Name: Key

Midterm 1

Math 256

Spring 2023

You have 50 minutes to complete this exam and turn it in. You may use a 3x5 inch two-sided handwritten index card and a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

You should expect to spend about one minute per question per point it's worth — there are 50 points possible on the exam and 50 minutes total.

Part I (6 points) Multiple choice. You don't need to show your work.

1. (2 points) The DE $\frac{y}{t}y' = 1$ is

A) ~~Linear.~~

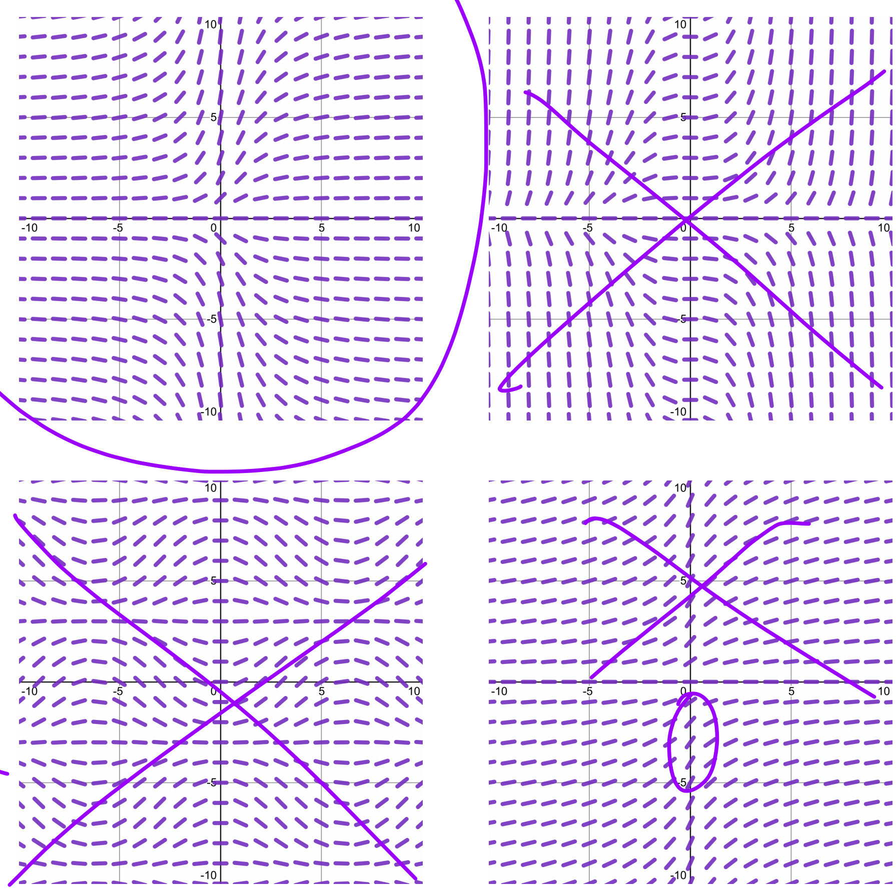
B) Separable.

C) ~~Exact.~~

D) None of the above.

$$y' = \frac{y}{(1+x^2)}$$

2. (2 points) Which of the following direction fields corresponds to $(1 + x^2)y' = y$? (Circle it)



$y' \neq 0$ when $x=0$

$y=0$ not equilibrium

$y' < 0$ when $y < 0$

3. (2 points) A second-order DE has a Wronskian of $W[y_1, y_2] = t^2e^{2t}$. Which of the following initial conditions would **not** have a solution?

A) $y(0) = 1$, $y'(0) = 4$.

$t=0$ bad

B) $y(2) = 1$, $y'(2) = 4$.

C) $y(-2) = 1$, $y'(-2) = 4$.

D) None of the above.

Part II (10 points) Short-answer. Explain your reasoning and show your work for each question.

