Math 1180

Midterm 1

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You have 75 minutes to complete this exam and turn it in. You may use a 3×5 inch two-sided handwritten index card, but no calculators. You may not consult the internet or other people, but if you have a question, please don't hesitate to ask me!

You should expect to spend a little more than one minute per question per point that it's worth: there are 70 points possible. Show enough work that there is no question about the mathematical process used to obtain your answers. Finally, please write your name on the **back** of the exam!

Multiple choice (each question has only one correct answer). You don't need to show your work.

1. (3 points) Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ with $\vec{v} \bullet \vec{w} \geq 4$. Which of the following must be true?

a) $||\vec{v}|| \geq 2$ and $||\vec{w}|| \geq 2$.

b) The angle between $\vec{v}$ and $\vec{w}$ is less than 90°.

c) Both $\vec{v}$ and $\vec{w}$ have positive first components.

d) All of the above.

Using the dot product as an angle formula, we have $||\vec{v}|| \cdot ||\vec{w}|| \cos(\theta) \geq 4$, and the only way for that to happen is $\cos(\theta) > 0$, so $\theta < 90°$. Neither a) nor c) must be true, and so the answer here is b). 2. (3 points) Let $P(w, s, m)$ be the yearly profit in dollars to a company when it sells w widgets per year, pays each employee s dollars per year, and spends m dollars per year on materials. Which of the following is a valid interpretation of

$$\frac{\partial P}{\partial m}(3000, 100000, 50000) = -200?$$

a) If the company makes 3000 widgets per year, pays each worker \$100000 per year, and spends \$50000 per year in materials, then each additional dollar it spends on materials per year results in approximately \$200 less yearly profit.

b) If the company makes 3000 widgets per year, pays each worker \$100000 per year, and spends \$50000 per year in materials, then its profit is decreasing by \$200 per year.

c) If the company's yearly profit is going down by approximately \$200 per additional dollar it spends on materials per year, then it is making 3000 widgets per year, paying each worker \$100000 per year, and spends \$50000 per year in materials.

d) If the company's profit is decreasing by \$200 per year, then it is making 3000 widgets per year, paying each worker \$100000 per year, and spends \$50000 per year in materials.

This partial derivative is talking about a change in profit *relative* to a change in materials spending, narrowing down the answer to a) or c). But c) has the cause and effect swapped — the interpretation of this partial is a). 3. (5 points) Let $f(x, y)$ be a differentiable function. The plane $3(x-1) + 2(y-1) = 0$ intersects the tangent plane to f at $(1, 1)$ to form a line. Which of the following vectors must be parallel to that line?

a) $\left\langle 3, 2, D_{\langle 3,2\rangle} f(1, 1)\right\rangle$.

b) $\langle 3, 2, \nabla f(1, 1) \bullet \langle 3, 2\rangle\rangle$.

c) $\left\langle -2, 3, D_{\langle -2,3\rangle} f(1, 1)\right\rangle$.

d) $\langle -2, 3, \nabla f(1, 1) \bullet \langle -2, 3\rangle\rangle$.

This is maybe the hardest conceptual question on the test. The crucial piece of information is that the intersection of this plane with the tangent plane makes a tangent line to the graph of f whose slope is given by a directional derivative. Since this plane has normal vector $\langle 3,2,0\rangle$, the xy-direction of the tangent line is *orthogonal* to $\langle 3,2\rangle$, which narrows us down to answers c) and d). Moving -2 in the x-direction and 3 in the y-direction while staying in the tangent plane moves us $-2f_x(1,1)+3f_y(1,1)$ in the z-direction by the equation of the tangent plane, but answer c) uses a directional derivative that normalizes the vector $\langle -2,3\rangle$ before taking a dot product, so its z-component wouldn't be correct. That narrows us down to d), but due to the difficulty and the finicky nature of this problem, I've decided to 1. give half credit for answer c) and 2. Make the problem out of 2.5 instead of 5 (so answer d) is effectively worth 2.5 points extra credit).

