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Balancing Unit Vectors

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Theorem A. *Let $x_1, \ldots, x_{2k+1}$ be unit vectors in a normed plane. Then there exist signs $\varepsilon_1, \ldots, \varepsilon_{2k+1} \in \{\pm 1\}$ such that $\|\sum_{i=1}^{2k+1} \varepsilon_i x_i\| \leqslant 1$.*

We use the method of proof of the above theorem to show the following point facility location result, generalizing Proposition 6.4 of Y. S. Kupitz and H. Martini (1997).

Theorem B. *Let $p_0, p_1, \ldots, p_n$ be distinct points in a normed plane such that for any $1 \leqslant i < j \leqslant n$ the closed angle $\angle p_i p_0 p_j$ contains a ray opposite some $\overrightarrow{p_0 p_k}$, $1 \leqslant k \leqslant n$. Then p_0 is a Fermat–Torricelli point of $\{p_0, p_1, \ldots, p_n\}$, i.e. $x = p_0$ minimizes $\sum_{i=0}^{n} \|x - p_i\|$.*

We also prove the following dynamic version of Theorem A.

Theorem C. *Let $x_1, x_2, \ldots$ be a sequence of unit vectors in a normed plane. Then there exist signs $\varepsilon_1, \varepsilon_2, \ldots \in \{\pm 1\}$ such that $\|\sum_{i=1}^{2k} \varepsilon_i x_i\| \leqslant 2$ for all $k \in \mathbf{N}$.*

Finally we discuss a variation of a two-player balancing game of J. Spencer (1977) related to Theorem C. © 2000 Academic Press

Key Words: vector balancing; balancing game; Fermat point; Fermat–Torricelli point; facilities location.

1. INTRODUCTION

In this note we consider balancing results for unit vectors related to work of Bárány and Grinberg [1], Spencer [6], and Peng and Yan [5]. We apply these results to generalize a point facility location result from the Euclidean plane [4] to general normed planes. Finally we consider a dynamical balancing problem for unit vectors in the form of a two-player perfect information game. Our results will mainly be in a normed plane X with norm $\|\cdot\|$ (except in Theorem 1.5, where higher-dimensional normed spaces are also considered).

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1.1. *Balancing Unit Vectors*

Bárány and Grinberg [1] proved the following:

THEOREM 1.1 [1]. *Let $x_1, x_2, ..., x_n$ be a sequence of vectors of norm $\leqslant 1$ in a d-dimensional normed space. Then there exist signs $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{\pm 1\}$ such that*

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leqslant d.$$

We sharpen this theorem for an odd number of *unit* vectors in a normed plane as follows.

THEOREM 1.2. *Let $x_1, ..., x_{2k+1}$ be unit vectors in a normed plane. Then there exist signs $\varepsilon_1, ..., \varepsilon_{2k+1} \in \{\pm 1\}$ such that*

$$\left\| \sum_{i=1}^{2k+1} \varepsilon_i x_i \right\| \leqslant 1.$$

This result is best possible in any norm, as is seen by letting $x_1 = x_2 = \cdots = x_{2k+1}$ be any unit vector. The proof of this theorem is in Section 2. The method of proof can also be used to generalize a result on Fermat–Torricelli points from the Euclidean plane to an arbitrary normed plane (Section 1.3).

Bárány and Grinberg also proved the following dynamic balancing theorem.

THEOREM 1.3 [1]. *Let $x_1, x_2, ...$ be a sequence of vectors of norm $\leqslant 1$ in a d-dimensional normed space. Then there exist signs $\varepsilon_1, \varepsilon_2, ... \in \{\pm 1\}$ such that for all $k \in \mathbf{N}$,*

compare to 1.1

$$\left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| \leqslant 2d.$$

Again, for unit vectors in a normed plane we sharpen this result as follows.

