IMC 2022

Second Day, August 4, 2022

Solutions

Problem 5. We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

(proposed by Mike Daas, Universiteit Leiden)

Hint: Call two connecting edges a *cherry*. Double-count cherries.

Solution. 1 Define a *cherry* to be a set of two distinct edges from K_{43} that have a vertex in common. We observe that a monochromatic triangle always contains three monochromatic cherries, and that a polychromatic triangle always contains one monochromatic cherry and two polychromatic cherries. Therefore we study the quantity $2M - P$, where M is the number of monochromatic cherries and P is the number of polychromatic cherries. By observing that every cherry is part of a unique triangle, we can split this quantity up into all the distinct triangles in K_{43}. By construction the contribution of a polychromatic triangle will vanish, whereas a monochromatic triangle will contribute 6. We conclude that

$$2M - P = 6 \cdot \{\text{number of monochromatic triangles}\}.$$

Consider any vertex v. Let M_v be the number of monochromatic cherries with central vertex v and P_v the number such polychromatic cherries. It then follows that

$$M_v = \frac{20 \cdot 19}{2} + \frac{22 \cdot 21}{2} = 421 \quad \text{and} \quad P_v = 20 \cdot 22 = 440.$$

In other words, for any vertex v it holds that $2M_v - P_v = 402$. Adding up all these contributions, we find that

$$2M - P = 43 \cdot 402.$$

We conclude that there are $43 \cdot 402/6 = 43 \cdot 67 = 2881$ monochromatic triangles in total. Since 2022 of these were blue, 859 must be red.

Problem 6. Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, ..., x_{p-1})$ of the numbers $(1, 2, ..., p-1)$ such that

$$x_1x_2 + x_2x_3 + ... + x_{p-2}x_{p-1} \equiv 2 \pmod{p}.$$

(proposed by Giorgi Arabidze, Tbilisi Free University, Georgia)

Hint:

Solution 1. We show such a permutation.

Let $x_i \equiv i^{-1} \pmod{p}$ for $i = 1, 2, \cdots, p-1$. Then

$$\sum_{i=1}^{p-2} x_i x_{i+1} \equiv \sum_{i=1}^{p-2} \frac{1}{i} \cdot \frac{1}{i+1} \equiv \sum_{i=1}^{p-2} \left(\frac{1}{i} - \frac{1}{i+1} \right) \equiv 1 - \frac{1}{p-1} \equiv \frac{p-2}{p-1} \equiv 2 \pmod{p}$$

Solution 2. We begin by noting that the identity permutation yields the value

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (p-2)(p-1) = 2 \cdot \binom{p}{3} \equiv 0 \pmod{p}$$

as soon as $p > 3$. The idea now is to perturb that permutation to obtain the desired value 2.

One thing we can do is to replace $(i, i+1, i+2, i+3)$ by $(i, i+2, i+1, i+3)$. Indeed, this will decrease the sum by 3. So if $p \equiv 2 \pmod{3}$, we can just take the permutation $(1, 3, 2, 4, 6, 5, 7, \ldots, p-4, p-2, p-3, p-1)$ i.e. exchanging $3k-1$ and $3k$ whenever $k = 1, 2, \ldots, \frac{p-2}{3}$. This means we decrease the sum $\frac{p-2}{3}$ times by 3, leading to a remaining sum of $-(p-2) \equiv 2 \pmod{p}$.

If $p \equiv 1 \pmod{3}$, this strategy does not work immediately. Instead, we can change $(1, 2, 3, 4, 5)$ to $(1, 4, 3, 2, 5)$ resulting in a decrement of the sum by 8. If we then exchange $3k$ and $3k+1$ for $k = 2, 3, \ldots, \frac{p-7}{3}$ as before, we get another $\frac{p-10}{3}$ times a decrement by 3, leading to a remaining sum of $-8 - \frac{p-10}{3} \cdot 3 \equiv 2 \pmod{p}$.

Of course this only works if $p \geq 13$. It thus remains to consider the cases $p = 3$ and $p = 7$ by hand. For $p = 3$, we just take $(1, 2)$ and for $p = 7$ we can take $(1, 4, 5, 2, 3, 6)$.

Problem 7. Let $A_1, A_2, \dots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_i A_j = -A_j A_i \quad \text{for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

(A matrix A is called idempotent if $A^2 = A$.)

