

# Homogeneous Equations

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**Ex**

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Does  $v_1 \dots v_3$  span  $\mathbb{R}^3$ ?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the matrix is already in row echelon form, it can span all of  $\mathbb{R}^3$ , but it does not tell how to get any individual vector. Note that if you have  $n$  vectors, at most, it could span  $\mathbb{R}^n$ . The dimension is the minimum amount of vectors required to span a space.

## Linearity of matrix multiplication

$A$  is a matrix,  $x, y$  are vectors, and  $\lambda$  is a scalar:

$$\begin{aligned} A(x + y) &= Ax + Ay \\ A(\lambda x) &= \lambda Ax \end{aligned}$$

## Homogeneous Systems

A homogeneous system is a system in the form of  $Ax = 0$ . All homogeneous systems has one trivial solution:  $x = 0$  and may have non-trivial solutions.

**Theorem 0.1** Any homogeneous system with more unknowns than equations has infinitely many solutions

Suppose we have a augmented matrix for a homogeneous system with more variables than equations with  $m$  columns and  $n$  rows. If  $m > n$ , there will be a free variable, and thus, infinitely many solutions.

$$[A \quad \vec{0}]$$

Suppose that  $Ax = b$  has a solution  $x_0$  and  $y$  is any solution to  $Ax = 0$ .  $x_0 + y$  is a solution to  $Ax = b$ .

$$\begin{aligned} A(x_0 + y) &= Ax_0 + Ay = b + 0 = b \\ A(x_0 - x_1) &= Ax_0 - Ax_1 = b - b = 0 \end{aligned}$$

## Linearly Independence

A set of vectors  $v_1 \dots v_m$  is linearly independent if in  $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$ , the only solution is  $\lambda_1 = \dots = \lambda_m = 0$ . This means that for  $A = [v_1 \dots v_m]$ , the only solution to  $Ax = 0$  is  $x = 0$ .

- a set of any one individual vector (except for the zero vector) is a set of linear independent vectors in  $\mathbb{R}^n$ .
- With a set of size two, any two vectors that are not scalar multiples of each other are linearly independent.
- In a set of  $m$  vectors in  $\mathbb{R}^n$ , a new linearly independent set can be created by adding a vector that is not in the span of the set.
- Any subset of a linearly independent set of vectors will also be linear dependent.

If  $v_1 \dots v_m$  are in  $\mathbb{R}^n$  and  $m > n$ , then  $v_1 \dots v_m$  is not linearly independent. Since there are more columns than rows, the row reduction of  $[v_1 \dots v_m]$  will have free variables, and thus a non-trivial solution to  $Ax = 0$ .

### Example 1

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{array}{ccccccc} \lambda_1 & + & 0 & + & 0 & = & 0 \\ 0 & + & \lambda_2 & + & 0 & = & 0 \\ 0 & + & 0 & + & \lambda_3 & = & 0 \end{array}$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

### Example 2

Is  $[1, 0, 0]^T$  and  $[0, 1, 0]^T$  linearly independent in  $\mathbb{R}^3$ ? Yes, when it is row reduced, it does not yield any free variables. They are also subsets of the linearly independent set in example 1, and are thus also linear independent.

**Example 3**

Is  $v_1 = [1, 2, 3]^T, v_2 = [2, 1, 0]^T, v_3 = [3, 3, 3]^T$  linearly independent?

$$v_1 + v_2 - v_3 = 0$$

Since there is a non-trivial solution for the homogeneous equation, this set of vectors are not linearly independent.