1. (12 points) For what value(s) of k does the following system of linear equations have **(i)** no solutions, **(ii)** exactly one solution, and **(iii)** infinitely many solutions? Please label any elementary row-operations so that someone could follow along with your work. **Make sure to answer all three parts! (i, ii, and iii)**

$$\begin{array}{rcrcrcr} 3x & + & 2y & + & 2z & = & 5 \\ -2x & - & y & - & 3z & = & k \\ & & y & - & 5z & = & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & 5 \\ -2 & -1 & -3 & k \\ 0 & 1 & -5 & 13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5+k \\ -2 & -1 & -3 & k \\ 0 & 1 & -5 & 13 \end{array}\right] \quad \vec{r}_1 \to \vec{r}_1 + \vec{r}_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5+k \\ 0 & 1 & -5 & 10+3k \\ 0 & 1 & -5 & 13 \end{array}\right] \quad \vec{r}_2 \to \vec{r}_2 + 2\vec{r}_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5+k \\ 0 & 1 & -5 & 10+3k \\ 0 & 0 & 0 & 3-3k \end{array}\right] \quad \vec{r}_3 \to \vec{r}_3 - \vec{r}_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & -5-2k \\ 0 & 1 & -5 & 10+3k \\ 0 & 0 & 0 & 3-3k \end{array}\right] \quad \vec{r}_1 \to \vec{r}_1 - \vec{r}_2$$

if this isn't zero, there are no solutions, so (i) is all real numbers other than 1

what about when $k=1$? then we have

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} & & \uparrow & \\ & & t & \end{array} \end{array}$$

$$\left.\begin{array}{l} x = -7-4t \\ y = 13+5t \\ z = t \end{array}\right\} \text{infinitely many solutions}$$

so: (i) all real numbers except 1

(ii) none

(iii) $k=1$

2. (25 points) The monthly profit P of a cell phone company (**in thousands of dollars**) is given by

$$P(x, y) = 2x^2 + y^2 - 4x + 4y - 2xy - 2,$$

where x is the number of cell phones produced (**in hundreds**) each month and y is the number of hours of advertising purchased each month. **Throughout this problem (part (a) through part (f)), assume that the company is currently producing 300 cell phones per month and purchasing 2 hours of advertising each month.**

(a) (2 points) What is the company's profit at their current level of production?

$P(3,2) = 4$, so \$4000/month

(b) (4 points) Find $P_x(3, 2)$ and interpret its meaning in the context of the company's profit. Include units.

$P_x(x,y) = 4x - 4 - 2y$

$P_x(3,2) = 4$

Currently, producing ~~spending~~ an additional ~~$120/month~~ 100 phones per month will resolt in an additional profit of approximately \$4000/month.

(c) (5 points) If the company decides to increase their advertising and production by an equal amount (that is, they increase x and y by an equal amount), then what is the instantaneous rate of change in profit?

This is $D_{\langle 1,1\rangle} P(3,2)$.

$\nabla P = \langle 4x - 4 - 2y,\ 2y + 4 - 2x \rangle$

$\nabla P(3,2) = \langle 4, 2 \rangle$

$D_{\langle 1,1\rangle} P(3,2) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \cdot \langle 4, 2 \rangle = \frac{6}{\sqrt{2}}$ (or $3\sqrt{2}$)

(Restating problem 2 for you so that you don't have to flip the page.) The monthly profit P of a cell phone company (**in thousands of dollars**) is given by

$$P(x, y) = 2x^2 + y^2 - 4x + 4y - 2xy - 2,$$

where x is the number of cell phones produced (**in hundreds**) each month and y is the number of hours of advertising purchased each month. **Throughout this problem (part (a) through part (f)), assume that the company is currently producing 300 cell phones per month and purchasing 2 hours of advertising each month.**

(d) (6 points) Find a unit vector in the direction in which P increases most rapidly from the company's current position. What is the instantaneous rate of change of P in that direction?

Normalize $\nabla P(3,2) = \langle 4,2 \rangle$, so $\frac{1}{\|\langle 4,2\rangle\|} \langle 4,2 \rangle = \frac{1}{\sqrt{20}} \langle 4,2 \rangle$.

