

Translating priors from linear mixed models to repeated-measures ANOVA and paired t tests

Supplement to *Bayes Factors for Mixed Models: Perspective on Responses*

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In van Doorn et al. (2021), we give Example 1 in which Bayes factors in mixed models with standardized effect size parameterization (Rouder et al. 2012) change when error variance decreases. We assumed a simple repeated-measures design with I participants responding K times in each of two conditions. The maximal model for these data is our Model 6 with random intercepts α_i and random slopes θ_i ,

$$y_{ijk} \sim \mathcal{N}(\mu + \sigma_\epsilon(\alpha_i + x_j(\nu + \theta_i)), \sigma_\epsilon^2)$$

$$\alpha_i \sim \mathcal{N}(0, g_\alpha)$$

$$\nu \sim \mathcal{N}(0, g_\nu)$$

$$\theta_i \sim \mathcal{N}(0, g_\theta)$$

$$g_\alpha \sim \mathcal{IG}(0.5, 0.5 r_\alpha^2)$$

$$g_\nu \sim \mathcal{IG}(0.5, 0.5 r_\nu^2)$$

$$g_\theta \sim \mathcal{IG}(0.5, 0.5 r_\theta^2)$$

$$(\mu, \sigma_\epsilon^2) = 1/\sigma_\epsilon^2.$$

1 Reduced error variance through aggregation

Because priors are placed on standardized effect sizes, a reduction of σ_ϵ increases prior plausibility of larger effect sizes. In our Example 1, measurement error decreases as the number of aggregated trials n increases, $\sigma'_\epsilon = \frac{\sigma_\epsilon}{\sqrt{n}}$,

$$y_{ijk} \sim \mathcal{N}(\mu + \sigma'_\epsilon(\alpha_i + x_j(\nu + \theta_i)), \sigma'^2_\epsilon)$$

$$\alpha_i \sim \mathcal{N}(0, g_\alpha)$$

$$\nu \sim \mathcal{N}(0, g_\nu)$$

$$\theta_i \sim \mathcal{N}(0, g_\theta).$$

This further implies the priors

$$\nu \sim \mathcal{N}(0, g_\nu/\sqrt{n})$$

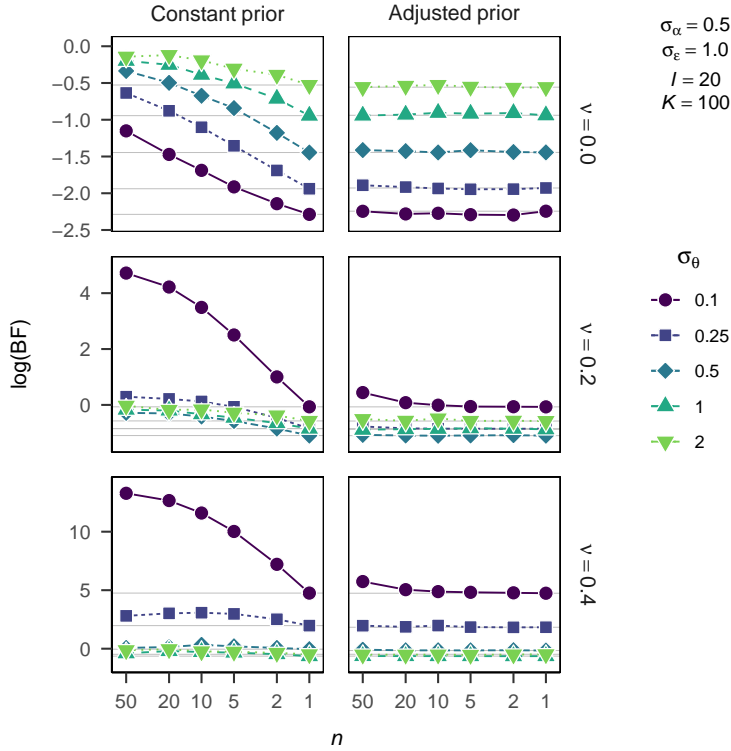
$$g_\alpha \sim \mathcal{IG}(0.5, 0.5 r_\alpha^2/\sqrt{n})$$

$$g_\theta \sim \mathcal{IG}(0.5, 0.5 r_\theta^2/\sqrt{n}).$$

Hence, to obtain equivalent Bayes factors the prior scales should be adjusted accordingly, $r'^2 = r^2\sqrt{n}$.

1.1 Simulation

To test whether this prior adjustment works as intended across all levels of aggregation, we conducted a small simulation for the balanced null comparison. We simulated $K = 100$ trials for $I = 20$ participants ($\mu = 1$; $\sigma_\alpha = 0.5$; $\nu = \{0, 0.2, 0.4\}$; $\sigma_\theta = \{0.1, 0.25, 0.5, 1, 2\}$; $\sigma_\epsilon = 1$). As in our Example 1, the data were generated deterministically with identical condition and participant means as well as standard errors across all levels of aggregation (n).



Horizontal lines represent logBF for each level of σ_θ with $n = 1$ (no aggregation) as reference. The results confirm that the prior adjustment works well. Only when an effect is present and the random slope variance σ_θ^2 is small, we observed a minor inflation of the Bayes factor for $n = 50$. This bias scaled with $\log BF$ and was negligible for small and inconsequential for large Bayes factors.

