Some Math Proofs in Portfolio Analysis Models

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1 Expectation, Variance and Covariance in Statistics

Definition. *The **variance** of a random variable X with expected value* $E(X) \equiv \mu_X$ *is defined as* $Var(X) \equiv \sigma_X^2 = E\left((X-\mu_X)^2\right)$*. The **covariance** between random variable Y and Z, with expected values* μ_Y *and* μ_Z*, is defined as* $Cov(Y,Z) \equiv \sigma_{YZ} = E\left((Y-\mu_Y)(Z-\mu_Z)\right)$*. The **correlation** between Y and Z is defined as*

$$Corr(Y,Z) \equiv \rho_{YZ} = \frac{Cov(Y,Z)}{\sqrt{Var(Y)Var(Z)}} = \frac{\sigma_{YZ}}{\sigma_Y \sigma_Z}.$$

*The square root of the variance of a random variable is called its **standard deviation** or **volatility**.*

Notice that $Var(X) = Cov(X,X)$.

A few facts about variances and covariances.

1. $Cov(Y,Z) = E(YZ) - E(Y)E(Z)$. In particular, $Var(X) = E(X^2) - (E(X))^2$.

2. For constants a, b, c, d, and random variables U, V, Y, Z,

$$\begin{aligned}
E(aU+bV) &= aE(U) + bE(V) \\
Var(aU) &= a^2 Var(U) \\
Var(aU+bV) &= a^2 Var(U) + b^2 Var(V) + 2ab\,Cov(U,V) \\
Cov(aU+bV, cY+dZ) &= ac\,Cov(U,Y) + bc\,Cov(V,Y) + ad\,Cov(U,Z) + bd\,Cov(V,Z)
\end{aligned}$$

2 Portfolio Analysis: Expectation, Variance, Covariance

Suppose that there are n risky assets, each of which has return r_i, weight w_i, expected return $E(r_i)$, variance σ_i^2. Covariance between asset i and j is σ_{ij}.

In matrix notation, matrix $E(\boldsymbol{r})$, $\boldsymbol{w}$, and $\mathbf{1}_{n\times 1}$ are $n \times 1$ column vectors, Σ is $n \times n$ variance-covariance matrix.

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \; E(\boldsymbol{r}) = \begin{pmatrix} E(r_1) \\ \vdots \\ E(r_n) \end{pmatrix}, \; \mathbf{1}_{n\times 1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \; \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

Two risky portfolios A, B are constructed using these n assets. Then the expected return of portfolio A is

$$E(r_A) = \sum_{i=1}^{n} w_i E(r_i) = \boldsymbol{w}^T E(\boldsymbol{r}) = E(\boldsymbol{r})^T \boldsymbol{w} \tag{1}$$

Proof. We intensively use the results in section 1.

$$\begin{aligned}
r_A &= \sum_{i=1}^{n} w_i r_i \\
E(r_A) &= E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i) \\
\boldsymbol{w}^T E(\boldsymbol{r}) &= \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix} \times \begin{pmatrix} E(r_1) \\ \vdots \\ E(r_n) \end{pmatrix} = \sum_{i=1}^{n} w_i E(r_i) = E(r_A) \\
E(\boldsymbol{r})^T \boldsymbol{w} &= \begin{pmatrix} E(r_1) & \cdots & E(r_n) \end{pmatrix} \times \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} w_i E(r_i) = E(r_A)
\end{aligned}$$

$\square$

The variance of return of portfolio A is

$$\sigma_A^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \sigma_{ij} = \boldsymbol{w}_A^T \Sigma \boldsymbol{w}_A \tag{2}$$

Proof. Notice that $Var(r_i) = Cov(r_i, r_i)$. In the following equations, I omit superscript "A" in its weight matrix.

$$\begin{aligned}
\sigma_A^2 &= Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \sigma_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
\boldsymbol{w}_A^T \Sigma \boldsymbol{w}_A &= \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix} \times \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{pmatrix} \times \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_{i=1}^{n} w_i \sigma_{i1} & \Sigma_{i=1}^{n} w_i \sigma_{i2} & \cdots & \Sigma_{i=1}^{n} w_i \sigma_{in} \end{pmatrix} \times \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\
&= w_1 \Sigma_{i=1}^{n} w_i \sigma_{i1} + w_2 \Sigma_{i=1}^{n} w_i \sigma_{i2} + \cdots + w_n \Sigma_{i=1}^{n} w_i \sigma_{in} \\
&= \sum_{j=1}^{n} w_j \sum_{i=1}^{n} w_i \sigma_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
&= \sigma_A^2
\end{aligned}$$

□

The covariance between portfolio A and B returns is

$$\sigma_{AB} = \sum_{i=1}^{n} w_i^A w_j^B \sigma_{i,j} = \boldsymbol{w}_A^T \Sigma \boldsymbol{w}_B \tag{3}$$

Superscript A and B represent the weights of assets in portfolio A and B.

