

# Matrix of a Linear Transformation

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## Linear Transformation

We say a transformation  $T : \mathbb{R}^n \mapsto \mathbb{R}^m$  is linear if it follows the rules of linearity. All matrices are linear transformations.

- for all  $u, v \in \mathbb{R}^n$ ,  $T(u + v) = T(u) + T(v)$
- for all scalars  $\lambda$  and  $u \in \mathbb{R}^n$ ,  $T(\lambda u) = \lambda T(u)$

Following these rules, we can conclude  $T(0) = 0$  and for all scalars,  $c, d$ , and vectors  $u, v$ ,  $T(cu + dv) = cT(u) + dT(v)$

$$T(c_1 u_1 + \cdots + c_k u_k) = c_1 T(u_1) + \cdots + c_k T(u_k)$$

$T$ 's value on the span of  $u_1 \dots u_n$  is determined by its values at  $u_1 \dots u_n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \dots$$

$$\begin{aligned} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \\ T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) &= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \end{aligned}$$

A linear transformation is entirely defined by how it affects the basis vectors. All linear transformations from  $T : \mathbb{R}^n \mapsto \mathbb{R}^m$  has a unique matrix that describes the transformation called the standard matrix. The standard matrix is composed of the transformations applied to the basis vectors.

$$A = [T(e_1) \quad T(e_2) \quad T(e_3) \quad \dots]$$

## Common Transformations

Reflection on the  $y = x$  line:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Counter clockwise rotation by the angle  $\theta$ :

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

## Properties

- A transformation is **one-to-one** if two different values of input always results in different outputs.
- A transform is **onto** if the range of a transformation is the same as the codomain

A linear transformation is one-to-one if the only answer to  $T_A(x) = 0$  is the trivial solution  $x = 0$ . This is equivalent to checking if the columns of the matrix is linearly independent.