Efficient Estimation of a Sparse Delay-Doppler Channel

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***Abstract*—Multiple wireless sensing tasks, e.g., radar detection for driver safety, involve estimating the "channel" or relationship between signal transmitted and received. In this paper, we focus specifically on the *delay-doppler channel*. This channel model has recently become relevant on the heels of the *mmWave* breakthrough, because the signals used experience a significant doppler effect. Additionally, *high resolution* delay-doppler estimation is often desirable, and one standard approach to achieving this is to use signals of large *bandwidth*, which is feasible in the mmWave realm. This approach, however, results in a tension with the desire for efficiency because, in particular, large bandwidth immediately implies that the signals in play live in a space of very high dimension N (e.g., $\sim 10^6$ in some applications), as per the Shannon-Nyquist sampling theorem.**

To address this, in this paper we propose a novel randomized algorithm for channel estimation in the k-sparse setting (e.g., k objects in radar detection), with sampling and space complexity on the order of $k(\log N)^2$, and arithmetic complexity on the order of $k(\log N)^3 + k^2$, for N sufficiently large.

To the best of our knowledge, the algorithm is the first of this nature. It seems to be extremely efficient, yet it is just a simple combination of three ingredients, two of which are well-known and widely used, namely digital chirp signals and discrete Gaussian filter functions, and the third being recent developments in Sparse Fast Fourier Transform algorithms.

***Index Terms*—channel estimation, signal processing algorithms.**

I. Introduction

The process of channel estimation is going on all around us. For instance, all our personal devices are constantly sending and receiving signals – in this case, intuitively speaking, the "channel" is simply the relationship between the signal transmitted and signal received, and devices compute or "estimate" some aspects of this relationship.

In this paper we focus on the *delay-doppler channel*. Our methods are quite general and may serve a variety of applications in wireless communication, but for clarity of exposition of key ideas we choose to discuss the specific application to radar detection. So let's start with an intuitive physical picture of the radar task. Sections I-A to I-C will closely follow standard references such as [1], [2].

A. Motivation and Intuitive Physical Picture

Consider the classical problem of estimating the position and velocity of some object of interest.

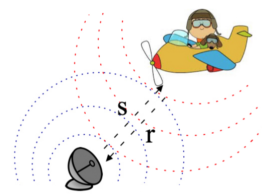


Fig. 1: Radar detection of object in the case of line of sight.

In practice, one approaches this problem using a device called a radar which emits electromagnetic waves in all directions around it – see Fig. 1. The emitted wave or "signal" travels through space and will be reflected back to the radar by certain materials, such as that of our object.

For the rest of this paper, we assume a *line of sight* between the radar and object – see Fig.1 for illustration. As a consequence, the reflection from the object back to the radar is strongest along this direction. Restricting to the line of sight, we denote the signal transmitted by s and the signal reflected/received by r.

The signals s and r are physically related, and this can be utilized to estimate position and velocity of the object of interest. More precisely, a "digital" computational procedure will yield the parameters of interest.

We will eventually describe certain computational challenges in the digital estimation step. However, to see exactly where these challenges come from, it will be to our advantage to first consider a mathematical model of the *continuous* (or *analog*) *channel* relationship just intuitively described, and the corresponding estimation problem.

B. Continuous Channel Model and Estimation Problem

In [1], [2] engineers tell us that we can model the transmitted and received signals s and r, as elements of the Hilbert space $L^2(\mathbb{R})$ of complex-valued L^2-functions of time, equipped with the standard inner product $\langle \cdot, \cdot \rangle$. We refer to elements of this space as *continuous* (or *analog*) *signals*.

It takes time for the signal transmitted to travel the distance to the object and back. This is modeled as a time delay. More precisely, let us denote the distance (often called *range*) between the radar and object by d_0 – see Fig. 2. Then,

$$r(t) = \alpha_0 \cdot s(t - t_0) + \text{Noise},$$

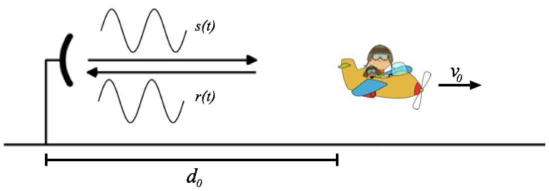


Fig. 2: Illustrating range and radial velocity of object.

where, $t_0 = \frac{2d_0}{c}$ with c denoting the speed of light. Moreover, α_0 is known as the *attenuation coefficient* and models loss of energy and, finally, the additive noise term that appears in the equation is intended to account for imprecisions in the model.

