Name: ______________________

Final Exam

Math 256

Spring 2023

You have 120 minutes to complete this exam and turn it in. You may use an 8.5×11 inch two-sided hand-written sheet of notes and a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

You should expect to spend a little over a minute per question per point it's worth — there are 90 points possible on the exam and 120 minutes total.

Part I (12 points) Multiple choice. You don't need to show your work.

1. (2 points) Suppose you solve a second-order DE and find a general solution of $y = c_1y_1 + c_2y_2$. The Wronskian $W[y_1, y_2]$ satisfies $W[y_1, y_2](2) = -1$ and $W[y_1, y_2](3) = 0$. Which of the following must be true?

A) $y = c_1y_1 + c_2y_2$ is not actually the general solution.

B) Given the initial conditions $y(2) = 1$, $y'(2) = 4$, we can solve for c_1 and c_2.

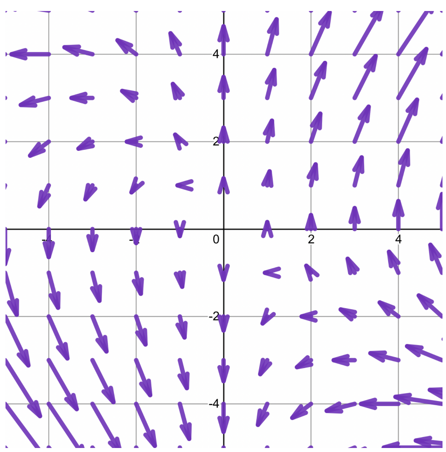
C) Given the initial conditions $y(3) = 1$, $y'(3) = 4$, we can solve for c_1 and c_2.

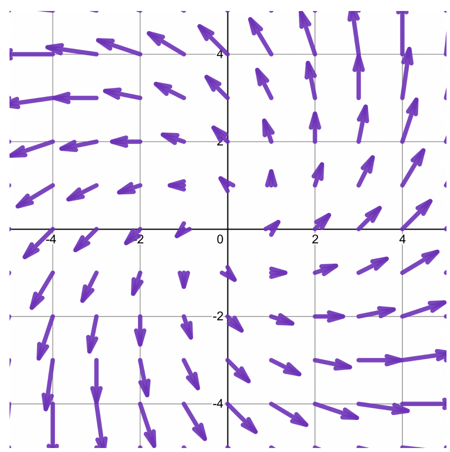
D) None of the above.

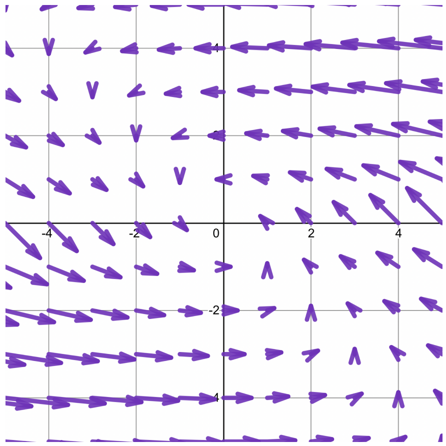
2. (2 points) Which of the following vector fields corresponds to the following system of DEs?

$$x' = \frac{1}{3}xy$$

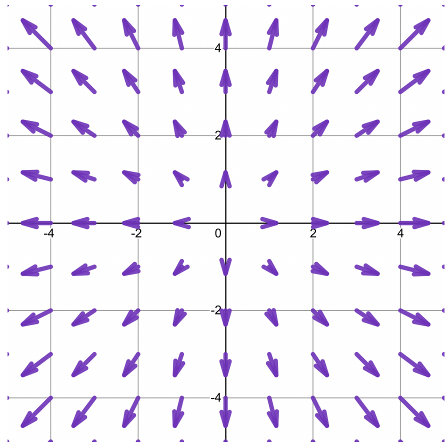
$$y' = x + y.$$







-4
-2
0
2
4
-2
-4



4
2
-4
-2
0
2
4
-2
-4

3. (2 points) One solution to a linear homogeneous DE is $y = 2\sin(t) + i\cos(t)$. Which of the following must be another solution?

A) $y = 2t\sin(t) + it\cos(t)$.

B) $y = \cos(t)$.

C) $y = \sin(t)\cos(t)$.

D) None of the above.

4. (2 points) Let $\mathbf{A}$ and $\mathbf{B}$ be matrices so that the product $\mathbf{AB}$ is defined. If $\mathbf{A}$ and $\mathbf{B}$ correspond to the functions f and g, so that $\mathbf{Ax} = f(\mathbf{x})$ and $\mathbf{Bx} = g(\mathbf{x})$ for any $\mathbf{x}$, then

A) $\mathbf{ABx} = f(\mathbf{x}) + g(\mathbf{x})$.

B) $\mathbf{ABx} = f(\mathbf{x})g(\mathbf{x})$.

C) $\mathbf{ABx} = f(g(\mathbf{x}))$.

D) None of the above.

5. (2 points) Suppose $\mathbf{A}$ is a 2×2 matrix with real entries and an eigenvalue of $\lambda_1 = 2i$, corresponding to an eigenvalue of $\mathbf{v_1} = \begin{bmatrix} 3+2i \\ -i \end{bmatrix}$. Which of the following is the other eigenvector and eigenvalue?

A) $\lambda_2 = -2i$, $\mathbf{v_2} = \begin{bmatrix} 3+2i \\ -i \end{bmatrix}$.

B) $\lambda_2 = 2i$, $\mathbf{v_2} = \begin{bmatrix} 3-2i \\ i \end{bmatrix}$.

