Pricing Options with Mathematical Models

6. Pricing deterministic payoffs

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Future value with constant interest rate

- Suppose you can lend money at annual interest rate r, so that

Present Value (PV)=\$1.00 $\rightarrow$ Future Value (FV) after 1 year=\$$(1+r)$

- Different conventions:

  - Simple interest: after T years, FV $= 1 + T \times r$
  - Interest compounded once a year: after T years, FV $= (1+r)^T$
  - Interest compounded n times a year: after m compounding periods,

$$\text{FV} = (1 + r/n)^m$$

- **Effective annual interest rate** r':

$$(1 + r/n)^n = 1 + r'$$

EXAMPLE: Quarterly compounding at **nominal** annual rate r = 8%,

$$(1 + 0.08/4)^4 = 1.0824 = 1 + 0.0824$$

Thus, the effective annual interest rate is 8.24%.

- **Continuous compounding**:

  - after one year, FV $= \lim_{n \to \infty} (1 + r/n)^n$ = e^r , e = 2.718 …
  - after T years, FV $= \lim_{n \to \infty} (1 + r/n)^{nT}$ = e^{rT}

EXAMPLE: r = 8%, e^r = 1.0833, r' = 8.33%

Price as Present Value

- **LAW OF ONE PRICE**: If two cash flows deliver the same payments in the future, they have the same price (value) today.
- **PRICE DEFINITION**: If one can guarantee having \$X(T) at time T by investing \$X(0) today, then, today's price of X(T) is X(0).
- For deterministic X(T), X(0) is called the **present value**, PV(X(T)).
- Thus, if one can invest at compounded rate of r/n, and T=m periods,

$$X(0) = PV\big(X(T)\big) = X(T)/(1 + r/n)^m$$

because this is equivalent to $X(0)(1 + r/n)^m = X(T)$.

- **Discount factor**: $1/(1 + r/n)^m$, e^{-rT}

sequence of payments

PV of cash flows

- Cash flow X(0), X(1), X(2), …, X(m), paid at compounding intervals:

$$PV = X(0) + \frac{X(1)}{\left(1+\frac{r}{n}\right)} + \frac{X(2)}{\left(1+\frac{r}{n}\right)^2} + \ldots + \frac{X(m)}{\left(1+\frac{r}{n}\right)^m}$$

ex. EMI

- When X(0)=0 and X(i) = X, we have a geometric series:
- $PV = X \cdot \left(\underbrace{\frac{1}{\left(1+\frac{r}{n}\right)} + \frac{1}{\left(1+\frac{r}{n}\right)^2} + \ldots + \frac{1}{\left(1+\frac{r}{n}\right)^m}}\right)$

$$= X \cdot \frac{1}{r/n} \cdot \left(1 - \frac{1}{\left(1+\frac{r}{n}\right)^m}\right)$$

Example

- We want to estimate the value of leasing a gold mine for 10 years. It is estimated that the mine will produce 10,000 ounces of gold per year, at a cost of \$200 per ounce, and that the gold will sell for \$400 per ounce. We also estimate that, if not invested in the mine, we could invest elsewhere at r=10% return per year.

Annual profit = 10,000 (400-200) = 2 million \$

$$PV = \sum_{k=1}^{10} \frac{2\ mil}{(1+0.1)^k} = 2 \cdot 10 \cdot \left(1 - \frac{1}{(1+0.1)^{10}}\right) = \boxed{12.29\text{ mil}}$$

Loan payments

averaged

- Suppose you take a loan with value PV = \$V, and the loan is supposed to be paid off (amortized) in equal amounts X over m periods at interest rate $\frac{r}{n}$.
- Inverting the PV formula, we get

amt. to be paid every period.

$$X = V \cdot \frac{\frac{r}{n}\left(1+\frac{r}{n}\right)^{m}}{\left(1+\frac{r}{n}\right)^{m}-1}$$

EXAMPLE 1. You take a 30-year loan on \$400,000, at annual rate of 8%, compounded monthly. What is the amount X of your monthly payments?

