Counting

Patrick Chen

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Product Rule

Suppose a procedure A can be divided into k tasks A_k. If there are n_k ways to do task k, then the amount of ways to complete the procedure is the product of the ways to do each task.

$$|A_1 \times A_2 \times \cdots \times A_k| = |A_1|\,|A_2|\ldots|A_k|$$

Counting functions

If set A has n elements and set B has m elements, then each elements in A can be mapped to an element in B, leading to m choices. With n total elements, the amount of functions that map from A to B is the

$$|f : A \mapsto B| = \underbrace{m \times m \times \cdots \times m}_{n} = m^n$$

For the set of injective functions, the first choice is m and the second is $(m-1)$.

$$|f : A \hookrightarrow B| = \underbrace{m(m-1)(m-2)\ldots(m-n+1)}_{n} = \frac{m!}{(m-n)!}$$

Sum Rule

Suppose a procedure can be done in k ways. If the kth way requires n_k without any overlap, then the amount of ways to do the procedure is the sum of all the ways.

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k|$$

If the intersection between A_i and A_j is not empty, then the intersection of the two sets is double counted. Thus to get an accurate count, the intersection must be subtracted.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

When there are more than two sets, all permutations of the sets have to be added or subtracted. If there is an even amount of sets in the intersection, the size will have to be subtracted, otherwise it will have to be added.

$$
\begin{aligned}
|A \cup B \cup C| = \quad & (|A| + |B| + |C|) \\
& - (|A \cap B| + |B \cap C| + |C \cap A|) \\
& + (|A \cap B \cap C|)
\end{aligned}
$$

Example

How many hexadecimal strings are there of length 2 or 3.

$$|S_{[2,3]}| = |S_2| + |S_3|$$
$$= 16^2 + 16^3$$

Example 2

How many bit strings of length 8 either start with 1 of end with 00?
Let A be the set of strings that start with 1
Let B be the set of strings that end with 00
The set A will be in the form $1_____$ and will have 7 binary choices
The set B will be in the form $_____00$ and will have 6 binary choices
The set $A \cup B$ will be in the form $1\_\_\_\_\_00$ and will have 5 binary choices

$$|A \cup B| = |A| + |B| - |A \cup B|$$
$$= 2^7 + 2^6 - 2^5$$

Division rule

Suppose a task can be done in n ways. For each way, there are d equivalent ways. There are $\frac{n}{d}$ distinct ways to do the task.

Example

How many different ways to arrange $1, 1, 2, 3, 4$ in a sequence?
There are 5! ways to arrange five distinct values, but there are 2! equivalent ways to arrange them because there are 2 ones in the sequence. Thus there are $\frac{5!}{2!} = 60$ ways to arrange them.

Example 2

How many ways are there to seat 5 people around a circular table? Rotations around to circular table are not considered distinct.
There are 5! ways to seat 5 people in a circular table and there are 5 possible rotations, thus there are $\frac{5!}{5} = 4! = 24$ distinct ways to seat 5 people around a circular table.

Pigeonhole Principle

If k is a positive integer and $k + 1$ more objects are place into k boxes, then there is at least one box containing two or more of the objects. Formally, if A and B be two countable sets such that $|A| > |B|$, then for all $f : A \mapsto B$, there are $a, b \in A$ such that $f(a) = f(b)$.

Generalized Pigeonhole Principle

If n objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{n}{k} \rceil$ objects.

$$\text{argmin}\left[n \mid \left\lceil \frac{n}{k} \right\rceil = c\right]$$

Example

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected

$$\text{argmin}\left[n \mid \left\lceil \frac{n}{4} \right\rceil = 3\right] = 9$$

Example 2

Show that among any $(n+1)$ positive integers (set A) not exceeding $2n$, there must be an integer that divides another integer.

$$\text{let } a_i = 2^{u_i} q_i \text{ for some } u_i, q_i$$
$$\text{where } q_i \text{ is odd,}$$
$$j \in \{1, 2, \ldots, n+1\}$$
$$u_i \in \mathbb{N}_0$$