The rate of change is the magnitude of the gradient, so $\sqrt{20}$.

(e) (4 points) Suppose the company decides to increase their advertising by 1 hour per month. If they want the instantaneous rate of change of profit to be 0 after this change, then how should their production of cell phones change? ↳ with respect to what...?

This question is ambiguous! :(I'm going to interpret it as the rate of change with respect to y, since that's the variable we're primarily increasing. We want a value of $x = a$ so that

$\underbrace{\frac{\partial}{\partial y} P(a, 3)}_{10-2a} = 0$ $\Rightarrow a = 5$, so increase by 200 phones per month.

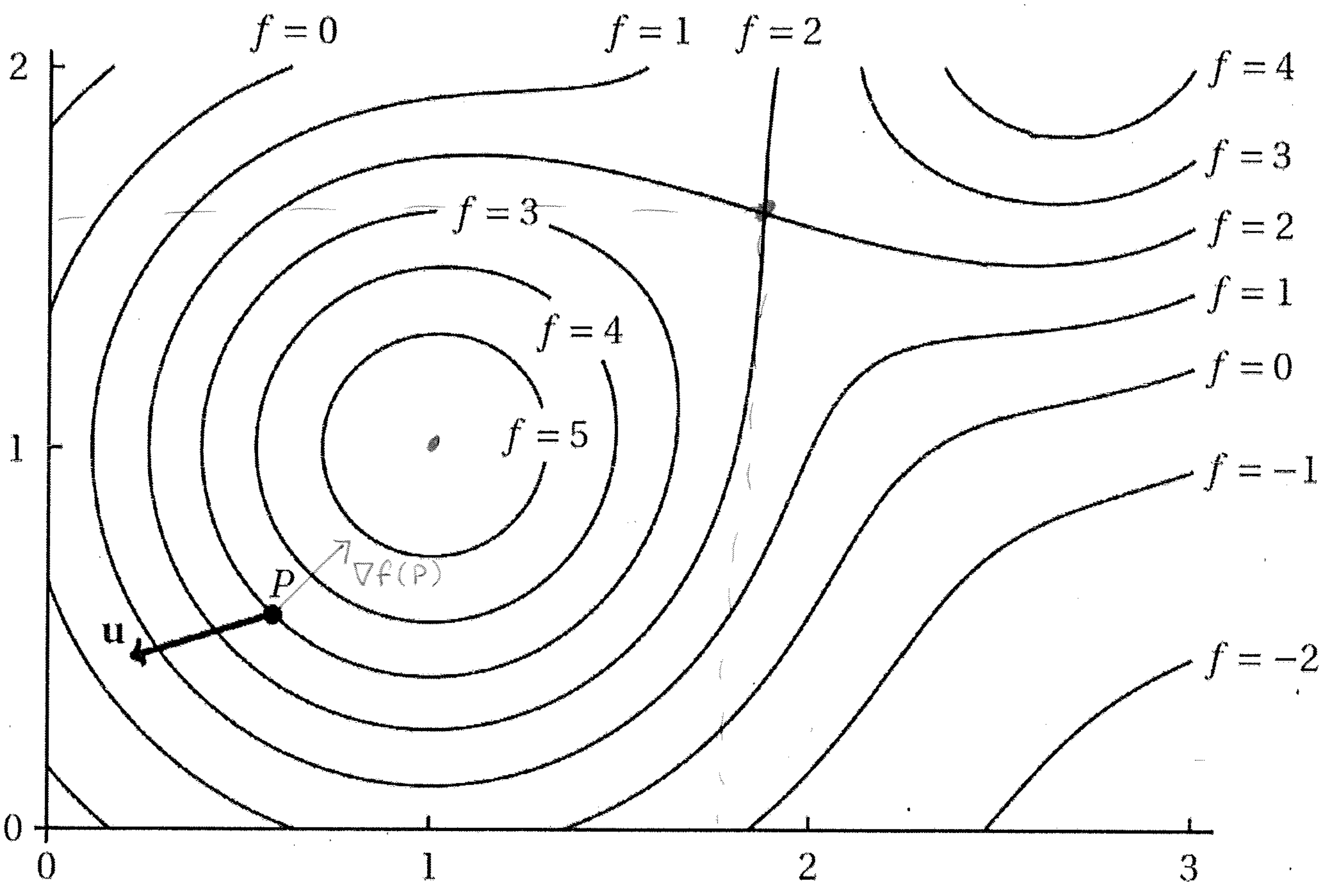
(f) (4 points) Write down the equation for the tangent plane to the graph of $P(x, y)$ at their current point. Then, use this tangent plane to estimate the profit if 350 cell phones are produced per month and 2.1 hours of advertising are purchased per month.

