Homework 3

April 24, 2021

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{R}^m, V)$ by

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + ... + x_m\mathbf{v}_m,$$

for $\mathbf{x} = \left\{ \begin{matrix} x_1 \\ . \\ . \\ . \\ x_m \end{matrix} \right\} \in \mathbb{R}^m$.

   (a) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ spanning V? Why?

   (b) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ being linearly independent? Why?

2. (a) Suppose $T \in \mathcal{L}(V, W)$ is injective and $\mathbf{v}_1, ..., \mathbf{v}_n$ is linearly independent in V. Prove that $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ is linearly independent in W.

   (b) Suppose $\mathbf{v}_1, ..., \mathbf{v}_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ spans $T(V)$.

   (c) Suppose V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U\cap$ Ker$(T) = \{0\}$ and $T(V) = T(U)$. Find a basis.

3. (a) Suppose V and W are both finite-dimensional. Prove that there exists an injective linear transfomation from V to W if and only if dim$V \leq$dim W.

   (b) Suppose V and W are both finite-dimensional. Prove that there exists an surjective linear transfomation from V onto W if and only if dim$V \geq$dim W.

   (c) Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that Ker$(T) = U$ if and only if dim$U \geq$dim$V-$dim$W$.

4. Let

$$\mathbf{b}_1 = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}, \mathbf{b}_2 = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}, \mathbf{b}_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ define by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3,$$

find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

5. Let

$$\mathbf{y}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \mathbf{y}_2 = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}, \mathbf{y}_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

and let $\mathcal{I}$ be the identity operator on $\mathbb{R}^3$.

(a) Find the coordinates of $\mathcal{I}(\mathbf{e}_1), \mathcal{I}(\mathbf{e}_2)$, and $\mathcal{I}(\mathbf{e}_3)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

(b) Find a matrix A such that $A\mathbf{x}$ is the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.