Polar Coordinates

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Polar coordinated describe a point by the angle and distance.

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

By some convention, the principal angle is $-\pi < \theta \leq \pi$ although, a principal angle of $0 \leq \theta < 2\pi$ is also common.

Negative Radius

A point with a $\theta \in (-\pi, \pi]$ and $r > 0$ can also be expressed with a negative radius.

$$\theta \mapsto \theta + \pi$$
$$r \mapsto -r$$

Example 1

$$r = \frac{1}{cos(\theta) + sin(\theta)}$$

$$\begin{aligned} r &= \frac{1}{cos(\theta) + sin(\theta)} \\ rcos(\theta) + rsin(\theta) &= 1 \\ x + y &= 1 \\ y &= 1 - x \end{aligned}$$

This is a straight line with slope -1 and y-intercept 1.

Example 2

$$\begin{aligned} r &= \sin\theta \\ r^2 &= r\sin\theta \\ x^2 + y^2 &= y \\ x^2 + y^2 - 2(\frac{1}{2}y) + (\frac{1}{2})^2 &= (\frac{1}{2})^2 \\ x^2 + (y - \frac{1}{2})^2 &= (\frac{1}{2})^2 \end{aligned}$$

This is a circle with radius $r = \frac{1}{2}$ and origin $O = (0, \frac{1}{2})$.

Shapes in polar Coordinates

Graphing

A graph of a polar equation $r = g(\theta)$ is the set of all points in the plane what satisfy the equation. θ does not need to be a principal angle and r does not need to be positive. Sometimes it is easier to convert a polar function into a Cartesian function.

Circle

A circle is described with $r(\theta) = c$ where c is a constant. The radius of this circle is $|c|$.

Polar Tangents

The derivative of a polar function can be calculated the same way a parametric function's derivative is calculated with $x = r\cos\theta$, and $y = r\sin\theta$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$