# Lista 3

November 23, 2018

## 1 Lista 3

2018/3
Aluno: Pedro Bandeira de Mello Martins
Disciplina: CPE773 - Otimização Convexa
Professor: Wallace A. Martins PEE/COPPE - UFRJ
Exercícios 6.1, 6.2, 6.3, 7.7 e 7.8 do livro texto.

```
In [1]: import sys
        if '../..' not in sys.path:
            sys.path.append('../..')

        import time
        import numpy as np
        from scipy import optimize

        from functions import functionObj, functionObj_multiDim
        from functions import exercise61, rosenbrock, exercise54
        from models.optimizers import ConjugateGradientAlgorithm,\
                                      FibonacciSearch,\
                                      InexactLineSearch,\
                                      BacktrackingLineSearch,\
                                      FletcherReevesAlgorithm,\
                                      QuasiNewtonAlgorithm,\
                                      SteepestDescentAlgorithm
        from run_exercises import run_exercise, plot_contour
```

### 1.1 Exercício 6.1 do Antoniou

Use ConjugateGradientAlgorithm to minimize $f(x) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x}$

```
In [2]: x_0 = np.zeros(16)
        f_x = exercise61

        print('-----------Non-linear Conjugate from Scipy-----------')

        f_x_obj = functionObj(f_x)
        start_time_scipy = time.process_time()
```

```
        x_min, f_min, _, _, _, = optimize.fmin_cg(f_x_obj, x_0, full_output=True)
        scipy_time_spent = time.process_time() - start_time_scipy
        x_min = f_x_obj.best_x
        f_min = f_x_obj.best_f
        print('x*: ', x_min)
        print('F(x*): ', f_min)
        print('Function evals: %d\nGradient evals: %d\nAll evals: %d'%(f_x_obj.fevals, f_x_obj
        print('Time: %f s'%scipy_time_spent)

        print('-----------ConjugateDescentAlgorithm-----------')
        f_x_obj = functionObj(f_x)

        opt = ConjugateGradientAlgorithm(f_x_obj,x_0,  1e3, xtol=1e-6)
        conju_start_time = time.process_time()
        opt.find_min()
        conjugate_spent_time = time.process_time() - conju_start_time
        x_min = f_x_obj.best_x._value
        f_min = f_x_obj.best_f._value
        print('x*: ', x_min)
        print('F(x*): ', f_min)
        print('Function evals: %d\nGradient evals: %d\nAll evals: %d'%(f_x_obj.fevals, f_x_obj
        print('Time: %f s'%conjugate_spent_time)
```

```
-----------Non-linear Conjugate from Scipy-----------
Optimization terminated successfully.
         Current function value: -0.119076
         Iterations: 17
         Function evaluations: 594
         Gradient evaluations: 33
x*:  [0.03423645132612283, 0.024233822238798948, 0.024233822278653335, 0.03423645137013724, -0
F(x*):  -0.11907560047253418
Function evals: 594
Gradient evals: 0
All evals: 594
Time: 0.412917 s
-----------ConjugateDescentAlgorithm-----------
x*:  [ 0.03423704  0.02423337  0.02423337  0.03423704 -0.00143237 -0.02088798
 -0.02088798 -0.00143237  0.03321913  0.02738606  0.02738606  0.03321913
 -0.00663057 -0.00486331 -0.00486331 -0.00663057]
F(x*):  -0.119075600509808
Function evals: 9
Gradient evals: 9
All evals: 18
Time: 0.059335 s
```

O algoritmo Conjugate Descent implementado encontrou o mesmo $f(x^*)$ que o algoritmo Non-Linear Conjugate implementado na biblioteca pública SciPy. Para uma função quadrática, o Conjugate Descent Algorithm deveria resolver o problema em uma avaliação de função.

## 1.2 Exercício 6.2

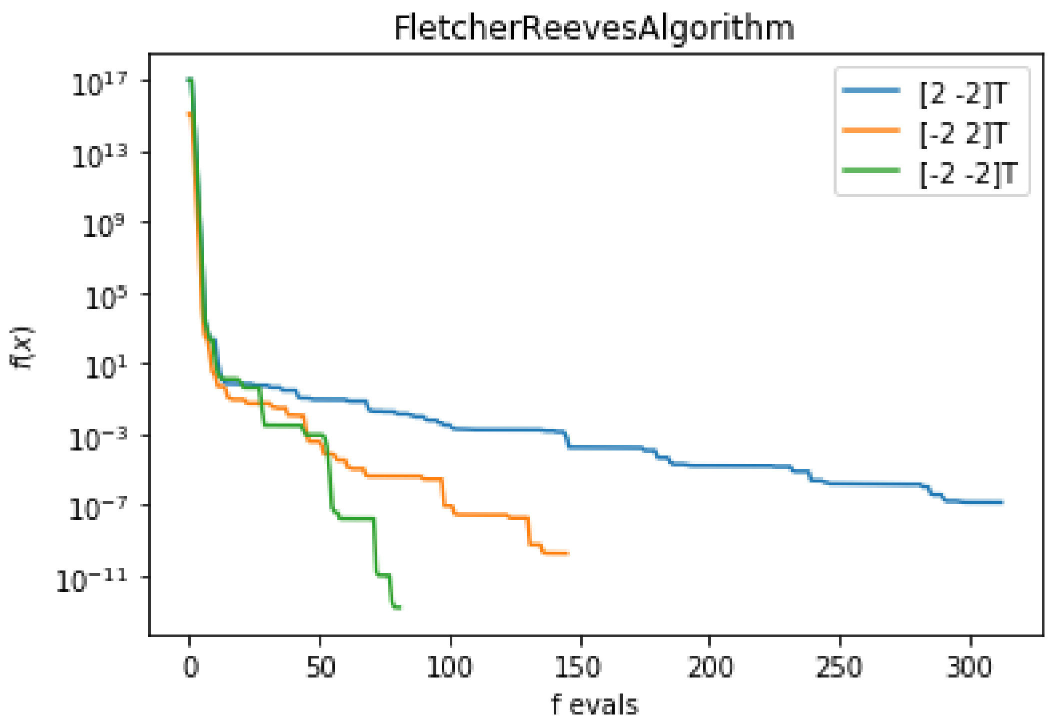
Use Fletcher-Reeves algorithm to find the minimizer of the Rosenbrock function.

