

Power Series

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A power series is a sum of powers. The output of a power series is a function.

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

The domain of the function is the set of all $x \in \mathbb{R}$ where the series converges. The power series is said to be centered on a . When $a = 0$, the series is a 0-centered power series. The interval of convergence is centered on a . The difference between the center and where it diverges is called the radius of convergence R .

Example

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n+1}$$

Using ratio test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{n+2}}{\frac{3^n x^n}{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3x(n+1)}{n+2} \right| \\ &= 3 \cdot |x| \end{aligned}$$

This series will absolutely converge when $3 \cdot |x| < 1$ and diverge when $3 \cdot |x| > 1$.

$$\begin{aligned} x &= \frac{1}{3} \\ \sum_{n=0}^{\infty} \frac{3^n \left(\frac{1}{3}\right)^n}{n+1} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

This will diverge because the $p = 1$.

$$x = -\frac{1}{3}$$

$$\sum_{n=0}^{\infty} \frac{3^n (-\frac{1}{3})^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

Using the alternating series test, this will converge. Thus the interval of convergence is $x \in [-\frac{1}{3}, \frac{1}{3})$

Example 2

Find the center and radius of convergence for the following series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{5^n n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-2)^{n+1}}{5^{n+1}(n+1)^2}}{\frac{(x-2)^n}{5^n n^2}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)n^2}{5(n+1)^2} \right| \\ &= \frac{|x-2|}{5} \end{aligned}$$

$$\frac{|x-2|}{5} < 1$$

$$|x-2| < 5$$

$$a = 2, R = 5$$

Example 3

$$\sum_{n=0}^{\infty} n! x^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| \\ &= \lim_{n \rightarrow \infty} |x|(n+1) \\ &= \infty \end{aligned}$$

This power series diverges for all non-zero values x

Example 4

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{|x|}{n+1} \\ &= 0 \end{aligned}$$

This power series converges for all finite values x

Functions as Power Series

Functions can be represented as power series. For example, $f(x) = \frac{1}{1-x}$ is represented by the following power series.

$$\sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

Using geometric series formula:

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \sum_{n=1}^{\infty} x^{n-1} \\ &= \frac{1}{1-x} \end{aligned}$$

Example

$$\begin{aligned} \frac{4x}{2-x} &= 4x \cdot \left(\frac{1}{2-x} \right) \\ &= \frac{4x}{2} \frac{1}{1 - (\frac{x}{2})} \\ &= 2x \left(\sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \right) \\ &= 2x \left(\sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \right) \end{aligned}$$

Example 2

Find the power series of $f(x) = \frac{1}{(1-x)^2}$

$$\begin{aligned}f(x) &= \frac{d}{dx} \left(\frac{1}{1-x} \right) \\&= \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) \\&= \sum_{n=0}^{\infty} \left(\frac{d}{dx} x^n \right) \\&= 0 + \sum_{n=1}^{\infty} (nx^{n-1}) \\&= \sum_{n=0}^{\infty} (n+1)x^n\end{aligned}$$

The radius of convergence is the same because by the ratio test, $\frac{n+2}{n+1}$ will approach 1 and not affect the outcome when multiplied.

Example 3

Find the power series of $f(x) = \ln(1-x)$.

$$\begin{aligned}f(x) &= \ln(1-x) \\f'(x) &= -\frac{1}{1-x} \\f'(x) &= -\left(\sum_{n=0}^{\infty} x^n \right) \\\int f'(x)dx &= -\int \left(\sum_{n=0}^{\infty} x^n \right) dx \\f(x) &= -\sum_{n=0}^{\infty} \left(\int x^n dx \right) \\f(x) &= -\sum_{n=0}^{\infty} \left(\frac{x^{n+1}}{n+1} \right) + c\end{aligned}$$

Finding the constant of integration:

$$\begin{aligned}f(0) &= -\sum_{n=0}^{\infty} \left(\frac{0^{n+1}}{n+1} \right) + c \\ \ln(1-0) &= -\sum_{n=0}^{\infty} (0) + c \\ 0 &= c\end{aligned}$$

Thus,

$$f(x) = -\sum_{n=0}^{\infty} \left(\frac{x^{n+1}}{n+1} \right)$$

$$f(x) = -\sum_{n=1}^{\infty} \left(\frac{x^n}{n} \right)$$

Taylor and MacLaurin Series

Suppose $f(x)$ has a zero-centered power series. $f(0)$ is the constant term of the power series. $f'(0)$ is the x term of the power series. $\frac{f''(0)}{2}$ is the x^2 term of the power series. In general, $\frac{1}{n!} \left[\frac{d^n}{dx^n} f \right](0)$ is the x^n term of the power series.

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 \dots$$

$$f'''(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$

The Taylor series is an extension of the MacLaurin Series where the series may not be centered at zero. When finding a Taylor series, it may be better to not simplify the expressions of the derivative to help when finding a pattern.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Common Taylor Series

$f(x)$	power series	domain
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$

Taylor Polynomial

A m^{th} order Taylor polynomial $T_m(x)$ is the partial sum of the Taylor series.

$$T_m(x) = \sum_{n=0}^m \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Taylor Remainder Theorem

The absolute difference between a function and its m^{th} Taylor polynomial is $R_m(x)$. It is always less than the next term of the Taylor series with the $(m+1)^{th}$ derivative of the function replaced with the supremum of the $(m+1)^{th}$ derivative of the function.

$$|f(x) - T_m(x)| = R_m(x) \leq \frac{M|x-a|^{m+1}}{(m+1)!}, \quad M \geq \sup|f^{(m+1)}(x)|$$

Example

Find the 0-centered power series for $f(x) = e^x$

$$f(0) = e^0 = 1$$

$$f'(0) = e^0 = 1$$

$$f'''(0) = e^0 = 1$$

$$f^{(n)}(0) = e^0 = 1$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Example 2