THEOREM 1.4. *Let $x_1, x_2, ...$ be a sequence of unit vectors in a normed plane. Then there exist signs $\varepsilon_1, \varepsilon_2, ... \in \{\pm 1\}$ such that for all $k \in \mathbf{N}$,*

$$\left\| \sum_{i=1}^{2k} \varepsilon_i x_i \right\| \leqslant 2.$$

In the Euclidean plane the upper bound 2 can be replaced by $\sqrt{2}$.

This result is best possible in the rectilinear plane with unit ball a parallelogram—let $x_{2i-1}=e_1$ and $x_{2i}=e_2$ for all $i \in \mathbf{N}$, where e_1 and e_2 are any adjacent vertices of the unit ball. See Section 3 for a proof of this theorem.

1.2. *Balancing Games*

Theorem 1.4 can be used to analyze the following variation of a two-player balancing game of Spencer. Fix $k \in \mathbf{N}$ and a normed space X. Let the starting position of the game be $p_0 = o \in X$. In round i, Player I chooses k unit vectors $x_1, ..., x_k$ in X, and then Player II chooses signs $\varepsilon_1, ..., \varepsilon_k \in \{\pm 1\}$. Then the position is adjusted to $p_i := p_{i-1} + \sum_{j=1}^{k} \varepsilon_j x_j$.

THEOREM 1.5. *In the above game, Player II can keep the sequence $(p_i)_{i \in \mathbf{N}}$ bounded iff X is at most two-dimensional and k is even. In fact, Player II can force $\|p_i\| \leqslant 2$ for all $i \in \mathbf{N}$.*

The proof is in Section 3. In [5] a vector balancing game with a buffer is considered. Theorem 1.5 readily implies Theorem 4 of [5] in the special case of unit vectors in a normed plane.

1.3. *Fermat–Torricelli Points*

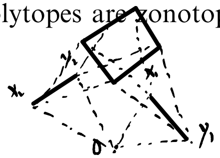
A point p in a normed space X is a *Fermat–Torricelli point* of $x_1, x_2, ..., x_n \in X$ if $x = p$ minimizes $x \mapsto \sum_{i=1}^{n} \|x_i - x\|$. See [4] for a survey on the problem of finding such points. It is well-known that in the Euclidean plane, if x_1 is in the convex hull of non-collinear $\{x_2, x_3, x_4\}$, then x_1 is the (unique) Fermat–Torricelli point of x_1, x_2, x_3, x_4. Cieslik [2] generalized this result to an arbitrary normed plane (where the Fermat–Torricelli point is not necessarily unique). There is also a generalization by Kupitz and Martini [4, Proposition 6.4] in another direction.

THEOREM. *Let $p_0, p_1, ..., p_{2m+1}$ be distinct points in the Euclidean plane such that for any distinct i and j the open angle $\angle p_i\, p_0\, p_j$ contains a ray opposite some $\overrightarrow{p_0\, p_k}$, $1 \leqslant k \leqslant 2m+1$. Then p_0 is the unique Fermat–Torricelli point of $\{p_0, p_1, ..., p_n\}$.*

We generalize this result as follows to an arbitrary normed plane.

THEOREM 1.6. *Let $p_0, p_1, ..., p_n$ be distinct points in a normed plane such that for any distinct i and j the closed angle $\angle p_i\, p_0\, p_j$ contains a ray opposite some $\overrightarrow{p_0\, p_k}$, $1 \leqslant k \leqslant n$. Then p_0 is a Fermat–Torricelli point of $\{p_0, p_1, ..., p_n\}$.*

The proof is in Section 2. Our seemingly weaker hypotheses easily imply that n must be odd. The proof in [4] of the Euclidean case uses rotations. Our proof for any norm shows that it is really an affine result. The correct affine tool turns out to be the fact that two-dimensional centrally symmetric polytopes are zonotopes.