$z = P(3,2) + P_x(3,2)(x-3) + P_y(3,2)(y-2)$

$= 4 + 4(x-3) + 2(y-2)$

$P(3.5, 2.1) \approx 4 + 4(0.5) + 2(0.1)$

$= 6.2$, i.e. \$6200


f = 0
f = 1
f = 2
f = 4
f = 3
f = 3
f = 2
f = 1
f = 4
f = 0
f = 5
f = −1
∇f(P)
P
u
f = −2
0
1
2
0
1
2
3

3. (12 points) Consider the contour plot for a function $f(x, y)$ shown below.

final exam contour.png

(a) (4 points) At point P, determine whether f_x and f_y are positive, negative, or 0. No justification necessary.

Both positive (since we're moving toward a higher level curve in both the positive x and y directions).

(b) (4 points) On the graph, sketch $\nabla f(x, y)$ at the point ~~Q~~ P. (I am only concerned with the direction of the vector, not the magnitude.) No justification necessary.

(c) (4 points) Do any points shown in the contour plot appear to correspond to local extrema of f? Justify your answer.

(1, 1) looks like a local max since everything nearby appears to be lower

Approximately (1.75, 1.6) looks to be a saddle point, since there are multiple directions we can move in and remain at z=2, and other directions increase while yet others decrease.

4. (12 points) Find an equation or set of parametric equations for each line or plane.

(a) (4 points) The tangent line to $f(x,y) = \frac{4x}{y^2}$ at $(x,y) = (-2,2)$ that is parallel to the yz-plane.

slope: $\frac{\partial f}{\partial y}(x,y) = -8xy^{-3}$

i.e. doesn't move in the x-direction

$\frac{\partial f}{\partial y}(-2,2) = -8(-2)\cdot\frac{1}{8} = 2$

so $\ell(t) = \langle -2, 2, \underbrace{-2}_{f(-2,2)} \rangle + t\langle 0, 1, 2\rangle$

(b) (4 points) The plane containing $P = (1,2,2)$, $Q = (3,-1,-2)$, and $R = (1,-1,-1)$.

$\vec{PQ} = \langle 2, -3, -4\rangle$

$\vec{PR} = \langle 0, -3, -3\rangle$

$\vec{PQ} \times \vec{PR} = \langle -3, 6, -6\rangle$

contains P, so one option is $-3(x-1) + 6(y-2) - 6(z-2) = 0$

(c) (4 points) The line that makes a right angle with the plane $z = -2x - 3y + 1$ at $(x,y) = (-3,4)$.

↳ parallel to the plane's normal vector

normal vector: $-2x - 3y - z = -1$, so $\langle -2, -3, -1\rangle$ works

passes through $(-3, 4, -2(-3) - 3(4) + 1) = (-3, 4, -5)$

$\ell(t) = \langle -2, -3, -1\rangle t + \langle -3, 4, -5\rangle$

5. (15 points) Consider the following matrix A and its reduced row echelon form R.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & 1 & 0 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (5 points) Find a basis for $\mathrm{Col}\,(A)$ and determine the rank of A. No justification necessary.

(b) (5 points) Find a basis for $\mathrm{Nul}\,(A)$.

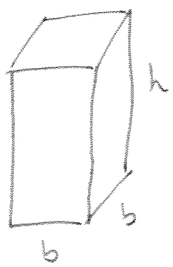
(c) (5 points) Suppose $\vec{b}$ is a vector in $\mathbb{R}^4$. What are the possibilities for the number of solutions $\vec{x}$ to the equation $A\vec{x} = \vec{b}$, where A is as above? Briefly justify your answer.

