Notes on Fourier Analysis

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1 Notation and Preliminaries

Let $\mathrm{i} = \sqrt{-1}$; the italicized variant i is reserved for indices. For $D \subseteq \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$, let $C^k(D)$ denote the set of functions from D to $\mathbb{R}$ that are k times continuously differentiable.

Definition 1.1. The length of an interval I, denoted as $\mu(I)$, is the difference between its right endpoint and its left. We shall restrict the definition of intervals to those with positive length.

Definition 1.2. Let $D \subseteq \mathbb{R}$. A function from D to $\mathbb{R}$ is said to be piecewise continuous if it is bounded and admits at most finitely many discontinuities.

Definition 1.3. A partition of $[a, b]$ is a finite subset of $[a, b]$ containing a and b. It is typically denoted as $a = x_0 < \cdots < x_n = b$.

Definition 1.4. A bounded function $f\colon [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for any $\epsilon > 0$ there exists a partition $P = \{x_0, \cdots, x_n\}$ of $[a, b]$ such that

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) < \epsilon,$$

where $a = x_0 < \cdots < x_n = b$. Here, we define

$$\begin{cases} \mathcal{U}(P, f) \coloneqq \sum_{i=1}^n \sup f[[x_{i-1}, x_i]] \cdot (x_i - x_{i-1}) \\ \mathcal{L}(P, f) \coloneqq \sum_{i=1}^n \inf f[[x_{i-1}, x_i]] \cdot (x_i - x_{i-1}). \end{cases}$$

Definition 1.5. If a bounded function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable, then $\sup U(P, f)$ and $\inf U(P, f)$ coincide, where the supremum and the infimum are both taken over partitions of $[a, b]$. This common value is defined as the Riemann integral of f over $[a, b]$, denoted as $\int_a^b f(x)\,\mathrm{d}x$.

Definition 1.6. A bounded function $f\colon [a, b] \to \mathbb{C}$ is said to be Riemann integrable if $\Re \circ f$ and $\Im \circ f$ are Riemann integrable. The Riemann integrable of f over $[a, b]$ is defined as $\int_a^b \Re f(x)\,\mathrm{d}x + \mathrm{i} \cdot \int_a^b \Im f(x)\,\mathrm{d}x$, denoted also as $\int_a^b f(x)\,\mathrm{d}x$.

Definition 1.7. The set of all Riemann integrable functions from $[a, b]$ to $\mathbb{C}$ is denoted as $\mathcal{R}([a, b])$.

Theorem 1.8 (Weierstrass' M-Test)**.** *Let $\{f_n(x)\}_{n=1}^\infty$ be a sequence of functions from $A \subseteq \mathbb{R}$ to $\mathbb{C}$. Suppose there exists a sequence $\{M_n\}_{n=1}^\infty$ of non-negative numbers such that $\sum_{n=1}^\infty M_n$ converges and $|f_n(x)| \leq M_n$ for all $n \in \mathbb{Z}_{>0}$ and all $x \in A$. Then, $\sum_{n=1}^\infty f_n(x)$ converges absolutely and uniformly on A.*

Proof. Let $T \coloneqq \sum_{n=1}^\infty M_n \geq 0$ and define $S(x) \coloneqq \sum_{n=1}^\infty f_n(x)$ formally. The latter converges pointwise absolutely. Denote $S_n(x) \coloneqq f_1(x) + \cdots + f_n(x)$.

Let $\epsilon > 0$. Because $M_1 + \cdots + M_N$ converges to T as $N \to \infty$, the partial sums can be arbitrarily close to T. In particular, fix $N \in \mathbb{Z}_{>0}$ such that $|T - (M_1 + \cdots + M_N)| < \epsilon$. Then,

$$\left| \sum_{n=1}^\infty f_n(x) - \sum_{n=1}^N f_n(x) \right| \leq \sum_{n=N+1}^\infty |f_n(x)| \leq \sum_{n=N+1}^\infty M_n < \epsilon.$$

The proof is complete. $\square$

2 Definition

We first define the structure on Riemann integrable functions.

Proposition 2.1. Let $f\colon [a,b] \to [0,+\infty)$ be continuous. Then, $\int_a^b f(x)\,\mathrm{d}x = 0$ implies f is identically zero.

