1. for all $x_0 + y_0 \in W$, $x_0 \in U$, $y_0 \in V$

since U is open. $\exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subseteq U$

since $\forall p \in B(x_0 + y_0, \varepsilon)$ can be expressed as $p = x_1 + y_0$ where $x_1 \in B(x_0, \varepsilon) \subseteq U, y_0 \subseteq V$

then $B(x_0 + y_0, \varepsilon) \subseteq W$

thus W is open.

2. for all $(x_0, y_0) \in R$, set $F(x,y) = f(x) g(y)$

since $f, g: \mathbb{R} \to \mathbb{R}$ is continuous,
$\forall x \in \mathbb{R} \ (\forall \varepsilon > 0, \exists \delta_x > 0$ s.t. $\forall |x' - x| < \delta_x, \ |f(x') - f(x)| < \varepsilon)$
$\forall y \in \mathbb{R} \ (\forall \varepsilon > 0, \exists \delta_y > 0$, s.t. $\forall |y' - y| < \delta_y, \ |g(y') - g(y)| < \varepsilon)$

then
$\forall (x,y) \in \mathbb{R}^2 \ (\forall \varepsilon > 0 \ \exists \delta = \min(\delta_x, \delta_y) > 0$, s.t. $\forall \|(x',y') - (x,y)\| < \delta$,

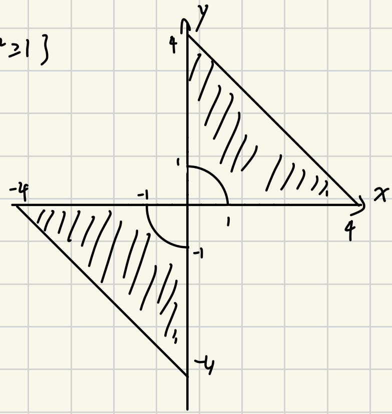
$$|F(x',y') - F(x,y)| = |f(x') g(y') - f(x) g(y)|$$
$$= |f(x') g(y') - f(x') g(y) + f(x') g(y) - f(x) g(y)|$$
$$\leq |f(x')| \, |g(y') - g(y)| + |g(y)| \, |f'(x) - f(x)|$$
$$< \varepsilon (|f(x')| + |g(y)|)$$

where $|f(x')|, |g(x)|$ are curb by a maximum since it's continuous.

thus $F(x,y)$ is continuous. , $(-\infty, 1)$ is open

thus $R = F^{-1}[(-\infty, 1)]$ is open.

3. $Q = \underset{\underset{A}{\shortparallel}}{\{(x,y) \mid xy \ge 0\}} \cap \underset{\underset{B}{\shortparallel}}{\{(x,y) \mid |x|+|y| \le 4\}} \cap \underset{\underset{C}{\shortparallel}}{\{(x,y) \mid x^2+y^2 \ge 1\}}$



$A^c = \{(x,y) \mid xy < 0\}$

for $(x_0, y_0) \in A^c$

$(x_0, y_0) \subseteq B[(x_0, y_0), r_0] \subseteq A^c$

where $r_0 = \min(x_0, y_0)$

$B^c = \{(x,y) \mid |x| + |y| > 4\}$

for $(x_1, y_1) \in B^c$

$(x_1, y_1) \subseteq B[(x_1, y_1), r_1] \subseteq B^c$

where $r_1 = \min\left(\frac{|x_1+y_1+4|}{\sqrt{2}}, \frac{|x_1+y_1-4|}{\sqrt{2}}, \frac{|x_1-y_1+4|}{\sqrt{2}}, \frac{|x_1-y_1-4|}{\sqrt{2}}\right)$

$C^c = \{(x,y) \mid x^2+y^2 < 1\}$ is a open ball

thus A, B, C is close, so is $Q = A \cap B \cap C$

$\partial Q \cap \{(x_1, 0) \mid x_1 \in \mathbb{R}\} = [0, 4] \times \{0\}$

4. S is not open. consider $(x,y) \in [0,1) \times \{0\}$, no neighbour is included.

S is closed. S is the union of countable many 2-cells.

S is not bounded. since as $x \to \infty$, y can $\to \infty$, there's no upper bound.

