Part I: Data Collection

The ten stocks I chose are from the Barron's 2021 stock pick:

1. GOOGL - Google
2. AAPL - Apple
3. BRK.B - Berkshire Hathaway
4. KO - Coca-Cola
5. ETN - Eaton Corporation
6. GS - Goldman Sachs
7. GHC - Graham Holdings Co
8. DIS - Walt Disney Co
9. MRK - Merck & Co
10. NEM - Newmont Corporation

I changed the eighth stock Madison Square Garden Entertainment Corp to Walt Disney Co because Madison Square only went IPO in 2020, which could cause inconvenience for our portfolio analysis.

I downloaded the past 24 months (2019.02.19 - 2021.02.19) data for the ten stocks from Yahoo Finance. The mean daily return and the mean annualized return are calculated based on the train set (first 16 months). The following is a table that summarizes the key information of each stock:

Stock	Mean Daily Return	Mean Annualized Return	Real Annualized Return (2019.2.19-2020.2.19)
Google	0.28%	100.58%	35.36%
Apple	0.34%	132.62%	91.81%
Berkshire Hathaway	0.12%	33.73%	11.45%
Coca-Cola	0.04%	10.64%	37.55%
Eaton Corporation	0.23%	76.87%	36.99%
Goldman Sachs	0.28%	99.95%	21.92%
Graham Holdings Co	0.24%	84.84%	-18.77%
Walt Disney Co	0.05%	12.76%	26.00%
Merck & Co.	-0.02%	-5.15%	6.30%
Newmont Corporation	0.03%	8.49%	39.17%

Part II: Estimation

In this part, we will calculate the daily return, estimate the expected return $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$. We calculate the daily returns of a stock by:

$$\text{daily return} = \frac{\text{(n+1)th day price - nth day price}}{\text{nth day price}}$$

The following table shows the first five daily returns of the ten stocks:

Date	GOOGL	AAPL	BRK-B	KO	ETN	GS	GHC	DIS	MRK	NEM
2019-02-20	-0.005255	0.006435	0.002285	0.006023	0.011675	-0.000352	0.008952	0.001498	0.002398	0.024906
2019-02-21	-0.014617	-0.005639	-0.004026	0.016851	-0.000376	-0.011279	0.006454	0.005366	0.005036	0.000565
2019-02-22	0.011185	0.011166	-0.016656	-0.012647	0.013803	-0.001833	0.003980	0.008400	0.011775	0.030217
2019-02-25	0.000690	0.007284	0.001981	-0.007509	0.001609	0.013520	0.007180	-0.014404	-0.004829	-0.010417
2019-02-26	0.004189	0.000574	-0.002027	-0.005563	-0.005314	0.001258	-0.018092	-0.000792	0.004479	-0.031856

Then, we estimate the expected return $\hat{\mu}$ from the first 16 months data. We calculate the expected return $\hat{\mu}$ by taking the mean of the first 16 month daily returns that we obtained above:

$$\hat{\mu} = \begin{bmatrix} 0.00094279 \\ 0.00246824 \\ -0.00020975 \\ 0.0004558 \\ 0.00083294 \\ 0.00055552 \\ -0.00159379 \\ 0.00047659 \\ 0.00017806 \\ 0.00183979 \end{bmatrix}$$

Finally, we estimate the covariance of the ten stocks by calculating the covariance matrix of the train set daily returns:

