section 2.6 Linear Transformations pb.1

Matrices as functions :

Recall that functions are ...

$$f : \underset{\text{domain}}{A} \longrightarrow \underset{\text{codomain}}{B}$$

- where $x \in A$ and $f(x) \in B$

let A be $m \times n$ matrix & $x \in \mathbb{R}^n$ a column with n entries ...

$Ax \in \mathbb{R}^m$ has m entries

Thus a rule matrix A provides defines a function w/ domain $\mathbb{R}^n$ & codomain $\mathbb{R}^m$

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\Rightarrow \quad T_A(x) = Ax$$

column vector from plugging column x into T_A

$$\Rightarrow \quad T_A\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{OR} \quad T_A = \left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

e.g.

Consider 2×3 matrix $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 4 & -3 \end{pmatrix}$

This defines function $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

for column vector $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, T_A is given by

$$\begin{aligned} T_A(x) &= Ax \\ &= \begin{pmatrix} 1 & -2 & 2 \\ 2 & 4 & -3 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{pmatrix} x - 2y + 2z \\ 2x + 4y - 3z \end{pmatrix} \end{aligned}$$

So T_A is the function given by equation

$$T_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y + 2z \\ 2x + 4y - 3z \end{pmatrix}$$

e.g.

let $A = \begin{pmatrix} 3 & 2 \\ -3 & 5 \end{pmatrix}$ $\quad x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

1. compute $T_A(x)$

$$\begin{aligned} T_A(x) &= Ax \\ &= \begin{pmatrix} 3 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 - 4 \\ -3 - 10 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -13 \end{pmatrix} \end{aligned}$$

2. $$\begin{aligned} T_A \begin{pmatrix} x \\ y \end{pmatrix} &= Ax \\ &= \begin{pmatrix} 3 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 3x + 2y \\ -3x + 5y \end{pmatrix}$$

3. domain : $\mathbb{R}^2$ target space : $\mathbb{R}^2$

→ **Matrix Transformation :**

we say $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a matrix transformation if ...

there exists $m \times n$ matrix A s.t.

$$T = T_A \quad \Rightarrow \quad T(x) = Ax$$

e.g.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 4y \\ x + 3y \\ x - y \end{pmatrix}$$ is a matrix transformation

Find matrix A s.t. $T = T_A$

$$T(x) = \begin{pmatrix} 2x - 4y \\ x + 3y \\ x - y \end{pmatrix}$$

let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$T_A = Ax = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ ex + fy \end{bmatrix}$$

$$\begin{bmatrix} ax + by \\ cx + dy \\ ex + fy \end{bmatrix} = \begin{pmatrix} 2x - 4y \\ x + 3y \\ x - y \end{pmatrix}$$

$$A = \begin{bmatrix} 2 & -4 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$$

$\Rightarrow$ **Properties:**

- $T_A(x+y) = T_A(x) + T_A(y)$

- $T_A(cx) = c\,T_A(x)$

$\longrightarrow$ Operations on Transformations:

Recall that if $f : A \to B$
$g : B \to C$

target space = domain

Then, we can compose f and g to get a new function $g \circ f : A \to C$

$$g \circ f(x) = g(f(x))$$

$\Rightarrow$ if A is $m \times k$, B is $k \times n$ $\Big\}$ AB is $m \times n$ matrix

$$T_{AB} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$\Rightarrow$ **Matrix Multiplication:**

$$T_{AB} = T_A \circ T_B$$

e.g.

let $\underbrace{A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{2 \times 3}$ $\quad$ $\underbrace{B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 5 \end{pmatrix}}_{3 \times 2}$

$$T_A \circ T_B = T_{AB}$$

$$= (AB)x$$

$$= \begin{pmatrix} 1-2+9 & 2+2+15 \\ 4-5+18 & 8+5+30 \end{pmatrix} x$$

$$= \underbrace{\begin{pmatrix} 8 & 19 \\ 17 & 43 \end{pmatrix}}_{\text{matrix composition}} x$$

$$T_A \circ T_B : \mathbb{R}^2 \to \mathbb{R}^2$$

The matrix multiplication => composition of transformation

=> Inverse matrix :

for $S : \mathbb{R}^n \to \mathbb{R}^n$ $\quad T : \mathbb{R}^n \to \mathbb{R}^n$

$$(S \circ T)x = x = (T \circ S)x \ , \quad x \in \mathbb{R}^n$$

we call S as T^{-1}

let A be an invertible matrix $(n \times n)$...

