Def. $U \subset \mathbb{R}^n$, $f: U \to \mathbb{R}$ is a C^k function on p

$\Leftrightarrow$ partial derivative

$$\frac{\partial^j f}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_j}}$$ exist on p for any $j \le k$

f is smooth $\Leftrightarrow$ f is C^∞ function

Def. $f: U \to \mathbb{R}$ is analytic on p

$\Leftrightarrow$

$$f(x) = f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \cdots + \frac{1}{k!} \sum_{i_1 \cdots i_k} \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}(p) \prod (x^{i_k} - p^{i_k})$$

is true in the neighbourhood of p.

Def. $S \subset \mathbb{R}^n$ is star-shaped about p

$\Leftrightarrow$ $\forall q \in S$, $\overline{pq} \in S$

Theorem. (Taylor's theorem with reminder)

$f: U \to \mathbb{R}$

where U is open f is C^∞ U is star-shaped about p in U

$\Rightarrow$

$\exists$ a family of function $g_1(x) \cdots g_n(x) \in C^\infty(U)$

s.t. $f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x)$ $\quad g_i(p) = \frac{\partial f}{\partial x^i}(p)$

Proof.

Def.

Tangent space of p is the space constructed by all vectors stares at p, denoted $T_p(\mathbb{R}^n)$

Any line go through p and direction along vector $v = \langle v^1, v^2, \cdots, v^n \rangle$
$c(t) = (p^1 + tv^1, \cdots, p^n + tv^n)$

Def.

directional derivative

$$D_v f = \lim_{t \to 0} \frac{f[c(t)] - f(p)}{t} = \frac{d}{dt}\Big|_{t=0} f(c(t))$$

by chain rule

$$D_v f = \sum_{i=1}^{n} \frac{d}{dt} c^i(0) \cdot \frac{\partial}{\partial x^i} f(p)$$

thus $D_v = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p$

Def.

U, V are neighbourhoods of p

(f, U) & (g, V) equivalence $\Leftrightarrow$

$\exists$ W is a neighbourhood of p, $W \subset U \cap V$ s.t. $f = g$ in W

Def.

germ of p on f is the set constracted by all the functions equivalent to f on p's neighbourhood.

the set constracted by all gems of p denoted C_p^∞

Def.

linear mapping $D: C_p^\infty \to \mathbb{R}$ is a derivation of p

$\Leftrightarrow$

D satisfies the Leibnize Rule:

$$D(fg) = D(f)\,g(p) + f(p)\cdot(Dg)$$

denote $\mathcal{D}_p(\mathbb{R}^n)$ is the set constracted by all the derivation on p

Lemma. $\phi: T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$

where $x \mapsto D_v = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p$ is a linear bijeetion

$$(e_1, e_2, \cdots, e_n) \mapsto \left(\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right)$$

thus $v = \langle v^1, v^2, \cdots, v^n \rangle$

can be rewritted as $v = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p$

Def.

a vector field is a function defined on open set $U \subset \mathbb{R}^n$ and give every $p \in U$ a vector $X_p \in T_p(\mathbb{R}^n)$

thus $X_p = \sum_i a^i \frac{\partial}{\partial x^i}\Big|_p$

Def. a vector field X of U is C^∞

$\Leftrightarrow \forall\, a^i$, a^i is C^∞

Lemma X is a C^∞ vector field on open set $U \subset \mathbb{R}^n$

f, g are C^∞ function on U

$\Rightarrow X(fg) = (Xf)g + f(Xg)$