Section 3.3, Diagonalization

Given a matrix A that is diagonal ... we cay say ...

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \Rightarrow A^k = \underbrace{\begin{pmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_{nn}^k \end{pmatrix}}_{\text{simply raise diag. by } k}$$

However given a $n \times n$ matrix that's not diagonal, it is much more complicated

Diagonalizable :

we say A is diagonalizable if matrix P is invert. & matrix D is diagonal s.t

$$A = PDP^{-1}$$ ← good for $A^k = PD^kP^{-1}$

How do we find such matrices P & D?

→ matrix called eigenvectors

Eigenvalues, Eigenvectors & Characteristic of Polynomial :

Let A be $n \times n$, we have eigenvector, v, s.t ...

$Av = \lambda v$, for some eigenvalue λ

=> the polynomial $c_A(x) = \det(xI_n - A)$ is called characteristic polynomial of A

1. λ is an eigenvalue of A if it is a root of $c_A(x)$

2. λ-eigenvectors are non-trivial solutions to the homogeneous system $(\lambda I_n - A)x = 0$

=> if λ is eigenvalue to $n \times n$ matrix A, we call basic solutions to homog. system basic eigenvectors...

$$(\lambda I_n - A)x = 0$$

Test for Diagonalizability:

A is only diagonizable if it has basic eigenvectors $v_1, v_2, \ldots, v_n$

=> if each eigenvector $v_1, v_2, \ldots, v_n$ have corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ ...

=> $P = [v_1 \; v_2 \; \cdots \; v_n]$ ← basic eigenvectors as its columns is invert.

we will have that $A = PDP^{-1}$, where ...

$$P = [v_1 \; v_2 \cdots v_n] \quad \& \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

* Diagonalization Procedure ...

1. compute $\underbrace{c_A(x)}_{\det(xI_n - A)}$ and factor to find eigenvalues

→ if not reducible, A is not diagonizable

2. for each eigenvalue λ, find basic eigenvector via homogeneous sol'tn $(\lambda I_n - A)x = 0$

3. if n = # of basic eigenvectors => A is diagonalizable

4. we set $P = [v_1 \ v_2 \ \cdots \ v_n]$ ← eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$ ← eigenvalues

e.g. $A = \begin{pmatrix} 4 & -6 \\ 1 & -1 \end{pmatrix}$

1. $c_A(x) = \det(xI_n - A)$

$$= \det\left(x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} x-4 & 6 \\ -1 & x+1 \end{bmatrix}\right)$$

$$= (x-4)(x+1) - (6)(-1)$$

$$= x^2 - 3x - 4 + 6$$

$$= x^2 - 3x + 2$$

$$= (x-1)(x-2), \quad \underbrace{\lambda_1 = 1 \quad \lambda_2 = 2}_{\text{eigenvalues}}$$

2. $(\lambda_i I_n - A)x = 0$

$\lambda_1 = 1, \quad (1 \cdot I_2 - A)x = 0$

$\Rightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix} \right) x = 0$

$\Rightarrow \left(\begin{bmatrix} -3 & 6 \\ -1 & 1 \end{bmatrix} \right) x = 0$

$\Rightarrow \left[\begin{array}{cc|c} -3 & 6 & 0 \\ -1 & 1 & 0 \end{array}\right] \cdots \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{array}{l} x_1 = 2t \\ x_2 = t \end{array} \Rightarrow v_1 = \begin{pmatrix} 2t \\ t \end{pmatrix} = t \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{eigenvector for } \lambda_1}$

$\lambda_2 = 2, \quad (2I_2 - A)x = 0$

$\Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right)$

$\Rightarrow \left[\begin{array}{cc|c} -2 & 6 & 0 \\ -1 & 3 & 0 \end{array}\right] = \cdots = \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{array}{l} x_1 = 3s \\ x_2 = s \end{array} \Rightarrow v_2 = \begin{pmatrix} 3s \\ s \end{pmatrix} = s \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\text{eigenvector } 2}$

3. 2 eigenvectors = n of A = 2 $\Rightarrow$ A is diagonal-izable

4. $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

Repeated Roots (m_λ) :

given characteristics of matrix A, if a root appears more than once ...

e.g. $C_A(x) = (x-1)^2(x+1)$

we'll call that a multiplicity of eigenvalue m_λ

if we find that for that eigenvalue, λ, if # of basic eigenvectors $(N_\lambda) = m_\lambda$

$\Rightarrow A$ is diagonalizable

you could add it into the steps

start w/ repeated roots since if $N_\lambda < m_\lambda \Rightarrow A$ is not diagonalizable