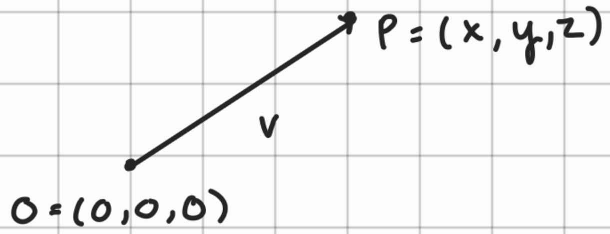
section 4.1 - 4.2, Vector Geometry

Vector → object w/ direction & magnitude/length

=> $\|v\|$ is length of vector, v

A vector, v, can be thought of w/ point $P = (x, y, z) \in \mathbb{R}^3$

- direction is based from origin $O = (0,0,0)$
- magnitude is length of arrow/vector
- $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, corresponds to point $(x, y, z) \in \mathbb{R}^3$



The length of a vector is $\|v\| = \sqrt{x^2 + y^2 + z^2}$

e.g.

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \longrightarrow \|v\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

→ Equal Vectors :

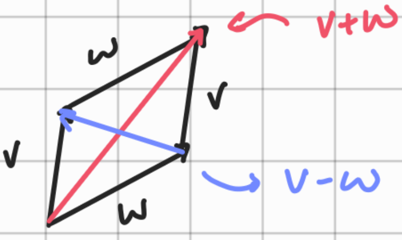
we say $v = w$ are equal if same $\begin{cases} \text{direction} \\ \text{length} \end{cases}$

They can start and end at different points

→ Operations on Vectors :

- $v + w = \begin{bmatrix} x+p \\ y+q \\ z+r \end{bmatrix}$

- $v - w = \begin{bmatrix} x-p \\ y-q \\ z-r \end{bmatrix}$



- Scaling v by constant, a...

$$+av = av = \begin{bmatrix} a\,x \\ a\,y \\ a\,z \end{bmatrix}$$

$$-av = |a|v$$

, points in opposite

e.g.

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ compute $2v - 3w = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$w = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 12 \\ -3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 \\ 4-(-3) \\ 6-6 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 0 \end{bmatrix}$$

→ Parallel vectors :

- if $v = aw$ for some $a \in \mathbb{R} \Rightarrow$ parallel

⇒ or if they have same / opposite direction

e.g.

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $w = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$, v & w are parallel

since $v = -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$

→ Unit Vector :

if $\|u\| = 1$, the vector is a unit vector, u.

=> for any vector, v ...

$$u = \frac{1}{\|v\|} \cdot v$$

, which is a unique unit vector w/ same direction as v

e.g.

find unit vector for $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$$u = \frac{1}{\|v\|} \cdot v \quad , \quad \|v\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$= \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \left[1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3} \right]^T$$

, same way to express column form

→ Geometric Vector from point to point:

vector $PQ = Q - P$

Q: Point Q

P: Point P

e.g.

$P = (1, -3, 5)$ find vector PQ ...

$Q = (2, 1, -2)$

$$PQ = Q - P$$

$$= \begin{bmatrix} 2 - 1 \\ 1 - (-3) \\ -2 - 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

Lines in $\mathbb{R}^3$:

Recall that 2 points can give a vector, we'll call d...

$d = PQ$, any vector that lies along line l is parallel to d

→ Vector Equation:

we call vector, d, a direction for l if...

$d = PQ$, for two points on the line (direction vector of line is not unique)

=> for point, P_0, and direction, d, on line l...
if $P = (x, y, z)$ is any other point on l...
↳ P_0P is parallel to d ; $P_0P = td$, $t \in \mathbb{R}$

=> vector equation is...

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{P_0}{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}} + t \underset{d}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$$

e.g.

find vector eq'tn of line passing through points
$P = (1, -2, 1)$
$Q = (2, 1, 3)$

① $d = \vec{PQ} = Q - P = \begin{bmatrix} 2-1 \\ 1-(-2) \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

② Choose P_0 ...