- $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible
- $(T_A)^{-1}$ is matrix transformation
- $(T_A)^{-1} = T_{A^{-1}}$

Recall :

we can determine if A is invert. (and find inverse) with ...

1. construct aug. matrix $[A \mid I_n]$

2. RREF fully

3. if you get $[I_n \mid B]$ => A is invert. & $A^{-1} = B$

4. if RREF not I_n => A not invert.

e.g.

let $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$. verify that T_A is invert.

Find formula for it's inverse

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right] \quad R_2 = R_2 - R_1$$

$$= \left[\begin{array}{cc|cc} 0 & 1 & 3 & -2 \\ 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 = R_1 - 2R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_2 <=> R_1$$

$A^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$, A is invertible

=> T_A is invertible

$$T_A^{-1} = T_{A^{-1}} = (A^{-1})\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} -x + y \\ 3x - 2y \end{pmatrix}$$

→ Linear Transformations :

if for any $x, y \in \mathbb{R}^n$ and any $c \in \mathbb{R}$ we have...

① $T(x+y) = T(x) + T(y)$

② $T(cx) = cT(x)$

=> Then $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation

e.g

verify that $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 4z + 2y \\ y - z + 2x \end{pmatrix}$ is linear

let $x = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $y = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ be columns in $\mathbb{R}^3$

so then, $x + y = \cdots = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$

check Condition ① :

$$T(x+y) = T(x) + T(y)$$

$$T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} 2(x_1+x_2) - 4(z_1+z_2) + 2(y_1+y_2) \\ (y_1+y_2) - (z_1+z_2) + 2(x_1+x_2) \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + 2x_2 - 4z_1 - 4z_2 + 2y_1 + 2y_2 \\ y_1 + y_2 - z_1 - z_2 + 2x_1 + 2x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 - 4z_1 + 2y_1 \\ y_1 - z_1 + 2x_1 \end{pmatrix} + \begin{pmatrix} 2x_2 - 4z_2 + 2y_2 \\ y_2 - z_2 + 2x_2 \end{pmatrix}$$

$$= T(x) + T(y)$$

Check Condition ② :

$$T(cx) = T\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(cx_1) - 4(cz_1) + 2(cy_1) \\ (cy_1) - (cz_1) + 2(cx_1) \end{pmatrix}$$

$$= \begin{pmatrix} c(2x_1 - 4z_1 + 2y_1) \\ c(y_1 - z_1 + 2x_1) \end{pmatrix}$$

$$= c\begin{pmatrix} \cdot\cdot & \cdot\cdot \end{pmatrix}$$

$$= cT(x)$$

$\Rightarrow$ T is linear transformation

$\Rightarrow$ Standard Matrix :

Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has $m \times n$ matrix A_T s.t.

$$T(x) = A_T x$$

, A_T has columns $T(e_1)\ T(e_2)\ T(e_3) \dots T(e_n)$

which is the standard matrix of T

e.g. $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 4z + 2y \\ y - z + 2x \end{pmatrix}$ is linear. Find its standard matrix A_T

for standard matrix only ← $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \Rightarrow A_T$ is 2×3

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$$A_T = \begin{pmatrix} 2 & 2 & -4 \\ 2 & 1 & -1 \end{pmatrix}$$

$\Rightarrow$ plug into $T(x) = Ax$
$= \dots$
$= T_A$

should get back $T(x)$, which means $T(x)$ is matrix transformation T_A
$\therefore$ linear

e.g. verify that transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$ is not linear.

To verify, we can check either conditions

① $T(x+y) = T(x) + T(y)$
② $T(cx) = cT(x)$

it is enough to just fail one condition

check ①:

let $x = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ $y = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$.

$$x + y = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

$$T(x+y) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_1+x_2)(y_1+y_2) \\ (x_1+x_2) + (y_1+y_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 \\ x_1 + x_2 + y_1 + y_2 \end{pmatrix}$$

$$T(x) + T(y) = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1y_1 \\ x_1 + y_1 \end{pmatrix} + \begin{pmatrix} x_2y_2 \\ x_2 + y_2 \end{pmatrix}$$

LHS $\neq$ RHS, Thus, not linear