$[A \mid \vec{b}]$ reduces to $[R \mid \vec{c}]$ for some vector $\vec{c}$; since R has rows of all zeros, there are either no solutions (if $\vec{c}_3$ and $\vec{c}_4$ aren't both zero) or infinitely many, with two free parameters, if they are (since there are two columns without leading 1s).

6. (15 points) A company wants to design a rectangular box with a square base (i.e., the top and bottom will be squares, and the other four sides will be rectangles) and a volume of 64 cubic inches. The top and bottom of the box will cost \$1 per square inch to produce, while the other four sides of the box will cost \$2 per square inch to produce. **Use the method of Lagrange multipliers** to determine the dimensions that will minimize the total cost of producing the box, along with this minimal cost.

$h = \frac{64}{b^2}$. Could do single-variable optimization but the question asks for Lagrange!

$V = b^2 h = 64$



cost: $C(b,h) = b^2 + b^2 + 2bh + 2bh + 2bh + 2bh$

$= 2b^2 + 8bh$

constraint: $g(b,h) = b^2 h - 64 = 0$

$\nabla C = \langle 4b + 8h, 8b \rangle$

$\nabla g = \langle 2bh, b^2 \rangle$

so: $4b + 8h = \lambda(2bh)$

$8b = \lambda(b^2) \rightarrow \lambda = \frac{8}{b}$ (can assume $b \neq 0$ since then $b^2h \neq$ ...

$4b + 8h = 16h$

$b = 2h$

$b^2 h = 64$

$4h^3 = 64$

$h = \sqrt[3]{16}$, $b = 2\sqrt[3]{16}$

Side length of square base: $2\sqrt[3]{16}$ **Height:** $\sqrt[3]{16}$ ~~**Volume:**~~ Cost? $4(16^{2/3}) + 16(16^{2/3})$

7. (16 points) Determine whether each set of vectors is linearly independent (I), linearly dependent (D), or if there is not enough information to determine whether the set of vectors is linearly dependent or independent (N). No justification necessary. Partial credit may be given for justification given for an incorrect answer.

(a) (2 points) The columns of a matrix representing a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following property: If $T(\vec{u}) = T(\vec{v})$, then $\vec{u}$ must be equal to $\vec{v}$. $\Rightarrow$ one-to-one

I by properties of one-to-one matrices

(b) (2 points) The columns of a square matrix with 0's down the main diagonal (starting in the top left and moving down and to the right).

N $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ doesn't work but $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does

(c) (2 points) The set $\{\vec{u}, \vec{v}\}$, where the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

N! $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ works, but $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ doesn't

(d) (2 points) The set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, where none of the vectors are $\vec{0}$ and $\underbrace{\vec{v_1} \bullet \vec{v_2} = \vec{v_1} \bullet \vec{v_3} = \vec{v_2} \bullet \vec{v_3}}_{\text{all mutually orthogonal, whatever } \mathbb{R}^n \text{ they live in}} = 0$.

I

(e) (2 points) The columns of the matrix representing the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $T\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

I T is onto, so also one-to-one b/c its matrix is square

(f) (2 points) The columns of an $n \times n$ matrix A, where $\det(AB) = 8$ for some $n \times n$ matrix B.

I $\det A \cdot \det B \neq 0$, so $\det A \neq 0$

(g) (2 points) The columns of a matrix with more columns than rows.

D too many vectors to be l.i.!

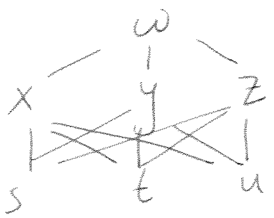
(h) (2 points) The columns of a square matrix whose rows are linearly dependent.

D non-invertible

8. (20 points) Find each partial derivative. Show your work.

(a) (6 points) Suppose $w = f(x,y,z)$, where $x = 2st^2u$, $y = su + \ln(t)$, and $z = s + t + u$. Use the information in the table below to evaluate $\frac{\partial w}{\partial t}$ at $(s,t,u) = (1,1,-2)$.

