Properties of Determinants

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Properties of Determinants

For $n \times n$ matrices A and B and scalars k:

$$\det(AB) = \det(A)\det(B)$$
$$\det(A) = \det(A^T)$$
$$\det(kA) = k^n \det(A)$$
$$A^{-1}\text{exists iif}\det(A) = 0$$

If the matrix A is invertible, then there is some combination of elementary matrices that when left multiplies with A, will give the identity.

$$\det(E_k \dots E_2 E_1 A) = \det(I)$$
$$\det(E_k)\dots\det(E_2)\det(E_1)\det(A) = 1$$

Since the determinant of elementary matrices are all non-zero, the determinant of A cannot be zero, thus A is invertible iff $\det A \neq 0$.

Efficient Computations of Determinants

Upper triangular matrices are matrices where the only non-zero values are either on or above the main diagonal. Everything under the main diagonals is zero. Lower triangular matrices are the opposite. The determinant of a upper triangular or lower triangular matrix is the product of the diagonals. This is evident from applying the cofactor definition of the determinant with the rows with all zeros except for the main diagonal.

Multiplying by a elementary matrix will modify it depending on what type of elementary matrix it is. Since the transpose of a matrix has the same determinant, it is possible to preform row operations on the transpose of the matrix. This corresponds with column operations on the original matrix.

1. If the row operation is interchanging two rows, then the determinant is negated.

$$\det(EA) = -\det(A)$$

2. If the row operation is multiplying a row by k, then the determinant will be multiplied by k.

$$\det(EA) = k\det(A)$$

3. If the row operation is adding or subtracting a multiple of another row, then the determinant will be unchanged.

$$\det(EA) = \det(A)$$

Example

$$\begin{aligned}
&\det \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} 4 & 0 & 6 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix} \\
&= -8
\end{aligned}$$

Example 2

$$\begin{aligned}
&\det \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 3 & 2 & 0 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \\ 0 & 2 & 3 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & -9 \end{bmatrix} \\
&= 9
\end{aligned}$$

Linearity of Determinants

The determinant of a set matrix where one column is the input to the transformation and all other columns are fixed is a linear transformation.

$$\begin{aligned}
T : \mathbb{R}^n &\mapsto \mathbb{R} \\
T(x) &= \det(v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_n) \\
&\text{where } v_1 \ldots v_{j-1} \text{ is constant,} \\
&\qquad v_{j+1} \ldots v_n \text{ is constant}
\end{aligned}$$

Proof of linearity:

$$
\begin{aligned}
T(x+y) &= \det(v_1, \ldots, v_{j-1}, x+y, v_{j+1}, \ldots, v_n) \\
&= \sum_{i=0}^{n} a_{ji}(-1)^{i+j} \det(A_{ij}) \\
&= \sum_{i=0}^{n} (x_i + y_i)(-1)^{i+j} \det(A_{ij}) \\
&= \sum_{i=0}^{n} \left(x_i(-1)^{i+j} \det(A_{ij}) + y_i(-1)^{i+j} \det(A_{ij}) \right) \\
&= \left(\sum_{i=0}^{n} x_i(-1)^{i+j} \det(A_{ij}) \right) + \left(\sum_{i=0}^{n} y_i(-1)^{i+j} \det(A_{ij}) \right) \\
&= \det(v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_n) + \det(v_1, \ldots, v_{j-1}, y, v_{j+1}, \ldots, v_n) \\
&= T(x) + T(y)
\end{aligned}
$$

$$
\begin{aligned}
T(\lambda x) =& \det(v_1, \ldots, v_{j-1}, \lambda x, v_{j+1}, \ldots, v_n) \\
=& \sum_{i=0}^{n} \lambda a_{ji}(-1)^{i+j} \det(A_{ij}) \\
=& \lambda \sum_{i=0}^{n} a_{ji}(-1)^{i+j} \det(A_{ij}) \\
=& \lambda \det(v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_n) \\
=& \lambda T(x)
\end{aligned}
$$

Example 3

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} &= \det \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} + \det \begin{bmatrix} 0 & 4 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} + (-2) \det \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

Example 4

$$
\begin{aligned}
\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} &= \det\begin{bmatrix} a & d & g \\ 0 & e & h \\ 0 & f & i \end{bmatrix} + \det\begin{bmatrix} 0 & d & g \\ b & e & h \\ 0 & f & i \end{bmatrix} + \det\begin{bmatrix} 0 & d & g \\ 0 & e & h \\ c & f & i \end{bmatrix} \\
&= \det\begin{bmatrix} a & d & g \\ 0 & 0 & h \\ 0 & 0 & i \end{bmatrix} + \det\begin{bmatrix} 0 & d & g \\ b & 0 & h \\ 0 & 0 & i \end{bmatrix} + \det\begin{bmatrix} 0 & d & g \\ 0 & 0 & h \\ c & 0 & i \end{bmatrix} \\
&+ \det\begin{bmatrix} a & 0 & g \\ 0 & e & h \\ 0 & 0 & i \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & g \\ b & e & h \\ 0 & 0 & i \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & g \\ 0 & e & h \\ c & 0 & i \end{bmatrix} \\
&+ \det\begin{bmatrix} a & 0 & g \\ 0 & 0 & h \\ 0 & f & i \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & g \\ b & 0 & h \\ 0 & f & i \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & g \\ 0 & 0 & h \\ c & f & i \end{bmatrix}
\end{aligned}
$$

After repeating this, in general, all columns will have only one non-zero value. If a entire row in a matrix is zero, then the determinant will be zero. Evaluating the determinant, the first column will have n choices for a non-zero determinant. The column will have $n-1$ choices for a non-zero determinant. In general the sequence will be $[n, n-1, n-2, \ldots, 3, 2, 1]$. Since recursively splitting matrices with linearity will give every possible combination of columns, it will contain every matrix with non-zero determinants (as well as a lot of determinant zero matrices). This means that we need to sum up $n!$ values to find the determinant.