Homework 2 Solutions

1. (10 points) *Spider on a wall*

a. Let X be the horizontal position and Y be the vertical position, or height. We have the following two equations:

$$\mathrm{P}\left(\text{behind the paint}\right) + \mathrm{P}\left(\text{not behind the paint}\right) = 1 \tag{1}$$
$$\mathrm{P}\left(\text{behind the paint}\right) = 2\mathrm{P}\left(\text{not behind the paint}\right) \tag{2}$$

P (not behind the paint) and P (behind the paint) are uniform, let c_1 and c_2 be their densities, resp.:

$$4c_1 = 2 \cdot 96c_2 \quad 4c_1 + 96c_2 = 1 \implies c_1 = 1/6, c_2 = 1/288 \tag{3}$$
$$f_{(X,Y)}(x, y) = \begin{cases} 1/6 \text{ if } (x, y) \in \text{paint} \\ 1/288 \text{ if } (x, y) \notin \text{paint} \end{cases} \tag{4}$$

b. This is just the marginal distribution of Y, it is supported on $(0, 10)$:

$$f_Y(y) = \int f_{(X,Y)}(x, y)dx = \begin{cases} \int_0^{10} \frac{1}{288}dx & = 5/144 \text{ if } y \notin (6, 8) \\ \int_{(0,4)\cup(6,10)} \frac{1}{288}dx + \int_{(4,6)} 1/6dx & = 13/36 \text{ if } y \in (6, 8) \end{cases} \tag{5}$$

c. Given that the spider is not under the paint there are three cases, its height is either below the paint, above the paint or at the paint level:

$$F_{Y|(X,Y)\notin(4,6)\times(6,8)}(y) = \mathrm{P}\left(Y < y|(X, Y) \notin (4, 6) \times (6, 8)\right) \tag{6}$$
$$= \frac{\mathrm{P}\left(Y < y, (X, Y) \notin (4, 6) \times (6, 8)\right)}{\mathrm{P}\left((X, Y) \notin (4, 6) \times (6, 8)\right)} \tag{7}$$
$$= \begin{cases} 30y/288 & \text{if } y \in (0, 6) \\ 180/288 + 24(y - 6)/288 & \text{if } y \in (6, 8) \\ 228/288 + 30(y - 8)/288 & \text{if } y \in (8, 10) \end{cases} \tag{8}$$

2. (10 points) *Pizza delivery*

a. If P and R represent the time until next call for those pizzerias then $\min(P, R)$ is the time until one of them receives a call. Assume P and R are independent. For $s > 0$:

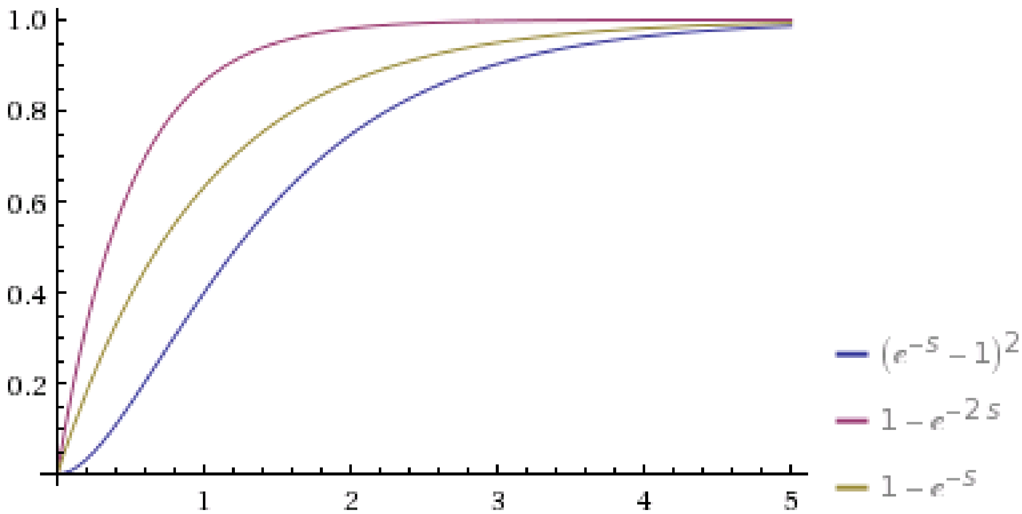
$$F_{\min}(s) = \mathrm{P}(\min(P, R) < s) \tag{9}$$
$$= \mathrm{P}\left(P < s \text{ or } R < s\right) \tag{10}$$
$$= 1 - \mathrm{P}\left(P > s \text{ and } R > s\right) \tag{11}$$
$$= 1 - \mathrm{P}\left(P > s\right)\mathrm{P}\left(R > s\right) \tag{12}$$
$$= 1 - [1 - F_P(s)][1 - F_R(s)] \tag{13}$$
$$= 1 - [1 - (1 - e^{-\lambda s})][1 - (1 - e^{-\lambda s})] \tag{14}$$
$$= 1 - e^{-2\lambda s} \tag{15}$$

