Portfolio Analysis

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FINA3351 – Spreadsheet Modelling in Finance

Roadmap for Today

1. Functions for Summary Statistics
2. Envelope Portfolio and Efficient Frontier using Black's Zero-Beta CAPM.
3. Envelope Portfolio and Efficient Frontier with Short-Sales Constraints
4. Alternatives to the Sample Variance-Covariance matrix

Functions for Summary Statistics

Excel Functions for Summary Statistics

- We use two HK stocks as an example to construct a portfolio.
- Two stocks: China Mobile (0941.HK) and China Unicom (0762.HK)
- Sample period: monthly data from December 2014 to December 2024
- Data is downloaded from Yahoo Finance and saved in the worksheet "2stocks_data" of "Lec6_Portfolio.xlsm".
- In the worksheet "2stocks", we estimate
  - expected returns using historical sample average: `AVERAGE` function
  - standard deviation of returns using <u>sample</u> standard deviation: `STDEV` or `STDEV.S` function
  - covariance of two returns using <u>sample</u> covariance: `COVARIANCE.S` function
  - correlation of two returns using sample correlation: `CORREL` function

Excel Functions for Summary Statistics

- In statistics, we distinguish between **sample** statistics and **population** statistics.
  - If we are dealing with a set of outcomes for the random variable, then we are dealing with a sample.
  - In data analysis, we choose unbiased estimations, which are sample estimations.

Excel Functions – Sample vs. Population

- **Excel functions for standard deviation**:
  - STDEV, STDEV.S (sample): $\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(x_t - \bar{x})^2}$
  - STDEV.P (population): $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})^2}$
- **Excel functions for variance**:
  - VAR, VAR.S (sample)
  - VAR.P (population)
- **Excel function for covariance**:
  - COVARIANCE.S (sample): $\frac{1}{n-1}\sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y})$
  - COVAR, COVARIANCE.P (population): $\frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y})$
- Excel function for correlation:
  - CORREL (sample and population): $\frac{\sum(x_t-\bar{x})(y_t-\bar{y})}{\sqrt{\sum(x_t-\bar{x})^2}\sqrt{\sum(y_t-\bar{y})^2}}$

