Homework 2 DS-GA 1002

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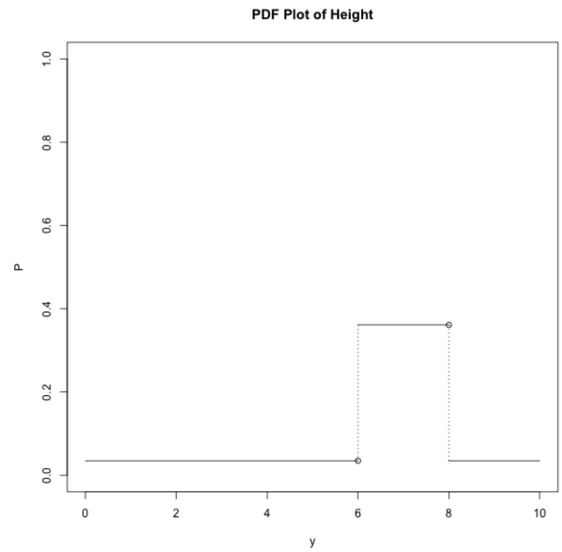
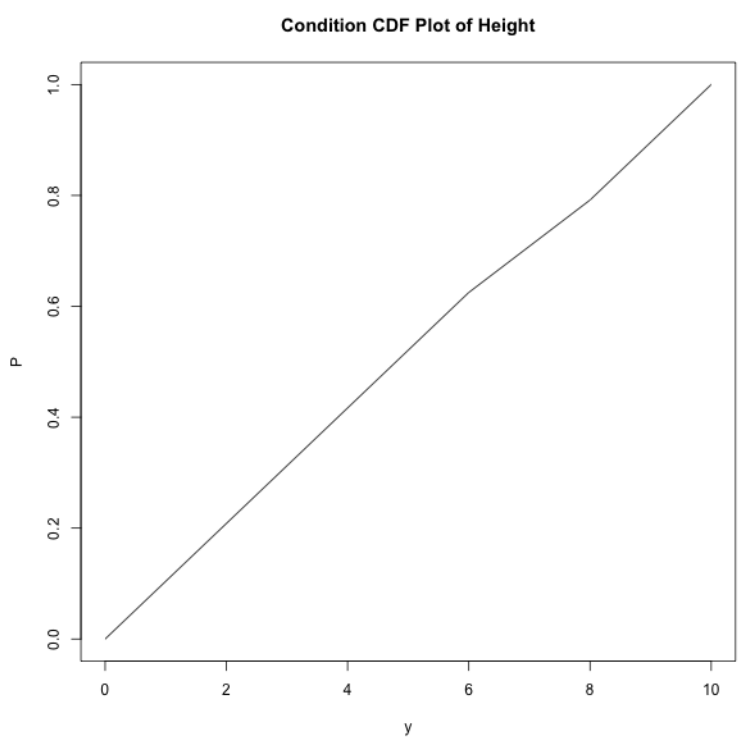
Problem 1.

(a) The position of spider can be modeled By F(X,Y). X is the horizontal position and Y is the height position.
Since we know that the spider stays twice the time under the painting area, and is uniformly distributed both inside and outside the painting area.
We have $P((X,Y) \in paint) = 2P((X,Y) \notin paint)$
Let $f_{X,Y}(x,y) = \begin{cases} c & (x,y) \in [4,6] \times [6,8] \\ d & otherwise \end{cases}$
Then we have $c \times 4 = 2 \times d \times 96$ and $4c = 192d, c = \frac{1}{6}, d = \frac{1}{288}$
Therefore
$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{6} & (x,y) \in [4,6] \times [6,8] \\ \frac{1}{288} & otherwise \end{cases}$$