$$\hat{\Sigma} = \begin{bmatrix}
4.4802^{-4} & 3.8567^{-4} & 2.7875^{-4} & 2.2278^{-4} & 3.8633^{-4} & 3.9802^{-4} & 2.4346^{-4} & 3.3591^{-4} & 2.1092^{-4} & 5.6115^{-5} \\
3.8567^{-4} & 5.4280^{-4} & 3.3284^{-4} & 2.6634^{-4} & 4.7669^{-4} & 4.6886^{-4} & 2.5960^{-4} & 3.6739^{-4} & 2.4699^{-4} & 8.0525^{-5} \\
2.7875^{-4} & 3.3284^{-4} & 3.5053^{-4} & 2.4947^{-4} & 4.4476^{-4} & 4.5287^{-4} & 3.3289^{-4} & 3.4316^{-4} & 2.1674^{-4} & 6.1062^{-5} \\
2.2278^{-4} & 2.6634^{-4} & 2.4947^{-4} & 3.3790^{-4} & 3.4165^{-4} & 3.2687^{-4} & 2.2633^{-4} & 2.8444^{-4} & 2.1156^{-4} & 1.2896^{-4} \\
3.8633^{-4} & 4.7669^{-4} & 4.4476^{-4} & 3.4165^{-4} & 7.9990^{-4} & 6.5861^{-4} & 5.1248^{-4} & 5.2488^{-4} & 2.8441^{-4} & 1.0028^{-4} \\
3.9802^{-4} & 4.6886^{-4} & 4.5287^{-4} & 3.2687^{-4} & 6.5861^{-4} & 7.6622^{-4} & 5.2215^{-4} & 5.2706^{-4} & 2.5697^{-4} & 8.6425^{-5} \\
2.4346^{-4} & 2.5960^{-4} & 3.3289^{-4} & 2.2633^{-4} & 5.1248^{-4} & 5.2215^{-4} & 8.4376^{-4} & 4.3894^{-4} & 1.6210^{-4} & 6.8598^{-5} \\
3.3591^{-4} & 3.6739^{-4} & 3.4316^{-4} & 2.8444^{-4} & 5.2488^{-4} & 5.2706^{-4} & 4.3894^{-4} & 6.3532^{-4} & 1.9828^{-4} & 1.1394^{-4} \\
2.1092^{-4} & 2.4699^{-4} & 2.1674^{-4} & 2.1156^{-4} & 2.8441^{-4} & 2.5697^{-4} & 1.6210^{-4} & 1.9828^{-4} & 3.3156^{-4} & 9.2516^{-5} \\
5.6115^{-5} & 8.0525^{-5} & 6.1062^{-5} & 1.2896^{-4} & 1.0028^{-4} & 8.6425^{-5} & 6.8598^{-5} & 1.1394^{-4} & 9.2516^{-5} & 5.7630^{-4}
\end{bmatrix}$$

Part III: Construction of Portfolios

We choose 11 different τ values: 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5 and calculate their corresponding optimal portfolio. From lecture, we know that

$$w_z = \Sigma^{-1}\mu - \frac{e^T\Sigma^{-1}\mu}{e^T\Sigma^{-1}e}\Sigma^{-1}e \text{ and } w_m = \frac{\Sigma^{-1}e}{e^T\Sigma^{-1}e}$$

Thus, we can plug in our estimates $\hat{\mu}$ and $\hat{\Sigma}$ in part II to get w_z and w_m:

$$w_m = \begin{bmatrix} 0.1935968 \\ -0.01536712 \\ 0.60260081 \\ 0.14325246 \\ -0.2280316 \\ -0.2758394 \\ 0.12103759 \\ 0.02361418 \\ 0.21190934 \\ 0.22322695 \end{bmatrix} \text{ and } w_z = \begin{bmatrix} -1.90898103 \\ 11.1540155 \\ -13.08456291 \\ 0.47226163 \\ 2.15462424 \\ 2.48308394 \\ -2.53673046 \\ 0.0827388 \\ -1.7187726 \\ 2.90232288 \end{bmatrix}$$