$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$

Use $\epsilon = 10^{-6}$ and three initial points:

$\mathbf{x}_0 = [-2\ \ 2]^T$ $\mathbf{x}_0 = [2\ \ -2]^T$ $\mathbf{x}_0 = [-2\ \ -2]^T$

```
In [3]: solution_62 = run_exercise(rosenbrock, opt=FletcherReevesAlgorithm, line_search=Fibona
```



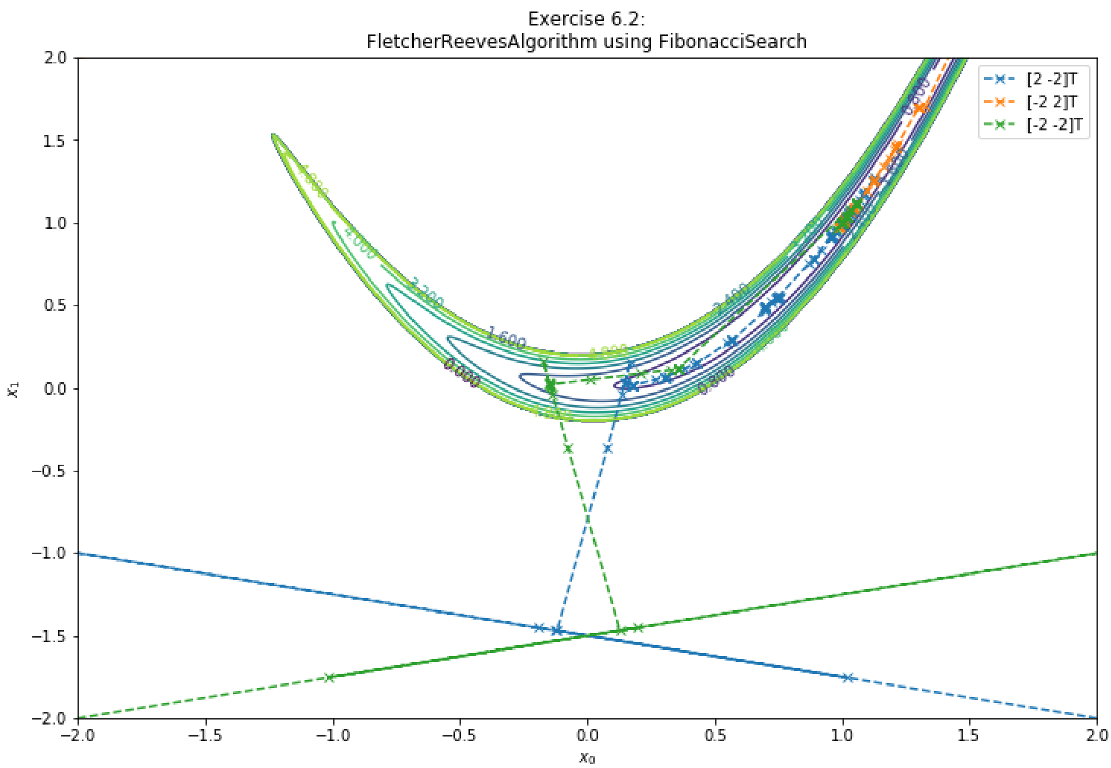

```
In [4]: solution_62[['best_x', 'best_f', 'fevals', 'grad_evals', 'nevals', 'run_time (s)']]
```

```
Out[4]:                                              best_x        best_f fevals  \
         [2 -2]T    [0.999622162281289, 0.9992424615518044]  1.431637e-07   2409
         [-2 2]T   [1.0000136448989594, 1.0000273266043576]  1.863174e-10   4818
         [-2 -2]T  [1.0000003977056775, 1.0000007978700378]  1.587742e-13    396

                   grad_evals nevals  run_time (s)
         [2 -2]T           73   2482      0.212877
         [-2 2]T          146   4964      0.390884
         [-2 -2]T          12    408      0.030569
```

O algoritmo de FletcherReeves utilizando a busca em linha por Fibonacci encontrou o mínimo para todos os $x_0$. $x_0 = [-2\ \ -2]^T$ atingiu o mínimo com menos avaliações e com menor erro em relação ao mínimo global $x^* = [1.0, 1.0]$. O gráfico em [3] apresenta só as avaliações que modificaram o menor $f(x)$ encontrado.

```
In [5]: results_1 = np.array(solution_62.all_best_x['[2 -2]T'])
        results_2 = np.array(solution_62.all_best_x['[-2 2]T'])
        results_3 = np.array(solution_62.all_best_x['[-2 -2]T'])
        name_1 = '[2 -2]T'
        name_2 = '[-2 2]T'
        name_3 = '[-2 -2]T'
        title = 'Exercise 6.2:\nFletcherReevesAlgorithm using FibonacciSearch'
        plot_contour(functionObj(rosenbrock),
                     region=([-2,2], [-2, 2]),
                     mask=5,
                     optimizers=[results_1, results_2, results_3],
                     names = [name_1, name_2, name_3],
                     title=title)
```




O gráfico em [5] mostra a evolução do melhor $f(x)$ encontrado para cada $x_0$ utilizando o algoritmo de Fletcher-Reeves com a busca em linha Fibonacci.

