# 拉阿伯（Raabe's）积分

$$R(a)=\int_a^{a+1}\ln\Gamma(x)\,dx$$

运用费曼（Feynmann's）技巧

$$R'(a)=\ln\Gamma(a+1)-\ln\Gamma(a)$$
$$=\ln a$$

$$R(a)=a\ln a-a+C\quad,\quad C=\int_0^1\ln\Gamma(x)\,dx$$

$$\int_0^1\ln\Gamma(x)\,dx=\int_0^1\ln\Gamma(1-x)\,dx$$
$$=\frac{1}{2}\int_0^1\ln\Gamma(x)\Gamma(1-x)\,dx$$
$$=\frac{1}{2}\int_0^1\ln\frac{\pi}{\sin\pi x}\,dx\qquad\text{(余元公式)}$$
$$=\frac{1}{2}\ln\pi-\frac{1}{2}\int_0^1\ln\sin\pi x\,dx=\frac{1}{2}\ln\pi-\frac{1}{2}I$$

$$I=\int_0^1\ln\sin\pi x\,dx\overset{\pi x=t}{=\!=\!=}\frac{1}{\pi}\int_0^{\pi}\ln\sin t\,dt$$
$$=\frac{1}{\pi}\left(\int_0^{\frac{\pi}{2}}\ln\sin t\,dt+\int_0^{\frac{\pi}{2}}\ln\cos t\,dt\right)$$
$$=\frac{1}{\pi}\frac{1}{2}\int_0^{\pi}\ln\frac{\sin t}{2}\,dt$$
$$=\frac{1}{\pi}\cdot\left(-\frac{\pi}{2}\ln 2\right)\cdot 2$$

$$C=\frac{\ln 2\pi}{2}$$

$$R(a)=\int_a^{a+1}\ln\Gamma(x)\,dx=\frac{\ln 2\pi}{2}+a\ln a-a\quad\left[\int_a^{a+k}\ln\Gamma(x)=\sum_{i=a}^{a+k-1}R(i)\right]$$

## 菲涅尔（Fresnel）积分

$$\int_0^{+\infty} \cos(x^k) = I_1$$

$$\int_0^{+\infty} \sin(x^k) = I_2$$

$$I_1 + iI_2 = \int_0^{\infty} e^{ix^k} dx = \frac{1}{k}\int_0^{\infty} t^{\frac{1}{k}-1} e^{it} dt$$

## 拉马努金主定理

$$\int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} \frac{\phi(n)}{n!} x^n dx = \Gamma(s)\,\phi(-s)$$

$$I_1 = \frac{1}{k}\,\Gamma\left(\frac{1}{k}\right) \cos\frac{\pi}{2k}$$

$$I_2 = \frac{1}{k}\,\Gamma\left(\frac{1}{k}\right) \sin\frac{\pi}{2k}$$

## Ahmed's 积分

定义

$$B(x)=\begin{cases}\dfrac{\arctan\sqrt{x}}{\sqrt{x}}, & x>0\\ 1, & x=0\\ \dfrac{\operatorname{arctanh}\sqrt{-x}}{\sqrt{-x}}, & -1<x<0\end{cases}$$

$$A(p,q,r)=\int_0^1\frac{B[q(px^2+1)]}{(r+1)px^2+1}\,dx$$

$$\tilde{A}(p,q,r)\equiv A(p,q,r)+B[r(q+1)]\,B\left(\frac{pq}{q+1}\right)$$ 称为关联形式

有以下性质:

(I) $\tilde{A}(p,q,r)=\tilde{A}(r,p,q)=\tilde{A}(q,r,p)$ $\quad p,q,r>0$ 三轮换对称性

(II) $A(p,q,0)=B[q(p+1)]+B(p)-B\left(\dfrac{pq}{q+1}\right)$ 特殊值

(III) $\sqrt{pqr}\,A(p,q,r)+\dfrac{1}{\sqrt{pqr}}A\left(\dfrac{1}{q},\dfrac{1}{p},\dfrac{1}{r}\right)$ 非对称型态 Ahmed's积分

$$=\frac{\pi}{2}\left(\arctan\sqrt{\frac{1}{r(q+1)}}+\arctan\sqrt{\frac{pr}{p+1}}\right)-\arctan\sqrt{p(r+1)}\cdot\arctan\sqrt{\frac{r+1}{qr}}$$

证明 (I)

