1. assume the matrix is $\begin{pmatrix} C_0 & C_1 \\ C_1 & C_2 \end{pmatrix}$

$$\begin{cases} C_0 + C_2 = a \\ C_0 C_2 - C_1^2 = b \end{cases}$$

because M is unique

$$\begin{cases} C_1 = C_2 = \frac{a}{2} \\ C_1 = \sqrt{\frac{a^2}{4} - b} \quad \text{or} \quad -\sqrt{\frac{a^2}{4} - b} = 0 \end{cases}$$

thus $\quad b = \frac{a^2}{4}$

$$(a,b) = (a, \frac{a^2}{4})$$

2. $$\sum_{k=1}^{\frac{n(n+1)}{2}} a_k = \sum_{k=1}^{n} (n+1-k) \cdot k = (n+1) \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$$

$$= (n+1) \cdot \frac{n(n+1)}{2} - \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{n(n+1)}{2} \left(n+1 - \frac{2n+1}{3} \right)$$

$$= \frac{1}{6} n(n+1)(n+2)$$

for $\quad \frac{n(n+1)}{2} < m < \frac{(n+1)(n+2)}{2}$

$$\frac{1}{6} n(n+1)(n+2) < \sum_{k=1}^{m} a_k < \frac{1}{6} (n+1)(n+2)(n+3)$$

$$\lim_{\frac{n(n+1)}{2} \to \infty} \frac{\frac{1}{6} n(n+1)(n+2)}{\left(\frac{n(n+1)}{2}\right)^2} \le I = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} a_k}{n^2} \le \lim_{\frac{n(n+1)(n+2)}{2} \to \infty} \frac{\frac{1}{6}(n+1)(n+2)(n+3)}{\left(\frac{(n+1)(n+2)}{2}\right)^2}$$

$$LHS = RHS = \lim_{n \to \infty} \frac{2^\alpha}{6} \cdot \frac{n^3}{n^{2\alpha}} = \beta$$

if $\alpha = \frac{3}{2}$, $\beta = \frac{\sqrt{2}}{3}$

if $\alpha > \frac{3}{2}$, $\beta = 0$

3.

when $n = 0, 1$, the theorem is clearly true

make the roots not equal to 0

assume when $n = k$, the theorem is true

So $P_n(x) = \pm a_k x^k \pm a_{k-1} x^{k-1} \pm \cdots \pm a_0$ have n distinct real roots

$x P_n(x) = \pm a_k x^{k+1} \pm a_{k-1} x^k \pm \cdots \pm a_0 x$ have $n+1$ distinct real roots

choose λ that $\lambda \neq 0$, $\lambda \in ($ sup (local minimum), inf (local maximum))

then we construct a $P_{n+1}(x)$ that have $n+1$ distinct real roots.

4.

$S(n) = (1 + P_1 + P_1^2 + \cdots + P_1^{e_1})(1 + P_2 + P_2^2 + \cdots + P_2^{e_2}) \cdots (1 + P_k + \cdots P_k^{e_k})$

$= 2n$

if e_1 is odd

then $\qquad p_1 + 1 \mid (\sum_{i=0}^{e_1} p_1^i)$

thus $\qquad p_1 + 1 \mid 2n$, $\quad 2n = 2(p_1 + 1)$

becaus $p_1 + 1 > 2$

one of $p_2 --- p_n$ divides $p_1 + 1$

$p_2 = p_1 + 1$
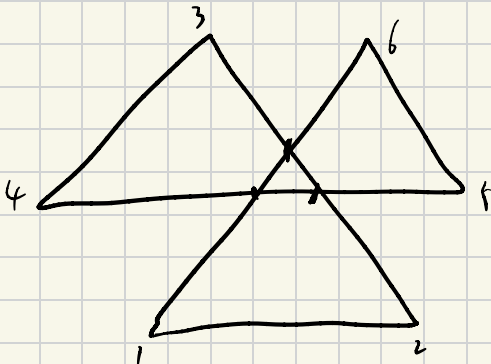
$p_2 = 3$, $p_1 = 2$

n , $\frac{n}{2}$, $\frac{n}{3}$, $\frac{n}{6}$ and 1 are distinct divisors of n

$S(n) \geq n + \frac{n}{2} + \frac{n}{3} + \frac{n}{6} + 1 = 2n + 1 > 2n$
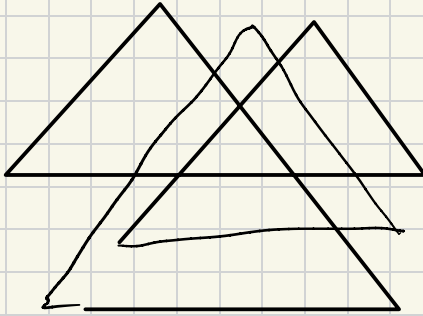
5.

when $n = 2$

the circumstance that fits the theorem is like as follows:

when   n = 3

broke   the   line   1-6   and   continue   the   procedure

adds   in   intersect   point.

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proved.