Next – and most interestingly – when the object is moving, the signal is also subject to the *Doppler effect*. We will adopt the standard *narrowband* assumption [1], [2], namely, denoting the bandwidth (i.e., length of Fourier support) of s by W and its carrier (i.e., central) frequency by f_c, we assume that $W \ll f_c$. Under this assumption, the doppler effect is modeled by a frequency shift [1], so that we have

$$r(t) = \alpha_0 \cdot e^{2\pi i f_0 t} s(t - t_0) + \text{Noise}, \tag{1}$$

where, $f_0 = -\frac{2f_c v_0}{c}$ with v_0 denoting the object's relative radial velocity – illustrated in Fig. 2. Note that in many contexts, the quantity f_0 is negligible and often ignored. However, in the mmWave setting [3], it may be on the order of *megahertz* (MHz) – even for relatively slow moving objects like cars.

In conclusion, given a good estimate of (t_0, f_0), we will be able to estimate range and radial velocity (d_0, v_0).

Finally, in general we may have more than one object of interest for which we would like to estimate range and radial velocity. In this case, the received signal is a *superposition* of reflections from each object, i.e., the relationship between r and s is given by

$$r(t) = \sum_{j=1}^{k} \alpha_j \cdot h_{t_j,f_j} s(t) + \text{Noise}, \tag{2}$$

where, for $(t_j, f_j) \in \mathbb{R}^2$, we denote by h_{t_j,f_j} the associated *continuous time-frequency shift* operator on $L^2(\mathbb{R})$ given by,

$$h_{t_j,f_j} s(t) = e^{2\pi i f_j t} s(t - t_j).$$

We will refer to Eq. (2) as the *continuous channel model* and refer to the parameter k in Eq. (2) as the *channel sparsity*. In the case of radar detection, k models the number of *targets* in the vicinity of the radar.

We can now formulate the following:

(P1) ***Continuous Estimation Problem:*** Estimate (t_j, f_j) for $j = 1, \ldots, k$.

Remark 1. *In theory, for k objects in generic position, their positions and velocities can be estimated using four non-coplanar radars and solving (P1) for each of them. For practical considerations, this may be extended to an array of radars (for instance, a phased array radar [4]).*

Remark 2. *In practice, we do not know the exact number of targets. One way to address this is to look for shifts (t_j, f_j) whose coefficient α_j is "significant". So, if k in Eq. (2) is a known upper bound on the number of targets, we would like to estimate (α_j, t_j, f_j) for $j = 1, \ldots, k$. For ease of exposition, we will work with the simpler (P1) for now.*

Now we are ready to move towards the promised digital computational procedure.

C. Digital Channel Model and Estimation Problem

$$\begin{tikzcd} s \arrow[r, "\text{Continuous Channel}"] & r \arrow[d, "\text{A-to-D}"] \\ S \arrow[u, "\text{D-to-A}"] \arrow[r, dotted, "\text{Digital Channel}"] & R \end{tikzcd}$$

Fig. 3: Signal life-cycle in DSP.

Digital signal processing (DSP) allows us to reduce (P1) to a finite dimensional linear algebra problem. For our purposes, digital signals are certain finite dimensional vectors, namely, elements of the space of N-periodic complex-valued functions on the integers (just as continuous signals are elements of $L^2(R)$), which we denote as $L^2(\mathbb{Z}_N)$[1]. This space comes with a standard inner product $\langle \cdot, \cdot \rangle$.

In DSP, the continuous signal transmitted s, begins life as a digital signal S, via a digital-to-analog process, and the received signal r ends up as a digital signal R, via an analog-to-digital process – see Fig. 3 for illustration. These processes can be modeled as linear maps,

- D-to-A : $L^2(\mathbb{Z}_N) \to L^2(\mathbb{R})$, and
- A-to-D : $L^2(\mathbb{R}) \to L^2(\mathbb{Z}_N)$.

