Matrix Algebra

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Addition and multiplication

When two matrices of the same size are added or subtracted, it is done element-wise.

$$\begin{bmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \\ \vdots & & \ddots \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & \dots \\ B_{21} & B_{22} & \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \dots \\ A_{21}+B_{21} & A_{22}+B_{22} & \\ \vdots & & \ddots \end{bmatrix}$$

When a matrix is multiplied by a scalar, every entry in the matrix is multiplied by the scalar

$$\lambda \begin{bmatrix} A_{11} & A_{12} & A_{13} & \\ A_{21} & A_{22} & A_{23} & \dots \\ A_{31} & A_{32} & A_{33} & \\ & \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} & \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} & \dots \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} & \\ & \vdots & & \ddots \end{bmatrix}$$

$$A + B = B + A \qquad \text{matrix addition commutes}$$
$$\lambda(A+B) = \lambda A + \lambda B \qquad \text{scalars are distributive}$$
$$(\lambda\mu)A = \lambda(\mu A)$$

Matrix Multiplication

When A is a $m \times k$ matrix and B is a $k \times n$ matrix, we define AB as:

$$\begin{bmatrix} Ab_1 & Ab_2 & \dots & AB_n \end{bmatrix}$$

Note that multiplication of matrices do not commute. That means that $AB \neq BA$ When A and B are multiplied, this is equivalent to composing their transformations. $AB \equiv T_A \circ T_B$

exmaple:

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Properties of matrix algebra

- Associativity $A(BC) = (AB)C$
- Left distributivity $A(B + C) = AB + AC$
- Right distributivity $(A + B)C = AC + BC$
- Scalar commutativity $\lambda(AB) = (\lambda A)B = A(\lambda B)$
- Identity $IA = AI = A$

Matrices are associative because multiplying by a matrix corresponds to linear transformations. Since function composition is associative, so is matrix multiplication. Distributivity is split between left and right distributivity because multiplication does not commute. Thus, multiplying from the left will distribute the matrix with the multiplication on the left, and vise versa for right.

Identity and Zero Matrix

A matrix composed with the basis vectors is called the identity matrix. This matrix is special because left multiplication and right multiplication by this matrix is equivalent to not doing anything.

$$I = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ & \vdots & & \ddots \end{bmatrix}$$

When a matrix is composed entirely of zeros, it is called the zero matrix because it acts like a zero in addition.

$$0 = \begin{bmatrix} 0 & 0 & \dots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix}$$