Find the Taylor series for $f(x) = \frac{1}{x^2}$ centered at $a = 2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-2}	$\frac{1}{2^2}$
1	$(-2)x^{-3}$	$\frac{-2}{2^3}$
2	$(-3)(-2)x^{-4}$	$\frac{(-3)(-2)}{2^4}$
3	$(-4)(-3)(-2)x^{-5}$	$\frac{(-4)(-3)(-2)}{2^4}$
n	$\frac{(-1)^n(n+1)!}{x^{n+2}}$	$\frac{(-1)^n(n+1)!}{2^{n+2}}$

$$f^{(n)}(2) = \frac{(-1)^n(n+1)!}{2^{n+2}}$$

$$\begin{aligned}\frac{1}{x^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{2^{n+2} n!} (x-2)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}} (x-2)^n\end{aligned}$$

Example 3

Find the Taylor series for $f(x) = \ln(x)$ centered at $a = 5$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!} (x-5)^n$$

n	$f^{(n)}(x)$	$f^{(n)}(5)$
0	$\ln(x)$	$\ln(5)$
1	$\frac{1}{x}$	$\frac{1}{5}$
2	$\frac{-1}{x^2}$	$\frac{-1}{5^2}$
3	$\frac{(-1)(-2)}{x^3}$	$\frac{(-1)(-2)}{5^3}$
4	$\frac{(-1)(-2)(-3)}{x^4}$	$\frac{(-1)(-2)(-3)}{5^4}$
n	$\frac{(-1)^{n-1}(n-1)!}{x^n}$	$\frac{(-1)^{n-1}(n-1)!}{5^n}$

$$\begin{aligned}f(x) &= \frac{\ln(5)}{0!} (x-5)^0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{5^n n!} (x-5)^n \\ &= \ln(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} (x-5)^n\end{aligned}$$

Example 4

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{2^n n!} &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x = \frac{1}{2} \\ &= e^x, \quad x = \frac{1}{2} \\ &= e^{\frac{1}{2}} \\ &= \sqrt{e}\end{aligned}$$

Example 5

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(9^n)(2n)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n \frac{\pi^{2n}}{3^{2n}}}{(2n)!} \\&= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} \\&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad x = \frac{\pi}{3} \\&= \cos(x), \quad x = \frac{\pi}{3} \\&= \cos\left(\frac{\pi}{3}\right) \\&= \frac{1}{2}\end{aligned}$$

Example 6

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{4^n (2n+1)!} &= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2^{2n+1})(2n+1)!} \\&= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \\&= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad x = \frac{\pi}{2} \\&= 2\pi \sin(x), \quad x = \frac{\pi}{2} \\&= 2\pi \sin\left(\frac{\pi}{2}\right) \\&= 2\pi\end{aligned}$$

Example 7

Find $T_4(x)$ about $x = 0$ for $f(x) = \cos(x)$ and estimate the upper bound on the error using the Taylor remainder estimate on the interval $[-2, 2]$.

$$\begin{aligned}T_4(x) &= \sum_{n=0}^4 \frac{f^{(n)}(0)}{n!} x^n \\&= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 \\&= 1 - \frac{x^2}{2} + \frac{x^4}{24}\end{aligned}$$

$$\begin{aligned}
|\cos(x) - T_4(x)| &\leq \frac{M|x-a|^{m+1}}{(m+1)!} \\
\frac{M|x-a|^{m+1}}{(m+1)!} &= \frac{M|x|^5}{5!} \\
\forall x \in [-2, 2], |x| &\leq 2 \\
|\cos(x) - T_4(x)| &\leq M \frac{2^5}{5!} \\
|\cos(x) - T_4(x)| &\leq M \frac{4}{15}
\end{aligned}$$

$$\begin{aligned}
M &\geq \sup_{[-2,2]} |f^{(m+1)}(x)| \\
\sup_{[-2,2]} |f^{(m+1)}(x)| &= \sup_{[-2,2]} |f^{(5)} \cos(x)| \\
&= \sup_{[-2,2]} |-\sin(x)| \\
&= 1 \\
\therefore M &= 1
\end{aligned}$$

$$|\cos(x) - T_4(x)| \leq \frac{4}{15}$$

Binomial Series

$$\begin{aligned}
(1+x)^m &= \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{m}x^m \\
\binom{n}{k} &= \frac{n!}{k!(n-k)!}
\end{aligned}$$

$$\begin{aligned}
(1+x)^m &= \sum_{n=0}^m \binom{m}{n} x^n \\
&= 1 + \sum_{n=1}^m \frac{k(k-1)(k-2)\dots(k-n+1)}{n!} x^n
\end{aligned}$$

Since for any term greater than m will have a zero factor,

$$(1+x)^k = \sum_{n=1}^{\infty} \binom{k}{n} x^n = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} x^n$$

Example

Use binomial series to find the zero-centered power series for $f(x) = \frac{1}{\sqrt{4+x}}$.

$$\begin{aligned}
\frac{1}{\sqrt{4+x}} &= (4+x)^{-\frac{1}{2}} \\
&= \frac{1}{2} \left(1 + \frac{x}{4}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2} (1+u)^{-\frac{1}{2}}, \quad u = \frac{x}{4} \\
&= \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} u^n\right) \\
&= \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} \frac{x^n}{4^n}\right) \\
&= \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2}) \dots (-\frac{1}{2} - n + 1)}{n!} \frac{x^n}{4^n}\right) \\
&= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5) \dots (2n-1)}{n! 2^n} \frac{x^n}{2^{2n}} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (1)(3)(5) \dots (2n-1)}{n! 2^{3n+1}} x^n
\end{aligned}$$