1、排序不等式利用

2、局部不等式，积分思想

1. prove $\sum_{i=1}^{n} \frac{x_i}{x_{i+1}} \geq \sum_{i=1}^{n} \frac{x_i+1}{x_{i+1}+1}$ (subscript mod n) $x_i \in [2017, 2019]$

Proof.

$$f(t) \equiv \sum_{i=1}^{n} \frac{x_i+t}{x_{i+1}+t}$$

$$f'(t) = \sum_{i=1}^{n} \frac{x_{i+1}-x_i}{(x_{i+1}+t)^2} = \sum_{i=1}^{n} \frac{x_{i+1}}{(x_{i+1}+t)^2} - \sum_{i=1}^{n} \frac{x_i}{(x_{i+1}+t)^2}$$

where $\sum_{i=1}^{n} x_{i+1} \cdot \frac{1}{(x_{i+1}+t)^2}$ is a Reserved Order

$\sum_{i=1}^{n} x_i \cdot \frac{1}{(x_{i+1}+t)^2}$ is a Random Order

Chebyshev's Inequality.

$a_1 \leq \cdots \leq a_n \quad b_1 \leq \cdots \leq b_n$

$$n\left(\sum a_i b_i\right) \geq \left(\sum a_i\right)\left(\sum b_i\right)$$ equal iff $a_1 = \cdots a_n$ or $b_1 = \cdots = b_n$

thus $f'(t) \leq 0$

Q.E.D.

2. Prove $\sum_{i=1}^{n} \frac{x_{i+1}}{x_i} \geq \sum_{i=1}^{n} \frac{x_i+1}{x_{i+1}+1}$ $x_i \in [2017, 2019]$

Proof.

$$\sum_{i=1}^{n} \frac{x_{i+1}}{x_i} - \sum_{i=1}^{n} \frac{x_i+1}{x_{i+1}+1}$$

$$= \sum_{i=1}^{n} \frac{(x_{i+1}-x_i)(x_i+x_{i+1}+1)}{x_i(x_{i+1}+1)}$$

The Definition of Definite Integral.

$$\int_a^b f(x)\,dx = \lim_{m\to\infty} \sum_{i=1}^{m} f(\xi_i)\cdot(t_i - t_{i-1})$$

Thus when x_{i+1} & x_i close enough.

$$\sum_{i=j}^{k} \frac{(x_{i+1}-x_i)(x_i+x_{i+1}+1)}{x_i(x_{i+1}+1)} = \int_{x_j}^{x_k} \frac{2x+1}{x(x+1)}\,dx = \ln \frac{x_k(x_k+1)}{x_j(x_j+1)}$$

Now prove such "Approximate" works:

for $\left(\frac{c}{a} - \frac{a+1}{c+1}\right) - \left(\frac{c}{b} - \frac{b+1}{c+1}\right) - \left(\frac{b}{a} - \frac{a+1}{b+1}\right)$

$$= (b-a)(c-b)\frac{1}{ab(b+1)(c+1)}\left[b(c-a)+b+c+1\right] \quad (1)$$

when $c>b>a$, $(1)>0$

$c<b<a$, $c \geq a-1$, $(1) \geq 0$

here we have $|a-c| \leq 2$

so for the second case, if $a-2 \leq c < a$
we can add a limitation that $c < b < \min\{a, c+1\}$
then (1) is always ≥ 0

then the question is Solved.

In another way to express the main idea of above

To find an Local Inequality:

for $b \geq a-2$, $a>0$, $b>0$

$$\frac{b}{a} - \frac{a+1}{b+1} \geq \ln \frac{b(b+1)}{a(a+1)}$$

then apply the Inequality, the question is solved.

The proof of the Inequality:

$$\ln \frac{b(b+1)}{a(a+1)} = \int_a^b \left(\frac{1}{x} + \frac{1}{x+1}\right) dx = \int_a^b \left(\frac{1}{x} + \frac{1}{a+b+1-x}\right) dx$$

$$\leq \int_a^b \left(\frac{1}{a} + \frac{1}{b+1}\right) dx = (b-a)\left(\frac{1}{a} + \frac{1}{b+1}\right) = \frac{b}{a} - \frac{a+1}{b+1}$$