For $A \in M_{n,n}(\mathbb{F})$, $tr(A) = \sum_{i=1}^{n} A_{i,i}$

$tr(A+B) = tr(A) + tr(B)$

$tr(\lambda A) = \lambda\, tr(A)$

$tr(AB) = tr(BA)$

proof

$$tr(AB) = \sum_{i=1}^{n} AB_{ii}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik}$$

$$= \sum_{k=1}^{n} BA_{ii}$$

example

$T: M_{n,n}(\mathbb{F}) \to M_{n,n}(\mathbb{F})$

$T(x) = AX - XA$ is not surjective

because $tr[T(x)] = tr(AX) - tr(XA) = 0$

$tr(A) = \sum \lambda_i \cdot \mu_i(\lambda)$

proof

$$P^{-1} A P = D = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots \lambda_1 & & & \\ & & \lambda_2 \cdots & & \\ & & & & \lambda_k \end{pmatrix}$$

$$tr(P^{-1}AP) = tr\, D = \sum \lambda_i \mu_A(\lambda_i)$$

$$LHS = tr[(P^{-1}A)P] = tr[P(P^{-1}A)] = tr(A)$$

geometric multiplicity

$$\dim \ker(A - \lambda_i I) = m_A(\lambda_i) = \mu_A(\lambda_i)$$

$$T: M_{n,n}(\mathbb{F}) \to M_{n,n}(\mathbb{F})$$

$$T(Y) = AY - YA$$

$$\exists\ X,\ Tr(X) = 0,\ X \notin Im(T)$$

proof:

suppose $W = \{ X \in M_{n,n}(\mathbb{F}) \mid Tr(x) = 0 \}$

$\dim(W) = n^2 - 1$

$\dim \ker(T) \geq 2$ $(A \in \ker T,\ I \in \ker(T))$

so $\dim(W) > \dim(T)$

Def $A \in M_{n,n}(\mathbb{F})$ nilpotent $\Leftrightarrow$ $k \in \mathbb{N}$, $A^k = 0$

$n$ nilpotency index of $A$ $\Leftrightarrow$ $A^n = 0$, $A^{n-1} \neq 0$