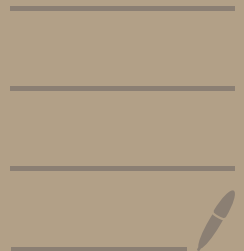


# Euclidean domain



Def.  $(R, +, \cdot)$  domain  $R$  is an Euclidean domain  
 if  $\exists f: R \setminus \{0\} \rightarrow \mathbb{N}_0$   
 s.t.  $\forall a \in R, b \in R \setminus \{0\} \exists q, r \in R$  s.t.  $a = qb + r$   
 where  $r = 0$  or  $f(r) < f(b)$   
 $f$  is called Euclidean function

example:  $(\mathbb{Z}, +, \cdot)$  is an Euclidean domain

Prop.  $(F, +, \cdot)$  field  $F$  is an Euclidean domain.

Prop.  $(R, +, \cdot)$  Euclidean domain  $\Rightarrow R$  is principle ideal domain

Def.  $(R, +, \cdot)$  domain  $N: R \rightarrow \mathbb{R}^{\geq 0}$  is a norm if  
 $N(x) = 0 \Leftrightarrow x = 0$   
 $N(x) = 1 \Leftrightarrow x$  is a unit  
 $\forall a, b \in R, N(ab) = N(a)N(b)$

example. Gauss integral ring  $\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}] = \{a+bi : a, b \in \mathbb{Z}\}$   
 $N(a+bi) = a^2 + b^2$

Prop.  $\mathbb{Z}[i]$  is an Euclidean domain.

Proof. suppose  $a = x+iy, b = z+iw \in \mathbb{Z}[i]$   
 $\exists a, b$  s.t.  $\frac{a}{b} = \frac{x+iy}{z+iw} = \frac{xz-yw}{z^2+w^2} + i \frac{xw+yz}{z^2+w^2} \in \mathbb{Q}[i]$

$\exists m, n \in \mathbb{Z}$  s.t.

$$\left| m - \frac{xz-yw}{z^2+w^2} \right| \leq \frac{1}{2}$$

$$\left| n - \frac{xw+yz}{z^2+w^2} \right| \leq \frac{1}{2}$$

( $m, n$  are the closest integer)

suppose  $q = m + in$ ,  $r = a - qb$

To prove  $r=0$  or  $N(r) < N(b)$

$$r = a - qb = b \left( \frac{a}{b} - q \right) = \left( m - \frac{xz - yw}{z^2 + w^2} \right) + i \left( n - \frac{xw + yz}{z^2 + w^2} \right)$$

( $r \neq 0$ )

thus  $N(r) \leq \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

$b \neq 0 \quad N(b) \geq 1 \geq N(r)$

Prop.  $(\mathbb{R}, +, \cdot)$  domain  $N: \mathbb{R} \rightarrow \mathbb{N}_0$  norm  $x \in \mathbb{R}$   
 $N(x)$  is a prime  $\Rightarrow x$  irreducible