64
64
63
63 A
60
58

57
57
55 B
$\boxed{54}$ 85 %
53

51
51
51 C
50
45

44 D
40

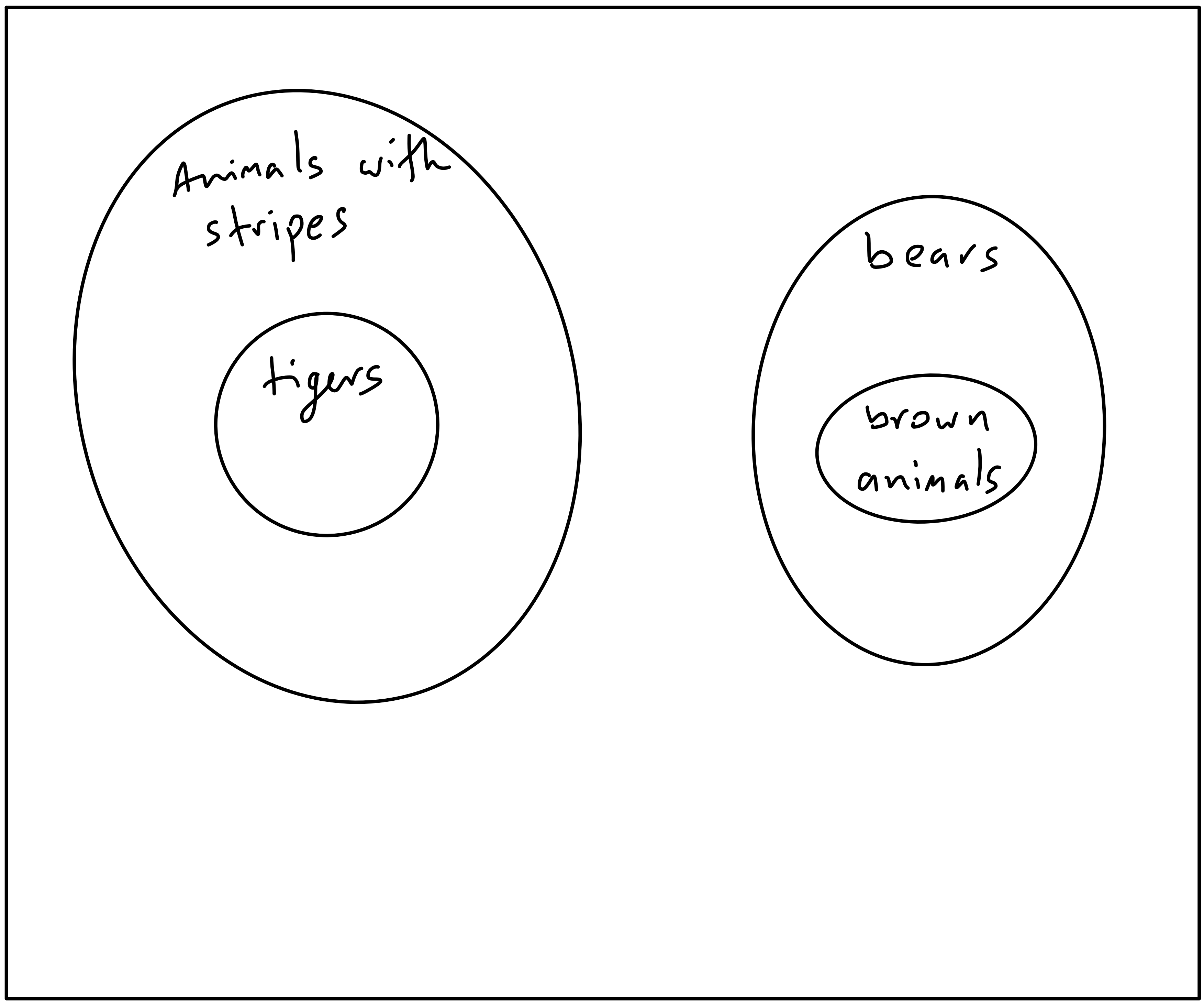
36 F
31

1. $p \rightarrow (q \wedge r)$

p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

② 1. All tigers have stripes.
2. Nothing with stripes is a bear.
3. All brown animals are bears.

No tigers are brown.



Therefore, the argument is valid.

(3) p : All tigers have stripes

q : Nothing with stripes is a bear

r : All brown animals are bears

s : No tigers are brown

$(p \wedge q \wedge r) \rightarrow s$

OR

p : you are a tiger

q : you have stripes

r : you are a bear

s : you are a brown animal

1. $p \rightarrow q$
2. $q \rightarrow \sim r$
3. $s \rightarrow r$

$$\frac{\begin{array}{l}1.\ p \rightarrow q\\ 2.\ q \rightarrow \sim r\\ 3.\ s \rightarrow r\end{array}}{p \rightarrow \sim s}$$

(or $s \rightarrow \sim p$)

(4) 1. You eat only if you are hungry
2. If you go to a restaurant, then you eat

You are hungry if you go to a restaurant.

