Monte Carlo Methods in Option Pricing

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FINA3351 – Spreadsheet Modelling in Finance

Roadmap

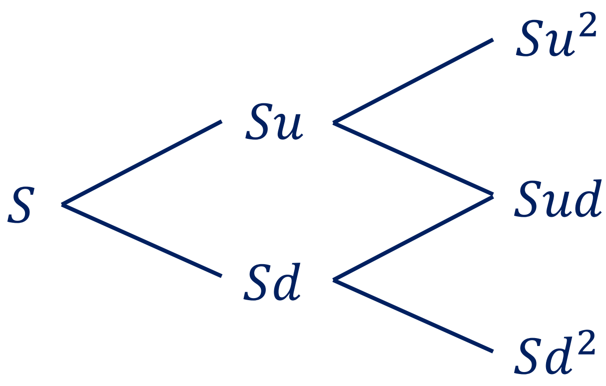
1. Monte Carlo Price of European and Asian Options Based on CRR Binomial Tree
2. Monte Carlo Price of European and Asian Options Based on Geometric Brownian Motion

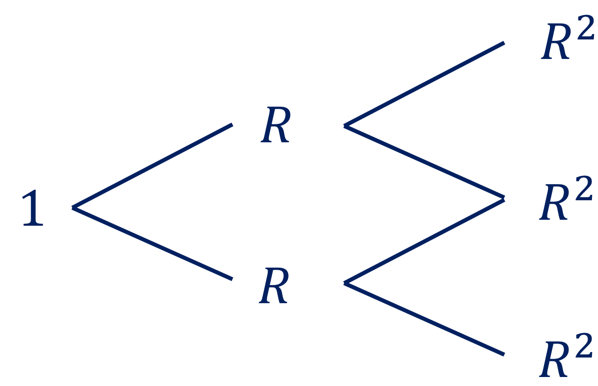
Path-dependent Options

- European and American options are path-independent options – options whose price only depends on the terminal price of the asset.
- Path-dependent options' price depends not only on the terminal price of the asset, but also on the path of the prices by which the terminal price was reached.
  - Examples: Asian options, barrier options
  - In general, a path-dependent option does not have an analytical price solution.
  - Monte Carlo method provides us with a handy numerical tool for pricing.
- Monte Carlo pricing of options depends on a simulation of the price path of the underlying asset.

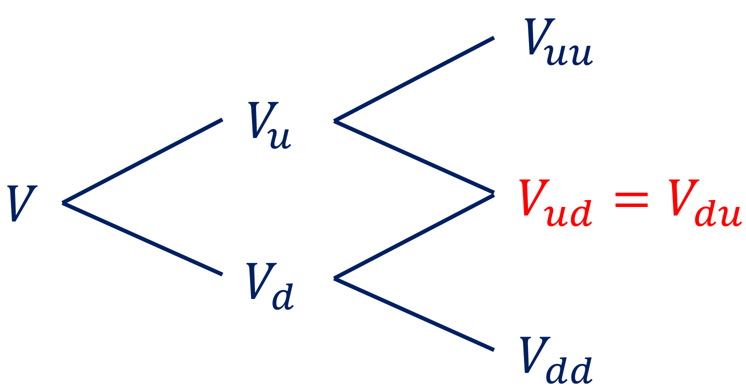
Path-dependent Options (Binomial Framework)

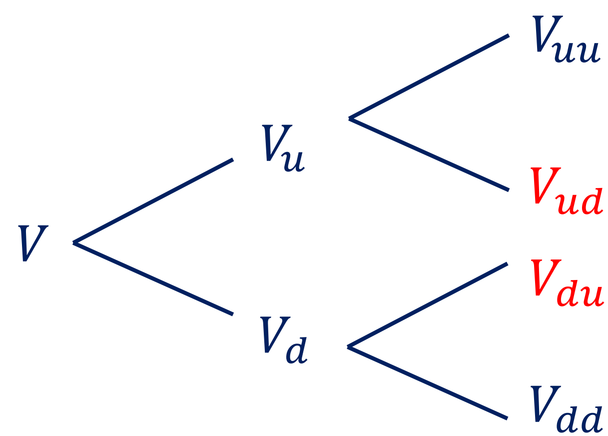
- Fundamental assets: stock and bond





- Path-independent vs path-dependent option payoffs





MCM Price of European and Asian Options Based on CRR Binomial Model

Two-Period Binomial Tree

- Suppose we have binomial framework with stock price S that grows at u or d in every period.
- Suppose the interest rate is $R = e^{r\Delta t}$ and continuously compounded dividend yield q.
- Risk-neutral probabilities are defined by $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$.
- Option price is the discounted expected value of the option payoffs.

$$V = e^{-r\Delta t}[pV_u + (1-p)V_d]$$

where V_u and V_d are option payoffs in up and down states.

Multiperiod Binomial Tree – European Option

- Multiperiod binomial setting:
- *u* and *d* do not change over time.
- $S(i,n)$, $i = 1, 2, \cdots, n$, denotes the date-*n* payoff of the asset in a state where there are i down moves in the binomial tree.
- At time 0, the asset price S_0 is known. At step n at node i $(= 0, 1, \ldots, n)$, the asset price is

$$S(i,n) = S_0 u^{n-i} d^i$$

- The terminal payoff of a European option is

$$V(i,n) = \max\big(\mathrm{optType} \cdot (S_0 u^{n-i} d^i - K), 0\big)$$

where optType equals 1 for call, and -1 for put.

