1. (12 points) For what value(s) of k does the following system of linear equations have **(i)** no solutions, **(ii)** exactly one solution, and **(iii)** infinitely many solutions? Please label any elementary row-operations so that someone could follow along with your work. **Make sure to answer all three parts! (i, ii, and iii)**

$$\begin{array}{rcrcrcr} 3x & + & 2y & + & 2z & = & 5 \\ -2x & - & y & - & 3z & = & k \\ & & y & - & 5z & = & 13 \end{array}$$

2. (25 points) The monthly profit P of a cell phone company (**in thousands of dollars**) is given by

$$P(x, y) = 2x^2 + y^2 - 4x + 4y - 2xy - 2,$$

where x is the number of cell phones produced (**in hundreds**) each month and y is the number of hours of advertising purchased each month. **Throughout this problem (part (a) through part (f)), assume that the company is currently producing 300 cell phones per month and purchasing 2 hours of advertising each month.**

(a) (2 points) What is the company's profit at their current level of production?

(b) (4 points) Find $P_x(3, 2)$ and interpret its meaning in the context of the company's profit. Include units.

(c) (5 points) If the company decides to increase their advertising and production by an equal amount (that is, they increase x and y by an equal amount), then what is the instantaneous rate of change in profit?

(Restating problem 2 for you so that you don't have to flip the page.) The monthly profit P of a cell phone company (**in thousands of dollars**) is given by

$$P(x, y) = 2x^2 + y^2 - 4x + 4y - 2xy - 2,$$

where x is the number of cell phones produced (**in hundreds**) each month and y is the number of hours of advertising purchased each month. **Throughout this problem (part (a) through part (f)), assume that the company is currently producing 300 cell phones per month and purchasing 2 hours of advertising each month.**

(d) (6 points) Find a unit vector in the direction in which P increases most rapidly from the company's current position. What is the instantaneous rate of change of P in that direction?

(e) (4 points) Suppose the company decides to increase their advertising by 1 hour per month. If they want the instantaneous rate of change of profit to be 0 after this change, then how should their production of cell phones change?

(f) (4 points) Write down the equation for the tangent plane to the graph of $P(x, y)$ at their current point. Then, use this tangent plane to estimate the profit if 350 cell phones are produced per month and 2.1 hours of advertising are purchased per month.

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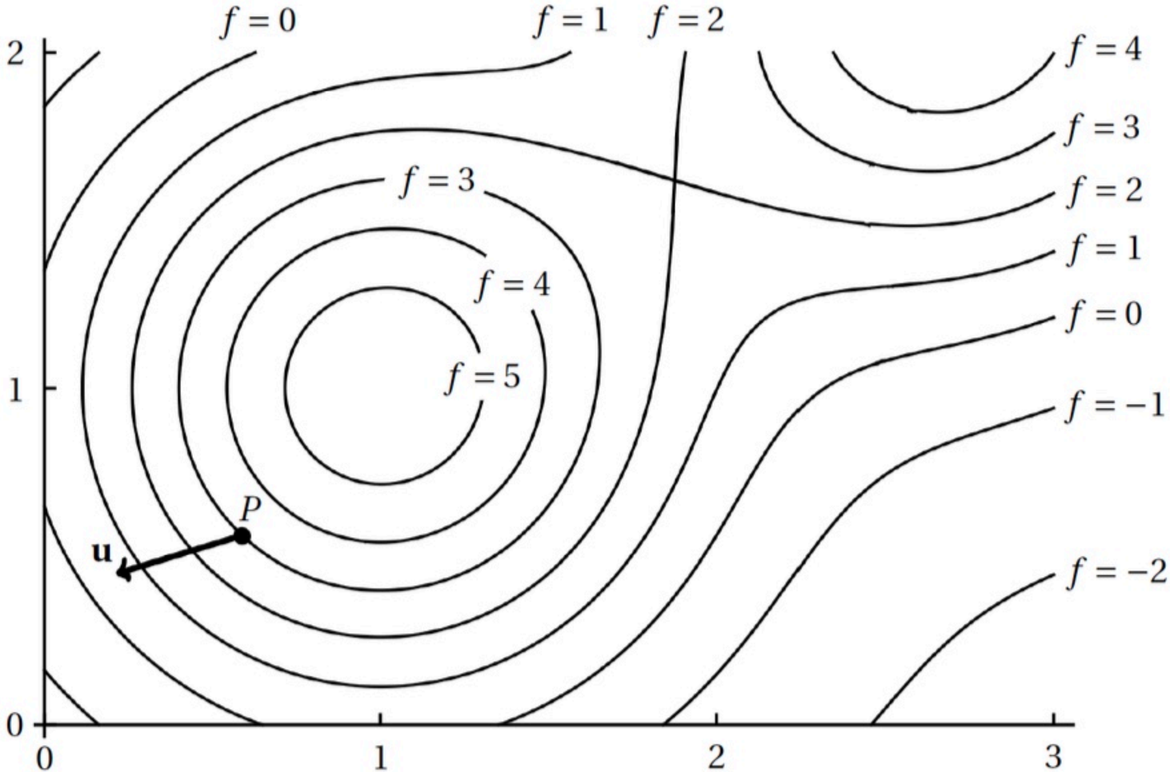

