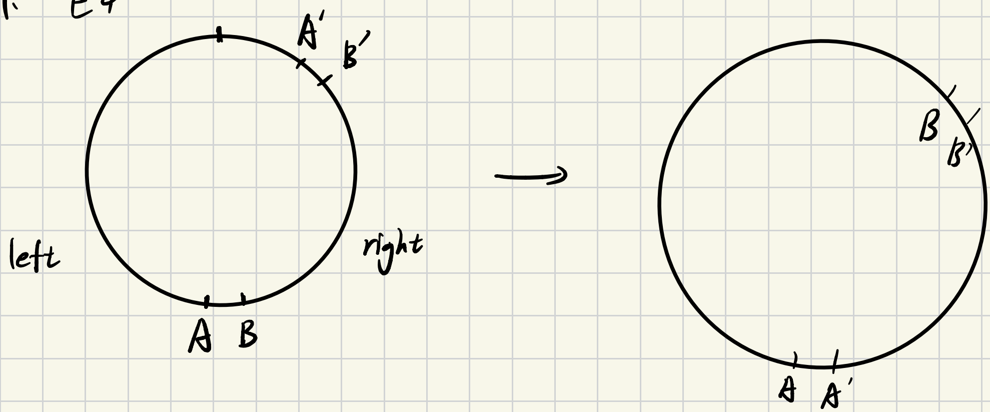
1. E4



A, B are hostiles

on A's right we can always find A' that is friendly

vice versa

the total couple of hostile can be reduced by changing A' and B's seat

E9.

$$f(x_1, x_2, \cdots x_5) = \sum_{i=1}^{5} (x_i - x_{i+2})^2 \quad x_6 = x_1, \ x_7 = x_2$$

suppose $y = x_4 < 0$

$$f_{new} - f_{old} = \cancel{(x_3' - x_1)^2} + \cancel{(x_5' - x_3')^2} + \cancel{(x_2' - x_2)^2} + \cancel{(x_4' - x_2)^2} + (x_4' - x_1)^2 - \cancel{(x_3 - x_1)^2} - \cancel{(x_5 - x_3)^2} - \cancel{(x_5 - x_2)^2} - \cancel{(x_4 - x_2)^2} - (x_4 - x_1)^2$$

$$= x_4(2x_3 + x_4 - 2x_1) + \text{blank} + x_4(2x_5 + x_4 - 2x_2) + (-2x_4)(-2x_2) + (-2x_4)(-2x_1)$$

$$= x_4(\cancel{2x_3} + x_4 - \cancel{2x_1} + \text{blank} + 2x_5 + x_4 - \cancel{2x_2} + \cancel{4x_2} + \cancel{4x_1})$$

$= x_4 (2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5)$

$= 2S x_4 < 0$

considering $s(i,j) = x_i + \cdots x_{j-1}$

$S = \{ s(i,j) \mid 1 \le i \le 5, j > i \}$

after one game play, only one element change and others stay invariant or change their position.

for example, choose x_3, x_4, x_5 $\quad x_4 < 0$

$s(4,5) = x_4 \Rightarrow -x_4$

$s(3,5) = x_3 + x_4 \Rightarrow x_3$

$s(3,4) = x_3 \Rightarrow x_3 + x_4$

$s(6,8) = x_6 + x_7 \Rightarrow x_1 + x_7$

so the step until stop is the number of negative elements of S

E 10

(a) without loss of generality, may as well set:

$\max S_i = a$

if $\max S_{i+1} > \max S_i = a$

then $|x - y| > a$

$x > a + y$ or $y > x + a$ contradiction then $\max S_{i+1} \le \max S_i$

the equel sign holds only when at least one element is 0

if $\max S_{i+4} = \max S_{i+3} = \max S_{i+2} = \max S_{i+1} = \max S_i$

then S_{i+2}, S_{i+1}, S_i have at least one zero and two same element

which can't be true

$\max S_{i+4} < \max S_i$

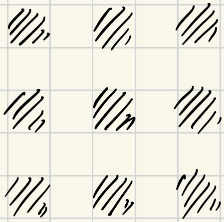
(b)

0	0	0	0	1	1
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	1
1	1	0	0	1	1
0	1	0	1	0	0
1	1	1	1	0	0
0	0	0	1	0	1
0	0	1	1	1	1

cycle

2.

