Localization of Ring

Def. $(R,+,\cdot)$ commutative, $S \subset R$

call S a multiplicative subset

if $1 \in S$

$\forall a,b \in S,\ ab \in S$ 	or 	is submonoid of $(R\setminus\{0\},\cdot)$

Def. $(R,+,\cdot)$ commutative S multiplicative subset

R localization of S, denote as $(S^{-1}R,+,\cdot)$, defined as

$$S^{-1}R = \{\frac{r}{s};\ r\in R, s\in S\}/\sim$$

where $\frac{r}{s} \sim \frac{r'}{s'} \iff \exists t \in S,\ t(rs'-r's)=0$

if $r,r' \in R$, $s,s' \in S$, define:

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs'+r's}{ss'}$$

$$\frac{r}{s}\cdot\frac{r'}{s'} = \frac{rr'}{ss'}$$

Prop. $(S^{-1}R,+,\cdot)$ is a commutative Ring

Proof. (I) to prove "$\sim$" is a relation of equivalence

suppose $r/s \sim r'/s'$, $\exists t \in S$, s.t.

$t(rs'-r's)=0$

thus $r/s = r'/s'$

if $r/s \sim r'/s'$ $r'/s' \sim r''/s''$

for $t,t' \in S$ s.t.

$t(rs'-r's)=0$ & $t'(r's''-r''s')=0$

thus $\exists\ t,t' \in S$, $t''=tt's'$

thus $(tt's')rs'' = t's''(trs') = t's''(tr's) = ts(t'r's'')$

$= ts(t'r''s') = (tt's')r''s$

thus $t''(rs''-r''s)=0$

(III) to prove $S^{-1}R$ is a Ring

for any $r/s \in S^{-1}R$

$$\frac{0}{1} + \frac{r}{s} = \frac{0 \cdot s + 1 \cdot r}{1 \cdot s} = \frac{r}{s}$$ (Identity)

$$\frac{1}{1} \cdot \frac{r}{s} = \frac{r}{s}$$

$$\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}\right) + \frac{r_3}{s_3} = \frac{r_1 s_2 s_3 + s_1 r_2 s_3 + s_1 s_2 r_3}{s_1 s_2 s_3} = \frac{r_1}{s_1} + \left(\frac{r_2}{s_2} + \frac{r_3}{s_3}\right)$$

(Associative law of multiplication)

$$\frac{r}{s} + \left(-\frac{r}{s}\right) = 0$$ (Inverse of addition)

$$\frac{r_1}{s_1} \cdot \left(\frac{r_2}{s_2} + \frac{r_3}{s_3}\right) = \frac{r_1(r_2 s_3 + r_3 s_2)}{s_1 s_2 s_3} = \frac{r_1 r_2 s_3}{s_1 s_2 s_3} + \frac{r_1 r_3 s_2}{s_1 s_2 s_3} = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} + \frac{r_1}{s_1} \cdot \frac{r_3}{s_3}$$

(Distribution law)

Prop. $(R, +, \cdot)$ integral ring S multiplicative subset

$$\frac{a}{b} \sim \frac{c}{d} \iff ad - bc = 0$$

Def. $(R, +, \cdot)$ an integral ring fraction field on R, denote as $Frac(R)$ defined as $S^{-1}(R)$, where $S = R \setminus \{0\}$ or sth. else

Prop. $Frac(R)$ is a field

Lemma. $p \trianglelefteq R$ is a prime ideal $\Rightarrow$ $S = R \setminus p$ is a multiplicative subset