Teacher: **Yanjie Li**

Course: **Linear Algebra in Control Theory**

Assignment Number: **2**

Disclosure date: May 19, 2021

Problem 1

Suppose $e_1, ..., e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, ..., e_m)$.

Problem 2

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ..., \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, ..., \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $C\left[-\pi, \pi\right]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f\left(x\right) g\left(x\right) dx.$$

Problem 3

On $\mathcal{P}_2\left(\mathbf{R}\right)$, consider the inner product given by

$$\langle p, q\rangle = \int_{0}^{1} p\left(x\right) q\left(x\right) dx$$

Apply the Gram–Schmidt Procedure to the basis 1, x, x^2 to produce an orthonormal basis of $\mathcal{P}_2\left(\mathbf{R}\right)$.

Problem 4

For each of the following, use the Gram-Schmidt process find an orthonormal basis for $R(A)$:

$$1. A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$$

where $R(A)$ is the linear space spanned by the columns of A.

Problem 5

Given $\mathbf{x}_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $\mathbf{x}_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonomal basis for $\mathbb{R}^4$ by finding an orthonomal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

[*Hint*: First find a basis for the null space and then use the Gram-Schmidt process.]

Problem 6

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)\, q(x)\, dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Pay Attention

a) Mark your class number, name and student number on the homework.
b) Please hand in your homework to your TA before class next Wednesday (May 26).
c) If you really cannot hand in your homework by the time mentioned above, please bring it to office D205a by yourself.

References

[1] Axler, S. (1997). Linear algebra done right. Springer Science Business Media.

[2] Lay, D. C. . Linear algebra and its applications. Academic Press.