2. ZONOGONS

A *zonotope* P in a d-dimensional vector space X is a Minkowski sum of line segments

$$P = [x_1, y_1] + [x_2, y_2] + \cdots + [x_n, y_n]$$

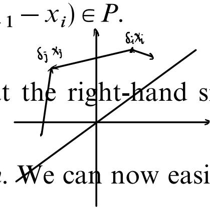
where $x_1, ..., x_n, y_1, ..., y_n \in X$. It is well-known that any centrally symmetric two-dimensional polytope (or polygon) is always a zonotope (or *zonogon*) [8, Example 7.14]. In particular, if $x_1, ..., x_n$ are consecutive edges of a $2n$-gon P symmetric around 0, then

$$P = \sum_{i=1}^{n} [(x_{i+1} - x_i)/2, (x_i - x_{i+1})/2] \tag{1}$$

where we take $x_{n+1} = -x_1$.

Lemma 2.1. *Let $n \in \mathbf{N}$ be odd and let P be a polygon with vertices $\pm x_1, ..., \pm x_n$ with $x_1, ..., x_n$ in this order on the boundary of P. Then*

$$\sum_{i=1}^{n} (-1)^i x_i = \tfrac{1}{2} \sum_{i=1}^{n} (-1)^{i+1} (x_{i+1} - x_i) \in P.$$



Proof. The equation is simple to verify. That the right-hand side is in P follows from (1). ∎

Note that Lemma 2.1 does not hold for even n. We can now easily prove Theorem 1.2.

$\sum_{i=1}^{2k+1} (-1)^i \delta_i x_i = \frac{1}{2} \sum_{i=1}^{2k+1} (-1)^{i+1} (\delta_{i+1} x_{i+1} - \delta_i x_i) \in P$

Proof of Theorem 1.2. Fix a line through the origin not containing any x_i. Fix one of the open half planes H bounded by this line. Then for each i, $\delta_i x_i \in H$ for some $\delta_i \in \{\pm 1\}$. We may renumber $x_1, ..., x_n$ such that $\delta_1 x_1, ..., \delta_n x_n$ occur in this order on $P = \operatorname{conv}\{\pm x_i\}$. Now take $\varepsilon_i = (-1)^i \delta_i$ and apply Lemma 2.1, noting that P is contained in the unit ball. ∎ ✓

Recall that the dual of a finite dimensional normed space X is the normed space of all linear functionals on X with norm $\|\phi\| = \max\{\phi(u): \|u\| = 1\}$. A *norming functional* ϕ of a non-zero $x \in X$ is a linear functional satisfying $\|\phi\| = 1$ and $\phi(x) = \|x\|$. Recall that by the separation theorem any non-zero $x \in X$ has a norming functional (see e.g. [7]).

The following lemma is well-known and easily proved. See [4] for the Euclidean case and [3] for the general case. We only need the second case of the lemma, but we also state the first case for the sake of completeness.

$\|x\|=1$ $x=\sum \lambda_i x_i$

LEMMA 2.2. *Let $p_0, p_1, ..., p_n$ be distinct points in a finite-dimensional normed space X.* $(\sum \phi_i)\, x = \lambda_i \|x_i\|$ if orthonormal

1. *Then p_0 is a Fermat–Torricelli point of $p_1, ..., p_n$ iff $p_i - p_0$ has a norming functional ϕ_i $(1 \leqslant i \leqslant n)$ such that $\sum_{i=1}^n \phi_i = o$,*

2. *and p_0 is a Fermat–Torricelli point of $p_0, p_1, ..., p_n$ iff $p_i - p_0$ has a norming functional ϕ_i $(1 \leqslant i \leqslant n)$ such that $\|\sum_{i=1}^n \phi_i\| \leqslant 1$.*

Proof of Theorem 1.6. By Lemma 2.2 it is sufficient to find norming functionals ϕ_i of $p_i - p_0$ such that $\|\sum_{i=1}^n \phi_i\| \leqslant 1$. We order $p_1, ..., p_n$ such that $\overrightarrow{p_0 p_1}, ..., \overrightarrow{p_0 p_n}$ are ordered counter-clockwise. If $p_0 \in [p_i, p_j]$ for some $1 \leqslant i < j \leqslant n$, we may choose $\phi_i = -\phi_j$. We may therefore assume that $p_0 \notin [p_i, p_j]$ for all distinct i, j. Thus for any i, the *open* angle $\angle p_i p_0 p_{i+1}$ contains a ray opposite some $\overrightarrow{p_0 p_k}$. We now show that necessarily n is odd and $k \equiv i + (n+1)/2 \pmod n$. Since each open angle contains at least one $-p_k$, each open angle contains exactly one such $-p_k$, say $-p_{k(i)}$. The line through p_0 and $p_{k(i)}$ cuts $\{p_1, ..., p_n\}$ in two open half planes: One half plane contains as many open angles as points p_i. Thus n is odd, and $k(i) \equiv i + (n+1)/2 \pmod n$.

