$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$(1+x^2) \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1)(n+2) a_{n+2} - 2a_n + (n-1) n a_n \right) x^n = 0$$

$$a_{n+2} = -\frac{(n-2)(n+1)}{(n+2)(n+1)} a_n$$

for even $a_2 = \frac{2}{2} a_0 = a_0 \quad a_4 = 0 \;\cdots\; a_{2k} = 0 \; (k \geq 2)$

for odd $n = 2k-1 \quad a_{2k+1} = -\frac{2k-3}{2k+1} a_{2k-1} = -\frac{\cancel{2k-3}}{2k+1}\left(-\frac{\cancel{2k-5}}{2k-1}\left(-\frac{2k-7}{\cancel{2k-3}} \cdots \frac{1}{\cancel{5}} \cdot \frac{-1}{\cancel{3}} a_1\right)\right)$

$$= (-1)^{k-1} \frac{1}{(4k^2-1)} a_1$$

$$y = \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n$$

$$= a_0 (1+x^2) + a_1 \left(x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4k^2-1} x^{2k+1} \right)$$

$$= a_0 y_1 + a_2 y_2$$

(b) $y_1(0) = 1 \quad y_2(0) = 0$

$y_1'(0) = 0 \quad y_2'(0) = 1$

(c) $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{2} x^n$$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$(1+x^2) \sum_{n=0}^{\infty} (n+1)(n+2) C_{n+2} x^n - 2 \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{2} x^n$$

$$\sum_{n=0}^{\infty} \left((n+1)(n+2) C_{n+2} - 2C_n + (n-1) n C_n - \frac{(-1)^n + 1}{2} \right) x^n = 0$$

$$(n+1)(n+2)\,C_{n+2} + (n+1)(n-2)\,C_n = \frac{(-1)^n+1}{2}$$

for $n = 2k$

$$(2k+2)(2k+1)\,C_{2k+2} + (2k-2)(2k+1)\,C_{2k} = (-1)^k$$

$$b_{k+1} + b_k = \frac{(-1)^k\,k}{2(2k+1)}$$

(d) ① $p(x) = \int \frac{1}{y_1(x)^2}\,dx = \frac{x}{2(1+x^2)} + \frac{1}{2}\arctan x$

$y_2(x) = \frac{x}{2} + \frac{1+x^2}{2}\arctan x$

② $W(y_1, y_2) = 1$

$$y_p = -u_1 \int u_2 \frac{1}{(1+x^2)^2}\,dx + u_2 \int u_1 \frac{1}{(1+x^2)^2}\,dx$$

$$= \frac{1}{4} + \frac{x}{2}\arctan x + \frac{1+x^2}{4}(\arctan x)^2$$

to fit $Y(0)=0 \quad Y'(0)=0$

$$Y(x) = y_p - \frac{1}{4}(1+x^2) = \frac{x}{2}\arctan x + \frac{1+x^2}{4}(\arctan x)^2 - \frac{x^2}{4}$$