Notes on Fourier Analysis

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Ver. 20241121

1 Notation and Preliminaries

Let $\mathrm{i} = \sqrt{-1}$; the italicized variant i is reserved for indices. For $D \subseteq \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$, let $C^k(D)$ denote the set of functions from D to $\mathbb{C}$ that are k times continuously differentiable. In addition, let $C^k(\mathbb{S}^1)$ denote the subset of functions in $C^k(\mathbb{R})$ that are 2π-periodic.

Definition 1.1. The length of an interval is the difference between its right endpoint and its left. We shall restrict the definition of intervals to those with positive length.

Definition 1.2. Let $D \subseteq \mathbb{R}$. A function from D to $\mathbb{R}$ is said to be piecewise continuous if it is bounded and admits at most finitely many discontinuities.

Definition 1.3. A partition of $[a, b]$ is a finite subset of $[a, b]$ containing a and b. It is typically denoted as $a = x_0 < \cdots < x_n = b$.

Definition 1.4. A bounded function $f\colon [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for any $\epsilon > 0$ there exists a partition $P = \{x_0, \cdots, x_n\}$ of $[a, b]$ such that

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) < \epsilon,$$

where $a = x_0 < \cdots < x_n = b$. Here, we define

$$\begin{cases} \mathcal{U}(P, f) \coloneqq \sum_{i=1}^{n} \sup f([x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\ \mathcal{L}(P, f) \coloneqq \sum_{i=1}^{n} \inf f([x_{i-1}, x_i]) \cdot (x_i - x_{i-1}). \end{cases}$$

Definition 1.5. If a bounded function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable, then $\sup U(P, f)$ and $\inf U(P, f)$ coincide, where the supremum and the infimum are both taken over partitions of $[a, b]$. This common value is defined as the Riemann integral of f over $[a, b]$, denoted as $\int_a^b f(x)\,\mathrm{d}x$.

Definition 1.6. A bounded function $f\colon [a, b] \to \mathbb{C}$ is said to be Riemann integrable if $\operatorname{Re} \circ f$ and $\operatorname{Im} \circ f$ are Riemann integrable. The Riemann integrable of f over $[a, b]$ is defined as $\int_a^b \operatorname{Re} f(x)\,\mathrm{d}x + \mathrm{i} \cdot \int_a^b \operatorname{Im} f(x)\,\mathrm{d}x$, denoted also as $\int_a^b f(x)\,\mathrm{d}x$.

Definition 1.7. The set of all Riemann integrable functions from $[a, b]$ to $\mathbb{C}$ is denoted as $\mathcal{R}([a, b])$. The set of 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$ that are Riemann integrable on $[-\pi, \pi]$ is denoted as $\mathcal{R}(\mathbb{S}^1)$.

The following results from analysis are useful in the context of these notes.

Theorem 1.8 (Weierstrass' M-Test)**.** *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions from $A \subseteq \mathbb{R}$ to $\mathbb{C}$. Suppose there exists a sequence $\{M_n\}_{n=1}^{\infty}$ of non-negative numbers such that $\sum_{n=1}^{\infty} M_n$ converges and $|f_n(x)| \leq M_n$ for all $n \in \mathbb{Z}_{>0}$ and all $x \in A$. Then, $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on A.*

Proof. Let $T \coloneqq \sum_{n=1}^{\infty} M_n \geq 0$ and define $S(x) \coloneqq \sum_{n=1}^{\infty} f_n(x)$ formally. The latter converges pointwise absolutely. Denote $S_n(x) \coloneqq f_1(x) + \cdots + f_n(x)$.

Let $\epsilon > 0$. Because $M_1 + \cdots + M_N$ converges to T as $N \to \infty$, the partial sums can be arbitrarily close to T. In particular, fix $N \in \mathbb{Z}_{>0}$ such that $|T - (M_1 + \cdots + M_N)| < \epsilon$. Then,

$$\left| \sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \sum_{n=N+1}^{\infty} |f_n(x)| \leq \sum_{n=N+1}^{\infty} M_n < \epsilon.$$

The proof is complete. ◻

Theorem 1.9 (Summation by Parts)**.** *Suppose [TODO]*

Theorem 1.10 (Dirichlet's Criterion)**.** *Suppose $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of reals tending to 0 and $\{b_n\}$ a sequence of complex numbers, all of whose partial sums are bounded in modulus uniformly by $M > 0$. Then $\sum a_n b_n$ converges.*

Proof. Let $S_n = a_1 b_1 + \cdots + a_n b_n$ and $B_n = b_1 + \cdots + b_n$. From the summation by part formula from Exercise 2.7(a) in Homework 1, one has $S_n = a_n B_n - \sum_{k=1}^{n-1}(a_{k+1} - a_k) \cdot B_k$. Clearly, $|a_n B_n| \leq a_1 M$ and the modulus of each term is bounded by $|a_{k+1} - a_k| \cdot |B_k| \leq M \cdot (a_k - a_{k+1})$, which telescopes. Then, S_n must converge. □

2 Fourier Series

Continuous functions on a compact interval form an inner product space, and the Fourier "basis" functions form an orthonormal collection whose span can approximate continuous functions by projecting thereto. This is true in greater generality, and the various senses of convergence of the Fourier series are discussed.

2.1 Definitions

Definition 2.1. Let $[a, b]$ be an interval. For $n \in \mathbb{Z}$, the n-th Fourier basis function on $[a, b]$ is defined as $e_n \coloneqq [a, b] \to \mathbb{C}$ via $e_n(x) \coloneqq \mathrm{e}^{2\pi \mathrm{i} n x / L}$, where $L = b - a > 0$.

Lemma 2.2. $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal set of vectors in $C^0([a, b])$ under the inner product

$$\langle f, g \rangle = \frac{1}{L} \int_a^b f(x)\overline{g(x)}\, \mathrm{d}x.$$

Proof. Suppose $m, n \in \mathbb{Z}$. Then,

$$\langle e_n, e_m \rangle = \frac{1}{L} \int_a^b \mathrm{e}^{2\pi \mathrm{i}(n-m)x/L}\, \mathrm{d}x.$$

When $n = m$, the integrand is 1 and $\langle e_n, e_n \rangle = 1/L \cdot L = 1$. Otherwise, the integral is

$$\langle e_n, e_m \rangle = \frac{1}{L} \cdot \frac{1}{2\pi \mathrm{i}(n-m)/L} \cdot \cancel{\mathrm{e}^{2\pi \mathrm{i}(n-m)x/L}\Big|_{x=a}^{b}} = 0.$$

The proof is finished. □

Definition 2.3. Let $f \in \mathcal{R}([a, b])$. The n-th Fourier coefficient of f, where $n \in \mathbb{Z}$, is defined as $\hat{f}(n) \coloneqq \frac{1}{L} \int_a^b f(x) \cdot \overline{e_n(x)}\, \mathrm{d}x$, where $L \coloneqq b - a$. We identify $\mathcal{R}(\mathbb{S}^1)$ with the subspace of $\mathcal{R}([-\pi, \pi])$ of functions whose endpoints at $-\pi$ and π coincide and define Fourier coefficients for functions in $\mathcal{R}(\mathbb{S}^1)$ by extension.

Note that the product of Riemann integrable functions remains Riemann integrable, and each $\overline{e_n} = e_{-n}$ is clearly Riemann integrable. Thus, $\hat{f}(n)$ is well-defined for all $n \in \mathbb{Z}$ and $f \in \mathcal{R}([a, b])$.

Definition 2.4. Let $f \in \mathcal{R}([a, b])$. The Fourier series of f is the formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e_n(x)$$

with an indeterminate $x \in \mathbb{R}$.

One typically writes

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e_n(x)$$

to denote that $f(x)$ has the Fourier series on the right-hand side of the $\sim$ relation.

Definition 2.5. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be a trigonometric series if it admits the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e_n(x) \qquad \text{for all } x \in \mathbb{R}$$

for some complex-valued sequence $\{c_n\}_{n=-\infty}^{\infty}$.

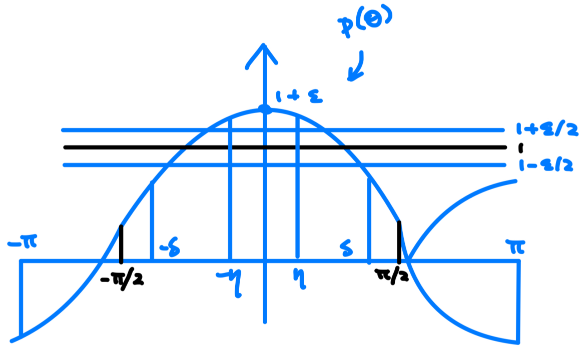

Figure 1: The plot of $p(\theta)$ in the proof of Theorem 2.8

Definition 2.6. A trigonometric polynomial p is a trigonometric series whose associated sequence $\{c_n\}_{n=-\infty}^{\infty}$ has all but finitely many zero terms. The degree of the trigonometric polynomial, denoted as $\deg p$, is defined as $\max_{n\in\mathbb{Z}} |n|$ subject to $c_n \neq 0$.

Corollary 2.7. Trigonometric polynomials are closed under addition, negation, and multiplication.

Proof. That trigonometric polynomials are closed under addition and negation is immediate. Suppose

$$f(x) = \sum_{n=-N}^{N} a_n \cdot e_n(x) \qquad \text{and} \qquad g(x) = \sum_{n=-N}^{N} b_n \cdot e_n(x)$$

are trigonometric polynomials, where $N \in \mathbb{Z}_{\geq 0}$. Then,

$$f(x) \cdot g(x) = \sum_{n=-N}^{N} \sum_{m=-N}^{N} a_n b_m \cdot e_n(x) e_m(x) = \sum_{n=-N}^{N} \sum_{m=-N}^{N} a_n b_m \cdot e_{m+n}(x) = \sum_{k=-2N}^{2N} \left(\sum_{n=\max\{-N,k-N\}}^{\min\{N,k+N\}} a_n b_{k-n} \right) \cdot e_k(x).$$

The proof is complete. □

Theorem 2.8. *Suppose $f \in \mathcal{R}(\mathbb{S}^1)$ is real-valued with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then, $f(\theta_0) = 0$ if f is continuous at θ_0.*

Proof. First, suppose f is real-valued. Without loss of generality, suppose $\theta = 0$ and $f(0) > 0$. Fix $0 < \delta \leq \pi/2$ such that $f(x) > f(0)/2$ whenever $|\theta| < \delta$. Let $p(\theta) \coloneqq \epsilon + \cos\theta$, which is a trigonometric polynomial, where $\epsilon > 0$ is chosen sufficiently small such that $|p(\theta)| < 1 - \epsilon/2$ whenever $\delta \leq |\theta| \leq \pi$. Fix $0 < \eta < \delta$ such that $p(\theta) \geq 1 + \epsilon/2$ whenever $|\theta| < \eta$. Define $p_k(\theta) \coloneqq p(\theta)^k$ for $k \in \mathbb{Z}_{\geq 0}$ and fix $B > 0$ such that $|f(\theta)| \leq B$ for all $\theta \in \mathbb{R}$.

We make three observations to estimate the integral $\int_a^b f(\theta) \cdot p_k(\theta)\, \mathrm{d}\theta$ by splitting the domain into three parts, where θ is assumed to satisfy $|\theta| < \eta$, $\eta < |\theta| < \delta$, and $\delta < |\theta| < \pi$ respectively.[1]

First, note that

$$\int_{|\theta|\leq\eta} f(\theta) \cdot p_k(\theta) \geq \int_{|\theta|\leq\eta} f(0)/2 \cdot (1+\epsilon)/2^k = \eta f(0) \cdot (1+\epsilon/2)^k,$$

where the right-hand side is unbounded as $k \in \mathbb{Z}_{\geq 0}$ varies.

For the second piece, it's enough to conclude

$$\int_{\eta\leq|\theta|\leq\delta} f(\theta) \cdot p_k(\theta)\, \mathrm{d}\theta \geq 0.$$

[1] We may modify the integrands of the three integrals so that the endpoints evaluate to 0; in this way, we do not change the value of each integral but can assume strict inequalities such as these in estimation.

Lastly, we have

$$\left|\int_{\delta\le|\theta|} f(\theta)\cdot p_k(\theta)\,\mathrm{d}\theta\right| \le \int_{\delta\le|\theta|} |f(\theta)|\cdot|p_k(\theta)|\,\mathrm{d}\theta \le (2\pi-\cancel{2\delta})B(1-\epsilon/2)^k,$$

where the right-hand side is bounded.

Hence, $\int_{-\pi}^{\pi} f(\theta)\cdot p_k(\theta)\,\mathrm{d}\theta$ is at least an unbounded number minus a bounded number. This integral, therefore, cannot tend to 0 as $k\to\infty$. However, since $p_k(\theta)$ is a trigonometric polynomial by induction on Corollary 2.7, we may write $p_k(\theta)=\sum_{n=S}^{T} c_n\cdot e_n$, and

$$\int_{-\pi}^{\pi} f(\theta)\cdot p_k(\theta)\,\mathrm{d}\theta = 2\pi\sum_{n=S}^{T} c_n\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cdot\overline{e_{-n}(\theta)}\,\mathrm{d}\theta\right)=0.$$

These integrals, then, must tend to 0. In particular, they cannot be unbounded, a contradiction. □

Proposition 2.9. Suppose $f\in C^0(\mathbb{S}^1)$ is real-valued with $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|<\infty$. Then,

$$f(x)=\lim_{N\to\infty}\sum_{n=-N}^{N}\hat{f}(n)\cdot e_n(x) \qquad \text{for all } x\in\mathbb{R},$$

and the convergence is uniform in x.

This is the first convergence result we have. To show this, we remark that periodicity is preserved by pointwise convergence.

Lemma 2.10. Let $P>0$. Suppose $\{f_n\}_{n=1}^{\infty}$ is a pointwise convergent sequence of P-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. Then, the limit is also P-periodic.

Proof. It is immediate that for all $x\in\mathbb{R}$, $f(x+P)-f(x)=\lim_{k\to\infty} f_k(x+P)-f_k(x)=\lim 0=0$. □

We now proceed to prove the proposition.

