Homework 1 DS-GA 1002

Yuhao Zhao
N17578783

September 21, 2015

Problem 1.

(a). If each of the friends decided to go independently, the number of people come can be modeled by a Binomial distribution, i.e Bin(5,0.1).
Therefore, the probability of three people come is $\binom{5}{3} \times 0.1^3 \times 0.9^2 \approx 0.0081$

b) Let S_i be the event that the friend i comes. The probability of no one comes is $1-P(U_i S_i)$, By the Union Bound theorem, $P(U_i S_i) \leq \sum_i P(S_i) = 0.1 \times 5 = 0.5$, $1 - P(U_i S_i) \geq 1 - 0.5 = 0.5$. Thus the lower bound is 0.5

Problem 2.

(a) Let X be the event that Joe has the disease, and Y be the event that the test is positive. In particular, X follows a Bernoulli distribution.
P(Y) = P(someone has the disease, test positive)+P(no one has the disease, test positive)
= (1- $\binom{10}{10}0.8^{10})0.9$ + $0.1\binom{10}{10}0.8^{10} \approx 0.8141$
Since P$(X|Y)$ × P(Y) = $P(Y|X) \times P(X)$
P$(X|Y) = \frac{P(Y|X)\times P(X)}{P(Y)} = \frac{0.9\times 0.2}{0.8141} \approx 0.2211$

(b) Let A be the event that test is positive, B be the event that Joe has the disease, and C be the event that fridge is broken. We need to find out the relation of $P(A, B|C)$ with $P(A|C) \times P(B|C)$
$P(A, B|C) = \frac{P(A,B,C)}{P(C)} = \frac{P(C)P(B|C)P(A|B,C)}{P(C)} = P(B|C)P(A|B, C)$
Thus, we just need to compare $P(A|C)$ and $P(A|B, C)$ $P(A|C) = 1, P(A|B, C) = 1$ Since $P(A|B, C) = P(A|C)$, the event A is independent of B given C.

(c) Define event A and C as part b.
$P(C|A) = \frac{P(A|C)P(C)}{P(A)} = \frac{1\times 0.4}{0.6\times 0.8141+0.4\times 1} \approx 0.4502$

Problem 3.

(a) The probability of breaking down for the first time in the k'th drive is $(\frac{3}{4})^{k'-1} \times \frac{1}{4} = \frac{3^{k'-1}}{4^{k'}}$
Let breaking down first time in the k'+k drive be event A. $P(A|E) = \frac{P(A,E)}{P(E)}$.
Since the break at t_i is independent from the break at t_j,$E \subset A$, $P(A,E) = P(A) = (\frac{3}{4})^{k+k'-1} \times \frac{1}{4}$, $\frac{P(A,E)}{P(E)} = \frac{(\frac{3}{4})^{k+k'-1} \times \frac{1}{4}}{(\frac{3}{4})^{k}} = \frac{3^{k'-1}}{4^{k'}}$
Therefore, the probability that the car breaks down in the k'th drive is equal to the probability of that it breaks down in th (k+k')th drive given E.
This implies that the distribution of waiting time until the car breaks is memoryless,i.e $P(T \leq t) = P(T \leq t_0 + t | T > t_0)$,

(b) P $= \binom{k}{n} \times (\frac{1}{4})^n \times (\frac{3}{4})^{k-n}$

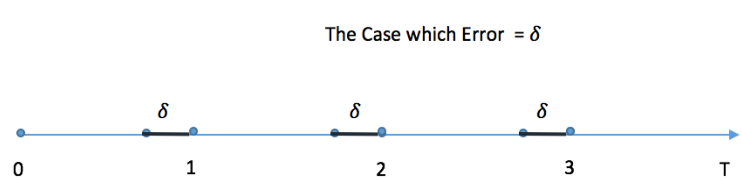
(c) P $= \binom{k-1}{n-1} \times (\frac{1}{4})^{n-1} \times (\frac{3}{4})^{k-n} \times \frac{1}{4}$

