Optimization

Patrick Chen

Oct 24, 2024

When solving optimization problems, it is often beneficial to draw a picture representing the problem. After translating the problem into math equations and solving, make sure the solutions are reasonable given the context.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.

$$2x + y = 2400$$
$$y = 2400 - 2x$$
$$A = x \cdot y$$
$$A(x) = x(2400 - 2x)$$
$$A(x) = -2x^2 + x2400$$
$$A'(x) = -4x + 2400$$
$$A'(x) = 0$$
$$-4x + 2400 = 0$$
$$x = \frac{-2400}{-4}$$
$$x = 600$$
$$2x + y = 2400$$
$$2(600) + y = 2400$$
$$y = 1200$$

Example 2

Find the area of the largest rectangle that can be inscribed in a semisircle of radius r.

$$x^2 + y^2 = r^2$$
$$A = 2xy$$

$$y = \sqrt{r^2 - x^2}$$
$$A = 2x\sqrt{r^2 - x^2}$$
$$A = \sqrt{4x^2(r^2 - x^2)}$$
$$A = \sqrt{4x^2 r^2 - 4x^4}$$

Since the maximum of $\sqrt{f(x)}$ is equal to $\sqrt{\text{max of } f(x)}$, we can find the max of $4x^2 r^2 - 4x^4$, then take a square root. This works in general for any increasing function.

$$f(x) = 4x^2 r^2 - 4x^4$$
$$f'(x) = 8xr^2 - 16x^3$$
$$f'(x) = 8x(r^2 - 2x^2)$$
$$f'(x) = 8x(r - x\sqrt{2})(r + x\sqrt{2})$$
$$f'(x) = -8x(x\sqrt{2} - r)(x\sqrt{2} + r)$$
$$x = 0, \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}$$

since x must be positive, $x = \frac{r}{\sqrt{2}}$.

Example 3

Find the point on the parabola $y^2 = 2x$ closest to $(1, 4)$.

$$y^2 = 2x$$
$$r = \sqrt{(x-1)^2 + (y-4)^2}$$

$$x = \frac{y^2}{2}$$
$$r = \sqrt{(\frac{1}{2}y^2 - 1)^2 + (y - 4)^2}$$
$$r = \sqrt{\frac{1}{4}y^4 - y^2 + 1 + y^2 - 8y + 16}$$
$$r = \sqrt{\frac{1}{4}y^4 - 8y + 17}$$
$$f(y) = \frac{1}{4}y^4 - 8y + 17$$
$$f'(y) = y^3 - 8$$
$$0 = y^3 - 8$$
$$8 = y^3$$
$$2 = y$$