Notes on the GSW function gsw_geostrophic_velocity (geo_strf,long,lat,p)

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This function **gsw_geostrophic_velocity**(geo_strf,long,lat,p) calculates the difference between the geostrophic velocity at pressure p and that at the reference pressure (where $p = p_{\text{ref}}$ dbar). The general expression for this velocity difference is

$$\mathbf{k}\times\nabla_{\text{surf}}\vartheta = f\left(\mathbf{v}-\mathbf{v}_{\text{ref}}\right),$$

where f is the Coriolis parameter and ϑ is the geostrophic streamfunction in a particular surface. Note that the type of geostrophic streamfunction should be carefully chosen to be appropriate to the surface in which it is being used. The reference pressure $p = p_{\text{ref}}$ dbar is the value that was selected in the calls to the various geostrophic streamfunctions whose outputs are used as "geo_strf" in **gsw_geostrophic_velocity**(geo_strf,long,lat,p).

Below we reproduce sections 3.27 – 3.30 of the TEOS-10 Manual (IOC *et al*. (2010)) which describe four different types of geostrophic streamfunction. Dynamic height anomaly is the geostrophic streamfunction for use in an isobaric surface, while the Montgomery geostrophic streamfunction is designed for use in a surface of constant specific volume anomaly. The remaining choices of geostrophic streamfunction, the Cunningham and the "isopycnal" geostrophic streamfunctions of McDougall and Klocker (2010) are designed for use in various types of "density" surfaces, and are not exact geostrophic streamfunctions in any particular surface. For these streamfunctions the argument "p" in the input to this function is the pressure on these "density" surfaces and so varies along each surface.

It is emphasized that the choice of geostrophic streamfunction must be made carefully to match the surface in which it is to be used. Figures 2 and 3 of McDougall and Klocker (2010) are reproduced below ("Eqn.(62)" on the figure refers to their "isopycnal" geostrophic streamfunction). These figures illustrate the errors in geostrophic velocity that are introduced when using a geostrophic streamfunction in a surface for which it is not designed. The corresponding r.m.s. errors for the Montgomery geostrophic streamfunction in each of these six approximately neutral surfaces are not shown in these figures, but are considerably larger, at approximately $4x10^{-3}\,\text{m s}^{-1}$, illustrating that while the Montgomery geostrophic streamfunction is the correct streamfunction in a specific volume anomaly surface, it is rather inaccurate when used in $\omega-$surfaces (Klocker *et al*. (2009a,b)), γ^{n} surfaces (Jackett and McDougall (1997)) and in potential density surfaces, such as σ_2 surfaces. The Cunningham geostrophic streamfunction is considerably more accurate than the Montgomery streamfunction, but is not as accurate as the "isopycnal" streamfunction of McDougall and Klocker (2010) in these six surfaces. Hence it is the geostrophic streamfunction **gsw_geo_strf_isopycnal_CT** which is recommended for use in potential density surfaces, in $\omega-$surfaces (Klocker *et al*. (2009a,b)) and in γ^{n} surfaces (Jackett and McDougall (1997)).

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org See sections 3.27, 3.32 and appendix A.30.

Jackett, D. R. and T. J. McDougall, 1997: A neutral density variable for the world's oceans. *Journal of Physical Oceanography*, **27**, 237-263.

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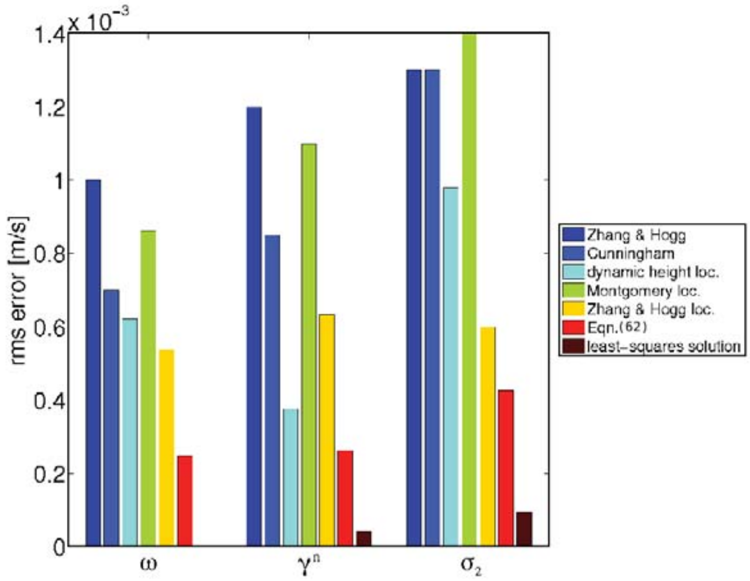

Figure 2. Rms errors of the geostrophic velocities calculated from the expressions of various geostrophic streamfunctions on an $\omega-$ surface, a γ^{n}-surface and a σ_2-surface. Each density surface has an average pressure of 1300 dbar.