As depicted in Fig. 3, the continuous channel relationship between s and r induces a digital channel relationship between the "trasmitted" and "received" signals, S and R.

Let T denote the duration of the signal s, and recall that W denotes its bandwidth[2]. Shannon, in his seminal work [5], showed that the space of such signals is essentially N dimensional for $N = TW$ (for simplicity, we assume TW is a positive integer). He also defined D-to-A and A-to-D maps for this N – for precise formulas see Section V-A. It then turns out that the digital channel relationship between R and S can be modeled as follows,

$$R[\tau] = \sum_{j=1}^{k} \alpha_j \cdot H_{\tau_j,\omega_j} S[\tau] + \text{Noise} \tag{3}$$

where, H_{τ_j,ω_j}, for $(\tau_j, \omega_j) \in \mathbb{Z}_N^2$, denotes the *discrete time-frequency shift* operator on $L^2(\mathbb{Z}_N)$ given by,

$$H_{\tau_j,\omega_j} S[\tau] = e^{2\pi i \frac{\omega_j \tau}{N}} S[\tau - \tau_j].$$

The justification for Eq. (3) comes from the fact that for certain shifts (t_0, f_0), there exists (τ_0, ω_0) in $\mathbb{Z}_N^2$ such that

$$\text{A-to-D} \circ h_{t_0,f_0} \circ \text{D-to-A} = H_{\tau_0,\omega_0}. \tag{4}$$

[1]$\mathbb{Z}_N$ denotes the ring of integers modulo N, namely the set $\{0, 1, \ldots, N-1\}$, together with addition and multiplication modulo N.

[2]As a consequence of the *uncertainty principle*, signals cannot be compactly supported both in time and frequency, i.e., *time-limited* and *bandlimited*. In practice signals are *essentially* time-limited and bandlimited

In particular, this is true for $t_0 \in \frac{1}{W}\mathbb{Z}$ and $f_0 \in \frac{1}{T}\mathbb{Z}$. We will refer to $\frac{1}{W}\mathbb{Z} \times \frac{1}{T}\mathbb{Z}$ as a *grid*, and $\frac{1}{W}, \frac{1}{T}$ as the *resolution* of the grid in time and frequency, respectively.

While the continuous shift (t_0, f_0) will never lie exactly on this grid, if the resolution is sufficiently small – equivalently, if $N = TW$ is sufficiently large – then it will be very close to a point on the grid. So we are willing to approximate (t_0, f_0) by the point on the grid closest to it, and assume the model in Eq. (3). In fact, state-of-the-art radars aim to resolve objects at a distance of as little as *centimeters* apart [6]. In this case, the model in Eq. (3) calls for bandwidth on the order of $10^9 Hz$, i.e. even for signal duration on the order of *milliseconds*, $N \sim 10^6$.

We refer to Eq. (3) as the *digital channel model* and can now formulate the following problem:

(P2) ***Digital Estimation Problem:*** Detect (τ_j, ω_j) for $j = 1, \ldots, k$.

We assume certain "Rules of the Game" for any estimation scheme to solve (P2). In describing these rules, we will refer to evaluating a signal $S \in L^2(\mathbb{Z}_N)$ at some $\tau \in \mathbb{Z}_N$, as *sampling* S. In addition, to distinguish a vector S from its representation as a tuple $(S[\tau])_{\tau \in \mathbb{Z}_N}$, for the reader's convenience, we refer to the latter as a *vector of samples*.

In this paper, we only consider estimation schemes which may involve

- A preprocessing step of choosing some family $\mathcal{S} \subseteq L^2(\mathbb{Z}_N)$ of *transmissible signals*.
- An estimation algorithm which may involve:
  - Picking the digital signal to transmit $S \in \mathcal{S}$,
  - Correlating and/or applying linear operators to vectors of values and, towards this end,
  - Storing samples of elements of $L^2(\mathbb{Z}_N)$, such as S and R, as needed.

Remark 3. *If in addition to lying on the grid, we assume, for instance, that* $t_0 \in [0, T]$ *and* $f_0 \in [-W/2, W/2]$, *then the continuous shift can be uniquely recovered from the discrete shift given by Eq.* (4).

