

Trig Substitution

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Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta \quad 0 \leq \theta < \frac{\pi}{2}$ $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Example 1

$$\int \frac{\sqrt{9 - x^2}}{x^2} dx$$

$$x = 3 \sin \theta$$

$$dx = 3 \cos \theta d\theta$$

$$\begin{aligned}
&= \int \frac{\sqrt{9 - 9 \sin^2 \theta}}{9 \sin^2 \theta} 3 \cos \theta \, d\theta \\
&= \int \frac{3 \sqrt{1 - \sin^2 \theta}}{9 \sin^2 \theta} 3 \cos \theta \, d\theta \\
&= \int \frac{3 \cos \theta}{9 \sin^2 \theta} 3 \cos \theta \, d\theta \\
&= \int \frac{\cos^2 \theta}{\sin^2 \theta} \, d\theta \\
&= \int \cot^2 \theta \, d\theta \\
&= \int \cot^2 \theta + 1 - 1 \, d\theta \\
&= (-\cot \theta) - \theta + c \\
&= -\cot(\sin^{-1}(\frac{x}{3})) - \sin^{-1}(\frac{x}{3}) + c \\
&= -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}(\frac{x}{3}) + c
\end{aligned}$$