With 12 months in a year, the number of periods is $m=30 \cdot 12=360$. The rate per period is $0.08/12=0.0067$. The value of the loan is $V=400{,}000$. We compute X= \$2,946 in monthly payments, approximately.

The loan balance is computed as follows:

- Before the end of the first month, balance $=400{,}000+0.0067 \cdot 400{,}000=402{,}680$
- After the first installment of \$2,946 is paid, balance = 402,680-2,946=399,734.
- Before the end of the second month, balance = 399,734(1+0.0067), and so on.

The future value corresponding to these payments thirty years from now is

$$400{,}000 \cdot (1 + 0.0067)^{360} = 4{,}426{,}747 .$$

EXAMPLE 2 (Loan fees). For mortgage products, there are usually two rates listed: the mortgage interest rate and the APR, or annual percentage rate. The latter rate includes the fees added to the loan amount and also paid through the monthly installments. Consider the previous example with the rate of 7.8% compounded monthly, and the APR of 8.00%. As computed above, the monthly payment at this APR is 2,946. Now, we use this monthly payment of 2,946 and the rate of 7.8/12=0.65% in the formula for V, to find that the total balance actually being paid is 407,851.10. This means that the total fees equal

$$407{,}851.10 - 400{,}000 = 7{,}851.10 \quad .$$

Perpetual annuity

- Pays amount X at the end of each period, for ever. If the interest rate per period is r,

$$\text{PV} = \sum_{k=1}^{\infty} \frac{X}{(1+r)^k} = \frac{X}{r}$$

Internal rate of return

- The rate r for which

$$0 = X(0) + \frac{X(1)}{\left(1+\frac{r}{n}\right)} + \frac{X(2)}{\left(1+\frac{r}{n}\right)^2} + \ldots + \frac{X(m)}{\left(1+\frac{r}{n}\right)^m}$$

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7. Bonds

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Bond yield

- **Yield to maturity** (YTM) of a bond is the internal rate of return of the bond, or the rate that makes the bond price equal to the present value of its future payments.
- Suppose the bond pays a **face value** V at maturity $T = m$ periods and n identical coupons a year in the amount of C/n, and its price today is P. Then, the bond's annualized yield corresponding to compounding n times a year is the value y that satisfies

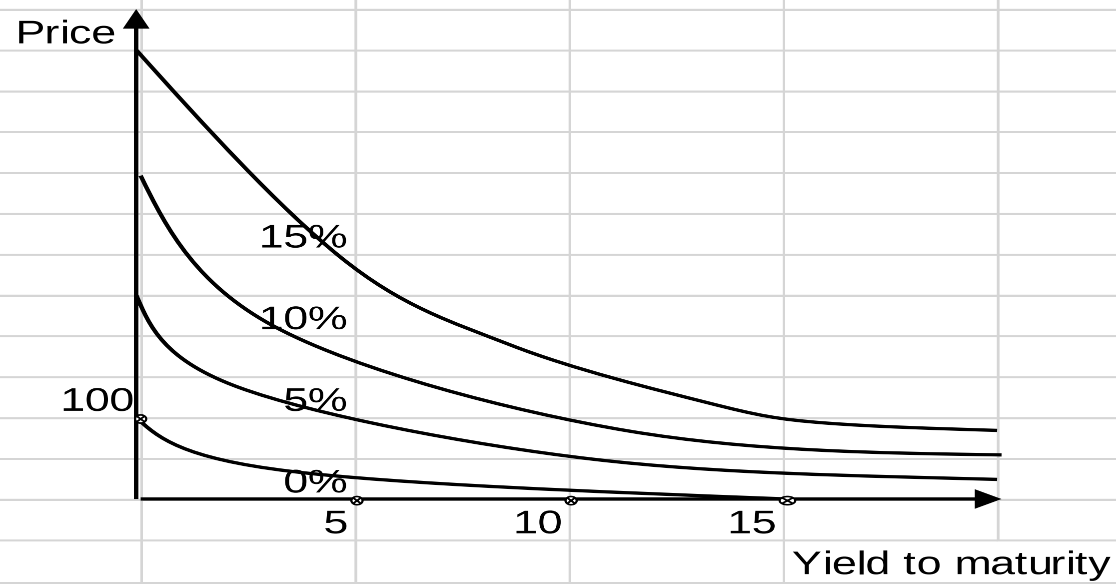
$$P = \overbrace{\frac{V}{\left(1+\frac{y}{n}\right)^m}}^{\text{face value}} + \overbrace{\sum_{k=1}^{m} \frac{C/n}{\left(1+\frac{y}{n}\right)^k}}^{\text{coupons}} = \frac{V}{\left(1+\frac{y}{n}\right)^m} + \frac{C}{y} \cdot \left(1 - \frac{1}{\left(1+\frac{y}{n}\right)^m} \right)$$

