Problem Session 2

1. *Triangle Island.* Example 2.8 from lecture notes 2.

2. *Sums.* Consider two random variables X, Y, and a function of the two $Y = f(X,Y) = X + Y$, namely their sum.
   a. Express the density of Z as a summation when both X and Y are discrete. Hint: express $X + Y = z$ as an intersection of two sets concerning X and Y separately.
   b. Express the density of Z as an integral when both X and Y are continuous. What happens when X and Y are independent?
   c. If X and Y are independent uniform on $[0,1]$ find the density in two ways by calculating the convolution formula found above and by drawing a diagram of (X,Y) over the unit square and considering $\mathrm{P}(Z \in dz)$.

3. *Discrete uniform - binomial.* Suppose there are $N+1$ boxes labeled by $b = 0,1,2,...,N$.Box b contains b black and $N-b$ white balls. A box is picked uniformly at random, and then n balls are drawn at random with replacement from whatever box is picked (the same box for each of the n draws). Let S_n denote the total number of black balls that appear among the n balls drawn.
   a. Find the distribution of S_n.
   b. For a fixed value of n, find the limiting distribution of S_n, the number of black balls that appear in n draws, as the number of boxes N tends to ∞.

4. *Continuous uniform - binomial.* Suppose U is uniformly distributed over $[0,1]$. Given $U = p$ let X_n denote the number of successes in Binomial(n,p).
   a. Find the distribution of X_n.
   b. Find the conditional distribution of U given $X_n = k$.
   c. Given n trials produced k successes, what is the probability that the next trial is a success?

5. *Sampling from the unit disc.* The unit disc is given by $\{(x,y)|x^2 + y^2 \leq 1\}$. Let (X,Y) be a point randomly chosen from the unit disc.
   a. Calculate the cdf and the pdf of X.
   b. Calculate the pdf of the radius: $R = \sqrt{X^2 + Y^2}$.

6. *Maximum of independent random variables.* Given a collection of n independent random variables: $(X_1, \ldots, X_n)$. Let $X_{\max} = \max(X_1, \ldots, X_n)$
   a. Calculate the cdf of $X_{\max}$.
   b. If all X_i has exponential distribution with rate λ_i, find the distribution of $X_{\min}$.