This is valid.

Let p be "you eat"
q be "you are hungry"
r be "you go to a restaurant".

Then $P_1 \equiv p \rightarrow q$

$P_2 \equiv r \rightarrow p$

$C \equiv r \rightarrow q$

p	q	r	P_1	P_2	C	$P_1 \wedge P_2$	$P_1 \wedge P_2 \to C$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	T	T	F	T
F	T	T	T	F	T	F	T
F	T	F	T	T	T	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

(5) You are hungry if you go to the restaurant.

$\equiv$ If you go to the restaurant, then you are hungry.

Converse: If you are hungry, then you go to the restaurant.

Inverse: If you don't go to the restaurant, then you are not hungry.

Contrapositive: If you are not hungry, then you do not go to the restaurant.

For any statement, the contrapositive is equivalent to it.

⑥ A : UO students who are currently taking 105.

B : UO students who have taken and passed 105.

$A \cup B$: UO students who have either taken 105 or are currently in 105.

$A \cap B$: UO students who are currently taking 105 but who have also taken and passed it before.

A': UO students who are not currently in 105.

B': UO students who have not passed 105.

7. Which is/are true:

i. $A \cup B = \emptyset$

ii. $\boxed{A \cap B = \emptyset}$

iii. $A' = \emptyset$

iv. $B' = \emptyset$

(8) C has 15 elements

D has 10

$C \cup D$ has 17

$n(C \cap D)$?

$$n(C \cup D) = n(C) + n(D) - n(C \cap D)$$

$$17 = 15 + 10 - n(C \cap D)$$

$$n(C \cap D) = 8.$$

Ex: ${}_7P_3$ "7 permute 3"

this is the number of ways to choose 3 objects from a group of 7 and then arrange them
(permute)

$${}_7P_3 = {}_7C_3\,(3!) = \frac{7!}{\cancel{3!}(7-3)!}\,(\cancel{3!})$$

$$= \frac{7!}{4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{\cancel{4\cdot 3\cdot 2\cdot 1}}$$

$$= 7\cdot 6\cdot 5$$

$$= 210.$$

$${}_7C_3 = \frac{7!}{3!\,(7-3)!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{3\cdot 2\cdot 1\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}$$

$$= \frac{7\cdot \cancel{6}\cdot 5}{\cancel{3\cdot 2}} = 7\cdot 5 = 35$$

2.5: Infinite Sets

We know that the number of elements in a set A is $n(A)$.

Def: Two sets A and B are equivalent, written $A \sim B$, if we can pair every element of A with a unique element of B, and vice versa.

Ex: $\{1, 2, 3\} \sim \{a, b, x\}$ because we have the pairing

$$\begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ a & b & x \end{array}$$

Theorem: If $A \sim B$, then $n(A) = n(B)$.

Ex: Let $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$. Let $E = \{0, 2, 4, 6, 8, \cdots\}$ be the set of positive even numbers. We would think that $n(\mathbb{N}) > n(E)$, but this isn't true!

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ | & | & | & | & | & | & \\ 0 & 2 & 4 & 6 & 8 & 10 & \cdots \end{array}$$

We're forced to conclude that $n(\mathbb{N}) = n(E)$.

Remember that $\mathbb{Q}$ is the set of rational numbers. We can list all of them like this:

	0	1	−1	2	−2	3	−3
1	$0/1$	$1/1$	$-1/1$	$2/1$	$-2/1$	$3/1$	$-3/1$
2	$0/2$	$1/2$	$-1/2$	$2/2$	$-2/2$	$3/2$	$-3/2$
3	$0/3$	$1/3$	$-1/3$	$2/3$	$-2/3$	$3/3$	$-3/3$
4	$0/4$	$1/4$	$-1/4$	$2/4$	$-2/4$	$3/4$	$-3/4$
5	$0/5$	$1/5$	$-1/5$	$2/5$	$-2/5$	$3/5$	$-3/5$
6	$0/6$	$1/6$	$-1/6$	$2/6$	$-2/6$	$3/6$	$-3/6$

0	1	2	3	4	5	6	7	8	9	...
|	|	|	|	|	|	|	|	|	|	
0,	1,	$1/2$,	-1,	$1/3$,	$-1/2$,	2,	$1/4$,	$-1/3$,	-2,	...

Therefore, $n(\mathbb{N}) = n(\mathbb{Q})$

Def: The first infinite cardinal is written $\aleph_0$ ("aleph-naught").

$n(\mathbb{N}) = \aleph_0$

$n(\mathbb{Q}) = \aleph_0$

Theorem: $n(\mathbb{R}) > \aleph_0$.