Portfolio Analysis

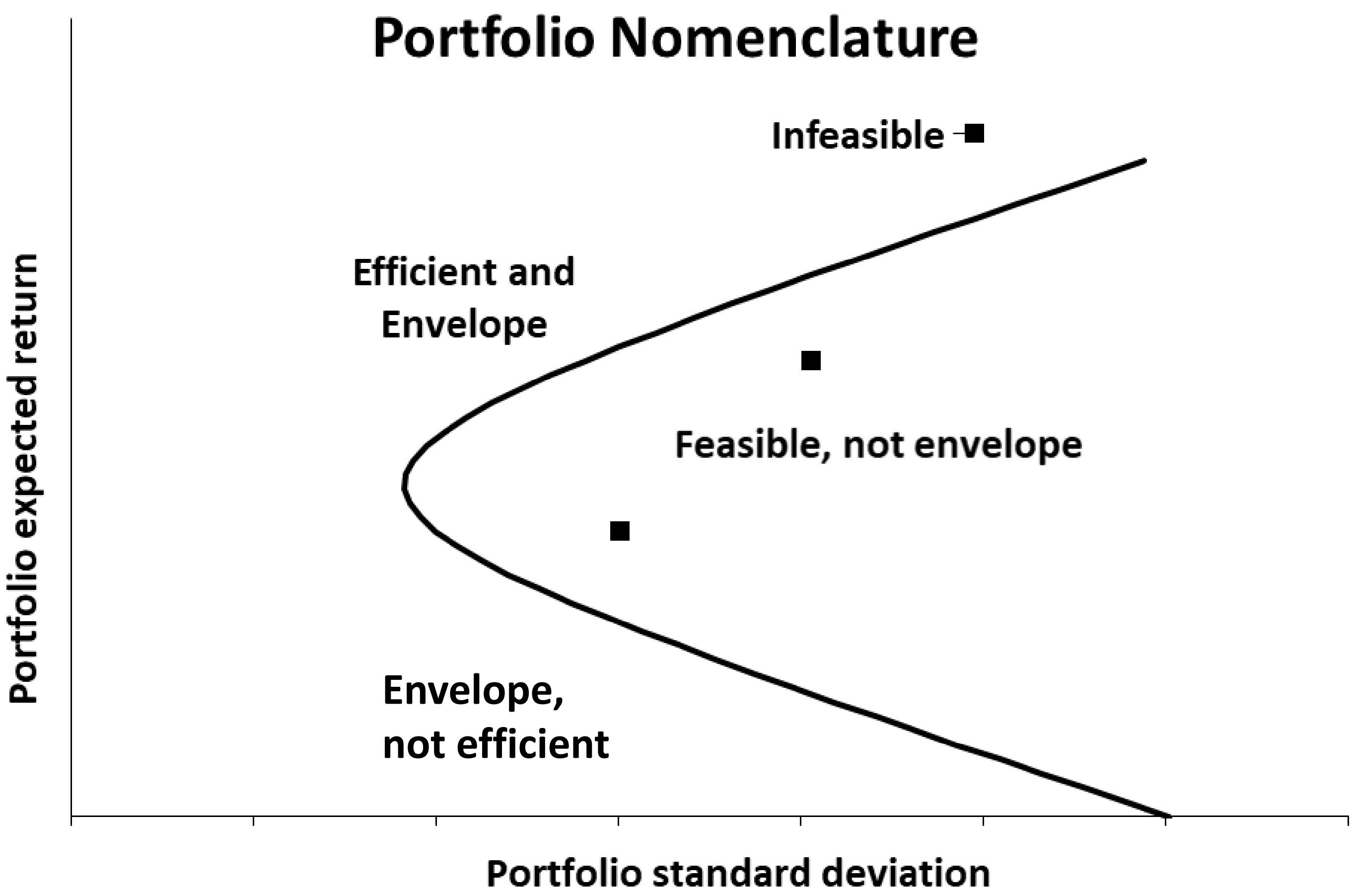
Portfolio of Two Risky Assets

- We apply the knowledge learned in FINA2320 (Investment and Portfolio Analysis) course in Excel.
- Start with the portfolio of 2 risky assets.
- Portfolio (*P*) consists of two risky assets *A* and *B*.
  - Weight of A is w_A, weight of B is w_B
  - $w_A + w_B = 1$
  - If there is no additional restrictions on weights, $w_A, w_B \in (-\infty, \infty)$.
- Return on the risky portfolio: $r_P = w_A r_A + w_B r_B$
- Expected return: $E(r_P) = w_A E(r_A) + w_B E(r_B)$
- Variance of return: $\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \mathrm{Cov}(r_A, r_B)$
- Go to Excel tab “2stocks_portfolio”.

Feasible, Envelope, Efficient Portfolios

- A *feasible* portfolio is any portfolio that can be constructed by the n assets, and weights sum to 1.
- A feasible portfolio is on the *envelope* of the feasible set if for a given expected return, it has minimum variance.
- A portfolio A is an *efficient* portfolio if it maximizes the expected return given the portfolio variance.

Feasible, Envelope, Efficient Portfolios



Preliminary Notation

❖ In the portfolio models, we make intensive use of Excel's matrix calculations and array functions.

❖ There are n risky assets, each of which has return r_i, weight w_i, expected return $E(r_i)$, variance σ_i^2. Covariance between asset i and j is σ_{ij}.

❖ In matrix notation,

- Matrix $E(\boldsymbol{r})$ is the column vector of expected returns of these assets.
- Matrix $\boldsymbol{w}$ is the column vector of weights of assets in a portfolio.
- Σ is $n \times n$ variance-covariance matrix.
- $\mathbf{1}$ is $n \times 1$ column vector where each element has value of 1.

