Math 1180 **Midterm 2** **Cruz Godar**

You have 75 minutes to complete this exam and turn it in. You may use a 3×5 inch two-sided handwritten index card, but no calculators. You may not consult the internet or other people, but if you have a question, please don't hesitate to ask me!

You should expect to spend a little more than one minute per question per point that it's worth: there are 70 points possible. Show enough work that there is no question about the mathematical process used to obtain your answers. Finally, please write your name on the **back** of the exam!

Multiple choice (each question has only one correct answer). You don't need to show your work.

1. (3 points) Let $f(x,y)$ be a differentiable function that is defined for all $(x,y) \in \mathbb{R}^2$. On which set is f guaranteed to attain a global maximum and a global minimum?

a) $\{(x,y) \in \mathbb{R}^2 \mid x^4 + y^4 \leq 3\}$. This is the only compact one — b) is unbounded and c) is closed.

b) $\{(x,y) \in \mathbb{R}^2 \mid |x-y| \leq 3\}$

c) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 3\}$

d) None of the above.

2. (3 points) Let A be a matrix with 3 rows and 4 columns. If B is a matrix so that BA has 5 rows and 4 columns, then what must be the shape of B?

a) 5×4.

b) 3×3.

c) 5×3. For the product to be defined, B must have 3 columns, and for it to be 5×4, B must have 5 rows.

d) 3×5.

3. (3 points) A system of 4 equations and 5 unknowns

a) Always has no solution.

b) Always has exactly one solution.

c) Always has infinitely many solutions.

d) None of the above. While many systems like this have a free parameter and therefore infinitely many solutions, we can easily include two equations like $x + y = 0$ and $x + y = 1$ to make there be no solutions, and similarly for a unique solution.

4. (3 points) Which of the following matrices is **not** in reduced row echelon form?

a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. This one doesn't have every leading 1 alone in its column.

c) $\begin{bmatrix} 1 & 43 & -10 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

5. (3 points) When considered as a function on column vectors, a 3×7 matrix A is a function

a) $\mathbb{R}^3 \to \mathbb{R}^7$.

b) $\mathbb{R}^7 \to \mathbb{R}^3$. The function takes in vectors in $\mathbb{R}^7$, since $A\vec{v}$ is only defined when $\vec{v}$ has length 7, and it outputs vectors in $\mathbb{R}^3$.

c) $\mathbb{R}^3 \to \mathbb{R}^3$.

d) $\mathbb{R}^7 \to \mathbb{R}^7$.

6. (3 points) Let A be a 3×3 matrix. Which of the following is true?

a) If A is in reduced row echelon form, then it must be the identity matrix.

b) If A is not invertible, then every equation $A\vec{x} = \vec{b}$ has either no solutions or infinitely many solutions. Only this one. a) isn't true, since A could have a row of all zeros. This one is true though — if $A\vec{x} = \mathbf{b}$ always has a unique solution for $\vec{x}$, then that means we always know $\vec{x}$ in terms of $\vec{b}$ — in other words, we can write $\vec{x} = B\vec{b}$ for some matrix B, and that matrix must be $B = A^{-1}$.

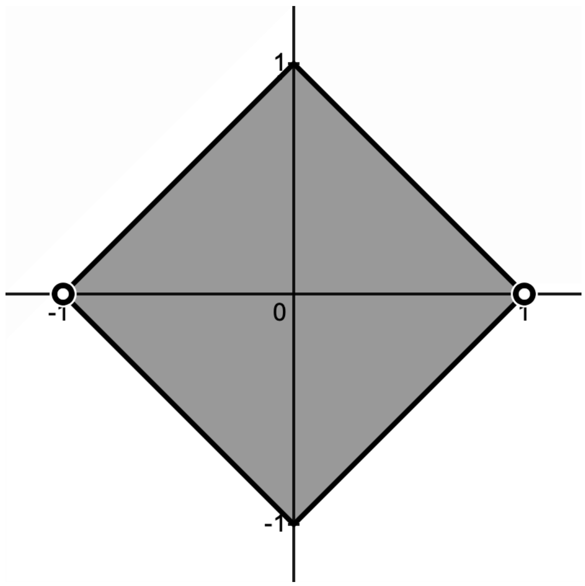
c) Both a) and b) are true.

d) Neither a) nor b) is true.