$f = 0$
$f = 1$
$f = 2$
$f = 4$
$f = 3$
$f = 2$
$f = 1$
$f = 0$
$f = -1$
$f = -2$
$f = 3$
$f = 4$
$f = 5$
P
u
2
1
0
0
1
2
3

3. (12 points) Consider the contour plot for a function $f(x, y)$ shown below.

final exam contour.png

(a) (4 points) At point P, determine whether f_x and f_y are positive, negative, or 0. No justification necessary.

(b) (4 points) On the graph, sketch $\nabla f(x, y)$ at the point Q. (I am only concerned with the direction of the vector, not the magnitude.) No justification necessary.

(c) (4 points) Do any points shown in the contour plot appear to correspond to local extrema of f? Justify your answer.

4. (12 points) Find an equation or set of parametric equations for each line or plane.

(a) (4 points) The tangent line to $f(x, y) = \frac{4x}{y^2}$ at $(x, y) = (-2, 2)$ that is parallel to the yz-plane.

(b) (4 points) The plane containing $P = (1, 2, 2)$, $Q = (3, -1, -2)$, and $R = (1, -1, -1)$.

(c) (4 points) The line that makes a right angle with the plane $z = -2x - 3y + 1$ at $(x, y) = (-3, 4)$.

5. (15 points) Consider the following matrix A and its reduced row echelon form R.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & 1 & 0 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (5 points) Find a basis for $\operatorname{Col}(A)$ and determine the rank of A. No justification necessary.

(b) (5 points) Find a basis for $\operatorname{Nul}(A)$.

(c) (5 points) Suppose $\vec{b}$ is a vector in $\mathbb{R}^4$. What are the possibilities for the number of solutions $\vec{x}$ to the equation $A\vec{x} = \vec{b}$, where A is as above? Briefly justify your answer.

6. (15 points) A company wants to design a rectangular box with a square base (i.e., the top and bottom will be squares, and the other four sides will be rectangles) and a volume of 64 cubic inches. The top and bottom of the box will cost \$1 per square inch to produce, while the other four sides of the box will cost \$2 per square inch to produce. **Use the method of Lagrange multipliers** to determine the dimensions that will minimize the total cost of producing the box, along with this minimal cost.

Side length of square base: **Height:** **Volume:**

7. (16 points) Determine whether each set of vectors is linearly independent (I), linearly dependent (D), or if there is not enough information to determine whether the set of vectors is linearly dependent or independent (N). No justification necessary. Partial credit may be given for justification given for an incorrect answer.

(a) (2 points) The columns of a matrix representing a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following property: If $T(\vec{u}) = T(\vec{v})$, then $\vec{u}$ must be equal to $\vec{v}$.

(b) (2 points) The columns of a square matrix with 0's down the main diagonal (starting in the top left and moving down and to the right).

(c) (2 points) The set $\{\vec{u}, \vec{v}\}$, where the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

(d) (2 points) The set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, where none of the vectors are $\vec{0}$ and $\vec{v_1} \cdot \vec{v_2} = \vec{v_1} \cdot \vec{v_3} = \vec{v_2} \cdot \vec{v_3} = 0$.

(e) (2 points) The columns of the matrix representing the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $T\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(f) (2 points) The columns of an $n \times n$ matrix A, where $\det(AB) = 8$ for some $n \times n$ matrix B.

(g) (2 points) The columns of a matrix with more columns than rows.

(h) (2 points) The columns of a square matrix whose rows are linearly dependent.

8. (20 points) Find each partial derivative. Show your work.

(a) (6 points) Suppose $w = f(x, y, z)$, where $x = 2st^2u$, $y = su + \ln(t)$, and $z = s + t + u$. Use the information in the table below to evaluate $\frac{\partial w}{\partial t}$ at $(s, t, u) = (1, 1, -2)$.

