Black-Scholes and Binomial Tree

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FINA3351 – Spreadsheet Modelling in Finance

Roadmap

1. Black-Scholes-Merton (BSM) Model for European Options
2. Delta in European Option (BSM Model) and Delta Hedging
3. Binomial Option Pricing Model
   - CRR Binomial Trees – European Options
   - CRR Binomial Trees – American Options

BSM Model for European Options

Black-Scholes Option Pricing Model (1)

- Black-Scholes Model: Underlying security does not pay dividend:

$$c = SN(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - SN(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

  - $c =$ Current price of the call option
  - $p =$ Current price of the put option
  - $S =$ Current stock price
  - $K =$ strike price for the option
  - $T =$ time to maturity

Black-Scholes Option Pricing Model (2)

- $N(d) =$ cumulative probability that a random draw from a standard normal distribution will be less than d $(-\infty < d < \infty)$.
  - $0 < N(d) < 1.$
  - $N(-d) = 1 - N(d).$
- $r =$ Risk-free interest rate (continuously compounded with the same maturity as the option)

  $=$ the expected return on the stock under risk-neutral probability
- $\sigma =$ Standard deviation of continuously compounded return on the stock.
- One key insight from BS analysis: stock beta is irrelevant.
- Time unit of T, r, and σ should be consistent.

Black-Scholes Option Pricing Model (3)

- Intuition for the following equation:

$$c = SN(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - SN(-d_1)$$

Call option:

- the buyer pays little premium to make money when stock price (S) appreciates; so, it's like financing a long position on stock by short-selling a bond (borrowing from banks)
- Investor has the option to surrender cash K in exchange for a share of stock at time T. Present value of K is Ke^{-rT}; present value of the stock is S.

Black-Scholes Option Pricing Model (4)

- Intuition for the following equation:

$$c = SN(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - SN(-d_1)$$

Put option:

- the buyer pays little premium to make money when stock price (S) drops; so, it's like short-selling the stock and using the cash to finance a long position on bond (save in banks)
- Investor has the option to surrender a share of stock in exchange for cash K at time T.

Put-Call Parity: No Dividend (1)

- We know that the initial investment (i.e., cash flow) of

(a) buying one call at strike price *K* and selling one put at strike price *K*

will equal the initial investment (i.e., cash flow) of

(b) buying a share of the stock (no dividend) and short-selling a risk-free bond with face value of *K* at *T*.

$$c - p = S - Ke^{-rT}$$

Proof:

$$\begin{aligned}
c &= SN(d_1) - Ke^{-rT}N(d_2) \\
p &= Ke^{-rT}N(-d_2) - SN(-d_1) \\
&= \big(1 - N(d_2)\big)Ke^{-rT} - S\big(1 - N(d_1)\big) \\
&= SN(d_1) - Ke^{-rT}N(d_2) - S + Ke^{-rT} \\
&= c - S + Ke^{-rT}
\end{aligned}$$

- We can use put-call parity to examine the calculations in Excel.

BS Model in Excel (No Dividend)

- See Excel file “Lec9_BlackScholes.xlsm” for examples of BS pricing model.
- Excel tab “BS”
- **Note**: in practice, option investors assume that there are 365 trading days per year. (This is different from the convention in equity models.)

Dividend Adjustment to Black-Scholes

- Dividend adjustment in 2 cases:
  1. The underlying security pays out a continuous dividend.
  2. Future dividends of underlying security are known with certainty.
- The principle underlying both cases is the same: options are priced on an adjusted underlying value which nets out the present value of dividends paid between the option purchase date and exercise date.

Continuous Dividend Payouts – BSM Model

- Continuous dividend assumption may seem odd for an individual stock.
- But an index can best be approximated by the assumption of a continuous dividend payout, since there are many stocks in the portfolio, and its components pay out their dividends throughout the year.

$$c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$
$$p = Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$$

$$\text{where } d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

  - $q =$ continuously compounded dividend yield
  - Se^{-qT} = The price of e^{-qT} share of the dividend-paying stock

- In VBA Module `BlackScholes`, we write a VBA function `BS` returns European option price on BSM model.

Put-Call Parity: With Dividend (2)

- Put-Call Parity:

$$c - p = Se^{-qT} - Ke^{-rT}$$

Proof:

$$\begin{aligned}
c &= Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) \\
p &= Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1) \\
&= \big(1 - N(d_2)\big)Ke^{-rT} - Se^{-qT}\big(1 - N(d_1)\big) \\
&= Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) - Se^{-qT} + Ke^{-rT} \\
&= c - Se^{-qT} + +Ke^{-rT}
\end{aligned}$$

- We can use put-call parity to examine the calculations in Excel.

Known Dividends Amount

- This is most commonly the case when a dividend has already been announced, but it can also happen because many stocks pay quite regular and relatively inflexible dividends.
- In Black-Scholes model, current stock price *S* should be replaced by the stock price minus the present value of the dividends anticipated before the option expiration date *T*.
- Two known dividend payments case:

Time 0 | Time t_1 | Time t_2 | Time T option expiration

Stock price = *S*
Stock price minus PV(dividends)
$= S - D_1 e^{-rt_1} - D_2 e^{-rt_2}$

D_1

D_2

call: $\max(S_T - K, 0)$
put: $\max(K - S_T, 0)$

- See Excel tab "BS_div2"

Example: NASDAQ 100 Option (1)