Proof. Without loss of generality, suppose f is 2π-periodic. Let $S_N(x)\coloneqq\sum_{n=-N}^{N}\hat{f}(n)\cdot e_n(x)$ be the N-th partial sum of the Fourier series of f, where $N\in\mathbb{Z}_{\ge 0}$. By Weierstrass' M-test, $\{S_N(x)\}$ converges absolutely and uniformly. Denote the limit as $g(x)$, the Fourier series of f which must be continuous. Hence,

$$\begin{aligned}\widehat{f-g}(n) &= \langle f,e_n\rangle-\langle g,e_n\rangle && \text{(Fubini)}\\ &= \hat{f}(n)-\sum_{m=-\infty}^{\infty}\hat{f}(m)\cdot\langle e_m,e_n\rangle\\ &= \hat{f}(n)-\sum_{m=-\infty}^{\infty}\hat{f}(m)\cdot\delta_{m,n}\\ &= 0,\end{aligned}$$

where we have denoted $\langle f,g\rangle=1/2\pi\cdot\int_{-\pi}^{\pi} f(x)\cdot\overline{g(x)}\,\mathrm{d}x$ for $f,g\in\mathcal{R}(\mathbb{S}^1)$.

The lemma implies that g is 2π-periodic as well. Then, $f-g$ is continuous and 2π-periodic, with all zero Fourier coefficients. Therefore, by Theorem 2.8, $f-g$ is identically zero. Therefore, f coincides with its Fourier series g. □

Here is a non-trivial application of Fourier series.

Proposition 2.11. $\sum_{n=1}^{\infty} 1/n^2=\pi^2/6$.

Proof. Extend $\tilde{f}(x)=|x|$ for $x\in[-\pi,\pi]$ to a 2π-periodic function $f\colon\mathbb{R}\to\mathbb{R}$. Then, f is continuous. Observe that for all non-zero $n\in\mathbb{Z}$,

$$\hat{f}(n)=\frac{1}{2\pi}\int_0^{\pi}\left(f(x)\cdot\mathrm{e}^{-\mathrm{i}nx}+f(-x)\,\mathrm{e}^{\mathrm{i}nx}\right)\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{\pi} x \,\mathrm{d}\left(\frac{1}{n}\sin nx\right) \\
&= \frac{1}{\pi n}\left(\cancel{x\sin nx\big|_{x=0}^{\pi}} - \int_0^{\pi} \sin nx \,\mathrm{d}x\right) \\
&= -\frac{1}{\pi n}\int_0^{\pi} \mathrm{d}\left(-\frac{1}{n}\cos nx\right) \\
&= \frac{1}{\pi n^2}(\cos n\pi - 1).
\end{aligned}
$$

It is obvious that $\hat{f}(0) = 1/2\pi \cdot 2 \cdot (1/2 \cdot \pi \cdot \pi) = \pi/2$.

Then,

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cdot \mathrm{e}^{inx} && ((-1)^n - 1 = -2 \cdot \mathbb{I}[2 \nmid n]) \\
&= \frac{\pi}{2} - \sum_{n=1,3,\dots} \frac{2}{\pi n^2} \cdot (\mathrm{e}^{inx} + \mathrm{e}^{-inx}) \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,\dots} \frac{\cos nx}{n^2}
\end{aligned}
$$

The coefficients are absolutely summable by the p-test since $p = 2 > 1$. Then, the Fourier series converges at 0 in particular, and $f(0) = 0$ implies that $\sum_{k=1}^{\infty} 1/(2k-1)^2 = (\pi/2)/(4/\pi) = \pi^2/8$. Now observe that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2} = (\pi^2/8)/(1 - 1/4) = \pi^2/6$. □

2.2 Convolutions

The concept of convolutions is fundamental to Fourier series and is applicable in greater generality in the context of functions.

Definition 2.12. Let $f, g \in \mathcal{R}(\mathbb{S}^1)$. Then, the convolution of f and g, denoted as $f * g \colon \mathbb{R} \to \mathbb{C}$, is defined for all $x \in \mathbb{R}$ as

$$(f * g)(x) \coloneqq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y.$$

The convolution is well-defined because Riemann integrable functions are closed under pointwise multiplication. The following is immediate.

Proposition 2.13. $*$ is commutative and bilinear over $\mathcal{R}(\mathbb{S}^1)$.

Proof. Let $f, g \in \mathcal{R}(\mathbb{S}^1)$ be arbitrary. To show commutativity, note that for all $x \in [-\pi, \pi]$,

$$2\pi \cdot (f * g)(x) = \int_{-\pi}^{\pi} f(t) \cdot g(x-t)\,\mathrm{d}t = \int_{x+\pi}^{x-\pi} f(x-t) \cdot g(t)\ -\mathrm{d}t = \int_{-\pi}^{\pi} g(t) \cdot f(x-t)\,\mathrm{d}t = 2\pi \cdot (g * f)(x).$$

To prove bilinearity, it is sufficient to show that $*$ is linear in the first component. Let $h \colon \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$ also. Then, for all $c \in \mathbb{C}$ and $x \in \mathbb{R}$,

$$2\pi \cdot ((c \cdot f + g) * h)(x) = \int_{-\pi}^{\pi} (c \cdot f(t) + g(t)) \cdot h(x-t)\,\mathrm{d}t = 2\pi \cdot (c \cdot (f * h)(x) + (g * h)(x)).$$

□

A useful approximation lemma is first presented before proving various properties of convolutions.

Lemma 2.14 (L_1 Approximation)**.** Suppose $f \in \mathcal{R}(\mathbb{S}^1)$ is real-valued. Then, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset C^0(\mathbb{S}^1)$ of real-valued functions such that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} |f_k(x) - f(x)| \, \mathrm{d}x = 0.$$

Further, there exists a constant $B > 0$ which uniformly bounds f and f_k in the sense that

$$|f|(\mathbb{R}) \cup \bigcup_{k=1}^{\infty} |f_k|(\mathbb{R}) \subset [-B, B].$$

Proof. Let $k \in \mathbb{Z}_{>0}$ and fix a partition $P = \{x_0, \cdots, x_N\}$ of $[-\pi, \pi]$ such that $U(P, f) - L(P, f) < 1/2k$. Fix $B > 0$ such that $f(\mathbb{R}) \subset [-B, B]$. Denote $I_n \coloneqq [x_{n-1}, x_n)$ for $1 \leq n < N$ and $I_N \coloneqq [x_{N-1}, x_N]$. Note that $I_1, \cdots, I_N$, whose endpoints coincide with P, partition $[-\pi, \pi]$.

Define the upper-bound step function $\tilde{f}_k(x) \coloneqq \sum_{n=1}^{N} \sup f([x_{n-1}, x_n]) \cdot \mathbb{I}[x \in I_n]$ on $[-\pi, \pi]$. Observe that $\tilde{f}_k(x) \geq f(x)$ always, and the partition has been chosen so that

$$\int_{-\pi}^{\pi} (\tilde{f}_k(x) - f(x)) \, \mathrm{d}x \leq U(P, f) - L(P, f) < \frac{1}{2k}.$$

Define $\delta \coloneqq \min\{\min_{1\leq i\leq N} \Delta x_i/3, 1/8Bk(N+1)\}$ and construct a 2π-periodic, continuous function $f_k \colon \mathbb{R} \to \mathbb{R}$ where, for all $x \in [-\pi, \pi]$,

$$f_k(x) = \begin{cases} \frac{\tilde{f}_k(x_0+\delta)}{\delta} \cdot (x - x_0) & \text{if } x_0 \leq x < x_0 + \delta \\ \tilde{f}_k(x) & \text{if } x_{n-1} + \delta \leq x < x_n - \delta \text{ for some } 1 \leq n \leq N \\ \frac{\tilde{f}_k(x_n+\delta) - \tilde{f}_k(x_n-\delta)}{2\delta} \cdot (x - x_n) + \frac{\tilde{f}_k(x_n+\delta) + \tilde{f}_k(x_n-\delta)}{2} & \text{if } x_n - \delta \leq x < x_n + \delta \text{ for some } 1 \leq n \leq N-1 \\ -\frac{\tilde{f}_k(x_N-\delta)}{\delta} \cdot (x - x_N) & \text{if } x_N - \delta \leq x \leq x_N. \end{cases}$$

In other words, one obtains $f_k(x)$ from $\tilde{f}_k(x)$ by connecting the endpoints of the partition with line segments to make $f(x)$ continuous and forcing $f_k(-\pi) = f_k(\pi) = 0$ without loss of generality for the restriction of periodicity. By construction, $f([-\pi, \pi]) = \tilde{f}_k([-\pi, \pi]) \subseteq f_k([-\pi, \pi]) \subset [-B, B]$. Then,

$$\begin{aligned} \int_{-\pi}^{\pi} |f_k(x) - f(x)| \, \mathrm{d}x &\leq \int_{-\pi}^{\pi} (\tilde{f}_k(x) - f(x)) \, \mathrm{d}x + \int_{-\pi}^{\pi} |f_k(x) - \tilde{f}_k(x)| \, \mathrm{d}x \\ &< \frac{1}{2k} + \sum_{n=0}^{N} \int_{\max\{x_n-\delta, -\pi\}}^{\min\{x_n+\delta, \pi\}} |f_k(x) - \tilde{f}(x)| \, \mathrm{d}x \\ &< \frac{1}{2k} + (N+1) \cdot 2\delta \cdot 2B \\ &< \frac{1}{k}. \end{aligned}$$

Hence, $\int_{-\pi}^{\pi} |f_k(x) - f(x)| \, \mathrm{d}x$ tends to 0 as $k \to \infty$ by the comparison test. □

Corollary 2.15. Suppose $f \in \mathcal{R}(\mathbb{S}^1)$. Then, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset C^0(\mathbb{S}^1)$ such that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} |f_k(x) - f(x)| \, \mathrm{d}x = 0.$$

Further, there exists a constant $B > 0$ which uniformly bounds f and f_k in the sense that

$$|f|(\mathbb{R}) \cup \bigcup_{k=1}^{\infty} |f_k|(\mathbb{R}) \subset [-B, B].$$

The proof is immediate by considering the real and imaginary parts separately and is hence omitted.

We now have sufficient machinery regarding several useful properties of the convolution. To establish properties regarding all Riemann integrable functions, we first restrict our attention to continuous such functions before applying the approximation lemma above for generalization.

Lemma 2.16. Suppose $f, g, h \in C^0(\mathbb{S}^1)$. Then,

- $f * g \in C^0(\mathbb{S}^1)$
- $(f * g) * h = f * (g * h)$
- $\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n)$ for all $n \in \mathbb{Z}$.

Proof. To see that $f * g$ is 2π-periodic, one notes, for all $x \in \mathbb{R}$,

$$(f * g)(x + 2\pi) - (f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cdot \cancel{(g(x + 2\pi - t) - g(x - t))} \,\mathrm{d}t = 0.$$

To see that $f * g$ is continuous, we show the stronger condition of uniform continuity. Fix $B > 0$ such that $|f|(\mathbb{R}) \cup |g|(\mathbb{R}) \subset [-B, B]$. Let $\epsilon > 0$ and fix $\delta > 0$ such that $|g(x) - g(y)| < \epsilon/B$ whenever $|x - y| < \delta$, where $x, y \in \mathbb{R}$ by the (uniform) continuity of g. Consequently, if $|x - y| < \delta$, then

$$|(f * g)(x) - (f * g)(y)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| \cdot |g(x - t) - g(y - t)| \,\mathrm{d}t < B \cdot \frac{\epsilon}{B} = \epsilon.$$

Associativity is similarly obtained by expanding

$$\begin{aligned}
4\pi^2((f * g) * h)(x) &= \int_{-\pi}^{\pi} 2\pi(f * g)(t) \cdot h(x - t) \,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s) \cdot g(t - s) \cdot h(x - t) \,\mathrm{d}s \,\mathrm{d}t && \text{(Fubini)} \\
&= \int_{-\pi}^{\pi} f(s) \cdot \left(\int_{-\pi}^{\pi} g(t - s) \cdot h(x - t) \,\mathrm{d}t \right) \mathrm{d}s && (v = t - s) \\
&= \int_{-\pi}^{\pi} f(s) \cdot \left(\int_{-\pi \cancel{- s}}^{\pi \cancel{- s}} g(v) \cdot h(x - s - v) \,\mathrm{d}v \right) \mathrm{d}s \\
&= \int_{-\pi}^{\pi} f(s) \cdot 2\pi(g * h)(x - s) \,\mathrm{d}s \\
&= 4\pi^2(f * (g * h))(x).
\end{aligned}$$

Lastly, for all $n \in \mathbb{Z}$, one has

$$\begin{aligned}
4\pi^2 \widehat{f * g}(n) &= \int_{-\pi}^{\pi} 2\pi(f * g)(x) \cdot \mathrm{e}^{-\mathrm{i}nx} \,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t) \cdot g(x - t) \cdot \mathrm{e}^{-\mathrm{i}nx} \,\mathrm{d}t \,\mathrm{d}x && \text{(Fubini)} \\
&= \int_{-\pi}^{\pi} f(t) \cdot \mathrm{e}^{-\mathrm{i}nt} \cdot \left(\int_{-\pi}^{\pi} g(x - t) \cdot \mathrm{e}^{-\mathrm{i}n(x - t)} \,\mathrm{d}x \right) \mathrm{d}t \\
&= \left(\int_{-\pi}^{\pi} f(t) \cdot \mathrm{e}^{-\mathrm{i}nt} \,\mathrm{d}t \right) \cdot \left(\int_{-\pi \cancel{- t}}^{\pi \cancel{- t}} g(x) \cdot \mathrm{e}^{-\mathrm{i}nx} \,\mathrm{d}x \right) \\
&= 2\pi \hat{f}(n) \cdot 2\pi \hat{g}(n).
\end{aligned}$$

The proof is finished. □

It is true, though not at all straightforward, that all these properties hold for f, g being more generally Riemann integrable rather than continuous. We first approximate the convolution and then derive the result of continuity.

Lemma 2.17. Suppose $f, g \in \mathcal{R}(\mathbb{S}^1)$. If $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are taken from Lemma 2.14 to approximate f and g respectively, then $f_k * g_k$ converges uniformly to $f * g$.

Proof. Let $\epsilon > 0$ and fix $K \in \mathbb{Z}_{>0}$ such that $\int_{-\pi}^{\pi} |f_k(t) - f(t)|\, \mathrm{d}t$ and $\int_{-\pi}^{\pi} |g_k(t) - g(t)|\, \mathrm{d}t$ are both less than ϵ whenever $k \geq K$. Then, for any such $k \geq K$, one has, for all $x \in \mathbb{R}$,

$$\begin{aligned}
|(f_k * g_k)(x) - (f * g)(x)| &\leq |((f_k - f) * g_k)(x)| + |(f * (g_k - g))(x)| \\
&\leq \frac{1}{2\pi}\left(\sup |g_k|(\mathbb{R}) \cdot \int_{-\pi}^{\pi} |f_k(t) - f(t)|\, \mathrm{d}t + \sup |f|(\mathbb{R}) \cdot \int_{-\pi}^{\pi} |g_k(t) - g(t)|\, \mathrm{d}t\right) \\
&\leq \frac{\max\{\sup |g_k|(\mathbb{R}), \sup |f|(\mathbb{R})\}}{2\pi} \cdot \epsilon,
\end{aligned}$$

so the convergence of $f_k * g_k$ to $f * g$ is uniform. ◻

Corollary 2.18. Suppose $f, g \in \mathcal{R}(\mathbb{S}^1)$. Then, $f * g \in C^0(\mathbb{S}^1)$.

