Chapter 8. Jordan Normal Form and Spectrum

8.1. Direct sum decomposition

Empty

8.2. Jordan normal form

Program: jordan.py

In [1]:
```
from sympy import *
from numpy.random import seed, permutation
from functools import reduce

A = diag(1, 2, 2, 2, 2, 3, 3, 3, 3, 3)
A[1, 2] = A[3, 4] = A[5, 6] = A[7, 8] = A[8, 9] = 1

seed(123)
for n in range(10):
    P = permutation(10)
    for i, j in [(P[2 * k], P[2 * k + 1]) for k in range(5)]:
        A[:, j] += A[:, i]
        A[i, :] -= A[j, :]

B = Lambda(S('lmd'), A - S('lmd') * eye(10))
x = Matrix(var('x0, x1, x2, x3, x4, x5, x6, x7, x8, x9'))
y = Matrix(var('y0, y1, y2, y3, y4, y5, y6, y7, y8, y9'))
z = Matrix(var('z0, z1, z2, z3, z4, z5, z6, z7, z8, z9'))
```

In [2]:
```
A
```

Out[2]:

$$\begin{bmatrix} 35 & 28 & 24 & 6 & 26 & 16 & -6 & 14 & 26 & 42 \\ -5 & -8 & -9 & -24 & -21 & -2 & -5 & -3 & -5 & -17 \\ 11 & 14 & 10 & 5 & 7 & 8 & 2 & 6 & 13 & 13 \\ 5 & 2 & 4 & 2 & 5 & 1 & -3 & 1 & 2 & 7 \\ 1 & 4 & 4 & 6 & 8 & 0 & -1 & 2 & 0 & 5 \\ 19 & 11 & 14 & 19 & 25 & 13 & 6 & 7 & 16 & 31 \\ 8 & 11 & 5 & 7 & 6 & 7 & 7 & 5 & 11 & 10 \\ 6 & 16 & 14 & 40 & 34 & 2 & 8 & 7 & 6 & 26 \\ -27 & -19 & -16 & -6 & -19 & -16 & -2 & -11 & -22 & -33 \\ -20 & -21 & -19 & -11 & -22 & -9 & 5 & -10 & -15 & -28 \end{bmatrix}$$

In [3]:
```
P = A.charpoly(); P
```

Out[3]:

$$\operatorname{PurePoly}\left(\lambda^{10} - 24\lambda^9 + 257\lambda^8 - 1616\lambda^7 + 6603\lambda^6 - 18304\lambda^5 + 34827\lambda^4 - 4485$$

In [4]:
```
factor(P.expr)
```

Out[4]: $$(\lambda - 3)^5 (\lambda - 2)^4 (\lambda - 1)$$

In [5]:
```
a1 = x.subs(solve(B(1) * x)); a1
```

Out[5]:
$$\begin{bmatrix} -\frac{x_9}{3} \\ \frac{5x_9}{3} \\ -x_9 \\ 0 \\ -\frac{x_9}{3} \\ \frac{8x_9}{3} \\ -\frac{x_9}{3} \\ -\frac{8x_9}{3} \\ -2x_9 \\ x_9 \end{bmatrix}$$

In [6]:
```
a2 = x.subs(solve(B(2) * x)); a2
```

Out[6]:
$$\begin{bmatrix} -\frac{x_8}{3} - x_9 \\ -\frac{5x_8}{12} + \frac{x_9}{4} \\ \frac{17x_8}{12} - \frac{5x_9}{4} \\ -\frac{3x_8}{4} + \frac{x_9}{4} \\ \frac{x_8}{12} - \frac{x_9}{4} \\ -\frac{29x_8}{12} + \frac{5x_9}{4} \\ \frac{4x_8}{3} - x_9 \\ \frac{5x_8}{6} - \frac{x_9}{2} \\ x_8 \\ x_9 \end{bmatrix}$$