### 1.3 Exercício 6.3

minimize $f(x) = 5x_1^2 - 9x_1x_2 + 4.075x_2^2 + x_1$

with $\mathbf{x}_0 = [1 \;\; 1]^T$ and $\epsilon = 3 \times 10^{-6}$

Solução encontrada utilizando máximo de 1000 iterações.

```
In [6]: solution_63 = run_exercise(exercise54, opt=ConjugateGradientAlgorithm, plot_charts=Fals
```

```
In [7]: f_x = functionObj(exercise54)
        opt = SteepestDescentAlgorithm(func = f_x, x_0 = np.array([1.0, 1.0]),  xtol=3e-6)
        sda_start_time = time.process_time()
        opt.find_min()
        sda_spent_time = time.process_time() - sda_start_time
        x_min = f_x.best_x
        f_min = f_x.best_f
        print(SteepestDescentAlgorithm.__name__)
        print('x*: ', x_min)
        print('F(x*): ', f_min)
        print('Function evals: %d\nGradient evals: %d\nAll evals: %d'%(f_x.fevals, f_x.grad_eva
        print('Time: %f s'%sda_spent_time)

        results_sda = np.array(f_x.all_best_x)
        name_sda = 'SteepestDescentAlgorithm without line search'
```

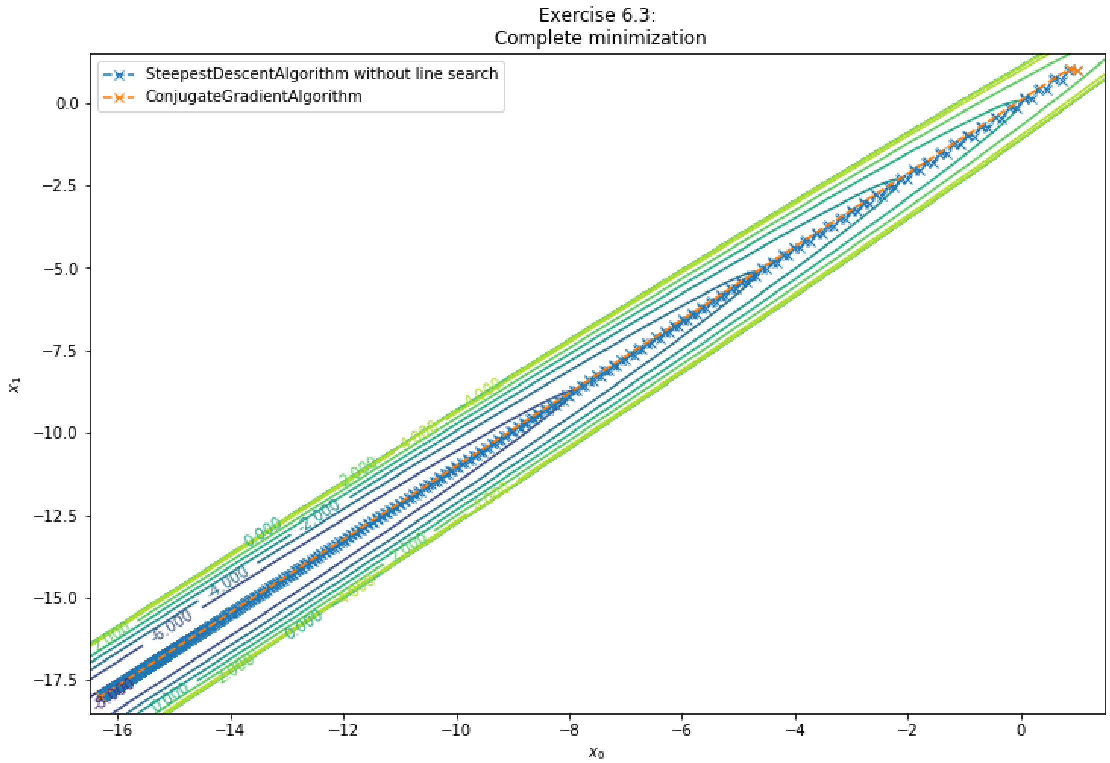
```
SteepestDescentAlgorithm
x*:  [-16.293973644671954, -17.993381119524216]
F(x*):  -8.14999888070452
Function evals: 2001
Gradient evals: 1000
All evals: 3001
Time: 0.821125 s
```

```
In [8]: f_x = functionObj(exercise54)
        opt = ConjugateGradientAlgorithm(func = f_x, x_0 = np.array([1.0, 1.0]),  xtol=3e-6)
        conju_start_time = time.process_time()
        opt.find_min()
        conjugate_spent_time = time.process_time() - conju_start_time
        x_min = f_x.best_x._value
        f_min = f_x.best_f._value
        print(ConjugateGradientAlgorithm.__name__)
        print('x*: ', x_min)
        print('F(x*): ', f_min)
        print('Function evals: %d\nGradient evals: %d\nAll evals: %d'%(f_x.fevals, f_x.grad_eva
        print('Time: %f s'%conjugate_spent_time)

        results_cga = np.array(f_x.all_best_x)
        name_cga = 'ConjugateGradientAlgorithm'
```

```
ConjugateGradientAlgorithm
x*:  [-16.3 -18. ]
F(x*):  -8.149999999999824
Function evals: 3
Gradient evals: 3
All evals: 6
Time: 0.021231 s
```

```
In [9]: title = 'Exercise 6.3:\n Complete minimization'
        results_cga = np.array(
            list(map(lambda x: list(map(lambda x: x if not hasattr(x, '_value') else x._value,
        plot_contour(f_x,
                     region=([-16.5,1.5], [-18.5, 1.5]),
                     mask=5,
                     optimizers=[results_sda, results_cga],
                     names=[name_sda, name_cga],
                     title=title)
```




