三次方程求根

1. Solution of [Cubic function with one variable]

$$Ax^3 + Bx^2 + Cx + D = 0$$

$$x^3 + bx^2 + cx + d = 0 \quad \left(b = \frac{B}{A},\ c = \frac{C}{A},\ d = \frac{D}{A} \right)$$

$$z^3 + pz + q = 0 \quad \left(p = c - \frac{b^2}{3} \quad q = \frac{2b^3}{27} - \frac{bc}{3} + d \right)$$

suppose u, v satisfy $3uv = -p$, $u^3 + v^3 = -q$

then $u + v$ is a solution of origin function

suppose $U = u^3 \quad V = v^3$,

then U, V are the solution of following function:

$$X^2 + qX - \frac{p^3}{27} = 0$$

solution of $x^3 + px + q = 0$

$$x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{2}\right)^3}}$$

$$x_2 = w\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + w\sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{2}\right)^3}}$$

$$x_3 = w^2\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + w^2\sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{2}\right)^3}}$$

where $w^3 = 1 \quad w = \frac{-1 + \sqrt{3}i}{2}$

1. $x_1 + x_2 + x_3 = 0 \qquad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = -\frac{p}{q} \qquad x_1 x_2 x_3 = -q$

2. [Discriminant] $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$

$\Delta > 0$, 1 real 2 complex

$\Delta = 0$, 3 real, of two are same

$\Delta < 0$, 3 unique real.

3. When $\Delta < 0$, the roots can be expressed as:

$$x_1 = 2\sqrt[3]{r}\cos\theta \qquad x_2 = 2\sqrt[3]{r}\cos\left(\theta + \frac{2}{3}\pi\right) \qquad x_3 = 2\sqrt[3]{r}\cos\left(\theta - \frac{2}{3}\pi\right)$$

where $r = \sqrt{-\left(\frac{p}{q}\right)^3} \qquad \theta = \frac{1}{3}\arccos\left(-\frac{q}{2r}\right)$