只需证明 $\tilde{A}(p,q,r)=\tilde{A}(q,r,p)$

$\forall\ p,q,r>0$

$$B(x)=\int_0^1\frac{1}{xt^2+1}\,dt=\frac{1}{x}\sqrt{x}\arctan\sqrt{x}\,t\Big|_{t=0}^{1}$$

$$A(p,q,r) = \int_0^1 \frac{1}{(r+1)px^2+1} \int_0^1 \frac{1}{q(px^2+1)y^2+1}\,dy\,dx$$

$$= \int_0^1\int_0^1 \underbrace{\frac{1}{qry^2+r+1}\left[\frac{r+1}{(r+1)px^2+1}\right.}_{I_1} - \left.\underbrace{\frac{qy^2}{pqy^2x^2+qy^2+1}}_{I_2}\right] dy\,dx$$

$$I_1 = B[p(r+1)] \cdot B\left(\frac{qr}{r+1}\right)$$

$$I_2 = \int_0^1 \sqrt{\frac{p}{q}}\,\frac{y}{\sqrt{qy^2+1}} \arctan\sqrt{\frac{pqy^2}{qy^2+1}}\,dy$$

$$= \frac{1}{\sqrt{pqr}} \int_0^1 \left(\arctan\sqrt{r(qy^2+1)}\right)' \arctan\sqrt{\frac{pqy^2}{qy^2+1}}\,dy$$

$$= \underbrace{\frac{1}{\sqrt{pqr}} \arctan\sqrt{r(qy^2+1)}\,\arctan\sqrt{\frac{pqy^2}{qy^2+1}}\,\Bigg|_0^1}_{\Updownarrow \atop B\left(\frac{pq}{q+1}\right)\cdot B[r(q+1)]} - \underbrace{\int_0^1 \frac{\arctan\sqrt{r(qy^2+1)}}{[(1+p)qy^2+1]\sqrt{r(qy^2+1)}}\,dy}_{\Updownarrow \atop A(q,r,p)}$$

Q.E.D.

证明 (2)

when $r \to 0^+$

$$\hat{A}(q,r,p) = \hat{A}(p,q,r) = A(p,q,r) + B[r(q+1)]\,B\left(\frac{pq}{q+1}\right)$$

$$= A(q,r,p) + B[p(r+1)]\,B\left(\frac{qr}{r+1}\right)$$

$$= \int_0^1 \frac{1}{(1+p)qy^2+1}\,dy + B(p) = B[q(p+1)] + B(p)$$

Q.E.D.

证明 (3) 略.

(Ⅲ) 的推论：

$$A\left(\frac{1}{\alpha},\alpha,1\right)=\frac{\pi}{2}\arctan\sqrt{\frac{1}{\alpha+1}}-\frac{1}{2}\left(\arctan\sqrt{\frac{2}{\alpha}}\right)^2$$

例：

$$\int_0^1\frac{\arctan\sqrt{x^2+1}}{(x^2+1)\sqrt{x^2+1}}\,dx=A(1,1,0)$$

$$\int_0^1\frac{\arctan\sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}}\,dx=A\left(\frac{1}{2},2,1\right)$$

$$\int_0^1\frac{\arctan\sqrt{1+2x^2}}{(1+3x^2)\sqrt{1+2x^2}}\,dx=\int_0^1\frac{B[q(px^2+1)]}{(r+1)px^2+1}\,dx=A\left(2,1,\frac{1}{2}\right)$$

$$=A\left(\frac{1}{2},2,1\right)$$

$$\iiint_{[0,1]^3}\frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}\ \underset{y=r\sin\phi}{\overset{x=r\cos\phi}{=\!=\!=}}\ 2\int_{[0,1]}\int_{[0,\frac{\pi}{4})}\int_0^{\sec\phi}\frac{1}{(1+r^2+z^2)^2}\,r\,dr\,d\theta\,dz$$

$$=2\int_{[0,1]}\int_{[0,\frac{\pi}{4}]}\frac{1}{2(1+z^2)}-\frac{1}{2(1+z^2+\sec^2\phi)}\,d\theta\,dz$$

$$=\int_{[0,1]}\frac{\operatorname{arccot}\sqrt{x^2+2}}{(x^2+1)\sqrt{x^2+2}}\,dx=I$$

$$I=\int_0^1\int_0^1\frac{1}{(x^2+1)[(\sqrt{x^2+2})^2+y^2]}\,dx\,dy=\int_0^1\int_0^1\frac{1}{(y^2+1)(x^2+y^2+2)}\,dx\,dy$$

$$=\frac{1}{2}\int_{[0,1]^2}\frac{1}{(x^2+1)(y^2+1)}\,dx\,dy=\frac{1}{2}\left(\int_0^1\frac{1}{x^2+1}\,dx\right)^2=\frac{\pi^2}{32}$$