### 1.3.1 a) Perform 1 iteration on ConjugateGradientAlgorithm and SteepestGradientAlgorithm

```
In [10]: f_x_sda = functionObj(exercise54)
         _=SteepestDescentAlgorithm(func = f_x_sda, x_0=np.array([1.0, 1.0]),  xtol=3e-6, maxI
         f_x_cga = functionObj(exercise54)
         _=ConjugateGradientAlgorithm(func=f_x_cga, x_0 = np.array([1.0, 1.0]),  xtol=3e-6, ma
```

### 1.3.2 b) Compare the results of the first iteration obtained by both algorithms

```
In [11]: print('Melhores resultados da primeira iteração do SteepestDescentAlgorithm:\n f(x) =
         print(' x = ', f_x_sda.best_x)
```

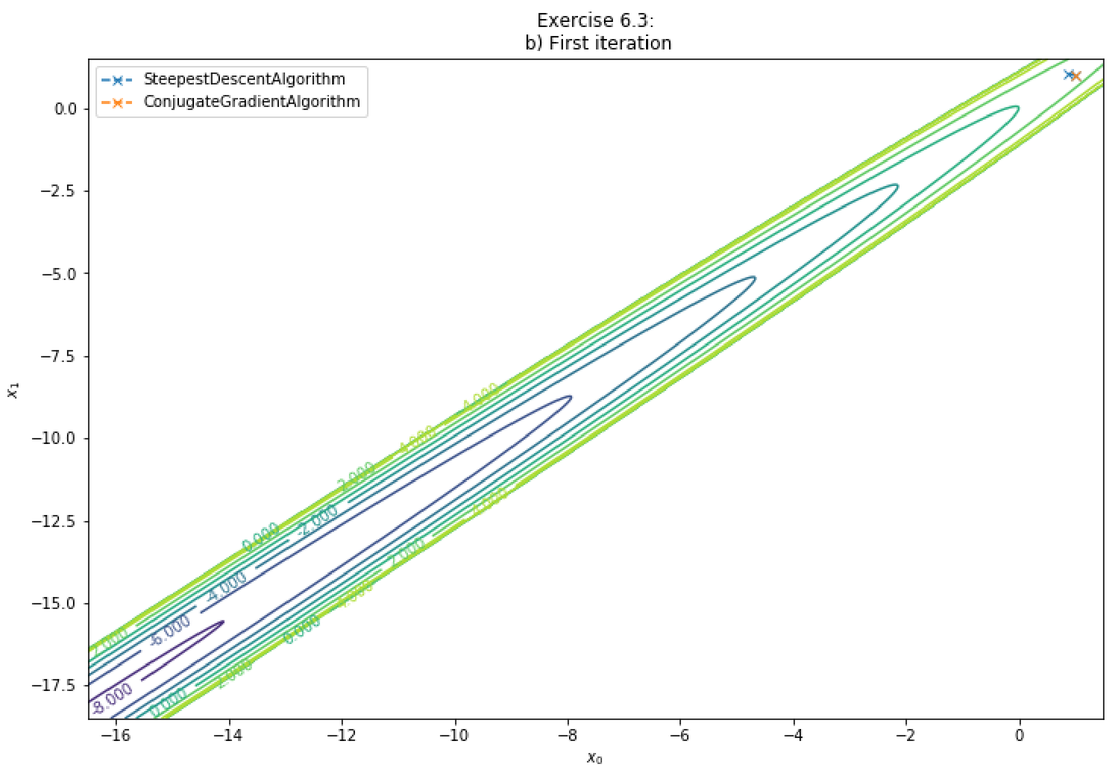
```
Melhores resultados da primeira iteração do SteepestDescentAlgorithm:
 f(x) = 0.929213099
```

```
x =  [0.876517183166723, 1.0524801971541426]
```

```
In [12]: print('Melhores resultados da primeira iteração do ConjugateGradientAlgorithm:\n f(x)
         print(' x = ', f_x_cga.best_x._value)
```

```
Melhores resultados da primeira iteração do ConjugateGradientAlgorithm:
 f(x) = 1.075000000
 x =  [1. 1.]
```

```
In [13]: title = 'Exercise 6.3:\n b) First iteration'
         results_sda = np.array(f_x_sda.all_best_x)
         results_cga = np.array(f_x_cga.all_best_x)
         results_cga = np.array(
             list(map(lambda x: list(map(lambda x: x if not hasattr(x, '_value') else x._value
         name_sda = 'SteepestDescentAlgorithm'
         name_cga = 'ConjugateGradientAlgorithm'
         plot_contour(f_x_sda,
                      region=([-16.5,1.5], [-18.5, 1.5]),
                      mask=5,
                      optimizers=[results_sda, results_cga],
                      names=[name_sda, name_cga],
                      title=title)
```




O SteepestDescentAlgorithm encontrou um mínimo menor na primeira iteração.