1. (4 points) Evaluate $\frac{\partial}{\partial x}[\log(\cos(xy))]$.

$$= \frac{1}{\cos(xy)} (-\sin(xy)) \cdot y$$

$t^2y'' - 4ty' + 6y = 0$

2. (6 points) One fundamental solution to ~~$ty'' - y' + 4t^3y = 0$~~ is ~~$y = \sin(t^2)$~~. What is the other fundamental solution?

$y = t^2$

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0$$

$$t^2v'' + (4t - 4t)v' = 0$$

$$t^2v'' = 0$$

$$v'' = 0$$

$$v' = c_1$$

$$v = c_1t + c_2 = t$$

$$y_2 = t \cdot t^2 = \boxed{t^3}$$

Part III (34 points) More involved questions with multiple parts.

1. (22 points) The population of rabbits on an island *measured in thousands* after t years is given by the function $P(t)$, which satisfies the DE $P' + t^2 P = t^2$.

a) (6 points) There are initially 500 rabbits when we start measuring (i.e. at time $t = 0$). Solve for the function P using integrating factors.

$$\mu(t) = \exp \int t^2 \, dt = e^{t^3/3}$$

$$\frac{d}{dt}\left[e^{t^3/3} P\right] = t^2 e^{t^3/3}$$

$$e^{t^3/3} P = \int t^2 e^{t^3/3} \, dt$$

$$P = e^{-t^3/3}\left(e^{t^3/3} + c\right)$$

$$P = 1 + ce^{-t^3/3}$$

$P(0) = \frac{1}{2}$, so $c = -\frac{1}{2}$

$$P = 1 - \frac{1}{2} e^{-t^3/3}$$

b) (6 points) Solve for P using separation of variables.

