all  $\gamma < 1$  provided  $x \neq y$ .

3.3. **Jacobi polynomials.** The family  $\{P_n^{(\alpha,\beta)}: n \in \mathbb{N}_0\}$  of Jacobi polynomials on [-1,1] is given in terms of two parameters  $\alpha$  and  $\beta$ . While the Jacobi polynomials can be defined for all  $\alpha, \beta \in \mathbb{R}$  via the rising Pochhammer symbol by, for  $n \in \mathbb{N}_0$  and  $x \in [-1,1]$ ,

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+\beta+1)_k (\alpha+k+1)_{n-k} \left(\frac{x-1}{2}\right)^k ,$$

see [19, Section 4.22], we only obtain classical orthogonal polynomials for  $\alpha, \beta > -1$ . In that regime, the Jacobi polynomials are the classical orthogonal polynomials on I = [-1, 1] with respect to the weight function  $W: (-1, 1) \to \mathbb{R}$  given by

$$W(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

subject to the standardisation

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n! \Gamma(\alpha+1)} .$$

The class of Jacobi polynomials is a rich class of orthogonal polynomials which gives rise, for instance, to the Legendre polynomials  $\{P_n : n \in \mathbb{N}_0\}$ , to the Chebyshev polynomials of the first kind  $\{T_n : n \in \mathbb{N}_0\}$  and to the Chebyshev polynomials of the second kind  $\{U_n : n \in \mathbb{N}_0\}$  through, for  $n \in \mathbb{N}_0$ ,

$$(3.37) P_n = P_n^{(0,0)}, T_n = \frac{n!\sqrt{\pi}}{\Gamma\left(n + \frac{1}{2}\right)} P_n^{\left(-\frac{1}{2}, -\frac{1}{2}\right)} \text{and} U_n = \frac{(n+1)!\sqrt{\pi}}{2\Gamma\left(n + \frac{3}{2}\right)} P_n^{\left(\frac{1}{2}, \frac{1}{2}\right)}.$$

If  $\alpha, \beta > -1$ , the weight function W is integrable on (-1,1) and, as derived for [19, 4.3.3], we obtain, for  $n \in \mathbb{N}$ ,

(3.38) 
$$M_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}.$$

Note that if  $\alpha, \beta \in \mathbb{R}$ , the above expression remains well-defined for sufficiently large  $n \in \mathbb{N}$ , and we can take (3.38) as the definition for  $M_n$  as long as  $n+1 > -\min(\alpha, \beta, \alpha + \beta)$  whenever we are outside the regime  $\alpha, \beta > -1$ . Moreover, for  $\alpha, \beta \in \mathbb{R}$  fixed, the Jacobi polynomials satisfy, for  $n \in \mathbb{N}_0$ ,

$$K_n = \frac{1}{2^n} \binom{2n + \alpha + \beta}{n} ,$$

see [19, 4.21.6], and they solve the differential equation, for  $x \in [-1, 1]$ ,

$$(1 - x^2) \frac{d^2 P_n^{(\alpha,\beta)}(x)}{dx^2} + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d P_n^{(\alpha,\beta)}(x)}{dx} + n(n + \alpha + \beta + 1) P_n^{(\alpha,\beta)}(x) = 0 ,$$

see [19, Theorem 4.2.2]. This implies that

$$Q(x) = 1 - x^2$$
,  $L(x) = \beta - \alpha - (\alpha + \beta + 2)x$  and  $\lambda_n = n(n + \alpha + \beta + 1)$ .

Similar to our study of the asymptotic error in the eigenfunction expansion for the Green's function associated with the Hermite polynomials and with the associated Laguerre polynomials, we could use Proposition 3.1 and Proposition 3.2 to show that for Jacobi polynomials, certainly if  $\alpha, \beta \geq 0$ , the limit moments on the diagonal satisfy the same recurrence relation as the weighted moments of the claimed limit function. However, this approach would require us to obtain asymptotic control, as  $n \to \infty$ , over

$$\int_{-1}^{1} (1-x)^{2\alpha} (1+x)^{2\beta} \left( P_n^{(\alpha,\beta)}(x) \right)^2 dx \quad \text{and} \quad \int_{-1}^{1} x (1-x)^{2\alpha} (1+x)^{2\beta} \left( P_n^{(\alpha,\beta)}(x) \right)^2 dx ,$$