In [7]:
```
b2 = y.subs(solve(B(2) * y - a2)); b2
```

Out[7]:

$$\left[\begin{matrix}2 y_{7} - 2 y_{8}\\\frac{y_{6}}{6} + \frac{y_{7}}{12} - \frac{17 y_{8}}{24} + \frac{11 y_{9}}{24}\\\frac{5 y_{6}}{6} + \frac{5 y_{7}}{12} - \frac{y_{8}}{24} - \frac{5 y_{9}}{24}\\- \frac{y_{6}}{2} + \frac{5 y_{7}}{4} - \frac{9 y_{8}}{8} + \frac{3 y_{9}}{8}\\- 2 y_{7} + \frac{7 y_{8}}{4} - \frac{5 y_{9}}{4}\\- y_{6} - \frac{5 y_{7}}{2} + y_{8} - y_{9}\\y_{6}\\y_{7}\\y_{8}\\y_{9}\end{matrix}\right]$$

In [8]:
```
a3 = x.subs(solve(B(3) * x)); a3
```

Out[8]:

$$\left[\begin{matrix}- 2 x_{9}\\- \frac{x_{8}}{3} + \frac{x_{9}}{3}\\- \frac{2 x_{8}}{3} + \frac{5 x_{9}}{3}\\\frac{x_{8}}{3} - \frac{4 x_{9}}{3}\\0\\- x_{8}\\- \frac{2 x_{8}}{3} + \frac{5 x_{9}}{3}\\\frac{2 x_{8}}{3} - \frac{2 x_{9}}{3}\\x_{8}\\x_{9}\end{matrix}\right]$$

In [9]:
```
b3 = y.subs(solve(B(3) * y - a3)); b3
```

Out[9]:

$$\left[\begin{matrix}- \frac{y_{6}}{2} + y_{7} - y_{8} - \frac{y_{9}}{2}\\- \frac{y_{7}}{2}\\\frac{7 y_{6}}{6} - \frac{y_{7}}{3} + \frac{y_{8}}{3} - \frac{y_{9}}{2}\\- \frac{2 y_{6}}{3} + \frac{y_{7}}{3} - \frac{y_{8}}{3}\\\frac{y_{6}}{6} - \frac{y_{7}}{3} + \frac{y_{8}}{3} - \frac{y_{9}}{2}\\- \frac{y_{6}}{3} - \frac{5 y_{7}}{6} - \frac{2 y_{8}}{3}\\y_{6}\\y_{7}\\y_{8}\\y_{9}\end{matrix}\right]$$

In [10]:
```
c3 = z.subs(solve(B(3) * z - b3)); c3
```

Out[10]:

$$\left[\begin{matrix}- \frac{15 z_{5}}{11} - \frac{21 z_{6}}{22} - \frac{3 z_{7}}{22} - \frac{21 z_{8}}{11} - \frac{z_{9}}{2}\\\frac{3 z_{5}}{11} + \frac{z_{6}}{11} - \frac{3 z_{7}}{11} + \frac{2 z_{8}}{11}\\- \frac{z_{5}}{11} + \frac{25 z_{6}}{22} - \frac{9 z_{7}}{22} + \frac{3 z_{8}}{11} - \frac{z_{9}}{2}\\- \frac{8 z_{5}}{11} - \frac{10 z_{6}}{11} - \frac{3 z_{7}}{11} - \frac{9 z_{8}}{11}\\z_{5} + \frac{z_{6}}{2} + \frac{z_{7}}{2} + z_{8} - \frac{z_{9}}{2}\\z_{5}\\z_{6}\\z_{7}\\z_{8}\\z_{9}\end{matrix}\right]$$

In [11]:
```
v0 = a1.subs({x9:1})
```

In [12]:
```
v1 = b2.subs({y6:1, y7:0, y8:0, y9:0})
v2 = B(2) * v1
```

In [13]:
```
v3 = b2.subs({y6:0, y7:1, y8:0, y9:0})
v4 = B(2) * v3
```

In [14]:
```
v5 = c3.subs({z5: 1, z6: 0, z7: 0, z8: 0,z9: 0})
v6 = B(3) * v5
v7 = B(3) * v6
```

In [15]:
```
v8 = b3.subs({y6: 1, y7: 0, y8: 0, y9: 0})
v9 = B(3) * v8
```

In [16]:
```
L = [v0, v1, v2, v3, v4, v5, v6, v7, v8, v9]
V = reduce(lambda x, y: x.row_join(y), L)
V**(-1) * A * V
```

Out[16]:

$$\left[\begin{matrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3\end{matrix}\right]$$

In [17]:
```
L = [v9, v8, v7, v6, v5, v4, v3, v2, v1, v0]
U = reduce(lambda x, y: x.row_join(y), L)
U**(-1) * A * U
```

