IMC Linear Algebra Questions

Problem 1. *(Day1-2023-P2) Let* A *,* B *and* C *be* $n \times n$ *matrices with complex entries satisfying* $A^2 = B^2 = C^2$ *and* $B^3 = ABC + 2I$*. Prove that* $A^6 = I$*.*

Problem 2. *(Day1-2022-P2) Let* n *be a positive integer. Find all* $n \times n$ *real matrices* A *with only real eigenvalues satisfying* $A + A^k = A^T$ *for some integer* $k \geq n$*. (*A^T *denotes the transpose of* A*.)*

Problem 3. *(Day1-2021-P1) Let* A *be a real* $n \times n$ *matrix such that* $A^3 = 0$*.*

(a) Prove that there is a unique real $n \times n$ *matrix* X *that satisfies the equation*

$$X + AX + XA^2 = A.$$

(b) Express X *in terms of* A*.*

Problem 4. *(Day2-2022-P7)*

Let $A_1, A_2, \ldots, A_k$ *be* $n \times n$ *idempotent complex matrices such that*

$$A_i A_j = -A_j A_i$$

for all $i \neq j$*. Prove that at least one of the given matrices has rank* $\leq \frac{n}{k}$ *. (A matrix* A *is called idempotent if* $A^2 = A$*.*

Problem 5. *(Day2-2021-P5) Let* A *be a real* $n \times n$ *matrix and suppose that for every positive integer* m *there exists a real symmetric matrix* B *such that*

$$2021B = A^m + B^2.$$

Prove that $|\det(A)| \leq 1$*.*

Problem 6. *(Day1-2020-P2) Let* A *and* B *be* $n \times n$ *real matrices such that* $rk(AB - BA + I) = 1$ *where* I *is the* $n \times n$ *identity matrix. Prove that*

$$trace(ABAB) - trace(A^2B^2) = \frac{1}{2}n(n-1)$$

($rk(M)$ *denotes the rank of matrix* M*, i.e., the maximum number of linearly independent columns in* M*.* $trace(M)$ *denotes the trace of* M*, that is the sum of diagonal elements in* M*.)*

References

[1] https://imc-math.org.uk

Day2-2015-P9

Problem 9. An $n \times n$ complex matrix A is called *t-normal* if $AA^t = A^tA$ where A^t is the transpose of A. For each n, determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

(10 points)

Day1-2015-P1

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(10 points)

Day2-2018-P6

Problem 6. Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal.

(10 points)

Day1-2017-P2

Problem 2. Let k and n be positive integers. A sequence $(A_1, \ldots, A_k)$ of $n \times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_iA_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

(10 points)

Day2-2019-P9

Problem 9. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2A$.

(10 points)

Day2-2017-P8

Problem 8. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where I_m is the $m \times m$ identity matrix.

Prove that A_n has $n + 1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

(10 points)

Day1-2019-P5

Problem 5. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

1. $\det(B) = 1$;
2. $AB = BA$;
3. $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here I denotes the $n \times n$ identity matrix.)

(10 points)

Day1-2018-P3

Problem 3. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

(10 points)

Day2-2016-P10

Problem 10. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \le \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup\limits_{\|x\|\le 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(10 points)

Day1-2017-P7

Problem 1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

(10 points)

(Day1-2011-P2) Does there exist a real 3×3 matrix A such that $tr(A) = 0$ and $A^2 + A^t = I$? ($tr(A)$ denotes the trace of A, A^t is the transpose of A, and I is the identity matrix.)

(Day1-2012-P2) Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(Day1-2013-P1) Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

(Day1-2014-P1) Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $trace(M) = a$ and $det(M) = b$.

(Day2-2014-P2) Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \le i < j \le n} a_{ii} a_{jj} \ge \sum_{1 \le i < j \le n} \lambda_i \lambda_j,$$

and determine all matrics for which equality holds.

Day 1-2009-P2

Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

Day2-2009-P3

Let $A, B \in M_n(\mathbb{C})$ be two $n \times n$ matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

Day1-2008-P6

Problem 6. For a permutation $\sigma = (i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$ define $D(\sigma) = \sum_{k=1}^{n} |i_k - k|$. Let $Q(n, d)$ be the number of permutations σ of $(1, 2, ..., n)$ with $d = D(\sigma)$. Prove that $Q(n, d)$ is even for $d \geq 2n$.

Day2-2008-P5

Problem 5. Let n be a positive integer, and consider the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $|\det A| = k^2$ for some integer k.

Day1-2007-P2

Problem 2. Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

Day1-2007-P3

Problem 3. Call a polynomial $P(x_1, \ldots, x_k)$ *good* if there exist 2×2 real matrices $A_1, \ldots, A_k$ such that

$$P(x_1, \ldots, x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good.
(A polynomial is homogeneous if each term has the same total degree.)

Day2-2007-P4

Problem 4. Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1\ldots n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Find $\det A$.

