Power Series

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A power series is a sum of powers. The output of a power series is a function.

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

The domain of the function is the set of all $x \in \mathbb{R}$ where the series converges. The power series is said to be centered on a. When $a = 0$, the series is a 0-centered power series. The interval of convergence is centered on a. The difference between the center and where it diverges is called the radius of convergence R.

Example

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1}$$

Using ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}x^{n+1}}{n+2}}{\frac{3^n x^n}{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{3x(n+1)}{n+2} \right| \\
&= 3 \cdot |x|
\end{aligned}$$

This series will absolutely converge when $3 \cdot |x| < 1$ and diverge when $3 \cdot |x| > 1$.

$$\begin{aligned}
x &= \frac{1}{3} \\
\sum_{n=0}^{\infty} \frac{3^n (\frac{1}{3})^n}{n+1} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}$$

This will diverge because the $p = 1$.

$$x = -\frac{1}{3}$$
$$\sum_{n=0}^{\infty} \frac{3^n(-\frac{1}{3})^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

Using the alternating series test, this will converge. Thus the interval of convergence is $x \in [-\frac{1}{3,} \frac{1}{3})$

Example 2

Find the center and radius of convergence for the following series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{5^n n^2}$$

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(x-2)^{n+1}}{5^{n+1}(n+1)^2}}{\frac{(x-2)^n}{5^n n^2}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(x-2)n^2}{5(n+1)^2}\right| \\
&= \frac{|x-2|}{5}
\end{aligned}$$

$$\begin{aligned}
\frac{|x-2|}{5} &< 1 \\
|x-2| &< 5
\end{aligned}$$

$$a = 2,\ R = 5$$

Example 3

$$\sum_{n=0}^{\infty} n! x^n$$

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| \\
&= \lim_{n\to\infty} |x|(n+1) \\
&= \infty
\end{aligned}$$

This power series diverges for all non-zero values x

Example 4

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n\to\infty} \frac{|x|}{n+1} \\
&= 0
\end{aligned}$$

This power series converges for all finite values x

Functions as Power Series

Functions can be represented as power series. For example, $f(x) = \frac{1}{1-x}$ is represented by the following power series.

$$\sum_{n=0}^{\infty} x^n, \ -1 < x < 1$$

Using geometric series formula:

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \sum_{n=1}^{\infty} x^{n-1} \\
&= \frac{1}{1-x}
\end{aligned}$$

Example

$$\begin{aligned}
\frac{4x}{2-x} &= 4x \cdot (\frac{1}{2-x}) \\
&= \frac{4x}{2} \frac{1}{1-(\frac{x}{2})} \\
&= 2x\Big(\sum_{n=0}^{\infty} (\frac{x}{2})^n\Big) \\
&= 2x\Big(\sum_{n=0}^{\infty} (\frac{x}{2})^n\Big)
\end{aligned}$$

Example 2

Find the power series of $f(x) = \frac{1}{(1-x)^2}$

$$\begin{aligned}
f(x) &= \frac{d}{dx}(\frac{1}{1-x}) \\
&= \frac{d}{dx}\Big(\sum_{n=0}^{\infty} x^n\Big) \\
&= \sum_{n=0}^{\infty}\Big(\frac{d}{dx}x^n\Big) \\
&= 0 + \sum_{n=1}^{\infty}(nx^{n-1}) \\
&= \sum_{n=0}^{\infty}(n+1)x^n
\end{aligned}$$

The radius of convergence is the same because by the ratio test, $\frac{n+2}{n+1}$ will approach 1 and not affect the outcome when multiplied.

