Linear Independence and Basis

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Linear Independence

If V is a vector space, we say that $v_1, \ldots, v_n \in V$ are linearly independent if the only solution to $c_1v_1 + \cdots + c_nv_n = 0$ has $c_1, \ldots c_n = 0$. For infinite dimensions, a set of vectors B in V is linearly independent if every finite subset of B is linearly independent.

The set $e_1, \ldots, e_n$ is maximally linearly independent in $\mathbb{R}^n$ because there is not anything that can be added to the set without causing it to become linearly dependent. $\mathbb{R}^n$ cannot be spanned by fewer than n vectors.

Basis for a Vector Space

A subset B is said to be a basis for a vector space V if B is linearly independent and B spans V. Every vector space has a basis. In $\mathbb{R}^n$, a $n \times n$ matrix A is invertible if the columns of A form a basis.

Spanning Set Theorem

if V is a vector space and $H = span\{v_1, \ldots, v_n\}$ then there is a subset $\{v_1, \ldots, v_n\}$ which is a basis for H.

To find this basis, we must filter the vectors that are not in span of all previous vectors. Suppose that after this process, the vectors are not linearly independent. This implies that there is some linear combination vectors that, when added, will equal zero and is not the trivial solution. By rearranging the equation, it is possible to show that there is some vector that is a linear combination of the other vectors and thus in the span of the other vectors. This implies that one of the vectors is a linear combination of the previous vectors. This contradicts with the algorithm used to produce these vectors so thus, the assumption that the vectors are linearly dependent is false. Below is an example of this.

$$
\begin{aligned}
c_1v_1 + c_3v_3 + c_5v_5 + c_8v_8 = 0 \\
c_8 = -\frac{c_1}{c_8}v_1 - \frac{c_2}{c_8}v_3 - \frac{c_5}{c_8}v_5
\end{aligned}
$$

To finding the basis for the columnspace of matrix A, we compute the reduced row echelon form of A and take the pivot columns of A.

Rowspace

If A is a $n \times m$ matrix, then the rowspace $row(A)$ is a subset of $\mathbb{R}^m$. Elementary row operations do not change the rowspace. Proving this for swapping and constant multiple is trivial.

$$\begin{aligned} r_i &\leftarrow r_i + cr_j \\ &span\{r_1, \ldots, r_i + cr_j, \ldots, r_n\} \\ (r_i + cr_j) &\in span\{r_1, \ldots, r_i + cr_j, \ldots, r_n\} \\ (r_i + cr_j) - cr_j &\in span\{r_1, \ldots, r_i + cr_j, \ldots, r_n\} \\ r_i &\in span\{r_1, \ldots, r_i + cr_j, \ldots, r_n\} \end{aligned}$$

Since r_i is still in the span after adding a multiple of a row, r_i is still in the span and this row operation doesn't change the span of the rows. The basis for the rowspace of A is the basis for the columnspace of A^T.

Example 1

Prove that $B = \{1, x, x^2, x^3 \ldots\}$ is linearly independent.

$$c_0 + c_1 x + x_2 x^2 + \cdots + c_n x^n = 0 \text{ for all values } x$$

This means that for every x, this polynomial is equal to zero. Every polynomial of degree $n > 0$ has at most n roots. A degree 0 polynomial can have infinite roots if it is the constant zero function. Since this equality must hold for every x, which there are infinite values of, the polynomial must be the zero polynomial with $c_0 \ldots c_n = 0$.

Example 2

The set $\{\sin x, \cos x\}$ is a subset of functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that this set of functions is linearly independent.

$$c \sin x + d \cos x = 0 \text{ for all values } x$$
$$\sin x = -\frac{d}{c} \cos x$$

Since this is not true for all values x, sin and cos are not scalar multiples of each other and thus not linearly dependent.