Proof: Suppose $n(\mathbb{R}) = n(\mathbb{N})$. Then we would have a pairing

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \\ | & | & | & | & | & \\ r_0 & r_1 & r_2 & r_3 & r_4 & \cdots \end{array}$$

real numbers

Now every real number r has a decimal expansion that we can write $r_0 = .a_0 a_1 a_2 a_3 a_4 \ldots$

For example, 32.1567912 has decimal $.1567912$.

So we have

$$r_0 = .a_0\, a_1\, a_2\, a_3\, a_4 \ldots$$
$$r_1 = .b_0\, b_1\, b_2\, b_3\, b_4 \ldots$$
$$r_2 = .c_0\, c_1\, c_2\, c_3\, c_4 \ldots$$
$$\vdots$$

Let $R = .a_0\, b_1\, c_2\, d_3\, e_4 \ldots$

Let S be the same, but with every digit shifted up by one.

So if $R = .321718,$

$S = .432829.$

Now S can not appear in the list. So we didn't list all the real numbers!

Question: is there a set A for which $n(\mathbb{N}) < n(A) < n(\mathbb{R})$?

It's impossible to say.

3.2: Basic Probability

Def: An experiment is a process by which an outcome is obtained. The sample space is the set of all possible outcomes, and an event is a subset of the sample space.

Ex: rolling a die. The experiment is rolling the die. The sample space is $\{1, 2, 3, 4, 5, 6\}$.

Some examples of events:

$\{1\}$ (you roll a 1)

$\{1, 2, 3\}$ (you roll a 1, 2, or 3)

$\{1, 2, 3, 4, 5, 6\}$ (you roll anything)

<u>Def</u>: An event is <u>certain</u> or <u>guaranteed</u> if it always occurs, and <u>impossible</u> if it never does.

<u>Ex</u>: $\{1, 2, 3, 4, 5, 6\}$ is certain

$\{7\}$ is impossible.

<u>Def</u>: The probability of an event E with a sample space S is

$$P(E) = \frac{n(E)}{n(S)}$$

if all outcomes in the sample space are equally likely. It's also written

$$P(E), \quad P_r(E), \quad \text{and} \quad \mathbb{P}(E).$$

<u>Def</u>: The odds of an event E occuring is $o(E) = n(E) : n(E')$.

Ex: We flip a coin. The sample space is $\{H, T\}$. If $E = \{H\}$, then

$$p(E) = \frac{n(E)}{n(S)} = \frac{1}{2} = .5.$$

Note that we could only do this because H and T are equally likely. The odds are $n(E) : n(E') = n(\{H\}) : n(\{T\})$

$= 1 : 1$.

Similarly, if $F = \{H, T\}$, then

$p(F) = \frac{2}{2} = 1$, and $o(F) = 2 : 0$.

Also, $p(\emptyset) = 0$ and $o(\emptyset) = 0:2$.

Ex: in real life, if you flip a coin 10 times, you might get 3 heads.

Def: The *relative frequency* of an event is the number of times in an experiment that an event occurs divided by the number of attempts.

