Simple group.

A group is non-trival & has no non-trival normal proper subgroup. called simple group.

suppose non-trival finite group G

$N_1 \lhd G$ is the normal proper subgroup of the greatest order

according to the 4^{th} isomorphism theorem

G/N_1 is shaped like H/N_1 where $N_1 \leq H \lhd G$

thus $H = G$ or $H = N_1$

thus G/N_1 is simple

if $N_1 \neq \{e\}$, continue the process above

until get: $G = N_0 \geq N_1 \geq N_2 \cdots \geq N_k = \{e\}$

where N_i is normal & N_i / N_{i+1} is simple

If $n \geq 5$ A_n is non-commutative simple group

Proof

non-commutativity:

$(123)(124) \neq (124)(123)$

Simplity:

$\exists$ lemma: $n \geq 3$, if $N \lhd A_n$ consist of All 3-cycle, $N = A_n$

thus prove $\forall N \lhd A_n$ consists of All 3-cycle

suppose $H \lhd A_n$, $H \neq \{(1)\}$

for $\sigma \in S_n$, if $\sigma(i) = i$, call it a fixed point of σ

suppose $\tau \neq (1)$ has the largest number of fixed point.

denoted $D(\tau)$

now prove $D(\tau) = n-3$ (H has one 3-cycle)

(I) $D(\tau) \neq n-2$, or τ is a odd-permutation.

(II) $D(\tau) \neq n-1$, or τ is trival

(III) if $D(\tau) < n-3$

(1) if $\tau = (1\,2\,3\,\cdots)\cdots$

then τ has at least 5 moving point

suppose $1\ 2\ 3\ 4\ 5$ are 5 moving points

$\sigma = (3\ 4\ 5) \in A_n$

by definition $\tau' = \sigma\tau\sigma^{-1} \in H$

$\tau_1 = \tau^{-1}\tau' \in H$

then $\tau_1(1) = 1$

thus $D(\tau_1) > D(\tau)$ contradiction

(2) if $\tau = (1\,2)(3\,4)\cdots$

$\tau_2 = \tau^{-1}\sigma\tau\sigma^{-1}$

$\tau_2(1) = 1 \quad \tau_2(2) = 2$

at most one fixed point changes under τ_2

thus $D(\tau_2) > D(\tau)$ contradiction.

now prove H contains all 3-cycle

for $(i, j, k) \in A_n$

suppose $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ i & j & k & x_4 & x_5 & \cdots & x_n \end{pmatrix}$ $\tau = (1\,2\,3)$

$(i\ j\ k) = \pi\tau\pi^{-1} \in H$

Q(uadrangle). E(quals). D(amn).

$n \geq 5$. A_n is the only non-trival normal proper subgroup.