IMC 2022

Second Day, August 4, 2022

Problem 5. We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

(10 points)

Problem 6. Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, ..., x_{p-1})$ of the numbers $(1, 2, ..., p-1)$ such that

$$x_1x_2 + x_2x_3 + ... + x_{p-2}x_{p-1} \equiv 2 \pmod{p}.$$

(10 points)

Problem 7. Let $A_1, A_2, \ldots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_iA_j = -A_jA_i \quad \text{for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

(A matrix A is called idempotent if $A^2 = A$.)

(10 points)

Problem 8. Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

(10 points)