Simple Harmonic Motion

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Oscillatory motion is a repetitive back and forth motion about a equilibrium position. Periodic motion is common at all length scales. The period (T) is the time to complete a full cycle. The frequency (f, ν) is the amount of times it oscillates per unit time. An example of oscillatory motion is a block on a spring.

Simple Harmonic Oscillation

The spring force is a restoring force that pulls the mass back towards the equilibrium point. Simple harmonic motion is a periodic motion. To have simple harmonic motion, the restoring force must be proportional to the displacement. Simple harmonic motion is always a sinusoidal curve.

$$\begin{aligned} F &= -kx \\ m\frac{d^2x}{dt^2} &= -kx \\ x(t) &= A\cos(\omega t + \Phi) \end{aligned}$$

- A is the amplitude
- ω is the angular frequency ($\omega = 2\pi f$)
- Φ is the phase constant

Angular Frequency

The period T is the time needed to complete one cycle.

$$\begin{aligned} x(t) &= A\cos(\omega t + \Phi) \\ &= A\cos(\omega t + \Phi + 2\pi) \\ x(t+T) &= x(t) \end{aligned}$$

thus $\omega t = 2\pi$

$$\omega = \frac{2\pi}{T} = 2\pi f$$

The angular frequency is measured in radians per second.

Phase

The initial phase determines where the in motion we set $t = 0$.

Velocity and Acceleration

$$\begin{aligned}
x(t) &= A\cos(\omega t + \Phi) \\
v(t) &= -A\omega \sin(\omega t + \Phi) \\
a(t) &= -A\omega^2 \cos(\omega t + \Phi) = -\omega^2 x(t) \\
v_{max} &= A\omega \\
a_{max} &= A\omega^2
\end{aligned}$$

The maximum velocity occurs when the position is at the equilibrium point. The maximum acceleration occurs when the position is farthest away from the equilibrium point. The phase and amplitude can be computed from the starting positions and velocities.

$$\begin{aligned}
t &= 0 \\
x_0 &= A\cos(\omega t + \Phi) \\
&= A\cos(\Phi) \\
v_0 &= -A\omega \sin(\omega t + \Phi) \\
&= -A\omega \sin(\Phi)
\end{aligned}$$

$$\begin{aligned}
\frac{v_0}{x_0} &= -\frac{A\omega \sin(\Phi)}{A\cos(\Phi)} \\
\frac{v_0}{x_0} &= -\omega \tan(\Phi) \\
\tan^{-1}(-\frac{v_0}{\omega x_0}) &= \Phi
\end{aligned}$$

$$\begin{aligned}
x_0^2 + (\frac{v_0}{\omega})^2 &= (-A\sin(\Phi))^2 + (A\cos(\Phi))^2 \\
x_0^2 + (\frac{v_0}{\omega})^2 &= A^2\sin^2(\Phi) + A^2\cos^2(\Phi) \\
x_0^2 + (\frac{v_0}{\omega})^2 &= A^2(\sin^2(\Phi) + \cos^2(\Phi)) \\
x_0^2 + (\frac{v_0}{\omega})^2 &= A^2 \\
\sqrt{x_0^2 + (\frac{v_0}{\omega})^2} &= A
\end{aligned}$$

$$\begin{aligned}
\Phi &= \tan^{-1}(\frac{v_0}{\omega x_0}) \\
A &= \sqrt{x_0^2 + (\frac{v_0}{\omega})^2}
\end{aligned}$$

Spring

$$
\begin{aligned}
F &= -kx = ma \\
a &= \frac{d^2x}{dt^2} = -\frac{k}{m}x \\
a &= -\omega^2 x \\
\omega^2 &= \frac{k}{m} \\
\omega &= \sqrt{\frac{k}{m}}
\end{aligned}
$$

The frequency of the oscillations depends on only the material properties of the object and not the amplitude

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \\
f &= \frac{\omega}{2\pi} = \frac{\sqrt{\frac{k}{m}}}{2\pi}
\end{aligned}
$$

Energy

The oscillator will have both kinetic and spring potential. There are no non-conservative forces in the system.

$$
\begin{aligned}
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \Phi) \\
U &= \frac{1}{2}kx^2 \\
&= \frac{1}{2}kA^2 \cos^2(\omega t + \Phi) \\
K + U &= \frac{1}{2}A^2(m\omega^2 \sin^2(\omega t + \Phi) + k\cos^2(\omega t + \Phi)) \\
&= \frac{1}{2}A^2(k\sin^2(\omega t + \Phi) + k\cos^2(\omega t + \Phi)) \\
&= \frac{1}{2}kA^2(\sin^2(\omega t + \Phi) + \cos^2(\omega t + \Phi)) \\
&= \frac{1}{2}kA^2
\end{aligned}
$$

This could also be written as:

$$
K + U = \frac{1}{2}mv_{max}^2
$$

Using these formulas, the max velocity and velocity function with respect to position can be calculated.

$$\begin{aligned}
E = \frac{1}{2}mv_{max}^2 &= \frac{1}{2}kA^2 \\
v_{max}^2 &= \frac{k}{m}A^2 \\
v_{max}^2 &= \omega^2 A^2 \\
v_{max} &= \sqrt{\omega^2 A^2} \\
v_{max} &= \omega A
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}kx^2 + \frac{1}{2}mv^2 &= \frac{1}{2}kA^2 \\
kx^2 + mv^2 &= kA^2 \\
mv^2 &= kA^2 - kx^2 \\
v^2 &= \frac{k}{m}(A^2 - x^2) \\
v &= \sqrt{\omega^2(A^2 - x^2)} \\
v &= \omega\sqrt{A^2 - x^2}
\end{aligned}$$

$$\begin{aligned}
v_{max} &= \omega A \\
v &= \omega\sqrt{A^2 - x^2}
\end{aligned}$$

Example 1

A 0.25 kg block is oscillating on a spring with $k = 4$ N/m. At $t = 0$ s, $V = -0.2$ m/s and $a = +0.5m/s^2$. Find its total energy and equation of motion.

$$\begin{aligned}
v &= -A\omega\sin(\omega t + \Phi) \\
-0.2 &= -A\sqrt{\frac{k}{m}}\sin(\Phi) \\
-0.2 &= -A\sqrt{\frac{4}{0.25}}\sin(\Phi) \\
-0.2 &= -4A\sin(\Phi)
\end{aligned}$$

$$\begin{aligned}
a &= -A\omega^2\cos(\omega t + \Phi) \\
0.5 &= -\frac{k}{m}A\cos(\Phi) \\
0.5 &= -\frac{4}{0.25}A\cos(\Phi) \\
0.5 &= -16A\cos(\Phi)
\end{aligned}$$

Divide velocity equation by acceleration equation:

$$
\begin{aligned}
-\frac{0.2}{0.5} &= \frac{-4A\sin(\Phi)}{-16A\cos(\Phi)} \\
-\frac{0.2}{0.5} &= \frac{\sin(\Phi)}{4\cos(\Phi)} \\
-1.6 &= \tan(\Phi) \\
\Phi &= 2.12 \text{ or } -1.01
\end{aligned}
$$

Using the sign from the velocity and acceleration, we can determine that $\Phi = 2.12$.

$$
\begin{aligned}
-0.2 &= -4A\sin(2.12) \\
A &= 0.059m
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{1}{2}kA^2 \\
&= \frac{1}{2}(4)(0.059)^2 \\
&= 0.007J
\end{aligned}
$$

$$
x(t) = 0.059\cos(4t + 2.12)E \qquad = 0.007J
$$