Parametric Function

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A parametric function are functions describing the position along some axis for a given parameter. Typically, these functions are continuous and the parameter is named t. A 2D parametric function can be plotted by varying the value of t and plotting the x and y positions. The direction of increasing t along the curve is called the orientation of the parametrization. This is typically indicated by arrows.

$$x = f(t) \quad y = g(t)$$

Any function $f : \mathbb{R} \mapsto \mathbb{R}$ can be trivially parameterized as $x = t$ and $y = f(t)$

Isolation

A plot for a parametric function may be plotted more easily by isolating and substituting t.

Example

$$x = \ln(t) \quad y = \frac{t^4 + 1}{2t^2}$$

$$\begin{aligned}
x &= \ln(t) \\
e^x &= t \\
y &= \frac{t^4 + 1}{2t^2} \\
&= \frac{(e^x)^4 + 1}{2(e^x)^2} \\
&= \frac{e^{4x} + 1}{2e^{2x}} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= \cosh(2x)
\end{aligned}$$

Parametric Differentiation

Parametric functions can be differentiated by dividing the y-derivative with respect to t by the x-derivative with respect to t. Let $y = h(x)$.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}h(x) \\
\frac{dy}{dt} &= \left(\frac{d}{dx}h(x)\right)\frac{dx}{dt} \\
\frac{d}{dx}h(x) &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\
\frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}
\end{aligned}$$

- if $\frac{dy}{dt}$ is zero and $\frac{dx}{dt}$ is not zero, it is a horizontal tangent.
- if $\frac{dx}{dt}$ is zero and $\frac{dy}{dt}$ is not zero, it is a vertical tangent.

Higher Derivatives

$$\begin{aligned}
\frac{d}{dt}(\frac{dy}{dx}) &= \frac{d}{dx}(\frac{dy}{dx}) \cdot \frac{dx}{dt} \\
&= \frac{d^2y}{dx^2}\frac{dx}{dt}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \\
\frac{d^ny}{dx^n} &= \frac{\frac{d}{dt}\frac{d^{n-1}y}{dx^{n-1}}}{\frac{dx}{dt}}
\end{aligned}$$

Parametric Integration

Parametric functions can be integrated as follows.

$$\int_a^b f(x)\ dx = \int_{t_0}^{t_1} y(t)\frac{dx}{dt}\ dt$$

Example

Integrate the following parametric function on the interval $x \in [0, \pi]$

$$x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t)$$

$$x = 0 \Rightarrow t = 0$$
$$x = \pi \Rightarrow t = \pi$$

$$\frac{d}{dt}x(t) = 1 - \cos(t)$$

$$\begin{aligned}
\int_0^\pi y(t)(\frac{dx}{dt})\ dt &= \int_0^\pi (1-\cos t)(1-\cos t)\ dx \\
&= \int_0^\pi 1 - 2\cos t + \cos^2 t\ dx \\
&= \int_0^\pi 1 - 2\cos t + \frac{1}{2}(1+\cos 2t)\ dx \\
&= \left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\cos 2t\right]_0^\pi \\
&= \frac{3}{2}\pi
\end{aligned}$$

Arclength

$$\int_c ds = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^1 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Surface Area of Revolution

about the x-axis

$$\int_I 2\pi r ds = \int_{t_0}^{t_1} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

about the y-axis

$$\int_I 2\pi r ds = \int_{t_0}^{t_1} 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$