Therefore the distribution of the minimum is exponential with parameter 2λ.

b. $\max(P,R)$ is the time until both of them receives a call. For $s > 0$:

$$\begin{aligned} F_{\max}(s) &= \mathrm{P}(\max(P,R) < s) &(16)\\ &= \mathrm{P}(P < s \text{ and } R < s) &(17)\\ &= \mathrm{P}(P < s)\,\mathrm{P}(R < s) &(18)\\ &= F_P(s)F_R(s) &(19)\\ &= (1 - e^{-\lambda s})^2 &(20)\\ & &(21)\end{aligned}$$

c. For $\lambda = 1$ the plots are given below. It makes sense as $\min(R,P) \leq R \leq \max(R,P)$ and $\min(R,P) \leq P \leq \max(R,P)$.



3. (10 points) *Cars*

a. The pmf of a binomial adds up to one, so

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^2 = 1. \tag{22}$$

Setting $p = 1/2$ we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n} = 1, \tag{23}$$

which proves the result.

b. Let us define the random variables T (number of total problems in a month), N_1 (number of problems I have) and N_2 (number of problems my sister has). To find the pmf of T we compute the probability of $T = n$. There are $n+1$ different ways to have n total mechanical problems. You can have k and your sister $n-k$, for $k = 0, 1, \ldots, n$. These are disjoint events so we can add them up.

$$p_T(n) = \sum_{k=0}^{n} \mathrm{P}(\text{I have } k \text{ problems} \cap \text{Sister has } n-k \text{ problems}) \tag{24}$$

$$= \sum_{k=0}^{n} \mathrm{P}(\text{I have } k \text{ problems})\,\mathrm{P}(\text{Sister has } n-k \text{ problems}) \quad \text{(by independence)} \tag{25}$$

$$= \sum_{k=0}^{n} p_{N_1}(k)\, p_{N_2}(n-k) \tag{26}$$

$$= \sum_{k=0}^{n} \frac{\lambda^k e^{-\lambda}}{k!} \frac{\lambda^{n-k} e^{-\lambda}}{(n-k)!} \tag{27}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{\lambda^n e^{-2\lambda n}}{n!} \tag{28}$$

$$= \frac{(2\lambda)^n\, e^{-2\lambda n}}{n!} \quad \text{using the result from (a).} \tag{29}$$

The total number of mechanical problems is distributed as Poisson random variable with parameter 2λ.

4. (10 points) *Race*

a. See Figure 1.

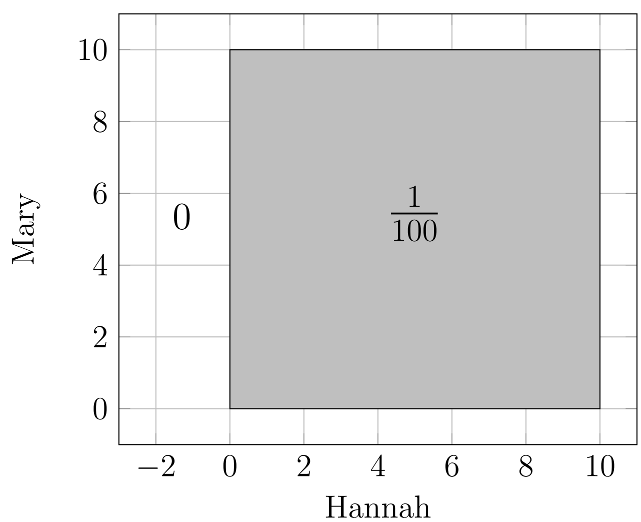


Figure 1: Joint pdf of the positions of Mary and Hannah.

b. See Figure 2.