Proof. Suppose for contradiction that $0 \in [a,b]$ and $f(0) > 0$ without loss of generality. Fix $\delta > 0$ such that $f(x) > f(0)/2$ for all $x \in (-\delta,\delta) \cap [a,b]$. Let $I \coloneqq [-\delta,\delta] \cap [a,b]$, which is an interval via straightforward verification when $0 = a$, $a < 0 < b$, and $0 = b$. Then,

$$\int_a^b f(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x + \int_{[a,b]\setminus I} f(x)\,\mathrm{d}x \geq \int_I \frac{f(0)}{2}\,\mathrm{d}x \geq \frac{f(0)\mu(I)}{2} > 0,$$

a contradiction. □

Corollary 2.2. The set $\mathcal{R}([a,b])$ of complex-valued, Riemann integrable functions on a segment $[a,b]$ is made into an infinite-dimensional inner product space with the inner product

$$\langle f, g\rangle \coloneqq \frac{1}{L}\int_a^b f(x)\cdot\overline{g(x)}\,\mathrm{d}x,$$

where $L = b - a > 0$.

Proof. We first show that the inner product is well-defined. Suppose $f, g \in \mathcal{R}([a,b])$. Firstly, $\langle f, f\rangle = \frac{1}{L}\int_a^b |f(x)|^2\,\mathrm{d}x$ is clearly non-negative. When it is equal to zero, $|f(x)|^2$ is identically zero, so $f(x)$ is identically zero. (Sesqui)-linearity and conjugate symmetry are straightforward, which concludes this proof. □

Definition 2.3. Let $[a,b]$ be an interval. For $n \in \mathbb{Z}$, the n-th Fourier basis function on $[a,b]$ is defined as $e_n \coloneqq [a,b] \to \mathbb{C}$ via $e_n(x) \coloneqq \mathrm{e}^{2\pi\mathrm{i}nx/L}$, where $L = b - a > 0$.

Lemma 2.4. $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal set of vectors in $\mathcal{R}([a,b])$.

Proof. Suppose $m, n \in \mathbb{Z}$. Then,

$$\langle e_n, e_m\rangle = \frac{1}{L}\int_a^b \mathrm{e}^{2\pi\mathrm{i}(n-m)x/L}\,\mathrm{d}x.$$

When $n = m$, the integrand is 1 and $\langle e_n, e_n\rangle = 1/L \cdot L = 1$. Otherwise, the integral is

$$\langle e_n, e_m\rangle = \frac{1}{L}\cdot\frac{1}{2\pi\mathrm{i}(n-m)/L}\cdot\cancel{\mathrm{e}^{2\pi\mathrm{i}(n-m)x/L}\Big|_{x=a}^{b}} = 0.$$

The proof is finished. □

Definition 2.5. Let $f \in \mathcal{R}([a,b])$. The n-th Fourier coefficient of f, where $n \in \mathbb{Z}$, is defined as $\hat{f}(n) \coloneqq \langle f, e_n\rangle$, where $L \coloneqq b - a$.

Definition 2.6. Let $f \in \mathcal{R}([a,b])$. The Fourier series of f is the formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\cdot e_n(x)$$

with an indeterminate $x \in \mathbb{R}$.

One typically writes

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\cdot e_n(x)$$

to denote that $f(x)$ has the Fourier series on the right-hand side of the $\sim$ relation.

Definition 2.7. A function $f\colon \mathbb{R} \to \mathbb{C}$ is said to be a trigonometric series if it admits the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\cdot e_n(x) \qquad \text{for all } x \in \mathbb{R}$$

for some complex-valued sequence $\{c_n\}_{n=-\infty}^{\infty}$.

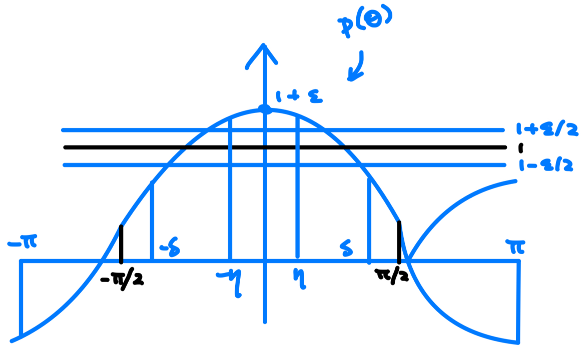

Figure 1: The plot of $p(\theta)$ in the proof of Theorem 2.10

Definition 2.8. A trigonometric polynomial p is a trigonometric series whose associated sequence $\{c_n\}_{n=-\infty}^{\infty}$ has all but finitely many zero terms. The degree of the trigonometric polynomial, denoted as $\deg p$, is defined as $\max_{n\in\mathbb{Z}} |n|$ subject to $c_n \neq 0$.

