Derivative

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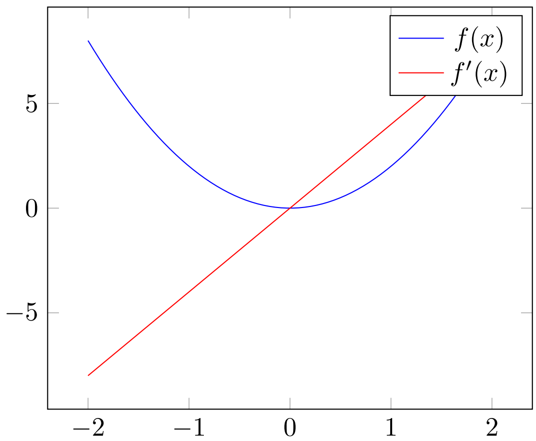
The Derivative Function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f : \mathbb{R} \mapsto \mathbb{R}$$
$$f' : \mathbb{R} \mapsto \mathbb{R}$$
$$f' : x \mapsto \text{ the slope of the function } f$$

$$f(x) = 2x^2$$

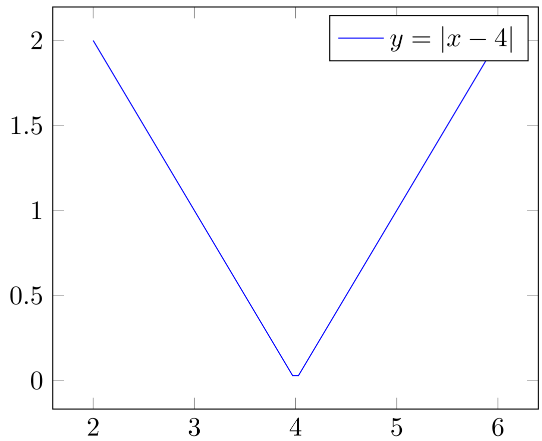


$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx} f(x)$$

A function f is differentiable at a if $f'(a)$ exists (and is finite). f is differentiable on (a, b) if f is differentiable at every point in (a, b).

Example

$$f(x) = |x - 4|$$



$$|x-4| = \begin{cases} x-4, & x-4 \geq 0 \\ -(x-4) & x-4 < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1, & x > 4 \\ -1 & x < 4 \\ \text{undefined} & x = 4 \end{cases}$$

$f'(0)$ is undefined because the left and right limits are not equal.

$$\begin{aligned} &\lim_{h \to 0^+} \frac{f(4+h) - f(4)}{h} \\ =&\lim_{h \to 0^+} \frac{|h|}{h} \\ &|h| \text{ and } h \text{ are positive} \\ =&1 \end{aligned}$$

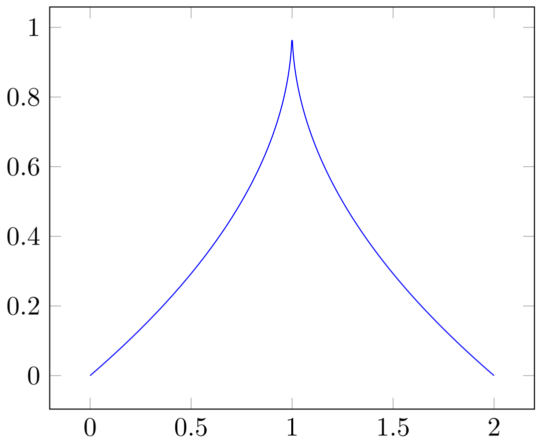
$$\begin{aligned} &\lim_{h \to 0^-} \frac{f(4+h) - f(4)}{h} \\ =&\lim_{h \to 0^-} \frac{|h|}{h} \\ &|h| \text{ is positive but } h \text{ is negative} \\ =&-1 \end{aligned}$$

Theorem 0.1 If f is differentiable at a, then f is continuous at a.

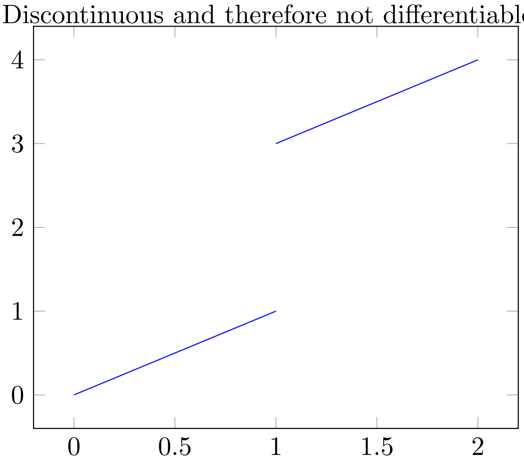
We saw this in the absolute value example. $|x - 4|$ is continuous but not differentiable at point (4, 0).

