自控 HW-8 自动化1班 190410102 方尧

1.证明：(1) $E(At) = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$

$E(A\tau) = I + A\tau + \frac{1}{2!}A^2\tau^2 + \frac{1}{3!}A^3\tau^3 + \cdots$

记 $E(At)\cdot E(A\tau) = H$，记 H_k 为 H 表达式中 A^k 项系数。

知 $H_0 = 1$，$H_1 = (t+\tau)$，$H_2 = \frac{1}{2!}\tau^2 + \frac{1}{2!}t^2 + t\cdot\tau = \frac{1}{2!}(t+\tau)^2$.

$H_k = 1\cdot\frac{1}{k!}\tau^k + t\cdot\frac{1}{(k-1)!}\tau^{k-1} + \frac{1}{2!}t^2\cdot\frac{1}{(k-2)!}\tau^{k-2} + \cdots + \frac{1}{k!}t^k\cdot 1$

$= \frac{1}{k!}\left[\tau^k + \frac{k!}{(k-1)!}t\cdot\tau^{k-1} + \frac{k!}{2!(k-2)!}t^2\tau^{k-2} + \cdots + \frac{k!}{i!(k-i)!}t^i\tau^{k-i} + \cdots\right.$

$\left. + \frac{k!}{(k-1)!}t^{k-1}\tau + t^k\right] = (\tau+t)^k\cdot\frac{1}{k!}$

故 $H = \sum_{k=0}^{\infty} A^k\cdot(\tau+t)^k\cdot\frac{1}{k!}$

即 $E(A(t+\tau)) = H = E(At)\cdot E(A\tau) = \sum_{k=0}^{\infty} A^k(\tau+t)^k\cdot\frac{1}{k!}$，得证.

(2) 有 $E(At) = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$

$\frac{d}{dt}E(At) = A + A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \cdots = \sum_{k=0}^{\infty} A^{k+1}t^k\cdot\frac{1}{k!}$

$= A\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right] = A\cdot\sum_{k=0}^{\infty}A^k\cdot t^k\cdot\frac{1}{k!} = AE(At)$

$= \left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right]A = \left(\sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k\right)A = E(At)\cdot A$

故 $\frac{dE(At)}{dt} = AE(At) = E(At)A$，得证.

2. $\Phi(t) = e^{At} = L^{-1}[(sI-A)^{-1}]$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{s-\lambda} & \frac{1}{(s-\lambda)^2} & \frac{1}{(s-\lambda)^3} \\ 0 & 0 & \frac{1}{s-\lambda} & \frac{1}{(s-\lambda)^2} \\ 0 & 0 & 0 & \frac{1}{s-\lambda} \end{bmatrix}$$，求拉氏反变换：

$$\Phi(t) = e^{At} = L^{-1}[(sI-A)^{-1}] = \begin{bmatrix} e^{\lambda t} & 0 & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & 0 & e^{\lambda t} \end{bmatrix}$$

3. $\begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} = \Phi(t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2e^{-2t} \\ -e^{-2t} \end{bmatrix} = \Phi(t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

设 $\Phi(t)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$，$\begin{cases} a-b=e^{-2t} \\ c-d=-e^{-2t} \end{cases}$，$\begin{cases} 2a-b=2e^{-2t} \\ 2c-d=-e^{-2t} \end{cases}$ 得 $\begin{cases} a=e^{-2t} \\ b=0 \\ c=0 \\ d=e^{-2t} \end{cases}$

即 $\Phi(t)=\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

又有 $\Phi(t)=e^{At}=L^{-1}[(SI-A)^{-1}]$

故 $(SI-A)^{-1}=L[\Phi(t)]=\begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$ 得 $A=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

故 $A=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$，$\Phi(t)=\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

