

Partial Fraction

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$$\int \frac{P(x)}{Q(x)} dx$$

If $P(x)$ and $Q(x)$ are polynomials.

- If $\deg(P) \geq \deg(Q)$, then use polynomial long division.
- If the degree of the numerator is less than the degree of the denominator and $Q(x)$ is factorizable into non-repeating linear terms, then:

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$$
$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}$$

- If the degree of the numerator is less than the degree of the denominator and the factorization of $Q(x)$ contains repeated roots, then there must be fractions for each of the degrees for that root

$$P(x) = 4$$
$$Q(x) = (x - 3)^3(x + 2)$$
$$\frac{4}{(x - 3)^3(x + 2)} = \frac{A_1}{(x - 3)} + \frac{A_2}{(x - 3)^2} + \frac{A_3}{(x - 3)^3} + \frac{A_4}{x + 2}$$

- If the denominator is a irreducible polynomial (polynomials with no real roots), instead of putting constants in the numerator, put a polynomial with one fewer degree in the numerator.

$$\frac{1}{(x - 1)(x^2 + x + 2)} = \frac{A_1}{x - 1} + \frac{A_2x + B_2}{x^2 + x + 2}$$

Since these equations are true for all values x , certain values of x can be strategically substituted to cancel the other factors and make solving easier. Every polynomial can be factored into linear and quadratic terms in the reals.

$$x^4 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$$
$$a = \sqrt{2}$$
$$b = -\sqrt{2}$$

Example 1

$$\int \frac{x^3 + x}{x - 1} dx$$

$$\begin{array}{r} x^2 - x + 2 \\ x - x \overline{) x^3 - x^2 + x} \\ \underline{x^3 - x^2} \\ -x^2 + x \\ \underline{x^2 - x} \\ 2x \\ \underline{2x - 2} \\ 2 \end{array}$$

$$\begin{aligned} \frac{x^3 + x}{x - 1} &= (x^2 - x + 2) + \frac{2}{x - 1} \\ \int \frac{x^3 + x}{x - 1} dx &= \int (x^2 - x + 2) + \frac{2}{x - 1} dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln |x - 1| + c \end{aligned}$$

Example 2

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$

$$\begin{aligned} 2x^3 + 3x^2 - 2x &= x(2x^2 + 3x - 2) \\ &= x(2x^2 + 4x - 1x - 2) \\ &= x(x + 2)(2x - 1) \\ \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} &= \frac{A_1}{x} + \frac{A_2}{2x + 2} + \frac{A_3}{x + 2} \\ &= \frac{A_1(2x + 2)(x + 2) + A_2x(x + 2) + A_3x(2x + 2)}{2x^3 + 3x^2 - 2x} \\ x^2 + 2x - 1 &= A_1(2x + 2)(x + 2) + A_2x(x + 2) + A_3x(2x + 2) \\ x^2 + 2x - 1 &= (2A_1 + A_2 + 2A_3)x^2 + (3A_1 + 2A_2 - A_3)x - 2A_1 \\ x^2 &= (2A_1 + A_2 + 2A_3)x^2 \\ 2x &= (3A_1 + 2A_2 - A_3)x \\ -1 &= -2A_1 \\ A_1 &= 0/2 \\ A_2 &= a \\ A_3 &= -1/10 \end{aligned}$$

Example 3

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$$

$$x^3 - x^2 - x + 1 \Big| \frac{x + 1}{x^4 - x^3 - x^2 + x}$$

$$\int x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \, dx$$

$$\begin{array}{r} x^2 \quad -1 \\ (x-1) \overline{) x^3 - x^2 - x + 1} \\ \underline{x^3 - x^2} \\ -x + 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

$$\int x + 1 + \frac{4x}{(x-1)(x^2-1)} dx$$

$$\begin{aligned}\frac{4x}{(x-1)(x^2-1)} &= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1} \\ &= A_1(x-1)() \\ 4x &= (A_1 + A_3)x^2 + (A_2 - 2A_3)x + (-A_1 + A_2 + A_3) \\ A_1 &= 1 \\ A_2 &= 2 \\ A_3 &= -1\end{aligned}$$

$$\begin{aligned} & \int x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + c \end{aligned}$$

Example 4

$$\int \frac{-2x^2 + 4x + 2}{(x-1)(x^2+3)} dx$$

$$\begin{aligned}\frac{A_1}{x-1} + \frac{A_2x+B_2}{x^2+3} &= \frac{A_1(x^2+3) + (A_2x+B_2)(x-1)}{(x-1)(x^2+3)} \\ -2x^2 + 4x + 2 &= A_1x^2 + 3A_1 + A_2x^2 + B_2x - A_2x - 1 \\ A_1 &= 1 \\ A_2 &= -3 \\ B_2 &= 1\end{aligned}$$

$$\begin{aligned}& \int \frac{1}{x-1} + \frac{-3x+1}{x^2+3} dx \\ &= \ln|x-1| + \int \frac{-3x}{x^2+3} + \frac{1}{x^2+3} dx \\ &= \ln|x-1| + \int \frac{-3x}{x^2+3} + \frac{1}{x^2+3} dx \\ &= \ln|x-1| - \frac{3}{2} \int \frac{1}{u} du + \int \frac{1}{x^2+3} dx \\ &= \ln|x-1| - \frac{3}{2} \ln|x^2+3| + \int \frac{1}{x^2+3} dx \\ &= \ln|x-1| - \frac{3}{2} \ln|x^2+3| + 1/3 \int \frac{1}{(\frac{x}{\sqrt{3}})^2 + 1} dx \\ &= \ln|x-1| - \frac{3}{2} \ln|x^2+3| + \sqrt{3}/3 \int \frac{1}{u^2+1} du \\ &= \ln|x-1| - \frac{3}{2} \ln|x^2+3| + \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\end{aligned}$$