section 2.6, Linear Transformations pt. 2

→ **Cross Product :**

same as cross product of vectors ....

$$v \times w = \det \begin{bmatrix} \hat{i} & x_1 & x_2 \\ \hat{j} & y_1 & y_2 \\ \hat{k} & z_1 & z_2 \end{bmatrix}$$

$$\Rightarrow \|v \times w\|^2 = \|v\|^2 \|w\|^2 - (v \cdot w)^2 = \|v\|^2 \|w\|^2 \sin^2\theta$$

as a cosequence we learn Area of parallelogram spanned by v & w is...

$$\text{Area parallelogram} = \|v \times w\|$$
$$= \|v\| \|w\| \sin\theta$$

e.g find area of parallelogram spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \; \& \; \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

let $v = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\text{Area} \,\square = \|v\| \|w\| \sin\theta$$

$$= \|v \times w\|$$

$$= \left\| \det \begin{pmatrix} \hat{i} & 1 & 2 \\ \hat{j} & -1 & 3 \\ \hat{k} & 2 & -1 \end{pmatrix} \right\|$$

$= \left\| \hat{i} \det\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} - \hat{j} \det\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} + \hat{k} \det\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \right\|$

$= \| \hat{i}(1-6) - \hat{j}(-1-4) + \hat{k}(3+2) \|$

$= \| -5\hat{i} + 5\hat{j} + 5\hat{k} \|$

$= \| (-5, 5, 5) \|$

$= \sqrt{(-5)^2 + 5^2 + 5^2}$

$= \sqrt{75}$

Recall :

for $n \times n$ matrix A, vector v is eigenvector if...

$Av = \lambda v$, λ is eigenvalue

let polynomial $c_A(x) = \det(x I_n - A)$ be the characteristic polynomial

$\Rightarrow$ λ is eigenvalue of A iff it is a root of $c_A(\lambda) = 0$

$\Rightarrow$ λ - eigenvectors are non-trivial solutions to homogeneous system $(\lambda I_n - A)x = 0$
(basic eigenvectors)

we say A is diagonizable if there exist invertible P & diagonal matrix D s.t.

$$A = PDP^{-1} \Rightarrow D = P^{-1}AP$$

$$D := \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \longleftrightarrow P := (V_1 \ V_2 \ \dots \ V_n)$$

e.g.

check if $A = \begin{pmatrix} 2 & 4 \\ 0 & -5 \end{pmatrix}$ is diagonalizable.
find invert. P and diagonal D s.t. $A = PDP^{-1}$

$$C_A(x) = \det(xI_2 - A)$$
$$= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & -5 \end{bmatrix}\right)$$
$$= \det \begin{bmatrix} x-2 & -4 \\ 0 & x+5 \end{bmatrix}$$
$$= (x-2)(x+5) - 0$$
$$= x^2 + 5x - 2x - 10$$
$$= x^2 + 3x - 10$$
$$= (x+5)(x-2)$$

$\lambda_1 = -5$, $\lambda_2 = 2$

$$(\lambda_1 I_n - A)x = 0 \qquad (\lambda_2 I_n - A)x = 0$$

$$\left(\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & -5 \end{bmatrix}\right)x = 0 \qquad \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & -5 \end{bmatrix}\right)x = 0$$

$$\begin{pmatrix} -7 & -4 \\ 0 & 0 \end{pmatrix} x = 0 \qquad \begin{pmatrix} 0 & -4 \\ 0 & 7 \end{pmatrix} x = 0$$

$$\left(\begin{array}{cc|c} -7 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} 0 & -4 & 0 \\ 0 & 7 & 0 \end{array}\right)$$

$x_1 = 7$?

$x_2 = 4$ $x_2 = 0$

$\Rightarrow V_1 = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ $\Rightarrow V_2 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P(2,5)$ $D = \begin{bmatrix} 7 & 1 \\ 4 & 0 \end{bmatrix}$

→ we already went over linear transformations & standard matrix

e.g.
let $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y + 7z \\ 2x - 4y + z \end{pmatrix}$, compute $T(e_1)$ $T(e_2)$ $T(e_3)$. Find A_T, the standard matrix for T

$T(e_1) = \cdots = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$T(e_2) = \cdots = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ $\Rightarrow$ $A_T = \begin{bmatrix} 1 & -2 & 7 \\ 2 & -4 & 1 \end{bmatrix}$

$T(e_3) = \cdots = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

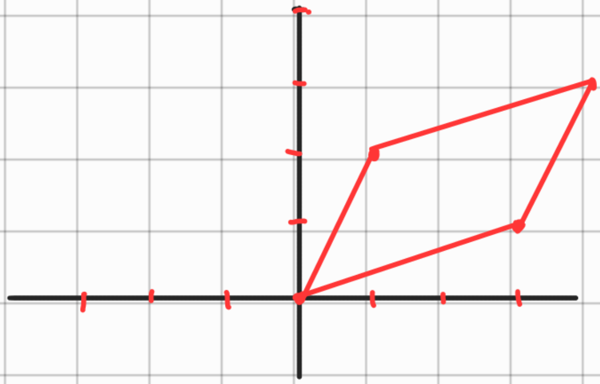
Tutorial Questions:

* 1. visualize each transformation. –

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are your base vertices

a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ x + 2y \end{pmatrix}$

$\Rightarrow T\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$



2. sketch the transformation $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x-2y\\x+4y\end{pmatrix}$ to unit square

Find the resulting area of parallelogram

sketch w/ base vertices

A_T will be used to find area

$$A_T = \begin{pmatrix}3 & -2\\1 & 4\end{pmatrix}$$