(b)
(i) For $y \in [0,6], F(Y) = P(Y \in [0,y]) = \int_0^y \int_0^{10} f_{X,Y}(u,v)dudv = \int_0^y \int_0^{10} \frac{1}{288}dudv = \frac{10}{288}y,$
the pdf is $\frac{dF}{dy} = \frac{10}{288}$
(ii) For $y \in [6,8], F(Y) = P(Y \in [6,y]) + P(Y \in [0,6]) = \int_6^y \int_0^{10} f_{X,Y}(u,v)dudv + \frac{60}{288}$
$\int_6^y \int_0^{10} f_{X,Y}(u,v)dudv = \int_6^y \int_0^{4} \frac{1}{288}dudv + \int_6^y \int_4^{6} \frac{1}{6}dudv + \int_6^y \int_6^{10} \frac{1}{288}dudv = \frac{13}{36}(y-6)$
Therefore, For $y \in [6,8]$ the pdf is $\frac{13}{36}$
(iii) For $y \in [8,10], F(Y) = P(Y \in [0,8]) + P(Y \in [8,y]) = \int_8^y \int_0^{10} f_{X,Y}(u,v)dudv + \frac{67}{72} = \frac{67}{72} + \int_8^y \int_0^{10} \frac{1}{288}dudv$
Therefore For $y \in [8,10]$, the pdf is $\frac{10}{288}$



(c) Let event A be that the spider is located outside the painting area.
$F_{X,Y|A} = \frac{p(X \leq x, Y \leq y, A)}{P(A)}$, $P(A) = \frac{1}{3}$
For $y \in [0,6], F(Y) = P(Y \in [0,y]) = \int_0^y \int_0^{10} 3f_{X,Y,A}(u,v)dudv = \int_0^y \int_0^{10} \frac{3}{288}dudv = \frac{30}{288}y,$
For $y \in [6,8], F(Y) = P(Y \in [6,y]) + P(Y \in [0,6]) = \int_6^y \int_0^{10} f_{X,Y,A}(u,v)dudv + \frac{180}{288}$
$\int_6^y \int_0^{10} f_{X,Y,A}(u,v)dudv = \int_6^y \int_0^4 3\frac{1}{288}dudv + \int_6^y \int_6^{10} 3\frac{1}{288}dudv = \frac{24}{288}(y-6)$
For $y \in [8,10], F(Y) = P(Y \in [0,8]) + P(Y \in [8,y]) = \int_8^y \int_0^{10} f_{X,Y,A}(u,v)dudv + \frac{228}{288} = \frac{228}{288} + \int_8^y \int_0^{10} 3\frac{1}{288}dudv = \frac{30}{288}(y-8) + \frac{228}{288}$

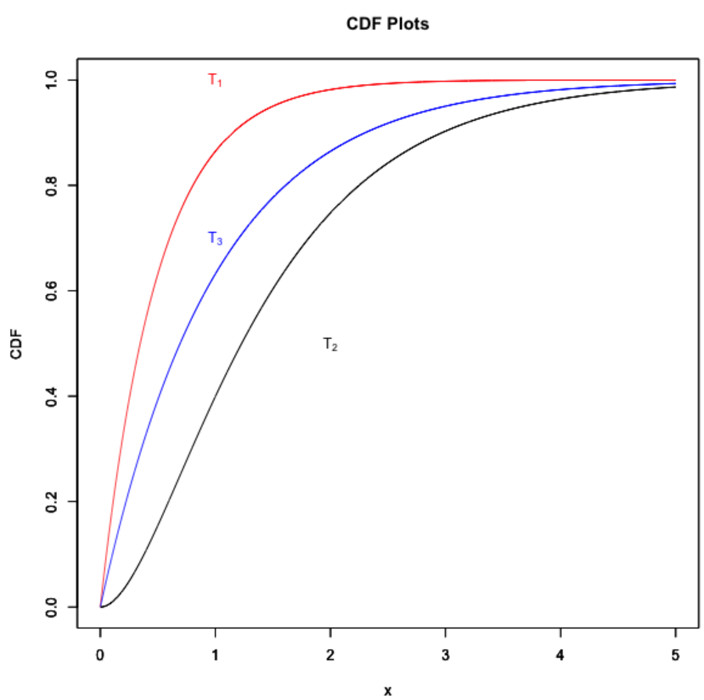


Problem 2.