Corollary 2.9. Trigonometric polynomials are closed under addition, negation, and multiplication.

Proof. That trigonometric polynomials are closed under addition and negation is immediate. Suppose

$$f(x) = \sum_{n=-N}^{N} a_n \cdot e_n(x) \qquad \text{and} \qquad g(x) = \sum_{n=-N}^{N} b_n \cdot e_n(x)$$

are trigonometric polynomials, where $N \in \mathbb{Z}_{\geq 0}$. Then,

$$f(x) \cdot g(x) = \sum_{n=-N}^{N} \sum_{m=-N}^{N} a_n b_m \cdot e_n(x) e_m(x) = \sum_{n=-N}^{N} \sum_{m=-N}^{N} a_n b_m \cdot e_{m+n}(x) = \sum_{k=-2N}^{2N} \left(\sum_{n=\max\{-N,k-N\}}^{\min\{N,k+N\}} a_n b_{k-n} \right) \cdot e_k(x).$$

The proof is complete. □

Theorem 2.10. *Let $f \in \mathcal{R}(\mathbb{R})$ be 2π-periodic with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then, $f(\theta_0) = 0$ if f is continuous at θ_0.*

Proof. First, suppose f is real-valued. Without loss of generality, suppose $\theta = 0$ and $f(0) > 0$. Fix $0 < \delta \leq \pi/2$ such that $f(x) > f(0)/2$ whenever $|\theta| < \delta$. Let $p(\theta) := \epsilon + \cos\theta$, which is a trigonometric polynomial, where $\epsilon > 0$ is chosen sufficiently small such that $|p(\theta)| < 1 - \epsilon/2$ whenever $\delta \leq |\theta| \leq \pi$. Fix $0 < \eta < \delta$ such that $p(\theta) \geq 1 + \epsilon/2$ whenever $|\theta| < \eta$. Define $p_k(\theta) := p(\theta)^k$ for $k \in \mathbb{Z}_{\geq 0}$ and fix $B > 0$ such that $|f(\theta)| \leq B$ for all $\theta \in \mathbb{R}$.

We make three observations to estimate the integral $\int_a^b f(\theta) \cdot p_k(\theta)\, \mathrm{d}\theta$ by splitting the domain into three parts, where θ is assumed to satisfy $|\theta| < \eta$, $\eta < |\theta| < \delta$, and $\delta < |\theta| < \pi$ respectively.[1]

First, note that

$$\int_{|\theta|\leq\eta} f(\theta) \cdot p_k(\theta) \geq \int_{|\theta|\leq\eta} f(0)/2 \cdot (1+\epsilon)/2^k = \eta f(0) \cdot (1+\epsilon/2)^k,$$

where the right-hand side is unbounded as $k \in \mathbb{Z}_{\geq 0}$ varies.

For the second piece, it's enough to conclude

$$\int_{\eta\leq|\theta|\leq\delta} f(\theta) \cdot p_k(\theta)\, \mathrm{d}\theta \geq 0.$$

[1] We may modify the integrands of the three integrals so that the endpoints evaluate to 0; in this way, we do not change the value of each integral but can assume strict inequalities such as these in estimation.

Lastly, we have

$$\left|\int_{\delta\le|\theta|} f(\theta)\cdot p_k(\theta)\,\mathrm{d}\theta\right| \le \int_{\delta\le|\theta|} |f(\theta)|\cdot|p_k(\theta)|\,\mathrm{d}\theta \le (2\pi - 2\delta)B(1-\epsilon/2)^k,$$

where the right-hand side is bounded.

Hence, $\int_{-\pi}^{\pi} f(\theta)\cdot p_k(\theta)\,\mathrm{d}\theta$ is at least an unbounded number minus a bounded number. This integral, therefore, cannot tend to 0 as $k \to \infty$. However, since $p_k(\theta)$ is a trigonometric polynomial by induction on Corollary 2.9, we may write $p_k(\theta) = \sum_{n=S}^{T} c_n \cdot e_n$, and

$$\int_{-\pi}^{\pi} f(\theta)\cdot p_k(\theta)\,\mathrm{d}\theta = 2\pi\sum_{n=S}^{T} c_n\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cdot\overline{e_{-n}(\theta)}\,\mathrm{d}\theta\right) = 0.$$

These integrals, then, must tend to 0. In particular, they cannot be unbounded, a contradiction. □

Proposition 2.11. Suppose $f\colon \mathbb{R}\to\mathbb{R}$ is periodic and continuous with $\sum_{n=-\infty}^{\infty}|\hat{f}(n)| < \infty$. Then,

$$f(x) = \lim_{N\to\infty}\sum_{n=-N}^{N}\hat{f}(n)\cdot e_n(x) \qquad \text{for all } x\in\mathbb{R},$$

and the convergence is uniform in x.

