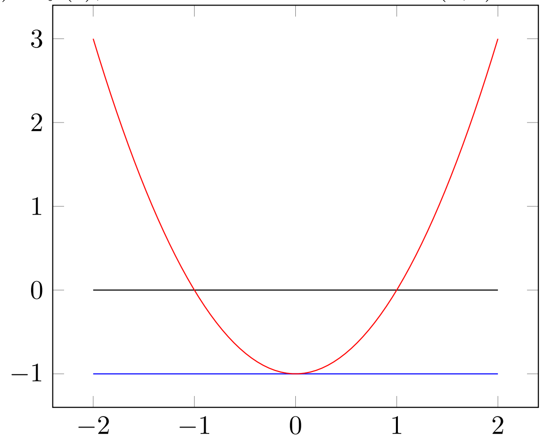
Rolle's Theorem, Mean Value Theorem, and Graphing

Patrick Chen

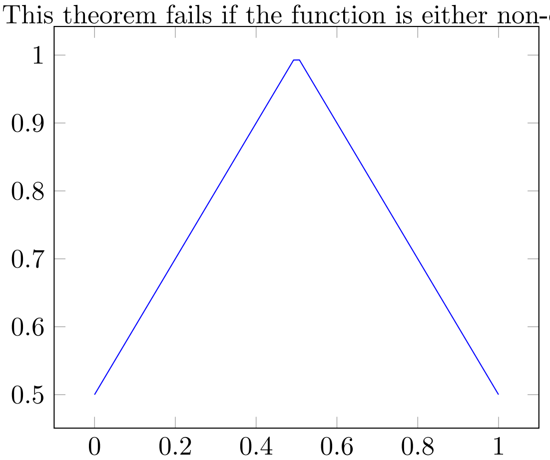
Oct 7, 2024

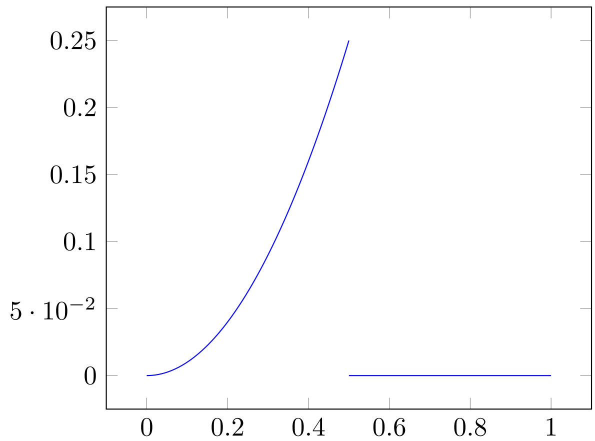
Rolle's theorem

if f is continuous on the closed interval $[a, b]$, f is differentiable on the open interval (a, b), and $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$



This theorem fails if the function is either non-differentiable or not continuous.





Example

Prove that $x^3 + x - 1 = 0$ has exactly one root.

To do this, we must prove that the function has at least one root, then show that there exists no more than one root.

$$
\begin{aligned}
f(0) &= 0^3 + 0 - 1 \\
f(0) &= -1 \\
f(1) &= 1^3 + 1 - 1 \\
f(1) &= 1
\end{aligned}
$$

Since polynomial functions are continuous, by the intermediate value theorem, there must exist a root in the interval $(0, 1)$.

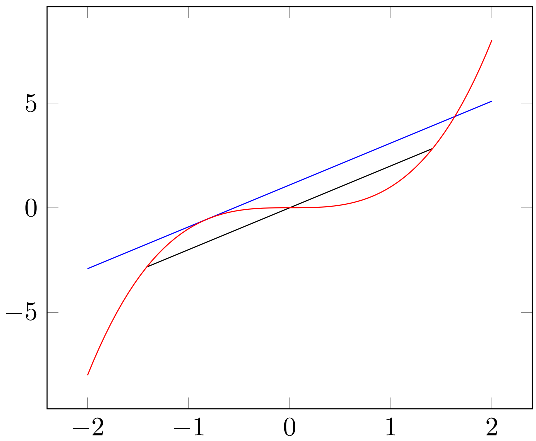
To prove that there does not exists two roots x_1, x_2, assume there are two roots. By Rolle's theorem, there exists a point c where $f'(c) = 0$. Since the derivative is never zero, this is a contradiction and there cannot be two roots.

$$
\begin{aligned}
f(x) &= x^3 + x - 1 \\
f'(x) &= 3x^2 + 1
\end{aligned}
$$

Mean Value Theorem

For a function f that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

In other words, if a function is continuous and differentiable on a interval, then there is a point in that interval where the slope is equal to the secant line formed by the two endpoints. The mean value theorem is a generalized version of Rolle's theorem.



Example

If $f(0) = -3$ and $f'(x) \leq 5$, for all x, how large can $f(2)$ possible be. Using the interval $[0, 2]$:

$$\begin{aligned}
f'(c) &= \frac{f(b) - f(a)}{b - a} \\
&= (f(2) - 3)/(2 - 0) \\
&= (f(2) + 3)/2 \\
f(2) &= 2f'(c) - 3 \\
f(2) &\leq 2(5) - 3 \\
f(2) &\leq 7
\end{aligned}$$

Derivative information

- if $f'(x) > 0$ on an interval, then f is increasing on that interval
- if $f'(x) < 0$ on an interval, then f is increasing on that interval

For a continuous function f and a critical value c

- if f' changes from positive to negative at c, then f has local max at c.
- if f' changes from negative to positive at c, then f has local min at c.
- if f' is positive to left and right of c, or negative to left and right of c, then f has no local maximum or minimum.

A function f is called concave upward on an interval I if all tangents lays below the function on the interval I.

- If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I
- If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I

A point (x, y) on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave up to concave down or from concave down to concave up at x. Just like the first derivative, if $f''(x) = 0$, then there is not always a inflection point.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example: Graphing

$$f(x) = x^{2/3}(6-x)^{1/3}$$
$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$
$$f''(x) = -\frac{8}{x^{4/3}(6-x)^{5/3}}$$

$$f(0) = 0$$
$$f(x) = 0 \Rightarrow x = 0, 6$$

f' is zero when $x = 4$ and f' is undefined when $x = 0, 6$.

	$x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$4 < x < 6$	$x = 6$	$6 < x$
f	$+$	0	$+$	$+$	$+$	0	-
		$m = \infty$		local max		$m = \infty$	
f’	-	DNE	$+$	0	-	DNE	-
f”	-	DNE	-	-	-	DNE	$+$