Orthogonal Sets

Patrick Chen

Dec 2, 2024

Orthogonal Sets

A orthogonal set if a set where every vector is orthogonal to each other. A orthonormal set is a orthogonal set where all the vectors have length 1.

Projection

The projection of a vector v onto a vector u, written $proj_u(v)$ is a scalar multiple of u and $v - proj_u(v)$ is orthogonal to u. If W is a subspace, the projection of a vector y onto W is written as $proj_W(y)$.

$$proj_u(v) = \frac{v \cdot u}{||u||^2}u$$

$$u \cdot (v - proj_u(v)) = u \cdot \left(v - \frac{v \cdot u}{||u||^2}u\right)$$

$$= u \cdot v - \frac{v \cdot u}{||u||^2}(u \cdot u)$$

$$= u \cdot v - v \cdot u$$

$$= \vec{0}$$

Suppose that W is a subspace of $\mathbb{R}^n$ with an orthogonal basis $v_1, \ldots, v_k$. Then any $y \in \mathbb{R}^n$ can be written as $\hat{y} + z$ where $\hat{y} \in W$ and $z \in W^\perp$. $\hat{y} = proj_{v1}(y) + \cdots + proj_{vk}(y)$ It is very important that the basis is orthogonal or else this will not work.

Gram-Schmidt process

Every subspace of $\mathbb{R}^n$ has an orthonormal basis. If $v_1, \ldots, v_i$ are linearly independent and $W_i = span\{v_1, \ldots, v_k\}$. A basis $B = \{u_1, \ldots, u_k\}$ can be computed:

$$u_1 = v_1$$
$$u_2 = v_2 - proj_{W1}(v_2)$$
$$u_3 = v_3 - proj_{W2}(v_3)$$
$$\vdots$$
$$u_k = v_k - proj_{W(k-1)}(v_k)$$

Example 1

Orthogonalize the basis and project $y = [2\ 1\ 3]^T$ onto V.

$$B = \{u, v\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$proj_u(v) = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v - proj_u(v) = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$V = \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$proj_V(y) = proj_{v1}(y) + proj_{v2}(y)$$

$$= \frac{6}{3}v_1 + \frac{1}{\frac{6}{9}}v_2$$

$$= 2v_1 + \frac{3}{2}v_2$$

$$= 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{2}\frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix}$$

$$z = y - proj_V(y)$$

$$= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$z \cdot v_1 = 0$$

$$z \cdot v_2 = 0$$

Example 2

Apply the Gram-Schmidt process to the following basis

$$U = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$$

u_1 and u_2 are already orthogonal, therefore $v_2 = u_2 - proj_{v1}(u_2) = u_2$

$$v_3 = u_3 - proj_{v1}(u_3) - proj_{v2}(u_3)$$

$$= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{7}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$