Propositional Logic

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A proposition is a declarative sentence with either a true or false value. Usually, propositions are denoted by lowercase letters: p, q, r

Operators

- Negation $\neg$ (NOT): $\neg p$ is the opposite of p.
- conjunction $\wedge$ (AND): $p \wedge q$ is true when both p and q are true.
- disjunction $\vee$ (OR): $p \vee q$ is true when either p or q is true.
- Exclusive disjunction $\oplus$ (XOR): $p \oplus q$ is true when exactly one of p and q is true but not both.
- Implication $\rightarrow$: conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.
  - if p, then q
  - q if p
  - p is sufficient for q
  - q unless $\neg p$
- Biconditional $\leftrightarrow$ (EQ): biconditional statement $p \leftrightarrow q$ is the formalism for $(p \rightarrow q) \wedge (q \rightarrow p)$.
  - p if and only if q

Manipulations of operators

Two statements are equivalent if both have the same truth value. Equivalence of statements is shown with the "$\equiv$" symbol.

Implication

If $p \rightarrow q$ then:

- converse: $q \rightarrow q$
- contrapositive $\neg q \rightarrow \neg p$
- inverse: $\neg p \rightarrow \neg q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Laws

- Identity: $p \wedge T \equiv p$ and $p \vee F \equiv p$
- Domination: $p \wedge F \equiv F$ and $p \vee T \equiv T$
- Idempotent: $p \wedge p \equiv p$ and $p \vee p \equiv p$
- Double Negation $\neg(\neg p) \equiv p$
- Commutative: $p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$
- Negation: $p \vee \neg p \equiv T$ and $p \wedge \neg p \equiv F$
- Associative: $(p \vee q) \vee r \equiv p \vee (q \vee r)$ and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distribution: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- De Morgan's: $\neg(p \vee q) \equiv \neg p \wedge \neg q$ and $\neg(p and q) \equiv \neg p \vee \neg q$
- Absorption: $p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$
- Implication: $p \to q \equiv \neg p \vee q$

Tautology

A tautology is a statement that is always true. If the statement $p \leftrightarrow q$ is a tautology, then $p \equiv q$

Example 1

Show that $p \to q = \neg q \to \neg p$

$$\begin{aligned} p \to q &= \neg p \vee q && \text{Implication} \\ &= q \vee \neg p && \text{Commutative} \\ &= \neg(\neg q) \vee (\neg p) && \text{Double Negative} \\ &= \neg q \to \neg p && \text{implies} \end{aligned}$$

Example 2

Show that $(p \to r) \wedge (q \to r) = (p \vee q) \to r$

$$\begin{aligned} (p \to r) \wedge (q \to r) &= (\neg p \vee r) \wedge (\neg q \vee r) && \text{Implication} \\ &= (\neg p \wedge \neg q) \vee r && \text{Distribution} \\ &= \neg(p \vee q) \vee r && \text{De Morgan's} \\ &= (p \vee q) \to r && \text{Implication} \end{aligned}$$

Propositional Functions

In zero order logic, there are no quantifiers (for all, there exists, etc.). In order to extent Propositional logic, propositional functions are needed.

Quantifiers

- Universal quantification of $P(x)$ is the proposition "$P(x)$ for all values of x in the domain". This is denoted by $\forall x P(x)$. If the proposition holds for all values in the domain, then $\forall x P(x)$ evaluates to true, but if even one value in the domain does not satisfy the proposition, then $\forall x P(x)$ evaluates to false.
- Existential quantifiers of $P(x)$ is the proposition "There exists an x in the domain such that $P(x)$". This is denoted by $\exists x P(x)$. If the proposition $P(x)$ is true for even a single value x in the domain, then $\exists x P(x)$ is true. If there are no values x in the domain that satisfies the proposition, then $\exists x P(x)$ is false.

Manipulations of quantifiers

A negation can be brought inside a quantifier by inverting the quantifier.

$$\neg \forall x P(x) = \exists x \neg P(x)$$
$$\neg \exists x P(x) = \forall x \neg P(x)$$

A negation can also be brought inside a propositional function.

$$\neg \forall x (x^2 > x) = \exists x \neg (x^2 > x) = \exists x (x^2 \leq x)$$

Example 3

Show that $\neg \forall x (P(x) \to Q(x)) = \exists x (P(x) \wedge \neg Q(x))$

$$\begin{aligned}
\neg \forall x (P(x) \to Q(x)) &= \neg \forall x (\neg P(x) \vee Q(x)) && \text{Implication} \\
&= \exists x \neg (\neg P(x) \vee Q(x)) && \text{Quantifier} \\
&= \exists x (P(x) \wedge \neg Q(x)) && \text{De Morgan's}
\end{aligned}$$

Nested Quantifiers

When quantifiers are nested, they are read left to right. Quantifiers of the same type commute, but quantifiers of different types do not commute. For example, let $P(x) =$ student x falls asleep during lecture y. $\exists x \forall y P(x)$ means that there is a student that falls asleep during all lectures. $\forall y \exists x P(x)$ means that in every lecture, there is a student that falls asleep.

$$\forall x \forall y P(x, y) = \forall y \forall x P(x, y)$$
$$\exists x \exists y P(x, y) = \exists y \exists x P(x, y)$$
$$\forall x \exists y P(x, y) \neq \exists y \forall x P(x, y)$$

Although the quantifiers do not commute, then can imply other propositions with nested quantifiers are true.

$$\exists x \forall y P(x, y) \to \forall y \exists x P(x, y)$$

Example 4

Is $\forall x \exists y (x^2 - y^2 = 1)$ true for the domain $x, y \in \mathbb{R}$

$$\text{If } x = \frac{1}{2}$$
$$\text{Then } y^2 = -\frac{3}{4}$$

Since no real number squared to a negative, this proposition is false.