对已知函数有以下性质

在 $x_1 < x_2 < x_3 \cdots < x_n$

$f(x_1) = a_{1,1}$ $\quad\cdots\cdots\quad$ $f(x_n) = a_{1,n}$

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

$f^{(b_1)}(x_1) = a_{b_1,1}$ $\qquad\qquad$ $f^{(b_n)}(x_n) = a_{b_n,n}$

可构建 $F(x) = f(x) - \sum_{i=0}^{k} a_i x^i \quad (k = \sum_{i=1}^{n} b_i)$

e.g.

1. $f(x) \in C[0,1] \quad f(x) \in D^2(0,1)$

$f(0) = 1, \quad f(\frac{1}{2}) = 2, \quad f(1) = 2$

证：$\exists \xi \in (0,1)$ s.t. $f''(\xi) = -4$

Proof. 构造 $F(x)$

$$F(x) = \begin{vmatrix} f(x) & x^2 & x & 1 \\ f(0) & 0 & 0 & 1 \\ f(\frac{1}{2}) & \frac{1}{4} & \frac{1}{2} & 1 \\ f(1) & 1 & 1 & 1 \end{vmatrix}$$

$F(0) = F(\frac{1}{2}) = F(1) = 0$

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$\exists x_1 \in (0, \frac{1}{2}), x_2 \in (\frac{1}{2}, 1)$ s.t. $f'(x_1) = f'(x_2) = 0$

$$F''(x) = \begin{vmatrix} f''(x) & 2 & 0 & 0 \\ f(0) & 0 & 0 & 1 \\ f(\frac{1}{2}) & \frac{1}{4} & \frac{1}{2} & 1 \\ f(1) & 1 & 1 & 1 \end{vmatrix} = -\frac{1}{4}f''(x) + 1$$

Q.E.D.

2. $f(x) \in D^3[0,1]$, $f(0) = -1$ $f'(0) = 0$ $f(1) = 0$

证 $\exists\ \xi \in (0,1)$ s.t. $f(x) = -1 + x^2 + \frac{x^2(x-1)}{6} f'''(\xi)$

Proof.

$$F(x) = \begin{vmatrix} f(x) & x^3 & x^2 & x & 1 \\ f(0) & 0 & 0 & 0 & 1 \\ f'(0) & 0 & 0 & 1 & 0 \\ f(1) & 1 & 1 & 1 & 1 \\ f(t) & t^3 & t^2 & t & 1 \end{vmatrix}$$

$$F(0) = F(1) = F(t) = 0$$

$$\exists\ x_1,\ x_2,\quad F'(x_1) = F'(x_2) = 0$$

$$\exists\ y_1,\ y_2,\quad F''(y_1) = F''(y_2) = 0$$

$$\exists\ \xi,\quad F'''(\xi) = 0$$

$$F'''(x) = f'''(x)\, t^2(t-1) - 6\left(f(t) + 1 - t^2\right)$$

$$0 = F'''(\xi) = f'''(\xi)\, t^2(t-1) - 6\left(f(t) + 1 - t^2\right)$$

Q.E.D.