From lecture, we know that $w_{opt} = \tau w_z + w_m$. Thus, we plug in our 11 τ values to get an optimal portfolio for each τ:

	tau	GOOGL	AAPL	BRK-B	KO	ETN	GS	GHC	DIS	MRK	NEM
0	0.00	0.193597	-0.015367	0.602601	0.143252	-0.228032	-0.275839	0.121038	0.023614	0.211909	0.223227
1	0.01	0.174507	0.096173	0.471755	0.147975	-0.206485	-0.251009	0.095670	0.024442	0.194722	0.252250
2	0.02	0.155417	0.207713	0.340910	0.152698	-0.184939	-0.226178	0.070303	0.025269	0.177534	0.281273
3	0.03	0.136327	0.319253	0.210064	0.157420	-0.163393	-0.201347	0.044936	0.026096	0.160346	0.310297
4	0.04	0.117238	0.430794	0.079218	0.162143	-0.141847	-0.176516	0.019568	0.026924	0.143158	0.339320
5	0.05	0.098148	0.542334	-0.051627	0.166866	-0.120300	-0.151685	-0.005799	0.027751	0.125971	0.368343
6	0.10	0.002699	1.100034	-0.705855	0.190479	-0.012569	-0.027531	-0.132635	0.031888	0.040032	0.513459
7	0.20	-0.188199	2.215436	-2.014312	0.237705	0.202893	0.220777	-0.386309	0.040162	-0.131845	0.803692
8	0.30	-0.379098	3.330838	-3.322768	0.284931	0.418356	0.469086	-0.639982	0.048436	-0.303722	1.093924
9	0.40	-0.569996	4.446239	-4.631224	0.332157	0.633818	0.717394	-0.893655	0.056710	-0.475600	1.384156
10	0.50	-0.760894	5.561641	-5.939681	0.379383	0.849281	0.965703	-1.147328	0.064984	-0.647477	1.674388

Notice that when $\tau = 0$, our portfolio is the minimum risk portfolio w_m.

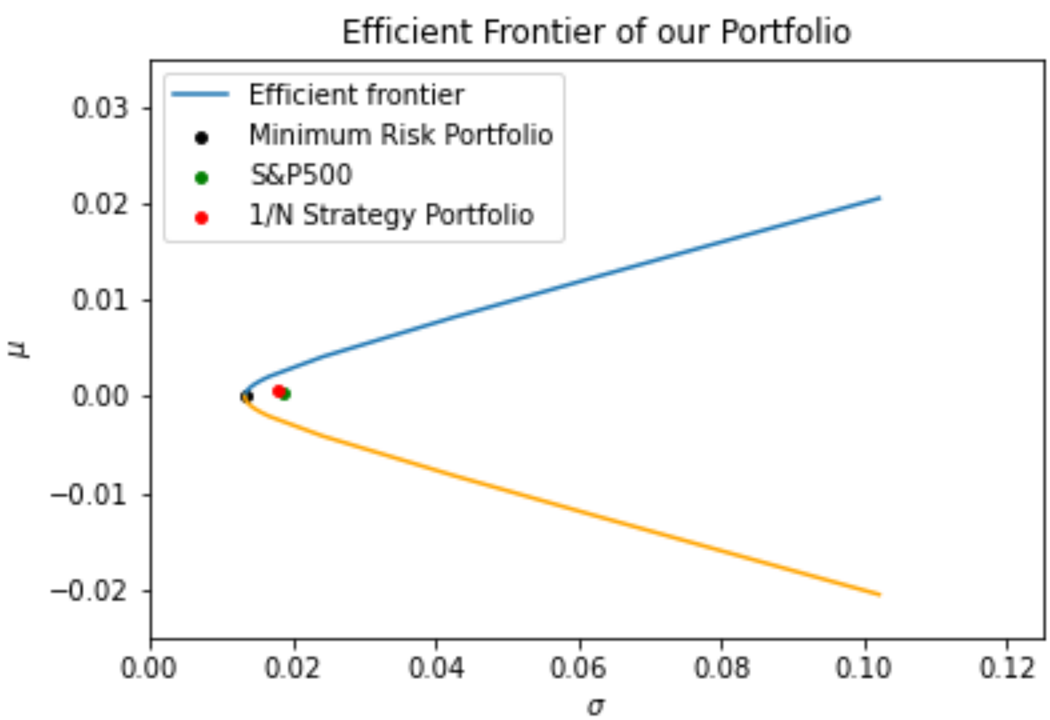
Part IV: In-sample Performance Evaluation