4. $X(0)=\Phi(0,t)X(t)=e^{-At}X(t)$

$e^{At}=L^{-1}[(SI-A)^{-1}]=\begin{bmatrix} \frac{2}{3}e^{t}+\frac{1}{3}e^{-2t} & \frac{1}{3}e^{t}-\frac{1}{3}e^{-2t} \\ \frac{2}{3}e^{t}-\frac{2}{3}e^{-2t} & \frac{1}{3}e^{t}+\frac{2}{3}e^{-2t} \end{bmatrix}$

$X(0)=e^{-At}\cdot X(t)=\begin{bmatrix} 3e^{-t}-e^{2t} \\ 3e^{-t}+2e^{2t} \end{bmatrix}$

5. $e^{At}=L^{-1}[(SI-A)^{-1}]=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$

$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\int_0^t \begin{bmatrix} 2e^{-(t-\tau)}-2e^{-2(t-\tau)} \\ -2e^{-(t-\tau)}+4e^{-2(t-\tau)} \end{bmatrix}d\tau=\begin{bmatrix} 1-2e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} \end{bmatrix}$

$X(t)=e^{At}\cdot X(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\begin{bmatrix} 1-e^{-t} \\ e^{-t} \end{bmatrix}$

$y=[1,0]X(t)=1-e^{-t}$，$t\geq 0$

6. $G=e^{AT}=\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}$

$H=\int_0^T e^{At}dt\cdot B=\int_0^T\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}dt\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}T & \frac{1}{2}T^2\\ 0 & T\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}T^2\\ T\end{bmatrix}$

故 $x(k+1)=\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}\cdot x(k)+\begin{bmatrix}\frac{1}{2}T^2\\ T\end{bmatrix}u(k)$

7. $y(k+2)+3y(k+1)+2y(k)=u(k)$

∵ $x_2(k+1)=y(k+2)$, $x_2(k)=y(k+1)$, $x_1(k)=y(k)$ 代入得

$x_2(k+1)+3x_2(k)+2x_1(k)=u(k)$ ①

$x_1(k+1)=y(k+1)=x_2(k)$ ②

整理得系统状态方程:

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u(k),\quad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$(zI-G)^{-1}=\begin{bmatrix}z+3 & 1\\ -2 & z\end{bmatrix}\frac{1}{(z+1)(z+2)}$$

$$(zI-G)^{-1}zx_0=\frac{1}{(z+1)(z+2)}\begin{bmatrix}z\\ z^2\end{bmatrix},\quad z^{-1}[(zI-G)^{-1}zx_0]=\begin{bmatrix}\cos k\pi-2^k\cos k\pi\\ -\cos k\pi+2^{k+1}\cos k\pi\end{bmatrix}$$

$$(zI-G)^{-1}HU(z)=\frac{1}{(z+1)(z+2)}\begin{bmatrix}\frac{z}{z-1}\\ \frac{z^2}{z-1}\end{bmatrix},\quad z^{-1}[(zI-G)^{-1}HU(z)]=\begin{bmatrix}-\frac{1}{2}\cos k\pi+\frac{1}{3}\cdot 2^k\cos k\pi+\frac{1}{6}\\ \frac{1}{2}\cos k\pi-\frac{2}{3}2^k\cos k\pi+\frac{1}{6}\end{bmatrix}$$

$$x(k)=\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}=z^{-1}[(zI-G)^{-1}zx_0]+z^{-1}[(zI-G)^{-1}HU(z)]=\begin{bmatrix}\frac{1}{2}\cos k\pi-\frac{2}{3}2^k\cos k\pi+\frac{1}{6}\\ -\frac{1}{2}\cos k\pi-\frac{1}{2}2^k\cos k\pi+\frac{1}{6}\end{bmatrix}$$

$y(k)=x_1(k)=\frac{1}{2}\cos k\pi-\frac{2}{3}2^k\cos k\pi+\frac{1}{6}$, $k\geq 0$

$y(0)=0$, $y(1)=1$, $y(2)=-2$, $y(3)=5$, ….