斐波那契数列的性质

Millin 级数

欲买桂花同载酒，终不似，少年游？

比处定义 $F_0=1$ ， $F_1=F_2=1$

$$F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

有以下性质：

(1) $$\lim_{n\to\infty}\frac{F_n}{F_{n+1}}=\frac{\sqrt{5}-1}{2}$$

(2) $$F_1+\cdots+F_n=F_{n+2}-1$$

(3) $$F_1+F_3+\cdots+F_{2n-1}=F_{2n}$$

(4) $$F_2+F_4+\cdots+F_{2n}=F_{2n+1}-1$$

(5) $$F_1^2+F_2^2+\cdots+F_n^2=F_n\cdot F_{n+1}$$

(6) 母函数 $$f(x)=\sum_{i=0}^{\infty}F_i x^i$$

$$=0+x+x^2+2x^3+3x^4+\cdots$$

$$f(x)=xf(x)+x^2f(x)$$

$$f(x)=\frac{1}{1-x-x^2}$$

(7) $$\sum_{i+j=n}\binom{i}{j}=F_{n+1}\quad,\quad\sum_{i\ge 0}\sum_{j\ge 0}\binom{n-i}{j}\binom{n-j}{i}=F_{2n+1}$$

再有再补充罢

Millin 级数：

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$$

Proof.

有以下引理：

Lemma.

1. $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

2. $F_{m+n} = F_{m-1} F_n + F_m F_{n+1}$

assert that $\sum_{n=0}^{k} 1/F_{2^n} = 3 - \dfrac{F_{2^k-1}}{F_{2^k}}$

when $k=1$, 成立

$$3 - \frac{F_{2^k-1}}{F_{2^k}} + \frac{1}{F_{2^{k+1}}} = 3 - \frac{\frac{F_{2^k-1} F_{2^k+1}}{F_{2^k}} - 1}{F_{2^{k+1}}}$$

lemma 2. $$= 3 - \frac{F_{2^k-1}\left(F_{2^k-1} + F_{2^k+1}\right) - 1}{F_{2^{k+1}}}$$

lemma 1. $$= 3 - \frac{F_{2^k-1}^2 + F_{2^k}^2}{F_{2^{k+1}}}$$

lemma 2. $$= 3 - \frac{F_{2^{k+1}-1}}{F_{2^{k+1}}}$$

then by Prop. (2). the equation is proved.

Q.E.D.