7. (4 points) Let $A = \begin{bmatrix} 2 & a \\ b & -3 \end{bmatrix}$ be a matrix with unknown entries a and b. If $A\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -16 \end{bmatrix}$, what are a and b?

We have $A\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2+5a \\ -b-15 \end{bmatrix}$, so $-2+5a = 13$ and $-b-15 = -16$, giving us $a = 3$ and $b = 1$.

8. (4 points) Let $C = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, |x| + |y| \le 1 \text{ and } |x| \ne 1 \right\}$ be a diamond without the corners at $(\pm 1, 0)$. Express the boundary of C in set builder notation.



The boundary is the edges, *including* the missing points: any circle around them contains points in the set and points out of it, so the boundary is $\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, |x| + |y| = 1 \right\}$.

9. (10 points) Let B be the matrix

$$B = \begin{bmatrix} -2 & 5 & 7 \\ -1 & 0 & 2 \\ -2 & 0 & 7 \end{bmatrix}.$$

Find B^{-1} by using row reduction, labeling each row operation you perform.

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 7 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 0 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 5 & 7 & 1 & 0 & 0 \\ -2 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \qquad \vec{r}_1 \leftrightarrow \vec{r}_2$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 & -2 & 1 \end{array}\right] \qquad \begin{array}{l} \vec{r}_2 \to \vec{r}_2 - 2\vec{r}_1 \\ \vec{r}_3 \to \vec{r}_3 - 2\vec{r}_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & -2 & 1 \end{array}\right] \qquad \vec{r}_2 \to \vec{r}_2 - \vec{r}_3$$

$$\left[\begin{array}{ccc|ccc} -3 & 0 & 0 & 0 & 7 & -2 \\ 0 & 5 & 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 0 & -2 & 1 \end{array}\right] \qquad \vec{r}_1 \to 3\vec{r}_1 - 2\vec{r}_3$$

The inverse is then

$$B^{-1} = \begin{bmatrix} 0 & -7/3 & 2/3 \\ 1/5 & 0 & -1/5 \\ 0 & -2/3 & 1/3 \end{bmatrix}.$$

10. (8 points) Let $f(x,y) = xe^y$. Find the maximum and minimum values of $z = f(x,y)$ on the set $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 2\}$.

We'll use Lagrange multipliers. We have $\nabla f = \langle e^y, xe^y \rangle$ and $\nabla g = \langle 2x, 2y \rangle$, so $e^y = 2\lambda x$ and $xe^y = 2\lambda y$. Eliminating λ, $\dfrac{e^y}{x} = \dfrac{xe^y}{y}$, and so $y = x^2$. Plugging that into $x^2 + y^2 = 2$, $y + y^2 = 2$, so $y^2 + y - 2 = (y+2)(y-1) = 0$. If $y = -2$, then $x = \pm\sqrt{-2}$ is not a real number, but if $y = 1$, then $x = \pm 1$, so we have the points $(1,1)$ and $(-1,1)$. The first gives $f(1,1) = e$ and the second is $f(-1,1) = -e$, so e is the maximum value and $-e$ is the minimum.

11. (14 points) Consider the system of equations

$$2x + y + z = 4$$

$$3x - y + 9z = 1$$

$$5x + 2y + 4z = 9$$

a) (2 points) Write the system as a matrix equation $A\vec{x} = \vec{b}$ for a matrix of numbers A, a vector of numbers $\vec{b}$, and a vector of variables $\vec{x}$ (this is not yet an augmented matrix).

