积分不等式

1. 狄利克雷积分
   概率论应用

---

2. 连接 $f(x)$ & $f'(x)$（3种方法）
   分部积分技巧

1. $f(x) \in C[0,1]$，证：

$$\int_0^1 \left(\int_0^1 |f(x)+f(y)|\,dy \right) dx \geq \int_0^1 |f(x)|\,dx$$

Proof.

有 狄利克雷积分 (Dirichlet) $\int_0^{+\infty} \frac{\sin z}{z}\,dz = \frac{\pi}{2}$

Lemma.

$$\int_0^{\infty} \frac{\sin az \cdot \sin bz}{z^2}\,dz = \frac{1}{4}(|a+b| - |a-b|)$$

Proof of lemma.

$$\text{LHS} = \int_0^{\infty} \frac{\cos[(a-b)z] - \cos[(a+b)z]}{2z^2}\,dz$$

$$\text{分部积分} = -0 - \frac{1}{2}\int_0^{+\infty} -\frac{1}{z}\left[-|a-b|\sin(|a-b|z) + |a+b|\sin(|a+b|z)\right]dz$$

$$= \frac{1}{2}\left[\int_0^{\infty} |a+b| \frac{\sin(|a+b|z)}{z}\,dz - \int_0^{\infty} |a-b| \frac{\sin(|a-b|z)}{z}\,dz\right]$$

$$\text{Dirichlet} = \frac{\pi}{4}(|a+b| - |a-b|)$$

Q.E.D.

Thus.

$$\int_0^1 dx \int_0^1 |f(x)+f(y)|\,dy = \int_0^1 dx \int_0^1 |f(x)-f(y)|\,dy + \int_0^1 dx \int_0^1 \left[\frac{4}{\pi}\int_0^{\infty} \frac{\sin(zf(x))\sin(zf(y))}{z^2}\,dz\right] dy$$

$$= \int_0^1 dx \int_0^1 |f(x)-f(y)|\,dy + \frac{4}{\pi}\int_0^{\infty}\left[\int_0^1 \frac{\sin(zf(x))}{z}\,dx\right]^2 dz$$

$$\geq \int_0^1 dx \int_0^1 |f(x)-f(y)|\,dy$$

Therefore,

$$\int_0^1 dx \int_0^1 |f(x)+f(y)|\,dy \geq \int_0^1 dx \int_0^1 \frac{1}{2}\left(|f(x)+f(y)|+|f(x)-f(y)|\right) dy$$

(三角不等式) $$\geq \int_0^1 dx \int_0^1 |f(x)|\,dy$$

$$= \int_0^1 |f(x)|\,dx$$

Q.E.D.

[用概率论 同分布 $E(|X+Y|) \geq E(|X-Y|)$ 可替代中间过程.]

2. $f \in C[0,a]$, $f(0)=0$

证. $$\left|\int_0^a f(x)\,dx\right| \leq \frac{M}{2}a^2 \qquad M = \max_{x\in[0,a]} |f'(a)|$$

Proof I. 拉中

$$\left|\int_0^a f(x)\,dx\right| \leq \int_0^a \underbrace{|f(x)|}_{=f'(\xi)(x-0)}\,dx \leq \int_0^a Mx\,dx = \frac{M}{2}a^2$$

Q.E.D.

Proof II. 逆用牛顿-莱布尼茨公式

$$\left|\int_0^a f(x)\,dx\right| \leq \int_0^a \underbrace{|f(x)|}_{=\left|f(0)+\int_0^x f'(x)\,dx\right|}\,dx \leq \int_0^a Mx\,dx = \frac{M}{2}a^2$$

Q.E.D.

Proof Ⅲ. 分部积分

$$\int_0^a f(x)\,dx = \int_0^a f(x)\,d(x-a)$$

$$= (x-a)f(x)\Big|_0^a - \int_0^a (x-a)f'(a)\,dx$$

$$= \int_0^a (a-x)f'(a)\,dx$$

则 $\left|\int_0^a f(x)\,dx\right| \leq \int_0^a (a-x)\,|f'(x)|\,dx = \frac{M^2}{2}$

Q.E.D.