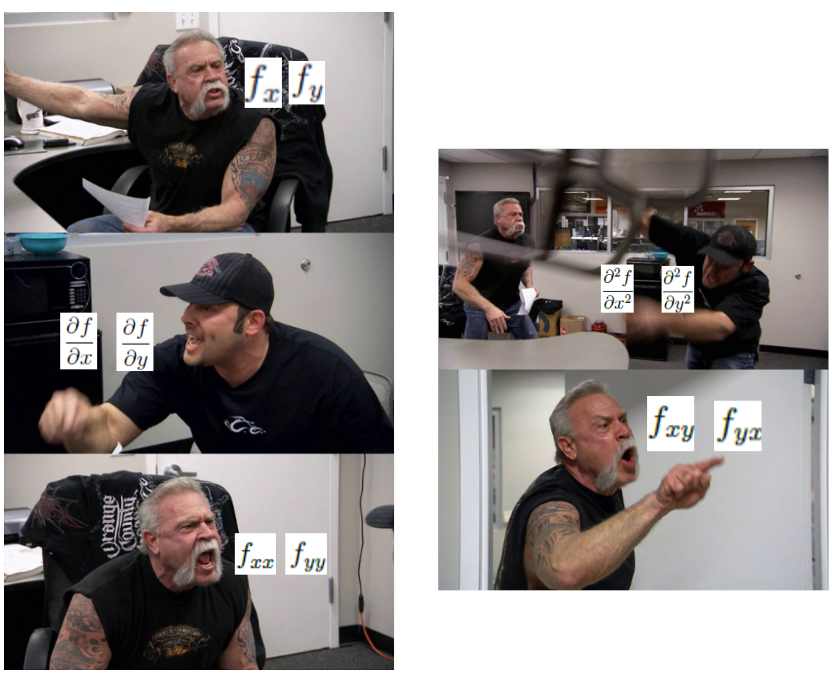
Introduction to Functions of Several Variables Exam 2

Please sign the following Honor Code statement.

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of academic integrity.

Signature: Anne Serkey

- There are 80 total points on this exam.
- **Please read all instructions carefully and ask for clarification if you are unsure about anything.**
- If you need extra space, you may write on the backs of the pages. In that case, please indicate that you have extra work on another page.
- You are not permitted to use any resources besides a "cheat sheet" of one side of an 8.5×11 inch (or smaller) sheet of paper. Please make sure your name is on your cheat sheet and turn it in with your exam. I will return your cheat sheet at a later date.
- Please give exact answers (e.g., $\sqrt{2}$ instead of 1.41) unless explicitly instructed not to.
- Even when justification is not necessary, partial credit may be given for work/explanations that do not lead to the correct answer.
- Take a deep breath and trust yourself. **I believe in you and am proud of you no matter how you perform on this exam.**



1. (18 points) Derivative Calculations. Show your work.

(a) (6 points) Find $f_y(x,y)$ for $f(x,y) = \frac{xe^y}{y^2+1}$.

$$f_y(x,y) = \frac{\partial/\partial y(xe^y)\cdot(y^2+1) - xe^y \frac{\partial}{\partial y}(y^2+1)}{(y^2+1)^2} = \frac{xe^y(y^2+1) - 2xye^y}{(y^2+1)^2}$$

$$= \boxed{\frac{xe^y(y^2+1-2y)}{(y^2+1)^2}}$$

(b) (6 points) Find $\partial z/\partial x$ for the surface defined by $\ln(x^2y) + 3e^{yz} - y^2 - xyz^2 = 0$.

Write $f(x,y,z) = \ln(x^2y) + 3e^{yz} - y^2 - xyz^2$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2y}\frac{\partial}{\partial x}(x^2y) - yz^2 = \frac{2xy}{x^2y} - yz^2 = \frac{2}{x} - yz^2$$

$$\frac{\partial f}{\partial z} = 3e^{yz}\frac{\partial}{\partial z}(yz) - 2xyz = 3ye^{yz} - 2xyz = y(3e^{yz} - 2xz)$$

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2/x - yz^2}{y(3e^{yz} - 2xz)} = \boxed{\frac{yz^2 - 2/x}{y(3e^{yz} - 2xz)}}$$

(c) (6 points) Find $f_{xy}(x,y)$ for $f(x,y) = y\sin(xy)$.

$$f_x(x,y) = y\cos(xy)\cdot\frac{\partial}{\partial x}(xy) = y^2\cos(xy)$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(y^2)\cdot\cos(xy) + y^2\cdot\frac{\partial}{\partial y}(\cos(xy))$$

$$= 2y\cos(xy) - y^2\sin(xy)\cdot\frac{\partial}{\partial y}(xy)$$

$$= \boxed{2y\cos(xy) - xy^2\sin(xy)}$$

2. (15 points) Suppose S is a surface defined by a function $z = f(x, y)$ with continuous first- and second-order partial derivatives over all of $\mathbb{R}^2$. Consider the following table of values of f and its derivatives at various points.

