Differential Equations

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An ordinary differential equations (ODE) is a type of differential equation (DE) that contain derivatives with respect to only a single independent variable. It is called ordinary because it only contains one independent variable. The order of an ODE the highest derivative in the differential equation. The solution for a differential equation is a function that satisfies the differential equation on some interval. The solution may or may not be unique. An autonomous differential equation is a differential equation that has no explicit terms containing the independent variable.
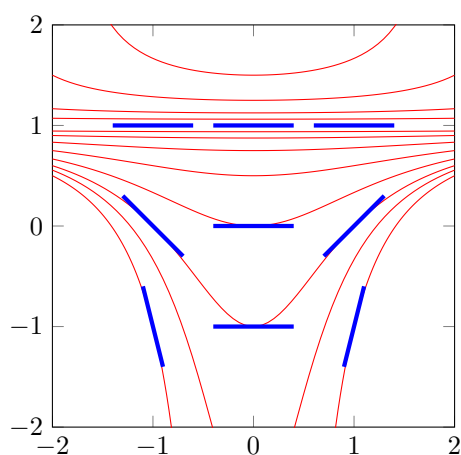
Shape of Differential Equation Graphs

It is possible to determine properties of the shape of a differential equation by examining the equation.

Example

$$y' = x(y - 1)^2$$

When this differential equation is evaluated at a point (x, y), the solution to the differential equation passing through the point (x, y) must be the slope found.

y x	-1	0	1
1	0	0	0
0	-1	0	1
-1	-4	0	4

Orthogonal trajectories

If $\frac{dy}{dx} = F(x, y)$, then the orthogonal trajectory is $\frac{dy}{dx} = -\frac{1}{F(x,y)}$

Example

$$y = ke^x$$
$$y' = ke^x$$

$$y' = -\frac{1}{ke^x}$$
$$\frac{dy}{dx} = -\frac{1}{y}$$
$$y \, dy = -dx$$
$$\frac{1}{2}y^2 = -x + c$$

Example 2

$$y = kx^2$$
$$y' = 2kx$$
$$k = \frac{y}{x^2}$$
$$y' = 2(\frac{y}{x^2})x$$
$$y' = \frac{2y}{x}$$

$$\frac{dy}{dx} = -\frac{1}{\frac{2y}{x}}$$
$$\frac{dy}{dx} = -\frac{x}{2y}$$
$$2y \, dy = -x \, dx$$
$$y^2 = -\frac{1}{2}x^2 + c$$
$$x^2 + 2y^2 = c$$

Guessing a Solution

If a differential equation is simple, it may be possible to correctly guess a solution. The guess and check method is to guess a generic function and check if there are any parameters of a function that will satisfy the differential equation.

Example

$$y'' - 3y' + 2y = 0$$

Let $y = e^{mx}$

$$y = e^{mx}$$
$$y' = me^{mx}$$
$$y'' = m^2 e^{mx}$$

$$y'' - 3y' + 2y = 0$$
$$m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$$
$$(m^2 - 3m + 2)e^{mx} = 0$$
$$(m^2 - 3m + 2) = 0$$
$$(m - 2)(m - 1) = 0$$
$$m = 1, 2$$
$$\therefore \; y = e^{2x}, y = e^{x}$$

Initial Value Problems

Initial value problems are differential equation problems that need to satisfy some initial conditions.

Example

A Person is jumping off a high place. If $g = 10$ m/s^2 how fast do they move at 2s if at time 0s, position is 10m and velocity is 0 m/s.

$$F_{net} = -F_g$$
$$ma = -mg$$
$$a = -g$$
$$x''(t) = -10 \text{ m/s}^2$$
$$x'(0) = 0 \text{ m/s}$$
$$x(0) = 10 \text{ m}$$

$$x''(t) = -10$$
$$x'(t) = \int -10 \; dt$$
$$x'(t) = -10t + c_1$$
$$x(t) = \int -10t + c_1 \; dt$$
$$x(t) = -5t^2 + c_1 t + c_2$$

$$x(0) = -5(0)^2 + c_1(0) + c_2$$
$$10 \text{ m} = c_2$$

$$x'(0) = -10(0) + c_1$$
$$0 \text{ m/s} = c_1$$

$$\therefore \ x(t) = -5t^2 + 10$$
$$x'(2) = -10(2) = -20 \text{ m/s}$$

Separable Differential Equation

A separable first order differential equation has the following form. In separable differential equations, the variables can be separated and put on different sides of the equal sign, then integrated.

$$\frac{dy}{dx} = f(x)g(y)$$

Example

$$y' = \frac{x}{y}$$
$$\frac{dy}{dx} = \frac{x}{y}$$
$$y\frac{dy}{dx} = x$$
$$\int y\frac{dy}{dx} \ dx = \int x \ dx$$
$$\int y \ dy = \int x \ dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$
$$y^2 = x^2 + c$$

Linear Differential Equations

A first order linear ode can be put in the following form.

$$y' + p(x)y = q(x)$$
$$I(x)y' + I(x)P(x)y = I(x)Q(x)$$

If we assume that this is a product rule,

$$\Big(I(x)y\Big)' = I(x)Q(x) \Rightarrow I(x)y' + I'(x)y = I(x)Q(x)$$

This is almost in the form of the previous equation, just with the restriction

$$\frac{d}{dx}I(x) = P(x)I(x)$$

$$\frac{dy}{dx} = P(x)y$$

$$\frac{1}{y}dy = P(x)dx$$

$$\ln|y| = \int P(x)\ dx$$

$$I(x) = Ae^{\int P(x)\ dx}$$

Thus any linear differential equation can be converted to the derivative of the product of two functions $I(x)$, $Q(x)$

Example

$$xy' + 3y = \frac{e^x}{x^2}$$

$$y' + \frac{3}{x}y = \frac{e^x}{x^3}$$

$$P(x) = \frac{3}{x}$$

$$Q(x) = \frac{e^x}{x^3}$$

$$I(x) = e^{\int \frac{3}{x}\ dx}$$

$$= e^{3\ln x}$$

$$= x^3$$

$$x^3 y' + \frac{3}{x}x^3 y = e^x$$

$$x^3 y' + 3x^2 y = e^x$$

$$(x^3 y)' = e^x$$

$$x^3 y = e^x + c$$

$$y = \frac{e^x}{x^3} + \frac{c}{x^3}$$

Example 2

$0 < x < \pi/2$

$$y' + \tan(x)y = \sin(x)$$
$$P(x) = \tan(x)$$
$$Q(x) = \sin(x)$$
$$I(x) = e^{\int \tan(x)\ dx}$$
$$= e^{-\ln(\cos(x))}$$
$$= \sec x$$
$$y'\sec x + (\sec x \tan x)y = \sec x \sin x$$
$$(y\sec x)' = \tan x$$
$$y\sec x = \ln(\sec x) + c$$
$$y = \cos(x)\ln(\sec x) + c\cos(x)$$