Improper Integrals

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Integrals to Infinity

When an integral has an infinity in its bounds, it is equal to the limit as the bound goes to infinity. If $f(x)$ is continuous on $[a, \infty)$, and $g(x)$ is continuous on $(-\infty, b]$, then:

$$\int_a^\infty f(x)\ dx = \lim_{n\to\infty} \int_a^n f(x)\ dx$$
$$\int_{-\infty}^b g(x)\ dx = \lim_{n\to-\infty} \int_n^b g(x)\ dx$$

If the limit exists, then the integral is said to be convergent. Otherwise, the integral is divergent. If both bounds are infinity, then it can be broken up into two integrals. If $f(x)$ is continuous on $(-\infty, \infty)$, the integral can be defined as:

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^a f(x)\ dx + \int_a^\infty f(x)\ dx$$

If both of the integrals are convergent, then the original integral is convergent. If any of the two integrals are divergent, then the original integral is divergent.

Example 1

$$\int_2^\infty \cos(x)\ dx$$
$$\begin{aligned} &= \lim_{n\to\infty} \int_2^n \cos(x)\ dx \\ &= \lim_{n\to\infty} \sin(x)\ \Big|_2^n \\ &= \lim_{n\to\infty} \sin(n) - \sin(2) \end{aligned}$$

Since $\sin(x)$ does not converge, the integral is divergent.

Example 2

$$\begin{aligned}\int_2^\infty \frac{1}{x}\ dx &= \lim_{n\to\infty}\int_2^n \frac{1}{x}\ dx\\ &= \lim_{n\to\infty} \ln|x|\ \Big|_2^n\\ &= \ln(\infty)-\ln(2)\\ &= \infty\end{aligned}$$

Since the limit is infinity, the integral is divergent.

Example 3

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx &= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\ dx + \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\ dx\\ &= \Big(\lim_{b\to\infty} \tan^{-1}(x)\ \Big|_0^b\Big) + \Big(\lim_{a\to-\infty} \tan^{-1}(x)\ \Big|_a^0\Big)\\ &= \Big(\lim_{b\to\infty} \tan^{-1}(b)-\tan^{-1}(0)\Big) + \Big(\lim_{a\to-\infty} \tan^{-1}(0)-\tan^{-1}(a)\Big)\\ &= \Big(\frac{\pi}{2}-0\Big)+\Big(0-(-\frac{\pi}{2})\Big)\\ &= \pi\end{aligned}$$

Type 1 p-integrals

$$\int_1^\infty \frac{1}{x^p}\ dx$$

This integral converges for all $p > 1$ and diverge for all $p \le 1$

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\ dx &= \int_1^\infty x^{-p}\ dx\\ &= \Big[\frac{1}{-p+1}x^{-p+1}\Big]_1^\infty\\ &= \frac{1}{1-p}\Big(\big(\lim_{b\to\infty} b^{p-1}\big)-1\Big)\end{aligned}$$

For $p > 1$, the limit results in ∞^{negative} and will converge to zero. If $p \le 1$, the limit will result in ∞^{positive} and will diverge. For improper Type 1 integrals, only the behaviour at infinity or negative infinity determines the convergence.

Integrals That Contain Infinity

If $f(x)$ is continuous on (a,b] then

$$\int_a^b f(x)\ dx = \lim_{c\to a^+}\int_c^b f(x)\ dx$$

If $f(x)$ is continuous on [a,b), then

$$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx$$

If there is a discontinuity or a infinity value in function on the domain being integrated, then the integral needs to be split on the discontinuity.

$$\int_a^d f(x)\ dx = \lim_{b \to x_{dis}^-} \int_a^b f(x)\ dx + \lim_{c \to x_{dis}^+} \int_c^d f(x)\ dx$$
$$\text{where } x_{dis} = \text{ the position of the discontinuity}$$

If either of the component integrals diverges, the original is said to be divergent.

Example

$$\begin{aligned}\int_0^1 \frac{1}{x^3}\ dx &= \lim_{x \to 0^+} \int_a^1 x^{-3}\ dx \\ &= \lim_{x \to 0^+} \frac{x^{-2}}{2}\ \Big|_a^1 \\ &= \lim_{a \to 0^+} -\frac{1}{2} + \frac{1}{2a^2} \\ &= \infty\end{aligned}$$

This integral diverges

Example 2

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\ dx &= \lim_{x \to 0^+} \int_a^1 x^{-\frac{1}{2}}\ dx \\ &= \lim_{x \to 0^+} 2\sqrt{x}\ \Big|_a^1 \\ &= \lim_{a \to 0^+} 2\sqrt{1} - 2\sqrt{a} \\ &= 2\sqrt{1} - 2\sqrt{0} \\ &= 2\end{aligned}$$

This integral converges

Type 2 p-integrals

type 2 p-integrals are of the following.

$$\int_0^a \frac{1}{x^p}\ dx$$

These integrals converge for all $p < 1$ and diverges to infinity for all $p \geq 1$.

Example 3

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\ dx = \lim_{a\to 0^-} \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\ dx + \lim_{b\to\infty} \int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}}\ dx$$

$$\begin{aligned} &\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\ dx \\ u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}} dx \\ \int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\ dx &= 2\int e^{-u}\ du \\ &= -2e^{-u} + c \\ &= -2e^{-\sqrt{x}} + c \end{aligned}$$

$$\begin{aligned} I &= \lim_{a\to 0^-} \Big[-2e^{-\sqrt{x}} \Big]_a^1 + \lim_{b\to\infty} \Big[-2e^{-\sqrt{x}} \Big]_1^b \\ &= (-2e^{-1} + 2e^0) + (-2e^{-\infty} + 2e^{-1}) \\ &= (-2e^{-1} + 2) + (0 + 2e^{-1}) \\ &= 2 \end{aligned}$$

Convergence

If $f(x) \geq g(x) > 0$ and $\int_a^\infty g(x)\ dx$ diverges then $\int_0^\infty f(x)\ dx$ also diverges.

Example

$$\int_5^\infty \frac{2 + cos(x)}{x}\ dx$$

$$\begin{aligned} \cos(x) &\in [-1, 1] \\ 2 + \cos(x) &\in [1, 3] \\ \frac{2+\cos(x)}{x} &\in [\frac{1}{x}, \frac{3}{x}] \end{aligned}$$

$$\begin{aligned} \int_5^\infty \frac{1}{x}\ dx \leq \int_5^\infty &\frac{2 + cos(x)}{x}\ dx \leq \int_5^\infty \frac{3}{x}\ dx \\ \infty \leq \int_5^\infty &\frac{2 + cos(x)}{x}\ dx \leq \infty \\ \therefore \int_5^\infty &\frac{2 + cos(x)}{x}\ dx \to \infty \end{aligned}$$