

# Diagonalization

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## Linear independence of eigenvectors

If  $A$  is a  $n \times n$  matrix and  $v_1, \dots, v_k$  are eigenvectors for distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ , then  $v_1, \dots, v_k$  are linearly independent.

### Proof

If there is exactly one eigenvector and one eigenspace, the eigenvectors are trivially linearly independent. Assume that  $\{v_1, \dots, v_i\}$  is linearly independent.

$$c_1 v_1 + \dots + c_i v_i + c_{i+1} v_{i+1} = \vec{0} \quad (1)$$

$$A(c_1 v_1 + \dots + c_i v_i + c_{i+1} v_{i+1}) = A \vec{0}$$

$$c_1 \lambda_1 v_1 + \dots + c_i \lambda_i v_i + c_{i+1} \lambda_{i+1} v_{i+1} = \vec{0} \quad (2)$$

Multiply (1) by  $\lambda_{i+1}$

$$c_1 \lambda_{i+1} v_1 + \dots + c_i \lambda_{i+1} v_i + c_{i+1} \lambda_{i+1} v_{i+1} = 0 \quad (3)$$

Subtract (3) from (2)

$$\begin{aligned} (c_1 \lambda_1 v_1 + \dots + c_i \lambda_i v_i + c_{i+1} \lambda_{i+1} v_{i+1}) - (c_1 \lambda_{i+1} v_1 + \dots + c_i \lambda_{i+1} v_i + c_{i+1} \lambda_{i+1} v_{i+1}) &= \vec{0} - \vec{0} \\ c_1 (\lambda_1 - \lambda_{i+1}) v_1 + \dots + c_i (\lambda_i - \lambda_{i+1}) v_i + c_{i+1} (\lambda_{i+1} - \lambda_{i+1}) v_{i+1} &= \vec{0} \\ c_1 (\lambda_1 - \lambda_{i+1}) v_1 + \dots + c_i (\lambda_i - \lambda_{i+1}) v_i &= \vec{0} \end{aligned}$$

Since  $\{v_1, \dots, v_i\}$  is linearly independent, then for all  $n$ ,  $c_n (\lambda_n - \lambda_{i+1}) = 0$ . Using the zero product property, either  $c_n = 0$  or  $\lambda_n - \lambda_{i+1} = 0$ , but since the eigenvalues are distinct,  $c_n$  must be equal to zero. Thus,

$$\begin{aligned} c_1, \dots, c_i &= 0 \\ c_1 v_1 + \dots + c_i v_i + c_{i+1} v_{i+1} &= \vec{0} \\ 0 v_1 + \dots + 0 v_i + c_{i+1} v_{i+1} &= \vec{0} \\ c_{i+1} v_{i+1} &= \vec{0} \\ c_{i+1} &= 0 \end{aligned}$$

Therefore  $\{v_1, \dots, v_i, v_{i+1}\}$  is linearly independent. Using the inductive hypothesis, every eigenvector belonging to distinct eigenvalues are linearly independent.

## Similarity

For two  $n \times n$  matrices  $A$  and  $B$ , we say that  $A$  is similar to  $B$  if there exists an invertible matrix  $P$  such that  $P^{-1}AP = B$ . If  $A$  and  $B$  are similar then they have the same characteristic polynomial and thus, have the same eigenvalues.

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\ &= \det(P^{-1}(A - \lambda I)P) \\ &= \det(P^{-1})\det((A - \lambda I)P) \\ &= \det(A - \lambda I) \end{aligned}$$

## Diagonalization

If a  $n \times n$  matrix  $A$  has  $n$  different eigenvalues then  $A$  is diagonalizable.  $A$  is similar to a diagonal matrix iff there are  $n$  linearly independent eigenvectors for  $A$ . In  $P^{-1}AP = D$ , the columns of  $P$  are the eigenvectors and the diagonals of  $D$  are the eigenvalues.

$$P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$
$$D = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

From the construction of  $D$ , we can show that  $P$  contains the eigenvectors.

$$\begin{aligned} P^{-1}AP &= D \\ AP &= PD \\ AP &= [Av_1 \quad Av_2 \quad \dots \quad Av_n] \\ PD &= [\lambda_1 v_1 \quad \lambda_2 v_2 \quad \dots \quad \lambda_n v_n] \end{aligned}$$

## Algorithm for Diagonal

1. Suppose  $\lambda_1, \dots, \lambda_k$  are eigenvalues of  $A$ .
2. For each eigenvalue, determine the basis for the eigenspace. If any eigenspace has a dimension less than its algebraic multiplicity, it is not diagonalizable.
3. Otherwise, collect all the bases and put them in the columns of a matrix  $P$ .
4.  $P$  will be invertible and  $AP = PD$  where  $D$  is a diagonal matrix that has eigenvalue corresponding with the columns of  $P$ .

## Powers of Diagonal Matrices

The powers of a diagonal matrix is the power of the elements in the diagonal matrix. If a matrix is diagonalizable, then the powers of that matrix can be easily computed by first diagonalizing

it and raising the diagonal matrix to the power.

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} d_1^k & 0 \\ 0 & d_2^k \end{bmatrix}$$

$$\begin{aligned} A^k &= (PDP^{-1})(PDP^{-1}) \dots (PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P) \dots (P^{-1}P)DP^{-1} \\ &= PDD \dots DP^{-1} \\ &= PD^kP^{-1} \end{aligned}$$

## Eigenvector Basis

Let  $B$  be a basis constructed from the eigenvectors of a transformation  $T$ .

$$\begin{aligned} [T]_B &= \begin{bmatrix} [T(v_1)]_B & [T(v_2)]_B & \dots & [T(v_n)]_B \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \dots & \lambda_n e_n \end{bmatrix} \end{aligned}$$

The standard matrix for a linear transformation with respect to the basis of eigenvectors is a diagonal matrix.

### Example 1

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ x_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, Ax_1 = 3x_1 \\ x_2 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, Ax_2 = 1x_2 \\ P &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ P^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ P^{-1}AP &= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

### Example 2

$$\begin{aligned}A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\T_A : \mathbb{R}^3 &\mapsto \mathbb{R}^3 \\A &= [T_A]_{\{e_1, e_2, e_3\}} \\B &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \\[T]_B &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

### Example 3

$$\begin{aligned}V &= \text{span}\{\sin x, \cos x\} \\dim(V) &= 2 \\f &\in V \\T(f) &= \frac{df}{dx} \\T : V &\mapsto V \\B &= \{\sin x, \cos x\} \\[T]_B &= \begin{bmatrix} [T(v_1)]_B & [T(v_2)]_B \end{bmatrix} \\&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\end{aligned}$$