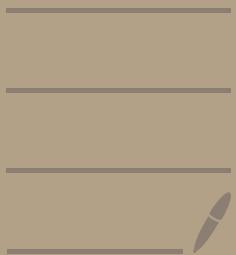


Localization of Ring



Def. $(R, +, \cdot)$ commutative, SCR
 call S a multiplicative subset
 if $1 \in S$ or $\forall a, b \in S, ab \in S$ is submonoid of $(R \setminus \{0\}, \cdot)$

Def. $(R, +, \cdot)$ commutative S multiplicative subset
 R localization of S , denote as $(S^{-1}R, +, \cdot)$, defined as
 $S^{-1}R = \left\{ \frac{r}{s} : r \in R, s \in S \right\} / \sim$
 where $\frac{r}{s} \sim \frac{r'}{s'} \Leftrightarrow \exists t \in S, t(rs' - r's) = 0$
 if $r, r' \in R, s, s' \in S$, define:
 $\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$
 $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$

Prop. $(S^{-1}R, +, \cdot)$ is a commutative Ring

Proof. (I) to prove " \sim " is a relation of equivalence

suppose $\frac{r}{s} \sim \frac{r'}{s'}, \exists t \in S$ st.
 $t(rs' - r's) = 0$

$$\text{thus } \frac{r}{s} = \frac{r'}{s'}$$

$$\text{if } \frac{r}{s} \sim \frac{r''}{s''}, \frac{r}{s} \sim \frac{r''}{s''}$$

for $t, t' \in S$ st.

$$t(rs' - r's) = 0 \quad \& \quad t'(r's'' - r''s') = 0$$

$$\text{thus } \exists t, t' \in S, t'' = tt's'$$

$$\begin{aligned} \text{thus } (tt's')rs'' &= t's''(trs') = t's''(tr's) = ts(t'r's'') \\ &= ts(t'r''s') = (tt's')r''s \end{aligned}$$

$$\text{thus } t''(rs'' - r''s) = 0$$

(III) to prove $S^{-1}R$ is a Ring

for any $r/s \in S^{-1}R$

$$\frac{0}{1} + \frac{r}{s} = \frac{0 \cdot s + 1 \cdot r}{1 \cdot s} = \frac{r}{s} \quad (\text{Identity})$$

$$\frac{1}{1} \cdot \frac{r}{s} = \frac{r}{s}$$

$$\left(\frac{r_1}{s_1} + \frac{r_2}{s_2} \right) + \frac{r_3}{s_3} = \frac{r_1 s_2 s_3 + s_1 r_2 s_3 + s_1 s_2 r_3}{s_1 s_2 s_3} = \frac{r_1}{s_1} + \left(\frac{r_2}{s_2} + \frac{r_3}{s_3} \right)$$

(Associative law of multiplication)

$$\frac{r}{s} + \left(-\frac{r}{s} \right) = 0 \quad (\text{Inverse of addition})$$

$$\frac{r_1}{s_1} \cdot \left(\frac{r_2}{s_2} + \frac{r_3}{s_3} \right) = \frac{r_1 (r_2 s_3 + r_3 s_2)}{s_1 s_2 s_3} = \frac{r_1 r_2 s_3}{s_1 s_2 s_3} + \frac{r_1 r_3 s_2}{s_1 s_2 s_3} = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} + \frac{r_1}{s_1} \cdot \frac{r_3}{s_3}$$

(Distribution law)

Prop. $(R, +, \cdot)$ integral ring S multiplicative subset

$$\frac{a}{b} \cap \frac{c}{d} \iff ad - bc = 0$$

Pet. $(R, +, \cdot)$ an integral ring fraction field on R , denote as $\text{Frac}(R)$ defined as $S^{-1}(R)$, where $S = R \setminus \{0\}$ or sth. else

Prop. $\text{Frac}(R)$ is a field

Lemma. $p \trianglelefteq R$ is a prime ideal $\Rightarrow S = R \setminus p$ is a multiplicative subset