

IMC 2023

Second Day, August 3, 2023 Solutions

Problem 6. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

- he chooses a row or a column of the matrix, and
- he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint: Construct an invariant quantity that does not change during Ivan's procedure.

Solution. We show that starting from $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$, Ivan cannot reach the matrix $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$.

Notice first that the allowed operations preserve the positivity of entries; all matrices Ivan can reach have only positive entries.

For every matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with positive entries, let $L(X) = \begin{pmatrix} \log_2 x_{11} & \log_2 x_{12} \\ \log_2 x_{21} & \log_2 x_{22} \end{pmatrix}$. By taking logarithms of the entries, the steps in Ivan game will be replaced by adding or subtracting a row or column to the other row. Such standard row and column operations preserve the determinant.

Hence, if the matrices in the game are $A = X_0, X_1, X_2, \dots$, then we have $\det L(A) = \det L(X_1) = \det L(X_2) = \dots$, and it suffices to verify that $\det L(A) \neq \det L(B)$.

Indeed,

$$\det L(A) = \log_2 2 \cdot \log_2 4 - \log_2 4 \cdot \log_2 3 = \log_2(4/3) > 0$$

and similarly $\det L(B) < 0$, so $\det L(A) \neq \det L(B)$.

Problem 7. Let V be the set of all continuous functions $f: [0, 1] \rightarrow \mathbb{R}$, differentiable on $(0, 1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0, 1)$ such that

$$f(\xi) + \alpha = f'(\xi).$$

(proposed by Mike Daas, Leiden University)

Hint: Find a function $h \in V$ such that $h' - h$ is constant, then apply Rolle's theorem to $f - h$.

Alternatively, you can apply Cauchy's mean value theorem with some auxiliary functions.

Solution 1. First consider the function

$$h(x) = \frac{e^x - 1}{e - 1}, \quad \text{which has the property that } h'(x) = \frac{e^x}{e - 1}.$$

Note that $h \in V$ and that $h'(x) - h(x) = 1/(e - 1)$ is constant. As such, $\alpha = 1/(e - 1)$ is the only possible value that could possibly satisfy the condition from the problem. For $f \in V$ arbitrary, let

$$g(x) = f(x)e^{-x} + h(-x), \quad \text{with } g(0) = 0 \quad \text{and also } g(1) = e^{-1} + \frac{e^{-1} - 1}{e - 1} = 0.$$

We compute that

$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} - h'(-x).$$

Now apply Rolle's Theorem to g on the interval $[0, 1]$; it yields some $\xi \in (0, 1)$ with the property that

$$g'(\xi) = 0 \implies f'(\xi)e^{-\xi} - f(\xi)e^{-\xi} - \frac{e^{-\xi}}{e - 1} = 0 \implies f'(\xi) = f(\xi) + \frac{1}{e - 1},$$

showing that $\alpha = 1/(e - 1)$ indeed satisfies the condition from the problem.

Solution 2. Notice that the expression $f'(x) - f(x)$ appears in the derivative of the function $F(x) = f(x) \cdot e^{-x}$: $F'(x) = (f'(x) - f(x))e^{-x}$.

Apply Cauchy's mean value theorem to $F(x)$ and the function $G(x) = -e^{-x}$. By the theorem, there is some $\xi \in (0, 1)$ such that

$$\begin{aligned} \frac{F'(\xi)}{G'(\xi)} &= \frac{F(1) - F(0)}{G(1) - G(0)} \\ f'(\xi) - f(\xi) &= \frac{e^{-1} - 0}{-e^{-1} + 1} = \frac{1}{e - 1}. \end{aligned}$$

This proves the required property for $a = \frac{1}{e - 1}$.