non linear, numerically solved.

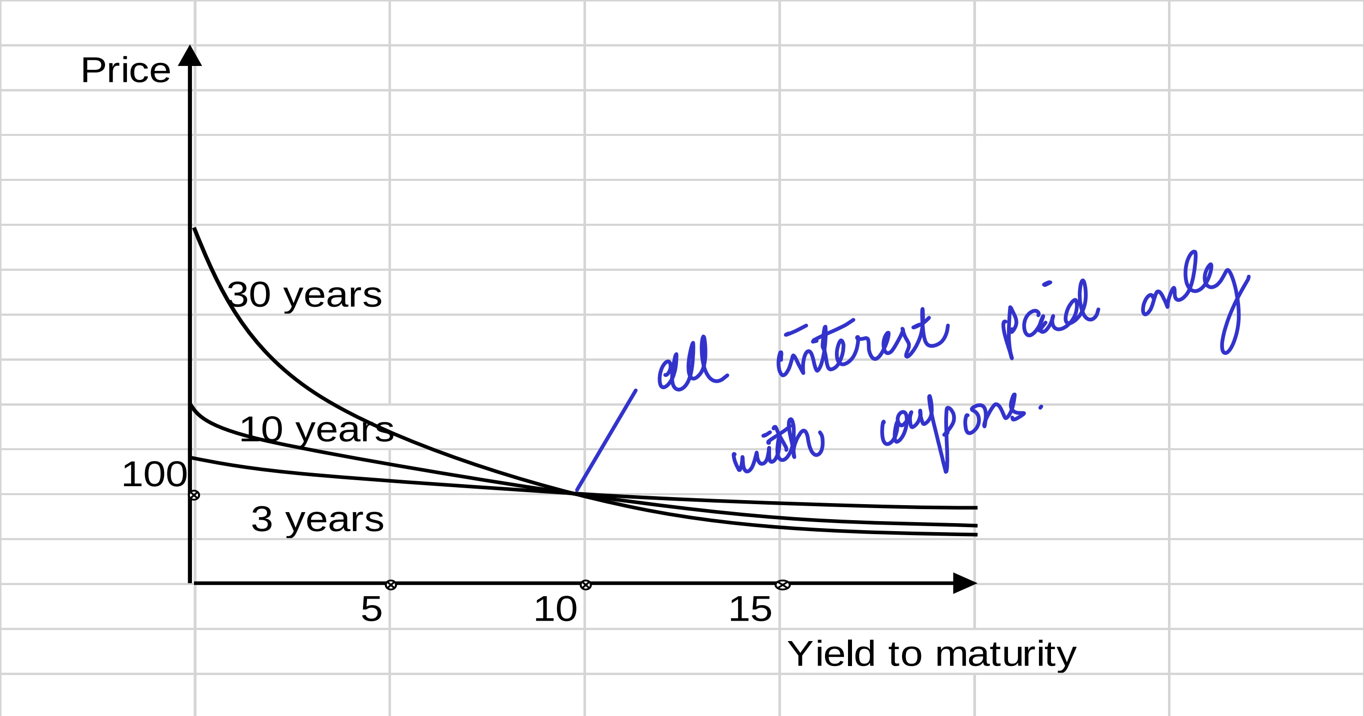
- Higher price corresponds to lower yield.

