section 5.1, Subspaces & Spanning sets

Consider the xy-plane (P_{xy}):

- sums and scalar multiples of vectors in P_{xy} are still in P_{xy}

However, not all planes in $\mathbb{R}^3$ have this property

→ **Subspace of $\mathbb{R}^n$:**

we may say a subset $U \subseteq \mathbb{R}^n$ is subspace of $\mathbb{R}^n$ if...

① $0 \in U$
② if $x, y \in U \Rightarrow x + y \in U$
③ $x \in U \Rightarrow kx \in U$, for any k

e.g.

we've seen $P_{xy} = \{(x, y, 0)^T : x, y \in \mathbb{R}\}$ is a subspace in $\mathbb{R}^3$

Determine if set $S = \{(x, y, xy)^T : x, y \in \mathbb{R}\}$ is subspace of $\mathbb{R}^3$

① $0 \in S$?

let $x = 0$, $y = 0$ $\Rightarrow S = (0, 0, 0)^T$ ✓

② $x, y \in S \Rightarrow x + y \in S$

let $x = \begin{pmatrix} x \\ y \\ xy \end{pmatrix}$ $y = \begin{pmatrix} a \\ b \\ ab \end{pmatrix}$ $\in S$

$x + y = \begin{pmatrix} x + a \\ y + b \\ xy + ab \end{pmatrix}$ X

$\Rightarrow$ does not satisfy condition ②

* Note: just like linear transformations, we can always suspect non-linear subsets by seeing non-linear terms.

e.g.

show that $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + 2y = 3z \right\}$ is a subspace of $\mathbb{R}^3$.

① $0 \in U$?

let $x = 0$ $y = 0$ $z = 0$

$(0) + 2(0) = 3(0) \Rightarrow 0 = 0$ ✓

② $x, y \in U \Rightarrow x + y \in U$

let $x = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $y = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$x + y = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z \end{pmatrix}$

. . .

✓

③ $x \in U \Rightarrow cx \in U$

→ **Important Subspaces :**

let A be $m \times n$ matrix

null space ← $\text{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$ is subspace of $\mathbb{R}^n$

image ↙ $\text{im}(A) = \{b \in \mathbb{R}^m : Ax = b, \ x \in \mathbb{R}^n\}$ is subspace of $\mathbb{R}^m$

e.g.

show that $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + 2y = 3z \right\}$ is a subspace of $\mathbb{R}^3$.

Now we can use something other then the conditions

$$x + 2y - 3z = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \{ x \in \mathbb{R}^3 : \begin{pmatrix} 1 & 2 & -3 \end{pmatrix} x = 0$$

$$= \text{null} \begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \hookrightarrow A = \begin{pmatrix} 1 & 2 & -3 \end{pmatrix}$$

e.g.

1. let $U = \left\{ [x \ y \ z \ w]^T \in \mathbb{R}^4 : \begin{matrix} 3x - 2y = z - w \\ x + y - z = 2w \end{matrix} \right\}$

we can say that in U

$$3x - 2y - z + w = 0$$
$$x + y - z - 2w = 0$$

$$\Rightarrow \begin{pmatrix} 3 & -2 & -1 & 1 \\ 1 & 1 & -1 & -2 \end{pmatrix} x = 0 \quad , x = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\Rightarrow \text{Null} \begin{pmatrix} 3 & -2 & -1 & 1 \\ 1 & 1 & -1 & -2 \end{pmatrix}$$

2.
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + z \\ 3x - 2y + z \end{pmatrix}$$

$$A_T = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \end{pmatrix}$$

$\longrightarrow$ **Spanning Sets :**

let $x_1 \ x_2 \ \cdots \ x_k$ be vectors in $\mathbb{R}^n$

Span of $x_1 \ \cdots \ x_k$ is the set of all possible linear combinations of x_i's

$$\{ c_1 x_1 + c_2 x_2 + \cdots + c_k x_k \}$$

$\Rightarrow$ denoted by $\{ x_1 \ x_2 \ \cdots \ x_k \}$

if V is subspace of $\mathbb{R}^n$ and $V = \text{span}\{x_1, x_2 \dots x_k\}$

$\Rightarrow$ V is spanned by $x_1\ x_2 \dots x_k$

OR

$\{x_1\ x_2 \dots x_k\}$ is a spanning set of V

$\Rightarrow$ for any finite set of vectors $S \subseteq \mathbb{R}^n$

span(S) is subspace of $\mathbb{R}^n$

- span(S) is smallest subspace of $\mathbb{R}^n$ containing vectors from S

e.g.

Find a spanning set for $U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x+y = 2z-w \\ x-4y+3z = -w \end{matrix} \right\}$

and determine if $v = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ is in U

Note : $U = \text{null}\begin{pmatrix} 1 & 1 & -2 & 1 \\ 1 & -4 & 3 & 1 \end{pmatrix}$

Solve $Ax = 0$ to find spanning set of U

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & -4 & 3 & 1 \end{bmatrix} \xrightarrow[\dots]{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{pmatrix} s-t \\ s \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

the basic sol'tns we get create the spanning set of U

To check V, we need linear combo

$$s\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}4\\1\\1\\3\end{pmatrix}$$

$s - t = 4$, this is not satisfied

$s = 1$

$s = 1$

$t = 3$

$\Rightarrow$ V is not in U

$\Rightarrow$ let A be $m \times n$ matrix w/ columns $c_1\, c_2 \ldots c_n$ and let $x_1\, x_2 \ldots x_k$ be basic solutions to $Ax = 0$

Then,

- $Ax = b$ has sol'tn if & only if $b \in \text{span}\{c_1\, c_2 \ldots c_n\}$
- $Ax = b$ has sol'tn for any b if & only if $\text{span}\{c_1\, c_2 \ldots c_n\} = \mathbb{R}^m$
- $\text{im}(A) = \text{span}\{c_1\, c_2 \ldots c_n\}$
- $\text{null}(A) = \text{span}\{x_1\, x_2 \ldots x_k\}$

e.g.

find spanning set for subspace...

$$U = \left\{ \begin{pmatrix} 4x + 3y - 2z \\ x - 5y - 3 \end{pmatrix} \in \mathbb{R}^2 : x, y, z \in \mathbb{R} \right\}$$

image of u $\Rightarrow$ $U = \text{im} \begin{pmatrix} 4 & 3 & -2 \\ 1 & -5 & -3 \end{pmatrix}$

$\Rightarrow$ $U = \text{span} \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right)$

e.g.

spanning set for null (A), where $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} x = 0$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \Rightarrow \begin{aligned} x &= 3s \\ y &= -2s \\ z &= s \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\downarrow$

$$\text{null}(A) = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

e.g.

spanning set for $P_{xy} \subseteq \mathbb{R}^3$ that is diff. from $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

any $a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}a\\b\\0\end{pmatrix}$

$\downarrow$

any vectors like this as a set works