# Inner Product

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# Dot product

The dot product  $u \cdot v$  is a special case of inner product  $\langle u, v \rangle$  on  $\mathbb{R}^n$ .

$$u \cdot v = u^T v = \sum_n u_n v_n$$

The dot product satisfies the following properties:

- 1.  $u \cdot v = v \cdot u$
- $2. \ u \cdot (v+w) = u \cdot v + u \cdot w$
- 3.  $(cu) \cdot v = c(u \cdot v)$
- $4. \ u \cdot u \ge 0$
- 5.  $u \cdot u = 0$  iff u = 0

Properties (4) and (5) are very important for dot products.

# Length and Angle

Let  $u, v \in \mathbb{R}^n$ .

- ||u|| represents the magnitude of u
- ullet r represents the distance between u and v
- ullet  $\theta$  represents the angle between u and v

$$||u|| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$r = ||u - v||$$

$$\cos(\theta) = \frac{u \cdot v}{||u|| \ ||v||}$$

#### Orthogonality

Using cosine law

$$\begin{aligned} ||u||^2 + ||v||^2 - 2||u|| ||v|| \cos(\theta) &= ||u - v||^2 \\ &= (u - v) \cdot (u - v) \\ &= (u \cdot u) - 2(u \cdot v) + (v \cdot v) \\ ||u||^2 + ||v||^2 - 2||u|| ||v|| \cos(\theta) &= ||u||^2 + ||v||^2 - 2(u \cdot v) \\ &- 2u \cdot v = -2||u|| ||v|| \cos(\theta) \\ &\frac{u \cdot v}{||u|| ||v||} = \cos(\theta) \end{aligned}$$

If  $\theta = \frac{\pi}{2}$ , then  $\cos \theta$  is zero. Two vectors are orthogonal to each other if their dot products are zero.

### Set of orthogonal vectors

A set of vectors  $\{v_1, \ldots, v_n\}$  are orthogonal if all vectors in the set are non-zero and all pairs of vectors have a dot product of zero. Often the  $e_1, \ldots, e_n$  basis vectors are used as orthogonal vectors. A set of orthogonal vectors are always linearly independent.

#### Proof of linear independence

Suppose there is a orthogonal set  $\{v_1, \ldots, v_k\}$  and  $c_1v_1 + \cdots + c_kv_k = 0$ 

$$(c_1v_1 + \dots + c_kv_k) \cdot v_1 = 0 \cdot v_1$$

$$c_1(v_1 \cdot v_1) + \dots + c_k(v_k \cdot v_1) = 0$$

$$c_1||v_1||^2 + 0 + \dots + 0 = 0$$

$$c_1||v_1||^2 = 0$$

$$c_1 = 0$$

By using this same process with each of the vectors in the orthogonal set, all of the coefficients must be equal to zero and thus, orthogonal sets are linearly independent.

#### **Orthogonal Basis**

The coordinates in an orthogonal basis with dot products. If  $w \in span\{v_1, \dots, v_k\}$ , then:

$$w = c_1 v_1 + \dots + c_k v_k$$

$$w \cdot v_i = c_1 (v_1 \cdot v_i) + \dots + c_i (v_i \cdot v_i) + \dots + c_k (v_k \cdot v_i)$$

$$w \cdot v_i = c_1(0) + \dots + c_i ||v_i||^2 + \dots + c_k(0)$$

$$w \cdot v_i = c_i ||v_i||^2$$

$$\frac{w \cdot v_i}{||v_i||^2} = c_i$$

Often orthonormal basis are used. Since vectors in an orthonormal basis are both mutually orthogonal and unit length,  $||v_i||^2 = 1$  and thus, the division is necessary.

$$c_i = w \cdot v_i$$

## Example 1

Find  $[w]_B$ 

$$B = \left\{ \begin{bmatrix} 1\\0\\1 \end{bmatrix}, \begin{bmatrix} -1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\2\\-1 \end{bmatrix} \right\}, \quad w = \begin{bmatrix} 2\\3\\1 \end{bmatrix}$$

$$||v_1|| = 2$$

$$||v_2|| = 3$$

$$||v_3|| = 6$$

$$\frac{w \cdot v_1}{||v_1||^2} = \frac{3}{2}$$

$$\frac{w \cdot v_2}{||v_2||^2} = \frac{2}{3}$$

$$\frac{w \cdot v_3}{||v_3||^2} = \frac{7}{6}$$

$$w = \frac{3}{2}v_1 + \frac{2}{3}v_2 + \frac{7}{6}v_3$$

$$[w]_B = \begin{bmatrix} \frac{3}{2}\\\frac{2}{3}\\\frac{7}{6} \end{bmatrix}$$

# Orthogonal complement

Suppose that W is a subspace of  $\mathbb{R}^n$ . The orthogonal complement of W, written  $W^{\perp}$  is defined as all vectors that are orthogonal to W.

$$W^{\perp} = \{u \in \mathbb{R}^n : u \cdot w = 0 \text{ for all } w \in W\}$$
  
$$W \cap W^{\perp} = \{0\}$$

The orthogonal complex is a subspace and as such is closed under addition and scalar multiplication. For any  $w \in W$  and  $u, v \in W^{\perp}$ :

$$w \cdot (u + v) = w \cdot u + w \cdot v$$

$$= 0 + 0$$

$$= 0$$

$$cw = w \cdot cu$$

$$= c(w \cdot u)$$

$$= c(0)$$

$$= 0$$

The intersection of W and  $W^{\perp}$  is  $\{\vec{0}\}$ .

$$w \in W$$
 
$$w \in W^{\perp}$$
 
$$w \cdot w = 0$$
 
$$\therefore w = \vec{0}$$

The vector space W is contained inside the orthogonal computed of the orthogonal complex of W If  $w \in W$  and  $u \in W^{\perp}$ ,  $w \cdot u = 0$  therefore  $w \in (W^{\perp})^{\perp}$ .

$$W\subseteq (W^\perp)^\perp$$