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ON THE PRODUCTS OF POWERS OF GENERALIZED DIRICHLET COMPONENTS WITH AN APPLICATION

by

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Key Words and Phrases: *Products of generalized Dirichlet variables, independence of products and ratios of generalized Dirichlet variables, generalized gamma variables, moments, moment generating functions, exact distributions, G and H-functions, tests for intraclass structure.*

ABSTRACT

In this article an attempt is made to prove that $T_1=\Pi_{i=1}^{N+1} Y_i^{s_i}$ and $T_2=(X_1+\ldots+X_{N+1})^{s_1+\cdots+s_{N+1}}$ are independently distributed where $X_1,\ldots,X_{N+1}$ are generalized gamma variates, $Y_i=X_i/(X_1+\ldots$ $\ldots+X_{N+1})$ and $s_1,\ldots,s_{N+1}$ are non-negative integers. It is shown that when the X_i's are certain two-parameter gamma variables T_1 and T_2 are independent and the independence does not hold good when the parameters are arbitrary. The density of T_1 is also obtained in terms of H-functions and Meijer's G-functions.

1. INTRODUCTION

In developing likelihood ratio tests concerning the structure of a covariance matrix in a Gaussian situation (Rogers and Young (1973)), it was noted that certain critical regions were equivalent to $Y_1^{k_1} Y_2^{k_2} Y_3^{k_3} Y_4^{k_4} < c$, where the k's depended on the hypothesis being tested and the dimension of the Gaussian distribution and Y_1,Y_2,Y_3 $(Y_4=1-Y_1-Y_2-Y_3)$ were the components of a corresponding Dirichlet distribution. Here we consider generalized gamma variates X_i (Stacy 1962), define a generalized Dirichlet variable by $Y_i = X_i/(X_1+\ldots+X_{N+1})$, $i = 1(1)N+1$, and attempt to prove that $\Pi_{i=1}^{N+1} Y_i^{s_i}$ is independent of $(X_1+\ldots+X_{N+1})^{s_1+\ldots+s_{N+1}}$. We succeed only when the X_i's are certain two-parameter gamma variables and ask whether there are other values of the parameters of the generalized gamma for which the independence holds. (It does not hold for arbitrary values.)

We then express the densities of $\Pi_{i=1}^{N+1} Y_i^{s_i}$ in terms of H-functions and Meijer's G-functions. Finally, we note the specialization to the original problem in the Gaussian situation (Rogers and Young (1973)).

2. MOMENTS AND INDEPENDENCE IN A RATIO

In the following, i ranges over $I_1=\{1,2,\ldots,N+1\}$; Σ and Π have index set I_1, unless otherwise indicated.

Definitions. The random variable X has a generalized gamma distribution (Stacy 1962) if $P(X<0)=0$ and for $x\geq 0, a, d, p$ positive constants, the density is given by

$$px^{d-1}\, e^{-(x/a)^p} /a^d\, \Gamma(d/p) .$$

Let X_i be independent generalized gamma random variables with corresponding parameters (a_i,d_i,p_i). Let $Y_i = X_i/(X_1+\ldots+X_{N+1})$. Then the vector $(Y_1,\ldots,Y_N)$ is said to have a generalized Dirichlet distribution (Craiu and Craiu 1969).

The following theorem is proved by straightforward techniques of integration.

Theorem 1. In the above definition, $Y_i \geq 0$, $Y_1+\ldots+Y_{N+1} = 1$ and the density of $(Y_1,\ldots,Y_N)$ is :

$$(a) \qquad [\Pi_{i=1}^{N+1} p_i y_i^{d_i-1} / a_i^{d_i-1}\Gamma(d_i/p_i)]I,$$

where

$$I = \int_0^\infty z^{d_i-1} \exp[-\Sigma(y_i z/a_i)^{p_i}]\, dz$$

$$= (1/p_{N+1})\ \Sigma_{j=0}^{\infty} (-1)^j \sum_{r_1+\ldots+r_N=j} [\Pi_{t=1}^{N}(y_t/a_t)^{p_t r_t} /r_t!] \times$$

$$(a_{N+1}/y_{N+1})^{(\Sigma d_i+\Sigma_{t=1}^{N} p_t r_t)}\Gamma[(\Sigma d_i+\Sigma_{t=1}^{N} p_t r_t)/p_{N+1}];$$