Now we show that no other α is possible. Choose f and F in such a way that $\frac{F'(x)}{G'(x)} = f'(x) - f(x) = \frac{1}{e - 1}$ is constant. That means

$$\begin{aligned} F'(x) &= \frac{G'(x)}{e - 1} = \frac{e^{-x}}{e - 1}, \\ F(x) &= \frac{1 - e^{-x}}{e - 1}, \\ f(x) &= F(x) \cdot e^x = \frac{e^x - 1}{e - 1}. \end{aligned}$$

With this choice we have $f(0) = 0$ and $f(1) = 1$, so $f' \in V$, and $f'(x) - f(x) \equiv \frac{1}{e - 1}$ for all x , so for this function the only possible value for α is $\frac{1}{e - 1}$.

Problem 8. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u, v)$ denote the distance between u and v , that is, the number of edges in the shortest path in T that connects u with v .

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} d(u, v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} \frac{1}{d(u, v)}.$$

Prove that

$$W(T) \cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

(proposed by Slobodan Filipovski, University of Primorska, Koper)

Hint: There are $n-1$ pairs u, v with $d(u, v) = 1$; in all other cases $d(u, v) \geq 2$.

Solution. Let $k = \binom{n}{2}$ and let $x_1 \leq x_2 \leq \dots \leq x_k$ be the distances between the pairs of vertices in the tree T . Thus

$$W(T) \cdot H(T) = (x_1 + x_2 + \dots + x_k) \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} \right).$$

Since the tree has exactly $n-1$ edges, there are exactly $n-1$ pairs of vertices at distance one, that is, $x_1 = x_2 = \dots = x_{n-1} = 1$. Thus

$$\begin{aligned} W(T) \cdot H(T) &= (n-1 + x_n + x_{n+1} + \dots + x_k) \cdot \left(n-1 + \frac{1}{x_n} + \frac{1}{x_{n+1}} + \dots + \frac{1}{x_k} \right) = \\ &= (n-1)^2 + (n-1) \left(\left(x_n + \frac{1}{x_n} \right) + \dots + \left(x_k + \frac{1}{x_k} \right) \right) + \\ &\quad + (x_n + \dots + x_k) \left(\frac{1}{x_n} + \dots + \frac{1}{x_k} \right). \end{aligned}$$

From Cauchy inequality we have

$$(x_n + \dots + x_k) \left(\frac{1}{x_n} + \dots + \frac{1}{x_k} \right) \geq (1+1+\dots+1)^2 = (k-n+1)^2 = \frac{(n-1)^2(n-2)^2}{4}.$$

The equality holds if and only if $x_n = x_{n+1} = \dots = x_k$.

Now we minimize the expression $\left(x_n + \frac{1}{x_n} \right) + \dots + \left(x_k + \frac{1}{x_k} \right)$, where $x_i \in [2, n-1]$.

It is clear that the minimal value is achieved for $x_n = x_{n+1} = \dots = x_k = 2$. Therefore we get

$$W(T) \cdot H(T) \geq (n-1)^2 + (n-1) \left(\left(2 + \frac{1}{2} \right) (k-n+1) \right) + \frac{(n-1)^2(n-2)^2}{4} = \frac{(n-1)^3(n+2)}{4}.$$

The equality holds for $x_1 = \dots = x_{n-1} = 1$ and $x_n = x_{n+1} = \dots = x_k = 2$, that is, the smallest value is achieved for the tree where $n-1$ pairs are at distance one, and the remaining $k-(n-1) = \frac{(n-1)(n-2)}{2}$ pairs are at distance two. The unique tree which satisfies these conditions is the star graph S_n . In this case it holds

$$W(S_n) \cdot H(S_n) = (n-1)^2 \cdot \frac{(n-1)(n+2)}{4} = \frac{(n-1)^3(n+2)}{4}.$$

Problem 9. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in \mathbb{R}^3 , with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint.

Find $\sup\{V \mid V \text{ is good}\}$.

(proposed by Josef Tkadlec and Arseniy Akopyan)

Hint: The two bodies can be replaced by a pair symmetric to the midpoint of the cube.

Solution. We prove that $\sup\{V \mid V \text{ is good}\} = 1/4$.

