Optimization

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When solving optimization problems, it is often beneficial to draw a picture representing the problem. After translating the problem into math equations and solving, make sure the solutions are reasonable given the context.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.

$$\begin{aligned}
2x + y &= 2400 \\
y &= 2400 - 2x \\
A &= x \cdot y \\
A(x) &= x(2400 - 2x) \\
A(x) &= -2x^2 + x2400 \\
A'(x) &= -4x + 2400 \\
A'(x) &= 0 \\
-4x + 2400 &= 0 \\
x &= \frac{-2400}{-4} \\
x &= 600 \\
2x + y &= 2400 \\
2(600) + y &= 2400 \\
y &= 1200
\end{aligned}$$

Example 2

Find the area of the largest rectangle that can be inscribed in a semisircle of radius r.

$$\begin{aligned}
x^2 + y^2 &= r^2 \\
A &= 2xy
\end{aligned}$$

$$
\begin{aligned}
y &= \sqrt{r^2 - x^2} \\
A &= 2x\sqrt{r^2 - x^2} \\
A &= \sqrt{4x^2(r^2 - x^2)} \\
A &= \sqrt{4x^2r^2 - 4x^4}
\end{aligned}
$$

Since the maximum of $\sqrt{f(x)}$ is equal to $\sqrt{\text{max of } f(x)}$, we can find the max of $4x^2r^2 - 4x^4$, then take a square root. This works in general for any increasing function.

$$
\begin{aligned}
f(x) &= 4x^2r^2 - 4x^4 \\
f'(x) &= 8xr^2 - 16x^3 \\
f'(x) &= 8x(r^2 - 2x^2) \\
f'(x) &= 8x(r - x\sqrt{2})(r + x\sqrt{2}) \\
f'(x) &= -8x(x\sqrt{2} - r)(x\sqrt{2} + r) \\
x &= 0, \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}
\end{aligned}
$$

since x must be positive, $x = \frac{r}{\sqrt{2}}$.

Example 3

Find the point on the parabola $y^2 = 2x$ closest to $(1, 4)$.

$$
\begin{aligned}
y^2 &= 2x \\
r &= \sqrt{(x-1)^2 + (y-4)^2}
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{y^2}{2} \\
r &= \sqrt{(\frac{1}{2}y^2 - 1)^2 + (y-4)^2} \\
r &= \sqrt{\frac{1}{4}y^4 - y^2 + 1 + y^2 - 8y + 16} \\
r &= \sqrt{\frac{1}{4}y^4 - 8y + 17} \\
f(y) &= \frac{1}{4}y^4 - 8y + 17 \\
f'(y) &= y^3 - 8 \\
0 &= y^3 - 8 \\
8 &= y^3 \\
2 &= y
\end{aligned}
$$