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, E(\boldsymbol{r}) = \begin{pmatrix} E(r_1) \\ \vdots \\ E(r_n) \end{pmatrix}, \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}, \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

Portfolio Mean, Variance, Covariance

- Two risky portfolios A and B are constructed using these n assets.
- The expected return $E(r_A)$ and variance of return σ_A^2 are

$$E(r_A) = \sum_{i=1}^{n} w_i E(r_i) = \boldsymbol{w}^T E(\boldsymbol{r}) = E(\boldsymbol{r})^T \boldsymbol{w}$$

(1 by n matrix) *
(n by 1 matrix) =>
1 by 1 matrix

$$\sigma_A^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1} w_i w_j \mathrm{Cov}(r_i, r_j) = \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

$$\text{and} \sum_{i=1}^{n} w_i = 1 \text{ (or } \boldsymbol{w}^T \boldsymbol{1} = 1 \text{ or } \boldsymbol{1}^T \boldsymbol{w} = 1\text{)}$$

(Superscript T represents transpose of a matrix.)

(1 by n matrix) *
(n by n matrix) *
(n by 1 matrix) =>
1 by 1 matrix

- Covariance between portfolio A and B is

$$\mathrm{Cov}(r_A, r_B) = \boldsymbol{w}_A^T \Sigma \boldsymbol{w}_B$$

Basic Matrix Operation

- Excel tab “Matrix” introduces Excel built-in functions to compute matrix multiplication, transpose, inverse matrix.
- `MMULT(array1,array2)` returns the matrix product of two arrays.

Note that this is an array function.

- `MINVERSE(array)` returns the inverse matrix for the matrix.
  - Matrix must have equal number of rows and columns, otherwise, it returns a #VALUE! error.
  - In linear algebra, A is an *n*-by-*n* square matrix. Denote its inverse matrix as A^{-1}, then

$$AA^{-1} = I_n$$

where I_n denotes the *n*-by-*n* identity matrix.

Envelope and Efficient Portfolios using Black's Zero-Beta CAPM

Black's Zero-Beta CAPM and Efficient Frontier

- Black's (1972) Zero-Beta CAPM paper provides us an analytical solution to the weight vector of efficient portfolios.
- The model is valid only when the weight vector does not have additional constraints.
- We will calculate the efficient portfolios and efficient frontier.
- Let's start with propositions on efficient portfolios and the CAPM.

Proposition 1: Envelope Portfolios

- **PROPOSITION 1:** Let c be a constant. We use the notation $E(\boldsymbol{r}) - c$ to denote the following column vector:

