1. for all $x_0 + y_0 \in W$, $x_0 \in U$, $y_0 \in V$

 since U is open. $\exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subseteq U$

 since $\forall p \in B(x_0 + y_0, \varepsilon)$ can be expressed as $p = x_1 + y_0$ where $x_1 \in B(x_0, \varepsilon) \subseteq U, y_0 \in V$

 then $B(x_0 + y_0, \varepsilon) \subseteq W$

 thus W is open.

2. for all $(x, y) \in \mathbb{R}$, set $F(x, y) = f(x) g(y)$

 since $f, g: \mathbb{R} \to \mathbb{R}$ is continuous,
 $\forall x \in \mathbb{R}$ $(\forall \varepsilon > 0. \exists \delta_x > 0$ s.t. $\forall |x' - x| < \delta_x. \ |f(x') - f(x)| < \varepsilon)$
 $\forall y \in \mathbb{R}$ $(\forall \varepsilon > 0. \exists \delta_y > 0$, s.t. $\forall |y' - y| < \delta_y. \ |g(y') - g(y)| < \varepsilon)$

 then
 $\forall (x, y) \in \mathbb{R}^2$ $(\forall \varepsilon > 0$ $\exists \delta = \min(\delta_{x_0}, \delta_y) > 0$, s.t. $\forall \|(x', y') - (x, y)\| < \delta$,
 $|F(x', y') - F(x, y)| = |f(x') g(y') - f(x) g(y)|$
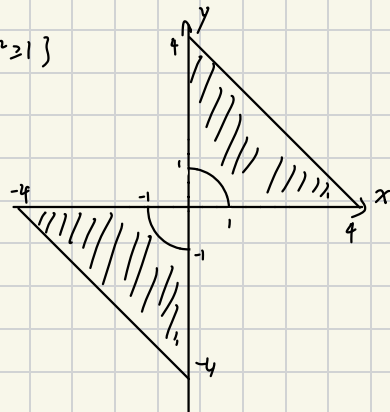 $$= |f(x') g(y') - f(x') g(y) + f(x') g(y) - f(x) g(y)|$$
 $$\leq |f(x')| \, |g(y') - g(y)| + |g(y)| \, |f(x) - f(x)|$$
 $$< \varepsilon (|f(x')| + |g(y)|) \quad \text{where } |f(x')|, |g(x)| \text{ are curb by a maximum since it's continuous.}$$

 thus $F(x, y)$ is continuous. , $(-\infty, 1)$ is open

 thus $R = F^{-1}[(-\infty, 1)]$ is open.

3. $Q = \{(x,y) \mid xy \geq 0\} \cap \{(x,y) \mid |x|+|y| \leq 4\} \cap \{(x,y) \mid x^2+y^2 \geq 1\}$

$\qquad \underset{\underset{A}{\shortparallel}}{} \qquad\qquad\qquad \underset{\underset{B}{\shortparallel}}{} \qquad\qquad\qquad \underset{\underset{C}{\shortparallel}}{}$

$A^c = \{(x,y) \mid xy < 0\}$

for $(x_0, y_0) \in A^c$

$\quad (x_0, y_0) \subseteq B[\,(x_0,y_0), r_o\,] \subseteq A^c$

$\qquad\qquad$ where $r_o = \min(x_0, y_0)$

$B^c = \{(x,y) \mid |x|+|y| > 4\}$
for $(x_1, y_1) \in B^c$

$\quad (x_1, y_1) \subseteq B[\,(x_1, y_1), r_1\,] \subseteq B^c$

$\qquad\qquad$ where $r_1 = \min\left(\frac{|x_1+y_1+4|}{\sqrt{2}}, \frac{|x+y-4|}{\sqrt{2}}, \frac{|x_1-y_1+4|}{\sqrt{2}}, \frac{|x_1-y_1-4|}{\sqrt{2}}\right)$

$C^c = \{(x,y) \mid x^2+y^2 < 1\}$ is a open ball

thus A, B, C is close, so is $Q = A \cap B \cap C$

$\partial Q \cap \{(x_1, 0) \mid x_1 \in \mathbb{R}\} = [0,4] \times \{0\}$

4. S is not open. consider $(x,y) \in [0,1) \times \{0\}$, no neighbour is included.

$\quad S$ is closed. S is the union of countable many 2-cells.

$\quad S$ is not bounded. since as $x \to \infty$, y can $\to \infty$, there's no upper bound.

