Multivariate Derivatives

Patrick Chen

Partial Derivative

In two dimensions, the partial derivative a of function $f(x, y)$ with respect to a variable is the one dimensional derivative with the other variable fixed.

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$\frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

In general, the partial derivative of a multivariate function $f(x_1, x_2, \ldots, x_n)$ with respect to x_i is the one dimensional derivative with all other independent variables fixed.

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(\ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots) - f(x_1, x_2, \ldots, x_n)}{h}$$

Partial derivatives are sometimes written using subscript notation.

$$\frac{\partial f}{\partial x} = f_x(x, y, \ldots)$$

Higher Partial Derivatives

Higher partial derivatives of a function is calculated the same way that regular higher derivatives are calculated.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial f}{\partial x}$$

When differentiating with respect to two different variables, it is written as follows. Note that the subscript notation and the Leibniz have reversed order.

$$f_{yx}(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y) = \frac{\partial^2 f}{\partial x \partial y}$$

Clairaut's Theorem: If the both mixed partial derivatives of a function is locally continuous, then the partial derivatives commute.

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 1

$$\begin{aligned}
g(x,y) &= xe^{x-y} \\
\frac{\partial g}{\partial x} &= e^{x-y} + xe^{x-y} \\
&= e^{x-y}(1+x) \\
\frac{\partial g}{\partial y} &= xe^{x-y}(-1) \\
&= -xe^{x-y}
\end{aligned}$$

Example 2

$$\begin{aligned}
f(x,y,z) &= \frac{z\arctan(xy)}{y} \\
\frac{\partial f}{\partial x} &= \frac{z}{y} \cdot \frac{1}{1+(xy)^2} \cdot y \\
&= \frac{z}{1+x^2y^2} \\
\frac{\partial f}{\partial y} &= z \cdot \frac{\left(\frac{\partial}{\partial y}\arctan(xy)\right)y - \arctan(xy)}{y^2} \\
&= \frac{z}{y^2}\left(\frac{xy}{1+x^2y^2} - \arctan(xy)\right) \\
\frac{\partial f}{\partial z} &= \frac{\arctan(xy)}{y}
\end{aligned}$$

Tangent Planes

A tangent line to $y = f(x)$ at a is a line passing through $(a, f(a))$ that is parallel to the graph of $f(x)$ at a.

$$y = \frac{df}{dx}x + b$$

A tangent plane to a function $z = f(x,y)$ passing through a point (x_0, y_0) is a plane such that is is passing the point $(a, b, f(a,b))$.

$$z = f_x(a,b)x + f_y(a,b)y + z_0$$

Thus

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

The tangent plane is also called the linearization of a function at a point. The first degree Taylor polynomial $T_1(x,y)$ is a tangent plane.

Example

Find the tangent plane to $z = 4x^2 + y^2$ at (x,y)=(1,2)

$$
\begin{aligned}
f(1,2) &= 8 \\
f_x(x,y) &= 8x \\
f_x(1,2) &= 8 \\
f_y(x,y) &= 2y \\
f_y(1,2) &= 4 \\
z &= 8x + 4y + z_0 \\
8 &= 8(1) + 4(2) + z_0 \\
z_0 &= -8 \\
z &= 8x + 4y - 8
\end{aligned}
$$

Differential

For low changes in x and y, the difference in z can be approximated with differentials.

$$
\begin{aligned}
\Delta z &\approx f_x(x,y)\Delta x + f_y(x,y)\Delta y \\
dz &= f_x(x,y)dx + f_y(x,y)dy
\end{aligned}
$$

Example

Given the function $u = \frac{t^4}{s^3}$, use the total differential to find the change in u if s decreases by 0.2 and t increases by 0.1 when $(s,t) = (2,1)$.

$$
\begin{aligned}
\Delta u &\approx f_s(s,y)\Delta s + f_t(s,t)\Delta t \\
f_s(s,t) &= -\frac{3t^4}{s^4} \\
f_t(s,t) &= \frac{4t^3}{s^3} \\
\Delta u &\approx -\frac{3t^4}{s^4}\Delta s + \frac{4t^3}{s^3}\Delta t \\
\Delta u &\approx -\frac{3(1)^4}{(2)^4}(-0.2) + \frac{4(1)^3}{(2)^3}(0.1) \\
\Delta u &\approx \frac{7}{80}
\end{aligned}
$$

Example 2

The volume of a cylinder is $V = \pi r^2 h$. What is the absolute error of the volume of a cylinder if the cylinder is measured to have a height of $14\text{m} \pm 0.2$ and a radius of $3\text{m} \pm 0.2$

$$\begin{aligned}
dV &= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\
&= 2\pi r h \ dr + \pi r^2 \ dh \\
&= 2\pi(3)(14) \ dt + \pi(3)^2 \ dh \\
&= 84\pi \ dr + 9\pi \ dh \\
\Delta V &\approx 84\pi(0.2) + 9\pi(0.2) \\
&\approx 18.6\pi \\
V &= \pi r^2 h \pm \text{err} \\
V &= \pi 3^2 \cdot 14 \pm \text{err} \\
V &= 126\pi \ \text{m}^3 \pm 18.6\pi \ \text{m}^3
\end{aligned}$$

