Review

Linear spaces (Vector Spaces).

Vector addition + Scalar Multiplication } closure properties.

Vector Space Axioms

A1. $x+y=y+x$

A2. $(x+y)+z=x+(y+z)$

A3. $(ab)x=a(bx)$

A4. $a(x+y)=ax+ay$

A5. $(a+b)x=ax+bx$

A6. $x+0=x$

A7. $x+(-x)=0$

A8. $1\cdot x=x$.

Subspace $\begin{cases} \text{nonempty subset.} \\ a x \in S, \quad \forall a, \ \forall x \in S \\ x + y \in S \quad \forall x, y \in S. \end{cases}$

$\mathrm{Span}(v_1, \cdots, v_n)$: all linear combinations of $v_1, v_2, \cdots, v_n$.

If $\mathrm{Span}(v_1, \cdots, v_n) = V$, $\{v_1, \cdots, v_n\}$ is a spanning set.

linearly independent or dependent.

$c_1 v_1 + \cdots + c_n v_n = 0 \overset{?}{\Rightarrow} c_1 = \cdots = c_n = 0$

HW 1:

1. (a). $\beta \vec{0} = \beta(x + (-x))$ (A7)

$= \beta x + \beta(-x)$ (A4)

$= \beta x - \beta x$

$= 0$.

(b). If $a = 0$. $0 \vec{x} = \vec{0}$.

If $a \neq 0$. $a \vec{x} = \vec{0}$ $\quad \frac{1}{a}(a \vec{x}) = \underbrace{\frac{1}{a} \cdot \vec{0}}_{} = \vec{0}$ (from (a))

$= (\frac{1}{a} \cdot a) \vec{x} = 1 \vec{x} = \vec{x}$

(A3) (A8)

2. (A1) and (A2) are satisfied.

A3. $(ab)\circ(x_1, x_2) = (abx_1, x_2)$

$a\circ(b\circ(x_1, x_2)) = a\circ(bx_1, x_2) = (abx_1, x_2)$ (A3) ✓.

A4. $a\circ[(x_1, x_2) + (y_1, y_2)] = a\circ[(x_1+y_1, x_2+y_2)] = (a(x_1+y_1), x_2+y_2)$

$a\circ(x_1, x_2) + a\circ(y_1, y_2) = (ax_1, x_2) + (ay_1, y_2) = (a(x_1+y_1), x_2+y_2)$ (A4) ✓

A5. $(a+b)\circ(x_1, x_2) = ((a+b)x_1, x_2)$

$a\circ(x_1, x_2) + b\circ(x_1, x_2) = (ax_1, x_2) + (bx_1, x_2) = ((a+b)x_1, 2x_2)$ (A5) ✗.

3. A1. $x\oplus y = xy \quad y\oplus x = yx = xy$. ✓

A2. $(x\oplus y)\oplus z = (xy)\oplus z = xyz$

$x\oplus(y\oplus z) = x\oplus(yz) = xyz$. ✓

A3. $(ab)\circ x = x^{ab}$

$a\circ(b\circ x) = a\circ(x^b) = (x^b)^a = x^{ab}$ ✓.

A4. $a\circ(x\oplus y) = a\circ(xy) = (xy)^a = x^a y^a$

$(a\circ x)\oplus(a\circ y) = (x^a)\oplus(y^a) = x^a y^a$ ✓

A5. $(a+b)\circ x = x^{(a+b)}$

$a\circ x\oplus b\circ x = x^a\oplus x^b = x^a\cdot x^b = x^{a+b}$ ✓

A6. $x \oplus \vec{0} = x$

$= x \cdot \vec{0} \Rightarrow \vec{0} = 1.$ ✓

A7. $x \oplus \vec{-x} = \vec{0} = 1$

$= x \cdot \vec{-x} = 1 \Rightarrow \vec{-x} = \frac{1}{x}.$ ✓

A8. $1 \circ x = x^{1} = x.$ ✓

4. Proof: $I_{n\times n} \in S$, $O_{n\times n} \in S$. S is a nonempty subset of $R^{n\times n}$.

If $B_1 \in S$, $\forall \partial$

$$A(\partial B_1) = \partial AB_1 = \partial B_1 A = (\partial B_1)\cdot A \Rightarrow \partial B_1 \in S.$$

