Homework 11 DS-GA 1002

Yuhao Zhao
Yz3085

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Problem 1.

a) $f := \sum_{i=1}^{n} a_i f_i$ and $f_i s$ are convex functions.
for $\forall x, y \in R$ and $\theta \in [0,1]$, $\theta f(x) + (1-\theta) f(y) = \theta \sum_{i=1}^{n} a_i f_i(x) + (1-\theta) \sum_{i=1}^{n} a_i f_i(y)$
$= \sum_{i=1}^{n} a_i (\theta f_i(x) + (1-\theta) f_i(y))$
Since $f_i s$ are convex, $\theta f_i(x) + (1-\theta) f_i(y) \geq f_i(\theta x + (1-\theta) y)$
if $a_i' s$ are non-negative, $a_i(\theta f_i(x) + (1-\theta) f_i(y)) \geq a_i f_i(\theta x + (1-\theta) y)$
Therefore, $RHS \geq \sum_{i=1}^{n} a_i f_i(\theta x + (1-\theta) y) = f(\theta x + (1-\theta) y)$
Thus, f is convex.

b) $g = \max_{1 \leq i \leq m} f_i(x)$
for $\forall x, y \in R$ and $\theta \in [0,1]$, $\theta g(x) + (1-\theta) g(y) = \max \theta f_i(x) + \max (1-\theta) f_i(y)$
We claim that $\sup(f+g) \leq \sup(g) + \sup(f)$
since $f \leq \sup f, g \leq \sup g, f + g \leq \sup(f) + \sup(g)$
$\sup(f+g) \leq \sup(\sup(f) + \sup(g)) = \sup(f) + \sup(g))$
Therefore, $\max \theta f_i(x) + \max(1-\theta) f_i(y) \geq \max \theta f_i(x) + (1-\theta) f_i(y) = g(\theta x + (1-\theta) y)$
Thus, g is convex

c) $h := \prod_{i=1}^{m} f_i$
counterexample: $f_1 = 1 - x$, $f_2 = 1 + x$
f_1, f_2 are convex since $f_1'' \equiv 0, f_2'' \equiv 0$.
$h = f_1 f_2 = 1 - x^2$, $h'' = -2 < 0$
h is not convex.

Problem 2.

Code:

```
def derivative_descent(x_ini, f, der_f, step, eps):
        if abs(der_f(x_ini)) > eps:
```

```
        return [x_ini] + derivative_descent(x_ini-step*der_f(x_ini),f,der_f,step,eps)
        else:
        return [x_ini]


def newton_method(x_ini, f, der_f, der2_f, eps):
        if abs(der_f(x_ini)) > eps:
        return [x_ini] + newton_method(x_ini- (der_f(x_ini)/der2_f(x_ini))
                                    ,f, der_f, der2_f, eps)
        else:
        return [x_ini]

def quadratic_approx(x, point, f, df, d2f):
        return f(point)+df(point)*(x- point) +0.5*d2f(point)*(x-point)**2
```

Problem 3.

a) The projection of a vector $x \in R^n$ on the positive orthant R^n_+ is :
1. x, if x $\in R^n_+$
2. The closest point $x^* \in R^n_+$ that minimize the distance between x and x^* which is:
$\min\limits_{x^* \in R^n_+} ||x - x^*|| = \min\limits_{x^* \in R^n_+} \sum(x_i - x^*_i)^2$
The solution is equivalent to minimize the difference in each coordinates. The projection is indeed
$x^*_i = I(x_i \geq 0)x_i$

b) The projection of a vector $x \in R^n$ on the unit l_2 ball B_{l_2} is :
1. x, if x $\in B_{l_2}$
2. The closest point $x^* \in B_{l_2}$ that minimize the distance between x and x^*
We claim that $x^* \in span(x)$:
If not, let the projection $x^* \notin span(x)$
by triangular inequality $||x^*|| + ||x - x^*|| \geq ||x||$, we know that $||x^*|| \leq 1$
we can find a vector $\tilde{x} \in span(x)$ and $||\tilde{x}|| = ||x^*||, \tilde{x} \in B_{l_2}$
Then $||x - \tilde{x}|| = ||x|| - ||\tilde{x}|| \leq ||x^*|| + ||x - x^*|| - ||\tilde{x}|| = ||x - x^*||$
Then x^* is not he closet vector to x in B_{l_2} which is a contradiction