which does not appear to be straightforward. Instead, we exploit the asymptotic formula which is known for the Jacobi polynomials more than we did use the asymptotic formulae for the Hermite polynomials and for the associated Laguerre polynomials. The benefits of this approach are that it links to the analysis performed in Section 2 and that it works for all  $\alpha, \beta \in \mathbb{R}$ . In particular, no extra care is needed when  $\alpha, \beta \in (-1, -\frac{1}{2}]$  to ensure integrability of the integrals we consider. The Jacobi polynomials admit the following asymptotic formula stated in [19, Theorem 8.21.8], which is due to Darboux [5].

**Theorem 3.14** (Darboux formula). Let  $\alpha, \beta \in \mathbb{R}$  be fixed. For  $\theta \in (0, \pi)$ , set

$$k(\theta) = \pi^{-\frac{1}{2}} \left( \sin\left(\frac{\theta}{2}\right) \right)^{-\alpha - \frac{1}{2}} \left( \cos\left(\frac{\theta}{2}\right) \right)^{-\beta - \frac{1}{2}}$$
.

We have, as  $n \to \infty$ ,

$$P_n^{(\alpha,\beta)}\left(\cos\left(\theta\right)\right) = n^{-\frac{1}{2}}k(\theta)\cos\left(\left(n + \frac{\alpha+\beta+1}{2}\right)\theta - \left(\alpha + \frac{1}{2}\right)\frac{\pi}{2}\right) + O\left(n^{-\frac{3}{2}}\right) \ ,$$

where the bound on the error term is uniform in  $\theta \in [\varepsilon, \pi - \varepsilon]$  for  $\varepsilon > 0$ .

To extend our considerations to  $\alpha, \beta \in \mathbb{R}$ , we further need to observe that the Jacobi polynomials satisfy the three-term recurrence relation [19, 4.5.1] for all  $\alpha, \beta \in \mathbb{R}$ . This implies that the proof of Proposition 3.4 can be adapted to give us sufficient control to deduce the off-diagonal convergence for all families of Jacobi polynomials.

Besides, as derived for [19, 4.21.7], we have the identity that, for all  $n \in \mathbb{N}$  and for all  $x \in [-1, 1]$ ,

(3.39) 
$$\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\alpha,\beta)}(x) = \frac{1}{2} (n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1,\beta+1)}(x) .$$

As previously, we still require two local uniform bounds to establish the on-diagonal convergence in Theorem 1.5, and one of the two bounds is obtained through a telescoping-like rearrangement relying on the next lemma.

**Lemma 3.15.** For  $\alpha, \beta \in \mathbb{R}$  fixed, we have, for all  $n \in \mathbb{N}$  with  $n \geq 2$  and for all  $x \in [-1, 1]$ ,

$$(2n + \alpha + \beta)(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)P_n^{(\alpha,\beta)}(x)$$

$$= (2n + \alpha + \beta)(n + \alpha + \beta + 1)(n + \alpha + \beta + 2)P_n^{(\alpha+1,\beta+1)}(x)$$

$$+ (2n + \alpha + \beta + 1)(\alpha - \beta)(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

$$- (2n + \alpha + \beta + 2)(n + \alpha)(n + \beta)P_{n-2}^{(\alpha+1,\beta+1)}(x) .$$

*Proof.* According to [17, §138. (14), (15)], we have, for all  $n \in \mathbb{N}$  and for all  $x \in [-1, 1]$ ,

(3.40) 
$$(2n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x) = (n + \alpha + \beta + 1)P_n^{(\alpha,\beta+1)}(x) + (n + \alpha)P_{n-1}^{(\alpha,\beta+1)}(x)$$
 as well as

From (3.41), it follows that, for  $n \in \mathbb{N}$ ,

$$(3.41) (2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x).$$

 $(2n + \alpha + \beta + 2)P_n^{(\alpha,\beta+1)}(x) = (n + \alpha + \beta + 2)P_n^{(\alpha+1,\beta+1)}(x) - (n + \beta + 1)P_{n-1}^{(\alpha+1,\beta+1)}(x)$  and, for  $n \in \mathbb{N}$  with  $n \ge 2$ ,

$$(2n+\alpha+\beta)P_{n-1}^{(\alpha,\beta+1)}(x) = (n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) - (n+\beta)P_{n-2}^{(\alpha+1,\beta+1)}(x) .$$

Substituting these two identities into (3.40) yields the claimed result.

