Characteristic Equation

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Finding eigenvalues

If λ is an eigenvalue for A, then $A - \lambda I = 0$ has a non-trivial solution and is not invertible. This is equivalent to finding when the determinant of $A - \lambda I$ is equal to zero. In a upper and lower triangular matrix, the eigenvalues are the values along the diagonal.

$$\det(A - \lambda I) = 0$$

This determinant will give a polynomial of degree n. This is called the characteristic polynomial.

- The eigenvalues of A are solutions of the characteristic equation of A.
- The algebraic multiplicity of an eigenvalue λ is degree of the λ root in the characteristic equation.
- The geometric multiplicity of λ is the dimension of the λ-eigenspace.
- The geometric multiplicity is always less than or equal to the algebraic multiplicity.

Example 1

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$
$$= (1-\lambda)^2$$

$\lambda = 1$ is the only eigenvalue and has a algebraic multiplicity of 2.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$u = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The dimension of the 1-eigenspace has a dimension of 1, thus a geometric multiplicity of 1.

Example 2

Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 1 & -7 & 3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -7 & 3 & 4 \\ 0 & 2-\lambda & 1 & -1 \\ 0 & 0 & 6-\lambda & 3 \\ 0 & 0 & 0 & -1-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = (1-\lambda)(2-\lambda)(6-\lambda)(-1-\lambda) = 0$$

$$\lambda = 1, 2, 6, -1$$