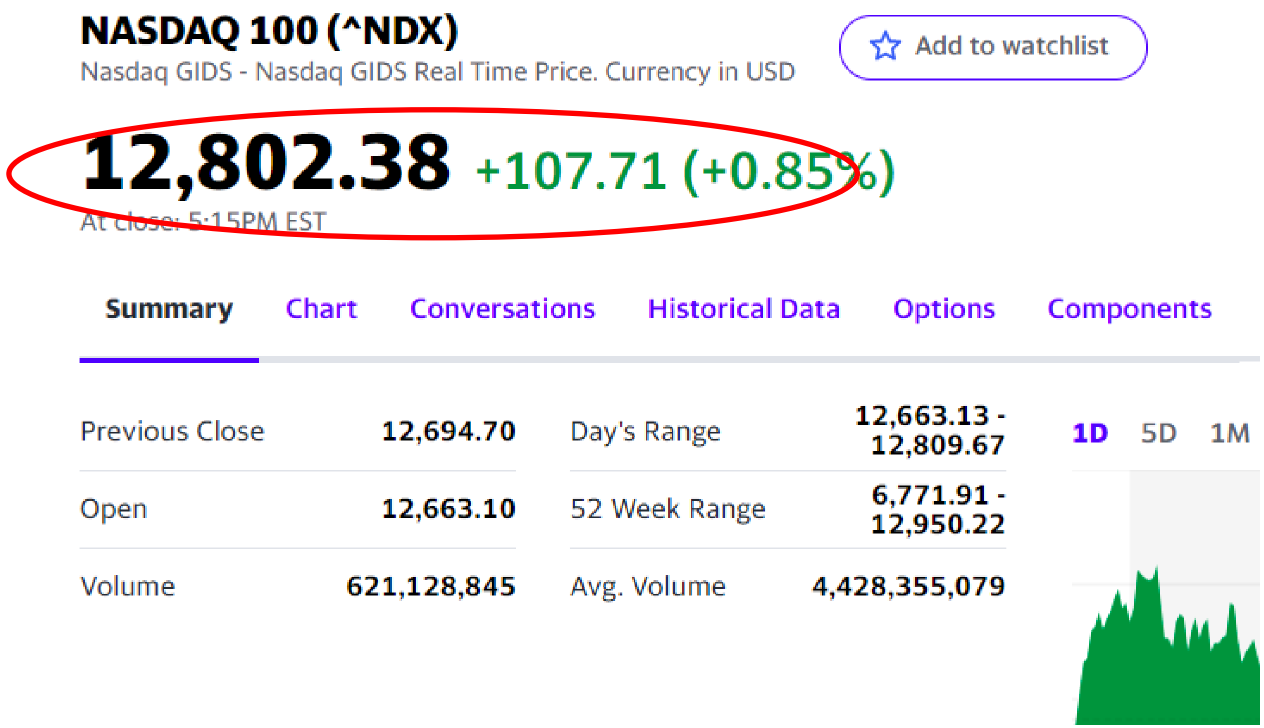
- Nasdaq 100 index (Symbol: NDX) is a stock market index which contains 100 of the largest non-financial securities (based on market capitalization) listed on the Nasdaq stock exchange.

https://www.nasdaq.com/market-activity/quotes/nasdaq-ndx-index

- The Nasdaq 100 index option contract has an underlying value that is equal to the full value of the level of the Nasdaq 100 index.
- Option style: European Option

Real Example: NASDAQ 100 Option (2)

- Now, let's use Black-Scholes model to price a Nasdaq 100 Index option.
- Yahoo!Finance reports the current market data of NDX options.
- The current index price = $12,802.38 on January 5, 2021.



Real Example: NASDAQ 100 Option (3)

- Let’s check the call option prices of NDX matures on January 20, 2021

January 20, 2021 | In The Money | Show: **List** | Straddle | Option Lookup

Calls for January 20, 2021

Contract Name	Last Trade Date	Strike	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
NDXP210120C12200000	2020-12-15 11:01AM EST	12,200.00	590.60	685.60	701.10	0.00	-	-	1	29.84%
NDXP210120C12250000	2020-12-28 9:40AM EST	12,250.00	690.53	643.50	658.70	0.00	-	1	2	29.30%
NDXP210120C12300000	2020-12-15 10:59AM EST	12,300.00	531.70	601.80	617.30	0.00	-	-	1	28.80%
NDXP210120C12350000	2020-12-15 10:58AM EST	12,350.00	498.20	561.00	575.70	0.00	-	-	1	28.20%
NDXP210120C12375000	2020-12-28 9:40AM EST	12,375.00	588.90	541.00	555.40	0.00	-	-	1	27.92%
NDXP210120C12400000	2020-12-15 11:00AM EST	12,400.00	458.90	521.00	535.30	0.00	-	-	1	27.64%
NDXP210120C12450000	2021-01-04 3:01PM EST	12,450.00	427.25	481.70	495.60	0.00	-	5	6	27.07%
NDXP210120C12500000	2020-12-15 10:52AM EST	12,500.00	401.65	443.50	456.90	0.00	-	-	1	26.51%
NDXP210120C12550000	2020-12-15 10:59AM EST	12,550.00	368.95	406.00	418.80	0.00	-	-	1	25.91%
NDXP210120C12600000	2021-01-04 2:14PM EST	12,600.00	319.25	370.00	382.00	0.00	-	1	3	25.34%
NDXP210120C12650000	2021-01-04 1:05PM EST	12,650.00	259.05	334.70	346.20	0.00	-	1	0	24.74%

Real Example: NASDAQ 100 Option (4)

- See “NDXoption” tab. How to set up the parameters?

1. **Drift term** = risk-free rate (stock return in a risk-neutral world) at annual frequency

- What risk-free rate should we use?

⇒ Latest US T-Bill rate with maturity close to the option’s maturity.

⇒ 4-week T-Bill rate on January 4, 2021 (data store on Tab: “DTB4WK”)

- Option price is not very sensitive to the drift term.

2. **Dividend yield** = 0.72% (from WSJ website)

(Yahoo!Finance provides dividend yield data for individual stocks, but not indices. You may use other data source for this estimation.)

2. **Volatility term** = 20.5% per annual, estimated using historical monthly return data of NDX in the past 3 years.

(Nasdaq 100 index data are from Yahoo!Finance. See tab “^NDX”)

2. **Time to maturity** = 15 days/365 = 0.0411 year

Real Example: NASDAQ 100 Option (5)

- Our calculation implies the price of \$324.84 , which is lower than the market price (average of bid and ask price) \$376.
- If you believe in our calculation, the market price is “more expensive” than our estimation.
- If you believe in the option market price, which parameter would you re-estimate?

$\Rightarrow$ Option price is sensitive to volatility parameter.

