Homework 2

April 14, 2021

1. For each subspace in (a)-(d), (1) find a basis, and (2) state the dimension.

   (a)
$$\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$$

   (a) $s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}-2\\1\\3\end{bmatrix}$ basis $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\3\end{bmatrix}\right\}$ dim = 2.

   (b)
$$\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

   (b) $a\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\-1\\1\\2\end{bmatrix} + c\begin{bmatrix}2\\0\\-3\\0\end{bmatrix}$ dim = 3.

   (c)
$$\left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

   (c) $a\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} + b\begin{bmatrix}-4\\-5\\0\\7\end{bmatrix} + c\begin{bmatrix}-2\\-4\\2\\6\end{bmatrix}$

   basis: $\left\{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}, \begin{bmatrix}-4\\-5\\0\\7\end{bmatrix}\right\}$ dim = 2

   (d)
$$\{(a, b, c, d) : a - 3b + c = 0\}$$

   $a = 3b - c$, $b = b$, $c = c$, $d = d$

   (d). $\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = b\cdot\begin{bmatrix}3\\1\\0\\0\end{bmatrix} + c\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

   basis: $\left\{\begin{bmatrix}3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ dim = 3.

2. Let
$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

   2. (a) $\begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{pmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 1 \\ 0 & -\frac{5}{2} & 5 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \to \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

   (a) Show that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent. $\to \begin{bmatrix}-4\\2\\1\end{bmatrix}$

   (b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

   (c) What is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$? dim = 2.

   (d) Give a geometric description of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.
   a plane spanned by x_1 and x_2.

3. Let
$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$$

   $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & a \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & a \end{bmatrix} \Rightarrow a = 6.$

If $\dim(\mathrm{span}(x_1, x_2, x_3)) = 2$, compute a.

4. V is a nonzero finite-dimensional vector spaces, and the vectors listed belong to V. Mark each statement True or False. Justify each answer (Prove it if True or give an anti-example if False).

   a. If there exists a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ that spans V, then dim $V \leq p$. ✓

   b. If there exists a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then ~~then~~ dim $V \geq p$. ✓

   c. If dim $V = p$, then there exists a spanning set of $p + 1$ vectors in V. ✓

   d. If there exists a linearly dependent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then dim $V \leq p$. X. in $\mathbb{R}^3$, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}$ $p=2$, $\dim \mathbb{R}^3 = 3$.

   e. If every set of p elements in V fails to span V, then dim $V > p$. ✓

   f. If $p \geq 2$ and dim $V = p$, then every set of $p - 1$ nonzero vectors is linearly independent. In $\mathbb{R}^3$, $p=3$, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}$ X

5. Without computing A, find bases for its four fundamental subspaces:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix}}_{C} \underbrace{\begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \\ \hline 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}}_{R} = (CR_1 \; CR_2 \; CR_3 \; CR_4)$$

Ax. linear combination of the columns of A

$A \in \mathbb{R}^{3\times 4}$

$$C(A) = \mathrm{span}\left\{ \begin{bmatrix}1\\6\\9\end{bmatrix}, 2\begin{bmatrix}1\\6\\9\end{bmatrix} + \begin{bmatrix}0\\1\\8\end{bmatrix}, 3\begin{bmatrix}1\\6\\9\end{bmatrix} + 2\begin{bmatrix}0\\1\\8\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}, 4\begin{bmatrix}1\\6\\9\end{bmatrix} + 3\begin{bmatrix}0\\1\\8\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

$$= \mathrm{span}\left(\begin{bmatrix}1\\6\\9\end{bmatrix}, \begin{bmatrix}0\\1\\8\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right). \quad \dim C(A) = 3.$$

$N(A)$: $Ax = 0 \Leftrightarrow C(Rx) = 0 \Leftrightarrow Rx = 0$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

I F

$N(A) = \mathrm{span}\left(\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}\right)$ $\dim N(A) = 1$.

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow C(A^T) = \mathrm{span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, 6\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, 9\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 8\begin{bmatrix}0\\1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\\2\end{bmatrix} \right)$$

$$= \mathrm{span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix} \right) \quad \dim C(A^T) = 3.$$

$\dim N(A^T) = 0$. $\quad N(A^T) = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.