1. $p(x) = \dfrac{1}{(1-x)(1-x^2)\cdots\cdots} = \sum_{n=0}^{\infty} p(n) \cdot x^n \qquad |x|<1$

$p_\Delta(x) = \dfrac{1}{(1-x^2)(1-x^3)\cdots} = \sum_{n=0}^{\infty} \alpha(n) \cdot x^n \qquad |x|<1$

$$\sum p(n) - p(n-1) x^n = \frac{1}{(1-x^2)\cdots} = \sum \alpha(n) x^n$$

2. $n=1$

$\inf \operatorname{rank}(A) = 0$

$n = 2$

$\inf \operatorname{rank}(A) = 2$

$n = 3$

$\det A = a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} \neq 0$

$\inf \operatorname{rank}(A) = 3$

$n > 3$

$\inf \operatorname{rank}(A) \geq 3$

now construct A that $\operatorname{rank} A = 3$

$$A = \begin{pmatrix} 0^2 & 1^2 & 2^2 & \cdots \\ & & & \\ \vdots & \cdots & \cdots & 0^2 \end{pmatrix} = \left((i-j)^2 \right)_{i,j=1}^{n}$$

$$= \underbrace{\begin{pmatrix} 1^2 \\ 2^2 \\ \vdots \\ n^2 \end{pmatrix} (1, 1, \cdots, 1)}_{\text{rank ① }=1} - 2 \underbrace{\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} (1\ 2\ \cdots\ n)}_{\text{rank ② }=1} + \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1^2, 2^2, \cdots, n^2)}_{\text{rank ③ }=1}$$

$$\text{rank}(A) \le \text{rank ①} + \text{rank ②} + \text{rank ③}$$
$$= 3$$

3. when $n=2$

Player A can win by select the identity

when $n=3$

Player A can win by select the 3-cycle i.e.: (123)

4. Lemma 1. $\lim_{t\to+\infty} f(t)$ doesn't exist or $\lim_{t\to+\infty} f(t) \neq +\infty$

Proof.

Assume $\lim_{t\to\infty} f(t) = +\infty$

then for $t > T_1 > 0$, $f(t) > 2$

for $t > T_2 > 0$, $f(t) > T_1$

hence

for $t > T_2$, $f'(t) > f(f(t)) > 2$

then

for $t > T_3$, $f(t) > t$

then

for $f(t) > t > T_3$, $f'(t) > f(f(t)) > f(t)$

$$\frac{f'(t)}{f(t)} > 1$$

$$\int_{T_3}^{t} \frac{f'(t)}{f(t)} dt > \int_{T_3}^{t} dt$$

$$\ln f(t) - \ln T_3 > t - T_3$$

then

$$f'(t) > f(f(t)) > T_3 e^{f(t) - T_3}$$

$$\int_{T_3}^{t} f'(t) e^{-f(t)} dt > \int_{T_3}^{t} T_3 e^{-T_3} dt$$

$$+\infty > e^{-f(T_3)} - e^{-f(t)} > (t - T_3) T_3 e^{-T_3} > +\infty$$ contradiction

Lemma 2. For all $t > 0$, $f(t) < t$

Proof.

⋮