Out[17]:

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

fig. 8.1 Relation between the eigenspace and generalized eigenspace for eigenvalue 1

fig8-1.py

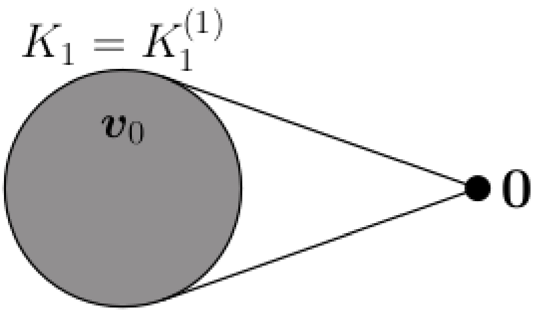


fig. 8.2 Relation between the eigenspace and generalized eigenspace for eigenvalue 2

fig8-2.py

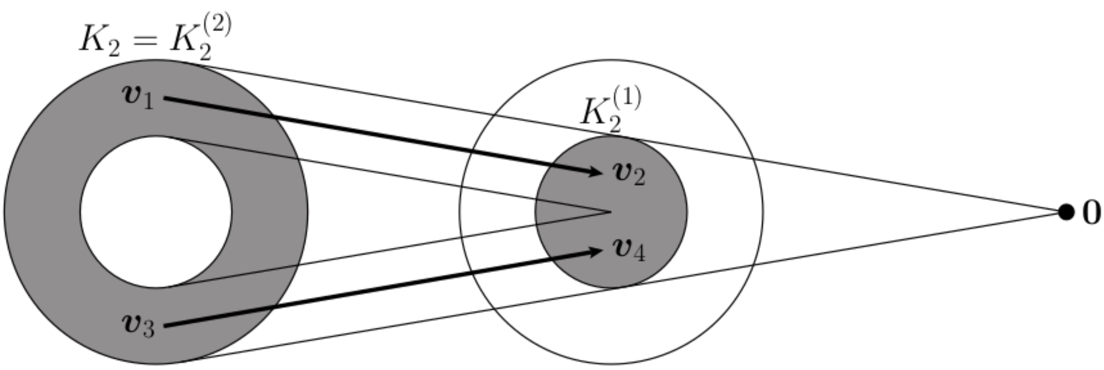
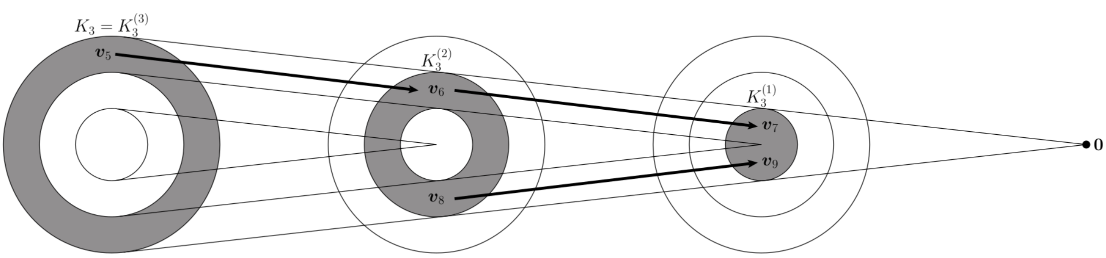


**fig. 8.3 Relation between the eigenspace and generalized eigenspace for eigenvalue 3*

fig8-3.py



Program: jordan2.py

In [18]:
```
from sympy import Matrix, diag
from numpy.random import permutation, seed

X = Matrix([[1, 1, 0], [0, 1, 0], [0, 0, 2]])
Y = Matrix([[2, 1, 0], [0, 2, 1], [0, 0, 2]])
Z = Matrix([[2, 1, 0], [0, 2, 0], [0, 0, 2]])

seed(2021)
while True:
    A = X.copy()
    while 0 in A:
        i, j, _  = permutation(3)
        A[:, j] += A[:, i]
        A[i, :] -= A[j, :]
        if max(abs(A)) >= 10:
            break
    if max(abs(A)) < 10:
        break
U, J = A.jordan_form()
print(f'A = {A}')
print(f'U = {U}')
print(f'U**(-1)*A*U = {J}')
C = U * diag(J[0, 0], J[1, 1], J[2, 2]) * U**(-1)
B = A - C
print(f'B = {B}')
print(f'C = {C}')
```

```
A = Matrix([[2, 4, 4], [-4, 3, -1], [2, -4, -1]])
U = Matrix([[24/7, -4/7, -1/2], [30/7, 1, -1], [-36/7, 0, 1]])
U**(-1)*A*U = Matrix([[1, 1, 0], [0, 1, 0], [0, 0, 2]])
B = Matrix([[8, 8, 12], [10, 10, 15], [-12, -12, -18]])
C = Matrix([[-6, -4, -8], [-14, -7, -16], [14, 8, 17]])
```

