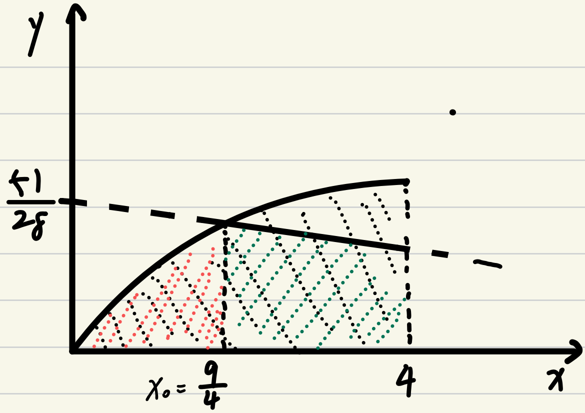
1. (a) $D = \{(x,y) \mid 0 \le y \le \sqrt{x} \le 2,\ \frac{x}{7} + y \le \frac{51}{28}\}$

$D_1 = D \cap [0, x_0) \times \mathbb{R}$

$D_2 = D \cap [x_0, 2) \times \mathbb{R}$

x_0 :

$\sqrt{x_0} = \frac{51}{28} - \frac{x_0}{7} \Rightarrow x_0 = \frac{9}{4}$



$\mathbb{P}\left(\frac{x}{7} + Y \le \frac{51}{28}\right) = \iint_D f_{X,Y}(x,y)\, dS$

$= \left(\iint_{D_1} + \iint_{D_2}\right) f_{X,Y}(x,y)\, ds$

$\iint_{D_1} f_{X,Y}(x,y)\, ds = \int_0^{x_0} \int_0^{\sqrt{x}} f(x,y)\, dy\, dx = \frac{459}{1792}$

$\iint_{D_2} f_{X,Y}(x,y)\, ds = \int_{x_0}^{4} \int_0^{\frac{51}{28} - \frac{x}{7}} f(x,y)\, dy\, dx = \frac{625}{1152}$

$\Rightarrow \mathbb{P}\left(\frac{x}{7} + Y \le \frac{51}{28}\right) = \frac{12881}{16128}$

(b) for $x \in [0, 4]$

$f_X(x) = \int_0^{\sqrt{x}} \frac{5}{84}(x+y)\, dy = \frac{5}{84} x\left(\sqrt{x} + \frac{1}{2}\right)$

otherwise

$f_X(x) = 0$

(c) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x^{\frac{3}{2}} + \frac{1}{2}x}$

(d) $\mathbb{E}[Y \mid X = x] = \int_0^{\sqrt{x}} y\, f_{Y|X}(y|x)\, dy = \frac{1}{3} \frac{3x + 2\sqrt{x}}{1 + 2\sqrt{x}}$

(e) suppose $t = \sqrt{x}$

$$\mathbb{E}[Y \mid X = x] = \frac{1}{3} \frac{3t^2 + 2t}{1 + 2t} = \frac{1}{3} \left(\frac{3}{2}t + \frac{1}{4} - \frac{1}{4} \frac{1}{1+2t} \right)$$

which is monotonically increasing on $t \in [0, 2]$

thus

$$\max \mathbb{E}[Y \mid X = x] = \frac{16}{15}$$

2. (a) $$\mathbb{E}(Y \mid X) = \int_x^{+\infty} y e^{x-y} \, dy = x + 1$$

$$Var(Y \mid X) = \int_x^{+\infty} (y - x - 1)^2 e^{x-y} \, dy$$

$$= \int_0^{+\infty} (t-1)^2 e^{-t} \, dt \qquad t = y - x$$

$$= \left(-t^2 e^{-t} - e^{-t} \right)_0^{+\infty}$$

$$= 1$$

(b) $$\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}[Y \mid X]] = 1 + \sqrt{\frac{2}{\pi}}$$

