Proposals for a cyclic cubic spline

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1 Definition of cubic splines

$$s(x) = \gamma_0 + \gamma_1 X + \gamma_2 X^2 + \gamma_3 X^3 + \beta_1 (X - t_1)_+^3 + \beta_2 (X - t_2)_+^3 + ... + \beta_k (X - t_k)_+^3 + \quad (1)$$

$$s(x) = \gamma_0 + \gamma_1 X + \gamma_2 X^2 + \gamma_3 X^3 + \sum_{j=1}^{k} \beta_j (X - t_j)_+^3 \quad (2)$$

For $x \leq t_1$

$$s(x) = \gamma_0 + \gamma_1 X + \gamma_2 X^2 + \gamma_3 X^3$$
$$s'(x) = \gamma_1 + 2\gamma_2 X + 3\gamma_3 X^2$$
$$s''(x) = 2\gamma_2 + 6\gamma_3 X$$

For $x \geq t_k$

$$s(x) = \gamma_0 + \gamma_1 X + \gamma_2 X^2 + \gamma_3 X^3 + \sum_{j=1}^{k} \beta_j (X - t_j)^3$$
$$s'(x) = \gamma_1 + 2\gamma_2 X + 3\gamma_3 X^2 + 3\sum_{j=1}^{k} \beta_j (X - t_j)^2$$
$$s''(x) = 2\gamma_2 + 6\gamma_3 X + 6\sum_{j=1}^{k} \beta_j (X - t_j)$$

2 Constraining Cubic splines to achieve the same estimated value at the beginning and end of the cycle

Imposing the constraints

$$s(x_{min}) = s(x_{max})$$

$$\gamma_0 + \gamma_1 X_{min} + \gamma_2 X_{min}^2 + \gamma_3 X_{min}^3 = \gamma_0 + \gamma_1 X_{max} + \gamma_2 X_{max}^2 + \gamma_3 X_{max}^3 + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3$$

$$\gamma_1 (X_{min} - X_{max}) = \gamma_2 (X_{max}^2 - X_{min}^2) + \gamma_3 (X_{max}^3 - X_{min}^3) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3$$

$$\gamma_1 = -\gamma_2 \frac{(X_{max}^2 - X_{min}^2)}{(X_{max} - X_{min})} - \gamma_3 \frac{(X_{max}^3 - X_{min}^3)}{(X_{max} - X_{min})} - \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3}{(X_{max} - X_{min})}$$

$$\gamma_1 = -\gamma_2 (X_{max} + X_{min}) - \gamma_3 (X_{max}^2 + X_{max} X_{min} + X_{min}^2) - \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3}{(X_{max} - X_{min})}$$

$$\gamma_1 = -\frac{1}{X_{max} - X_{min}} \left(\gamma_2 (X_{max}^2 - X_{min}^2) + \gamma_3 (X_{max}^3 - X_{min}^3) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \right)$$

Additionally constraining

$$s'(x_{min}) = s'(x_{max})$$

$$\gamma_1 + 2\gamma_2 X_{min} + 3\gamma_3 X_{min}^2 = \gamma_1 + 2\gamma_2 X_{max} + 3\gamma_3 X_{max}^2 + 3 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2$$

$$\gamma_2 (X_{min} - X_{max}) = 3/2 \gamma_3 (X_{max}^2 - X_{min}^2) + 3/2 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2$$

$$\gamma_2 = -3/2 \gamma_3 \frac{(X_{max}^2 - X_{min}^2)}{(X_{max} - X_{min})} - 3/2 \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{(X_{max} - X_{min})}$$

$$\gamma_2 = -\frac{3}{2(X_{max} - X_{min})} \left(\gamma_3 (X_{max}^2 - X_{min}^2) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2) \right)$$

$$\gamma_2 = -3/2 \gamma_3 (X_{max} + X_{min}) - 3/2 \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{(X_{max} - X_{min})}$$

Finaly, we have to constrain

$$s''(x_{min}) = s''(x_{max})$$

$$2\gamma_2 + 6\gamma_3 X_{min} = 2\gamma_2 + 6\gamma_3 X_{max} + 6 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)$$

$$\gamma_3 (X_{min} - X_{max}) = \sum_{j=1}^{k} \beta_j (X_{max} - t_j)$$

Deriving γ_3

$$\gamma_3 = \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{min} - X_{max}} = -\frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}}$$

Summarizing the results

$$\gamma_3 = -\frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} \tag{3}$$

$$\gamma_2 = -\frac{3}{2(X_{max} - X_{min})} \left(\gamma_3 (X_{max}^2 - X_{min}^2) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2 \right) \tag{4}$$

$$\gamma_1 = -\frac{1}{X_{max} - X_{min}} \left(\gamma_2 (X_{max}^2 - X_{min}^2) + \gamma_3 (X_{max}^3 - X_{min}^3) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \right) \tag{5}$$

We already have γ_3 expressed as a fnuction of β_j.
Deriving γ_2 as a function of β_j

$$\gamma_2 = -\frac{3}{2(X_{max} - X_{min})} \left(\gamma_3 (X_{max}^2 - X_{min}^2) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2 \right)$$

$$\gamma_3 = -\frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}}$$

$$\gamma_2 = -\frac{3}{2(X_{max} - X_{min})} \left(\left(-\frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} \right) (X_{max}^2 - X_{min}^2) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2 \right)$$

$$\gamma_2 = \frac{3(X_{max}^2 - X_{min}^2) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2(X_{max} - X_{min})^2} - \frac{3 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2(X_{max} - X_{min})}$$

$$\gamma_2 = \frac{3(X_{max} + X_{min}) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2(X_{max} - X_{min})} - \frac{3 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2(X_{max} - X_{min})}$$

