Sequences

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Sequences

A sequence is a function from a subset of integers to a set S. Usually, the subset is the set of natural numbers $\mathbb{N}$ or positive integers $\mathbb{Z}^+$. A finite sequence is called a string. Below, the expression on the left denotes a infinite sequence and the expression on the right denotes a finite sequence.

$$\{a_n\} \quad \{a_n\}_{n=A}^{B}$$

- a_n is a term in the sequence
- A is where the sequence begins
- B is where the sequence ends.

Progression

A geometric progression is

$$\{a \cdot r^k\}_{k=0}^{n} = \{a, a \cdot r, a \cdot r^2, \ldots, a \cdot r^n\}$$

A arithmetic progression is:

$$\{a + kd\}_{k=0}^{n} = \{a, a + d, a + 2d, \ldots, a + nd\}$$

Recurrences

A recurrence relation is a sequence where the next term depends on the previous values.

$$a_{n+1} = f(a_n)$$

For a recursive sequence to have a unique closed form formula, there must be initial conditions for the sequence.

Summations

Given a sequence $\{a_k\}_{k=0}^{n}$, the partial sum of the sequence is denoted by S_n.

$$S_n = \sum_{k=0}^{n} a_k$$

The index of the sum can be changes.

$$\sum_{i=m}^{n} a_i = \sum_{i=m+k}^{n+k} a_{i-k} \qquad \sum_{i=m}^{n} a_i = \sum_{i=m-k}^{n-k} a_{i+k}$$

Summations of Geometric Progression

The sum of a geometric progression is:

$$\sum_{k=0}^{n} ar^k$$

If $r = 1$, then

$$\begin{aligned} S_n &= \sum_{k=0}^{n} a \\ &= (n+1)a \end{aligned}$$

If $r \neq 1$, then

$$\begin{aligned} S_n &= \sum_{k=0}^{n} ar^k \\ rS_n &= \sum_{k=0}^{n} ar^{k+1} \\ rS_n &= \sum_{k=1}^{n+1} ar^k \\ &= \left(\sum_{k=1}^{n} ar^k\right) + ar^{n+1} \\ &= \left(\sum_{k=1}^{n} ar^k\right) + ar^{n+1} \\ &= \left(\sum_{k=0}^{n} ar^k\right) + a(r^{n+1} - 1) \\ rS_n &= S_n + a(r^{n+1} - 1) \\ S_n(r-1) &= a(r^{n+1} - 1) \\ S_n &= \frac{a(r^{n+1} - 1)}{r - 1} \end{aligned}$$

Thus,

$$\sum_{k=0}^{n} ar^k = \begin{cases} \frac{a(r^{n+1}-1)}{r-1}, & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

When the sum is to infinity:

$$\begin{aligned}
\sum_{k=0}^{\infty} ar^k &= \lim_{n\to\infty} \sum_{k=0}^{n} ar^k \\
&= \lim_{n\to\infty} \begin{cases} \frac{a(r^{n+1}-1)}{r-1}, & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases} \\
&= \begin{cases} \frac{-a}{r-1}, & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases} \\
&= \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}
\end{aligned}$$

Common Summations

There are a closed form solutions for many summations

$$\begin{aligned}
\sum_{i=1}^{n} c &= cn \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left(\frac{n(n+1)}{2}\right)^2 \\
\sum_{k=0}^{n} ar^k &= \frac{a(r^{n+1}-1)}{r-1} \\
\sum_{k=0}^{\infty} x^k &= \frac{1}{1-x} \\
\sum_{k=1}^{\infty} kx^{k-1} &= \frac{1}{(1-x)^2}
\end{aligned}$$

Double Sum

For a sequence with two indices $\{a_{i,j}\}_{i,j\in\mathbb{Z}}$, the sum is written as a double sum.

$$\sum_i \sum_j a_{i,j}$$

For finite sums, the order of evaluation does not matter and can be exchanged.

$$\sum_i \sum_j a_{i,j} = \sum_j \sum_i a_{i,j}$$

Cardinality

For finite sets, the cardinality is the amount of elements in the set. For infinite set, if there is a bijection from one set to another, then they have the same cardinality. If there is an injection from a set A to a set B, then the cardinality of A is less than or equal to B

$$A \hookrightarrow B \Rightarrow |A| \leq |B|$$

Countable Sets

A set A is countable if either A is finite or there is a bijection $f : A \mapsto \mathbb{Z}^+$. This is equivalent to being able to list the elements of the set, indexed by $\mathbb{Z}^+$.

- Any subset of countable sets is countable.
- The union of two countable sets is countable.

If a set is not countable, then it is called uncountable. The cardinality of a countable set is denoted by $\aleph_0$.

Example 1

The set of odd positive integers A is a countable set.

$$f : \mathbb{Z}^+ \mapsto A,\ \ f(n) = 2n + 1$$

Since f is bijective, the cardinality of A and $\mathbb{Z}^+$ is the same and thus, it is countable.

Example 2

Is $|\mathbb{Z}| = \mathbb{Z}^+$?

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots, n, -n, \ldots\}$$

Since there is an integer indexed list of elements for the set $\mathbb{Z}$, it is a countable set.

Example 3

Prove that the set of real numbers in between zero and one $\mathbb{R}_{(0,1)}$ is an uncountable set.

Assume that (0,1) is countable and $d_{i,j}$ is an digit in base ten

$$\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\
r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\
&\vdots
\end{aligned}$$

Let $r = 0.d_1d_2d_3d_4\ldots$ be a new number such that

$$d_i = \begin{cases} 4, & \text{if } d_{ii} \neq 4 \\ 5, & \text{if } d_{ii} = 4 \end{cases}$$

r cannot be equal to any number in the list since $d_i \neq d_{ii}$. Thus it is not in the list of numbers, therefore, the set of real numbers from zero to one is not countable.