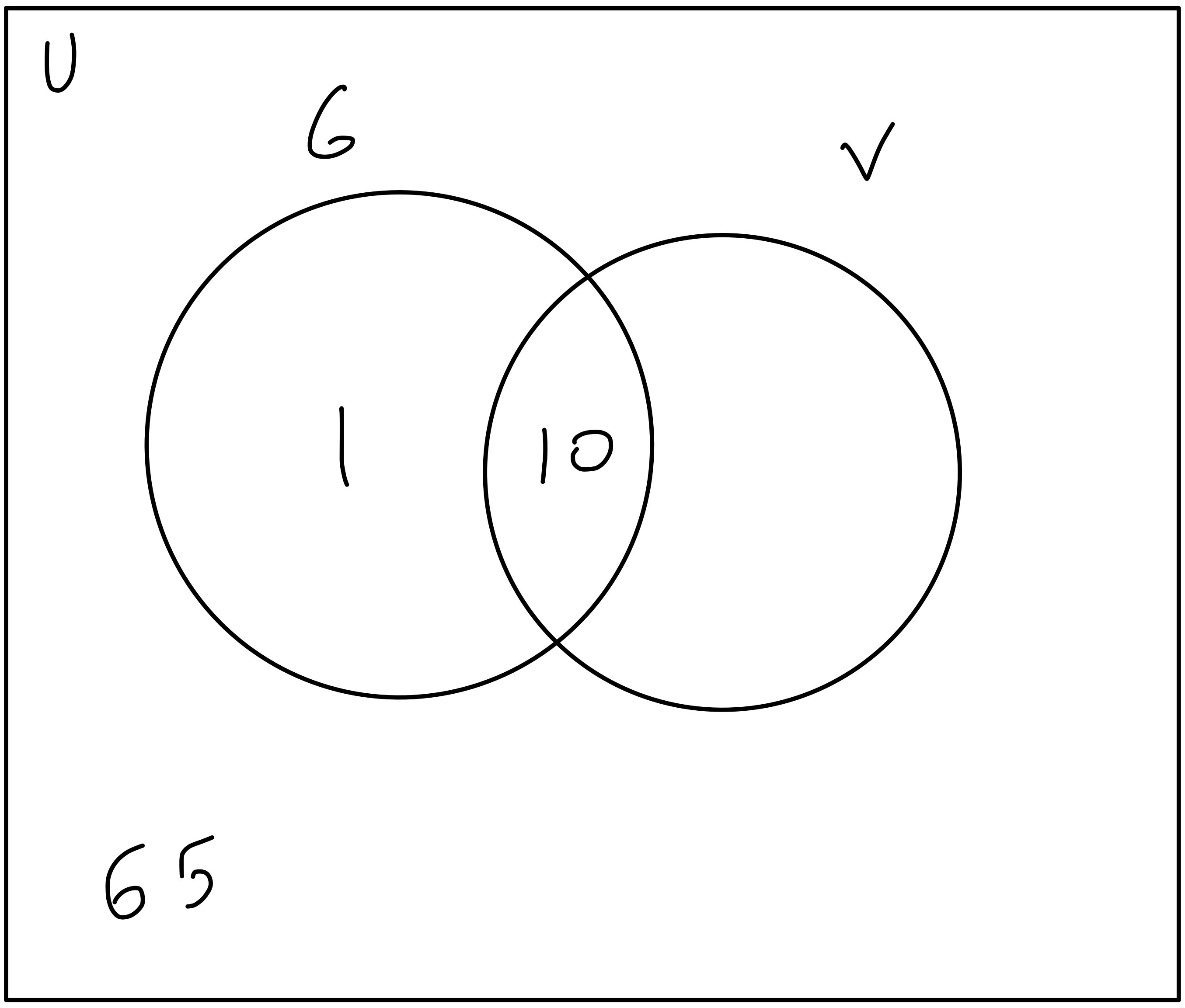
Ex: if you flip a coin 10 times and get 3 heads, then the relative frequency of heads is $3/10 = .3$.

Theorem (The Law of Large Numbers):

If an experiment is repeated a large number of times, the relative frequency of an event is approximately equal to the probability of the event.

Solutions to Quiz 3

(1)



