IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

  (1) w consists of n letters, all of them are from the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$;

  (2) w contains an even number of letters a;

  (3) w contains an even number of letters b.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

<div align="right">Armend Sh. Shabani, University of Prishtina</div>

Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.
    Prove that

$$\mathrm{trace}(ABAB) - \mathrm{trace}(A^2B^2) = \frac{1}{2}\mathrm{n}(\mathrm{n} - 1).$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. $\mathrm{trace}(\mathrm{M})$ denotes the trace of M, that is the sum of diagonal elements in M.)

<div align="right">Rustam Turdibaev, V. I. Romanovskiy Institute of Mathematics</div>

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0, 1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1 - \varepsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset \mathbb{R}^d$ of at most α points, i.e., $T = \{\sum_{x \in X} t_x x \mid t_x \geq 0, \sum_{x \in X} t_x = 1\}$. For a real λ, put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

<div align="right">Fedor Petrov, St. Petersburg State University</div>

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x + 1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1 - t) \geqslant p(t)$ for $0 \leqslant t \leqslant 1/2$.

<div align="right">Daniil Klyuev, St. Petersburg State University</div>

$$p(x+1) - p(x) = x^{100}$$

$$p(x) - p(x-1) = (x-1)^{100}$$

$$\vdots$$

$$p(2) - p(1) = 1^{100}$$

$$p(x+1) = \sum_{i=1}^{x} i^{100}$$