### 1.3.3 c) Compare the results of the second iteration obtained by both algorithms.

In [14]:
```
f_x_sda = functionObj(exercise54)
_=SteepestDescentAlgorithm(func = f_x_sda, x_0=np.array([1.0, 1.0]),  xtol=3e-6, maxI
f_x_cga = functionObj(exercise54)
_=ConjugateGradientAlgorithm(func=f_x_cga, x_0 = np.array([1.0, 1.0]),  xtol=3e-6, ma
```

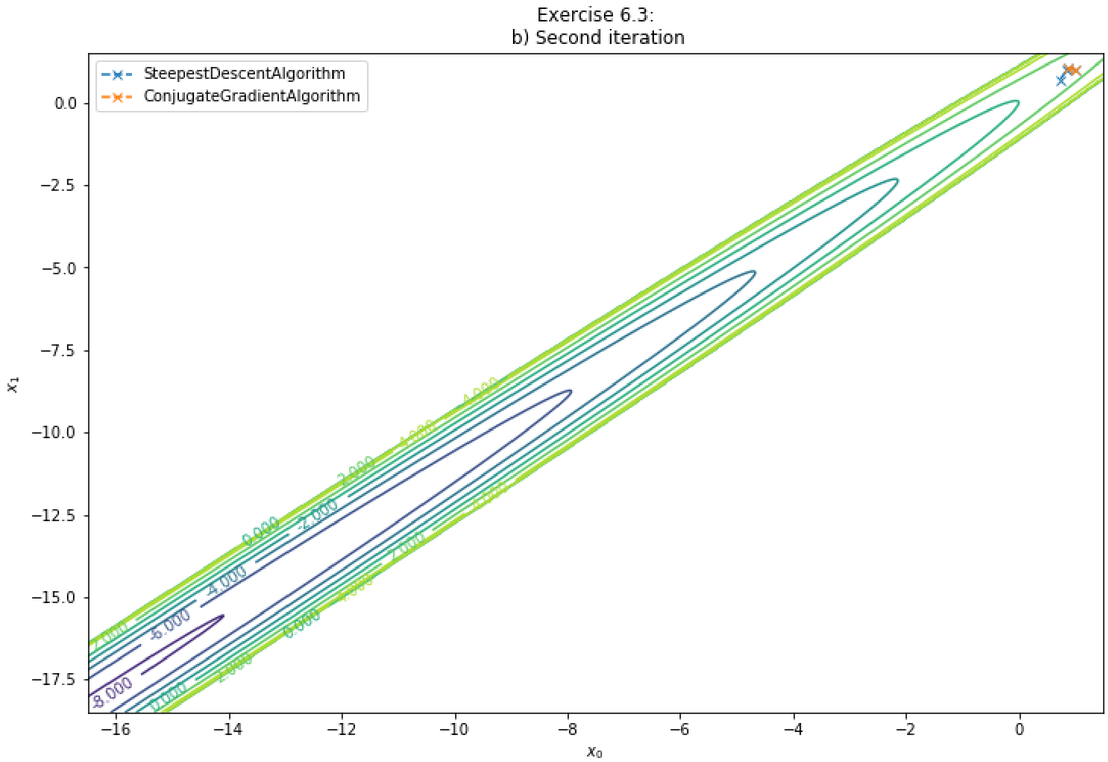
In [15]:
```
print('Melhores resultados da segunda iteração do SteepestDescentAlgorithm:\n f(x) = %
print(' x = ', f_x_sda.best_x)
```

```
Melhores resultados da segunda iteração do SteepestDescentAlgorithm:
 f(x) = 0.785730136
 x =  [0.7266001755240615, 0.699734296818468]
```

In [16]:
```
print('Melhores resultados da segunda iteração do ConjugateGradientAlgorithm:\n f(x) =
print(' x = ', f_x_cga.best_x._value)
```

```
Melhores resultados da segunda iteração do ConjugateGradientAlgorithm:
 f(x) = 0.929213099
 x =  [0.87651718 1.0524802 ]
```

In [17]:
```
title = 'Exercise 6.3:\n b) Second iteration'
results_sda = np.array(f_x_sda.all_best_x)
results_cga = np.array(f_x_cga.all_best_x)
results_cga = np.array(
    list(map(lambda x: list(map(lambda x: x if not hasattr(x, '_value') else x._value
name_sda = 'SteepestDescentAlgorithm'
name_cga = 'ConjugateGradientAlgorithm'
plot_contour(f_x_sda,
             region=([-16.5,1.5], [-18.5, 1.5]),
             mask=5,
             optimizers=[results_sda, results_cga],
             names=[name_sda, name_cga],
             title=title)
```




Novamente o SteepestDescentAlgorithm encontrou um mínimo menor na segunda iteração. Porém, podemos ver pelo gráfico de [9] e os resultados em [8] que o ConjugateGradientAlgorithm encontra o mínimo da função na terceira iteração, enquanto o SteepestDescentAlgorithm necessita de mais iterações.

### 1.4 Exercício 7.7

Implement a quasi-Newton algorithm based on the DFP formula and minimize:

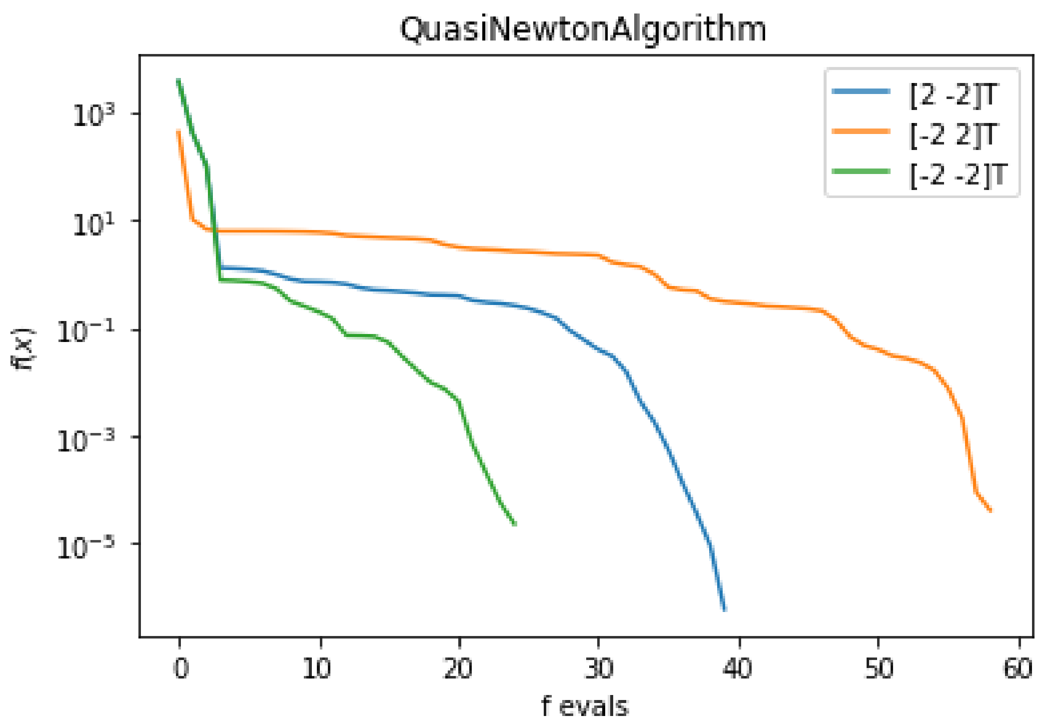
$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$

Use $\epsilon = 10^{-6}$ and three initial points:

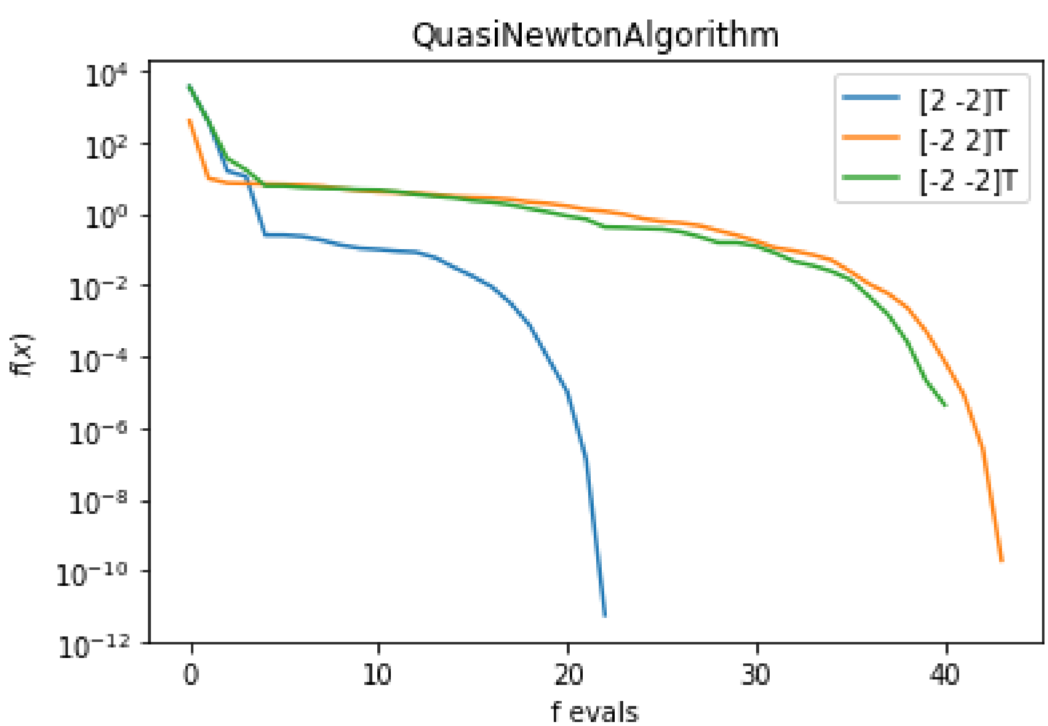
$\mathbf{x}_0 = [-2\ \ 2]^T$ $\mathbf{x}_0 = [2\ \ -2]^T$ $\mathbf{x}_0 = [-2\ \ -2]^T$

Foram escolhidos os mesmos pontos iniciais que no exercício 6.2, para podermos comparar os resultados.