Proof. That $f * g$ is continuous is immediate from the preceding lemma coupled with the continuity of each $f_k * g_k$. Periodicity follows from the same argument as in Lemma 2.16. ◻

We also show that uniform convergence implies L^1 convergence.

Lemma 2.19. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$ that are Riemann integrable on $[-\pi, \pi]$. If $\{f_k\}$ converges uniformly to $f : \mathbb{R} \to \mathbb{C}$, then $\lim_{k \to \infty} \int_{-\pi}^{\pi} |f_k(x) - f(x)|\, \mathrm{d}x = 0$.

Proof. Let $\epsilon > 0$ and fix $K \in \mathbb{Z}_{>0}$ such that $|f_k(x) - f(x)| < \epsilon/2\pi$ for all $x \in \mathbb{R}$ and $k \geq K$. Then, for all such $k \geq K$ one has

$$\int_{-\pi}^{\pi} |f_k(x) - f(x)|\, \mathrm{d}x < 2\pi \cdot \frac{\epsilon}{2\pi} = \epsilon.$$

The proof is complete. ◻

Proposition 2.20. Suppose $f, g, h : \mathbb{R} \to \mathbb{C}$ are 2π-periodic and Riemann integrable on $[-\pi, \pi]$. Then,

- $(f * g) * h = f * (g * h)$
- $\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n)$ for all $n \in \mathbb{Z}$.

Proof. Fix sequences of functions $\{f_k\}$, $\{g_k\}$, and $\{h_k\}$ for f, g, and h respectively from Lemma 2.14. To show associativity, observe that for all $x \in \mathbb{R}$,

$$\begin{aligned}
&2\pi\, |(f * g) * h - (f_k * g_k) * h_k|\,(x) \\
&\leq 2\pi \cdot |(f * g) * (h - h_k)|\,(x) + 2\pi \cdot |(f * g - f_k * g_k) * h_k|\,(x) \\
&\leq \int_{-\pi}^{\pi} \underbrace{|(f * g)(t)|}_{\text{cont. hence bounded}}\ |(h - h_k)(x - t)|\, \mathrm{d}t + \int_{-\pi}^{\pi} |(f * g - f_k * g_k)(t)| \cdot \underbrace{|h_k(x - t)|}_{\text{bounded}}\, \mathrm{d}t \\
&\leq C \cdot \int_{-\pi}^{\pi} |h(x - t) - h_k(x - t)|\, \mathrm{d}t + C \cdot \int_{-\pi}^{\pi} |(f * g)(x) - (f_k * g_k)(t)|\, \mathrm{d}t,
\end{aligned}$$

where $C \coloneqq \max\{\sup |f * g|(\mathbb{R}), \sup |h_k|(\mathbb{R})\}$. Observe that $\int_{-\pi}^{\pi} |h(x - t) - h_k(x - t)|\, \mathrm{d}t = \int_{-\pi}^{\pi} |h(t) - h_k(t)|\, \mathrm{d}t$ tends to 0 by construction. Further, the uniform convergence of $f_k * g_k$ to $f * g$ implies $\int_{-\pi}^{\pi} |(f * g)(t) - (f_k * g_k)(t)|\, \mathrm{d}t \to 0$ by Lemmata 2.17 and 2.19. Then, both terms must converge to 0, and $|(f * g) * h - (f_k * g_k) * h_k| \to 0$.

By the same reasoning, $|f * (g * h) - f_k * (g_k * h_k)| = |(g * h) * f - (g_k * h_k) * f_k| \to 0$. Therefore, $|(f * g) * h - f * (g * h)| \leq |(f * g) * h - (f_k * g_k) * h_k| + |f_k * (g_k * h_k) - f * (g * h)| \to 0$ as $k \to \infty$. Note that $|(f * g) * h - f * (g * h)|$ is a constant w.r.t. k and hence must be 0.

To show the second item, fix $n \in \mathbb{Z}$ and first consider $\left|\widehat{f_k - f}(n)\right| \leq 1/2\pi \cdot \int_{-\pi}^{\pi} |f_k(x) - f(x)| \cdot \cancel{\left|e^{-inx}\right|} \,\mathrm{d}x$, which tends to 0 by construction; similarly, $|\widehat{g_k - g}(n)| \to 0$ as $k \to \infty$. So,

$$\left|\widehat{f * g - f_k * g_k}(n)\right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |(f * g - f_k * g_k)(x)| \,\mathrm{d}x \to 0$$

as has been shown when proving the first item, and

$$|\hat{f}_k(n) \cdot \hat{g}_k(n) - \hat{f}(n) \cdot \hat{g}(n)| \leq |\hat{f}_k(n)| \cdot |\widehat{g_k - g}(n)| + \left|\widehat{f_k - f}(n)\right| \cdot |\hat{g}(n)| \to 0$$

because $\hat{f}_k(n)$ is bounded uniformly in k and $|\hat{g}(n)|$ is a constant in k.

Therefore,

$$\left|\widehat{f * g}(n) - \hat{f}(n) \cdot \hat{g}(n)\right| \leq \left|\widehat{f * g - f_k * g_k}(n)\right| + \left|\hat{f}_k(n) \cdot \hat{g}_k(n) - \hat{f}(n) \cdot \hat{g}(n)\right| \to 0$$

as $k \to \infty$. Since the left-hand side is a constant w.r.t. k, it must be 0. □

While tedious, the same techniques apply over and over again. The properties of commutativity, bilinearity, and associative are no surprise. It is however noteworthy that the convolution of integrable functions is necessarily continuous. Convolutions truly "smoothen" functions.

2.3 Kernels

We now define some kernels—sequences of functions commonly used to convolve with a given function. A prototypical family of kernels, known as the Dirichlet kernels, are defined as follows.

Definition 2.21. For $N \in \mathbb{Z}_{\geq 0}$, the N-th Dirichlet kernel, denoted as $D_N : \mathbb{R} \to \mathbb{C}$, is the trigonometric polynomial defined as $D_N(x) \coloneqq \sum_{n=-N}^{N} e^{inx}$.

We first provide a closed-form expression.

Proposition 2.22. Let $N \in \mathbb{Z}_{\geq 0}$. Then N-th Dirichlet kernel is

$$D_N(x) = \frac{\sin((N + 1/2)x)}{\sin(x/2)} \qquad \text{for all } x \neq 0.$$

Proof. We sum the finite geometric series

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{inx} \\ &= e^{i(-N)x} \cdot \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} \\ &= \frac{e^{-i(N+1/2)x} - e^{i(N+1/2)x}}{e^{i(-1/2)x - e^{i(1/2)x}}} \\ &= \frac{\sin((N + 1/2)x)}{\sin(x/2)}. \end{aligned}$$

The proof is finished. □

The Dirichlet kernels naturally appear when considering the partial sums of a Fourier series.

Definition 2.23. Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. For $N \in \mathbb{Z}_{\geq 0}$, let $S_N(f) : \mathbb{R} \to \mathbb{C}$ be the trigonometric polynomial defined as

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) \cdot e^{inx} \qquad \text{for all } x \in \mathbb{R}.$$

Proposition 2.24. Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. Then, $S_N(f) = f * D_N$ for all $N \in \mathbb{Z}_{\geq 0}$.

Proof. For all $x \in \mathbb{R}$, one has

$$S_N(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \cdot \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n(x-t)}\, \mathrm{d}t = (f * D_N)(x).$$

The proof is complete. □

We use the term "kernels" synonymously with functions. Some reasonable properties of (sequences of) kernels are quite commonplace, and we call such kernels well-behaved or an approximation to the identity.

Definition 2.25. A sequence of 2π-periodic functions $\{K_n\}_{n=1}^{\infty}$ from $\mathbb{R} \to \mathbb{C}$, also called kernels, are said to be well-behaved or to approximate the identity if

(−) For all $n \in \mathbb{Z}_{>0}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\, \mathrm{d}x = 1.$$

(=) There exists $M > 0$ such that for all $n \in \mathbb{Z}_{>0}$,

$$\int_{-\pi}^{\pi} |K_n(x)|\, \mathrm{d}x \leq M.$$

(≡) For all $\delta > 0$,

$$\lim_{n\to\infty}\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, \mathrm{d}x = 0$$

Note that the second item is a consequence of the first for non-negatively-valued kernels, which we shall also frequently encounter. In this case, one can view the well-behaved kernels as distributions on a circle that eventually peak "infinitely" at 0—approximating the Dirac δ function, in an informal sense. The utility of such kernels is seen in the following theorem.

Theorem 2.26 (Approximation to the Identity). *Let $\{K_n\}_{n=1}^{\infty}$ be a family of well-behaved kernels and suppose $f\colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and integrable on $[-\pi, \pi]$. Then, for all $x \in \mathbb{R}$ where f is continuous,*

$$\lim_{n\to\infty}(f * K_n)(x) = f(x).$$

Further, if f is continuous everywhere, then the convergence $f * K_n \to f$ is uniform.

Proof. Suppose $x \in \mathbb{R}$ is given, where f is continuous at x. Let $B > 0$ where $f(\mathbb{R}) \subset [-B, B]$. Fix $M > 0$ from item (=) of Definition 2.25.

Let $\epsilon > 0$ be arbitrary. Fix $\delta > 0$ such that $|f(x-y) - f(x)| < \epsilon/2M$ whenever $|y| \leq \delta$. Fix also $N \in \mathbb{Z}_{>0}$ such that $\int_{\delta \leq |y| \leq \pi} |K_n(y)|\, \mathrm{d}y < \epsilon/4B$ whenever $n \geq N$. Then, for all such $n \geq N$,

$$\begin{aligned}
|(f * K_n)(x) - f(x)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} K_n(y) \cdot f(x-y)\, \mathrm{d}y - \int_{-\pi}^{\pi} K_n(y) \cdot f(x)\, \mathrm{d}y\right| && (-)\\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(y)| \cdot |f(x-y) - f(x)|\, \mathrm{d}y\\
&= \frac{1}{2\pi}\int_{|y|\leq\delta} |K_n(y)| \cdot |f(x-y) - f(x)|\, \mathrm{d}y + \frac{1}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)| \cdot |f(x-y) - f(x)|\, \mathrm{d}y\\
&\leq \frac{M}{2\pi} \cdot \frac{\epsilon}{2M} + \frac{2B}{2\pi} \cdot \frac{\epsilon}{4B} && (1/2\pi < 1)\\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

This concludes the first part of the proof. For the second part, suppose f is continuous and hence uniformly continuous. Then, the choice of $\delta > 0$ can be made independent of x, and the desired bound by ϵ still holds. Therefore, the convergence is uniform in this case. □

2.4 The Cesàro Sum

Unfortunately, the Dirichlet kernels are not well-behaved. In fact, $\int_{-\pi}^{\pi} |D_N(x)| \, \mathrm{d}x$ grows at least logarithmically.

Lemma 2.27. Denote $L_N = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |D_N(\theta)| \, \mathrm{d}\theta$ for positive integers $N \in \mathbb{Z}_{>0}$. Then, $L_N \geq \dfrac{4}{\pi^2}(\ln N - 1)$.

Proof. Observe that since each $D_N(\cdot)$ is even,

$$\begin{aligned}
L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)| \, \mathrm{d}\theta &= \frac{1}{\pi}\int_{0}^{\pi} |D_N(\theta)| \, \mathrm{d}\theta \geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin((N+1/2)\theta)|}{\theta} \, \mathrm{d}\theta && (\sin(\theta/2) < \theta/2 \text{ for positive } \theta) \\
&= \frac{2}{\pi}\int_0^{(N+1/2)\pi} \frac{|\sin u|}{u} \, \mathrm{d}u && (u = (N+1/2)\theta) \\
&\geq \frac{2}{\pi}\int_{\pi}^{N\pi} \frac{|\sin u|}{u} \, \mathrm{d}u = \frac{2}{\pi}\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u} \, \mathrm{d}u \geq \frac{2}{\pi}\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{(k+1)\pi} \, \mathrm{d}u \\
&= \frac{2}{\pi}\sum_{k=2}^{N} \frac{2}{k\pi} = \frac{4}{\pi^2}\left(\sum_{k=1}^{N} \frac{1}{k} - \frac{1}{1}\right) \geq \frac{4}{\pi^2}(\ln N - 1)
\end{aligned}$$

as claimed. □

If they were well-behaved, their approximation to the identity can be used to investigate the convergence of Fourier series with significant aid. We may then consider other senses in which the Fourier series converge, which may correspond to other kernels which are well-behaved. This is indeed the case with regard to the Cesàro sum.

Definition 2.28. Suppose $\{c_k\}_{k=1}^{\infty}$ is a sequence of complex numbers. The formal sum $\sum_{k=1}^{\infty} c_k$ is said to be Cesàro summable to $\lim_{n\to\infty} \sigma_n$ if the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges, where $\sigma_n \coloneqq (S_1 + \cdots + S_n)/n$ and $S_n \coloneqq c_1 + \cdots + c_n$ for $n \in \mathbb{Z}_{>0}$.

Cesàro summability is more general than the convergence of partial sums.

Proposition 2.29. Suppose the series $\sum_{k=1}^{\infty} c_k$ of complex numbers converges to $s \in \mathbb{C}$. Then, $\sum_{k=1}^{\infty} c_k$ is Cesàro summable to s.

Proof. Fix $B' > 0$ such that $|S_n| \leq B'$ for all n, and let $B \coloneqq B' + |s| > 0$ be such that $|S_n - s| \leq B$ for all n.