(proposed by Danila Belousov, Novosibirsk)

Hint: Consider the trace and the rank of A.

Solution 1.

Lemma. For any idempotent matrix B

$$\operatorname{tr}(B) = \operatorname{rank}(B)$$

Proof. Observe that an idempotent matrix satisfies the equation $\lambda(1-\lambda) = 0$. Hence the minimal polynomial is a product of linear factors and the matrix is diagonalizable. Therefore, the rank of the matrix equals the number of non-zero eigenvalues. Since the matrix has eigenvalues 0 or 1, this provides that the trace is equal to the number of unity eigenvalues, or non-zero eigenvalues.

It can be shown that $\sum\limits_{i=1}^{k} A_i$ is also an idempotent. Indeed,

$$\left(\sum_{i=1}^{k} A_i\right)^2 = \sum_{i=1}^{k} A_i^2 + \sum_{i \neq j} (A_i A_j + A_j A_i) = \sum_{i=1}^{k} A_i$$

Applying the lemma one can obtain

$$\sum_{i=1}^{k} rank(A_i) = \sum_{i=1}^{k} tr(A_i) = tr\left(\sum_{i=1}^{k} A_i\right) = rank\left(\sum_{i=1}^{k} A_i\right) \leqslant n$$

The required inequality follows.

Solution 2. We first prove that for idempotents A, B with $AB = -BA$ we already must have $AB = BA = 0$. Indeed, it is clear that $ABx = BAx = 0$ for $x \in \ker(A)$ so it suffices to prove the same for $x \in \operatorname{im}(A)$, i.e. when $Ax = x$. But then writing $Bx = y$ we have $Ay = -y$ i.e. $y = -Ay = -A^2 y = Ay = -y$ and hence $y = 0$ so that again $ABx = BAx = 0$.

Henceforth, we can assume the stronger condition $A_i A_j = 0$ for all $i \neq j$. We next claim that all the image spaces V_i of A_i are linearly independent. This will imply the claim, since then the sum of their dimensions can be at most n, and so one of them has to be $\leq \frac{n}{k}$. Now, for the sake of contradiction, suppose that $\sum_i v_i = 0$ with $v_i \in V_i$ and w.l.o.g. $v_1 \neq 0$. But then

$$0 = A_1(v_1 + \cdots + v_k) = v_1 + A_1 v_2 + \cdots + A_1 v_k = v_1 + A_1 A_2 v_2 + \cdots + A_1 A_k v_k = v_1$$

since $A_1 A_i = 0$ for all i.

Remark. Here is a different argument for $AB = BA = 0$, without eigenvectors: multiplying by A and using its idempotence and the super-commutativity , we have

$$-BA = AB = A^2 B = AAB = -ABA = BAA = BA^2 = BA$$

thus $BA = 0$.

Problem 8. Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

(proposed by Fedor Petrov, St. Petersburg)

Hint:

Solution 1. We prove that

$$E(m) = \frac{2kn}{n+k-1} - 2\frac{k!n!}{(k+n-1)!}.$$

Let $A_1, \ldots, A_n$ be blue points. Fix $i \in \{1, \ldots, n\}$. Enumerate our $n+k$ points starting from a blue point A_i counterclockwise as $A_i, X_{1,i}, X_{2,i}, \ldots, X_{(n+k-1),i}$. Denote the minimal index j for which the point $X_{j,i}$ is blue as $m(i)$. So, $A_i X_{m(i),i}$ is a side of the convex hull of blue points. Denote by b_i the following random variable:

$$b_i = \begin{cases} 1, & \text{if the chord } A_i X_{m(i),i} \text{ contains a side of } F \\ 0, & \text{otherwise.} \end{cases}$$

