Optimization

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When solving optimization problems, it is often beneficial to draw a picture representing the problem. After translating the problem into math equations and solving, make sure the solutions are reasonable given the context.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.

$$2x + y = 2400$$

$$y = 2400 - 2x$$

$$A = x \cdot y$$

$$A(x) = x(2400 - 2x)$$

$$A(x) = -2x^{2} + x2400$$

$$A'(x) = -4x + 2400$$

$$A'(x) = 0$$

$$-4x + 2400 = 0$$

$$x = \frac{-2400}{-4}$$

$$x = 600$$

$$2x + y = 2400$$

$$2(600) + y = 2400$$

$$y = 1200$$

Example 2

Find the area of the largest rectangle that can be inscribed in a semisircle of radius r.

$$x^2 + y^2 = r^2$$
$$A = 2xy$$

$$y = \sqrt{r^2 - x^2}$$

$$A = 2x\sqrt{r^2 - x^2}$$

$$A = \sqrt{4x^2(r^2 - x^2)}$$

$$A = \sqrt{4x^2r^2 - 4x^4}$$

Since the maximum of $\sqrt{f(x)}$ is equal to $\sqrt{\max \text{ of } f(x)}$, we can find the max of $4x^2r^2 - 4x^4$, then take a square root. This works in general for any increasing function.

$$f(x) = 4x^{2}r^{2} - 4x^{4}$$

$$f'(x) = 8xr^{2} - 16x^{3}$$

$$f'(x) = 8x(r^{2} - 2x^{2})$$

$$f'(x) = 8x(r - x\sqrt{2})(r + x\sqrt{2})$$

$$f'(x) = -8x(x\sqrt{2} - r)(x\sqrt{2} + r)$$

$$x = 0, \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}$$

since x must be positive, $x = \frac{r}{\sqrt{2}}$.

Example 3

Find the point on the parabola $y^2 = 2x$ closest to (1,4).

 $y^2 = 2x$

$$r = \sqrt{(x-1)^2 + (y-4)^2}$$

$$x = \frac{y^2}{2}$$

$$r = \sqrt{(\frac{1}{2}y^2 - 1)^2 + (y-4)^2}$$

$$r = \sqrt{\frac{1}{4}y^4 - y^2 + 1 + y^2 - 8y + 16}$$

$$r = \sqrt{\frac{1}{4}y^4 - 8y + 17}$$

$$f(y) = \frac{1}{4}y^4 - 8y + 17$$

$$f'(y) = y^3 - 8$$

$$0 = y^3 - 8$$

$$8 = y^3$$

$$2 = y$$