$$\int \frac{1}{1+x^2}\,dx = \int \frac{i}{2}\left(\frac{1}{x-i} + \frac{1}{x+i}\right)dx$$

$$= \frac{i}{2}\ln\left(\frac{x-i}{x+i}\right) + C_0$$

$$= \frac{i}{2}\ln e^{2i\arctan\frac{1}{x}} + C_0$$

$$= \arctan x + C$$

$$\int x^i\,dx = \frac{1}{1+i}x^{1+i} + C = \int \cos(\ln x)\,dx + i\int \sin(\ln x)\,dx$$

$$= x(1-i)\left(\cos(\ln x) + i\sin(\ln x)\right) + C$$

$Re = Re$ $\qquad Im = Im$

If $f(x,u)$ is continuous on $[a,+\infty) \times [\alpha,\beta]$

$$I(u) = \int_a^{+\infty}$$

Frullani Integral

proof 1. $\displaystyle\int_0^{+\infty} \frac{f(ax)-f(bx)}{x} = \int_a^b \left(\int_0^{+\infty} f'(xt)\,dx\right)dt$

$$= \int_a^b [f(+\infty) - f(0)] \cdot \frac{1}{t}\, dt$$

$$= [f(+\infty) - f(0)] \ln\left(\frac{a}{b}\right)$$

proof 2

$$\int_\delta^\Delta \frac{f(ax)}{x} dx - \int_\delta^\Delta \frac{f(bx)}{x} dx$$

$$= \int_{a\delta}^{a\Delta} \frac{f(z)}{z} dz - \int_{b\delta}^{b\Delta} \frac{f(z)}{z} dz$$

$$= \int_{a\delta}^{b\delta} \frac{f(z)}{z} dz - \int_{a\Delta}^{b\Delta} \frac{f(z)}{z} dz$$

$$= -f(\xi) \ln\frac{a}{b} + f(\mu) \ln\frac{a}{b}$$

$$= [f(+\infty) - f(0)] \ln\frac{a}{b}$$

if $f(0)$ or $f(+\infty)$ doesn't exist

$$I = \ln\frac{a}{b} \cdot f(0) \quad \text{or} \quad \ln\frac{a}{b} \cdot f(+\infty)$$

$$\frac{1}{1+x^{2024}} = \sum_{k=0}^{2023} \frac{A_k}{x-e^{\frac{(2k+1)\pi i}{2024}}}$$

$$1 = \sum_{i=0}^{2023}\left[\prod_{k\neq i}\left(x-e^{\frac{(2k+1)}{2026}\pi i}\right)\right]A_k$$

when $x = e^{\frac{(2j+1)}{2024}\pi i}$

$$1 = \left[\prod_{k\neq j}\left(x-e^{\frac{(2k+1)}{2024}\pi i}\right)\right]A_j$$

$$= \frac{x^{2024}+1}{x-e^{\frac{(2j+1)}{2024}\pi i}}A_j$$

$$A_j = \frac{1}{2024\,x^{2023}} = \frac{\frac{x}{x^{2024}}}{2024} = -\frac{e^{\frac{(2j+1)}{2024}\pi i}}{2024}$$

$$\sum_{k=0}^{2023}\frac{A_k}{x-e^{\frac{(2k+1)}{2024}\pi i}} = \sum_{k=0}^{2023}\left(-\frac{e^{\frac{(2k+1)}{2024}\pi i}}{2024}\right)\frac{1}{x-e^{\frac{(2k+1)}{2024}\pi i}}$$

$$= \frac{1}{2024}\sum_{k=0}^{2023}\frac{1}{1-\frac{x}{e^{\frac{2k+1}{2024}\pi i}}}$$

$$= \frac{1}{2024}\sum_{k=0}^{2023}\sum_{m=0}^{+\infty}\left(x\cdot e^{-\frac{2k+1}{2024}\pi i}\right)^m$$

$$= \frac{1}{2024}\sum_{m=0}^{+\infty}\sum_{k=0}^{2023}\left(x\,e^{-\frac{2k+1}{2024}\pi i}\right)^m$$

首 $= \left(x e^{-\frac{1}{2024}\pi i}\right)^m$

公比 $= e^{-\frac{2}{2024}\pi i m}$

$$x^m e^{-\frac{1}{2024}\pi i}\cdot\frac{e^{-2\pi i m}-1}{e^{-\frac{1}{1012}\pi m i}-1}$$

$$\int \frac{1}{1+x^{2024}} =$$

$$\int \sin(mx) \cdot \frac{e^{-ax}-e^{-bx}}{x} dx \overset{①}{=} \int_0^{+\infty} \sin(mx) \int_b^a e^{-tx}\, dt\, dx$$

② $I(m) = \int \sin(mx) \frac{e^{-ax}-e^{-bx}}{x} dx$

$I(0) = 0$

$I'(m) = \int \cos(mx) \cdot (e^{-ax} - e^{-bx}) dx$

It's equivalent to say that if right/left multiply a matrix with determinant 1 the determinant is unchanged

$$\begin{cases} x\lambda_1 + a_n = 0 \\ -\lambda_1 + x\lambda_2 + a_{n-1} = 0 \\ \vdots \\ -\lambda_{n-2} + x\lambda_{n-1} + a_2 = 0 \\ -\lambda_{n-1} + a_1 = 0 \end{cases}$$

$$\lambda_1 = -\frac{a_n}{x}$$