Partial Fraction

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$$\int \frac{P(x)}{Q(x)}\,dx$$

If $P(x)$ and $Q(x)$ are polynomials.

- If $deg(P) \geq deg(Q)$, then use polynomial long division.
- If the degree of the numerator is less than the degree of the denominator and $Q(x)$ is factorizable into non-repeating linear terms, then:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\ldots(a_nxb_n)$$
$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nxb_n}$$

- If the degree of the numerator is less than the degree of the denominator and the factorization of $Q(x)$ contains repeated roots, then there must be fractions for each of the degrees for that root

$$P(x) = 4$$
$$Q(x) = (x - 3)^3(x + 2)$$
$$\frac{4}{(x-3)^3(x+2)} = \frac{A_1}{(x-3)} + \frac{A_2}{(x-3)^2} + \frac{A_3}{(x-3)^3} + \frac{A_4}{x+2}$$

- If the denominator is a irreducible polynomial (polynomials with no real roots), instead of putting constants in the numerator, put a polynomial with one fewer degree in the numerator.

$$\frac{1}{(x-1)(x^2+x+2)} = \frac{A_1}{x-1} + \frac{A_2x + B_2}{x^2+x+2}$$

Since these equations are true for all values x, certain values of x can be strategically substituted to cancel the other factors and make solving easier. Every polynomial can be factored into linear and quadratic terms in the reals.

$$x^4 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$$
$$a = \sqrt{2}$$
$$b = -\sqrt{2}$$

Example 1

$$\int \frac{x^3 + x}{x - 1}\ dx$$

$$
\begin{array}{rllll}
 & x^2 & -x & +2 & \\ \cline{2-5}
x - x\,| & x^3 & & +x & \\
 & x^3 & -x^2 & & \\ \cline{2-4}
 & & -x^2 & +x & \\
 & & x^2 & -x & \\ \cline{3-4}
 & & & 2x & \\
 & & & 2x & -2 \\ \cline{4-5}
 & & & & 2
\end{array}
$$

$$
\begin{aligned}
\frac{x^3 + x}{x - 1} &= (x^2 - x + 2) + \frac{2}{x - 1} \\
\int \frac{x^3 + x}{x - 1}\ dx &= \int (x^2 - x + 2) + \frac{2}{x - 1}\ dx \\
&= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x - 1| + c
\end{aligned}
$$

Example 2

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\ dx$$

$$
\begin{aligned}
2x^3 + 3x^2 - 2x &= x(2x^2 + 3x - 2) \\
&= x(2x^2 + 4x - 1x - 2) \\
&= x(x + 2)(2x - 1) \\
\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} &= \frac{A_1}{x} + \frac{A_2}{2x + 2} + \frac{A_3}{x + 2} \\
&= \frac{A_1(2x + 2)(x + 2) + A_2x(x + 2) + A_3x(2x + 2)}{2x^3 + 3x^2 - 2x} \\
x^2 + 2x - 1 &= A_1(2x + 2)(x + 2) + A_2x(x + 2) + A_3x(2x + 2) \\
x^2 + 2x - 1 &= (2A_1 + A_2 + 2A_3)x^2 + (3A_1 + 2A_2 - A_3)x - 2A_1
\end{aligned}
$$

$$
\begin{aligned}
x^2 &= (2A_1 + A_2 + 2A_3)x^2 \\
2x &= (3A_1 + 2A_2 - A_3)x \\
-1 &= -2A_1 \\
A_1 &= 0/2 \\
A_2 &= a \\
A_3 &= -1/10
\end{aligned}
$$

Example 3

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\ dx$$

$$\begin{array}{r|llllll}
 & x & +1 & & & & \\ \cline{2-6}
x^3 - x^2 - x + 1 & x^4 & & -2x^2 & +4x & +1 \\
 & x^4 & -x^3 & -x^2 & +x & \\ \cline{2-6}
 & & x^3 & -x^2 & +3x & +1 \\
 & & x^3 & -x^2 & -x & +1 \\ \cline{3-6}
 & & & & 2x &
\end{array}$$

$$\int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}\ dx$$

$$\begin{array}{r|llll}
 & x^2 & & -1 & \\ \cline{2-5}
(x-1) & x^3 & -x^2 & -x & +1 \\
 & x^3 & -x^2 & & \\ \cline{2-5}
 & & & -x & +1 \\
 & & & -x & +1 \\ \cline{4-5}
 & & & & 0
\end{array}$$

$$\int x + 1 + \frac{4x}{(x-1)(x^2-1)}\ dx$$

$$\begin{aligned}
\frac{4x}{(x-1)(x^2-1)} &= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1} \\
&= A_1(x-1)() \\
4x &= (A_1 + A_3)x^2 + (A_2 - 2A_3)x + (-A_1 + A_2 + A_3) \\
A_1 &= 1 \\
A_2 &= 2 \\
A_3 &= -1
\end{aligned}$$

$$\begin{aligned}
&\int x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\ dx \\
&= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + c
\end{aligned}$$

Example 4

$$\int \frac{-2x^2+4x+2}{(x-1)(x^2+3)}\ dx$$

$$\begin{aligned}
\frac{A_1}{x-1}+\frac{A_2x+B_2}{(x^2+3)} &= \frac{A_1(x^2+3)+(A_2x+B_2)(x-1)}{(x-1)(x^2+3)}\\
-2x^2+4x+2 &= A_1x^2+3A_1+A_2x^2+B_2x-A_2x-1\\
A_1 &= 1\\
A_2 &= -3\\
B_2 &= 1
\end{aligned}$$

$$\begin{aligned}
&\int \frac{1}{x-1}+\frac{-3x+1}{x^2+3}\ dx\\
&= \ln|x-1|+\int \frac{-3x}{x^2+3}+\frac{1}{x^2+3}\ dx\\
&= \ln|x-1|+\int \frac{-3x}{x^2+3}+\frac{1}{x^2+3}\ dx\\
&= \ln|x-1|-\frac{3}{2}\int \frac{1}{u}\ du+\int \frac{1}{x^2+3}\ dx\\
&= \ln|x-1|-\frac{3}{2}\ln|x^2+3|+\int \frac{1}{x^2+3}\ dx\\
&= \ln|x-1|-\frac{3}{2}\ln|x^2+3|+1/3\int \frac{1}{(\frac{x}{\sqrt{3}})^2+1}\ dx\\
&= \ln|x-1|-\frac{3}{2}\ln|x^2+3|+\sqrt{3}/3\int \frac{1}{u^2+1}\ dx\\
&= \ln|x-1|-\frac{3}{2}\ln|x^2+3|+\frac{\sqrt{3}}{3}\tan^{-1}(\frac{x}{\sqrt{3}})
\end{aligned}$$