It is now possible to choose norming functionals ϕ_i of each $p_i - p_0$ such that $\phi_1, -\phi_{m+1}, \phi_2, -\phi_{m+2}, ...$ are consecutive vectors on the unit circle in the dual normed plane. It is therefore sufficient to prove that in any normed plane, if we choose unit vectors $x_1, ..., x_n$ such that $x_1, ..., x_n, -x_1, ..., -x_n$ are in this order on the unit circle, then $\|\sum_{k=1}^n (-1)^k x_k\| \leqslant 1$. This follows at once from Lemma 2.1. ∎

3. ONLINE BALANCING

Proof of Theorem 1.5. $\Rightarrow$ We assume that some inner product structure has been fixed on X.

If k is odd then in round i Player I chooses the k unit vectors all to be the same unit vector, orthogonal to p_{i-1}. Then, independent of the choice of signs by Player II, the Euclidean norm of p_i grows $> c\sqrt{i}$.

If k is even and X is at least three-dimensional, Player I finds unit vectors e_1 and e_2 such that e_1, e_2, p_{i-1} are mutually orthogonal, then in round i takes e_1 for the first $k-1$ unit vectors, and e_2 for the last unit vector. Again the Euclidean norm of p_i will grow $> c\sqrt{i}$.

$\Leftarrow$ follows immediately from Lemmas 3.1 and 3.2 below. ∎

Proof of Theorem 1.4. Follows immediately from the following two lemmas. ∎

LEMMA 3.1. *Let w, a, b be vectors in a normed plane such that $\|w\| \leqslant 2$, $\|a\| = \|b\| = 1$. Then there exist signs $\delta, \varepsilon \in \{\pm 1\}$ such that $\|w + \delta a + \varepsilon b\| \leqslant 2$.*

Proof. If $a = \pm b$, then the lemma is trivial. So assume that a and b are linearly independent. Let $w = \lambda a + \mu b$. Without loss of generality we assume that $\lambda, \mu \geqslant 0$, and show that $\|w - a - b\| \leqslant 2$.

If $\lambda = 0$, then $0 \leqslant \mu \leqslant 2$ and $\|(\lambda - 1)a + (\mu - 1)b\| \leqslant \|a\| + \|(\mu - 1)b\| \leqslant 2$. So we may assume that $\lambda > 0$, and similarly, $\mu > 0$. Then we can write $a = -(\mu/\lambda)\, b + (1/\lambda) w$. Taking norms we obtain $1 = \|a\| \leqslant \mu/\lambda + 2/\lambda$, and therefore, $\lambda - \mu \leqslant 2$. Similarly, $\mu - \lambda \leqslant 2$. So we already have $|(\lambda - 1) - (\mu - 1)| \leqslant 2$. If furthermore $\lambda + \mu \leqslant 4$, we also obtain $|(\lambda - 1) + (\mu - 1)| \leqslant 2$, giving $\|(\lambda - 1)a + (\mu - 1)b\| \leqslant |\lambda - 1| + |\mu - 1| \leqslant 2$.