If $B_1, B_2 \in S$,

$$A(B_1+B_2) = AB_1 + AB_2 = B_1A + B_2A = (B_1+B_2)A \Rightarrow (B_1+B_2) \in S.$$

5. Proof:

(a). For any $x \in V$, since $(x_1, \dots, x_k)$ is a spanning set,

$$x = a_1 x_1 + \dots + a_k x_k + 0 \cdot x_{k+1}$$

$\Rightarrow \{x_1, \dots, x_k, x_{k+1}\}$ is also a spanning set.

(b) May or May not.

If x_k is a linear combination of the others, we still have a spanning set; Otherwise, we fail to have a spanning set.

Review Cont.

Basis of a vector space: $\begin{cases} \text{Span}(v_1, \dots, v_n) = V. \\ v_1, \dots, v_n \text{ are linearly independent.} \end{cases}$

$\dim V = n$

5 properties ☆ $\begin{cases} \text{i)} \\ \text{ii)} \\ \text{iii)} \\ \text{iv)} \\ \text{v)} \end{cases}$ ☆

Four subspaces of a matrix $A \in R^{m \times n}$.

$C(A) \in R^m$, $N(A) \in R^n$, $C(A^T) \in R^n$ $N(A^T) \in R^m$.

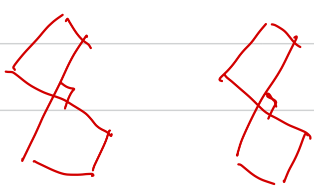
Fundemental Theorem of Linear Algebra.

$\dim C(A) = \dim C(A^T) = r$

$\dim N(A) = n - r$, $\dim N(A^T) = m - r$

$N(A) = C(A^T)^\perp$ $N(A^T) = C(A)^\perp$

Big picture.



reduced row echelon form

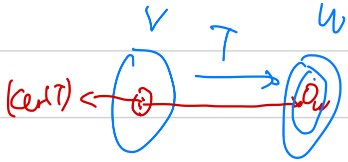
$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix}$$

$Ax=0$

HW2: On the draft.

Review Cont.

Linear Transformations.



$T(\lambda v) = \lambda T(v)$, $T(v_1+v_2) = T(v_1) + T(v_2)$. $T \in \mathcal{L}(V, W)$

Linear operator, $T \in \mathcal{L}(V)$

核. $\ker(T) = \{ v \in V \mid T(v) = 0_W \}$ Null T.

像 $T(S) = \{ w \in W \mid w = T(v) \text{ for some } v \in S \}$.

值域 $T(V)$, Range T.

Injection (单射) one-to-one.

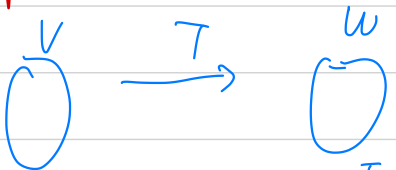
$T(u)=T(v)$ implies $u=v$ $\Leftrightarrow$ $\ker(T)=\{0\}$.

Surjection (满射) onto. $T(V)=W$.

Fundemental Theorem of Linear Transformations.

$$\dim V = \dim \ker(T) + \dim T(V).$$ 证明: ✓

Matrix Representations of Linear Tramformations.



$E=\{v_1,\dots,v_n\}$ $\qquad$ $F=\{w_1,\dots,w_m\}$

$$v\in V \xrightarrow{T} w=T(v)\in W$$

$$\updownarrow$$

$$x=[v]_E\in R^n \xrightarrow{A} y=Ax=[T(v)]_F\in R^m.$$

$$[T(v)]_F = A\cdot[v]_E.$$

Equivalent Matrix. Similar Matrix

Another basis of V

$E' = \{v_1', \dots, v_n'\}$.

