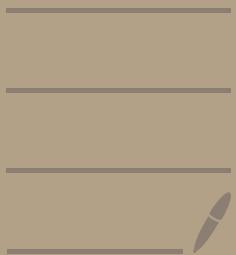


三次方程求根



1. Solution of [Cubic function with one variable]

$$Ax^3 + Bx^2 + Cx + D = 0$$

$$x^3 + bx^2 + cx + d = 0 \quad (b = \frac{B}{A}, c = \frac{C}{A}, d = \frac{D}{A})$$

$$z^3 + pz + q = 0 \quad (p = c - \frac{b^2}{3}, q = \frac{2b^3}{27} - \frac{bc}{3} + d)$$

suppose u, v satisfy $3uv = -p, u^3 + v^3 = -q$

then $u+v$ is a solution of origin function.

$$\text{suppose } U = u^3 \quad V = v^3$$

then U, V are the solution of following function:

$$x^2 + qx - \frac{p^3}{27} = 0$$

solution of $x^3 + px + q = 0$

$$x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3}}$$

$$x_2 = w^3 \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3}} + w^3 \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3}}$$

$$x_3 = w^2 \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3}} + w^2 \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3}}$$

$$\text{where } w^3 = 1 \quad w = \frac{-1 + \sqrt{3}i}{2}$$

$$1. \quad x_1 + x_2 + x_3 = 0 \quad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = -\frac{p}{q} \quad x_1 x_2 x_3 = -q$$

$$2. [\text{Discriminant}] \quad \Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p^3}{27}\right)^3 \quad \Delta > 0, 1 \text{ real, 2 complex}$$

$$\Delta = 0, 3 \text{ real, or two are same}$$

$$\Delta < 0, 3 \text{ unique real.}$$

3. When $\Delta < 0$, the roots can be expressed as:

$$x_1 = 2 \sqrt[3]{r} \cos \theta \quad x_2 = 2 \sqrt[3]{r} \cos (\theta + \frac{2}{3}\pi) \quad x_3 = 2 \sqrt[3]{r} \cos (\theta - \frac{2}{3}\pi)$$

$$\text{where } r = \sqrt{-\left(\frac{P}{q}\right)^3} \quad \theta = \frac{1}{3} \arccos \left(-\frac{q}{2r} \right)$$