

Derivative

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The Derivative Function

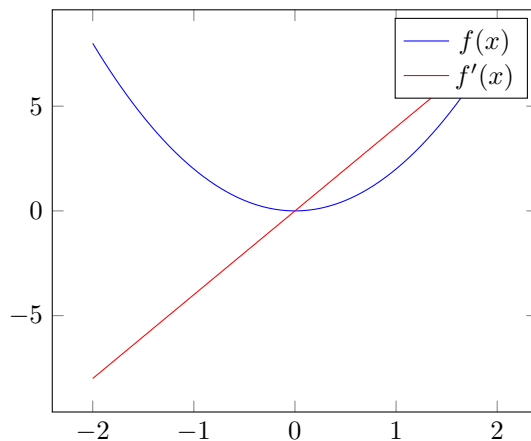
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f : \mathbb{R} \mapsto \mathbb{R}$$

$$f' : \mathbb{R} \mapsto \mathbb{R}$$

$$f' : x \mapsto \text{the slope of the function } f$$

$$f(x) = 2x^2$$

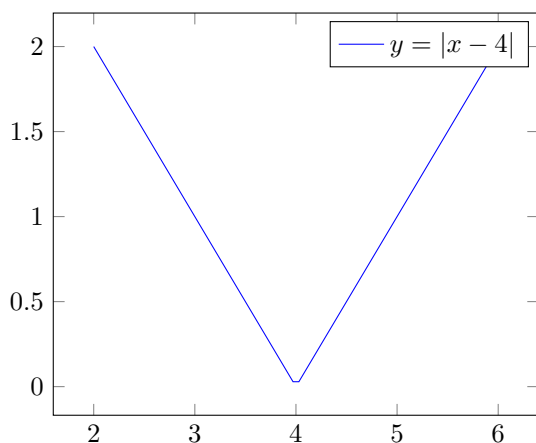


$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f(x)$$

A function f is differentiable at a if $f'(a)$ exists (and is finite). f is differentiable on (a, b) if f is differentiable at every point in (a, b) .

Example

$$f(x) = |x - 4|$$



$$|x - 4| = \begin{cases} x - 4, & x - 4 \geq 0 \\ -(x - 4) & x - 4 < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1, & x > 4 \\ -1 & x < 4 \\ \text{undefined} & x = 4 \end{cases}$$

$f'(0)$ is undefined because the left and right limits are not equal.

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \frac{f(4 + h) - f(4)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|h|}{h} \\ & \quad |h| \text{ and } h \text{ are positive} \\ &= 1 \end{aligned}$$

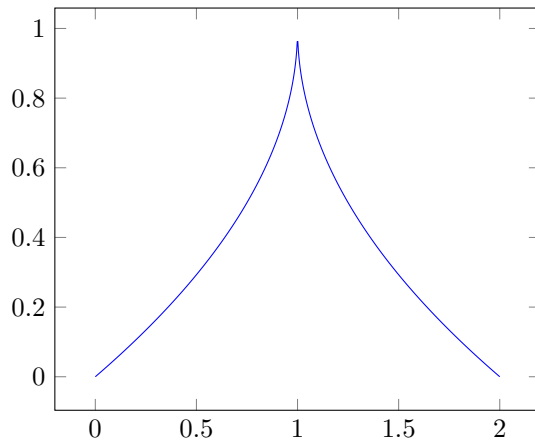
$$\begin{aligned} & \lim_{h \rightarrow 0^-} \frac{f(4 + h) - f(4)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ & \quad |h| \text{ is positive but } h \text{ is negative} \\ &= -1 \end{aligned}$$

Theorem 0.1 If f is differentiable at a , then f is continuous at a .

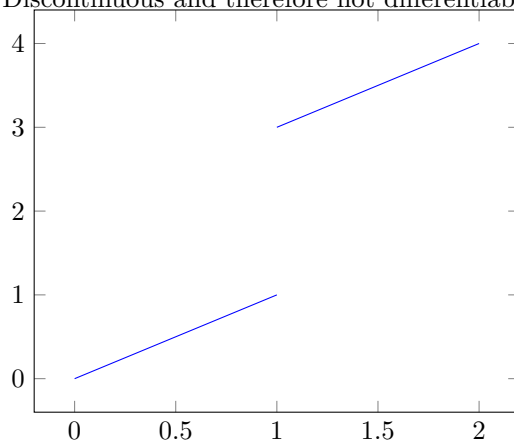
We saw this in the absolute value example. $|x - 4|$ is continuous but not differentiable at point $(4, 0)$.

There are many ways a function cannot be differentiable.