(x,y,z)	$f(x,y,z)$	$f_x(x,y,z)$	$f_y(x,y,z)$	$f_z(x,y,z)$
$(1,1,-2)$	-1	3	-4	2
$(-4,-2,0)$	-3	2	-4	1



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = 2(-8) - 4(1) + 1(1) = \boxed{-19}$$

$\frac{\partial x}{\partial t} = 4stu$ $\quad \frac{\partial x}{\partial t}(1,1,-2) = -8$

$\frac{\partial y}{\partial t} = \frac{1}{t}$ $\quad \frac{\partial y}{\partial t}(1,1,-2) = 1$

$\frac{\partial z}{\partial t} = 1$

At $(s,t,u) = (1,1,-2)$, $(x,y,z) = (-4,-2,0)$, and since we're given partials like $f_x(x,y,z)$, we need to use that point.

(b) (8 points) Find $f_{xy}(x,y)$ for $f(x,y) = \ln\left(\frac{xy}{x+y}\right)$. $= \ln(xy) - \ln(x+y)$

$= \ln(x) + \ln(y) - \ln(x+y)$.

Not necessary, just convenient!

$f_x(x,y) = \frac{1}{x} - \frac{1}{x+y}$

$f_{xy}(x,y) = +(x+y)^{-2}$

(c) **(6 points)** Find $\partial z/\partial y$ for the surface defined by $e^{zy} + xz^2 = 6xy^4z^3 + y^2$ at the point $(1, 1, 0)$.

9. (20 points) True/False. Determine whether each statement is true or false. No justification necessary. Partial credit may be given for justification given for an incorrect answer. Each part is worth two points.

(a) **True/False:** If (x_0, y_0) is a critical point of $f(x, y)$ and the linear transformation represented by $\underbrace{\begin{bmatrix} f_{xx}(x_0, y_0) & f_{yx}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}}$ $\underline{\text{preserves the orientation of } \mathbb{R}^2}$, then f has a saddle point at (x_0, y_0).

$\hookrightarrow$ i.e. det > 0

$\det = f_{xx} f_{yy} - \underbrace{f_{yx} f_{xy}}_{= f_{xy}^2} = D(x,y)$

Technically not true in general!

False! We'd need $D < 0$

(b) **True/False:** If $f(x, y)$ is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \vec{0}$, then there must exist unit vectors $\vec{u}$ and $\vec{v}$ such that $D_{\vec{u}} f(x_0, y_0) > 0$ and $D_{\vec{v}} f(x_0, y_0) < 0$.

True: there's a non-flat line in the tangent plane to f at (x_0, y_0), so we can move both up and down.

(c) **True/False:** The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x - 4y - 2 \\ 4y + 8 \end{bmatrix}$ is a linear transformation.

False! That -2 messes it all up, as does the $+8$.

E.g. $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, so $T(2 \cdot \vec{0}) \neq 2 \cdot T(\vec{0})$.

(d) True/False: If $f(x,y)$ is a differentiable function and (x_0, y_0) is a point in the domain of f such that $D_{\vec{u}} f(x_0, y_0) \leq f_x(x_0, y_0)$ for all unit vectors $\vec{u}$, then it must be the case that $f_y(x_0, y_0) = 0$.

True. That inequality means the steepest direction of increase is in the x-direction, so the gradient is $\nabla f = \langle f_x(x_0, y_0), 0 \rangle$.

(e) True/False: If a differentiable function $f(x,y)$ constrained to a smooth curve $g(x,y) = 0$ attains an absolute maximum at (x_0, y_0), then $\nabla g(x_0, y_0)$ is tangent to the level curve of f at (x_0, y_0).

False. That level curve and the curve $g(x,y)=0$ are tangent, not ∇g (which is orthogonal to g's level curve).

(f) True/False: The level curves of the function $f(x,y) = \frac{1}{1-x^2-y^2}$ are circles centered at the origin.

$c = \dfrac{1}{1-x^2-y^2}$

$1-x^2-y^2 = \frac{1}{c} \rightarrow x^2+y^2 = 1 - \frac{1}{c}$ ✓

True!

(g) True/False: For any nonzero scalars a, b, c, the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ -5 & 3 & c \end{bmatrix}$ is invertible.