5. (a) $D^o = \mathbb{R}^d / \partial D$, suppose $t \in Q$

$\quad$ then all $z \in \{t + x \mid x \in \mathbb{R}^d / \partial D\} = S$ can be expressed.

consider $z_0 \in \{ t + x \mid x \in \partial D \} = \partial S$

since O is open. $\exists r > 0$, $B(t, r) \subseteq O$

$\{ t + x \mid x \in \partial D \}$ is the boundary of $\{ t + x \mid x \in \mathbb{R}^d / \partial D \}$

thus $\exists (a_n)_n \in S$, $\forall \varepsilon > 0$. $\exists N$, $\forall n \geq N$, $\| a_n - z_0 \| < \varepsilon$

choose the smallest $n' \geq N$ for $\varepsilon = r$, set $a_{n'} = b \in S$

$\| b - z_0 \| < r$ $\qquad b = t + x_0$ $\qquad z_0 \in \{ x_0 + t' \mid t' \in B(t, r) \}$

thus z_0 can be expressed as $z = x + y$

(b) use the outcome of (a) where set $A = O = D$

6. The statement equivalent to.

If $A \cap U = \phi$, then U is ϕ or not open.

U is ϕ is obvious.

U is not open:

If U is open & $A \cap U = \phi$, contradict to $f(a)$

1. (a) $H_0: \theta = 1$ $H_1: \theta = \theta, < 1$

$$L(\theta_o; x) = \left(\prod_{i=1}^{n} 4x_i^3\right) \cdot \exp\left(-\sum_{i=1}^{n} x_i^4\right)$$

$$L(\theta_1; x) = \left(\prod_{i=1}^{n} \frac{4x_i^3}{\theta_1}\right) \cdot \exp\left(-\frac{1}{\theta_1}\sum_{i=1}^{n} x_i^4\right)$$

$$C = \left\{ x \;\middle|\; \frac{\exp\left[-\left(\frac{1}{\theta_1}-1\right)\sum_{i=1}^{n} x_i^4\right]}{\theta_1^n} > k \right\}$$

where $\;P(X \in C \mid H_0) = \alpha$

$$P\left(\exp\left[-\left(\frac{1}{\theta_1}-1\right)\frac{\chi_{2n}^2}{2}\right] > k\theta_1^n\right) = P\left(\chi_{2n}^2 < \frac{2\theta_1}{\theta_1-1}\ln(k\theta_1^n)\right) = \alpha$$

$$C = \left\{ x \;\middle|\; \sum_{i=1}^{n} x_i^4 \leq \tfrac{1}{2}\chi_{2n,\alpha}^2 \right\}$$

(b) p-value $= P(\chi_{10}^2 \leq 3.589) = 0.042$

(c) $P(\chi_{10}^2 \leq 2k_1) = \tfrac{1}{2}\alpha$ $P(\chi_{10}^2 \geq 2k_2) = \tfrac{1}{2}\alpha$

$k_1 = 1.623$ $k_2 = 10.24$

(d) $1.623 < \sum_{i=1}^{6} x_i^4 = 1.794 < 10.24$

   thus   retain.   (b) is   rejected   since its   one-tailed.

2. (a) $\ell(\lambda; x) = -n\lambda + \left(\sum x_i\right)\ln\lambda - \sum \ln(x_i)!$

   $\frac{\partial}{\partial\lambda}\ell(\lambda; x) = -n + \frac{1}{\lambda}\sum x_i$

   $\hat{\lambda} = \arg\max_{\lambda} \ell = \frac{1}{n}\sum x_i = 5.05$

$X_i \sim Possion (\lambda)$

for large n

$\hat{\lambda} = \frac{1}{n} \sum X_i \sim N(\lambda, \frac{\lambda}{n})$

$\left| \frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\hat{\lambda}}{n}}} \right| < 1.96$

95% CI : $(4.15, 6.13)$

(b) $\sum X_i = 101$

$\sigma = \sqrt{80}$ $\mu = 80$

$p = 2 \left[1 - \Phi \left(\frac{101 - 80}{\sqrt{80}} \right) \right] = 0.0188$ refuse.

(c) $S_{-w} = 101$ $S_{-s} = 63$

$E_{-w} = E_{-s} = 81.5$

$\sum \frac{(O_k - E_k)^2}{E_k} = 9.33 \sim \chi_1^2$

use $\alpha = 0.01$ $9.33 > 3.84$. unreasonable.

(d) under $H_0 : \lambda = \mu = \theta$

$\ell [(\theta, \theta)] = -2n\lambda_0 + (S_w + S_s) \ln \theta - \ln \prod X_i \prod y_i$

$\frac{\partial}{\partial \theta} \ell = -2n + (S_w + S_s)/\theta = 0$

$\hat{\theta} = 4.075$

under $H_1 : \lambda \neq \mu$

$$L[(\lambda, \mu)] = -n(\lambda + \mu) + S_w \ln\lambda + S_s \ln\mu - \ln \prod x_i \prod y_j$$

$$\begin{cases} \dfrac{\partial}{\partial \lambda} L = -n + \dfrac{S_w}{\lambda} = 0 \\[2mm] \dfrac{\partial}{\partial \mu} L = -n + \dfrac{S_s}{\mu} = 0 \end{cases} \implies \begin{cases} \hat{\lambda} = 5.05 \\[2mm] \hat{\mu} = 31 \end{cases}$$

$$2 \left[L[(\hat{\lambda}, \hat{\mu})] - L[(\hat{\theta} \cdot \hat{\theta})] \right] = 2 \left[S_w \ln \right.$$