Cramer's Rule and Volume

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Cramer's Rule

For an $n \times n$ matrix A and any vector $b \in \mathbb{R}^n$, we define $\det(A)_i(b)$ as the determinant of the matrix A with the ith column replaced by b. If A is invertible, then the solution of $Ax = b$ is given by the following formula

$$x_i = \frac{\det(A)_i(b)}{\det(A)}$$

We can use Cramer's rule to get a explicit formula for the inverse

$$\begin{aligned} \operatorname{adj}(A) &= (C_{ij})^T \\ A^{-1} &= \frac{\operatorname{adj}(A)}{\det(A)} \end{aligned}$$

Example 1

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2 \\ \det \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} &= 1 \\ \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} &= 2 \\ \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} &= 3 \end{aligned}$$

$$\begin{aligned} x_1 &= 1/2 \\ x_2 &= 1 \\ x_3 &= 3/2 \end{aligned}$$

Example 2

Compute A^{-1} using Cramer's rule

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C_{11} = 1$$
$$C_{12} = 0$$
$$C_{13} = 1$$
$$C_{21} = 1$$
$$C_{22} = 2$$
$$C_{23} = -1$$
$$C_{31} = -1$$
$$C_{32} = 2$$
$$C_{33} = 1$$

$$\mathrm{adj}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\mathrm{adj} A}{\det A}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Relation with area

Suppose that S is a region in $\mathbb{R}^2$ with finite area and $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a linear transformation with a standard matrix A. The area of $T(S)$ is equal to $|\det A|$ times the area of S. The same is true for volumes in $\mathbb{R}^3$ and general n-volumes in $\mathbb{R}^n$.