Problem Solving Strategies I

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1 Problems to be discussed in lecture

1.1 Invariance Principle

n odd

n even

1.1.1 Problem 1

Suppose the positive integer n is odd. First Al writes the numbers $1, 2, \ldots, 2n$ on the blackboard. Then he picks any two numbers a, b erases them, and writes, instead, $|a - b|$. Prove that an odd number will remain at the end.

① O-2 E+1 $S = (2n+1)n$

② O E-1

③ O E-1

$f(n) = \binom{n}{1} f(n-1) + \binom{n}{2} f(n-2) + \cdots + \binom{n}{n} f(0)$

1.1.2 Problem 2

(National Iranian Mathematical Competition for University Students) Let k be a fixed natural number. There are $n \geq 4$ people in a "lottery"-game where each of them has some coins. In each step of the game 4 of them, which satisfy the following conditions, are chosen randomly.

1. the total number of coins of the first, and the second person+ 2k > the total number of coins of the third, and the fourth person ,
2. and the third, and the fourth person each has at least k coins.

$$\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

Then, from the third and the fourth person k coins are taken from each, and k of them given to the first and the other k coins are given to the second person. Show that this game cannot continue for ever, i.e., after finite number of steps the condition (1) or (2) is not valid for any four persons.

$$A = \begin{array}{c|c} & 1 \quad 2 \quad 3 \quad \cdots \quad n \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \end{array}$$

1.2 Algebraic Technique in Combinatorics

1.2.1 Problem 3

(Odd-Town) The town of Odd-town has a population of n people. Inhabitants of Odd-town like to form clubs. Every club has an odd number of members, and every pair of clubs share an even number of members (possibly none). Show that there are at most n clubs in this town

$A \cdot x = 0$

$a_{ij} = \begin{cases} 0 & \text{in the club} \\ 1 & \text{not in the club} \end{cases}$

1.2.2 Problem 4

(National Iranian Mathematical Competition for University Students) Suppose you have n stones where n is odd number. If you remove any one of them, the remaining stones can be divided into two equal-sized sets with equal weights. Show that all stones have the same weight.

$$\begin{pmatrix} 0 & \pm1 & \pm1 & \pm1 & \pm1 & \pm1 \\ \pm1 & 0 & & & & \\ \vdots & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = 0$$

$\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \cdots -$

$- \binom{n}{n-1} 1! + \binom{n}{n} 0!$

$-n+1$

1.3 Convexity and Inequalities

Problem 5

(AM-GM inequality) Prove that for any positive real numbers $a_1, \dots, a_n$, we have

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}.$$

Moreover, the equality happens iff $a_1 = \cdots = a_n$.

Problem 6

Show that if A, B and C are angles of a triangle, then

$$\sin A + \sin B + \sin C \geq \frac{3\sqrt{3}}{2}.$$

Problem 7

(Holder-Inequality) If p, q are positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$ and x_i, y_i are also positive real numbers for $i = 1, \dots, n$, then

$$(\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}} (\sum_{i=1}^{n} y_i^q)^{\frac{1}{q}} \geq \sum_{i=1}^{n} x_i y_i.$$

Moreover, equality holds iff there is a constant $\lambda > 0$ such that $(x_1, \dots, x_n) = \lambda(y_1, \dots, y_n)$.

Problem 8

Let $0 < a_i < \pi$ for $i = 1, \dots, n$ and set $a = \frac{a_1 + \dots + a_n}{n}$. Show that

$$\prod (\frac{\sin a_i}{a_i}) \leq (\frac{\sin a}{a})^n.$$

2 Problems

1. Suppose not all four integers a, b, c, d are equal. Start with (a, b, c, d) and repeatedly replace (a, b, c, d) by $(a - b, b - c, c - d, d - a)$. Then at least one number of the quadruple will eventually become arbitrarily large.

2. (Putnam 2008) Start with a sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $gcd(a_j, a_k)$ and $lcm(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

3. (IMO 1986) To each vertex of a pentagon, we assign an integer x_i with sum $\sum x_i > 0$. If x, y, z are the numbers assigned to three successive vertices and if $y < 0$, then we replace (x, y, z) by $(x+y, -y, y+z)$. This step is repeated as long as there is a $y < 0$. Decide if the algorithm always stops.