We observe that, due to the asymptotics (3.10) for the Gamma function, as  $n \to \infty$ ,

$$M_n \sim \frac{2^{\alpha+\beta}}{n}$$
.

In particular, the two local uniform bounds derived below are sufficient for our purposes.

**Lemma 3.16.** Fix  $\varepsilon \in \mathbb{R}$  with  $\varepsilon > 0$  and fix  $\alpha, \beta \in \mathbb{R}$ . The family

$$\left\{ N \sum_{n=N+1}^{\infty} \frac{P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)}{n} : N \in \mathbb{N} \text{ and } x,y \in [-1+\varepsilon,1-\varepsilon] \right\}$$

is uniformly bounded.

*Proof.* As a consequence of the Darboux formula, see Theorem 3.14, there exists a positive constant  $C \in \mathbb{R}$  such that, for all  $n \in \mathbb{N}$  and for all  $x, y \in [-1 + \varepsilon, 1 - \varepsilon]$ ,

$$\left| P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) \right| \le Cn^{-1}$$
.

We conclude that, for all  $N \in \mathbb{N}$ ,

$$\left| N \sum_{n=N+1}^{\infty} \frac{P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)}{n} \right| \le N \sum_{n=N+1}^{\infty} \frac{C}{n^2} \le C ,$$

as required.

**Lemma 3.17.** Fix  $\varepsilon \in \mathbb{R}$  with  $\varepsilon > 0$  and fix  $\alpha, \beta \in \mathbb{R}$ . The family

$$\left\{ N \sum_{n=N+1}^{\infty} P_n^{(\alpha,\beta)}(x) P_{n-1}^{(\alpha+1,\beta+1)}(x) : N \in \mathbb{N} \text{ and } x \in [-1+\varepsilon, 1-\varepsilon] \right\}$$

is uniformly bounded.

*Proof.* Since the Darboux formula by itself does not provide sufficient control to deduce the desired uniform boundedness, we first employ a similar telescoping-like trick as before and then apply the Darboux formula. Using Lemma 3.15, we write, for  $n \in \mathbb{N}$  with  $2n + \alpha + \beta > 0$  and for  $x \in [-1, 1]$ ,

$$P_{n}^{(\alpha,\beta)}(x)P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

$$= \frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n}^{(\alpha+1,\beta+1)}(x)P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

$$+ \frac{(\alpha-\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\left(P_{n-1}^{(\alpha+1,\beta+1)}(x)\right)^{2}$$

$$- \frac{(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-2}^{(\alpha+1,\beta+1)}(x)P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

According to the Darboux formula, there exists a positive constant  $C \in \mathbb{R}$  such that, for all  $N \in \mathbb{N}$  with  $2N + \alpha + \beta > 0$  and for all  $x \in [-1 + \varepsilon, 1 - \varepsilon]$ ,

$$\left| N \sum_{n=N+1}^{\infty} \frac{(\alpha - \beta)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)} \left( P_{n-1}^{(\alpha+1,\beta+1)}(x) \right)^2 \right| \le N \sum_{n=N+1}^{\infty} \frac{C}{n^2} \le C.$$

For the remaining two series arising from (3.42), we observe that, as  $n \to \infty$ ,

$$\frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} - \frac{(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)} \sim \frac{\alpha+\beta+2}{4n}$$

and that, due to the Darboux formula, the constant  $C \in \mathbb{R}$  can be chosen to satisfy, for all  $N \in \mathbb{N}$  with  $2N + \alpha + \beta > 0$  and for all  $x \in [-1 + \varepsilon, 1 - \varepsilon]$ ,

$$\left| \frac{(N+\alpha+1)(N+\beta+1)}{(2N+\alpha+\beta+2)(2N+\alpha+\beta+3)} P_{N-1}^{(\alpha+1,\beta+1)}(x) P_N^{(\alpha+1,\beta+1)}(x) \right| \le CN^{-1}$$

as well as

$$\left| N \sum_{n=N+1}^{\infty} \frac{(\alpha+\beta+2) P_n^{(\alpha+1,\beta+1)}(x) P_{n-1}^{(\alpha+1,\beta+1)}(x)}{4n} \right| \le C.$$