Define analogously k random variables $r_1, \ldots, r_k$ for the red points. Clearly,

$$m = b_1 + \ldots + b_n + r_1 + \ldots + r_k. \qquad (\heartsuit)$$

We proceed with computing the expectation of each b_i and r_j. Note that $b_i = 0$ if and only if all red points lie on the side of the line $A_i X_{m(i),i}$. This happens either if $m(i) = 1$, *i.e., the point* $X_{i,1}$ *is blue* (which happens with probability $\frac{n-1}{k+n-1}$), or if $i = k+1$, *points* $X_{1,i}, \ldots, X_{k,i}$ *are red, and points* $X_{k+1,i}, \ldots, X_{k+n-1,i}$ *are blue* (which happens with probability $1/\binom{k+n-1}{k}$, since all subsets of size k of $\{1, 2, \ldots, n+k-1\}$ have equal probabilities to correspond to the indices of red points between $X_{1,i}, \ldots, X_{n+k-1,i}$). Thus the expectation of b_i equals $1 - \frac{n-1}{k+n-1} - 1/\binom{k+n-1}{k} = \frac{k}{n+k-1} - \frac{k!(n-1)!}{(k+n-1)!}$. Analogously, the expectation of r_j equals $\frac{n}{n+k-1} - \frac{n!(k-1)!}{(k+n-1)!}$. It remains to use $(\heartsuit)$ and linearity of expectation.

Solution 2. Let $C_1, \ldots, C_{n+k}$ be the colours of the points, scanned counterclockwise from a fixed point on the circle. We consider the sequence as cyclic (so C_{n+k} is also adjacent to C_1). There are two cases: Either (i) all red points appear contiguously, followed by all blue points contiguously, or (ii) the red and blue points alternate at least twice. It can be seen that in the second case, m is exactly equal to the number of colour changes in the C_i sequence: For example, if C_i is red and C_{i+1} is blue, then the intersection of the red chord from C_i to the next red point with the blue chord from C_{i+1} to the previous blue point is a vertex of F, and every vertex is of this form. Case (i) is exceptional, as we have two colour changes, but $m = 0$, so it is 2 less than the number of changes in that case.

Now observe that the distribution of C_i is purely combinatorial: Each of the $\binom{n+k}{n,k}$ distributions of colours is equally likely (for example, because we can generate the distribution by first choosing all $n+k$ points on the circle, and then assigning colours uniformly). In particular the probability that $C_i C_{i+1}$ is a colour change is exactly $\frac{2nk}{(n+k)(n+k-1)}$, and by linearity of expectation, the total expected number of color changes (including $i = n+k$) is $n+k$ times this, i.e. $\frac{2nk}{n+k-1}$.

To get the expected value of m, we must subtract from the above 2 times the probability of case (i). Exactly $n+k$ of the $\binom{n+k}{n,k}$ distributions belong to case (i), so we must subtract $2(n+k)\binom{n+k}{n,k}^{-1} = 2\frac{n!k!}{(n+k-1)!}$, as claimed.

Solution 3. Let $A_1, \dots, A_n$ be the blue points and $B_1, \dots, B_k$ be the red points. For every pair of blue points A_i, A_j, $1 \leqslant i < j \leqslant n$, we evaluate the probability p that A_iA_j contains a side of F (it obviously does not depend on the choice of i and j). By q denote the analogous probability for the red points. Then by linearity of expectation we have $\mathbb{E}m = \binom{n}{2}p + \binom{k}{2}q$.

We proceed with finding p. Without loss of generality $i = 1$, $j = 2$. Let the length of the circle be 1, and the length of arc A_1A_2 (counterclockwise from A_1 to A_2) be x. Then x is uniformly distributed on $[0, 1]$. Then A_1A_2 contains a side of F if

(i) all blue points are on the same side of A_1A_2, but

(ii) the red points are not on the same side of A_1A_2.

The probability of (i) is $x^{n-2} + (1-x)^{n-2}$. The probability of (ii) is $1 - (x^k + (1-x)^{n-k})$.

Thus, using Beta function value $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = B(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ for positive integers a, b

$$p = \int_0^1 (x^{n-2} + (1-x)^{n-2})(1 - (x^k + (1-x)^{n-k}))dx = \frac{2}{n-1} - \frac{2}{n+k-1} - 2B(n-1, k+1)$$
$$= \frac{2}{n-1} - \frac{2}{n+k-1} - 2\frac{(n-2)!k!}{(n+k-1)!}.$$

Next,

$$\binom{n}{2}p = n - \frac{n(n-1)}{n+k-1} - \frac{n!k!}{(n+k-1)!} = \frac{nk}{n+k-1} - \frac{n!k!}{(n+k-1)!},$$

and by symmetry $\binom{k}{2}q$ takes the same value (that is in agreement with the observation that red and blue sides of F alternate).