Since $a_j \leq 2n$, then $q_j \leq 2n$ thus the set $(q_1, \ldots, q_{n+1})$ is $n+1$ odd numbers.
Since there are n odd numbers less than $2n$,
by the pigeonhole principle $q_r = q_s$ for some $1 \leq r \leq n+1$ and $1 \leq s \leq n+1$.
Thus $a_r = 2^{u_r} q_r, a_s = 2^{u_s} q_s$.
If $u_r \leq u_s$ then $a_r \mid a_s$ otherwise $a_s \mid a_r$

Permutations

A permutation is an ordered arrangement of a set of objects. In general, any set with size n has $n!$ permutations. The r-permutation of a set A is a permutation of a subset of A of size r. The total amount of r-permutations of size n is $\frac{n!}{(n-r)!}$

$$P(n, r) = \frac{n!}{(n-r)!}$$

Example

For the set A, how many permutations contain the string abc?

$$A = \{a, b, c, d, e, f, g\}$$

This is equivalent to asking how many permutations are there when abc is considered one element.

$$A' = \{abc, d, e, f, g\}$$

Thus there are 5! permutations.

Combinations

An r-combination of elements of a set A is an unordered selection of r elements from A.

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 1

7 women and 9 men are on the faculty in cs department, how many ways are there to select a committee of 5 members such that there is at least one woman on the committee
committee with no women: $\binom{9}{5}$
arbitrary committee: $\binom{16}{5}$

$$\binom{16}{5} - \binom{9}{5}$$

Example 2

7 women and 9 men are on the faculty in cs department, how many ways are there to select a committee of 5 members such that there is at least one woman and one man on the committee
committee with no women: $\binom{9}{5}$
committee with no men: $\binom{9}{5}$
arbitrary committee: $\binom{16}{5}$

$$\binom{16}{5} - \binom{9}{5} - \binom{7}{5}$$

Example 3

Prove that for $n \geq 0$ and $0 \leq r \leq n$ that $C(n,r) = C(n, n-r)$

$$
\begin{aligned}
C(n, n-r) &= \frac{n!}{(n-r)!(n-(n-r))!} \\
&= \frac{n!}{(n-r)!r!} \\
&= C(n,r)
\end{aligned}
$$

Binomial Theorem

$$(x+y)^n = (x+y)(x+y)\ldots(x+y)$$

All terms in the product are of the form $x^j y^{n-j}$ where $0 \leq j \leq n$. There are $\binom{n}{n-j}$ ways to choose $(n-j)$ y terms from n factors. Therefore, the coefficient of $x^j y^{j-n}$ is $\binom{n}{n-j}$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

Interesting Conclusions

- $x = y = 1$ leads to

$$(x+y)^n = \sum_{j=0}^{n}\binom{n}{j}1^{n-j}1^j$$

$$2^n = \sum_{j=0}^{n}\binom{n}{j}$$

The sum of chooses is equal to a base two exponential.

- $x = 1$, $y = -1$ leads to

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (-1)^j$$

$$0 = \sum_{j=0}^{n} \binom{n}{j} (-1)^j$$

The alternating sum of chooses is equal to zero.

Pascel's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(k + (n-k))(n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

Pascals Triangle

Pascal's triangle is defined as follows

$$\binom{0}{0}$$
$$\binom{1}{0} \binom{1}{1}$$
$$\binom{2}{0} \binom{2}{1} \binom{2}{2}$$
$$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$$
$$\vdots$$

Using pascal's identity, its possible to express the next term as a sum of the previous terms.

$$\frac{\binom{n-1}{k-1}\binom{n}{k-1}}{\binom{n}{k}}$$

Vandermonde's Identity

For $r \leq \min(m, n)$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

The left hand side is choosing r from $m + n$
The right hand side is choosing k objects from the second bin and the remaining objects from the right bin

Identity 2

For $r \leq n$

$$\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$$

$$\binom{k+1}{r+1} = \binom{k}{r} + \binom{k}{r+1} \qquad \text{Pascal's Identity}$$

$$\binom{k}{r} = \binom{k+1}{r+1} - \binom{k}{r+1}$$

$$\sum_{k=r}^{n} \binom{k}{r} = \sum_{k=r}^{n} \left(\binom{k+1}{r+1} - \binom{k}{r+1} \right)$$

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1} - \binom{r}{r+1} \qquad \text{Alternating Series}$$

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$