Example 3

Find the power series of $f(x) = \ln(1-x)$.

$$\begin{aligned}
f(x) &= \ln(1-x) \\
f'(x) &= -\frac{1}{1-x} \\
f'(x) &= -\Big(\sum_{n=0}^{\infty} x^n\Big) \\
\int f'(x)dx &= -\int\Big(\sum_{n=0}^{\infty} x^n\Big)dx \\
f(x) &= -\sum_{n=0}^{\infty}\Big(\int x^n dx\Big) \\
f(x) &= -\sum_{n=0}^{\infty}\Big(\frac{x^{n+1}}{n+1}\Big) + c
\end{aligned}$$

Finding the constant of integration:

$$\begin{aligned}
f(0) &= -\sum_{n=0}^{\infty}(\frac{0^{n+1}}{n+1}) + c \\
\ln(1-0) &= -\sum_{n=0}^{\infty}(0) + c \\
0 &= c
\end{aligned}$$

Thus,

$$f(x) = -\sum_{n=0}^{\infty}(\frac{x^{n+1}}{n+1})$$
$$f(x) = -\sum_{n=1}^{\infty}(\frac{x^{n}}{n})$$

Taylor and MacLaurin Series

Suppose $f(x)$ has a zero-centered power series. $f(0)$ is the constant term of the power series. $f'(0)$ is the x term of the power series. $\frac{f''(0)}{2}$ is the x^2 term of the power series. In general, $\frac{1}{n!}[\frac{d^n}{dx^n}f](0)$ is the x^n term of the power series.

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$
$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 \dots$$
$$f'''(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 \dots$$

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$\text{where } f^{(n)}(x) = \frac{d^n}{dx^n}f(x)$$

The Taylor series is an extension of the MacLaurin Series where the series may not be centered at zero. When finding a Taylor series, it may be better to not simplify the expressions of the derivative to help when finding a pattern.

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Common Taylor Series

$f(x)$	power series	domain
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n}!$	$x \in \mathbb{R}$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}!$	$x \in \mathbb{R}$

Taylor Polynomial

A m^{th} order Taylor polynomial $T_m(x)$ is the partial sum of the Taylor series.

$$T_m(x) = \sum_{n=0}^{m}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Remainder Theorem

The absolute difference between a function and its m^{th} Taylor polynomial is $R_m(x)$. It is always less than the next term of the Taylor series with the $(m+1)^{th}$ derivative of the function replaced with the supremum of the $(m+1)^{th}$ derivative of the function.

$$|f(x) - T_m(x)| = R_m(x) \leq \frac{M|x-a|^{m+1}}{(m+1)!}, \quad M \geq sup|f^{(m+1)}(x)|$$

Example

Find the 0-centered power series for $f(x) = e^x$

$$\begin{aligned} f(0) &= e^0 = 1 \\ f'(0) &= e^0 = 1 \\ f'''(0) &= e^0 = 1 \\ f^{(n)}(0) &= e^0 = 1 \end{aligned}$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Example 2

Find the Taylor series for $f(x) = \frac{1}{x^2}$ centered at $a = 2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-2}	$\frac{1}{2^2}$
1	$(-2)x^{-3}$	$\frac{-2}{2^3}$
2	$(-3)(-2)x^{-4}$	$\frac{(-3)(-2)}{2^4}$
3	$(-4)(-3)(-2)x^{-5}$	$\frac{(-4)(-3)(-2)}{2^4}$
n	$\frac{(-1)^n(n+1)!}{x^{n+2}}$	$\frac{(-1)^n(n+1)!}{2^{n+2}}$

$$f^{(n)}(2) = \frac{(-1)^n(n+1)!}{2^{n+2}}$$

$$
\begin{aligned}
\frac{1}{x^2} &= \sum_{n=0}^{\infty} \frac{\frac{(-1)^n (n+1)!}{2^{n+2}}}{n!} (x-2)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}} (x-2)^n
\end{aligned}
$$

Example 3

Find the Taylor series for $f(x) = \ln(x)$ centered at $a = 5$.

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!} (x-5)^n
$$

n	$f^{(n)}(x)$	$f^{(n)}(5)$
0	$\ln(x)$	$\ln(5)$
1	$\frac{1}{x}$	$\frac{1}{5}$
2	$\frac{-1}{x^2}$	$\frac{-1}{5^2}$
3	$\frac{(-1)(-2)}{x^3}$	$\frac{(-1)(-2)}{5^3}$
4	$\frac{(-1)(-2)(-3)}{x^4}$	$\frac{(-1)(-2)(-3)}{5^4}$
n	$\frac{(-1)^{n-1}(n-1)!}{x^n}$	$\frac{(-1)^{n-1}(n-1)!}{5^n}$

$$
\begin{aligned}
f(x) &= \frac{\ln(5)}{0!}(x-5)^0 + \sum_{n=1}^{\infty} \frac{\frac{(-1)^{n-1}(n-1)!}{5^n}}{n!} (x-5)^n \\
&= \ln(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} (x-5)^n
\end{aligned}
$$