2 Full aggregation

When aggregating each participant's data to a single observation per cell the data can be analyzed in two ways: By modeling participants' (1) cell means using a one-way repeated-measures ANOVA, or (2) cell mean differences using a paired t -test.

2.1 Repeated-measures ANOVA

When data are fully aggregated data (i.e., $n = K$), the random slopes variance σ_θ^2 folds into the error variance σ_ϵ^2 . That is, Model 6 reduces to Model 4,

$$\begin{aligned}\bar{y}_{ij.} &\sim \mathcal{N}(\mu + \sigma'_\epsilon(\alpha_i + x_j\nu), \sigma_\epsilon'^2 + \sigma_\theta^2/2) \\ \alpha_i &\sim \mathcal{N}(0, g_\alpha\sqrt{\sigma_\theta^2}) \\ \nu &\sim \mathcal{N}(0, g_\nu\sqrt{\sigma_\theta^2/2}).\end{aligned}$$

The random slopes variance σ_θ^2 is scaled by the orthonormal effect coding, $\pm\sqrt{2}/2$. Compared to partial aggregation, the adjustment for the fixed effect requires an additional factor that depends on a weighted ratio of random variance σ_θ^2 and error variance σ_ϵ^2 ,

$$\begin{aligned}\sqrt{\sigma_\epsilon^2/K + \sigma_\theta^2/2} &= \sigma_\epsilon/\sqrt{K}\sqrt{1 + \frac{K\sigma_\theta^2}{2\sigma_\epsilon^2}} \\ &= \sigma_\epsilon/\sqrt{K}\sqrt{\frac{2\sigma_\epsilon^2 + K\sigma_\theta^2}{2\sigma_\epsilon^2}} \\ &= \sigma_\epsilon/\sqrt{K}\sqrt{\frac{2/K + \sigma_\theta^2/\sigma_\epsilon^2}{2/K}}\end{aligned}$$

For random intercepts, the additional correction factor is obtained by marginalizing over the dummy coded random effect, yielding a weight of 1 for the random slope variance.

2.2 t Test

For the paired t -test the prior adjustment additionally must account for the different model parameterization (± 0.5 vs. $\pm\sqrt{2}/2$),

$$\begin{aligned}\bar{y}_{ij\cdot} - \bar{y}_{i\cdot} &\sim \mathcal{N}(\mu + \sigma'_\epsilon(\alpha_i + 0.5\nu), \sigma'^2_\epsilon) \\ &\quad - \mathcal{N}(\mu + \sigma'_\epsilon(\alpha_i - 0.5\nu), \sigma'^2_\epsilon) \\ &= \mathcal{N}(\sigma'_\epsilon\nu, 2\sigma'^2_\epsilon)\end{aligned}$$

Rescaling the prior on ν to the orthonormal scale, $\sqrt{2}\nu$, yields the same adjustment as for the ANOVA prior,

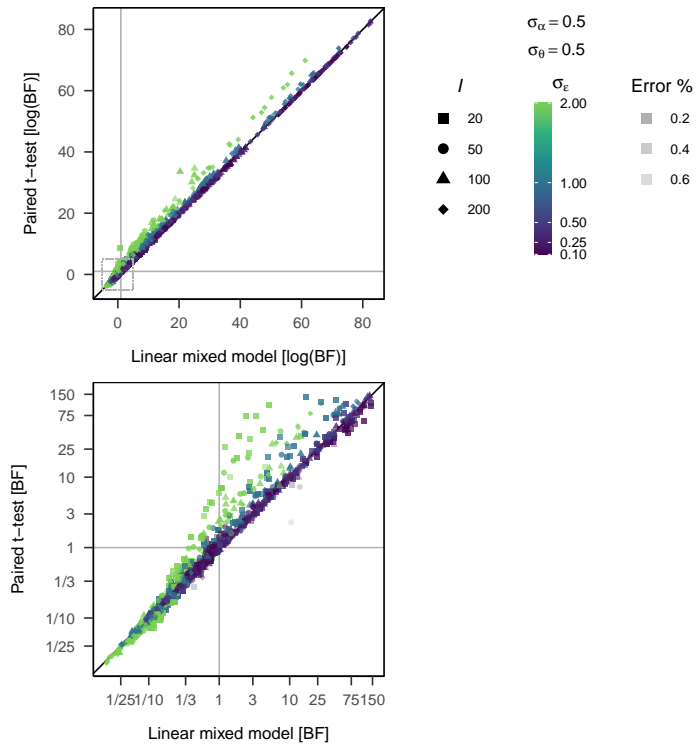
$$\begin{aligned}\sqrt{\sigma_\epsilon^2/K + \sigma_\theta^2/2} &= \sigma_\epsilon/\sqrt{K} \sqrt{1 + \frac{K\sigma_\theta^2}{2\sigma_\epsilon^2}} \\ &= \sigma_\epsilon/\sqrt{K} \sqrt{\frac{2\sigma_\epsilon^2 + K\sigma_\theta^2}{2\sigma_\epsilon^2}} \\ &= \sigma_\epsilon/\sqrt{K} \sqrt{\frac{2/K + \sigma_\theta^2/\sigma_\epsilon^2}{2/K}}.\end{aligned}$$

Note that in the linear mixed model σ_θ^2 is characterized by a probability distribution, which prohibits a translation of the prior in terms of an adjusted scaling factor. To test whether the adjustment can be approximated by using a point estimate, we conducted a simulation for balanced null comparisons.