8.3. Jordan decomposition and matrix power

Untitled.ipynb

In [1]:
```
from sympy import Matrix, S
J = Matrix([[S('a'), 0, 0], [1, S('a'), 0], [0, 1, S('a')]])
J
```

Out[1]:
$$\begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$$

In [2]:
```
J**2
```

Out[2]:
$$\begin{bmatrix} a^2 & 0 & 0 \\ 2a & a^2 & 0 \\ 1 & 2a & a^2 \end{bmatrix}$$

In [3]:
```
J**3
```

Out[3]:

$$\begin{bmatrix} a^3 & 0 & 0 \\ 3a^2 & a^3 & 0 \\ 3a & 3a^2 & a^3 \end{bmatrix}$$

In [4]:

```
J**S('k')
```

Out[4]:

$$\begin{bmatrix} a^k & 0 & 0 \\ a^{k-1}k & a^k & 0 \\ \frac{a^{k-2}k(k-1)}{2} & a^{k-1}k & a^k \end{bmatrix}$$

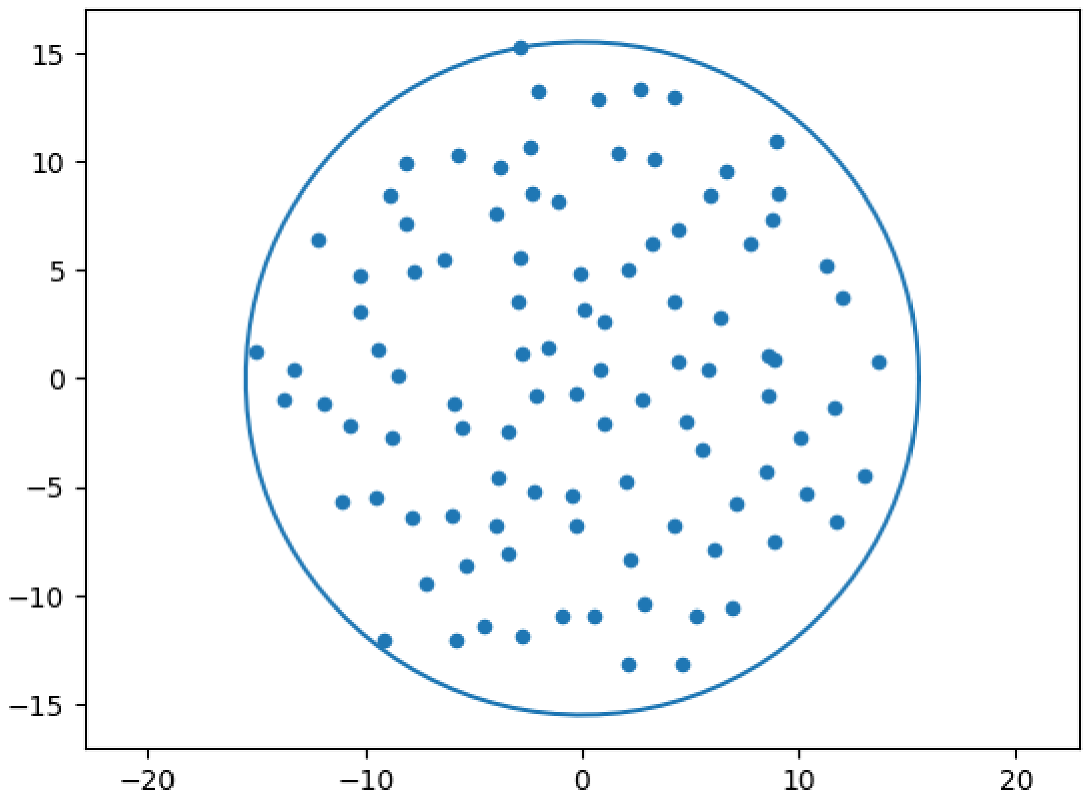
8.4.Spectrum of matrix

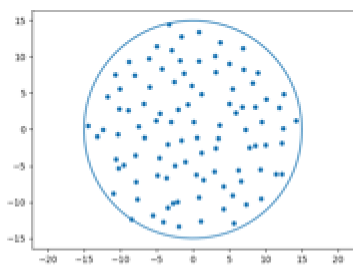
Program: spectrum.py

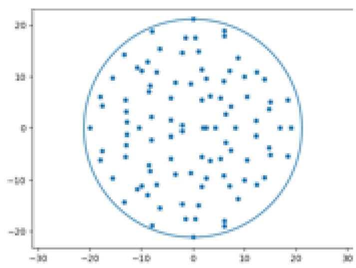
