FORECASTING HOMEWORK 3

1) Consider the $MA(1)$ model $x_t = \varepsilon_t + \beta\varepsilon_{t-1}$, where $\{\varepsilon_t\}$ is zero mean white noise.

A) Use the formula $\rho_1 = Corr(x_t, x_{t-1}) = \beta \dfrac{var\ \varepsilon_t}{var\ x_t}$ to show that $\rho_1 = \dfrac{\beta}{1+\beta^2}$.

B) Using the result from Part A), determine the maximum possible value for ρ_1 for the $MA(1)$ model. At what value of β is this maximum attained?

C) Based on your answer to Part B), is the $MA(1)$ model capable of producing very smooth (i.e., very highly autocorrelated) realizations?

D) What is the correlation between x_t and x_{t-1} if $\{x_t\}$ is the first difference of a white noise series?

2) Granger, p. 56, Problem 2.

3) Granger, p. 56, Problem 3.

4) Granger, p. 57, Problem 4.

5) Granger, p. 62, Problem 2.

6) Granger, p. 63, Problem 3.

7) Granger, p. 63, Problem 4.

$$y_{t+2} = \varepsilon_{t+2} + b\varepsilon_{t+1},$$

where ε_{t+2} is the effect of yet a further news item and $b\varepsilon_{t+1}$ reflects the reassessment of the earlier piece of news. If a sequence of unexpected news items keep affecting the price on the market and their complete impact takes several days to work out, the price change series will be well represented by a moving average model.

For a further example, consider a maternity hospital. Let ε_t be the number of new patients that arrive on day t and suppose that this is a white noise series, so that the number arriving one day is unrelated to the number coming the next day. Suppose that typically 10% stay just one day, 50% two days, 30% three days, and 10% four days. The number of patients leaving the hospital on day t, x_t, will then be given by

$$x_t = 0.1\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.3\varepsilon_{t-3} + 0.1\varepsilon_{t-4} + \eta_t,$$

where η_t is a white noise error term, because of random variations about the typical situation, so that $x_t \sim$ MA(4). Thus, one could predict how many patients will be leaving in one, two, three, or four days time but not in seven days time.

Exercises

1. $-0.50 \quad 0.40 \; -0.81 \quad 0.00 \; -0.24 \; -0.36 \; -0.39 \quad 0.79 \; 0.98 \; -0.17$
$0.72 \; -0.45 \quad 0.52 \; -0.76 \; -0.18 \; -0.83 \quad 0.57 \; -0.86 \; 0.71 \; -0.72$

   are the first twenty terms of a white noise series ε_t, $t = 1, \ldots, 20$. Using these values, generate the MA(2) series x_t given by

$$x_t = \tfrac{1}{3}(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2})$$

   for $t = 3, \ldots, 20$. From the plots of ε_t and x_t against time, which do you consider to be the smoother in appearance?

2. You are told that the monthly change in sales of paperback books in a certain store obeys an MA(2) model of the form

$$x_t = \varepsilon_t + 0.6\varepsilon_{t-1} + 0.3\varepsilon_{t-2}$$

   and are given the following recent values for x_t: $x_{20} = 180$, $x_{21} = -120$, $x_{22} = 90$, $x_{23} = 10$.
   (a) What forecast would you have made for x_{20} and x_{21} at time $t = 19$ if you had assumed $\varepsilon_{17} = -10$, $\varepsilon_{18} = 30$, $\varepsilon_{19} = 70$?
   (b) Using these same values for $\varepsilon_{17}, \varepsilon_{18}$, and ε_{19}, what do you forecast for x_{24}, x_{25}, x_{26}, and x_{27} at time $t = 23$?

3. Suppose that you are unsure whether a certain series has been generated by an MA(1) or an MA(2) model. How would the value of the

second autocorrelation coefficient ρ_2 help you choose between these alternative models?

4. Prove that the two series x_t and y_t, generated by

$$x_t = \varepsilon_t + 0.8\varepsilon_{t-1}$$

   and

$$y_t = \eta_t + 1.25\eta_{t-1},$$

   where ε_t, η_t are each zero-mean white noise series and $\text{var}(\varepsilon_t) = 1$, $\text{var}(\eta_t) = 0.64$, have identical variances and autocorrelation sequences ρ_k, $k = 0, 1, 2, \ldots$. Prove that if z_t is generated by

$$z_t = \theta_t + a\theta_{t-1} + b\theta_{t-2},$$

   where θ_t is white noise, and if $b \neq 0$, then z_t cannot have the same autocorrelation sequence as x_t and y_t.

3.3 AUTOREGRESSIVE MODELS

An alternative way of producing a series with more structure than white noise is from an iterative generating equation of the form

$$x_t = \alpha x_{t-1} + \varepsilon_t \tag{9}$$

where ε_t is zero-mean white noise. Given the sequence ε_t, $t = 1, 2, \ldots, n$ and a starting value x_0, the series x_t is formed by repeated application of (9). For example, with $\alpha = 0.5$, $x_0 = 0.16$, we have the values shown in Table 3.1D.

