

Hyperbolic Trigonometric Functions

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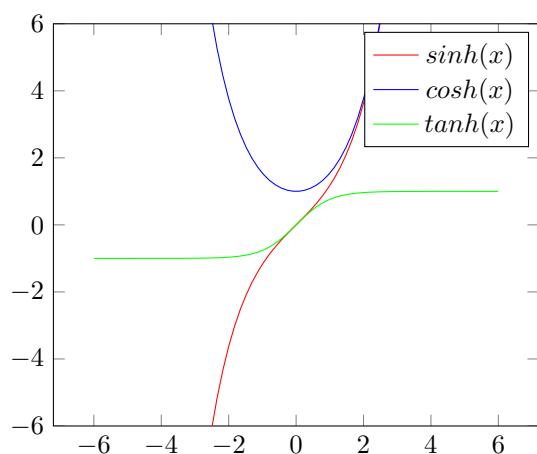
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Hyperbolic trigonometric functions describe points on a hyperbola. The graph of a hyperbola satisfies $x^2 - y^2 = 1$ and points on the hyperbola can be described as $(\cosh x, \sinh x)$ in the same way points on a circle can be described as $(\cos t, \sin t)$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$



$$\cosh^2 x - \sinh^2 x = 1$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$

$$\frac{(e^{2x} + 2e^x e^{-x} + e^{-2x}) - (e^{2x} - 2e^x e^{-x} + e^{-2x})}{4}$$

$$\frac{2 - (-2)}{4}$$

$$1$$

Identities

$$\begin{aligned}\sinh(-x) &= -\sinh x \\ \cosh(-x) &= \cosh x \\ \cosh^2 x - \sinh^2 x &= 1 \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y\end{aligned}$$

$$\begin{aligned}1 - \tanh^2 x &= \operatorname{sech}^2 x \\ \frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} &= \operatorname{sech}^2 x \\ \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} &= \operatorname{sech}^2 x \\ \frac{1}{\cosh^2 x} &= \operatorname{sech}^2 x \\ \operatorname{sech}^2 x &= \operatorname{sech}^2 x\end{aligned}$$

Derivatives of Hyperbolic Trigonometric Functions

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \cosh x \\ \frac{d}{dx}(\cosh x) &= \sinh x \\ \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x \\ \frac{d}{dx}(\operatorname{csch} x) &= -\operatorname{csch} x \coth x \\ \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x \\ \frac{d}{dx}(\coth x) &= -\operatorname{csch}^2 x\end{aligned}$$

$$\begin{aligned}y &= \sinh x \\ \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{2}\right)(e^x - e^{-x}) \\ &= \left(\frac{1}{2}\right)\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) \\ &= \left(\frac{1}{2}\right)(e^x - (-e^{-x})) \\ &= \left(\frac{1}{2}\right)(e^x + e^{-x}) \\ &= \cosh x\end{aligned}$$

One to one and derivatives

If a function's derivative is always positive or always negative for all points, then it is one-to-one. Since $\frac{d}{dx} \sinh x = \cosh x$, and $\cosh x$ is always positive, \sinh is always positive and one-to-one.

Inverse

Inverse of $\sinh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$y = \frac{e^x - e^{-x}}{2}$$

$$2y = e^x - e^{-x}$$

$$2y = e^x(1 - e^{-2x})$$

$$2ye^{-x} = 1 - e^{-2x}$$

$$2ye^{-x} + e^{-2x} = 1$$

$$(e^{-x})^2 + 2y(e^{-x}) - 1 = 0$$

$$e^{-x} = -y \pm \sqrt{y^2 + 1}$$

$$x = -\ln(-y \pm \sqrt{y^2 + 1})$$

$$x = -\ln(-y - \sqrt{y^2 + 1})$$