Proof. The proof is similar.

$$\begin{aligned}
\sigma_{AB} &= Cov\left(\sum_{i=1}^{n} w_i^A r_i, \sum_{i=1}^{n} w_i^B r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i^A w_j^B \sigma_{ij} \\
\boldsymbol{w}_A^T \Sigma \boldsymbol{w}_B &= \begin{pmatrix} w_1^A & \cdots & w_n^A \end{pmatrix} \times \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{pmatrix} \times \begin{pmatrix} w_1^B \\ \vdots \\ w_n^B \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_{i=1}^n w_i^A \sigma_{i1} & \Sigma_{i=1}^n w_i^A \sigma_{i2} & \cdots & \Sigma_{i=1}^n w_i^A \sigma_{in} \end{pmatrix} \times \begin{pmatrix} w_1^B \\ \vdots \\ w_n^B \end{pmatrix} \\
&= w_1^B \Sigma_{i=1}^n w_i^A \sigma_{i1} + w_2^B \Sigma_{i=1}^n w_i^A \sigma_{i2} + \cdots + w_n^B \Sigma_{i=1}^n w_i^A \sigma_{in} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i^A w_j^B \sigma_{ij} \\
&= \sigma_{AB}
\end{aligned}$$

□

3 Markowitz Model

3.1 Optimal risky portfolio

In the lecture, we state that if two portfolios A and B are constructed by these n assets, and both are on the efficient frontier, then the optimal risky portfolio, denoted by P can be constructed using A and B with weights:

$$\begin{aligned}
w_P(A) &= \frac{[E(r_A) - r_f]\sigma_B^2 - [E(r_B) - r_f]\sigma_{AB}}{[E(r_B) - r_f]\sigma_A^2 + [E(r_A) - r_f]\sigma_B^2 - [E(r_A) + E(r_B) - 2r_f]\sigma_{AB}} \\
w_P(B) &= 1 - w_P(A)
\end{aligned} \tag{4}$$

Proof. To simplify the notations in the following proof, we denote $\mu_A = E(r_A) - r_f$ and

$\mu_B = E(r_B) - r_f$.

Since $Var(r_A - r_f) = Var(r_A) = \sigma_A^2$, $Var(r_B - r_f) = Var(r_B) = \sigma_B^2$, and $Cov(r_A - r_f, r_B - r_f) = Cov(r_A, r_B) = \sigma_{AB}$, the Sharpe ratio of optimal risky portfolio P can be written as

$$S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{w_A\mu_A + w_B\mu_B}{\sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}}}$$

Weight w_A and w_B are the solutions of the following optimization equation:

$$\begin{array}{ll} & \max\limits_{(w_A, w_B)} S = \dfrac{E(r_P) - r_f}{\sigma_P} \\ \text{s.t.} & w_A + w_B = 1 \end{array}$$

We incorporate the constraint by replacing $w_B = 1 - w_A$ in the objective function. Then w_A is solved by setting the first-order derivative of $S = \frac{E(r_P)-r_f}{\sigma_P}$ equal to 0. That is,

$$\frac{dS}{dw_A} = d\left(\frac{w_A\mu_A + w_B\mu_B}{\sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}}}\right) / dw_A = 0$$

To simplify the notations, we denote the numerator and denominator of S function as

$$\begin{aligned} f(w_A) &= E(r_P) - r_f = w_A\mu_A + w_B\mu_B = (\mu_A - \mu_B)w_A + \mu_B \\ g(w_A) &= \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}} = \sqrt{w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)\sigma_{AB}} \end{aligned}$$

Then, the first-order derivative of $f(w_A)$ and $g(w_A)$ are

$$\begin{aligned} f'(w_A) &= \mu_A - \mu_B \\ g'(w_A) &= \frac{w_A\sigma_A^2 - (1 - w_A)\sigma_B^2 + (1 - 2w_A)\sigma_{AB}}{\sqrt{w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A}} \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{dS}{dw_A} = \frac{f'(w_A)g(w_A) - f(w_A)g'(w_A)}{(g(w_A))^2} = 0 \\ &f'(w_A)g(w_A) = f(w_A)g'(w_A) \end{aligned}$$

$$
\begin{aligned}
& (\mu_A - \mu_B)\sqrt{w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A} \\
= \; & [(\mu_A - \mu_B)w_A + \mu_B]\left(\frac{w_A\sigma_A^2 - (1-w_A)\sigma_B^2 + (1-2w_A)\sigma_{AB}}{\sqrt{w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A}}\right)
\end{aligned}
$$

Rearrange above equation and we can solve the w_A as

$$
w_A = \frac{\mu_A\sigma_B^2 - \mu_B\sigma_A^2}{\mu_A\sigma_B^2 + \mu_B\sigma_A^2 - (\mu_A + \mu_B)\sigma_{AB}}
$$

