Introduction to Algorithms: Section 3.2 exercises

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1 3.2-1

Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $max\{f(n), g(n)\} \in \Theta(f(n) + g(n))$

Solution:

$$max\{f(n), g(n)\} = \begin{cases} f(n), & \text{if } f(n) > g(n) \\ g(n), & \text{otherwise} \end{cases} \tag{1}$$

$max\{f(n), g(n)\}$ grows at least as fast as $f(n)$, as if $f(n)$ is returned, it is larger than $g(n)$. Thus $max\{f(n), g(n)\} \in \Omega(f(n))$. And because $f(n)$ and $g(n)$ are both nonnegative, $f(n) + g(n)$ is larger than $f(n)$ or $g(n)$ individually. So it can be said that $max\{f(n), g(n)\} \in O(f(n) + g(n))$, because $max\{f(n), g(n)\}$ can grow no faster than $f(n) + g(n)$. And because Θ-notation characterizes the rate of growth from a constant factor above, and a constant factor below. It can be said that $max\{f(n), g(n)\} \in \Theta(f(n) + g(n))$, given $O(f(n) + g(n))$ and $\Omega(f(n))$

2 3.2-2

Explain why the statement, "The running time of algorithm A is at least $O(n^2)$," is meaningless

3 3.2-3

Is $2^{\mathrm{n}+1} \in O(2^{\mathrm{n}})$? Is $2^{2\mathrm{n}} \in O(2^{\mathrm{n}})$?

4 3.2-4

Prove Theorem 3.1:
For any two functions $f(n)$ and $g(n)$, we have $f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

5 3.2-5

Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$

6 3.2-6

Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set

7 3.2-7

We can extend our notation to the case of two parameters n and m that can go to ∞ independently at different rates. For a given function $g(n,m)$, we denote by $O(g(n,m))$ the set of functions

$O(g(n,m)) \in \{f(n,m)$: there exist positive constants c, n_0, and m_0 such that $0 \leq f(n,m) \leq cg(n,m)$ for all $n \geq n_0$ or $m \geq m_0\}$

Give corresponding definitions for $\Omega(g(n,m))$ and $\Theta(g(n,m))$

8 Additional Notes

This is still a work in progress