Determinants

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A $n \times n$ matrix A will map the unit vectors in $\mathbb{R}^n$ to columns of A, which is also in $\mathbb{R}^n$. The n-dimensional equivalent to volume of a shape, when mapped through the matrix, will be scaled by $det\ A$. If $det\ A = 0$, then the matrix A is not invertible.

2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ 0 & -\frac{cb}{a} + d \end{bmatrix} \ (-\frac{c}{a} \text{ R1} + \text{R2})$$

A is invertible iff $a \neq 0$ and $-\frac{cb}{a} + d \neq 0$. The determinant of A is $ad - bc$ then

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The area of a parallelogram formed by $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ is equal to $|det\ A|$

If a edge of the parallelogram is not on the x-axis, it can be rotated onto the x-axis with a rotation matrix. Rotation matrices preserves the determinant so the determinant is unchanged after being rotated.

3x3 Matrix

The absolute value of the determinant of a 3×3 matrix is the volume of a parallelepiped formed by the three columns in $\mathbb{R}^3$. A parallelepiped with v_1, v_2 in the xy-plane forming a parallelogram, and $v_3 = [x_3, y_3, z_3]^T$ will have a volume $z_3 *$ (area of parallelogram). Any parallelepiped can be rotated to have a plane laying in the xy-plane.

$$det \begin{bmatrix} a & b & x_3 \\ c & d & y_3 \\ 0 & 0 & z_3 \end{bmatrix} = z_3 \cdot det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

General Matrix

The determinant can be generalized to higher dimensions by rotating one "cell" into a (n-1) volume without a x_n component, calculating the determinant for the first (n-1) vectors, then multiplying by the last vector's x_n component.

General Determinant Formula

Suppose the matrix A is a $n \times n$ matrix and A_{ij} is the matrix A with the ith row and the jth column removed. A_{ij} will be a size $(n-1) \times (n-1)$ matrix. The (i,j)-cofactor of A is the determinant of A_{ij} multiplied by one or negative one depending on if the sum of the row index and column index if even.

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

The determinant is defined as

$$\begin{aligned} \det(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in} \\ &= \sum_{j=0}^{n} a_{ji}C_{ji} \end{aligned}$$

The determinant can also be found by summing across the columns instead of the rows.

$$\begin{aligned} \det(A) &= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj} \\ &= \sum_{i=0}^{n} a_{ji}C_{ji} \end{aligned}$$

The determinant is well defined when summing across any row or column.

Example

Computing the determinant of a matrix across the first row

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \\ \det(A) &= (1)(1)\det\left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\right) + (-1)(-1)\det\left(\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}\right) + (1)(2)\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\right) \\ &= 1 - 5 + 2(-2) \\ &= -8 \end{aligned}$$

Computing the determinant of the same matrix across the second column

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \\ \det(A) &= (-1)(-1)\det\left(\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}\right) + (1)(1)\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right) + (-1)(0)\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) \\ &= (-5) + (-3) + 0 \\ &= -8 \end{aligned}$$

Example 2

$$A = \begin{bmatrix} 1 & 2 & -5 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Finding the determinant across the fourth row

$$\begin{aligned} \det(A) &= 0 + 0 + 0 + (1)(3)\det(\begin{bmatrix} 1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}) \\ &= 3(0 + 0 + 1(\det(\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}))) \\ &= 3(1(3)) \\ &= 9 \end{aligned}$$