

Indeterminate Forms

Patrick Chen

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Indeterminate forms are forms where substituting the limit variable will not give an answer.

Indeterminate Fractions

$$\frac{0}{0} \qquad \qquad \qquad \frac{\infty}{\infty}$$

L'Hospital's rule is a rule that can be used if the numerator and denominator of a fraction are both zero or both infinite.

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0$$

or

$$\lim_{x \rightarrow a} f(x) = \infty \text{ and } \lim_{x \rightarrow a} g(x) = \infty$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example 1

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^x}{x^2} &= \frac{\infty}{\infty} \\ &\stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2x} \\ &\stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty \end{aligned}$$

Indeterminate products

$$0 \cdot \infty$$

When an indeterminate product is found, it is often favorable to rewrite it as a fraction.

$$\frac{0}{(\frac{1}{\infty})} = \frac{0}{0} \qquad \qquad \qquad \frac{\infty}{(\frac{1}{0})} = \frac{\infty}{\infty}$$

Example 2

$$\begin{aligned}\lim_{x \rightarrow 0^+} x e^{\frac{1}{x}} &= 0 \cdot \infty \\&= \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \\&\stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} \\&= \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} \\&= e^{\frac{1}{0}} \\&= e^{\infty} \\&= \infty\end{aligned}$$

Example 3

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln x &= 0 \cdot -\infty \\&= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \\&\stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\&= \lim_{x \rightarrow 0^+} (-x) \\&= 0\end{aligned}$$

Indeterminate Powers

$$0^0 \quad \infty^0 \quad 1^\infty$$

Indeterminate powers can be converted into products by applying a logarithm.

$$\lim_{x \rightarrow n} a^b = e^{\left(\lim_{x \rightarrow n} b \cdot \ln a \right)}$$

Example 4

$$L = \lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} = 1^\infty$$

$$\ln L = \lim_{x \rightarrow 0^+} \cot x \ln(1 + \sin 4x) = \infty \cdot 0$$

$$\ln L = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

$$\ln L \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\left(\frac{4 \cos 4x}{1 + \sin 4x} \right)}{1 + \tan^2 x}$$

$$\ln L = \frac{\left(\frac{4 \cos 0}{1 + \sin 0} \right)}{1 + \tan^2 0}$$

$$\ln L = \frac{4}{1}$$

$$\ln L = 4$$

$$L = e^4$$