$$\begin{pmatrix} E(r_1) - c \\ \vdots \\ E(r_n) - c \end{pmatrix}$$

Let the vector z solve the system of simultaneous linear equations

$$E(\boldsymbol{r}) - c = \Sigma z.$$

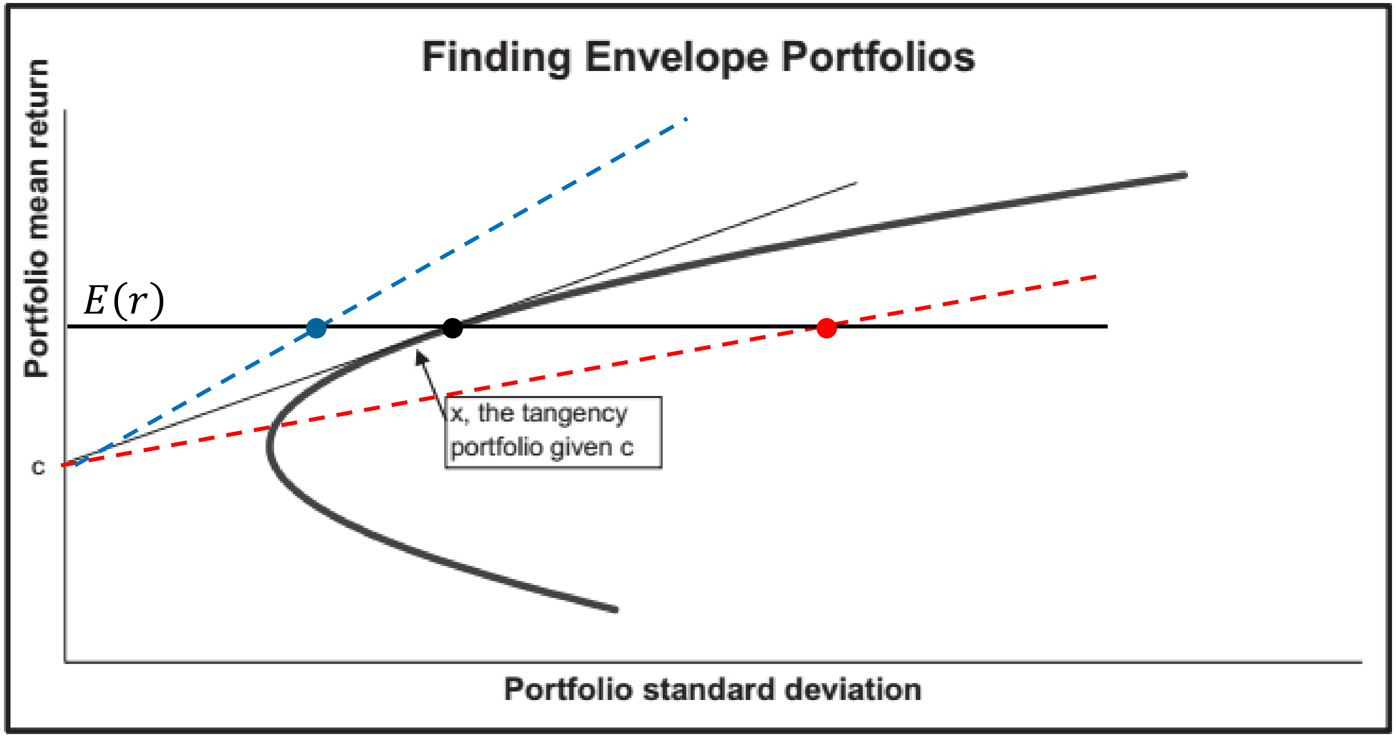
Then this solution produces a portfolio x on the envelope of the feasible set in the following manner:

$$z = \Sigma^{-1}(E(\boldsymbol{r}) - c),$$
$$x = (x_1 \quad \cdots \quad x_n)^T$$

where $x_i = \frac{z_i}{\sum_{j=1}^{n} z_j}$. Further, all envelope portfolios are of this form.

Intuition of the Proposition 1

- Math proof is provided in the handout “Lec6_Portfolio_math_proofs”.
- ***Intuition***:
  - Suppose we pick a constant c and we try to find an efficient portfolio x for which there is a tangency between c and the feasible set:



Intuition of the Proposition 1

- The formal proof is provided in the handout.
- ***Intuition***:
  - Suppose we pick a constant c and we try to find an efficient portfolio x for which there is a tangency between c and the feasible set.
  - The tangency portfolio x maximizes the slope

$$\frac{x^T[E(\boldsymbol{r})-c]}{\sqrt{x^T\Sigma x}}$$

and $\sum_{i=1}^{n} x_i = 1$.

  - The solution in the proposition is found by solving this maximization question.
  - This proposition also tells us that if x is any envelope portfolio, then there must exist a constant c and a vector z such that

$$z = \Sigma^{-1}(E(\boldsymbol{r})-c), \qquad x_i = \frac{z_i}{\sum_{j=1}^{n} z_j}.$$

Proposition 2: Two Mutual Fund Theorem

- **PROPOSITION 2:** By a theorem first proved by Black (1972), any two envelope portfolios are enough to establish the whole envelope.
- Given two envelope portfolios $x = (x_1, \cdots, x_n)^T$ and $y = (y_1, \cdots, y_n)^T$, then for any real number a,

$$ax + (1 - a)y$$

is also an envelope portfolio.

- The formal proof is provided in the handout.

Steps to Construct Envelope Frontier

❖ **How to construct envelope frontier based on the proposition?**

Step 1: Have the input list of all the securities: $E(r_i)$, σ_i^2, Σ.

Step 2: Choose two arbitrary constants, c_1 and c_2.

Find two envelope portfolios A and B with these two constants.

Take A for example.

