## Vector Spaces

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A vector space V is a non-empty set together with a binary function + and unary functions c for every  $c \in \mathbb{R}$  satisfying the following axioms. Here, bold plus (+) represents addition between the abstract objects in the set, regular plus (+) represents addition between the scalars and zero  $(\vec{\mathbf{0}})$  represents a abstract zero element in the set that behaves like a additive identity. The abstract addition operator is often just regular addition but it doesn't have to be. The same is true for the zero element.

- 1. The vector space is closed under addition
- 2. Addition is associative
- 3. Addition is commutative
- 4. there exists  $\vec{0} \in V$  such that  $v + \vec{0} = v$  for all  $v \in V$
- 5. for every  $v \in V$ , there exists  $-v \in V$  such that  $v + (-v) = \vec{0}$
- 6. for every  $c \in \mathbb{R}$  and  $u \in V$ ,  $cu \in V$
- 7. for every  $c \in \mathbb{R}$  and  $u, v \in V$ , c(u+v) = cu + cv
- 8. for every  $c, d \in \mathbb{R}$  and  $u \in V$ , (c+d)u = cu + du
- 9. for every  $c, d \in \mathbb{R}$  and  $u \in V$ , c(du) = (cd)u
- 10. for every  $v \in V$ , 1v = v

Side note: A space that satisfies 1-5 is called a abelian group.

# Real Valued Functions as Vector Spaces

Let V any function from X to  $\mathbb{R}$  where X is any set. Define addition on V with inputs f, g in V to be f(x) + g(x). Given a c in  $\mathbb{R}$  and  $f: X \mapsto \mathbb{R}$ , define (cf) to be cf(x).

- Axioms 1 and 6 are immediate.
- Axioms 2 and 3 follow from the definition of addition being the sum of the outputs of the two functions. The outputs are in  $\mathbb{R}$  where addition is associative and commutative, thus addition is both associative and commutative in this vector space.

• Axiom 4:  $\vec{0}$  can be defined as the constant function that returns zero for all inputs x.

$$g + 0 = g(x) + f(x)$$

$$= g(x) + 0$$

$$= g(x)$$

$$= g$$

• Axiom 5: -f can be defined as the function whose outputs are the negation of f for all points x

$$(-f) = -1 \cdot f(x)$$
  
 $f + (-f) = f(x) + -f(x)$   
 $= 0$ 

• Axiom 7:

$$c(f+g) = c(f(x) + g(x))$$
$$= cf(x) + cg(x)$$
$$= cf + cg$$

• Axiom 8:  $\mathbb{R}$ .

$$(c+d)u = (c+d)u(x)$$
$$= cu(x) + du(x)$$
$$= cu + du$$

• Axiom 9: The unary function c can be defined as  $c \cdot f(x)$ .

$$c(df) = c(df(x))$$
$$= (cd)f(x)$$
$$= (cd)f$$

• Axiom 10: The scalar 1 is the identity of multiplication in  $\mathbb{R}$ .

$$1f = 1 * f(x)$$
$$= f(x)$$
$$= f$$

# Subspaces

if V is a vector space and W is a non-empty subset of V then we call W a subspace of V if W is closed under addition and scalar multiplication. If W is a subspace of V, then W itself is a vector space.

### Example 1

Prove that  $\vec{0}$  is unique.

Suppose there are two zeros,  $\vec{0}, \vec{0}'$ .

$$\vec{0} + \vec{0}' = \vec{0}$$
 axiom 4  

$$\vec{0} + \vec{0}' = \vec{0}' + \vec{0}$$
 axiom 2  

$$\vec{0}' + \vec{0} = \vec{0}'$$
 axiom 4  

$$\vec{0} = \vec{0}'$$

# Example 2

prove that -(-v) = v.

$$-(-v) = v$$
$$0 = v + (-v)$$
$$0 = -v + -(-v)$$
$$v = -(-v)$$

### Example 3

Subspaces of  $\mathbb{R}^2$ .

- $\mathbb{R}^2$  is a subspace of  $\mathbb{R}^2$ .
- Lines through the origin is a subspace of  $\mathbb{R}^2$ .
- $\vec{0}$  is a subspace of  $\mathbb{R}^2$ .

#### Example 4

Let  $V_n$  be the set of all polynomials with degree less than or equal to n.  $V_n$  is closed under addition because adding two polynomials doesn't increase the degree. Similarly, it is closed in scalar multiplication. This proves the  $V_n$  is a subspace.