Indeterminate Forms

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Indeterminate forms are forms where substituting the limit variable will not give an answer.

Indeterminate Fractions

$$\frac{0}{0} \qquad\qquad \frac{\infty}{\infty}$$

L'Hospital's rule is a rule that can be used if the numerator and denominator of a fraction are both zero or both infinite.

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$
$$\text{or}$$
$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty$$
$$\text{then}$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 1

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$$
$$\stackrel{H}{=} \lim_{x\to\infty} \frac{e^x}{2x}$$
$$\stackrel{H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

Indeterminate products

$$0 \cdot \infty$$

When a indeterminate product is found, it is often favorable to rewrite it as a fraction.

$$\frac{0}{(\frac{1}{\infty})} = \frac{0}{0} \qquad\qquad \frac{\infty}{(\frac{1}{0})} = \frac{\infty}{\infty}$$

Example 2

$$\begin{aligned}
&\lim_{x\to 0^+} xe^{\frac{1}{x}} = 0\cdot\infty \\
&= \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \\
&\overset{H}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} e^{\frac{1}{x}} \\
&= e^{\frac{1}{0}} \\
&= e^{\infty} \\
&= \infty
\end{aligned}$$

Example 3

$$\begin{aligned}
&\lim_{x\to 0^+} x\ln x = 0\cdot -\infty \\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} (-x) \\
&= 0
\end{aligned}$$

Indeterminate Powers

$$0^0 \qquad \infty^0 \qquad 1^\infty$$

Indeterminate powers can be converted into products by applying a logarithm.

$$\lim_{x\to n} a^b = e^{\left(\lim_{x\to n} b\cdot\ln a\right)}$$

Example 4

$$
\begin{aligned}
L &= \lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = 1^{\infty} \\
\ln L &= \lim_{x\to 0^+} \cot x \ln(1+\sin 4x) = \infty \cdot 0 \\
\ln L &= \lim_{x\to 0^+} \frac{ln(1+\sin 4x)}{\tan x} \\
\ln L &\overset{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{4\cos 4x}{1+\sin 4x}\right)}{1+\tan^2 x} \\
\ln L &= \frac{\left(\frac{4\cos 0}{1+\sin 0}\right)}{1+\tan^2 0} \\
\ln L &= \frac{4}{1} \\
\ln L &= 4 \\
L &= e^4
\end{aligned}
$$