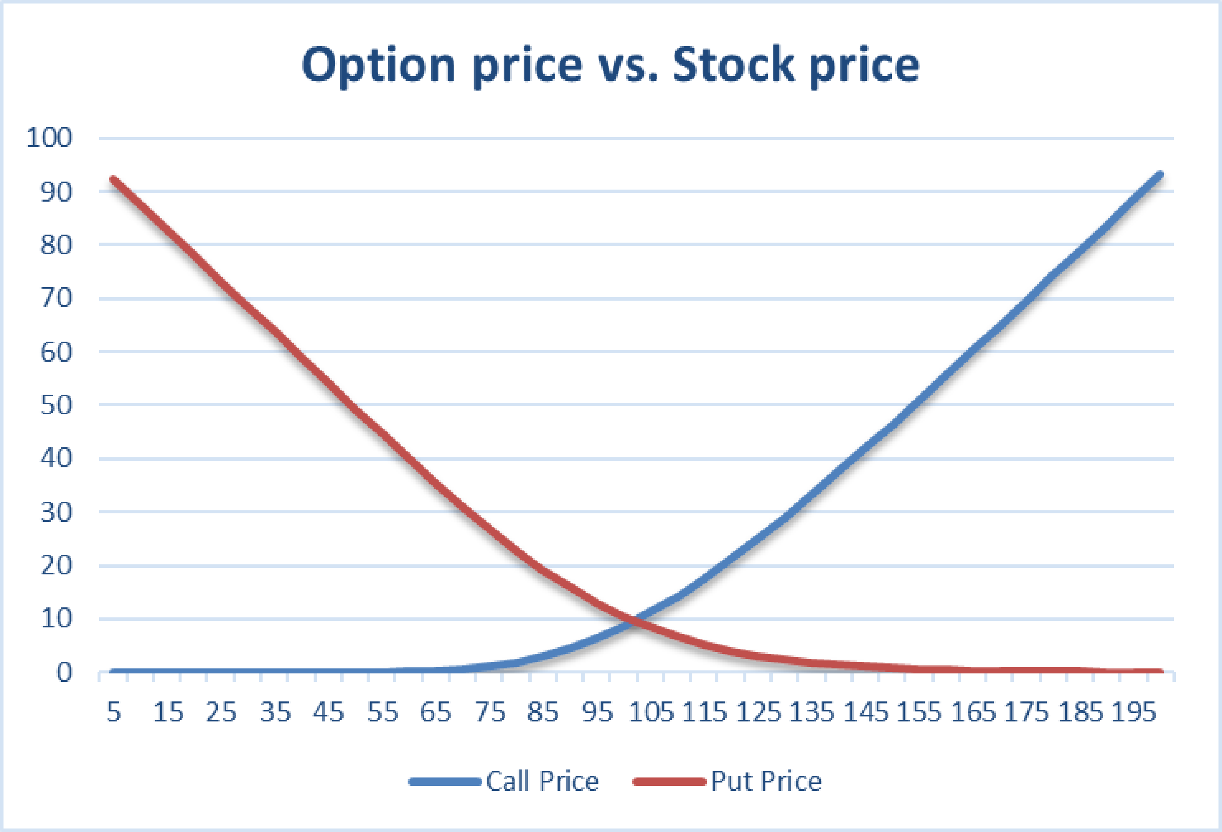
$\Rightarrow$ Option price increases with volatility of underlying stock returns.

$\Rightarrow$ It is very likely that the volatility estimation is lower from market’s view.

Option Delta and Delta Strategy

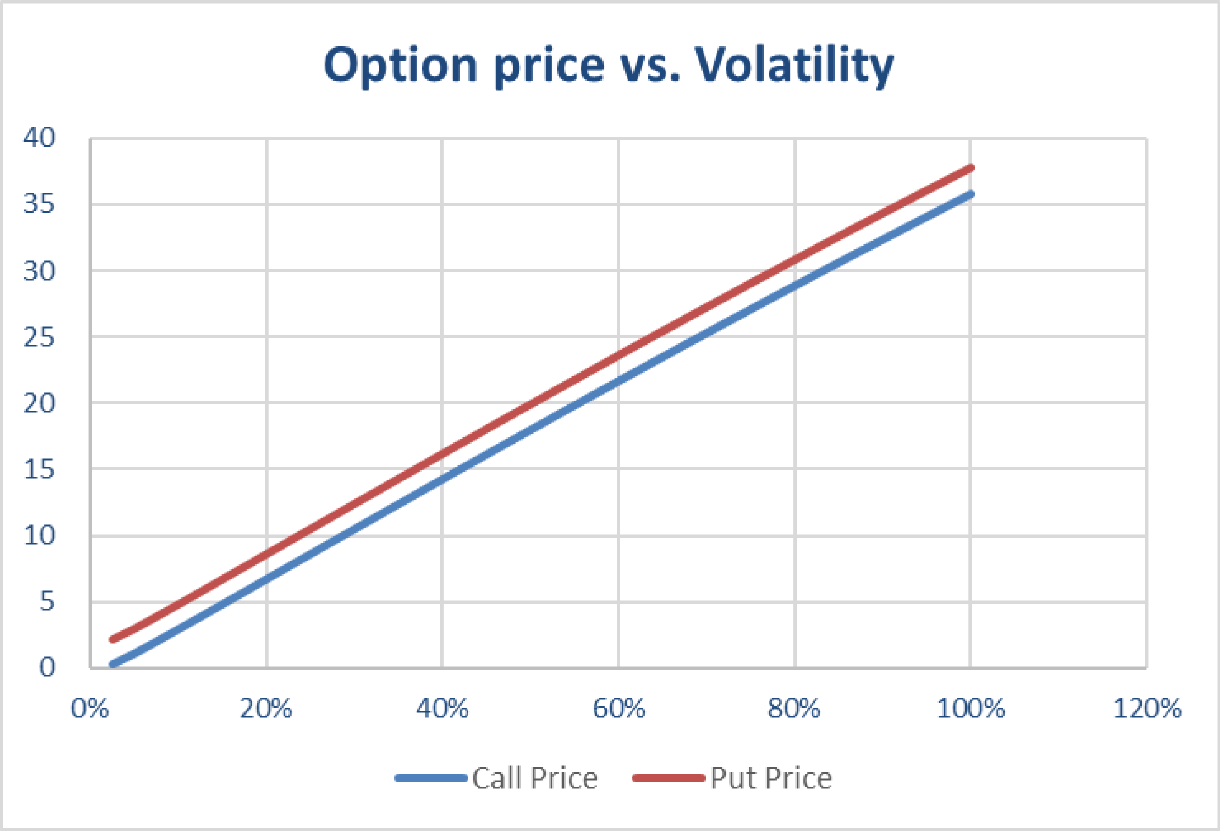
How Do European Option Prices React to Input Changes: Stock Price

