Orthogonal Sets

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Orthogonal Sets

A orthogonal set if a set where every vector is orthogonal to each other. A orthonormal set is a orthogonal set where all the vectors have length 1.

Projection

The projection of a vector v onto a vector u, written $proj_u(v)$ is a scalar multiple of u and $v - proj_u(v)$ is orthogonal to u. If W is a subspace, the projection of a vector y onto W is written as $proj_W(y)$.

$$\begin{aligned}
proj_u(v) &= \frac{v \cdot u}{||u||^2} u \\
u \cdot (v - proj_u(v)) &= u \cdot \left(v - \frac{v \cdot u}{||u||^2} u\right) \\
&= u \cdot v - \frac{v \cdot u}{||u||^2}(u \cdot u) \\
&= u \cdot v - v \cdot u \\
&= \vec{0}
\end{aligned}$$

Suppose that W is a subspace of $\mathbb{R}^n$ with an orthogonal basis $v_1, \ldots, v_k$. Then any $y \in \mathbb{R}^n$ can be written as $\hat{y} + z$ where $\hat{y} \in W$ and $z \in W^\perp$. $\hat{y} = proj_{v1}(y) + \cdots + proj_{vk}(y)$ It is very important that the basis is orthogonal or else this will not work.

Gram-Schmidt process

Every subspace of $\mathbb{R}^n$ has an orthonormal basis. If $v_1, \ldots, v_i$ are linearly independent and $W_i = span\{v_1, \ldots, v_k\}$. A basis $B = \{u_1, \ldots, u_k\}$ can be computed:

$$\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_2 - proj_{W1}(v_2) \\
u_3 &= v_3 - proj_{W2}(v_3) \\
&\vdots \\
u_k &= v_k - proj_{W(k-1)}(v_k)
\end{aligned}$$

Example 1

Orthogonalize the basis and project $y = [2\ 1\ 3]^T$ onto V.

$$
\begin{aligned}
B = \{u, v\} &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \\
proj_u(v) &= \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
v - proj_u(v) &= \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \\
V = \{v_1, v_2\} &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\
proj_V(y) &= proj_{v1}(y) + proj_{v2}(y) \\
&= \frac{6}{3} v_1 + \frac{1}{\frac{6}{9}} v_2 \\
&= 2v_1 + \frac{3}{2} v_2 \\
&= 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{2} \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
&= 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \\
z &= y - proj_V(y) \\
&= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \\
z \cdot v_1 &= 0 \\
z \cdot v_2 &= 0
\end{aligned}
$$

Example 2

Apply the Gram-Schmidt process to the following basis

$$U = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$$

u_1 and u_2 are already orthogonal, therefore $v_2 = u_2 - proj_{v1}(u_2) = u_2$

$$\begin{aligned} v_3 &= u_3 - proj_{v1}(u_3) - proj_{v2}(u_3) \\ &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{7}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{bmatrix} \end{aligned}$$