Advanced Problem Solving II

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1 Problems to be discussed in lecture

$\frac{(90+99)1\cancel{0}}{7}5$

1.1 Pigeonhole Principle

The **Pigeonhole Principle** simply says that if one wants to put pigeons in holes, and there are more pigeons than the holes, then one of the holes has to contain more than one pigeon. Or **more generally**, if more than nk items are placed into k boxes, then at least one of the boxes contains at least $k+1$ of the items.

1.1.1 Problem 1

$2^{10} = 1024 \qquad 10 \leq S \leq 945$

(IMO 1972) Prove that from a set of ten distinct two-digit numbers, it is possible to select two disjoint subsets whose elements have the same sum.

$10 \leq a_1 \leq a_2 \leq a_3 \cdots \leq a_{10} \leq 99$

1.1.2 Problem 2

2 3 5

(National Iranian Competition for University Students) Let m be a positive integer such that $gcd(m, 30) = 1$, i.e., the greatest common factor of m and 30 is 1. Show that there are infinitely many terms in the following sequence

$$1, 11, 111, 1111, 11111, \ldots$$

which are divisible by m.

$\frac{10^n - 1}{9} \equiv 0 \mod p^n$

$10^n \equiv 1 \mod p^n$

by Euler,

1.2 Polynomials Methods for Computing Determinants

1.2.1 Problem 3 (Vandermonde Determinant)

Let $x_1, \ldots, x_n$ be non-zero real numbers and set

$$A = \begin{pmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & x_n & \ldots & x_n^{n-1} \end{pmatrix}.$$

$a^{\varphi(p^n)} \equiv 1$

where

$\{ a^n \mod p^n \}$ create an Abel group

Compute $\det(A)$.

so $a^{k\varphi(p^n)} \equiv 1$

so $M_n = n\,\varphi(p_1^{t_1}) \cdot \varphi(p_2^{t_2}) \cdots \varphi(p_n^{t_n})$ where $m = p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}$

$(7 \leq p_1 \leq p_2 \leq \cdots p_n,\ t_1 t_2 t_3 \cdots t_n \neq 0)$

$p(a_1, a_2, \ldots, a_n) = a_1 \cdots a_n$

1.2.2 Problem 4

Let $a_1, \ldots, a_n$ be non-zero real numbers and set

$(1+a_1)\cdot(1+a_2)\cdots(1+a_n)\cdot$

$$A = \begin{pmatrix} 1+a_1 & 1 & \ldots & 1 \\ 1 & 1+a_2 & \ldots & 1 \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1+a_n \end{pmatrix}$$

Show that

$$\det(A) = a_1 \ldots a_n \left(1 + \frac{1}{a_1} + \ldots + \frac{1}{a_n}\right).$$

$+ \quad a_2 \cdots a_n \left(1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$

2 Problems

1. Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Then there always exists a subset of these numbers with sum divisible by n.

2. Let n be a positive integer which is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

2^m

3. There are m cards on a table, and each of them is labelled with one of the numbers $1, \ldots, m$. Prove that if the sum of the labels of any subset of the cards is not a multiple of $m+1$, then each card is labeled by the same number.

$1 \le \cdots \le m$

References

[1] Arthur Engel, Problem-Solving Strategies, Springer, 1998.

[2] Bamdad R. Yahaghi, Iranian Mathematics Competitions, 1973–2007.

[3] Razvan Gelca and Titu Andreescu, Putnum and beyound, springer.

1.1.1 Problem 1.

suppose the ten distinct two-digit numbers

$$10 \le a_1 < a_2 < a_3 \cdots < a_{10} \le 99$$

thus the range of the sum of a subset is:

$$10 \le S \le 90+91+92+\cdots+99 = 945$$

where the number of choices of a nonvoid proper subset equal to $2^{10}-2 = 1022 \ge 945+1-10$

therefore, there must exist two subsets whose sum is equal.

if such two subsets have joint elements, by simultaneously removing the joint elements, we create two disjoint subsets whose elements have the same sum.

and clearly, none of the subsets equals ϕ or $\{a_i \mid i \in \{1,2,3\cdots 10\}\}$

In conclusion, there must be at least two disjoint subsets whose elements have the same sum.

To construct n vector on $\mathbb{R}^k$ s.t. any $k+1$ of them don't on the same hyperplane

$$f: \mathbb{R} \to \mathbb{R}^k$$

$$f(x) = (1, x, x^2, \cdots, x^{k-1})$$

there are $k+1$ distinct nonzero real numbers t_1 t_2 $\cdots$ t_{k+1}

suppose

$$A = \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_{k+1}) \end{pmatrix} = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{k-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_{k+1}^{k-1} \end{pmatrix}$$

if $k+1$ vectors $f(t_1)$ $f(t_2)$ $\cdots$ $f(t_{k+1})$ is on the same hyperplane α

where $\alpha: c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = c_0$

then

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{k-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{k-1} \\ \vdots & & & & \vdots \\ 1 & \cdots & & & t_{k+1}^{k-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} c_0 \\ c_0 \\ \vdots \\ c_0 \end{pmatrix}$$

so rank $A \le r$

because $\det A' = \det \begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{k-1} \\ \vdots & & & & \vdots \\ 1 & t_k & t_k^2 & \cdots & t_k^{k-1} \end{vmatrix} = \prod_{1 \le i < j \le k} (x_j - x_i) \neq 0$

rank $A = r$

$f(t_1)$ $f(t_2)$ $\cdots$ $f(t_{k+1})$ are linearly dependent.

notice that

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = A'^{-1} \begin{pmatrix} c_0 \\ c_0 \\ \vdots \\ c_0 \end{pmatrix}_{k\times 1}$$

thus

$$\begin{pmatrix} A' \\ f(t_{k+1}) \end{pmatrix} A'^{-1} \begin{pmatrix} c_0 \\ c_0 \\ \vdots \\ c_0 \end{pmatrix}_{k\times 1} = \begin{pmatrix} c_0 \\ c_0 \\ \vdots \\ c_0 \end{pmatrix}_{(k+1)\times 1}$$

$$f(t_{k+1}) \cdot A'^{-1} \begin{pmatrix} c_0 \\ \vdots \\ c_0 \end{pmatrix}_{k\times 1} = c_0$$

write $f(t_{k+1})$ as $\lambda_1 f(t_1) + \lambda_2 f(t_2) + \lambda_3 f(t_3) + \cdots + \lambda_k f(t_k)$

so $\lambda_1 + \lambda_2 + \cdots \lambda_k = 1$

so $\lambda_1 \cdot 1 + \lambda_2 \cdot 1 + \cdots + \lambda_k \cdot 1 = 1$.

$\lambda_1 \cdot t_1 + \lambda_2 t_2 + \cdots + \lambda_k t_k = t_{k+1}$

$\lambda_1 t_1^2 + \lambda_2 t_2^2 + \cdots + \lambda_k t_k^2 = t_{k+1}^2$

$\vdots$

notice that

$$t_{k+1} = (\lambda_1 t_1^2 + \lambda_2 t_2^2 + \cdots \lambda_k t_k^2)^{\frac{1}{2}} \geq (\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n)^1 = t_{k+1}$$

the inequality is sharp when $t_1 = t_2 = t_3 \cdots = t_n$

contradiction

so, for such f, for any large $n \geq k+1$, there not $k+1$ vectors on the same hyperplane.