A relationship such as (9) is known as a difference equation and has a solution of the form

$$x_t = x_0\alpha^t + (\varepsilon_t + \alpha\varepsilon_{t-1} + \alpha^2\varepsilon_{t-2} + \cdots + \alpha^{t-1}\varepsilon_1). \tag{10}$$

That this is a solution to (9) can be seen by writing (10) as

$$x_t = \alpha[x_0\alpha^{t-1} + (\varepsilon_{t-1} + \alpha\varepsilon_{t-2} + \alpha^2\varepsilon_{t-3} + \cdots + \alpha^{t-2}\varepsilon_1)] + \varepsilon_t$$

TABLE 3.1D

t	ε_t	$0.5x_{t-1}$	x_t	t	ε_t	$0.5x_{t-1}$	x_t
0	—	—	0.16	4	−0.18	0.075	−0.105
1	−0.12	0.08	−0.04	5	−0.04	−0.05	−0.09
2	0.20	−0.02	0.18	6	0.00	−0.045	−0.045
3	0.06	0.09	0.15				

and that

$$x_n = 50, \qquad x_{n-1} = 40;$$

then

$$f_{n,1} = 0.5 \times 50 + 0.2 \times 40 = 33,$$
$$f_{n,2} = 0.5 \times 33 + 0.2 \times 50 = 26.5,$$
$$f_{n,3} = 0.5 \times 26.5 + 0.2 \times 33 = 19.85,$$
$$f_{n,4} = 0.5 \times 19.85 + 0.2 \times 26.5 = 15.225,$$
$$f_{n,5} = 0.5 \times 15.225 + 0.2 \times 19.85 = 11.58,$$
$$\vdots$$

The time series properties of the AR(p) process are quite difficult to determine, but it can be shown that they greatly depend on the largest root of the equation

$$Z^p = \alpha_1 Z^{p-1} + \alpha_2 Z^{p-2} + \cdots + \alpha_{p-1} Z + \alpha_p.$$

Let θ be this largest root and denote by $|\theta|$ the absolute size of this root, so that a positive value is taken if θ is negative and the modulus if θ is a complex root. Theory states that if $|\theta| < 1$, then x_t will not be an explosive series, but if $|\theta| > 1$, it will be explosive. Further, corr(x_t, x_{t-k}) is well approximated for k not small by $A\theta^k$ if θ real and positive or will approximately lie in the region $\pm A|\theta|^k$ otherwise, where A is some constant. This helps determine the shape of the plot of this correlation against k, which is a particularly useful diagram and will be discussed in Section 3.6.

Exercises

1. If you are told that the number of accidents in a factory in month t obeys an AR(1) model of the form

$$x_t = 10 + 0.7x_{t-1} + \varepsilon_t,$$

where ε_t is zero-mean white noise, and that $x_{19} = 20$, $x_{20} = 24$, $x_{21} = 16$, what forecasts would you make
   (a) for x_{22}, x_{23}, and x_{24} at time $t = 21$?
   (b) for x_{23} and x_{24} at time $t = 20$?
2. A company economist believes that a rival firm's quarterly advertising expenditures obey the AR(3) model

$$x_t = 0.2x_{t-1} + 0.6x_{t-2} - 0.3x_{t-3} + \varepsilon_t.$$

If the values for the last three quarters are $x_{10} = 180$, $x_{11} = 140$, and $x_{12} = 200$, forecast expenditures for the next three quarters, $t = 13$, 14, and 15.
3. A time series is constructed on a computer by an AR(1) model. You are told that the series has zero mean, that the second autocorrelation is $\rho_2 = 0.64$, and that $x_{50} = 10$. Forecast x_{51} and x_{52}. Why would knowing the value of ρ_3 help forecast x_{51} but not x_{52}?
4. If a series x_t is generated by

$$x_t = 0.8x_{t-1} + \varepsilon_t,$$

where ε_t is zero-mean white noise with variance 25, find the variances of the one-, two-, and three-step forecast errors, i.e., var($e_{n,h}$), $h = 1, 2$, and 3.

3.4 MIXED AUTOREGRESSIVE-MOVING AVERAGE MODELS

A much wider class of models can be produced by mixing an autoregressive and a moving average, so that x_t is generated by

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \cdots + \alpha_p x_{t-p} + y_t, \tag{16}$$

where y_t is a moving average of order q,

$$y_t = \varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + \cdots + b_q\varepsilon_{t-q}.$$

This is a mixed autoregressive-moving average model of order p, q, denoted ARMA(p, q). A specific example is the ARMA(1, 1) generating process

$$x_t = 0.5x_{t-1} + \varepsilon_t + 0.3\varepsilon_{t-1},$$

so that given a starting value for x_0 and the white noise sequence ε_t, x_t is formed iteratively.

Forecasting is straightforward by just using the rules given in the previous two sections, so that in the ARMA(1, 1) example, x_{n+1} is formed by

$$x_{n+1} = 0.5x_n + \varepsilon_{n+1} + 0.3\varepsilon_n.$$

Then

$$f_{n,1} = 0.5x_n + 0.3\varepsilon_n = 0.5x_n + 0.3(x_n - f_{n-1,1})$$

by noting that the one-step forecast error is

$$e_{n,1} = \varepsilon_{n+1},$$