1) Construct a new vector using expected return vector:

$$\begin{pmatrix} E(r_1) - c_1 \\ \vdots \\ E(r_n) - c_1 \end{pmatrix}$$

Steps to Construct Envelope Frontier

2) Let Σ^{-1} denote the inverse matrix of variance-covariance matrix Σ

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} E(r_1) - c_1 \\ \vdots \\ E(r_n) - c_1 \end{pmatrix}$$

3) Scale this weight vector to make sum of weights equal to 1:

$$\begin{pmatrix} w_1^A \\ \vdots \\ w_n^A \end{pmatrix} = \frac{1}{\sum_{j=1}^{n} z_j} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

This is the weight vector of portfolio A.

4) Find weight vector of portfolio B using constant number c_2. B is also on the envelope.

5) Based on two mutual fund theorem, entire envelope is just all the combination of A and B.

Optimal Risky Portfolio

- If you only want to construct optimal risky portfolio, it is easier:

Step 1: Have the input list of all the securities: $E(r_i)$, σ_i^2, Σ, r_f.

Step 2: Choose only one constant number, $c = r_f$ (risk-free rate).

Then same as in previous slide,

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} E(r_1) - r_f \\ \vdots \\ E(r_n) - r_f \end{pmatrix}$$

Scale this weight vector to make sum of weights equal to 1:

$$\begin{pmatrix} w_1^P \\ \vdots \\ w_n^P \end{pmatrix} = \frac{1}{\sum_{j=1}^{n} z_j} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

This is the weight vector of optimal risky portfolio P.

Optimal Risky Portfolio

- If you are familiar with matrix operation, previous slide can be written in the following equation:

$$\boldsymbol{w}_P = \frac{\Sigma^{-1}[E(\boldsymbol{r}) - r_f\boldsymbol{1}]}{\boldsymbol{1}^T\Sigma^{-1}[E(\boldsymbol{r}) - r_f\boldsymbol{1}]} \left(= \frac{\Sigma^{-1}[E(\boldsymbol{r}) - r_f\boldsymbol{1}]}{\text{sum(numerator)}} \right)$$

  - $\boldsymbol{w}_P$, $E(\boldsymbol{r})$, $\boldsymbol{1}$ are n-dimensional column vectors, Σ is $n \times n$ variance-covariance matrix.
  - Note: $\boldsymbol{1}^T\Sigma^{-1}[E(\boldsymbol{r}) - r_f\boldsymbol{1}]$ is to sum up all elements in the numerator.
  - Superscript T represents matrix transpose operation.

Global Minimum Variance Portfolio

- Global minimum variance portfolio (GMVP) has smallest variance among all feasible portfolios.
- We cannot use a constant c to find GMVP.
- It is determined by variance-covariance matrix only.
- Weights of all risky assets in GMVP can be computed by

$$\boldsymbol{w}_{\text{GMVP}} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \left(= \frac{\Sigma^{-1}\mathbf{1}}{\text{sum(numerator)}} \right)$$

Efficient Portfolio and Frontier in Excel

- See tab “Frontier-Analytic” as an example.
- Note: All roads lead to Rome - the precise values of c are irrelevant for determining the frontier.

A Data Example

- Excel tab “7stocks” construct optimal risky portfolio (P) and GMVP using the analytical solution.
- **Data**: 10-year monthly (adjusted closed) price data for 7 stocks used in Lecture 5.
- Variance-covariance matrix is estimated using:
  - sample unbiased estimators
  - constant correlation method
- We write five VBA functions to facilitate calculations:
  - `UnitRowVector`
  - `UnitColVector`
  - `VarCov`
  - `VarCovConst`
  - `CorrMat`

See VBA Module `Functions`.

Calculate Optimal Risky Portfolio

- In this example, since stock data are saved in columns, it is easier to put $E(r_i)$'s in a row (using Autofill).
- Then we can modify the analytical equation for the weight vector of optimal risky portfolio:

$$\boldsymbol{w}_{\text{row}} = (w_1, \cdots, w_n) = \frac{[E(\boldsymbol{r}) - r_f \mathbf{1}_{\text{row}}]\Sigma^{-1}}{\text{sum(numerator)}}$$

  - $\boldsymbol{w}_{\text{row}}$ is $1 \times n$ row vector of weights
  - $E(\boldsymbol{r})$ is $1 \times n$ row vector of expected returns
  - $\mathbf{1}_{\text{row}} = (1, \cdots, 1)$ is n-dimensional row vector of 1's
  - Σ is $n \times n$ variance-covariance matrix
  - Note: sum(numerator) is to sum up all elements in the numerator.

