Dimensions

Patrick Chen

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Dimension

Suppose that V has a finite spanning set. Every basis of V is finite and has the same size. The size of any such basis is called the dimension of V, written $\dim(V)$. If V does not have a finite spanning set, then V is said to be infinite dimensional.

Suppose $B = \{v_1 \ldots v_n\}$ is a basis for V. Take $w_1 \ldots w_k$ is a linearly independent set in V.

$$[w_1]_B, \ldots, [w_k]_B \in \mathbb{R}^n$$

Thus k must be less than or equal to n because w is linearly independent.

$$c_1[w_1]_B + \ldots c_k[w_k]_B = 0$$

1. A set of more than $\dim(V)$ vectors cannot be linearly independent

2. A set of less than $\dim(V)$ cannot be a spanning set.

3. Every linearly independent set of size $\dim(V)$ is a basis of V

4. Every spanning set of size $\dim(V)$ is a basis of V

5. Every subspace of V has a dimension less than or equal to $\dim(V)$

The dimension of a subspace of a vector space is less than or equal to the dimension of the parent space.

Coordinates as Linear Transformations

Suppose that $B = \{v_1 \ldots v_n\}$ is a basis for V. The map $T : V \mapsto \mathbb{R}^n$ given by $T(v) = [v]_B$ is a linear transformation which is one-to-one and onto. These types of linear transformations are called isomorphisms. Two vector spaces that are isomorphisms are indistinguishable in terms of vector spaces. Every basis with the same dimension is isomorphic to each other.

Dimension of different spaces

The dimension of the column space is the rank of the matrix. The dimension of the nullspace is the nullity of the matrix. The sum of the rank and nullity is the number of columns of the matrix.

$$A : m \times n$$
$$rank(A) + nullity(A) = n$$

The dimension of the column space is the amount of pivots columns of a matrix. The size of the basis of the nullspace is the amount of free variables in the reduced row echelon form of the matrix. Since a column in the reduced row echelon can only have a pivot or be a free variable, the amount of columns is the sum of the rank and nullity of the matrix.

$$dim(Col(A)) = dim(Row(A))$$

Since the column space is the columns that contain the pivots of A and the row space has a basis of the non-zero rows in the reduced row echelon form of A, their dimensions will be equal.

Example 1

Find the dimension of: vectors with n components, polynomials with degree less than or equal to n, matrices of size 2×2, matrices of size 3×6, matrices of size 6×3.

1. $\dim(\mathbb{R}^n) = n$

2. $\dim(P_n) = n + 1$

3. $\dim(M_{2,2}) = 4$

4. $\dim(M_{3,6}) = 18$

5. $\dim(M_{6,3}) = 18$