Inference

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Inference

We denote a inference as one (or more) premises leading to a conclusion. If $p_1, \ldots, p_n$ are premises and q is a conclusion, then an inference is $(p_1 \wedge \cdots \wedge p_n) \to q$.

$$\frac{p_1 \qquad \cdots \qquad p_n}{q}$$

If there are propositional functions, then

$$\frac{\forall x P(x)}{P(c) \text{ for all } c \text{ in the domain}} \quad \frac{\exists x P(x)}{P(c) \text{ for some } c \text{ in the domain}}$$

- Modus Ponens

$$\frac{p \to q \qquad p}{q}$$

- Modus Tollens

$$\frac{p \to q \qquad \neg q}{\neg p}$$

- Hypothetical Syllogism

$$\frac{p \to q \qquad q \to r}{p \to r}$$

- Disjunctive Syllogism

$$\frac{p \vee q \qquad \neg p}{q}$$

- Addition

$$\frac{p}{p \vee q}$$

- Simplification

$$\frac{p \wedge q}{q}$$

- Conjunction

$$\frac{p \qquad q}{p \wedge q}$$

- Resolution

$$\frac{p \vee q \qquad \neg p \vee r}{q \vee r}$$

Example

$$\frac{\neg p \wedge q \qquad r \to p \qquad \neg r \to s \qquad s \to t}{t}$$

Since $\neg p$ is true from $r \to p$ we see that $\neg r$ must be true.

$$\frac{\neg p \wedge q \qquad r \to p}{\neg r}$$

From $\neg r \to s$ and $\neg r$, we see that s must be true (modus ponens).

$$\frac{\neg r \qquad \neg r \to s}{s}$$

From $s \to t$ and s we conclude that t must be true.

$$\frac{s \qquad s \to t}{t}$$

Example 2

$$\frac{(p \wedge q) \vee r \qquad r \to s}{p \vee s}$$

If r is false, then $(p \wedge q) \vee r$, $p \wedge q$ must be T. Therefore, p is true and hence $p \vee s$ is true.

$$\frac{\frac{\frac{\neg r \qquad (p \wedge q) \vee r}{p \wedge q}}{p}}{p \vee s}$$

If r is true, then s is true from $r \to s$, therefore $p \vee s$ is true.

$$\frac{\frac{r \qquad r \to s}{s}}{p \vee s}$$

Example 3

Determine whether the following argument is valid.

$$\frac{p \to r \qquad q \to r \qquad \neg r}{p \vee q}$$

This argument is not valid.

$$\frac{\frac{\frac{p \to r \qquad \neg r}{\neg p} \qquad \frac{q \to r \qquad \neg r}{\neg q}}{\neg p \wedge \neg q}}{\neg(p \vee q)}$$

Proofs

Nomenclature

- Theorem: and important mathematical result

- Proposition: less important mathematical result
- Lemma a result that is needed to prove a theorem
- Corollary: a result that directly follows from a theorem

Methods of Proofs

- Direct proof: use all lines of reasoning. In a direct proof, we show that $P(c) \to Q(c)$ for any arbitrary c in the domain. We start with a hypothesis $P(c)$ and work to show that $Q(c)$ is true.
- Proof by contraposition: proving the contraposition. Since $P(c) \to Q(c) \equiv \neg Q(c) \to \neg P(c)$, we can prove that $\neg Q(c) \to \neg P(c)$.
- Proof by contradiction: assume that the theorem is false, then use lines of reasoning until there is a contradiction. If we wish to prove that $P(c) \to Q(c)$, for some c in the domain, we want to show that $P(c) \wedge \neg Q(c)$ is false.
- Proof by cases: proving all cases of a theorem. If we have as statement that can be expressed as multiple cases $P(c) \equiv P_1(c) \vee P_2(c) \vee \dots P_n(c)$, then we need to show that all possible cases is true, or equivalently $(P_1(c) \to Q(c)) \wedge \dots \wedge (P_n(c) \to Q(c))$.

Example

$P(n) = n$ is odd
$Q(n) = n^2$ is odd
Prove that $P(n) \to Q(n)$
If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$.
$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2k' + 1$

Example 2

For integers m and n, show that if nm is even, then either m or n is even.
Using contraposition, if m and n is odd, then mn is odd.

$$\begin{aligned}
m &= 2k_1 + 1 \\
n &= 2k_2 + 1 \\
mn &= (2k_1 + 1)(2k_2 + 1) \\
&= 4k_1k_2 + 2k_1 + 2k_2 + 1 \\
&= 2(2k_1k_2 + k_1 + k_2) + 1 \\
&= 2k' + 1 \\
\text{where } k' &= 2k_1k_2 + k_1 + k_2
\end{aligned}$$

Example 3

For all real number x, prove that $x \leq |x|$.
$P(x) \equiv \forall x(x \leq |x|) \equiv (\forall x < 0\ (x \leq |x|)) \wedge (\forall x \geq 0\ (x \leq |x|))$
If $x < 0$ then negative $<$ positive.
If $x \geq 0$, then $|x| = x \leq x$ therefore $x \leq |x|$

Example 4

Prove that $\sqrt{2}$ is irrational. Suppose that $\sqrt{2}$ is rational. Then:

$$\sqrt{2} = \frac{a}{b}$$

where a and b are coprime (no common factors).

$$\begin{aligned} b\sqrt{2} &= a \\ 2b^2 &= a^2 \end{aligned}$$

Since $2b^2$ is even, a^2 is even, thus $a = 2k$.

$$\begin{aligned} 2b^2 &= (2k)^2 \\ &= 4k^2 \\ b^2 &= 2k^2 \end{aligned}$$

Thus, b is also even and have a common factor of 2. This is a contradiction, therefore the assumption is false. Thus, $\sqrt{2}$ must be irrational.