Efficient Frontier

From lecture, we know that $\mu_{opt} = \mu^T w_{opt}$ and $\sigma^2_{opt} = w^T_{opt} \Sigma w_{opt}$. Thus, we can graph the efficient frontier based on the 11 τ values. We can also calculate the return μ and risk σ of the S&P500, minimum risk and 1/N strategy portfolio:

$$\mu_{sp500} = 0.00051189,\ \sigma_{sp500} = 0.01852099672$$
$$\mu_m = 7.0868^{-6},\ \sigma_m = 0.0133$$
$$\mu_{1/N} = 0.0005946,\ \sigma_{1/N} = 0.01789$$

Then, we can graph them on the efficient frontier:



We can see that the 1/N strategy and S&P500 have very similar daily risk and returns. They are all riskier and provide higher returns than the minimum risk portfolio. However, they are not on the efficient frontier, which means they are not the optimal portfolio.

Portfolio Comparison

We have 11 different optimal portfolios with different τ values, the 1/N strategy, and the S&P500(market portfolio) to compare. In the following, I will compare these portfolios based on their mean daily return, daily standard deviation, cumulative return, beta, Sharpe Ratio, Treynor Ratio, and Jensen's alpha.

Since this an is in-sample evaluation, all the data is generated from the first 16 months performance of the portfolios. As we described above, mean daily returns are the optimal return($\mu_{opt} = \mu^T w_{opt}$) of the 11 optimal portfolios. The standard deviations($\sigma^2_{opt} = w^T_{opt} \Sigma w_{opt}$) are the optimal portfolio daily standard deviations. The cumulative return is obtained by compounding the 16 months daily return.

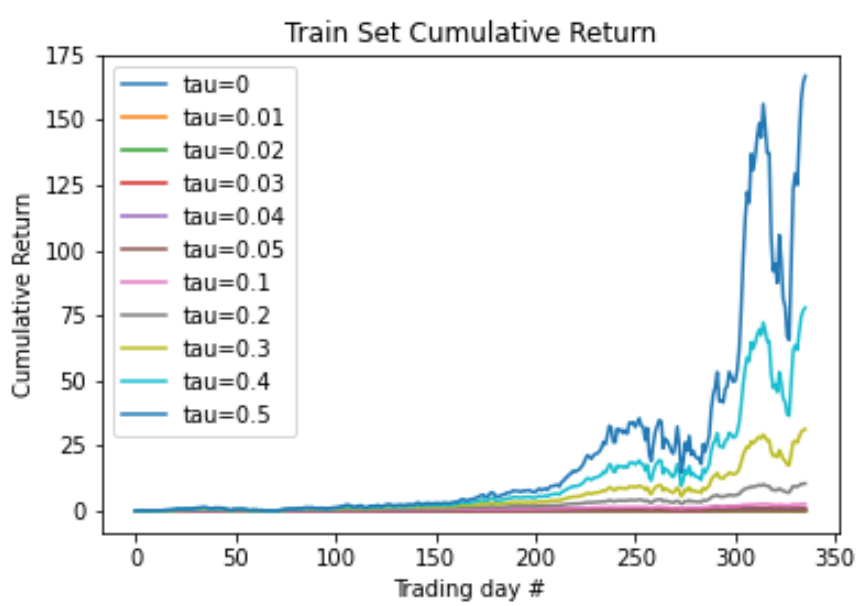
Strategy	Mean Daily Return	Standard Deviation	Cumulative Return	Beta	Sharpe Ratio	Treynor Ratio	Jensen's Alpha
$\tau = 0$	0.0007%	1.33%	-2.71%	0.53	-0.1478	-0.0049	0.0038
$\tau = 0.01$	0.0417%	1.35%	11.59%	0.55	-0.1249	-0.0040	0.0042
$\tau = 0.02$	0.0827%	1.39%	27.80%	0.57	-0.1019	-0.0032	0.0047
$\tau = 0.03$	0.1237%	1.46%	46.15%	0.59	-0.0790	-0.0024	0.0052
$\tau = 0.04$	0.1646%	1.56%	66.90%	0.62	-0.0560	-0.0016	0.0056
$\tau = 0.05$	0.2056%	1.67%	90.33%	0.64	-0.0331	-0.0009	0.0061
$\tau = 0.1$	0.4105%	2.42%	259.40%	0.74	0.0816	0.0020	0.0083
$\tau = 0.2$	0.8203%	4.26%	1054.41%	0.94	0.3110	0.0059	0.0129
$\tau = 0.3$	1.2301%	6.22%	3132.00%	1.14	0.5405	0.0084	0.0174
$\tau = 0.4$	1.6399%	8.21%	7790.83%	1.35	0.7699	0.0102	0.0219
$\tau = 0.5$	2.0498%	10.21%	16674.82%	1.55	0.9993	0.0115	0.0265
1/N	0.0595%	1.79%	15.70%	0.93	-0.1149	-0.0022	0.0052
S&P500	0.0512%	1.85%	12.07%	1.00	-0.1155	-0.0021	0.0000

