柯西行列式

$(1+x)^{\frac{1}{x}}$ 在 $x=0$ 处

泰勒展开

1. Cauchy determinant

for a matrix $C=[C_{ij}]$ $\quad C_{ij}=\frac{1}{x_i-y_i}$ $\quad x_i \neq y_i$

$$\det C=\frac{\prod_{1\le i<j\le n}\frac{1}{x_i-x_j}\prod_{1\le i<j\le n}\frac{1}{y_i-y_j}}{\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{x_i-y_j}}$$

Proof. .

提取因子法、

for A :

$$|A|=\begin{vmatrix} \frac{1}{a_1-b_1} & \cdots & \frac{1}{a_1-b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n-b_1} & \cdots & \frac{1}{a_n-b_n} \end{vmatrix}$$

对第 i 行乘 $\prod_{j=1}^{n}(a_i+b_j)$ 以构造 $|B|=\prod_{i,j=1}^{n}(a_i-b_j)|A|$

$$|B|=\begin{vmatrix} \frac{\prod_j(a_1-b_j)}{a_1-b_1} & \cdots & \frac{\prod_j(a_1-b_j)}{a_1-b_n} \\ \vdots & & \vdots \\ \frac{\prod_j(a_i-b_j)}{a_i-b_1} & \cdots & \frac{\prod_j(a_i-b_j)}{a_i-b_n} \\ \vdots & & \vdots \\ \frac{\prod_j(a_n-b_j)}{a_n-b_1} & \cdots & \frac{\prod_j(a_n-b_j)}{a_n-b_n} \end{vmatrix}$$

if $\exists\ a_i=a_j$ $(i\neq j)$，则 $|B|=0$

同理若 $b_i=b_j$ $(i\neq j)$，则 $|B|=0$

则 $|B|=k\prod_{1\le i<j\le n}(a_i-a_j)\prod_{1\le i<j\le n}(b_i-b_j)$

为了确定 k 的值，令 $a_i=b_i$ 此时

$$|B| = \begin{vmatrix} \prod\limits_{j\neq 1}(a_1-b_j) & & 0 \\ & \ddots & \\ 0 & & \prod\limits_{j\neq n}(a_n-b_j) \end{vmatrix} = \prod_{i\neq j}(a_i-b_j) = \prod_{1\le i<j\le n}(a_i-a_j)\prod_{1\le i<j\le n}(b_i-b_j)$$

则 $k=1$

因此

$$|A| = \frac{\prod\limits_{1\le i<j\le n}(a_i-a_j)\prod\limits_{1\le i<j\le n}(b_i-b_j)}{\prod\limits_{i,j=1}^{n}(a_i-b_j)}$$

Q.E.D.

2、 $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}$

$$= e^{\frac{1}{x}\sum_{i=1}^{\infty}(-1)^{i+1}x^{i}\frac{1}{i}}$$

$$= e^{1+(-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3+\cdots)}$$

$$= e\cdot\left[1+(-\frac{1}{2}x+\frac{1}{3}x^2)+\frac{1}{2}(-\frac{1}{2}x+\frac{1}{3}x^2)^2+\cdots\right]$$

$$= e\left[1-\frac{1}{2}x+\frac{11}{24}x^2-\frac{7}{16}x^3+o(x^3)\right]$$