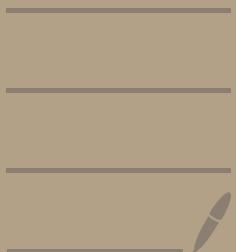


# Laplace 方法

(含 Strey 公式)



Laplace 方法主要用于解决形如以下类型的极限

$$\lim_{n \rightarrow \infty} \int_a^b \varphi(x) e^{nh(x)} dx$$

也可变形为：

$$\lim_{n \rightarrow \infty} \int_a^b \varphi(x) f^h(x) dx \quad [h(x) = \ln f(x)]$$

### Theorem 1

- (I)  $\exists n_0 \in \mathbb{N}$ ,  $\forall n \geq n_0$ ,  $\varphi(x) e^{nh(x)}$  在  $[a, b]$  上可积
- (II)  $\exists \xi$ .  $h(\xi) > h(x) \quad x \in [a, b]$ ,
- (III)  $\exists$  neighbour of  $\xi$ ,  $h''(\xi) < h''(x)$  continuous
- (IV)  $\varphi(x)$  continuous at  $x = \xi$ ,  $\varphi(\xi) \neq 0$

$$n \rightarrow \infty, \quad \int_a^b \varphi(x) e^{nh(x)} dx \sim \varphi(\xi) \sqrt{-\frac{2\pi}{nh''(\xi)}} e^{nh(\xi)}$$

### Theorem 2

当  $\xi = a$  且  $h'(a) = 0$

$$\int_a^b \varphi(x) e^{nh(x)} dx \sim \frac{1}{2} \varphi(\xi) \sqrt{-\frac{2\pi}{nh''(\xi)}} e^{nh(\xi)}$$

当  $\xi = a$  且  $h'(a) < 0$

$$\int_a^b \varphi(x) e^{nh(x)} dx \sim -\frac{\varphi(\xi)}{nh'(\xi)} e^{nh(\xi)}$$

以下是对全纯多元函数的 Laplace 方法.

$$\int_a^b \varphi(t) e^{\lambda f(t)} dt \quad \text{满足:}$$

1. 在  $\lambda = \lambda_0$  绝对收敛

2.  $f$  在  $t_0 \in [a, b]$  取得绝对最大

3. 存在  $B(t_0, \delta)$ ,  $f, \varphi$  在其上有一致收敛的 Taylor 级数

(I).  $t_0 \in (a, b)$ ,  $f'(t_0) \neq 0$ .

$$\int_a^b \varphi(t) \cdot e^{\lambda f(t)} dt \sim e^{\lambda f(t_0)} \sum_{n=0}^{\infty} a_n T(n+\frac{1}{2}) \lambda^{(n+\frac{1}{2})}$$

$a_n$  为函数  $\psi(T) := \varphi[t(T)] \cdot t'(T)$  的系数

$$\psi(T) = \sum_{n=0}^{\infty} a_n T^n$$

$t(T)$  为  $T(t) = \sqrt{f(t) - f(t_0)}$  的逆函数 (在  $t < t_0$  时取负支,  $t > t_0$  时正分支)

可由 Bürmann - Lagrange 级数确定:

$$t(T) = \sum_{n=0}^{\infty} d_n T^n$$

$$d_0 = t_0, \quad d_n = \lim_{t \rightarrow t_0} \frac{1}{n!} \left( \frac{d}{dt} \right)^{n-1} \left( \frac{t-t_0}{T(t)} \right)^n$$

$$= \lim_{t \rightarrow t_0} \frac{1}{n!} \left( \frac{d}{dt} \right)^{n-1} \left[ \sum_{k=0}^{\infty} -\frac{f^{(k+2)}(t_0)}{(k+2)!} (t-t_0)^k \right]^{-\frac{n}{2}}$$

(II)  $t_0 = a$ ,  $f'(a) \neq 0$

$$\int_a^b \varphi(t) e^{\lambda f(t)} dt \sim e^{\lambda f(a)} \sum_{n=0}^{\infty} a_n T(n+1) \lambda^{-(n+1)}$$

$$\text{IV) } t_0 \in [a, b] \quad f'(t_0) = \dots = f^{(p)}(t_0) = 0, \quad f^{(p+1)}(t_0) \neq 0$$

这是最一般的情况. 在证明过程中换元时令  $f(t_0) - f(t) = T^p$

由此可验证 Stirling 公式.

$$\begin{aligned} \Gamma(\lambda + 1) &= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \cdot \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \cdot n^{1-2k}\right) \\ &= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \dots\right) \end{aligned}$$