

# Sequences

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## Sequences

A sequence is a function from a subset of integers to a set  $S$ . Usually, the subset is the set of natural numbers  $\mathbb{N}$  or positive integers  $\mathbb{Z}^+$ . A finite sequence is called a string. Below, the expression on the left denotes a infinite sequence and the expression on the right denotes a finite sequence.

$$\{a_n\} \quad \{a_n\}_{n=A}^B$$

- $a_n$  is a term in the sequence
- $A$  is where the sequence begins
- $B$  is where the sequence ends.

## Progression

A geometric progression is

$$\{a \cdot r^k\}_{k=0}^n = \{a, a \cdot r, a \cdot r^2, \dots, a \cdot r^n\}$$

A arithmetic progression is:

$$\{a + kd\}_{k=0}^n = \{a, a + d, a + 2d, \dots, a + nd\}$$

## Recurrences

A recurrence relation is a sequence where the next term depends on the previous values.

$$a_{n+1} = f(a_n)$$

For a recursive sequence to have a unique closed form formula, there must be initial conditions for the sequence.

## Summations

Given a sequence  $\{a_k\}_{k=0}^n$ , the partial sum of the sequence is denoted by  $S_n$ .

$$S_n = \sum_{k=0}^n a_k$$

The index of the sum can be changes.

$$\sum_{i=m}^n a_i = \sum_{i=m+k}^{n+k} a_{i-k} \qquad \sum_{i=m}^n a_i = \sum_{i=m-k}^{n-k} a_{i+k}$$

## Summations of Geometric Progression

The sum of a geometric progression is:

$$\sum_{k=0}^n ar^k$$

If  $r = 1$ , then

$$\begin{aligned} S_n &= \sum_{k=0}^n a \\ &= (n+1)a \end{aligned}$$

If  $r \neq 1$ , then

$$\begin{aligned} S_n &= \sum_{k=0}^n ar^k \\ rS_n &= \sum_{k=0}^n ar^{k+1} \\ rS_n &= \sum_{k=1}^{n+1} ar^k \\ &= \left( \sum_{k=1}^n ar^k \right) + ar^{n+1} \\ &= \left( \sum_{k=1}^n ar^k \right) + ar^{n+1} \\ &= \left( \sum_{k=0}^n ar^k \right) + a(r^{n+1} - 1) \\ rS_n &= S_n + a(r^{n+1} - 1) \\ S_n(r-1) &= a(r^{n+1} - 1) \\ S_n &= \frac{a(r^{n+1} - 1)}{r-1} \end{aligned}$$

Thus,

$$\sum_{k=0}^n ar^k = \begin{cases} \frac{a(r^{n+1}-1)}{r-1}, & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

When the sum is to infinity:

$$\begin{aligned}
\sum_{k=0}^{\infty} ar^k &= \lim_{n \rightarrow \infty} \sum_{k=0}^n ar^k \\
&= \lim_{n \rightarrow \infty} \begin{cases} \frac{a(r^{n+1}-1)}{r-1}, & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases} \\
&= \begin{cases} \frac{-a}{r-1}, & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases} \\
&= \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}
\end{aligned}$$

## Common Summations

There are a closed form solutions for many summations

$$\begin{aligned}
\sum_{i=1}^n c &= cn \\
\sum_{i=1}^n i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^n i^3 &= \left( \frac{n(n+1)}{2} \right)^2 \\
\sum_{k=0}^n ar^k &= \frac{a(r^{n+1}-1)}{r-1} \\
\sum_{k=0}^{\infty} x^k &= \frac{1}{1-x} \\
\sum_{k=1}^{\infty} kx^{k-1} &= \frac{1}{(1-x)^2}
\end{aligned}$$

## Double Sum

For a sequence with two indices  $\{a_{i,j}\}_{i,j \in \mathbb{Z}}$ , the sum is written as a double sum.

$$\sum_i \sum_j a_{i,j}$$

For finite sums, the order of evaluation does not matter and can be exchanged.

$$\sum_i \sum_j a_{i,j} = \sum_j \sum_i a_{i,j}$$

## Cardinality

For finite sets, the cardinality is the amount of elements in the set. For infinite set, if there is a bijection from one set to another, then they have the same cardinality. If there is an injection from a set  $A$  to a set  $B$ , then the cardinality of  $A$  is less than or equal to  $B$

$$A \hookrightarrow B \Rightarrow |A| \leq |B|$$

## Countable Sets

A set  $A$  is countable if either  $A$  is finite or there is a bijection  $f : A \mapsto \mathbb{Z}^+$ . This is equivalent to being able to list the elements of the set, indexed by  $\mathbb{Z}^+$ .

- Any subset of countable sets is countable.
- The union of two countable sets is countable.

If a set is not countable, then it is called uncountable. The cardinality of a countable set is denoted by  $\aleph_0$ .

### Example 1

The set of odd positive integers  $A$  is a countable set.

$$f : \mathbb{Z}^+ \mapsto A, \quad f(n) = 2n + 1$$

Since  $f$  is bijective, the cardinality of  $A$  and  $\mathbb{Z}^+$  is the same and thus, it is countable.

### Example 2

Is  $|\mathbb{Z}| = \mathbb{Z}^+$ ?

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots, n, -n, \dots\}$$

Since there is an integer indexed list of elements for the set  $\mathbb{Z}$ , it is a countable set.

### Example 3

Prove that the set of real numbers in between zero and one  $\mathbb{R}_{(0,1)}$  is an uncountable set.

Assume that  $(0,1)$  is countable and  $d_{i,j}$  is a digit in base ten

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14} \dots$$

$$r_2 = 0.d_{21}d_{22}d_{23}d_{24} \dots$$

$$r_3 = 0.d_{31}d_{32}d_{33}d_{34} \dots$$

$$r_4 = 0.d_{41}d_{42}d_{43}d_{44} \dots$$

$$\vdots$$

Let  $r = 0.d_1d_2d_3d_4 \dots$  be a new number such that

$$d_i = \begin{cases} 4, & \text{if } d_{ii} \neq 4 \\ 5, & \text{if } d_{ii} = 4 \end{cases}$$

$r$  cannot be equal to any number in the list since  $d_i \neq d_{ii}$ . Thus it is not in the list of numbers, therefore, the set of real numbers from zero to one is not countable.