Sequences

Patrick Chen

Jan 13, 2025

A sequence is a ordered list of real values. The list may or may not be infinite length. A sequence can be expressed either explicitly where there is a formula for the nth term or recursively where each term relies on the previous terms. A sequence is convergent if it approaches a value when the limit is taken, otherwise it is divergent.

$$\left(\lim_{n\to\infty} a_n = L\right) \Rightarrow \text{ convergent}$$
$$\left(\lim_{n\to\infty} a_n \neq L\right) \Rightarrow \text{ divergent}$$

For a recursive sequence, if the limit exists, it must be a fixed point of the recursion relation.

$$\left(\lim_{n\to\infty} a_{n+1} = f(a_n)\right) \Rightarrow \left(f(L) = L\right)$$

Example

For the following series, find the limit assuming that it exists.

$$\begin{aligned} a_{n+1} &= \sqrt{a_n + 2} \\ a_1 &= 0 \end{aligned}$$

$$\begin{aligned} L &= \sqrt{L+2} \\ L^2 &= L + 2 \\ L^2 - L - 2 &= 0 \\ (L-2)(L+1) &= 0 \\ L &= 2, -1 \end{aligned}$$

Since -1 can not be the result of a square root in the domain of real numbers, $L = 2$

Series

A series is an infinite sum of a sequence.

$$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=1}^{m} a_n$$

If the limit exists, then the series is convergent. Otherwise the series is divergent. A m-th partial sum S_m is the sum containing the first m vales.

$$S_m = \sum_{n=1}^{m} a_n$$
$$S = S_\infty = \lim_{m \to \infty} S_m$$

Geometric Series

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(1-r^n)}{1-r}$$

Convergence:

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{a}{1-r}(1-r^n) \\ &= \frac{a}{1-r}(1 - \lim_{n\to\infty} r^n)\end{aligned}$$

If $|r| < 1$, then the limit will be zero and the result is $\frac{a}{1-r}$. If $|r| > 1$, then the limit will diverge to infinity. If $r = 1$, then the sum will add a constant an infinite amount of times and will diverge to infinity. If $r = -1$, the sum will osculate and not converge.

P-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

The p-series converge for all $p > 1$ and diverge for all $p \leq 1$

Alternating Series

An alternating series is a series that alternates between adding and subtracting a term. If $b_n > 0$ then:

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if for all n, $a < a_n < b$ for some values a, b. A sequence is monotonic increasing if $a_n \leq a_{n+1}$ and is monotonic decreasing if $a_n \geq a_{n+1}$. A sequence that is both bonded and monotonic will converge.

Example

Use induction to show that the sequence $a_n = \sqrt{a_n + 2}$, $a_1 = 0$ is monotonic increasing.
Base Case:

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \sqrt{a_1 + 2} \\ &= \sqrt{0 + 2} \\ &= \sqrt{2} \\ 0 &\leq \sqrt{2} \\ \therefore a_1 &\leq a_2 \end{aligned}$$

Assume $0 \leq a_k \leq a_{k+1}$
$2 \leq a_k + 2 \leq a_{k+1}$
Since square roots is monotonic increasing, $\sqrt{2} \leq \sqrt{a_k + 2} \leq \sqrt{a_{k+1} + 2}$
Thus, $0 \leq a_{k+1} \leq a_{k+2}$ for all k

Example 2

$$\sum_{n=1}^{\infty} \ln(\frac{n}{n+1})$$

$$\begin{aligned} \sum_{n=1}^{\infty} \ln(\frac{n}{n+1}) &= \sum_{n=1}^{\infty} \ln(n) - \ln(n+1) \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} \ln(n) - \ln(n+1) \\ &= \lim_{m\to\infty} \Big(\ln(1) - \ln(2)\Big) + \Big(\ln(2) - \ln(3)\Big) + \cdots + \ln(m+1) \\ &= \lim_{m\to\infty} \ln(1) - \ln(m+1) \\ &= -\infty \end{aligned}$$

Therefore, this series diverges.

Absolute and Conditional Convergence

If a positive series converges, then a series with any sign change will also converge.

$$\Big(\sum_{n=1}^{\infty} |a_n| \text{ converges}\Big) \Rightarrow \Big(\sum_{n=1}^{\infty} a_n \text{ converges}\Big)$$

Assume $\sum |a_n|$ converges.

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

By comparison test, $\sum(a_n + |a_n|)$ will converge if $\sum |a_n|$ converges. Since $\sum(a_n + |a_n|)$ and $\sum |a_n|$ both converges, the difference between them will also converge.

$$\sum(a_n + |a_n|) - \sum |a_n| = \sum a_n$$

If $\sum |a_n|$ diverges but $\sum a_n$ converges, it is said to be conditionally convergent.

