$a^2+b^2+c^2\le 3$

proof. $abc+2\ge ab+bc+ac$

Proof.

set $f(a,b,c) = abc - \sum_{cyc} ab + 2$,

$a\ge b\ge c$, $a^2+b^2+c^2 = n \le 3$, $a\ge\sqrt{\frac{n}{3}}$

$$\frac{\partial f}{\partial a} = 2-b-c$$

$$b^2+c^2 = n-a^2 \le \frac{2n}{3} \Longrightarrow \frac{(b+c)^2}{2} \le b^2+c^2 \le \frac{2n}{3} \Rightarrow b+c\le\sqrt{\frac{4n}{3}}$$

thus $\frac{\partial f}{\partial a}\ge 0$

$$f_{min}(a,b,c) = f\left(\sqrt{\frac{n}{3}},\sqrt{\frac{n}{3}},\sqrt{\frac{n}{3}}\right) = \left(\frac{n}{3}\right)^{\frac{3}{2}} -n+2 \ge 0$$

Q.E.D.

$x_1^2+\cdots+x_n^2\le n$

proof. $2+(n-2)\prod x_i \ge \sum_i \prod_{j\ne i} x_j$

Proof.

上题同样解法，此处为另一种解法

equivalent to $F(x_i) = 2\prod\frac{1}{x_i} + (n-2) - \sum\frac{1}{x_i}$

set $x_3^2+\cdots+x_n^2 = R \le n$

$$F = \frac{1}{x_3\cdots x_n}\left(\frac{1}{x_1x_2}\right) - \frac{1}{x_1} - \frac{1}{x_2} + (n-2) - \left(\frac{1}{x_3}+\cdots+\frac{1}{x_n}\right)$$

set. , $x_1^2+x_2^2 = S \le n$. $T = \frac{1}{x_3\cdots x_n}$

$$f(x_1) = T\frac{1}{x_1x_2} - \frac{1}{x_1} - \frac{1}{x_2} \qquad x_2 = \sqrt{5-x_1^2}$$

where $x_2\, dx_2 = -x_1\, dx_1$

$$f'(x_1) = \frac{x_1 - x_2}{x_2} \quad \frac{T(x_1+x_2) - (x_1^2 + x_1x_2 + x_2^2)}{x_1^2x_2^2}$$

notice that:

$$1 \le \frac{(x_1^2 + x_1x_2 + x_2^2)^2}{(x_1+x_2)^2\,(x_1^2+x_2^2)} \le \frac{9}{8}$$

thus $(x_1^2 + x_1x_2 + x_2^2) \le \frac{3}{2\sqrt{2}}\sqrt{5}\,(x_1+x_2)$

if $T \ge \frac{3}{2\sqrt{2}}\sqrt{5} \Rightarrow f(x_1) \quad (0, \sqrt{\frac{5}{2}})\downarrow \quad (\sqrt{\frac{5}{2}}, \sqrt{5})\uparrow$

where $(x_1^2 + x_2^2)\, x_3^2 \cdots x_n^2 \le \left(\frac{x_1^2 + \cdots x_n^2}{n-1}\right)^{n-1} \le e < 3 < \frac{32}{9}$

thus $x_1 = x_2 \cdots \cdots = x_n$ minimum

Q.E.D.