(b) $$[p^N\Gamma(\Sigma d_i/p)\ \Pi y_i^{d_i-1}]/[\ \Pi a_i^{d_i}\ \Gamma(d_i/p)\ \{\Sigma(y_i/a_i)^p\}^{\Sigma d_i/p}\ ],$$

when $p_1 = \dots = p_{N+1} = p$;

(c) $$[p^N\ \Gamma(\Sigma d_i/p)\ \Pi y_i^{d_i-1}]/[\ \Pi\Gamma(d_i/p)\ (\Sigma y_i^p)^{\Sigma d_i/p}]\ ,$$

when $p_1 = \dots = p_{N+1} = p$ and $a_1 = \dots = a_{N+1} = a$;

(d) $$[\ \Pi y_i^{d_i-1}/\Gamma(d_i)]\Gamma(\Sigma d_i)\ ,$$

when $p_1 = \dots = p_{N+1} = 1$ and $a_1 = \dots = a_{N+1} = a$.

Of course, when the p_i are all p, we are dealing with independent variables having the same shape parameter and when this $p= 1$, the X_i are just two-parameter gamma variables. The density in *(d)* is that of the ordinary Dirichlet distribution and its derivation from gamma variables is well-known (Johnson and Kotz 1972).

The joint moments of $(Y_1,\dots,Y_{N+1})$ are given by $E[\Pi Y_i^{s_i}]$ where s_i are non-negative integers. In considering the independence of $\Pi Y_i^{s_i} = (\Pi X_i^{s_i})/(\Sigma X_i)^{\Sigma s_i}$ and its denominator, we use the following result given by Hogg (1951).

THEOREM 2. If U, V are positive random variables with joint moment generating function $M(\mu,t) = E[exp(\mu U + tV)]$ in a neighborhood of the origin, then V and U/V are stochastically independent iff

$$\frac{\partial^k M(0,t)}{\partial\mu^k}\frac{\partial^k M(0,0)}{\partial t^k} = \frac{\partial^k M(0,0)}{\partial\mu^k}\ \frac{\partial^k M(0,t)}{\partial t^k},\ for\ k= 0(1)\infty. \tag{2.1}$$

It follows that $E[(U/V)^k] = E[U^k]/E[V^k]$ for $k= 0(1)\infty$. Now *(2.1)* is equivalent to $E[U^k\ exp\ tV]E[V^k] = E[U^k]E[V^k\ exp\ tV]$ which in our general case reduces to

$$E[exp\ tW^{\Sigma s_i}]E[Z^{k\Sigma s_i}] = E[Z^{k\Sigma s_i}\ exp\ tZ^{\Sigma s_i}\ ]\ ,$$

where $Z = X_1+\dots+X_{N+1}$ as in *Theorem 1*, and W is a like sum of $N+1$

independent generalized gamma random variables with parameters (a_i, d_i+ks_i, p_i).

Comparing coefficients of t in the series expansions of the generating functions, we see that this equation will hold iff

$$(j\Sigma s_i)! \sum_{\Sigma r_i=j\Sigma s_i} \Pi\beta(r_i;a_i,d_i+ks_i,p_i)\ (k\Sigma s_i)! \sum_{\Sigma t_i=k\Sigma s_i} \Pi\beta(t_i;a_i,d_i,p_i)$$

$$= [(j+k)\Sigma s_i]! \sum_{\Sigma r_i=(j+k)\Sigma s_i} \Pi\beta(r_i;a_i,d_i,p_i)$$

for all non-negative integers $j, k, s_1,\ldots, s_{N+1}$, where $\beta(r;a,d,p)= a^r\Gamma[(d+r)/p]/\Gamma(d/p)r!$.