Let $\epsilon > 0$ be arbitrary. Fix $K \in \mathbb{Z}_{>0}$ such that for all $k \geq K$, $|S_k - s| < \epsilon/2$. Let $N \coloneqq \max\{\lceil 2BK/\epsilon \rceil, K\} \in \mathbb{Z}_{>0}$. Then, for all $n \geq N \geq K$, one has

$$\begin{aligned}
|\sigma_n - s| &\leq \frac{1}{n}(|S_1 - s| + \cdots + |S_K - s|) + \frac{1}{n}(|S_{K+1} - s| + \cdots + |S_n - s|) \\
&< \frac{K}{n}\cdot B + \frac{n-K}{n}\cdot \epsilon/2 && (n \geq N \geq 2BK/\epsilon \Rightarrow K/n \cdot B \leq \epsilon/2) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

The proof is complete. □

The converse is not always true. For instance, $\sum_{n=1}^{\infty} (-1)^n$ is Cesàro summable to $-1/2$ while the partial sums diverge.

A theorem of Tauber states that, with suitable conditions on the summands, the Cesàro sum coincides with the limit of the partial sums.

Theorem 2.30 (Tauber). *If $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to $\sigma \in \mathbb{C}$ and $|c_n| = o(1/n)$ (that is, $nc_n \to 0$), then $\sum c_n$ converges to σ.*

Proof. Fix $B > 0$ such that $n \cdot |c_n| \leq B$, and hence $|c_n| \leq B/n \leq B$, for all n.

Let $\epsilon > 0$ be arbitrary. Fix $K_1, K_2 \in \mathbb{Z}_{>0}$ such that, respectively, $k \cdot |c_k| < \epsilon/4$ for all $k \geq K_1$ and $|\sigma_k - \sigma| < \epsilon/2$ for all $k \geq K_2$; then, define $K \coloneqq \max\{K_1, K_2\}$ and $N \coloneqq \max\{\lceil 4K^2B/\epsilon \rceil, K\}$.

Then, for all $n \geq N \geq K$, $S_n - \sigma_n = \frac{1}{n}c_2 + \cdots + \frac{n-1}{n}c_n$, and

$$\begin{aligned} |S_n - \sigma| &\leq |\sigma_n - \sigma| + |S_n - \sigma_n| \\ &< \frac{\epsilon}{2} + \left(\frac{0}{n} \cdot |c_1| + \cdots + \frac{K-1}{n} \cdot |c_K|\right) + \left(\frac{K}{n^2} \cdot n\,|c_{K+1}| + \cdots + \frac{n-1}{n^2} \cdot n\,|c_n|\right) \\ &< \frac{\epsilon}{2} + \underbrace{\frac{K}{n} \cdot K \cdot B}_{\leq \epsilon/4} + \underbrace{\frac{1}{n} \cdot (n-K)}_{<1} \cdot \frac{\epsilon}{4}. \end{aligned}$$

The first underbraced portion is at most $\epsilon/4$ because we have chosen $n \geq N \geq 4K^2B/\epsilon$. Therefore, $|S_n - \sigma| < \epsilon$. □

The series $\sum_{n=1}^{\infty}(-1)^n$ is Cesàro summable to $-1/2$, but the partial sums are alternately -1 and 0 and do not converge. This fact, combined with Proposition 2.29, shows that Cesàro summability is strictly more general.

We may now consider the Cesàro sum in the context of Fourier series, that is, summing the Fourier series in the sense of Cesàro. One may reasonably expect better convergence results in the sense of Cesàro, and this is indeed the case.

Let $\sigma_N(f)(x)$ denote the N-th Cesàro mean of the Fourier series of x. Then,

$$\sigma_N(f) = \frac{1}{N}(S_0 + \cdots + S_{N-1})(f) = \frac{1}{N}(f * D_0 + \cdots + f * D_{N-1}) = f * \frac{1}{N}(D_0 + \cdots + D_{N-1}).$$

Here, $\{\frac{1}{N}(D_0 + \cdots + D_{N-1})\}$ is the kernels corresponding to the Cesàro sum of the Fourier series.

Definition 2.31. For $N \in \mathbb{Z}_{>0}$, the N-th Fejér kernel, denoted as $F_N : \mathbb{R} \to \mathbb{C}$, is the trigonometric polynomial defined as $F_N \coloneqq \frac{1}{N}(D_0 + \cdots + D_{N-1})$.

Proposition 2.32. Let $N \in \mathbb{Z}_{>0}$. Then, the N-th Fejèr kernel has the closed-form expression

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)},$$

which holds for all $x \in \mathbb{R}\backslash 2\pi\mathbb{Z}$.

Proof. Recall that $D_N(x) = (\omega^{-N} - \omega^{N+1})/(1-\omega)$. Then,

$$\begin{aligned} NF_N(x) &= \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega} \\ &= \frac{1}{1-\omega} \cdot \left(\frac{1 - 1/\omega^N}{1 - 1/\omega} - \omega \cdot \frac{1-\omega^N}{1-\omega}\right) \\ &= \frac{1}{1-\omega} \cdot \left(\frac{\omega - \omega^{-N+1}}{\omega - 1} - \frac{\omega - \omega^{N+1}}{1-\omega}\right) \\ &= \frac{\omega \cdot (\omega^{-N/2 \cdot 2} - 2 + \omega^{N/2 \cdot 2})}{\omega \cdot (\omega^{1/2} - \omega^{-1/2})^2} \\ &= \frac{\sin^2(Nx/2)}{\sin^2(x/2)} \end{aligned}$$

as desired. Thus, $F_N(x) = 1/N \cdot \sin^2(Nx/2)/\sin^2(x/2)$. □

Lemma 2.33. The Fejèr kernels are well-behaved.

Proof. (–) For all $N \in \mathbb{Z}_{>0}$, one has

$$\int_{-\pi}^{\pi} F_N(x)\,\mathrm{d}x = \frac{1}{N}\sum_{n=0}^{N} \hat{D}_n(0) = 1.$$

(=) Observe that $F_N(x) \geq 0$ for all $x \in (-\pi, \pi)$, so (−) implies (=).

(≡) Let $\delta > 0$. Note that F_N is even and $\sin^2(x/2) = (1 - \cos x)/2$ is increasing on $x \in [0, \pi]$. Then, $F_N(x) = 1/N \cdot \sin^2(Nx/2)/\sin^2(x/2) \leq 1/NC_\delta$ for all $x \in [\delta, \pi]$, where $C_\delta = \sin^2(\delta/2) > 0$. Thus,

$$\int_{\delta \leq |x| \leq \pi} |F_N(x)| \, \mathrm{d}x = 2 \int_\delta^\pi F_N(x) \, \mathrm{d}x \leq \frac{1}{C_\delta \pi \cdot N} \to 0.$$

□

Corollary 2.34. Suppose $f \colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and integrable on $[-\pi, \pi]$. Then, $\sigma_N(f)(x) \to f(x)$ if f is continuous at $x \in \mathbb{R}$. If, further, f is continuous, then $\sigma_N(f) \to f$ uniformly.

This follows immediately from the application of Theorem 2.26. Incidentally, this piece of machinery lends us a much more straightforward proof of Theorem 2.8, generalized to include complex-valued functions.

Theorem 2.35 (Theorem 2.8 Generalized)**.** *Let $f \colon \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$ with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then, $f(\theta_0) = 0$ if f is continuous at θ_0.*

Proof. Note that $\hat{f}(n) = 0$, so $S_N(f)$ is always the zero function, and so is $\sigma_N(f)$. If f is continuous at $\theta_0 \in \mathbb{R}$, then $\lim_{N\to\infty} \sigma_N(f)(\theta_0) = 0 = f(\theta_0)$. □

Concerning the convergence of Fourier series, one could note the following more generally about approximations to the identity.

Lemma 2.36. Suppose $\{K_n\}_{n=1}^\infty$ is an approximation to the identity with each $K_n(\cdot)$ even. Let $f \colon \mathbb{R} \to \mathbb{C}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. Then, $(f * K_n)(x)$ tends to $(f(x^+) + f(x^-))/2$ if both limits $f(x^+) \coloneqq \lim_{t\to x^+} f(t)$ and $f(x^-) \coloneqq \lim_{t \to x^-} f(t)$ exist.

Proof. Let $\epsilon > 0$. Fix $B > 0$ such that $|f(t)| \leq B$ for all t and $\int_{-\pi}^{\pi} |K_n(t)| \, \mathrm{d}t \leq B$ for all n. Then, fix $\delta \in (0, \pi)$ such that both $|f(x-h) - f(x^-)|$ and $|f(x+h) - f(x^+)|$ are less than $\epsilon/2B$ whenever $h \in (0, \delta)$. Finally, fix $N \in \mathbb{Z}_{>0}$ such that $\int_{\delta \leq |t| \leq \pi} |K_n(t)| \, \mathrm{d}t < \epsilon/2B$ whenever $n \geq N$.

Since each $K_n(\cdot)$ is even, $1/2\pi \cdot \int_{-\pi}^0 K_n(t) \, \mathrm{d}t = 1/2\pi \cdot \int_0^\pi K_n(t) \, \mathrm{d}t = 1/2$. Then, for all such $n \geq N$,

$$\begin{aligned}\left|(f * K_n)(x) - \frac{f(x^+) + f(x^-)}{2}\right| &\leq \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t) \cdot f(x-t) \, \mathrm{d}t - \frac{1}{2\pi}\int_0^\pi K_n(t) \cdot f(x^-) \, \mathrm{d}t - \frac{1}{2\pi}\int_{-\pi}^0 K_n(t) \cdot f(x^+) \, \mathrm{d}t\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^0 |K_n(t)| \cdot \left|f(x-t) - f(x^+)\right| \, \mathrm{d}t + \frac{1}{2\pi}\int_0^\pi |K_n(t)| \cdot \left|f(x-t) - f(x^-)\right| \, \mathrm{d}t.\end{aligned}$$

Note that the blue portion may be split into

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^0 |K_n(t)| \cdot \left|f(x-t) - f(x^+)\right| \, \mathrm{d}t &= \frac{1}{2\pi}\int_{-\pi}^{-\delta} |K_n(t)| \cdot \overbrace{\left|f(x-t) - f(x^+)\right|}^{\leq 2B} \, \mathrm{d}t + \frac{1}{2\pi}\int_{-\delta}^0 |K_n(t)| \cdot \overbrace{\left|f(x-t) - f(x^+)\right|}^{<\epsilon/4} \, \mathrm{d}t \\ &\leq \frac{1}{2\pi} \cdot 2B \cdot \frac{1}{2} \cdot \frac{\epsilon}{2B} + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)| \, \mathrm{d}t\right) \cdot \frac{\epsilon}{2B} \\ &\leq \underbrace{\frac{1}{\pi}}_{<1} \left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) < \frac{\epsilon}{2}.\end{aligned}$$

By an identical argument, the red portion is less than $\epsilon/2$ as well, and $\left|(f * K_n)(x) - \frac{f(x^+)+f(x^-)}{2}\right|$ is less than ϵ. □

2.5 The Abel Sum

In even greater generality, one can sum a series in the sense of Abel.

Definition 2.37. Suppose $\{c_n\}_{n=1}^{\infty}$ is a sequence of complex numbers. The formal sum $\sum_{n=1}^{\infty} c_n$ is said to be Abel summable to $\lim_{r\to 1^-} A_r$ if the limit exists, where $A_r = \sum_{n=1}^{\infty} c_n r^n$.

Summation in the sense of Abel is even more general than in that of Cesàro. We first establish some useful observations.

Lemma 2.38. Suppose $\sum_{n=1}^{\infty} c_n$ is a series of complex numbers summable to 0 in the sense of Cesàro. Then, both c_n and S_n are $o(n)$; that is, both c_n/n and S_n/n tend to 0.

Proof. Note that

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{n\sigma_n - (n-1)\sigma_{n-1}}{n} = \lim_{n\to\infty} \sigma_n - \left(\lim_{n\to\infty} \frac{n-1}{n}\right) \cdot \lim_{n\to\infty} \sigma_{n-1} = 0.$$

Consequently,

$$\lim_{n\to\infty} \frac{c_n}{n} = \lim_{n\to\infty} \frac{S_n - S_{n-1}}{n} = 0 - \left(\lim_{n\to\infty} \frac{n-1}{n}\right) \cdot \lim_{n\to\infty} \frac{S_{n-1}}{n-1} = -1 \cdot 0 = 0$$

as well. □

Lemma 2.39. Suppose $\sum_{n=1}^{\infty} c_n$ is a series of complex numbers summable to 0 in the sense of Cesàro. Then, the series sums to 0 in the sense of Abel.

The estimation necessary in this analysis hinges on the summation by parts formula which connects c_n's with its cumulative sums.

Proof. Because σ_n is a convergent sequence, fix $B > 0$ such that $|\sigma_n| \leq B$.

First, we establish that $\sum_{n=1}^{\infty} n\sigma_n r^n$ converges for all $r \in (0,1)$. Noting that $\sum_{n=1}^{\infty} nr^n = r/(1-r)^2$ for $r \in (0,1)$, one has

$$\sum_{n=1}^{\infty} n\sigma_n r^n \leq \sum_{n=1}^{\infty} nr^n \cdot |\sigma_n| \leq B \cdot \frac{r}{(1-r)^2} < +\infty.$$

Now, summation by parts gives the observation that

$$\sum_{n=1}^{N} c_n r^n = (1 = 1 - r + r) \cdot r^N S_N - 0 - \sum_{n=1}^{N-1} (r^{n+1} - r^n) \cdot S_n = r^{N+1} S_N + (1-r) \sum_{n=1}^{N} S_n r^n,$$

and applying again with $\{S_n\}$ as a choice of the $\{c_n\}$ above,

$$\sum_{n=1}^{N} c_n r^n = r^{N+1} S_N + r^{N+1} \cdot (1-r) \sum_{n=1}^{N} S_n + (1-r)^2 \sum_{n=1}^{N} n\sigma_n r^n.$$

As $N \to \infty$, $r^{N+1} S_N = (Nr^{N+1}) \cdot (S_N/N) \to 0 \cdot 0 = 0$ and $r^{N+1} \cdot (1-r) \cdot N\sigma_N = Nr^{N+1} \cdot (1-r) \cdot \sigma_N \to 0 \cdot (1-r) \cdot 0 = 0$.