$$Var(Y) = \mathbb{E}[Var[Y \mid X]] + Var[\mathbb{E}(Y \mid X)]$$

$$= 2 - \frac{2}{\pi}$$

3. (a) $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy$

$= \int_0^3 dx \int_x^{3x} kx\,dy$

$= \int_0^3 2kx^2\,dx = \frac{2}{3}kx^3\Big|_0^3 = 18k = 1$

$\Rightarrow k = \frac{1}{18}$

(b) $S = \{(x,y) \mid \frac{1}{2} < x < 3,\ x < y < 3x\}$

$S' = \{(x,y) \mid \frac{1}{2} < x < 3,\ x < y < 3x^2,\ y > x^2\}$

$S_1 = S' \cap [\frac{1}{2}, 1) \times \mathbb{R}$ $\quad S_2 = S' \cap [1,3] \times \mathbb{R}$

$\mathbb{P}(Y > X^2 \mid X > \frac{1}{2}) = \dfrac{\mathbb{P}(Y > X^2, X > \frac{1}{2})}{\mathbb{P}(X > \frac{1}{2})}$

$\mathbb{P}(X > \frac{1}{2}) = \iint_S f_{X,Y}(x,y)\,dA$

$= \int_{\frac{1}{2}}^3 dx \int_x^{3x} \frac{1}{18}x\,dy = \frac{215}{216}$

$\mathbb{P}(X > \frac{1}{2}, Y > X^2) = \iint_{S'} f_{X,Y}(x,y)\,dA$

$= (\iint_{S_1} + \iint_{S_2}) f_{X,Y}(x,y)\,dA$

$\iint_{S_1} f_{X,Y}\,dA = \frac{29}{1152}$ $\qquad \iint_{S_2} f_{X,Y}\,dA = \frac{26}{27}$

thus $\mathbb{P}(Y > X^2, X > \frac{1}{2}) = \frac{1133}{1152}$

(c) $D_1 = \{(x,y) \mid x<y<3x,\ 0<y<3\}$

$D_2 = \{(x,y) \mid x<y<3x,\ 3<y<9\}$

on D_1

$$f_Y(y) = \int_{D_{1,x}} \frac{1}{18} x \, dx = \frac{2}{81} y^2$$

on D_2

$$f_Y(y) = \int_{D_{2,x}} \frac{1}{18} x \, dx = \frac{1}{4} - \frac{y^2}{324}$$

otherwise

$$f_Y(y) = 0$$

(d)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{9}{4} \cdot \frac{x}{y^2} & (x,y) \in D_1 \\ 18 \cdot \frac{x}{81-y^2} & (x,y) \in D_2 \\ 0 & \text{otherwise.} \end{cases}$$

1. (a) a_n converges to L

then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^+$, s.t. $\forall n > N$, $|a_n - L| < \frac{\varepsilon}{2}$

$$\varepsilon > |a_{n+p} - L| + |L - a_n| > |a_{n+p} - a_n|$$

thus $\lim_{n\to\infty} |a_{n+p} - a_n| = 0$ for all p

(b) $a_n = \ln n$

2. $$|x_{n+r} - x_n| < \sum_{i=0}^{n+r-1} |x_{n+i+1} - x_{n+i}| \le (1 + q + \cdots + q^{r-1}) |x_{n+1} - x_n|$$
$$\le \frac{1-q^r}{1-q} \cdot q^{n-1} |x_2 - x_1|$$

$\forall \varepsilon > 0$, choose N s.t. $\frac{1-q^r}{1-q} \cdot q^{N-1} |x_2 - x_1| < \varepsilon$

thus $\{x_n\}_1$ is cauchy

3. b_n is convergent on $\mathbb{R}$, thus its cauchy

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^+$ s.t. $\forall n, m > N$, $|b_n - b_m| < (\frac{\varepsilon}{5})^2$

then for a_n, $n \ge N$

$$|a_n - a_m| \le \sqrt[5]{|b_n - b_m|} < \varepsilon$$

thus a_n is cauchy on $\mathbb{R}$, thus is convergent.

