

# Subspaces

Patrick Chen

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The dimension of a vector space is the amount of free variables that exist.

## Nullspace

For a  $n \times m$  matrix  $A$ , the set of solutions for  $Ax = 0$  is a subspace of  $\mathbb{R}^m$  called the nullspace of  $A$ , written as  $Nul(A)$ . The nullspace is a subspace, because it is closed under addition and scalar multiplication.

$$\begin{aligned}Ax &= 0 \\Ay &= 0 \\A(x + y) &= 0 + 0 \\Ax + Ay &= 0\end{aligned}$$

$$\begin{aligned}Ax &= 0 \\A(cx) &= cAx \\&= c0 \\&= 0\end{aligned}$$

## Span as a Subspace

The span of  $v_1 \dots v_k \in \mathbb{R}^n$  is a subspace. Zero is in the span of this set because all the vectors can have coefficients zero.

$$\begin{aligned}(c_1v_1 + \dots + c_kv_k) + (d_1v_1 + \dots + d_kv_k) &= (c_1 + d_1)v_1 + \dots + (c_k + d_k)v_k \\c(d_1v_1 + \dots + d_kv_k) &= (cd_1) + \dots + (cd_k)v_k\end{aligned}$$

## Columnspace

For a  $n \times m$  matrix  $A$ , the column space of  $A$  is a subspace of  $\mathbb{R}^n$  equal to the span of all the columns of  $A$ . Column space is written as  $Col(A)$ .

## Transformations Between Vector Spaces

Suppose that  $V$  and  $W$  are vector spaces and  $T : V \mapsto W$  is a function from  $V$  to  $W$ . The transformation  $T$  is called a linear transformation from  $V$  to  $W$  if for all  $u, v \in V$  and  $c \in \mathbb{R}$ ,

$$\begin{aligned}T(u + v) &= T(u) + T(v) \\T(cu) &= cT(u)\end{aligned}$$

## Kernel and Range

Suppose that  $T : V \mapsto W$  is a linear transformation. The kernel ( $ker$ ) of a linear transformation is a generalization of the nullspace of a matrix for linear transformations that are not matrix transformations. Likewise, the range ( $rng$ ) is the generalization for the column space. When working with matrices, the nullspace is the same as the kernel and the column space is the same as the range.

$$\begin{aligned}ker(T) &= \{v \in V \mid T(v) = 0\} \\rng(T) &= \{T(v) \mid v \in V\}\end{aligned}$$

The kernel and range of a linear transformation is a subspace.

$$\begin{aligned}0 &\in ker(T) \\u, v &\in ker(T) \\T(u) + T(v) &= 0 + 0 \\&= 0 \in ker(T) \\cT(u) &= c \cdot 0 \\&= 0 \in ker(T)\end{aligned}$$

$$\begin{aligned}0 &\in rng(T) \\T(u), T(v) &\in rng(T) \\T(u) + T(v) &= T(u + v) \in rng(T) \\cT(u) &= T(cu) \in rng(T)\end{aligned}$$

### Example 1

Find the null space of the following matrix.

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \\
 rref(A) &= \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \\
 \begin{array}{cccccccl}
 1x_1 & +0x_2 & +0x_3 & +\frac{3}{2}x_4 & -\frac{1}{2}x_5 & +\frac{3}{2}x_6 & = & 0 \\
 0x_1 & +1x_2 & +0x_3 & -\frac{3}{4}x_4 & +\frac{1}{4}x_5 & +\frac{5}{4}x_6 & = & 0 \\
 0x_1 & +0x_2 & +1x_3 & +0x_4 & +1x_5 & +1x_6 & = & 0
 \end{array} \\
 x &= \begin{bmatrix} -\frac{3}{2}x_4 + \frac{1}{2}x_5 - \frac{3}{2}x_6 \\ \frac{3}{4}x_4 - \frac{1}{4}x_5 - \frac{5}{4}x_6 \\ -x_5 - x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{4} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} -\frac{3}{2} \\ -\frac{5}{4} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_6
 \end{aligned}$$

### Example 2

Suppose that  $V$  is the vector space of all real-valued polynomials.

$$\begin{aligned}
 T : V &\mapsto V \\
 T(p) &= xp
 \end{aligned}$$

$$\begin{aligned}
 p, q &\in V \\
 T(p+q) &= x(p+q) \\
 &= xp + xq \\
 &= T(p) + T(q) \\
 T(cp) &= x(cp) \\
 &= c xp \\
 &= cT(p)
 \end{aligned}$$

$$\begin{aligned}
 \ker(T) &= \{0\} \\
 \text{rng}(T) &= \text{all polynomials with a zero constant term}
 \end{aligned}$$

$$D : V \mapsto V$$

$$D(p) = \frac{dp}{dx}$$

$$\begin{aligned} D(p+q) &= \frac{d(p+q)}{dx} \\ &= \frac{dp}{dx} + \frac{dq}{dx} \\ &= D(p) + D(q) \end{aligned}$$

$$\begin{aligned} D(cp) &= \frac{d(cp)}{dx} \\ &= c \frac{dp}{dx} \\ &= cD(p) \end{aligned}$$

$$\ker(D) = \text{constants}$$

$$\text{rng}(D) = V$$