$n(G) = 11$

$n(V) = 34$

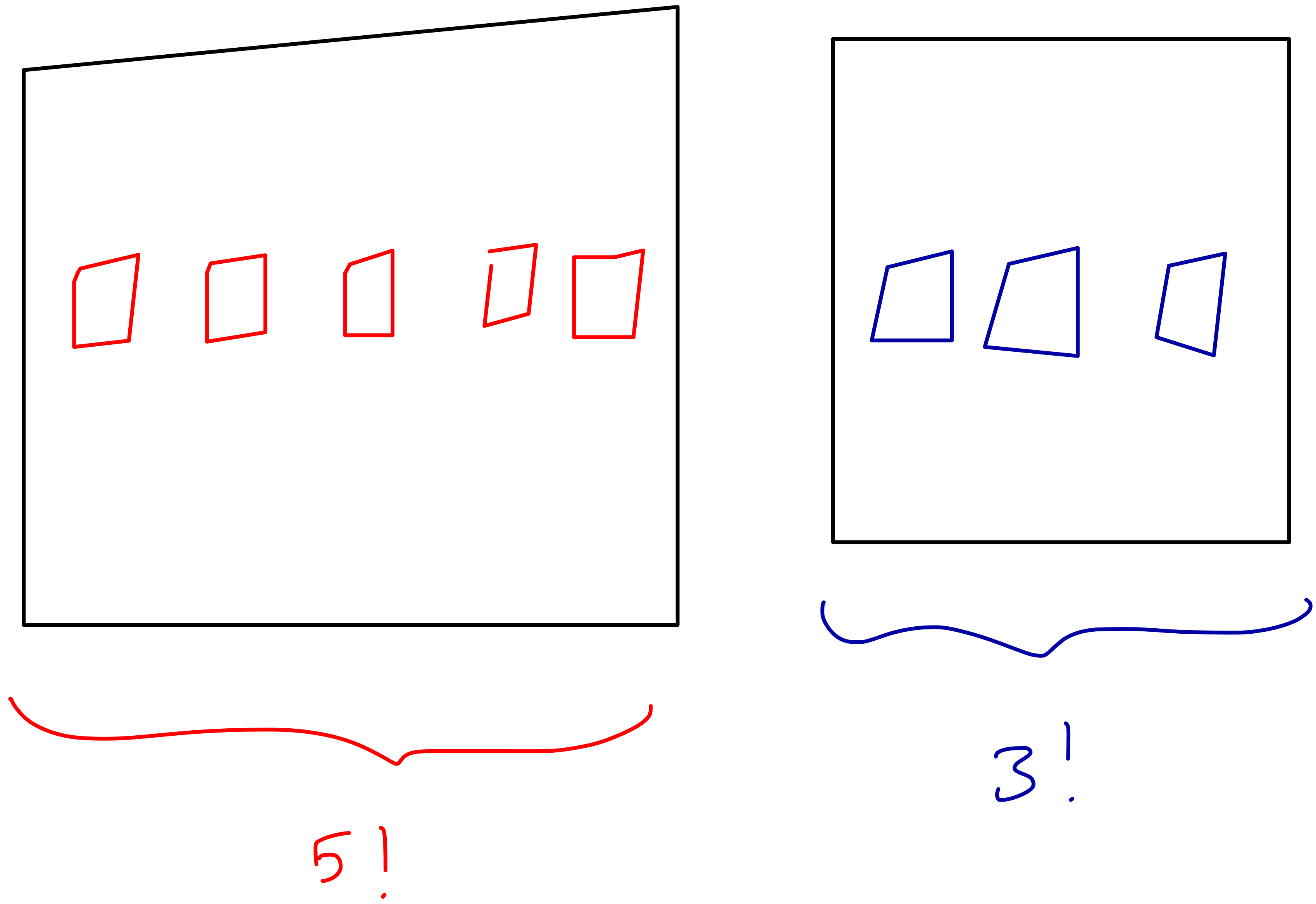
$n(U) = 100$

$n(G' \cap V') = 65$

$$\underbrace{n(G \cup V)}_{\substack{100-65 \\ 35}} = \underbrace{n(G)}_{11} + \underbrace{n(V)}_{34} - n(G \cap V)$$

$\Longrightarrow n(G \cap V) = 10$

②



$$\Rightarrow 5!\,3! = 720$$

③

$$_{10}C_3 = \frac{10!}{3!\,(10-3)!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}}{3\cdot 2\cdot 1\cdot \cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}$$

$$= \frac{10\cdot \overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}}{\cancel{3}\cdot\cancel{2}}$$

$$= 120.$$

Comment: Let E be an event. Then

$$0 \leq p(E) \leq 1.$$

$$0\% \leq p(E) \leq 100\%.$$

Theorem: If the sample space is S, then $p(S) = 1$ and $p(\emptyset) = 0$.

Def: Two events E and F are mutually exclusive if they cannot occur at the same time: if $E \cap F = \emptyset$.

Ex: rolling a 5 and rolling less than a 4 are mutually exclusive.

Notice that $\{5\} \cap \{1,2,3\} = \{\} = \emptyset$.

Ex: You flip two coins. If E is the event of getting two tails and F is the event of getting at least one tail, find $p(E)$, $p(F)$, and whether E and F are mutually exclusive.

Can we say the sample space is

$S = \{HH, HT, TT\}$?

Yes — but it's a bad idea, since HT can be reached in two ways — head then tail or tail then head.

Instead, let's write

$$S = \{HH, HT, TH, TT\}.$$

Whenever possible, write S so that every element has an equal probability.

Now we need to write E and F as subsets of S.

$$E = \{TT\}$$

$$F = \{HT, TH, TT\}$$

$$p(E) = \frac{n(E)}{n(S)} = \frac{1}{4} = .25$$

$$p(F) = \frac{n(F)}{n(S)} = \frac{3}{4} = .75$$

$E \cap F = \{TT\} \neq \emptyset$, so E and F are <u>not</u> mutually exclusive.

Ex: rolling two dice.

	die 2: 1	2	3	4	5	6
die 1: 1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

2 3 4 5 6 7 8 9 10 11 12

E: event of rolling between a 7 and 9 with two dice.

$$p(E) = \frac{4+5+6}{36} = \frac{15}{36}.$$

Theorem: Let E and F be events.

1. $p(E \cup F) = p(E) + p(F) - p(E \cap F)$.
2. If E and F are mutually exclusive, then $p(E \cup F) = p(E) + p(F)$.
3. $p(E') = 1 - p(E)$.

Ex: Find the probability that a random card from a 52-card deck is

1. the king of diamonds. E
2. not the king of diamonds. F
3. a king or a diamond. G

$E = \{KD\}$ $\qquad$ $F = E'$, so

$p(E) = 1/52$. $\qquad$ $p(F) = 1 - p(E) = 51/52$.

(3) Let K be the event of getting a king (so $K = \{KC, KD, KH, KS\}$), and D the event of getting a diamond (so $D = \{2D, 3D, \ldots, KD, AD\}$).