Price-yield curve

- Terminology: ‘a 10% five-year bond’ is a bond that pays 10% of its face value per year, for five years, plus the face value at maturity.



Price yield curves for 30-year bonds with various coupon rates

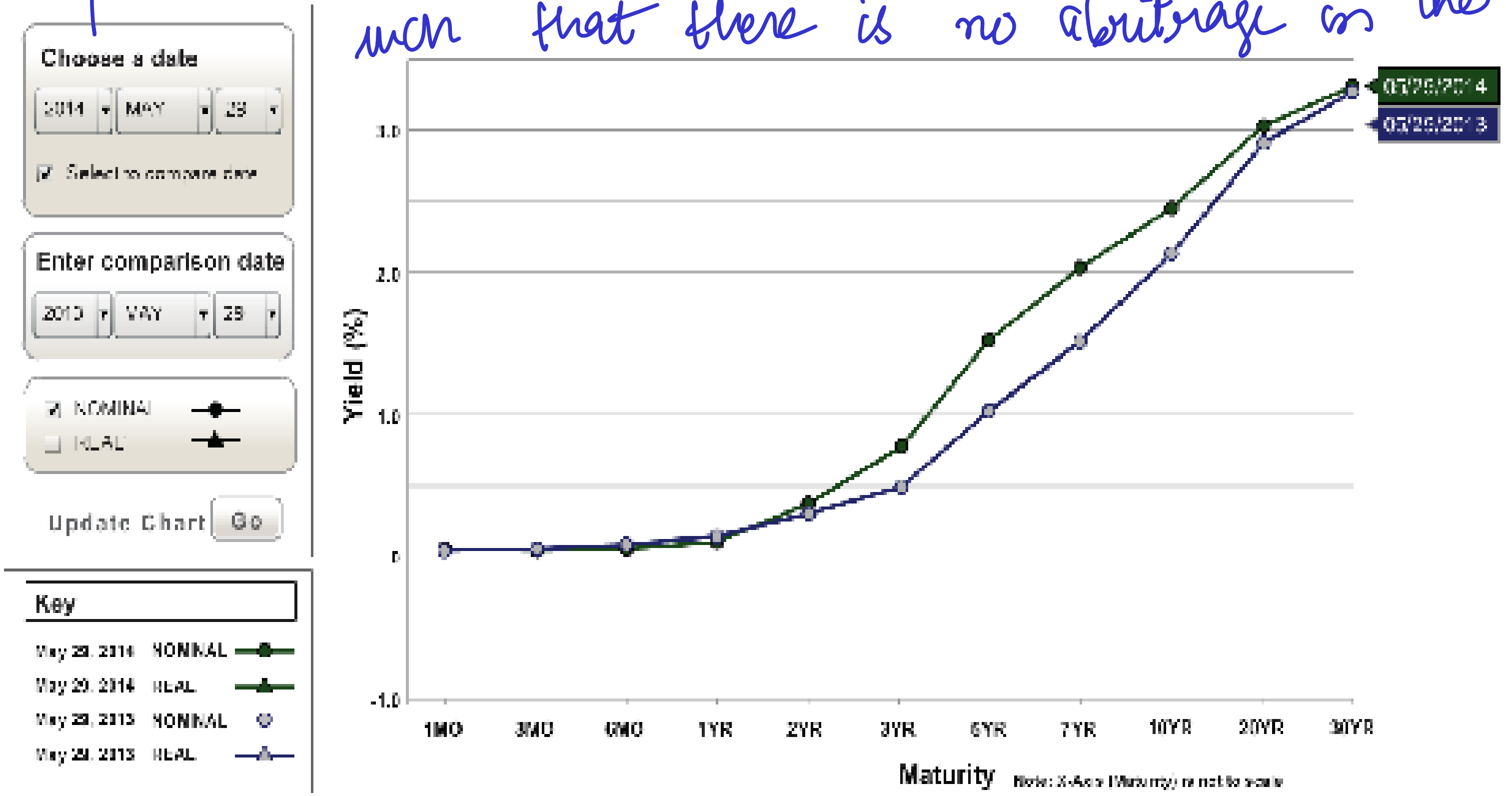


Price yield curves for 10% coupon bonds with various maturities

- Why do they all intersect at the point (10, 100) ?
- Hint: Set $C = y \cdot V$ in the formula for P. We say 'the bond trades at par'.

