Problem Solving Strategies III

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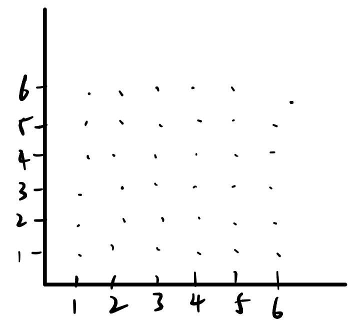

1 Problems to be discussed in lecture

1.1 Generating Functions

1.1.1 Problem 1

$\frac{1}{11}(1+x+\cdots x^{10}) = (p_1 + p_2 x + p_3 x^2 + \cdots p_6 x^5)(q_1 + q_2 x + \cdots q_6 x^5)$

(Warm up) Is it possible that the sum of the outcomes of two dice (not necessarily fair) with the usual numbers $\{1, 2, \ldots, 6\}$ on their sides has a uniform distribution on $\{2, \ldots, 12\}$?

1.1.2 Problem 2

(USAMO 1996) Is there a subset X of integers such that every integer n can be rewritten uniquely as $a + 2b$ for some $a, b \in X$.

$\frac{1}{1-x} = f(x)\, f(x^2)$

1.1.3 Problem 3

(Putnum) Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which

1 2 3 4 5

1 0 1 0 1

1 1 1 1 1 1

$(1+x)^N p(x) \overset{mod}{=} p(x) + x^N p(x)$

$$S_N = S_0 \cup \{N + a : a \in S_0\}.$$

1.1.4 Problem 4

(From American Mathematical Monthly)Prove that for every natural number m, we have

$$\sum_{i=0}^{m} \binom{m}{i} 2^{m-i} \binom{i}{\lfloor i/2 \rfloor} = \binom{2m+1}{m}.$$

$= c\left((1+x)\sum_{i=0}^{n}\binom{m}{i} 2^{m-i}(x+\tfrac{1}{x})^i\right)$

$= c\left((1+x)(2+x+\tfrac{1}{x})^n\right)$

$= [(1+x)(x+\tfrac{1}{x})^n]$'s constant

$= c\left((1+x)(\sqrt{x}+\tfrac{1}{\sqrt{x}})^{2m}\right)$

$= \binom{2m}{m} + \binom{2m}{m+1}$

$= \binom{2m+1}{m+1}$

1.1.5 Problem 5

For a positive integer n, denote by $s(n)$ the number of choices $+$ or $-$ such that $\pm 1 \pm 2 \cdots \pm n = 0$. Show that

$x = \frac{\pi t}{n}$

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos x \cos 2x \cos 3x \cdots \cos nx \, dx$$

1.1.6 Problem 6

Find the number of subsets of $\{1, 2, \ldots, 2024\}$ with the sum of the elements is divisible by 11.

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$[(z+z^{-1})(z+z^{-2})\cdots(z^n+z^{-n})] = S(n) + \sum c_k z^k$

$= 2^n \cos x \cos 2x \cdots \cos nx$

$\Rightarrow \int_0^{n} 2^n \cos x \cos 2x \cdots \cos nx\, dx = 2\pi S(n)$

compute $$\sum_{i=0}^{m} (-1)^i \binom{n}{i}$$

$$= \sum_{i=0}^{m} (-1)^i \frac{n!}{i!\,(n-i)!}$$

$$= \sum_{i=0}^{m} (-1)^i \frac{1}{B(i, n-i)}$$

Q.6. let $f(z) = (1+z)(1+z^2) \cdots (1+z^{2024})$

then the coefficient of x^k is the number of subsets with sum of the elements equals to k

denote this number by $s(k)$

thus $f(z) = \sum_{i=0}^{1+2+3+\cdots 2024} s(i) \cdot x^i$

denote the 11 root of $x^{11} = 1$ by $\lambda_1, \lambda_2 \cdots \lambda_{11}$

where $\lambda_{11} = 1$

$$\sum_{j=1}^{11} f(\lambda_j) = \sum_{i=0}^{1+2+3+\cdots+2024} s(i)\left(\lambda_1^i + \lambda_2^i + \lambda_3^i + \cdots + \lambda_{11}^i\right)$$

where $\{\lambda_1^i, \lambda_2^i, \cdots, \lambda_{11}^i\} = \{\lambda_1, \lambda_2, \lambda_3 \cdots, \lambda_{11}\}$ for $i \neq 11t$ $t \in N^+$

