Name: ______________________

Midterm 1 Key

Math 256

Spring 2023

You have 50 minutes to complete this exam and turn it in. You may use a 3x5 inch two-sided handwritten index card and a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

You should expect to spend about one minute per question per point it's worth — there are 50 points possible on the exam and 50 minutes total.

Part I (9 points) Multiple choice. You don't need to show your work.

1. (3 points) The DE $\frac{y}{t}y' = 1$ is

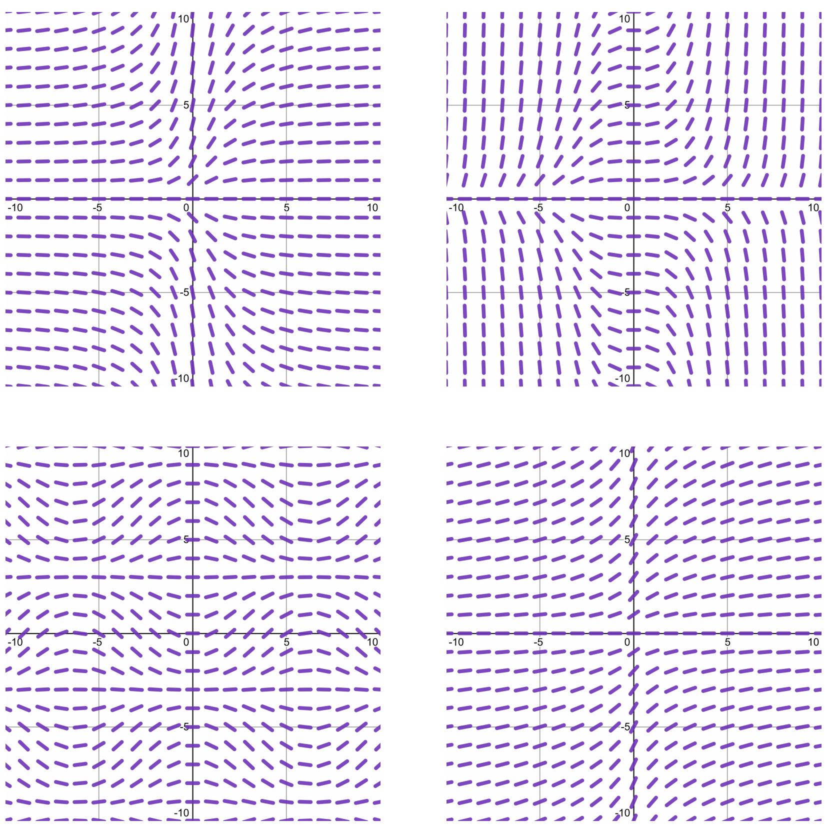
   A) Linear.

   B) **Separable.** Answer C is also acceptable, since you could multiply both sides by t, then subtract t from both sides.

   C) Exact.

   D) None of the above.

2. (3 points) Which of the following direction fields corresponds to $(1 + x^2)y' = y$? (Circle it)



The top-left one. There are many ways to approach this, but notice that when $x = 0$ and $y \neq 0$, y' is nonzero and has sign equal to y. That only matches the top-left field.

Part II (9 points) Short-answer. Explain your reasoning and show your work for each question.

1. (3 points) Evaluate $\dfrac{\partial}{\partial x}\left[\log(\cos(xy))\right]$.

$$\frac{\partial}{\partial x}\left[\log(\cos(xy))\right] = \frac{1}{\cos(xy)}\left(-\sin(xy)\right)(y).$$

Part III (32 points) More involved questions with multiple parts.

1. (22 points) The population of rabbits on an island *measured in thousands* after t years is given by the function $P(t)$, which satisfies the DE $P' + t^2 P = t^2$. There are initially 500 rabbits when we start measuring (i.e. at time $t = 0$).

a) (6 points) Solve for the function P using integrating factors.

We have $\mu(t) = \exp \int t^2\, dt = e^{t^3/3}$, and so the DE collapses to $\frac{d}{dt}\left[e^{t^3/3}P\right] = t^2 e^{t^3/3}$. Integrating the right side results in $e^{t^3/3} + c$ by u-sub, and so $P = 1 + ce^{-t^3/3}$. Since $P(0) = 0.5$ (remember P is in thousands), $P = 1 - 0.5e^{-t^3/3}$.

b) (6 points) Solve for P using separation of variables.

We have $P' = t^2(1-P)$, so the DE separates as $\frac{1}{1-P} = t^2\, dt$. Integrating, $-\log(1-P) = \frac{t^3}{3} + c$, so $P = 1 + ce^{-t^3/3}$. We solve for c identically to part a), so $P = 1 - 0.5e^{-t^3/3}$

c) (4 points) What will the rabbit population settle down to over time?

This is $\lim_{t\to\infty} P(t)$, so 1. Since P is measured in thousands, that means 1000 rabbits.

d) (6 points) Sketch a direction field for the DE below for $-2 \le t \le 2$ and $-2 \le P \le 2$. Draw your solution curve from parts a) and b).