Induction and Recursion

Patrick Chen

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Induction

Suppose we with to prove a proposition $P(n)$. If it is true for a base case $P(1)$ and for all k, $P(k)$ implies $P(k+1)$ is true, then it is true for all n.

- Basis step: $P(1)$
- Inductive step: $P(k) \to P(k+1)$

Strong induction is a form of induction where in the inductive step, we assume that all $P(i)$ where $i \leq k$ is true. Strong induction is mathematically equivalent to regular induction.

- Basis step: $P(m)$
- Inductive step: $\Big(\forall i : m \leq i \leq k,\ P(i)\Big) \to P(k+1)$

Well Ordering Principle

The well ordering principle is an axiom says that every non-empty set of non-negative integers has a smallest element. The well ordering principle is equivalent to assuming that induction is true.

Proof of Induction

Assume the well ordering principle.

$$P(1) \quad P(k) \to P(k+1)$$

Let $\mathcal{C} = \{n \in \mathbb{N}^+ | P(n) \text{ is not true}\}$
Assume $\mathcal{C}$ is non-empty
By the well ordering principle, pick the smallest element k of $\mathcal{C}$
Since $P(1)$ is true, $k > 1$, thus $k-1$ is a positive integer not in $\mathcal{C}$.
Thus $P(k-1)$ is true.
By the inductive hypothesis $P(k)$ is true.
This contradicts that $P(k) \in \mathcal{C}$
Thus $\mathcal{C}$ is empty.
Thus proving the base case and the inductive hypothesis is equivalent to proving for all elements.

Structural Induction

- Basis step: Prove the result for the basis element
- Prove that if the result holds for each element in the construction of the new element, it holds for the new construction.

Example

Prove that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

$$P(n) = \Big(\sum_{j=1}^{n} j = \frac{n(n+1)}{2}\Big)$$

Base case:

$$\begin{aligned} P(1) &= \Big(\sum_{j=1}^{1} j = \frac{1(1+1)}{2}\Big) \\ &= (1 = 1) \\ &= T \end{aligned}$$

Inductive step:
Assume $P(k)$ is true

$$\begin{aligned} p(k+1) &= \Big(\sum_{j=1}^{k+1} j = \frac{(k+1)((k+1)+1)}{2}\Big) \\ &= \Big((k+1) + \sum_{j=1}^{k} j = \frac{(k+1)(k+2)}{2}\Big) \\ &= \Big((k+1) + \frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2}\Big) \\ &= \Big(\frac{2(k+1)+k(k+1)}{2} = \frac{(k+1)(k+2)}{2}\Big) \\ &= \Big(\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}\Big) \\ &= T \end{aligned}$$

Therefore, $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ for all $n \geq 1$

Example 2

Prove that $21 \mid 4^{n+1} + 5^{2n-1}$
$21 \mid n$ if both $3 \mid n$ and $7 \mid n$

$$P(n) = (3 \mid 4^{n+1} + 5^{2n-1})$$

Base case:

$$\begin{aligned} P(1) &= (3 \mid 4^{1+1} + 5^{2(1)-1}) \\ &= (3 \mid 21) \\ &= T \end{aligned}$$

Inductive Step

$$\begin{aligned}P(k+1) &= (3 \mid 4^{k+1+1} + 5^{2(k+1)-1}) \\ &= (3 \mid 4^{k+2} + 5^{2k+1}) \\ &= (3 \mid 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1}) \\ &= (3 \mid \underbrace{3 \cdot 4^{k+1} + 24 \cdot 5^{2k+1}}_{e_1} + \underbrace{4^{k+1} + 5^{2k-1}}_{e_2})\end{aligned}$$

3 divides e_1 trivially and 3 divides e_1 by inductive hypothesis. Therefore $3 \mid 4^{n+1} + 5^{2n-1}$

$$Q(n) = (7 \mid 4^{n+1} + 5^{2n-1})$$

Base case:

$$\begin{aligned}Q(1) &= (7 \mid 21) \\ &= 1\end{aligned}$$

Inductive Step

$$\begin{aligned}Q(k+1) &= (7 \mid 4^{(k+1)+1} + 5^{2(k+1)-1}) \\ &= (7 \mid 4^{k+2} + 5^{2k+1}) \\ &= (7 \mid 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1}) \\ &= (7 \mid 4 \cdot 4^{k+1} + 4 \cdot 5^{2k-1} + 21 \cdot 5^{2k-1}) \\ &= (7 \mid 4\underbrace{(\cdot 4^{k+1} + \cdot 5^{2k-1})}_{e_3} + \underbrace{21 \cdot 5^{2k-1}}_{e_4})\end{aligned}$$

7 divides e_3 by inductive hypothesis and 7 divides e_4 trivially
Since $3 \mid 4^{n+1} + 5^{2n-1}$ and $(7 \mid 4^{n+1} + 5^{2n-1})$, then $21 \mid 4^{n+1} + 5^{2n-1}$

Example 3

Prove that every integer $n \geq 2$ is a product of primes.

$$p(n) = (\text{every } n \geq 2 \text{ is a product of primes})$$

$P(2)$ is trivially true.
Suppose that $P(i)$ is true for all $2 \leq i \leq k$
If $k+1$ is prime, then it is trivially a product of itself
If $k+1$ is composite, $k+1 = ab$ for $2 \leq a, b < k+1$
By the inductive hypothesis, a and b are a product of prime numbers
Therefore $k+1$ is also a product of primes

Example 4

Using the well ordering principle, prove that $\sum_{i=1}^{n} n = \frac{n(n+1)}{2}$ for all $n \geq 1$
Let $C = \{x \geq 1 \mid \neg P(n)\}$

Let $k \in C$ be the smallest element
$P(k)$ is not true but $P(k-1)$ is true.

$$\sum_{i=1}^{k-1} i = \frac{(k-1)k}{2}$$
$$k + \sum_{i=1}^{k-1} i = k + \frac{(k-1)k}{2}$$
$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

This contradicts that $P(k)$ is false.
Thus C is empty.

Example 5

Every well-formed formula for compound propositions contains an equal number of left and right parentheses.

- T, F have no parentheses
- Given well formed compound propositions p, q, let l_p and r_p be the number of left and right parentheses in p and similarly l_q and r_q be for q.
  Assume $l_p = r_p$ and $l_q = r_q$.
  - $(\neg p)$ has parentheses $l = l_p + 1$, $r = r_p + 1$
  - $(p \vee q)$ has parentheses $l = l_p + l_q + 1$, $r = r_p + r_q + 1$
  - $(p \wedge q)$ has parentheses $l = l_p + l_q + 1$, $r = r_p + r_q + 1$
  - $(p \to q)$ has parentheses $l = l_p + l_q + 1$, $r = r_p + r_q + 1$
  - $(p \leftrightarrow q)$ has parentheses $l = l_p + l_q + 1$, $r = r_p + r_q + 1$

  In all cases, $l = r$

Thus for all compound propositions, there is an equal number of left and right parentheses.