Differential Equations

Patrick Chen

April 20, 2025

An ordinary differential equations (ODE) is a type of differential equation (DE) that contain derivatives with respect to only a single independent variable. It is called ordinary because it only contains one independent variable. The order of an ODE the highest derivative in the differential equation. The solution for a differential equation is a function that satisfies the differential equation on some interval. The solution may or may not be unique. An autonomous differential equation is a differential equation that has no explicit terms containing the independent variable.

Shape of Differential Equation Graphs

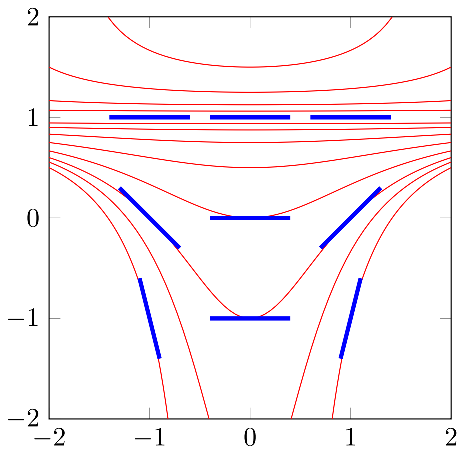
It is possible to determine properties of the shape of a differential equation by examining the equation.

Example

$$y' = x(y-1)^2$$

When this differential equation is evaluated at a point (x, y), the solution to the differential equation passing through the point (x, y) must be the slope found.

y x	-1	0	1
1	0	0	0
0	-1	0	1
-1	-4	0	4



Orthogonal trajectories

If $\frac{dy}{dx} = F(x, y)$, then the orthogonal trajectory is $\frac{dy}{dx} = -\frac{1}{F(x,y)}$

Example

$$y = ke^x$$
$$y' = ke^x$$

$$y' = -\frac{1}{ke^x}$$
$$\frac{dy}{dx} = -\frac{1}{y}$$
$$y\ dy = -dx$$
$$\frac{1}{2}y^2 = -x + c$$

Example 2

$$y = kx^2$$
$$y' = 2kx$$
$$k = \frac{y}{x^2}$$
$$y' = 2(\frac{y}{x^2})x$$
$$y' = \frac{2y}{x}$$

$$\frac{dy}{dx} = -\frac{1}{\frac{2y}{x}}$$
$$\frac{dy}{dx} = -\frac{x}{2y}$$
$$2y\ dy = -x\ dx$$
$$y^2 = -\frac{1}{2}x^2 + c$$
$$x^2 + 2y^2 = c$$

Guessing a Solution

If a differential equation is simple, it may be possible to correctly guess a solution. The guess and check method is to guess a generic function and check if there are any parameters of a function that will satisfy the differential equation.

Example

$$y'' - 3y' + 2y = 0$$

Let $y = e^{mx}$

$$\begin{aligned} y &= e^{mx} \\ y' &= me^{mx} \\ y'' &= m^2 e^{mx} \end{aligned}$$

$$\begin{aligned} y'' - 3y' + 2y &= 0 \\ m^2 e^{mx} - 3me^{mx} + 2e^{mx} &= 0 \\ (m^2 - 3m + 2)e^{mx} &= 0 \\ (m^2 - 3m + 2) &= 0 \\ (m-2)(m-1) &= 0 \\ m &= 1, 2 \\ \therefore\ y &= e^{2x}, y = e^x \end{aligned}$$

Initial Value Problems

Initial value problems are differential equation problems that need to satisfy some initial conditions.

Example

A Person is jumping off a high place. If $g = 10$ m/s^2 how fast do they move at 2s if at time 0s, position is 10m and velocity is 0 m/s.

$$\begin{aligned} F_{net} &= -F_g \\ ma &= -mg \\ a &= -g \\ x''(t) &= -10 \text{ m/s}^2 \\ x'(0) &= 0 \text{ m/s} \\ x(0) &= 10 \text{ m} \end{aligned}$$

$$\begin{aligned} x''(t) &= -10 \\ x'(t) &= \int -10\ dt \\ x'(t) &= -10t + c_1 \\ x(t) &= \int -10t + c_1\ dt \\ x(t) &= -5t^2 + c_1 t + c_2 \end{aligned}$$

$$
\begin{aligned}
x(0) &= -5(0)^2 + c_1(0) + c_2 \\
10 \text{ m} &= c_2
\end{aligned}
$$

$$
\begin{aligned}
x'(0) &= -10(0) + c_1 \\
0 \text{ m/s} &= c_1
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ x(t) &= -5t^2 + 10 \\
x'(2) &= -10(2) = -20 \text{ m/s}
\end{aligned}
$$

Separable Differential Equation

A separable first order differential equation has the following form. In separable differential equations, the variables can be separated and put on different sides of the equal sign, then integrated.

$$\frac{dy}{dx} = f(x)g(y)$$

Example

$$
\begin{aligned}
y' &= \frac{x}{y} \\
\frac{dy}{dx} &= \frac{x}{y} \\
y\frac{dy}{dx} &= x \\
\int y\frac{dy}{dx}\ dx &= \int x\ dx \\
\int y\ dy &= \int x\ dx \\
\frac{1}{2}y^2 &= \frac{1}{2}x^2 + c \\
y^2 &= x^2 + c
\end{aligned}
$$

Linear Differential Equations

A first order linear ode can be put in the following form.

$$
\begin{aligned}
y' + p(x)y &= q(x) \\
I(x)y' + I(x)P(x)y &= I(x)Q(x)
\end{aligned}
$$

If we assume that this is a product rule,

$$\Big(I(x)y\Big)' = I(x)Q(x) \Rightarrow I(x)y' + I'(x)y = I(x)Q(x)$$

This is almost in the form of the previous equation, just with the restriction

$$
\begin{aligned}
\frac{d}{dx} I(x) &= P(x) I(x) \\
\frac{dy}{dx} &= P(x) y \\
\frac{1}{y} dy &= P(x) dx \\
\ln |y| &= \int P(x)\ dx \\
I(x) &= A e^{\int P(x)\ dx}
\end{aligned}
$$

Thus any linear differential equation can be converted to the derivative of the product of two functions $I(x)$, $Q(x)$

Example

$$
\begin{aligned}
xy' + 3y &= \frac{e^x}{x^2} \\
y' + \frac{3}{x} y &= \frac{e^x}{x^3} \\
P(x) &= \frac{3}{x} \\
Q(x) &= \frac{e^x}{x^3} \\
I(x) &= e^{\int \frac{3}{x}\ dx} \\
&= e^{3 \ln x} \\
&= x^3 \\
x^3 y' + \frac{3}{x} x^3 y &= e^x \\
x^3 y' + 3x^2 y &= e^x \\
(x^3 y)' &= e^x \\
x^3 y &= e^x + c \\
y &= \frac{e^x}{x^3} + \frac{c}{x^3}
\end{aligned}
$$

Example 2

$0 < x < \pi/2$

$$
\begin{aligned}
y' + \tan(x)y &= \sin(x) \\
P(x) &= \tan(x) \\
Q(x) &= \sin(x) \\
I(x) &= e^{\int \tan(x)\ dx} \\
&= e^{-\ln(\cos(x))} \\
&= \sec x \\
y' \sec x + (\sec x \tan x)y &= \sec x \sin x \\
(y \sec x)' &= \tan x \\
y \sec x &= \ln(\sec x) + c \\
y &= \cos(x)\ln(\sec x) + c\cos(x)
\end{aligned}
$$