When p_i are all 1 and a_i are all a, we can use another result of Stacy (1962):

$$\sum_{\Sigma r_i=m} \Pi\beta(r_i;a,d_i,1) = a^m\ \Gamma(\Sigma d_i+m)/\Gamma(\Sigma d_i)m!\ .$$

It is now an easy exercise to show the desired equivalence of the two sums and we have proven (essentially) the theorem stated below. By taking $N+1=2$, $k=1$, it is seen that the equality does not hold for arbitrary a_i, d_i, p_i. Are there other sets of these parameters for which the equality does hold?

THEOREM 3. Let X_i be independent (generalized) gamma random variables with parameters $(a_i=c, d_i, p_i=1)$. Then

(i) for all non-negative integers $s_1,\ldots,s_{N+1}$,

$$\Pi X_i^{s_i}/(\Sigma X_i)^{\Sigma s_i} \quad \text{and} \quad (\Sigma X_i)^{\Sigma s_i}$$

are independent random variables;

(ii) for $Y_i= X_i/(X_1+\ldots+X_{N+1})$, the joint moments of $Y_1,\ldots,Y_{N+1}$ are given by

$$E[\Pi Y_i^{s_i}] = E[\Pi X_i^{s_i}]/E[(\Sigma X_i)^{\Sigma s_i}] = \Gamma(\Sigma d_i)\Pi\Gamma(d_i+s_i)/[\Gamma\{\Sigma(d_i+s_i)\}\Pi\Gamma(d_i)];$$

(iii) the Mellin transform of $\Pi Y_i^{k_i}$, for k_i non-negative, is given by

$$E[(\Pi Y_i^{k_i})^{s-1}] = E[\Pi Y_i^{k_i(s-1)}]= g(s,k_1,\ldots,k_{N+1},d_1,\ldots,d_{N+1})$$

$$= \Gamma(\Sigma d_i)\ \Pi\Gamma\{d_i+k_i(s-1)\}/[\Gamma\{\Sigma(d_i+k_i(s-1))\}\ \Pi\Gamma(d_i)] \quad ,$$

where $Re\ s > 0$.

The result in *(ii)* is not new since, when $s_{N+1}=0$, the ratio there is the set of joint moments of the ordinary Dirichlet random variable $(Y_1,\ldots,Y_N)$ (Johnson and Kotz 1972). The independence result in *(i)* is a generalization of a result of Shenton and Bowman (1970) who proved the independence when all $s_i = 1$.

Using the result *(iii)* in *Theorem 2* and the inversion formula for the Mellin transform, we get immediately:

THEOREM 4. If $(Y_1,\ldots,Y_N)$ is a Dirichlet random variable with parameters $d_1,\ldots,d_{N+1}$, and $k_1,\ldots,k_{N+1}$ are non-negative, then the density of $\Pi Y_i^{k_i}$, where $Y_{N+1} = 1-Y_1-\ldots-Y_N$, is given in terms of an H-function as

$$\frac{\Gamma(\Sigma d_i)}{\Pi\Gamma(d_i)}\ H^{N+1,\,0}_{1\,,\,N+1}\left[x\ \middle|\ \begin{matrix}(\Sigma d_i-\Sigma k_i,\Sigma k_i)\\ (d_1-k_1,k_1),\ldots,(d_{N+1}-k_{N+1},k_{N+1})\end{matrix}\right] \quad ;$$

the H-function is the Mellin inversion

$$(1/2\pi i)\int_L g(s,k_1,\ldots,k_{N+1},d_1,\ldots,d_{N+1})x^{-s}\,ds \ .$$

Mathai (1971) has given the details on the H-function.