- See tab “BS chart”. X-axis = stock price



How Do European Option Prices React to Input Changes: Volatility

- X-axis = stock volatility



Delta (Δ)

- In previous slide, we observed that the call/put option price changes with stock price.
- The sensitivity of option price to stock price changes is defined as “Delta”.
- We set out delta of options defined on an underlying which pays a continuous dividend.
- Black-Scholes-Merton (BSM) model:

$$c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Delta (Δ)

- Delta: represents the sensitivity of option price to stock price changes:

$$\Delta_{call} = \frac{\partial c}{\partial S} = \frac{\partial\left(Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)\right)}{\partial S} = e^{-qT}N(d_1) > 0$$

$$\Delta_{put} = \frac{\partial p}{\partial S} = \frac{\partial\left(Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1)\right)}{\partial S} = -e^{-qT}N(-d_1) < 0$$

where $d_1 = \frac{\ln(S/K)+\left(r-q+\sigma^2/2\right)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$

  - The price change of a call option is positively related to the price of its underlying asset (stock) – higher stock price means better profits from call option.
  - The price change of a put option is negatively related to the price of its underlying asset (stock) – lower stock price means better profits from put option.

Delta in Excel

- See Excel tab “BS Delta” that computes Delta in Excel.
- See VBA Module `BlackScholes`.
- VBA function `BSDelta` returns the delta of an option based on the BSM model.

Delta Hedging (1)

- Delta hedging is a fundamental technique in option pricing.
- The idea is to replicate an option by a portfolio of stocks and bonds, with the portfolio proportions determined by the Black-Scholes-Merton formula.

$$\Delta_{call} = e^{-qT}N(d_1)$$

$$\Delta_{put} = -e^{-qT}N(-d_1)$$

$$c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) = S\Delta_{call} - Ke^{-rT}N(d_2)$$

$$p = -Se^{-qT}N(-d_1) + Ke^{-rT}N(-d_2) = S\Delta_{put} + Ke^{-rT}N(-d_2)$$

$$\text{where } d_1 = \frac{\ln(S/K) + \left(r - q + \sigma^2/2\right)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

Delta Hedging (1)

- BSM formula:

$$\Delta_{call} = e^{-qT}N(d_1)$$

$$\Delta_{put} = -e^{-qT}N(-d_1)$$

$$c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) = S\Delta_{call} - Ke^{-rT}N(d_2)$$

$$p = -Se^{-qT}N(-d_1) + Ke^{-rT}N(-d_2) = S\Delta_{put} + Ke^{-rT}N(-d_2)$$

$$\text{where } d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

- Call option can be perfectly replicated using a portfolio with Δ_{Call} shares of stock and $-Ke^{-rT}N(d_2)$ amount of bonds, if investor can rebalance portfolio continuously.
- Put option can be perfectly replicated using a portfolio with Δ_{put} shares of stock and $Ke^{-rT}N(-d_2)$ amount of bonds, if investor can rebalance portfolio continuously.

Delta Hedging (2)

- In practice, however, investors can only do the delta hedging <u>periodically</u>.
- Delta hedging portfolio payoff is different from the option payoff.
- See tab “Delta Hedging” as an example.

Delta Hedging (3)

- Suppose we decide to replicate an at-the-money European call option that has 12 weeks to run until expiration.
- The stock on which the option is written has $S_0 = \$50$ and strike price $K = \$50$, the interest rate is $r = 4\%$, dividend yield is $q = 0$, stock volatility is $\sigma = 40\%$.
- We decide to create this option by replicating, on a week-to-week basis, the BS option-pricing formula using delta hedging.
- Stock price is simulated based on geometric Brownian motion under risk-neutral probability:

$$\frac{\Delta S}{S} = (r - q)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$S_{t-\Delta t} = S_t \exp\left((r - q - \sigma^2/2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}\right)$$

where t is time-to-maturity, Δt = 1 week = 1/52 year.

(After Δt, time-to-maturity decreases by Δt.)

Delta Hedging (4)

- Black-Scholes-Merton model:

$$c_t = S_t \Delta_{\text{call},t} - Ke^{-rT} N\left(d_{2,t}\right)$$

- At the beginning, 12 weeks before the option's expiration, we determine our stock/bond portfolio using BS model:
  - $t = 12/52$ year
  - $S_t = 50$
  - $\Delta_{\text{Call},t} = 0.5573$ shares of stock (using `BSDelta` function)
  - Value of stock is $S_t \Delta_{\text{Call},t} = 50 \times 0.5573 = 27.86$
  - $-Ke^{-rt} N\left(d_{2,t}\right) = -23.82$ amount of bonds
  - Portfolio value is $27.86 - 23.82 = 4.044$.

Delta Hedging (5)

- One week later, 11 weeks to maturity, we simulate the stock price based on GBM model.
- In a trial,
  - $t = 11/52$ year
  - $S_t = 50\exp\left((r - q - \sigma^2/2)\delta t + \sigma\varepsilon\sqrt{\Delta t}\right) = 45.93$
  - $\Delta_{Call,t} = 0.373$ shares of stock (using `BSDelta` function)
  - Investor should purchase $0.373 - 0.5573 = -0.1843$ share at price 45.93. Total cost is $-0.1843 \times 45.93 = -8.464$.
  - Bond value is calculated from the assumption of self-financing portfolio – when investor rebalance the hedging portfolio at time t, portfolio value does not change!

$$\text{Bond} = (-23.82) \times e^{r\Delta t} - (-8.464) = -15.38$$

Bond holder pays interest at the rate of *r* over 1 week.

