Inner Product

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Dot product

The dot product $u \cdot v$ is a special case of inner product $\langle u, v \rangle$ on $\mathbb{R}^n$.

$$u \cdot v = u^T v = \sum_n u_n v_n$$

The dot product satisfies the following properties:

1. $u \cdot v = v \cdot u$
2. $u \cdot (v + w) = u \cdot v + u \cdot w$
3. $(cu) \cdot v = c(u \cdot v)$
4. $u \cdot u \geq 0$
5. $u \cdot u = 0$ iff $u = 0$

Properties (4) and(5) are very important for dot products.

Length and Angle

Let $u, v \in \mathbb{R}^n$.

- $||u||$ represents the magnitude of u
- r represents the distance between u and v
- θ represents the angle between u and v

$$\begin{aligned} ||u|| &= \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \\ r &= ||u - v|| \\ \cos(\theta) &= \frac{u \cdot v}{||u|| \; ||v||} \end{aligned}$$

Orthogonality

Using cosine law

$$\begin{aligned}
||u||^2 + ||v||^2 - 2||u||||v||\cos(\theta) &= ||u-v||^2 \\
&= (u-v)\cdot(u-v) \\
&= (u\cdot u) - 2(u\cdot v) + (v\cdot v) \\
||u||^2 + ||v||^2 - 2||u||||v||\cos(\theta) &= ||u||^2 + ||v||^2 - 2(u\cdot v) \\
-2u\cdot v &= -2||u||||v||\cos(\theta) \\
\frac{u\cdot v}{||u||||v||} &= \cos(\theta)
\end{aligned}$$

If $\theta = \frac{\pi}{2}$, then $\cos\theta$ is zero. Two vectors are orthogonal to each other if their dot products are zero.

Set of orthogonal vectors

A set of vectors $\{v_1, \dots, v_n\}$ are orthogonal if all vectors in the set are non-zero and all pairs of vectors have a dot product of zero. Often the $e_1, \dots, e_n$ basis vectors are used as orthogonal vectors. A set of orthogonal vectors are always linearly independent.

Proof of linear independence

Suppose there is a orthogonal set $\{v_1, \dots, v_k\}$ and $c_1v_1 + \cdots + c_kv_k = 0$

$$\begin{aligned}
(c_1v_1 + \cdots + c_kv_k)\cdot v_1 &= 0\cdot v_1 \\
c_1(v_1\cdot v_1) + \cdots + c_k(v_k\cdot v_1) &= 0 \\
c_1||v_1||^2 + 0 + \cdots + 0 &= 0 \\
c_1||v_1||^2 &= 0 \\
c_1 &= 0
\end{aligned}$$

By using this same process with each of the vectors in the orthogonal set, all of the coefficients must be equal to zero and thus, orthogonal sets are linearly independent.

Orthogonal Basis

The coordinates in an orthogonal basis with dot products. If $w \in span\{v_1, \dots, v_k\}$, then:

$$\begin{aligned}
w &= c_1v_1 + \cdots + c_kv_k \\
w\cdot v_i &= c_1(v_1\cdot v_i) + \cdots + c_i(v_i\cdot v_i) + \cdots + c_k(v_k\cdot v_i) \\
w\cdot v_i &= c_1(0) + \cdots + c_i||v_i||^2 + \cdots + c_k(0) \\
w\cdot v_i &= c_i||v_i||^2 \\
\frac{w\cdot v_i}{||v_i||^2} &= c_i
\end{aligned}$$

Often orthonormal basis are used. Since vectors in an orthonormal basis are both mutually orthogonal and unit length, $||v_i||^2 = 1$ and thus, the division is necessary.

$$c_i = w\cdot v_i$$

Example 1

Find $[w]_B$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}, \quad w = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
||v_1|| &= 2 \\
||v_2|| &= 3 \\
||v_3|| &= 6 \\
\frac{w \cdot v_1}{||v_1||^2} &= \frac{3}{2} \\
\frac{w \cdot v_2}{||v_2||^2} &= \frac{2}{3} \\
\frac{w \cdot v_3}{||v_3||^2} &= \frac{7}{6} \\
w &= \frac{3}{2}v_1 + \frac{2}{3}v_2 + \frac{7}{6}v_3 \\
[w]_B &= \begin{bmatrix} \frac{3}{2} \\ \frac{2}{3} \\ \frac{7}{6} \end{bmatrix}
\end{aligned}$$

Orthogonal complement

Suppose that W is a subspace of $\mathbb{R}^n$. The orthogonal complement of W, written $W^\perp$ is defined as all vectors that are orthogonal to W.

$$\begin{aligned}
W^\perp &= \{u \in \mathbb{R}^n : u \cdot w = 0 \text{ for all } w \in W\} \\
W \cap W^\perp &= \{0\}
\end{aligned}$$

The orthogonal complex is a subspace and as such is closed under addition and scalar multiplication. For any $w \in W$ and $u, v \in W^\perp$:

$$\begin{aligned}
w \cdot (u + v) &= w \cdot u + w \cdot v \\
&= 0 + 0 \\
&= 0 \\
cw &= w \cdot cu \\
&= c(w \cdot u) \\
&= c(0) \\
&= 0
\end{aligned}$$

The intersection of W and $W^\perp$ is $\{\vec{0}\}$.

$$\begin{aligned} w &\in W \\ w &\in W^\perp \\ w \cdot w &= 0 \\ \therefore w &= \vec{0} \end{aligned}$$

The vector space W is contained inside the orthogonal computed of the orthogonal complex of W If $w \in W$ and $u \in W^\perp$, $w \cdot u = 0$ therefore $w \in (W^\perp)^\perp$.

$$W \subseteq (W^\perp)^\perp$$