

Sample Midterm

1. *Short questions* Answer with a single sentence or a mathematical statement.
 - a. Can the variance of a random variable $2 \leq X \leq 7$ be greater than 25?
 - b. A random variable has mean 10 and variance 25. Could this be a binomial variable?
 - c. Let $\text{Var}(X + Y) = a$ for two independent random variables X and Y . What is the variance of $2X - 2Y$?
 - d. W, X, Y and Z are independent zero-mean Gaussian random variables. What is $P(4X + 3Y < Z + W)$? (Hint: no integrations are necessary!).
 - e. X and Y are independent random variables. X is a uniform random variable with range $[0, 2]$, Y is an exponential random variable with mean $1/2$. What is $E(Xe^{-Y})$? (Hint: $\int e^x dx = e^x$)
 - f. X is uniform on $[0, 1]$ and $P(A|X = x) = x^2$. What is $P(A)$?
 - g. Let X be a uniform random variable on $\{-1, 0, 1\}$ and let $Y = X^2$. Are X and Y uncorrelated?
 - h. Let X be a uniform random variable on $\{-1, 0, 1\}$ and let $Y = X^2$. Are X and Y independent?
 - i. X_1, X_2, \dots is a sequence of random variables with mean 1 and variance $1/n$. Does the sequence converge?
 - j. X_1, X_2, \dots is a sequence of random variables with mean 1 and variance 1. Does the distribution of $(X_i - 1)/\sqrt{n}$ converge to a Gaussian distribution?
 - k. If a sequence of random variables X_1, X_2, \dots converges to a random variable X in distribution, is X a good estimator for X_n as $n \rightarrow \infty$?
 - l. If X and Y are random variables with marginal pdfs that are nonzero between 1 and 3, what is $E\left(E\left(\frac{1}{f_X(X)}|Y\right)\right)$ equal to?
2. *Monty Hall problem* You're on a game show, and you're given the choice of three doors: behind one door there is a car; behind each of the others there is a goat. When you pick a door, the host will open one of the two remaining doors to reveal a goat (he knows where the car is). You are then allowed to switch doors.
 - a. Assume that the car is placed behind each door with the same probability. Define a probabilistic model for the problem by finding a suitable but small sample space and describing the σ -algebra and the probability measure. Do *not* model your initial decision probabilistically.
 - b. Assuming that you value cars more than goats, should you switch?
 - c. Suppose you decide whether to switch using an unbiased coin flip. If you win the car what is the probability that you switched?
 - d. If you win a car, you are invited to play the game again the day after. What is the expected number of cars that you win?

3. *Family* Assume that whenever a couple has a kid, the kid is female with probability p and has red hair with probability q (we will assume that sex and hair color are independent). The couple decides to have as many kids as necessary until they have two daughters.
 - a. What is the expected number of kids that they will have?
 - b. What is the expected number of red-haired kids that they will have?
 - c. What is the distribution of the number of kids they will have if exactly one of them is red haired? (You do not need to compute any sums. Just write out the expression.)
4. *Late* Peter and Paula agree to meet at a restaurant at noon. If it is raining they each arrive independently at a time that is uniformly distributed in $[11 \text{ am}, 1 \text{ pm}]$. If it's not raining they each arrive independently at a time that is uniformly distributed in $[11:30 \text{ am}, 12:30 \text{ pm}]$. The probability of rain is $1/3$.
 - a. Draw the joint pdf of the time of arrival of Peter and Paula. If it is a piecewise constant function just label each part separately.
 - b. What is the probability that one of them arrives late?
 - c. Both arrive late. What is the probability that it is raining?
 - d. Are the arrivals of Peter and Paula uncorrelated?
 - e. Are the arrivals of Peter and Paula independent?
5. *Overbooking* An airline hires you to calibrate their overbooking policy for an airplane that has 300 seats. Looking at previous data it seems that customers only show up 90% of the time.
 - a. If the company sells n tickets, what is the standard deviation of the number of passengers that show up if each passenger shows up independently?
 - b. How many tickets should be sold so that the probability that all the passengers that show up are able to travel is at least 0.95? Use an approximation if necessary, but explain your assumptions.
 - c. Assume that conditioned on the plane being full, each passenger has the same probability of being left out. What is the probability that a particular passenger will not be able to get on the plane if the company sells 301 tickets?
 - d. The airline charges \$600 per ticket, but must give any passenger back their money back and an additional \$2000 dollars if they show up, but can't travel. What is the expected amount of money that the airline makes if they sell 301 tickets?
6. *Christmas* Johnny is about to do his Christmas shopping. He goes to a store which has turkey around 30% of the time during Christmas. If there is no turkey, he will buy chicken. From past years, Johnny knows that it takes him on average 4 hours to cook a turkey with a standard deviation of 30 minutes, whereas it only takes him 2 hours on average to cook a chicken with a standard deviation of 15 minutes.
 - a. What is the expected time that Johnny will spend cooking?
 - b. Bound the probability that Johnny cooks for more than 5 hours using the answer to the previous question.
 - c. Obtain a better bound for the probability that Johnny cooks for more than 5 hours.

- d. If B is a random variable that equals 1 if Johnny cooks turkey and 0 otherwise. What is the covariance between B and the time Johnny spends cooking? Interpret the result.