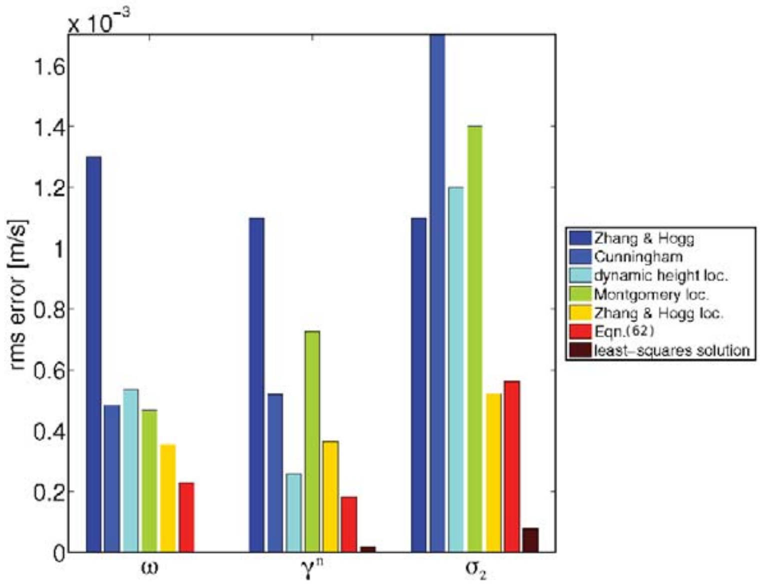

Figure 3. Rms errors of the geostrophic velocities calculated from the expressions of various geostrophic streamfunctions on an $\omega-$ surface, a γ^{n}-surface and a σ_2-surface. Each density surface has an average pressure of 800 dbar.

Here follows sections 3.27 – 3.30 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.27 Dynamic height anomaly

The dynamic height anomaly Ψ with respect to the sea surface is given by

$$\Psi = -\int_{P_0}^{P} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right],\Theta\left[p'\right],p'\right) dP', \quad \text{where} \quad \hat{\delta}\left(S_{\mathrm{A}},\Theta,p\right) = \hat{v}\left(S_{\mathrm{A}},\Theta,p\right) - \hat{v}\left(S_{\mathrm{SO}},0°\mathrm{C},p\right). \qquad (3.27.1)$$

This is the geostrophic streamfunction for the flow at pressure P with respect to the flow at the sea surface and $\hat{\delta}$ is the specific volume anomaly. Thus the two-dimensional gradient of Ψ in the P pressure surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ at P and at the sea surface $\mathbf{v}_0$ according to

$$\mathbf{k}\times\nabla_P\Psi \;=\; f\mathbf{v} - f\mathbf{v}_0 . \qquad (3.27.2)$$

Dynamic height anomaly is also commonly called the "geopotential anomaly". The specific volume anomaly, $\hat{\delta}$ in the vertical integral in Eqn. (3.27.1) can be replaced with specific volume $\hat{v}$ without affecting the isobaric gradient of the resulting streamfunction. That is, this substitution does not affect Eqn. (3.27.2) because the additional term is a function only of pressure. Traditionally it was important to use $\hat{\delta}$ in preference to $\hat{v}$ as it was more accurate with computer code which worked with single-precision variables. Since computers now regularly employ double-precision, this issue has been overcome and consequently either $\hat{\delta}$ or $\hat{v}$ can be used in the integrand of Eqn. (3.27.1), so making it either the "dynamic height anomaly" or the "dynamic height". As in the case of Eqn. (3.24.2), so also the dynamic height anomaly Eqn. (3.27.1) has not assumed that the gravitational acceleration is constant and so Eqn. (3.27.2) applies even when the gravitational acceleration is taken to vary in both the vertical and in the horizontal.

