Data Science Math Skills
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The Binomial Theorem and Bayes' Theorem

Video companion

1 Introduction

Binomial theorem used when there are two possible outcomes—a success or a non-success, for example, flipping a coin—heads are a success, binary outcome.

Not limited to fair coins, where the probability of success is 0.5. Probability can be any value > 0 and < 1.

2 Binomial theorem

Probability of s successes in n trials, when probability of 1 success is p:

$$\underbrace{\binom{n}{s}}_{\text{\#number of success events}} \underbrace{p^{\overbrace{s}^{\text{\#successes}}}}_{\text{P(success)}} \underbrace{(1-p)^{\overbrace{n-s}^{\text{\#failures}}}}_{\text{P(failure)}}$$

where n is the number of independent trials (with replacement), s is the number of successes, and p is the probability of one success

Example: 72 heads out of 100 coin tosses of a fair coin

$$n = 100$$
$$s = 72$$
$$p = 0.5$$

$$\binom{100}{72}(0.5)^{72}(1-0.5)^{100-72}$$
$$= \binom{100}{72}(0.5)^{72}(0.5)^{28} = 3.94 \times 10^{-6}$$

3 With Bayes' theorem

Question: Is it more likely a fair coin ($p = 0.5$) heads or a bent coin ($p = 0.55$) heads?

$$
\begin{aligned}
&P(\text{fair coin} \mid 72 \text{ heads}/100) \\
&= \frac{P(72 \text{ heads}/100 \mid \text{fair coin})P(\text{fair coin})}{P(72 \text{ heads}/100 \mid \text{fair coin})P(\text{fair coin}) + P(72 \text{ heads}/100 \mid \text{bent coin})P(\text{bent coin})} \\
&= \frac{(3.94 \times 10^{-6})(1/2)}{(3.94 \times 10^{-6})(1/2) + (1.972 \times 10^{-4})(1/2)} \\
&= 1.96\%
\end{aligned}
$$

(assuming it is equally likely that the coin is fair or bent)

Therefore, there is $< 2\%$ probability that the coin is fair and $> 98\%$ probability that the coin is bent.

Bayes' theorem together with binomial theorem can tell us the probability of a process given data that we have observed.