D. Existing Methods and Computational Challenges.

1) Matched Filter and Pseudorandom Method: Given the "Rules of the Game" in Section I-C, one reasonable solution that one might come up with may be as follows.

Design a signal S such that, for $R = H_{\tau_0,\omega_0}S$, the inner product $\langle R, H_{\tau,\omega}S\rangle$ is maximized at $(\tau, \omega) = (\tau_0, \omega_0)$. Equivalently, by observing that the operators $H_{\tau,\omega}$ are unitary, we try to design a signal S such that $\langle S, H_{\tau,\omega}S\rangle$ is maximized at $(\tau, \omega) = (0, 0)$. This type of procedure is essentially a *matched filter* algorithm. Since *pseudorandom* signal satisfies this requirement, it may also be referred to as the *pseudorandom* method [7].

For convenience, we introduce some standard notation [1].

Definition I.1. *We define the* <u>*ambiguity function*</u> *of* S *against* R, *denoted* $\mathcal{A}[S, R]$, *on the* $\mathbb{Z}_N^2$ *plane as follows*

$$\mathcal{A}[S, R](\tau, \omega) = \langle H_{\tau,\omega}S, R\rangle \qquad \tau, \omega \in \mathbb{Z}_N^2.$$

Assuming the model in Eq. (3), since $\mathcal{A}$ is bilinear on $L^2(\mathbb{Z}_N)^2$, we will be able to solve the *digital estimation problem* by finding k peaks of $\mathcal{A}[S, R]$, for a pseudorandom signal S, under minimal assumptions on noise and k [8].

Without any additional information, this will involve estimating N^2 correlations, i.e., naively, this will require $O(N^3)$ arithmetic operations. However, it is well-known that we can compute N of these correlations simultaneously – namely, $\mathcal{A}[S, R]$ can be evaluated in $O(N \log N)$ operations on any line in the $\mathbb{Z}_N^2$ plane[3], and consequently on the entire plane in $O(N^2 \log N)$ operations.

For $N \sim 10^6$, one might hope – and expect – that under realistic assumptions on noise and targets, one can do much better than $O(N^2 \log N)$ operations.

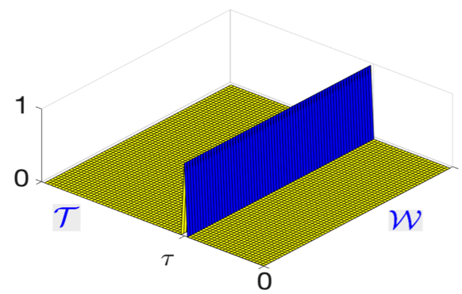


Fig. 4: $\mathcal{A}[S, H_{\tau,\omega}S]$ for chirp signal $S = S_{\mathcal{T}}$.

2) Incidence Method: This improvement, studied in [7]–[10], is based on the following observation.

If, instead of being maximized at $(\tau, \omega) = (0, 0)$, the ambiguity function of S against itself is supported on a line, for instance, $\mathcal{W} = \{(0, \omega) : \omega \in \mathbb{Z}_N\}$, for $R = H_{\tau_0,\omega_0}S$ we can detect τ_0, by computing $\mathcal{A}[S, R]$ just on, say, the line $\mathcal{T}$, see Fig.4. In other words, we can detect τ_0 from just N values of the ambiguity function – we achieve half of the goal in just $O(N \log N)$ operations. In this way, we can shave off a factor of N from the arithmetic complexity of solving the digital estimation problem.

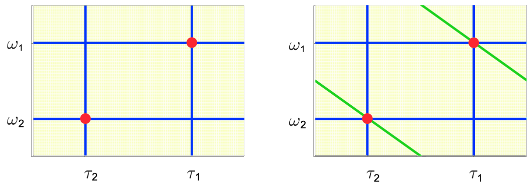
For convenience, let's denote the signal, just described as $S_{\mathcal{W}}$. More generally, for arbitrary slope $a \in \mathbb{Z}_N$, there is a signal S_{L_a} – in fact an orthonormal basis of such signals – whose ambiguity $\mathcal{A}[S_{L_a}, S_{L_a}]$ is supported on the line $L_a = \{(\tau, a\tau) : \tau \in \mathbb{Z}_N\}$ (Theorem V.1). Moreover, for distinct lines L, M, their cross-correlation is small, $|\langle S_L, S_M\rangle| \leq \frac{1}{\sqrt{N}}$ [7]. These signals are known as *chirps*.

