Pricing Options with Mathematical Models

8. Model independent pricing relations: forwards, futures and swaps

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Pricing forward contracts

initial time

- Consider a forward contract on asset S, starting at t, with payoff at T equal to

  S(T) – F(t)

- Here, F(t) is the **forward price**, decided at t and paid at T.
- QUESTION: What is the value of F(t) that makes time t value of the contract equal to zero?
- Suppose $1.00 invested/borrowed at risk-free rate at time t results in payoff $B(t,T) at time T.

risk free asset Bond $= e^{r(T-t)}$

- CLAIM: There is no arbitrage if and only if

sell the asset today

$$F(t) = B(t,T)S(t),$$ and put in bank.

that is, if the forward price is equal to the time T value of one share worth invested at the risk-free rate at time t.

- Suppose first $F(t) > B(t,T)S(t)$:

 - At t: borrow S(t) to buy one share, and go short in the forward contract;

- At T: deliver the share, receive $F(t)$, which is more than enough to cover the debt of $B(t,T)S(t)$. Arbitrage!

simple.

- Suppose now $F(t) < B(t,T)S(t)$:

 - At t: sell short one share, invest S(t) risk-free, long the forward contract.

- At T: have more than enough in savings to pay for $F(t)$ for one share and close the short position.

- Suppose S pays deterministic dividends between t and T, with present value $D(t)$. Then,

$$F(\mathrm{t}) = \mathrm{B}(\mathrm{t},\mathrm{T})(\mathrm{S}(\mathrm{t}) - \mathrm{D}(\mathrm{t}))\ .$$

as we miss the dividends on stock.

- Suppose dividends will be paid between t and T, continuously at a constant rate q. Then,

$$F(\mathrm{t}) = \mathrm{B}(\mathrm{t},\mathrm{T})\, e^{-q(T-t)}\, \mathrm{S}(\mathrm{t})\ .$$

EXAMPLE: S(t)=100, a dividend of 5.65 is paid in 6 months. The 1-year continuous interest rate is 10%, and the 6-month continuous annualized rate is 7.41%.The price of the 1-year forward contract should be

$$F(\mathrm{t}) = e^{0.1}\,(100 - e^{-0.0741/2} \cdot 5.65) = 104.5$$

Suppose that the price is instead $F(\mathrm{t}) = 104$.

- At t: long the forward, sell one share, buy the 6-month bond in the amount of $e^{-0.0741/2} \cdot 5.65 = 5.4445$; invest the remaining balance, 100-5.4445=94.5555, in the 1-year bond.

- At 6 months from t : receive 5.65 from the 6-month bond and pay the dividend of 5.65.

- At 1 year from t : receive $e^{0.1} \cdot 94.5555 = 104.5$ from the 1-year bond; pay $F(\mathrm{t}) = 104$ for one share, and deliver the share to cover the short position; keep 104.5-104 = 0.5, as profit. Arbitrage!

risky (for me)

EXAMPLE: **Forward contract on foreign currency.** Let S(t) denote the current price in dollars of one unit of the foreign currency. We denote by r_f (r) the foreign (domestic) risk-free rate, with continuous compounding. The foreign interest is equivalent to continuously paid dividends, so we guess that

$$F(\mathrm{t}) = \underbrace{e^{(r- r_f)(T-t)}}_{\text{Dividend}} \mathrm{S(t)} \,.$$

If, for example, $F(\mathrm{t}) < e^{(r- r_f)(T-t)} \mathrm{S(t)}$:

- At time t: long the forward, borrow $e^{- r_f(T-t)}$ units of foreign currency and invest its value in dollars $e^{- r_f(T-t)} \mathrm{S(t)}$ at rate r.

- At time T: use part of the amount $e^{(r- r_f)(T-t)} \mathrm{S(t)} > F(t)$ from the domestic risk-free investment to pay $F(t)$ for one unit of foreign currency in the forward contract, and deliver that unit to cover the foreign debt. There is still extra money left. Arbitrage! Similarly if $e^{(r- r_f)(T-t)} \mathrm{S(t)} < F(t)$.

Futures

- Main difference relative to forwards: **marked to market** daily.
- The daily profit/loss is deposited to/taken out of the **margin account**:

Total profit/loss for a contract starting at t, ignoring the margin interest rate, using F(T)=S(T),

cancellation.

= [F(t+1)-F(~~t~~)] + … + [F(~~T~~)-F(T-1)] = $\boxed{\text{S(T)-F(t)}}$

- CLAIM: If the interest rate is deterministic, futures price F(t) is equal to the corresponding forward price.
- REPLICATION: At t=0, go long $e^{-r(T-1)}$ futures; at t=1, increase to $e^{-r(T-2)}$ futures, …, at t=T-1, increase to 1 future contract.

Profit/loss in period (k,k+1) = $[F(k+1)-F(k)]e^{-r(T-(k+1))}$, the time T value of which is= $[F(k+1)-F(k)]$. Thus, time T profit/loss is S(T)-F(0), the same as for a forward contract.

Swaps pricing

- The payoff of the party receiving the floating rate L and paying the fixed rate R at time T_i is

$$C_i := \Delta T[L(T_{i-1}, T_i) - R]$$

- LIBOR rate is, by definition,

$$L(T_{i-1}, T_i) := \frac{1 - P(T_{i-1}, T_i)}{\Delta T P(T_{i-1}, T_i)} \quad ,$$

implying

$$C_i = \frac{1}{P(T_{i-1}, T_i)} - (1 + R\Delta T)$$

- The value at time $t < T_0$ of the constant amount $(1 + R\Delta T)$ paid at time $T_i > t$ is $(1 + R\Delta T)P(t, T_i)$. (Why?)