□

3.2 Global minimum variance portfolio (GMVP)

If two portfolios A and B are constructed by these n assets, and both are on the efficient frontier, then the global minimum variance portfolio, denoted by G can be constructed using A and B with weights:

$$
\begin{aligned}
w_G(A) &= \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} \\
w_G(B) &= 1 - w_G(A)
\end{aligned} \tag{5}
$$

Proof. Weights of A and B in GMVP are the solution of the following optimization equation:

$$
\begin{aligned}
& \min_{(w_A, w_B)} \sigma_G^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB} \\
\text{s.t.} \quad & w_A + w_B = 1
\end{aligned}
$$

We incorporate the constraint by replacing $w_B = 1 - w_A$ in the objective function. Then w_A is solved by setting the first-order derivative of $\sigma_G^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}$ equal to 0. That is,

$$
\begin{aligned}
0 &= \frac{d\left(w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A(1-w_A)\sigma_{AB}\right)}{dw_A} \\
w_A &= \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}
\end{aligned}
$$

$\square$

4 Black's (1972) Zero-Beta CAPM

Black's (1972) Zero-Beta CAPM model provides the analytical solutions to the weights of efficient portfolios.

Proposition 1. *Let c be a constant. We use the notation $E(r)-c$ to denote the following column vector:*

$$E(r) - c = \begin{pmatrix} E(r_1) - c \\ E(r_2) - c \\ \vdots \\ E(r_n) - c \end{pmatrix}$$

Let the vector z solve the system of simultaneous linear equations $E(r) - c = \Sigma z$. Then this solution produces a portfolio x on the envelope of the feasible set in the following manner:

$$\begin{aligned} z &= \Sigma^{-1}[E(r) - c] \\ x &= (x_1, \ldots, x_n)^T \end{aligned}$$

where

$$x_i = \frac{z_i}{\sum_{j=1}^{n} z_j}$$

Furthermore, all envelope portfolios are of this form.

Proof. A portfolio x is on the envelope of the feasible set of portfolios if and only if it lies on the tangency of a line connecting some point c on the y-axis to the feasible set. Such a portfolio must either maximize or minimize the ratio $\frac{x^T[E(r)-c]}{\sigma_x}$, where $x^T[E(r)-c]$ is the vector product which gives the portfolio's expected excess return over c, and $\sigma_x = \sqrt{x^T \Sigma x}$ is the portfolio's standard deviation. Take first-order derivative of this ratio with respect to x. This gives:

$$\begin{aligned} d\left[\frac{x^T(E(r)-c)}{\sqrt{x^T\Sigma x}}\right]/dx &= \frac{(E(r)-c)\sqrt{x^T\Sigma x} - x^T(E(r)-c)\frac{\Sigma x}{\sqrt{x^T\Sigma x}}}{x^T\Sigma x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} \\ E(r)-c &= \frac{x^T(E(r)-c)}{x^T\Sigma x}\Sigma x \end{aligned}$$

Notice that $\frac{x^T(E(r)-c)}{x^T\Sigma x}$ is a number. We denote it by λ. Then writing $z=\lambda x$, we see that a portfolio is efficient if and only if it solves the system $E(r)-c=\Sigma z$. Normalizing z so that its coordinates add to 1 gives the desired result. □

By a theorem first proved by Black (1972), any two envelope portfolios are enough to establish the whole envelope.

Proposition 2. *The convex combination of any two envelope portfolios is on the envelope of the feasible set.*

Proof. Let x and y be portfolios on the envelope. Then, they are feasible portfolios of risky assets. A portfolio x is feasible if and only if the weights of the portfolio add up to 1; i.e., $\sum_{i=1}^n x_i = 1$, where n is the number of risky assets. Suppose that λ is some number. Then it is clear that $z=\lambda x+(1-\lambda)y$ also satisfies that $\sum_{i=1}^n z_i = 1$. That is, z, the convex combination of feasible portfolio x and y, is feasible.
By Proposition 1, x and y are envelope portfolios. It follows that there exist two vectors, z_x and z_y, and two constants c_x and c_y, such that:

- x is the normalized-to-unity vector of z_x; i.e., $x_i = \frac{z_{xi}}{\sum_{j=1}^n z_{xj}}$
- y is the normalized-to-unity vector of z_y; i.e., $y_i = \frac{z_{yi}}{\sum_{j=1}^n z_{yj}}$
- $E(r)-c_x=\Sigma z_x$ and $E(r)-c_y=\Sigma z_y$.

Furthermore, since z maximizes the ratio $\frac{z^T(E(r)-c)}{\sigma_z}$, it follows that any normalization of z also maximizes this ratio. With no loss in generality, therefore, we assume that z

sums to 1.

It follows that for any real number a, the portfolio $az_x + (1-a)z_y$ solves the system $E(r) - [ac_x + (1-a)c_y] = \Sigma\,[az_x + (1-a)z_y]$. This proves our claim. $\square$