Let's consider the case $k > 1$, e.g. $k = 2$.

If $S = S_{\mathcal{W}} + S_{\mathcal{T}}$ and $R = \sum_{j=1}^{2} H_{\tau_j,\omega_j}S$, we can detect $\{\tau_1, \tau_2\}$ from $\mathcal{A}[S_{\mathcal{W}}, R]$, and $\{\omega_1, \omega_2\}$ from $\mathcal{A}[S_{\mathcal{T}}, R]$ in $O(N \log N)$ operations. This, however leads to a matching issue – see Fig.5 for illustration. The matching issue can be remedied by using $S = S_{\mathcal{W}} + S_{\mathcal{T}} + S_L$, for a third line L of known slope, and computing points of triple incidence.

In fact, for k points (τ_j, ω_j) in generic position, three transversal lines K, L, M should suffice by Lemma V.1.

[3] $\mathcal{A}[S, R]$ can be expressed as a certain convolution [7] on any line in the $\mathbb{Z}_N^2$ plane. It then follows by the *Convolution Theorem* and the *Fast Fourier Transform* algorithm that the ambiguity function can be evaluated at these N points in $O(N \log N)$ operations



(a) Two lines may not suffice to identify true shifts. (b) Three lines suffice to identify true shifts.

Fig. 5: True shifts $\{(\tau_1, \omega_1), (\tau_2, \omega_2)\}$

The arithmetic complexity of the *Incidence Method* (IM) is $O(N \log N + k^2)$ – computing (for instance) the k peaks of $\mathcal{A}(S_K, R)$ on L, $\mathcal{A}(S_L, R)$ on K, $\mathcal{A}(S_M, R)$ on L in $O(N \log N)$ operations, and solving the matching issue by computing points of triple incidence in $O(k^2)$ operations.

However, it still seems that for relevant, practical levels of noise and number of targets, one can do better than this. For instance, in FMCW radar [11] for automotive systems, while bandwidth is on the order of GHz, the largest observed frequency shifts can typically be assumed to be on the order of tens of MHz. This is exploited to require fewer samples, in this case $\sim 10^4$ samples per millisecond rather than $\sim 10^6$.

If the bound on the number of targets $k \ll N$, under reasonable assumptions on noise, we may hope to do still better than these methods. For instance, [12] demonstrates an algorithm with sample complexity sublinear in N. Here we present an approach that leads to an algorithm with sampling, space *and* arithmetic complexity sublinear in N (Theorem III.1).

II. A Sublinear Algorithm

In order to explain the idea behind our suggested improvement, we first introduce another *sparse estimation problem*.

Consider $S \in L^2(\mathbb{Z}_N)$ such that

$$S[\tau] = \sum_{j=1}^{k} \alpha_j e^{\frac{2\pi i \omega_j \tau}{N}} + \text{Noise}.$$

If we denote the *discrete fourier transform* of S as $\mathcal{F}S$, then $\mathcal{F}S[\omega_j] = \alpha_j$. If $k \ll N$, then we say that $\mathcal{F}S$ is essentially k-sparse, and the problem of detecting $\omega_j \in \mathbb{Z}_N$ is one version of what is known as the *sparse Fast Fourier Transform (SFFT) problem*. Under certain assumptions on noise, it is known that this problem can be solved with sampling and arithmetic complexity sublinear in N. There is a large body of work on algorithms that achieve this, including [13], [14].

The improvement we suggest, called *Sparse Channel Estimation* or *SCE* for short, is based on the fact that if $k \ll N$ then estimating, for instance, k time shifts using the Incidence Method is an SFFT problem, by Theorem V.2.

While the Sparse Channel Estimation algorithm (SCE) does not assume any particular choice of SFFT algorithm, generally speaking, there are certain standard components to such algorithms, which we summarize below.

