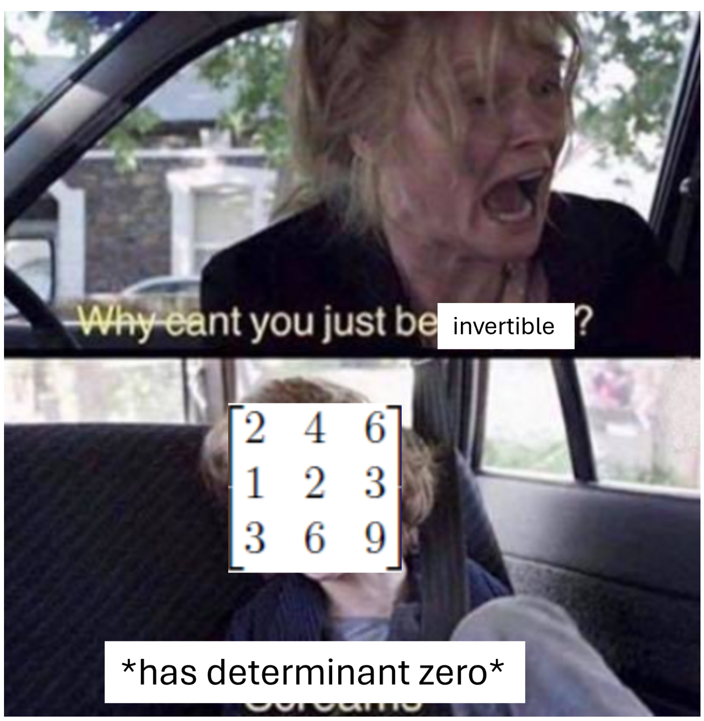
Introduction to Functions of Several Variables Exam 1

Please sign the following Honor Code statement.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of academic integrity.

Signature: May Trix

- There are 75 total points on this exam.
- If you need extra space, you may write on the backs of the pages. In that case, please indicate that you have extra work on another page.
- You are not permitted to use any resources besides a "cheat sheet" of one side of an 8.5×11 inch (or smaller) sheet of paper. Please make sure your name is on your cheat sheet and turn it in with your exam. I will return your cheat sheet at a later date.
- Please give exact answers (e.g., $\sqrt{2}$ instead of 1.41) unless explicitly instructed not to.
- **Please read all instructions carefully and ask for clarification if you are unsure about anything.**



1. (15 points) Short Answer. Briefly show your work (unless explicitly told not to).

(a) (3 points) For what value(s) of a are $\langle -4, a, -3\rangle$ and $\langle -3, 2, -4\rangle$ orthogonal?

$(-4)(-3) + 2a - 3(-4) = 0 \rightarrow 12 + 2a + 12 = 0 \rightarrow 2a = -24$

$a = -12$

(b) (3 points) Write down the matrix representing the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying $T\left(\vec{i}\right) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$, $T\left(\vec{j}\right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, and $T\left(\vec{k}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. No justification necessary.

$$\boxed{\begin{bmatrix} -5 & 3 & 2 \\ 4 & -4 & 5 \end{bmatrix}}$$

(c) (3 points) If M is a 4×10 matrix and rank$(M) = 3$, then what is the dimension of Nul(M)? In other words, how many vectors are in any basis for Nul(M)? No justification necessary.

$10 - 3 = 7$

See the Rank-Nullity Theorem

(d) (3 points) Suppose A is a 2×2 matrix such that $A^{-1} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$. Find a vector $\vec{x}$ in $\mathbb{R}^2$ such that $A\vec{x} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$, or explain why no such $\vec{x}$ exists.

$$\vec{x} = A^{-1}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} -11 \\ -20 \end{bmatrix}$$

(e) (3 points) For what value of b does the linear transformation represented by $A = \begin{bmatrix} 3 & b \\ 1 & -4 \end{bmatrix}$ stretch space by a factor of 7 and reverse (or flip) the orientation of $\mathbb{R}^2$?

Want $\det(A) = -7 \rightarrow 3(-4) - b = -7 \rightarrow -12 - b = -7$

$\rightarrow -b = 5$

$\rightarrow \boxed{b = -5}$

2. (16 points) For each statement, write exactly one of the following. You do not need to justify your answers.

- Write "I" if the given set of vectors **must be** linearly independent.
- Write "D" if the given set of vectors **must be** linearly dependent.
- Write "N" if there is not enough information to determine whether the set of vectors is linearly independent or dependent.

(a) (2 points) The columns of a 3×3 matrix M such that $M \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$.

D The linear transformation represented by M isn't one-to-one.

