余面积公式

C_1

余面积公式：

设 $f: \mathbb{R}^n \to \mathbb{R}$ 连续可微 n 元函数，$\|\nabla f\|$ 和

g 是 $f^{-1}(t)$ $t \in [a,b]$ 上的连续函数

$\Omega = \Phi(D \times [a,b])$，$D \subset \mathbb{R}^{n-1}$，则：

$$\int_a^b dt \int \cdots \int_{f^{-1}(t)} g \frac{ds}{\|\nabla f\|} = \int \cdots \int_{f^{-1}([a,b])} g \, dx_1 \cdots dx_n$$

证明涉及隐函数定理：

Proof.

$f(x_1, \cdots x_n) - t = 0$ 存在局部解 $x_n = \varphi_t(x_1, \cdots, x_{n-1})$

则 $\frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t} - 1 = 0 \iff \frac{\partial x_n}{\partial t} = \left(\frac{\partial f}{\partial x_n}\right)^{-1}$

变量替换：

$$\Phi(x_1, \cdots x_{n-1}, t) = (x_1, \cdots, x_{n-1}, \varphi_t(x_1, \cdots x_{n-1}))$$

$$\det J\Phi = \frac{\partial \varphi_t}{\partial t} = \left(\frac{\partial f}{\partial x_n}\right)^{-1}$$

记 $\Omega = \Phi(D \times [a,b])$，则

$$\Theta(\Omega) = \int \cdots \int_{D \times [a,b]} \left|\frac{\partial f}{\partial x_n}\right|^{-1} dx_1 \cdots dx_{n-1} \, dt$$

$$= \int_a^b dt \int \cdots \int_D \left|\frac{\partial f}{\partial x_n}\right|^{-1} dx_1 \cdots dx_{n-1}$$

再考虑 $f^{-1}(t) \subset \mathbb{R}^n$ 上的曲线积分.

$$\frac{\partial \varphi_t}{\partial x_i} = -\frac{\partial f}{\partial x_i} \left(\frac{\partial f}{\partial x_n}\right)^{-1} \quad i = 1, \cdots n-1$$

从而

$$1 + \|\nabla \varphi_t\|^2 = 1 + \sum_{i=1}^{n-1} \left(\frac{\partial \varphi_t}{\partial x_i}\right)^2 = \sum_{i=1}^{n} \left(\frac{\partial \varphi_t}{\partial x_i}\right)^2 = \left(\frac{\partial f}{\partial x_n}\right)^{-2} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2$$

$$= \|\nabla f\|^2 \left(\frac{\partial f}{\partial x_n}\right)^{-2}$$

由第一型曲面积分的面积公式 $S = \int \cdots \int_D \sqrt{1 + \|\nabla \varphi_t\|^2} \, dx_1 \, dx_2 \cdots dx_{n-1}$

$$\int \cdots \int_{\Omega \cap f^{-1}(t)} g \frac{ds}{\|\nabla f\|} = \int \cdots \int_D g \frac{\sqrt{1 + \|\nabla \varphi_t\|}}{\|\nabla f\|} dx_1 \cdots dx_{n-1} = \int \cdots \int_D g \left|\frac{\partial f}{\partial x_n}\right|^{-1} dx_1 \cdots dx_{n-1}$$

$$\int_a^b dt \int\cdots\int_{f^{-1}(t)} g\,\frac{dS}{\|\nabla f\|} = \int_a^b dt \int\cdots\int_{\Omega\cap f^{-1}(t)} g\,\frac{dS}{\|\nabla f\|}$$

$$= \int_a^b \int\cdots\int_D g \left|\frac{\partial f}{\partial x_n}\right|^{-1} dx_1\cdots dx_{n-1}\,dt$$

$$= \int\cdots\int_{f^{-1}([a,b])} g \; dx_1 \cdots dx_{n-1}\,dx_n$$

Q.E.D.

C_2 : "余面积公式" on Zhihu.

for $\oint_{(L)} P\,dx + Q\,dy = \int_z \left(\oint_{(L_z)} \mathrm{rot}\,\vec{F} \cdot \frac{\nabla f \times d\vec{r}}{\|\nabla f\|^2} \right) dz$

where $\vec{F} = (P(x,y), Q(x,y))$ $\vec{r} = (x,y)$

L_z为 L 化为 $F(x,y) = z$ 的隐函数后得到的曲线

C_3:

用这种方法类似地处理概率分布函数.

$$P(Z \le z) = \iint_{f(x,y)\le z} g(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{z} \left[\int_{L_u} g(x,y)\,\frac{dy}{f_x'} \right] du$$

$$= -\int_{-\infty}^{z} \left[\int_{L_u} g(x,y)\,\frac{dx}{f_y'} \right] du$$