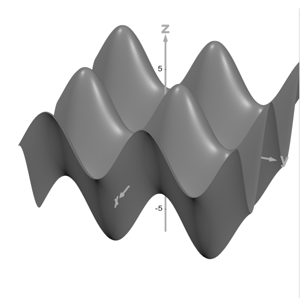
4. (6 points) For each of the following equations, write its letter below its 3D plot and its level curves. Some of the level curves will not be matched.

$$(\textbf{A})\ f(x,y)=\frac{1}{5}\left(x^2-y^2-2xy\right)$$

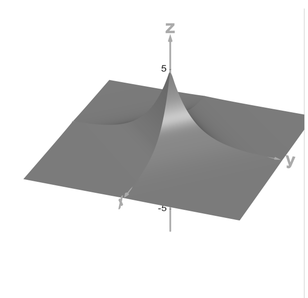
$$(\textbf{B})\ g(x,y)=y\sin(x)$$

$$(\textbf{C})\ h(x,y)=5\left(2^{-|x|-|y|}\right)$$

$$(\textbf{D})\ k(x,y)=3\sin(x)+2\sin(y)$$



D



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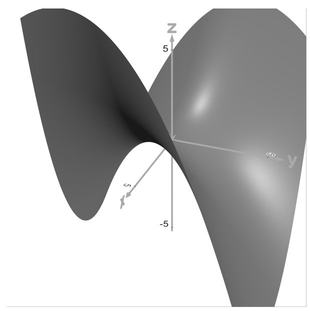
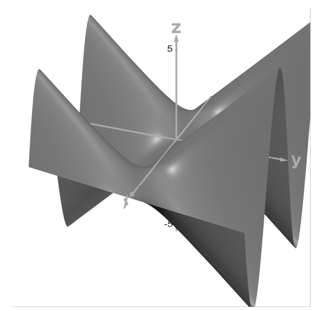


A



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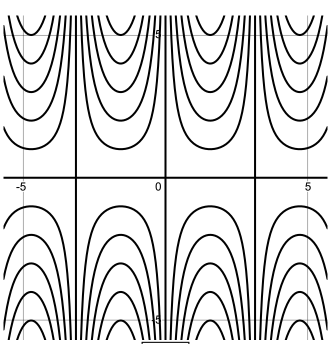

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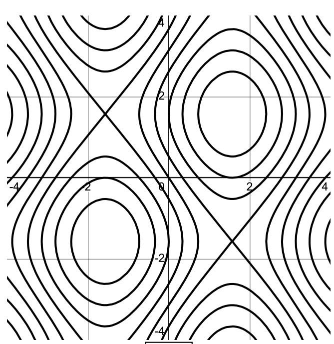

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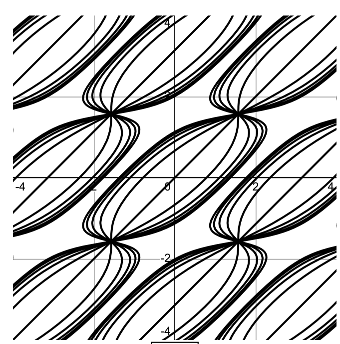
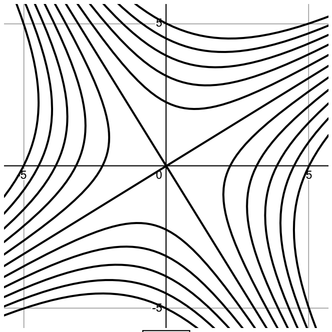
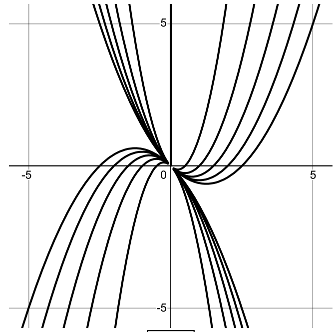
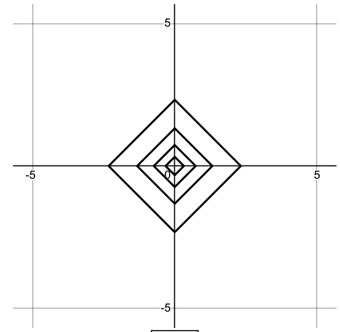




