Action of Group

Def. S is a set, a symmetric group on S, denote as $(\text{Perm}(S), \circ)$, is

$$\text{Perm}(S) = \{ f : S \mapsto S, f \text{ is bijective} \}$$

Prop. $\phi : (G, \cdot) \to (\text{Perm}(G), \circ)$, ϕ is a homomorphism

Proof: $x, y \in G$ $(\phi_x \circ \phi_y)(z) = x(yz) = (xy)z = \phi_{xy}(z)$
thus $\phi_x \circ \phi_y = \phi_{xy}$

Def. group $(G, \cdot)$ set S $\phi : G \to \text{Perm}(S)$

if ϕ is a group homomorphism $\Rightarrow$
ϕ is the group action on S of G

Def. $x \in G$, $\phi_x \in \text{Perm}(G)$, $y \in G$

$$\phi_x(y) = xyx^{-1}$$

$\phi : G \to \text{Perm}(G)$, for $x \in G$, $\phi(x) = \phi_x$ called conjugate action

Prop. conjugate action of G is a group action of G itself

Def. a inner automorphism of G generated by $x \in G$ refers to $\phi_x : G \to G$, for $y \in G$.

$$\phi_x(y) = xyx^{-1}$$

other automorphism on G is called outer automorphism.

Prop. group action $\phi : (G, \cdot) \to (\mathrm{Perm}(S), \circ)$

if denote $\phi_x(s)$ by $x \cdot s$

then equivalently, the property of group action are:

$$\forall s \in S, \ e \cdot s = s$$

$$\forall x, y \in G, \ x \cdot (y \cdot s) = (xy) \cdot s$$

Def. $\phi : (G, \cdot) \to (\mathrm{Perm}(S), \circ)$ group action.

for $s \in S$, orbit of s:

$$\mathrm{Orb}(s) = \{ s' \in S : \exists x \in G, s' = xs \} = \{ xs : x \in G \}$$

stabilizer:

$$\mathrm{Stab}(s) = \{ x \in G : xs = s \}$$

Prop. $\mathrm{Orb}(s) = \mathrm{Orb}(s')$ or $\mathrm{Orb}(s) \cap \mathrm{Orb}(s') = \phi$

Prop. $\mathrm{Stab}(s) < G$

Prop. $\phi : (G, \cdot) \to (\mathrm{Perm}(S), \circ)$ a group action.

for $s \in S$, there exist a bijection between $G / \mathrm{Stab}(s)$ & $\mathrm{Orb}(s)$

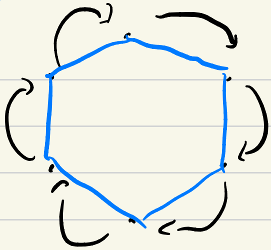
if G is finite, $|G| = |\mathrm{Stab}(s)| \cdot |\mathrm{Orb}(s)|$

Example. a dihedral group D_{2n} (二面体群) is formed by all symmetric transformations of regular n-polygon on itself

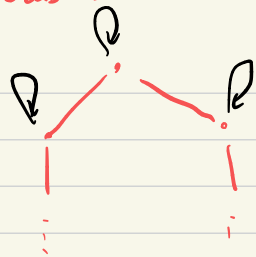
$|\mathrm{Orb}(s)| = n$ $\quad |\mathrm{Stab}(s)| = 2$

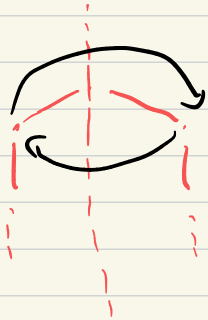
$|D_{2n}| = |\mathrm{Orb}(s)| \cdot |\mathrm{Stab}(s)| = 2n$

Orb(s)



Stab(s)





axis of symmetry