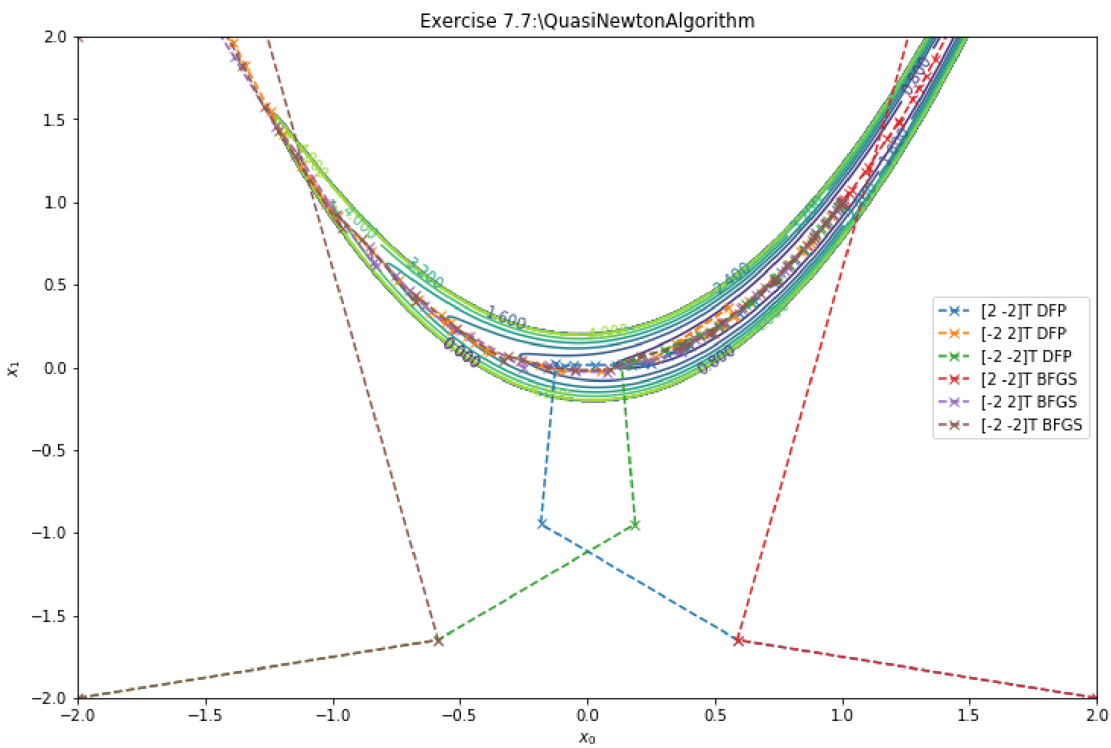
```
In [18]: solution_77_DFP = run_exercise(rosenbrock, QuasiNewtonAlgorithm, formula='DFP')
```




```
In [19]: solution_77_BFGS = run_exercise(rosenbrock, QuasiNewtonAlgorithm, formula='BFGS')
```



```
In [20]: results_1 = np.array(solution_77_DFP.all_best_x['[2 -2]T'])
         results_2 = np.array(solution_77_DFP.all_best_x['[-2 2]T'])
         results_3 = np.array(solution_77_DFP.all_best_x['[-2 -2]T'])
         results_4 = np.array(solution_77_BFGS.all_best_x['[2 -2]T'])
         results_5 = np.array(solution_77_BFGS.all_best_x['[-2 2]T'])
         results_6 = np.array(solution_77_BFGS.all_best_x['[-2 -2]T'])
         name_1 = '[2 -2]T DFP'
         name_2 = '[-2 2]T DFP'
         name_3 = '[-2 -2]T DFP'
         name_4 = '[2 -2]T BFGS'
         name_5 = '[-2 2]T BFGS'
         name_6 = '[-2 -2]T BFGS'
         title = 'Exercise 7.7:\QuasiNewtonAlgorithm'
         plot_contour(functionObj(rosenbrock),
                      region=([-2,2], [-2, 2]),
                      mask=5,
                      optimizers=[results_1, results_2, results_3, results_4, results_5, resul
                      names = [name_1, name_2, name_3, name_4, name_5, name_6],
                      title=title)
```



### 1.4.1 Compare the results with those obtained in Exercise 6.2

**Resultado do QuasiNewton com DFP na minimização da função de Rosenbrock.**

```
In [21]: solution_77_DFP[['best_x', 'best_f', 'fevals', 'grad_evals', 'nevals', 'run_time (s)']]

Out[21]:                                              best_x        best_f fevals  \
         [2 -2]T   [1.0006104751973983, 1.0012711872028497]  6.213231e-07     56
         [-2 2]T    [0.9944433187890923, 0.989249583948909]  4.190373e-05     80
         [-2 -2]T  [0.9957373840446976, 0.9917225990189015]  2.344431e-05     34

                  grad_evals nevals  run_time (s)
         [2 -2]T        2039   2095      1.155622
         [-2 2]T        2058   2138      1.089518
         [-2 -2]T       2025   2059      1.066176
```

**Resultado do QuasiNewton com BFGS na minimização da função de Rosenbrock.**

```
In [22]: solution_77_BFGS[['best_x', 'best_f', 'fevals', 'grad_evals', 'nevals', 'run_time (s)']]

Out[22]:                                              best_x        best_f fevals  \
         [2 -2]T    [0.9999978890274238, 0.999995887097943]  5.645148e-12     37
         [-2 2]T   [0.9999904366992792, 0.9999798639392066]  1.933760e-10     54
         [-2 -2]T  [1.0002026307799345, 1.0001969799541746]  4.380893e-06     49

                  grad_evals nevals  run_time (s)
         [2 -2]T        2027   2064      1.069981
         [-2 2]T        2044   2098      1.132062
         [-2 -2]T       2040   2089      1.141904
```

**Resultado do Fletcher-Reeves Algorithm com Fibonacci Search na minimização da função de Rosenbrock.**

```
In [23]: solution_62[['best_x', 'best_f', 'fevals', 'grad_evals', 'nevals', 'run_time (s)']]

Out[23]:                                              best_x        best_f fevals  \
         [2 -2]T    [0.999622162281289, 0.9992424615518044]  1.431637e-07   2409
         [-2 2]T   [1.0000136448989594, 1.0000273266043576]  1.863174e-10   4818
         [-2 -2]T  [1.0000003977056775, 1.0000007978700378]  1.587742e-13    396

                  grad_evals nevals  run_time (s)
         [2 -2]T          73   2482      0.212877
         [-2 2]T         146   4964      0.390884
         [-2 -2]T         12    408      0.030569
```

Com essas tabelas, podemos verificar que o Fletcher-Reeves encontra os mínimos em menos tempo de processamento que os algoritmos de QuasiNewton, porém com maior quantidade de avaliações de função.