Before proving this foundational proposition, we remark that periodicity is preserved by uniform convergence.

Lemma 2.12. Let $P > 0$. Suppose $\{f_n\}_{n=1}^{\infty}$ is a uniformly convergent sequence of P-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. Then, the limit is also P-periodic.

Proof. Let $\epsilon > 0$. Fix $N \in \mathbb{Z}_{>0}$ such that $|f_n(x) - f(x)| < \epsilon/2$ for all $n \in \mathbb{Z}_{>} N$ and all $x\in\mathbb{R}$. Then, for all $x\in\mathbb{R}$,

$$|f(x+P) - f(x)| \le |f(x+P) - f_n(x+P)| + |f_n(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The proof is complete. □

We now proceed to prove the proposition.

Proof. Without loss of generality, suppose f is 2π-periodic. Let $S_N(x) \coloneqq \sum_{n=-N}^{N}\hat{f}(n)\cdot e_n(x)$ be the N-th partial sum of the Fourier series of f, where $N\in\mathbb{Z}_{\ge 0}$. By Weierstrass' M-test, $\{S_N(x)\}$ converges absolutely and uniformly. Denote the limit as $g(x)$, the Fourier series of f which must be continuous. Hence,

$$\begin{aligned}
\widehat{f-g}(n) &= \langle f, e_n\rangle - \langle g, e_n\rangle && \text{(Fubini)}\\
&= \hat{f}(n) - \sum_{m=-\infty}^{\infty}\hat{f}(m)\cdot\langle e_m, e_n\rangle\\
&= \hat{f}(n) - \sum_{m=-\infty}^{\infty}\hat{f}(m)\cdot\delta_{m,n}\\
&= 0.
\end{aligned}$$

The lemma implies that g is 2π-periodic as well. Then, $f-g$ is continuous and 2π-periodic, with all zero Fourier coefficients. Therefore, by Theorem 2.10, $f-g$ is identically zero. Therefore, f coincides with its Fourier series g. □

Here is a non-trivial application of Fourier series.

Proposition 2.13. $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Proof. Extend $\tilde{f}(x) = |x|$ for $x \in [-\pi, \pi]$ to a 2π-periodic function $f\colon \mathbb{R} \to \mathbb{R}$. Then, f is continuous. Observe that for all non-zero $n \in \mathbb{Z}$,

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi} \int_0^\pi \left(f(x) \cdot \mathrm{e}^{-\mathrm{i}nx} + f(-x)\, \mathrm{e}^{\mathrm{i}nx} \right) \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\pi x \,\mathrm{d}\left(\frac{1}{n} \sin nx \right) \\
&= \frac{1}{\pi n} \left(\cancel{x \sin nx\Big|_{x=0}^{\pi}} - \int_0^\pi \sin nx \,\mathrm{d}x \right) \\
&= -\frac{1}{\pi n} \int_0^\pi \mathrm{d}\left(-\frac{1}{n} \cos nx \right) \\
&= \frac{1}{\pi n^2} (\cos n\pi - 1).
\end{aligned}$$

It is obvious that $\hat{f}(0) = 1/2\pi \cdot 2 \cdot (1/2 \cdot \pi \cdot \pi) = \pi/2$.

Then,

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cdot \mathrm{e}^{\mathrm{i}nx} && ((-1)^n - 1 = -2 \cdot \mathbb{I}[2 \nmid n]) \\
&= \frac{\pi}{2} - \sum_{n=1,3,\ldots} \frac{2}{\pi n^2} \cdot (\mathrm{e}^{\mathrm{i}nx} + \mathrm{e}^{-\mathrm{i}nx}) \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,\ldots} \frac{\cos nx}{n^2}
\end{aligned}$$

In particular, $f(0) = 0$ implies that $\sum_{k=1}^{\infty} 1/(2k-1)^2 = (\pi/2)/(4/\pi) = \pi^2/8$. Now observe that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2} = (\pi^2/8)/(1 - 1/4) = \pi^2/6$. ◻