Distinguished subgroup.

Def. $\forall a \in G,\ aN = Na \iff N \trianglelefteq G$, N is a distinguished subgroup

Prop. $N \trianglelefteq G$, $(aN)\cdot(bN) = (ab)\cdot N$ is well-defined.

Prop. $N \trianglelefteq G \Rightarrow (G/N, \cdot)$ is a group
whose identity is N, the inverse of aN is $a^{-1}N$

Lemma. $N < G$,

$$\left.\begin{array}{l} \forall a \in G,\ aNa^{-1} \subset N \\ \quad \text{or equivalently} \\ \forall a \in G,\ \forall n \in N,\ ana^{-1} \in N \end{array}\right\} \Rightarrow N \trianglelefteq G$$

Prop. $(N_i)_{i\in I}$ is a family of distinguished subgroup

$\bigcap N_i \trianglelefteq G$

Prop. $\{e\} \trianglelefteq G$, $G \trianglelefteq G$

all subgroup of a Abelian group is distinguished

Theorem : First theorem of group isomorphism

$f: G \to G' \Rightarrow \ker(f) \trianglelefteq G$, $G/\ker(f) \cong \mathrm{im}(f)$

specially, f is epimorphism $\Rightarrow G/\ker(f) \cong G'$
f is homomorphism $\Rightarrow G/\{e\} \cong G \cong \mathrm{im}(f)$
G is finite group $\Rightarrow \dfrac{|G|}{|\ker f|} = |\mathrm{im} f|$

Theorem Second theorem of group isomorphism

$N \triangleleft G, H < G \Rightarrow H \cap N \triangleleft H, N \triangleleft HN, H/(H \cap N) \cong HN/N$

Theorem Third theorem of group isomorphism.

$N \triangleleft G, M \triangleleft G, M < N \Rightarrow N/M \triangleleft G/M$

$(G/M)/(N/M) \cong G/N$