section 4.2 - 4.3, Vector Geometry pt. 2

Planes in $\mathbb{R}^3$:

as mentioned before, a plane is determined by the following sets of data...

- 3 points which do not all lie on same line
- 2 non-parallel vectors & a point
- an orthogonal axis & a point
- an orthogonal vector & a point

Recall these 2 def'ns ...

① vector n is normal vector for plane P if
=> all vectors in P are orthogonal to n

② plane w/ normal vector $n = (a, b, c)^T$ passing through $P_0 = (x_0\ y_0\ z_0)$ has eq'tn given by ...

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

→ **Cross Product:**

let $v = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ & $w = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$

cross product of $v \times w = \det \begin{bmatrix} \hat{i} & x_1 & x_2 \\ \hat{j} & y_1 & y_2 \\ \hat{k} & z_1 & z_2 \end{bmatrix}$

$\Rightarrow e_1 = \hat{i} \quad e_2 = \hat{j} \quad e_3 = \hat{k}$

$\Rightarrow$ expanded upon first column

e.g.

let $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, find $v \times w$

$$v \times w = \det \begin{bmatrix} \hat{i} & 1 & 2 \\ \hat{j} & 1 & 1 \\ \hat{k} & 1 & 2 \end{bmatrix}$$

$$= \hat{i} \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \hat{j} \det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \hat{k} \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \hat{i}(2-1) - \cancelto{0}{\hat{j}(2-2)} + \hat{k}(1-2)$$

$$= \hat{i} - \hat{k}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ * $v \times w$ is orthogonal to both v & w

e.g.

$v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $w = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, compute $v \times w$ & verify that it's orthogonal to both v & w

$$v \times w = \det \begin{pmatrix} \hat{i} & 1 & 2 \\ \hat{j} & 2 & 0 \\ \hat{k} & -1 & 1 \end{pmatrix}$$

$$= \hat{i} \det \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} - \hat{j} \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} + \hat{k} \det \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

$$= \hat{i}(2+0) - \hat{j}(1+2) + \hat{k}(0-4)$$

$$= 2\hat{i} - 3\hat{j} - 4\hat{k}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix}$$

$$v \cdot (v \times w) = (2) + (-6) + (4) = 0 \checkmark$$

$$w \cdot (v \times w) = (4) + 0 + (-4) = 0 \checkmark$$

→ **Now putting everything together...**

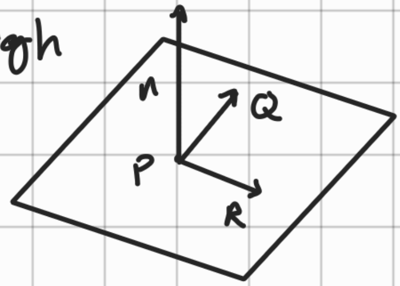
e.g

find eq'tn of the plane through points:

$P = (1, 2, 1)$

$Q = (2, 0, 1)$

$R = (3, -1, 2)$



Express it in the form $ax + by + cz = d$

① $\vec{PQ} = \dots = (1, -2, 0)^T$

$\vec{PR} = \dots = (2\ -3, 1)^T$, both lie on the plane

② $n = \vec{PQ} \times \vec{PR}$, since n is orthogonal to both $\vec{PQ}$ & $\vec{PR}$

$$= \det \begin{pmatrix} \hat{i} & 1 & 2 \\ \hat{j} & -2 & -3 \\ \hat{k} & 0 & 1 \end{pmatrix}$$

$$= \dots$$

$$= \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

③ Choose any of 3 points P, Q or R as our P_0

$P_0 = R = (3, -1, 2)$

④ eq'tn of plane....

$(-2)(x-3) + (-1)(y-(-1)) + (1)(z-2) = 0$

...

$-2x - y + z = -3$, $n = (-2, -1, 1)^T$
$P_0 = (3, -1, 2)$

=> Properties of Cross Product :

- $v \times w$ is orthogonal to v & w
- $v \times 0 = 0 = 0 \times v$
- $v \times v = 0$
- v & w are parallel if only if $v \times w = 0$
- $v \times w = -(w \times v)$
- $(kv) \times w = k(v \times w) = v \times (kw)$
- $(u + v) \times w = (u \times w) + (v \times w)$
- $u \times (v + w) = (u \times v) + (u \times w)$

Tutorial Questions

* 6. The area of a Δ w/ vertices $(1, -1, 2)$ $(2, 1, 1)$ $(0, 1, -1)$

Let $A = (1\ {-1}\ 2)$
$B = (2\ 1\ 1)$
$C = (0\ 1\ {-1})$

① get cross product and length

$\vec{AB} = (1, 2, -1)$

$\vec{AC} = (-1, 2, -3)$

$$\vec{AB} \times \vec{AC} = \det \begin{pmatrix} \hat{i} & 1 & -1 \\ \hat{j} & 2 & 2 \\ \hat{k} & -1 & -3 \end{pmatrix}$$

$$= \hat{i} \det \begin{pmatrix} 2 & 2 \\ -1 & -3 \end{pmatrix} - \hat{j} \det \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} + \hat{k} \det \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$$

$$= \hat{i}(-6+2) - \hat{j}(-3-1) + \hat{k}(2+2)$$

$$= -4\hat{i} + 3\hat{j} + 4\hat{k}$$

$$= \begin{pmatrix} -4 \\ 3 \\ 4 \end{pmatrix}$$

$$\| \vec{AB} \times \vec{AC} \| = \sqrt{(-4)^2 + 3^2 + 4^2}$$

$$= \sqrt{16 + 9 + 16}$$

$$= \sqrt{41}$$

② $\text{Area}_\Delta = \dfrac{\| \vec{AB} \times \vec{AC} \|}{2} = \dfrac{\sqrt{41}}{2}$