Notes on the function gsw_enthalpy_diff_CT_exact(SA,CT,p_shallow,p_deep)

This function, **gsw_enthalpy_diff_CT_exact**(SA,CT,p_shallow,p_deep), returns the difference between the specific enthalpy of two seawater parcels, both having the same Absolute Salinity and Conservative Temperature, but having different pressures. This function uses the full TEOS-10 Gibbs function $g\left(S_{\mathrm{A}},t,p\right)$ of IOC *et al*. (2010), being the sum of the IAPWS-09 and IAPWS-08 Gibbs functions.

This function is simply two calls to each of two GSW functions as follows,

```
t_shallow = gsw_t_from_CT(SA,CT,p_shallow);
t_deep    = gsw_t_from_CT(SA,CT,p_deep);
enthalpy_diff_CT_exact = gsw_enthalpy_t_exact(SA,t_deep,p_deep) - ...
                         gsw_enthalpy_t_exact(SA,t_shallow,p_shallow);
```

References

IAPWS, 2008: Release on the IAPWS Formulation 2008 for the Thermodynamic Properties of Seawater. The International Association for the Properties of Water and Steam. Berlin, Germany, September 2008, available from www.iapws.org. This Release is referred to in the text as **IAPWS-08**.

IAPWS, 2009: Supplementary Release on a Computationally Efficient Thermodynamic Formulation for Liquid Water for Oceanographic Use. The International Association for the Properties of Water and Steam. Doorwerth, The Netherlands, September 2009, available from http://www.iapws.org. This Release is referred to in the text as **IAPWS-09**.

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Below, for motivation and for reference, is section 3.32 of the TEOS-10 Manual (IOC *et al*. (2010))

3.32 Pressure to height conversion

The vertical integral of the hydrostatic equation ($g = -v\,P_z$) can be written as

$$\begin{aligned}\int_0^z g\left(z'\right)dz' \;&=\; \Phi^0 - \int_{P_0}^{P} v\left(p'\right)dP' \;=\; -\int_{P_0}^{P} \hat{v}\left(S_{\mathrm{SO}},0°\mathrm{C},p'\right)dP' + \Psi \,+ \Phi^0 \\ &=\; -\,\hat{h}\left(S_{\mathrm{SO}},0°\mathrm{C},p\right) \,+\, \Psi \,+ \Phi^0,\end{aligned} \qquad (3.32.1)$$

where the dynamic height anomaly Ψ is expressed in terms of the specific volume anomaly $\hat{\delta} = \hat{v}\left(S_{\mathrm{A}},\Theta,p\right) - \hat{v}\left(S_{\mathrm{SO}},0°\mathrm{C},p\right)$ by

$$\Psi \;=\; -\int_{P_0}^{P} \hat{\delta}\left(p'\right)dP', \qquad (3.32.2)$$

where $P_0 = 101\,325\,\mathrm{Pa}$ is the standard atmosphere pressure. Writing the gravitational acceleration of Eqn. (D.3) as $g = g\left(\phi,z\right) = g\left(\phi,0\right)\left(1-\gamma z\right)$, the left-hand side of Eqn.

(3.32.1) becomes $g(\phi,0)\left(z-\tfrac{1}{2}\gamma z^2\right)$, and using the 48-term expression for the specific enthalpy of Standard Seawater, Eqn. (3.32.1) becomes

$$\hat{h}^{48}\left(S_{\mathrm{SO}},0°\mathrm{C},p\right)-\Psi-\Phi^0+g\left(\phi,0\right)\left(z-\tfrac{1}{2}\gamma z^2\right)=0\ . \qquad (3.32.3)$$

This is the equation that is solved for height z in the GSW function **gsw_z_from_p**. It is traditional to ignore $\Psi+\Phi^0$ when converting between pressure and height, and this can be done by simply calling this function with only two arguments, as in **gsw_z_from_p**(p,lat). Ignoring $\Psi+\Phi^0$ makes a difference to z of up to 4m at 5000 dbar. Note that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to be $\Psi+\Phi^0$ and this is used in the solution of Eqn. (3.32.3). Dynamic height anomaly Ψ can be evaluated using the GSW function **gsw_geo_strf_dyn_height**. The GSW function **gsw_p_from_z** is the exact inverse function of **gsw_z_from_p**; these functions yield outputs that are consistent with each other to machine precision.

When vertically integrating the hydrostatic equation $P_z=-g\rho$ in the context of an ocean model where Absolute Salinity S_{A} and Conservative Temperature Θ are piecewise constant in the vertical, the geopotential (Eqn. (3.24.2))

$$\Phi=\int_0^z g\left(z'\right)dz' = \Phi^0-\int_{P_0}^{P} v\left(p'\right)dP', \qquad (3.32.4)$$

can be evaluated as a series of exact differences. If there are a series of layers of index i separated by pressures p^i and p^{i+1} (with $p^{i+1}>p^i$) then the integral can be expressed (making use of (3.7.5), namely $h_P|_{S_{\mathrm{A}},\Theta}=\hat{h}_P=v$) as a sum over n layers of the differences in specific enthalpy so that

$$\Phi=\Phi^0-\int_{P_0}^{P} v\left(p'\right)dP' = \Phi^0-\sum_{i=1}^{n}\left[\hat{h}\left(S_{\mathrm{A}}^i,\Theta^i,p^{i+1}\right)-\hat{h}\left(S_{\mathrm{A}}^i,\Theta^i,p^i\right)\right]. \qquad (3.32.5)$$

The difference in enthalpy at two different pressures for given values of S_{A} and Θ is available in the GSW Oceanographic Toolbox via the function **gsw_enthalpy_diff**. The summation of a series of such differences in enthalpy occurs in the GSW functions to evaluate two geostrophic streamfunctions from piecewise-constant vertical property profiles, **gsw_geo_strf_dyn_height_pc** and **gsw_geo_strf_isopycnal_pc**.