Point	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{yy}(x,y)$	$f_{xy}(x,y)$
P_1	-1	3	-2	4	3	2
P_2	3	0	0	5	-4	4
P_3	-3	0	0	1	4	-2
P_4	-2	0	0	5	4	1
P_5	-3	0	0	-3	-2	1

(a) (10 points) Classify each point as **exactly one** of the following: (i) not a critical point, (ii) saddle point, (iii) local maximum, (iv) local minimum, or (v) not enough information to determine. No justification necessary.

- P_1: Not a critical point — $f_x \neq 0$ and $f_y \neq 0$ (although we only need one of these)
- P_2: $D = 5\cdot(-4) - 4^2 = -36 < 0 \rightarrow$ saddle point
- P_3: $D = 1\cdot 4 - (-2)^2 = 0 \rightarrow$ not enough information
- P_4: $D = 5\cdot 4 - 1^2 = 19 > 0$, and $f_{xx} > 0 \rightarrow$ local min
- P_5: $D = (-3)\cdot(-2) - 1^2 = 5 > 0$, and $f_{xx} < 0 \rightarrow$ local max

(b) (5 points) Suppose $P_1 = (-5, 4)$. Find an equation for the tangent plane to f at P_1 and use this tangent plane to approximate $f(-4.7, 4.5)$. Briefly show your work.

$$z = f(-5,4) + f_x(-5,4)(x+5) + f_y(-5,4)(y-4)$$

$$\boxed{z = -1 + 3(x+5) - 2(y-4)}$$

$$f(-4.7, 4.5) \approx -1 + 3(-4.7+5) - 2(4.5-4) = -1 + 0.9 - 1 = \boxed{-1.1}$$

3. (17 points) The temperature T (in °F) in Bailey's apartment is a function of his location (x, y). He deduces that $\partial T/\partial x = -1$ and $\partial T/\partial y = 4$ at his current position. Assume that T has continuous partial derivatives.

(a) (5 points) Suppose Bailey is cold, so he wants to walk in the direction in which the temperature is increasing most quickly. Find a unit vector in the direction in which he should move. Briefly show your work.

Direction is given by $\nabla f(x,y) = \langle -1, 4 \rangle$. Need a unit vector $\vec{u}$ in this direction:

$$\vec{u} = \frac{1}{\sqrt{(-1)^2+4^2}}\langle -1, 4\rangle = \frac{1}{\sqrt{17}}\langle -1, 4\rangle = \boxed{\left\langle -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle}$$

(b) (2 points) If Bailey instead walks in a direction that is orthogonal to the direction from (a) (i.e., orthogonal to the direction in which the temperature is increasing most rapidly), then what is the instantaneous rate of change of temperature in that direction? No justification necessary.

0 This is one of the amazing properties of the gradient!

(c) (5 points) What is the instantaneous rate of change of T in the direction of $\vec{v} = \langle -3, 4 \rangle$? Show your work.

Need a unit vector $\vec{u}$ in direction of $\vec{v}$. $\|\vec{v}\| = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{5}\langle -3, 4\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$$

$$D_{\vec{u}} f(x,y) = \langle -1, 4\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = \frac{3}{5} + \frac{16}{5} = \frac{19}{5}$$

(d) (5 points) The x-coordinate $x(t)$ and y-coordinate $y(t)$ of Bailey's position at time t (in seconds) are both functions of t. Suppose that $dy/dt = 2$ when he is at his current position. If the temperature is increasing at a rate of 4°F/second, then what is dx/dt at this moment? Briefly show your work.