(a) Let X be the time until Pat receives a call, Y be the time until Robbie receives a call. If we assume that the customer has no preference between the two restaurant and order pizzas independently, let the time until one of them receives a call be T_1. $P(0 < T_1 < t) = 1 - P(X > t, Y > t)$
Since X,Y are independent by assumption, $P(X > t, Y > t) = P(X > t) \times P(Y > t) = \int_t^\infty \lambda e^{-\lambda x} dx \int_t^\infty \lambda e^{-\lambda y} dy = (-e^{-\lambda x}|_t^\infty)^2 = e^{-2\lambda t}$
$P(0 < T_1 < t) = 1 - P(X > t, Y > t) = 1 - e^{-2\lambda t}$

(b) Making the same assumption as part (a), let T_2 be the time until both of them have received a call. If both of them received the call until t, the possibility should be X received at any time within [0,t], and Y received at any time within [x,t] as well as the symmetric case of Y received first.
$P(T_2 < t) = 2 \times \int_0^t \int_x^t \lambda e^{-\lambda x} \lambda e^{-\lambda y} dy dx = 2 \int_0^t \lambda e^{-\lambda x}(-e^{-\lambda y}|_x^t) dx = 2 \int_0^t \lambda e^{-2\lambda x} - \lambda e^{-\lambda t} e^{-\lambda x} dx$
$= 2(-\frac{e^{-2\lambda x}}{2}|_0^t + e^{-\lambda t} e^{-\lambda x}|_0^t) = e^{-2\lambda t} - 2e^{-\lambda t} + 1$

(c)The result is reasonable. Let the individual waiting time distribution be T_3. At any given time, the probability of at least one of them receive a call should be greater than that of both of them receive a call. The corresponding CDF plots are identical to this fact, in particular T_1 lies above T_2. The event Pat receives a call at time t is contained in the event that at least one of them receive a call. Meanwhile, it contain the event that both of them receive call. In the plot we observed that $T_2(t) < T_3(t) < T_1(t), \forall t > 0$, this is identical to the fact.



Problem 3.

(a) Let X be a Binomial(n,$\frac{1}{2}$), and X_k be the events that X = k. Since $X_i's$ are disjoint events and $\bigcup X_k$ is the whole sample space defined by X. By the axiom of probability: $P(\bigcup_{k=0}^{n} X_k) = \sum_{k=0}^{n} P(X_k) = 1$

$\sum_{k=0}^{n} P(X_k) = \sum_{k=0}^{n} \binom{n}{k}(\frac{1}{2})^k(\frac{1}{2})^{n-k} = \frac{\sum_{k=0}^{n}\binom{n}{k}}{2^n} = 1$

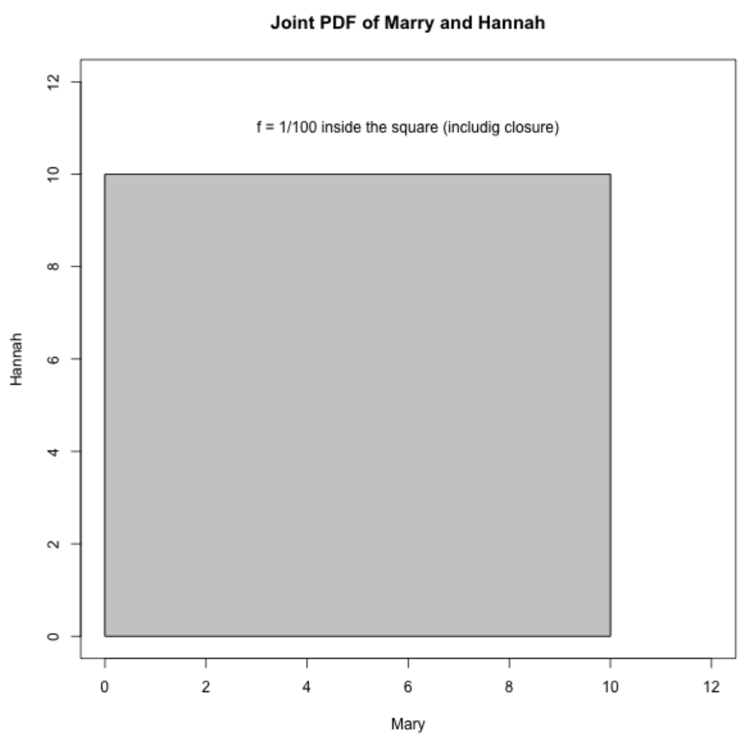
Therefore, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