Calculating GMVP

- We can also modify the analytical equation for the weight vector of GMVP.
- If $\boldsymbol{w} = (w_1, \cdots, w_n)$ is defined as a row vector:

$$\boldsymbol{w}_{\text{row}} = (w_1, \cdots, w_n) = \frac{\mathbf{1}_{\text{row}}\Sigma^{-1}}{\text{sum(numerator)}}$$

  - $\boldsymbol{w}_{\text{row}}$ is $1 \times n$ weight vector
  - $\mathbf{1}_{\text{row}} = (1,1, \cdots, 1)$ is n-dimensional row vector of 1's.
- VBA functions `UnitRowVector` and `UnitColVector` return $\mathbf{1}_{\text{row}}$ and $\mathbf{1}_{\text{col}}$ with user specified number of elements.

Alternatives to the Sample Variance-Covariance

Sample Variance-Covariance Matrix

Method 1: Use sample variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

- Excel functions `Var.S`, `Covariance.S`, `Correl` return sample variance, covariance, and correlation of two data series.
- The drawback is that the number of estimations is large.

Variance-Covariance Matrix – Constant Correlation

Method 2: Constant correlation model computes the Var-Cov matrix by assuming that asset returns have the same correlation coefficient.

Usually, constant correlation is estimated as the average correlation of the assets in question, or simply choose 0.

Denote correlation between asset *i* and *j* returns is ρ_{ij}. Since covariance is $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, in constant coefficient model:

$$\sigma_{ij} = \begin{cases} \sigma_i^2, & \text{if } i = j \\ \rho\sigma_i\sigma_j, & \text{if } i \neq j \end{cases}$$

σ_i^2 is the sample variance of asset *i*.

ρ is constant correlation.

Envelope and Efficient Portfolios with Short-Sales Constraints

If There is Short-Sales Constraint...

- If there are constraints on portfolio weights,
  - analytical solutions in Black's zero-beta CAPM does not work anymore.
  - Two mutual fund theorem doesn't hold.
- We will use Markowitz Model and **Excel Solver** to calculate envelope efficient portfolios and frontier.
- Some practical constraints:
  - Mutual fund investment policy
  - No short sales
  - Maintain a minimal level of expected dividend yield
  - Role out certain industries, countries, etc.
  - ……

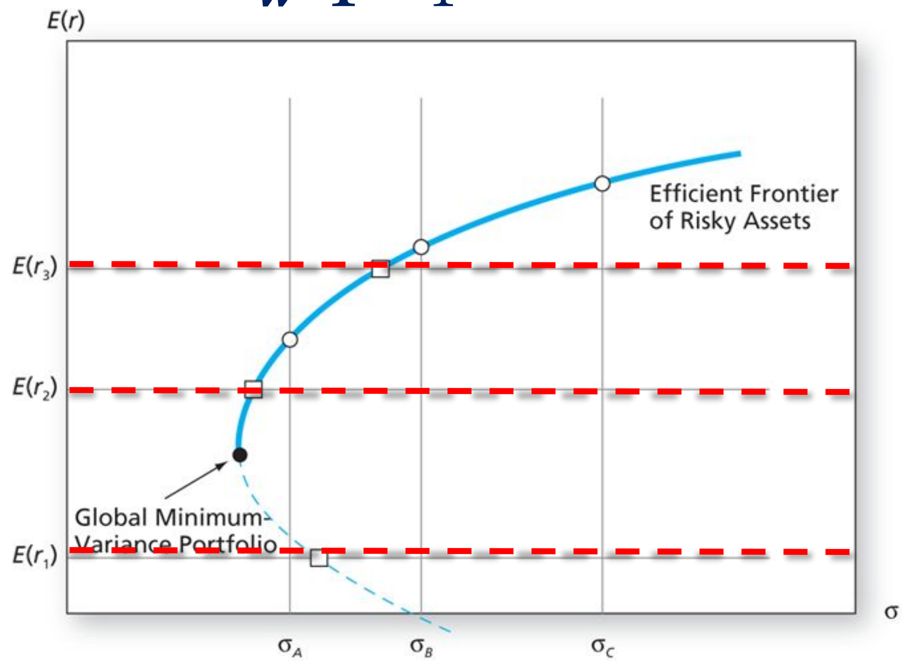
Minimum Variance Frontier (Envelope)

