Integrals

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Oct 28, 2024

Antiderivatives

A function F is called an antiderivative or indefinite integral of f on an interval I if $F'(x) = f(x)$ for all x in I. Since adding a constant doesn't change the derivative of a function, the addition of a constant is often required for integration.

$$F'(x) = f(x)$$
$$F(x) = \int f(x)\ dx$$

Table of Antiderivatives

function	antiderivative
x^n when $n \neq -1$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\sec^2 x$	$\tan x + c$
$\sec x \tan x$	$\sec + c$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
$\frac{1}{\sqrt{1+x^2}}$	$\tan^{-1} x + c$
a^x	$\frac{a^x}{\ln a} + c$

Linearity of Integral

Just like derivatives, integrals are linear.

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$$
$$\int cf(x)\ dx = c\int f(x)\ dx$$

Example 1

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x} \qquad\qquad g(0) = 2$$

$$\begin{aligned}
g(x) &= \int 4\sin x + \frac{2x^5 - \sqrt{x}}{x}\ dx \\
&= \int 4\sin x\ dx + \int \frac{2x^5}{x}\ dx - \int \frac{\sqrt{x}}{x}\ dx \\
&= 4\int \sin x\ dx + 2\int x^4\ dx - \int x^{-\frac{1}{2}}\ dx \\
&= 4(-\cos x) + 2(\frac{x^5}{5}) - 2\sqrt{x} + c \\
&= -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + c
\end{aligned}$$

$$\begin{aligned}
g(0) &= -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + c \\
2 &= -4\cos 0 + \frac{2}{5}0^5 - 2\sqrt{0} + c \\
2 &= -4 + c \\
6 &= c \\
g(x) &= -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + 6
\end{aligned}$$

Example 2

$$\begin{aligned}
f''(x) &= 12x^2 + 6x - 4 \\
f(0) &= 4 \\
f(1) &= 1
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \int 12x^2 + 6x - 4\ dx \\
f'(x) &= 4x^3 + 3x^2 - 4x + c_1 \\
f(x) &= \int 4x^3 + 3x^2 - 4x + c_1\ dx \\
f(x) &= x^4 + x^3 - 2x^2 + c_1 x + c_2
\end{aligned}$$

$$
\begin{aligned}
f(0) &= 0 + 0 - 0 + 0 + c_2 \\
4 &= c_2 \\
f(1) &= 1 + 1 - 2(1) + c_1 + 4 \\
1 &= c_1 + 4 \\
-3 &= c_1 \\
f(x) &= x^4 + x^3 - x^2 - 3x + 4
\end{aligned}
$$

Sigma Notation

Sigma notation is a compact way to write a sum of many numbers.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

The index of the sum can be changes.

$$\sum_{i=m}^{n} a_i = \sum_{i=m+k}^{n+k} a_{i-k} \qquad \sum_{i=m}^{n} a_i = \sum_{i=m-k}^{n-k} a_{i+k}$$

Summations are linear.

$$
\begin{aligned}
\sum_{i=1}^{n} ca_i &= c\sum_{i=1}^{n} a_i \\
\sum_{i=m}^{n} (a_i + b_i) &= \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i
\end{aligned}
$$

There are a closed form solutions for many summations

$$
\begin{aligned}
\sum_{i=1}^{n} c &= cn \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}
$$

Example 3

$$\begin{aligned}
f(x) &= x^2 \\
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}(\frac{i}{n})^2 &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=i}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \\
&= \lim_{n\to\infty} \frac{n^2(2+3n^{-1}+n^{-2})}{6n^2} \\
&= \lim_{n\to\infty} \frac{2+3n^{-1}+n^{-2}}{6} \\
&= \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

Estimating Area

The area under a curve can be estimated with rectangles. The area in the interval $[0,1]$ can be estimated by the following formula. L_n is the area with the rectangles having a height determined by the left most part of the rectangle and R_n has height determined by the right most part. The height of the rectangle doesn't have to be the left most or right most point, it can be any point in between the bounds of the rectangle. M_n is the sum using the mid point of each rectangle.

$$\begin{aligned}
L_n &= \sum_{i=1}^{n} \frac{1}{n} \cdot f(\frac{i}{n}) \\
R_n &= \sum_{i=1}^{n} \frac{1}{n} \cdot f(\frac{i-1}{n}) \\
M_n &= \sum_{i=1}^{n} \frac{1}{n} \cdot f(\frac{2i+1}{2})
\end{aligned}$$

The precise area under a function is the limit as the rectangle width approaches zero.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

$$R_n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(\frac{i}{n})$$

In general, if f is continuous and positive on a interval $[a, b]$, then the area under the graph of $f(x)$ on the interval is given by the following formulas.

