Multivariate Integration

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April 2, 2025

Riemann Sum

For a function $f : \mathbb{R} \mapsto \mathbb{R}$ of one independent variable, the area under a function can be expressed as a Riemann sum.

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

For multiple independent variables, $f : \mathbb{R}^2 \mapsto \mathbb{R}$ can be expressed as a multidimensional Riemann sum. The volume under a function of two independent variables in the domain $a \leq x \leq b$ and $c \leq y \leq d$ can be expressed as follows.

$$A = \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x\Delta y$$

$$\Delta x = \frac{b-a}{m}$$

$$\Delta y = \frac{d-c}{n}$$

By convention, $+x$ is described as right, $-x$ is described as left, $+y$ is described as upper and $-y$ is described as lower. Interval notation can be extended into multiple dimensions by defining $\mathbf{r} \in [a, b] \times [c, d]$ as $x \in [a, b]$, $y \in [c, d]$.

Definite Integrals

$$\int_R f(x,y)dA = \iint_R f(x,y)\ dx\ dy = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_1, y_1)\Delta x\Delta y$$

Given that $f(x,y)$ and $g(x,y)$ are continuous function on disjoint rectangular regions R, R_1:

- $\int_R f(x,y)\ dA$ is the volume below the surface
- $\int_{R,R_1} f(x,y)\ dA = \int_R f(x,y)\ dA + \int_{R_1} f(x,y)\ dA$
- $\int_R f(x,y) + g(x,y)\ dA = \int_R f(x,y)\ dA + \int_R g(x,y)\ dA$
- $\int_R kf(x,y)\ dA = k\int_R f(x,y)\ dA$
- $f(x,y) \leq g(x,y)$ for all $(x,y) \in R$ implies that $\int_R f(x,y)\ dA \leq \int_R g(x,y)\ dA$

Fubini's Theorem

If $f(x, y)$ is continuous on a finite rectangle $R = \{a \leq x \leq y,\ c \leq y \leq d\}$ then the integral commutes.

$$\int_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dy\ dx = \int_c^d \int_a^b f(x, y)\ dx\ dy$$

When evaluating an integral, integrate from the inside out. Since the outer variable is constant when integrating along the inside variable's domain, the inner integral can be evaluated like a integral of one dimension.

Separable Integrals

An integral over a rectangular region R is a separable integral if the integrand has the form $f(x)g(y)$.

$$\int_a^b \int_c^d f(x)g(y)\ dy\ dx = \int_a^b f(x) \cdot \Big(\int_c^d g(y)\ dy \Big)\ dx = \Big(\int_a^b f(x)\ dx \Big) \Big(\int_c^d g(y)\ dy \Big)$$

Example

Approximate the volume under the surface $z = xy^2$ on the rectangular region given by $x \in [1, 5]$, $y \in [2, 8]$ using two sub-intervals in the x direction and three in the y direction. Use upper left endpoints.

$$m = 2 \Rightarrow \Delta x = \frac{5 - 1}{2} = 2$$
$$n = 3 \Rightarrow \Delta y = \frac{8 - 2}{3} = 2$$

$$x_1 = 1, \quad x_2 = 3$$
$$y_1 = 4, \quad y_2 = 6, \quad y_3 = 8$$

$$A = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x \Delta y$$
$$A = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) 2 \cdot 2$$
$$A = 4(f(1, 4) + f(1, 6) + f(1, 8) + f(3, 4) + f(3, 6) + f(3, 8))$$
$$A = 4(1(4^2) + 1(6)^2 + 1(8^2) + 3(4^2) + 3(6)^2 + 3(8^2))$$
$$A = 1856$$

Example 2

$$
\begin{aligned}
\int x\cos(xy)\ dA, \quad R=[0,\pi]\times[1,2] \\
\int_0^{\pi}\int_1^2 x\cos(xy)\ dy\ dx &= \int_0^{\pi}\Big[\sin(xy)\Big]_{y=1}^{2}\ dx \\
&= \int_0^{\pi}\sin(2x)-\sin(x)\ dx \\
&= \Big[-\frac{1}{2}\cos(2x)+\cos(x)\Big]_{x=0}^{\pi} \\
&= -2
\end{aligned}
$$

Example 3

$$
\begin{aligned}
\int_{[-2,2]\times[0,1]} x(1-y^2)\ dA &= \int_{-2}^{2}\int_0^1 x(1-y^2)\ dy\ dx \\
&= \int_{-2}^{2} x\int_0^1 (1-y^2)\ dy\ dx \\
&= \int_{-2}^{2} x\Big[y-\frac{1}{3}y^3\Big]_{y=0}^{1} dx \\
&= \int_{-2}^{2} x(\frac{2}{3})dx \\
&= (\frac{2}{3})\int_{-2}^{2} xdx \\
&= (\frac{2}{3})(0)
\end{aligned}
$$