- An <u>envelope portfolio</u> is the portfolio of risky assets that gives the lowest variance of return of all portfolios having the same expected return.
- **Optimization problem (Eqn-1):**

$$\min_{\boldsymbol{w}=(w_1,\cdots,w_n)^T} \sigma_A^2 = \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

$$\text{s.t.} \qquad \boldsymbol{w}^T E(\boldsymbol{r}) = E(r_A),$$

$$\boldsymbol{w}^T \boldsymbol{1} = 1$$



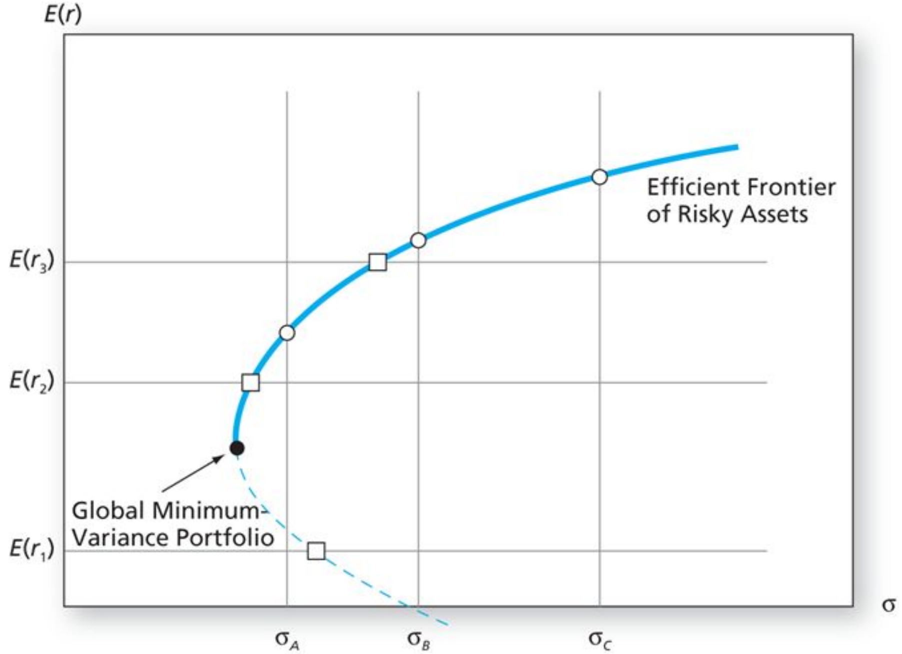
Efficient Frontier

Global Minimum Variance Portfolio (GMVP):

$$\min_{\boldsymbol{w}=(w_1,\cdots,w_n)^T} \sigma_{GMVP}^2 = \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

$$\text{s.t.} \quad \boldsymbol{w}^T \mathbf{1} = 1$$

❖ Efficient frontier is the part above GMVP. Efficient portfolio is the portfolios on the efficient frontier.



