Vector Spaces

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A vector space V is a non-empty set together with a binary function $+$ and unary functions c for every $c \in \mathbb{R}$ satisfying the following axioms. Here, bold plus ($\boldsymbol{+}$) represents addition between the abstract objects in the set, regular plus ($+$) represents addition between the scalars and zero ($\vec{\mathbf{0}}$) represents a abstract zero element in the set that behaves like a additive identity. The abstract addition operator is often just regular addition but it doesn't have to be. The same is true for the zero element.

1. The vector space is closed under addition
2. Addition is associative
3. Addition is commutative
4. there exists $\vec{\mathbf{0}} \in V$ such that $v \boldsymbol{+} \vec{\mathbf{0}} = v$ for all $v \in V$
5. for every $v \in V$, there exists $-v \in V$ such that $v \boldsymbol{+} (-v) = \vec{\mathbf{0}}$
6. for every $c \in \mathbb{R}$ and $u \in V$, $cu \in V$
7. for every $c \in \mathbb{R}$ and $u, v \in V$, $c(u \boldsymbol{+} v) = cu \boldsymbol{+} cv$
8. for every $c, d \in \mathbb{R}$ and $u \in V$, $(c + d)u = cu \boldsymbol{+} du$
9. for every $c, d \in \mathbb{R}$ and $u \in V$, $c(du) = (cd)u$
10. for every $v \in V$, $1v = v$

Side note: A space that satisfies 1-5 is called a abelian group.

Real Valued Functions as Vector Spaces

Let V any function from X to $\mathbb{R}$ where X is any set. Define addition on V with inputs f, g in V to be $f(x) + g(x)$. Given a c in $\mathbb{R}$ and $f : X \mapsto \mathbb{R}$, define (cf) to be $cf(x)$.

- Axioms 1 and 6 are immediate.
- Axioms 2 and 3 follow from the definition of addition being the sum of the outputs of the two functions. The outputs are in $\mathbb{R}$ where addition is associative and commutative, thus addition is both associative and commutative in this vector space.

- Axiom 4: $\vec{0}$ can be defined as the constant function that returns zero for all inputs x.

$$\begin{aligned}
g + 0 &= g(x) + f(x) \\
&= g(x) + 0 \\
&= g(x) \\
&= g
\end{aligned}$$

- Axiom 5: $-f$ can be defined as the function whose outputs are the negation of f for all points x

$$\begin{aligned}
(-f) &= -1 \cdot f(x) \\
f + (-f) &= f(x) + -f(x) \\
&= 0
\end{aligned}$$

- Axiom 7:

$$\begin{aligned}
c(f + g) &= c(f(x) + g(x)) \\
&= cf(x) + cg(x) \\
&= cf + cg
\end{aligned}$$

- Axiom 8: $\mathbb{R}$.

$$\begin{aligned}
(c + d)u &= (c + d)u(x) \\
&= cu(x) + du(x) \\
&= cu + du
\end{aligned}$$

- Axiom 9: The unary function c can be defined as $c \cdot f(x)$.

$$\begin{aligned}
c(df) &= c(df(x)) \\
&= (cd)f(x) \\
&= (cd)f
\end{aligned}$$

- Axiom 10: The scalar 1 is the identity of multiplication in $\mathbb{R}$.

$$\begin{aligned}
1f &= 1 * f(x) \\
&= f(x) \\
&= f
\end{aligned}$$

Subspaces

if V is a vector space and W is a non-empty subset of V then we call W a subspace of V if W is closed under addition and scalar multiplication. If W is a subspace of V, then W itself is a vector space.

Example 1

Prove that $\vec{0}$ is unique.

Suppose there are two zeros, $\vec{0}, \vec{0}'$.

$$\begin{aligned}
\vec{0} + \vec{0}' &= \vec{0} && \text{axiom 4} \\
\vec{0} + \vec{0}' &= \vec{0}' + \vec{0} && \text{axiom 2} \\
\vec{0}' + \vec{0} &= \vec{0}' && \text{axiom 4} \\
\vec{0} &= \vec{0}'
\end{aligned}$$

Example 2

prove that $-(-v) = v$.

$$\begin{aligned}
-(-v) &= v \\
0 &= v + (-v) \\
0 &= -v + -(-v) \\
v &= -(-v)
\end{aligned}$$

Example 3

Subspaces of $\mathbb{R}^2$.

- $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.
- Lines through the origin is a subspace of $\mathbb{R}^2$.
- $\vec{0}$ is a subspace of $\mathbb{R}^2$.

Example 4

Let V_n be the set of all polynomials with degree less than or equal to n. V_n is closed under addition because adding two polynomials doesn't increase the degree. Similarly, it is closed in scalar multiplication. This proves the V_n is a subspace.