4. for $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$

$$0 < \sum_{n=1}^{\infty} \frac{1}{n^3+n} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

absolute converge.

for $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$

$a_n = \frac{1}{n-\sqrt{n}}$ monotonically decreasing and positive

when $n \to \infty$ $a_n = 0$

thus a_n converge

for $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n-\sqrt{n}} \right|$

$$= \sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

thus not absolute converge.

5. $$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\left(\frac{2}{3}\right)^{\lfloor \frac{n+1}{2} \rfloor} \left(\frac{4}{3}\right)^{\lfloor \frac{n}{2} \rfloor} \right)^{\frac{1}{n}} = \lim_{n\to\infty} (c_n)^{\frac{1}{n}} \quad \frac{2\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}$$

$R = \frac{3\sqrt{2}}{4}$

6. (a) $\sum_{i=3k+1}^{3j} c_i = 0$ $\qquad a_n \xrightarrow{n\to\infty} 0$

$\sum_{n=1}^{\infty} c_n$ converges.

(b) $c_{3k+1} + c_{3k+2} + c_{3k+3} = 10(k+1)^{-1}$

$\sum_{n=1}^{\infty} c_n^3 \quad n \quad \sum_{k=1}^{\infty} 10(k+1)^{-1}$ diverges

thus $\sum_{n=1}^{\infty} c_n^3$ diverges.

(c) $C_{3k+1} + C_{3k+2} + C_{3k+3} = 34(k+1)^{-\frac{5}{3}}$

$\sum_{n=1}^{\infty} C_n^5 \sim \sum_{k=1}^{\infty} 34 (k+1)^{-\frac{5}{3}}$, $\frac{5}{3} > 1$, converges.

thus $\sum_{n=1}^{\infty} C_n^5$ converges.

7. $\lim_{n\to\infty} (p_n)^{\frac{1}{n}} = \lim_{n\to\infty} (n \ln n)^{\frac{1}{n}} = 1$

$R = 1$

1. (a) $\nabla r^n = n \cdot r^{n-2} \cdot \vec{r}$

$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla^2 \frac{1}{r} = \sum_{cyc} \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right)$$

$$= \sum_{cyc} -\frac{1}{r^3} + \frac{3x^2}{r^5} = -\frac{3}{r^3} + 3\frac{r^2}{r^5} = 0 \quad (\vec{r} \neq \vec{0})$$

(b) suppose $f(x,y,z) = 2x^2 z - 4xy - 6x - 10$

$$\nabla f = (4xz - 4y - 6,\ -4x,\ 2x^2)$$

the tagent plane:

$$\nabla F(1,-4,0) \cdot (\vec{r} - (1,-4,0)) = 0$$

$$\Rightarrow \quad 5x - 2y + z - 13 = 0$$

2 (a) $$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & xy^3 & -x^2yz \end{vmatrix} = (-x^2z - 0)\vec{i} + [2xyz - (-2xyz)]\vec{j} + (y^3 - xz^2)\vec{k}$$

$$\nabla \times (\nabla \times A) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2z & 4xyz & y^3 - xz^2 \end{vmatrix} = (-4xy + 3y^2)\vec{i} + (-x^2 + z^2)\vec{j} + 4yz\,\vec{k}$$

$$\nabla^2 A = 2xy\,\vec{i} + 6xy\,\vec{j} + (-2yz)\,\vec{k}$$

$\nabla \cdot A = yz^2 + 3xy^2 - x^2y$

$\nabla(\nabla \cdot A) = (3y^2 - 2xy)\vec{i} + (6xy - x^2 + z^2)\vec{j} + 2yz\,\vec{k}$

$$\begin{cases} 3y^2 - 2xy - 2xy = 3y^2 - 4xy \\ 6xy - x^2 + z^2 - 6xy = -x^2 + z^2 \\ 2yz + 2yz = 4yz \end{cases}$$

