# Derivative

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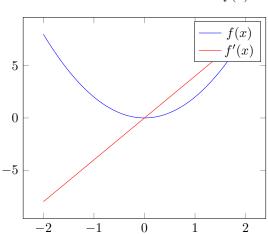
## The Derivative Function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f : \mathbb{R} \to \mathbb{R}$$

 $f': \mathbb{R} \mapsto \mathbb{R}$ 

 $f': x \mapsto$  the slope of the function f

$$f(x) = 2x^2$$

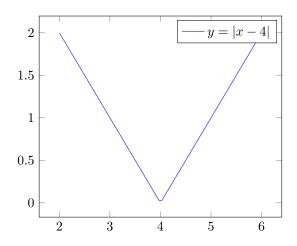


$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f(x)$$

A function f is differentiable at a if f'(a) exists (and is finite). f is differentiable on (a,b) if f is differentiable at every point in (a, b).

## Example

$$f(x) = |x - 4|$$



$$|x-4| = \begin{cases} x-4, & x-4 \ge 0 \\ -(x-4) & x-4 < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1, & x > 4 \\ -1 & x < 4 \\ \text{undefined} & x = 4 \end{cases}$$

f'(0) is undefined because the left and right limits are not equal.

$$\lim_{h \to 0^+} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0^+} \frac{|h|}{h}$$

$$|h| \text{ and } h \text{ are positive}$$

$$= 1$$

$$\lim_{h \to 0^{-}} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0^{-}} \frac{|h|}{h}$$

$$|h| \text{ is positive but } h \text{ is negative}$$

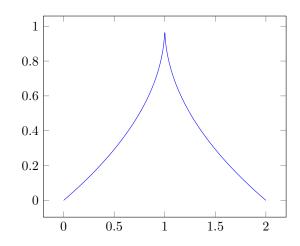
$$= -1$$

## **Theorem 0.1** If f is differentiable at a, then f is continuous at a.

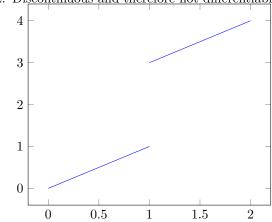
We saw this in the absolute value example. |x-4| is continuous but not differentiable at point (4, 0).

There are many ways a function cannot be differentiable.

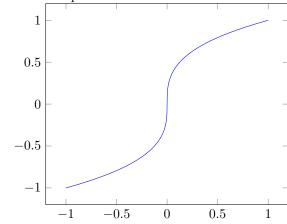
1. Cusp







# 3. Infinite slope



## **Higher Derivatives**

The derivative of the derivative of a function is called the second derivative of the function.

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d}{dx}\frac{d}{dx}\dots\frac{d}{dx}\frac{dy}{dx} = \frac{d^ny}{dx^n}$$

## **Derivative Rules**

## Constant Multiple, Addition, and Subtraction

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

### Product and Quotient rules

When two functions are multiplied together (product rule):

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x))g(x)$$
$$(fg)' = f'g + fg'$$

When one function is divided by another function (quotient rule):

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x))g(x) - f(x)\frac{d}{dx}(g(x))}{g(x)^2}$$
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

#### Chain rule

For a function g that is differentiable at x and a function f that is differentiable at f(x), the derivative of  $f \circ g$  can be found using the chain rule.

$$y = f(g(x))$$
$$y' = f'(g(x)) \cdot g'(x)$$

Example: let  $y = \sqrt{x^2 + 1}$ 

$$g(x) = x^2 + 1 \Rightarrow g'(x) = 2x$$
$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$
$$y' = \frac{1}{2\sqrt{x^2 + 1}} 2x = \frac{x}{\sqrt{x^2 + 1}}$$

## Implicit Differentiation

Implicit differentiation is the process of treating a y like the function f(x). This can be useful when it is difficult or impossible to isolate for y in a equation. The derivative of a curve can be found by using implicit differentiation then solving for y'.

Example:

$$x^{3} + y^{3} = 6xy$$

$$\downarrow \downarrow$$

$$x^{3} + f(x)^{3} = 6xf(x)$$

$$(y^3)' = (f(x)^3)'$$
$$= 3f(x) \cdot f'(x)$$
$$= 3y \cdot y'$$

$$x^{3} + y^{3} = 6xy$$

$$3x^{2} + 3y^{2}y' = 6(xy' + 1y)$$

$$3x^{2} + 3y^{2}y' = 6xy' + 6y$$

$$3x^{2} - 6y = 6xy' - 3y^{2}y'$$

$$3x^{2} - 6y = (6x - 3y^{2})y'$$

$$\frac{3x^{2} - 6y}{6x - 3y^{2}} = y'$$

This means the slope at the point (x,y) is  $\frac{3x^2-6y}{6x-3y^2}$ 

## **Inverse Functions**

If f is a differentiable one-to-one function, the derivative of the inverse function can be found with the following formula.

$$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$$

Proof:

$$y = f^{-1}(x)$$

$$f(y) = f(f^{-1}(x))$$

$$f(y) = x$$

$$f'(y)y' = 1 \qquad \text{implicit differentiation}$$

$$y' = \frac{1}{f'(y)}$$

$$y' = \frac{1}{f'(f^{-1}(x))} \qquad \text{substitute } y$$

# Derivatives of functions

## Polynomial

The constant function:

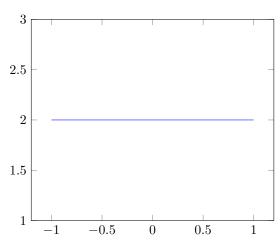
$$f(x) = c$$
$$\frac{d}{dx}(c) = 0$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

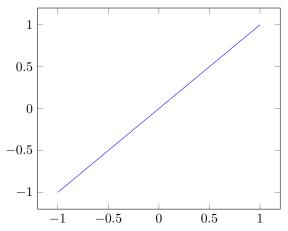
$$= 0$$



Linear (degree one) function:

$$f(x) = x$$

$$f(x) = x$$
$$\frac{d}{dx}(x) = 1$$



Quadratic:

$$f(x) = x^2$$
$$\frac{d}{dx}(x^2) = 2x$$

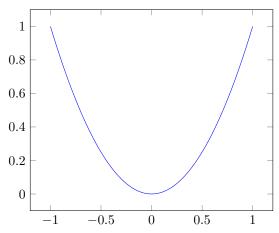
$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$



General:

$$f(x) = x^{n}$$

$$\frac{d}{dx}(x^{n}) = nx^{n-1}$$

## Exponential

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{e^{(x+h) - e^x}}{h}$$

$$\lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$\lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

The derivative of  $e^x$  is itself.

$$\frac{d}{dx}e^x = e^x$$

For an exponent with a different base, convert to base e using logarithm rules.

$$a^{x} = e^{\ln(a^{x})} = e^{x \ln(a)}$$
$$(a^{x})' = e^{x \ln(a)} \ln(a) = a^{x} \cdot \ln(a)$$

## Logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$y = \log_a x$$

$$a^y = x$$

$$(\ln a)a^y y' = 1$$

$$y' = \frac{1}{(a^y)\ln a}$$

$$y' = \frac{1}{x \ln a}$$

## **Trigonometric Functions**

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\begin{split} & \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x) \cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x) \sin(h)}{h} \\ &= \sin(x) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} \\ &= \sin(x)(0) + \cos(x)(1) \\ &= \cos(x) \end{split}$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 1 \qquad \qquad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

#### **Inverse Trigonometric Function**

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1 + x^2}$$