c. Let us define the random variable $D = H - M$ where H = Hannah and M = Mary. For $d \leq 0$,

$$F_D(d) = \mathrm{P}(D \leq d) = 0. \tag{30}$$

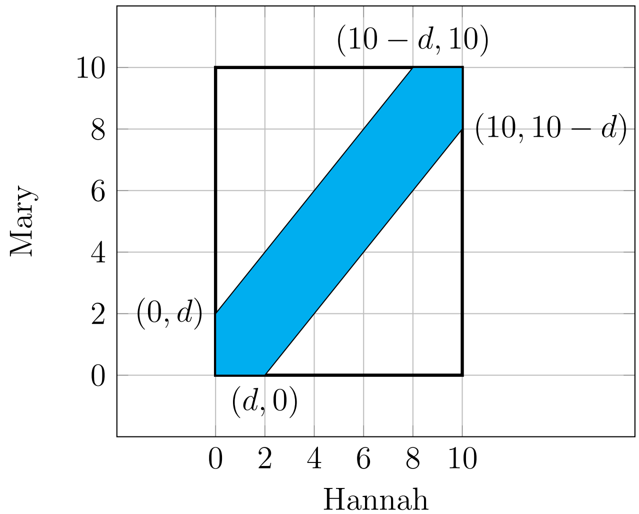


Figure 2: Area representing the event *the distance between them is smaller than d.*

For $0 < d \leq 10$

$$\begin{aligned}
F_D(d) &= \mathrm{P}(D \leq d) && (31)\\
&= \int_{h=0}^{d} \int_{m=0}^{h+d} f_{H,M}(h,m)\, \mathrm{d}m\, \mathrm{d}h + \int_{h=d}^{10-d} \int_{m=h-d}^{h+d} f_{H,M}(h,m)\, \mathrm{d}m\, \mathrm{d}h && (32)\\
&\quad + \int_{h=10-d}^{10} \int_{m=h-d}^{10} f_{H,M}(h,m)\, \mathrm{d}m\, \mathrm{d}h && (33)\\
&= \frac{1}{100}\left(\int_{h=0}^{d} (h+d)\, \mathrm{d}h + \int_{h=d}^{10-d} 2d\, \mathrm{d}h + \int_{h=10-d}^{10} (10-h+d)\, \mathrm{d}h\right) && (34)\\
&= \frac{1}{100}\left(\frac{d^2}{2} + d^2 + 2d(10-2d) + d(10+d) - \frac{100-(10-d)^2}{2}\right) && (35)\\
&= \frac{20d - d^2}{100}. && (36)
\end{aligned}$$

For $d > 10$,

$$F_D(d) = \mathrm{P}(D \leq d) = 1. \tag{37}$$

The pdf is obtained by differentiating the cdf:

$$f_D(d) = \begin{cases} 0 & \text{if } d < 0, \\ \frac{10-d}{50} & \text{if } d < 0, \\ 0 & \text{if } d > 10. \end{cases} \tag{38}$$

The plot is shown in Figure 3

5. (10 points) *Cheating at coin flips*

a. The model makes sense because you think that Marvin is using a coin that is more prone to be heads than tails, but you are uncertain about the exact bias so you just assume it

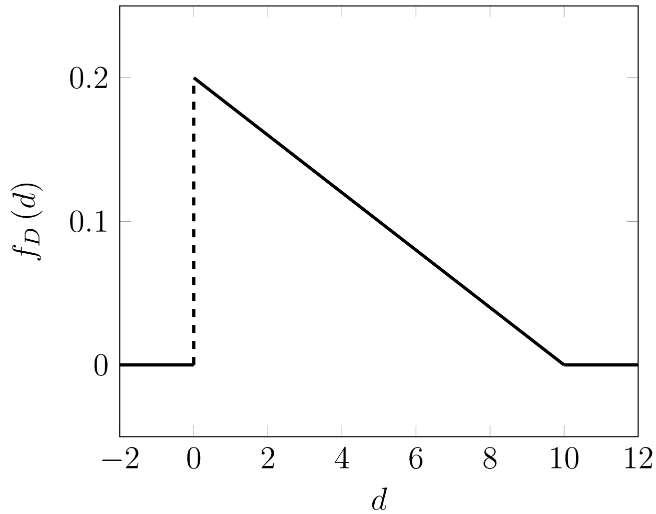


Figure 3: Probability density function of D.