Another basis of W

$F' = \{w_1', \dots, w_n'\}$.

$[v_1', \dots, v_n'] = [v_1, \dots, v_n] P$

$[w_1', \dots, w_n'] = [w_1, \dots, w_n] Q$.

$\Downarrow$

$v = [v_1', \dots, v_n'][v]_{E'} = [v_1, \dots, v_n][v]_E$

$= [v_1, \dots, v_n] P [v]_{E'}$

$\Rightarrow [v]_E = P[v]_{E'}$

$[T(v)]_F = Q[T(v)]_{F'}$

$$[T(v)]_{F'} = Q^{-1}[T(v)]_F = Q^{-1}A \cdot [v]_E = \underbrace{Q^{-1}AP}_{B}[v]_{E'}.$$

If $V = W$. $Q = P$.

$S^{-1}AS$

Hw3.

2\. (a). Let $c_1 T(v_1) + \cdots + c_n T(v_n) = 0$.

$\Rightarrow T(c_1 v_1 + \cdots + c_n v_n) = 0$

Since T is injective, $\ker(T) = \{0\}$, $\Rightarrow c_1 v_1 + \cdots + c_n v_n = 0$

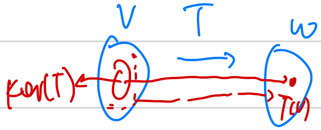
Since $v_1, \cdots, v_n$ is linearly independent, $c_1 = \cdots = c_n = 0$.

(b). To prove $T(V) = \mathrm{Span}(T(v_1), \cdots, T(v_n))$, it is suffice to show that for any $w \in T(V)$, w is a linear combination of $T(v_1), \cdots, T(v_n)$.

Suppose $w \in T(V)$. There exists $v \in V$ such that $w = T(v)$. Then we have $v = c_1 v_1 + \cdots + c_n v_n$ for some $c_1, \cdots, c_n$ since $V = \mathrm{span}(v_1, \cdots, v_n)$

$$w = T(v) = T(c_1 v_1 + \cdots + c_n v_n)$$
$$= c_1 T(v_1) + \cdots + c_n T(v_n).$$

(c). Idea.



Proof. Assume $\dim V = n$. $\ker(T)$ is a subspace of V. If $\ker(T) = \{0\}$, $U = V$. If $\ker(T) \neq \{0\}$, it has a basis $\{v_1, \cdots, v_r\}$, $r \leq n$, which can be extended to a basis of V, $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$.

Consider the space spanned by $\{v_{r+1}, \cdots, v_r\}$, denoted U.

Let $v \in U \cap \ker(T)$, $v = c_1 v_1 + \cdots + c_r v_r \in \ker(T)$

$= c_{r+1} v_{r+1} + \cdots + c_n v_n \in U$.

We have $c_1 v_1 + \cdots + c_r v_r - c_{r+1} v_{r+1} - \cdots - c_n v_n = 0$

Since $v_1, \cdots, v_n$ is linearly independent, $c_1 = \cdots = c_n = 0$. $\Rightarrow v = 0$.

$\Rightarrow U \cap \ker(T) = \{0\}$.

For any $v \in V$, $v = c_1 v_1 + \cdots + c_n v_n$.

$$T(v) = c_1 \underset{\overset{\shortparallel}{0}}{T(v_1)} + \cdots + c_r \underset{\overset{\shortparallel}{0}}{T(v_r)} + \underline{c_{r+1} T(v_{r+1}) + \cdots + c_n T(v_n)}$$

$$= T(\underbrace{c_{r+1} v_{r+1} + \cdots + c_n v_n}_{\in U}) \in T(U)$$

$\Rightarrow T(V) \subset T(U)$.