1) *$k = 1$ subroutine* – This is a randomized subsampling scheme, and a corresponding computational procedure with arithmetic and space complexity linear in the number of samples, for detecting the only frequency in the signal, if any, that is significant, i.e. with coefficient $|\alpha_1| \geq \mu$. $\mu > 0$ is a parameter of the algorithm. Theorem III.1, assumes an implementation very similar to that in [15].
2) *Discrete filter functions* – These are elements of $L^2(\mathbb{Z}_N)$ that are essentially localized in time and frequency (the formal notion for this used in this work, is provided in Definition V.1), and allow us to reduce to the case $k = 1$ with "few" samples and arithmetic operations [16]. Their existence and construction is mathematically highly non-trivial (Theorem V.1).
3) *Pseudorandom spectral permutation* – This provides us a way to sample S by which, intuitively speaking, we can assume that the k frequencies ω_i are uniformly spread in $\mathbb{Z}_N$, with no additional cost to sampling or arithmetic complexity, see [13] for more details.

These are used in combination to solve the SFFT problem.

In Sparse Channel Estimation (SCE), a direct modification provides a subsampling scheme for R, which we will refer to as Sampling$_{SCE}$, and corresponding computational procedure to estimate the relevant ambiguity functions, which we will refer to simply as SFFT$_\mu$.

We now present the pseudocode for this algorithm.

Algorithm 1 Sparse Channel Estimation (SCE$_\mu$)

Input: Channel sparsity k.

1: Randomly choose transversal lines K, L, M
2: Transmit $S = S_K + S_L + S_M$.
3: Sample R using Sampling$_{SCE}$ $\rightarrow \mathcal{R}$.
4: Locate peaks of $\mathcal{A}(S_K, R)$ on L using $\mathcal{R}$ and SFFT$_\mu$
    $\rightarrow \kappa_i \in \mathbb{Z}_N^2$, $i \in \{1, \ldots k\}$.
5: Locate peaks of $\mathcal{A}(S_L, R)$ on K using $\mathcal{R}$ and SFFT$_\mu$
    $\rightarrow \ell_j \in \mathbb{Z}_N^2$, $j \in \{1, \ldots k\}$.
6: Locate peaks of $\mathcal{A}(S_M, R)$ on L using $\mathcal{R}$ and SFFT$_\mu$
    $\rightarrow m_{j'} \in \mathbb{Z}_N^2$, $j' \in \{1, \ldots k\}$.
7: *Matching:* Find points of triple incidence, e.g.

$$\{\kappa_i + \ell_j\} \cap \{\kappa_i + m_{j'}\}.$$

Output: Return every shift (τ, ω) found in Step 7.

III. Guarantees for Sparse Channel Estimation (SCE)

We now analyze the performance of SCE. We do so using certain standard measures to quantify the quality of such an algorithm, namely, *probability of detection (PD)* and *probability of false alarm (PFA)* [17].

Definition III.1. *The probability of detection (PD) is the probability that the j^{th} target, (τ_j, ω_j), is returned by the algorithm.*

Definition III.2. *The probability of false alarm (PFA) is the probability that the j^{th} shift returned by the algorithm, does not correspond to any target.*

We provide a guarantee for *Sparse Channel Estimation* (SCE), namely Theorem III.1, which holds true under the following assumptions.

- We make a sparsity assumption on the number of targets. **A1** (*Sparsity*): $k \ll N$ is a constant, i.e. independent of N.
- We also make the following assumption on the coefficients α_j and their distribution. **A2** (ε*-targets*): There is some $A > 0$ and $\varepsilon \in (0, 1)$ such that $(\alpha_1, \ldots, \alpha_k)$ is drawn uniformly at random from the following set:

$$B_\varepsilon = \left\{ x \in \mathbb{C}^k \colon \|x\|^2 = A \text{ and } \min_{x_j \neq 0} |x_j| \geq \varepsilon \cdot \sqrt{A/k} \right\}.$$

- The final assumptions that we make are on the distribution of the noise ν_n. **A3** (*Subgaussian*): We assume ν_τ are i.i.d, mean zero and *subgaussian* random variables [18] with subgaussian parameter σ^2/N.

We use a standard definition of *signal-to-noise ratio* or *SNR*, namely, SNR $= A/\sigma^2$.