A





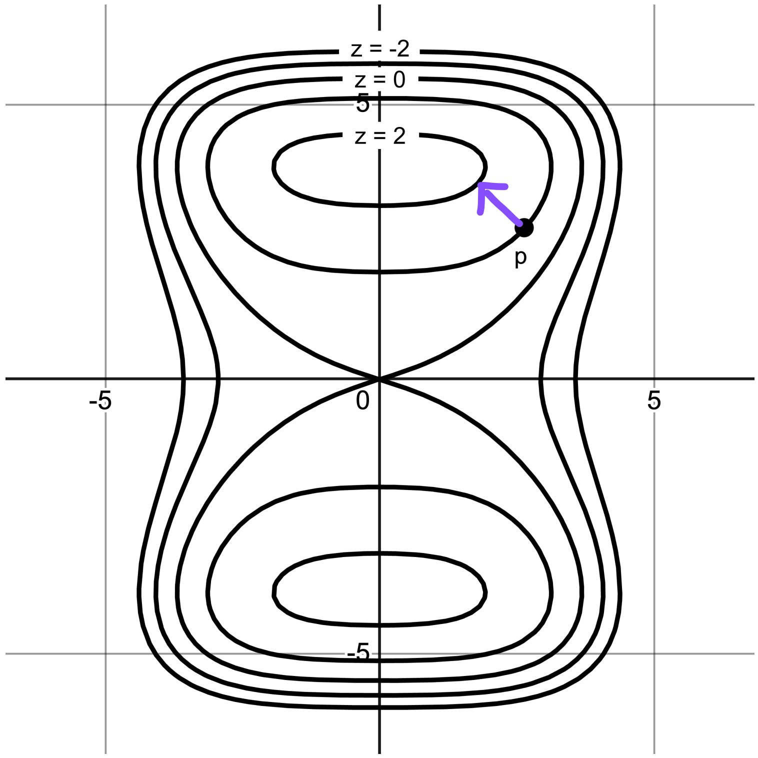
C

Apologies for the formatting of the answers in the boxes here. To briefly explain how we might arrive at these, $k(x,y) = 3\sin(x) + 2\sin(y)$ is an egg-carton-like function whose ilk we've seen on homeworks and is the first 3D plot; the absolute values in h result in the sharp diamond edges in the second; f has a saddle point at $(0,0)$, since setting $x = 0$ gives $-\frac{1}{5}y^2$ and $y = 0$ gives $\frac{1}{5}x^2$, which have opposite concavities, and the plot is therefore the third one, and $g(x,y) = y\sin(x)$ is the only graph that's zero along either axis, forcing it to be the third one.

We can mostly match graphs to level curves at this point, with the possible exception of matching $f(x,y)$ to either the bottom-left or bottom-middle set of level curves. One way to determine that it's the bottom-left one is that all of the level curves in the bottom-middle plot pass through $(0,0)$, and that isn't the case for f: we can see from its graph that not all planes $z = c$ intersect a point whose xy-coordinates are $(0,0)$, or alternatively, plugging in $(0,0)$ to $f(x,y)$ outputs 0, so only that one level curve can pass through $(0,0)$.

5. (2 points) Below is a plot of level curves of a function $f(x,y)$. Draw a vector starting from the point p in the same direction as $\nabla f(p)$.

The gradient points in the direction of greatest increase, so it should point orthogonal to the level curve in the direction of the $z = 2$ curve.



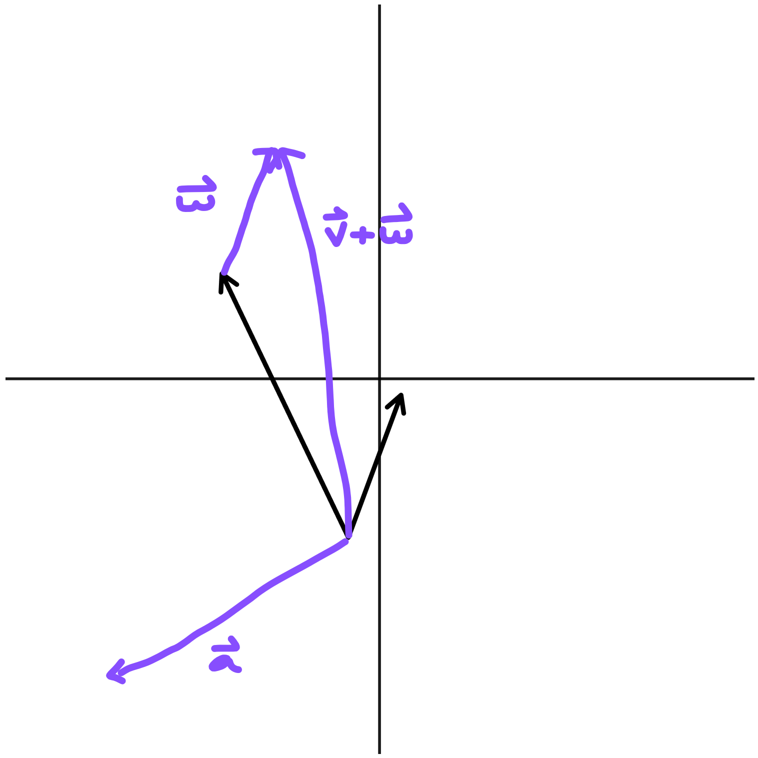
6. (4 points) Below are vectors $\vec{v}$ (on the left) and $\vec{w}$ (on the right). Both are in the xy-plane.