When all k_i are positive integers (as in the next section), the Mellin transform of *Theorem 2* becomes

$$(2.2)\qquad [(2\pi)^{N/2}\ \{\Pi k_i^{d_i-k_i-\frac{1}{2}}/\Gamma(d_i)\}\{\Gamma(D)/K^{D-K-\frac{1}{2}}\}\ \Pi_{r_i=0}^{k_i-1}\Gamma(s-1+d_i/k_i+r_i/k_i)\ \times$$

$$(\Pi k_i^{k_i}/K^K)^s]/\ \Pi_{r=0}^{K-1}\Gamma(s-1+D/K+r/K) \ ,$$

where $D = \Sigma d_i$, $K = \Sigma k_i$. The inversion formula, Erdélyi (1953), then gives as the density of $T = \Pi Y_i^{k_i}$:

$$(2.3)\qquad B\ G^{K,0}_{K,K}\left[cy\ \middle|\ \begin{matrix}a_1,\ldots,a_K\\ b_{11},\ldots,b_{1k_1};b_{21},\ldots;\ldots,b_{N+1,k_{N+1}}\end{matrix}\right] ,$$

where

$$B = [(2\pi)^{N/2}\ \Gamma(D)K^{K-D+\frac{1}{2}}\ /\ \Pi\Gamma(d_i)k_i^{k_i-d_i+\frac{1}{2}}]\ ,\quad c = \Pi k_i^{k_i}/K^K \ ,$$

$$a_r = (D+r)/K - 1\ ,\ r = 0(1)K-1,\ b_{ir_i} = (d_i+r_i)/k_i - 1,\ r_i = 0(1)k_i - 1,$$

$$i = 1(1)N+1, \text{ and } 0 < y < 1/c\ .$$

This may be expanded in a series, following the method of Mathai (1971). Our example will be seen to be a special case.

If in *Theorem 2* we first take $k_{N+1}=0$, we then have the Mellin transform of the product of powers of dependent Beta variates $Y_1,\dots,Y_N$, *i.e.*, the Dirichlet components. In fact, if some other k_i is zero we get a product on different components but the forms of the transform and the corresponding density are the same. This generalizes Mathai's (1971) result on products of independent Beta variables; we can, of course, take the other $k_i = 1$. See also Springer and Thompson (1970). Further, we can use the same technique to find the density of $\Pi_{j=1}^{M}\Pi_{i=1}^{N_j+1} Y_{ij}^{k_{ij}}$ where $(Y_{1j},\dots,Y_{N_j,j})$, $j=1(1)M$ have independent Dirichlet distributions and k_{ij} are non-negative. This density can also be calculated as the Mellin convolution of the corresponding G-functions of type (2.3) (Sneddon (1972)).

3. TESTS WHEN A NORMAL COVARIANCE HAS INTRACLASS STRUCTURE

In another paper (Rogers and Young (1973)), we have derived various likelihood ratio tests on the components of a normal covariance matrix Σ which has intraclass structure of arbitrary order. Let U be a kp-dimensional normal random variable whose $\Sigma=(\Sigma_{ij})$ is such that $\Sigma_{ii} = A(p\times p)$, $\Sigma_{ij} = B(p\times p)$, $i\neq j$, where the diagonal elements of $A[B]$ are all $a_1[b_1]$ and the off diagonal elements are all $a_2[b_2]$. Letting ∇ be any known orthogonal matrix with first row $1/\sqrt{k}$ and Δ be any known orthogonal matrix with first row $1/\sqrt{p}$, $X = (\nabla\otimes\Delta)U$, we get $X_1,\dots,X_{kp}$ independent normal variables with means Θ_j, $j=1(1)kp$ and variances

$$\sigma_1^2 = a_1 + (p-1)a_2 + (k-1)(b_1+(p-1)b_2) = \tau_1$$

$$\sigma_j^2 = a_1 - a_2 + (k-1)(b_1-b_2) = \tau_2,\ j\in I_2$$

$$= a_1+(p-1)a_2 - (b_1+(p-1)b_2) = \tau_3,\ j\in I_3$$

$$= (a_1-a_2)-(b_1-b_2) = \tau_4,\ j\in I_4$$

where

$$I_2 = \{2,3,\ldots,p\}$$
$$I_3 = \{p+1,\ 2p+1,\ldots,(k-1)p+1\}$$
$$I_4 = \{p+2,\ p+3,\ldots,\ 2p,\ 2p+2,\ \ldots,\ kp\}\ .$$