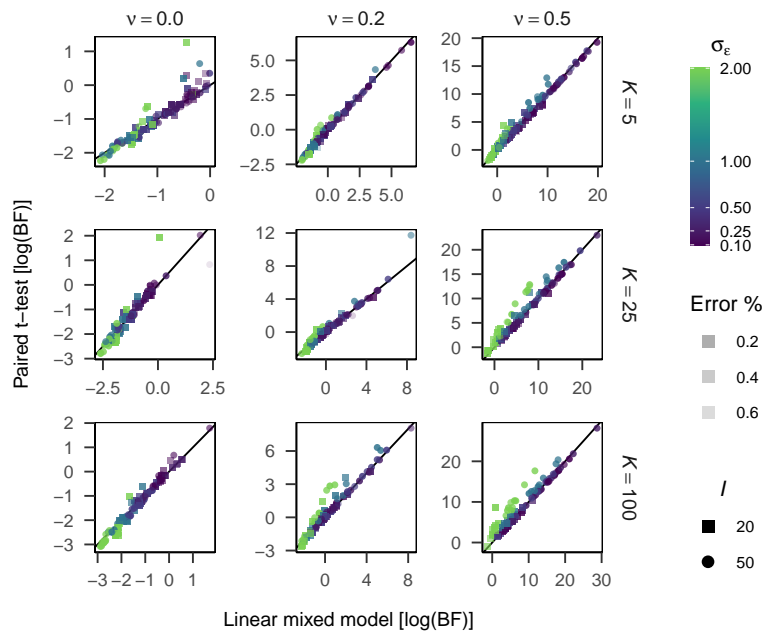
2.3 Simulation

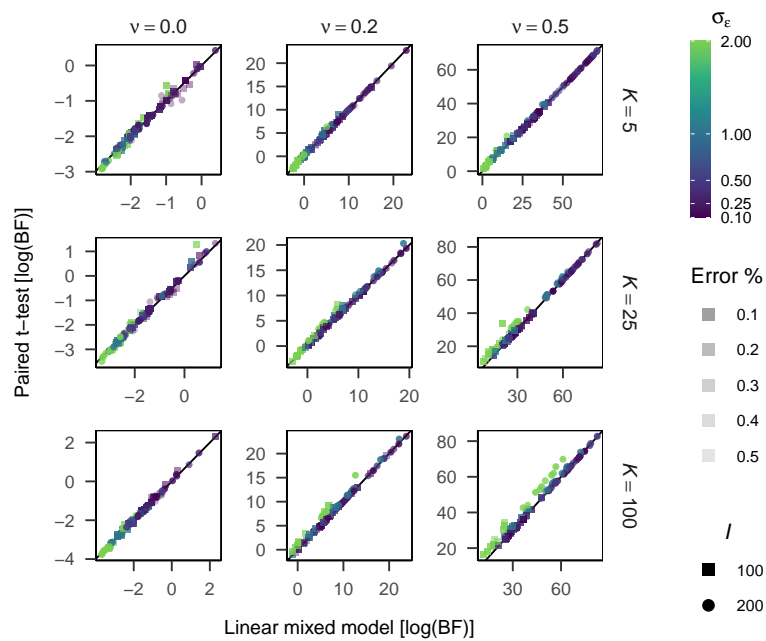
We randomly simulated $K = \{5, 25, 100\}$ responses for $I = \{20, 50, 100, 200\}$ participants ($\mu = 1$; $\sigma_\alpha = 0.5$; $\nu = \{0, 0.2, 0.5\}$; $\sigma_\theta = 0.5$; $\sigma_\epsilon = \{0.1, 0.25, 0.5, 1, 2\}$) 10 times each.

