Cauchy-Schwarz 不等式

数列

$$\sum a_n^2 \sum b_n^2 \geq \left(\sum a_n b_n\right)^2$$

概率论

$$E(X^2)E(Y^2) \geq [E(XY)]^2$$

Hölder 不等式

$\forall p > 1$，$\frac{1}{p}+\frac{1}{q}=1$．$f \in L^p(\Omega)$ $g \in L^q(\Omega)$

有
$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx\right)^{\frac{1}{q}}$$

Carlson (卡尔松) 不等式

$\forall (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$：

$$\left(\sum_{j=1}^{n}\left(\prod_{i=1}^{m}|a_{ij}|\right)\right)^m \leq \prod_{i=1}^{m}\left(\sum_{j=1}^{n}|a_{ij}|^m\right)$$

等价于

$\forall A, B \in \mathbb{R}^{m\times n}$

$$\det(A\cdot A^T)\cdot\det(B\cdot B^T) \geq (\det(AB^T))^2$$

解题思路：

若 $f(x) \in D$　$f(a)=0$　可转化为以下信息：

(Ⅰ) $f(x) = \int_a^x f'(x)\,dx$

(Ⅱ) $\int_a^b f^k(x)\,dg(x) = f^k(b)g(b) - \int_a^b k f^{k-1}(x) f'(x) g(x)\,dx$　$(g(b)=0)$

(Ⅲ) $f(x) = f'(\xi)(x-a)$

ex 1. $f(x) \in D[0,1]$ $f(0)=0$,

证明: $\int_0^1 f^2(x)\,dx \le \frac{1}{2}\int_0^1 f'^2(x)\,dx$

proof:

$$f^2(x) = \left(\int_0^x f'(x)\,dx\right)^2 \le \int_0^x 1^2\,dx \cdot \int_0^x f'^2(x)\,dx$$

$$\le x\int_0^1 f'^2(x)\,dx$$

$$\int_0^1 f^2(x)\,dx \le \left(\int_0^1 f'^2(x)\,dx\right)\cdot\int_0^1 x\,dx = \frac{1}{2}\int_0^1 f'^2(x)\,dx$$

Q.E.D.

ex.2. $f(x) \in D[0,1]$ $f(0)=f(1)=0$

证明: $\int_0^1 f^2(x)\,dx \le \frac{1}{8}\int_0^1 f'^2(x)\,dx$

Proof.

$x\in[0,\frac{1}{2}]$ $$f^2(x) = \left(\int_0^x f'(x)\,dx\right)^2 \le \int_0^x 1^2\,dx \cdot \int_0^{\frac{1}{2}} f'^2(x)\,dx$$

$$= x\int_0^{\frac{1}{2}} f'^2(x)\,dx$$

$x\in[\frac{1}{2},1]$ $$f^2(x) = \left(\int_x^1 f'(x)\,dx\right)^2 \le \int_x^1 1^2\,dx \cdot \int_{\frac{1}{2}}^1 f'^2(x)\,dx$$

$$= (1-x)\int_{\frac{1}{2}}^1 f'^2(x)\,dx$$

$$\int_0^1 f^2(x)\,dx \le \int_0^{\frac{1}{2}} x\,dx \cdot \int_0^{\frac{1}{2}} f'^2(x)\,dx + \int_{\frac{1}{2}}^1 (1-x)\,dx \cdot \int_{\frac{1}{2}}^1 f'^2(x)\,dx$$

$$= \frac{1}{8}\int_0^1 f'^2(x)\,dx$$

Q.E.D.

ex 3. $f(x) \in D[0,1]$ $f(0)=0$

证明: $\int_0^1 f^2(x)\,dx \leq 4\int_0^1 (1-x)^2 f'^2(x)\,dx$

Proof.

$$\int_0^1 f^2(x)\,dx = \int_0^1 f^2(x)\,d(x-1) = f^2(x)(x-1)\Big|_0^1 + \int_0^1 2f\cdot f'(1-x)\,dx$$

$$\left[\int_0^1 f^2(x)\,dx\right]^2 \leq 4\int_0^1 f^2\,dx \int_0^1 f'(1-x)^2\,dx$$

Q.E.D.