Let $\epsilon > 0$ be arbitrary and fix $N \in \mathbb{Z}_{>0}$ such that $|\sigma_n| < \epsilon/2$ whenever $n \geq N$. Further, fix $\delta \in (0,1)$ such that $|1-r| < \epsilon/2BN$ whenever $r \in (1-\delta, 1)$. Then, for all $r \in (1-\delta, 1)$,

$$\begin{aligned}
(1-r)^2 \left|\sum_{n=1}^{\infty} nr^n \sigma_n\right| &\leq (1-r)^2 \left|\sum_{n=1}^{N} nr^n \sigma_n\right| + (1-r)^2 \left|\sum_{n=N+1}^{\infty} nr^n \sigma_n\right| \\
&\leq (1-r)^2 \left|\sum_{n=1}^{N} Nr^n B\right| + (1-r)^2 \left|\sum_{n=N+1}^{\infty} nr^n \epsilon/2\right| \\
&= (1-r)^2 \cdot N \cdot r \frac{1-r^N}{1-r} \cdot B + (1-r)^2 \cdot \frac{r}{(1-r)^2} \cdot \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

so $\lim_{r\to 1^-} (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n = 0$, and $\sum c_n r^n = 0 + 0 + \lim_{r\to 1^-} (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n = 0$. □

Observe that $\sum_{n=1}^{\infty}(-1)^n \cdot n$ is Abel summable to

$$\lim_{r\to 1^-}\left(\sum_{n=1}^{\infty} n(-r)^n = -\frac{r}{(1-(-r))^2}\right) = -\frac{1}{4}.$$

However, it cannot be Cesàro summable since $(-1)^n n/n \not\to 0$.

3 Some Linear Algebra

Some definitions are reproduced. Note that some conventions, with respect to inner product spaces, may differ.

Definition 3.1. An F-vector space, where $F \in \{\mathbb{R}, \mathbb{C}\}$, is a triple $(V, +, \cdot)$ with $+\colon V \times V \to V$ and $\cdot\colon F \times V \to V$, where **(i)** $(V, +)$ is an abelian group, **(ii)** $1 \cdot v = v$, **(iii)** $a \cdot (b \cdot v) = (a \cdot b) \cdot v$, **(iv)** $(a + b) \cdot v = a \cdot v + b \cdot v$, **(v)** $c \cdot (v + w) = c \cdot v + c \cdot w$ for all $a, b, c \in F$ and $v, w \in V$.

Definition 3.2. An inner product (otherwise known as a positive-semidefinite Hermitian form) over an F-vector space V, where $F \in \{\mathbb{R}, \mathbb{C}\}$, is a map $\langle\cdot,\cdot\rangle\colon V \times V \to F$ such that **(i)** $\langle c \cdot u + v, w\rangle = c \cdot \langle u, w\rangle + \langle v, w\rangle$, **(ii)** $\overline{\langle v, w\rangle} = \langle w, v\rangle$, and **(iii)** $\langle v, v\rangle \geq 0$ for all $c \in F$ and $u, v, w \in V$.

In particular, the inner product need not be strictly positive-definite, in the sense that there may exist $v \in V\backslash\{0\}$ such that $\langle v, v\rangle = 0$.

Definition 3.3. Every inner product over an F-vector space, where $F \in \{\mathbb{R}, \mathbb{C}\}$, induces a map $\|\cdot\| : V \to [0, +\infty)$ via $v \mapsto \sqrt{\langle v, v\rangle}$.

Proposition 3.4. Suppose $\langle\cdot,\cdot\rangle$ is an inner product over an F-vector space V, where $F \in \{\mathbb{R}, \mathbb{C}\}$. Then, for all $v, w \in V$

(−) Pythagorean Theorem: If $\langle v, w\rangle = 0$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.

(=) Cauchy-Schwarz Inequality: $|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$.

(≡) Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$.

Proof. (−) If $\langle v, w\rangle = 0$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} = \|v\|^2 + \|w\|^2$.

(=) Observe that for all $s \in \mathbb{R}$, $0 \leq \|v + (s\langle v, w\rangle) \cdot w\|^2 = \|v\|^2 + s^2 \cdot |\langle v, w\rangle|^2 \cdot \|w\|^2 + 2\operatorname{Re}(\overbrace{\langle v, (s\langle v, w\rangle)w\rangle = s \cdot \overline{\langle v, w\rangle} \cdot \langle v, w\rangle}^{\in\mathbb{R}}) = \|v\|^2 + |\langle v, w\rangle|^2 \cdot s \cdot (s \cdot \|w\|^2 + 2)$.

In case $\|w\| = 0$, then $\|v\|^2 + 2\,|\langle v, w\rangle|^2 \cdot s \geq 0$ for any $s \in \mathbb{R}$. If $\langle v, w\rangle \neq 0$, then $\|v\|^2 + 2\,|\langle v, w\rangle|^2 \cdot s$ is negative for sufficiently negative s, which is a contradiction. In this case, $\langle v, w\rangle = 0$ and $0 \leq 0$ holds.

Now suppose $\|w\| > 0$. Take $s = -1/\|w\|^2$ so that $0 \leq \|v\|^2 + |\langle v, w\rangle|^2 \cdot (-1/\|w\|^2) \cdot 1$, or $|\langle v, w\rangle|^2 \leq \|v\|^2 \cdot \|w\|^2$.

(≡) Finally, leveraging (=), one has $2\operatorname{Re}\langle v, w\rangle \leq 2\,|\langle v, w\rangle| \leq 2\,\|v\| \cdot \|w\|$. Adding $\|v\|^2 + \|w\|^2$ to both ends of the inequality, $\|v + w\|^2 \leq (\|v\| + \|w\|)^2$, so $\|v + w\| \leq \|v\| + \|w\|$. □

These familiar inequalities hold even when $\langle\cdot,\cdot\rangle$ is not strictly positive-definite.

One particular example is $\mathcal{R}([-\pi, \pi])$ (as well as the subset $\mathcal{R}(\mathbb{S}^1)$ of those functions whose endpoints coincide in value), which is not strictly positive-definite.

Definition 3.5. Define the positive-semidefinite inner product $\langle\cdot,\cdot\rangle\colon \mathcal{R}(\mathbb{S}^1) \times \mathcal{R}(\mathbb{S}^1) \to \mathbb{C}$ by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot \overline{g(x)}\,\mathrm{d}x.$$

This positive-semidefinite inner product $\langle\cdot,\cdot\rangle$ then induces a semi-norm $\|\cdot\| : \mathcal{R}(\mathbb{S}^1) \to [0, +\infty)$ which maps $f \mapsto \sqrt{\langle f, f\rangle}$. Note that $\chi_{2\pi\mathbb{Z}} \in \mathcal{R}(\mathbb{S}^1)$ has semi-norm 0, even though it's not the zero function. Regardless, Lemma 2.2 holds and $\{e_n\}_{n=-\infty}^{\infty}$ remains an orthonormal collection of vectors in $\mathcal{R}(\mathbb{S}^1)$.

Definition 3.6. For $N \in \mathbb{Z}_{\geq 0}$, denote the subspace $V_N := \operatorname{span}\{e_n\}_{n=-N}^{N}$, where $V_0 \subset V_1 \subset \cdots \subset C^{\infty}(\mathbb{S}^1) \subset \mathcal{R}(\mathbb{S}^1)$.

Fourier series are particularly special because they are precisely the limit of the orthogonal projections onto V_N's under various appropriate senses of convergence. We first make this idea precise.

Proposition 3.7. Let $f \in \mathcal{R}(\mathbb{S}^1)$. Then, $S_N(f)$ is the orthogonal projection of f onto V_N for any $N \in \mathbb{Z}_{\geq 0}$; that is, $\|f - S_N(f)\| = \min_{p \in V_N} \|f - p\|$.

Proof. For all $n \in \mathbb{Z}$ with $|n| \leq N$, $\langle f - S_N(f), e_n \rangle = \langle f, e_n \rangle - \langle S_N(f), e_n \rangle = \hat{f}(n) - \hat{f}(n) = 0$, so $f - S_N(f)$ is orthogonal to V_N and hence any particular $(S_N(f) - p) \in V_N$. Applying the Pythagorean theorem, $\|f - p\|^2 = \|f - S_N(f)\|^2 + \|S_N(f) - p\|^2 \leq \|f - S_N(f)\|^2$. Further, "=" holds in the preceding inequality iff $\|S_N(f) - p\|^2$, which is true when $p = S_N(f)$. □

For *any* integrable $f \in \mathcal{R}(\mathbb{S}^1)$, then, there is some sense whereby the Fourier series converges.

Proposition 3.8 (L^2 Convergence)**.** Let $f \in \mathcal{R}(\mathbb{S}^1)$. Then, $\|S_N(f) - f\| \to 0$ as $N \to \infty$.

Proof. Choose a sequence of continuous functions $\{f_k\}_{n=1}^{\infty} \subset C^0(\mathbb{S}^1)$ that approximate f in the sense of Lemma 2.14, namely, with $\int_{-\pi}^{\pi} |f_k(x) - f(x)| \, \mathrm{d}x \to 0$ as $k \to \infty$, with some uniform bound $B > 0$ such that $|f_k(x)|$ and $|f(x)|$ are both at most B for all k and x.

Let $\epsilon > 0$. Fix a sufficiently large $k \in \mathbb{Z}_{>0}$ such that $\int_{-\pi}^{\pi} |f_k(x) - f(x)| < \epsilon/4B$. Then, the uniform approximation in N of $\sigma_N(f_k)$ to $f_k \in C^0(\mathbb{S}^1)$ (Corollary 2.34) affords a uniform upper bound $B' > 0$ of $\{\sigma_N(f_k)\}_N$ and f_k in modulus, and by extension some $N_0 \in \mathbb{Z}_{>0}$ such that $|\sigma_N(f_k)(x) - f_k(x)| < \epsilon/4B'$ whenever $N \geq N_0$. Then, because $\sigma_N(f_k) \in V_N$ for any such $N \geq N_0$,

$$\begin{aligned}
\|f - S_N(f)\|^2 &\leq \|f - \sigma_N(f_k)\|^2 && \text{(Proposition 3.7)} \\
&\leq 2\,\|f - f_k\|^2 + 2\,\|f_k - \sigma_N(f_k)\|^2 && ((x+y)^2 \leq 2(x^2+y^2) \text{ for } x, y \geq 0) \\
&\leq \frac{2}{2\pi} \int_{-\pi}^{\pi} |f(x) - f_k(x)| \cdot |f(x) - f_k(x)| \, \mathrm{d}x + \frac{2}{2\pi} \int_{-\pi}^{\pi} |f_k(x) - \sigma_N(f_k)(x)| \cdot |f_k(x) - \sigma_N(f_k)(x)| \, \mathrm{d}x \\
&\leq \frac{2}{2\pi} \cdot 2B \cdot \frac{\epsilon}{4B} + \frac{2}{2\pi} \cdot 2B' \cdot \frac{\epsilon}{4B'} = \epsilon/\pi < \epsilon.
\end{aligned}$$

Thus, $\lim_{N\to\infty} \|S_N(f) - f\|^2 = \lim_{N\to\infty} \|S_N(f) - f\| = 0$. □

We will now consider the space $\ell^2(\mathbb{Z})$, corresponding naturally to some well-behaved Fourier coefficients.

Definition 3.9. Define the map $\|\cdot\| \colon \mathbb{C}^{\mathbb{Z}} \to [0, +\infty]$ by $\{c_n\}_{n=-\infty}^{\infty} \mapsto \sqrt{\sum_{n=-\infty}^{\infty} |c_n|^2}$. Let $\ell^2(\mathbb{Z}) \subset \mathbb{C}^{\mathbb{Z}}$ be the subset of those sequences $\{c_n\}_{n=-\infty}^{\infty}$ such that $\|c\| < +\infty$.

We will define $\|\cdot\|$ more map more generally into the extended reals with $\sqrt{\infty} \coloneqq \infty$. Since the series involved have non-negative terms, this choice is sensible to work with by monotone convergence.

Proposition 3.10. $\ell^2(\mathbb{Z})$ is a vector subspace of $\mathbb{C}^{\mathbb{Z}}$ and $\langle \cdot, \cdot \rangle$ which sends $(\{a_n\}, \{b_n\}) \mapsto \sum a_n \overline{b}_n$ is a positive-definite inner product, and induces the norm $\|\cdot\|$, on $\ell^2(\mathbb{Z})$.

Proof. First, observe that for any $\{a_n\}, \{b_n\} \in \ell^2(\mathbb{Z})$, $|\sum a_n \cdot \overline{b}_n| \leq \sum |a_n \cdot \overline{b}_n| = \sum |a_n| \cdot |b_n| \leq (\sum |a_n|^2 + \sum |b_n|^2)/2 < +\infty$, so $\langle \cdot, \cdot \rangle$ is a well-defined map.

We now show closure under + and $\cdot$. Indeed, if $\{a_n\}, \{b_n\} \in \ell^2(\mathbb{Z})$, then $\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\operatorname{Re} \sum a_n \overline{b}_n$. Because $\operatorname{Re} \sum a_n \overline{b}_n \leq |\sum a_n \overline{b}_n| < +\infty$, $\|a + b\|^2 < +\infty$. And for any $c \in \mathbb{C}$, $\|c \cdot a\|^2 = \sum |c \cdot a_n|^2 = \left|c^2\right| \cdot \sum {a_n}^2 < +\infty$. Finally, linearity in the first component of $\langle \cdot, \cdot \rangle$ is immediate from the linearity of the series, and conjugate symmetry is straightforward as $\overline{\langle a, b \rangle} = \overline{\sum a_n b_n} = \sum \overline{a_n \overline{b}_n} = \sum b_n \overline{a}_n = \langle b_n, a_n \rangle$. □

Definition 3.11. An F-vector space V equipped with a positive-definite inner product $\langle \cdot, \cdot \rangle$, where $F \in \{\mathbb{R}, \mathbb{C}\}$, is said to be complete if every $\|\cdot\|$-Cauchy sequence $\|\cdot\|$-converges in V.

Note that the inner product is specified as positive-definite. Indeed, if not, then such a limit, when it exists, is not in general unique.

Proposition 3.12. $\ell^2(\mathbb{Z})$ is complete.

Proof. Suppose $\{c_\bullet^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence of elements in $\ell^2(\mathbb{Z})$. Let $\epsilon > 0$ be arbitrary and fix $K \in \mathbb{Z}_{>0}$ so that $\sum_n \left|c_n^{(k)} - c_n^{(l)}\right|^2 = \left\|c^{(k)} - c^{(l)}\right\|^2 < \epsilon$ whenever $k, l \geq K$ are sufficiently large. In particular, each term is bounded by $\left|c_{n_0}^{(k)} - c_{n_0}^{(l)}\right|^2 \leq \sum_n \left|c_n^{(k)} - c_n^{(l)}\right|^2 < \epsilon$. Hence, $\{c_{n_0}^{(k)}\}_{k=1}^\infty$ is Cauchy and has a limit denoted as $c_{n_0}^{(\infty)} \in \mathbb{C}$.