is uniform between 1/2 (fair coin) and 1 (coin that always lands heads). To compute the probability of heads, we integrate over all the possible values of the Bernouilli parameter. Let C be the outcome of the coin flip and B the parameter of the Bernouilli. Since C is uniform between 1/2 and 1 $f_B(b)$ is equal to 2 for $1/2 \leq b \leq 1$ and zero otherwise.

$$P(\text{heads}) = p_C(1) = \int_{b=1/2}^{1} f_B(b)\, p_{C|B}(1|b)\, \mathrm{d}b = \int_{b=1/2}^{1} 2\, b\, \mathrm{d}b = 1 - \frac{1}{4} = \frac{3}{4}, \tag{39}$$

$$P(\text{tails}) = 1 - P(\text{heads}) = \frac{1}{4}. \tag{40}$$

b. If the coin flip is tails, then the conditional pdf on the bias of the coin flip conditioned on this equals

$$f_{B|C}(b|0) = \frac{f_B(b)\, p_{C|B}(0)}{p_C(0)} \tag{41}$$

$$= \frac{f_B(b)(1-b)}{1/4} = \begin{cases} 8(1-b) & 1/2 \leq b \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

Note that the conditional pdf is more skewed towards 1/2, as shown on Figure 4. It makes sense because the coin flip is tails. Intuitively the model should be adjusted towards the coin flip being less biased.

If the coin flip is heads, then the conditional pdf on the bias of the coin flip conditioned on this equals

$$f_{B|C}(b|1) = \frac{f_B(b)\, p_{C|B}(1)}{p_C(1)} \tag{43}$$

$$= \frac{f_B(b)\, b}{3/4} = \begin{cases} \frac{8\,b}{3} & 1/2 \leq b \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

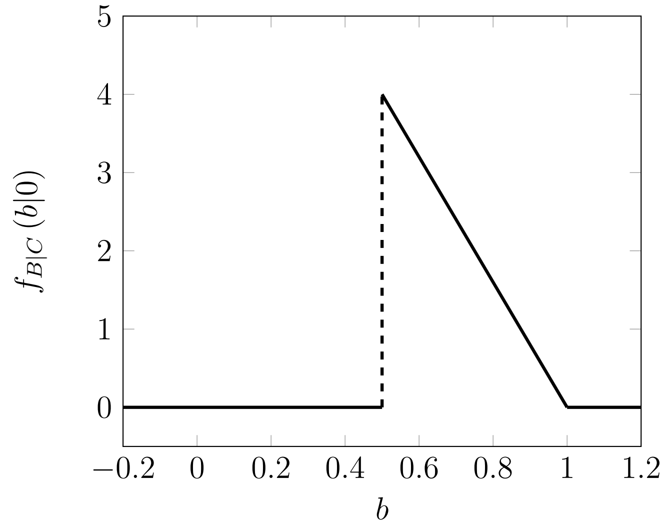


Figure 4: Conditional pdf of the bias of the coin flip given tails.

In this case the conditional pdf is more skewed towards $1/2$, as shown on Figure 5. Intuitively, you are adjusting your model by incorporating some evidence that supports that the coin might be biased towards heads.

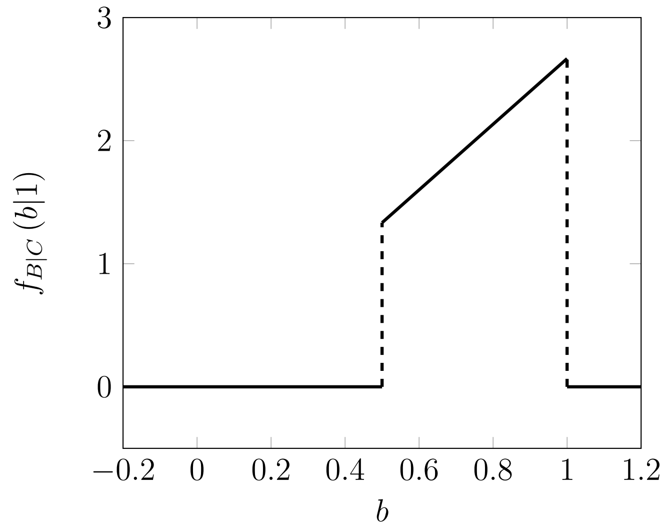


Figure 5: Conditional pdf of the bias of the coin flip given tails.