There are many ways a function cannot be differentiable.

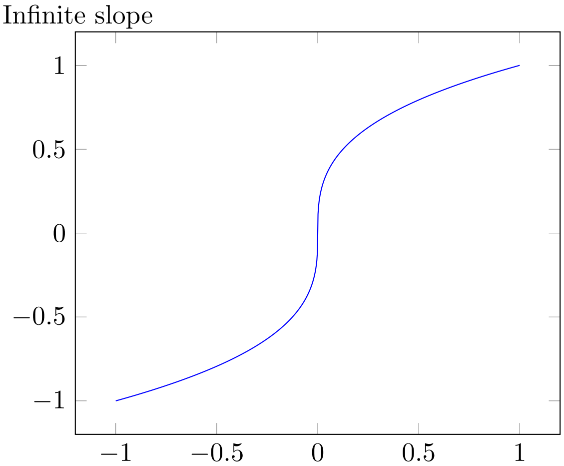
1. Cusp



2. Discontinuous and therefore not differentiable



3. Infinite slope



Higher Derivatives

The derivative of the derivative of a function is called the second derivative of the function.

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}\frac{dy}{dx} = \frac{d^ny}{dx^n}$$

Derivative Rules

Constant Multiple, Addition, and Subtraction

$$\begin{aligned}
\frac{d}{dx}(cf(x)) &= c\frac{d}{dx}(f(x)) \\
\frac{d}{dx}(f(x)+g(x)) &= \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) \\
\frac{d}{dx}(f(x)-g(x)) &= \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))
\end{aligned}$$

Product and Quotient rules

When two functions are multiplied together (product rule):

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= f(x)\frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x))g(x) \\
(fg)' &= f'g + fg'
\end{aligned}$$

When one function is divided by another function (quotient rule):

$$\begin{aligned}
\frac{d}{dx}(\frac{f(x)}{g(x)}) &= \frac{\frac{d}{dx}(f(x))g(x) - f(x)\frac{d}{dx}(g(x))}{g(x)^2} \\
(\frac{f}{g})' &= \frac{f'g - fg'}{g^2}
\end{aligned}$$

Chain rule

For a function g that is differentiable at x and a function f that is differentiable at $f(x)$, the derivative of $f \circ g$ can be found using the chain rule.

$$\begin{aligned}
y &= f(g(x)) \\
y' &= f'(g(x)) \cdot g'(x)
\end{aligned}$$

Example: let $y = \sqrt{x^2+1}$

$$\begin{aligned}
g(x) = x^2 + 1 &\Rightarrow g'(x) = 2x \\
f(x) = \sqrt{x} &\Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \\
y' = \frac{1}{2\sqrt{x^2+1}}2x &= \frac{x}{\sqrt{x^2+1}}
\end{aligned}$$

Implicit Differentiation

Implicit differentiation is the process of treating a y like the function $f(x)$. This can be useful when it is difficult or impossible to isolate for y in a equation. The derivative of a curve can be found by using implicit differentiation then solving for y'.

Example:

$$x^3 + y^3 = 6xy$$
$$\Downarrow$$
$$x^3 + f(x)^3 = 6xf(x)$$

$$\begin{aligned}(y^3)' &= (f(x)^3)' \\ &= 3f(x) \cdot f'(x) \\ &= 3y \cdot y'\end{aligned}$$

$$\begin{aligned}x^3 + y^3 &= 6xy \\ 3x^2 + 3y^2y' &= 6(xy' + 1y) \\ 3x^2 + 3y^2y' &= 6xy' + 6y \\ 3x^2 - 6y &= 6xy' - 3y^2y' \\ 3x^2 - 6y &= (6x - 3y^2)y' \\ \frac{3x^2 - 6y}{6x - 3y^2} &= y'\end{aligned}$$

This means the slope at the point (x, y) is $\frac{3x^2-6y}{6x-3y^2}$

Inverse Functions

If f is a differentiable one-to-one function, the derivative of the inverse function can be found with the following formula.

$$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$$

Proof:

$$\begin{aligned}y &= f^{-1}(x) \\ f(y) &= f(f^{-1}(x)) \\ f(y) &= x \\ f'(y)y' &= 1 && \text{implicit differentiation} \\ y' &= \frac{1}{f'(y)} \\ y' &= \frac{1}{f'(f^{-1}(x))} && \text{substitute } y\end{aligned}$$