a) (2 points) Draw $\vec{v} + \vec{w}$.

Here, we have to remember to shift one to start at the end of the other.

b) (2 points) Draw a vector $\vec{a}$ in the plane so that $\vec{v} \times \vec{a}$ sticks up orthogonally out of the page, and $||\vec{v} \times \vec{a}|| = ||\vec{v}||^2$.

Using the right-hand rule, $\vec{a}$ should be counterclockwise from $\vec{v}$, and since $||\vec{v} \times \vec{a}|| = ||\vec{v}|| \cdot ||\vec{a}|| \sin(\theta)$, we should make them orthogonal.



Longer questions. Show all your work!

7. (12 points) Let $f(x,y) = e^x \sin\left(\frac{\pi}{2}\left(x+y^2\right)\right)$.

a) (6 points) Find $\dfrac{\partial^2 f}{\partial y\,\partial x}(0,1)$.

Since there's only a single y, I'd personally choose to use Clairaut's theorem and do $\frac{\partial}{\partial y}$ first, even though the notation is asking us to the x partial first. We have

$$\begin{aligned}
f_y(x,y) &= e^x \cos\left(\frac{\pi}{2}\left(x+y^2\right)\right)(\pi y)\\
f_{xy}(x,y) = f_{yx}(x,y) &= e^x \cos\left(\frac{\pi}{2}\left(x+y^2\right)\right)(\pi y) - e^x \sin\left(\frac{\pi}{2}\left(x+y^2\right)\right)(\pi y)\cdot\frac{\pi}{2}\\
f_{yx}(0,1) &= \cos\left(\frac{\pi}{2}\right)(\pi) - \sin\left(\frac{\pi}{2}\right)(\pi)\cdot\frac{\pi}{2}\\
&= 0 - (\pi)\cdot\frac{\pi}{2}\\
&= -\frac{\pi^2}{2}.
\end{aligned}$$

b) (6 points) Find $D_{\vec{u}} f(0,1)$ for $\vec{u} = \langle 3, 4 \rangle$.

This requires us to find the other derivative so that we can find the gradient. We have

$$\begin{aligned} f_x(x,y) &= e^x \sin\left(\frac{\pi}{2}\left(x + y^2\right)\right) + e^x \cos\left(\frac{\pi}{2}\left(x + y^2\right)\right) \cdot \frac{\pi}{2} \\ f_x(0,1) &= \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{2} \\ &= 1. \end{aligned}$$

Since $f_y(0,1) = 0$, $\nabla f(0,1) = \langle 1, 0 \rangle$. We can find the directional derivative by normalizing $\vec{u}$ and taking a dot product:

$$\begin{aligned} D_{\vec{u}} f(0,1) &= \nabla f(0,1) \bullet \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \langle 1, 0 \rangle \bullet \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \frac{3}{5}. \end{aligned}$$

8. (5 points) Let C be the solid cylinder in $\mathbb{R}^3$ with radius 4 that is parallel to the y-axis, and whose center axis passes through $(1, 2, -3)$. Write C in set builder notation.

The points in the cylinder are those with $(x-1)^2 + (z+3)^2 \le 4^2$, and in set builder notation, that's

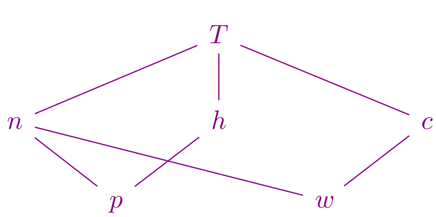
$$\left\{(x,y,z) \in \mathbb{R}^3 \mid (x-1)^2 + (z+3)^2 \le 4^2\right\}.$$

9. (6 points) The number of hours it takes a student to complete a weekly homework assignment is a function $T(n, h, c)$ of the number of problems n on the homework, the number of hours h that they spend in class and in office hours that week, and the number of friends c that they collaborate with.

