

Laplace 方法

(含 Stiny 公式)



Laplace 方法主要用于解决形如以下类形的极限

$$\lim_{n \rightarrow \infty} \int_a^b \varphi(x) e^{nh(x)} dx$$

也可变形为:

$$\lim_{n \rightarrow \infty} \int_a^b \varphi(x) f^n(x) dx \quad [h(x) = \ln f(x)]$$

Theorem 1

(I) $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $\varphi(x) e^{nh(x)}$ 在 $[a, b]$ 上可积

(II) $\exists \xi$, $h(\xi) > h(x)$ $x \in [a, b]$,

(III) \exists neighbour of ξ , $h''(\xi) < 0$ $h''(x)$ continuous

(IV) $\varphi(x)$ continuous at $x = \xi$, $\varphi(\xi) \neq 0$

$$n \rightarrow \infty, \quad \int_a^b \varphi(x) e^{nh(x)} dx \sim \varphi(\xi) \sqrt{-\frac{2\pi}{nh''(\xi)}} e^{nh(\xi)}$$

Theorem 2

当 $\xi = a$ 且 $h'(a) = 0$

$$\int_a^b \varphi(x) e^{nh(x)} dx \sim \frac{1}{2} \varphi(\xi) \sqrt{-\frac{2\pi}{nh''(\xi)}} e^{nh(\xi)}$$

当 $\xi = a$ 且 $h'(a) < 0$

$$\int_a^b \varphi(x) e^{nh(x)} dx \sim -\frac{\varphi(\xi)}{nh'(\xi)} e^{nh(\xi)}$$

以下是对全纯多元函数的 Laplace 方法.

$$\int_a^b \varphi(t) e^{\lambda f(t)} dt \quad \text{满足:}$$

1. 在 $\lambda = \lambda_0$ 绝对收敛
2. f 在 $t_0 \in [a, b]$ 取得绝对最大
3. 存在 $B(t_0, \delta)$, f, φ 在其上有一致收敛的 Taylor 级数

(I). $t_0 \in (a, b)$, $f'(t_0) \neq 0$.

$$\int_a^b \varphi(t) \cdot e^{\lambda f(t)} dt \sim e^{\lambda f(t_0)} \sum_{n=0}^{\infty} a_n \Gamma(n + \frac{1}{2}) \lambda^{-(n+\frac{1}{2})}$$

a_n 为函数 $\psi(T) := \varphi[t(T)] \cdot t'(T)$ 的泰勒系数

$$\psi(T) = \sum_{n=0}^{\infty} a_n T^n$$

$t(T)$ 为 $T(t) = \sqrt{f(t) - f(t_0)}$ 的逆函数 (在 $t < t_0$ 时取负分支, $t > t_0$ 时正分支)

可由 Bürmann - Lagrange 级数确定:

$$t(T) = \sum_{n=0}^{\infty} d_n T^n$$

$$d_0 = t_0, \quad d_n = \lim_{t \rightarrow t_0} \frac{1}{n!} \left(\frac{d}{dt} \right)^{n-1} \left(\frac{t - t_0}{T(t)} \right)^n$$

$$= \lim_{t \rightarrow t_0} \frac{1}{n!} \left(\frac{d}{dt} \right)^{n-1} \left[\sum_{k=0}^{\infty} - \frac{f^{(k+2)}(t_0)}{(k+2)!} (t - t_0)^k \right]^{-\frac{n}{2}}$$

(II) $t_0 = a$, $f'(a) \neq 0$

$$\int_a^b \varphi(t) e^{\lambda f(t)} dt \sim e^{\lambda f(a)} \sum_{n=0}^{\infty} a_n \Gamma(n+1) \lambda^{-(n+1)}$$

$$(III) \quad t_0 \in [a, b] \quad f'(t_0) = \dots = f^{(p)}(t_0) = 0, \quad f^{(p+1)}(t_0) \neq 0$$

这是最一般的情况，在证明过程中换元时令 $f(t_0) - f(t) = T^p$

由此可验证 Stirling 公式：

$$\begin{aligned} \Gamma(\lambda+1) &= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \cdot \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \cdot n^{1-2k}\right) \\ &= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \dots\right) \end{aligned}$$