1. $a = e^{i\alpha}$ $b = e^{i\beta}$

$= \cos\alpha + i\sin\alpha$ $= \cos\beta + i\sin\beta$

$$\mathrm{Im}(a+b+a\bar{b}) = (\sin\alpha\cos\beta - \sin\beta\cos\alpha) + \sin\alpha + \sin\beta = 0$$

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2} = 0$$

$$\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} = 0$$

$(1, a)$, $(b, 1)$, $(a, -a)$

2. $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} i\ln i\right) - \frac{1}{2}\ln n$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} i\ln i - \frac{n^2}{2}\ln n\right)$$

$$= \lim_{n\to\infty} \frac{(n+1)\ln(n+1)}{2n+1} + \frac{1}{2}\,\frac{-(n+1)^2\ln(n+1) + n^2\ln n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{n^2\ln n - (n^2-1)\ln(n+1)}{2(2n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n\ln\left(1-\frac{1}{n+1}\right)^n + \ln(n+1)}{2n+1} = -\frac{1}{4}$$

3. $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$ has the eigenvalue of n

so $n = k^2$ $\quad k \in \mathbb{N}^+$

$$A = \begin{pmatrix} B_0 & B_1 & \cdots & B_{k-1} \\ B_0 & B_1 & \cdots & B_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_0 & B_1 & \cdots & B_{k-1} \end{pmatrix}$$ where $B_l \in M_{k\times k}$ $\quad b_{ij}=1$ iff $j-i \equiv l \pmod{k}$

for example $B_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $\quad B_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 1 & \cdots & \cdots & 0 \end{pmatrix}$

then $(B_0 \cdots B_{k-1}) \cdot \begin{pmatrix} B_0 \\ \vdots \\ B_{k-1} \end{pmatrix} = J_k$

4.

X_m denote the subset of G of products of the form $g_1 \cdots g_m$

Lemma. $\forall j = 1, \cdots n$ $\forall a, b \in X_j$, the ratio $a^{-1}b$ is contained in H.

Prove the lemma by induction.

I) Begin with the case $j=1$

By The Drawer Principle, $\exists k < l$ s.t.

$(w_{k+1}, \cdots, w_{k+n-1})$ and $(w_{l+1}, \cdots, w_{l+n-1})$ equal

If $w_{k+m} = w_{l+m}$ $\forall m \in \mathbb{N}^+$, w is periodic eventually.

thus, $\exists m > 0$ $w_{k+m} \neq w_{l+m}$

then $w_{k+m-i} = w_{l+m-i}$ $\forall i = 1, 2, \cdots n-1$

then $x = w_{k+m-n+1} \cdots w_{k+m}$ $y = w_{l+m-n+1} \cdots w_{l+m}$

$x^{-1}y$, $y^{-1}H \in H$

thus $g^{-1}h$, $h^{-1}g \in H$

II) Induction step from $j-1$ to j

$a \in X_j$ a $g(h)$ - element iff $\exists a_1 \in X$, $a = g(h)\, a_1$

all $g(h)$ - element constitute a equivalent class thus $v = w_{k+m-n+j} \cdots w_{k+m} w_{k+m+1} \cdots w_{k+m+j-1}$

$u = w_{l+m-n+j} \cdots w_{l+m} w_{l+m+1} \cdots w_{l+m+j-1}$

since $\{w_{k+m}, w_{l+m}\} = \{g, h\}$, $w_{k+m-i} = w_{l+m-i}$ $\forall$ $i = 1, 2, \cdots, n-j$

$u^{-1}v$ is a ratio of g-element & h-element.

Q.E.D.

By the Lemma the decription is true.

8.