5. (a) $D^o = \mathbb{R}^d / \partial D$, suppose $t \in O$

then all $z \in \{t + x \mid x \in \mathbb{R}^d / \partial D\} = S$ can be expressed.

consider $z_0 \in \{t+x \mid x \in \partial D\} = \partial S$

since O is open. $\exists r > 0$, $B(t, r) \subseteq O$

$\{t+x \mid x \in \partial D\}$ is the boundary of $\{t+x \mid x \in \mathbb{R}^d / \partial D\}$

thus $\exists (a_n)_n \in S$, $\forall \varepsilon > 0$. $\exists N$, $\forall n \geq N$, $\| a_n - z_0 \| < \varepsilon$

choose the smallest $n' \geq N$ for $\varepsilon = r$, set $a_{n'} = b \in S$

$\|b - z_0\| <$ $\quad b = t + x_0 \quad z_0 \in \{x_0 + t' \mid t' \in B(t, r)\}$

thus z_0 can be expressed as $z = x+y$

(b) use the outcome of (a) where set $A = O = D$

6. The statement equivalent to:

If $A \cap U = \phi$, then U is ϕ or not open.

U is ϕ is obvious.

U is not open:

If U is open & $A \cap U = \phi$, contradict to $f(a)$

1. (a) $H_0: \theta = 1 \qquad H_1: \theta = \theta_1 < 1$

$$\mathcal{L}(\theta_0; X) = \left(\prod_{i=1}^{n} 4x_i^3\right) \cdot \exp\left(-\sum_{i=1}^{n} x_i^4\right)$$

$$\mathcal{L}(\theta_1; X) = \left(\prod_{i=1}^{n} \frac{4x_i^3}{\theta_1}\right) \cdot \exp\left(-\frac{1}{\theta_1}\sum_{i=1}^{n} x_i^4\right)$$

$$C = \left\{ X \,\middle|\, \frac{\exp\left[-\left(\frac{1}{\theta_1}-1\right)\sum_{i=1}^{n} X_i^4\right]}{\theta_1^n} > k \right\}$$

where $\mathbb{P}(X \in C \mid H_0) = \alpha$

$$\mathbb{P}\left(\exp\left[-\left(\frac{1}{\theta_1}-1\right)\frac{\chi^2_{2n}}{2}\right] > k\theta_1^n\right) = \mathbb{P}\left(\chi^2_{2n} < \frac{2\theta_1}{\theta_1 - 1}\ln(k\theta_1^n)\right) = \alpha$$

$$C = \left\{ X \,\middle|\, \sum_{i=1}^{n} x_i^4 \leq \frac{1}{2}\chi^2_{2n,\alpha} \right\}$$

(b) p-value $= \mathbb{P}(\chi^2_{10} \leq 3.589) = 0.042$

(c) $\mathbb{P}(\chi^2_{10} \leq 2k_1) = \frac{1}{2}\alpha \qquad \mathbb{P}(\chi^2_{10} \geq 2k_2) = \frac{1}{2}\alpha$

$k_1 = 1.623 \qquad k_2 = 10.24$

(d) $1.623 < \sum_{i=1}^{5} x_i^4 = 1.794 < 10.24$

thus retain. (b) is rejected since its one-tailed.

2. (a) $l(\lambda; X) = -n\lambda + \left(\sum x_i\right)\ln\lambda - \sum \ln(x_i)!$

$$\frac{\partial}{\partial\lambda} l(\lambda; X) = -n + \frac{1}{\lambda}\sum x_i$$

$$\hat{\lambda} = \arg\max_{\lambda} l = \frac{1}{n}\sum x_i = 5.05$$

$X_i \sim Possion(\lambda)$

for large n

$$\hat{\lambda} = \frac{1}{n}\sum X_i \sim N(\lambda, \frac{\lambda}{n})$$

$$\left| \frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} \right| < 1.96$$

95% CI : $(4.15 , 6.13)$

(b) $\sum x_i = 101$

$\sigma = \sqrt{80}$ $\mu = 80$

$p = 2\left[1 - \Phi\left(\frac{101-80}{\sqrt{80}}\right)\right] = 0.0188$ refuse.

(c) $S_{-w} = 101$ $S_{-s} = 63$

$E_{-w} = E_{-s} = 81.5$

$$\sum \frac{(O_k - E_k)^2}{E_k} = 9.33 \sim \chi_1^2$$

use $\alpha = 0.05$ $9.33 > 3.84 = \chi^2_{1,0.95}$ unreasonable.

