

Arc Length

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The length of a curve can be approximated by sampling it and connecting the points. The length is the limit as the samples go to infinity.

$$\begin{aligned} L_{line} &= \sqrt{(P_{ix} - P_{(i-1)x})^2 + (P_{iy} - P_{(i-1)y})^2} \\ L_{line} &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ L &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{(\Delta x)^2 \left(1 + \frac{(f(x_i) - f(x_{i-1})))^2}{(\Delta x)^2}\right)} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\Delta x^2 (1 + f'(x_i)^2)} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \sqrt{1 + f'(x_i)^2} \end{aligned}$$

From this, the arc length of a curve is:

$$L = \int_a^b \sqrt{1 + f'(x)^2} \, dx$$

Example 1

Find the arc length of $x^{\frac{3}{2}}$ on the interval $[0, 28]$.

$$\begin{aligned}\int_0^{28} \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} dx &= \int_0^{28} \sqrt{1 + \frac{9}{4}x} dx \\ u &= 1 + \frac{9}{4}x \\ du &= \frac{9}{4}dx \\ I &= \frac{4}{9} \int_1^{64} u^{1/2} du \\ &= \frac{8}{27} u^{\frac{3}{2}} \Big|_1^{64} \\ &= \frac{8}{27} (64)^{\frac{3}{2}} - \frac{8}{27} (1)^{\frac{3}{2}} \\ &= \frac{8}{27} 512 - \frac{8}{27} \\ &= \frac{488}{27}\end{aligned}$$

Example 2

Find the arc length of $x^2 - \frac{1}{8} \ln x$ on the interval $[1, e]$.

$$\begin{aligned}\int_1^e \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2} dx &= \int_1^e \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}} dx \\ &= \int_1^e \sqrt{\frac{(1 + 16x^2)^2}{64x^2}} dx \\ &= \int_1^e \frac{1 + 16x^2}{8x} dx \\ &= \int_1^e \frac{1}{8x} + 2x dx \\ &= \frac{1}{8} \ln x + x^2 \Big|_1^e \\ &= \left(\frac{1}{8} \ln e + e^2\right) - \left(\frac{1}{8} \ln 1 + 1^2\right) \\ &= \left(\frac{1}{8} + e^2\right) - 1 \\ &= e^2 - \frac{7}{8}\end{aligned}$$

Example 3

Find the arc length of $g(x) = \int_{\pi/4}^x \sqrt{\sec^8(t) - 1} \, dt$ on the interval $[\frac{\pi}{4}, \frac{\pi}{3}]$.

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + (\sqrt{\sec^8(t) - 1})^2} \, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + \sec^8(t) - 1} \, dx \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^4(t) \, dx \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x \sec^2 x \, dx \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\tan^2 x + 1) \sec^2 x \, dx \\ u &= \tan x \\ du &= \sec^2 x \\ &= \int_1^{\sqrt{3}} u^2 + 1 \, du \\ &= \frac{1}{3}u^3 + u \Big|_1^{\sqrt{3}} \\ &= \left(\frac{1}{3}(\sqrt{3})^3 + \sqrt{3}\right) - \left(\frac{1}{3}(1)^3 + 1\right) \\ &= (\sqrt{3} + \sqrt{3}) - \frac{4}{3} \\ &= 2\sqrt{3} - \frac{4}{3} \end{aligned}$$