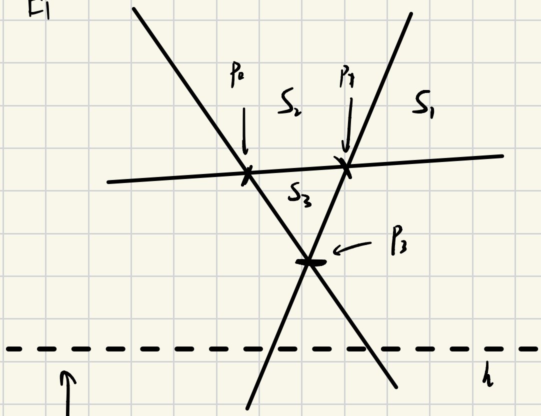
1.



2x2 tiles always cover one black

1x4 tiles always cover 0 or 2 black

3. E_1

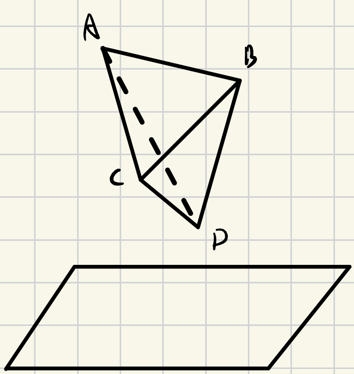


P_i is the deepest point of S_i

parts without a deepest point divide the horizontal line h with $n+1$ parts

number of parts $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2}$

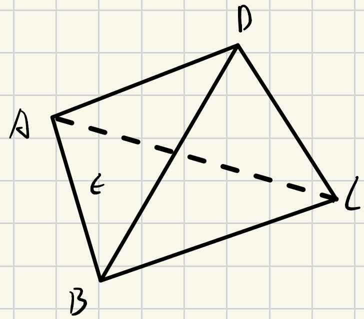
similarly for the space



Number of parts

$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$

E_2



D is the vertices that has the smallest distance to ϵ

$T = ABCD$ is formed by $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$

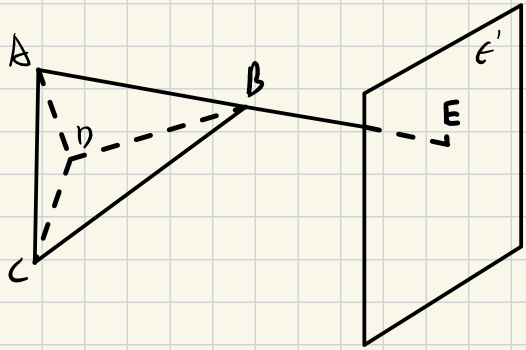
if ϵ' cuts T, then ϵ' have to cut at least one of AD, BD, CD, causing D no longer the point with smallest distance.

for most such ϵ, it divides the space into H_1 and H_2 and has a smallest-distance point on both sides

if there are 4 special ϵ that only have vertices on one side, assume they are $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$

then, ϵ_1 ϵ_2 ϵ_3 ϵ_4 form a tetrahedron $ABCD$

$n \geq 5$, so there is another ϵ' that cut one edge of $ABCD$, suppose it cut AB in E



then D and E lie on different sides of ABC

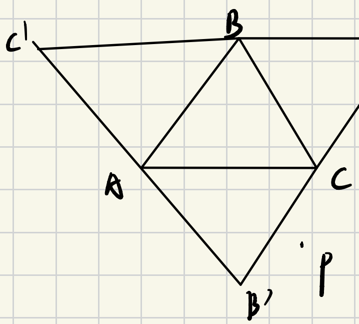
contradiction.

