Sets

Patrick Chen

Jan 20, 2025

Sets

A set is an unordered collections of distinct objects. They are represented by curly brackets. An empty set is denoted by $\emptyset$. If a is a member of the set A, then it is denoted by $a \in A$. If a set is small, then the roster method is usually used. The roster method is just writing all of the members. If there is a pattern, then ellipses can be used. The members of a set can be described by set builder notation.

$$A = \{a_1,\ a_2,\ \dots\}$$

Tuples

A n-tuple $(a_1, a_2, \dots, a_n)$ is an ordered collection of n elements.

Common Sets

- Natural Numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- Integers $\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$
- Positive Integers $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$
- Rational $\mathbb{Q}^+ = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$
- Real $\mathbb{R}$
- Positive Real $\mathbb{R}^+$
- Universal Set U

The Universal set is the set of all elements in a bounding set.

Set Builder Notation

Let $\mathbb{K}$ be a set and P be a proposition.

$$A = \{x \in \mathbb{K} \mid P\}$$

A represents a set that contain all the elements of $\mathbb{K}$ such that the proposition P holds. $\mathbb{K}$ is considered the bounding set.

Interval Notation

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
[a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\}
\end{aligned}$$

Set operations

- Equal: $A = B$ if and only if $\forall x(x \in A \leftrightarrow x \in B)$

$$\begin{aligned}
\{1, 3, 5, 7\} &= \{1, 1, 1, 5, 5, 7, 7, 4\} \\
\{1, 2, 3\} &\neq \{\{1, 2\}, 3\} \\
\emptyset &\notin \{1, 2, 3, 4, 5\}
\end{aligned}$$

- Subset: $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$
  - $A \subseteq A$
  - If $A = B$ then $A \subseteq B$

$$\begin{aligned}
\{1, 3, 5, 7\} &\subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
\{1\} &\subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
\{\{1, 2\}\} &\subseteq \{\{1, 2\}, 3\} \\
\{1, 3\} &\not\subseteq \{\{1, 2\}, 3\} \\
\emptyset &\subseteq \{1, 2, 3, 4, 5\}
\end{aligned}$$

- Proper subset: $A \subset B$ if and only if $(A \subseteq B) \wedge (A \neq B)$
- Cardinality: $|A|$. The cardinality is the size of the set.

$$\begin{aligned}
|\emptyset| &= 0 \\
|\{1, 2, 2, 4\}| &= 3 \\
|\{\{1, 2\}, 3\}| &= 2
\end{aligned}$$

- Power set: $\mathcal{P}(A)$ The set of all possible unique subsets. $\mathcal{P}(A) = \{B \mid B \subseteq A\}$

$$\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

- Cartesian product: The Cartesian product $A \times B$ is the collection of all pairs (a, b) with $a \in A$ and $b \in B$. Elements of the Cartesian product are tuples.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

  The Cartesian product can be extended to work on multiple sets

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1,\ a_2 \in A_2,\ \ldots\ ,\ a_n \in A_n\}$$

- Union: The union of A and B is the elements that are in either or both sets.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

- Intersection: The intersection of A and B is the elements that both sets.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

- Two sets are disjoint if they have no shared elements.

$$A \cap B = \emptyset$$

- The difference between A and B is all the elements in A that are not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

- The complement of a set A is everything that is not in A.

$$\overline{A} = U - A$$

- Symmetric Difference: the symmetric difference of A and B are all the elements in exactly one set but not both.
$$A \oplus B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$$

Example

Show that $A - B = A \cap \overline{B}$

$$\begin{aligned} x \in (A - B) &= x \in A \wedge x \notin B \\ &= x \in A \wedge x \in \overline{B} \\ &= x \in A \cap \overline{B} \end{aligned}$$

Set Identities

- Identity

$$\begin{aligned} A \cup \emptyset &= A \\ A \cap U &= A \end{aligned}$$

- Domination:

$$\begin{aligned} A \cup U &= U \\ A \cap \emptyset &= \emptyset \end{aligned}$$

- Commutative:

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

- Associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Complementation:

$$\overline{\overline{A}} = A$$

- DeMorgan's:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

- Distributive

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Negation:

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

- Absorption

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

Example

Show that $A \oplus B = (A - B) \cup (B - A)$

$$\begin{aligned}
(A - B) \cup (B - A) &= (A \cap \overline{B}) \cup (B \cap \overline{A}) && \text{Definition} \\
&= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) && \text{Distributive} \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) && \text{Distributive} \\
&= ((A \cup B) \cap U) \cap (U \cap (\overline{B} \cup \overline{A})) \\
&= (A \cup B) \cap (\overline{B} \cup \overline{A}) \\
&= (A \cup B) \cap \overline{A \cap B} && \text{DeMorgan} \\
&= (A \cup B) - (A \cap B) \\
&= A \oplus B && \text{Definition}
\end{aligned}$$

Union Notations

Let $A_1, \ldots, A_n$ be sets. The union and intersections of many sets can be written with the big union and big intersection notation.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \cdots \cup A_n$$
$$\bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n$$

Multiset

A multiset is a generalization of set which allows repetition. The notation we use for showing notation of an element is $n.a$ where a is the element and n is the amount of notation.

$$A = \{a_1.n_1,\ a_2.n_2\ \ldots\}$$