Subspaces

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The dimension of a vector space is the amount of free variables that exist.

Nullspace

For a $n \times m$ matrix A, the set of solutions for $Ax = 0$ is a subspace of $\mathbb{R}^m$ called the nullspace of A, written as $Nul(A)$. The nullspace is a subspace, because it is closed under addition and scalar multiplication.

$$\begin{aligned} Ax &= 0 \\ Ay &= 0 \\ A(x+y) &= 0 + 0 \\ Ax + Ay &= 0 \end{aligned}$$

$$\begin{aligned} Ax &= 0 \\ A(cx) &= cAx \\ &= c0 \\ &= 0 \end{aligned}$$

Span as a Subspace

The span of $v_1 \dots v_k \in \mathbb{R}^n$ is a subspace. Zero is in the span of this set because all the vectors can have coefficients zero.

$$\begin{aligned} (c_1v_1 + \cdots + c_kv_k) + (d_1v_1 + \cdots + d_kv_k) &= (c_1 + d_1)v_1 + \cdots + (c_1 + d_1)v_k \\ c(d_1v_1 + \dots d_kv_k) &= (cd_1) + \cdots + (cd_k)v_k \end{aligned}$$

Columnspace

For a $n \times m$ matrix A, the column space of A is a subspace of $\mathbb{R}^n$ equal to the span of all the columns of A. Column space is written as $Col(A)$.

Transformations Between Vector Spaces

Suppose that V and W are vector spaces and $T : V \mapsto W$ is a function from V to W The transformation T is called a linear transformation from V to W if for all $u, v \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned} T(u+v) &= T(u) + T(v) \\ T(cu) &= cT(u) \end{aligned}$$

Kernel and Range

Suppose that $T : V \mapsto W$ is a linear transformation. The kernel (ker) of a linear transformation is a generation of the nullspace of a matrix for linear transformations that are not matrix transformations. Likewise, the range (rng) is the generalization for the column space. When working with matrices, the nullspace is the same as the kernel and the column space is the same as the range.

$$\begin{aligned} ker(T) &= \{v \in V \mid T(v) = 0\} \\ rng(T) &= \{T(v) \mid v \in V\} \end{aligned}$$

The kernel and range of a linear transformation is a subspace.

$$\begin{aligned} 0 &\in ker(T) \\ u, v &\in ker(T) \\ T(u) + T(v) &= 0 + 0 \\ &= 0 \in ker(T) \\ cT(u) &= c \cdot 0 \\ &= 0 \in ker(T) \end{aligned}$$

$$\begin{aligned} 0 &\in rng(T) \\ T(u), T(v) &\in rng(T) \\ T(u) + T(v) &= T(u+v) \in rng(T) \\ cT(u) &= T(cu) \in rng(T) \end{aligned}$$

Example 1

Find the null space of the following matrix.

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & \frac{5}{4} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} 1x_1 & +0x_2 & +0x_3 & +\frac{3}{2}x_4 & -\frac{1}{2}x_5 & +\frac{3}{2}x_6 & =0 \\ 0x_1 & +1x_2 & +0x_3 & -\frac{3}{4}x_4 & +\frac{1}{4}x_5 & +\frac{5}{4}x_6 & =0 \\ 0x_1 & +0x_2 & +1x_3 & +0x_4 & +1x_5 & +1x_6 & =0 \end{matrix}$$

$$x = \begin{bmatrix} -\frac{3}{2}x_4 + \frac{1}{2}x_5 - \frac{3}{2}x_6 \\ \frac{3}{4}x_4 - \frac{1}{4}x_5 - \frac{5}{4}x_6 \\ -x_5 - x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{4} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} -\frac{3}{2} \\ -\frac{5}{4} \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_6$$

Example 2

Suppose that V is the vector space of all real-valued polynomials.

$$\begin{aligned} T &: V \mapsto V \\ T(p) &= xp \end{aligned}$$

$$\begin{aligned} p, q &\in V \\ T(p+q) &= x(p+q) \\ &= xp + xq \\ &= T(p) + T(q) \\ T(cp) &= x(cp) \\ &= cxp \\ &= cT(p) \end{aligned}$$

$$\begin{aligned} ker(T) &= \{0\} \\ rng(T) &= \text{all polynomials with a zero constant term} \end{aligned}$$

$$D : V \mapsto V$$
$$D(p) = \frac{dp}{dx}$$

$$\begin{aligned} D(p+q) &= \frac{d(p+q)}{dx} \\ &= \frac{dp}{dx} + \frac{dq}{dx} \\ &= D(p) + D(q) \\ D(cp) &= \frac{d(cp)}{dx} \\ &= c\frac{dp}{dx} \\ &= cD(p) \end{aligned}$$

$$\begin{aligned} &ker(D) = \text{constants} \\ &rng(D) = V \end{aligned}$$