Then $G = K \cup D$, so

$$p(G) = p(K \cup D) = p(K) + p(D) - p(K \cap D) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}.$$

<u>Comment</u>: Recall the basic combinatorics flowchart —

1. If you're arranging objects and not drawing a subset of them, use a factorial.

2. If you're drawing a subset and the order doesn't matter, use ${}_nC_k$.

3. If you're drawing a subset and the order does matter, use ${}_nP_k$.

4. If you're doing multiple of these things, find them individually and multiply them.

Ex: You select three people at random.

What is the probability that at least two have the same birthday?

$S = \{$ all possible lists of three days $\}$.

$n(S) = ?$

Use combinatorics! There are 365 possibilities for each day and 3 days to choose from, so there are

$365 \cdot 365 \cdot 365$ possible sets of three birthdays.

$E = \{$ lists of three birthdays where two are the same $\}$

what is $n(E)$?

An element of E looks like:

<u>March 3</u> <u>August 24</u> <u>March 3</u>

How could two of the three entries be the same?

- the first and second are the same
  $\hookrightarrow 365 \cdot 1 \cdot 365$
- the first and third are the same
  $\hookrightarrow 365 \cdot 365 \cdot 1$
- the second and third are the same
  $\hookrightarrow 365 \cdot 365 \cdot 1$

But we've counted some of the elements of E multiple times!

- there are 365 elements we counted two times too many

$n(E) = 365^2 + 365^2 + 365^2 - 2 \cdot 365$

This was hard!

What about $n(E')$?

E' is the set of lists of 3 days where none of the three is the same.

An element of E' looks like

$\underline{\quad\quad}_{365} \quad \underline{\quad\quad}_{364} \quad \underline{\quad\quad}_{363}$

So $n(E') = 365 \cdot 364 \cdot 363$

So $p(E') = \frac{n(E')}{n(S)} = \frac{365 \cdot 364 \cdot 363}{365 \cdot 365 \cdot 365}$

$= .992$

Therefore, $p(E) = 1 - p(E') = .008$

<u>Ex</u>: What is the chance of being dealt 4 aces in a 5-card hand from a randomly shuffled deck?

The sample space is the set S of all 5-card hands. The order doesn't matter, so $n(S) = {}_{52}C_5 = \frac{52!}{5!\,(52-5)!}$

$$= \frac{52!}{5!\,(52-5)!} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot \cancel{50}^{10} \cdot 49 \cdot \cancel{48}^{2}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2}} = 2598960$$

E is the set of all five-card hands that contain 4 aces. Order doesn't matter, so the only variation is what th 5th card is. There are 48 cards left to possibly fill the hand, so there are 48 possible hands with four aces.

So $p(E) = 48/2598960 \approx .0000185$

Ex: What is the probability of being dealt 5 hearts?

We already know n(S), so we just need to find n(E), where E is the set of 5-heart hands.

There are 13 hearts to choose from, and order doesn't matter, so

$$n(E) = {}_{13}C_5 = \frac{13!}{5!\,8!} = \frac{13 \cdot \cancel{12}^{2} \cdot 11 \cdot \cancel{10} \cdot 9}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

$$= 13 \cdot 11 \cdot 9 = 1287$$

So $p(E) = \frac{1287}{2598960} = .0005$

Ex: Find the probability of getting four-of-a-kind.

There are 13 "numbers" with which four-of-a-kind is possible. With the same reasoning as the 4 aces example, there are 48 possibilities for the fifth card. In total, $n(E) = 13 \cdot 48$

$= 624$

So $p(E) = \dfrac{624}{2598969} = .00024$

3.5: Expected Value

Comment: this is how we'll discuss averages.

Def: Consider an experiment with some sample space S. The expected value of the experiment is the approximate value you should expect to receive on average when performing the experiment a large number of times.

To calculate it, make a table with one column for every outcome. The first row contains the probability, the second contains the value, and the third row is the product of the first two. Then the expected value is the sum of the third row.

Ex: find the expected value of rolling a 6-sided die.

$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$
1	2	3	4	5	6
$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$

$$EV = 1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6$$

$$= 21/6 = 7/2 = 3.5.$$

Note that 3.5 is not a value you can roll.

Ex: You've on a game show and you've won \$10000 so far. You have the option to pick 1 of 6 briefcases at random and open it. They contain:

\$1   \$1000   \$10000

\$1000   \$0   Lose everything

Should you open a briefcase?

We can find the expected value of opening a briefcase.

1/6	1/6	2/6	1/6	1/6
0	1	1000	10000	-10000
0	1/6	333	1667	-1667

$\Rightarrow$ \$334

Since $334 > 0$, you gain money on average by opening a briefcase.

Comment: Expected value only is meaningful on experiments that have numbers as outputs.

Ex: the expected value of flipping a coin is $\frac{1}{2}H + \frac{1}{2}T$

Comment: If we denote the outcome of an experiment by X, the expected value is written $E[X]$ or $\mathbb{E}[X]$.

