Row Reduction

Patrick Chen

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Finding solutions to systems of linear equations:

$$\begin{array}{ccccccc} & & 2y & + & 3z & = & -4 \\ 2x & + & 6y & + & 2z & = & 4 \\ 3x & & & + & z & = & 3 \end{array}$$

An augmented matrix representing the system of linear equations:

$$\begin{bmatrix} 0 & 2 & 2 & -4 \\ 2 & 6 & 2 & 4 \\ 3 & 0 & 1 & 3 \end{bmatrix}$$

The coefficient matrix representing the system of linear equations:

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix}$$

Rules for manipulating augmented matrices

- interchange rows
- multiply an entire row by a non-zero constant
- add a constant multiple of another row

The row reduction algorithm is as follows:

1. divide the first row by the first element, causing the first element to be one. This will make the first row have the form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \dots & a_{1m} & b_1 \end{bmatrix}$$

2. subtract multiples of the first row from every row below such that the first element of those rows is zero. This should make the matrix take the following form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & \dots & a_{2m} & b_2 \\ 0 & a_{32} & a_{33} & & a_{3m} & b_3 \\ & \vdots & & \ddots & & \\ 0 & a_{n2} & a_{n3} & & a_{nm} & b_n \end{bmatrix}$$

3. repeat for the second element, then the third, etc. afterward the matrix should take the form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & & a_{1m} & b_1 \\ 0 & 1 & a_{23} & \dots & a_{2m} & b_2 \\ 0 & 0 & 1 & & a_{3m} & b_3 \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & & 1 & b_n \end{bmatrix}$$

4. subtract multiples of the second row from each row above such the second column results in zero. Repeat for third row and third column, then fourth, etc. This should result in the matrix taking the form:

$$\begin{bmatrix} 1 & 0 & 0 & & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & b_2 \\ 0 & 0 & 1 & & 0 & b_3 \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & & 1 & b_n \end{bmatrix}$$

Row Reduction example

$$\begin{aligned}
\begin{bmatrix} 0 & 2 & 2 & -4 \\ 2 & 6 & 2 & 4 \\ 3 & 0 & 1 & 3 \end{bmatrix} &= \begin{bmatrix} 2 & 6 & 2 & 4 \\ 0 & 2 & 2 & -4 \\ 3 & 0 & 1 & 3 \end{bmatrix} \text{swap row (1), (2)} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 2 & 2 & -4 \\ 3 & 0 & 1 & 3 \end{bmatrix} \text{divide row (1) by 2} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 2 & 2 & -4 \\ 0 & -9 & -2 & -3 \end{bmatrix} \text{subtract } 3 \times \text{ row (1) from row (3)} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & -9 & -2 & -3 \end{bmatrix} \text{divide row (2) by 2} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 7 & -21 \end{bmatrix} \text{add } 9 \times \text{ row (2) to row (3)} \\
&= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \text{divide row (3) by 7} \\
&= \begin{bmatrix} 1 & 0 & -2 & 8 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \text{subtract } 3 \times \text{ row (2) from row (1)} \\
&= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \text{add } 2 \times \text{ row (3) to row (1), subtract row (3) from row (2)}
\end{aligned}$$

A matrix in is row echelon form if:

1. All zero rows are grouped contiguously at the bottom of the matrix
2. In any two consecutive row which are both non-zero, the non-zero entry of the upper row is to the left of the non-zero entry of the lower row

If all leading entries in non-zero rows are 1, it is in **reduced** row echelon form.

Example 2

$$\begin{array}{ccccccc} & & y & + & 3z & = & 4 \\ 2x & - & y & + & z & = & 2 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ 2 & -1 & 1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$
$$\begin{array}{cccccc} x & & + & 2z & = & 3 \\ & y & + & 3z & = & 4 \end{array}$$

$$\begin{aligned} \therefore x =& 3 - 2t \\ y =& 4 - 3t \\ z =& t \\ & \text{where } t \in \mathbb{R} \end{aligned}$$

Since there is one free variable, there are infinite solutions residing on a line. If there are two free variables, there will be infinite solutions residing on a plane. For three free variables, a volume. etc.
Sometimes a matrix may have no solutions. The following matrix has the row $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, which is equivalent to the non-sense expression $0x + 0y + 0z = 1$.

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$