so there are at most such special plane

so the number of tetrahedra $\geq \frac{(2n-3)}{4}$

E 3.

assume a largest ABC



if P is outside A'B'C'

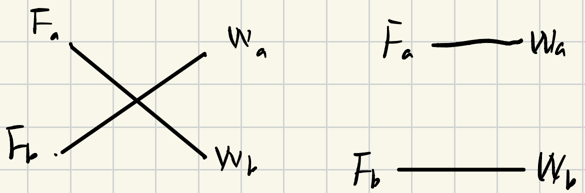
then P and two point of ABC can form a larger triangle

then all point are in A'B'C'

E 4 consider $f_k : F \to W$ $(k = 1, 2, \cdots, n!)$

where f_j construct the smallest total road length.

if f_j has two intersecting road



then we find a smaller total-road-length f_i

contradiction

E 9 suppose there exist such a quadruple

choosing the solution with smallest x^2+y^2

let (a, b, c, d) be such solution

$$a^2 + b^2 = 3(c^2 + d^2)$$

$$3 \mid (a^2 + b^2)$$

let $a = 3k_1 + d_1$ $\quad d_1, d_2 \in \{0, 1, 2\}$

$b = 3k_2 + d_2$

$$a^2 + b^2 = 9k_1^2 + 6k_1d_1 + d_1^2 + 9k_2^2 + 6k_2d_2 + d_2^2$$

$$\equiv d_1^2 + d_2^2 \pmod 3$$

$d_1 = d_2 = 0$

therefore

$3 \mid a$, $3 \mid b$

$a^2+b^2 = 9(k_1^2+k_2^2) = 3(c^2+d^2)$

(c, d, k_1, k_2) will be an new solution with smaller x^2+y^2

contradiction.

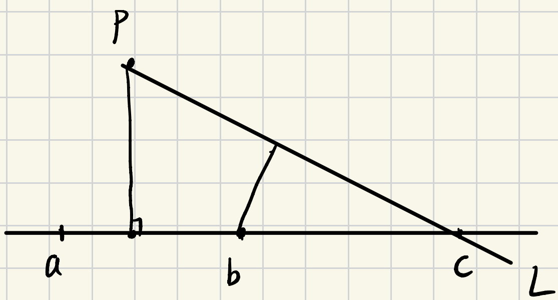
E10.

A infinite set S of points in the plane has the property that:

Any line through two of them passes a third.

If all the points don't lie on a line

assume the pair (p, L) that p has the smallest distance to L



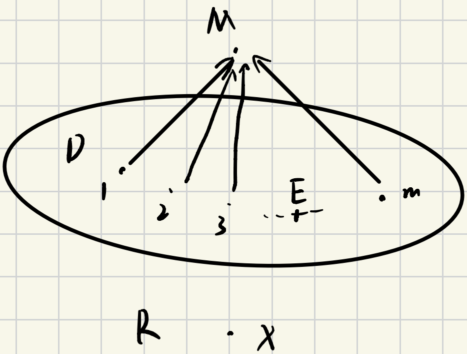
assume a, b construct L, so there is a point c on L

then (b, pc) is a smaller-distance one contradiction

E 11.

Assume M is the city that has m roads leading to it

m is the maximum number of direct roads leading into any city.



let D be the set of the m cities leading to M

let R be the set of the cities apare from M and cities in D

If $R = \phi$, the theorem obviously holds true

If $R \neq \phi$, for $X \in R$, if there is not connection

$X \to E \to M$ where $E \in D$

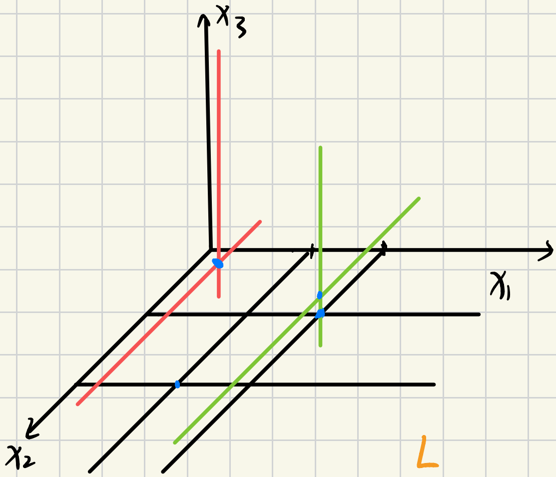
then M and all the cities in D lead to X

the number of roads lead to X would be $m+1 > m$

contradiction.

