$A \in M_{m\times n}$ $\quad tr(A\cdot A^T) = \sum_{i,j} a_{ij}^2$

"$A = 0$" $\Leftrightarrow$ "$tr(A\cdot A^T) = 0$"

$A_{m\times n}\cdot B_{n\times s} = 0 \Rightarrow r(A) + r(B) \le n$

Proof.

$A(\beta_1, \cdots \beta_s) = (0, \cdots 0) \Rightarrow A\beta_i = 0$

$\beta_i \in$ span $\{Ax=0\}$ $\quad$ rank $B \le$ rank $\ker A = n -$ rank A

Q.E.D.