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 9 \\ 5 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

b) (8 points) Write the system as an augmented matrix and row reduce it to reduced row echelon form, labeling each row operation you perform.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 3 & -1 & 9 & 1 \\ 5 & 2 & 4 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 1 & -2 & 8 & -3 \\ 5 & 2 & 4 & 9 \end{array}\right] \qquad \vec{r_2} \to \vec{r_2} - \vec{r_1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 8 & -3 \\ 2 & 1 & 1 & 4 \\ 5 & 2 & 4 & 9 \end{array}\right] \qquad \vec{r_1} \leftrightarrow \vec{r_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 8 & -3 \\ 0 & 5 & -15 & 10 \\ 0 & 12 & -36 & 24 \end{array}\right] \qquad \begin{array}{l} \vec{r_2} \to \vec{r_2} - 2\vec{r_1} \\ \vec{r_3} \to \vec{r_3} - 5\vec{r_1} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 8 & -3 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -3 & 2 \end{array}\right] \qquad \begin{array}{l} \vec{r_2} \to \frac{1}{5}\vec{r_2} \\ \vec{r_3} \to \frac{1}{12}\vec{r_3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 8 & -3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \vec{r_3} \to \vec{r_3} - \vec{r_2}$$

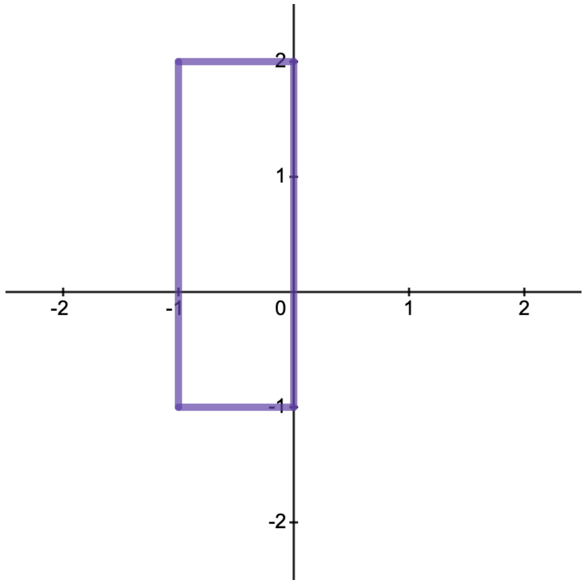
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \vec{r_1} \to \vec{r_1} + 2\vec{r_2}$$

c) (4 points) Write the solution $\vec{x}$ to the matrix equation in part a. If there is no solution, briefly explain why. If there are infinitely many solutions, use one or more free parameters.

We have

$$\vec{x} = \begin{bmatrix} 1 - 2t \\ 2 + 3t \\ 3 \end{bmatrix}.$$

12. (12 points) Let $g(x,y) = x^2 + y^4 + 4xy$ be defined on the rectangle R with edges at $x = -1$, $x = 0$, $y = -1$, and $y = 2$, as shown.



Find the maximum and minimum values of g on R. Note that R is hollow: you do not need to consider g inside of it.

We need to handle one line at a time, since this isn't a single equation that would allow us to use Lagrange multipliers.

$y = 2$: $g(x,2) = x^2 + 16 + 8x$, and $\frac{d}{dx}[g(x,2)] = 2x + 8$, which is zero when $x = -4$. However, the point $(-4,2)$ isn't in R, so we ignore it.

$y = -1$: $g(x,-1) = x^2 + 1 - 4x$, and $\frac{d}{dx}[g(x,-1)] = 2x - 4$, which is zero when $x = 2$. Again, outside R, so we throw it out.

$x = -1$: $g(-1,y) = 1 + y^4 - 4y$, and $\frac{d}{dy}[g(-1,y)] = 4y^3 - 4$, which is zero when $y = 1$. That gives us the point $(-1,1)$.

$x = 0$: $g(0,y) = y^4$, and $\frac{d}{dy}[g(0,y)] = 4y^3$, which is zero when $y = 0$. That gives us the point $(0,0)$.

We consider these points, and also the corners. We have

$$g(-1,1) = -2$$
$$g(0,0) = 0$$
$$g(-1,2) = 9$$
$$g(0,2) = 16$$
$$g(-1,-1) = 6$$
$$g(0,-1) = 1.$$

In total, the maximum value is 16, and the minimum -2.

Name: ______________________