Euclidean domain

Def. $(R,+,\cdot)$ domain R is an Euclidean domain
if $\exists f: R\setminus\{0\} \to \mathbb{N}_0$
s.t. $\forall a \in R, b \in R\setminus\{0\}$ $\exists q, r \in R$ s.t. $a = qb + r$
where $r = 0$ or $f(r) < f(b)$
f is called Euclidean function

example: $(\mathbb{Z}, +, \cdot)$ is an Euclidean domain

Prop. $(F, +, \cdot)$ field F is an Euclidean domain.

Prop. $(R, +, \cdot)$ Euclidean domain $\Rightarrow$ R is principle ideal domain

Def. $(R, +, \cdot)$ domain $N: R \to \mathbb{R}^{\geq 0}$ is a norm if
$N(x) = 0 \Leftrightarrow x = 0$
$N(x) = 1 \Leftrightarrow x$ is an unit
$\forall a, b \in R$, $N(ab) = N(a)N(b)$

example. Gauss integral ring $\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}] = \{a+bi : a, b \in \mathbb{Z}\}$
$N(a+bi) = a^2 + b^2$

Prop. $\mathbb{Z}[i]$ is an Euclidean domain.

Proof. suppose $a = x + iy$ $b = z + iw \in \mathbb{Z}[i]$
$\exists a, b$ s.t. $\frac{a}{b} = \frac{x+iy}{z+iw} = \frac{xz-yw}{z^2+w^2} + i\frac{xw+yz}{z^2+w^2} \in \mathbb{Q}[i]$

$\exists m, n \in \mathbb{Z}$ s.t.
$$\left| m - \frac{xz-yw}{z^2+w^2} \right| \leq \frac{1}{2}$$
$$\left| n - \frac{xw+yz}{z^2+w^2} \right| \leq \frac{1}{2}$$
(m, n are the closest integer)

suppose $q = m + in$, $r = a - qb$

To prove $r=0$ or $N(r) < N(b)$

$$r = a - qb = b\left(\frac{a}{b} - q\right) = \left(m - \frac{xz - yw}{z^2+w^2}\right) + i\left(n - \frac{xw+yz}{z^2+w^2}\right)$$

$(r \neq 0)$

thus $N(r) \leq (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$

$b \neq 0$ $N(b) \geq 1 \geq N(r)$

Prop. $(R, +, \cdot)$ domain $N: R \to \mathbb{N}_0$ norm $x \in R$

$N(x)$ is a prime $\Rightarrow$ x irreducible