Considering terms of the original series which correspond to  $n \in \mathbb{N}$  with  $2n + \alpha + \beta \leq 0$  separately, we conclude there exists a positive constant  $D \in \mathbb{R}$  such that, for all  $N \in \mathbb{N}$  and  $x \in [-1 + \varepsilon, 1 - \varepsilon]$ ,

$$\left| N \sum_{n=N+1}^{\infty} P_n^{(\alpha,\beta)}(x) P_{n-1}^{(\alpha+1,\beta+1)}(x) \right| \le 3C + D.$$

This establishes the claimed uniform boundedness.

We are now in a position to prove the proposition stated below. While it has some overlap with Theorem 1.1, for most  $\alpha, \beta \in \mathbb{R}$  the result is not a straightforward application of Theorem 1.1. This is easily seen by noting that

$$\pi\left(n+\frac{\alpha+\beta+1}{2}\right)$$

need not satisfy the asymptotics given in Proposition 2.1 as  $n \to \infty$ . In particular, we move outside the setting of separated homogeneous boundary conditions. The reason for having delayed the proof of the next proposition until now is that it seems more convenient to use Lemma 3.17 for the second local uniform bound as opposed to carefully integrating by parts as in the proof of Proposition 2.3 and as for establishing [10, Lemma 4.2].

**Proposition 3.18.** Fix  $\alpha, \beta \in \mathbb{R}$ . We have, for  $\theta \in (0, \pi)$ ,

$$\lim_{N\to\infty} N \sum_{n=N+1}^{\infty} \frac{2\left(\cos\left(\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}\right)\right)^2}{n^2} = 1 \ .$$

*Proof.* For  $N \in \mathbb{N}_0$ , let  $S_N : (-1,1) \to \mathbb{R}$  be defined by

$$S_N(x) = N \sum_{n=N+1}^{\infty} \frac{\left(P_n^{(\alpha,\beta)}(x)\right)^2}{n}.$$

Due to the identity (3.39) for the derivative of Jacobi polynomials, it is a consequence of Lemma 3.17 that, for all  $x \in (-1, 1)$ ,

$$S'_{N}(x) = N \sum_{n=N+1}^{\infty} \frac{n+\alpha+\beta+1}{n} P_{n}^{(\alpha,\beta)}(x) P_{n-1}^{(\alpha+1,\beta+1)}(x) .$$

The Arzelà-Ascoli theorem together with Lemma 3.16 as well as Lemma 3.17 then imply that the functions  $S_N$  converge locally uniformly on the interval (-1,1) as  $N \to \infty$ . From the Darboux formula, it follows that with  $u_n: [0,\pi] \to \mathbb{R}$  for  $n \in \mathbb{N}_0$  given by

$$u_n(\theta) = \cos\left(\left(n + \frac{\alpha + \beta + 1}{2}\right)\theta - \left(\alpha + \frac{1}{2}\right)\frac{\pi}{2}\right),$$

the functions  $R_N:(0,\pi)\to\mathbb{R}$  defined by

$$R_N(\theta) = N \sum_{n=N+1}^{\infty} \frac{2 \left(u_n(\theta)\right)^2}{n^2}$$

converge locally uniformly on  $(0, \pi)$  as  $N \to \infty$ . Hence, the pointwise limit function is continuous on  $(0, \pi)$  and we can identify it through a moment argument. Set, for  $n \in \mathbb{N}_0$ ,

$$\nu_n = \left(n + \frac{\alpha + \beta + 1}{2}\right)^2.$$

Proceeding as in the proof of Proposition 2.3, we obtain, for all  $n, k \in \mathbb{N}_0$ ,

