Continuity

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Limits

$$\lim_{x \to a^-} f(x) = l$$

This means that we can make f(x) as close to l as we want, provided that we choose a x close enough to a to the left. If both left limit and right limit exists and are the same, we say that the function has a limit.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x)$$

Continuity

Consider

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 0 & x = 2 \end{cases} \qquad g(x) = \begin{cases} x^2 & x \neq 2 \\ undefined & otherwise \end{cases} \qquad h(x) = x^2$$

When $f(x)$, $g(x)$, and $h(x)$ is approached from either the left or the right, it is equal to 4.

$$\lim_{x \to 2^-} f(x) = 4$$
$$\lim_{x \to 2^+} f(x) = 4$$

For a function to be continuous at a point, the limit of the function at the point must exist and be equal to the function evaluated at the point.

$$\lim_{x \to a} f(x) = f(a)$$

Directional Continuity

A function if continuous from the right at a point x if the right limit is equal to the function evaluated at x. Likewise with left continuity.

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{left continuous}$$
$$\lim_{x \to a^-} f(x) = f(a) \qquad \text{right continuous}$$

Continuity Rules

If two functions are continuous at a and c is a constant, the following is also continuous

1. $f(a) + g(a)$
2. $f(a) - g(a)$
3. $cf(a)$
4. $f(a)g(a)$
5. $\frac{f(a)}{g(a)}$ if $g(a) \neq 0$
6. $[f \circ g](a)$

The following types of functions are continuous at every number in their domains:

- polynomial
- trigonometric functions
- exponential functions
- rational functions (when denominator is non-zero)
- root functions
- inverse trigonometric functions
- logarithmic functions

Intermediate Value Theorem

Theorem 0.1 – Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq (b)$. Then there exist a number c in (a, b) such that $f(c) = n$.

In other words, if a continuous function is evaluated at two points, any point in between the outputs at those points are also outputs of the function. Note that IVT requires a closed interval, $(a, b]$ and $[a, b)$ is not enough to satisfy the IVT.
Consider the following function. It is continuous on the interval $[-3, 0)$ and $f(-3) = -\frac{1}{3}$ and $f(0) = 3$ but there is no x such that $f(x) = 2$.

$$f(x) = \begin{cases} 3, & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Example

$$\ln(1 + \cos(x))$$

ln is continuous when the input is in its domain (positive numbers) and $1 + \cos(x)$ is continuous for all numbers.

$$1 + \cos(x) > 0$$

Since $\cos(x)$ ranges from -1 to 1, $1 + \cos(x)$ has a range of $[0, 2]$. This is problematic because 0 is in the range of $1 + \cos(x)$ but not in the domain of $\ln(x)$, thus there is a discontinuity in $\ln(1 + \cos(x))$.

Example 2

let $f(x) = e^{-x} - x$. Show that the function has at least one root on the interval $[0, 1]$.

$$f(0) = e^{-0} - 0 = 1$$
$$f(1) = e^{-1} - 1 = \frac{1}{e} - 1 \approx -0.63$$

By the intermediate value theorem, $f(1) < 0 < f(0)$, thus there exist some input N of the function such that $f(N) = 0$

Example 3

Show that $4x^3 - 6x^2 = 2 - 3x$ has at lease one root.

$$4x^3 - 6x^2 = 2 - 3x$$
$$4x^3 - 6x^2 + 3x - 2 = 0$$
$$\text{let } f(x) = 4x^3 - 6x^2 + 3x - 2$$

$$f(0) = 4(0)^3 - 6(0)^2 + 3(0) - 2$$
$$f(0) = -2$$
$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2$$
$$f(2) = 32 - 24 + 6 - 2$$
$$f(2) = 12$$

By the intermediate value theorem, there exist some value N in the interval $[0, 2]$ there $f(x) = 0$.