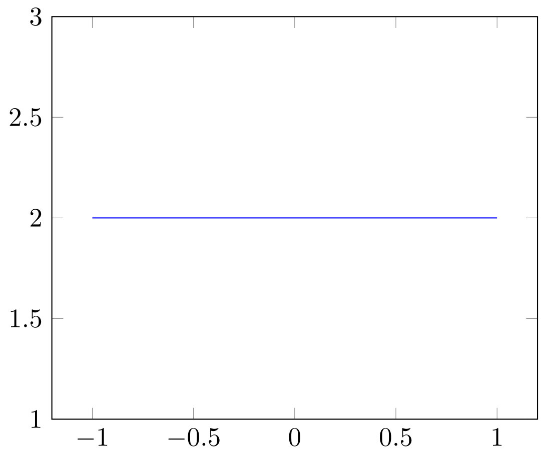
Derivatives of functions

Polynomial

The constant function:

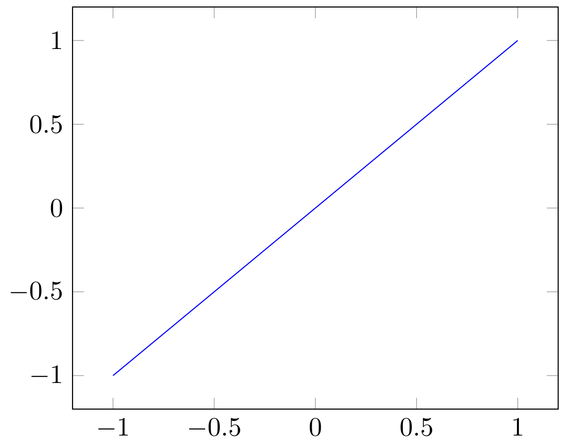
$$f(x) = c$$
$$\frac{d}{dx}(c) = 0$$

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{c-c}{h} \\
&= \lim_{h\to 0} \frac{0}{h} \\
&=0
\end{aligned}$$



Linear (degree one) function:

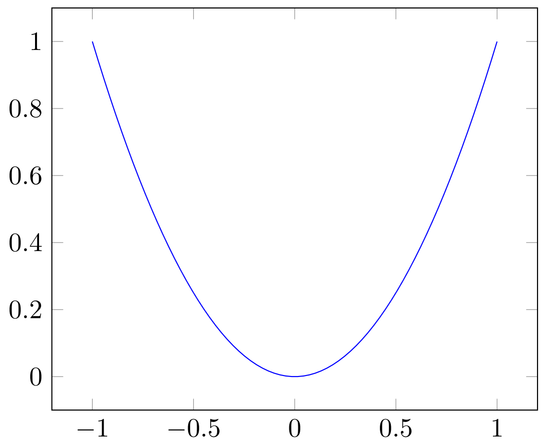
$$f(x) = x$$
$$\frac{d}{dx}(x) = 1$$



Quadratic:

$$f(x) = x^2$$
$$\frac{d}{dx}(x^2) = 2x$$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} 2x + h$$
$$= 2x$$



General:

$$f(x) = x^n$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Exponential

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
\lim_{h\to 0} \frac{e^(x+h)-e^x}{h} \\
\lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
\lim_{h\to 0} \frac{e^x(e^h-1)}{h} \\
e^x \lim_{h\to 0} \frac{e^h-1}{h}
\end{aligned}$$

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1$$

The derivative of e^x is itself.

$$\frac{d}{dx}e^x = e^x$$

For an exponent with a different base, convert to base e using logarithm rules.

$$\begin{aligned}
a^x = e^{\ln(a^x)} = e^{x\ln(a)} \\
(a^x)' = e^{x\ln(a)}\ln(a) = a^x \cdot \ln(a)
\end{aligned}$$

Logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\begin{aligned}
y &= \log_a x \\
a^y &= x \\
(\ln a)a^y y' &= 1 \\
y' &= \frac{1}{(a^y)\ln a} \\
y' &= \frac{1}{x\ln a}
\end{aligned}$$

Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\begin{aligned}
&\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\
&=\lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}\\
&=\sin(x)\lim_{h\to 0}\frac{(\cos(h)-1)}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&=\sin(x)(0)+\cos(x)(1)\\
&=\cos(x)
\end{aligned}$$

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=1 \qquad \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

Inverse Trigonometric Function

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$