

# Propositional Logic

Patrick Chen

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A proposition is a declarative sentence with either a true or false value. Usually, propositions are denoted by lowercase letters:  $p, q, r$

## Operators

- Negation  $\neg$  (NOT):  $\neg p$  is the opposite of  $p$ .
- conjunction  $\wedge$  (AND):  $p \wedge q$  is true when both  $p$  and  $q$  are true.
- disjunction  $\vee$  (OR):  $p \vee q$  is true when either  $p$  or  $q$  is true.
- Exclusive disjunction  $\oplus$  (XOR):  $p \oplus q$  is true when exactly one of  $p$  and  $q$  is true but not both.
- Implication  $\rightarrow$ : conditional statement  $p \rightarrow q$  is false when  $p$  is true and  $q$  is false, and true otherwise.
  - if  $p$ , then  $q$
  - $q$  if  $p$
  - $p$  is sufficient for  $q$
  - $q$  unless  $\neg p$
- Biconditional  $\leftrightarrow$  (EQ): biconditional statement  $p \leftrightarrow q$  is the formalism for  $(p \rightarrow q) \wedge (q \rightarrow p)$ .
  - $p$  if and only if  $q$

## Manipulations of operators

Two statements are equivalent if both have the same truth value. Equivalence of statements is shown with the " $\equiv$ " symbol.

## Implication

If  $p \rightarrow q$  then:

- converse:  $q \rightarrow p$
- contrapositive  $\neg q \rightarrow \neg p$
- inverse:  $\neg p \rightarrow \neg q$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

## Laws

- Identity:  $p \wedge T \equiv p$  and  $p \vee F \equiv p$
- Domination:  $p \wedge F \equiv F$  and  $p \vee T \equiv T$
- Idempotent:  $p \wedge p \equiv p$  and  $p \vee p \equiv p$
- Double Negation  $\neg(\neg p) \equiv p$
- Commutative:  $p \wedge q \equiv q \wedge p$  and  $p \vee q \equiv q \vee p$
- Negation:  $p \vee \neg p \equiv T$  and  $p \wedge \neg p \equiv F$
- Associative:  $(p \vee q) \vee r \equiv p \vee (q \vee r)$  and  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distribution:  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$  and  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- De Morgan's:  $\neg(p \vee q) \equiv \neg p \wedge \neg q$  and  $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- Absorption:  $p \vee (p \wedge q) \equiv p$  and  $p \wedge (p \vee q) \equiv p$
- Implication:  $p \rightarrow q \equiv \neg p \vee q$

## Tautology

A tautology is a statement that is always true. If the statement  $p \leftrightarrow q$  is a tautology, then  $p \equiv q$

### Example 1

Show that  $p \rightarrow q = \neg q \rightarrow \neg p$

$$\begin{aligned} p \rightarrow q &= \neg p \vee q && \text{Implication} \\ &= q \vee \neg p && \text{Commutative} \\ &= \neg(\neg q) \vee (\neg p) && \text{Double Negative} \\ &= \neg q \rightarrow \neg p && \text{implies} \end{aligned}$$

### Example 2

Show that  $(p \rightarrow r) \wedge (q \rightarrow r) = (p \vee q) \rightarrow r$

$$\begin{aligned} (p \rightarrow r) \wedge (q \rightarrow r) &= (\neg p \vee r) \wedge (\neg q \vee r) && \text{Implication} \\ &= (\neg p \wedge \neg q) \vee r && \text{Distribution} \\ &= \neg(p \vee q) \vee r && \text{De Morgan's} \\ &= (p \vee q) \rightarrow r && \text{Implication} \end{aligned}$$

## Propositional Functions

In zero order logic, there are no quantifiers (for all, there exists, etc.). In order to extend Propositional logic, propositional functions are needed.

## Quantifiers

- Universal quantification of  $P(x)$  is the proposition " $P(x)$  for all values of  $x$  in the domain". This is denoted by  $\forall xP(x)$ . If the proposition holds for all values in the domain, then  $\forall xP(x)$  evaluates to true, but if even one value in the domain does not satisfy the proposition, then  $\forall xP(x)$  evaluates to false.
- Existential quantifiers of  $P(x)$  is the proposition "There exists an  $x$  in the domain such that  $P(x)$ ". This is denoted by  $\exists xP(x)$ . If the proposition  $P(x)$  is true for even a single value  $x$  in the domain, then  $\exists xP(x)$  is true. If there are no values  $x$  in the domain that satisfies the proposition, then  $\exists xP(x)$  is false.

## Manipulations of quantifiers

A negation can be brought inside a quantifier by inverting the quantifier.

$$\begin{aligned}\neg\forall xP(x) &= \exists x\neg P(x) \\ \neg\exists xP(x) &= \forall x\neg P(x)\end{aligned}$$

A negation can also be brought inside a propositional function.

$$\neg\forall x(x^2 > x) = \exists x\neg(x^2 > x) = \exists x(x^2 \leq x)$$

## Example 3

Show that  $\neg\forall x(P(x) \rightarrow Q(x)) = \exists x(P(x) \wedge \neg Q(x))$

$$\begin{aligned}\neg\forall x(P(x) \rightarrow Q(x)) &= \neg\forall x(\neg P(x) \vee Q(x)) && \text{Implication} \\ &= \exists x\neg(\neg P(x) \vee Q(x)) && \text{Quantifier} \\ &= \exists x(P(x) \wedge \neg Q(x)) && \text{De Morgan's}\end{aligned}$$

## Nested Quantifiers

When quantifiers are nested, they are read left to right. Quantifiers of the same type commute, but quantifiers of different types do not commute. For example, let  $P(x)$  = student  $x$  falls asleep during lecture  $y$ .  $\exists x\forall yP(x)$  means that there is a student that falls asleep during all lectures.  $\forall y\exists xP(x)$  means that in every lecture, there is a student that falls asleep.

$$\begin{aligned}\forall x\forall yP(x, y) &= \forall y\forall xP(x, y) \\ \exists x\exists yP(x, y) &= \exists y\exists xP(x, y) \\ \forall x\exists yP(x, y) &\neq \exists y\forall xP(x, y)\end{aligned}$$

Although the quantifiers do not commute, then can imply other propositions with nested quantifiers are true.

$$\exists x\forall yP(x, y) \rightarrow \forall y\exists xP(x, y)$$

**Example 4**

Is  $\forall x \exists y (x^2 - y^2 = 1)$  true for the domain  $x, y \in \mathbb{R}$

$$\text{If } x = \frac{1}{2}$$

$$\text{Then } y^2 = -\frac{3}{4}$$

Since no real number squared to a negative, this proposition is false.