$$\nu_n \int_0^{\pi} \theta^k (u_n(\theta))^2 d\theta = -\theta^k u_n(\theta) u_n'(\theta) \Big|_0^{\pi} + \int_0^{\pi} \theta^k (u_n'(\theta))^2 d\theta + \int_0^{\pi} k \theta^{k-1} u_n(\theta) u_n'(\theta) d\theta$$

as well as

$$\int_0^{\pi} \theta^k (u'_n(\theta))^2 d\theta + \nu_n \int_0^{\pi} \theta^k (u_n(\theta))^2 d\theta = \frac{\pi^{k+1} \left( (u'_n(\pi))^2 + \nu_n (u_n(\pi))^2 \right)}{k+1} = \frac{\pi^{k+1} \nu_n}{k+1}.$$

Since we have, for  $k \in \mathbb{N}_0$  fixed and as  $n \to \infty$ ,

$$\theta^k u_n(\theta) u_n'(\theta) \big|_0^\pi = O(n)$$
 and  $\int_0^\pi k \theta^{k-1} u_n(\theta) u_n'(\theta) d\theta = O(1)$ ,

it follows that, for  $k \in \mathbb{N}_0$  fixed and as  $n \to \infty$ .

$$2\nu_n \int_0^{\pi} \theta^k (u_n(\theta))^2 d\theta = \frac{\pi^{k+1}\nu_n}{k+1} + O(n).$$

Observing that  $\nu_n = O(n^2)$  as  $n \to \infty$ , we deduce

$$\int_0^{\pi} \frac{2\theta^k (u_n(\theta))^2}{n^2} d\theta = \frac{\pi^{k+1}}{(k+1)n^2} + O(n^{-3}) ,$$

which by Fubini's theorem implies that, for all  $k \in \mathbb{N}_0$ 

$$\lim_{N \to \infty} \int_0^{\pi} \theta^k R_N(\theta) d\theta = \frac{\pi^{k+1}}{k+1} = \int_0^{\pi} \theta^k d\theta.$$

As the limit moments agree with the moments of the claimed limit function, the result follows by continuity.  $\Box$ 

We close by proving Theorem 1.5 on the asymptotic error in the eigenfunction expansion for the Green's function associated with the Jacobi polynomials and by showing that [11, Theorem 1.5] is a special case thereof.

Proof of Theorem 1.5. Recall that due to the asymptotics (3.10) for the Gamma function, we have, as  $n \to \infty$ ,

$$M_n \sim \frac{2^{\alpha+\beta}}{n}$$
.

Moreover, we note that, for  $\theta \in (0, \pi)$  and  $x = \cos(\theta)$ .

$$\left(\sin\left(\frac{\theta}{2}\right)\right)^{-2\alpha - 1} \left(\cos\left(\frac{\theta}{2}\right)\right)^{-2\beta - 1} = 2^{\alpha + \beta + 1} (1 - x)^{-\alpha - \frac{1}{2}} (1 + x)^{-\beta - \frac{1}{2}} .$$

The Darboux formula and Proposition 3.18 then yield, for all  $x \in (-1,1)$ ,

$$\lim_{N \to \infty} N \sum_{n=N+1}^{\infty} \frac{\left(P_n^{(\alpha,\beta)}(x)\right)^2}{M_n \lambda_n} = \frac{(1-x)^{-\alpha-\frac{1}{2}}(1+x)^{-\beta-\frac{1}{2}}}{\pi} ,$$

which establishes the claimed convergence on the diagonal. For the convergence away from the diagonal, we exploit a shifted Christoffel–Darboux type formula which ensures that all terms are well-defined. For the Jacobi polynomials, we have, by Stirling's formula (3.9) and as  $n \to \infty$ ,

$$K_n = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n n! \Gamma(n + \alpha + \beta + 1)} \sim \frac{2^{n + \alpha + \beta}}{\sqrt{\pi n}}.$$

Using the Darboux formula, we deduce that there exists a constant  $C \in \mathbb{R}$  such that, for sufficiently large  $n \in \mathbb{N}$  and for  $x, y \in (-1, 1)$ ,

$$\left| \frac{K_n}{K_{n+1} M_n} \frac{D_{n+1}(x,y)}{\lambda_n} \right| \le C n^{-2} .$$

