Functions

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Jan 23, 2025

A function is a mapping between two sets that maps an element from the input set to exactly one element of the output set. If f is a function that takes an element of the set A as the input and outputs a element of the set B, then it is written as $f : A \mapsto B$

Representations

- Implicit representation

$$\begin{aligned} f &: \mathbb{R} \mapsto \mathbb{R} \\ y &= f(x) \\ f(x) &= x^2 \\ y - x^2 &= 0 \end{aligned}$$

- Set representation

$$\{(a, 3), (b, 1), (c, 4), (d, 3)\}$$

Terminology

For a function $f : A \mapsto B$:

- Domain: The domain is A
- Codomain: The codomain is B
- Image: For any $x \in A$, $f(x)$ is called the image of A.
- Preimage: If for a $y \in B$, there is an $x \in A$ such that $f(x) = y$, then x is called the preimage of y.
- Range: The range is the set of all possible outputs of f.
- Image of a set: The image of a set $X \subseteq A$ is a set with all the elements mapped through the function.
$$f(X) = \{f(x) \mid x \in X\}$$

a function is real-valued if codomain is $\mathbb{R}$ a function is integer-valued if codomain is $\mathbb{Z}$

Operations

- Two functions f and g are equal if:
  - f and g have the same domain.
  - f and g have the same codomain.
  - $f(x) = g(x)$ for all values x in the domain.
- Addition: $(f+g)(x) = f(x) + g(x)$
- Multiplication: $(fg)(x) = f(x)g(x)$

Types of Functions

- A function f is increasing if
$$\forall x, y \in \mathbb{R} : (x \leq y) \rightarrow (f(x) \leq f(y))$$
- A function f is strictly increasing if
$$\forall x, y \in \mathbb{R} : (x < y) \rightarrow (f(x) < f(y))$$
- A function f is decreasing if
$$\forall x, y \in \mathbb{R} : (x \leq y) \rightarrow (f(x) \geq f(y))$$
- A function f is strictly decreasing if
$$\forall x, y \in \mathbb{R} : (x < y) \rightarrow (f(x) > f(y))$$
- A function is injective (one-to-one) if for all values in the range, there is only one value in the domain that maps to it.
$$f(a) = f(b) \Rightarrow a = b$$
  Every strictly increasing function f is injective.
  Proof:
  Suppose $x \neq y$. Since $x \neq y$, $x < y$ or $x > y$
  If $x < y$, then $f(x) < f(y)$ since f is strictly increasing. Thus $f(x) \neq f(y)$.
  If $x > y$, then $f(x) > f(y)$ since f is strictly increasing. Thus $f(x) \neq f(y)$.
- A function $f : A \mapsto B$ is surjective (onto) if the range is the entire codomain
$$\forall b \in B\ \exists a \in A\ (f(a) = b)$$
- A function is bijective if it is both injective and surjective.

Bijective Functions

For a function $f : A \mapsto B$, the following properties hold.

- range of f is codomain B. This is because bijective functions are surjective and by definition, onto.
- Cardinality of domain and codomain is same size $|A| = |B|$. This is because f is a one-to-one correspondence.
- the cardinality of preimage of each $b \in B$ is one. Since f is injective, the preimage is unique.

Function Composition

The composition of functions $f : A \mapsto B$ and $g : B \mapsto C$ is defined as

$$\begin{aligned} f \circ g : A &\mapsto C \\ x &\mapsto g(f(x)) \end{aligned}$$

Inverse Functions

A function $g : B \mapsto A$ is said to be the inverse if a function $f : A \mapsto B$ if their composition is the identity. An inverse function of f only exists if F is bijective.

$$(g \circ f)(x) = id(x) = x$$

Graph of a function

The graph of a function is a set or ordered pair

$$\{(a, f(a)) \mid a \in A\}$$

Important Functions

- Floor function $f : \mathbb{R} \mapsto \mathbb{Z}$ is the largest integer that is less than or equal to the input.

$$f(x) = \lfloor x \rfloor$$

- Ceiling function $f : \mathbb{R} \mapsto \mathbb{Z}$ is the smallest integer that is greater than or equal to the input.

$$f(x) = \lceil x \rceil$$

- Factorial $f : \mathbb{N} \mapsto \mathbb{Z}^+$

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ x \cdot f(x-1) & \text{otherwise)} \end{cases}$$

Example

Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ for all $x \in \mathbb{R}$
let $x = n + \epsilon$
If $0 \leq \epsilon < \frac{1}{2}$

$$\begin{aligned} \lfloor 2x \rfloor &= \lfloor 2n + 2\epsilon \rfloor \\ &= 2n + \lfloor 2\epsilon \rfloor \\ &= 2n \end{aligned}$$

$$\begin{aligned} \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor &= \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + \frac{1}{2} \rfloor \\ &= n + \lfloor \epsilon \rfloor + n + \lfloor \epsilon + \frac{1}{2} \rfloor \\ &= 2n \end{aligned}$$

If $\frac{1}{2} \leq \epsilon < 1$

$$\begin{aligned}
\lfloor 2x \rfloor &= \lfloor 2n + 2\epsilon \rfloor \\
&= 2n + \lfloor 2\epsilon \rfloor \\
&= 2n + 1
\end{aligned}$$

$$\begin{aligned}
\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor &= n + \lfloor \epsilon \rfloor + n + \lfloor \epsilon + \frac{1}{2} \rfloor \\
&= 2n + 1
\end{aligned}$$