In [1]:

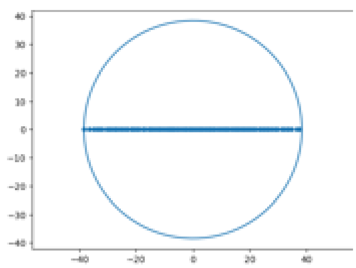
```
from numpy import matrix, pi, sin, cos, linspace
from numpy.random import normal
from numpy.linalg import eig, eigh
import matplotlib.pylab as plt

N = 100
B = normal(0, 1, (N, N, 2))
A = matrix(B[:, :, 0] + 1j * B[:, :, 1])
Real = A + A.conj()
Hermite = A + A.H
PositiveSemiDefinite = A * A.H
PositiveComponents = abs(A)
Unitary = matrix(eigh(Hermite)[1])

X = A
#X = Real
#X = Hermite
#X = PositiveSemiDefinite
#X = PositiveComponents
#X = Unitary
Lmd = eig(X)[0]
r = max(abs(Lmd))
T = linspace(0, 2 * pi, 100)
plt.axis('equal')
plt.plot(r * cos(T), r * sin(T))
plt.scatter(Lmd.real, Lmd.imag, s=20)
```

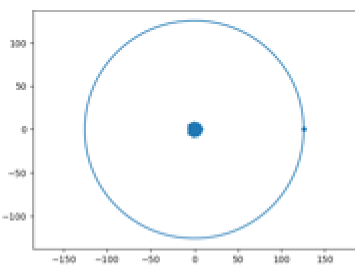
Out[1]: <matplotlib.collections.PathCollection at 0x7f52e7fb20>