Example 4

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^n n!} &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x = \frac{1}{2} \\
&= e^x,\ x = \frac{1}{2} \\
&= e^{\frac{1}{2}} \\
&= \sqrt{e}
\end{aligned}
$$

Example 5

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(9^n)(2n)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n \frac{\pi^{2n}}{3^{2n}}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{3})^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \;\; x = \frac{\pi}{3} \\
&= \cos(x), \;\; x = \frac{\pi}{3} \\
&= \cos(\frac{\pi}{3}) \\
&= \frac{1}{2}
\end{aligned}$$

Example 6

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{4^n (2n+1)!} &= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2^{2n+1})(2n+1)!} \\
&= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \\
&= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \;\; x = \frac{\pi}{2} \\
&= 2\pi \sin(x), \;\; x = \frac{\pi}{2} \\
&= 2\pi \sin(\frac{\pi}{2}) \\
&= 2\pi
\end{aligned}$$

Example 7

Find $T_4(x)$ about $x = 0$ for $f(x) = \cos(x)$ and estimate the upper bound on the error using the Taylor remainder estimate on the interval $[-2, 2]$.

$$\begin{aligned}
T_4(x) &= \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{6} x^3 + \frac{f^{(4)}(0)}{24} x^4 \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{24}
\end{aligned}$$

$$
\begin{aligned}
|\cos(x) - T_4(x)| &\leq \frac{M|x-a|^{m+1}}{(m+1)!} \\
\frac{M|x-a|^{m+1}}{(m+1)!} &= \frac{M|x|^5}{5} \\
\forall x \in [-2,2],&\ \ |x| \leq 2 \\
|\cos(x) - T_4(x)| &\leq M\frac{2^5}{5!} \\
|\cos(x) - T_4(x)| &\leq M\frac{4}{15}
\end{aligned}
$$

$$
\begin{aligned}
M &\geq sup_{[-2,2]}|f^{(m+1)}(x)| \\
sup_{[-2,2]}|f^{(m+1)}(x)| &= sup_{[-2,2]}|f^{(5)}\cos(x)| \\
&= sup_{[-2,2]}|-\sin(x)| \\
&= 1 \\
\therefore M &= 1
\end{aligned}
$$

$$
|\cos(x) - T_4(x)| \leq \frac{4}{15}
$$

Binomial Series

$$
\begin{aligned}
(1+x)^m &= \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m \\
\binom{n}{k} &= \frac{n!}{k!(n-k)!}
\end{aligned}
$$

$$
\begin{aligned}
(1+x)^m &= \sum_{n=0}^{m} \binom{m}{n} x^n \\
&= 1 + \sum_{n=1}^{m} \frac{k(k-1)(k-1)(k-3)\dots(k-n+1)}{n!}x^n
\end{aligned}
$$

Since for any term greater than m will have a zero factor,

$$
(1+x)^k = \sum_{n=1}^{\infty} \binom{k}{n} x^n = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!}
$$

Example

Use binomial series to find the zero-centered power series for $f(x) = \frac{1}{\sqrt{4+x}}$.

$$
\begin{aligned}
\frac{1}{\sqrt{4+x}} &= (4+x)^{-\frac{1}{2}} \\
&= \frac{1}{2}(1+\frac{x}{4})^{-\frac{1}{2}} \\
&= \frac{1}{2}(1+u)^{-\frac{1}{2}},\ u = \frac{x}{4} \\
&= \frac{1}{2}\Big(1 + \sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} u^n\Big) \\
&= \frac{1}{2}\Big(1 + \sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^n}{4^n}\Big) \\
&= \frac{1}{2}\Big(1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{1}{2}-n+1)}{n!} \frac{x^n}{4^n}\Big) \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5)\dots(2n-1)}{n!2^n} \frac{x^n}{2^{2n}} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n\ (1)(3)(5)\dots(2n-1)}{n!\ 2^{3n+1}} x^n
\end{aligned}
$$