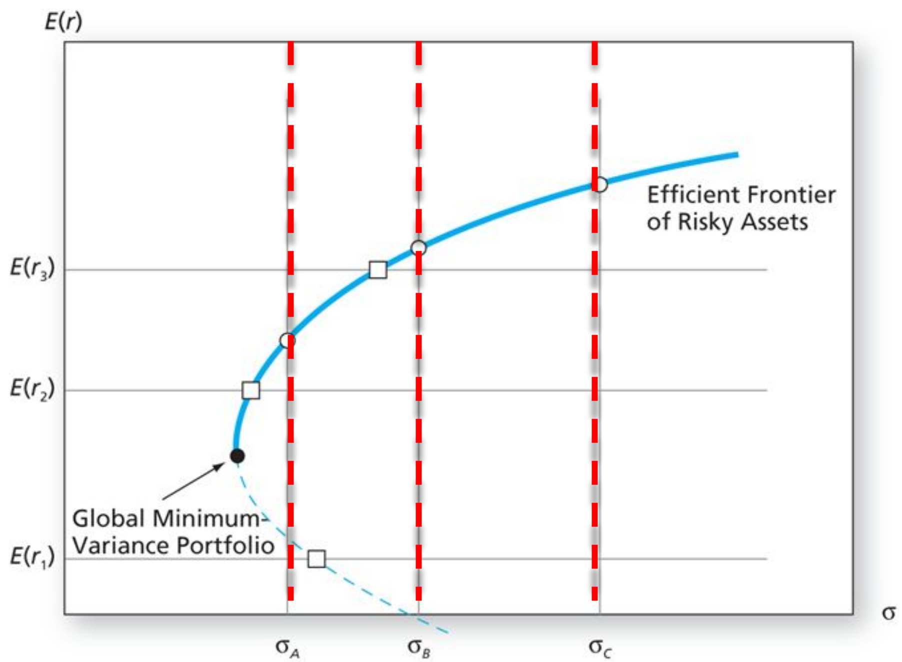
Maximum Expectation Frontier (Efficient)

- An <u>efficient portfolio</u> is the portfolio of risky assets that gives the highest expected return of all portfolios having the same variance.
- **Optimization problem (Eqn-2):**

$$\max_{\boldsymbol{w}=(w_1,\cdots,w_n)^T} E(r_A) = \boldsymbol{w}^T E(\boldsymbol{r})$$

$$\text{s.t.} \quad \boldsymbol{w}^T \Sigma \boldsymbol{w} = \sigma_A^2,$$

$$\boldsymbol{w}^T \mathbf{1} = 1$$



How To Construct Optimal Risky Portfolio and GMVP

Step 1: Have the input list of all the securities: $E(r_i), \sigma_i^2, \Sigma, r_f$.

Step 2: The expectation and variance of return of a risky portfolio is then:

$$E(r) = \boldsymbol{w}^T E(\boldsymbol{r})$$
$$\sigma^2 = \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

Step 3: <u>Optimal risky portfolio</u> *P* is the one that maximizes the Sharpe ratio. GMVP is the one that minimizes the portfolio variance.

$$\max_{\boldsymbol{w}_P = (w_1, \cdots, w_n)^T} \text{Sharpe ratio} = \frac{E(r) - r_f}{\sigma}$$

$$\min_{\boldsymbol{w}_{GMVP} = (w_1, \cdots, w_n)^T} \sigma^2 = \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

$$\text{s.t.} \quad \boldsymbol{w}^T \mathbf{1} = 1$$

Portfolio Construction Using Excel Solver

- See Excel tab "Portfolio-Solver" as an example.
- The optimization problem in Markowitz model can be solved using Excel's Solver.
- Limitation: every time we solve an optimization problem, we have to setup Solver again. It is not convenient if we need to solve optimization problems for multiple times.

If There is Short-Sales Constraint...

❖ If there is short-sales constraint, additional constraints are

$$w_i \geq 0, \text{ for all } i$$

must be added to Markowitz optimization problem.

❖ Two mutual fund theorem does NOT hold anymore.

❖ To construct the efficient frontier, you need to set many target expected returns. For each target expectation, we solve the optimization problem (Eqn-1).

That is, you have use Excel Solver many times!

❖ Any way to decrease our workload?

$\Rightarrow$ Yes! Use VBA and record Solver setting.

$\Rightarrow$ Go to tab “Frontier-ShortSale” and VBA module “ShortSale”.

If There is Short-Sales Constraint...

- To learn how to write VBA codes for Excel Solver setting, you can record Excel Solver (sub "`solve`").
- Then, write a sub procedure `RepeatSolve` to iterate value of target expected return. For every target value, call `solve` to find weights of Fund 1, 2, and 3.