3.6: Conditional Probability

Ex: you know the probability of rolling two sixes in a row on six-sided dice is $(1/6)(1/6) = 1/36$. But what is chance if we know that the first die rolled higher than a 3?

Def: Let A and B be events. The probability of A given B is

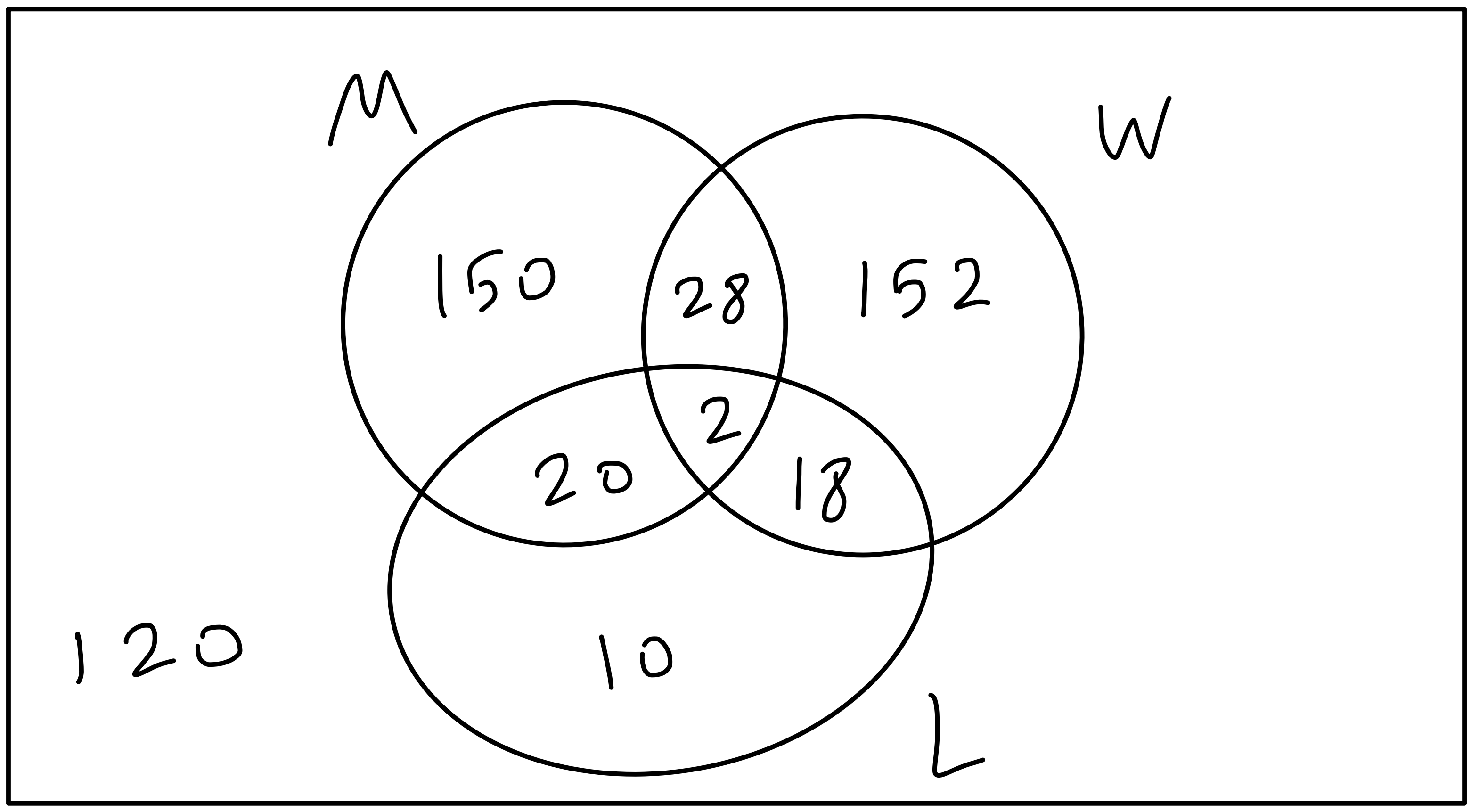
$$p(A \mid B) = \frac{n(A \cap B)}{n(B)}.$$

We're effectively reducing the sample space to B.

Ex: If A is rolling a six and B is rolling higher than a 3, $A = \{6\}$ and $B = \{4, 5, 6\}$, so $A \cap B = \{6\}$.

Therefore, $p(A \mid B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{1}{3}$.

Ex: 200 people use macOS computers, 200 Windows, etc. from HW3.



The chance that you own a Linux computer given that own a macOS one is $p(L|M) = \frac{n(L \cap M)}{n(M)} = \frac{22}{200} \approx .1$.

The chance that you own a Linux computer at all is $50/500 = .1$.