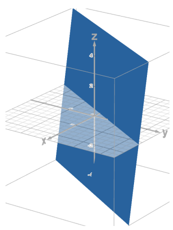
$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt}$$

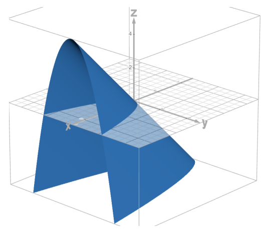
$$4 = -1\cdot\frac{dx}{dt} + 4\cdot 2 \rightarrow 4 = -\frac{dx}{dt} + 8 \rightarrow -4 = -\frac{dx}{dt} \rightarrow \boxed{4 = \frac{dx}{dt}}$$

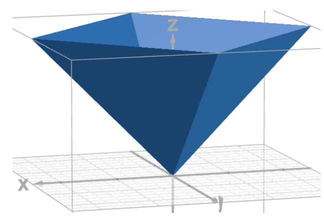
4. (10 points) Match each function to its graph and contour plot. No justification needed.

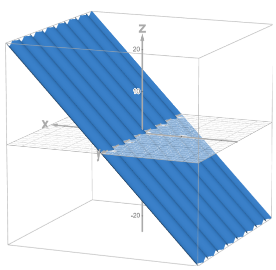
Function	Graph	Contour Plot
$f_1(x,y) = x + \cos(y)$	d	v
$f_2(x,y) = x - y^2$	b	ii
$f_3(x,y) = \frac{x^2+y^2+1}{x^2+y^2}$	e	i
$f_4(x,y) = -5x + 3y - 3$	a	iii
$f_5(x,y) = \lvert x\rvert + \lvert y\rvert$	c	iv

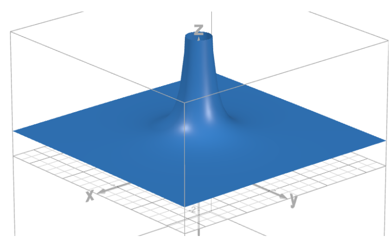
Graphs:

(a) 

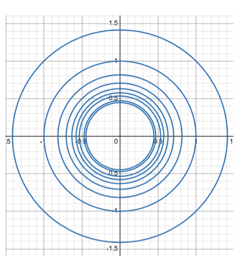
(b) 

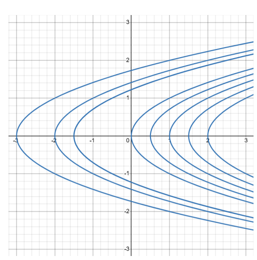
(c) 

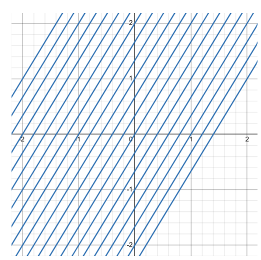
(d) 

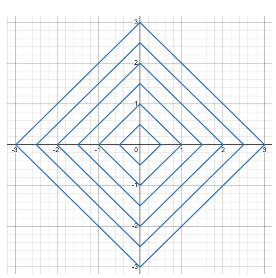
(e) 

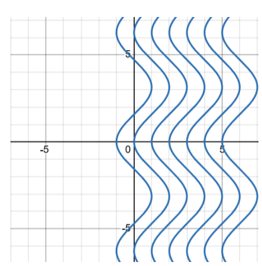
Contour Plots:

(i) 

(ii) 

(iii) 

(iv) 

(v) 

5. (20 points) Short Answer. Clearly answer each question. Assume that $f(x, y)$ is continuous with continuous first- and second-order partial derivatives over $\mathbb{R}^2$. **The parts of this problem are independent of one another.**

(a) (3 points) Suppose we restrict the domain of f to some subset D of $\mathbb{R}^2$. For which of the following choices of D is f guaranteed to attain an absolute maximum value and an absolute minimum value over D? **Select all that apply.** You do not need to justify your answers.

Not bounded

(i) $x^2 + y^2 \geq 25$ **(iii)** $x^2 + y^2 \leq 25$ **(v)** $0 \leq x \leq 3$ and $0 \leq y \leq 5$

(ii) $x^2 + y^2 < 25$ **(iv)** $x^2 + y^2 = 25$ **(vi)** $0 < x < 3$ and $0 < y < 5$

Not closed   Not Closed

(b) (3 points) Suppose $f_x(x, y) = 3x^2y + 2y^2 - 1$ and $f_y(x, y) = x^3 + 2cxy$, where c is some constant. What must the value of c be? Briefly show your work. (**Hint:** Mixed second-order partial derivatives.)