  - Portfolio value is $45.93 \times 0.373 - 15.38 = 1.755$.

Delta Hedging (6)

- In each successive week (t years to maturity), portfolio values are calculated in the same way until maturity date.
- On maturity date, investor liquidates the hedging portfolio.
  - In a trial, we simulate the $\Delta_{\text{Call},t} = 0.0029$ share when $t = 1/52$ year.
  - On maturity date, investor does not need to rebalance the portfolio anymore, she sells the stock at the price of 44.84 (simulated). Total cash flow is

$$44.84 \times 0.0029 = 0.132$$

  - Bond value is 0.02 when $t = 1/52$ year.
  - On maturity date, bond value $= 0.02e^{r\Delta t} = 0.020$
  - Liquidation value is $0.132 + 0.02 = 0.152$
  - We compare it with call option payoff $\max\{\mathrm{S}_0 - \mathrm{K}, 0\}$. Here S_0 denotes the stock price at expiration date (with zero year to maturity).
  - Since we have rebalanced only weekly, our hedging portfolio payoff is slightly different from call option payoff.

Greeks for American Options

- We do not have closed-form solutions for American option prices; thus, there is no way we can have analytical solution for the Greeks of these options.
- We report Greeks using Black-Scholes-Merton model.

Binomial Option Pricing

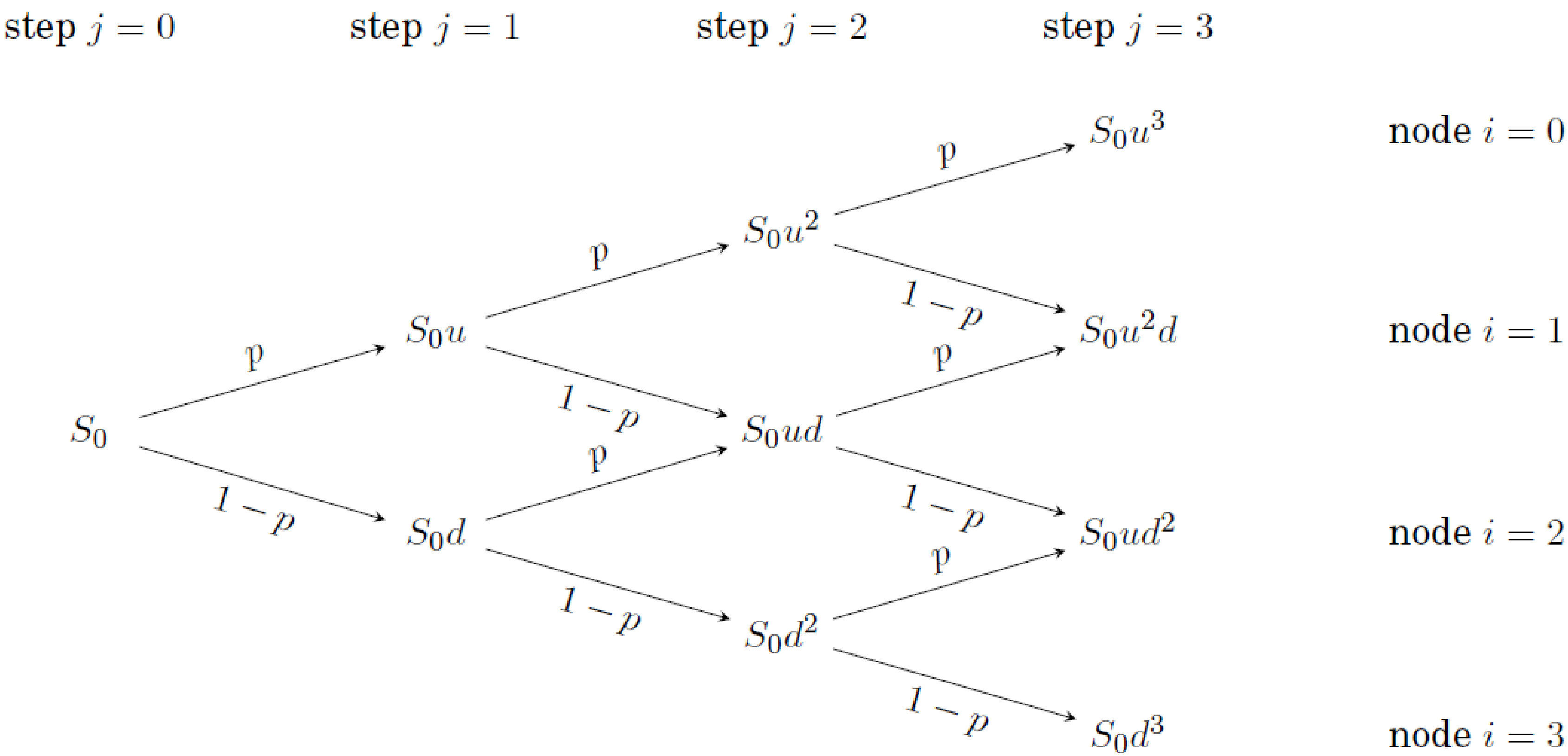
Binomial Option Pricing Model

- Binomial options pricing model (BOPM) is a "discrete-time" model for the valuation of options.
- Compare to Black-Scholes model, although BOPM is computationally slower, it handle a variety of conditions for which other models cannot easily be applied.
- For example, it is used to value American options.
- There is no close-form solutions for BOPM (BS model has!). We usually implement the model using computer software.
- Option price in binomial tree is computed based on replication and no arbitrage.

Binomial Pricing Model

- **Assumptions**:
  - There are two "fundamental" assets: a stock and a bond.
  - There is also a derivative asset, an option written on the stock.
  - There are n periods and $n+1$ dates: today is date 0, and maturity date is T from now.
  - Time step is $\Delta t = T/n$.
  - At each node, there are two possible states: "up" and "down"
  - Total return of the stock in "up" state is u; total return of the stock in "down" state is d.