2.3.0.1 Mixed model vs. t-test

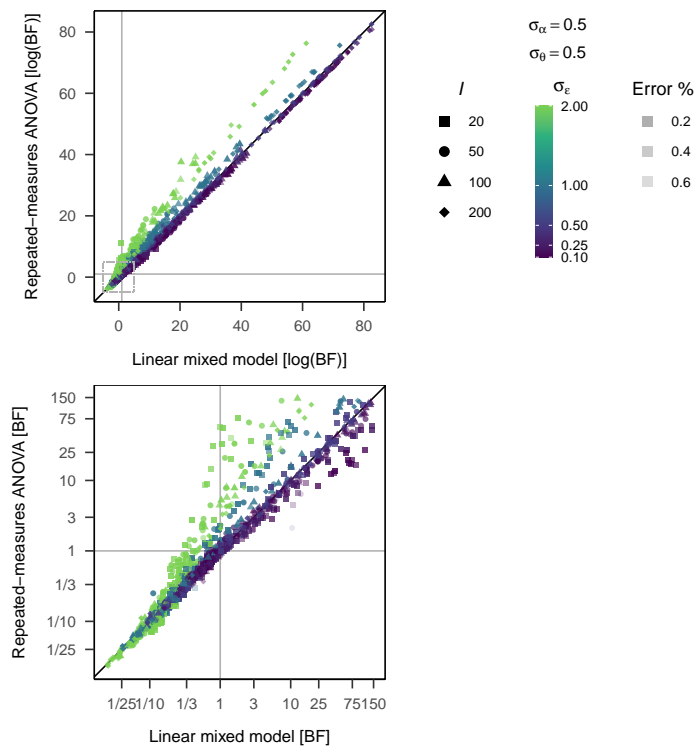


Results split by all varied factors

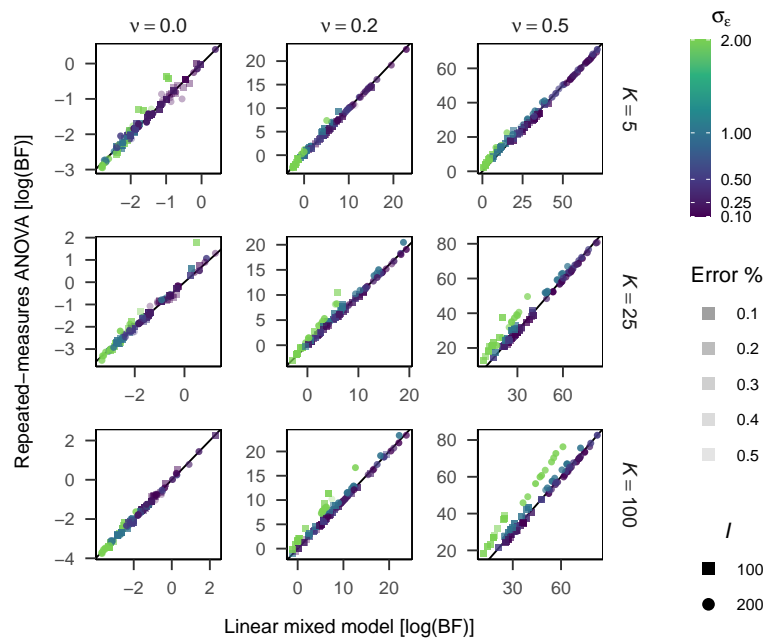
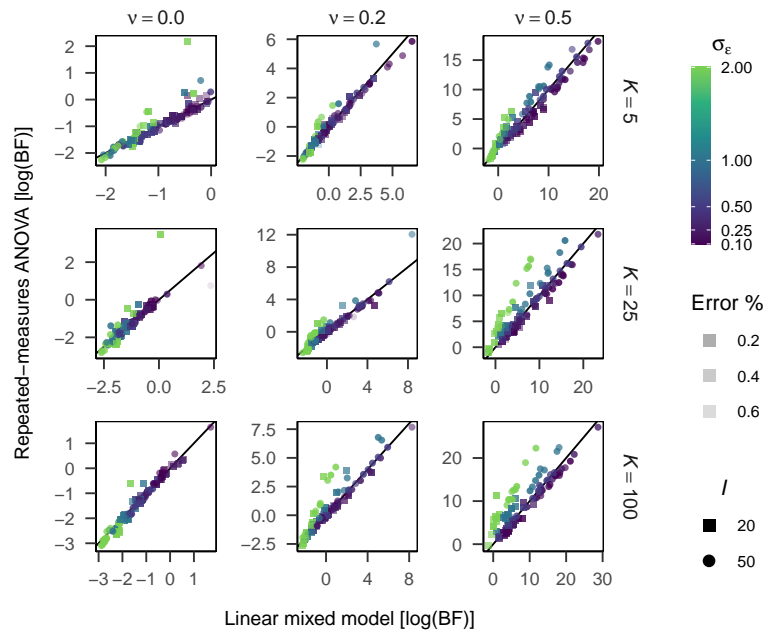