(d) The probability of breaking down first time in the k'th drive is :
$\prod_1^{k'-1} 2^{-n} \times (1-2^{-k'}) = (1-2^{-k'}) \times 2^{-\frac{k'(k'-1)}{2}} \quad eqn.1$
The probability of breaking down first time in the k+k' the drive given E is:
$\frac{\prod_1^{k'+k-1} 2^{-n} \times (1-2^{-k'-k})}{\prod_1^{k} 2^{-n}} = \frac{2^{\frac{-(k'+k)(k'+k-1)}{2}} \times (1-2^{-k'-k})}{2^{\frac{-(1+k)k}{2}}} \quad eqn.2$
We notice that in eqn.2 there is a cross term $k'k$, while in eqn.1 there is no k terms. eqn.2 and eqn.1 are not equal. This means that the distribution of waiting time until the car breaks under this model is not memoryless. This model is more realistic, since the car is more probable to break with longer driving time

Problem 4.

(a) Let T be the number of hours it takes to get a reading indicating the particle has decayed, and X be the actual time when the particle decayed.
$P(T=t) = P(X \in [t-1, t]) = \int_{t-1}^{t} \lambda e^{-\lambda x} dx = -e^{-\lambda x}|_{t-1}^{t} = e^{-\lambda(t-1)} - e^{-\lambda t}$

(b) The distribution of error $D, P(D \in [0, \delta])$ is just the sum of all probability that the decay happened δ hours ($\delta < 1$) prior to any integer time T :



$P(D \in [0,\delta]) = \sum_{n=1}^{\infty} \int_{n-\delta}^{n} \lambda e^{-\lambda x} dx = \sum_{n=1}^{\infty} (-e^{-\lambda x}|_{n-\delta}^{n}) = \sum_{n=1}^{\infty} (e^{-\lambda(n-\delta)} - e^{-n\lambda})$
$= \sum_{n=1}^{\infty} (e^{-\lambda n}(e^{\lambda\delta} - 1)) = (e^{\lambda\delta} - 1) \sum_{n=1}^{\infty} (e^{-\lambda})^n$
Since for exponential distribution $\lambda > 0, e^{-\lambda} < 1$. Thus $\sum_{n=1}^{\infty}(e^{-\lambda})^n$ is summable which is $\lim_{n\to\infty} (e^{-\lambda}) \frac{1-e^{-n\lambda}}{1-e^{-\lambda}} = \frac{e^{-\lambda}}{1-e^{-\lambda}}$
Therefore, $P(D \in [0,\delta]) = \frac{e^{-\lambda}}{1-e^{-\lambda}}(e^{\lambda\delta} - 1)$
The pdf of E is the derivative of cdf which is : $\frac{\lambda e^{-\lambda(1-\delta)}}{1-e^{-\lambda}}$

Problem 5.

(a) The CDF of W is $P(W < w) = P(F_Y(Y) < w)$
Since F_Y is invertible, $P(F_Y(Y) < w) = P(Y < F_Y^{-1}(w))$, this is , by definition of CDF, $F(F_Y^{-1}(w)) = w$.
Therefore we have $P(W < w) = w$

(b) Let $T = F_X^{-1}(W)$, the CDF of T is $P(T < t) = P(F_X^{-1}(W) < t) = P(W < F_X(t))$. Since we know the cdf of W is w. $P(W < F_X(t)) = F_X(t)$. The CDF of T and CDF of X are identical.

(c) Since F_Y is a CDF, it's non-decreasing, and right continuous. If F_Y is not invertible, that means for a fixed constant w all the values of y such that $F_Y(y) =$ w belong to a closed interval [a(w), b(w)], F_Y is flat in this interval. We can still do the above even though F_Y is not invertible. In particular, we can first omit the interval where F_Y is not invertible and find the inverse. In the interval [a(w),b(w)], the inverse function is just a discontinuous jump from a(w) to b(w).