(d) under $H_0 : \lambda = \mu = \theta$

$$l[(\theta, \theta)] = -2n\lambda_0 + (S_w + S_s)\ln\theta - \ln \prod x_i \prod y_i$$

$$\frac{\partial}{\partial\theta} l = -2n + (S_w + S_s)/\theta = 0$$

$$\hat{\theta} = 4.075$$

under $H_1: \lambda \neq \mu$

$$l[(\lambda, \mu)] = -n(\lambda + \mu) + S_w \ln \lambda + S_s \ln \mu - \ln \prod x_i \prod y_j$$

$$\begin{cases} \frac{\partial}{\partial \lambda} l = -n + S_w / \lambda = 0 \\ \frac{\partial}{\partial \mu} l = -n + S_s / \mu = 0 \end{cases} \Rightarrow \begin{cases} \hat{\lambda} = 5.05 \\ \hat{\mu} = 3.1 \end{cases}$$

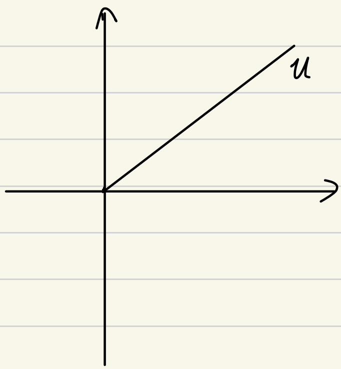
$$2[\,l[(\hat{\lambda}, \hat{\mu})] - l[(\hat{\theta}, \hat{\theta})]\,] = 2[S_w \ln(\frac{\hat{\lambda}}{\hat{\theta}}) + S_y (\frac{\hat{\mu}}{\hat{\theta}})]$$

$$= 9.4 \sim \chi^2_1$$

$9.4 > \chi^2_{1, 0.95} = 3.84$. refuse. H_0 same as (c)

1. (a) i)

$$\frac{y}{x} = \tan w$$



ii) $$e_u = \frac{(v\sin w,\ v\cos w,\ 0)}{v} = (\sin w,\ \cos w,\ 0)$$

$$e_v = (\sin w,\ \cos w,\ 0)$$

$$e_w = \frac{(uv\cos w,\ -uw\sin w,\ 1)}{\sqrt{1+u^2v^2}}$$

$dV = 0$ for $|\det[r_u, r_v, r_w]| = 0$

b)

$$\varepsilon_{ijk}\,\varepsilon_{jlm}\,\varepsilon_{lni} = (\delta_{im}\delta_{kl} - \delta_{il}\delta_{km})\,\varepsilon_{lni}$$

$$= \varepsilon_{knm}$$

2、 a) $\rho_b = -\nabla \cdot P$ $\sigma_b = P \cdot \vec{n}$

$= -\frac{1}{r}\frac{\partial}{\partial r}(rP)$ $= aR^2$

$= -3\alpha r$

b) $P = \varepsilon_0 \chi_e E$

$D = \varepsilon_0 E + P = \frac{1+\chi_e}{\chi_e} P$

$\rho_f = \frac{1+\chi_e}{\chi_e} \nabla \cdot P = \frac{3\alpha r(1+\chi_e)}{\chi_e}$

3.a) $B_z(r) = \left(\int_c \frac{\mu_0 I \cdot dI \times \hat{r}}{4\pi}\right)_z$

$= \frac{\mu_0 I}{4\pi}\left[\left(\frac{3}{4}\int_{x^2+y^2=R^2} dI \times \hat{r}\right)_{\hat{z}} + \underbrace{\left(2\int_{\{0\}\times[0,R]} dI \times \hat{r}\right)_{\hat{z}}}_{=0}\right]$

$(dI \times \hat{r})_{\hat{z}} = \frac{R^2\, d\theta}{(R^2+z^2)^{3/2}}$

$B_z(r) = \frac{3\mu_0 I}{16\pi}\int_0^{2\pi} \frac{R^2\, d\varphi}{(R^2+z^2)^{3/2}} = \frac{3}{8}\frac{\mu_0 I R^2}{(R^2+z^2)^{3/2}}$

b) $B_z(0) = \frac{3}{8}\frac{\mu_0 I}{R}$

4. a) $\nabla \cdot B = \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z$

$= 0$ where Magnetic field always satisfies $\nabla \cdot B = 0$

b) $\Phi_m = \int_0^{2\pi} \int_0^{R} \beta (R-r)^2 r \, dr \, d\theta = \frac{1}{6} \beta \pi R^3$

By divergence Theorem. & $\nabla \cdot B = 0$

$\Phi_{sphere} + (-\Phi_m) = 0 \Rightarrow \Phi_{sphere} = \frac{1}{6} \beta \pi R^3$

c) $\oint_{\partial D} A \cdot dl = \iint_D (\nabla \times A) \cdot \vec{n} \, ds$ (Stokes' Theorem)

$= \iint_D B \cdot \vec{n} \, ds = \Phi_m$

$\oint_{\partial D} A \cdot dl = \int_0^{2\pi} A \cdot R \hat{\theta} \, d\theta = 2\pi R \, A \cdot \hat{\theta}$

$A \cdot \hat{\theta} = \frac{\beta R^3}{12}$