Inverse and Logarithms

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Inverse Functions

A function f is called one-to-one function if it never takes on the same value twice. If for every horizontal lines, there are no horizontal lines that intersect the graph more than once, then the function satisfies the horizontal line test. If a function satisfies the horizontal line test, it is one-to-one.

Let f be a one-to-one function with domain A and range B. Its inverse f^-1 has domain of B and range A (inverse functions have inverted domain and ranges).

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$
$$f : A \mapsto B \Leftrightarrow f^{-1} : B \mapsto A$$

Theorem 0.1 Cancellation equations:

$$f^{-1}(f(x)) = x \text{ for every x in A}$$
$$f(f^{-1}(y)) = y \text{ for every y in B}$$

Example: let $f(x) = x^3$. Find $f^{-1}(x)$

$$\begin{aligned} f^{-1}(x) &= \sqrt[3]{x} \\ f^{-1}(f(x)) &= x \\ \sqrt[3]{x^3} &= x \\ x &= x \end{aligned}$$

$$\begin{aligned} f(f^{-1}(y)) &= y \\ (\sqrt[3]{y})^3 &= y \\ y &= y \end{aligned}$$

If you have the graph of f, f^{-1} is the function f reflected upon the x=y line. This is equivalent to swapping x and y for every point on the graph. The inverse of a function can be found algebraically by following these steps:

1. write $y = f(x)$
2. solve this equation for x in terms of y
3. interchange x and y

Logarithmic Functions

If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either strictly increasing or decreasing and therefore it is one-to-one and has an inverse function $f^{-1} = \log_b\ y$.

$$b^x = y \Leftrightarrow log_b y = x$$
$$\log_b(b^x) = x \text{ for every } x \in \mathbb{R}$$
$$b^{\log_b y} = y \text{ for every } y > 0$$

Laws of Logarithms

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b(\frac{x}{y}) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b(x)$

Natural Logarithms

The natural logarithm is a special logarithm with a base of e.

$$\ln(x) = \log_e(x)$$
$$\text{where } e \approx 2.718$$

$$e^x = y \Leftrightarrow \ln y = x$$
$$\ln(e^x) = x \text{ for every } x \in \mathbb{R}$$
$$e^{\ln y} = y \text{ for every } y > 0$$

Log change of base formula

$$\frac{\log_b a}{\log_b c} = \log_c a$$