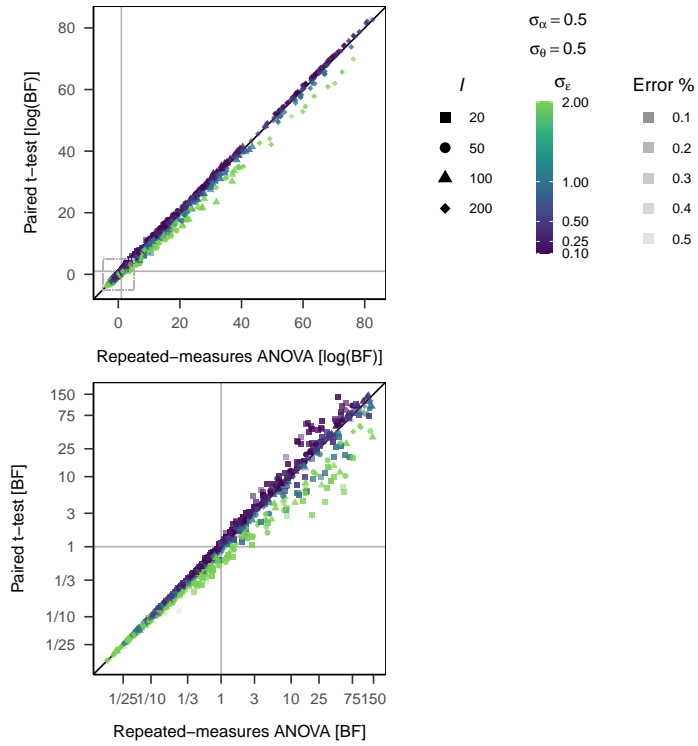
2.3.0.2 Mixed model vs. RM-ANOVA



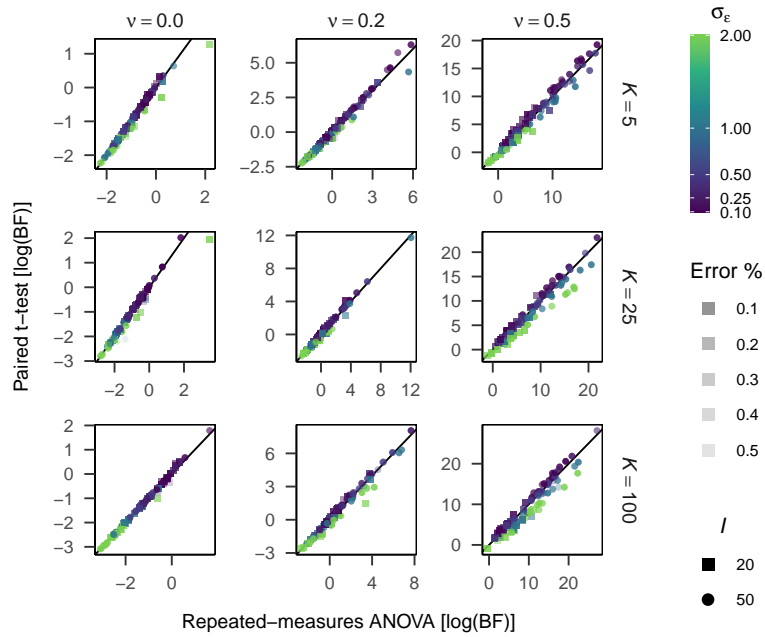
Results split by all varied factors

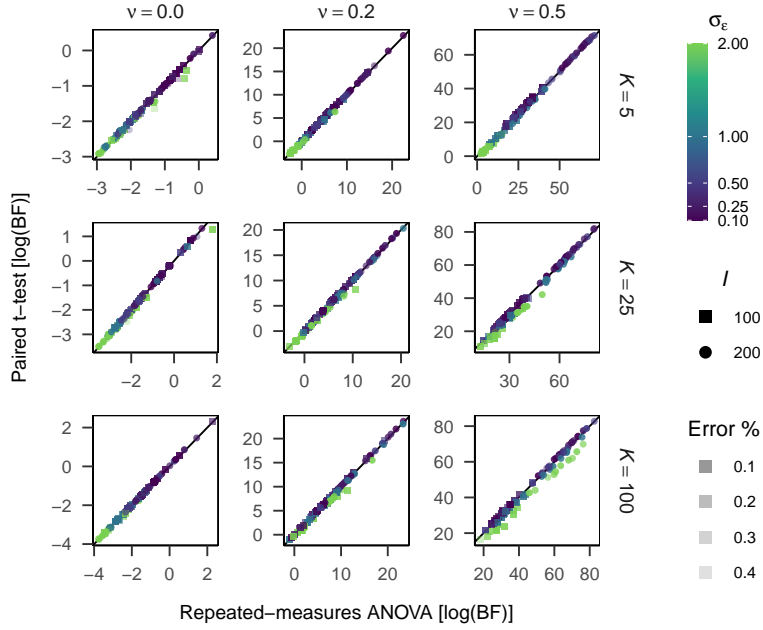


2.3.0.3 RM-ANOVA vs. t-test



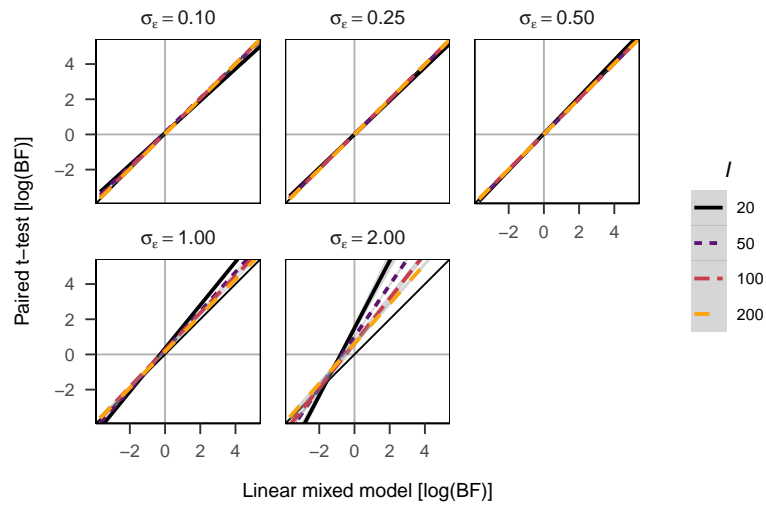
Results split by all varied factors



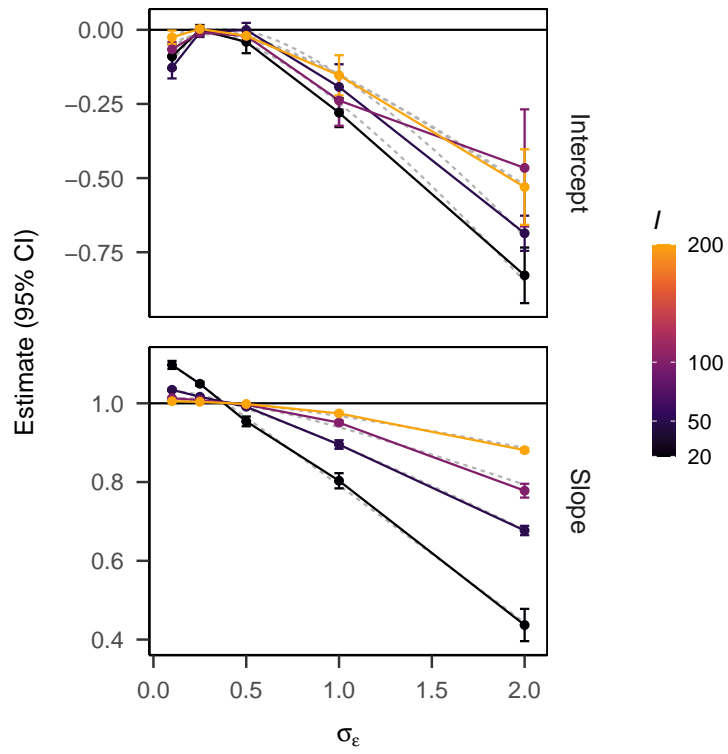


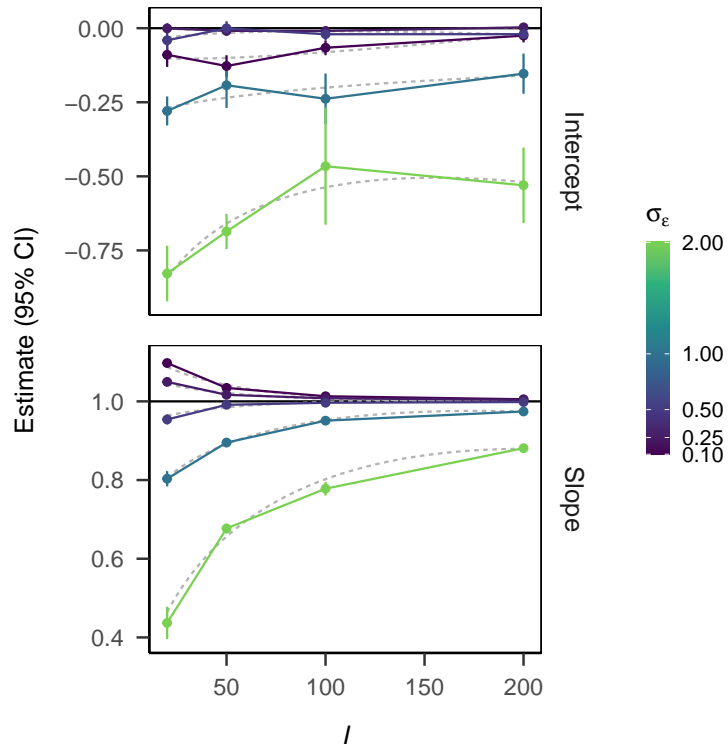
When the error variance σ_ϵ^2 is small or the sample size I is large, the adjustments works well. However, compared to the linear mixed model, both aggregate analyses produced diverging Bayes factors. As illustrated by the following trend lines, the divergence increased as (1) the difference in random slope and error variance increased and (2) the number of participants decreased. The divergence of Bayes factors was more pronounced in the repeated-measures ANOVA than in the paired t -test, because priors on both fixed and random effects require adjustment.

