Hyperbolic Trigonometric Functions

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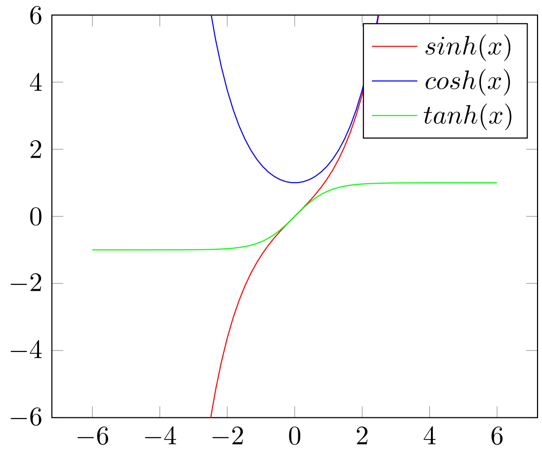
Hyperbolic trigonometric functions describe points on a hyperbola. The graph of a hyperbola satisfies $x^2 - y^2 = 1$ and points on the hyperbola can be described as $(\cosh x, \sinh x)$ in the same way points on a circle can be described as $(\cos t, \sin t)$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$



$$\cosh^2 x - \sinh^2 x = 1$$
$$(\frac{e^x + e^{-x}}{2})^2 - (\frac{e^x - e^{-x}}{2})^2$$
$$\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$
$$\frac{(e^{2x} + 2e^x e^{-x} + e^{-2x}) - (e^{2x} - 2e^x e^{-x} + e^{-2x})}{4}$$
$$\frac{2 - (-2)}{4}$$
$$1$$

Identities

$$\begin{aligned}
\sinh(-x) &= -\sinh x \\
\cosh(-x) &= \cosh x \\
\cosh^2 x - \sinh^2 x &= 1 \\
1 - \tanh^2 x &= \text{sech}^2 x \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y
\end{aligned}$$

$$\begin{aligned}
1 - \tanh^2 x &= \text{sech}^2 x \\
\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} &= \text{sech}^2 x \\
\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} &= \text{sech}^2 x \\
\frac{1}{\cosh^2 x} &= \text{sech}^2 x \\
\text{sech}^2 x &= \text{sech}^2 x
\end{aligned}$$

Derivatives of Hyperbolic Trigonometric Functions

$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \sinh x \\
\frac{d}{dx}(\tanh x) &= \text{sech}^2 x \\
\frac{d}{dx}(\text{csch} x) &= -\text{csch} x \coth x \\
\frac{d}{dx}(\text{sech} x) &= -\text{sech} x \tanh x \\
\frac{d}{dx}(\coth x) &= -\text{csch}^2 x
\end{aligned}$$

$$\begin{aligned}
y &= \sinh x \\
\frac{dy}{dx} &= \frac{d}{dx}(\frac{1}{2})(e^x - e^{-x}) \\
&= (\frac{1}{2})(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}) \\
&= (\frac{1}{2})(e^x - (-e^{-x})) \\
&= (\frac{1}{2})(e^x + e^{-x}) \\
&= \cosh x
\end{aligned}$$

One to one and derivatives

If a function's derivative is always positive or always negative for all points, then it is one-to-one. Since $\frac{d}{dx}\sinh x = \cosh x$, and $\cosh x$ is always positive, sinh is always positive and one-to-one.

Inverse

Inverse of $\sinh x$

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
y &= \frac{e^x - e^{-x}}{2} \\
2y &= e^x - e^{-x} \\
2y &= e^x(1 - e^{-2x}) \\
2ye^{-x} &= 1 - e^{-2x} \\
2ye^{-x} + e^{-2x} &= 1 \\
(e^{-x})^2 + 2y(e^{-x}) - 1 &= 0 \\
e^{-x} &= -y \pm \sqrt{y^2 + 1} \\
x &= -\ln\big(-y \pm \sqrt{y^2+1}\big) \\
x &= -\ln\big(-y - \sqrt{y^2+1}\big)
\end{aligned}$$