For a random sample $X_1,\ldots,X_n$, the log likelihood is

$$(3.1)\qquad (-n/2)\{kp\,\ln 2\pi + \ln|\Sigma_X| + \Sigma_{\alpha=1}^{n}(X_\alpha-\Theta)'\Sigma_X^{-1}(X_\alpha-\Theta)/n\}\ ,$$

where $\Theta=(\Theta_1,\ldots,\Theta_{kp})'$ and $\Sigma_X=(\nabla\otimes\Delta)\Sigma(\nabla'\otimes\Delta')$ is diagonal with elements $\sigma_1^2,\ldots,\sigma_{kp}^2$. Noting that the maximum likelihood estimate *(MLE)* $\hat{\Theta}_j=\Theta_j$ or $\Sigma_{\alpha=1}^{n} X_{j\alpha}/n$ according as Θ_j is known or unknown, the maximum of *(3.1)* is the maximum of

$$(3.2)\qquad (-n/2)\{kp\,\ln 2\pi + \ln|\Sigma_X| + tr\Sigma_X^{-1} S\},$$

where S has diagonal elements $s_j^2 = \Sigma_{\alpha=1}^{n}(X_{j\alpha}-\hat{\Theta}_j)^2/n$, $j=1(1)kp$. Note that these s_j^2 are independent chisquares with degrees of freedom $2f=n$ or $n-1$ according as Θ is known or unknown. Let $S_1= s_1^2$, $S_2 = \Sigma_{j\epsilon I_2} s_j^2$, $S_3= \Sigma_{j\epsilon I_3} s_j^2$, $S_4 = \Sigma_{j\epsilon I_4} s_j^2$. Then *(3.2)* can be written as

$$(3.3)\qquad (-n/2)\{kp\,\ln 2\pi + \ln\tau_1 + (p-1)\ln\tau_2 + (k-1)\ln\tau_3 + (p-1)(k-1)\ln\tau_4 + \Sigma_{i=1}^{4} S_i/\tau_i\}\ .$$

It is clear that the *MLE*'s are $\hat{\tau}_1 = S_1$, $\hat{\tau}_2 = S_2/(p-1)$, $\hat{\tau}_3 = S_3/(k-1)$, $\hat{\tau}_4 = S_4/(p-1)(k-1)$. Hence the maximum of *(3.1)* is

$$(3.4)\qquad (-n/2)\{kp\,\ln 2\pi + kp + \ln S_1 + (p-1)\ln S_2/(p-1) + (k-1)\ln S_3/(k-1) + (p-1)(k-1)\ln S_4/(p-1)(k-1)\}\ .$$

One hypothesis of interest is whether or not the components of U are independent; *i.e.*, $a_2= b_2 = b_1 = 0$. This is equivalent to the hypothesis H_0: $\tau_1 = \tau_2 = \tau_3 = \tau_4$ and then the log likelihood *(3.3)* is

$$(3.5)\qquad (-n/2)\{kp\,\ln 2\pi + kp\,\ln\tau_1 + \Sigma_{i=1}^{4} S_i/\tau_1\}\ .$$

The maximum of *(3.5)* is

$$(3.6)\qquad (-n/2)\{kp\ln 2\pi + kp + kp\ln \Sigma_{i=1}^{4} S_i/kp\} \quad ,$$

and the likelihood ratio test of H_0 has critical region given by (3.6) to (3.4):

$$\log \lambda = (n/2)\{\ln S_1 + (p-1)\ln S_2/(p-1) + (k-1)\ln S_3/(k-1)$$
$$+ (p-1)(k-1)\ln S_4/(p-1)(k-1) - kp\ln \Sigma_{i=1}^{4} S_i/kp\} < c_1$$

or, equivalently,

$$R = [\{S_1 S_2^{p-1} S_3^{k-1} S_4^{(p-1)(k-1)}\}/(S_1+S_2+S_3+S_4)^{kp}] < c \; .$$

