IMC 2014, Blagoevgrad, Bulgaria

Day 1, July 31, 2014

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\mathrm{trace}(M) = a$ and $\det(M) = b$.

(10 points)

Problem 2. Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots).$$

Find all pairs (α, β) of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{n} a_k}{n^\alpha} = \beta$.

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Problem 3. Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \dots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_1 x \pm a_0$$

has n distinct real roots.

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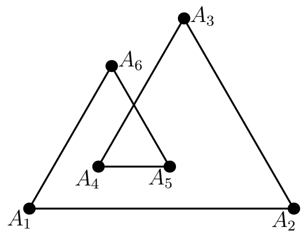
Problem 4. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorization with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number. 奇

A number n is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

(10 points)

$$2n = (1+p_1+p_1^2+\cdots+p_1^{e_1})(1+p_2+\cdots+p_2^{e_2})\cdots(1+\cdots+p_k^{e_k})$$

Problem 5. Let $A_1A_2\ldots A_{3n}$ be a closed broken line consisting of $3n$ line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \dots, 3n$, the triangle $A_iA_{i+1}A_{i+2}$ has counterclockwise orientation and $\angle A_iA_{i+1}A_{i+2} = 60°$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$.



(10 points)

1. suppose $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$ is such a unique symmetry matrix

then $a = a_1 + a_3$ and $b = a_1 a_3 - a_2^2$

where a_1 and a_3 can change their values and make another new matrix, making M not unique

so $a_1 = a_3$

now $a = 2a_1$ $b = a_1^2 - a_2^2$

so $b = \frac{a^2}{4} - a_2^2 \leq \frac{a^2}{4}$

so all the pairs are $b \leq \frac{a^2}{4}$

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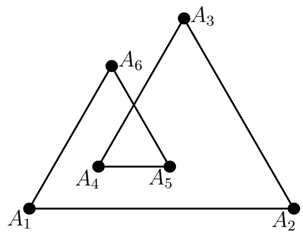

(10 points)

$$\sum_{k=1}^{\frac{n(n+1)}{2}} a_k = n\cdot 1 + (n-1)\cdot 2 + (n-2)\cdot 3 + \cdots + 1\cdot n$$

$$= \sum_{i=1}^{n} (n+1-i)\cdot i$$

$$= \sum_{i=1}^{n} i\,(n+1) - \sum_{i=1}^{n} i^2$$

$$= (n+1)\cdot\frac{n(n+1)}{2} - \frac{1}{6} n\,(2n+1)(n+1)$$

$$= n(n+1)\,\frac{n+2}{6}$$

so $$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_k}{n^{\alpha}} = \lim_{\frac{n(n+1)}{2}\to\infty} \frac{\sum_{i=1}^{\frac{n(n+1)}{2}} a_k}{\left[\frac{n(n+1)}{2}\right]^{\alpha}} = \lim_{\frac{n(n+1)}{2}\to\infty} \frac{n(n+1)\frac{n+2}{6}}{\left[\frac{n(n+1)}{2}\right]^{\alpha}}$$

when $\alpha = \frac{3}{2}$

$LHS = 3\sqrt{2}$

$\alpha > \frac{3}{2}$

$LHS = 0$

4. the sum of the divisor of $n = (1+p_1+p_1^2+\cdots+p_1^{e_1})\cdot(1+p_2+p_2^2+\cdots+p_2^{e_2})\cdots$

$$= \prod_{i=1}^{k}\sum_{j=0}^{e_i} p_i^j$$

so $\prod_{i=1}^{k}\sum_{j=0}^{e_i} p_i^j = 2n$

suppose n is even

then $p_1 = 2$

$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_n^{e_n}$

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$(x-x_1)(x-x_2) \cdots (x-x_3)$

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$a_1x \pm a_0 = 0$

has n distinct real roots.

(10 points)

$n + \frac{n}{2} + 2$

Problem 4. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorization with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number. odd

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$2n = \underline{(1 + p_1 + p_1^2 + \cdots + p_1^{e_1})}(1 + p_2 + \cdots + p_2^{e_2}) \cdots (1 + \cdots + p_k^{e_k})$

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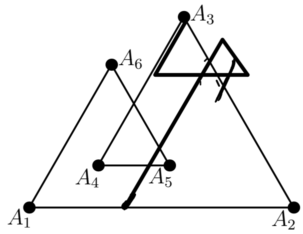
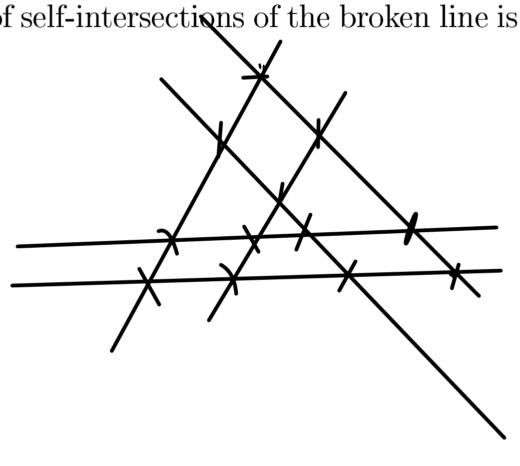


(10 points)

$\frac{3n-3}{2}$

$\frac{3}{2} - 2 + 1$



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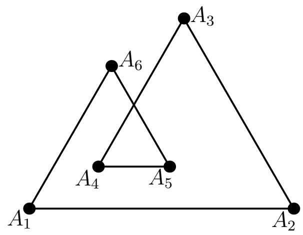


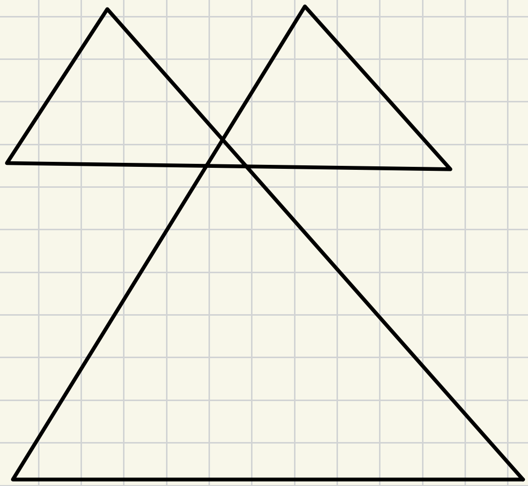
(10 points)

by induction

when $n=1,2$, the assumption is true

suppose when $n=k$, the assumption is true

when $n=k+1$



$n=2$

let the new broken line goes like k, and let the last three lines cross $3n$ line when k is even; $3n-1$ line when k is odd.

$$\frac{3}{2}(n+1)^2 - 2(n+1)+1 - \left(\frac{3}{2}n^2 - 2n + 1\right)$$

$= 3n + \frac{1}{2}$, which follows as above.