

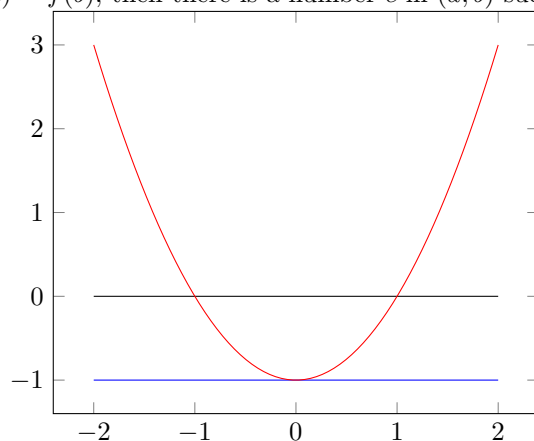
# Rolle's Theorem, Mean Value Theorem, and Graphing

Patrick Chen

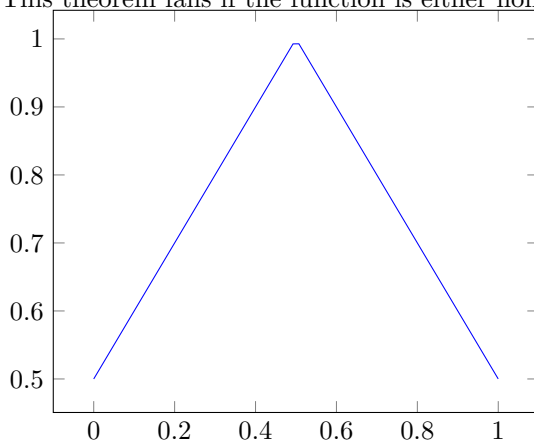
Oct 7, 2024

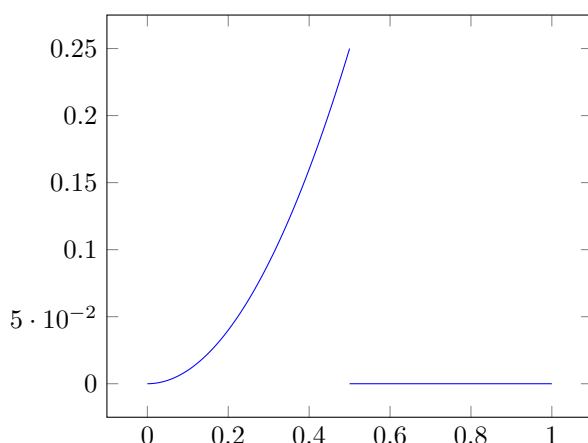
## Rolle's theorem

if  $f$  is continuous on the closed interval  $[a, b]$ ,  $f$  is differentiable on the open interval  $(a, b)$ , and  $f(a) = f(b)$ , then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$



This theorem fails if the function is either non-differentiable or not continuous.





### Example

Prove that  $x^3 + x - 1 = 0$  has exactly one root.

To do this, we must prove that the function has at least one root, then show that there exists no more than one root.

$$f(0) = 0^3 + 0 - 1$$

$$f(0) = -1$$

$$f(1) = 1^3 + 1 - 1$$

$$f(1) = 1$$

Since polynomial functions are continuous, by the intermediate value theorem, there must exist a root in the interval  $(0, 1)$ .

To prove that there does not exist two roots  $x_1, x_2$ , assume there are two roots. By Rolle's theorem, there exists a point  $c$  where  $f'(c) = 0$ . Since the derivative is never zero, this is a contradiction and there cannot be two roots.

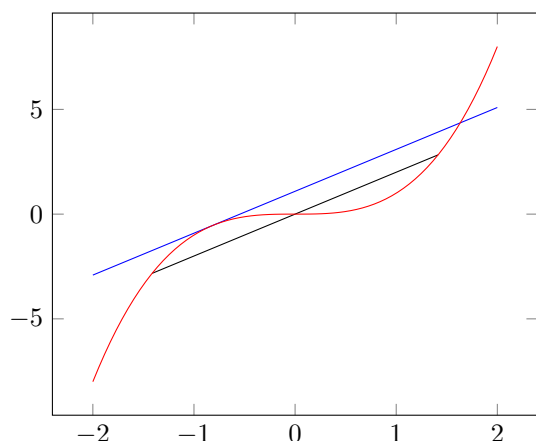
$$f(x) = x^3 + x - 1$$

$$f'(x) = 3x^2 + 1$$

## Mean Value Theorem

For a function  $f$  that is continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ , then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

In other words, if a function is continuous and differentiable on an interval, then there is a point in that interval where the slope is equal to the secant line formed by the two endpoints. The mean value theorem is a generalized version of Rolle's theorem.



### Example

If  $f(0) = -3$  and  $f'(x) \leq 5$ , for all  $x$ , how large can  $f(2)$  possible be. Using the interval  $[0, 2]$ :

$$\begin{aligned}
 f'(c) &= \frac{f(b) - f(a)}{b - a} \\
 &= (f(2) - 3)/(2 - 0) \\
 &= (f(2) + 3)/2 \\
 f(2) &= 2f'(c) - 3 \\
 f(2) &\leq 2(5) - 3 \\
 f(2) &\leq 7
 \end{aligned}$$

### Derivative information

- if  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval
- if  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval

For a continuous function  $f$  and a critical value  $c$

- if  $f'$  changes from positive to negative at  $c$ , then  $f$  has local max at  $c$ .
- if  $f'$  changes from negative to positive at  $c$ , then  $f$  has local min at  $c$ .
- if  $f'$  is positive to left and right of  $c$ , or negative to left and right of  $c$ , then  $f$  has no local maximum or minimum.

A function  $f$  is called concave upward on an interval  $I$  if all tangents lies below the function on the interval  $I$ .

- If  $f''(x) > 0$  on an interval  $I$ , then the graph of  $f$  is concave upward on  $I$
- If  $f''(x) < 0$  on an interval  $I$ , then the graph of  $f$  is concave downward on  $I$

A point  $(x, y)$  on a curve  $y = f(x)$  is called an inflection point if  $f$  is continuous there and the curve changes from concave up to concave down or from concave down to concave up at  $x$ . Just like the first derivative, if  $f''(x) = 0$ , then there is not always a inflection point.

- If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $c$ .
- If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $c$ .

### Example: Graphing

$$f(x) = x^{2/3}(6-x)^{1/3}$$

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$

$$f''(x) = -\frac{8}{x^{4/3}(6-x)^{5/3}}$$

$$f(0) = 0$$

$$f(x) = 0 \Rightarrow x = 0, 6$$

$f'$  is zero when  $x = 4$  and  $f'$  is undefined when  $x = 0, 6$ .

	$x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$4 < x < 6$	$x = 6$	$6 < x$
f	+	0 $m = \infty$	+	+ local max	+	0 $m = \infty$	-
f'	-	DNE	+	0	-	DNE	-
f''	-	DNE	-	-	-	DNE	+

