1. Calculate $\displaystyle\sum_{n=1}^{\infty}\left[n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2} + \cdots\right) - \frac{1}{2n}\right]$

Solution:

$$\text{LHS} = \sum_{n=1}^{\infty}\left(n\sum_{j=0}^{\infty}\frac{(-1)^j}{(n+j)^2} - \frac{1}{2n}\right)$$

$$= \sum_{n=1}^{\infty}\left(n\sum_{j=0}^{\infty}(-1)^j\int_0^{\infty} x\,e^{-(n+j)x}\,dx - \frac{1}{2n}\right)\quad(\mathrm{I})$$

$$= \sum_{n=1}^{\infty}\left(n\int_0^{\infty} x\,e^{-nx}\frac{1}{1+e^{-x}}\,dx - \frac{1}{2n}\right)$$

$$= \sum_{n=1}^{\infty} n\int_0^{\infty} x\,e^{-nx}\left(\frac{1}{1+e^{-x}} - \frac{1}{2}\right)dx\quad(\mathrm{I})$$

$$= \int_0^{\infty}\left(\frac{1}{1+e^{-x}} - \frac{1}{2}\right)x\left(\sum_{n=1}^{\infty} n\,e^{-nx}\right)dx$$

$$= \int_0^{\infty}\left(\frac{1}{1+e^{-x}} - \frac{1}{2}\right)\frac{x\,e^x}{(e^x-1)^2}\,dx\quad(\mathrm{II})$$

$$= \int_0^{\infty}\frac{1}{2}\cdot\frac{x\,e^{-x}}{1-e^{-2x}}\,dx$$

$$= \frac{1}{2}\cdot\int_0^{\infty} x\,e^{-x}\sum_{n=0}^{\infty}e^{-2nx}\,dx\quad\text{(convergent Taylor)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\int_0^{\infty} x\,e^{-(2n+1)x}\,dx$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{16}$$

$(\mathrm{I})\ \Gamma(s) = \int_0^{+\infty} e^{-x}\cdot x^{s-1}\,dx = (s-1)!$

$\underset{\underline{x=yx}}{=\!=} \int_0^{+\infty} e^{-yx}\,y^{s-1}x^{s-1}\cdot y\,dx$

thus $\dfrac{\Gamma(s)}{y^s} = \int_0^{+\infty} e^{-yx}\cdot x^{s-1}\,dx$

$(\mathrm{II})\ \displaystyle\sum_{i=1}^{n} i\cdot x^i$

$= x\dfrac{d}{dx}\displaystyle\sum_{i=1}^{n} x^i$

$$f(x) = x e^{x^2} \sinh x + e^x \sinh x$$

$$= x e^{x^2} \sinh x + \sinh x \cdot \sinh hx + \sinh x \cdot \cosh x$$

where $g^{(2k+1)}(0) = 0$ for g even.

$$f^{(2021)}(x) = (\sinh x \cdot \cosh x)^{(2021)}$$

$$= (\sinh x \cdot \cos ix)^{(2021)}$$

$$= \tfrac{1}{2}\left[\sinh(x + ix) + \sinh(x - ix)\right]^{(2021)}$$

$$= \tfrac{1}{2}\left[(i+1)^{2021} + (1-i)^{2021}\right]$$

$$= \tfrac{1}{2}(\sqrt{2})^{2021}\left(e^{\frac{\pi}{4}i \cdot 2021} + e^{-\frac{\pi}{4}i \cdot 2021}\right)$$

$$= \tfrac{1}{2}(\sqrt{2})^{2021} \cosh (i\pi \cdot 2021)$$

$$= -2^{1010}.$$