Theorem III.1. *Let $\mu = \kappa \cdot \varepsilon\sqrt{A/k}$ for some confidence parameter $\kappa \in (0, 1)$.*
Then, under the sparsity, ε-targets, and subgaussian assumptions there is an implementation of SCE$_\mu$ which takes

1) $c_1 k (\log N)^2 (\varepsilon^2 SNR)^{-1}$ *samples,*
2) $c_2 k (\log N)^2 (\varepsilon^2 SNR)^{-1}$ *bits of memory, and*
3) $c_3 k (\log N)^3 (\varepsilon^2 SNR)^{-1} + k^2$ *arithmetic operations*

for which PD $\to 1$ and PFA $\to 0$ as $N \to \infty$, where c_1, c_2, c_3 are constants independent of ε, SNR and N.

Remark 4. *A3 is satisfied, for instance, by the standard assumption of additive white Gaussian noise (AWGN) [2], namely, that ν_τ are i.i.d, mean zero and Gaussian random variables with variance σ^2/N.*

Remark 5. *The proof of Theorem III.1 confirms that for $\kappa = 1/2$ and a reasonable choice of constants c_1, c_2, c_3, the rate of convergence of PD $\to 1$ and PFA $\to 0$ is at least polynomial in N. The proof is omitted here due to space constraints, but will appear in the journal version of this paper.*

IV. Numerical Results

While we do not present the proof of Theorem III.1 in this paper, for the reader's benefit we now provide the experimentally observed convergence rates for PD and PFA in a specific case, see Figs. 6 and 7.

In addition, we provide numerical comparisons for time and space complexity of SCE and the Incidence method (IM) in a specific case, see Table I.

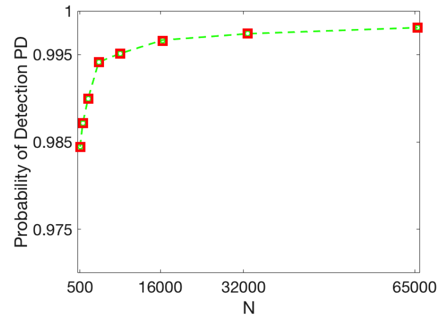


Fig. 6: Experimentally observed convergence rate for PD, $k = 5$, SNR $=$ 10dB, 500 random trials.

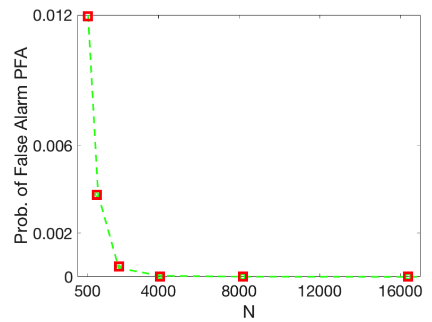


Fig. 7: Experimentally observed convergence rate for PFA, $k = 5$, SNR $=$ 10dB, 500 random trials.

	SCE		IM	
N	Samples	Time (sec)	Samples	Time (sec)
2048	2048	0.1423	2048	0.0370
4096	4096	0.1250	4096	0.0740
8192	5468	0.1595	8192	0.1620
16,384	6242	0.1789	16,384	0.3230
32,768	7064	0.1992	32,768	0.5790
65,536	7934	0.2314	65,536	1.2010

TABLE I: Numerical comparison of time and sample complexity, $k = 50$, SNR $=$ 10dB, 500 random trials.

V. Conclusion

We provide a novel algorithm that essentially breaks the relationship between sampling, space, arithmetic complexity and $N = TW$. This result is the outcome of a simple combination ingredients: digital chirp signals, discrete Gaussian filter functions, and lastly, recent developments in Sparse Fast Fourier Transform algorithms.

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Appendix

A. Digital to Analog and Analog to Digital.

For $r \in L^2(\mathbb{R})$, let $R =$ A-to-D(r). Then,

$$R[\tau] = \sum_{m \in \mathbb{Z}} r\left(\frac{\tau}{W} + mT\right) \qquad n \in \mathbb{Z}. \tag{5}$$

For $S \in L^2(\mathbb{Z}_N)$, let $s = \text{D-to-A}(S)$. Then,

$$s(t) = e^{if_ct}s_0(t) \tag{6}$$

where,

- $s_0(t) = \sum_{\tau\in\mathbb{Z}_N} S[\tau]\cdot \text{sinc}_W(t-\frac{\tau}{W})$ and,
- $\text{sinc}_W(t) = \frac{\sin(2\pi Wt)}{2\pi Wt}$.