I am going to discuss how I obtain the beta of each portfolio. Firstly, we need to get the risk-free rate. I downloaded the past 24 months US 10 Treasury Bill yield from the Federal Reserve website and used it as the risk-free rate. Since the risk-free rate we downloaded is the annual risk-free rate, we need to calculate the daily risk-free rate by $rf_d = (rf_a + 1)^{\frac{1}{365}} - 1$. According to the CAPM, we know that $r_p = r_f + \beta(r_m - r_f)$. Rearrange the equation, we get $r_p - r_f = \beta(r_m - r_f)$. Since each day has a different risk-free rate, portfolio return, and market return, we need to run a linear regression on the model $r_p - r_f = \beta(r_m - r_f)$, to get an estimate for β. Notice that this model has no intercept, so we need to force intercept to 0 when we are fitting the linear regression model.

After we get the beta for each portfolio, we can calculate:
1. Sharpe Ratio = $\dfrac{E(R_p - R_f)}{\sigma_{excess}}$: the ratio between excess return and excess standard deviation;
2. Treynor Ratio = $\dfrac{r_p - r_f}{\beta_p}$: the ratio between excess return and beta;
3. Jensen's Alpha = $R_p - [R_f + \beta(R_M - R_f)]$: the excess return that exceeds the expectation of CAPM.

Comment

Since τ is a risk aversion factor, higher τ means higher risk tolerance. Therefore, we can see from our table that the return and standard deviation increase as the τ value increases. The cumulative return also increases as the value of τ increases. Notice that there are some very very high cumulative returns at the end, because of the bold and highly risk-taking preference(large τ values).



Those strategies with high τ values had extremely high returns during a bull market and were hurt the most in a market dip. Notice that there were large drawdowns in those portfolios in the 2020 March market crash. The 1/N strategy and the market portfolio have daily returns and cumulative returns between $\tau = 0.01$ and $\tau = 0.02$ optimal portfolio, but they have a higher standard deviation(risk) than them.

The betas of optimal portfolios of $\tau = 0$ to $\tau = 0.2$ are lower than the market portfolio($\beta = 1$). The betas of optimal portfolios of $\tau = 0.3$ to $\tau = 0.5$ are higher than the market portfolio($\beta = 1$). 1/N strategy has a beta a little bit lower than the market portfolio. The Sharpe Ratio increases from negative to positive as the increase of τ values. We generally prefer higher Sharpe Ratio, because it means that we can get more excess return from one unit of risk. Therefore, the optimal portfolio of $\tau = 0.5$ is the best portfolio among the 13 choices. Notice that 1/N strategy and the market portfolio have a Sharpe ratio between $\tau = 0.01$ and $\tau = 0.02$.

