Diagonalization

Patrick Chen

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Linear independence of eigenvectors

If A is a $n \times n$ matrix and $v_1, \ldots, v_k$ are eigenvectors for distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $v_1, \ldots, v_k$ are linearly independent.

Proof

If there is exactly one eigenvector and one eigenspace, the eigenvectors are trivially linearly independent. Assume that $\{v_1, \ldots, v_i\}$ is linearly independent.

$$\begin{aligned} c_1 v_1 + \cdots + c_i v_i + c_{i+1} v_{i+i} &= \vec{0} \qquad (1) \\ A(c_1 v_1 + \cdots + c_i v_i + c_{i+1} v_{i+i}) &= A\vec{0} \\ c_1 \lambda_1 v_1 + \cdots + c_i \lambda_i v_i + c_{i+1} \lambda_{i+1} v_{i+1} &= \vec{0} \qquad (2) \end{aligned}$$

Multiply (1) by λ_{i+1}

$$c_1 \lambda_{i+1} v_1 + \cdots + c_i \lambda_{i+1} v_i + c_{i+1} \lambda_{i+1} v_{i+1} = 0 \qquad (3)$$

Subtract (3) from (2)

$$\begin{aligned} (c_1 \lambda_1 v_1 + \cdots + c_i \lambda_i v_i + c_{i+1} \lambda_{i+1} v_{i+1}) - (c_1 \lambda_{i+1} v_1 + \cdots + c_i \lambda_{i+1} v_i + c_{i+1} \lambda_{i+1} v_{i+1}) &= \vec{0} - \vec{0} \\ c_1(\lambda_1 - \lambda_{i+1}) v_1 + \cdots + c_i(\lambda_i - \lambda_{i+1}) v_i + c_{i+1}(\lambda_{i+1} - \lambda_{i+1}) v_{i+1} &= \vec{0} \\ c_1(\lambda_1 - \lambda_{i+1}) v_1 + \cdots + c_i(\lambda_i - \lambda_{i+1}) v_i &= \vec{0} \end{aligned}$$

Since $\{v_1, \ldots, v_i\}$ is linearly independent, then for all n, $c_n(\lambda_n - \lambda_{i+1}) = 0$. Using the zero product property, either $c_n = 0$ or $\lambda_n - \lambda_{i+1} = 0$, but since the eigenvalues are distinct, c_n must be equal to zero. Thus,

$$\begin{aligned} c_1, \ldots, c_i &= 0 \\ c_1 v_1 + \cdots + c_i v_i + c_{i+1} v_{i+i} &= \vec{0} \\ 0 v_1 + \cdots + 0 v_i + c_{i+1} v_{i+i} &= \vec{0} \\ c_{i+1} v_{i+i} &= \vec{0} \\ c_{1+i} &= 0 \end{aligned}$$

Therefore $\{v_1, \ldots, v_i, v_{i+1}\}$ is linearly independent. Using the inductive hypothesis, every eigenvector belonging to distinct eigenvalues are linearly independent.

Similarity

For two $n \times n$ matrices A and B, we sat that A is similar to B if there exists a invertible matrix P such that $P^{-1}AP = B$. If A and B are similar then they have the same characteristic polynomial and thus, have the same eigenvalues.

$$\begin{aligned} det(B - \lambda I) &= det(P^{-1}AP - \lambda I) \\ &= det(P^{-1}(A - \lambda I)P) \\ &= det(P^{-1})det((A - \lambda I)det(P) \\ &= det(A - \lambda I) \end{aligned}$$

Diagonalization

If a $n \times n$ matrix A have n different eigenvalues then A is diagonalizable. A is similar to a diagonal matrix iff there are n linearly independent eigenvectors for A. In $P^{-1}AP = D$, the columns of P are the eigenvectors and the diagonals of D are the eigenvalues.

$$\begin{aligned} P &= \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \\ D &= \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix} \end{aligned}$$

From the construction of D, we can show that P contains the eigenvectors.

$$\begin{aligned} P^{-1}AP &= D \\ AP &= PD \\ AP &= \begin{bmatrix} Av_1 & Av_2 & \dots & Av_n \end{bmatrix} \\ PD &= \begin{bmatrix} \lambda_q v_1 & \lambda_2 v_2 & \dots & \lambda_n v_n \end{bmatrix} \end{aligned}$$

Algorithm for Diagonal

1. Suppose $\lambda_1, \dots, \lambda_k$ are eigenvalues of A.
2. For each eigenvalue, determine the basis for the eigenspace. If any eigenspace has a dimension less than it's algebraic multiplicity, it is not diagonalizable.
3. Otherwise, collect all the bases and put them in the columns of a matrix P.
4. P will be invertible and $AP = PD$ where D is a diagonal matrix that has eigenvalue corresponding with the columns of P.

Powers of Diagonal Matrices

The powers of a diagonal matrix is the power of the elements in the diagonal matrix. If a matrix is diagonalizable, then the powers of that matrix can be easily computed by first diagonalizing it and raising the diagonal matrix to the power.

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$
$$D^k = \begin{bmatrix} d_1^k & 0 \\ 0 & d_2^k \end{bmatrix}$$

$$\begin{aligned} A^k &= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)\dots(P^{-1}P)DP^{-1} \\ &= PDD\dots DP^{-1}) \\ &= PD^kP^{-1}) \end{aligned}$$

Eigenvector Basis

Let B be a basis constructed from the eigenvectors of a transformation T.

$$\begin{aligned} [T]_B &= \begin{bmatrix} [T(v_1)]_B & [T(v_1)]_B & \dots & [T(v_n)]_B \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \dots & \lambda_n e_n \end{bmatrix} \end{aligned}$$

The standard matrix for a linear transformation with respect to the basis of eigenvectors is a diagonal matrix.

Example 1

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, Ax_1 = 3x_1$$
$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, Ax_2 = 1x_2$$
$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
T_A &: \mathbb{R}^3 \mapsto \mathbb{R}^3 \\
A &= [T_A]_{\{e1,e2,e3\}} \\
B &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \\
[T]_B &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Example 3

$$
\begin{aligned}
V &= span\{\sin x, \cos x\} \\
dim(V) &= 2 \\
f &\in V \\
T(f) &= \frac{df}{dx} \\
T &: V \mapsto V \\
B &= \{\sin x, \cos x\} \\
[T]_B &= \begin{bmatrix} [T(v_1)]_B & [T(v_2)]_B \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned}
$$