Denote $S_N(k,l) \coloneqq \sum_{n=-N}^N \left|c_n^{(k)} - c_n^{(l)}\right|^2$ with $N \in \mathbb{Z}_{>0}$ and $k, l \in \mathbb{Z}_{>0} \cup \{\infty\}$, so that $\left\|c_\bullet^{(k)} - c_\bullet^{(l)}\right\|^2 = \sup_N S_N(k,l)$. Now fix a sufficiently large $k \geq K$. Then, $S_N(k,K), S_N(k,K+1), \cdots$ are all smaller than ϵ and hence $\lim_l S_N(k,l) \leq \epsilon$ for all N. Taking the supremum then gives

$$\sup_{N>0} \lim_{l\to\infty} S_N(k,l) = \sup_{N>0} \sum_{n=-N}^{N} \left|c_n^{(k)} - \lim_{l\to\infty} c_n^{(l)}\right|^2 = \sup_{N>0} \sum_{n=-N}^{N} \left|c_n^{(k)} - c_n^{(\infty)}\right|^2 = \left\|c_\bullet^{(k)} - c_\bullet^{(\infty)}\right\|^2 \leq \epsilon,$$

so $\left\|c_\bullet^{(k)} - c_\bullet^{(\infty)}\right\|^2$ and hence $\left\|c_\bullet^{(k)} - c_\bullet^{(\infty)}\right\|$ tend to 0 as $k \to \infty$. [2]

Finally, $\left\|c_\bullet^{(\infty)} - c_\bullet^{(K)}\right\| < \infty \Rightarrow c_\bullet^{(\infty)} - c_\bullet^{(K)} \in \ell^2(\mathbb{Z})$ and $c_\bullet^{(K)} \in \ell^2(\mathbb{Z})$ imply that the sum $c_\bullet^{(\infty)} \in \ell^2(\mathbb{Z})$. □

Corollary 3.13 (Parseval)**.** Let $f \in \mathcal{R}(\mathbb{S}^1)$. Then, $\|f\| = \|\hat{f}(\cdot)\|$.

Note that the left-hand side is the norm induced by the L^2 inner product and the right-hand side by the ℓ^2 inner product.

Proof. For all $N \in \mathbb{Z}_{>0}$, $f - S_N(f)$ is orthogonal to $V_N \ni S_N(f)$, so $\|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2$. Note that by the Pythagorean theorem,

$$\|S_N(f)\|^2 = \left\|\sum_{n=-N}^{N} \hat{f}(n) \cdot e_n\right\|^2 = \sum_{n=-N}^{N} |\hat{f}(n)|^2 \cdot \cancel{\|e_n\|^2}.$$

So $\|\hat{f}(\cdot)\| = \lim_{N\to\infty} S_N(f) = \|f\|^2 - \lim_{N\to\infty} \|f - S_N(f)\|^2 = \|f\|^2$. □

The first equal sign in the last equality above certifies the formal series in $\|\hat{f}(\cdot)\|$ converges by definition. We are careful here since the preceding corollary is qualified to establish the following rigorously:
Proposition 3.14. Define $\mathcal{F} \colon \mathcal{R}(\mathbb{S}^1) \to \mathbb{C}^{\mathbb{Z}}$ by $f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$. Then $\mathcal{F}$ is a linear isometry into $\ell^2(\mathbb{Z})$.

Proof. Because for every $f \in \mathcal{R}(\mathbb{S}^1)$, $\|\hat{f}(\cdot)\| = \|f\| < \infty$, $\hat{f}(\cdot) \in \ell^2(\mathbb{Z})$. That is, $\mathcal{F}$ maps into $\ell^2(\mathbb{Z})$ while preserving the norm.

We finish the proof by first restating the polarization identity $\langle v, w\rangle = \frac{1}{4}\sum_{k=0}^3 \mathrm{i}^k \|v + \mathrm{i}\cdot w\|^2$ for all v, w in some inner product space, as easily verifiable by expanding the right-hand side. One then concludes

$$\langle f, g\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \|f + \mathrm{i}\cdot g\|^2 = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \cdot \left(\quad \|f + \mathrm{i}\cdot g\|^2 = \left\|\widehat{f + \mathrm{i}\cdot g}\right\|^2 = \left\|\hat{f} + \mathrm{i}\cdot\hat{g}\right\|^2 \quad \right) = \langle \hat{f}(\cdot), \hat{g}(\cdot)\rangle.$$

The proof is finished. □

The following is an immediate corollary.
Lemma 3.15 (Riemann-Lebesgue)**.** Let $f \in \mathcal{R}(\mathbb{S}^1)$. Then, $\hat{f} \to 0$ as $|n| \to \infty$.

Proof. Since $\sum |\hat{f}(n)|^2 < \infty$, $|\hat{f}(n)|$ and hence $\hat{f}(n)$ tend to 0. □

[2]Note that the first equality above, read from right to left, justifies that the limit $\lim_l S_N(k,l)$ is well-defined.

We are now equipped with enough machinery to tackle a *local* result regarding the convergence of the Fourier series of a function.

Theorem 3.16. *Suppose $f \in \mathcal{R}(\mathbb{S}^1)$ is differentiable at $x_0 \in \mathbb{R}$, then $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$.*

Proof. Without loss of generality, let $x_0 \in [-\pi, \pi]$. Consider the function $F\colon [-\pi,\pi] \to \mathbb{C}$ defined as

$$F(t) \coloneqq \begin{cases} \dfrac{f(x_0 - t) - f(x_0)}{t} & \text{if } t \neq 0 \\ -f'(x_0) & \text{otherwise.} \end{cases}$$

Since f is differentiable at x_0, F is continuous at 0 by construction. It is thus bounded and integrable on some open interval $(-\Delta, \Delta)$ containing 0, where $0 < \Delta < \pi$. For $\Delta \le |t| \le \pi$, $F(t)$ is the product of two integrable functions $f(x_0 - t) - f(x_0)$ and $1/t$ in t, so $F(t)$ is also integrable when $\Delta \le |t| \le \pi$. Thus, F is integrable on the entire interval $[-\pi, \pi]$.

Also define $G\colon [-\pi,\pi] \to \mathbb{C}$ as $G(t) \coloneqq \lim_{\tau \to t} \tau/\sin(\tau/2)$, a continuous, positively-valued function which is also integrable on $[-\pi, \pi]$. This can be obtained by similar reasoning, noting that $\lim_{\tau \to 0} \tau/\sin(\tau/2) = 2$ is finite.

Noting that $S_N(f)(x_0) = (D_N * f)(x_0)$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,\mathrm{d}t = 1$,

$$\begin{aligned} S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t) \cdot (f(x_0 - t) - f(x_0))\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin((N + 1/2)t) \cdot \frac{t}{\sin(t/2)} \cdot F(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin((N + 1/2)t) \cdot G(t) \cdot F(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\underbrace{F(t)G(t)\cos(t/2)}_{\text{Integrable}} \cdot \sin(Nt) + \underbrace{F(t)G(t)\sin(t/2)}_{\text{Integrable}}\cos(Nt)\Big)\,\mathrm{d}t \\ &\to 0, \end{aligned}$$

where the last step follows from the Riemann-Lebesgue lemma (Lemma 3.15). The proof is finished. □

Corollary 3.17. Let $f, g \in \mathcal{R}(\mathbb{S}^1)$ and $x_0 \in \mathbb{R}$. If f and g agree on an open interval $I \ni x_0$ containing x_0, then $S_N(f)(x_0) - S_N(g)(x_0) \to 0$ as $N \to \infty$.

Proof. Since $(f - g)(x) = 0$ whenever $x \in I \ni x_0$, $f - g \in \mathcal{R}(\mathbb{S}^1)$ is differentiable at x_0 and hence $S_N(f - g)(x_0) = S_N(f)(x_0) - S_N(g)(x_0) \to 0$. □

Finally, the Riemann-Lebesgue lemma in fact gives stronger guarantees of the decay of Fourier coefficients of C^k functions.

Proposition 3.18. Let $f \in C^k(\mathbb{S}^1)$. Then, $\hat{f}(n) = o(1/|n|^k)$, that is, $|n|^k \hat{f}(n)$ goes to 0 as $|n| \to \infty$.

Proof. Because $f \in C^k$, one has that $\widehat{f^{(k)}}(n) = (in)^k \hat{f}(n)$. Then,

$$\left|n^k \hat{f}(n)\right| = \left|\widehat{f^{(k)}}(n)\right| \to 0$$

as $|n| \to \infty$ by the Riemann-Lebesgue lemma since $f^{(k)} \in C^0 \subset \mathcal{R}$. □

4 Some Applications of Fourier Series

4.1 The Isoperimetric Inequality

Given a string of finite length, which shape it can enclose has the largest area? The answer seems to be the circle. This geometric problem does not seem to relate immediately to Fourier series. Here, an elegant argument is provided. To do so, however, some elementary notions of length and area must be decreed; a complete exposition is beyond the scope of these notes.

Definition 4.1. A normalized simple closed curve parametrized by arc-length is a 2π-periodic map $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma \in C^1(\mathbb{R})$, $\|\gamma'(t)\|_{\mathbb{R}^2} = 1$, and that $\gamma|_{[0,2\pi)}$ is injective.

With the above definition, we are effectively considering "shapes with circumference 2π." To show that the answer is a circle (of radius 1), we hope to show that the area of the "shape" is at most $\pi \cdot 1^2 = \pi$.

The area must also be defined in concrete terms. As far as these notes are concerned, we will content ourselves with the familiar "definition" from vector calculus that arises from Green's theorem. Since $\mathrm{d}(x \wedge \mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y$ and $\mathrm{d}(-y \wedge \mathrm{d}x) = -\mathrm{d}y \wedge \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y$:

Definition 4.2. Let γ be a normalized simple closed curve parametrized by arc-length. The area A enclosed by γ is defined as

$$A = \frac{1}{2}\left|\int_0^{2\pi} (x(t)y'(t) - x'(t)y(t))\,\mathrm{d}t\right|.$$

We take the absolute value since we haven't specified a particular handedness of γ; without it, A may be positive or negative.

Theorem 4.3. *Let γ be a normalized simple closed curve parametrized by arc-length. Then, the area A enclosed by γ is at most π. Further, if $A = \pi$, then the image of γ is a circle in $\mathbb{R}^2$.*

Proof. Write $(x(t), y(t)) \coloneqq \gamma(t)$. Then, $x, y \in C^1(\mathbb{S}^1)$. Denote $a_n \coloneqq \hat{x}(n)$ and $b_n \coloneqq \hat{y}(n)$ for $n \in \mathbb{Z}$. Integrating by parts, we have $\widehat{x'}(n) = \mathrm{i}n \cdot a_n$ and similarly $\widehat{y'}(n) = \mathrm{i}n \cdot b_n$.

The condition that $|\gamma'(t)| = 1$ can be translated to $x'(t)^2 + y'(t)^2 = 1$ for all t. Integrating 1,

$$\frac{1}{2\pi}\left(\int_0^{2\pi} x'(t)^2\,\mathrm{d}t + \int_0^{2\pi} y'(t)^2\,\mathrm{d}t\right) = 1.$$

Since x, y and hence x', y' are all real-valued, this implies by Parseval's identity that

$$\left\|\widehat{x'}\right\|^2_{\ell^2(\mathbb{Z})} + \left\|\widehat{y'}\right\|^2_{\ell^2(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} \left(|\mathrm{i}n \cdot a_n|^2 + |\mathrm{i}n \cdot b_n|^2\right) = \sum_{n=-\infty}^{\infty} n^2 \cdot \left(|a_n|^2 + |b_n|^2\right) = 1.$$

Meanwhile,

$$\begin{aligned}
A &= \pi \left|\langle x, \overline{y'}\rangle_{\mathcal{R}(\mathbb{S}^1)} - \langle x', \overline{y}\rangle_{\mathcal{R}(\mathbb{S}^1)}\right| \\
&= \pi \left|\langle x, y'\rangle_{\mathcal{R}(\mathbb{S}^1)} - \langle x', y\rangle_{\mathcal{R}(\mathbb{S}^1)}\right| && \text{(Real-valued } y', y) \\
&= \pi \left|\langle \hat{x}, \widehat{y'}\rangle_{\ell^2(\mathbb{Z})} - \langle \widehat{x'}, \hat{y}\rangle_{\ell^2(\mathbb{Z})}\right| && \text{(Parseval, bilinear form)} \\
&= \pi \left|\sum_{n=-\infty}^{\infty} (a_n \mathrm{i}(-n) b_{-n} - \mathrm{i}(-n) a_{-n} b_n)\right| && (f \text{ is real-valued} \Rightarrow \overline{\hat{f}(n)} = \hat{f}(-n)) \\
&= \pi \left|\sum_{n=-\infty}^{\infty} n \cdot (a_n \overline{b_n} - \overline{a_n} b_n)\right| && \textit{(Ibid)}
\end{aligned}$$

Because $\left|a_n\overline{b_n} - \overline{a_n}b_n\right| \le \left|a_n\overline{b_n}\right| + |\overline{a_n}b_n| = 2|a_n| \cdot |b_n| \le |a_n|^2 + |b_n|^2$ and $|n| \le \left|n^2\right|$ for integral n, we have $A \le \pi \sum n^2 \cdot (|a_n|^2 + |b_n|^2) = \pi$.

Now suppose equality holds. $|n| = \left|n^2\right|$ iff $n \in \{-1, 0, 1\}$, so a_n and b_n must both vanish for $|n| \ge 2$. Further, $a_{-1} = \overline{a_1}$ and $b_{-1} = \overline{b_1}$, so $\sum n^2 \cdot (|a_n|^2 + |b_n|^2) = |a_{-1}|^2 + |b_{-1}|^2 + 0 + |a_1|^2 + |b_1|^2 = 2(|a_1|^2 + |b_1|^2) = 1$. Meanwhile, rearranging the (in)equality in blue gives $(|a_n| - |b_n|)^2 = 0$, so $|a_n| = |b_n|$ for all non-zero n. Specifically, $|a_1| = |b_1|$, so $|a_1| = |b_1| = \sqrt{1/2 \cdot 1/2} = 1/2$. Thus, write $a_1 = \mathrm{e}^{\mathrm{i}\alpha}/2$ and $b_1 = \mathrm{e}^{\mathrm{i}\beta}/2$ for some $\alpha, \beta \in [-\pi, \pi)$. Now, $1 = 2\left|a_1\overline{b_1} - \overline{a_1}b_1\right| = 2/4 \cdot \left|\mathrm{e}^{\mathrm{i}\alpha} \cdot \mathrm{e}^{-\mathrm{i}\beta} - \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta}\right| = |2\mathrm{i} \cdot \sin(\alpha - \beta)|/2$. Thus, $\sin(\alpha - \beta) = \pm 1$ and $\alpha - \beta = \pm\pi/2$. Now $x(t) = a_0 + (\mathrm{e}^{-\mathrm{i}(\alpha+t)} + \mathrm{e}^{\mathrm{i}(\alpha+t)})/2 = a_0 + \cos(\alpha + t)$ and similarly $y(t) = b_0 + \cos(\beta + t) = b_0 + \cos(\alpha + t \mp \pi/2) = b_0 \pm \sin(\alpha + t)$. In either case, (x, y) specify a circle. $\square$

5 Fourier Transform

The Fourier transform is the continuous analog of the Fourier series. With the convention $L = 1$, the coefficients are continuously indexed and replaced by an improper integral. However, care must be taken to make the integrals meaningful. We consider two important classes of functions.

