求和

例:

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{i}{2^{i+j}(i+j)}$$
$$=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\cancel{i+j}}{2^{i+j}\cancel{(i+j)}}$$
$$=\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{2^i}\right)^2$$
$$=\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]^2$$

例: $T=\{(a,b,c)\in\mathbb{N}^3 : a,b,c$ 构成三角形三边长$\}$

则 $\sum_{(a,b,c)\in T} A_{a,b,c} = \sum_{\substack{(x,y,z)\in T \\ \text{且奇偶性相同}}} A_{\frac{x+y}{2},\frac{y+z}{2},\frac{x+z}{2}}$

显然

不等式

Cauchy 不等式:

$\forall\ n\in\mathbb{N}$, $(a_1,\cdots\cdots a_n),(b_1,\cdots b_n)\in\mathbb{R}^n$

$$\sum_{i=1}^{n}a_i^2\cdot\sum_{i=1}^{n}b_i^2\geq\left(\sum_{i=1}^{n}ab\right)^2$$

取等当且仅当两向量线性相关

Young 不等式

$\forall\ a, b \geq 0,\ \frac{1}{p} + \frac{1}{q} = 1 \quad p > 1:$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.

$$p + q = pq$$

$$RHS = \frac{q a^p + p a^q}{pq} = \frac{q a^p + p a^q}{p+q} \geq \left(a^{p+q} \cdot b^{p+q}\right)^{\frac{1}{p+q}} = ab$$

Q.E.D.

Holder 不等式

$\forall\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1,\ a_1, a_2, \cdots, a_n \geq 0,\ b_1, b_2, \cdots, b_n \geq 0$

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}$$

Proof.

denote $a_k' = \dfrac{a_k}{\left(\sum_{i=1}^{n} a_k^p\right)^{\frac{1}{p}}}$ $\qquad b_k' = \dfrac{b_k}{\left(\sum_{i=1}^{n} b_k^q\right)^{\frac{1}{q}}}$

where $\sum_{k=1}^{n} a_k' = \sum_{k=1}^{n} b_k'$.

by the Young inequality.

$$\sum_{k=1}^{n} a_k' b_k' \leq \sum_{k=1}^{n} \left[\frac{(a_k')^p}{p} + \frac{(b_k')^q}{q} \right] = \frac{1}{p} + \frac{1}{q} = 1$$

均值不等式

$\forall\ a_1, \cdots, a_n > 0$

$$f(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n}\right)^{\frac{1}{r}} & r \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n} & r = 0 \end{cases}$$

单调递增

Proof.

$$\lim_{r\to 0} f(r) = \exp\left(\lim_{r\to 0} \frac{\ln \frac{a_1^r + \cdots + a_n^r}{n}}{r}\right)$$

$$= \exp\left(\lim_{r\to 0} \frac{\cancel{n}}{a_1^r + \cdots + a_n^r} \cdot \frac{\ln a_1 \cdot a_1^r + \cdots + \ln a_n \cdot a_n^r}{\cancel{n}}\right)$$

$$= (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

$$[\ln f(r)]' = \left(-\frac{1}{r^2}\right) \cdot \ln \frac{a_1^r + \cdots + a_n^r}{n} + \frac{1}{r} \cdot \frac{\cancel{n}}{a_1^r + \cdots + a_n^r} \cdot \frac{\ln a_1 \cdot a_1^r + \cdots + \ln a_n \cdot a_n^r}{\cancel{n}}$$

$$= \frac{1}{r^2}\left(r \cdot \frac{\ln a_1 \cdot a_1^r + \cdots + \ln a_n \cdot a_n^r}{a_1^r + \cdots + a_n^r} - \ln \frac{a_1^r + \cdots + a_n^r}{n}\right)$$

$$= \frac{1}{r^2}\left(\frac{a_1^r \ln a_1^r + \cdots + a_n^r \ln a_n^r}{a_1^r + \cdots + a_n^r} - \ln \frac{a_1^r + \cdots + a_n^r}{n}\right)$$

by the Jenson inequality.

$g(x) = x\ln x$, $g''(x) \geq 0$ g concave upward.

$$\frac{1}{n}\left(a_1^r \ln a_1^r + \cdots + a_n^r \ln a_n^r\right) \geq \frac{a_1^r + \cdots + a_n^r}{n} \ln \frac{a_1^r + \cdots + a_n^r}{n}$$

thus $f(x)$ monotonically increasing.

Q.E.D.

Bernouli 不等式

$x_1, \cdots x_n \geq -1$ 且 x_i x_j 同号

$$(1+x_1)\cdots(1+x_n) \geq 1+x_1+\cdots+x_n$$

Proof.

when $1 \leq n \leq 2$, obvious.

assume for $n \leq l$, the statement is true.

for $n = l+1$

$$(1+x_1)\cdots(1+x_l)(1+x_{l+1}) \geq (1+x_1+\cdots x_l)(1+x_{l+1})$$
$$= 1+x_1+\cdots+x_{l+1}+x_{l+1}(x_1+\cdots x_n)$$
$$\geq 1+x_1+\cdots+x_{l+1}$$

Q.E.D.

Jensen 不等式

$\lambda_i \geq 0 \qquad \sum \lambda_i = 1$

对下凸函数 f. $\qquad f(\sum \lambda_i x_i) \leq \sum \lambda_i f(x_i)$

上凸函数 f $\qquad$ vice versa.

排序不等式

$a_1 \leq \cdots \leq a_n$. $b_1 \leq \cdots \leq b_n$

$c_1, \cdots c_n$ 为 b_n 重排,

则 逆序 $\leq$ 乱序 $\leq$ 正序

Proof.

suppose π makes $\sum a_i b_{\pi(i)}$ takes the maximum

if $\pi \neq I$, suppose i is the first number s.t. $\pi(i) = j > i$
thus $\exists$ $k > i$ s.t. $\pi(k) = i$

thus $(a_k - a_i)(b_j - b_i) \geq 0 \Rightarrow a_i b_j + a_k b_i \leq a_k b_j + a_i b_i$
contradiction.

similarly suppose σ makes the minimum

Q.E.D.

Chebyshev 不等式.

a_i b_i 单调递增列

$$\sum a_i b_{n+1-i} \le \frac{1}{n} \sum a_i \sum b_i \le \sum a_i b_i$$

Proof. n次排序不等式

裂项

例 $$\frac{1}{1+x^4} = \frac{1}{(1+x^2)^2 - 2x^2} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$