B. Underlying Algebraic Structure.

a) Chirp Signals: These can be constructed as shared eigenfunctions of certain commuting operators coming from commutative subgroups of the *Heisenberg-Weyl* group G_{HW} [7], [9], [10], [19], [20].

Formulas for Chirp Signals:

- For $a \in \mathbb{Z}_N$ and the line $L = \{(\tau, a\tau)\colon \tau \in \mathbb{Z}_N\}$, we have the following orthonormal basis of chirps:
$$\mathcal{B}_L = \left\{S_L^b[\tau] = \frac{1}{\sqrt{N}}\, e^{\frac{2\pi i}{N}\left(\frac{a}{2}\tau^2+b\tau\right)}\colon b\in\mathbb{Z}_N\right\}.$$
- For the $\mathcal{W} = \{(0,\omega)\colon \omega\in\mathbb{Z}_N\}$, we have the following orthonormal basis of chirps
$$\mathcal{B}_\mathcal{W} = \{S_\mathcal{W}^\tau[n] = \delta_\tau\}.$$

Theorem V.1. *For $S_L \in \mathcal{B}_L$*

$$|\langle H_{\tau,\omega}S_L, S_L\rangle| = \begin{cases}1 & \text{if } (\tau,\omega)\in L\\ 0 & \text{otherwise}\end{cases}$$

Lemma V.1. *For k points $\{p_1,\ldots,p_k\}$ in $\mathbb{Z}_N^2$, and three distinct pairwise transversal lines L, M, K with slopes chosen uniformly at random, the triple intersections of the shifted lines*

$$p_i + L,\ p_i+M,\ p_i+K \qquad\qquad i = 1,\ldots,k$$

uniquely identify the points with probability at least $1-\frac{k^3}{N-2}$.

b) Reduction to Sparse FFT:

Theorem V.2. *Given $S\in L^2(\mathbb{Z}_N)$, lines L and M:*

- $L = \{(\tau, a_1\tau)\colon \tau\in\mathbb{Z}_N\}$.
- $M = \{(\tau, a_2\tau)\colon \tau\in\mathbb{Z}_N\}$.

the values of the ambiguity function of S against the chirp S_L^b on the shifted line $M' = M + (0,\omega)$, $\omega\in Z_N$ are given by the following Fourier coefficients,

$$\mathcal{A}_{S,S_L^b}[(\tau,a_2\tau)+(0,\omega)] = |\mathcal{F}(S\cdot\overline{S_L})[b+\omega+(a_2-a_1)\tau]|$$

where, S_L is the chirp $S_L = S_L^0$.

c) Discrete Filter Functions:

Definition V.1 ((k,k',δ)-family of filters)**.** *We will refer to $\{F_N :\in L^2(\mathbb{Z}_N)\}_N$ as a (k,k',δ)-family of filters if*

1. $F_N[\tau] = O(1/N)$ *for* $n \ge k'$
2. $\mathcal{F}F_N[\omega] = O(1/N)$ *for* $\omega \ge N/k$
3. $\delta \le \mathcal{F}F_N[\omega]\le 1$ *for* $\omega \le \dfrac{N}{k\sqrt{\log N\cdot\log(1/\delta)}}$.

Theorem V.3 (Discrete Gaussian)**.** *Let $G \in L^2(\mathbb{Z}_N)$ be defined as follows,*

$$G[\tau] = \sum_{m\in\mathbb{Z}} e^{-\pi\left(\frac{\tau}{\sqrt{N}}+m\sqrt{N}\right)^2}. \tag{7}$$

Then, G is an eigenvector of the discrete fourier transform.

Note that the only significant term in the expression in Eq. (7) corresponds to $m = 0$. We then have the following corollary.

Corollary V.1 (Discrete Gaussian Filters)**.** *There exists a $(k, k\log N, \delta)$-family of filters for every $0\le\delta<1$, namely*

$$F_N[\tau] = e^{-\pi\left(\frac{\tau^2}{k^2\log N}\right)}.$$

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