- The value of this asset is given by

$$V_0 = e^{-rT} \sum_{i=0}^{n} \binom{n}{i} p^{n-i} (1-p)^i V(i,n)$$

MCM Price of European Option on CRR Tree

- See Excel file "Lecture10_MCMOptionPricing.xlsm".
- Sheet "Euro_Binomial" contains the MCM simulation of this European put based on 6-period CRR binomial tree.
- We simulate the states of two steps (up or down) for 3 and 100 times.
- The option price is the average discounted value of option final payoffs.
- We also compare simulated price with Black-Scholes price.

VBA Function for MCM Pricing - CRR Tree

- We compare option pricing results using MCM simulation (VBA function: `MCMEuroBin`) and Black-Scholes Model (VBA function: `BS`)
- VBA function `MCMEuroBin` is given in Module “Binomial_MCM”.
- Main principles:
  - Price paths are generated by using the risk-neutral probabilities.
  - In `MCMEuroBin`, for example, the price of the stock moves “up” if the random-number generator is less than p and moves “down” if the random-number generator is greater than p.
  - Effectively, therefore, the risk-neutral probabilities $\{p, 1\text{-}p\}$ of each price path are incorporated into the price path itself.
  - Value of the option using Monte Carlo is determined by the discounted value of the simple average of all results over the price paths generated.

Asian Options

- An Asian option is an option whose payoff depends in some way on the average price of the asset over a period prior to option expiration.
- Two common kinds of Asian options:

a) Average price option: the option's payoff is:

$$\max[\text{optType} \times (\text{Average asset price} - K), 0]$$

b) Average strike option: option's strike price is:

$$\max[\text{optType} \times (S_T - \text{Average asset price}), 0]$$

- Average is calculated in two ways:

a) Arithmetic average $A(S,T) = \frac{1}{n}\sum_{i=1}^{n} S_{i\Delta t}$

b) Geometric average $G(S,T) = \sqrt[n]{\prod_{i=1}^{n} S_{i\Delta t}}$

where time 0 to expiration time *T* is evenly divided into *n* sub-intervals with time length $\Delta t = T/n$.

- Asian option is path-dependent. We must simulate entire time path of the binomial tree to price the option.

Asian Options – An Example

- Sheet "Asian_Binomial" contains an example of Asian put with 6 steps.
- The option has 6 months to maturity. The average price is computed at the end of each month.
- $\text{Option terminal payoff} = \max[\text{optType} \times (\text{Average asset price} - K), 0]$
- Average asset price is arithmetic average of stock prices at step 1 to 6.
- Option price is calculated as the expected payoff at maturity T, discounted by risk-free rate over the same period, which is e^{-rT}.

VBA Function for Asian Option – CRR Tree

- Tab "Asian_Binomial" shows option pricing results using MCM simulation based on CRR Binomial Tree model (VBA function: `MCMAsianBin`)
- VBA function `MCMAsianBin` is given in Module "Binomial_MCM".
- Main principles:
  - Price paths are generated by using the risk-neutral probabilities. In `MCMAsianBin`.
  - In each run, state ("up" or "down") at each step is simulated and corresponding stock price is calculated. Average stock price from step 1 to *n* is calculated based on the time-series simulation of the stock prices.
  - Option's terminal payoff is calculated as
    $$\max[\text{optType} \times (\text{Average asset price} - K), 0]$$
  - The value of the option using Monte Carlo is determined by the average of simulated option terminal payoffs discounted by the interest rate.

MCM Price of European and Asian Options Based on GBM Model

European Option: GBM Model

- We assume that stocks prices follow Geometric Brownian motion under the risk-neutral measure:

$$\frac{dS}{S} = (r - q)dt + \sigma\varepsilon\sqrt{dt}$$

  - S_0 is current stock price.
  - Option's time to maturity is *T*.
  - In the risk-neutral world, drift term is the risk-free interest rate *r* less the continuous dividend yield *q*, that is by $(r - q)$.
  - σ is still the volatility term of the underlying stock.
- The option payoff depends on the stock price at maturity

$$S_T = S_0 e^{(r-q-0.5\sigma^2)T+\sigma\varepsilon\sqrt{T}}, \varepsilon \sim N(0,1).$$

- See sheet "Euro_GBM".
- VBA function `MCMEuroGBM` is given in Module "GBM_MCM".

Asian Option: GBM Model

- We still use "arithmetic average price" option as an example.
- Sheet "Asian_GBM" contains an example of Asian put with 6 steps and 100 runs.
- $\text{Option terminal payoff} = \max[\text{optType} \times (A(S,T) - K), 0]$
- Average asset price is arithmetic average of stock prices.
- Option price is calculated as the expected payoff at maturity *T*, discounted by risk-free rate over the same period, that is e^{-rT}.
- VBA function `MCMAsianGBM` is given in Module "GBM_MCM".