Inverse of a Matrix

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Powers of a matrix

When a square matrix is raised to a power A^n, this is equivalent to multiplying by the matrix n times. When a matrix is raised to the zeroth power, it is equal to the identity matrix.

$$\begin{aligned} A^2 &= AA \\ A^n &= AA^{n-1} \\ A^0 &= I \end{aligned}$$

Transpose

The transpose of a matrix is the matrix flipped along the main diagonal.

$$\begin{gathered} (A+B)^T = A^T + B^T \\ (AB)^T = B^T A^T \\ (\lambda A)^T = \lambda A^T \\ T_A : \mathbb{R}^n \mapsto \mathbb{R}^m \Leftrightarrow T_{A^T} : \mathbb{R}^m \mapsto \mathbb{R}^n \\ T_A \circ T_{A^T} : \mathbb{R}^n \mapsto \mathbb{R}^n \Leftrightarrow T_{A^T} \circ T_A : \mathbb{R}^m \mapsto \mathbb{R}^m \end{gathered}$$

A matrix is called symmetric if $A = A^T$

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 3 & 1 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 4 & 1 \end{bmatrix}$$

Inverse of a matrix

We say a $n \times n$ matrix A is invertible if there exists a matrix B such that $AB = BA = I$. The matrix B is the inverse of A and can be written as A^{-1}. If A is invertible, then A is one-to-one and onto. The inverse of a matrix is always unique if it exists.

$$
\begin{aligned}
BAC &= B(AC) = BI = B \\
BAC &= (BA)C = IC = C \\
BAC &= B = C
\end{aligned}
$$

The inverse of the inverse of matrix A is A.

$$
\begin{aligned}
(A^{-1})^{-1} &= A \\
(A^{-1})^{-1} &= B \\
A^{-1}(A^{-1})^{-1} &= A^{-1}B \\
I &= A^{-1}B \\
AI &= AA^{-1}B \\
A &= B
\end{aligned}
$$

When two matrices are multiplied, then inverted, it is equivalent to reversing the order of matrices and inverting all of them.

$$
\begin{aligned}
(ABC)^{-1} &= C^{-1}B^{-1}A^{-1} \\
(ABC)(ABC)^{-1} &= (ABC)(C^{-1}B^{-1}A^{-1}) \\
I &= AB(CC^{-1})B^{-1}A^{-1} \\
I &= ABIB^{-1}A^{-1} \\
I &= ABB^{-1}A^{-1} \\
I &= AIA^{-1} \\
I &= AA^{-1} \\
I &= I
\end{aligned}
$$

Transpose and invert commute with each other.

$$(A^T)^{-1} = (A^{-1})^T = A^{-T}$$

For a two by two matrix, If $ad - bc = 0$, then the matrix is not invertible.

$$
\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
ad - bc &= 0 \\
ad &= cd \\
\frac{a}{c} &= \frac{b}{d}
\end{aligned}
$$

Note that this means that ac and bd are linearly dependent.

Suppose A is an invertible matrix

$$\begin{aligned} Ax &= 0 \\ A^{-1}Ax &= A^{-1}0 \\ x &= 0 \end{aligned}$$

Since there is only one solution to the homogeneous equation, A is one-to-one.

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Since all vectors in $\mathbb{R}^n$ has a vector in $\mathbb{R}^n$ that maps to it, $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is onto.

Elementary Matrices

A elementary matrix is one row operation applied to the identity matrix.

- For interchanging rows, the matrix E will be its own inverse because swapping a row twice is the same as not swapping at all. Multiplying by this matrix will interchange rows on the matrix being multiplied.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For multiplying by a non zero scalar, the inverse of matrix E will be the identity matrix with the row multiplied by the reciprocal of the multiplied scalar. Multiplying by this matrix will result in a row being multiplied by the non zero scalar.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For adding a multiple of another row, the inverse of E will be the that row subtracted by the multiply of the row. Multiplying by this matrix will result in a multiply a row being added to the row.

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Left multiplying by elementary matrices is identical to preforming row operations. If by left multiplying elementary matrices, some matrix A can be transformed into the identity matrix, then the series of elementary matrices is the inverse of A.

$$E_n \dots E_2 E_1 A = I$$
$$E_n \dots E_2 E_1 I = A^{-1}$$

If the matrix $[A \ I]$ is row reduced, and A is invertible, then the reduced row echelon form will be $[I \ A^{-1}]$. If A is not invertible, then the row reduced form will not contain the identity matrix.