粒子势垒安定理

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微分算符 $D \equiv D_x \equiv \frac{d}{dx}$

则 $e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!}$

$$e^D f = \sum_{n=0}^{\infty} \frac{D^n}{n!} = f(x+1)$$

延拓至 $\mathbb{R}$

$$e^{tD} f = f(x+t)$$

则

$$e^{aD} \cdot e^{bD} = \sum_{n=0}^{\infty} \frac{(aD)^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{(bD)^n}{n!}$$

$$= \sum_{n=0}^{\infty} D^n \sum_{k=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \quad \text{(Cauchy Convolution)}$$

$$= \sum_{n=0}^{\infty} \frac{D^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \cdot a^k \cdot b^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{D^n}{n!} (a+b)^n = e^{(a+b) \cdot D}$$

差分 $\Delta f = f(x+1) - f(x) = (e^D - 1) f$

$$\Delta \equiv e^D - 1$$

则 $\Delta^{-1} = \frac{1}{e^D - 1} = \frac{1}{D} \cdot \frac{D}{e^D - 1} = \frac{1}{D} \cdot \sum_{n=0}^{\infty} \frac{B_n}{n!} D^n$

(Bernouli Number)

$$\sum = \Delta^{-1} \qquad \int = D^{-1}$$

则在收敛的情况下，

$$\sum = \int - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} D^{2n-1} \quad \text{(奇项}B_n\text{除}B_1\text{都为}0)$$

$$\sum_{a<n<b} f(n) = \int_a^b f(x)\,dx - \frac{f(b)-f(a)}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \left[f^{(2n-1)}(x) \right]_{x=a}^{b}$$

上式即为欧拉-麦克劳林(Euler-Mclaughlin)公式

拉马努金主定理

使用幂函数的 Laplace 变换

$$\mathcal{L}\{x^{s-1}\}(e^{D}) = \int_0^{\infty} x^{s-1} \exp(-x e^{D})\,dx$$

$$= \Gamma(s) \cdot e^{-sD}$$

$$\int_0^{\infty} x^{s-1} \exp(-x e^{D}) \circ \phi(0)\,dx = \Gamma(s)\, e^{-sD} \circ \phi(0)$$

$$\int_0^{\infty} x^{s-1} \sum_{n=0}^{\infty} \frac{e^{nD}\phi}{n!} (-x)^n dx = \Gamma(s)\, \phi(-s)$$

$$\int_0^{\infty} x^{s-1} \sum_{n=0}^{\infty} \frac{\phi(n)}{n!} (-x)^n dx = \Gamma(s)\, \phi(-s)$$

应用、

(I) 扩展菲涅尔积分、

$$I_k = \int_0^{\infty} \sin x^k \, dx \overset{x^k = t}{=\!=} \frac{1}{k}\int_0^{\infty} t^{s-1} \sin t \, dt$$

$$= Im \left\{ \frac{1}{k}\int_0^{\infty} t^{s-1} e^{it} \, dt \right\}$$

$$= \Gamma\left(\frac{1}{k}+1\right) \sin\frac{\pi}{2k}$$

同理: $J_k = \int_0^{\infty} \cos x^k \, dx = \Gamma\left(\frac{1}{k}+1\right) \cos\frac{\pi}{2k}$

(II) 有理函数积分、

$$I_n = \int_0^{\infty} \frac{1}{1+x^n} \, dx \overset{t=x^n}{=\!=} \frac{1}{n}\int_0^{\infty} \frac{t^{\frac{1}{n}-1}}{1+t} \, dt$$

$$= \frac{1}{n}\int_0^{\infty} t^{\frac{1}{n}-1} \sum_{k=0}^{\infty} \frac{(-t)^n}{n!} \Gamma(1+n) \, dt$$

$$= \frac{1}{n} \Gamma(s) \Gamma(1-s) \quad \text{(余元公式)}$$

$$= \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$

(III) 对数相关积分

双伽马函数 $\psi(n) \equiv \frac{\Gamma'}{\Gamma}(n)$

由伽马函数的无穷乘积式

$$\psi(n) = \gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+n}\right) = -\gamma + H_{n-1}$$

则 $H_n = \psi(n+1) + \gamma$

考虑积分 $I(s) = \int_0^\infty x^{s-1} \frac{\ln(1+x)}{1+x} dx \qquad R(s) \in (0,1)$

$$= \int_0^\infty x^{s-1} \sum_{n=0}^\infty [-H_n (-x)^n] dx$$

$$= \int_0^\infty x^{s-1} \sum_{n=0}^\infty \left[-\left(\frac{\gamma n! + \Gamma'(1+n)}{n!}\right)(-x)^n\right] dx$$

$$= -\Gamma(s)\left[\gamma(s)! + \Gamma'(1-s)\right]$$

$$= -\Gamma(s)\Gamma(1-s)\left(\gamma + \psi(1-s)\right)$$

$$= -\frac{\pi}{\sin \pi s}\left(\gamma + \psi(1-s)\right)$$

(IV) 双伽马函数积分

$$I(s) = \int_0^\infty \frac{\psi(1+x)+\gamma}{x^{2-s}} dx \qquad R(s) \in (0,1)$$

$$\psi(1+x) = -\gamma + \sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+x}\right)$$

$$= -\gamma + \sum_{k=1}^\infty \frac{x}{k^2} \frac{1}{1+\frac{x}{k}}$$

$$= -\gamma + \sum_{k=1}^\infty \frac{x}{k^2} \sum_{n=0}^\infty (-1)^n \frac{x^n}{k^n}$$

$$= -\gamma + \sum_{n=0}^\infty (-1)^n x^{n+1} \sum_{k=1}^\infty \frac{1}{k^{n+2}}$$

$$= -\gamma + x \sum_{n=0}^\infty \zeta(n+2)(-x)^n$$

则 $I(s) = \frac{\pi}{\sin \pi s} \zeta(2-s)$

(V) ζ 函数 & 伯努利数

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$$

等价于 $$\mathcal{M}\left\{\frac{1}{e^x-1}\right\}(s) = \Gamma(s)\zeta(s)$$

$$\frac{1}{e^x-1} = \sum_{n=0}^{\infty} \frac{\zeta(-n)}{n!}(-x)^n$$

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-)^{n+1} n\, \zeta(1-n) \frac{x^n}{n!}$$

则 $$\zeta(1-2n) = -\frac{B_{2n}}{2n} \qquad n \in \mathbb{N}$$