

# Parametric Function

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A parametric function are functions describing the position along some axis for a given parameter. Typically, these functions are continuous and the parameter is named  $t$ . A 2D parametric function can be plotted by varying the value of  $t$  and plotting the  $x$  and  $y$  positions. The direction of increasing  $t$  along the curve is called the orientation of the parametrization. This is typically indicated by arrows.

$$x = f(t) \quad y = g(t)$$

Any function  $f : \mathbb{R} \mapsto \mathbb{R}$  can be trivially parameterized as  $x = t$  and  $y = f(t)$

## Isolation

A plot for a parametric function may be plotted more easily by isolating and substituting  $t$ .

## Example

$$x = \ln(t) \quad y = \frac{t^4 + 1}{2t^2}$$

$$x = \ln(t)$$

$$e^x = t$$

$$y = \frac{t^4 + 1}{2t^2}$$

$$= \frac{(e^x)^4 + 1}{2(e^x)^2}$$

$$= \frac{e^{4x} + 1}{2e^{2x}}$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh(2x)$$

## Parametric Differentiation

Parametric functions can be differentiated by dividing the  $y$ -derivative with respect to  $t$  by the  $x$ -derivative with respect to  $t$ . Let  $y = h(x)$ .

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt} h(x) \\ \frac{dy}{dt} &= \left( \frac{d}{dx} h(x) \right) \frac{dx}{dt} \\ \frac{d}{dx} h(x) &= \frac{\left( \frac{dy}{dt} \right)}{\left( \frac{dx}{dt} \right)} \\ \frac{dy}{dx} &= \frac{\left( \frac{dy}{dt} \right)}{\left( \frac{dx}{dt} \right)}\end{aligned}$$

- if  $\frac{dy}{dt}$  is zero and  $\frac{dx}{dt}$  is not zero, it is a horizontal tangent.
- if  $\frac{dx}{dt}$  is zero and  $\frac{dy}{dt}$  is not zero, it is a vertical tangent.

## Higher Derivatives

$$\begin{aligned}\frac{d}{dt} \left( \frac{dy}{dx} \right) &= \frac{d}{dx} \left( \frac{dy}{dx} \right) \cdot \frac{dx}{dt} \\ &= \frac{d^2 y}{dx^2} \frac{dx}{dt}\end{aligned}$$

Thus

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} \\ \frac{d^n y}{dx^n} &= \frac{\frac{d}{dt} \frac{d^{n-1} y}{dx^{n-1}}}{\frac{dx}{dt}}\end{aligned}$$

## Parametric Integration

Parametric functions can be integrated as follows.

$$\int_a^b f(x) \, dx = \int_{t_0}^{t_1} y(t) \frac{dx}{dt} \, dt$$

### Example

Integrate the following parametric function on the interval  $x \in [0, \pi]$

$$x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t)$$

$$x = 0 \Rightarrow t = 0$$

$$x = \pi \Rightarrow t = \pi$$

$$\frac{d}{dt}x(t) = 1 - \cos(t)$$

$$\begin{aligned} \int_0^\pi y(t) \left( \frac{dx}{dt} \right) dt &= \int_0^\pi (1 - \cos t)(1 - \cos t) dx \\ &= \int_0^\pi 1 - 2 \cos t + \cos^2 t dx \\ &= \int_0^\pi 1 - 2 \cos t + \frac{1}{2}(1 + \cos 2t) dx \\ &= \left[ \frac{3}{2}t - 2 \sin t + \frac{1}{4} \cos 2t \right]_0^\pi \\ &= \frac{3}{2}\pi \end{aligned}$$

## Arclength

$$\int_c ds = \int_{t_0}^{t_1} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt$$

## Surface Area of Revolution

about the x-axis

$$\int_I 2\pi r ds = \int_{t_0}^{t_1} 2\pi y(t) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt$$

about the y-axis

$$\int_I 2\pi r ds = \int_{t_0}^{t_1} 2\pi x(t) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt$$