Another way to evaluate it

$$
\begin{aligned}
\int_{[-2,2]\times[0,1]} x(1-y^2)\ dA &= \int_{-2}^{2}\int_0^1 x(1-y^2)\ dy\ dx \\
&= \Big(\int_{-2}^{2} x\ dx\Big)\Big(\int_0^1 (1-y^2)\ dy\Big) \\
&= (0)\int_0^1 (1-y^2)\ dy \\
&= 0
\end{aligned}
$$

Non rectangular domains

Type 1 Regions

Type 1 regions can be cut into continuous strips in the y directions. If the upper bound is given by $h(x)$ and the lower bound is given by $g(x)$, then the integral can be found by integrating over the y axis then the x axis.

$$A = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\ dy\ dx$$

Type 2 Regions

Type 2 regions are regions that can be cut into continuous strips in the x direction. The integral can be evaluated similarly to type 1 regions

$$A = \int_c^d \int_{p(y)}^{q(y)} f(x, y)\ dx\ dy$$

Other Regions

If a region is nether type 1 nor type 2, it can be broken up into components that are either type 1 or type 2. To find the overall integral, evaluate the integrals of the subregions and sum the result.

$$\int_{R,R_1} f(x, y) dA = \int_R f(x, y) dA + \int_{R_1} f(x, y) dA$$

Example 1

Let $D = \{(x, y) \mid x^2 + y^2 \leq 4, y \geq 0\}$ be a semicircle that of radius 2 on the $+x, +y$ and $-x, +y$ quadrant. Evaluate $\int_D x^2 y\ dA$

$$\begin{aligned} x^2 + y^2 &= 4 \\ y_{\min} &= 0 \\ y_{\max} &= \sqrt{4 - x^2} \end{aligned}$$

$$\begin{aligned} \int_D x^2 y\ dA &= \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} x^2 y\ dy\ dx \\ &= \int_{-2}^{2} \left[\frac{1}{2} x^2 y^2\right]_{y=0}^{\sqrt{4-x^2}} dy\ dx \\ &= \int_{-2}^{2} \frac{1}{2} x^2 \left(\sqrt{4 - x^2}\right)^2 dy\ dx \\ &= \frac{1}{2} \int_{-2}^{2} x^2 (4 - x^2)\ dx \\ &= \frac{1}{2} \int_{-2}^{2} 4x^2 - x^4\ dx \\ &= \left[\frac{2x^3}{3} - \frac{x^5}{10}\right]_{x=-2}^{2} \\ &= \frac{64}{15} \end{aligned}$$

Example 2

Evaluate the integral of $f(x,y) = x^2 - y$ on the region enclosed by $y = x^2 + 3$ and $y = 4x^2$.

$$\begin{aligned} x^2 + 3 &= 4x^2 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Thus the endpoints on the x axis is -1 and 1.

$$\begin{aligned} \int_D f(x,y)\ dA &= \int_{-1}^{1} \int_{4x^2}^{x^2+3} x^2 - y\ dy\ dx \\ &= \int_{-1}^{1} \left[x^2 y - \frac{y^2}{2} \right]_{y=4x^2}^{x^2+3} dx \\ &= \int_{-1}^{1} \left(x^2(x^2+3) - \frac{(x^2+3)^2}{2} \right) - \left(x^2(4x^2) - \frac{(4x^2)^2}{2} \right) dx \\ &= \int_{-1}^{1} (x^4 + 3x^2 - \frac{x^4}{2} - \frac{6x^2}{2} - \frac{9}{2}) - (4x^4 - 8x^4)\ dx \\ &= \int_{-1}^{1} \frac{x^4}{2} - \frac{9}{2} + 4x^4\ dx \\ &= \int_{-1}^{1} \frac{9x^4}{2} - \frac{9}{2}\ dx \\ &= \frac{9}{2} \left[\frac{x^5}{5} - x \right]_{x=-1}^{1} \\ &= -\frac{36}{5} \end{aligned}$$

Area of a Shape

The area can of a shape R be computed by integrating 1 over the domain.

$$A_R = \int_R 1\ dA$$

Average Height of a 3D function

The average height is the integral of the function divided by the area

$$Z_{\text{avg}} = \frac{\int_D f(x,y)\ dA}{\int_D 1\ dA}$$

Volume Between Two Surfaces

The volume between two surfaces can be computed in the same way area between two functions can be computed

$$A = \int_R f(x,y) - g(x,y)\ dA$$
$$\text{where} f(x,y) \geq g(x,y)\ \forall (x,y) \in R$$