## Coxeter 积分

定义: $$I_c = \int_0^{\theta_0} \arctan\sqrt{\frac{\cos\theta+1}{a\cos\theta+b}}\, d\theta$$

$$I(\alpha,\beta,\phi) = \int_0^{\phi} \arctan\left(\frac{\cos\theta}{\alpha\sqrt{\beta^2-\sin^2\theta}}\right) d\theta$$

性质:

(I) $$I_c = 2\, I\left(\sqrt{a}, \sqrt{\frac{a+b}{2a}}, \frac{\theta_0}{2}\right)$$

(II) $$I(\alpha,\beta,\phi) = \frac{\tan\tilde{\phi}}{\alpha} A\left[(1-\beta^2)\tan^2\tilde{\phi}, \frac{1}{\alpha^2\beta^2}, \frac{\beta^2}{1-\beta^2}\right]$$

$$\tilde{\phi} = \arcsin\frac{\sin\phi}{\beta}$$

(III) $$I_c = 2\sqrt{\frac{1-\cos\theta_0}{a\cos\theta_0+b}}\; A\left(\frac{a-b}{2}\cdot\frac{1-\cos\theta_0}{a\cos\theta_0+b}, \frac{2}{a+b}, \frac{a+b}{a-b}\right)$$

例题:

(I) $$\int_0^{\frac{\pi}{2}} \arccos\frac{\cos\theta}{1+2\cos\theta}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \arctan\sqrt{\frac{1+\cos\theta}{1+3\cos\theta}}\, d\theta \qquad a=3 \quad b=1 \quad \theta_0=\frac{\pi}{2}$$

$$= 4\cdot 1\cdot A\left(1, \frac{1}{2}, 2\right)$$ 可计算

(II) $$\int_0^{\frac{\pi}{2}} \arccos\sqrt{\frac{\cos\theta}{1+2\cos\theta}}\, d\theta$$

$$= 4\frac{1}{\sqrt{5}} A\left(\frac{1}{5}, \frac{1}{2}, 2\right)$$ 无法计算

## Watson's tripe integral.

$$I_1 = \frac{1}{\pi^3}\int_{[0,\pi]^3} \frac{1}{1-\cos u\cos v\cos w}\, du\, dv\, dw$$

$$I_2 = \frac{1}{\pi^3}\int_{[0,\pi]^3} \frac{1}{3-\cos u\cos v-\cos v\cos w-\cos u\cos w}\, dv\, du\, dw$$

$$I_3 = \frac{1}{\pi^3}\int_{[0,\pi]^3} \frac{1}{3-\cos u-\cos v-\cos w}\, du\, dv\, dw$$

解 $I_1$:

$$\int_0^{\pi} \frac{dx}{1+b\cos x} = \frac{\pi}{\sqrt{1-b^2}}$$

$$I_1 = \frac{1}{\pi^2}\int_{[0,\pi]} \frac{1}{\sqrt{1-b^2}}\, dv\, dw$$

$$(1-b^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})\cdots(-\frac{1}{2}-n+1)}{n!}(-1)^n b^n$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} b^n$$

$$I_1 = \frac{1}{\pi^2}\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\left(\int_0^{\pi} \cos^{2n} v\, dv\right)^2$$

$$= \sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^3$$

运用 第一类完全椭圆积分的展开式

$$[K(k)]^2 = \frac{\pi^2}{4}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^3 (2k\cdot k')^{2n}$$

$k$为椭圆积分模量

$k'$为互补模量，$k'=\sqrt{1-k^2}$

$$\frac{\Gamma^4(\frac{1}{4})}{16\pi} = \left[K\left(\frac{\sqrt{2}}{2}\right)\right]^2 = \frac{\pi^2}{4}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^3$$

$$I_1 = \frac{\Gamma^4(\frac{1}{4})}{4}$$

解决 $I_2$：

定理：

(I) $K(k) = K'(k') = \int_0^{\infty}\frac{dt}{\sqrt{(1+t^2)(1+k'^2t^2)}}$

(II) $K'(k) = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\left[\psi(n+1)-\psi(n+\frac{1}{2})-\ln k\right]\cdot k^{2n}$

