# Problem Solving Strategies IV

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## 1 Problems to be discussed in lecture

### 1.1 Additive Subgroups of the Real Line and Density

**Problem 1**

Find all continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(x + \sqrt{5}) = f(x + \sqrt{3})$ for all $x$.

**Problem 2**

Prove that that for any point $x_0 \in [-1, 1]$ there is a sub-sequence of the sequence $\{\sin n\}_{n\in\mathbb{N}}$ such that converges to $x_0$. In other words, show that the set $\{\sin n : n \in \mathbb{N}\}$ is dense in $[-1, 1]$.

**Problem 3**

Let $\alpha > 0$ be a irrational number. Show that the set $\{n\alpha - \lfloor n\alpha \rfloor : n \in \mathbb{N}\}$ is dense in $[0, 1]$.

**Problem 4**

Show that infinitely many power of 2 starts with the digit 7.

### 1.2 Trace, Eigenvalues and Characteristic Polynomials

#### 1.2.1 Problem 5

(The Spectral Mapping Theorem) Let $A \in M_n(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) be the eigenvalues of $A$. Suppose $P$ is a polynomial with complex coefficients. Show that the eigenvalues of $P(A)$ are $P(\lambda_1), \ldots, P(\lambda_n)$.

#### 1.2.2 Problem 6

(Cayley-Hamilton Theorem) Let $A \in M_n(\mathbb{C})$ and $P_A(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of $A$. Show that $P_A(A) = 0$.

**Problem 7**

Let $A$ and $B$ be $2 \times 2$ matrices with complex entries with determinant equal to 1. Show that

$\lambda_1 + \frac{1}{\lambda_1}$

$$tr(AB) - tr(A)tr(B) + tr(AB^{-1}) = 0.$$

$\lambda_2 + \frac{1}{\lambda_2}$

$A^2B^2 + I - (A^2+I)(B^2+I) +$

$\lambda_1\lambda_2 + \frac{1}{\lambda_1\lambda_2}$

$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$

### 1.2.3 Problem 8

Let $A$ and $B$ be $3 \times 3$ complex matrices. Show that

$$det(AB - BA) = \frac{tr((AB - BA)^3)}{3}.$$

### 1.2.4 Problem 9

Let $A \in M_n(\mathbb{C})$ with $tr(A^i) = 0$ for all $i \in \mathbb{N}$. Show that $A$ is an nilpotent matrix, that is, there is a $m \in \mathbb{N}$ such that $A^m = 0$.

### 1.2.5 Problem 10

(National Iranian Competition for University Students) Let A be an $n \times n$ matrix with real entries. Prove that

$$\det(A) = \begin{pmatrix} trc(A) & 1 & 0 & \cdots & \cdots & 0 \\ trc(A^2) & trc(A) & 2 & 0 & \cdots & 0 \\ trc(A^3) & trc(A^2) & trc(A) & 3 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ trc(A^{n-1}) & trc(A^{n-2}) & \cdots & \cdots & tr(A) & n-1 \\ trc(A^n) & trc(A^{n-1}) & \cdots & \cdots & tr(A^2) & tr(A) \end{pmatrix}$$

### 1.2.6 Problem 11

(Putnum Competition) Let $A$ be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer $k$, let $A^{[k]}$ be the matrix obtained by raising each entry to the $k$th power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

# References


[1] Arthur Engel, Problem-Solving Strategies, Springer, 1998.

[2] Bamdad R. Yahaghi, Iranian Mathematics Competitions, 1973–2007.

[3] American Mathematical Monthly.

[4] The Putnam Mathematical Competition.

[5] Razvan Gelca and Titu Andreescu, Putnum and beyound, springer.

Proof 1.

$$P_A(x) \doteq \prod (x - \lambda_i)$$

if $A \in D$, $A = Q \Lambda Q^{-1}$

$$P_A(A) = Q P(\Lambda) Q^{-1}$$ wherer $P(\lambda_j) = \prod (\lambda_j - \lambda_i) = 0$

$$= 0$$

if $A \notin D$, because $D$ is dense, there exist $A_n \to A$

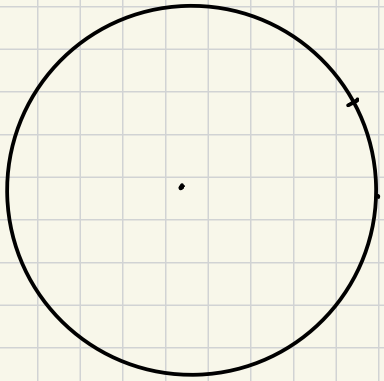
s.t.

$$P_{A_n}(x) \to P_A(x)$$

$$0 = P_{A_n}(A) \to P_A(0)$$

$$P_A(A) = 0$$

2. Proof.



for $1 \le k \le N+1$

divide the circle into $N$ parts

so there are at least two point in same part, suppose they are $m, n$ and $m > n$

when $N \to \infty$

$k(m-n) \equiv$ the circle $(\bmod\ 2\pi)$

8. Proof. $C = AB - BA$

$$P_C(x) = x^3 - \operatorname{trace}(C)\, x^2 + c_0 x - \det C$$

$$P_C(C) = C^3 + c_0 x - \det C = 0$$

$$\operatorname{trace}(C^3) = 3 \det C$$

9 Proof.

$A^n$ have eigenvalues $\lambda_i^n$

thus

$$P_A(x) = \prod_{i=1}^{n} (x - \lambda_i) = x^n - \sum_{i=1}^{n} \lambda_i\, x^{n-1} + \sum_{\substack{i=1, j=1 \\ i \ne j}}^{n} \lambda_i \lambda_j\, x^{n-2} + \cdots$$

$$= x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} \cdots$$

where $\sigma_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{trace}(A) = 0$

$\sigma_1^2 - 2\sigma_2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = \operatorname{trace}(A^2) = 0$

$\vdots$

) ) 2 2 3 3 4

$\{\lambda_1, \cdots, \lambda_n\}$, $\{y_1, \cdots, y_k\}$

$m_1\ y_1$

$y_i \ne y_j$

thus

$p_A(x) = x^n$

[illegible] $m_2 \quad g_2$ [illegible]

$\vdots$

[illegible] $m_k \quad g_k$ [illegible]

so there exist $m \le n$, $A^m = 0$