Matrix of a Linear Transformation

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Linear Transformation

We say a transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear if it follows the rules of linearity. All matrices are linear transformations.

- for all $u, v \in \mathbb{R}^n, T(u + v) = T(u) + T(v)$
- for all scalars λ and $u \in \mathbb{R}^n, T(\lambda u) = \lambda T(u)$

Following these rules, we can conclude $T(0) = 0$ and for all scalars, c, d, and vectors u, v, $T(cu + dv) = cT(u) + dT(v)$

$$T(c_1u_1 + \cdots + c_ku_k) = c_1T(u_1) + \cdots + c_kT(u_k)$$

T's value on the span of $u_1 \ldots u_n$ is determined by its values at $u_1 \ldots u_n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \qquad \ldots$$

$$\begin{aligned} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= x_1e_1 + x_2e_2 + \cdots + x_ne_n \\ T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) &= T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n) \end{aligned}$$

A linear transformation is entirely defined by how it affects the basis vectors. All linear transformations from $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ has a unique matrix that describes the transformation called the standard matrix. The standard matrix is composed of the transformations applied to the basis vectors.

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & \ldots \end{bmatrix}$$

Common Transformations

Reflection on the $y = x$ line:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Counter clockwise rotation by the angle θ:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Properties

- A transformation is **one-to-one** if two different values of input always results in different outputs.
- A transform is **onto** if the range of a transformation is the same as the codomain

A linear transformation is one-to-one if the only answer to $T_A(x) = 0$ is the trivial solution $x = 0$. This is equivalent to checking if the columns of the matrix is linearly independent.