Swaps pricing (continued)

- As for the first term, we claim that the value at time $t < T_0$ of the payoff $1/P(T_{i-1}, T_i)$ paid at time T_i is equal to $P(t, T_{i-1})$. Indeed, if we invest $P(t, T_{i-1})$ at time t in buying a bond with maturity T_{i-1}, we get 1 dollar at time T_{i-1}, with which we can buy exactly $1/P(T_{i-1}, T_i)$ of bonds with maturity T_i, and hence collect $1/P(T_{i-1}, T_i)$ at time T_i. Altogether, the price at time t is

$$C_i(t) = P(t, T_{i-1}) - (1 + R\Delta T)P(t, T_i)$$

Therefore, the price $S(t)$ of the swap at time t is

$$S(t) = \sum_{i=1}^{n} C_i(t) = \sum_{i=1}^{n} [P(t, T_{i-1}) - (1 + R\Delta T)P(t, T_i)]$$

$$S(t) = P(t, T_0) - R\Delta T \sum_{i=1}^{n} P(t, T_i) - P(t, T_n)$$

Swaps pricing (continued)

$$S(t) = P(t, T_0) - R\Delta T \sum_{i=1}^{n} P(t, T_i) - P(t, T_n) \quad .$$

- The **swap rate** R is the fixed rate such that the cost of entering the swap at the initial time 0 is equal to zero. We get

$$R = \frac{P(0, T_0) - P(0, T_n)}{\Delta T \sum_{i=1}^{n} P(0, T_i)}$$

Example

- Consider a swap investor who receives a fixed rate of 10% in semiannually paid coupons and pays the six-month LIBOR. The swap still has 9 months left to maturity. At the last resetting date, the six-month LIBOR was 6%. The continuous three-month and nine-month rates are 5% and 7%, respectively. Nominal principal is 10,000. In the above notation, T_0 is 3 months in the past, T_1 is 3 months from now, and T_2 is 9 months from now, $\Delta T = 0.5$. We want to find the value of the investor's position today.

- The payoff 3 months from now, on one unit of the notional principal, is

$$C_1 = \frac{1}{P(T_0, T_1)} - (1 + R\Delta T)$$

and 9 months from now it is

$$C_2 = \frac{1}{P(T_1, T_2)} - (1 + R\Delta T)$$

Example (continued)

- The first payoff's price is obtained by multiplying it by the price of the 3-month bond, and is equal to

$$C_1(t) = \left[\frac{1}{P(T_0,T_1)} - (1+R\Delta T)\right] e^{-0.05\cdot 0.25}$$

- The bond price $P = P(T_0,T_1)$ can be found from $P(1+\Delta T L) = 1$, where $L = 0.06$ is the LIBOR rate. We get $P = 1/1.03$, and $C_1(t) = -0.0198$.

- The price of C_2 is found to be

$$C_2(t) = P(t,T_1) - (1+R\Delta T)P(t,T_2) = e^{-0.05\cdot 0.25} - (1+0.1\cdot 0.5)e^{-0.07\cdot 0.75} = -0.0087$$

Example (continued)

- Altogether, the value of the swap for the short position is

$$C(t) = C_1(t) + C_2(t) = -0.0285$$

- Since our investor is long the swap, his value is

$$10,000 \cdot 0.0285 = 285$$

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9. Model independent pricing relations: options

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- Notation:

- European call and put prices at time t, $c(t), p(t)$
- American call and put prices at time t, $C(t), P(t)$

- RELATION 1: $c(t) \leq C(t) \leq S(t)$
- RELATION 2: $p(t) \leq P(t) \leq K$
- RELATION 3: $p(t) \leq Ke^{-r(T-t)}$
- RELATION 4: $c(t) \geq S(t) - Ke^{-r(T-t)}$, if S pays no dividends

- Suppose not: $c(t) + Ke^{-r(T-t)} < S(t)$; sell short one share and have more than enough money to buy one call and invest $Ke^{-r(T-t)}$ at rate r.

At T: If $S(T) > K$, exercise the option by buying S(T) for K;

If $S(T) \leq K$, buy stock from your invested cash.

- RELATION 5: $p(t) \geq Ke^{-r(T-t)} - S(t)$

- RELATION 6: $c(t) = C(t)$, if S pays no dividends.

- Suppose not: $c(t) < C(t)$

1. At t : Sell $C(t)$ and have more than enough to buy c(t);
2. If C exercised at τ : have to pay $S(\tau) - K$, which is possible by selling c, because $c(t) \geq S(t) - Ke^{-r(T-t)} \geq S(t) - K$;
3. If C never exercised, there is no obligation to cover.

Arbitrage!

- COROLLARY: An American call on an asset that pays no dividends should not be exercised early.

- Indeed, it is better to sell it than to exercise it: $C(t) \geq S(t) - K$.

- What if there are dividends?
- What about the American put option?

- RELATION 7**, Put-Call Parity**:

$$c(t) + Ke^{-r(T-t)} = p(t) + S(t)$$

1. Portfolio A: buy c(t) and invest discounted K at risk-free rate;
2. Portfolio B: buy put and one share.

- If $S(T) > K$, both portfolios worth $S(T)$ at time T.
- If $S(T) \leq K$, both portfolios worth K at time T.

- RELATION 8:

$$\mathrm{S(t) - K} \leq C(t) - P(t) \leq S(t) - Ke^{-r(T-t)}$$

- The RH side follows from put-call parity and $P(t) \geq p(t), C(t) = c(t)$.
- For the LHS, suppose not: $\mathrm{S(t) + P(t)} > C(t) + K$.

1. At t : Sell the LHS and have more than enough to buy the RHS;
2. If P exercised at τ : use the invested cash to pay K for $S(\tau)$;
3. If P never exercised, exercise C at maturity.