$$P' = t^2(1-P)$$

$$\frac{1}{1-P} P' = t^2$$

$$\int \frac{1}{1-P} \, dP = \int t^2 \, dt$$

$$-\log(1-P) = \frac{t^3}{3} + c$$

$$1 - P = e^{-t^3/3 - c}$$

$$P = 1 - e^{-\frac{t^3}{3} - c}$$

$$P(0) = \frac{1}{2} = 1 - e^{-c}$$

$$e^{-c} = \frac{1}{2}$$

$$c = -\log\left(\frac{1}{2}\right)$$

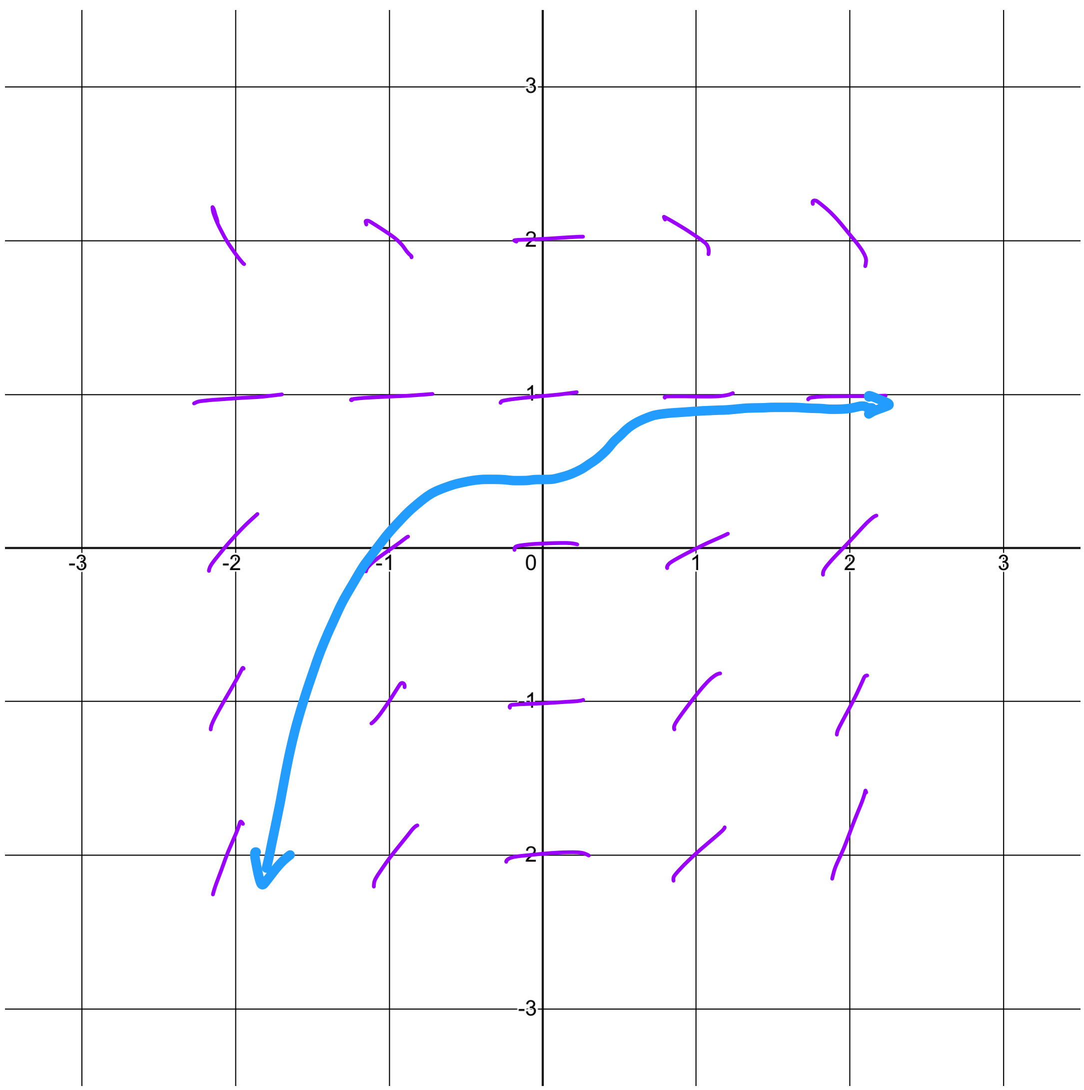
$$P = 1 - \frac{1}{2} e^{-t^3/3}$$

c) (4 points) What will the rabbit population settle down to over time?

$\lim_{t \to \infty} P(t) = 1$, so 1000 rabbits

d) (6 points) Sketch a direction field for the DE below for $-3 \leq t \leq 3$ and $-3 \leq P \leq 3$. Draw your solution curve from parts a) and b).

$P' = t^2(1-P)$



2. (12 points) By changing the coefficients in a DE just slightly, we can produce wildly different solutions — let's see this in action.

a) (4 points) Find the general solution to $y'' + 6y' + 8y = 0$.

$r^2 + 6r + 8 = 0$

$(r+2)(r+4) = 0$

$y = c_1 e^{-2t} + c_2 e^{-4t}$

b) (4 points) Find the general solution to $y'' + 6y' + 9y = 0$.

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$y = c_1 e^{-3t} + c_2 t e^{-3t}$

c) (4 points) Find the general solution to $y'' + 6y' + 10y = 0$.

$$r^2 + 6r + 10 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i$$

$$y = c_1 e^{-3t} \cos(t) + c_2 e^{-3t} \sin(t)$$

d) (2 points extra credit) Solve all of the previous DEs for the particular solutions with $y(0) = 0$ and $y'(0) = 6$. What are the values of $y(1)$ for all three?

1. $c_1 + c_2 = 0, \quad -2c_1 - 4c_2 = 6$

$$-2c_2 = 6$$

$$c_2 = -3$$

$$c_1 = 3$$

$$y = 3e^{-2t} - 3e^{-4t}$$

$$y(1) = 3e^{-2} - 3e^{-4}$$

2. $c_1 = 0, \ c_2 = 6$

$$y = 6te^{-3t}$$

$$y(1) = 6e^{-3}$$

3. $c_1 = 0, \ c_2 = 6$

$$y(1) = 6e^{-3} \sin(1).$$