1. $1+\frac{1}{x^2}$ 的换元

$$\left(1+\frac{1}{x^2}\right)dx = d\left(x-\frac{1}{x}\right)$$

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2. 使用 $\delta-\varepsilon$ 语言

以及换元处理破坏点.

1. calculate $\displaystyle\int_0^{+\infty} \frac{1}{1-x^2+x^4} \ln \frac{x^2}{1-x^2+x^4}\, dx$

$$I = \int_0^{+\infty} \frac{t^2}{1-t^2+t^4} \ln \frac{t^2}{1-t^2+t^4}\, dt$$

$$= \frac{1}{2}\int_0^{+\infty} \frac{1+t^2}{1-t^2+t^4} \ln \frac{t^2}{1-t^2+t^4}\, dt$$

$$= \frac{1}{2}\int_0^{+\infty} \frac{1+\frac{1}{x^2}}{\frac{1}{x^2}-1+x^2} \ln \frac{1}{\frac{1}{x^2}-1+x^2}\, (dx)$$

$$= \frac{1}{2}\int_0^{+\infty} \frac{1}{(x-\frac{1}{x})^2+1} \ln \frac{1}{(x-\frac{1}{x})^2+1}\, d(x-\tfrac{1}{x})$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\ln(1+u^2)}{1+u^2}\, du$$

$$= -\int_0^{+\infty} \frac{\ln(1+u^2)}{1+u^2}\, du$$

$$= 2\int_0^{\frac{\pi}{2}} \ln \sin x\, dx = -\pi \ln 2$$

2. find the value of $\displaystyle\lim_{n\to\infty} n\int_0^1 \frac{x^{n-2}}{x^{2n}+x^n+1}\, dx$

$$L = \lim_{n\to\infty} n\int_0^1 \frac{x^{n-2}}{x^{2n}+x^n+1}\, dx \qquad (t = x^n)$$

$$= \lim_{n\to\infty} \int_0^1 \frac{1}{\sqrt[n]{t}\,(t^2+t+1)}\, dt$$

Analysys the Singular Point (这里指的是破坏点、)

as $n\to\infty$ $$\frac{1}{\sqrt[n]{t}} = \begin{cases} 1 & 0<t\le 1 \\ 0 & t=0 \end{cases}$$

use δ- ε language to describe [(insulate)] the point.

$$\int_0^1 \frac{1}{\sqrt[n]{t}} \frac{1}{t^2+t+1} dt = \underbrace{\int_0^{\varepsilon_n} \frac{1}{\sqrt[n]{t}} \frac{1}{t^2+t+1} dt}_{\Downarrow \; \mathbb{I}} + \underbrace{\int_{\varepsilon_n}^1 \text{same} \; dt}_{\Downarrow \; \mathbb{J}}$$

suppose $\varepsilon_n^{\frac{1}{n}} \to 1$

$$0 \le \mathbb{I} \le \int_0^{\varepsilon_n} \frac{1}{t^{\frac{1}{n}}} dt = \frac{1}{-\frac{1}{n}+1} t^{-\frac{1}{n}+1} \Big|_0^{\varepsilon_n}$$

$$= - \frac{1}{\frac{1}{n}+1} \varepsilon_n^{-\frac{1}{n}} \varepsilon_n \to 0 \quad (n \to \infty)$$

$$\mathbb{I} = 0$$

$$\mathbb{J} = \frac{1}{t_n^{\frac{1}{n}}} \int_{\varepsilon_n}^1 \frac{1}{t^2+t+1} dt$$ (mean value theorem of Integrals)

$$\xrightarrow{n \to \infty} \int_0^1 \frac{1}{t^2+t+1} dt = P$$

$$P = \int_0^1 \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}} d(t+\tfrac{1}{2})$$

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2+\frac{3}{4}} du$$

$$= \sqrt{\frac{3}{4}} \arctan u \Big|_{\frac{1}{2}}^{\frac{3}{2}} = \frac{\sqrt{3}\pi}{9}$$