In the remaining case $\lambda + \mu \geqslant 4$ we write $(\lambda - 1)a + (\mu - 1)b$ as a non-negative linear combination

$$(\lambda - 1)a + (\mu - 1)b = \frac{\lambda + \mu - 4}{\lambda + \mu - 2}(\lambda a + \mu b) + \frac{2 + \lambda - \mu}{\lambda + \mu - 2}\, a + \frac{2 - \lambda + \mu}{\lambda + \mu - 2}\, b,$$

and apply the triangle inequality:

$$\|(\lambda - 1)a + (\mu - 1)b\| \leqslant 2\frac{\lambda + \mu - 4}{\lambda + \mu - 2} + \frac{2 + \lambda - \mu}{\lambda + \mu - 2} + \frac{2 - \lambda + \mu}{\lambda + \mu - 2} = 2. \quad ∎$$

LEMMA 3.2. *Let w, a, b be vectors in the Euclidean plane such that $\|w\| \leqslant \sqrt{2}$, $\|a\| = \|b\| = 1$. Then there exist signs $\delta, \varepsilon \in \{\pm 1\}$ such that $\|w + \delta a + \varepsilon b\| \leqslant \sqrt{2}$.*

Proof. Note that $a + b \perp a - b$. Write $p = a + b$, $q = a - b$. Let m be the midpoint of pq, and L the perpendicular bisector of pq. Assume without loss that $\|p\| \geqslant \|q\|$ and that w is inside $\angle poq$. We now show that $\|w - p\| \leqslant \sqrt{2}$ or $\|w - q\| \leqslant \sqrt{2}$. Note that as w varies, $\min(\|w - p\|, \|w - q\|)$ is maximized on L. Let L and op intersect in c (between o and p), and L and the circle with centre o and radius $\sqrt{2}$ in d (inside $\angle poq$). See Fig. 1. Then clearly

$$\max_{\|w\| \leqslant \sqrt{2}} \min(\|w - p\|, \|w - q\|) = \max(\|p - c\|, \|p - d\|),$$

and we have to show $\|p - c\| \leqslant \sqrt{2}$ and $\|p - d\| \leqslant \sqrt{2}$. Since $\|p\| \geqslant \|q\|$, we have $\angle opq \leqslant 45°$ and $\|p - c\| = \sec \angle opq \leqslant \sqrt{2}$. Since c is between o and p, we have $\angle omd \geqslant 90°$, hence $\|m - d\|^2 \leqslant \|d\|^2 - \|m\|^2 = 2 - 1$, and $\|p - d\|^2 = \|p - m\|^2 + \|m - d\|^2 \leqslant 1 + 1$. ∎

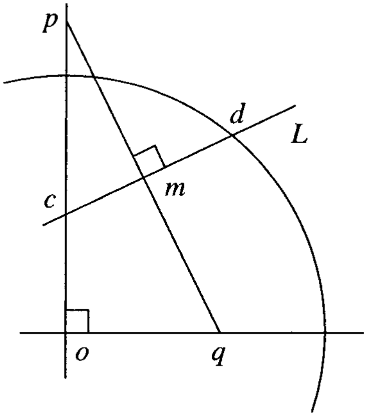


FIGURE 1

4. CONCLUDING REMARKS

It would be interesting to find higher dimensional generalizations of our results and methods. We only make the following remarks.

Perhaps there is an analogue of Theorem 1.2 with an upper bound of $d-1$ for n unit vectors in a d-dimensional normed space where $n \not\equiv d \pmod 2$. This would be best possible, as the standard unit vectors in the d-dimensional space with the L_1 norm show.

Regarding Theorem 1.4, it is not even clear what the best upper bound in Theorem 1.3 should be. Bárány and Grinberg [1] claim that they can replace $2d$ by $2d-1$. On the other hand, the upper bound cannot be smaller than d, as the d-dimensional L_1 space shows [1]. As the negative part of Theorem 1.5 and the results of [5] show, an online method would have to have a (sufficiently large) buffer where Player II can put vectors supplied by Player I and take them out in any order.

We finally remark that a naive generalization of Theorem 1.6 is not possible, even in Euclidean 3-space. For example, using Lemma 2.2 it can be shown that for a regular simplex with vertices x_i ($i=1, ..., 4$) there exists a point x_5 in the interior of the simplex such that x_5 is not a Fermat–Torricelli point of $\{x_1, ..., x_5\}$—we may take any x_5 sufficiently near a vertex.

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