Homework 3

April 30, 2021

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{R}^m, V)$ by

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + ... + x_m\mathbf{v}_m,$$

$T(\mathbb{R}^m) = \mathrm{span}(v_1, \cdots, v_m) = V.$

for $\mathbf{x} = \begin{Bmatrix} x_1 \\ . \\ . \\ . \\ x_m \end{Bmatrix} \in \mathbb{R}^m$.

Surjective. $T(\mathbb{R}^m) = V$.

   (a) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ spanning V? Why?

   (b) What property of T corresponds to $\mathbf{v}_1, ..., \mathbf{v}_m$ being linearly independent? Why? Injective. If $T(x)=0$, $x_1v_1+\cdots+x_mv_m=0$. Since $v_1, \cdots, v_m$ are linearly independent, $x_1=\cdots=x_m=0$. $\Rightarrow x=\vec{0} \Rightarrow \ker(T)=\{0\}$. $\Rightarrow T$ is injective.

2. (a) Suppose $T \in \mathcal{L}(V, W)$ is injective and $\mathbf{v}_1, ..., \mathbf{v}_n$ is linearly independent in V. Prove that $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ is linearly independent in W.

   (b) Suppose $\mathbf{v}_1, ..., \mathbf{v}_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $T(\mathbf{v}_1), ...T(\mathbf{v}_n)$ spans $T(V)$.

   (c) Suppose V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U\cap$ Ker(T) $= \{0\}$ and $T(V) = T(U)$. Find a basis.

3. (a) Suppose V and W are both finite-dimensional. Prove that there exists an injective linear transfomation from V to W if and only if dim$V$ $\leq$dim W.

   (b) Suppose V and W are both finite-dimensional. Prove that there exists an surjective linear transfomation from V onto W if and only if dim$V$ $\geq$dim W.

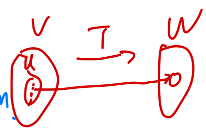
   (c) Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that Ker$(T) = U$ if and only if dim$U$ $\geq$dim$V-$dim$W$.

$r \geq n-m \Leftrightarrow m \geq n-r$

Idea:

$T \in \mathcal{L}(V, W)$

dim$V=n$



Let dim$U=r$. dim$W=m$

$V: \{v_1, \cdots, v_n\}$

$T(v_1) \in W \quad T(v_n) \in W$.

a basis of U, $\{v_1, \cdots, v_r\} \rightarrow$ a basis of V $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$

$T(v_1)=0 \quad T(v_r)=0 \quad T(v_{r+1})=w_1$

Let $\{w_1, \cdots, w_m\}$ be a basis of W.

$\forall v \in V$, $v = c_1v_1 + \cdots + c_nv_n$

1) $T(U)=\{0\}$. $U \subset \ker(T)$

$T(v) = c_1T(v_1) + \cdots + c_nT(v_n)$

2) For $v \in V$, if $T(v)=0$, $v = c_1v_1+\cdots+c_rv_r + c_{r+1}v_{r+1}+\cdots+c_nv_n$

$0=T(v)=c_1T(v_1)+\cdots+c_rT(v_r)+c_{r+1}T(v_{r+1})+\cdots+c_nT(v_n)$

$T(v_n)=w_{n-r}$

$m \geq n-r$

$= C_{r+1} w_1 + \cdots + C_n w_{n-r} = 0$

$w_1, \cdots, w_{n-r}$ linearly independent $C_{r+1} = \cdots = C_n = 0$

$v = c_1 v_1 + \cdots + c_r v_r \in U$.

$\ker(T) \subset U$.

$$A = \left[[L(e_1)]_{\{b_1,b_2,b_3\}} \quad [L(e_2)]_{\{b_1,b_2,b_3\}} \right]$$

$$L(e_1) = (b_1 \; b_2 \; b_3)\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad L(e_2) = (b_1 \; b_2 \; b_3)\begin{bmatrix}0\\1\\1\end{bmatrix} \qquad A = \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}$$

$x_1 = 1,\ x_2 = 0$ $\qquad$ $x_1 = 0 \ \ x_2 = 1$

4. Let

$$\mathbf{b}_1 = \begin{Bmatrix}1\\1\\0\end{Bmatrix}, \mathbf{b}_2 = \begin{Bmatrix}1\\0\\1\end{Bmatrix}, \mathbf{b}_3 = \begin{Bmatrix}0\\1\\1\end{Bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ define by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3,$$

$x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$

e_1 $\quad x_1 = 1$, $x_2 = 0$

5. Let

$$\mathbf{y}_1 = \begin{Bmatrix}1\\1\\1\end{Bmatrix}, \mathbf{y}_2 = \begin{Bmatrix}1\\1\\0\end{Bmatrix}, \mathbf{y}_3 = \begin{Bmatrix}1\\0\\0\end{Bmatrix}$$

and let $\mathcal{I}$ be the identity operator on $\mathbb{R}^3$.

(a) Find the coordinates of $\mathcal{I}(\mathbf{e}_1)$, $\mathcal{I}(\mathbf{e}_2)$, and $\mathcal{I}(\mathbf{e}_3)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

$''e_1 \quad ''e_2 \quad ''e_3$

(b) Find a matrix A such that $A\mathbf{x}$ is the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

$y = Ax \ ; \ T$

$T(x) = Ax$

$[T(x)]_{(y_1, y_2, y_3)}$

$\begin{bmatrix}x_1 + x_2\\x_3\\x_1\end{bmatrix}$

(a) $\quad \mathcal{I}(e_1) = e_1 = (y_1, y_2 \; y_3)\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\mathcal{I}(e_2) = e_2 = (y_1, y_2 \; y_3)\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\mathcal{I}(e_3) = e_3 = (y_1 \; y_2 \; y_3)\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$x \xrightarrow{T} Ax$

$(T(x))$

(b). $\quad A = \begin{pmatrix}0 & 0 & 1\\0 & 1 & -1\\1 & -1 & 0\end{pmatrix}$

$T(e_1) = \begin{bmatrix}1\\0\\1\end{bmatrix} = (y_1, y_2, y_3)\left(\quad\right)$