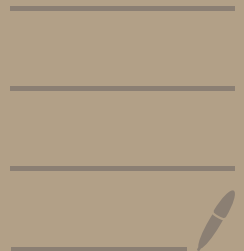


Localization of Ring



Def. $(R, +, \cdot)$ commutative, $S \subset R$

call S a multiplicative subset

if $1 \in S$ or is submonoid of $(R, \{0\}, \cdot)$
 $\forall a, b \in S, ab \in S$

Def. $(R, +, \cdot)$ commutative S multiplicative subset
 R localization of S , denote as $(S^{-1}R, +, \cdot)$, defined as
 $S^{-1}R = \{\frac{r}{s} : r \in R, s \in S\} / \sim$

where $\frac{r}{s} \sim \frac{r'}{s'} \iff \exists t \in S, t(rs' - r's) = 0$

if $r, r' \in R, s, s' \in S$, define:

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$$

$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

Prop. $(S^{-1}R, +, \cdot)$ is a commutative Ring

Proof. (I) to prove " \sim " is a relation of equivalence

suppose $\frac{r}{s} \sim \frac{r'}{s'}, \exists t \in S, \text{ s.t.}$
 $t(rs' - r's) = 0$

thus $r/s = r'/s'$

if $r/s \sim r'/s', r'/s' \sim r''/s''$

for $t, t' \in S$ s.t.

$$t(rs' - r's) = 0 \quad \& \quad t'(r's'' - r''s') = 0$$

thus $\exists t, t' \in S, t'' = tt' \text{ s.t.}$

$$\begin{aligned} \text{thus } (tt's')rs'' &= t's''(trs') = t's''(tr's) = ts(tt'r's'') \\ &= ts(tt'r's'') = (tt's')r''s'' \end{aligned}$$

thus $t''(rs'' - r''s) = 0$

(III) to prove $S^{-1}R$ is a Ring

for any $r/s \in S^{-1}R$

$$\frac{0}{1} + \frac{r}{s} = \frac{0 \cdot s + 1 \cdot r}{1 \cdot s} = \frac{r}{s} \quad (\text{Identity})$$

$$\frac{1}{1} \cdot \frac{r}{s} = \frac{r}{s}$$

$$\left(\frac{r_1}{s_1} + \frac{r_2}{s_2} \right) + \frac{r_3}{s_3} = \frac{r_1 s_2 s_3 + s_1 r_2 s_3 + s_1 s_2 r_3}{s_1 s_2 s_3} = \frac{r_1}{s_1} + \left(\frac{r_2}{s_2} + \frac{r_3}{s_3} \right)$$

(Associative law of multiplication)

$$\frac{r}{s} + \left(-\frac{r}{s} \right) = 0 \quad (\text{Inverse of addition})$$

$$\frac{r_1}{s_1} \cdot \left(\frac{r_2}{s_2} + \frac{r_3}{s_3} \right) = \frac{r_1 (r_2 s_3 + r_3 s_2)}{s_1 s_2 s_3} = \frac{r_1 r_2 s_3}{s_1 s_2 s_3} + \frac{r_1 r_3 s_2}{s_1 s_2 s_3} = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} + \frac{r_1}{s_1} \cdot \frac{r_3}{s_3}$$

(Distribution law)

Prop. $(R, +, \cdot)$ integral ring S multiplicative subset
 $\frac{a}{b} \sim \frac{c}{d} \iff ad - bc = 0$

Def. $(R, +, \cdot)$ an integral ring fraction field on R , denote as $\text{Frac}(R)$
defined as $S^{-1}(R)$, where $S = R \setminus \{0\}$ or sth. else

Prop. $\text{Frac}(R)$ is a field

Lemma. $\mathfrak{p} \nsubseteq R$ is a prime ideal $\Rightarrow S = R \setminus \mathfrak{p}$ is a multiplicative subset