Deriving γ_1 as a function of β_j

$$\gamma_1 = -\frac{1}{X_{max} - X_{min}} \left(\gamma_2(X_{max}^2 - X_{min}^2) + \gamma_3(X_{max}^3 - X_{min}^3) + \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \right)$$

$$\begin{aligned}\gamma_1 = &- \frac{1}{X_{max} - X_{min}} \Bigg(\\ &(X_{max}^2 - X_{min}^2) \left(\frac{3(X_{max} + X_{min}) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2(X_{max} - X_{min})} - \frac{3 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2(X_{max} - X_{min})} \right) + \\ &- \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} (X_{max}^3 - X_{min}^3) \\ &+ \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \Bigg)\end{aligned}$$

$$\begin{aligned}\gamma_1 = &-\frac{1}{X_{max} - X_{min}} \Bigg(\\ &(X_{max}^2 - X_{min}^2) \left(\frac{3(X_{max} + X_{min}) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2(X_{max} - X_{min})} - \frac{3 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2(X_{max} - X_{min})} \right) + \\ &\qquad - \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} (X_{max}^3 - X_{min}^3) + \\ &\qquad\qquad \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \Bigg)\end{aligned}$$

$$\begin{aligned}\gamma_1 = &- \frac{1}{X_{max} - X_{min}} \Bigg(\\ &\frac{3(X_{max} + X_{min})^2 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2} - \frac{3(X_{max} + X_{min}) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2} + \\ &- \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} (X_{max}^3 - X_{min}^3) \\ &+ \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \Bigg)\end{aligned}$$

$$
\begin{aligned}
s(x) =& \gamma_0 + \gamma_1 X + \gamma_2 X^2 + \gamma_3 X^3 + \sum_{j=1}^{k} \beta_j (x - t_j)_+^3 \\
&= \gamma_0 + \\
&- \frac{X}{X_{max} - X_{min}} \Bigg(\\
&\frac{3(X_{max} + X_{min})^2 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2} + \\
&- \frac{3(X_{max} + X_{min}) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2} + \\
&- \frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} (X_{max}^3 - X_{min}^3) \\
&+ \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^3 \Bigg) \\
&+ \left(\frac{3(X_{max} + X_{min}) \sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{2(X_{max} - X_{min})} - \frac{3 \sum_{j=1}^{k} \beta_j (X_{max} - t_j)^2}{2(X_{max} - X_{min})} \right) X^2 \\
&- \left(\frac{\sum_{j=1}^{k} \beta_j (X_{max} - t_j)}{X_{max} - X_{min}} \right) X^3 \\
&+ \sum_{j=1}^{k} \beta_j (x - t_j)_+^3
\end{aligned}
$$

$$
s(x) = \gamma_0 + \sum_{j=1}^{k} \beta_j s_j(x)
$$

$$
s_j(x) = a_j x + b_j x^2 + c_j x^3 + (x - t_j)_+^3
$$

with

$$
\begin{aligned}
a_j = & - \frac{1}{X_{max} - X_{min}} \Bigg(\\
& \frac{3(X_{max} + X_{min})^2 (X_{max} - t_j)}{2} + \\
& - \frac{3(X_{max} + X_{min})(X_{max} - t_j)^2}{2} + \\
& - \frac{(X_{max} - t_j)}{X_{max} - X_{min}} (X_{max}^3 - X_{min}^3) \\
& + (X_{max} - t_j)^3 \Bigg) = \\
& - \frac{1}{X_{max} - X_{min}} \Bigg(\\
& \frac{X_{max}^2 + X_{min}^2 + 4X_{max}X_{min}}{2} (X_{max} - t_j) + \\
& - \frac{3(X_{max} + X_{min})}{2} (X_{max} - t_j)^2 + \\
& + (X_{max} - t_j)^3 \Bigg)
\end{aligned}
$$

$$
b_j = \frac{3(X_{max} + X_{min})(X_{max} - t_j)}{2(X_{max} - X_{min})} - \frac{3(X_{max} - t_j)^2}{2(X_{max} - X_{min})}
$$

$$
c_j = - \frac{(X_{max} - t_j)}{X_{max} - X_{min}}
$$

Assuming that the period of x is defined on $[0, T]$ (i.e., $X_{min} = 0, X_{max} = T$) a_j, b_j and c_j simplify to

$$
\begin{aligned}
a_j &= -\frac{1}{T} \Big(\frac{3T^2(T - t_j)}{2} - \frac{3T(T - t_j)^2}{2} - \frac{T^3(T - t_j)}{T} + (T - t_j)^3 \Big) = \\
&= -\frac{3T(T - t_j)}{2} + \frac{3(T - t_j)^2}{2} + \frac{T(T - t_j)}{1} - \frac{(T - t_j)^3}{T} - \frac{T(T - t_j)}{2} + \frac{3(T - t_j)^2}{2} - \frac{(T - t_j)^3}{T} \\
b_j &= \frac{3(T - t_j)}{2} - \frac{3(T - t_j)^2}{2T} \\
c_j &= -\frac{T - t_j}{T},
\end{aligned}
$$

which are the same values reported by Zhang et al., 2000 (but note that c_j was reported omitting the minus sign in the original paper, which should be regarded as a typo, as the equations produce a periodic cubic spline using the formulas reported above.)