1. 积分求最佳思想
在复数不等式中的运用
+ Possion公式利用

2. 分子 分母化简
求积换序

1. $\alpha_i \in \mathbb{C}$, $i = 1, \cdots m$, prove $\exists\, J \subset \{1, \cdots m\}$ s.t. $\left|\sum_{i\in J} \alpha\right| \geq \frac{1}{\pi}\sum_{i=1}^{m} |\alpha_i|$

Proof.

$$a_k = x_k + i y_k \qquad x_k, y_k \in \mathbb{R}, \quad k = 1, 2, \cdots m$$

$$f(a) := \sum_{k=1}^{m} \max\{x \cdot x_k + y \cdot y_k, 0\} \qquad a = x + iy, \quad x, y \in \mathbb{R}$$

$$\frac{1}{2\pi}\int_{|a|=1} f(a)\, dS = \frac{1}{2\pi}\sum_{k=1}^{m}\int_{|a|=1} \max\{x\cdot x_k + y\cdot y_k, 0\}\, dS$$

[Possion's formula]

$$= \frac{1}{2\pi}\sum_{k=1}^{m}\int_{|a|=1} \max\{y\sqrt{x_k^2+y_k^2}, 0\}\, dS$$

$$= \frac{1}{2\pi}\sum_{k=1}^{m}\int_{x^2+y^2=1,\, y>0} \sqrt{x_k^2+y_k^2}\cdot y\, dS$$

$$= \frac{1}{\pi}\sum_{k=1}^{m}\sqrt{x_k^2+y_k^2}$$

by mean value theorem of integral.

$$\exists\, |a_0| = 1 \quad \text{s.t.} \quad f(a_0) = \frac{1}{\pi}\sum_{k=1}^{m} |a_k| \qquad a_0 = x_0 + i y_0 \quad x_0, y_0 \in \mathbb{R}$$

Define $J = \{k \mid x_k \cdot x_0 + y_k \cdot y_0 \geq 0, \ k = 1, 2, \cdots, m\}$

then

$$f(a_0) = \sum_{k=1}^{m} \max\{x_k \cdot x_0 + y_k \cdot y_0, 0\}$$

$$= \sum_{k=1}^{m} (x_k \cdot x_0 + y_k \cdot y_0)$$

$$= \left(\sum_{k\in J} x_k\right)\cdot x_0 + \left(\sum_{k\in J} y_k\right) y_0 \leq \sqrt{\left(\sum_{k\in J} x_k\right)^2 + \left(\sum_{k\in J} y_k\right)^2}\cdot\sqrt{x_0^2+y_0^2} \quad (\text{Cauchy})$$

$$= \left|\sum_{k\in J} a_k\right|$$

Q.E.D.

2. calculate $\dfrac{\sqrt{10+\sqrt{1}}+\sqrt{10+\sqrt{2}}+\cdots+\sqrt{10+\sqrt{99}}}{\sqrt{10-\sqrt{1}}+\sqrt{10-\sqrt{2}}+\cdots+\sqrt{10-\sqrt{99}}}$

Numerator $= \sum_{k=1}^{n^2-1} \sqrt{n+\sqrt{k}}$ Denominator $= \sum_{k=1}^{n^2-1} \sqrt{n-\sqrt{k}}$

$$\left(\sqrt{n+\sqrt{k}} - \sqrt{n-\sqrt{k}}\right)^2 = 2n - 2\sqrt{n^2-k}$$

$$\sum_{k=1}^{n^2-1} \sqrt{n+\sqrt{k}} - \sum_{k=1}^{n^2-1} \sqrt{n-\sqrt{k}} = \sqrt{2} \sum_{k=1}^{n^2-1} \sqrt{n-\sqrt{n^2-k}} = \sqrt{2} \sum_{k=1}^{n^2-1} \sqrt{n-\sqrt{k}}$$

thus LHS $= \sqrt{2}+1$