### 1.5 Exercício 7.8

minimize $f(x) = 5x_1^2 - 9x_1x_2 + 4.075x_2^2 + x_1$
with $\mathbf{x}_0 = [0 \ \ 0]^T$ and $\epsilon = 3 \times 10^-7$

```
In [24]: f_x = functionObj(exercise54)
         opt = QuasiNewtonAlgorithm(func = f_x, x_0 = np.array([0.0, 0.0]), formula='BFGS', xto
         conju_start_time = time.process_time()
         opt.find_min()
         conjugate_spent_time = time.process_time() - conju_start_time
         x_min = f_x.best_x
         f_min = f_x.best_f
         print(QuasiNewtonAlgorithm.__name__ + ' with BFGS')
         print('x*: ', x_min)
         print('F(x*): ', f_min)
         print('Function evals: %d\nGradient evals: %d\nAll evals: %d'%(f_x.fevals, f_x.grad_ev
         print('Time: %f s'%conjugate_spent_time)
         result_bfgs = np.array(f_x.all_best_x)
         name_bfgs = '[0.0, 0.0] BFGS'
```

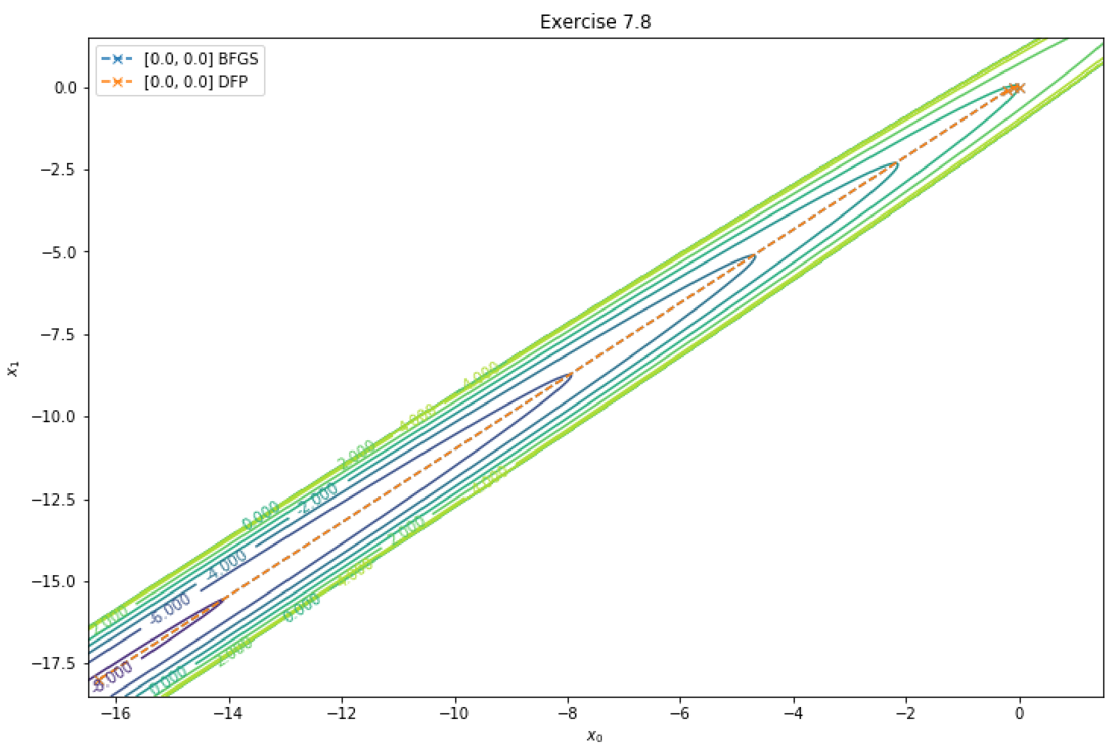
```
QuasiNewtonAlgorithm with BFGS
x*:  [-16.299999999999624, -17.99999999999958]
F(x*):  -8.149999999999988
Function evals: 5
Gradient evals: 2004
All evals: 2009
Time: 1.352772 s
```

```
In [25]: f_x = functionObj(exercise54)
         opt = QuasiNewtonAlgorithm(func = f_x, x_0 = np.array([0.0, 0.0]), formula='DFP', xtol
         conju_start_time = time.process_time()
         opt.find_min()
         conjugate_spent_time = time.process_time() - conju_start_time
         x_min = f_x.best_x
         f_min = f_x.best_f
         print(QuasiNewtonAlgorithm.__name__ + ' with DFP')
         print('x*: ', x_min)
         print('F(x*): ', f_min)
         print('Function evals: %d\nGradient evals: %d\nAll evals: %d'%(f_x.fevals, f_x.grad_ev
         print('Time: %f s'%conjugate_spent_time)
         result_DFP = np.array(f_x.all_best_x)
         name_DFP = '[0.0, 0.0] DFP'
```

```
QuasiNewtonAlgorithm with DFP
x*:  [-16.300000000000043, -18.000000000000046]
F(x*):  -8.149999999999725
Function evals: 5
Gradient evals: 2004
```

```
All evals: 2009
Time: 1.495161 s
```

```
In [26]: title = 'Exercise 7.8'
         results_cga = np.array(
             list(map(lambda x: list(map(lambda x: x if not hasattr(x, '_value') else x._value
         plot_contour(f_x,
                      region=([-16.5,1.5], [-18.5, 1.5]),
                      mask=5,
                      optimizers=[result_bfgs, result_DFP],
                      names=[name_bfgs, name_DFP],
                      title=title)
```




As duas modificações do algoritmo funcionam de forma quase igual. Inclusive utilizaram a mesma quantidade de avaliações de funções (5) e obtiveram resultados muito parecidos.