1. $\begin{cases} A^2B = ABC + 2I \\ BC^2 = ABC + 2I \end{cases}$

$$A(AB - BC) = 2I$$

$$(AB - BC)(-C) = 2I$$

$$A = -C$$

$$B^3 = -ABA + 2I$$

$$B^4 = -(AB)^2 + 2B = -(BA)^2 + 2B$$

$$(AB + BA)(AB - BA) = 0$$

$$AB = BA$$

$$B^3 = I$$

$$A^6 = (B^2)^3 = (B^3)^2 = I$$

2. $A + A^k = A^T$

$$A^T + (A^T)^k = A$$

$$\cancel{A} + A^k + (A + A^k)^k = \cancel{A}$$

$$A^k(I + (I + A^{k-1})^k) = 0$$

$p(x) = x^k(1 + (1+x^{k-1})^k)$ is annihilating polynomial for A

where $(1+x^{k-1})^k \geq 0$

all the eigenvalue at A is 0 ; A is nilpotent matrix

$A^n = 0 \Rightarrow A^k = 0$

thus

$A = A^T$

all real symetric matrix can be diagonaled

thus $A = 0$

3.

$$\begin{cases} XA^2 + AXA^2 = 0 \iff AXA^2 + A^2XA^2 = 0 \\ A^2X + A^2XA^2 = 0 \end{cases}$$

$$AXA^2 = A^2X \Rightarrow A^2X = XA^2 \Rightarrow A^2X = XA^2 = 0$$

thus

$$AX = A^2$$

$$X = A - A^2$$

4. the minimal polynomial is $x^2 - x = 0$

so A_i have only eigenvalue 0 and 1

therefore rank A = trace A

$$(\sum A)^2 = \sum A^2 + \sum A_iA_j + A_jA_i = \sum A_i$$

$$(A_1 \;\cdots\; A_n)\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} A_1A_1 & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

5. $A^m = -B^2 + 2021B$

when $m=1$, we can see that A is symmetric

write A as

$$A = T \Lambda T^{-1}$$

$$\Lambda^m = -K^2 + 2021K$$

$$\lambda^m = -k^2 + 2021k \leq \left(\frac{2021}{2}\right)^2$$

$m \to +\infty$

$|\lambda| \leq 1$

$$|\det(A)| = \left|\prod \lambda\right| \leq 1$$

6. $T = \text{trace}((AB)^2) - \text{trace}(A^2B^2) = \frac{1}{2}\text{trace}[(AB-BA)^2]$

$\text{rank}(AB-BA+I) = 1 \Rightarrow \dim\ker(AB-BA+I) = n-1$

thus $AB-BA$ have the eigenvalue -1 with the multiplicity $n-1$

where $\text{trace}(AB-BA) = 0 = \sum_{i=1}^{n} \lambda_i = -n+1+\lambda$

thus $AB-BA$ have the eigenvalue $n-1$

$$(AB-BA)^2 = P^{-1} J^2 P = P^{-1} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & (n-1)^2 \end{pmatrix} P$$

$$T = \frac{1}{2}\text{trace}[(AB-BA)^2] = \frac{1}{2}\cdot[(n-1)+(n-1)^2] = \frac{1}{2}n(n-1)$$

7.

$$A \cdot A^t = A^t A$$

write A as

$$A = V \Sigma V^T$$

$$A \cdot A^t = U \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_r^2 \end{bmatrix} U^T$$

$$A^t \cdot A = V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_r^2 \end{bmatrix} V^T$$

$$U = V$$

$$A = U \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} U^T$$

A is symmetric

$$\dim A = \frac{n(n+1)}{2}$$

8.

$$I + AB^{-1} + BA^{-1} + I = I + A^{-1}B + B^{-1}A + I = I$$

$$B + A + BA^{-1}B = A + B + AB^{-1}A = 0$$

$$BA^{-1}B = AB^{-1}A$$

$$\frac{\det(B)^2}{\det(A)} = \frac{\det(A)^2}{\det(B)}$$

$$\det(B)^3 = \det(A)^3$$

$$\det B = \det A$$

if complex

$$\left(\frac{\det B}{\det A}\right)^3 = 1 \quad , \quad \frac{\det B}{\det A} = 1 \ , \ e^{i \cdot \frac{2}{3}\pi} \text{ or } e^{i \frac{4}{3}\pi}$$

9.

$$V = (V_1 \quad V_2 \quad \cdots \quad V_k)$$

$$V^T \cdot V = A_{k\times k} = \begin{pmatrix} V_1^2 & V_1\cdot V_2 & & \\ V_2\cdot V_1 & V_2^2 & & \\ & & \ddots & \\ & & & V_k^2 \end{pmatrix}$$

$\text{rank}(A) =$

10

$$V = (A_1 \quad A_2 \quad \cdots \quad A_k).$$

$$U = V^T \cdot V = \begin{pmatrix} A_1^2 & & & \\ & A_2^2 & & \\ & & \ddots & \\ & & & A_n^2 \end{pmatrix}$$

$$k \le \operatorname{rank} U \le \operatorname{rank} V \le n$$

when $n = k$, an example is than

$$A_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \quad \cdots \quad A_n = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$T \begin{bmatrix} \lambda_1^2+\lambda_1 & & \\ & \ddots & \\ & & \lambda_n^2+\lambda_n \end{bmatrix} T^{-1} =$$

11.

$$ABA^{-1} = B^2 + B = C$$

thus C and B have same eigenvalue

for x_i is an eigenvector of B under λ_i

$$Cx = B^2 x_i + B x_i = (\lambda_i^2 + \lambda_i) x_i$$

thus

$\{\lambda_1^2+\lambda_1, \lambda_2^2+\lambda_2, \cdots, \lambda_n^2+\lambda_n\}$ is a permutation of $\{\lambda_1, \cdots, \lambda_n\}$

under f: $f(x) = x^2 + x$, $\{\lambda_1, \cdots \lambda_n\}$

12 $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$p(A_1) = (x+1)(x-1) = x^2 - 1$

$p(x) = (x-1)(x+1)$

A_1

$-$

$-$

$A_2 \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} A_1 & I_2 \\ I_2 & A_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$

d_2

$A_2 \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 & I_2 \\ I_2 & A_1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = 0 \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$

$p(x) = (x-2)(x-1)^2 x$