Arc Length

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The length of a curve can be approximated by sampling it and connecting the points. The length is the limit as the samples go to infinity.

$$
\begin{aligned}
L_{line} &= \sqrt{(P_{ix} - P_{(i-1)x})^2 + (P_{iy} - P_{(i-1)y})^2} \\
L_{line} &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\
L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x^2(1 + \frac{(f(x_i) - f(x_{i-1}))}{\Delta x}))} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta x^2(1 + f'(x_i)^2)} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \sqrt{(1 + f'(x_i)^2)}
\end{aligned}
$$

From this, the arc length of a curve is:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

Example 1

Find the arc length of $x^{\frac{3}{2}}$ on the interval $[0, 28]$.

$$\begin{aligned}
\int_0^{28} \sqrt{1 + (\frac{3}{2}x^{\frac{1}{2}})^2}\ dx &= \int_0^{28} \sqrt{1 + \frac{9}{4}x}\ dx \\
u &= 1 + \frac{9}{4}x \\
du &= \frac{9}{4}dx \\
I &= \frac{4}{9}\int_1^{64} u^{1/2}\ du \\
&= \frac{8}{27}u^{\frac{3}{2}}\ \Big|_1^{64} \\
&= \frac{8}{27}(64)^{\frac{3}{2}} - \frac{8}{27}(1)^{\frac{3}{2}} \\
&= \frac{8}{27}512 - \frac{8}{27} \\
&= \frac{488}{27}
\end{aligned}$$

Example 2

Find the arc length of $x^2 - \frac{1}{8}\ln x$ on the interval $[1, e]$.

$$\begin{aligned}
\int_1^{e} \sqrt{1 + (2x - \frac{1}{8x})^2}\ dx &= \int_1^{e} \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}}\ dx \\
&= \int_1^{e} \sqrt{\frac{(1 + 16x^2)^2}{64x^2}}\ dx \\
&= \int_1^{e} \frac{1 + 16x^2}{8x}\ dx \\
&= \int_1^{e} \frac{1}{8x} + 2x\ dx \\
&= \frac{1}{8}\ln x + x^2\ \Big|_1^{e} \\
&= (\frac{1}{8}\ln e + e^2) - (\frac{1}{8}\ln 1 + 1^2) \\
&= (\frac{1}{8} + e^2) - 1 \\
&= e^2 - \frac{7}{8}
\end{aligned}$$

Example 3

Find the arc length of $g(x) = \int_{\pi/4}^{x} \sqrt{sec^8(t) - 1}\ dt$ on the interval $[\frac{\pi}{4,} \frac{\pi}{3}]$.

$$
\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + (\sqrt{\sec^8(t) - 1})^2}\ dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + \sec^8(t) - 1}\ dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^4(t)\ dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x \sec^2 x\ dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (tan^2 x + 1) \sec^2 x\ dx \\
u &= \tan x \\
du &= \sec^2 x \\
&= \int_{1}^{\sqrt{3}} u^2 + 1\ du \\
&= \frac{1}{3}u^3 + u\ \Big|_{1}^{\sqrt{3}} \\
&= (\frac{1}{3}(\sqrt{3})^3 + \sqrt{3}) - (\frac{1}{3}(1)^3 + 1) \\
&= (\sqrt{3} + \sqrt{3}) - \frac{4}{3} \\
&= 2\sqrt{3} - \frac{4}{3}
\end{aligned}
$$