Ring

Def. $(R,+,\cdot)$ is a ring $\Leftrightarrow$
$(R,+)$ is a Abel group
$(R,\cdot)$ is a monoid
$\forall a,b,c \in R$, $a(b+c) = ab+ac$
$\forall a,b,c \in R$, $(a+b)\cdot c = ac+bc$

Additionally,
$(R,+,\cdot)$ is a commutative ring $\Leftrightarrow$
$\forall$ $a,b \in R$, $ab=ba$

example. $(M(n,\mathbb{R}),+,\cdot)$ $n*n$ matrix ring

Def. $(R^{\times},\cdot)$ is a group constructed by all invertible element in R

if $(R^{\times},\cdot) = R\setminus\{0\} \Rightarrow (R,+,\cdot)$ is a division ring

if $(R,+,\cdot)$ is a commutative division ring $\Rightarrow (R,+,\cdot)$ is a field.

Prop. $(R,+,\cdot)$ is a field $\Leftrightarrow$
$(R,+)$ is Abelian
$(R\setminus\{0\},\cdot)$ is Abelian
Multiplication is distributive to addition.

Def. $S \subset R$, S is a subring $\Leftrightarrow$
$0,1 \in S$
$\forall$ $a,b \in S$, $a+b, ab \in S$
$\forall$ $a \in S$, $-a \in S$

Def.

$$\langle A \rangle = \bigcap \{ S \subset R : S \supset A,\ S < R \}$$

Prop. $A \subset R$, $\langle A \rangle < R$

Def. $((R_i, +_i, \cdot_i))_{i \in I}$ their direct product $(\prod_{i \in I} R_i, +, \cdot)$
for $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} R_i$

$$(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i +_i y_i)_{i \in I}$$
$$(x_i)_{i \in I} \cdot (y_i)_{i \in I} = (x_i \cdot_i y_i)_{i \in I}$$

Prop. their direct product is a Ring