True! We can use a and b to clear -5 and 3 in row reduction.

(h) True/False: If M is a matrix such that $\text{Col}(M) = \{\vec{0}\}$, then M must be a zero matrix.

True. If image $M = \{\vec{0}\}$, then the image M only vector M outputs is $\vec{0}$, meaning $M\vec{x} = \vec{0}$ for all $\vec{x}$. Only the zero matrix does that!

(i) True/False: For any $n \times n$ matrices A and B, $(A+B)(A-B) = A^2 - B^2$.

False! This is $A^2 + \underbrace{BA - AB}_{\neq 0} - B^2$ since matrix multiplication isn't commutative

(j) True/False: For any three vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$, we have $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \cdot \vec{w})$. ?!

False, I guess? If those are dot products, this is undefined since you can't dot a vector and a number.
If we're supposed to contextually interpret it as scalar multiplication, it's false if $\vec{w}$ and $\vec{u}$ aren't linearly independent.

10. (24 points) Short Answer. Show your work/briefly justify your answers unless explicitly told not to.

(a) (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying $T\left(\vec{i}\right) = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$ and $T\left(\vec{j}\right) = c\vec{i}$ for some scalar c. If T stretches space by a factor of 3 and reverses the orientation of $\mathbb{R}^2$, then write down the matrix M representing T. Your matrix should **not** have any variables in it.

$\det(M) = -3$

$T(\vec{i}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = M\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$ ← first column

$T(\vec{j}) = c\vec{i} = \begin{bmatrix} c \\ 0 \end{bmatrix}$ ← second column

$M = \begin{bmatrix} 4 & c \\ -9 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -\frac{1}{3} \\ -9 & 0 \end{bmatrix}$

$\det M = -3 = 4 \cdot 0 - c(-9)$, so $c = -\frac{1}{3}$

(b) (6 points) Find all critical points of $f(x, y) = \frac{x^2}{2} + 2x - 3xy + y^3 + 7$.

$\nabla f = \langle x + 2 - 3y, -3x + 3y^2 \rangle = \vec{0}$

$x = y^2$

$y^2 - 3y + 2 = 0$

$(y-2)(y-1) = 0$

$y = 2 \qquad y = 1$

$x = 2^2 \qquad x = 1^2$

So: $(4, 2)$ and $(1, 1)$.

(c) (3 points) Suppose $T : \mathbb{R}^7 \to \mathbb{R}^3$ is a linear transformation. What are all of the possible dimensions of the null space of T? Briefly justify your answer.

Whoops- I should have told you to skip this one! It won't be on the final, but if you're curious, null(T) is defined as $\{\vec{x} \in \mathbb{R}^7 \mid T(\vec{x}) = \vec{0}\}$. T can send at most 3 vectors to linearly independent outputs (since they're in $\mathbb{R}^3$), so we can find $\boxed{\text{between 4 and 7}}$ l.i. vectors that T sends to $\vec{0}$.

domain

(d) (5 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying $T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}7\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}4\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\-5\end{bmatrix}$. Find $T\left(\vec{j}\right)$.

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}7/2\\1\end{bmatrix}$, so $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}4\\1\end{bmatrix} - 4\begin{bmatrix}1\\0\end{bmatrix}\right)$

$= \begin{bmatrix}-4\\-5\end{bmatrix} - 4\begin{bmatrix}7/2\\1\end{bmatrix}$

$= \begin{bmatrix}-18\\-9\end{bmatrix}.$

(e) (3 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $M = \begin{bmatrix}4 & -2\\-5 & -5\end{bmatrix}$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $N = \begin{bmatrix}5 & -4\\3 & -2\end{bmatrix}$. Find the matrix representing the linear transformation $U \circ T$ (i.e., the transformation that first applies T and then applies U).

Matrix multiplication is function composition!! :)

Matrix for $U \circ T$ is $NM = \begin{bmatrix}5 & -4\\3 & -2\end{bmatrix}\begin{bmatrix}4 & -2\\-5 & -5\end{bmatrix} = \begin{bmatrix}40 & 10\\22 & 16\end{bmatrix}.$