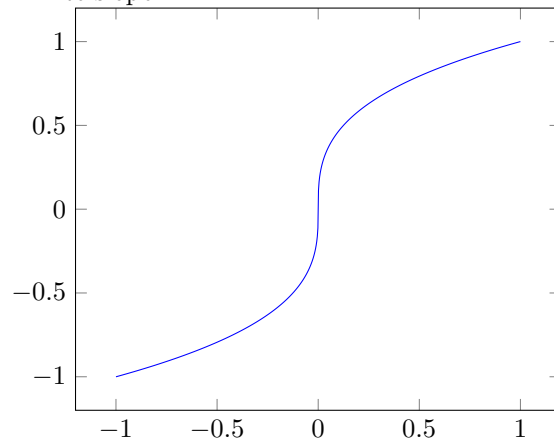
1. Cusp



2. Discontinuous and therefore not differentiable



3. Infinite slope



Higher Derivatives

The derivative of the derivative of a function is called the second derivative of the function.

$$\frac{d}{dx} \frac{dy}{dx} = \frac{d^2 y}{dx^2}$$

$$\frac{d}{dx} \frac{d}{dx} \cdots \frac{d}{dx} \frac{dy}{dx} = \frac{d^n y}{dx^n}$$

Derivative Rules

Constant Multiple, Addition, and Subtraction

$$\begin{aligned}\frac{d}{dx}(cf(x)) &= c \frac{d}{dx}(f(x)) \\ \frac{d}{dx}(f(x) + g(x)) &= \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) \\ \frac{d}{dx}(f(x) - g(x)) &= \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))\end{aligned}$$

Product and Quotient rules

When two functions are multiplied together (product rule):

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f(x) \frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x))g(x) \\ (fg)' &= f'g + fg'\end{aligned}$$

When one function is divided by another function (quotient rule):

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{\frac{d}{dx}(f(x))g(x) - f(x)\frac{d}{dx}(g(x))}{g(x)^2} \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2}\end{aligned}$$

Chain rule

For a function g that is differentiable at x and a function f that is differentiable at $f(x)$, the derivative of $f \circ g$ can be found using the chain rule.

$$\begin{aligned}y &= f(g(x)) \\ y' &= f'(g(x)) \cdot g'(x)\end{aligned}$$

Example: let $y = \sqrt{x^2 + 1}$

$$\begin{aligned}g(x) &= x^2 + 1 \Rightarrow g'(x) = 2x \\ f(x) &= \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \\ y' &= \frac{1}{2\sqrt{x^2 + 1}} 2x = \frac{x}{\sqrt{x^2 + 1}}\end{aligned}$$

Implicit Differentiation

Implicit differentiation is the process of treating a y like the function $f(x)$. This can be useful when it is difficult or impossible to isolate for y in an equation. The derivative of a curve can be found by using implicit differentiation then solving for y' .

Example:

$$\begin{aligned}x^3 + y^3 &= 6xy \\ \Downarrow \\ x^3 + f(x)^3 &= 6xf(x)\end{aligned}$$

$$\begin{aligned}(y^3)' &= (f(x)^3)' \\ &= 3f(x) \cdot f'(x) \\ &= 3y \cdot y'\end{aligned}$$

$$\begin{aligned}x^3 + y^3 &= 6xy \\ 3x^2 + 3y^2y' &= 6(xy' + 1y) \\ 3x^2 + 3y^2y' &= 6xy' + 6y \\ 3x^2 - 6y &= 6xy' - 3y^2y' \\ 3x^2 - 6y &= (6x - 3y^2)y' \\ \frac{3x^2 - 6y}{6x - 3y^2} &= y'\end{aligned}$$

This means the slope at the point (x, y) is $\frac{3x^2 - 6y}{6x - 3y^2}$

Inverse Functions

If f is a differentiable one-to-one function, the derivative of the inverse function can be found with the following formula.

$$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$$

Proof:

$$\begin{aligned}y &= f^{-1}(x) \\ f(y) &= f(f^{-1}(x)) \\ f(y) &= x \\ f'(y)y' &= 1 && \text{implicit differentiation} \\ y' &= \frac{1}{f'(y)} \\ y' &= \frac{1}{f'(f^{-1}(x))} && \text{substitute } y\end{aligned}$$