Treynor Ratio measures how much excess return was generated for each unit of risk taken on by a portfolio. As τ value increase, the Treynor Ratio of the optimal portfolio increases as well. This means that we get more excess returns for each unit of risk when we are more risk-taking. Therefore, the optimal portfolio of $\tau = 0.5$ is the best, too, based on the Treynor Ratio. Notice that 1/N strategy and the market portfolio have a Treynor ratio between $\tau = 0.03$ and $\tau = 0.04$.

Jensen's alpha is used to determine the excess return of a portfolio over the theoretical expected return(CAPM return). As τ value increase, the Jensen's alpha of the optimal portfolio increases as well. This means that we get more excess return over the theoretical expected return when we are more risk-taking. Therefore, the optimal portfolio of $\tau = 0.5$ is the best portfolio, too, based on Jensen's alpha. Notice that Jensen's alpha of the market portfolio(S&P500) is 0. The Jensen's alpha of the 1/N strategy is between $\tau = 0.03$ and $\tau = 0.04$.

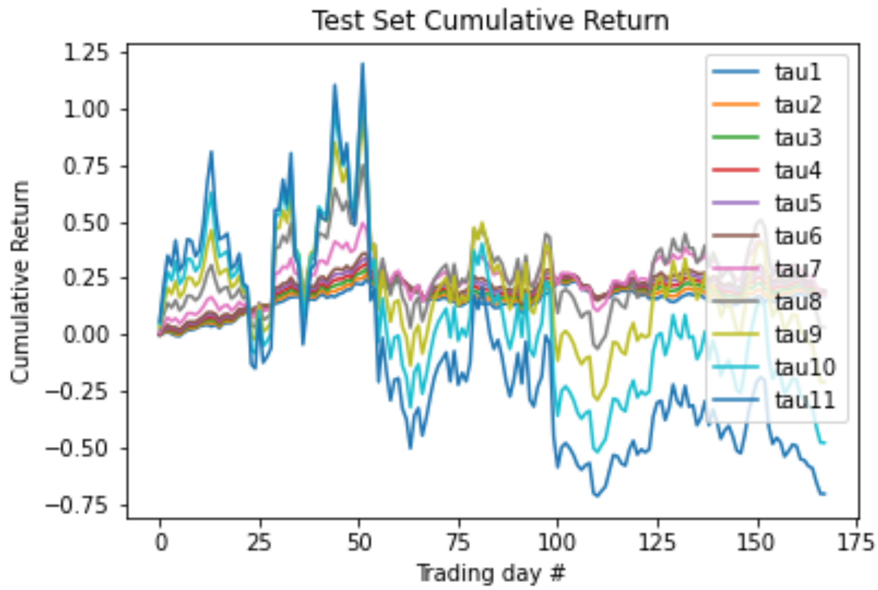
Part V: Out-of-Sample Performance Evaluation

In this part, we use the optimal portfolio(weight) we constructed on the test set and again compare them based on their mean daily return, daily standard deviation, cumulative return, beta, Sharpe Ratio, Treynor Ratio, and Jensen's alpha.

