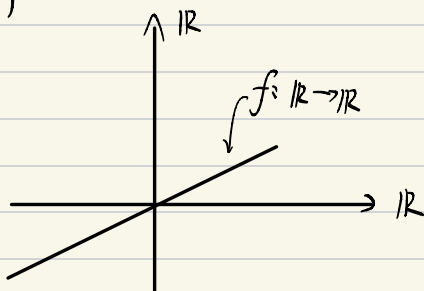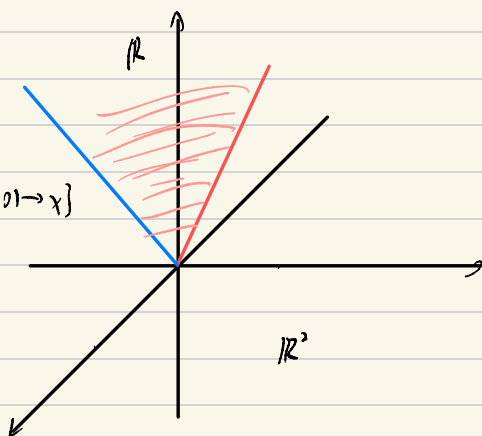$$T_p(\mathbb{R}^n) \cong D_p(\mathbb{R}^n) \qquad v \mapsto \sum_i v^i \frac{\partial}{\partial x^i}\big|_p$$

$$V^\vee = \text{Hom}(V, \mathbb{R})$$

$$\{1\} \cong \{f: x \to x\}$$

$$\{(1,0), (0,1)\}$$
$$\cong \{f: (0,x) \to x, \; g(x,0) \to x\}$$

$$Af = \sum_{\sigma \in S_k} \text{sgn}(\sigma)\, \sigma f \qquad \text{for } f \in L_k$$

$$f \wedge g = \frac{1}{k!\, l!} A(f \otimes g)$$

for $\; k = l = 1$

$$f \wedge g (x_1, x_2) = A(f \otimes g)(x_1, x_2) = f(x_1)\, g(x_2) \; - \; f(x_2)\, g(x_1)$$

it can be see that $\quad (a^1 \wedge \cdots \wedge a^k)(v_1, \cdots v_k) = \det[a^i(v_j)]$

External Algebra.

$$A_*(V) = \bigoplus_{k=0}^{n} A_k(V) \qquad (\dim V = n)$$

for $n = 2$

$$A_*(V) = A_0(V) \oplus A_1(V) \oplus A_2(V)$$

$$= \mathbb{R} \oplus \text{span}\{f:(x,y) \to x, \; g:(x,y) \to y\} \oplus \text{span}\{F:(x_1,y_1)\times(x_2,y_2) \to x_1 y_2 - x_2 y_1\}$$

Notice, if $V = \text{span}\{(0,1)\}$ $V = \text{span}\{(1,0)\}$
$$\mathbb{R}^2 = V \oplus V$$

$I = (i_1, \cdots i_k)$ $1 \le i_1 < \cdots < i_k \le n$ is   a base of $A_k(V)$

proof:
  for $g = \sum_I f(e_I) \alpha^I$

$$g(e_J) = \sum_I f(e_I) \alpha^I e_J = f(e_J)$$

where the following lemma is needed:

$$a^I e_J = \delta^I_J$$

can prove this, by observe that $a^I e_J = \det[(a^i e_j)]$

Q.E.D.

Covector filed (differential 1 - form)

$$w : U \rightarrow \bigcup_{p \in V} T_p^* (\mathbb{R}^n) \ , \ p \mapsto w_p = \sum_I a_I(p) \cdot (dx^I)_p \in T_q^*(\mathbb{R}^n)$$

$\downarrow$ k-form

$$w : U \rightarrow A_k \left(\bigcup_{p \in V} T_p (\mathbb{R}^n) \right) \quad \text{base} \quad (dx^I)_p$$

$$T_p^* (\mathbb{R}^n) = \text{span} \{ dx^1, \cdots dx^n \} \quad \text{dual} \quad \text{to} \quad \text{span} \{ \frac{\partial}{\partial x_1}\big|_p, \cdots \}$$

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

$$\Omega^* (V) = \bigoplus_{k=0}^n \Omega^k (V)$$

$$w(X) = \sum_i a_i b^i$$

exterior derivate of k - form

$$dw = \sum_I d a_I \wedge dx^I = \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(V)$$

(I) $d(w \wedge T) = (dw) \wedge T + (-1)^{deg(w)} w \wedge dT$

(II) $d^2 = 0$

(III) $(df)(X) = Xf$

f 的 外导数 是 梯度 (gradient)

1 - 形式 的 外导数 是 旋度 (Curl)

2 - 形式 的 外导数 是 散度 (divergence)

Topology Space

Second Countable

Hausdorff

Locally Euclidean

$\Bigg\}$ Topological Manifold

Topological Manifold

Maximal Atlas

$\Big\}$ Smooth (C^∞) Manifold.