Homogeneous Equations

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Ex

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Does $v_1 \ldots v_3$ span $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the matrix is already in row echelon form, it can span all of $\mathbb{R}^3$, but it does not tell how to get any individual vector. Note that if you have n vectors, at most, it could span $\mathbb{R}^n$. The dimension is the minimum amount of vectors required to span a space.

Linearity of matrix multiplication

A is a matrix, x, y are vectors, and λ is a scalar:

$$A(x + y) = Ax + Ay$$
$$A(\lambda x) = \lambda Ax$$

Homogeneous Systems

A homogeneous system is a system in the form of $Ax = 0$. All homogeneous systems has one trivial solution: $x = 0$ and may have non-trivial solutions.

Theorem 0.1 Any homogeneous system with more unknowns than equations has infinitely many solutions

Suppose we have a augmented matrix for a homogeneous system with more variables than equations with m columns and n rows. If $m > n$, there will be a free variable, and thus, infinitely many solutions.

$$\begin{bmatrix} A & \vec{0} \end{bmatrix}$$

Suppose that $Ax = b$ has a solution x_0 and y is any solution to $Ax = 0$. $x_0 + y$ is a solution to $Ax = b$.

$$A(x_0 + y) = Ax_0 + Ay = b + 0 = b$$
$$A(x_0 - x_1) = Ax_0 - Ax_1 = b - b = 0$$

Linearly Independence

A set of vectors $v_1 \dots v_m$ is linearly independent if in $\lambda_1 v_1 + \cdots + \lambda_m v_m = 0$, the only solution is $\lambda_1 = \cdots = \lambda_m = 0$. This means that for $A = [v_1 \dots v_m]$, the only solution to $Ax = 0$ is $x = 0$.

- a set of any one individual vector (except for the zero vector) is a set of linear independent vectors in $\mathbb{R}^n$.
- With a set of size two, any two vectors that are not scalar multiples of each other are linearly independent.
- In a set of m vectors in $\mathbb{R}^n$, a new linearly independent set can be created by adding a vector that is not in the span of the set.
- Any subset of a linearly independent set of vectors will also be linear dependent.

If $v_1 \dots v_m$ are in $\mathbb{R}^n$ and $m > n$, then $v_1 \dots v_m$ is not linearly independent. Since there are more columns than rows, the row reduction of $[v_1 \dots v_m]$ will have free variables, and thus a non-trivial solution to $Ax = 0$.

Example 1

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{array}{ccccccc} \lambda_1 & + & 0 & + & 0 & = & 0 \\ 0 & + & \lambda_2 & + & 0 & = & 0 \\ 0 & + & 0 & + & \lambda_2 & = & 0 \end{array}$$

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Example 2

Is $[1, 0, 0]^T$ and $[0, 1, 0]^T$ linearly independent in $\mathbb{R}^3$? Yes, when it is row reduced, it does not yield any free variables. They are also subsets of the linearly independent set in example 1, and are thus also linear independent.

Example 3

Is $v_1 = [1, 2, 3]^T, v_2 = [2, 1, 0]^T, v_3 = [3, 3, 3]^T$ linearly independent?

$$v_1 + v_2 - v_3 = 0$$

Since there is a non-trivial solution for the homogeneous equation, this set of vectors are not linearly independent.