Multivariate Functions

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Single Variables Functions

Single variable functions have one independent variable and one dependent variable. They can be graphed in $\mathbb{R}^2$.

Multi Variable Functions

Functions with n independent variables and 1 dependent variables can be graphed in $\mathbb{R}^{n+1}$. In three dimensions with the axis (x, y, z), the axis is usually ordered by the right hand rule. For $z = f(x, y)$, the domain is a set of (x, y) pairs. The range is some interval on z axis.

$$f : \mathbb{R}^2 \mapsto \mathbb{R}$$

Example

Find the domain of the following functions

- $z = \sqrt{4 - x^2 - y^2}$

  $$\begin{aligned} 4 - x^2 - y^2 &> 0 \\ 2^2 &> x^2 + y^2 \end{aligned}$$

  The domain is the inside and border of a circle on the xy-plane with a radius of 2.

  $$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$$

- $z = \ln(x^2 - y^2 - 1)$

  $$\begin{aligned} x^2 - y^2 - 1 &> 0 \\ x^2 - y^2 &> 1 \end{aligned}$$

  This is a graph of a hyperbola.

Level Sets

A level set (also called contours) is a set of points in the domain where the function has the same value. Since a function can only result in one output, level sets cannot cross each other.

$$L_k = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = k\}$$

Limits and Continuity in 3D

For a function of two variables $f(x, y)$, a limit exists if for all directions, the limit approaches the same value L. A function is continuous at a point (a, b) if when the limit as (x, y) approaches (a, b), $f(a, b) = f(x, y)$.

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Most of the single variable rules apply to the multi variable limits except for H'Lopital's rule. H'Lopital's rule only applies to single variables.

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \quad \lim_{(x,y)\to(a,b)} g(x, y) = G$$

$$\lim_{(x,y)\to(a,b)} kf(x, y) = kL$$
$$\lim_{(x,y)\to(a,b)} f(x, y) + g(x, y) = L + G$$
$$\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = LG$$
$$\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{G}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

When x is fixed to 0

$$\lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

When y is fixed to 0

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Since the directional limits do not agree with each other, the limit does not exist.

Example 2

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

Fix $x = 0$

$$\lim_{y\to 0} \frac{(0)y}{(0)^2 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Fix $y = 0$

$$\lim_{x\to 0} \frac{x(0)}{x^2 + (0)^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Fix $x = y$

$$\lim_{x\to 0} \frac{xx}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Since the directional limits do not agree with each other, the limit does not exist.

Example 3

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$$

Fix $x = 0$

$$\lim_{y\to 0} \frac{(0)y^2}{(0)^2 + y^4} = \lim_{y\to 0} \frac{0}{y^4} = 0$$

Fix $y = 0$

$$\lim_{x\to 0} \frac{x(0)^2}{x^2 + (0)^4} = \lim_{x\to 0} 0/x^2 = 0$$

Fix $y = mx$

$$\lim_{x\to 0} \frac{x(mx)^2}{x^2 + m^4x^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2(1 + m^4x^2)} = \lim_{x\to 0} \frac{m^2x}{1 + m^4x^2} = \frac{0}{1} = 0$$

Fix $x = y^2$

$$\lim_{y\to 0} \frac{y^2y^2}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

Even though every directional limit from linear directions is zero, since there is one path that doesn't agree with the other limits, the limit doesn't exist.

Multi Variable Squeeze Theorem

Given $f(x, y) \leq g(x, y) \leq h(x, y)$ and that the limit of $f(x, y)$ and $h(x, y)$ approaches the same finite value, the limit of $g(x, y)$ must approach the same finite value.

$$\left(\lim_{(x,y)\to(a,b)} f(x, y) = L \text{ and } \lim_{(x,y)\to(a,b)} h(x, y) = L\right) \Rightarrow \lim_{(x,y)\to(a,b)} g(x, y) = L$$

If $0 \leq |g(x, y) - L| \leq h(x, y)$ and $h(x, y)$ approaches zero, then $g(x, y) = L$.

Example 4

$$f(x,y) = \frac{xy^2}{x^2+y^2}$$

$$\begin{aligned}
0 &\leq |f(x,y)| \leq \left|\frac{xy^2}{x^2+y^2}\right| \\
0 &\leq |f(x,y)| \leq \frac{|x|y^2}{x^2+y^2} \\
0 &\leq |f(x,y)| \leq \frac{|x|(x^2+y^2)}{x^2+y^2} \\
0 &\leq |f(x,y)| \leq |x|
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq \lim_{(x,y)\to(0,0)} |f(x,y)| \leq \lim_{x\to 0} |x| \\
0 &\leq \lim_{(x,y)\to(0,0)} |f(x,y)| \leq 0
\end{aligned}$$

Therefore by squeeze theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$