23

$A^2 = B^2 = C^2$, $B^3 = ABC + 2I$

$A(AB - BC) = 2I$

$-(AB - BC)C = 2I$

A, C is the inverse of $\frac{1}{2}(AB - BC)$

so $A = -C$

$(AB - BC)(AB + BC)$

$= \cancel{(AB)^2} - \cancel{(BC)^2} + AB^2C - BCAB$

$= -AB^2A + BA^2B$

$= 0$

$AB - BC \neq 0$, $AB + BC = 0$

$AB = BA$

so $B^3 = I$

$A^6 = B^6 = I$

23

$$f(7x+1) = 49f(x)$$
$$f''(7x+1) = f''(x),$$

$$\lambda = 7\lambda+1$$
$$\lambda = -\frac{1}{6}$$

$$f''(x-\tfrac{1}{6}) = f''(7x-\tfrac{1}{6})$$

$$f''(x) = C_1$$

$$f(x) = \tfrac{1}{2}C_1x^2 + C_2x + C_3$$

$$49(\tfrac{1}{2}C_1x^2 + C_2x + C_3) = \tfrac{1}{2}C_1(49x^2+14x+1) + C_2(7x+1) + C_3$$

$$\begin{cases} 49C_2 = 7C_1 + 7C_2 \\ 49C_3 = \tfrac{1}{2}C_1 + C_2 + C_3 \end{cases}$$

$$\Rightarrow \begin{cases} C_1 = 6C_2 \\ C_3 = \tfrac{1}{12}C_2 \end{cases}$$

$$f(x) = 3C_2x^2 + C_2x + \tfrac{1}{12}C_2$$
$$= \tfrac{1}{12}(36C_2x^2 + 12C_2x + C_2)$$
$$= C(6x+1)^2$$

23

$$P(x,y)\,P(z,t) = P(xz - yt,\ xt + yz)$$

$$P(x,y) = (x+iy)^n (x-iy)^m$$

$$= \left(\sqrt{x^2+y^2}\right)^n e^{i\varphi n} \left(\sqrt{x^2+y^2}\right)^m e^{i\theta m}$$

where $\tan\varphi = \frac{y}{x}$ $\quad \tan\theta = -\frac{y}{x}$

suppose $\varphi = -\theta$

so $P(x,y) = (x^2+y^2)^{\frac{n+m}{2}} e^{i\varphi(n-m)}$

because $P(x,y)$ is a polynomial with real coefficients

so $n-m=0$

so $P(x,y) = (x^2+y^2)^k$

23

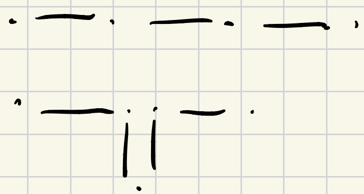
$$a_i = i^k + i$$

$$a^{p-1} \equiv 1 \pmod p$$

$$\prod_{i=1}^{p-1} (i^k + i) \equiv (p-1)! \pmod p$$

$$\prod_{i=1}^{p-1} (i^{k-1} + 1) \equiv 1$$

23



if a tree with n vertices has only one branch with 1 length.

then $\sum d$

21

$$XA^2 + AXA^2 = 0$$

$$A^2X + A^2XA^2 = 0$$

$$A^2X = 0$$

$$X + AX = A$$

$$AX = A^2$$

$$X = A - A^2$$

21

total number of conditions $= \binom{k+a}{k} \cdot \binom{k+n+a}{n}$

number of $\min(Y) > \max(X) = \binom{k+n+a}{n+k}$

$$P(\min(Y) > \max(X)) = \frac{\binom{k+n+a}{n+k}}{\binom{k+a}{k} \cdot \binom{k+n+a}{n}}$$

$$= \frac{k!\,\cancel{a!}\,\cancel{(k+a)!}\,n!}{\cancel{(k+a)!}\,\cancel{(k+n+a)!}} \cdot \frac{\cancel{(k+n+a)!}}{\cancel{a!}\,(k+n)!}$$

$$= \frac{1}{\binom{k+n}{k}}$$

21

$$\frac{1+n}{n}$$

$$a_0 = 0$$

$$a_{n+1} = d$$

20 $\quad \operatorname{rank}(AB-BA+I) = 1$

so $\dim[\ker(AB-BA)] = n-1$

suppose $X = AB-BA$

so $\lambda_1 = -1, \quad \mu_X(-1) = n-1$

$\lambda_2 = n-1, \quad \mu_X(n-1) = 1$

so $$P^{-1}XP = D = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & n-1 \end{pmatrix}$$

so $$P^{-1}X^2P = D^2 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & (n-1)^2 \end{pmatrix}$$

$$\operatorname{tr}(P^{-1}X^2P) = \operatorname{tr}(D^2) = n(n-1)$$

where $\text{LHS} = \operatorname{tr}(X^2)$

$= \operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2)$

19.1.5

$$(A^2-4B^2)(A^4+4A^2B^2+16B^4) = 2019(A^2-4B^2)$$

$$A^6 - 64B^6 = 2019(A^2-4B^2)$$

$$A^6 - 2019A^2 = (4B^2)^3 - 2019\cdot(4B^2)$$

so $A^2 = 4B^2$?

then $48B^4 = 2019$

contradiction