REPARTIȚIA DIRICHLET GENERALIZATĂ

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Fie $x_1, x_2, \ldots, x_{k+1}$ variabile aleatoare independente care au repartițiile gamma generalizate

$$f(x_i, a_i, \alpha_i, \beta_i) = \frac{\alpha_i}{(a_i)^{\beta_i/\alpha_i}\,\Gamma(\beta_i/\alpha_i)}\, x_i^{\beta_i-1} \exp\left(-\frac{x_i^{\alpha_i}}{a_i^{\alpha_i}}\right).$$

$$x_i > 0,\ \alpha_i > 0,\ \beta_i > 0$$

Fie

$$y_i = \frac{x_i}{x_1 + \ldots + x_{k+1}}, \qquad i = 1, \ldots, \mathrm{k}$$

Ne propunem să determinăm repartiția vectorului $(y_1, \ldots, y_k)$, pe care o vom numi *repartiția Dirichlet generalizată.*

Repartiția lui $(x_1, \ldots, x_{k+1})$ este

$$\prod_{i=1}^{k+1} f(x_i) = \prod_{i=1}^{k+1} \frac{\alpha_i}{(a_i)^{\beta_i/\alpha_i}\,\Gamma(\beta_i/\alpha_i)}\, x_i^{\beta_i-1} \exp\left(-\frac{x_i^{\alpha_i}}{x_i^{\alpha_i}}\right).$$

Se face schimbarea de variabile

$$\begin{aligned} &x_1 + \ldots + x_{k+1} = z \\ &x_1 = y_1 z \\ &x_2 = y_2 z \\ &\vdots \qquad \vdots \\ &x_k = y_k z, \quad y_i \geqslant 0, \quad y_1 + y_2 + \ldots + y_k \leqslant 1. \end{aligned}$$

Cu aceasta repartiția lui $(y_1 \ldots y_k)$ devine

$$h(y_1, \ldots, y_k) = A \int_0^\infty \prod_{i=1}^{k} y_i^{\beta_i-1}\, z^{\sum\limits_{i=1}^{k+1} \beta_i - 1} (1 - y_1 - \ldots - y_k)^{\beta_{k+1}-1} \times$$

$$\times \exp - \left(\left(\frac{y_1 z}{a_1}\right)^{\alpha_1} + \left(\frac{y_2 z}{a_2}\right)^{\alpha_2} + \cdots + \left(\frac{y_k z}{a_k}\right)^{\alpha_k} + \left(\frac{z(1 - y_1 - \ldots - y_k)}{a_{k+1}}\right)^{\alpha_{k+1}}\right) dz$$

unde

$$A = \prod_{1}^{k+1} \frac{\alpha_i}{a_i^{\beta_i/\alpha_i}\,\Gamma(\beta_i/\alpha_i)}$$

sau,

$$h(y_1,\dots,y_k) = \prod_{i=1}^{k+1} \frac{\alpha_i}{a_i^{\beta_i/\alpha_i}\,\Gamma(\beta_i/\alpha_i)} \cdot \prod_{i=1}^{k} y_i^{\beta_i-1}(1-y_1-\dots-y_k)^{\beta_{k+1}-1} \times$$

$$\times \int_0^\infty z^{\sum_1^{k+1}\beta_i-1} \exp -\left(\left(\frac{y_1 z}{a_1}\right)^{\alpha_1} + \left(\frac{y_2 z}{a_2}\right)^{\alpha_2} + \cdots + \left(\frac{y_k z}{a_k}\right)^{\alpha_k} + \right.$$

$$\left. + \left(\frac{z(1-y_1\dots-y_k)}{a_{k+1}}\right)^{\alpha_{k+1}}\right) dz.$$

În cazul particular în care

$$\alpha_1 = \alpha_2 = \dots = \alpha_{k+1} = 1$$

$$a_1 = a_2 = \dots = a_{k+1} = 1$$

se obţine

$$h(y_1,\dots,y_k) = \frac{1}{\Gamma(\beta_1)\dots\Gamma(\beta_{k+1})} \prod_{i=1}^{k} y_i^{\beta_i-1}(1-y_1-\dots-y_k)^{\beta_{k+1}-1} \times$$

$$\times \int_0^\infty z^{\sum_1^{k+1}\beta_i-1} e^{-z}\, dz$$

sau

$$h(y_1,\dots,y_k) = \frac{\Gamma(\beta_1+\dots+\beta_{k+1})}{\Gamma(\beta_1)\dots\Gamma(\beta_{k+1})} \prod_{i=1}^{k} y_i^{\beta_i-1}(1-y_1-\dots-y_k)^{\beta_{k+1}-1}$$

adică repartiţia simplă a lui Dirichlet.

În cazul în care

$$\alpha_1 = \alpha_2 = \dots = \alpha_{k+1} = \alpha, \quad (1) \text{ devine}$$

$$(1')\ h(y_1,\dots,y_k) = \frac{\alpha^{k+1}}{\prod_{i=1}^{k} \alpha_i^{\beta_i/\alpha}\,\Gamma\left(\frac{\beta_i}{\alpha}\right)} \prod_{i=1}^{k} y_i^{\beta_i-1}(1-y_1-\dots-y_k)^{\beta_{k+1}-1} \times$$

$$\times \int_0^\infty z^{\sum_1^{k+1}\beta_i-1} \exp - z^\alpha \left[\left(\frac{y_1}{a_1}\right)^\alpha + \left(\frac{y_2}{a_2}\right)^\alpha + \cdots + \left(\frac{1-y_1-\dots-y_k}{a_{k+1}}\right)^\alpha\right] dz.$$

Cu schimbarea de variabilă

$$z = u^{\frac{1}{\alpha}} \left[\left(\frac{y_1}{a_1}\right)^\alpha + \cdots + \left(\frac{y_k}{a_k}\right)^\alpha + \left(\frac{1-y_1-\dots-y_k}{a_{k+1}}\right)^\alpha\right]^{-\frac{1}{\alpha}}$$

(1') devine

$$h(y_1,\ldots,y_k) = \frac{\alpha^k}{\prod\limits_{i=1}^{k+1} a_i^{\beta_i/\alpha}\Gamma\left(\frac{\beta_i}{\alpha}\right)} \prod_{i=1}^{k} y_i^{\beta_i-1}(1-y_1-\ldots-y_k)^{\beta_{k+1}-1} \times$$

$$\times\left[\left(\frac{y_1}{a_1}\right)^\alpha + \cdots + \left(\frac{y_k}{a_k}\right)^\alpha + \left(\frac{1-y_1-\ldots-y_k}{a_{k+1}}\right)^\alpha\right]^{-\frac{\sum\limits_1^{k+1}\beta_i}{\alpha}} \times$$

$$\times\int_0^\infty u^{\frac{\sum\limits_1^{k+1}\beta_i}{\alpha}-1} e^{-u}\,du, \quad \alpha > 0$$

$$= \frac{\alpha^k\,\Gamma\left(\frac{\beta_1+\beta_2+\ldots+\beta_{k+1}}{\alpha}\right)}{\prod\limits_{i=1}^{k+1} a_i^{\beta_i/\alpha}\,\Gamma(\beta_i/\alpha)} \prod_{i=1}^{k} y_i^{\beta_i-1}(1-y_1-\ldots-y_k)^{\beta_{k+1}-1} \times$$

$$\times\left(\left(\frac{y_1}{a_1}\right)^\alpha + \cdots + \left(\frac{y_k}{a_k}\right)^\alpha + \left(\frac{1-y_1-\ldots-y_k}{a_{k+1}}\right)^\alpha\right)^{-\frac{\sum\limits_1^{k+1}\beta_i}{\alpha}}$$

Observaţie: În cazul $k = 1$, (1) devine repartiţia β generalizată studiată în [2].

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ОБОБЩЕННОЕ РАСПРЕДЕЛЕНИЕ ДИРИХЛЕ

КРАТКОЕ СОДЕРЖАНИЕ

В работе определяется и изучается распределение Дирихле в интегральной форме

THE GENERALIZED DIRECHLET REPARTITION

ABSTRACT

In this paper one define and one utilise the Dirichlet repartition in an integral form.