Let $Y_i = S_i/(S_1+S_2+S_3+S_4)$; then

$$R = Y_1 Y_2^{p-1} Y_3^{k-1} Y_4^{(p-1)(k-1)} \quad ,$$

where $Y_4 = 1-Y_1-Y_2-Y_3$ and (Y_1,Y_2,Y_3) is a Dirichlet random variable with $d_1 = f$, $d_2 = (p-1)f$, $d_3 = (k-1)f$, $d_4 = (k-1)(p-1)f$. The transform of R is given by (2.2) with $k_1 = 1$, $k_2 = p-1$, $k_3 = k-1$, $k_4 = (k-1)(p-1)$:

$$\{(2\pi)^{3/2}(p-1)^{kpf-kf-kp-k-1}\,(k-1)^{kpf-pf-pk-p-1}\,(kp)^{kp-kpf+\frac{1}{2}}\,\Gamma(kpf)$$

$$\div\, \Gamma(f)\Gamma((p-1)f)\Gamma((k-1)f)\Gamma((p-1)(k-1)f)\}[\,\Pi_{r=0}^{p-2}\,\Gamma(s-1+f+r/(p-1))$$

$$\times\Pi_{r=0}^{k-2}\Gamma(s-1+r/(k-1))\;\Pi_{r=0}^{(p-1)(k-1)-1}\Gamma(s-1+f+r/(p-1)(k-1))$$

$$\times((p-1)^{k(p-1)}\,(k-1)^{p(k-1)}/(kp)^{kp})^{s} \div \Pi_{r=1}^{kp-1}\Gamma(s-1+f+r/kp)\;] \; .$$

The corresponding G-function for the ratio in square brackets above is:

$$G_{\alpha,\alpha}^{\alpha,0}\left[cy\;\middle|\begin{matrix} a_1,\ldots,a_K \\ b_{10},\ldots,b_{1,p-2};b_{20},\ldots,b_{2p-2};b_{30},\ldots,b_{3,(p-1)(k-1)-1}\end{matrix}\right] \; ,$$

where

$$\alpha = kp-1,\; K = kp,\; D = kpf,\; c = (p-1)^{k(p-1)}(k-1)^{p(k-1)}/(kp)^{kp} \; ,$$

$$a_r = f-1+r/kp \;,\; r = 1(1)K-1,$$

$$b_{1r_1} = f-1 + r_1/(p-1) \text{ , } r_1 = 0(1)p-2 \text{ ,}$$

$$b_{2r_2} = f-1 + r_2/(k-1) \text{ , } r_2 = 0(1)k-2 \text{ ,}$$

$$b_{3r_3} = f-1 + r_3/(p-1)(k-1) \text{ , } r_3 = 0(1)(p-1)(k-1) -1.$$

When all the b's are ordered, it will be seen that the distribution of R is that of a multiple of products of independent Beta variables (Mathai(1971)). We will illustrate with the simpler case $k=p$. For ordering purposes, the b's can be considered as 0, $0,0$, $p-1$, $p-1$, $2(p-1)$, $2(p-1)$, $\ldots$, $(p-2)(p-1)$, $(p-2)(p-1)$, $1,2,3,\ldots,(p-1)^2-1$. Let S_m be the finite sequence $m(p-1)+1$, $m(p-1)+2$, $\ldots$, $m(p-1)+(p-2)$; $(m+1)(p-1)$, $(m+1)(p-1)$, $(m+1)(p-1)$. Then the last set of constants (b's) can be written in nondecreasing order as $0,0,0,S_0,S_1,\ldots,S_{p-3}$, p^2-3p+3, p^2-3p+4, $\ldots$, p^2-2p. It is then an easy exercise to verify that, for α_r the r-th "smallest" value, $\alpha_r < \alpha_r + \beta_r = f-1 + r/p^2$, $r=1(1)p^2-1$. Thus when $k=p$, the Mellin transform of R is proportional to

