IMC 2015, Blagoevgrad, Bulgaria

Day 1, July 29, 2015

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(10 points)

4

16

Problem 2. For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

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1 2 3 4 8

$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$

1 2 2 4 8

$n + f(n) = 2 \cdot 2^{[\log_2 n]} - 1$

When does equality hold?

(10 points)

$\frac{n(n+1)}{2} + \sum f(n) = 2 \sum 2^{[\log_2 n]} - n$

Problem 3. Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$.

Determine whether or not $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$ is a rational number.

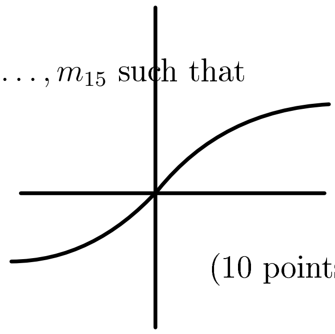
$2x^2 - 5x + 2 = 0$

(10 points)

$(2x-1)(x-2)$

Problem 4. Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$



(10 points)

Problem 5. Let $n \geq 2$, let $A_1, A_2, \ldots, A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of $A_1, A_2, \ldots, A_{n+1}$. Prove that $\angle A_i B A_j > 90^\circ$ holds for at least n pairs (i,j) with $1 \leq i < j \leq n+1$.

(10 points)

$$\begin{pmatrix} a_{i,1} \\ a_{i,2} \\ a_{i,3} \\ a_{i,4} \\ \vdots \\ a_{i,n} \end{pmatrix} \cdot \begin{pmatrix} a_{j,1} \\ a_{j,2} \\ \vdots \\ a_{j,n} \end{pmatrix}^T < 0$$

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

$$(A+B)(A^{-1} + B^{-1}) = I \Rightarrow AB^{-1} + BA^{-1} = -I$$

$$(A^{-1} + B^{-1})(A+B) = I \Rightarrow A^{-1}B + B^{-1}A = -I$$

$$(AB^{-1})^2 + (AB^{-1}) + I = 0$$

$$(AB^{-1})^3 = I$$

$$\det A = \det B$$

P_3

$$F_n = 2^n - 2^{-n}$$

$$\sum_{i=0}^{+\infty} \frac{1}{F(2^i)} = \sum_{i=0}^{+\infty} \frac{1}{2^{2^i} - 2^{-2^i}}$$

$$= \sum_{i=0}^{+\infty} \cdot 2^{2^i} \frac{1}{2^{2^{i+1}} - 1}$$

$$= \sum_{i=0}^{+\infty} \sum_{k=0}^{+\infty} 2^{-(2k+1)\cdot 2^{i}}$$

	2^{-1}	2^{-2}	2^{-3}	------
$k=0$	1	1	0	
$k=1$	0	0	1	
$k=2$	0	0	.---	
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