(x, y, z)	$f(x, y, z)$	$f_x(x, y, z)$	$f_y(x, y, z)$	$f_z(x, y, z)$
$(1, 1, -2)$	-1	3	-4	2
$(-4, -2, 0)$	-3	2	-4	1

(b) (8 points) Find $f_{xy}(x, y)$ for $f(x, y) = \ln\left(\frac{xy}{x+y}\right)$.

(c) (6 points) Find $\partial z/\partial y$ for the surface defined by $e^{zy} + xz^2 = 6xy^4z^3 + y^2$ at the point $(1, 1, 0)$.

9. (20 points) True/False. Determine whether each statement is true or false. No justification necessary. Partial credit may be given for justification given for an incorrect answer. Each part is worth two points.

(a) True/False: If (x_0, y_0) is a critical point of $f(x, y)$ and the linear transformation represented by $\begin{bmatrix} f_{xx}(x_0, y_0) & f_{yx}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$ preserves the orientation of $\mathbb{R}^2$, then f has a saddle point at (x_0, y_0).

(b) True/False: If $f(x, y)$ is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \vec{0}$, then there must exist unit vectors $\vec{u}$ and $\vec{v}$ such that $D_{\vec{u}} f(x_0, y_0) > 0$ and $D_{\vec{v}} f(x_0, y_0) < 0$.

(c) True/False: The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x - 4y - 2 \\ 4y + 8 \end{bmatrix}$ is a linear transformation.

(d) True/False: If $f(x, y)$ is a differentiable function and (x_0, y_0) is a point in the domain of f such that $D_{\vec{u}} f(x_0, y_0) \leq f_x(x_0, y_0)$ for all unit vectors $\vec{u}$, then it must be the case that $f_y(x_0, y_0) = 0$.

(e) True/False: If a differentiable function $f(x, y)$ constrained to a smooth curve $g(x, y) = 0$ attains an absolute maximum at (x_0, y_0), then $\nabla g(x_0, y_0)$ is tangent to the level curve of f at (x_0, y_0).

(f) True/False: The level curves of the function $f(x, y) = \frac{1}{1-x^2-y^2}$ are circles centered at the origin.

(g) True/False: For any nonzero scalars a, b, c, the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ -5 & 3 & c \end{bmatrix}$ is invertible.

(h) True/False: If M is a matrix such that $\text{Col}(M) = \left\{\vec{0}\right\}$, then M must be a zero matrix.

(i) True/False: For any $n \times n$ matrices A and B, $(A + B)(A - B) = A^2 - B^2$.

(j) True/False: For any three vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$, we have $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \cdot \vec{w})$.

10. (24 points) Short Answer. Show your work/briefly justify your answers unless explicitly told not to.

(a) (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying $T\left(\vec{i}\right) = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$ and $T\left(\vec{j}\right) = c\vec{i}$ for some scalar c. If T stretches space by a factor of 3 and reverses the orientation of $\mathbb{R}^2$, then write down the matrix M representing T. Your matrix should **not** have any variables in it.

(b) (6 points) Find all critical points of $f(x, y) = \frac{x^2}{2} + 2x - 3xy + y^3 + 7$.

(c) (3 points) Suppose $T : \mathbb{R}^7 \to \mathbb{R}^3$ is a linear transformation. What are all of the possible dimensions of the null space of T? Briefly justify your answer.

(d) (5 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$. Find $T\left(\vec{j}\right)$.

(e) (3 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $M = \begin{bmatrix} 4 & -2 \\ -5 & -5 \end{bmatrix}$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $N = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$. Find the matrix representing the linear transformation $U \circ T$ (i.e., the transformation that first applies T and then applies U).

(f) (3 points) Suppose you want to find the equation $z = a_0 + a_1 x + a_2 y$ that best fits the data below. This is equivalent to finding the best approximation $\vec{z}$ to a solution to an equation of the form $A\vec{x} = \vec{b}$. What are the matrices A and $\vec{b}$? No justification necessary.

x	y	z
2	2	9
4	3	-8
6	4	0
8	5	-9

11. (1 point) Bonus Question. After this class, which subject do you prefer: linear algebra or multivariable calculus? No justification necessary, but I would love to know why. If you have any other reflections on the course, I would love to hear them here! (Focus on finishing the rest of the exam, though – If you're procrastinating the other problems by writing this, then go back to the other problems please!)