$P_0 = P = (1,-2,1) \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

OR
Q

e.g.

find 2 direction vectors for line through
$P = (1,5,-3)$
$Q = (2,1,5)$

$d_1 = \vec{PQ} = Q - P = \begin{bmatrix} 2-1 \\ 1-5 \\ 5-(-3) \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 8 \end{bmatrix}$

$d_2 = \vec{QP} = P - Q = \begin{bmatrix} 1-2 \\ 5-1 \\ -3-5 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -8 \end{bmatrix}$

→ **Dot Product :**

for vectors $v = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ & $w = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ ...

$$v \cdot w = x_1x_2 + y_1y_2 + z_1z_2$$

⇒ Properties :

- $v \cdot w = w \cdot v$
- $v \cdot 0 = 0$, same as $0\vec{v} = 0$

- $v \cdot v = \|v\|^2$
- $u \cdot (v+w) = u \cdot v + u \cdot w$
- $(kv) \cdot w = k(v \cdot w) = v \cdot (kw)$

e.g.

Compute $v \cdot w$, $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$

$$v \cdot w = (1)(-2) + (2)(3) + (-1)(-4)$$
$$= 8$$

=> Angle between vectors :

$$v \cdot w = \|v\| \|w\| \cos\theta$$

$$\cos\theta = \frac{v \cdot w}{\|v\| \|w\|}$$

e.g.

find angle between the vectors $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

① $v \cdot w = (1)(-1) + (\sqrt{2})(0) + (1)(-1)$
$= -2$

② $\|v\| = \sqrt{1^2 + (\sqrt{2})^2 + 1^2}$
$= \sqrt{4}$
$= 2$

$\|w\| = \sqrt{(-1)^2 + 0^2 + (-1)^2}$
$= \sqrt{2}$

③ $\cos\theta = \dfrac{v \cdot w}{\|v\| \|w\|}$
$= \dfrac{-2}{2(\sqrt{2})}$

$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
$= \frac{3}{4}\pi \cdot \frac{180}{\pi}$
$= 135°$

→ **Orthogonal :**

if $v \cdot w = 0$ => orthogonal

=> so thus $v = 0$ OR $w = 0$ OR $\theta = \frac{\pi}{2}$

e.g.

verify vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ are orthogonal.

$$v \cdot w = (1)(1) + (1)(1) + (-1)(2)$$
$$= 2 - 2$$
$$= 0 \quad \checkmark$$

, thus orthogonal

→ **Projection onto :**

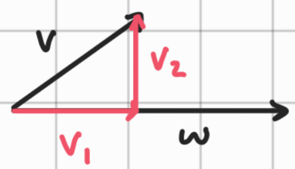
for any vector v, $v = v_1 + v_2$ ...

v_1 => parallel to w , $v_1 = aw$

v_2 => orthogonal to w , $v_2 \cdot w = 0$

so we call v_1 a projection of v onto w ...

$v_1 = \text{proj}_w v$,



=> $\text{proj}_w v = \frac{v \cdot w}{\|w\|^2}(w)$, $\|w\|^2 = w \cdot w$

=> $v_2 = v - \text{proj}_w v$, $v_1 = \text{proj}_w v$

e.g.

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$. find $\text{proj}_v w$ ...

express $w = w_1 + w_2$, where w_1 is parallel to v & w_2 is orthogonal to v

$$w_1 = \text{proj}_v w = \frac{w \cdot v}{\|v\|^2}(v) = \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

$$w \cdot v = (1)(2) + (1)(1) + (1)(2) = 2 + 1 + 2 = 5$$

$$\|v\|^2 = v \cdot v = (1)(1) + (1)(1) + (1)(1) = 3$$

$$w_2 = w - \text{proj}_v w = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

Planes in $\mathbb{R}^3$:

There are many ways to describe a plane on the $\underbrace{\text{xy-plane}}_{(x,y,0)}$, but more specifically (data uniquely specify a plane)