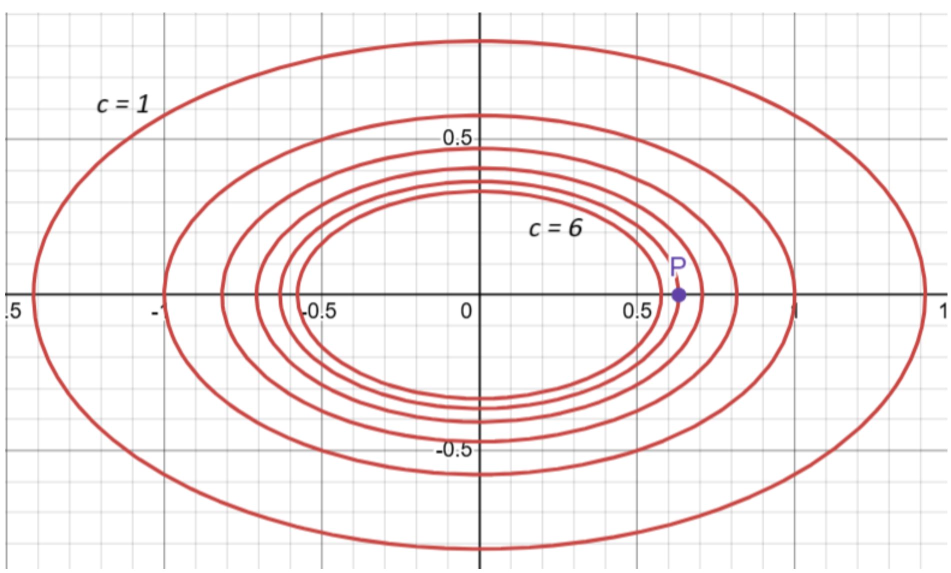
$f_{xy}(x,y) = 3x^2 + 4y$

$f_{yx}(x,y) = 3x^2 + 2cy$

Clairaut's Theorem $\Rightarrow f_{xy}(x,y) = f_{yx}(x,y)$.

Therefore, must have $4 = 2c \rightarrow \boxed{2 = c}$

(c) (3 points) A contour plot for $c = 1, 2, ..., 6$ of a continuous function $z = f(x, y)$ is shown below. The outermost curve corresponds to $c = 1$, the innermost curve corresponds to $c = 6$, and the other curves are sequentially in-between. Do you expect f_x to be positive or negative at P? Briefly justify your answer.



Negative

Based on the level curves, it appears that the z-values are decreasing as the x-values increase at P.

(d) (5 points) Find all critical points of $f(x,y) = 2x^2 - xy^2 + y^2 - 3$. Show your work.

$f_x(x,y) = 4x - y^2$

$f_y(x,y) = -2xy + 2y$

$4x - y^2 = 0$

$4x = y^2$

$x = \frac{y^2}{4}$

$-2xy + 2y = 0$

$-2\left(\frac{y^2}{4}\right)y + 2y = 0$

$-\frac{y^3}{2} + 2y = 0$

$y\left(-\frac{y^2}{2} + 2\right) = 0$

$y = 0 \quad -\frac{y^2}{2} + 2 = 0$

$x = 0 \quad 2 = \frac{y^2}{2}$

$4 = y^2$

$x = 1 \leftarrow \pm 2 = y$

$$\boxed{(0,0) \quad (1,-2) \quad (1,2)}$$

(e) (3 points) Suppose the line parallel to the xz-plane and tangent to the graph of f at $(x,y) = (0,5)$ is given by $x = t, y = 5$, and $z = -4 - 3t$. Find $f(0,5)$ and **either** $f_x(0,5)$ or $f_y(0,5)$. (You don't have enough information to find both of these partial derivatives, but you can find one of them.) No justification necessary.

This line goes through the point $(0,5,-4)$ and has direction vector $\langle 1,0,-3\rangle$. Therefore, $f(0,5) = -4$ and $f_x(0,5) = -3$

(f) (3 points) Suppose $f_x(-1,-5) = 2$ and $f_y(-1,-5) = 4$. Is there a unit vector $\vec{u}$ such that $D_{\vec{u}} f(-1,-5) \geq 5$? Justify your answer.

Maximum value of $D_{\vec{u}} f(-1,-5)$ is $\|\nabla f(-1,-5)\| = \|\langle 2,4\rangle\| = \sqrt{2^2+4^2} = \sqrt{20} < 5$

NO