Definition 5.1. A function $f: \mathbb{R} \to \mathbb{C}$ is said to decrease moderately if f is continuous and, for some $A > 0$, $|f(x)| \leq A/(1+x^2)$ for all $x \in \mathbb{R}$. The set of all functions decreasing moderately is denoted with $\mathcal{M}(\mathbb{R})$.

That is, a continuous function decreases moderately iff $x^2 f(x)$ is bounded. This is useful because the p-test applies and allows for a well-defined improper integral. Of course, one could replace 2 with any $p > 1$.

Proposition 5.2. $\mathcal{M}(\mathbb{R})$ is a subspace of $\mathbb{C}^{\mathbb{R}}$.

Proof. Let $f, g \in \mathcal{M}(\mathbb{R})$ with $|f(x)| \leq A/(1+x^2)$ and $|g(x)| \leq B/(1+x^2)$. Then, $|f(x)+g(x)| \leq (A+B)/(1+x^2)$ and hence $f+g \in \mathcal{M}(\mathbb{R})$. In addition, whenever $c \in \mathbb{C}$, $|c \cdot f(x)| \leq \max\{|c| \cdot A, 1\}/(1+x^2)$. ◻

We now define the Fourier transform.

Definition 5.3. Let $f \in \mathcal{M}(\mathbb{R})$ decrease moderately. Then, the Fourier transform of f, denoted as $\mathcal{F}(f): \mathbb{R} \to \mathbb{C}$ or $\mathcal{F}_x(f(x))$, or simply $\hat{f}$ when no ambiguity arises, is defined by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x) \cdot \mathrm{e}^{-2\pi \mathrm{i} \xi x} \, \mathrm{d}x.$$

Proposition 5.4. The Fourier transform is well-defined.

Proof. Let $I_T := \int_{-T}^{T} f(x) \cdot \mathrm{e}^{-2\pi \mathrm{i} \xi x} \, \mathrm{d}x$. Then, for all $\epsilon > 0$,

$$\int_{|x| \geq M} \frac{\mathrm{d}x}{x^2} = 2 \lim_{T \to \infty} \left(\frac{1}{M} - \frac{1}{T}\right) = 2/M < \epsilon$$

when M is sufficiently large. It follows that $\{I_T\} \subset \mathbb{C}$ is Cauchy and hence converges. ◻

Note that the Fourier transform of a function $f: \mathbb{R} \to \mathbb{C}$ is well-defined, more generally, whenever $|f|$ is integrable over $\mathbb{R}$.

The improper integral has some useful properties on $\mathcal{M}(\mathbb{R})$.

Proposition 5.5. Let $f, g \in \mathcal{M}(\mathbb{R})$ and $c \in \mathbb{C}$. Then,

- $\int_{-\infty}^{\infty} (c \cdot f + g)(x) \, \mathrm{d}x = c \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x + \int_{-\infty}^{\infty} g(x) \, \mathrm{d}x$.
- $\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} f(x-h) \, \mathrm{d}x$ for all $h \in \mathbb{R}$.
- For all $\delta > 0$, $\delta \int_{-\infty}^{\infty} f(\delta x) \, \mathrm{d}x = \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x$.
- As $h \to 0$, $\int_{-\infty}^{\infty} |f(x-h) - f(x)| \, \mathrm{d}x \to 0$.

Proof. Item 1 follows from the linearity of the limit and the definite Riemann integral. For item 2, first suppose $h > 0$. Then,

$$\left| \int_{-T}^{T} f(x) \, \mathrm{d}x - \int_{-T}^{T} f(x-h) \, \mathrm{d}x \right| \leq \int_{-T-h}^{-T} |f(x)| \, \mathrm{d}x + \int_{T-h}^{T} |f(x)| \, \mathrm{d}x.$$

Both terms can be made arbitrarily small for sufficiently large T, so the limit tends to 0 as desired. For $h < 0$, it is sufficient to note that $x \mapsto f(x+h)$ is also in $\mathcal{M}(\mathbb{R})$ since $\frac{1+x^2}{1+(x+h)^2}$ is bounded from above, so we may apply the proven part to $f(x+h)$ with $-h > 0$. Item 3 is a simple change-of-variables and is therefore omitted. The first three items all hold, more generally, whenever f and g are absolutely integrable over $\mathbb{R}$.

We prove the final item more carefully. We first claim that there exists a constant $C > 0$ such that for all $|h| < 1$, $(1 + x^2) \cdot |f(x - h)| \leq C$. Indeed, fixing $A > 0$ such that $(1 + x^2) \cdot |f(x)| \leq A$,

$$\begin{aligned}(1 + x^2) \cdot |f(x-h)| &= (1 + (x-h)^2 + 2hx - h^2) \cdot |f(x-h)| \\ &\leq A + 2\cancel{|h|}\,|(x - h + h) \cdot f(x-h)| \\ &\leq A + 2\sup_{x \in \mathbb{R}} |x \cdot f(x)| + 2\cancel{|h|} \cdot \sup |f|\,(\mathbb{R}).\end{aligned}$$

so taking $C = A + 2\sup_{x\in\mathbb{R}} |x \cdot f(x)| + 2\sup |f|\,(\mathbb{R})$ suffices.

Let $\epsilon > 0$ be arbitrary. Fix $T > 0$ so large that $\int_{|x| \geq T-1} |f(x)|\,\mathrm{d}x < \epsilon/3$. Subsequently, fix $\delta \in (0, 1)$ so small that for all $|h| < \delta$ and $|x| \leq T$, $|f(x - h) - f(x)| < \epsilon/6T$, which is possible since f uniformly continuous on $[-T-1, T+1]$. Then, for all $|h| < \delta < 1$,

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,\mathrm{d}x &= \int_{-T}^{T} |f(x-h) - f(x)|\,\mathrm{d}x + \int_{|x|\geq T} |f(x-h) - f(x)|\,\mathrm{d}x \\ &\leq 2T \cdot \frac{\epsilon}{6T} + \int_{|x|\geq T} |f(x-h)|\,\mathrm{d}x + \int_{|x|\geq T} |f(x)|\,\mathrm{d}x \\ &< \frac{\epsilon}{3} + \int_{|x|\geq T-1} |f(x)|\,\mathrm{d}x + \int_{|x|\geq T-1} |f(x)|\,\mathrm{d}x \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

as desired. □

While moderately decreasing functions already have well-defined Fourier transforms, it is not immediately clear that the "inversion formula," analogous to the formal Fourier series, is meaningful. In particular, $\hat{f}$ may not decrease moderately. We therefore restrict our attention temporarily to a narrower class of functions.

Definition 5.6. A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is said to decrease rapidly if $\left|x^m f^{(n)}(x)\right|$ is bounded in $x \in \mathbb{R}$ for any $m, n \in \mathbb{Z}_{\geq 0}$. The set of all rapidly decreasing functions is denoted as $\mathcal{S}(\mathbb{R})$, the Schwartz class.

Corollary 5.7. $\mathcal{S}(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$.

Proof. Suppose $f \in \mathcal{S}(\mathbb{R})$ and note that $\left|(1 + x^2) f(x)\right| \leq |f(x)| + \left|x^2 f(x)\right|$. Both terms are bounded: taking $m = n = 0$, the first term is bounded; taking $m = 2$ and $n = 0$, the second term is also bounded. □

Corollary 5.8. If $f \in \mathcal{S}(\mathbb{R})$, $p(x) \in \mathbb{R}[x]$, and $n \in \mathbb{Z}_{\geq 0}$, then $p \cdot f \in \mathcal{S}(\mathbb{R})$ and $f^{(n)} \in \mathcal{S}(\mathbb{R})$.

Proof. For the first part, note that $xf(x)$ decreases rapidly in x by definition, since $\mathbb{Z}_{\geq 1} \subset \mathbb{Z}_{\geq 0}$. The same argument applies to the second part. □

Proposition 5.9. Rapidly decreasing functions are uniformly continuous.

Proof. Let $\epsilon > 0$ and define $\delta \coloneqq \epsilon/(1 + \sup |f'|\,(\mathbb{R}))$. Then, for all $x < y$ with $y - x \in (0, \delta)$, fix according to the mean value some $c_{x,y} \in [x, y]$ such that $f'(c_{x,y}) \cdot (x - y) = f(x) - f(y)$. Then,

$$|f(x) - f(y)| \leq \left|f'(c_{x,y})\right| \cdot |x - y| < (1 + \sup |f'|\,(\mathbb{R})) \cdot \frac{\epsilon}{(1 + \sup |f'|\,(\mathbb{R}))} = \epsilon$$

as desired. □

We now have some machinery to discuss a few properties of the Fourier transform.

Proposition 5.10. Let $f \in \mathcal{S}(\mathbb{R})$. Then, the Fourier transform maps:

(1) $f(x + h) \mapsto \hat{f}(\xi) \cdot \mathrm{e}^{2\pi i \xi h}$ for all $h \in \mathbb{R}$.

(2) $f(x) \cdot \mathrm{e}^{-2\pi i h x} \mapsto \hat{f}(\xi + h)$ for all $h \in \mathbb{R}$.

(3) $f(\delta x) \mapsto \delta^{-1}\hat{f}(\delta^{-1}\xi)$ for all $\delta > 0$.

(4) $f'(x) \mapsto 2\pi\mathrm{i}\xi\hat{f}(\xi)$.

(5) $-2\pi\mathrm{i}xf(x) \mapsto (\hat{f})'(\xi)$.

Proof. For item (1), observe that

$$\mathcal{F}_x(f(x+h))(\xi) = \int_{-\infty}^{\infty} f(x+h) \cdot \mathrm{e}^{2\pi\mathrm{i}\xi h} \cdot \mathrm{e}^{-2\pi\mathrm{i}\xi(x+h)}\,\mathrm{d}x = \mathrm{e}^{2\pi\mathrm{i}\xi h} \cdot \hat{f}(\xi).$$

For item (2),

$$\mathcal{F}_x(f(x) \cdot \mathrm{e}^{-2\pi\mathrm{i}hx})(\xi) = \int_{-\infty}^{\infty} f(x) \cdot \mathrm{e}^{-2\pi\mathrm{i}(\xi+h)x}\,\mathrm{d}x = \hat{f}(\xi+h).$$

For item (3),

$$\mathcal{F}_x(f(\delta x))(\xi) = \int_{-\infty}^{\infty} f(\delta x) \cdot \mathrm{e}^{-2\pi\mathrm{i}(\delta^{-1}\xi)(\delta x)}\,\mathrm{d}x = \delta^{-1} \cdot \int_{-\infty}^{\infty} f(x) \cdot \mathrm{e}^{-2\pi\mathrm{i}(\delta^{-1}\xi)x}\,\mathrm{d}x = \delta^{-1} \cdot \hat{f}(\delta^{-1}\xi).$$

Note that items (1) through (3) hold more generally for any absolutely integrable f over $\mathbb{R}$.

For item (4),

$$\mathcal{F}(f')(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}f(x) = \cancel{f(x) \cdot \mathrm{e}^{-2\pi\mathrm{i}\xi x}\Big|_{x=-\infty}^{\infty}} - \int_{-\infty}^{\infty} f(x) \cdot (-2\pi\mathrm{i}\xi) \cdot \mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}x = 2\pi\mathrm{i}\xi\hat{f}(\xi),$$

where the boundary terms cancel because Schwartz functions are $o(1)$. Note that item (4) holds more generally for any absolutely integrable and differentiable f over $\mathbb{R}$.

For item (5), let $\epsilon > 0$ and define

$$\delta = \frac{\epsilon}{1 + \int_{-\infty}^{\infty} 4\pi^2\,|x^2 f(x)|\,\mathrm{d}x}.$$

For all $x, \xi \in \mathbb{R}$ and all $h \in \mathbb{R}\backslash\{0\}$, fix $c_{x,h,\xi}$ or simply c between ξ and $\xi+h$ such that $h\cdot(-2\pi\mathrm{i}x\mathrm{e}^{-2\pi\mathrm{i}cx}) = \mathrm{e}^{-2\pi\mathrm{i}(\xi+h)x} - \mathrm{e}^{-2\pi\mathrm{i}\xi x}$. Subsequently, fix $d_{x,h,\xi}$ or simply d between ξ and c such that $(c-\xi)\cdot(-2\pi\mathrm{i}x\mathrm{e}^{-2\pi\mathrm{i}dx}) = \mathrm{e}^{-2\pi\mathrm{i}cx} - \mathrm{e}^{-2\pi\mathrm{i}\xi x}$. Then, for all $h \in (-\delta,\delta)\backslash\{0\}$,

$$\begin{aligned}
\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} - \int_{-\infty}^{\infty} f(x)\cdot(-2\pi\mathrm{i}x)\cdot\mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}x &= \int_{-\infty}^{\infty} f(x)\cdot\frac{\mathrm{e}^{-2\pi\mathrm{i}(\xi+h)x}-\mathrm{e}^{-2\pi\mathrm{i}\xi x}}{h}\,\mathrm{d}x - \int_{-\infty}^{\infty} f(x)\cdot(-2\pi\mathrm{i}x)\cdot\mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x)\cdot(-2\pi\mathrm{i}x)\cdot(\mathrm{e}^{-2\pi\mathrm{i}cx}-\mathrm{e}^{-2\pi\mathrm{i}\xi x})\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x)\cdot(-4\pi^2x^2)\cdot(c-\xi)\cdot\mathrm{e}^{-2\pi\mathrm{i}dx}\,\mathrm{d}x,
\end{aligned}$$

so

$$\left|\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} - \int_{-\infty}^{\infty} f(x)\cdot(-2\pi\mathrm{i}x)\cdot\mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}x\right| \le \int_{-\infty}^{\infty} 4\pi^2\left|x^2f(x)\right|\mathrm{d}x\cdot\big(|c-\xi| \le |h| < \delta\big) < \epsilon$$

as desired. Note that item (5) holds more generally for any f where $|f(x)|$ and $x^2\,|f(x)|$ are each integrable over $x \in \mathbb{R}$. $\square$

Direct estimations are typically not hard since the decay and smoothness of Schwartz class functions lend us a lot of machinery to bound any quantity.

