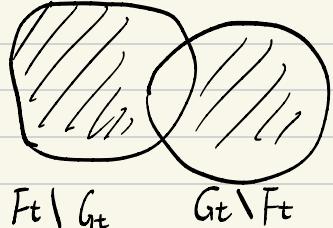



1. X, Y 独立同分布, 证 $E(|X-Y|) \leq E(|X+Y|)$

Proof.

$$\text{证 } E(|X-Y|) = \int_{-\infty}^{\infty} P[(F_t \setminus G_t) \cup (G_t \setminus F_t)] dt$$

$$\text{where } F_t = [X > t] \quad G_t = [Y > t]$$



$$\begin{aligned} \text{RHS} &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \mathbb{1}_{t \in [\min(X, Y), \max(X, Y)]} dP dt \\ &= \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \mathbb{1}_{t \in [\min(X, Y), \max(X, Y)]} dt dP \\ &= \int_{\mathbb{R}^2} [\max(X, Y) - \min(X, Y)] dP \end{aligned}$$

$$\text{where } \max(X, Y) = \frac{x+y+|x-y|}{2} \quad \min(X, Y) = \frac{x+y-|x-y|}{2}$$

$$\text{RHS} = \int_{\mathbb{R}^2} |X-Y| dP = E(|X-Y|)$$

$$\text{则 } E(|X-Y|) = \int_{-\infty}^{\infty} P(F_t) P(F_{-t}) + [1 - P(F_t)][1 - P(F_{-t})] dt$$

$$\begin{aligned} E(|X+Y|) &= 2 \int_{-\infty}^{\infty} P(F_t) - [P(F_t)]^2 dt \\ &= 2 \int_{-\infty}^{\infty} P(F_{-t}) - [P(F_{-t})]^2 dt \end{aligned}$$

$$E(|X-Y|) - E(|X+Y|)$$

$$= \int_{-\infty}^{\infty} - [P(F_t) + P(F_{-t}) - 1]^2 dt \leq 0$$

Q. E. D.