Divergence Test

If the Limit of a sequence converges to a non-zero constant, then the series will diverge. If the term goes to zero, this doesn't imply it will converge.

$$\Big(\lim_{m\to\infty} a_m \neq 0\Big) \Rightarrow \Big(\sum_{n=1}^{\infty} a_n \text{ diverges}\Big)$$

Proof:

$$\begin{aligned} a_n &= S_m - S_{m-1} \\ L &= \lim_{m\to\infty} S_m \end{aligned}$$

$$\begin{aligned} \lim_{m\to\infty} S_m - S_{m-1} &= \lim_{m\to\infty} a_m \\ L - L &= \lim_{m\to\infty} a_m \\ 0 &= \lim_{m\to\infty} a_m \end{aligned}$$

Integral Test

The integral test works by comparing the area under the graph of a function $f(n) = a_n$ with the area of a summation. If a $f(n)$ is positive, continuous, and monotonic decreasing, then the convergence of the integral of $f(x)$ will imply the convergence of the series. The $f(x)$ only needs to be monotonic decreasing after a certain point a for this test to work.

$$\Big(\int_a^{\infty} f(x)\,dx \text{ converges}\Big) \Rightarrow \Big(\sum_{n=a}^{\infty} a_n \text{ converges}\Big)$$

Example

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$$

$$\text{Let } f(x) = \frac{1}{x\ln(x)}$$

$f(x)$ is positive for values of $n \geq 2$ and continuous for $n \geq 2$. Since the denominator is always growing, then $f(x)$ is monotonic decreasing.

$$\begin{aligned} &\int_2^{\infty} \frac{1}{x\ln(x)}\,dx \\ u &= \ln(x) \\ du &= \frac{1}{x}dx \\ &\int_{\ln(2)}^{\infty} \frac{1}{u}\,du \end{aligned}$$

Since this integral diverges, the series will also diverge.

Comparison Test

If there are to series A and B with $0 \leq a_n \leq b_n$, then if A diverges to infinity, then B will diverge to infinity. If B converges to a finite value and a_n is monotonic, then A will converge because a_n is monotonic and bounded.

Limit Comparison Test

For two series A and B, if the limit of the ratio between terms L is a non-zero finite value, then A will converge if and only if B converges and A will diverge if and only if B diverges.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L$$

Alternating Series Test

An alternating series converges if b_n is monotonic decreasing and the limit goes to zero.

$$b_n \geq b_{n+1}, \quad \lim_{n \to \infty} b_n = 0$$

If the limit does not go to zero, the divergence test shows that it diverges. If the terms are not monotonic decreasing, it may or may not converge.

Ratio Test

In a geometric series, the next term is a ratio of the previous term. Since the geometric series is absolutely convergent, any sequence that behaves like geometric series will be absolutely convergent.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

Given a series, if the limit of the absolute value of the ratio between consecutive terms is less than 1, then it will converge. If it is greater than 1, it will diverge. The ratio test does not provide any useful information if the ratio is equal to 1.

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = L$$

$$\begin{aligned} \big(L = 1\big) &\Rightarrow \text{ No useful information} \\ \big(L < 1\big) &\Rightarrow \text{ Convergence} \\ \big(L > 1\big) &\Rightarrow \text{ Divergence} \end{aligned}$$

Root Test

The root test is more powerful than the ratio test but is much more painful to work with.

$$\begin{aligned} \sqrt[n]{|a_n|} &= |a_n|^{\frac{1}{n}} \\ &= |a|^{\frac{1}{n}} |r|^{\frac{n-1}{n}} \\ \lim_{n \to \infty} |a_n|^{\frac{1}{n}} &= |r| \end{aligned}$$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$\begin{aligned}
\big(L = 1\big) &\Rightarrow \text{ No useful information} \\
\big(L < 1\big) &\Rightarrow \text{ Convergence} \\
\big(L > 1\big) &\Rightarrow \text{ Divergence}
\end{aligned}$$

Example

Does the following series converge? Use the ratio test.

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{2n!}$$

$$\begin{aligned}
|\frac{a_{n+1}}{a_n}| &= \frac{\frac{3\cdot 5\cdot 7\cdots(2(n+1)+1)}{(2(n+1))!}}{\frac{3\cdot 5\cdot 7\cdots(2n+1)}{2n!}} \\
&= \frac{\frac{3\cdot 5\cdot 7\cdots(2n+1)\cdot(2n+3)}{(n+1)(n+2)\cdot(2n)!}}{\frac{3\cdot 5\cdot 7\cdots(2n+1)}{2n!}} \\
&= \frac{2n+3}{(n+1)(n+2)} \\
\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| &= \lim_{n\to\infty} \frac{2n+3}{(n+1)(n+2)} \\
&= 0
\end{aligned}$$

Since the limit goes to zero, this sequence is absolutely convergent.

Example 2

Does the following series converge? Use the root test.

$$\sum_{n=1}^{\infty} \Big(\frac{n^2+2n+1}{3n^2+n}\Big)^{2n}$$

$$
\begin{aligned}
a_n &= \left(\frac{n^2+2n+1}{3n^2+n}\right)^{2n} \\
\sqrt[n]{a_n} &= \left(\left(\frac{n^2+2n+1}{3n^2+n}\right)^{2n}\right)^{\frac{1}{n}} \\
&= \left(\frac{n^2+2n+1}{3n^2+n}\right)^{2} \\
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \left(\frac{n^2+2n+1}{3n^2+n}\right)^{2} \\
&= \left(\lim_{n\to\infty} \frac{n^2+2n+1}{3n^2+n}\right)^{2} \\
&= \left(\frac{1}{3}\right)^{2} \\
&= \frac{1}{9}
\end{aligned}
$$

Thus, this series is absolutely convergent.