(b) (2 points) A set of vectors that forms a basis for $\mathbb{R}^5$.

I By definition, a basis is linearly independent.

(c) (2 points) The columns of a 7×7 matrix A, where B is an invertible 7×7 matrix and $\det(AB) = 0$.

D B invertible $\Rightarrow \det(B) \neq 0$. Since $0 = \det(AB) = \det(A)\det(B)$, and $\det(B) \neq 0$, must have $\det(A) = 0$. Thus, linearly dependent.

(d) (2 points) The columns of a 4×2 matrix of rank 2.

I If the columns were linearly dependent, the rank would be either 0 or 1 since there are only two columns.

(e) (2 points) The set of vectors $\{\vec{v}, \vec{w}, \vec{0}\}$ in $\mathbb{R}^3$, where $\vec{v}$ and $\vec{w}$ are nonzero vectors.

D Any set of vectors containing $\vec{0}$ is linearly dependent.

(f) (2 points) The set $\left\{ \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right\}$, where $ad = bc$.

D The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has determinant 0.

(g) (2 points) A set of vectors in $\mathbb{R}^2$ whose span is all of $\mathbb{R}^2$.

N Ex: $\{\vec{i}, \vec{j}\}$ is independent, but $\{\vec{i}, \vec{j}, \vec{0}\}$ is dependent

(h) (2 points) The set $\left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \right\}$, where a, b, and c are all nonzero scalars.

I No vector can be written as a linear combination of the other two because of the zeros.

3. (15 points) Find the line $y = a_0 + a_1 x$ that best fits the data below. **Show your work**. If the columns of the matrix that you attain are linearly independent, you may use this fact without explicitly justifying it.

x	y
-1	-6
0	8
1	4

Want to solve following system:

$$a_0 - a_1 = -6$$
$$a_0 = 8$$
$$a_0 + a_1 = 4$$

However, this system has no solution.

Write $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}$. Need to find best approximation $\vec{z}$ to a solution to $A\vec{x} = \vec{b}$. Columns of A are linearly independent, so

$\vec{z} = (A^T A)^{-1} A^T \vec{b}$.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow (A^T A)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\vec{z} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\vec{b} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 + 8/3 + 4/3 \\ 3 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Thus, the line of best fit is $\boxed{y = 2 + 5x}$

4. (14 points) For the following system of linear equations: Write down the augmented matrix for the system (1 point; no justification necessary), perform elementary row operations to row-reduce your matrix **until it is in reduced row echelon form** (10 points; show your work by labeling your elementary row operations), and interpret your reduced matrix from to find all solutions of the system, or determine that no solutions exist (3 points; no justification necessary).

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 8 \\ 2x & + & 2y & + & 9z & = & 7 \\ & & y & + & 5z & = & -2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{R_3 \to 2R_3 - R_2} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 15R_3} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 3R_3} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

RREF!

$x = 5$

$y = 3$

$z = -1$

$\boxed{(5, 3, -1)}$

5. (14 points) Consider the points $P = (1, -2, 0)$ and $Q = (5, 0, -4)$ in $\mathbb{R}^3$.

(a) (2 points) Find the displacement vector $\overrightarrow{PQ}$. No justification necessary.

$$\overrightarrow{PQ} = \langle 5-1, 0-(-2), -4-0 \rangle = \boxed{\langle 4, 2, -4 \rangle}$$

(b) (3 points) Find a set of parametric equations for the line through P and Q in $\mathbb{R}^3$. No justification necessary.

$x = 1 + 4t$
$y = -2 + 2t$
$z = -4t$

OR

$x = 5 + 4t$
$y = 2t$
$z = -4 - 4t$

(c) (4 points) Find the equation of the form $ax+by+cz = d$ for the plane containing the point $R = (6, -4, -5)$ and orthogonal to $\overrightarrow{PQ}$. Briefly show your work.

$4(x-6) + 2(y+4) - 4(z+5) = 0$
$4x - 24 + 2y + 8 - 4z - 20 = 0$
$4x + 2y - 4z - 36 = 0$

$$\boxed{4x + 2y - 4z = 36}$$

(d) (6 points) Find the point of intersection of the line from (b) and the plane from (c), or determine that they do not intersect. **Show your work**.

$4(1+4t) + 2(-2+2t) - 4(-4t) = 36$
$4 + 16t - 4 + 4t + 16t = 36$
$36t = 36$
$t = 1$

$x = 1 + 4 \cdot 1 = 5$
$y = -2 + 2 \cdot 1 = 0$
$z = -4 \cdot 1 = -4$

$(5, 0, -4)$