Since the Jacobi polynomials satisfy the three-term recurrence relation [19, 4.5.1] for all  $\alpha, \beta \in \mathbb{R}$ , we can adapt the proof of Proposition 3.4, by choosing the starting point of the sum large enough to ensure that all summands are well-defined, to show that, for sufficiently large  $N \in \mathbb{N}$ ,

$$(3.44) (3.44) = \sum_{n=N+1}^{\infty} \frac{P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)}{M_n \lambda_n}$$

$$= \sum_{n=N}^{\infty} \frac{K_n}{K_{n+1} M_n} D_{n+1}(x,y) \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) - \frac{K_N}{K_{N+1} M_N} \frac{D_{N+1}(x,y)}{\lambda_N} .$$

We further have, as  $n \to \infty$ ,

$$\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} = \frac{2n+\alpha+\beta+2}{n(n+1)(n+\alpha+\beta+1)(n+\alpha+\beta+2)} \sim \frac{2}{n^3}.$$

Hence, by the Darboux formula, the constant  $C \in \mathbb{R}$  can be chosen such that, for sufficiently large  $n \in \mathbb{N}$  and for  $x, y \in (-1, 1)$ ,

$$\left| \frac{K_n}{K_{n+1}M_n} D_{n+1}(x,y) \left( \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right) \right| \le Cn^{-3} ,$$

which together with (3.43) and (3.44) implies that, for  $x \neq y$  and for all  $\gamma < 2$ 

$$\lim_{N \to \infty} N^{\gamma} \sum_{n=N+1}^{\infty} \frac{P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)}{M_n \lambda_n} = 0 ,$$

as claimed.  $\Box$ 

Note that Corollary 1.6 is an immediate consequence of Theorem 1.5 because of (3.37), which gives, for  $n \in \mathbb{N}$  and  $x, y \in (-1, 1)$ ,

$$\frac{T_n(x)T_n(y)}{n^2} = \frac{\pi}{2} \left( \frac{n!(2n)(n-1)!}{\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2} \frac{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(y)}{n^2} \right)$$

as well as

$$\frac{U_n(x)U_n(y)}{n(n+2)} = \frac{\pi}{2} \left( \frac{n!(2n+2)(n+1)!}{4\left(\Gamma\left(n+\frac{3}{2}\right)\right)^2} \frac{P_n^{(\frac{1}{2},\frac{1}{2})}(x)P_n^{(\frac{1}{2},\frac{1}{2})}(y)}{n(n+2)} \right) .$$

From Theorem 1.5, we can further deduce [11, Theorem 1.5] which states that, for  $x, y \in [-1, 1]$ ,

$$\lim_{N \to \infty} N \sum_{n=N}^{\infty} \frac{2n+1}{2} \int_{-1}^{x} P_n(z) dz \int_{-1}^{y} P_n(z) dz = \begin{cases} \frac{\sqrt{1-x^2}}{\pi} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Due to (3.37) and (3.39), we have, for  $n \in \mathbb{N}$  and  $x \in [-1, 1]$ ,

$$\frac{n-1}{2} \int_{-1}^{x} P_{n-1}(z) dz = P_n^{(-1,-1)}(x) ,$$

which yields, for  $N \in \mathbb{N}$ ,

$$N\sum_{n=N}^{\infty} \frac{2n+1}{2} \int_{-1}^{x} P_n(z) dz \int_{-1}^{y} P_n(z) dz = N\sum_{n=N+1}^{\infty} \frac{2(2n-1)}{(n-1)^2} P_n^{(-1,-1)}(x) P_n^{(-1,-1)}(y) .$$

This together with Theorem 1.5 implies the previous result since, for  $n \in \mathbb{N}$  with  $n \geq 2$ ,

$$\frac{2n!(2n-1)(n-2)!}{(n-1)!(n-1)!}\frac{1}{n(n-1)} = \frac{2(2n-1)}{(n-1)^2} ,$$

and as  $P_n^{(-1,-1)}(-1) = P_n^{(-1,-1)}(1) = 0$  for all  $n \in \mathbb{N}$ .

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