

# Determinants

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Oct 10, 2024

A  $n \times n$  matrix  $A$  will map the unit vectors in  $\mathbb{R}^n$  to columns of  $A$ , which is also in  $\mathbb{R}^n$ . The  $n$ -dimensional equivalent to volume of a shape, when mapped through the matrix, will be scaled by  $\det A$ . If  $\det A = 0$ , then the matrix  $A$  is not invertible.

## 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ 0 & -\frac{cb}{a} + d \end{bmatrix} \quad \left(-\frac{c}{a} R1 + R2\right)$$

$A$  is invertible iff  $a \neq 0$  and  $-\frac{cb}{a} + d \neq 0$ . The determinant of  $A$  is  $ad - bc$  then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The area of a parallelogram formed by  $\begin{bmatrix} a \\ c \end{bmatrix}$  and  $\begin{bmatrix} b \\ d \end{bmatrix}$  is equal to  $|\det A|$

If a edge of the parallelogram is not on the x-axis, it can be rotated onto the x-axis with a rotation matrix. Rotation matrices preserves the determinant so the determinant is unchanged after being rotated.

## 3x3 Matrix

The absolute value of the determinant of a  $3 \times 3$  matrix is the volume of a parallelepiped formed by the three columns in  $\mathbb{R}^3$ . A parallelepiped with  $v_1, v_2$  in the xy-plane forming a parallelogram, and  $v_3 = [x_3, y_3, z_3]^T$  will have a volume  $z_3 \cdot (\text{area of parallelogram})$ . Any parallelepiped can be rotated to have a plane laying in the xy-plane.

$$\det \begin{bmatrix} a & b & x_3 \\ c & d & y_3 \\ 0 & 0 & z_3 \end{bmatrix} = z_3 \cdot \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

## General Matrix

The determinant can be generalized to higher dimensions by rotating one "cell" into a (n-1) volume without a  $x_n$  component, calculating the determinant for the first (n-1) vectors, then multiplying by the last vector's  $x_n$  component.

## General Determinant Formula

Suppose the matrix  $A$  is a  $n \times n$  matrix and  $A_{ij}$  is the matrix  $A$  with the  $i$ th row and the  $j$ th column removed.  $A_{ij}$  will be a size  $(n-1) \times (n-1)$  matrix. The  $(i, j)$ -cofactor of  $A$  is the determinant of  $A_{ij}$  multiplied by one or negative one depending on if the sum of the row index and column index is even.

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

The determinant is defined as

$$\begin{aligned} \det(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in} \\ &= \sum_{j=1}^n a_{ji}C_{ji} \end{aligned}$$

The determinant can also be found by summing across the columns instead of the rows.

$$\begin{aligned} \det(A) &= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj} \\ &= \sum_{i=1}^n a_{ji}C_{ji} \end{aligned}$$

The determinant is well defined when summing across any row or column.

### Example

Computing the determinant of a matrix across the first row

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \\ \det(A) &= (1)(1) \det \left( \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right) + (-1)(-1) \det \left( \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \right) + (1)(2) \det \left( \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \right) \\ &= 1 - 5 + 2(-2) \\ &= -8 \end{aligned}$$

Computing the determinant of the same matrix across the second column

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \\ \det(A) &= (-1)(-1) \det \left( \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \right) + (1)(1) \det \left( \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) + (-1)(0) \det \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) \\ &= (-5) + (-3) + 0 \\ &= -8 \end{aligned}$$

### Example 2

$$A = \begin{bmatrix} 1 & 2 & -5 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Finding the determinant across the fourth row

$$\begin{aligned} \det(A) &= 0 + 0 + 0 + (1)(3) \det \begin{bmatrix} 1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 3(0 + 0 + 1(\det \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix})) \\ &= 3(1(3)) \\ &= 9 \end{aligned}$$