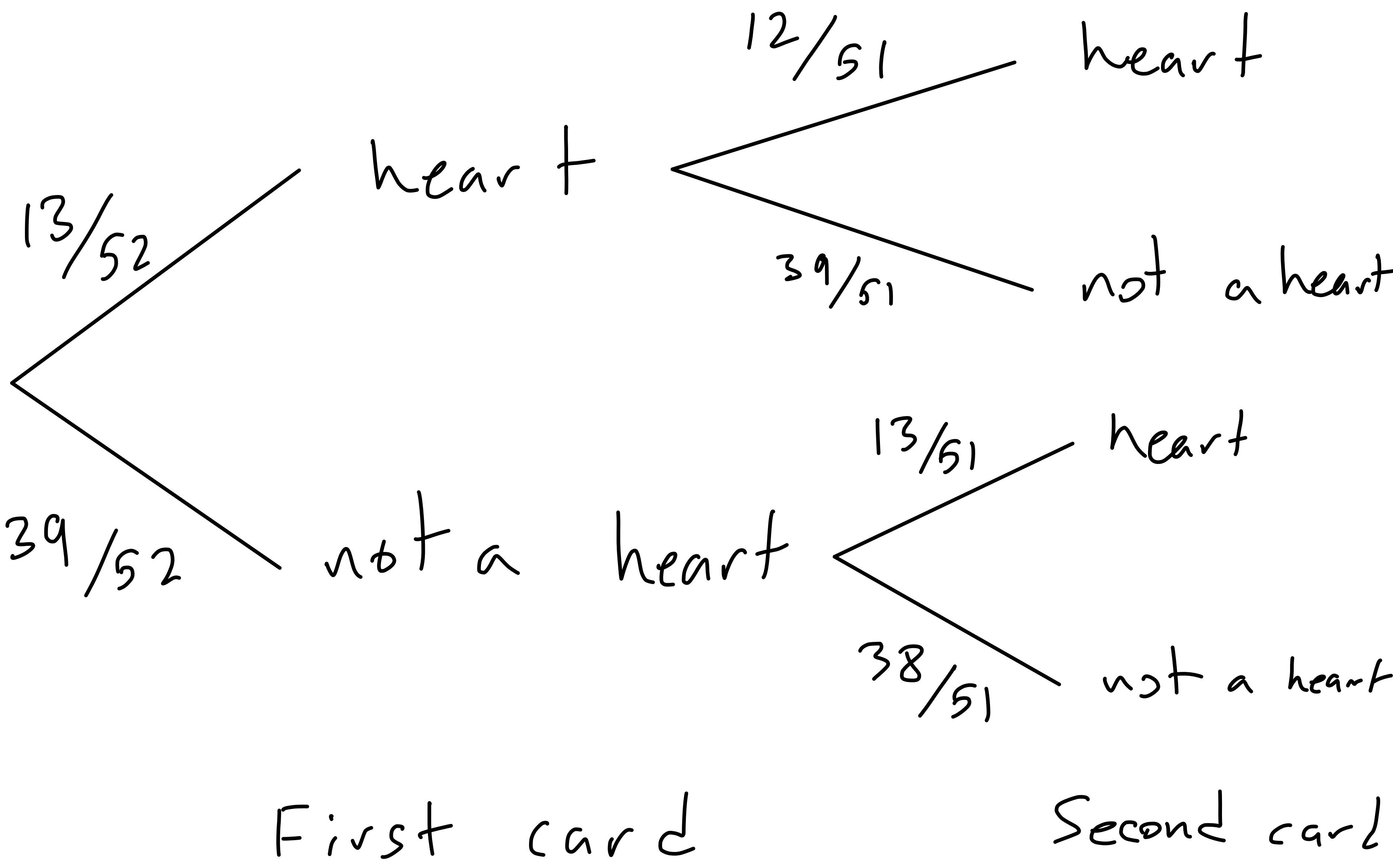
The chance that you own a macOS and Windows computer, given that you own a Linux one, is $p(M \cap W | L) = \frac{n(M \cap W \cap L)}{n(L)}$

$= 2/50 = 1/25 = .04$

The chance that you own a macOS and Windows computer without any other information is $30/500 = 3/50 = .06$

Def: A tree diagram is a method to list every possible outcome of an experiment and its probability.

Ex: A tree diagram for being dealt two hearts in a row.



- To use tree diagrams to find conditional probabilities, start at the condition.

Ex: The probability of being dealt a heart for the second card, given that the first card was not a heart, is $13/51$.

- To use a tree with and statements, multiply successive branches.

Ex: The chance of being dealt two hearts in a row is $13/52 \cdot 12/51 \approx .06$.

- To use a tree with or statements, add the probabilities.

Ex: What is the probability of either being dealt two hearts in a row or two nonhearts in a row?

It's $13/52 \cdot 12/51 + 39/52 \cdot 38/51$

$\approx .62$.

Comment: When making a tree diagram, every split must be mutually exclusive!