(III) $K(k)\cdot K(k') = \frac{\pi}{4}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^3\left[3\psi(n+1)-3\psi(n+\frac{1}{2})-2\ln 2k\cdot k'\right](2kk')^{2n}$

(IV) 三阶奇异值 $K(k_3) = \frac{3^{\frac{1}{4}}\Gamma^3(\frac{1}{3})}{2^{\frac{7}{3}}\pi}$，$\frac{K(k_3')}{K(k_3)} = \sqrt{3}$，$k_3 = \frac{\sqrt{6}-\sqrt{2}}{4}$

$$\begin{cases} x = \tan\frac{u}{2} \\ y = \tan\frac{v}{2} \\ z = \tan\frac{w}{2} \end{cases} \qquad |J| = 8\,\frac{1}{(1+x^2)(1+y^2)(1+z^2)}$$

$$I_2 = \frac{2}{\pi^3}\int_{[0,+\infty)^3}\frac{1}{x^2+y^2+z^2+x^2y^2+y^2z^2+x^2z^2}\,dx\,dy\,dz$$

$$\begin{cases} x = \rho\sin\theta\cos\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\theta \end{cases} \qquad |J| = \rho^2\sin\theta$$

$$I_2 = \frac{2}{\pi^3}\iint_{[0,\frac{\pi}{2}]^2}\int_0^{+\infty}\frac{\rho^2\sin\theta}{\rho^2+\rho^4(\sin^4\theta\sin^2\varphi\cos^2\varphi+\sin^2\theta\cos^2\theta)}\,d\rho\,d\theta\,d\varphi$$

$$= \frac{1}{\pi^2}\iint_{[0,\frac{\pi}{2}]^2}\frac{1}{\sqrt{\cos^2\theta+\frac{1}{4}\sin^2\theta\sin^2 2\varphi}}\,d\theta\,d\varphi$$

$$t = \tan\theta$$

$$= \frac{1}{\pi^2}\int_0^{\frac{\pi}{2}} K'\left(\tfrac{1}{2}\sin 2\varphi\right)d\varphi \qquad \left[k = \tfrac{1}{2}\sin 2\varphi\right]$$

$$= \frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\left[\psi(n+1)-\psi(n+\tfrac{1}{2})-\ln k\right]k^{2n}\,d\varphi$$ (附录二)

令 $y(\varepsilon) = \dfrac{\Gamma(n+\frac{1}{2}+\varepsilon)}{\Gamma(n+1+\varepsilon)}k^{2n+\varepsilon}$, $\ln y = \ln\Gamma(n+\frac{1}{2}+\varepsilon) - \ln\Gamma(n+1+\varepsilon) + (2n+\varepsilon)\ln k$

$$\frac{\partial}{\partial\varepsilon}\ln y = \psi(n+\tfrac{1}{2}+\varepsilon) - \psi(n+1+\varepsilon) + \ln\varepsilon$$

$$y'(\varepsilon) = y(\varepsilon)\cdot\frac{\partial}{\partial\varepsilon}\ln y = \frac{\Gamma(n+\frac{1}{2}+\varepsilon)}{\Gamma(n+1+\varepsilon)}k^{2n+\varepsilon}\left[\psi(n+\tfrac{1}{2}+\varepsilon)-\psi(n+1+\varepsilon)+\ln k\right]$$

$$\varepsilon\to 0^+ ,\quad y' = \frac{\sqrt{\pi}\,(2n-1)!!}{(2n)!!}\cdot k^{2n}\left[\psi(n+\tfrac{1}{2})-\psi(n+1)+\ln k\right]$$

$$I_2 = -\frac{1}{\pi^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{\partial}{\partial\varepsilon}\left(\frac{\Gamma(n+\frac{1}{2}+\varepsilon)}{\Gamma(n+1+\varepsilon)}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{2^{2n+\varepsilon}}\frac{\Gamma(n+\frac{\varepsilon}{2}+\frac{1}{2})}{\Gamma(n+\frac{\varepsilon}{2}+1)}\right)$$

$$= \frac{1}{4\pi}\sum_{n=0}^{\infty}\left(\left[\frac{(2n-1)!!}{(2n)!!}\right]^3\left[3\psi(n+1)-3\psi(n+\tfrac{1}{2})-2\ln 2\right)\cdot\left(\tfrac{1}{2}\right)^{2n}$$

运用性质三，使 $2kk'=\frac{1}{2}$，得 $k=k_3$

再利用性质四

$$I_2 = \frac{3\Gamma^6(\frac{1}{3})}{2^{\frac{14}{3}}\pi^4}$$