Surface Area of a Revolution

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The surface area of a rotated shape can be approximated by the frustum of a cone.

$$A_{frustum} = \frac{2\pi(R+r)s}{2}$$

where

- R is the larger radius
- r is the smaller radius
- s is the side length

When integrating, $R = r + dr \Rightarrow R \approx r$, thus $\frac{r+R}{2} \approx r$.

$$SA = \int_I 2\pi r \; ds$$

- When Revolving around the x axis:

$$\int_I 2\pi r \; ds = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2} \; dx$$

- When Revolving around the y axis:

$$\int_I 2\pi r \; ds = \int_a^b 2\pi x\sqrt{1+(f'(x))^2} \; dx$$

Example

Find the surface area of the revolution of the curve $y = x^3$ on the interval $[1, 2]$ about the x-axis.

$$f(x) = x^3$$

$$
\begin{aligned}
\int_1^2 2\pi f(x)\sqrt{1+(f'(x))^2}\ dx &= 2\pi\int_1^2 x^3\sqrt{1+(3x^2)^2}\ dx \\
&= 2\pi\int_1^2 x^3\sqrt{1+9x^4}\ dx \\
u &= 1+9x^4 \\
du &= 36x^3 \\
I &= \frac{1}{18}\pi\int_{10}^{145} u^{\frac{1}{2}}\ dx \\
I &= \frac{1}{18}\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{10}^{145} \\
I &= \frac{1}{27}\pi(145^{\frac{3}{2}}-10^{\frac{3}{2}})
\end{aligned}
$$

Example

Find the surface area of a revolution of the curve $y = x^2$ on the interval $x \in [0,2]$ about the y-axis.

$$f(x) = x^2$$

$$
\begin{aligned}
\int_0^2 2\pi x\sqrt{1+(f'(x))^2}\ dx &= \int_0^2 2\pi x\sqrt{1+(2x)^2}\ dx \\
&= \int_0^2 2\pi x\sqrt{1+4x^2}\ dx \\
u &= 1+4x^2 \\
du &= 8x \\
I &= \frac{\pi}{4}\int_1^{17} u^{\frac{1}{2}}\ dx \\
&= \frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{17} \\
&= \frac{\pi}{6}(17^{\frac{3}{2}}-1)
\end{aligned}
$$

Now in terms of y

$$f(y) = \sqrt{y}, \quad y \in [0,4]$$

$$
\begin{aligned}
\int_0^4 2\pi f(y)\sqrt{1+(f'(y))^2}\ dy &= \int_0^4 2\pi\sqrt{y}\sqrt{1+(\frac{1}{2\sqrt{y}})^2}\ dy \\
&= \int_0^4 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\ dy \\
&= \int_0^4 2\pi\sqrt{y+\frac{1}{4}}\ dy \\
&= \frac{4\pi}{3}[(y+\frac{1}{4})^3/2]_0^4 \\
&= \frac{\pi}{6}(17^{\frac{3}{2}}-1)
\end{aligned}
$$