C) $\lambda_2 = 2i$, $\mathbf{v_2} = \begin{bmatrix} -3+2i \\ -i \end{bmatrix}$.

D) $\lambda_2 = -2i$, $\mathbf{v_2} = \begin{bmatrix} 3-2i \\ i \end{bmatrix}$.

6. (2 points) Let $\mathbf{A}$ be an $n \times n$ matrix. Which of the following is **not** equal to $\det \mathbf{A}$?

A) The product of the eigenvalues of $\mathbf{A}$.

B) The number of linearly independent rows of $\mathbf{A}$.

C) The factor by which $\mathbf{A}$ scales volume from input to output.

D) None of the above.

Part II (18 points) Individual problems. Show your work for each question.

7. (6 points) Consider the system of DEs

$$x' = x\sin(y)$$
$$y' = x^2.$$

Solve the system and find the critical points.

8. (6 points) One fundamental solution to $2t^2y'' + ty' - y = 0$ is $y = t$. What is the other fundamental solution?

9. (6 points) Find the general solution to $(1 + 2xy) + x^2 y' = 0$.

Part III (60 points) More involved questions with multiple parts.

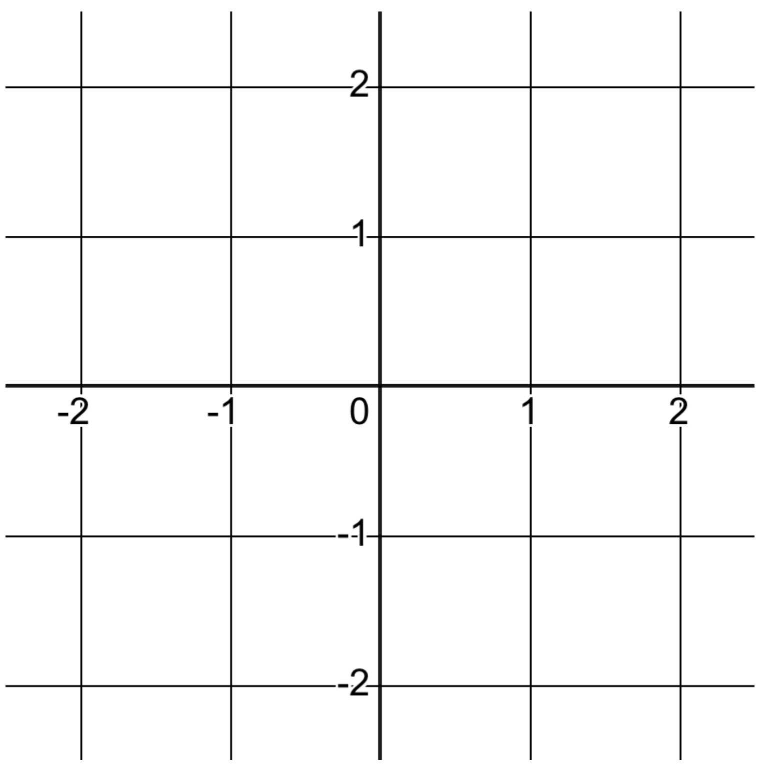
12. (16 points) A tank has a capacity of $2\,\mathrm{L}$. At time $t = 0$, we begin pouring in a brine solution with a concentration of $1\,\dfrac{\mathrm{kg}}{\mathrm{L}}$ at a rate of $1\,\dfrac{\mathrm{L}}{\mathrm{s}}$, and we also drain the well-mixed solution at a rate of $1\,\dfrac{\mathrm{L}}{\mathrm{s}}$.

a) (2 points) If $Q(t)$ is the mass of salt in the tank after t seconds, then $Q' = 1 - \frac{1}{2}Q$. Show how to find this DE from the problem statement.

b) (6 points) Find the general solution to the DE from part a) by using integrating factors.

c) (2 points) The tank initially contains no salt. Find the particular solution to the DE.

d) (6 points) Sketch a direction field for the DE for $-2 \leq t \leq 2$ and $-2 \leq Q \leq 2$. Also draw the particular solution from part c).



13. (10 points) Consider the system of equations

$$
\begin{aligned}
2x_1 + 3x_2 &= 2 \\
x_1 - x_2 &= 1.
\end{aligned}
$$

a) (2 points) Write the system in the form $\mathbf{A}\mathbf{x} = \mathbf{b}$ for a matrix $\mathbf{A}$ and vector $\mathbf{b}$.

b) (6 points) Solve the system for $\mathbf{x}$ by row reducing the augmented matrix $\left[\mathbf{A} \mid \mathbf{b}\right]$.

c) (2 points) Find the determinant of $\mathbf{A}$.

14. (14 points) Consider the DE $y'' + 2y' + y = 0$.

a) (4 points) Find the general solution to the DE.

b) (4 points) Find the general solution to $y'' + 2y' + y = e^{2t}$ using undetermined coefficients.

c) (6 points) Find the general solution to $y'' + 2y' + y = e^{-t}$ using variation of parameters.

15. (20 points) Consider the system of DEs

$$x_1' = -2x_1 - 3x_2$$
$$x_2' = x_1 + 2x_2$$

a) (2 points) Write the system in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for a matrix $\mathbf{A}$.

b) (4 points) Solve for the eigenvalues of $\mathbf{A}$.

c) (6 points) Solve for the eigenvectors of $\mathbf{A}$.

d) (2 points) Find the general solution to the system.

e) (4 points) Given $x_1(0) = 1$ and $x_2(0) = 3$, find the particular solution to the system.