Homework 1 Solutions

1. (2 points) *Stranded.*

a. The number of friends that decide to come is a binomial with parameters $n = 5$ and $p = 0.1$. So using the expression for the pmf of a binomial:

$$\mathrm{P}\left(3 \text{ friends come}\right) = \binom{5}{3} 0.1^3\, 0.9^2 = 0.81\%. \tag{1}$$

b. Since we just know the probability of the individual events, but nothing about their intersections, we use the union bound. Let S_i be the event that friend i comes:

$$\begin{aligned}\mathrm{P}\left(\text{stranded}\right) &= \mathrm{P}\left(\cap_{i=1}^{5} S_i^c\right) = 1 - \mathrm{P}\left(\left(\cap_{i=1}^{5} S_i^c\right)^c\right) = 1 - \mathrm{P}\left(\cup_{i=1}^{5} S_i\right) \quad \text{by De Morgan's laws} \\ &\geq 1 - \sum_{i=1}^{5} \mathrm{P}\left(S_i\right) = \frac{1}{2}. \end{aligned} \tag{2}$$

So no matter what, you have a probability of 1/2 of being rescued.

2. (2 points) *Army camp.*

a. Let J be the event that Joe has the disease, $+$ be that the test is positive, N that no soldier has the disease and A that at least one soldier has the disease. Note that $N = A^c$.

$$\mathrm{P}\left(J|+\right) = \frac{\mathrm{P}\left(J,+\right)}{\mathrm{P}\left(+\right)} = \frac{\mathrm{P}\left(J,+\right)}{\mathrm{P}\left(+,N\right) + \mathrm{P}\left(+,A\right)} \tag{3}$$

$$= \frac{\mathrm{P}\left(+|J\right)\mathrm{P}\left(J\right)}{\mathrm{P}\left(+|N\right)\mathrm{P}\left(N\right) + \mathrm{P}\left(+|A\right)\mathrm{P}\left(A\right)} \tag{4}$$

$$= \frac{0.9 \cdot 0.2}{0.1 \cdot 0.8^{10} + 0.9 \cdot (1 - 0.8^{10})} \approx 0.2211 \tag{5}$$

$$\tag{6}$$

Here $\mathrm{P}\left(+|A\right) = \mathrm{P}\left(+|J\right)$ holds because for the purposes of the test what matters is whether the sample has the disease or not. In other words within this context A and J give us the same information.

b. Now let B be the event that the fridge is broken. Under the new measure the event $+$ is the whole sample space, therefore J and $+$ are independent given B.

c. If the fridge is not broken then $\mathrm{P}\left(+|B^c\right)$ is just what we calculated above as $P(+)$ which is 0.8141.

$$\mathrm{P}\left(B|+\right) = \frac{\mathrm{P}\left(B,+\right)}{\mathrm{P}\left(+\right)} = \frac{\mathrm{P}\left(B,+\right)}{\mathrm{P}\left(B,+\right) + \mathrm{P}\left(B^c,+\right)} \tag{7}$$

$$= \frac{\mathrm{P}\left(+|B\right)\mathrm{P}\left(B\right)}{\mathrm{P}\left(+|B\right)\mathrm{P}\left(B\right) + \mathrm{P}\left(+|B^c\right)\mathrm{P}\left(B^c\right)} \tag{8}$$

$$= \frac{1.0 \cdot 0.4}{1.0 \cdot 0.4 + 0.8141 \cdot 0.6} \approx 0.45 \tag{9}$$

3. (2 points) *Old car.*

a. They are the same so the process is memoryless:

$$\mathrm{P}\left(\text{car breaks down in the } k'\text{th drive}\right) = 0.75^{k'-1}0.25 \tag{10}$$

$$\mathrm{P}\left(\text{car breaks down in the } k+k'\text{th drive}|E\right) = \frac{\mathrm{P}\left(\text{car breaks down in the } k+k'\text{th drive}\right)}{\mathrm{P}\left(E\right)} \tag{11}$$

$$= \frac{0.75^{k+k'-1}0.25}{0.75^{k}} = 0.75^{k'-1}0.25 \tag{12}$$

b. n successes in Binomial$(k, 0.25)$ (success is the event that the car breaks)

$$\binom{k}{n}0.75^{k-n}0.25^{n} \tag{13}$$

c. $n-1$ successes in Binomial$(k-1, 0.25)$ times one more success

$$\binom{k-1}{n-1}0.75^{k-n-1}0.25^{n-1}0.25 \tag{14}$$

d. Similar to part a. but no more memoryless.

$$\mathrm{P}\left(\text{breaks in the } k'\text{th drive}\right) = \prod_{i=1}^{k'-1} 2^{-i}(1-2^{k'}) = (1-2^{k'}) \cdot 2^{-\sum_{i=1}^{k'-1} i} \tag{15}$$

$$= (1-2^{k'}) \cdot 2^{-(k'-1)k'/2} \tag{16}$$

$$\mathrm{P}\left(\text{breaks in the } k+k'\text{th drive}|E\right) = \frac{(1-2^{k+k'}) \cdot 2^{-(k+k'-1)(k+k')/2}}{2^{-k(k+1)/2}} \tag{17}$$

$$\tag{18}$$

4. (2 points) *Radioactive decay.*

a. Let D be the time the particle takes to decay. The pmf of the reading $R = \lceil D \rceil$ is a geometric of parameter $1-e^{-\lambda}$,

$$\mathrm{P}\left(R=r\right) = \mathrm{P}\left(r-1 \leq D < r\right) = \int_{r-1}^{r} \lambda\, e^{-\lambda x}\mathrm{d}x = e^{-\lambda(r-1)} - e^{-\lambda r} \tag{19}$$

$$= \left(e^{-\lambda}\right)^{r-1}\left(1-e^{-\lambda}\right) \quad \text{for } r = 1, 2, 3, \ldots \tag{20}$$