3-Period Binomial Tree: Stock Tree

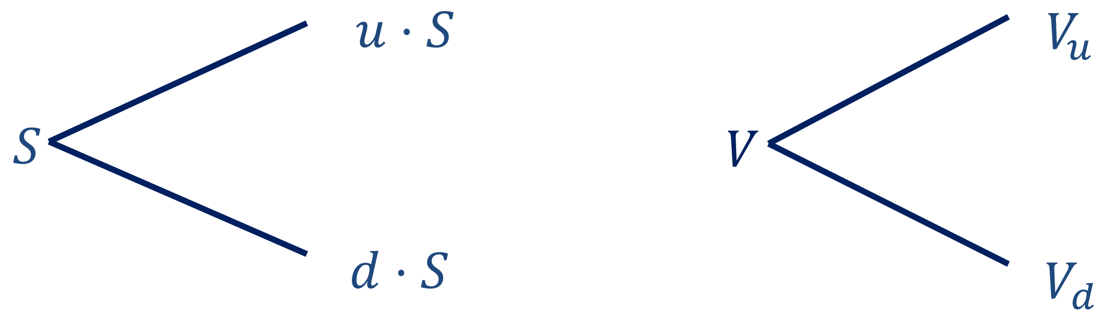


At time 0, the asset price S_0 is known. At step j we consider $j + 1$ asset prices. At node i $(= 0, 1, \ldots, j)$ the asset price is

$$S(i, j) = S_0 u^{j-i} d^i$$

Option Tree

- At a node, if the stock tree and corresponding option tree are as follows:



where V is the current option payoff, V_u and V_d are option payoffs in "up" and "down" states.

- The current option payoff using **risk-neutral valuation**:

$$V = e^{-r\Delta t}[pV_u + (1-p)V_d]$$

where the risk-neutral probability that "up" state occurs is

$$p = \frac{e^{(r-q)\Delta \mathrm{t}} - d}{u - d}$$

- Different binomial tree models specify u and d differently.

Risk-neutral Valuation

- We assume that we live in a risk-free world.
- The concept is that we do not worry about any uncertainty due to stock price ups and downs.
- As a result, we discount all cash flows using the risk-free rate.
- To price options using the risk-free-rate-discounted cash flows is risk-neutral valuation.
- Probability used is risk-neutral probability (not physical measure of probability).

Cox, Ross, Rubinstein's (CRR) Binomial Trees – European Options

CRR Binomial Trees – Step 1

- Binomial trees are used to approximate the movements in the price of a stock or other asset.
- In each small interval of time (Δt) the dividend-paying stock price *S* (dividend yield $= q$) is assumed

a) to move up by a proportional amount $u = e^{\sigma\sqrt{\Delta t}}$ with probability p; or

b) to move down by a proportional amount $d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$ with probability $1 - p$.

- For an arbitrage-free market, we require that

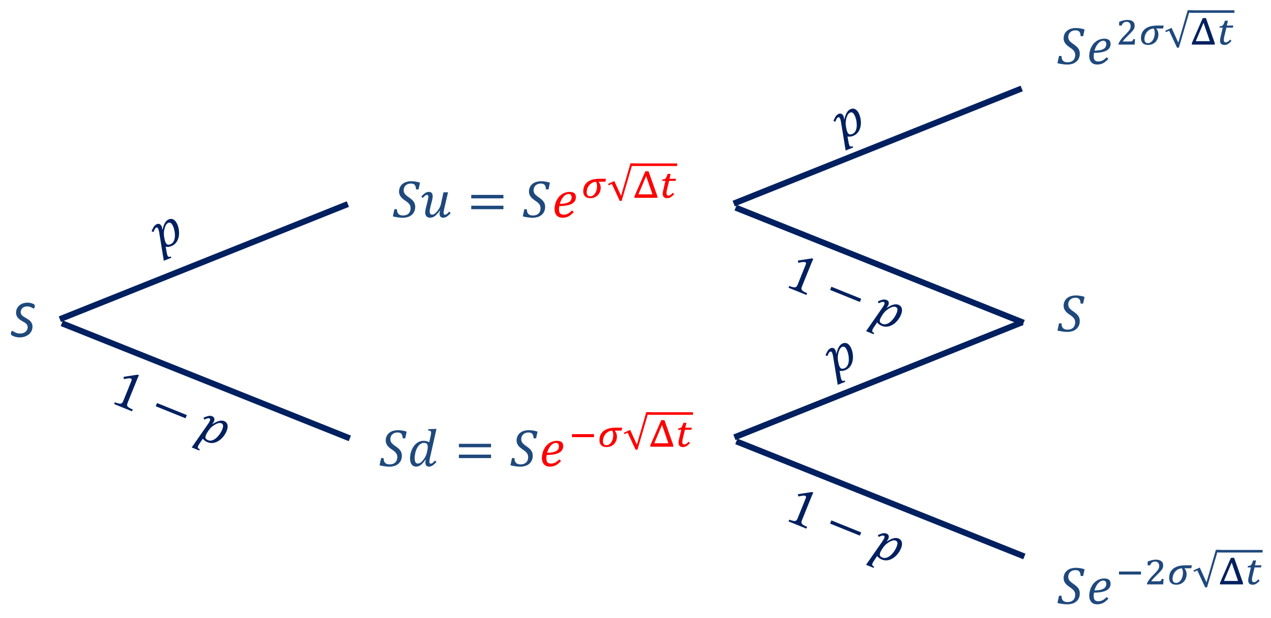
$$0 \leq p = \frac{e^{(r-q)\Delta t} - d}{u - d} \leq 1$$

That is,

$$e^{-\sigma\sqrt{\Delta t}} \leq e^{(r-q)\Delta t} \leq u = e^{\sigma\sqrt{\Delta t}}$$

