

# Continuity

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## Limits

$$\lim_{x \rightarrow a^-} f(x) = l$$

This means that we can make  $f(x)$  as close to  $l$  as we want, provided that we choose a  $x$  close enough to  $a$  to the left. If both left limit and right limit exists and are the same, we say that the function has a limit.

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x)$$

## Continuity

Consider

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 0 & x = 2 \end{cases} \quad g(x) = \begin{cases} x^2 & x \neq 2 \\ \text{undefined} & \text{otherwise} \end{cases} \quad h(x) = x^2$$

When  $f(x)$ ,  $g(x)$ , and  $h(x)$  is approached from either the left or the right, it is equal to 4.

$$\lim_{x \rightarrow 2^-} f(x) = 4$$

$$\lim_{x \rightarrow 2^+} f(x) = 4$$

For a function to be continuous at a point, the limit of the function at the point must exist and be equal to the function evaluated at the point.

$$\lim_{x \rightarrow a} f(x) = f(a)$$

## Directional Continuity

A function is continuous from the right at a point  $x$  if the right limit is equal to the function evaluated at  $x$ . Likewise with left continuity.

$$\begin{array}{ll} \lim_{x \rightarrow a^+} f(x) = f(a) & \text{left continuous} \\ \lim_{x \rightarrow a^-} f(x) = f(a) & \text{right continuous} \end{array}$$

## Continuity Rules

If two functions are continuous at  $a$  and  $c$  is a constant, the following is also continuous

1.  $f(a) + g(a)$
2.  $f(a) - g(a)$
3.  $cf(a)$
4.  $f(a)g(a)$
5.  $\frac{f(a)}{g(a)}$  if  $g(a) \neq 0$
6.  $[f \circ g](a)$

The following types of functions are continuous at every number in their domains:

- polynomial
- trigonometric functions
- exponential functions
- rational functions (when denominator is non-zero)
- root functions
- inverse trigonometric functions
- logarithmic functions

## Intermediate Value Theorem

**Theorem 0.1 – Intermediate Value Theorem** Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exist a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .

In other words, if a continuous function is evaluated at two points, any point in between the outputs at those points are also outputs of the function. Note that IVT requires a closed interval,  $(a, b)$  and  $[a, b]$  is not enough to satisfy the IVT.

Consider the following function. It is continuous on the interval  $[-3, 0)$  and  $f(-3) = -\frac{1}{3}$  and  $f(0) = 3$  but there is no  $x$  such that  $f(x) = 2$ .

$$f(x) = \begin{cases} 3, & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

## Example

$$\ln(1 + \cos(x))$$

$\ln$  is continuous when the input is in its domain (positive numbers) and  $1 + \cos(x)$  is continuous for all numbers.

$$1 + \cos(x) > 0$$

Since  $\cos(x)$  ranges from  $-1$  to  $1$ ,  $1 + \cos(x)$  has a range of  $[0, 2]$ . This is problematic because  $0$  is in the range of  $1 + \cos(x)$  but not in the domain of  $\ln(x)$ , thus there is a discontinuity in  $\ln(1 + \cos(x))$ .

**Example 2**

let  $f(x) = e^{-x} - x$ . Show that the function has at least one root on the interval  $[0, 1]$ .

$$f(0) = e^{-0} - 0 = 1$$

$$f(1) = e^{-1} - 1 = \frac{1}{e} - 1 \approx -0.63$$

By the intermediate value theorem,  $f(1) < 0 < f(0)$ , thus there exist some input  $N$  of the function such that  $f(N) = 0$

**Example 3**

Show that  $4x^3 - 6x^2 = 2 - 3x$  has at lease one root.

$$4x^3 - 6x^2 = 2 - 3x$$

$$4x^3 - 6x^2 + 3x - 2 = 0$$

$$\text{let } f(x) = 4x^3 - 6x^2 + 3x - 2$$

$$f(0) = 4(0)^3 - 6(0)^2 + 3(0) - 2$$

$$f(0) = -2$$

$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2$$

$$f(2) = 32 - 24 + 6 - 2$$

$$f(2) = 12$$

By the intermediate value theorem, there exist some value  $N$  in the interval  $[0, 2]$  there  $f(x) = 0$ .