Strategy	Mean Daily Return	Standard Deviation	Cumulative Return	Beta	Sharpe Ratio	Treynor Ratio	Jensen's Alpha
$\tau = 0$	0.0997%	1.07%	17.08%	0.66	-0.0595	-0.0010	0.0028
$\tau = 0.01$	0.1042%	1.12%	17.87%	0.76	-0.0553	-0.0008	0.0028
$\tau = 0.02$	0.1087%	1.24%	18.50%	0.85	-0.0511	-0.0006	0.0029
$\tau = 0.03$	0.1133%	1.40%	18.97%	0.94	-0.0468	-0.0005	0.0030
$\tau = 0.04$	0.1178%	1.61%	19.28%	1.04	-0.0426	-0.0004	0.0030
$\tau = 0.05$	0.1224%	1.83%	19.43%	1.13	-0.0383	-0.0004	0.0031
$\tau = 0.1$	0.1451%	3.11%	17.75%	1.60	-0.0171	-0.0001	0.0034
$\tau = 0.2$	0.1905%	0.06%	3.34%	2.53	0.0253	0.0001	0.0041
$\tau = 0.3$	0.2360%	8.71%	-21.04%	3.46	0.0677	0.0002	0.0048
$\tau = 0.4$	0.2814%	11.56%	-47.77%	4.39	0.1101	0.0003	0.0054
$\tau = 0.5$	0.3268%	14.42%	-70.35%	5.32	0.1526	0.0003	0.0061
1/N	0.1997%	1.05%	38.48%	0.93	0.0339	0.0004	0.0038
S&P500	0.1414%	1.07%	25.64%	1.00	-0.0119	-0.0002	0.0000

Comment

Since τ is a risk aversion factor, higher τ means higher risk tolerance. Thus, again, we can see from our table that the return and standard deviation increases as the τ value increases. The cumulative return first increases with the increase of τ values. After $\tau = 0.05$, the cumulative return decreases with the increase of τ. Those high-risk optimal portfolios were impacted the most in the market dip from August to October 2020 and could not recover after that. The 1/N strategy and the market portfolio give pretty good daily mean return and cumulative return comparing to the optimal portfolio, but they have higher standard deviations.



The betas of optimal portfolio of $\tau = 0$ to $\tau = 0.03$ are lower than the market portfolio($\beta = 1$). The betas of optimal portfolio of $\tau = 0.04$ to $\tau = 0.5$ are higher than the market portfolio($\beta = 1$). 1/N strategy has a beta a little bit lower than the market portfolio. The Sharpe Ratio increases from negative to positive with the increase of τ values. We generally prefer a higher Sharpe Ratio, because it means that we can get more excess return from one unit of risk. Therefore, the optimal portfolio of $\tau = 0.5$ is the best portfolio among the 13 choices.

Treynor Ratio measures how much excess return was generated for each unit of risk taken on by a portfolio. As τ value increases, the Treynor Ratio of the optimal portfolio increases as well. This means that we get more excess returns for each unit of risk when we are more risk-taking. Therefore, the 1/N strategy is the best portfolio, based on the Treynor Ratio. Notice that the market portfolio has a Treynor ratio between $\tau = 0.05$ and $\tau = 0.1$.

Jensen's alpha is used to determine the excess return of a portfolio over the theoretical expected return(CAPM return). As τ value increases, Jensen's alpha of the optimal portfolio increases as well. This means that we get more excess return over the theoretical expected return when we are more risk-taking. Therefore, the optimal portfolio of $\tau = 0.5$ is the best portfolio, too, based on Jensen's alpha. Notice that the Jensen's alpha of the market portfolio(S&P500) is 0. The Jensen's alpha of the 1/N strategy is between $\tau = 0.01$ and $\tau = 0.02$.

At last, I am going to discuss and compare the train set result and the test set result. The test set has a higher mean daily return than the train set. This is because, in the train set, there was a large market crash in March 2020, which reduces the mean daily return by a lot. However, the train set and the test set have a similar standard deviation in general. The train set cumulative returns outperform the test set cumulative returns greatly. This is simply because the train set has a duration of 16 months, while the test set only has a duration of 8 months. The test set low-risk optimal portfolios have very similar betas as the train set. However, the test set high-risk optimal portfolios have very large betas. This is because the test set has a lot of high daily portfolio premiums comparing to the train set. The train set generally has higher Sharpe Ratios and Treynor Ratios than the test set, which means that the test set experiences a higher risk for each unit of excess return. The train set risky optimal portfolios generate higher alpha than the test set risky optimal portfolios. This means that the train set risky optimal portfolios outperform the benchmark much more than the test set risky optimal portfolios.