- 3 points ; all do not lie on same plane
- 2 non-parallel vectors and a point
- Orthogonal line and a point
- Orthogonal vector and a point

→ Normal vector :

let P be a plane ... vector n is a normal vector for P if

⇒ all vectors lying on P are orthogonal to n

e.g.

vector $n = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is normal vector for xy-plane

$$n \cdot v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}}_{\text{xy-plane}} = 0 \quad \checkmark$$

⇒ Equation of a Plane :

for $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ passing through point $P_0 = (x_0, y_0, z_0)$ has eq'tn

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

e.g.

find eq'tn of the plane w/ normal vector $n = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ passing through point $(1, 0, -3)$

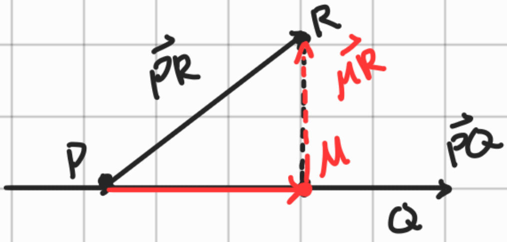
$$(2)(x-1) + (-3)(y-0) + (1)(z+3) = 0$$

$$2x - 2 - 3y + z + 3 = 0$$

$$2x - 3y + z + 1 = 0$$

Tutorial Questions

*4. Find point on line passing through P & Q which is closest to R



① find projection of $\vec{PR}$ onto d

$\vec{PR} = R - P = \cdots = (-3, 0, 4)$

$d = (1, 2, 2)$
$= \vec{PQ}$

$$\text{proj}_d \vec{PR} = \frac{\vec{PR} \cdot d}{\|d\|^2} (d)$$

$$= \frac{(-3) + 0 + 8}{(1) + 4 + 4} (1, 2, 2)$$

$$= \frac{5}{9} (1, 2, 2)$$

② use projection to add it to equation on line

$$l(t_0) = P + t_0 d \qquad t_0 = 5/9$$

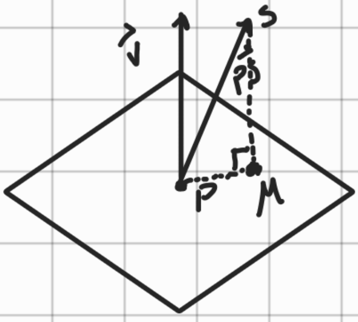
$$= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1t \\ 2t \\ 2t \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 1t \\ -1 + 2t \\ -2 + 2t \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 5/9 \\ -1 + 10/9 \\ -2 + 10/9 \end{bmatrix} = \begin{bmatrix} 23/9 \\ 1/9 \\ -8/9 \end{bmatrix}$$

closest point

3. find point on the plane containing P w/ normal vector $v = \vec{QR} = \dots = (-4,-2,2)$ which is closest to $S = (2,-4,1)$



① eq'tn of plane ...

$(-4)(x-2) + (-2)(y-(-1)) + (2)(z-(-2)) = 0$

$-4x + 8 - 2y - 2 + 2z + 4 = 0$

$P = (2,-1,-2)$

$-4x - 2y + 2z = -10$

② There exists a t_0 s.t. the point $M = L(t_0)$ for $L(t) = S + t\vec{v}$

$= (2\ -4\ \ 1) + t(-4,-2,2)$

so ...

$$L(t_0) = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} -4t_0 \\ -2t_0 \\ 2t_0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 4t_0 \\ -4 - 2t_0 \\ 1 + 2t_0 \end{bmatrix}$$

③ plug back into eq'tn of plane and solve for t_0

$-4[2 - 4t_0] - 2[-4 - 2t_0] + 2[1 + 2t_0] = -10$

$\cancel{-8} + 16t_0 \cancel{+8} + 4t_0 + 2 + 4t_0 = -10$

$24t_0 = -10 - 2$

$t_0 = \frac{-12}{24}$

$= -\frac{1}{2}$

④ plug t_0 into $L(t_0)$ ...

$$M = L(t_0) = (4, -3, 0)$$ //