(b)Let N be the number of mechanical problem that the two people will encounter per month. We assume that the cars encounter problems independently.

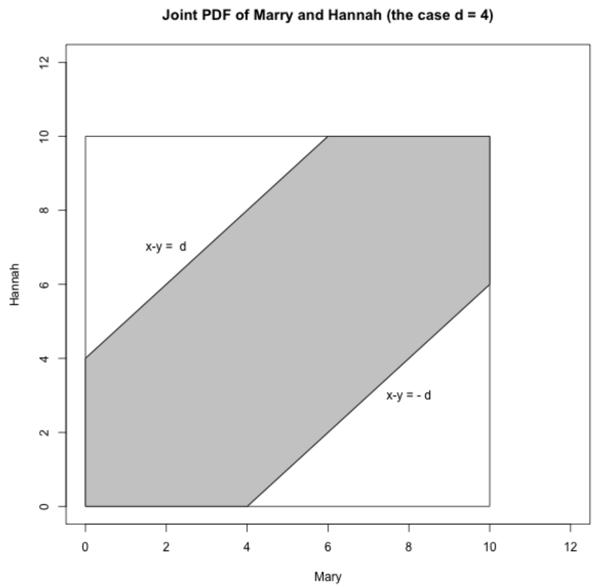
$P(N = n) = \sum_{k=0}^{n} \frac{\lambda^k}{k!}e^{-\lambda}\frac{\lambda^{n-k}}{(n-k)!}e^{-\lambda} = e^{-2\lambda}\sum_{k=0}^{n}\frac{\lambda^n}{k!(n-k)!} = e^{-2\lambda}\frac{\lambda^n}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} = e^{-2\lambda}\frac{\lambda^n}{n!}2^n = e^{-2\lambda}\frac{(2\lambda)^n}{n!}$

Problem 4.

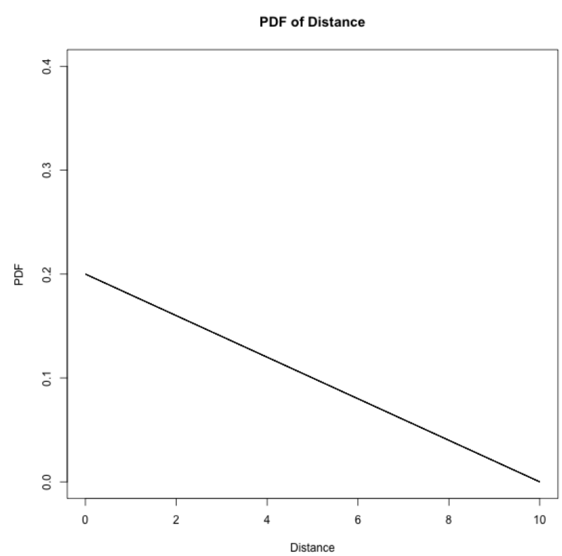
(a) Let X, Y be the positions of Mary and Hannah respectively. They are independently and uniformly distributed over [0,10]$\times[0,10]$. The joint pdf $f_{X,Y} = \frac{1}{100}$ for $(X,Y) \in [0,10]\times[0,10]$ and 0 otherwise.



(b)



(c) Let the distance be D. $P(D \in [0,d])$ is just the percentage of the shaded region for any give d.
$P(D \in [0,d]) = \frac{100-(10-d)^2}{100} = \frac{20d-d^2}{100}$
the pdf is $f(d) = \frac{20-2d}{100}$



Problem 5.

