Notes on the function gsw_geo_strf_Cunningham(SA,CT,p,p_ref)

Notes written 9th April 2011

This function, **gsw_geo_strf_Cunningham**(SA,CT,p,p_ref) evaluates the Cunningham geostrophic streamfunction (Cunningham, 2000) $\Pi\left(S_{\mathrm{A}}, \Theta, p, p_{\mathrm{ref}}\right)$ relative to a reference pressure p_ref. It uses the computationally-efficient 48-term rational function expression of McDougall and Barker (2013) for the specific volume of seawater $\hat{v}\left(S_{\mathrm{A}}, \Theta, p\right)$ in terms of Absolute Salinity S_{A}, Conservative Temperature Θ and pressure p. This 48-term rational function expression for density is discussed in McDougall and Barker (2013) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al*. (2010)). For dynamical oceanography we may take the 48-term rational function expression for density as essentially reflecting the full accuracy of TEOS-10.

The input variables SA,CT,p are either a single vertical cast or a series of such vertical casts. The input variable p_{ref} is a single positive scalar reference pressure (in dbar). When p_ref is zero, **gsw_geo_strf_Cunningham**(SA,CT,p,p_ref) returns the Cunningham geostrophic streamfunction with respect to the sea surface, otherwise, the function returns the Cunningham geostrophic streamfunction with respect to the (deep) reference pressure p_ref.

The specific volume anomaly we use is $\hat{\delta}\left(S_{\mathrm{A}}, \Theta, p\right)$ defined with respect to the constant reference values $S_{\mathrm{SO}} \equiv 35.165\,04 \mathrm{~g} \mathrm{~kg}^{-1}$ and $\Theta=0^{\circ} \mathrm{C}$ as

$$\hat{\delta}\left(S_{\mathrm{A}}, \Theta, p\right)=\hat{v}\left(S_{\mathrm{A}}, \Theta, p\right)-\hat{v}\left(S_{\mathrm{SO}}, 0^{\circ} \mathrm{C}, p\right), \tag{1}$$

and the Cunningham geostrophic streamfunction Π is defined in terms of the dynamic height anomaly Ψ by (from Eqn. (3.29.1) of the TEOS-10 Manual (IOC *et al*., 2010)

$$\begin{aligned} \Pi &= \hat{h}\left(S_{\mathrm{A}}, \Theta, p\right)-\hat{h}\left(S_{\mathrm{A}}, \Theta, 0\right)-\int_{P_0}^{P} \hat{v}\left(S_{\mathrm{A}}\left(p^{\prime}\right), \Theta\left(p^{\prime}\right), p^{\prime}\right) d P^{\prime} \\ &= \hat{h}\left(S_{\mathrm{A}}, \Theta, p\right)-c_p^0 \Theta-\hat{h}\left(S_{\mathrm{SO}}, 0^{\circ} \mathrm{C}, p\right)+\Psi . \end{aligned} \tag{2}$$

The code **gsw_geo_strf_Cunningham**(SA,CT,p,p_ref) operates by evaluating the second line of Eqn. (2), and the dynamic height anomaly Ψ is found from **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref). The first enthalpy, $\hat{h}\left(S_{\mathrm{A}}, \Theta, p\right)$, on the second line of Eqn. (2) is evaluated from the GSW function **gsw_enthalpy** and the other enthalpy, $\hat{h}\left(S_{\mathrm{SO}}, 0^{\circ} \mathrm{C}, p\right)$, is found from **gsw_enthalpy_SSO_0_p**(p) which is a streamlined version **gsw_enthalpy**, simplified because the Conservative Temperature argument is zero.

When p_ref is set to $0 \mathrm{~dbar}$, Π is the Cunningham geostrophic streamfunction relative to the sea surface, and when p_ref is set to a (deep) reference pressure, Π is the Cunningham geostrophic streamfunction relative to this reference pressure surface.

References

Cunningham, S. A., 2000: Circulation and volume flux of the North Atlantic using synoptic hydrographic data in a Bernoulli inverse. *J. Marine Res.*, **58**, 1-35.

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org See section 3.27 and appendix A.30.