The dynamic height anomaly Ψ should be quoted in units of $\mathrm{m^2\ s^{-2}}$. These are the units in which the GSW Toolbox (appendix N) outputs dynamic height anomaly in the function **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref). When the last argument of this function, p_ref, is other than zero, the function returns the dynamic height anomaly with respect to a (deep) reference pressure p_ref, rather than with respect to P_0 (i.e. zero dbar sea pressure) as in Eqn. (3.27.1). In this case the lateral isobaric gradient of the streamfunction represents the geostrophic velocity difference relative to the (deep) p_{ref} pressure surface, that is,

$$\mathbf{k}\times\nabla_P\Psi \;=\; f\mathbf{v} - f\mathbf{v}_{\mathrm{ref}} . \qquad (3.27.3)$$

Note that the integration in Eqn. (3.27.1) of specific volume anomaly with pressure must be done with pressure in Pa (not dbar) in order to have the resultant isobaric gradient, $\nabla_P\Psi$, in the usual units, being the product of the Coriolis parameter (units of $\mathrm{s^{-1}}$) and the velocity (units of $\mathrm{m\,s^{-1}}$). The GSW function **gsw_steric_height**(SA,CT,p,p_ref) returns Ψ divided by the constant gravitational acceleration $g_0 = 9.7963\ \mathrm{m\,s^{-2}}$. Hence steric height remains proportional to an exact geostrophic streamfunction but the spatial variation of the gravitational acceleration ensures that it cannot be exactly equal to the height of an isobaric surface above a geopotential surface.

3.28 Montgomery geostrophic streamfunction

The Montgomery "acceleration potential" (Montgomery, 1937) Ψ^{M} defined by

$$\Psi^{\mathrm{M}} = \left(P - P_0\right)\hat{\delta} - \int_{P_0}^{P} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right],\Theta\left[p'\right],p'\right) dP' = \left(P - P_0\right)\hat{\delta} + \Psi \tag{3.28.1}$$

is the geostrophic streamfunction for the flow in the specific volume anomaly surface $\hat{\delta}\left(S_{\mathrm{A}},\Theta,p\right) = \hat{\delta}_1$ relative to the flow at $P = P_0$ (that is, at $p = 0\,\mathrm{dbar}$). Thus the two-dimensional gradient of Ψ^{M} in the $\hat{\delta}_1$ specific volume anomaly surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ in the $\hat{\delta} = \hat{\delta}_1$ surface and at the sea surface $\mathbf{v}_0$ according to

$$\mathbf{k}\times\nabla_{\hat{\delta}_1}\Psi^{\mathrm{M}} = f\mathbf{v} - f\mathbf{v}_0 \qquad \text{or} \qquad \nabla_{\hat{\delta}_1}\Psi^{\mathrm{M}} = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right). \tag{3.28.2}$$

The definition, Eqn. (3.28.1), of the Montgomery geostrophic streamfunction applies to all choices of the reference values $\tilde{\tilde{S}}_{\mathrm{A}}$ and $\tilde{\tilde{\Theta}}$ in the definition, Eqn. (3.7.3), of the specific volume anomaly. By carefully choosing these reference values the specific volume anomaly surface can be made to closely approximate the neutral tangent plane (McDougall and Jackett (2007)).

It is not uncommon to read of authors using the Montgomery geostrophic streamfunction, Eqn. (3.28.1), as a geostrophic streamfunction in surfaces other than specific volume anomaly surfaces. This incurs errors that should be recognized. For example, the gradient of the Montgomery geostrophic streamfunction, Eqn. (3.28.1), in a neutral tangent plane becomes (instead of Eqn. (3.28.2) in the $\hat{\delta} = \hat{\delta}_1$ surface)

$$\nabla_n\Psi^{\mathrm{M}} = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right) + \left(P - P_0\right)\nabla_n\hat{\delta}\,, \tag{3.28.3}$$

where the last term represents an error arising from using the Montgomery streamfunction in a surface other than the surface for which it was derived.