Moreover, $c = c(w)$ is a function of the week w of the semester, $h = h(p)$ is a function of the pages of material p covered that week, and $n = n(w, p)$ is a function of both w and p. Given the following derivatives for a particular student at a point in the semester, how many additional hours of homework should this student expect per additional page of material covered?

$$\frac{\partial T}{\partial n} = 1 \qquad \frac{\partial T}{\partial h} = -\frac{1}{2} \qquad \frac{\partial T}{\partial c} = -2 \qquad \frac{\mathrm{d}c}{\mathrm{d}w} = -\frac{1}{3} \qquad \frac{\mathrm{d}h}{\mathrm{d}p} = \frac{1}{2} \qquad \frac{\partial n}{\partial w} = -\frac{1}{2} \qquad \frac{\partial n}{\partial p} = 2$$

There's a lot to parse out here, but what we're being asked to do is find $\frac{\partial T}{\partial p}$. Drawing out the dependency tree,



and so by the multivariable chain rule,

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial n}\frac{\partial n}{\partial p} + \frac{\partial T}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}p} = \frac{7}{4}.$$

10. (12 points) Let $p = (1, 0, -6)$, $q = (0, 2, -2)$, and $r = (3, 0, -2)$.

a) (6 points) There is only one plane that passes through p, q, and r. Find an equation for it.

We'll start by finding the normal vector, which we can use the cross product for: the vector from p to q is $\langle -1, 2, 4\rangle$, and the vector from p to r is $\langle 2, 0, 4\rangle$. Their cross product is $\langle 8, 12, -4\rangle$, and so using the point-slope-like formula for a plane, the formula for this one is

$$8(x-1) + 12y - 4(z+6) = 0,$$

or equivalently

$$2x + 3y - z = 8.$$

b) (6 points) The line $l(t)$ is parallel to the vector $\langle 1, 2, -1\rangle$ and passes through the point $(0, -1, 7)$. Find the intersection of $l(t)$ with the plane from part a) or show the two do not intersect.

The equation for this line is

$$l(t) = \langle 0 - 1, 7\rangle + \langle 1, 2, -1\rangle\, t,$$

so to intersect it with the plane, we set $x = t$, $y = -1 + 2t$, and $z = 7 - t$, giving

$$2t + 3(-1 + 2t) - (7 - t) = 8.$$

That results in $t = 2$, so they do intersect, and it's at the point $(2, -1 + 2(2), 7 - 2) = (2, 3, 5)$.

11. (12 points) Let $f(x, y) = 3x^2 + y^3 - 6xy$.

a) (2 points) Find ∇f.

Taking the partials, $\nabla f = \left\langle 6x - 6y, 3y^2 - 6x\right\rangle$.

b) (4 points) Find the tangent plane to f at $(1, 1)$ and use it to approximate $f(0, 0)$.

Plugging in $(1, 1)$ gives $f(1, 1) = -2$ and $\nabla f(1, 1) = \langle 0, -3\rangle$, so the equation of the tangent plane is

$$z = -2 - 3(y - 1).$$

Its approximation is $f(0, 0) \approx -2 - 3(0 - 1) = 1$. The actual value is $f(0, 0) = 0$, so it's not the best approximation.

c) (3 points) Find the critical points of f.

Solving $\nabla f = \vec{0}$, we have $6x - 6y = 0$, which means $x = y$, and $3y^2 - 6x = 0$, which we can rewrite as $x(x - 2) = 0$. The critical points are then $(0, 0)$ and $(2, 2)$.

d) (3 points) Classify the critical points of f as local maxima, local minima, or saddle points.

The mixed partials are $f_{xx} = 6$, $f_{xy} = -6$, and $f_{yy} = 6y$, so the discriminant is $D(x, y) = 36y - 36$. That means $D(0, 0) < 0$, so it's a saddle point, and $D(2, 2) > 0$. Since $f_{xx}(2, 2) = 6 > 0$, that means $(2, 2)$ is a local min.