McDougall T. J. and P. M. Barker, 2013: A computationally efficient 48-term expression for the density of seawater in terms of Conservative Temperature, and related properties of seawater. To be submitted to *Journal of Atmospheric and Ocean Technology.*

Here follows section 3.29 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.29 Cunningham geostrophic streamfunction

Cunningham (2000) and Alderson and Killworth (2005), following Saunders (1995) and Killworth (1986), suggested that a suitable streamfunction on a density surface in a compressible ocean would be the difference between the Bernoulli function $\mathcal{B}$ and potential enthalpy h^0. Since the kinetic energy per unit mass, $0.5\mathbf{u}\cdot\mathbf{u}$, is a tiny component of the Bernoulli function, it was ignored and Cunningham (2000) essentially proposed the streamfunction $\Pi+\Phi^0$ (see his equation (12)), where

$$\begin{aligned}\Pi &\equiv \mathcal{B} - h^0 - \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - \Phi^0 \\ &= h - h^0 + \Phi - \Phi^0 \\ &= \hat{h}(S_\mathrm{A},\Theta,p) - \hat{h}(S_\mathrm{A},\Theta,0) - \int_{P_0}^{P} \hat{v}\big(S_\mathrm{A}(p'),\Theta(p'),p'\big)\,dP'.\end{aligned} \qquad (3.29.1)$$

The last line of this equation has used the hydrostatic equation $P_z = -g\rho$ to express $\Phi \approx gz$ in terms of the vertical pressure integral of specific volume and the height of the sea surface where the geopotential is Φ^0. The difference between enthalpy and potential enthalpy $h - h^0$ in this equation has been named "dynamic enthalpy" by Young (2010).

The definition of potential enthalpy, Eqn. (3.2.1), is used to rewrite the last line of Eqn. (3.29.1), showing that Cunningham's Π is also equal to

$$\Pi = -\int_{P_0}^{P} \hat{v}\big(S_\mathrm{A}(p'),\Theta(p'),p'\big) - \hat{v}\big(S_\mathrm{A},\Theta,p'\big)\,dP'. \qquad (3.29.2)$$

In this form it appears very similar to the expression, Eqn. (3.27.1), for dynamic height anomaly, the only difference being that in Eqn. (3.27.1) the pressure-independent values of Absolute Salinity and Conservative Temperature were S_SO and $0°\mathrm{C}$ whereas here they are the local values on the surface, S_A and Θ. While these local values of Absolute Salinity and Conservative Temperature are constant during the pressure integral in Eqn. (3.29.2), they do vary with latitude and longitude along any "density" surface.

The gradient of Π along the neutral tangent plane is

$$\nabla_n\Pi \approx \left\{\tfrac{1}{\rho}\nabla_z P - \nabla\Phi_0\right\} - \tfrac{1}{2}\rho^{-1}T_\mathrm{b}^{\Theta}\left(P-P_0\right)^2\nabla_n\Theta, \qquad (3.29.3)$$

(from McDougall and Klocker (2010)) so that the error in $\nabla_n\Pi$ in using Π as the geostrophic streamfunction is approximately $-\tfrac{1}{2}\rho^{-1}T_\mathrm{b}^{\Theta}\left(P-P_0\right)^2\nabla_n\Theta$. When using the Cunningham streamfunction Π in a potential density surface, the error in $\nabla_\sigma\Pi$ is approximately $-\tfrac{1}{2}\rho^{-1}T_\mathrm{b}^{\Theta}\left(P-P_0\right)\left(2P_\mathrm{r}-P-P_0\right)\nabla_\sigma\Theta$. The Cunningham geostrophic streamfunction should be quoted in units of $\text{m}^2\ \text{s}^{-2}$ and is available in the GSW Oceanographic Toolbox as the function **gsw_geo_strf_Cunningham**(SA,CT,p,p_ref). When the last argument of this function, p_ref, is other than zero, the function returns the Cunningham geostrophic streamfunction with respect to a (deep) reference sea pressure p_ref, rather than with respect to $p = 0\ \text{dbar}$ (i.e. $P = P_0$) as in Eqn. (3.29.1).