2.3.0.4 Mixed model vs. t-test

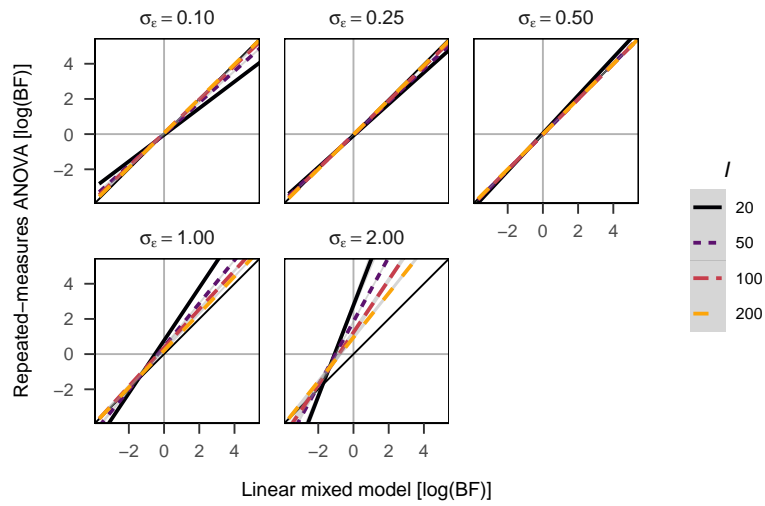


Intercepts and slopes as a function of I and σ_ϵ

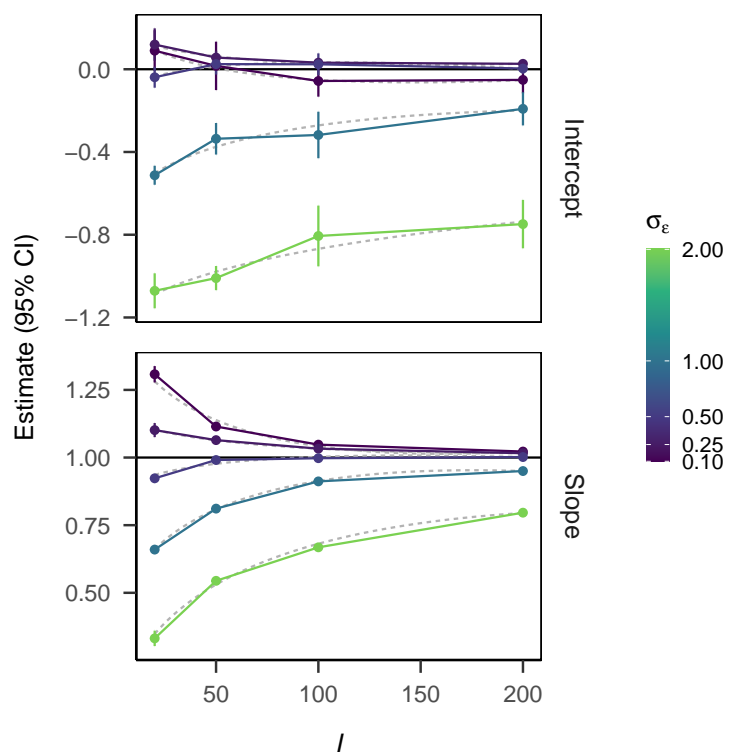
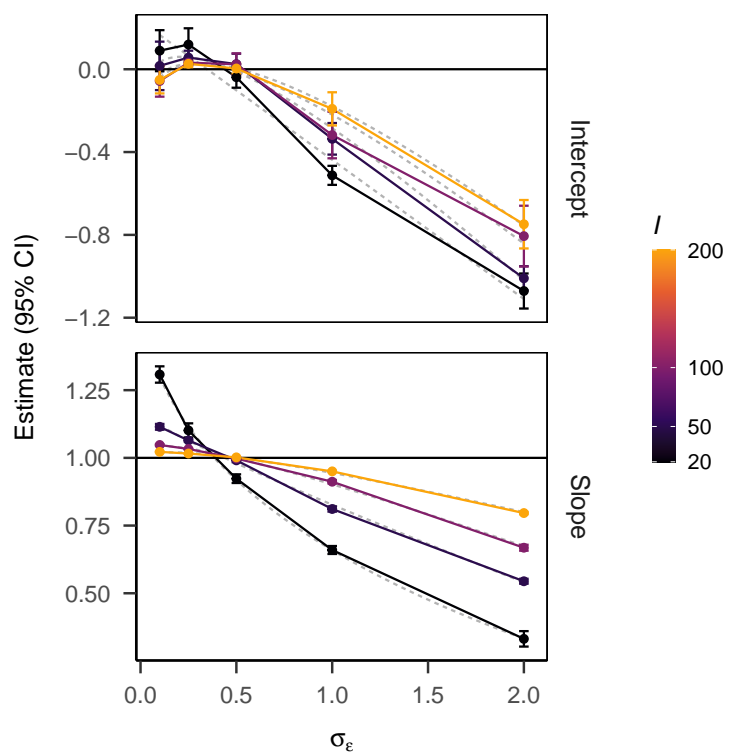




2.3.0.5 Mixed model vs. RM-ANOVA



Intercepts and slopes as a function of I and σ_ϵ

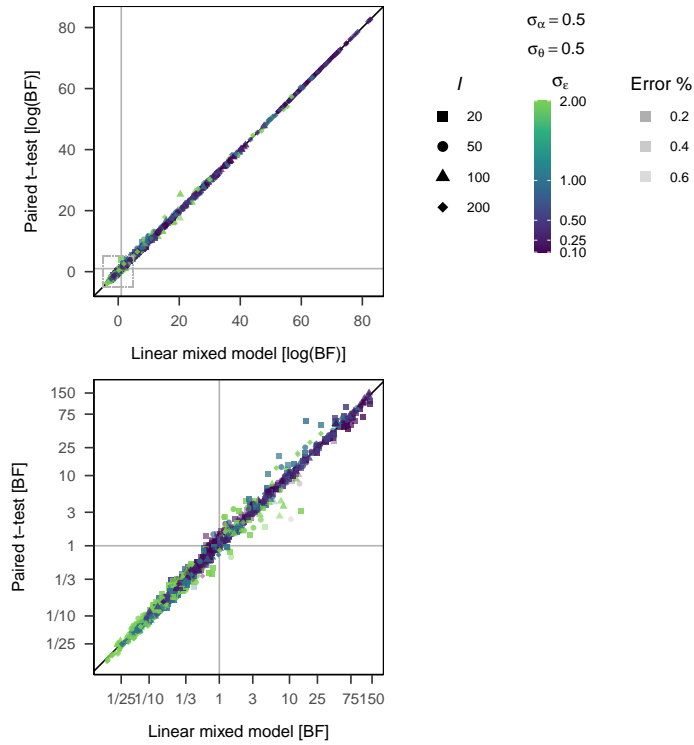


We examined how intercepts and slopes of the trend lines varied with error variance and number of participants. Using AIC and BIC, we compared several regression models that expressed the Bayes factors from the full mixed model (weighted by the reciprocal of their estimation error) as a function of Bayes factors from the corresponding paired t -test, sample size I , and error variance σ_ϵ^2 . We compared the same set of models for the Bayes factors from repeated-measures ANOVA and found the same winning model. The Bayes factors predicted by the selected model,