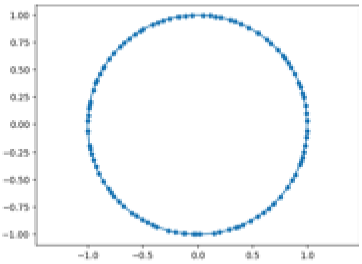



A


Real

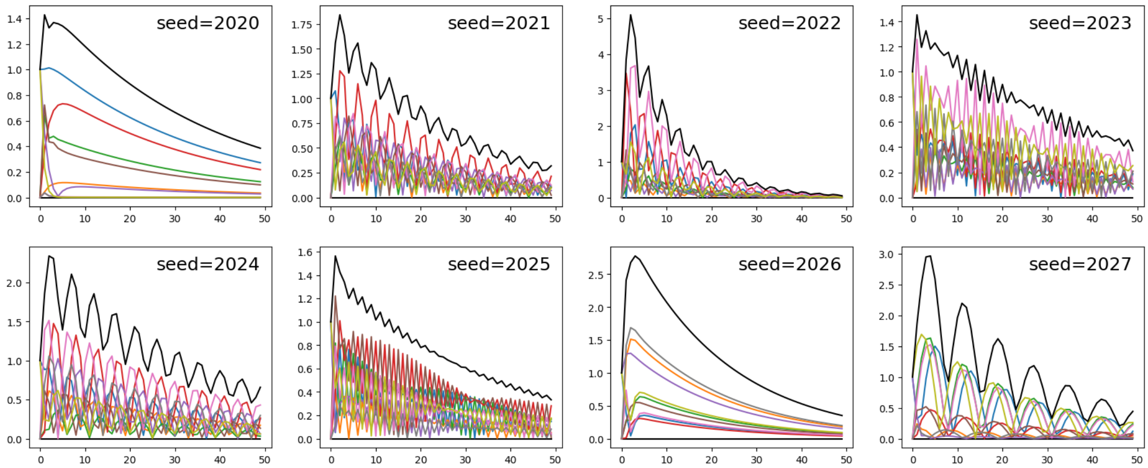

Hermite


PositiveComponents


Unitary

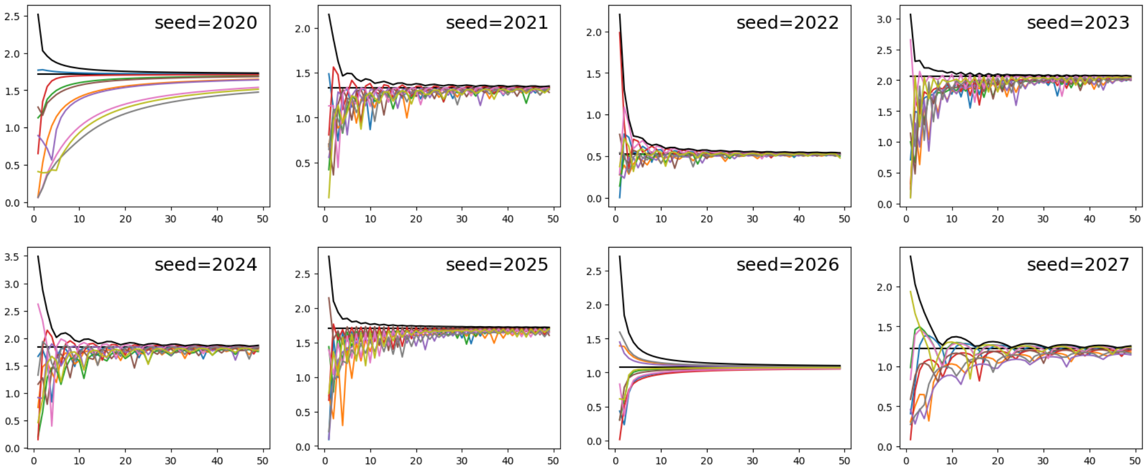
Programs: norm.py

In [1]:
```
from numpy import matrix
from numpy.linalg import eig, norm
from numpy.random import normal, seed
import matplotlib.pyplot as plt

def power(m, s):
    seed(s)
    A = matrix(normal(0, 2, (m, m)))
    lmd = max(abs(eig(A)[0])) + 0.1
    X = range(50)
    P = [(A / lmd)**n for n in X]
    Y = [norm(B, 2) for B in P]
    plt.plot([X[0], X[-1]], [0, 0], c='k')
    for i in range(m):
        for j in range(m):
            plt.plot(X, [abs(B[i, j]) for B in P])
    plt.plot(X, Y, c='k')
    plt.text(max(X), max(Y), f'seed={s}',
             size=18, ha='right', va='top')


plt.figure(figsize=(20, 8))
n = 241
for s in [2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027]:
    plt.subplot(n)
    power(3, s)
    n += 1
```



Program: gelfand.py

In [1]:
```python
from numpy import matrix
from numpy.linalg import eig, norm
from numpy.random import normal, seed
import matplotlib.pyplot as plt

def gelfand(m, s):
    seed(s)
    A = matrix(normal(0, 1, (m, m)))
    lmd = max(abs(eig(A)[0]))
    X = range(1, 50)
    P = [A**n for n in range(50)]
    Y = [norm(P[n], 2)**(1 / n) for n in X]
    plt.plot([X[0], X[-1]], [lmd, lmd], c='k')
    for i in range(m):
        for j in range(m):
            plt.plot(X, [abs(P[n][i, j])**(1 / n) for n in X])
    plt.plot(X, Y, c='k')
    plt.text(max(X), max(Y), f'seed={s}',
             size=18, ha='right', va='top')

plt.figure(figsize=(20, 8))
n = 241
for s in [2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027]:
    plt.subplot(n)
    gelfand(3, s)
    n += 1
```



8.5. Perron-Frobenius theorem

Empty

In [ ]: