# General Dirichlet Mean Ratios

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Using notation from the previous document, let

$$m_j^0 = m_j(\alpha, \lambda) \qquad \text{and} \qquad m_j^i = m_j(\alpha^{(i)}, \lambda)$$

The recursion is

$$m_j^0 = \frac{\alpha_j}{\lambda_j A} \sum_{i=1}^{k} \lambda_i m_i^j$$

where $A = \sum_{i=1}^{k} \alpha_i$. Multiply both sides of the recursion by $A$ and sum over $j$.

$$A \sum_{j=1}^{k} m_j^0 = \sum_{j=1}^{k} \frac{\alpha_j}{\lambda_j} \sum_{i=1}^{k} \lambda_i m_i^j$$

Simplify the left-hand side and separate the double sum on the right hand side into the cases where $i$ is equal to and not equal to $j$.

$$A = \sum_{i=1}^{k} \alpha_i m_i^i + \sum_{i=1}^{k} \sum_{j \neq i} \frac{\alpha_j \lambda_i}{\lambda_j} m_i^j$$

Write $m_i^i$ as one minus the sum of the other means from the same distribution.

$$A = \sum_{i=1}^{k} \alpha_i \Big( 1 - \sum_{j \neq i} m_j^i \Big) + \sum_{i=1}^{k} \sum_{j \neq i} \frac{\alpha_j \lambda_i}{\lambda_j} m_i^j$$

Subtract $A$ from both sides and exchange the order of summation of the second sum.

$$0 = -\sum_{i=1}^{k} \sum_{j \neq i} \alpha_i m_j^i + \sum_{i=1}^{k} \sum_{j \neq i} \frac{\alpha_i \lambda_j}{\lambda_i} m_j^i$$

Combine sums and simplify.

$$0 = \sum_{i=1}^{k} \sum_{j \neq i} \frac{\alpha_i}{\lambda_i} m_j^i (\lambda_j - \lambda_i)$$

Rewrite combining terms for the same $i$ and $j$.

$$0 = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \left[ \Big( \frac{\alpha_i}{\lambda_i} m_j^i - \frac{\alpha_j}{\lambda_j} m_i^j \Big) (\lambda_j - \lambda_i) \right]$$

This derivation shows that the entire sum equals zero. Simulation suggests that each term in the sum equals zero, or that

$$\frac{m_i^j}{m_j^i} = \frac{\alpha_i / \lambda_j}{\alpha_j / \lambda_i}$$

More information beyond the sum constraints and the recursion equation is needed to demonstrate this last equation.