$$\log \text{BF}_{\text{LMM}} = b_1 \sigma_\epsilon + b_2 \sqrt{\sigma_\epsilon} + b_3 \sqrt{I} + b_4 \sqrt{I} \sigma_\epsilon + \log \text{BF} (1 + b_5 \sigma_\epsilon^2 + b_6 \log I + b_7 \sigma_\epsilon^2 \log I),$$

largely offset divergences between linear mixed model and aggregate analyses.

2.3.0.6 Mixed model vs. t-test



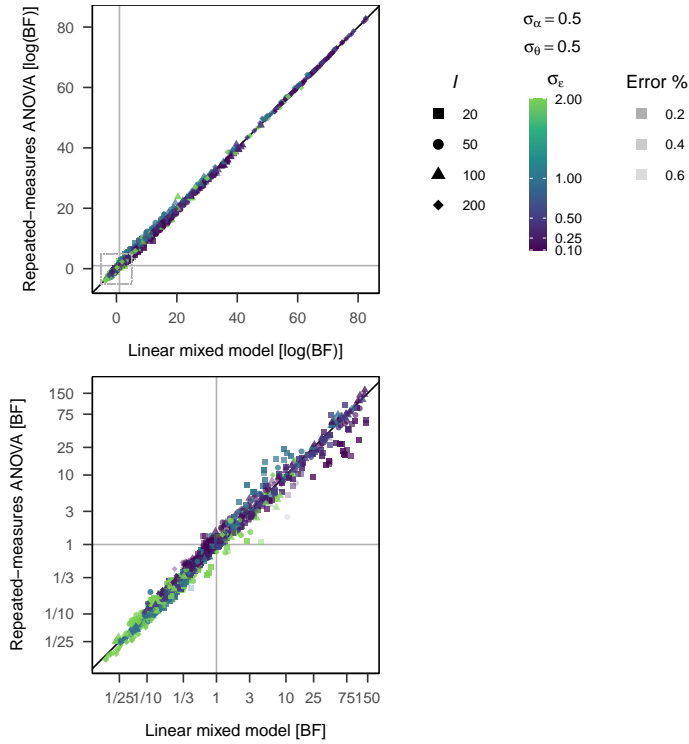
Estimated regression coefficients

(#tab:bias-correction-ttest-coef)

Results for regression of log BF from linear mixed models on log BF from paired t-tests.

Predictor	b	95% CI	$t(1793)$	p
σ_ϵ	-0.835	[-0.949, -0.721]	-14.32	< .001
$\sqrt{\sigma_\epsilon}$	0.551	[0.419, 0.684]	8.16	< .001
\sqrt{I}	-0.009	[-0.015, -0.003]	-2.89	.004
$\sigma_\epsilon \times \sqrt{I}$	0.019	[0.013, 0.025]	6.35	< .001
$\log \text{BF} \times \sigma_\epsilon^2$	-0.239	[-0.246, -0.231]	-64.16	< .001
$\log \text{BF} \times \log I$	0.001	[0.001, 0.001]	11.65	< .001
$\log \text{BF} \times \sigma_\epsilon^2 \times \log I$	0.039	[0.038, 0.041]	53.87	< .001

2.3.0.7 Mixed model vs. ANOVA



Estimated regression coefficients

(#tab:bias-correction-anova-coef)

Results for regression of log BF from linear mixed models on log BF from repeated-measures ANOVAs.

Predictor	b	95% CI	$t(1793)$	p
σ_ϵ	-0.871	[-1.042, -0.700]	-9.99	< .001
$\sqrt{\sigma_\epsilon}$	0.374	[0.176, 0.571]	3.71	< .001
\sqrt{I}	0.017	[0.008, 0.026]	3.79	< .001
$\sigma_\epsilon \times \sqrt{I}$	0.005	[-0.004, 0.014]	1.04	.300
$\log \text{BF} \times \sigma_\epsilon^2$	-0.316	[-0.325, -0.307]	-68.01	< .001
$\log \text{BF} \times \log I$	0.003	[0.003, 0.004]	19.74	< .001
$\log \text{BF} \times \sigma_\epsilon^2 \times \log I$	0.049	[0.047, 0.051]	53.89	< .001

- Rouder, Jeffrey N., Richard D. Morey, Paul L. Speckman, and Jordan M. Province. 2012. "Default Bayes Factors for ANOVA Designs." *Journal of Mathematical Psychology* 56 (5): 356–74. <https://doi.org/10.1016/j.jmp.2012.08.001>.
- van Doorn, Johnny, Frederik Aust, Julia M. Haaf, Angelika M. Stefan, and Eric-Jan Wagenmakers. 2021. "Bayes Factors for Mixed Models." *Computational Brain & Behavior*. <https://doi.org/10.1007/s42113-021-00113-2>.