Day1-2006-P3

Problem 3. Let A be an $n \times n$-matrix with integer entries and $b_1, \ldots, b_k$ be integers satisfying $\det A = b_1 \cdot \ldots \cdot b_k$. Prove that there exist $n \times n$-matrices $B_1, \ldots, B_k$ with integer entries such that $A = B_1 \cdot \ldots \cdot B_k$ and $\det B_i = b_i$ for all $i = 1, \ldots, k$.

Day2-2006-P4

Problem 4. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \ldots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \le i, j \le n+1$. Prove that $v_1, \ldots, v_{n+1}$ are linearly dependent over the rationals.
(20 points)

Day2-2006-P6

Problem 6. Let A_i, B_i, S_i $(i = 1, 2, 3)$ be invertible real 2×2 matrices such that

(1) not all A_i have a common real eigenvector;

(2) $A_i = S_i^{-1} B_i S_i$ for all $i = 1, 2, 3$;

(3) $A_1 A_2 A_3 = B_1 B_2 B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Prove that there is an invertible real 2×2 matrix S such that $A_i = S^{-1} B_i S$ for all $i = 1, 2, 3$.

IMC Linear Algebra Questions 2000-2005

Problem 1. (Day1-2005-P1) Let A be the $n \times n$ matrix, whose $(i,\, j)^{\text{th}}$ entry is $i+j$ for all $i,\, j = 1,\, 2,\, \ldots,\, n$. What is the rank of A?

Problem 2. (Day2-2005-P3) In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X,\, Y \in V, \quad \operatorname{trace}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

Problem 3. (Day2-2004-P1) Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{1}$$

Find BA.

Problem 4. (Day2-2004-P4) For $n \geq 1$ let M be an $n \times n$ complex matrix with distinct eigenvalues $\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_k$, with multiplicities $m_1,\, m_2,\, \ldots,\, m_k$, respectively. Consider the linear operator L_M defined by $L_M(X) = MX + XM^T$, for any complex $n \times n$ matrix X. Find its eigenvalues and their multiplicities. (M^T denotes the transpose of M; that is, if $M = (m_{k,\,l})$, then $M^T = (m_{l,\,k})$.)

Problem 5. (Day2-2004-P6) For $n \geq 0$ define matrices A_n and B_n as follows: $A_0 = B_0 = (1)$ and for every $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}. \tag{2}$$

Denote the sum of all elements of a matrix M by $S(M)$. Prove that $S(A_n^{k-1}) = S(A_k^{n-1})$ for every $n,\, k \geq 1$.

Problem 6. (Day1-2003-P3) Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.)

Problem 7. (Day2-2003-P1) Let A and B be $n \times n$ real matrices such that $AB + A + B = 0$. Prove that $AB = BA$.

Problem 8. (Day1-2002-P6) For an $n \times n$ matrix M with real entries let $\|M\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|_2}{\|x\|_2}$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n \times n$ matrix A with real entries satisfies $\|A^k - A^{k-1}\| \le \frac{1}{2002k}$ for all positive integers k. Prove that $\|A^k\| \le 2002$ for all positive integers k.

Problem 9. (Day2-2002-P1) Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & \text{if} \quad i \neq j, \\ 2, & \text{if} \quad i = j. \end{cases} \tag{3}$$

Problem 10. (Day2-2002-P5) Let A be an $n \times n$ matrix with complex entries and suppose that $n > 1$. Prove that

$$A\overline{A} = I_n \iff \exists S \in GL_n(\mathbb{C}) \quad \text{such that} \quad A = S\overline{S}^{-1}.$$

(If $A = [a_{ij}]$ then $\overline{A} = [\overline{a_{ij}}]$, where $\overline{a_{ij}}$ is the complex conjugate of a_{ij}; $GL_n(\mathbb{C})$ denotes the set of all $n \times n$ invertible matrices with complex entries, and I_n is the identity matrix.)

Problem 11. (Day1-2001-P1) Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left-to-right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Problem 12. (Day1-2001-P5) Let A be an $n \times n$ complex matrix such that $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

Problem 13. (Day2-2001-P4) Let $A = (a_{k,l})$, $k, l = 1, \ldots, n$ be an $n \times n$ matrix such that for each $m \in \{1, \ldots, n\}$ and $1 \le j_1 < \ldots < j_m \le n$ the determinant of the matrix (a_{j_k, j_l}), $k, l = 1, \ldots, m$, is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix

$$\left(a_{\sigma(k), \sigma(l)}\right), \ k, l = 1, \ldots, n$$

has all of its nonzero elements above the diagonal.

Problem 14. (Day1-2000-P3) A and B are square complex matrices of the same size and

$$\operatorname{rank}(AB - BA) = 1.$$

Show that $(AB - BA)^2 = 0$.

Problem 15. (Day2-2000-P6) For an $m \times m$ real matrix A, e^A is defined as $\sum_{n=0}^{\infty} \frac{1}{n!} A^n$. (The sum is convergent for all matrices.) Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and B, $p(e^{AB})$ is nilpotent if and only if $p(e^{BA})$ is nilpotent. (A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)

References

[1] `https://imc-math.org.uk`