Derivatives of functions

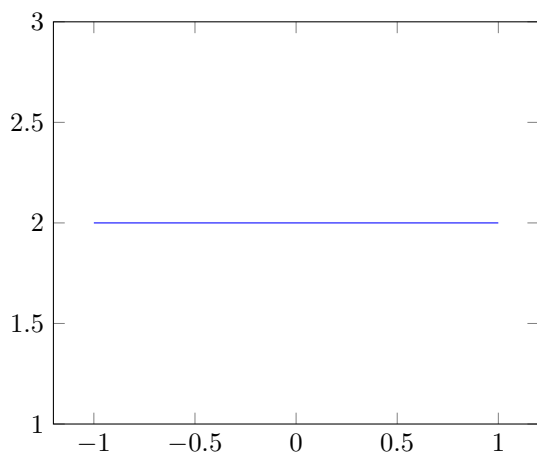
Polynomial

The constant function:

$$f(x) = c$$

$$\frac{d}{dx}(c) = 0$$

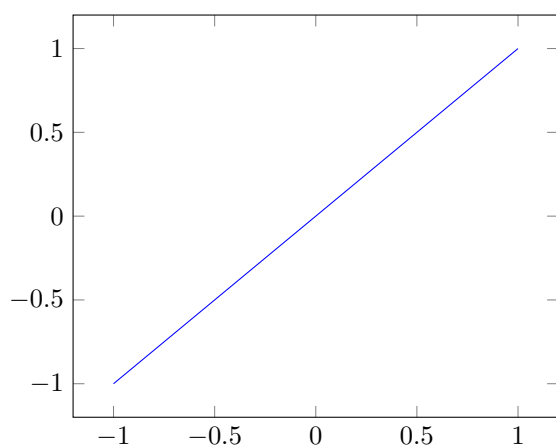
$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} \\ &= 0 \end{aligned}$$



Linear (degree one) function:

$$f(x) = x$$

$$\frac{d}{dx}(x) = 1$$

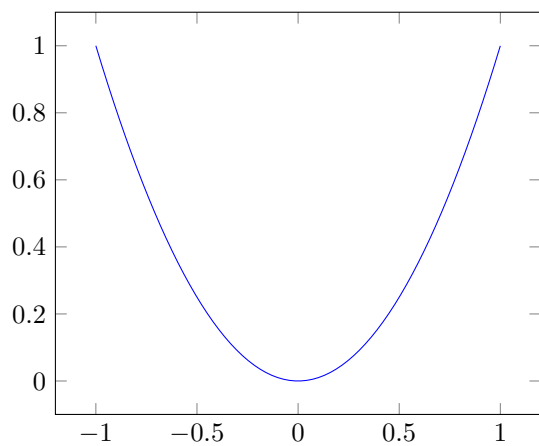


Quadratic:

$$f(x) = x^2$$

$$\frac{d}{dx}(x^2) = 2x$$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} 2x + h \\ &= 2x \end{aligned}$$



General:

$$f(x) = x^n$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Exponential

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \lim_{h \rightarrow 0} \frac{e^{(x+h)} - e^x}{h} \\ \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \\ \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} \\ e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h}\end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

The derivative of e^x is itself.

$$\frac{d}{dx} e^x = e^x$$

For an exponent with a different base, convert to base e using logarithm rules.

$$\begin{aligned}a^x &= e^{\ln(a^x)} = e^{x \ln(a)} \\ (a^x)' &= e^{x \ln(a)} \ln(a) = a^x \cdot \ln(a)\end{aligned}$$

Logarithm

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\begin{aligned}y &= \log_a x \\ a^y &= x \\ (\ln a) a^y y' &= 1 \\ y' &= \frac{1}{(a^y) \ln a} \\ y' &= \frac{1}{x \ln a}\end{aligned}$$

Trigonometric Functions

$$\begin{aligned}\frac{d}{dx} \sin(x) &= \cos(x) \\ \frac{d}{dx} \cos(x) &= -\sin(x) \\ \frac{d}{dx} \tan(x) &= \sec^2(x) \\ \frac{d}{dx} \csc(x) &= -\csc(x) \cot(x) \\ \frac{d}{dx} \sec(x) &= \sec(x) \tan(x) \\ \frac{d}{dx} \cot(x) &= -\csc^2(x)\end{aligned}$$

$$\begin{aligned}& \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) - \sin(x)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x) \sin(h)}{h} \\&= \sin(x) \lim_{h \rightarrow 0} \frac{(\cos(h) - 1)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \sin(x)(0) + \cos(x)(1) \\&= \cos(x)\end{aligned}$$

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 1$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Inverse Trigonometric Function

$$\begin{aligned}\frac{d}{dx} \sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \cos^{-1}(x) &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \tan^{-1}(x) &= \frac{1}{1+x^2} \\ \frac{d}{dx} \sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx} \csc^{-1}(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx} \cot^{-1}(x) &= -\frac{1}{1+x^2}\end{aligned}$$