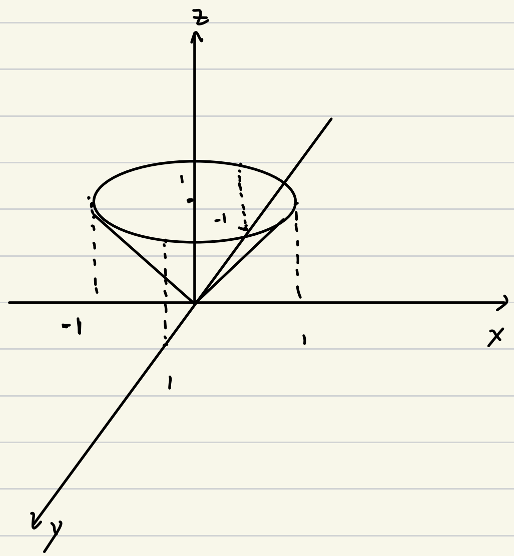
thus $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$

(b) there is maybe singular points

according to "Residue Theorem"

$$\oint_C f(z)\,dz = 2\pi i \sum_k \mathrm{Res}(f, z_k)$$

3 (a)



(b) $\iiint_V z\, dV = \int_0^1 z\, dz \iint_{S_z} dA$

$S_{z_0} = \{(x,y,z) \mid z = z_0\} \cap V$

$= \int_0^1 z\,(\pi z^2)\, dz$

$= \frac{\pi}{4}$

(b) $\oint_{\partial A} A \cdot dr = \int_0^{2\pi} (\sin^2 t,\ 2-\cos t - 3\sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$

[substitute $r = (\cos t, \sin t, 0)$]

$= \int_0^{2\pi} [-\cancel{\sin^3 t} + \cancel{2\cos t} - \cos^2 t - 3\, \cancel{\sin t \cos t}]\, dt$

$= -\pi$

$(\nabla \times A) \cdot \vec{n} = (-2y-1)z$

$\iint_S (\nabla \times A) \cdot \vec{n}\, dS = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (-2\sin\alpha \sin\beta - 1) \cos\alpha \cdot \sin\alpha\, d\alpha\, d\beta$

[substitute $x = \sin\alpha \cos\beta$ $y = \sin\alpha \sin\beta$ $z = \cos\beta$]

$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} -\frac{1}{2} \sin 2\alpha\, d\alpha\, d\beta$

$= -\pi$

4 (a) $Q = \int_0^R \rho(r') 4\pi r'^2\, dr' = 12\pi\alpha R^4 B(3,2) = \pi\alpha R^4$

(b)

$$E(r) = \frac{1}{4\pi\varepsilon_0} \iiint \rho(r') \frac{\vec{r} - \vec{r}'}{(r-r')^3}\, dV$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint \rho(r') \frac{1}{(r-r')^3} (-x', -y', r-z')\, dV$$

$$= E_x(r) + E_y(r) + E_z(r)$$

substitute:

$x' = r' \sin\alpha \cos\beta$ $y' = r' \sin\alpha \cos\beta$ $z' = r' \cos\alpha$

$dV = r'^2 \sin\alpha\, dr'\, d\alpha\, d\beta$

E_x, E_y :

$$E_x = \frac{1}{4\pi\varepsilon_0} \int_0^R \int_0^\pi \int_0^{2\pi} \rho(r') \frac{r - r'\cos\alpha}{(r^2 + r'^2 - 2rr'\cos\alpha)^{\frac{3}{2}}} \sin\alpha \, d\beta \, d\alpha \, dr$$

$$= \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} d\beta \int_0^R \rho(r') r^2 \int_0^\pi \frac{r - r'\cos\alpha}{(r^2 + r'^2 - 2rr'\cos\alpha)^{\frac{3}{2}}} d\alpha \, dr$$

$$= 0 \qquad r > R$$

same for E_y, and the situation $0 < r \le R$

$$E_z = \frac{1}{4\pi\varepsilon_0} 2\pi \int_0^R \rho(r') r'^2 \, dr' = \frac{\alpha R^4}{4\varepsilon r^2} \qquad r \ge R$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int_0^r \rho(r') 4\pi r'^2 \, dr' = \pi \alpha r^3 (4R - 3r) \quad 0 < r < R$$

$$E(\vec{r}) = \begin{cases} \frac{\alpha r(4R - 3r)}{4\varepsilon_0} \hat{z} & 0 < r \le R \\ \frac{\alpha R^4}{4\varepsilon_0 r^2} \hat{z} & R < r \end{cases}$$

