Tensor of matrix (Kronecker product)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Rightarrow A \otimes B = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

Dimensionally reduction tensor product

denote $(A \otimes B)(C \otimes D) = EF$ by

$$\begin{bmatrix} A^1{}_1 B^1{}_1 & A^1{}_1 B^1{}_2 & A^1{}_2 B^1{}_1 & A^1{}_2 B^1{}_2 \\ A^1{}_1 B^2{}_1 & A^1{}_1 B^2{}_2 & A^1{}_2 B^2{}_1 & A^1{}_2 B^2{}_2 \\ A^2{}_1 B^1{}_1 & A^2{}_1 B^1{}_2 & A^2{}_2 B^1{}_1 & A^2{}_2 B^1{}_2 \\ A^2{}_1 B^2{}_1 & A^2{}_1 B^2{}_2 & A^2{}_2 B^2{}_1 & A^2{}_2 B^2{}_2 \end{bmatrix} \cdot \begin{bmatrix} \text{censored} \end{bmatrix} = EF$$

$$\begin{aligned} (EF)^2{}_3 &= A^1{}_1 B^2{}_1 C^1{}_2 D^1{}_1 + A^1{}_1 B^2{}_2 C^1{}_2 D^2{}_1 + \\ &\quad A^1{}_2 B^2{}_1 C^2{}_2 D^1{}_1 + A^1{}_2 B^2{}_2 C^2{}_2 D^2{}_1 \\ &= A^1{}_\rho B^2{}_\sigma C^\rho{}_2 D^\sigma{}_1 \end{aligned}$$

Generalize

$$(EF)^2{}_3 \rightarrow (EF)^{12}{}_{21} = A^1{}_\rho B^2{}_\sigma C^\rho{}_2 D^\sigma{}_1$$

$$(EF)^{\mu\nu}{}_{\rho\sigma} \equiv A^\mu{}_\tau B^\nu{}_\eta C^\tau{}_\rho D^\eta{}_\sigma$$

$$\begin{aligned} [(A \otimes B)(C \otimes D)]^{\mu\nu}{}_{\rho\sigma} \equiv (EF)^{\mu\nu}{}_{\rho\sigma} &\equiv A^\mu{}_\tau B^\nu{}_\eta C^\tau{}_\rho D^\eta{}_\sigma \\ &= E^{\mu\nu}{}_{\tau\eta} F^{\tau\eta}{}_{\rho\sigma} \end{aligned}$$

$$(A \otimes B)^{\mu\nu}{}_{\tau\eta} = A^\mu{}_\tau B^\nu{}_\eta$$

Prop. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Proof:

$$(AC) \otimes (BD) \rightarrow T^{\mu\nu}{}_{\rho\sigma}$$
$$(A \otimes B) \rightarrow E^{\mu\nu}{}_{\rho\sigma}$$
$$(C \otimes D) \rightarrow F^{\mu\nu}{}_{\rho\sigma}$$
$$(A \otimes B)(C \otimes D) \rightarrow M^{\mu\nu}{}_{\rho\sigma}$$

$$\begin{aligned}
(A \otimes B)(C \otimes D) \rightarrow & [(A \otimes B)(C \otimes D)]^{\mu\nu}{}_{\rho\sigma} \\
&\equiv (A \otimes B)^{\mu\nu}{}_{\tau\eta} (C \otimes D)^{\tau\eta}{}_{\rho\sigma} \\
&\equiv A^{\mu}{}_{\tau} B^{\nu}{}_{\eta} C^{\tau}{}_{\rho} D^{\eta}{}_{\sigma} \\
&= A^{\mu}{}_{\tau} C^{\tau}{}_{\rho} B^{\nu}{}_{\eta} D^{\eta}{}_{\sigma} \\
&= (AC)^{\mu}{}_{\rho} (BD)^{\nu}{}_{\sigma} \\
&= [(AC) \otimes (BD)]^{\mu\nu}{}_{\rho\sigma} \rightarrow (AC) \otimes (BD)
\end{aligned}$$