Irreducible polynomial

Prop. (Eisenstein test)

$(R,+,\cdot)$ u-f-d $F = \mathrm{Frac}(R)$ $f(x) \in R[x]$ $\deg(f) \neq 0$

$f(x) = a_n x^n + \cdots + a_0$

if $\exists$ p prime $\in R$ s.t.

$p \nmid a_n$ & $p \mid a_{n-1}$ ... $p \mid a_0$ & $p^2 \nmid a_0$

$\Rightarrow$ $f(x)$ irreducible on $F[x]$

furthermore, if $\mathrm{cont}(f) = 1$, $f(x)$ irreducible on $R[x]$

Proof. If $f(x)$ irreducible on $F[x]$

$\exists$ $g(x) \cdot h(x) \in F[x]$, s.t.

$g(x) = b_0 + \cdots + b_m x^n$

$h(x) = c_0 + \cdots + c_l x^l$

$f(x) = g(x) \cdot h(x)$

suppose $p \mid b_0$ $p \nmid c_0$ $p \mid b_k$ for $0 \le k < l \le n-1$

when $k = l$

$p \mid a_l = b_0 c_l + \cdots + b_l c_0$

thus $p \mid b_l c_0 \Rightarrow p \mid b_l$

because $p \nmid a_n$. $a_n = b_0 c_n + \cdots + b_n c_0$

thus $p \nmid b_n$ thus $m = n$

$h(x) = c \in F \setminus \{0\}$

$f(x) = c g(x)$

however, c is an unit of F and $F[x]$

thus $f(x)$ irreducible on $F[x]$ contradiction

example $f(x) = x^3 + 20x^2 - 6x + 2$

$R = \mathbb{Z}$ $p = 2$. $f(x)$ irreducible on $Q[x] = \mathrm{Frac}(R)$

$\Rightarrow f(x)$ irreducible on $\mathbb{Z}[x]$