Zhang and Hogg (1992) subtracted an arbitrary pressure offset, $\left(\bar{P} - P_0\right)$, from $\left(P - P_0\right)$ in the first term in Eqn. (3.28.1), so defining the modified Montgomery streamfunction

$$\Psi^{\text{Z-H}} = \left(P-\bar{P}\right)\hat{\delta} - \int_{P_0}^{P} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right],\Theta\left[p'\right],p'\right) dP'. \tag{3.28.4}$$

The gradient of $\Psi^{\text{Z-H}}$ in a neutral tangent plane becomes

$$\nabla_n\Psi^{\text{Z-H}} = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right) + \left(P-\bar{P}\right)\nabla_n\hat{\delta}, \tag{3.28.5}$$

where the last term can be made significantly smaller than the corresponding term in Eqn. (3.28.3) by choosing the constant pressure $\bar{P}$ to be close to the average pressure on the surface. This term can be further minimized by suitably choosing the constant reference values $\tilde{\tilde{S}}_{\mathrm{A}}$ and $\tilde{\tilde{\Theta}}$ in the definition, Eqn. (3.7.3), of specific volume anomaly $\tilde{\tilde{\delta}}$ so that this surface more closely approximates the neutral tangent plane (McDougall (1989)). This improvement is available because it can be shown that

$$\rho\nabla_n\tilde{\tilde{\delta}} = -\left[\hat{\kappa}\left(S_{\mathrm{A}},\Theta,p\right) - \hat{\kappa}\left(\tilde{\tilde{S}}_{\mathrm{A}},\tilde{\tilde{\Theta}},p\right)\right]\nabla_n P \approx T_b^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\nabla_n P. \tag{3.28.6}$$

The last term in Eqn. (3.28.5) is then approximately

$$\left(P-\bar{P}\right)\nabla_n\tilde{\tilde{\delta}} \approx \tfrac{1}{2}\rho^{-1}T_b^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\nabla_n\left(P-\bar{P}\right)^2 \tag{3.28.7}$$

and hence suitable choices of $\bar{P}$, $\tilde{\tilde{S}}_{\mathrm{A}}$ and $\tilde{\tilde{\Theta}}$ can reduce the last term in Eqn. (3.28.5) that represents the error in interpreting the Montgomery geostrophic streamfunction, Eqn. (3.28.4), as the geostrophic streamfunction in a surface that is more neutral than a specific volume anomaly surface.

The Montgomery geostrophic streamfunction should be quoted in units of $\mathrm{m^2\,s^{-2}}$. These are the units in which the GSW Toolbox outputs the Montgomery geostrophic streamfunction in the function **gsw_geo_strf_Montgomery**(SA,CT,p,p_ref). When the last argument of this function, p_ref, is other than zero, the function returns the Montgomery geostrophic streamfunction with respect to a (deep) reference sea pressure p_ref, rather than with respect to $p = 0$ dbar (i.e. $P = P_0$) as in Eqn. (3.28.1).

3.29 Cunningham geostrophic streamfunction

Cunningham (2000) and Alderson and Killworth (2005), following Saunders (1995) and Killworth (1986), suggested that a suitable streamfunction on a density surface in a compressible ocean would be the difference between the Bernoulli function $\mathcal{B}$ and potential enthalpy h^0. Since the kinetic energy per unit mass, $\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}$, is a tiny component of the Bernoulli function, it was ignored and Cunningham (2000) essentially proposed the streamfunction $\Pi + \Phi^0$ (see his equation (12)), where

$$\begin{aligned}\Pi \;&\equiv \mathcal{B} - h^0 - \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - \Phi^0 \\ &= h - h^0 + \Phi - \Phi^0 \\ &= \hat{h}(S_A, \Theta, p) - \hat{h}(S_A, \Theta, 0) - \int_{P_0}^{P} \hat{v}\big(S_A(p'), \Theta(p'), p'\big)\, dP'.\end{aligned} \qquad (3.29.1)$$

The last line of this equation has used the hydrostatic equation $P_z = -g\rho$ to express $\Phi \approx gz$ in terms of the vertical pressure integral of specific volume and the height of the sea surface where the geopotential is Φ^0. The difference between enthalpy and potential enthalpy $h - h^0$ in this equation has been named "dynamic enthalpy" by Young (2010).