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
A = R_n &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a + i\Delta x) \\
A = L_n &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a + (i-1)\Delta x) \\
A = M_n &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a + x_i^*)
\end{aligned}
$$

Definite Integral

The definite integral on the interval $[a, b]$ is the signed area under a graph. If $f(x)$ is negative, then the area is considered negative. In other words, the integral on a interval is the area of every region where $f(x)$ is above the x-axis minus the area of every region where $f(x)$ is below the x-axis.

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example 4

$$
\begin{aligned}
\int_0^3 (x-1)\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} f(0 + \frac{3i}{n}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} (\frac{3i}{n} - 1) \\
&= \lim_{n\to\infty} \frac{3}{n} \Big(\frac{3}{n} \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \Big) \\
&= \lim_{n\to\infty} \frac{3}{n} (\frac{3}{n} \frac{n(n+1)}{2} - n) \\
&= \lim_{n\to\infty} \frac{3}{n} (\frac{3n+3}{2} - \frac{2n}{2}) \\
&= \lim_{n\to\infty} \frac{3}{n} (\frac{n+3}{2}) \\
&= \lim_{n\to\infty} \frac{3n+9}{2n} \\
&= \frac{3}{2}
\end{aligned}
$$

Properties of Definite Integrals

Definite integrals are linear

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$
$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

The integral of a constant is just the constant times the length of the bounds of the integral.

$$\int_a^b c\ dx = c(b-a)$$

Inverting the bounds of a integral will negate the result. This implies that integrals with both bounds equal will result in zero.

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$
$$\int_a^a f(x)\ dx = 0$$

Integrals can be combined if they share a common bound with the same function.

$$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$

Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and $F(x)$ is the antiderivative of f, then the area under the function f on the interval $[a, b]$ is equal to the antiderivative evaluated on b minus the antiderivative evaluated at a.

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

If $f(t)$ is continuous on $[a, b]$ and g is continuous on $[a, b]$ and differentiable on (a, b), then the derivative with respect to x of the definite integral with respect to t of $f(t)$ from a to x is the function f.

$$g(x) = \int_a^x f(t)\ dt$$
$$g'(x) = f(t)$$

Example 5

$$\begin{aligned}\int_3^6 \frac{1}{x}\ dx &= \ln(|x|)\ \Big|_3^6 \\ &= \ln(|6|) - \ln(|3|) \\ &= (\ln 2 + \ln 3) - \ln 3 \\ &= \ln 2\end{aligned}$$

Example 6

$$\begin{aligned}
\int_1^3 e^{-2x} &= -1/2e^{-2x} \Big|_1^3 \\
&= (-\frac{1}{2}e^{-6}) - (-\frac{1}{2}e^{-2}) \\
&= -\frac{1}{2}e^{-6} + \frac{1}{2}e^{-2}
\end{aligned}$$

Example 7

Find $g'(x)$.

$$\begin{aligned}
g(x) &= \int_2^{x^3} \sin(\ln t)\ dt \\
h(x) &= \int_2^{x} \sin(\ln t)\ dt \\
g(x) &= h(x^3) \\
g'(x) &= 3x^2 h'(x^3) \\
g'(x) &= 3x^2 \sin(\ln x^3)
\end{aligned}$$

Area between curves

If $f(x) \geq g(x)$, then

$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b f(x) - g(x)\ dx$$

If $f(x)$ is not greater or equal to $g(x)$, then

$$\int_a^b g(x)\ dx - \int_a^b f(x)\ dx = \int_a^b g(x) - f(x)\ dx$$

If f and g intersect, then split the integral at so the bounds are at the intersection point, then apply the previous rules.

Symmetry

If an even function is integrated on symmetric bounds, the result will be twice the are under the curve from zero to the first bound.

$$\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$$

If an odd function is integrated on symmetric bounds, the result will be zero.

$$\int_{-a}^{a} f(x)\ dx = 0$$