定义：

前向差分： $\Delta f(x_0) = f(x_0+1) - f(x_0)$

n阶前向差分： $\Delta^n f(x_0) = \Delta^{n-1} f(x_0+1) - \Delta^{n-1} f(x_0)$

$$\Delta^0 f(x_0) = f(x_0)$$

牛顿逆级数：

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k f(0)$$

其中 $\binom{x}{k} = \dfrac{x(x-1)\cdots(x-k+1)}{k!}$ （本质为牛顿插值）

通过牛顿多项式将离散求和延拓至连续

令 $S(x) = \sum_{n=0}^{x} f(n)$

则 $S(x) = \sum_{k=1}^{n} f(k) + \sum_{k=n+1}^{x+n} f(k) - \sum_{k=x+1}^{x+n} f(k)$

$$= \sum_{k=1}^{n} [f(k) - f(n+k)] + \sum_{k=1}^{x} f(n+k)$$

当 $n \to +\infty$ 时

$$S(x) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} [f(k) - f(n+k)] + \sum_{k=1}^{x} f(n+k) \right\}$$

其中 $\sum_{k=1}^{x} f(n+k) = \sum_{k=1}^{x}\left[f(n)+\sum_{i=0}^{k-1}\Delta f(n+i)\right]$

$= x f(n) + \sum_{k=1}^{x}\sum_{i=0}^{k-1}\left[\Delta f(n) + \sum_{j=0}^{i-1}\Delta^2 f(j)\right]$

$\vdots$

$= \sum_{k=1}^{\infty}\left[\binom{x}{k}\Delta^{k-1} f(n)\right]$

$$S(x) = \sum_{k=1}^{\infty}\left[f(k) - f(x+k)\right] + \underbrace{\sum_{k=1}^{\infty}\binom{x}{k}\Delta^{k-1} f(n)}_{\text{牛顿逆级数}}$$

例:

$$H_n = \sum_{i=1}^{n}\frac{1}{i} = \sum_{i=1}^{\infty}\left[f(i) - f(i+n)\right]$$

$$\overset{n\to+\infty}{=\!=\!=} \ln n + \gamma$$