1. If n is even:

$h'(x) = nx^{n-1} + a$, monotonically increasing on $\mathbb{R}$

then by Rolle's Theorem, at most exist two zeroes.

If n is odd.

$h'(x) = n x^{n-1} + a = \frac{1}{n} (x^{n-1} + \frac{a}{n})$

, by the case of n is even, $h'(x)$ has at most 2 zeroes.

suppose $h'(x_+) = h'(x_-) = 0$

then by Rolle's Theorem, at most exist x_1 x_2 s.t. $h'(x_-) = 0$
x_2 x_3 s.t. $h'(x_+) = 0$

thus at most 3 zeroes.

2. $f'' = \alpha(\alpha-1)x^{\alpha-2}$ where $\alpha x^{\alpha-2} \geq 0$ for $x \in [0,+\infty)$

thus if $\alpha-1 \geq 0$, f convex
$\alpha-1 < 0$, f concave.

3. suppose $f(x) = \frac{1}{1+x} - \frac{1}{1+bx}$

$$f'(x) = -\frac{1}{(1+x)^2} + \frac{b}{(1+bx)^2}$$

$$= \frac{b + \cancel{2bx} + bx^2 - 1 - \cancel{2bx} - b^2x^2}{(1+x)^2(1+bx)^2} = \frac{(b-1)(1-bx^2)}{(1+x)^2(1+bx)^2}$$

thus increase in $(0, \frac{\sqrt{b}}{b})$, decrease in $(\frac{\sqrt{b}}{b}, +\infty)$

thus $\frac{\sqrt{b}-1}{\sqrt{b}+1} = f(\frac{\sqrt{b}}{b}) \geq f(x)$

4. If f' is not a constant

then f' is not continuous.

suppose x_0 is a jump discontinuity

then $f'_-(x_0) \neq f'_+(x_0)$ f is not differentiable at $x = x_0$

contradiction.

5. when $t \in (0,1)$, the condition implies g is convex.
when $t \in (-\infty, 0) \cup (1, +\infty)$, the condition implies g is concave.

only functions like $g(x) = ax + b$ is concave & convex at the same time

6. $LHS = \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h}$ L' Hospital (since $f''(x)$ continuous)

$= f''(x_0)$

7. (a) Induction

for $n=1$, equation holds

suppose for all $n \le l$, the equation holds

for $n = l+1$

$$f\left(\frac{1}{2^l}\sum_{k=1}^{2^l} x_k\right) \le \frac{1}{2^l}\sum_{k=1}^{2^l} f(x_k)$$

express $x_k = (y_{2k-1} + y_{2k})/2$

then

$$f\left(\frac{1}{2^{l+1}}\sum_{k=1}^{2^{l+1}} y_k\right) \le \frac{1}{2^l}\sum_{k=1}^{2^l} f\left(\frac{y_{2k-1}+y_{2k}}{2}\right) \le \frac{1}{2^{l+1}}\sum_{k=1}^{2^{l+1}} f(y_k)$$

(b) choose $y_1 = \cdots = y_m = x$ $y_{m+1} + \cdots y_{2^k} = y$. then obvious.

(c) $x = x_0+h$ $y = x_0 - h$ $h>0$

then $\frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} \ge 0$ $\Rightarrow h \to 0$. $f''(x_0) \ge 0$ $\forall x_0 \in \mathbb{R}$

thus convex