We will use the unit cube $U = [-1/2, 1/2]^3$.

For $\varepsilon \rightarrow 0$, the axis-parallel boxes $X = [-1/2, -\varepsilon] \times [-1/2, -\varepsilon] \times [-1/2, 1/2]$ and $Y = [\varepsilon, 1/2] \times [\varepsilon, 1/2] \times [-1/2, 1/2]$ show that $\sup\{V\} \geq 1/4$.

To prove the other bound, consider two admissible convex bodies X, Y . For any point $P = [x, y, z] \in U$ with $xyz \neq 0$, let $\bar{P} = \{\pm x, \pm y, \pm z\}$ be the set consisting of 8 points (the original P and its 7 “symmetric” points). If for each such P we have $|\bar{P} \cap (X \cup Y)| \leq 4$, then the conclusion follows by integrating. Suppose otherwise and let P be a point with $|\bar{P} \cap (X \cup Y)| \geq 5$. Below we will complete the proof by arguing that:

- (1) we can replace one of the two bodies (the “thick” one) with the reflection of the other body about the origin, and
- (2) for such symmetric pairs of bodies we in fact have $|\bar{P} \cap (X \cup Y)| \leq 4$, for all P .

To prove Claim (1), we say that a convex body is *thick* if each of its three projections contains the origin. We claim that one of the two bodies X, Y is thick. This is a short casework on the 8 points of \bar{P} . Since $|\bar{P} \cap (X \cup Y)| \geq 5$, by pigeonhole principle, we find a pair of points in $\bar{P} \cap (X \cup Y)$ symmetric about the origin. If both points belong to one body (say to X), then by convexity of X the origin belongs to X , thus X is thick. Otherwise, label \bar{P} as $ABCDA'B'C'D'$. Wlog $A \in X, C' \in Y$ is the pair of points in \bar{P} symmetric about the origin. Wlog at least 3 points of \bar{P} belong to X . Since X, Y have disjoint projections, we have $C, B', D' \notin X$, so wlog $B, D \in X$. Then Y can contain no other point of \bar{P} (apart from C'), so X must contain at least 4 points of \bar{P} and thus $A' \in X$. But then each projection of X contains the origin, so X is indeed thick.

Note that if X is thick then none of the three projections of Y contains the origin. Consider the reflection $Y' = -Y$ of Y about the origin. Then (Y, Y') is an admissible pair with the same volume as (X, Y) : the two bodies Y and Y' clearly have equal volumes V and they have disjoint projections (by convexity, since the projections of Y miss the origin). This proves Claim (1).

Claim (2) follows from a similar small casework on the 8-tuple \bar{P} : For contradiction, suppose $|\bar{P} \cap Y'| = |\bar{P} \cap Y| \geq 3$. Wlog $A \in Y'$. Then $C' \in Y$, so $C, B', D' \notin Y'$, so wlog $B, D \in Y'$. Then $B', D' \in Y$, a contradiction with (Y, Y') being admissible.

Remark. There are more examples with $V \rightarrow 1/4$, e.g. X a union of two triangular pyramids with base ACD' – one with apex D , one with apex at the origin (and Y symmetric with X about the origin).

Remark. The word “convex” matters. E.g., in a $3 \times 3 \times 3$ cube, one can set X to be a $2 \times 2 \times 2$ sub-cube, and Y to be the (non-convex) 3D L-shape consisting of 7 unit cubes. This shows that without convexity we have $V \geq 7/27 > 1/4$.

Problem 10. For every positive integer n , let $f(n), g(n)$ be the minimal positive integers such that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} = \frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n) > n^{0.999n}$.

(proposed by Fedor Petrov, St. Petersburg State University)

Solution. We show that there does exist such a number n .

Let $\varepsilon = 10^{-10}$. Call a prime p *special*, if for certain $k \in \{1, 2, \dots, p-1\}$ there exist at least $\varepsilon \cdot k$ positive integers $j \leq k$ for which p divides $f(j)$.

