Invertible Matrix Theorem

Patrick Chen

Oct 7, 2024

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there is a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that S(T(x)) = x and T(S(x)) = x for all $x \in \mathbb{R}^n$. If the transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then the standard matrix for T is invertible.

If A is an $n \times n$ matrix, then the following are equivalent:

- a) A is invertible
- b) A can be row reduced to the identity matrix.
- c) A has n pivot positions.
- d) Ax = 0 has only the trivial solution.
- \bullet e) The columns of A form a linearly independent set.
- f) The linear transformation $x \mapsto Ax$ is one-to-one.
- g) The columns of A span \mathbb{R}^n .
- h) the linear transformation $x \mapsto Ax$ is onto
- i) there is an $n \times n$ matrix C such that CA = I
- j) there is an $n \times n$ matrix D such that AD = I
- k) A^T is invertible

Important note: since A^T is invertible, anything you can say about a column of a invertible matrix, you can also say about the rows of a matrix.