Note $T(U) \subset T(V)$. $\Rightarrow T(V) = T(U)$.

3. $\dim V = \dim \ker(T) + \dim T(V)$.

Let $\{v_1, \cdots, v_n\}$ be a basis of V.

Let $\{w_1, \cdots, w_m\}$ be a basis of W.

(a). $\Rightarrow$ If T is injective, $\ker(T) = \{0\}$.

$\dim V = \dim T(V) \leq \dim W$.

$\Leftarrow$ If $\dim V \leq \dim W$, $n \leq m$.

$\{v_1, \dots, v_n\}$

$\downarrow \quad \downarrow$

$\{w_1, \dots, w_n, \dots, w_m\}$

$\boxed{\text{Define } T \in \mathcal{L}(V, W),\ T(v_1) = w_1, \dots, T(v_n) = w_n.}$

If $T(v) = 0$, for $v = c_1 v_1 + \dots + c_n v_n \in V$.

$$T(v) = c_1 T(v_1) + \dots + c_n T(v_n)$$
$$= c_1 w_1 + \dots + c_n w_n = 0$$

Since $w_1, \dots, w_n$ is linearly independent, $c_1 = \dots = c_n = 0$.

We have $v = 0$. $\Rightarrow \ker(T) = 0$. T is injective.

(b). $\Rightarrow$. If T is surjective, $T(V) = W$.

$$\dim V = \dim \ker(T) + \dim T(V)$$
$$= \dim \ker(T) + \dim W \geq \dim W.$$

$\Leftarrow$. If $\dim V \geq \dim W$, $n \geq m$.

$\{v_1, \dots, v_m, \dots, v_n\}$ is a basis of V.

$\downarrow \quad \downarrow \quad \downarrow$

$\{w_1, \dots, w_m\}$ $\quad 0$. is a basis of W.

Define $T \in \mathcal{L}(V, W)$: $T(v_1) = w_1, \cdots, T(v_m) = w_m, T(v_{m+1}) = \cdots = T(v_n) = 0$.

For any $w \in W$, assume $w = c_1 w_1 + \cdots + c_m w_m$.

$= c_1 T(v_1) + \cdots + c_m T(v_m)$

$= T(\underbrace{c_1 v_1 + \cdots + c_m v_m}_{\in V})$.

$\Rightarrow T$ is surjective.

(c) $\Rightarrow$ $\dim V = \dim \ker(T) + \dim T(V)$

$= \dim U + \dim T(V)$

$\leq \dim U + \dim W \Rightarrow \dim U \geq \dim V - \dim W$

$\Leftarrow$ U is a subspace of V. Let $\{v_1, \cdots, v_r\}$ be a basis of U,

which can be extended to a basis of V, $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$.

Also let $\{w_1, \cdots, w_m\}$ be a basis of W. $r \geq n-m$ $m \geq n-r$

$\boxed{\text{Define } T \in \mathcal{L}(V, W): T(v_1) = \cdots = T(v_r) = 0, \quad T(v_{r+1}) = w_1, \cdots, T(v_n) = w_{n-r}}$ $\Rightarrow U \subset \ker(T)$

For any $v = c_1 v_1 + \cdots + c_n v_n$, if $T(v) = 0$,

$0 = T(c_1 v_1 + \cdots + c_n v_n) = c_1 \underset{0}{T(v_1)} + \cdots + c_r \underset{0}{T(v_r)} + c_{r+1} \underset{w_1}{T(v_{r+1})} + \cdots + c_n \underset{w_{n-r}}{T(v_n)}$

$= c_{r+1} w_1 + \cdots + c_n w_{n-r} = 0$

Since $w_1, \cdots, w_{n-r}$ is linearly independent, $c_{r+1} = \cdots = c_n = 0$. $v = c_1 v_1 + \cdots + c_r v_r \in U$

$\Rightarrow \ker(T) \subset U$.