Monoid

1.1

Def: Monoid 幺半群

$(S, *)$ is a monoid $\Leftrightarrow$ this binary operation satisfies:

(1) law of association

$\forall x, y, z \in S$, $x*(y*z) = (x*y)*z$

(2) have Identity

$\exists e \in S$, $\forall x \in S$, $e*x = x*e = x$

In addition:

$(S, *)$ is commutative monoid $\Leftrightarrow$ this operation satisfies:

law of commutation:

$\forall x, y \in S$, $x*y = y*x$

Prop. Identity of a monoid is unique.

Proof. assume e_1 e_2 are Identities of monoid $(S, \cdot)$

then $e_1 = e_1 \cdot e_2 = e_2$

Def. $(T, \cdot)$ is a submonoid of $(S, \cdot)$ $\Leftrightarrow$

(1) T is closed under this operation

(2) $e \in T$

Def. $f: S \to T$ is a map, $(S, \cdot)(T, *)$ are monoids

f is a monoid homomorphism $\Leftrightarrow$

$\forall\ x, y \in S,\ f(x \cdot y) = f(x) * f(y)$ &

$f(e) = e'$

where e and e' are Identities of S, T

Example: $f: (\mathbb{N}, +) \to (S, \cdot)$ where $(S, \cdot)$ is monoid

suppose $f(n) = x^n$, $f(0) = x^0 = e$

then $f(m+n) = x^{m+n} = x^m \cdot x^n = f(m) \cdot f(n)$

Def. $(S, \cdot)$ is a monoid, $A \subset S$

$\langle A \rangle = \bigcap \{ T \subset S : T \supset A,\ T \text{ is monoid} \}$

Prop. $(S,\cdot)$ is a monoid, $A \subset S$, $\langle A \rangle$ is a monoid
therefore it's also the smallest submonoid that contain A

Proof.
(1) for all T, $e \in T \Rightarrow e \in \langle A \rangle$
(2) for any $x, y \in \langle A \rangle$. $x, y \in T$, $x \cdot y \in T$
thus $x \cdot y \in \langle A \rangle$

Def. suppose $(S, \cdot)$ $(T, *)$ are monoids and $f: S \mapsto T$ a map

f is monoid isomorphism $\Leftrightarrow$

(1) f is bijective
(2) f is monoid homomorphism

ProP. $f: (S, \cdot) \to (T, *)$ is a monoid isomorphism $\Leftrightarrow$
$f^{-1}: (T, *) \to (S, \cdot)$ is a monoid isomorphism