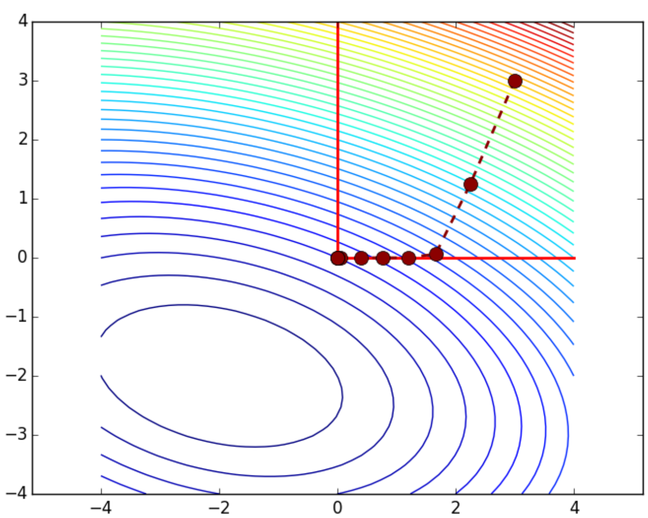
We claim also that $x^* \in \partial B_{l_2}$(Boundary of the set):
if not, let the projection $\tilde{x} \in B^\circ_{l_2}$(Interior of the set) and $\tilde{x} \in span(x)$
$||\tilde{x}|| \leq 1$, we can find $x^* \in span(x)$, and $||x^*|| = 1$
$||x - x^*|| \leq ||x - \tilde{x}||$ This is a contradiction.
Therefore, solution is $\frac{x}{||x||_2}$ if $x \notin B_{l_2}$ and x if $x \in B_{l_2}$

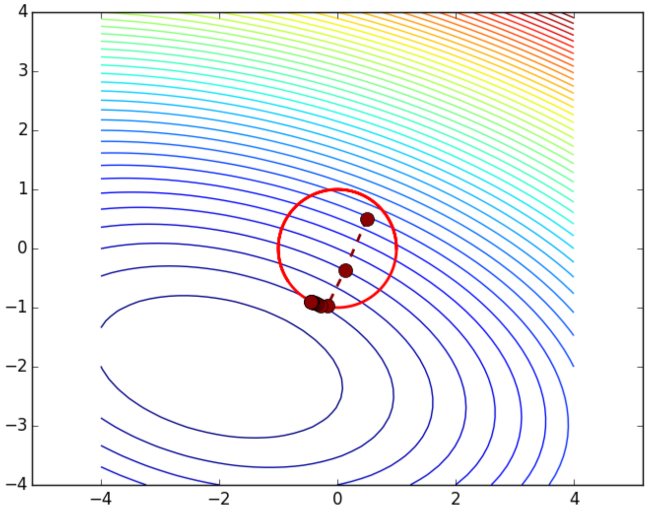
c) Code:

```
def projected_gradient_descent(x_ini, f, grad, step, eps, proj, n_iter):
    curr = proj(x_ini)
    res = np.matrix(curr)
    n = 0
    if np.dot(grad(curr),grad(curr)) <= eps:
           return res
    else:
           while  np.dot(grad(curr),grad(curr)) > eps:
           curr = proj(curr - step*grad(curr))
           res = np.vstack((res,curr))
           n += 1
           if n==10:
                return res

def projection_positive(x):
    dim = len(x)
    return np.array((x>=0)* x)

def projection_l2(x):
    norm = np.sqrt(np.dot(x,x))
    return x/norm if norm>1 else x
```





d) The projection positive will converge to (0,0) since it's the closet solution in the positive orthant to the global solution (it lies on the smallest contour line). The projection l2 will converge to $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$