4. (Fisher Inequality) Let $C = \{A_1, \ldots, A_r\}$ be a collection of distinct subsets of $\{1, \ldots, n\}$ such that every pairwise intersection $A_i \cap A_j$ has size t, where t is some fixed integer between 1 and n. Prove that $|C| \leq n$.

5. Let $A_1, \ldots, A_r$ be a collection of distinct subsets of $\{1, \ldots, n\}$ such that all $|A_i|$ are even, and also all $|A_i \cap A_j|$ are even for $i \neq j$. How big can r be, in terms of n?

6. (The generalized AM-GM inequality) Let λ_i for $i = 1, \ldots, n$ be positive integrers such that $\lambda_1 + \ldots + \lambda_n = 1$. Show that

$$\lambda_1 a_1 + \ldots + \lambda_n a_n \geq a_1^{\lambda_1} \ldots a_n^{\lambda_n}$$

Moreover, the equality happens iff $a_1 = \cdots = a_n$.

References

[1] Arthur Engel, Problem-Solving Strategies, Springer, 1998.

[2] Bamdad R. Yahaghi, Iranian Mathematics Competitions, 1973–2007.

[3] Razvan Gelca and Titu Andreescu, Putnum and beyound, springer.

assume that a_i is the mass of ith rock

$$A_{1,1}a_1 + A_{1,2}a_2 + A_{1,3}a_3 + \cdots + A_{1,n}a_n = 0$$
$$A_{2,1}a_1 + A_{2,2}a_2 + A_{2,3}a_3 + \cdots + A_{2,n}a_n = 0$$
$$\vdots$$
$$A_{n,1}a_1 + A_{n,2}a_2 + \cdots + A_{n,n}a_n = 0$$

where ① $A_{i,i} = 0$ for $i \in \{1, 2, 3 \cdots n\}$

② $A_{i,j} \in \{1, -1\}$ for $i \neq j$, $i, j \in \{1, 2, 3 \cdots n\}$

③ $\sum_{k=1}^{n} A_{i,k} = 0$ for $i \in \{1, 2, 3 \cdots n\}$

so the equation set can be write as

$$\begin{pmatrix} 0 & \pm1 & \pm1 & \cdots & \pm1 \\ \pm1 & 0 & \pm1 & \cdots & \vdots \\ \vdots & & \ddots & & \vdots \\ \pm1 & \cdots & & & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = A_{n\times n} \cdot X = 0$$

where $A_{i,j} \in \{-1, 1\}$

$\sum_{k=1}^{n} A_{i,k} = 0$ for $i \in \{1, 2, 3 \cdots n\}$

take one minor matrix of order k

$$B_{(n-1)\times(n-1)} = \begin{pmatrix} 0 & \pm1 & \pm1 & \pm1 & \cdots & \pm1 \\ \pm1 & 0 & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \pm1 & \cdots & & & & 0 \end{pmatrix}$$

suppose

$$B' = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & & & 0 \end{pmatrix}$$

$$B' \equiv B \pmod 2$$

$$B'^2 = \begin{pmatrix} n-1 & n-2 & n-2 & \cdots & n-2 \\ n-2 & n-1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ n-2 & \cdots & \cdots & & n-1 \end{pmatrix}$$

$\det(B'^2) \equiv \det I \equiv 1 \pmod 2$

$(\det B')^2 \equiv 1 \pmod 2$, $\det B' \equiv 1 \pmod 2$

$\det B \equiv \det B' \equiv 1 \pmod 2$

$\det B \neq 0$

so $\operatorname{rank} A \geq n-1$

because $\sum_{k=1}^{n} A_{i,k} = 0$

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} A_{1,k} \\ \sum_{k=1}^{n} A_{2,k} \\ \vdots \\ \sum_{k=1}^{n} A_{n,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

so 0 is an eigenvalue and $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector

so A is not full rank

$\operatorname{rank} A = n-1 \Rightarrow \dim \ker (A - 0 \cdot I) = 1$

the multiplicity of eigenvalue 0 is 1

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is the only eigenvector of 0

so the solution of $Ax = 0$ is $x = c \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

which implies that every rock has same mass