Row Reduction

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Finding solutions to systems of linear equations:

$$\begin{array}{ccccccc} & & 2y & + & 3z & = & -4 \\ 2x & + & 6y & + & 2z & = & 4 \\ 3x & & & + & z & = & 3 \end{array}$$

An augmented matrix representing the system of linear equations:

$$\begin{bmatrix} 0 & 2 & 2 & -4 \\ 2 & 6 & 2 & 4 \\ 3 & 0 & 1 & 3 \end{bmatrix}$$

The coefficient matrix representing the system of linear equations:

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix}$$

The row reduction algorithm is as follows:

1. multiply the first row by the reciprical of the first element. This will make the first row have the form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \dots & a_{1m} & b_1 \end{bmatrix}$$

2. subtract multiples of the first row from every row below such that the first element of those rows is zero. This should make the matrix take the following form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & \dots & a_{2m} & b_2 \\ 0 & a_{32} & a_{33} & & a_{3m} & b_3 \\ & \vdots & & \ddots & & \\ 0 & a_{n2} & a_{n3} & & a_{nm} & b_n \end{bmatrix}$$

3. repeat for the second element, then the third, etc. afterward the matrix should take the form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & & a_{1m} & b_1 \\ 0 & 1 & a_{23} & \dots & a_{2m} & b_2 \\ 0 & 0 & 1 & & a_{3m} & b_3 \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & & 1 & b_n \end{bmatrix}$$

4. subtract multiples of the second row from each row above such the second column results in zero. Repeat for third row and third column, then fourth, etc. This should result in the matrix taking the form:

$$\begin{bmatrix} 1 & 0 & 0 & & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & b_2 \\ 0 & 0 & 1 & & 0 & b_3 \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & & 1 & b_n \end{bmatrix}$$

Row Reduction example

$$\begin{aligned}
\begin{bmatrix} 0 & 2 & 2 & -4 \\ 2 & 6 & 2 & 4 \\ 3 & 0 & 1 & 3 \end{bmatrix} &= \begin{bmatrix} 2 & 6 & 2 & 4 \\ 0 & 2 & 2 & -4 \\ 3 & 0 & 1 & 3 \end{bmatrix} \text{swap row (1), (2)} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 2 & 2 & -4 \\ 3 & 0 & 1 & 3 \end{bmatrix} \text{divide row (1) by 2} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 2 & 2 & -4 \\ 0 & -9 & -2 & -3 \end{bmatrix} \text{subtract } 3 \times \text{row (1) from row (3)} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & -9 & -2 & -3 \end{bmatrix} \text{divide row (2) by 2} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 7 & -21 \end{bmatrix} \text{add } 9 \times \text{row (2) to row (3)} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \text{divide row (3) by 7} \\
&= \begin{bmatrix} 1 & 0 & -2 & 8 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \text{subtract } 3 \times \text{row (2) from row (1)} \\
&= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \text{add } 2 \times \text{row (3) to row (1), subtract row (3) from row (2)}
\end{aligned}$$