

# Cramer's Rule and Volume

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Oct 28, 2024

## Cramer's Rule

For an  $n \times n$  matrix  $A$  and any vector  $b \in \mathbb{R}^n$ , we define  $\det(A)_i(b)$  as the determinant of the matrix  $A$  with the  $i$ th column replaced by  $b$ . If  $A$  is invertible, then the solution of  $Ax = b$  is given by the following formula

$$x_i = \frac{\det(A)_i(b)}{\det(A)}$$

We can use Cramer's rule to get an explicit formula for the inverse

$$\begin{aligned}\text{adj}(A) &= (C_{ij})^T \\ A^{-1} &= \frac{\text{adj}(A)}{\det(A)}\end{aligned}$$

## Example 1

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned}\det(A) &= 2 \\ \det \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} &= 1 \\ \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} &= 2 \\ \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} &= 3\end{aligned}$$

$$x_1 = 1/2$$

$$x_2 = 1$$

$$x_3 = 3/2$$

## Example 2

Compute  $A^{-1}$  using Cramer's rule

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$C_{11} = 1$$

$$C_{12} = 0$$

$$C_{13} = 1$$

$$C_{21} = 1$$

$$C_{22} = 2$$

$$C_{23} = -1$$

$$C_{31} = -1$$

$$C_{32} = 2$$

$$C_{33} = 1$$

$$\text{adj}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{\det A}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

## Relation with area

Suppose that  $S$  is a region in  $\mathbb{R}^2$  with finite area and  $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$  is a linear transformation with a standard matrix  $A$ . The area of  $T(S)$  is equal to  $|\det A|$  times the area of  $S$ . The same is true for volumes in  $\mathbb{R}^3$  and general  $n$ -volumes in  $\mathbb{R}^n$ .