so there exist $X \to E \to M$, the theorem holds true

E12



for L parallel to x_1x_2 - plane

assume there are t rooks,

and these rooks dominate t_1 rows in x_1, t_2 rows in x_2

obviously, $t \geq t_1$, $t \geq t_2$

without loss of generality, assume $t_1 \geq t_2$

thus there are $(n-t_1)(n-t_2)$ cubes that are failed to dominate.

which must be dominate in x_3 direction.

consider n layers parallel to x_1x_3 - plane

1. where $n-t_1$ of these doesn't contain rook from L

these $n-t_1$ layers must have $(n-t_1)(n-t_2)$ roots in total

2. where t_1 of these layers have at least one rook from L,

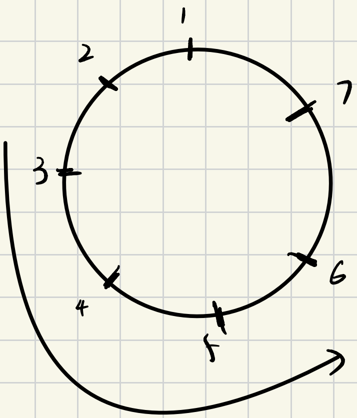
they have to dominate layers parallel to x_2x_3

by symmetry, these layers must have t rooks in each so the total number of rooks is

$$R \geq (n-t_1)(n-t_2) + t_1 t_1 \geq (n-t_1)^2 + t_1^2 = \frac{n^2}{2} + \frac{(2t_1-n)^2}{2}$$

so the smallest number is $\begin{cases} \frac{n^2}{2} & \text{when } n=2k \\ \frac{n^2+1}{2} & \text{when } n=2k-1 \end{cases}$

E 13.



assume dwarf No. i has the maximal amount x

before starting to share his milk

after a turn, No. i gets $\frac{\sum^{6} x_k}{6}$

thus

$$x = \frac{1}{6} \sum^{6} x_k \quad \text{where} \quad x_k \leq x$$

thus $x_1 = x_2 = \text{----} \; x_6 = x$

so each dwarf share same amount of milk

the distribution is like:

$$0 \quad \frac{x}{6} \quad \frac{2x}{6} \quad \frac{3x}{6} \quad \frac{4x}{6} \quad \frac{5x}{6} \quad \frac{6x}{6}$$

$x = \frac{6}{7}$

E 15 choose 1983 pairwise distinct positive integers < 100000 such that no three are in arithmetic progression.

Using Greedy Algorithm: start with 0

each step choose and add smallest integer which is not in arithmetic progression

(0) 0

(1) 0, 1 (2 makes arithmetic, choose 3)

(2) 0, 1, 3 (0+3), 4 (1+3) (choose 9)

(3) 0, 1, 3, 4, 9 (0+9), 10 (1+9), 12 (3+9), 13 (4+9)

(choose 27)

now proof at i-th end, we choose 3^i

$i \le 3$, the conjecture holds true.

assume for $i = i$, the conjecture is true

when $i = i+1$

the biggest number in i-th end is $\frac{3^i - 1}{2}$

the biggest number in $i+1$-th end is $\frac{3^{i+1} - 1}{2}$

so for $i+1$-th end, choose 3^{i+1}

therefore, write above sequence in ternary

0, 1, 10, 11, 100, 110, 111, 1000, ----

so if there are n number in the series, it will cover all the number 1, 2, 3 --- n write in binary and caculated in ternary.