Lemma. There exist only finitely many special primes.

Proof. Let p be a special prime number, and p divides $f(j)$ for at least $\varepsilon \cdot k$ values of $j \in \{1, 2, \dots, k\}$. Note that if p divides $f(j)$ and $f(j+r)$, then p divides

$$(j+r)! \left(\frac{f(j+r)}{g(j+r)} - \frac{f(j)}{g(j)} \right) = 1 + (j+r) + (j+r)(j+r-1) + \dots + (j+r) \dots (j+2)$$

that is a polynomial of degree $r-1$ with respect to j . Thus, for fixed j it equals to 0 modulo p for at most $r-1$ values of j . Look at our $\geq \varepsilon \cdot k$ values of $j \in \{1, 2, \dots, k\}$ and consider the gaps between consecutive j 's. The number of such gaps which are greater than $2/\varepsilon$ does not exceed $\varepsilon \cdot k/2$ (since the total sum of gaps is less than k). Therefore, at least $\varepsilon \cdot k/2 - 1$ gaps are at most $2/\varepsilon$. But the number of such small gaps is bounded from above by a constant (not depending on k) by the above observation. Therefore, k is bounded, and, since p divides $f(1)f(2)\dots f(k)$, p is bounded too.

Now we want to bound the product $g(1)g(2)\dots g(n)$ (for a large integer n) from below. Let $p \leq n$ be a non-special prime. Our nearest goal is to prove that

$$\nu_p(g(1)g(2)\dots g(n)) \geq (1 - \varepsilon)\nu_p(1! \cdot 2! \cdot \dots \cdot n!) \quad (1)$$

Partition the numbers $p, p+1, \dots, n$ onto the intervals of length p (except possibly the last interval which may be shorter): $\{p, p+1, \dots, 2p-1\}, \dots, \{p[n/p], \dots, n\}$. Note that in every interval $\Delta = [a \cdot p, a \cdot p+k]$, all factorials $x!$ with $x \in \Delta$ have the same p -adic valuation, denote it $T = \nu_p((ap)!)$. We claim that at least $(1 - \varepsilon)(k+1)$ valuations of $g(x)$, $x \in \Delta$, are equal to the same number T . Indeed, if $j = 0$ or $1 \leq j \leq k$ and $f(j)$ is not divisible by p , then

$$\frac{1}{(ap)!} + \frac{1}{(ap+1)!} + \dots + \frac{1}{(ap+j)!} = \frac{1}{(ap)!} \cdot \frac{A}{B}$$

where $A \equiv f(j) \pmod{p}$, $B \equiv g(j) \pmod{p}$, so, this sum has the same p -adic valuation as $1/(ap)!$, which is strictly less than that of the sum $\sum_{i=0}^{ap-1} 1/i!$, that yields $\nu_p(g(ap+j)) = \nu_p((ap)!)$. Using this for every segment Δ , we get (1).

Now, using (1) for all non-special primes, we get

$$A \cdot g(1)g(2)\dots g(n) \geq (1! \cdot 2! \cdot \dots \cdot n!)^{1-\varepsilon},$$

where $A = \prod_{p,k} p^{\nu_p(g(k))}$, p runs over non-special primes, k from 1 to n . Since $\nu_p(g(k)) \leq \nu_p(k!) = \sum_{i=1}^{\infty} \lfloor k/p^i \rfloor \leq k$, we get

$$A \leq \left(\prod_p p \right)^{1+2+\dots+n} \leq C^{n^2}$$

for some constant C . But if we had $g(n) \leq n^{0.999n} \leq e^n n^{0.999}$ for all n , then

$$\log(A \cdot g(1)g(2)\dots g(n)) \leq O(n^2) + 0.999 \log(1! \cdot 2! \cdot \dots \cdot n!) < (1 - \varepsilon) \log(1! \cdot 2! \cdot \dots \cdot n!)$$

for large n , a contradiction.