CRR Binomial Trees – Step 2

- $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$



Pricing European Options in Excel

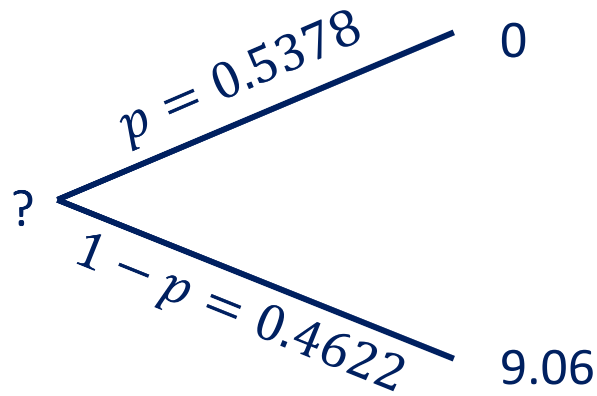
- We develop a model to estimate the price of European options (both call and put) on stocks with continuous dividend yields using 2-step CRR binomial tree.
- Note: different model has different definition of *u* and *d*.
- See sheet “Option2step” in Excel file “Lec9_BOPM.xlsm”

Pricing European Options – Backward from Step 2 (1)

- Given the stock price and stock payoff tree, we first compute

$$u = e^{\sigma\sqrt{\Delta t}}, d = 1/u, p = \left(e^{(r-q)\Delta t} - d\right)/(u - d).$$

- It's easy to compute the payoff in steps 2, but how about step 1? Look at cell "C26", we use <u>backward inference</u>:



- Since we know the probability now and it is European option, we can easily compute

$$\begin{aligned} \text{option payoff} &= e^{-r\Delta t}[pV_u + (1-p)V_d] \\ &= e^{-r\Delta t}(0.5378 \times 0 + 0.4622 \times 9.06) = 4.11 \end{aligned}$$

Pricing European Options – Backward from Step 2 (2)

- As a result, we get that the following:
  - Put option with strike = 50, it's price (p) = 1.86
  - Call option with strike = 50, it's price (c) = 3.08

Pricing European Options on Stocks with Dividends – 9 Steps

- Now, let's try 9-step CRR binomial tree for better approximation.
- See sheet "Option9step".
- As a result, we get that the following:
  - Put option with strike = 50, it's price (p) = 2.25
  - Call option with strike = 50, it's price (c) = 3.47

VBA Function for European Option

- VBA function `binomial_euro` in module `BOPM` returns European call/put option price based on CRR binomial tree.
- Let's use our earlier example for illustration:

Call or Put (c or p)	optType	p
Stock price	S0	50
Strike price	K	50
Time to maturity	T	0.5
Riskfree rate	r	8.0%
Dividend yield	q	3.0%
Volatilty	v	20.0%
Number of steps	n	9

Binomial Tree Vs. Black-Scholes

- We know that for European options, BS model is the continuous-time solution, and our binomial model is the discrete-time approximation.
- Check the "VBA_Euro" tab.
- Black-Scholes prices: c = 3.39 and p = 2.18.
- Summary:

Option	Binomial 2 steps	Binomial 9 steps	Binomial 15 steps	BS model
Euro Call	3.08	3.47	3.44	3.39
Euro Put	1.86	2.25	2.22	2.18

- Our summary supports that BS price is asymptotic value of Binomial price when step $n \to \infty$.

Cox, Ross, Rubinstein's (CRR) Binomial Trees – American Options

Using Binomial Tress to Price American Options (1)

- In the final step, the payoffs of American options are the same as the payoffs of European options.
- So, pricing American options is similar to what we have done for European options as we all start from the final step's payoffs.
- In any step before the final step, the risk-neutral investor makes the decision between

(a) exercise the option now, and

(b) wait for the next step.

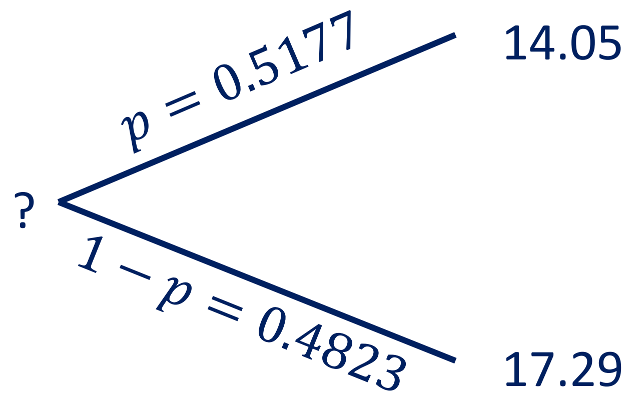
- Investor compare

(a) the payoff from exercise, and

(b) the expected payoff for waiting.

Using Binomial Tress to Price American Options (2)

- Check the “Option9step” tab for American options.
- Take the bottom triangle of step 8 (cell “J57”) as the example, we find the following



- As shown earlier, the claim in this node is 15.54, i.e., if the investor choose not to exercise the option now, the expected payoff is \$15.54 (present value).

Using Binomial Tress to Price American Options (3)

- Meanwhile, the current stock price is 34.29 (see "J28"). So, if the investor can make positive payoff by exercising the option at this node, the payoff is 50 – 34.29 = 15.71.
- So, the investor's choice is between

(a) exercise the put now = 15.71 and

(b) wait for the next step = 15.54.

- Obviously, the investor will choose (a).
- So, in each node, we just need to ask the Excel to compute the maximum of

(a) the positive payoff by exercising the option now and

(b) the expected payoff of waiting.

Using Binomial Tress to Price American Options (4)

- As a result, we get that the following:
  - Call option with strike = 50, it’s price = 3.47
  - Put option with strike = 50, it’s price = 2.37

Binomial 9-step	Call	Put
European	3.47	2.25
American	3.47	2.37

- The above table highlights the fact that the price of American option is always higher than or equal to European option (the option to exercise anytime is valuable).