so $1983 = (11110\ 111111)_2$

1983	1
991	1
495	1
247	1
123	1
61	1
30	0
15	1
7	1
3	1
1	1

caculate it in ternary

$(11110\ 111\ 111)_3 = 87844 < 100000$

E 16. divide the question into two parts

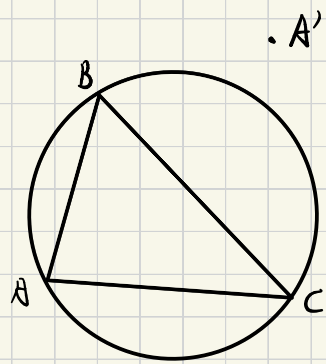
(a) the maximal circle covers the n-gon

(b) the maximal circle pass through three consecutive vertices

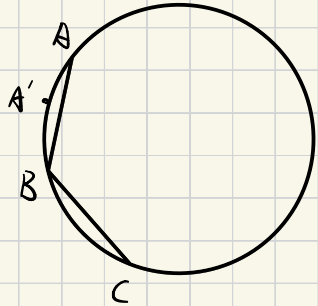
(a) if A' is not in the circle

A' and two of A, B, C can create

a bigger circle, contradiction



(b) if A' is between B and C



if A' is on the circle

A', B, C is consecutive

if A' is not on the circle

$A'BC$ form a bigger circle

contradiction.

E 17. consider $(\sqrt{2}-1)k$ where k is the smallest integer that make $\sqrt{2}\cdot k$ an integer

$\left[(\sqrt{2}-1)k\right]\sqrt{2} = 2k - \sqrt{2}\cdot k$, both side of the equality is integer

$(\sqrt{2}-1)k$ also make $\sqrt{2}\left[(\sqrt{2}-1)k\right]$ an integer

however

$(\sqrt{2}-1)k < k$

contraddiction.

4.

E5 $a, b \in N$ coprime $ax - by = 1$ for some $x, y \in N$

proof. $ax - by \pmod a \equiv -by$

$ax - by \pmod b \equiv ax$

by ~~box~~ theorem, make $ax - by \equiv 1 \pmod{a \text{ or } b}$

E6 for $n \geq (p-1)(q-1)+1$, every sequence of n integers contains either a monotonically increasing subsequence of length p or a monotonically decreasing subsequence of length q.

proof.

Denote the maximum length L_m of a monotonically increasing sequence with last element m and vice versa. for R_m

if $a \neq b$

$L_a \neq L_b$ since either $a > b$ or $b > a$

then there are $(p-1)(q-1)$ boxes, with n pearls

contradiction.

Ramsey Numbers Sum-free Sets Theorem of I. Schur

p points in space with no four lying in the same plane, joining each points by a segment, we get a complete graph G_p

- colour each edge with one of n colour, we call it n-coloring of the G_p
- If G_p contains a triangle with same color, we call G_p contains a monchromatic G_3

counter-example of E_{13}

consider G_{16}, a abel group with generating elements a, b, c, d where $a+a = b+b = c+c = d+d = 0$

divide G into three sum-free subsets:

$A_1 = \{a, b, c, d, a+b+c+d\}$

$A_2 = \{a+b, a+c, c+d, a+b+c, b+c+d\}$

$A_3 = \{b+c, a+d, b+d, a+c+d, a+b+d\}$

color A_1 A_2 A_3 with color $1, 2, 3$

let $x+y$ denote segment xy

then $(x+y)+(y+z) = x+z$, these three is not in same group

Ramsey's theorem:

If $q_1, q_2 \cdots, q_n \geq 2$ are integers, there is a minimal number $R(q_1, \cdots q_n)$, such that. for $p \geq R(q_1 \cdots q_n)$, for at least one $i = 1, \cdots n$, G_p contains at least one monochromatic G_{q_i}.

$$R_n(3) = R(3, 3 \cdots 3) \leq \lfloor en! \rfloor + 1 = n!\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$$