We remark that the Fourier transform maps $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. It is in fact a linear isometry that, in particular, maps onto $\mathcal{S}(\mathbb{R})$, which we will prove later.

Proposition 5.11. Let $f \in \mathcal{S}(\mathbb{R})$. Then, $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Proof. For all $n, m \in \mathbb{Z}_{\geq 0}$, the Fourier transform maps $(-2\pi \mathrm{i}x)^n \cdot f(x) \mapsto (\hat{f})^{(n)}(\xi)$ and hence $(\frac{\mathrm{d}}{\mathrm{d}x})^m(-2\pi \mathrm{i}x)^n \cdot f(x) \mapsto (2\pi \mathrm{i}\xi)^m (\hat{f})^{(n)}(\xi)$. That is,

$$\mathcal{F}_x \left(\frac{1}{(2\pi \mathrm{i})^m} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^m (-2\pi \mathrm{i}x)^n \cdot f(x)) \right)(\xi) = \xi^m (\hat{f})^{(n)}(\xi).$$

The Fourier transform of a Schwartz class is bounded because it is, for instance, moderately decreasing. Therefore, we have a bound $\sup_{\xi \in \mathbb{R}} \left| \xi^m \cdot (\hat{f})^{(n)}(\xi) \right| < \infty$ and thus $\hat{f} \in \mathcal{S}(\mathbb{R})$. □

Note that the Fourier transform of a function of moderate decrease need not be of moderate decrease.

We will continue the development of machinery for the inversion formula. Gaussians are particularly important from this aspect. One particular normalized example is $\mathrm{e}^{-\pi x^2}$, a pdf that integrates to 1.

Proposition 5.12. The rapidly decreasing function $\mathrm{e}^{-\pi x^2}$ equals its own Fourier transform.

As we have seen, the general strategy can be that of differential equations. That is, we will leverage properties of the Fourier transform to derive a differential equation which $\hat{f}$ must satisfy and therefore determine $\hat{f}$.

Proof. Denote $f(x) = \mathrm{e}^{-\pi x^2}$. One can compute, e.g., with polar coordinates of the integral squared, that

$$\hat{f}(0) = \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2} \,\mathrm{d}x = 1.$$

Further, the derivative of $\hat{f}$ must be

$$(\hat{f})'(\xi) = \mathcal{F}_x(-2\pi \mathrm{i}x f(x))(\xi) = \mathrm{i} \cdot \mathcal{F}_x \left(\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{-\pi x^2} \right) = \mathrm{i} \cdot (2\pi \mathrm{i}\xi) \cdot \hat{f}(\xi) = -2\pi\xi \hat{f}(\xi).$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left(\mathrm{e}^{\pi \xi^2} \cdot \hat{f}(\xi) \right) = \mathrm{e}^{\pi x^2} \left(-2\pi\xi \hat{f}(\xi) + (\hat{f})'(\xi) \right) = \mathrm{e}^{\pi x^2} \cdot 0 = 0,$$

so the parenthesized portion, which is smooth by construction in ξ, must be a constant $C \in \mathbb{C}$; that is, $\hat{f}(\xi) = C \cdot \mathrm{e}^{-\pi \xi^2}$. Because $\hat{f}(0) = C = 1$, $\hat{f}(\xi) = \mathrm{e}^{-\pi \xi^2} = f(\xi)$. □

Our earlier approach in dealing with Fourier series was primarily through the machinery of convolution with an approximate identity of well-behaved kernels. An appropriately chosen family of kernels, namely, the Fèjer kernels, represents the Fourier series; similarly we aim to construct a family of kernels that corresponds to the Fourier transform.

An integrable "bump function" such as $\mathrm{e}^{-\pi x^2}$ can be used to generate a family of well-behaved kernels quite straightforwardly. One can approximate the Dirac delta function, the identity for convolution, with Gaussian distributions. Specifically, we choose to index them as

$$K_\delta(x) \coloneqq \delta^{-1/2} \cdot \mathrm{e}^{-\pi x^2/\delta}.$$

We have replaced x with $x/\sqrt{\delta}$, so that as $\delta \to 0^+$ we "squeeze the bump thinner" while also making it taller in the process due to normalization. Because $K_1(x) = \mathrm{e}^{-\pi x^2}$ equals its own Fourier transform, by some calculation one has

$$G_\delta(t) \coloneqq \mathrm{e}^{-\pi \delta t^2}$$

has the Fourier transform $\hat{G}_\delta = K_\delta$.

Proposition 5.13. The family of kernels $\{K_\delta\}_{\delta > 0}$ is well-behaved in the following sense:

(–) $\int_{-\infty}^{\infty} K_\delta(x) \,\mathrm{d}x = 1$ for all $\delta > 0$.

(=) $\int_{-\infty}^{\infty} |K_\delta(x)| \,\mathrm{d}x \leq M$ for all $\delta > 0$ uniformly, where $M > 0$ is a constant.

(≡) $\int_{|x| \geq \eta} |K_\delta(x)| \,\mathrm{d}x \to 0$ as $\delta \to 0^+$ for all $\eta > 0$.

Proof. For (−), observe that

$$\delta^{-1/2} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{-\pi(\delta^{-1/2}x)^2}\,\mathrm{d}x = \delta^{-1/2} \cdot (\delta^{-1/2})^{-1} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\,\mathrm{d}x = 1.$$

Because each $K_\delta(\cdot) > 0$, we have (−) implies (=) immediately. Finally, let $\eta > 0$. Then,

$$\delta^{-1/2} \cdot \int_{|x|\geq\eta} \mathrm{e}^{-\pi(\delta^{-1/2}x)^2}\,\mathrm{d}x = \int_{|x|\geq\delta^{-1/2}\eta} \mathrm{e}^{-\pi x^2}\,\mathrm{d}x.$$

As $\delta \to 0^+$, $\delta^{-1/2}\eta \to \infty$ so the integral vanishes indeed. □

We therefore have uniform convergence similar to the case for functions on the circle. For the sake of completeness, we define the convolution and state some properties. For rigor, in particular, we will establish the interchange of derivatives and integrals over $\mathbb{R}$ in the following sense.

Proposition 5.14. Let $f\colon \mathbb{R}^2 \to \mathbb{C}$ be of rapid decrease in the sense that $f(\cdot, y), f(x, \cdot) \in \mathcal{S}(\mathbb{R})$ for all $x, y \in \mathbb{R}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f_x(x,y)\,\mathrm{d}y.$$

As before, we will appeal to the mean value theorem heavily. We note the following observation to dispose of any dependence due to its use.

Lemma 5.15. Let $f\colon \mathbb{R}^2 \to \mathbb{C}$ be of rapid decrease in the sense that $f(\cdot, y), f(x, \cdot) \in \mathcal{S}(\mathbb{R})$ for all $x, y \in \mathbb{R}$. Then, there exists a constant $C > 0$ such that for all $h \in [-1, 1]$ and all $x, y \in \mathbb{R}$,

$$|f(x+h, y)| \leq \frac{C}{1+y^2}.$$

Proof. First, fix $A > 0$ so that $|f(x, y)| \leq \frac{A}{(1+x^2)(1+y^2)}$. Then,

$$|f(x+h,y)| \leq \frac{A}{(1+(x+h)^2)(1+y^2)} \leq \frac{A}{1+(x+h)^2} \cdot \frac{1}{1+y^2} \leq \frac{A}{1+y^2}$$

as desired. □

Definition 5.16. Suppose $f, g \in \mathcal{S}(\mathbb{R})$. Then, the convolution of f and g, denoted as $f * g\colon \mathbb{R} \to \mathbb{C}$, is defined as via

$$(f * g)(y) = \int_{-\infty}^{\infty} f(x) \cdot g(y-x)\,\mathrm{d}x.$$

Proposition 5.17. The convolution $*$ is a symmetric, bilinear map into $\mathcal{S}(\mathbb{R})$.

The proof follows the same argument as in the case of functions on the circle and is therefore omitted. We obtain the following corollary to the fact that $\{K_\delta\}$ is well-behaved.

Corollary 5.18. Suppose $f \in \mathcal{S}(\mathbb{R})$. Then, $f * K_\delta \to f$ uniformly as $\delta \to 0^+$.

Proof. One can estimate $(f * K_\delta)(x) - f(x)$ via

$$|(f * K_\delta)(x) - f(x) \cdot 1| \leq \int_{-\infty}^{\infty} |K_\delta(y)| \cdot |f(x-y) - f(x)|\,\mathrm{d}y.$$

When $|y|$ is small, the absolute difference is small; when $|y|$ is large, the absolute difference is bounded by some constant and property (≡) allows this term to be arbitrarily small as desired. A complete argument is omitted. □

An important aspect that bridges the gap between the transform itself and convolution with this family is what is known as the multiplication formula for the Fourier transform. This result is a straightforward conclusion formally by interchanging the integrals in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) \cdot g(y) \cdot \mathrm{e}^{-2\pi \mathrm{i} xy}\,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) \cdot g(y) \cdot \mathrm{e}^{-2\pi \mathrm{i} xy}\,\mathrm{d}y\,\mathrm{d}x,$$

with the two sides being $\int_{-\infty}^{\infty} \hat{f}(y)g(y)\,\mathrm{d}y$ and $\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,\mathrm{d}x$, respectively. A careful justification requires defining integrals over cubes in $\mathbb{R}^2$ which is not a part of the prerequisites. One may justify this by Fubini's theorem, perhaps restricting the iterated integrals to expanding compact sets.

Proposition 5.19 (Multiplication Formula)**.** Let $f, g \in \mathcal{S}(\mathbb{R})$. Then,

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,\mathrm{d}x.$$

The proof is omitted.

We now have the machinery to tackle the Fourier inversion formula.

Theorem 5.20 (Fourier Inversion)**.** *Let $f \in \mathcal{S}(\mathbb{R})$. Then, for all $x \in \mathbb{R}$,*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi.$$

Proof. For $x \in \mathbb{R}$, denote $f_x(y) \coloneqq f(x+y)$. Then,

$$\hat{f_x}(\xi) = \mathrm{e}^{2\pi \mathrm{i}\xi x} \cdot \hat{f}(\xi),$$

so

$$f(x) = f_x(0) = \lim_{\delta\to 0^+} (f_x * K_\delta)(0) = \lim_{\delta\to 0^+} \int f_x(\xi)\cdot \hat{G}_\delta(0-\xi)\,\mathrm{d}\xi = \lim_{\delta\to 0^+} \int \hat{f_x}(\xi)\cdot G_\delta(\xi)\,\mathrm{d}\xi = \lim_{\delta\to 0^+} \int \hat{f}(\xi)\cdot \mathrm{e}^{2\pi \mathrm{i}\xi x}\cdot \mathrm{e}^{-\pi\delta\xi^2}\,\mathrm{d}\xi.$$

It is now sufficient to justify interchanging the limit and the integral. Let $\epsilon > 0$ and fix $T > 0$ such that $\int_{|\xi|\geq T} |\hat{f}(\xi)|\,\mathrm{d}\xi < \epsilon/2$ by virtue of $\hat{f} \in \mathcal{S}(\mathbb{R})$. Subsequently, fix $\delta_0 > 0$ such that $1 - \mathrm{e}^{-\pi\delta_0 T^2} < \epsilon/(1 + 2\int_{-\infty}^{\infty} |\hat{f}(\xi)|\,\mathrm{d}\xi)$. Then, for all $\delta \in (0, \delta_0)$,

$$\begin{aligned}
\left| f(x) - \int_{-\infty}^{\infty} \hat{f}(\xi)\cdot \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi \right| &\leq \int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right| \cdot \cancel{\left|\mathrm{e}^{2\pi \mathrm{i}\xi x}\right|} \cdot (1 - \mathrm{e}^{-\pi\delta\xi^2})\,\mathrm{d}\xi \\
&= \int_{-T}^{T} \left|\hat{f}(\xi)\right| \cdot (1 - \mathrm{e}^{-\pi\delta\xi^2})\,\mathrm{d}\xi + \int_{|\xi|\geq T} \left|\hat{f}(\xi)\right| \cdot \underbrace{(1 - \mathrm{e}^{-\pi\delta\xi^2})}_{<1}\,\mathrm{d}\xi \\
&\leq \int_{|\xi|\leq T<\infty} \left|\hat{f}(\xi)\right|\,\mathrm{d}\xi \cdot \frac{\epsilon}{1 + 2\int_{-\infty}^{\infty} |\hat{f}(\xi)|\,\mathrm{d}\xi} + \frac{\epsilon}{2} \\
&< \epsilon,
\end{aligned}$$

which completes the proof. □

Definition 5.21. Suppose $f \in \mathcal{S}(\mathbb{R})$. The inverse Fourier transform of f, denoted as $\mathcal{F}^{-1}(f)$ or $\mathcal{F}_\xi^{-1}(f(\xi))$, is defined as

$$\mathcal{F}^{-1}(f)(x) \coloneqq \int_{-\infty}^{\infty} f(\xi)\cdot \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi.$$

This defines a linear map $\mathcal{F}^{-1}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

The notation is partially justified so far, in the sense that $\mathcal{F}^{-1}$ so defined is known to be a one-sided inverse. To show the side, note that "the inverse is off only by a negative sign," in the sense that $\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x)$. This is helpful for establishing $\mathcal{F}$ as an automorphism of a vector space. It is in fact, as we later show, a linear isometry.

Proposition 5.22. $\mathcal{F}$ is a bijection from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$ with $\mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F} = \iota$.

Proof. That $\mathcal{F}^{-1} \circ \mathcal{F} = \iota$ has been shown. For the other side, we first note that reflection commutes with the Fourier transform: $\mathcal{F}(f)(-\xi) = \mathcal{F}_x(f(-x))(\xi)$ by change-of-variables. Then, $\mathcal{F}_t(\mathcal{F}^{-1}(f)(t))(x) = \mathcal{F}_t(\mathcal{F}(f)(-t))(x) = \mathcal{F}_t(\mathcal{F}(f)(t))(-x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$ as desired. Therefore, $\mathcal{F}^{-1}$ is indeed an inverse function of $\mathcal{F}$, concluding the proof. □