The definition of potential enthalpy, Eqn. (3.2.1), is used to rewrite the last line of Eqn. (3.29.1), showing that Cunningham's Π is also equal to

$$\begin{aligned}\Pi \;&= -\int_{P_0}^{P} \hat{v}\big(S_A(p'), \Theta(p'), p'\big) - \hat{v}\big(S_A, \Theta, p'\big)\, dP' \\ &= \Psi - \hat{h}\big(S_{SO}, 0°\text{C}, p\big) + \hat{h}\big(S_A, \Theta, p\big) - c_p^0\Theta\,.\end{aligned} \qquad (3.29.2)$$

The first line of this equations appears very similar to the expression, Eqn. (3.27.1), for dynamic height anomaly, the only difference being that in Eqn. (3.27.1) the pressure-independent values of Absolute Salinity and Conservative Temperature were S_{SO} and $0°$C whereas here they are the local values on the surface, S_A and Θ. While these local values of Absolute Salinity and Conservative Temperature are constant during the pressure integral in Eqn. (3.29.2), they do vary with latitude and longitude along any "density" surface.

The gradient of Π along the neutral tangent plane is

$$\nabla_n\Pi \;\approx\; \big\{\tfrac{1}{\rho}\nabla_z P - \nabla\Phi_0\big\} - \tfrac{1}{2}\rho^{-1}T_b^{\Theta}\left(P - P_0\right)^2 \nabla_n\Theta, \qquad (3.29.3)$$

(from McDougall and Klocker (2010)) so that the error in $\nabla_n\Pi$ in using Π as the geostrophic streamfunction is approximately $-\tfrac{1}{2}\rho^{-1}T_b^{\Theta}\left(P - P_0\right)^2\nabla_n\Theta$. When using the Cunningham streamfunction Π in a potential density surface, the error in $\nabla_\sigma\Pi$ is approximately $-\tfrac{1}{2}\rho^{-1}T_b^{\Theta}\left(P - P_0\right)\left(2P_r - P - P_0\right)\nabla_\sigma\Theta$. The Cunningham geostrophic streamfunction should be quoted in units of $\text{m}^2\,\text{s}^{-2}$ and is available in the GSW Oceanographic Toolbox as the function **gsw_geo_strf_Cunningham**(SA,CT,p,p_ref). When the last argument of this function, p_ref, is other than zero, the function returns the Cunningham geostrophic streamfunction with respect to a (deep) reference sea pressure p_ref, rather than with respect to $p = 0$ dbar (i.e. $P = P_0$) as in Eqn. (3.29.1).

3.30 Geostrophic streamfunction in an approximately neutral surface

In order to evaluate a relatively accurate expression for the geostrophic streamfunction in an approximately neutral surface a suitable reference seawater parcel $\left(\tilde{\tilde{S}}_{\mathrm{A}}, \tilde{\tilde{\Theta}}, \tilde{\tilde{p}}\right)$ is selected from the approximately neutral surface that one is considering, and the specific volume anomaly $\tilde{\tilde{\delta}}$ is defined as in (3.7.3) above. The approximate geostrophic streamfunction φ^{n} is given by (from McDougall and Klocker (2010))

$$\varphi^{\mathrm{n}} = \tfrac{1}{2}\left(P - \tilde{\tilde{P}}\right)\tilde{\tilde{\delta}}\left(S_{\mathrm{A}}, \Theta, p\right) - \tfrac{1}{12}\rho^{-1}T_{\mathrm{b}}^{\Theta}\left(\Theta - \tilde{\tilde{\Theta}}\right)\left(P - \tilde{\tilde{P}}\right)^{2} - \int_{P_0}^{P} \tilde{\tilde{\delta}}\, dP'. \qquad (3.30.1)$$

This expression is more accurate than the Montgomery and Cunningham geostrophic streamfunctions when used in potential density surfaces, in the ω-surfaces of Klocker *et al.* (2009a,b) and in the Neutral Density surfaces of Jackett and McDougall (1997). That is, in these surfaces $\nabla_n \varphi^{\mathrm{n}} \approx \frac{1}{\rho}\nabla_z P - \nabla\Phi_0 = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right)$ to a very good approximation. In Eqn. (3.30.1) $\rho^{-1}T_{\mathrm{b}}^{\Theta}$ is taken to be the constant value $2.7x10^{-15}\mathrm{K}^{-1}(\mathrm{Pa})^{-2}\mathrm{m}^{2}\mathrm{s}^{-2}$. This approximate isopycnal geostrophic streamfunction of McDougall and Klocker (2010) is available as the function **gsw_geo_strf_isopycnal** in the GSW Toolbox. When the last argument of this function, p_ref, is other than zero, the function returns the isopycnal geostrophic streamfunction with respect to a (deep) reference sea pressure p_ref, rather than with respect to the sea surface at $p = 0$ dbar (i.e. $P = P_0$) as in Eqn. (3.30.1).