You can arrive at the same conclusion by realizing that the probability of the particle having decayed within any interval is $1-e^{-\lambda}$ combined with the fact that the exponential distribution is memoryless.

b. Let E be the error, clearly $0 \leq E \leq 1$. Its cdf is

$$
\begin{aligned}
F_E(x) &= \mathrm{P}(E \leq x) && (21)\\
&= \mathrm{P}(\lceil D \rceil - D \leq x) && (22)\\
&= \mathrm{P}\left(\cup_{i=1}^{\infty} \{i - x \leq D \leq i\}\right) \quad \text{union of disjoint events} && (23)\\
&= \sum_{i=1}^{\infty} \mathrm{P}(i - x \leq D \leq i) && (24)\\
&= \sum_{i=1}^{\infty} \int_{i-x}^{i} e^{-\lambda x} \mathrm{d}x && (25)\\
&= \sum_{i=1}^{\infty} e^{-\lambda(i-x)} - e^{-\lambda i} && (26)\\
&= \left(e^{\lambda x} - 1\right) \sum_{i=1}^{\infty} e^{-\lambda i} && (27)\\
&= \frac{e^{-\lambda}\left(e^{\lambda x} - 1\right)}{1 - e^{-\lambda}} = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}. && (28)
\end{aligned}
$$

Differentiating we obtain

$$
f_E(x) = \begin{cases} \frac{\lambda e^{\lambda x}}{e^{\lambda} - 1} & 0 \leq r \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{29}
$$

5. (2 points) *Generating random variables.*

a. Applying the definition of cdf,

$$
\begin{aligned}
F_W(w) &= \mathrm{P}(W \leq w) && (30)\\
&= \mathrm{P}(F_Y(Y) \leq w). && (31)
\end{aligned}
$$

If F_Y is invertible, then $a \leq b$ is equivalent to $F_Y(a) \leq F_Y(b)$. Let us prove it by showing that each statement implies the other one.

(1) If $a \leq b$ then $F_Y(a) \leq F_Y(b)$ because cdfs are non increasing.

(2) If $F_Y(a) \leq F_Y(b)$ then either $F_Y(a) = F_Y(b)$ which implies $a = b$ because F_Y is invertible or $F_Y(a) < F_Y(b)$ which implies $a \leq b$ because cdfs are non increasing.

As a result, $F_Y(Y) \leq w$ implies $Y \leq F_Y^{-1}(w)$, so

$$
\begin{aligned}
F_W(w) &= \mathrm{P}\left(Y \leq F_Y^{-1}(w)\right) && (32)\\
&= F_Y\left(F_Y^{-1}(w)\right) && (33)\\
&= w, \quad 0 \leq w \leq 1. && (34)
\end{aligned}
$$

It turns out that W is a uniform random variable between 0 and 1.

b. Set $A = F_X^{-1}(W)$,

$$
\begin{aligned}
F_A(a) &= \mathrm{P}(A \leq a) && (35)\\
&= \mathrm{P}\left(F_X^{-1}(W) \leq a\right) && (36)\\
&= \mathrm{P}(W \leq F_X(a)) && (37)\\
&= F_X(a) \quad \text{from (a).} && (38)
\end{aligned}
$$

You have generated a random variable with cdf F_X!

c. Consider the set

$$S_w := \{y \mid F_Y(y) = w\}. \tag{39}$$

S is an interval because F_y is non increasing so there cannot be two values $a < b$ such that $F_Y(a) = F_Y(b)$ and $F_Y(c) \neq F_Y(a)$ for any $a < c < b$. Since it is the pre-image of the closed set $\{w\}$ and F_Y is continuous, then it is a closed interval. In particular, we can define

$$g(w) := \max_y S_w, \tag{40}$$

i.e. the largest element in S_w. Now, recall from (a) that

$$\begin{align} F_W(w) &= \mathrm{P}(W \leq w) \tag{41} \\ &= \mathrm{P}(F_Y(Y) \leq w). \tag{42} \end{align}$$

Now we show that $F_Y(a) \leq w$ is equivalent to $a \leq g(w)$ by showing that each statement implies the other one.

(1) If $F_Y(a) \leq w$ then either $F_Y(a) < w$, which directly implies that $a < g(w)$ because F_Y is non increasing, or $F_Y(a) = w$ which implies $a < g(w)$ by definition of g.

(2) If $a \leq g(w)$ then $F_Y(a) \leq w$ because F_Y is non increasing.

Finally,

$$\begin{align} F_W(w) &= \mathrm{P}(Y \leq g(w)) \tag{43} \\ &= F_Y(g(w)) \tag{44} \\ &= w, \quad 0 \leq w \leq 1. \tag{45} \end{align}$$

So you are OK.