(a) Let C be the bias of the coin, C $\sim$ Unif($\frac{1}{2}$,1), and D be the outcome of the coin flip. The parameter of the distribution of outcome depends on the bias. In Particular, $P(D = heads|C = c) = c$
$P(D = heads) = \int_{0.5}^{1} P(D = heads|C = c)f(c)dc = \int_{0.5}^{1} c \times 2dc = 1 - \frac{1}{4} = \frac{3}{4}$
$P(D = tails) = \frac{1}{4}$

(b) $f(c|D=heads) = \frac{f_c(c)P_{D|C}(D=heads|C=c)}{P_D(D=heads)} = \frac{2c}{3/4} = \frac{8c}{3}$

$f(c|D=tails) = \frac{f_c(c)P_{D|C}(D=tailss|C=c)}{P_D(D=tails)} = \frac{2(1-c)}{1/4} = 8(1-c)$

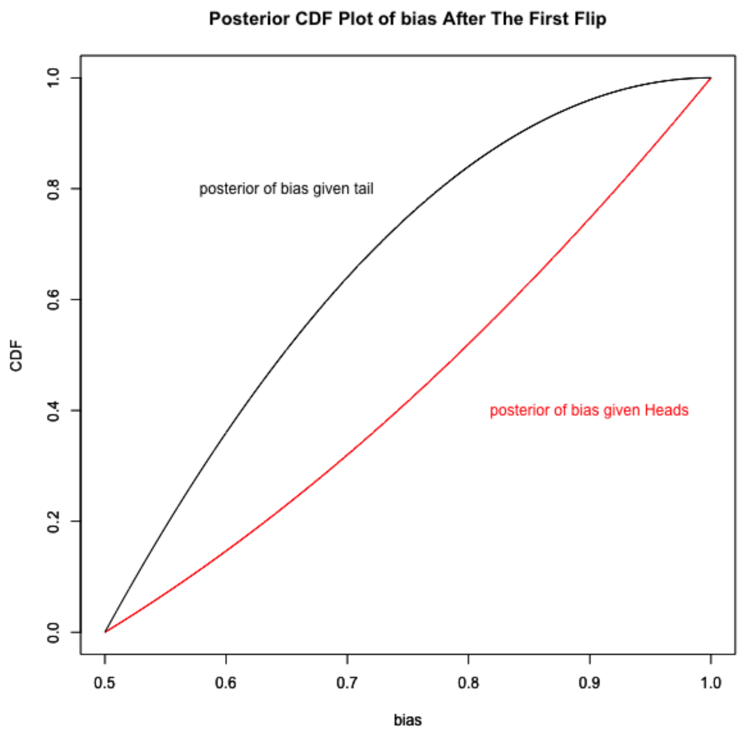
The posterior of bias given getting a head is $\frac{8c}{3}$, Thus the cdf is $\frac{8}{6}c^2 - \frac{1}{3}, 0.5 \le c \le 1$

The posterior of bias given getting a tail is $8-8c$, Thus the cdf is $8c-4c^2-3, 0.5 \le c \le 1$

If we continue the flip:

i) For D = heads, $f_{i+1}(c|D=heads) = \frac{f_{c,i}(c)P_{D|C}(D=heads|C=c)}{P_D(D=heads)} = \frac{f_{c,i}(c)c}{\int_{0.5}^{1} cf_{c,i}(c)dc}$

ii) For D = tails, $f_{i+1}(c|D=tails) = \frac{f_{c,i}(c)P_{D|,i=C}(D=tailss|C=c)}{P_D(D=tails)} = \frac{f_{c,i}(c)(1-c)}{\int_{0.5}^{1} f_{c,i}(c)(1-c)dc}$



This is reasonable, because if a head is observed the bias is more likely to be distributed close to 1.Therefore the CDF of the posterior is convex. If a tail is observed, the bias is more likely to be distributed close to 0.5. Thus the CDF is concave.