Paul Bendich and Daniel Egger
Duke University

Fast Growth, Slow Growth: How Logarithms and Exponents Are Related

Video companion

1 Introduction

$2^3 = 8$

$\log_2(8) = 3$ $\qquad$ $\sqrt[3]{8} = 2$

Logarithm means "raised to what power?"

If the question is "what power of two is $2 \cdot 2 \cdot 2 = 8$?" then the answer is the logarithm to the base two of eight, which is $\log_2(8) = 3$.

Two general forms

$$b^x = N \qquad \text{"exponential form"}$$
$$x = \log_b(N) \qquad \text{"logarithmic form"}$$

Examples

If $b = 2$, $x = 3$, and $N = 8$:

$$2^3 = 8$$
$$3 = \log_2(8)$$

If $b = 2$, $x = 4$, and $N = 16$:

$$2^4 = 16$$
$$4 = \log_2(16)$$

2 Logs of one

Recall that raising any number to the power of zero is one, $b^0 = 1$. Therefore, the log, to any base, of one is zero.

$$\log_2(1) = 0 \qquad 2^0 = 1$$
$$\log_{10}(1) = 0 \qquad 10^0 = 1$$
$$\log_{20}(1) = 0 \qquad 20^0 = 1$$

3 General rules

1. Product rule
   $\log(xy) = \log(x) + \log(y)$

2. Quotient rule
   $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

3. Power and root rule
   $\log(x^n) = n\log(x)$

Examples

$$\begin{aligned}
\log_b(35) &= \log_b(5) + \log_b(7) \\
&= \log_b(70) - \log_b(2) \\
\log_2\left(\frac{16}{4}\right) &= \log_2(16) - \log_2(4) = 4 - 2 = 2 \\
\log_2(1000)^{\frac{1}{3}} &= \frac{1}{3}\log_2(1000) \\
\log_{10}(7)^5 &= 5\log_{10} 7 \\
\log_b(x)^{-1} &= -\log_b(x)
\end{aligned}$$

$$\begin{aligned}
\log_b x^2 y^{-3} &= \log_b x^2 + \log_b y^{-3} \\
&= 2\log_b x - 3\log_b y \\
\log_b \frac{x^2}{y^{-\frac{1}{2}}} &= \log_b x^2 - \log_b y^{-\frac{1}{2}} \\
&= 2\log_b x + \frac{1}{2}\log_b y
\end{aligned}$$

4 Problem-solving technique

Problem-solving technique: Treat both sides of an equation as though they were exponents.

$$x = y \qquad\qquad z^x = z^y$$

Example

$$\begin{aligned}
\log_2\left(\frac{39x}{(x-5)}\right) &= 4 \\
2^{\log_2\left(\frac{39x}{(x-5)}\right)} &= 2^4 \\
\frac{39x}{(x-5)} &= 16 \\
39x &= 16x - 80 \\
23x &= -80 \\
x &= -\frac{80}{23}
\end{aligned}$$