Complex Eigenvalues

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Complex Numbers

The imaginary unit i is a number defined to be a number such that $i^2 = -1$. Complex numbers take the form $z = a + bi$.

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$
- Conjugate: $\overline{a + bi} = a - bi$
- Magnitude: $||z|| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$
- Division: $\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{a^2+b^2}$

Complex addition and multiplication are both associative and commutative. Complex numbers are useful because every non-constant complex polynomial of degree n has n (possibly repeated) root when considering complex polynomials.

$$
\begin{aligned}
r &= \sqrt{a^2 + b^2} \\
a &= r\cos(\theta) \\
b &= r\sin(\theta) \\
z &= r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}
\end{aligned}
$$

Complex Numbers can also be represented as angle and magnitude in what is called polar coordinates. The angle is measured from the x-axis, counter-clockwise.

Complex Eigenvalues

$\mathbb{C}^n$ is a complex vector space where addition and scalar multiplication is done component-wise. For a complex $n \times n$ matrix A, we say $\lambda \in \mathbb{C}$ is a eigenvalue of A if for some non-zero $u \in \mathbb{C}^n$, $Au = \lambda u$. Linear systems, matrices, determinants, eigenvalues, and diagonalization all remain unchanged when extending from $\mathbb{R}^n$ to $\mathbb{C}^n$.

General Diagonalization for Complex Matrices

A matrix A is only diagonalizable when the algebraic multiplicity equals the geometric multiplicity. Every symmetric real matrix has only real valued eigenvalues and is diagonalizable. If the matrix has n different eigenvalues, then A is diagonalizable.

General Notion of an eigenvalue

Suppose that $T : V \mapsto V$ is a linear transformation defined on a vector space V. If $T(v) = \lambda v$ for some non-zero $v \in V$, we say that λ is a eigenvalue for T and v is a eigenvector for λ.

Example 1

$$\begin{aligned} A &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ det(A - \lambda I) &= \lambda^2 + 1 \\ \lambda &= \pm i \end{aligned}$$

$$\begin{aligned} \lambda &= i \\ A - \lambda I &= \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \\ rref(A - \lambda I) &= \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \\ z_1 - iz_2 &= 0 \\ z_1 &= iz_2 \\ z &= t \begin{bmatrix} i \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lambda &= -i \\ A - \lambda I &= \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \\ rref(A - \lambda I) &= \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \\ z_1 + iz_2 &= 0 \\ z_1 &= -iz_2 \\ z &= t \begin{bmatrix} -i \\ 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
P &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \\
P^{-1} &= \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \\
D &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\
A &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \\
&= \frac{1}{2i} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}
\end{aligned}
$$

Example 2

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \\
\lambda = 1, 2, 3
\end{aligned}
$$

Since there are three distinct eigenvalues and the matrix A is 3×3, the matrix is diagonalizable.

Example 3

Suppose that there is a transformation T from infinitely differentiable function (C^∞) to infinitely different functions. Find the eigenvalues for T.

$$
\begin{aligned}
T : C^\infty &\mapsto C^\infty \\
T(f) &= \frac{df}{dx}
\end{aligned}
$$

$$
\begin{aligned}
\frac{df}{dx} &= \lambda f \\
f &= ke^{\lambda x}
\end{aligned}
$$

Every $\lambda \in \mathbb{C}$ is a eigenvalue

Example 4

Suppose that there is a linear transformation T from polynomials to polynomials. Find the eigenvalues for T.

$$
\begin{aligned}
T : P &\mapsto P \\
T(p) &= xp
\end{aligned}
$$

$$T(p) = \lambda p p = a_0 + a_1 x + \dots a_n x^n$$
$$T(p) = a_0 x + a_1 x^2 + \dots a_n x^{n+1}$$

If $\lambda \neq 0$, then $a_0 = 0$, and all the other coefficients are zero by induction. If $\lambda = 0$, then $p = 0$, thus there are no eigenvalues.