Approx

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Integral Approximation

$$
\begin{aligned}
S_m &= \sum_{n=1}^{m} a_n \\
R_m &= \sum_{n=m+1}^{\infty} a_n \\
S_\infty &= S_m + R_m \\
S_\infty &\approx S_m + \int_m^\infty f(x)\ dx \\
\int_{m+1}^\infty f(x)\ dx \leq S - S_m &\leq \int_m^\infty f(x)\ dx
\end{aligned}
$$

If $f(x)$ is positive continuous and monotonic decreasing then

$$\sum_{n=1}^{\infty} a_n \leq S_m + \int_m^\infty f(x)\ dx$$

Example

Approximate $\sum_{n=1}^{\infty} \frac{1}{n^2}$

1. using only the fourth partial sum.
2. within an error of 0.03.

$$
\begin{aligned}
S - S_4 = R_4 &\leq \int_4^\infty 1/x^2\ dx \\
&\leq \Big[-1/x \Big]_4^\infty \\
&\leq -\frac{1}{\infty} - -\frac{1}{4} \\
S - S_4 &\leq \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
S - S_m &\leq 0.03 \\
S - S_m \leq \int_m^\infty \frac{1}{x^2}\, dx &\leq 0.03 \\
\left[-\frac{1}{x} \right]_m^\infty &\leq 0.03 \\
\frac{1}{m} &\leq 0.03 \\
m &\geq \frac{1}{0.03} \\
m &\geq 33.\bar{3}
\end{aligned}
$$

Since we are looking for the minimum possible approximation (that has to be a whole number), $m = 34$.

Alternating Series Approximation

An alternating series that has terms whose absolute value is monotonic decreasing will oscillate between being greater than and being less than the convergent value.

$$
|S - S_m| = \left| \sum_{n=m+1}^{\infty} (-1)^n b_n \right| \leq b_{m+1}
$$

For something like $b_n = \frac{1}{n^2}$, just using 5 terms will give only an error of $\frac{1}{5^2} = 0.04$. Since alternating series converge much faster, integral approximations not needed as often.