IMC 2013, Blagoevgrad, Bulgaria

Day 1, August 8, 2013

Problem 1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

(10 points)

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

(10 points)

Problem 3. There are $2n$ students in a school ($n \in \mathbb{N}$, $n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

(10 points)

Problem 4. Let $n \geq 3$ and let $x_1, \ldots, x_n$ be nonnegative real numbers. Define $A = \sum_{i=1}^{n} x_i$, $B = \sum_{i=1}^{n} x_i^2$ and $C = \sum_{i=1}^{n} x_i^3$. Prove that

$$(n + 1)A^2 B + (n - 2)B^2 \geq A^4 + (2n - 2)AC.$$

(10 points)

Problem 5. Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^{\infty} a_n^p$ converges if and only if p is not a prime?

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(I)

(2)

$$\left[f(x) \cos x \right]'' = f''(x) \cos x - 2 f'(x) \sin x - \cos x \cdot f(x)$$

$$\left[f(x) \cdot \frac{1}{\cos x} \right]'' = f''(x) \cdot \frac{1}{\cos x} + 2 f'(x) \frac{\sin x}{\cos^2 x} + \frac{2 - \cos^2 x}{\cos^3 x} f(x)$$

$$+ \left[f(x) \cos x \right]'' + \cos^2 x \left[f(x) \cdot \frac{1}{\cos x} \right]'' = f''(x) \left(\cos x + \frac{1}{\cos x} \right) - \left(\cos x - \frac{2 - \cos^2 x}{\cos x} \right) f(x)$$

$$\frac{1 + \cos^2 x}{\cos x} f''(x) + \frac{1 - \cos^2 x}{\cos x} f(x)$$

(3)　　　assume　　there　are　　m　trips

assume　$A_{m,n}$　where

$$A_{i,j} = \begin{cases} 0 & \text{in the trip} \\ 1 & \text{not in the trip} \end{cases}$$

$$(A^T \cdot A)_{ij} = \begin{cases} \text{number of trips } i,j \text{ go together} & i \neq j \\ \text{number of trips } i \text{ go} & i = j \end{cases}$$

to meet the requirement,

$$(A^T A)_{ij} \geq 1 \qquad \text{if } i \neq j$$

$$\sum_{i=1}^{m} (A^T A)_{ij} = m \cdot n$$

(4)

$A = \sigma_1$

$B = \sigma_1^2 - 2\sigma_2$

$C = \sigma_1^3 - 3\sigma_2\sigma_1 + \sigma_3$

$(n+1)\,\sigma_1^2\left(\sigma_1^2 - 2\sigma_2\right) + (n-2)\left(\sigma_1^2 - 2\sigma_2\right)^2 - \sigma_1^4 - (2n-2)\sigma_1\left(\sigma_1^3 - 3\sigma_1\sigma_2 + \sigma_3\right)$

$= (n+1)\left(\sigma_1^4 - 2\sigma_1^2\sigma_2\right) + (n-2)\left(\sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_2^2\right) - \sigma_1^4 - (2n-2)\left(\sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_1\sigma_3\right)$

$= (2n-4)\,\sigma_2^2 - (n-1)\,\sigma_1\sigma_3$

when $n=3$

$(x_1 x_2 + x_2 x_3 + x_1 x_3)^2 \geq (x_1 + x_2 + x_3)\cdot x_1 x_2 x_3$

$x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2 +$

(f)
$$a_n = \rho_n e^{i\theta_n}$$

(f)
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