Yield curve (term structure of interest rates)

further we have to model these graphs moving in time s.t. that there is no arbitrage in the market.



Spot rates and arbitrage

- Spot rate = yield of a **zero-coupon bond** (pure discount bond)
- **Arbitrage** (of strong kind) = making positive sure profit with zero investment
- EXAMPLE: A 6-month zero-coupon bond with face value 100 trades at 98.00. A coupon bond that pays 3.00 in 6 months and 103 in 12 months trades at 101.505. What should be the yield of the 1-year zero coupon bond with face value 100?
- REPLICATION: Find a combination of the traded bonds to replicate exactly the payoff of the 1-year bond.
- BUY: one coupon bond; SELL (short): 0.03 units of the 6-month bond
- COST = 101.505-0.03·98 = 98.565
- In 6 months: pay 3.00 for the short bond, receive 3.00 as a coupon
- In 12 months: receive 103.00
- $98.565 = \frac{103}{1+r}$, $\boxed{r = 4.4996\ \%}$; Otherwise arbitrage!

Alternative computation

- First, compute the 6-month spot rate:

$$98 = \frac{100}{(1+y)^{1/2}} \rightarrow y = 4.1233\%$$

- Then, compute the one-year rate from

coupon. fixed. as 1 year.

$$101.505 = \overbrace{\frac{3}{(1+0.041233)^{1/2}}}^{\text{coupon.}} + \frac{\overbrace{103}^{\text{fixed.}}}{\boxed{1+r}} \rightarrow r = 4.4996\ \%$$

1/2 year.

Arbitrage if mispriced

- Suppose the 1-year bond price is 95.00 instead of

$$\frac{100}{1+\underline{0.044996}} = 95.6942$$

get

r

- BUY CHEAP, SELL EXPENSIVE:
  - buy the 1-year bond
  - go short 100/103 of the portfolio that $\boxed{\text{replicates}}$ it:

    sell short 100/103 units of the coupon bond;

    buy 0.03·100/103 units of the 6-month bond.

- this results in initial profit of *95.6942*

coupon

from 6-month.

- After 6 months: have to pay 3·100/103, and receive the same amount
- After 1 year: have to pay 100 and receive 100 → from

to coupon bond.

- Total profit: *95.6942 – 95.00* → arbitrage!

Forward rates

- r_k = annualized spot rate for k periods from now
- Annualized forward rate between the i-th and j-th period, compounding n times a year:

bond j   bond i   j>i

$$\left(1 + r_j/n\right)^j = \left(1 + r_i/n\right)^i \left(1 + f_{i,j}/n\right)^{j-i}$$

- EXAMPLE: The 1-year zero c. bond trades at 95, and the 2-year z.c. bond trades at 89, compounding done once a year.

$95 \cdot (1 + r_1) = 100 \quad \rightarrow \quad r_1 = 5.2632\%$

$89 \cdot (1 + r_2)^2 = 100 \quad \rightarrow \quad r_2 = 5.9998\%$

$1.052632 \cdot \left(1 + f_{1,2}\right) = (1.059998)^2 \quad \rightarrow \quad f_{1,2} = 6.7416\%$

- Suppose you believe $f_{1,2}$ is too high:

  \- buy one 2-year bond and sell short 89/95 units of the 1-year bond, at zero cost.
- After 1 year: have to pay 89/95·100 = 93.6842
- After 2 years: receive 100 for the second year return of 6.7416%
- If, after 1 year, the 1-year spot rate is, indeed, less than 6.7416%, sell the 1-year bond and receive more than 93.6842, and make ~~arbitrage~~ profit.

forward rate: by trading multiple bonds we make as if we invest some money in the future and get some interest on it after some time. This interest rate is called forward rate.