$$\{\Gamma(f-1+s)\Gamma(f-1+s)\Gamma(s-1+f) \div \Gamma(f-1+1/p^2+s)\Gamma(f-1+2/p^2+s)\Gamma(f-1+3/p^2$$

$$+s)\} \{(\Pi_{r=1}^{p-2}\Gamma(f-1+r/(p-1)^2+s))(\Gamma(f-1+1/(p-1)+s))^3$$

$$\div \Pi_{r=4}^{p+4}\Gamma(f-1+r/p^2+s)\} \{(\Pi_{r=1}^{p-2}\Gamma(f-1+(p-1+r)/(p-1)^2+s))$$

$$\times(\Gamma(f-1+2/(p-1)+s))^3 \div \Pi_{r=p+5}^{2p+5}\Gamma(f-1+r/p^2+s)\} \ldots$$

$$\times\{(\Pi_{r=1}^{p-2}\Gamma(f-1+((p-3)(p-1)+r)/(p-1)^2+s))(\Gamma(f-1+(p-2)/(p-1)+s))^3$$

$$\div \Pi_{r=(p-3)p+p+3}^{(p-3)p+2p+1}\Gamma(f-1+r/p^2+s)\}\{\Pi_{r=1}^{p-2}\Gamma(f-1+((p-2)(p-1)+r)/(p-1)^2$$

$$+s) \div \Pi_{r=p^2-p+1}^{p^2-1}\Gamma(f-1+r/p^2+s)\}\times((p-1)^{2p(p-1)}/(p^2)^{p^2})^{s-1} .$$

We relabel the constants so that this last form can be written as

$$\{\Pi_{r=1}^{p^2-1}\Gamma(\alpha_r+s)/\Gamma(\alpha_r+\beta_r+s)\}\{(p-1)^{2p(p-1)}/(p^2)^{p^2}\}^{s-1} .$$

Thus R is $(p-1)^{2p(p-1)}/(p^2)^{p^2}$ times the product of p^2-1 independent

Beta variables with parameters $\alpha_r > -1, \beta_r > 0$ and the series expansion of the density can be obtained from the paper of Mathai(1971). Also, an approximation to the density can be obtained using the work of Tukey and Wilks (1946).

Another hypothesis of interest is $B = 0$, *i.e.*, $\tau_1 = \tau_2$ and $\tau_2 = \tau_4$. The likelihood ratio criterion is equivalent to

$$\{S_1 S_3^{k-1} S_2^{p-1} S_4^{(k-1)(p-1)}\} / \{(S_1+S_3)^k (S_2+S_4)^{k(p-1)}\} .$$

By similar realignments, one can show that this ratio is distributed as a multiple of a product of independent Beta variables. Additional tests are given in Rogers and Young (1973).

Rogers (1964) obtained similar results for the likelihood ratio test of the equality of one-dimensional normal variances, but did not proceed to the G-function. It is obvious that similar results will be obtained when testing various equalities of $\phi_1, \phi_2, \phi_3, \ldots$ when a set of independent normal variates are such that a subset of them have variances equal to ϕ_1, another subset of them have variances equal to ϕ_2, and so on.

RÉSUMÉ

Dans cet article, une tentative est faite pour démontrer que $T_1 = \Pi_{i=1}^{N+1} Y_i^{s_i}$ et $T_2 = (X_1+\ldots+X_{N+1})^{s_1+\ldots+s_{N+1}}$ sont indépendemment distribuées, dans lesquelles $X_1, \ldots, X_{N+1}$ sont des variables distribuées selon des fonctions gamma généralisées, $Y_i = X_i/(X_1+\ldots+X_{N+1})$, et $s_1, \ldots, s_{N+1}$ sont des nombres entiers non-négatifs. Il est démontré que lorsque les X_i sont distribués selon certaines fonctions gamma à deux paramètres, T_1 et T_2 sont indépendantes, et l'indépendance ne subsiste plus quand les paramètres sont arbitraires. La densité de T_1 est aussi obtenue en terme de fonctions H de fonctions G de Meijer.

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