section 2.4, The inverse of Matrix

given $n \times n$ matrix A, it's **invertible** if there is $n \times n$ matrix B s.t. ...

$AB = BA = I_n \Rightarrow B = A^{-1}$ there is inverse of A

e.g.

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\therefore B = A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Properties :

let A, B be $n \times n$ matrices ...

before it was $AB = BA = I_n$

1. it is only sufficient to check $AB = I_n$
2. if A is invert. $\Rightarrow$ so is A^{-1}, $(A^{-1})^{-1} = A$

   A inverse of B $\Leftrightarrow$ B inverse of A
3. if A is invert. & $AB = AC \Rightarrow B = C$
4. if A is invert. $\Rightarrow$ so is A^T, $(A^T)^{-1} = (A^{-1})^T$
5. if A & B are invert. $\Rightarrow$ AB is invert., $(AB)^{-1} = B^{-1}A^{-1}$

Determining Invert. & Computing Inverses:

we can determine if A is invert. (and find inverse) with...

1. construct aug. matrix $[A \mid I_n]$
2. RREF fully
3. if you get $[I_n \mid B]$ => A is invert. & $A^{-1} = B$
4. if RREF not I_n => A not invert.

invert. if and only if rank(A) = n

e.g. determine if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ -2 & -4 & -5 \end{bmatrix}$ is invertible, find A^{-1}

Find $[A \mid I_n]$ => RREF

$$[A \mid I_n] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -2 & -4 & -5 & 0 & 0 & 1 \end{array}\right] \dots \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -2 & 3 \\ 0 & 1 & 0 & -7 & 1 & -3 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

RREF = I_3

=> A is invertible & $A^{-1} = \begin{bmatrix} 9 & -2 & 3 \\ -7 & 1 & -3 \\ 2 & 0 & 1 \end{bmatrix}$

Another easier way (if 2×2):

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invert. if $ad - bc \neq 0$

- if $ad - bc \neq 0$ => $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solving Systems w/ Inverses :

given $n \times n$ system of eq'ths w/ matrix form $Ax = b$ ...

if A is invert. => we can solve system as...

$$x = A^{-1}b$$