Evaluating Integrals

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The rules of derivatives can be inverted into rules of integrals.

Substitution Rule

The substitution rule (also called u-substitution) of the integral is the reverse of the chain rule for derivative. The function f must be continuous on the range of $g(x) = u$.

$$\begin{aligned} (F(u(x)))' &= f(u(x))u'(x) \\ F(u(x)) &= \int f(u(x))u'(x)\ dx \end{aligned}$$

When using substitution for definite integrals, the bounds needs to be changed to account for the change in integration variable. The bounds can be found by substituting the old bounds in for x and solving for u.

$$\int_a^b f(u(x))u'(x)\ dx = \int_{u(a)}^{u(b)} f(u)\ du$$

Example 1

Evaluate $\int 2x\sqrt{1+x^2}\ dx$

$$\begin{aligned} u &= 1 + x^2 \\ \frac{du}{dx} &= 2x \\ du &= 2x\ dx \\ \int 2x\sqrt{1+x^2}\ dx &= \int \sqrt{u}\ 2x\ dx \\ &= \int \sqrt{u}\ du \\ &= \int u^{\frac{1}{2}}\ du \\ &= \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c \end{aligned}$$

Example 2

Evaluate $\int x^3 \cos(x^4+2)\ dx$

$$\begin{aligned}
u &= x^4 + 2 \\
du &= 4x^3 dx \\
\int x^3 \cos(x^4+2)\ dx &= \int \frac{4x^3}{4} \cos(u)\ dx \\
&= \frac{1}{4} \int \cos(u)\ du \\
&= \frac{1}{4} \sin(u) + c \\
&= \frac{1}{4} \sin(x^4+2) + c
\end{aligned}$$

Example 3

Evaluate $\int \tan x\ dx$

$$\begin{aligned}
u &= \cos x \\
du &= -\sin x dx \\
\int \tan x\ dx &= \int \frac{\sin x}{\cos x}\ dx \\
&= \int -\frac{1}{u}\ du \\
&= -\ln|u| \\
&= -\ln|\cos x|
\end{aligned}$$

Example 4

Evaluate $\int_0^1 \frac{1}{x+4}\ dx$.

$$\begin{aligned}
u &= x + 4 \\
du &= dx \\
\int_0^1 \frac{1}{x+4}\ dx &= \int_{(0)+4}^{(1)+4} \frac{1}{u}\ du \\
&= \int_4^5 \frac{1}{u}\ du \\
&= \ln|u|\ \Big|_4^5 \\
&= \ln|5| - \ln|4| \\
&= \ln(\frac{5}{4})
\end{aligned}$$