General Dirichlet Integer Moments

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Exact moment calculations for integer α

Case: $k = 2$, $\alpha_1 = m$, $\alpha_2 = n$

$$f(x|m, n, \lambda_1, \lambda_2) = \frac{\Gamma(m+n)\lambda_1^m \lambda_2^n}{\Gamma(m)\Gamma(n)} \frac{x^{m-1}(1-x)^{n-1}}{((\lambda_1 - \lambda_2)x + \lambda_2)^{m+n}}$$

The expected value of X is the following:

$$\mathsf{E}(X) = \int_0^1 \frac{\Gamma(m+n)\lambda_1^m \lambda_2^n}{\Gamma(m)\Gamma(n)} \frac{x^m(1-x)^{n-1}}{\big((\lambda_1 - \lambda_2)x + \lambda_2\big)^{m+n}} \, \mathrm{d}x$$

Carry out the substitution $u = (\lambda_1 - \lambda_2)x + \lambda_2$ where $\mathrm{d}u = (\lambda_1 - \lambda_2)\mathrm{d}x$, $x = (u - \lambda_2)/(\lambda_1 - \lambda_2)$, and $1 - x = -(u - \lambda_1)/(\lambda_1 - \lambda_2)$.

$$\mathsf{E}(X) = \frac{\Gamma(m+n)\lambda_1^m \lambda_2^n}{\Gamma(m)\Gamma(n)(\lambda_1 - \lambda_2)^{m+n}} \int_{\lambda_2}^{\lambda_1} \frac{(-1)^{n-1}(u-\lambda_2)^m (u-\lambda_1)^{n-1}}{u^{m+n}} \, \mathrm{d}u$$

After binomial expansion, this expression follows.

$$\mathsf{E}(X) = \frac{\Gamma(m+n)\lambda_1^m \lambda_2^n}{\Gamma(m)\Gamma(n)(\lambda_1 - \lambda_2)^{m+n}} \sum_{i=0}^{m} \sum_{j=0}^{n-1} \binom{m}{i} \binom{n-1}{j} (-1)^{m-i-j} \lambda_1^{n-1-j} \lambda_2^{m-i} \int_{\lambda_2}^{\lambda_1} u^{i+j-m-n} \, \mathrm{d}u$$

In the very last term of the double sum, the exponent of u equals -1. Otherwise, integration just raises the exponent by one.

$$\begin{aligned} \mathsf{E}(X) &= \frac{\Gamma(m+n)\lambda_1^m \lambda_2^n}{\Gamma(m)\Gamma(n)(\lambda_1 - \lambda_2)^{m+n}} \\ &\quad \times \left[\sum_{i=0}^{m} \sum_{\substack{j=0 \\ (i,j) \neq (m,n-1)}}^{n-1} \left\{ \binom{m}{i} \binom{n-1}{j} (-1)^{m-i-j} \lambda_1^{n-1-j} \lambda_2^{m-i} \left(\frac{u^{i+j-m-n+1}}{i+j-m-n+1} \Bigg|_{u=\lambda_2}^{\lambda_1} \right\} + \log(u) \Bigg|_{\lambda_2}^{\lambda_1} \right] \end{aligned}$$

This simplifies to

$$\begin{aligned} \mathsf{E}(X) &= \frac{\Gamma(m+n)\lambda_1^m \lambda_2^n}{\Gamma(m)\Gamma(n)(\lambda_1 - \lambda_2)^{m+n}} \\ &\quad \times \left[\sum_{i=0}^{m} \sum_{\substack{j=0 \\ (i,j) \neq (m,n-1)}}^{n-1} \left\{ \binom{m}{i} \binom{n-1}{j} (-1)^{m-i-j+1} \frac{(\lambda_2/\lambda_1)^{(m-i)} - (\lambda_1/\lambda_2)^{n-1-j}}{(m-i) + (n-1-j)} \right\} + \log\left(\frac{\lambda_1}{\lambda_2}\right) \right] \end{aligned}$$

Change the order of both sums and let $r = \lambda_1/\lambda_2$.

$$
\begin{aligned}
\mathsf{E}(X) \quad = \quad & \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \left(\frac{r}{r-1}\right)^m \left(\frac{1}{r-1}\right)^n \\
& \times \left[\sum_{i=0}^{m} \sum_{\substack{j=0 \\ (i,j)\neq(0,0)}}^{n-1} \left\{ \binom{m}{i} \binom{n-1}{j} (-1)^{n+i+j} \frac{r^{-i} - r^{j}}{i+j} \right\} + \log\left(\frac{\lambda_1}{\lambda_2}\right) \right]
\end{aligned}
$$