$\lambda_1 + \lambda_2 + \cdots + \lambda_{11} = 0$

therefore:

$$\sum_{j=1}^{11} f(\lambda_j) = 11 \sum_{i=1}^{184} s(11i)$$

where $f(\lambda_{11}) = (1+1)(1+1^2) \cdots (1+1^{2024}) = 2^{2024}$

$f(\lambda_1) = f(\lambda_2) = \cdots f(\lambda_{10}) = \left[(1+\lambda_1)(1+\lambda_2) \cdots (1+\lambda_{11})\right]^{184}$

notice that

$$x^{11} - 1 = (x-\lambda_1)(x-\lambda_2) \cdots (x-\lambda_{11})$$

when $x = -1$

$-2 = (-1)^{11}(\lambda_1+1)(\lambda_2+1) \cdots (\lambda_{11}+1)$

$2 = (\lambda_1+1)(\lambda_2+1) \cdots (\lambda_{11}+1)$

to sum up

$$\sum_{i=1}^{184} S(11i) = \frac{10\cdot 2^{184} + 2^{2024}}{11}$$

Q 2. consider positive integer

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad (|x|<1)$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots \qquad (|x^2|<1)$$

suppose

$$f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} = \frac{1}{1-x-x^2+x^3} = \sum_{n=0}^{\infty} C_n x^n$$

$$x f(x) = \sum_{n=1}^{\infty} C_{n-1} x^n = C_0 x + C_1 x^2 + \cdots$$

$$x^2 f(x) = \sum_{n=2}^{\infty} C_{n-2} x^n = C_0 x^2 + C_1 x^3 + \cdots$$

$$x^3 f(x) = \sum_{n=3}^{\infty} C_{n-3} x^n = C_0 x^3 + C_1 x^4 + \cdots$$

$$f(x) - x f(x) - x^2 f(x) + x^3 f(x) = (1 - x - x^2 + x^3) f(x) = 1$$

$$= C_0 + (C_1 - C_0)x + (C_2 - C_1 - C_0)x^2 + (C_3 - C_2 - C_1 + C_0)x^3$$
$$+ \sum_{n=1}^{\infty} (C_{n+3} - C_{n+2} - C_{n+1} + C_n) x^{n+3}$$

thus $C_0 = 1$ $\quad C_1 = 1$ $\quad C_2 = 2$ $\quad C_3 = 2$

$$C_{n+3} - C_{n+2} = C_{n+1} - C_n \qquad (n \geq 0)$$

therefore

$$C_{2k+3} - C_{2k+2} = C_{2k+1} - C_{2k} = C_1 - C_0 = 0$$

$$C_{2k+4} - C_{2k+3} = C_{2k+2} - C_{2k+1} = C_2 - C_1 = 1$$

$C_n = \lfloor \frac{n}{2} \rfloor + 1$

$f(x) = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots$

to sum up, for positive integer only

only subspace $\{1\}$ satisfy the condition. (whose coefficione is 1)

设 $P_k(x) = \sum C_i x^i$ $\begin{cases} C_i = 0 & \text{if } i \text{不在 } S_k \text{ 中} \\ C_i = 1 & \text{if } i \text{ 在 } S_k \text{ 中} \end{cases}$

E.g.:

$P_k(x) = x^2 + x^3 + x^6$ $S_k = \{2, 3, 6\}$

$P_{k+1}(x) = x^2 + x^4 + x^6 + x^7$ $S_k = \{2, 4, 6, 7\}$

可以看出

$P_{k+1}(x) \equiv (1+x) P_k(x)$ (系数 mod 2)

所以

$P_k(x) \equiv (1+x)^k P_0(x)$

题目要求

$P_n(x) = (1 + x^n) P_0(x)$ (系数 mod 2)

当 $k = 2^t$ $t \in \{1, 2, 3 \cdots\}$ 时，显然成立