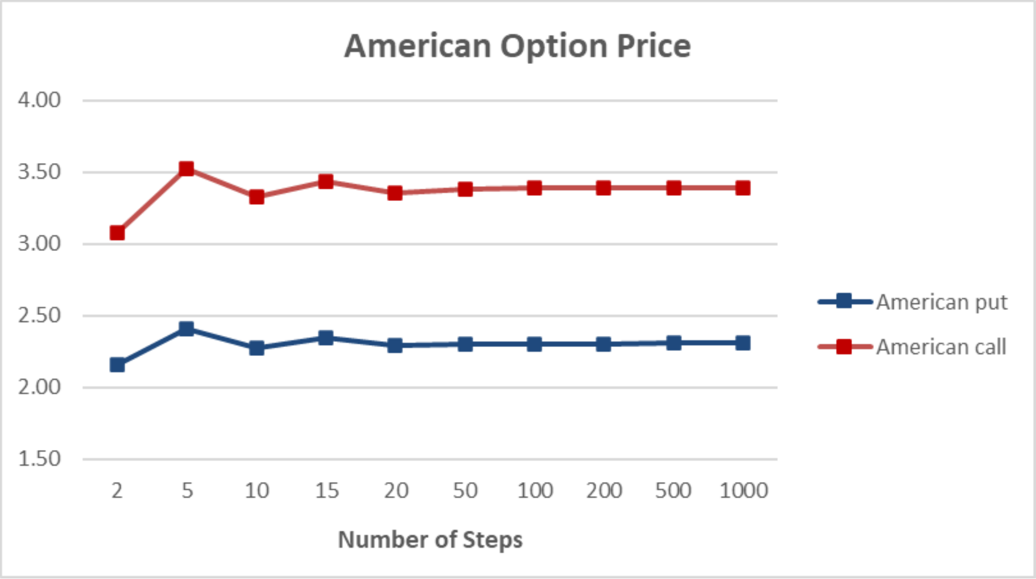
VBA for American Option Pricing (1)

- VBA function `binomial` in module `BOPM` returns either European or American call/put option price based on CRR binomial tree.
- Let's use our earlier example for illustration:

euro or amer	**optStyle**	amer
Call or Put (c or p)	optType	p
Stock price	S0	50
Strike price	K	50
Time to maturity	T	0.5
Riskfree rate	r	8.0%
Dividend yield	q	3.0%
Volatilty	v	20.0%
Number of steps	n	9

The Approximation of American Option Prices

❖ We now check the change of American call and put option prices with step number in the CRR binomial tree.



Example: AAPL Option (1)

- Now, let's use CRR Binomial model to price an Apple Inc. (AAPL) option.
- Options on single stocks are usually American option.
- Yahoo!Finance reports the current market data of AAPL options.
- The current index price = $132.05 on January 8, 2021.

Example: AAPL Option (2)

❖ The current stock price = $132.05 on January 8, 2021.

Apple Inc. (AAPL)

NasdaqGS - NasdaqGS Real Time Price. Currency in USD

Add to watchlist

132.05 +1.13 (+0.86%)

At close: January 8 4:00PM EST

Summary | Company Outlook | Chart | Conversations | Statist

Previous Close	130.92	Market Cap	2.222T
Open	132.43	Beta (5Y Monthly)	1.28
Bid	131.90 x 3000	PE Ratio (TTM)	40.26
Ask	131.76 x 1800	EPS (TTM)	3.28
Day's Range	130.23 - 132.63	Earnings Date	Jan 25, 2021 - Jan 31, 2021
52 Week Range	53.15 - 138.79	Forward Dividend & Yield	0.82 (0.62%)
Volume	105,158,245	Ex-Dividend Date	Nov 05, 2020
Avg. Volume	115,195,412	1y Target Est	129.84

Example: AAPL Option (3)

- Let's check the put option prices with strike price K = $130 that matures on April 16, 2021.

Puts for April 16, 2021

Contract Name	Last Trade Date	Strike	Last Price	Bid	Ask	Change	% Change	Volume	Open Interest	Implied Volatility
AAPL210416P00055000	2021-01-07 1:55PM EST	55.00	0.11	0.08	0.15	+0.01	+10.00%	1	582	73.63%
AAPL210416P00060000	2021-01-07 9:51AM EST	60.00	0.16	0.12	0.18	+0.01	+6.67%	4	1,018	69.34%
AAPL210416P00062500	2021-01-08 3:50PM EST	62.50	0.19	0.14	0.20	0.00	-	20	172	67.19%
AAPL210416P00065000	2021-01-04 10:34AM EST	65.00	0.24	0.16	0.23	0.00	-	14	53	65.23%
AAPL210416P00126250	2021-01-08 3:32PM EST	126.25	7.55	7.35	7.50	-0.20	-2.58%	165	1,118	38.24%
AAPL210416P00127500	2021-01-08 3:24PM EST	127.50	8.10	7.90	8.05	-0.30	-3.57%	67	1,663	38.15%
AAPL210416P00128750	2021-01-08 2:52PM EST	128.75	8.86	8.50	8.65	+0.06	+0.68%	80	980	38.16%
AAPL210416P00130000	2021-01-08 3:29PM EST	130.00	9.19	9.10	9.25	-0.46	-4.77%	271	1,944	38.08%
AAPL210416P00131250	2021-01-08 1:33PM EST	131.25	9.95	9.75	9.90	-0.55	-5.24%	104	787	38.09%
AAPL210416P00132500	2021-01-08 3:55PM EST	132.50	10.50	10.40	10.55	-0.50	-4.55%	221	834	38.01%
AAPL210416P00133750	2021-01-08 1:59PM EST	133.75	11.80	11.05	11.25	-0.05	-0.42%	24	1,593	38.04%

Example: AAPL Option (4)

- See "AAPLoption" tab. How to set up the parameters?

1. **Drift term** = risk-free rate (stock return in a risk-neutral world) at annual frequency

- What risk-free rate should we use?

⇒ Latest US T-Bill rate with maturity similar to the option's maturity.

⇒ 3-Month T-Bill rate on January 7, 2021 (data store on Tab: "T-Bill")

- Option price is not very sensitive to the drift term.

2. **Dividend yield** = 0.62% per annum (from Yahoo!Finance)
3. **Volatility term** = 33.54% per annum, estimated using historical monthly return data of AAPL in the past 3 years.
4. **Time to maturity** = 98 days/365 = 0.2685 year

Example: AAPL Option (5)

- Our calculation implies the price of $8.185, which is lower than the market price (average of bid and ask price) $9.18.
- If you believe in our calculation, the market price is “more expensive” than our estimation.
- If you believe in the option market price, which parameter would you re-estimate?

$\Rightarrow$ It is very likely that your volatility estimation is <u>lower</u> from market’s view.