Ideal

Def. $A \subset R$ (A), which is called the ideal generated by A, is
$$(A) = \bigcap \{ I \subset R : I \supset A, I \triangleleft R \}$$

Prop. $(A) \triangleleft R$

Def. $a \in R$, $(a) = (\{a\})$ is call the principle Ideal generated by a
$a_1, a_2 \cdots a_n \in R$, $(a_1, a_2, \cdots, a_n) = (\{a_1, a_2 \cdots a_n\})$ is called: finitely generated Ideal.

Prop. $(R, +, \cdot)$ is a commutative Ring, $a \in R \Rightarrow$
$(a) = Ra$
for $a_1, a_2 \cdots a_n \in R \Rightarrow$
$(a_1, a_2 \cdots a_n) = Ra_1 + \cdots Ra_n = \{ r_1 a_1 + \cdots r_n a_n : r_1, \cdots r_n \in R \}$

Def. $I, J \triangleleft R$, $I + J = \{ a+b : a \in I, b \in J \}$

Prop. $I + J \triangleleft R$

Prop. $I + J = (I \cup J)$

Def. $(R, +, \cdot)$ is a commutative ring. $I, J \triangleleft R$, $IJ = (\{ ab : a \in I, b \in J \})$

Prop $IJ = \{ a_1 b_1 + \cdots + a_n b_n : a_1 \cdots a_n \in I, b_1 \cdots b_n \in J \}$

Prop. $I, J, K \triangleleft R \Rightarrow$
1. $I + J = J + I$
2. $I + (J + K) = (I + J) + K$
3. $I(J+K) = IJ + IK$
4. $I(JK) = (IJ)K$
5. $I = RI = IR$

Proof. 3. $I(J+K) \supset I(J+\{0\}) = IJ$ & $I(J+K) \supset IK \Rightarrow$

$I(J+K) \supset IJ + IK$

meanwhile.

$\sum_i (a_i(b_i+c_i)) \in I(J+K)$

$\sum_i (a_i(b_i+c_i)) = \sum_i a_i b_i + \sum_i a_i c_i \in IJ + IK$

thus $I(J+K) \subset IJ + IK$

$I(J+K) = IJ + IK$

Lemma. $(R,+,\cdot)$ a commutative Ring, $I, J \lhd R \Rightarrow$

$IJ \subset I \cap J \subset I+J$

Lemma. $(I \cap J)(I+J) \subset IJ$

Prop. $I \cap (J+K) \supset I \cap J + I \cap K$

particularly, if $J \subset K$

$I \cap (J+K) = I \cap J + I \cap K$

Def. $(R,+,\cdot)$ commutative, $I, J \lhd R$, I, J are coprime $\Leftrightarrow$

$I+J = R$

Prop. I, J are coprime $\Leftrightarrow$ $\exists\, i \in I, j \in J$, $i+j=1$

Proof. if $I+J=R$

then $1 \in R = I, J$, thus $\exists\, i \in I, j \in J$, $i+j=1$

if $\exists\, i \in I, j \in J$, $i+j=1$

for $r \in R$, $r = r(i+j) = ri + rj \in RI + RJ = I+J$

Prop. I, J coprime. $\Rightarrow$ $IJ = I \cap J$

Prop. $(R,+,\cdot)$ $(R',+,\cdot)$ commutative, $f:(R,+,\cdot)\to(R',+,\cdot)$ homomorphism

$I' \triangleleft R' \Rightarrow f^{-1}(I') \triangleleft R$

Prop. 2.10.