Multivariate Chain Rule

For 2 variables

$$\begin{aligned}
z &= f(x, y) \\
x &= g(t) \\
y &= h(t)
\end{aligned}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

For n variables

$$\begin{aligned}
f &: \mathbb{R} \mapsto \mathbb{R}^n \\
g &: \mathbb{R}^n \mapsto \mathbb{R} \\
\mathbf{y} &= f(x) \\
z &= g(\mathbf{y})
\end{aligned}$$

$$\frac{\partial x}{\partial z} = \sum_i^n \frac{\partial x}{\partial y_i}\frac{\partial y_i}{\partial z}$$

When evaluating derivatives of multivariate function, it may be advantages to draw tree diagrams showing the dependencies of the functions.

Implicit Differentiation

In single variable

$$y = \frac{d}{dx} f(x) \Rightarrow \frac{d}{dx} g(y) = g'(y)\frac{dy}{dx}$$

Given a function $F(x, y) = k$ where F is differentiable and k is a constant. We can assume that y can be locally represented as a function of x.

$$\begin{aligned}
w &= F(x, y) \\
y &= f(x) \\
\frac{dw}{dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \\
0 &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \\
\frac{\partial F}{\partial y}\frac{dy}{dx} &= -\frac{\partial F}{\partial x} \\
\frac{dy}{dx} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \\
\frac{dy}{dx} &= -\frac{F_x}{F_y}
\end{aligned}$$

For multiple variables $F(x_1, \ldots, x_n) = k$:

$$\frac{\partial x_i}{\partial x_j} = -\frac{F_{x_j}}{F_{x_i}}$$

All relations can be made to be in that form by subtracting one side form the other.

$$\Big(f(x_1, x_2, \ldots) = g(x_1, x_2, \ldots)\Big) \Rightarrow \Big(f(x_1, x_2, \ldots) - g(x_1, x_2, \ldots) = 0\Big)$$

Example

Find $\frac{dy}{dx}$ for $e^{xy^2} = x - 3y$

$$\begin{aligned}
e^{xy^2} &= x - 3y \\
F(x, y) &= x - 3y - e^{xy^2} = 0 \\
F_x(x, y) &= 1 - y^2 e^{xy^2} \\
F_y(x, y) &= -3 - 2xye^{xy^2} \\
\frac{dy}{dx} &= -\frac{1 - y^2 e^{xy^2}}{-3 - 2xye^{xy^2}} \\
&= \frac{1 - y^2 e^{xy^2}}{3 + 2xye^{xy^2}}
\end{aligned}$$

Example 2

Find $\frac{\partial z}{\partial y}$ for $\cos(z^2 + xy) = \ln(x - z)$.

$$\begin{aligned}
F(x, y) &= \ln(x - z) - \cos(z^2 + xy) \\
F_y &= x \sin(z^2 + xy) \\
F_z &= -\frac{1}{x - z} + 2z \sin(z^2 + xy) \\
\frac{\partial z}{\partial y} &= -\frac{x \sin(z^2 + xy)}{-\frac{1}{x-z} + 2z \sin(z^2 + xy)} \\
&= \frac{x \sin(z^2 + xy)}{\frac{1}{x-z} - 2z \sin(z^2 + xy)}
\end{aligned}$$

Gradient

The gradient if function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a vector in $\mathbb{R}^n$ such that the components of the vector are the derivative of f with respect to the component. Geometrically, the gradient vector points in the direction f increases in. Vectors orthogonal to the gradient are directions where the value of f doesn't change.

$$\nabla f(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Directional Derivatives

The directional derivative $D_{\hat{\mathbf{u}}}$ of a function $f(x, y)$ in the direction of a unit vector $\hat{u}$ is the dot product between $\hat{u}$ and the gradient of the function.

$$D_{\hat{\mathbf{u}}} f(x, y) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\hat{\mathbf{u}}) - f(\mathbf{x})}{h} = \nabla f \cdot \hat{\mathbf{u}}$$

For two variables:

$$
\begin{aligned}
D_{\hat{\mathbf{u}}} f(x, y) &= \lim_{h \to 0} \frac{f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2)}{h} \\
&= \lim_{h \to 0} \frac{f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2 + hu_2) + f(x_1, x_2 + hu_2) - f(x_1, x_2)}{h} \\
&= \frac{f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2 + hu_2)}{h} + \frac{f(x_1, x_2 + hu_2) - f(x_1, x_2)}{h} \\
&= u_1 \frac{f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2 + hu_2)}{hu_1} + u_2 \frac{f(x_1, x_2 + hu_2) - f(x_1, x_2)}{hu_2} \\
&= u_1 \frac{\partial f}{\partial x_1} + u_2 \frac{\partial f}{\partial x_2} \\
&= u \cdot \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} \\
&= u \cdot \nabla f
\end{aligned}
$$

In general, this works for any number of variables.

$$D_{\hat{\mathbf{u}}} f = u \cdot \nabla f$$