A bound on the m-triviality of knots
Part II: Computational evidence

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Abstract

In [A. Carmi and E. Cohen, *A bound on the m-triviality of knots Part I: Jones-Vassiliev unknot detection*, arXiv:...] we established a uniform barrier for Jones-Vassiliev truncations: if a nontrivial knot K is J_m-trivial (its Birman–Lin layers $c_1, \ldots, c_{m-1}$ vanish), then $m \leq \mathscr{N}(K)$, with the proof passing through extended shadows, m-invisible crossing flips via clasper calculus, and a symbol-level skein that forces a last-row fingerprint for c_0 and a shadow criterion (property (P)) [Thm. 4.10, Cor. 4.11, Thm. 5.1]. As a corollary, the Jones polynomial detects the unknot [Thm. 5.3]. In this companion note we provide large-scale computational evidence for the barrier and uncover a sharper empirical cap. Using a finite packaging of the Birman-Lin layers, the Jones-Vassiliev polynomial (JVP), realized in the rank-two $\mathbb{Z}[p]$-module $\mathbb{Z}[p][x]/(x^2-px-1)$, we convert tabulated Jones polynomials $V_K(t)$ to the finite-type layers a_q, b_q (and $c_q = a_q + b_q$) by the substitutions $t^{1/2} = x$, $t^{-1/2} = x - p$, followed by the reduction $x^2 = px + 1$. Applying this to all 352,152,252 prime knots with $\mathscr{N} \leq 19$ (Knot Atlas for $\mathscr{N} \leq 10$, Dartmouth for $11 \leq \mathscr{N} \leq 19$), we find no violation of the uniform barrier and, moreover, observe the stricter inequality

$$m \leq \left\lfloor \frac{\mathscr{N}(K)}{2} \right\rfloor + 1 \quad (< \mathscr{N}(K))$$

throughout this range. We also validate, on model cubes, the symbol-skein recursion and the last-row fingerprint mechanism that underpins the theoretical obstruction, and we record min-laws under connected sum within the JVP framework.

1 Introduction

Part I. In Part I of this work [1] we proved a diagram-local obstruction to prolonged Jones-Vassiliev triviality. Decorating a fixed shadow by degree-m_j claspers makes the j-th crossing flip invisible below order m_j; organizing these m-invisible flips with a weighted GH filtration produces m-flat families on which the Birman-Lin layers $c_1, \ldots, c_{m-1}$ are constant. A symbol-level skein for the layers then forces a last-row fingerprint for the free term c_0: smoothing any prescribed $m-1$ crossings yields exactly m components. The shadow lemma converts this fingerprint into global nugatoriness when m exceeds the number of varied crossings, giving the uniform barrier $m \leq \mathscr{N}(K)$ and, as a consequence, Jones detection of the unknot.

Part II. Here we stress-test the barrier at scale and provide a computational companion to the symbol calculus developed in Part I. Our goals are threefold.

1. **A finite, computable packaging of the Birman-Lin layers.** We introduce the Jones-Vassiliev polynomial (JVP): in the ring $R[x]/(x^2 - px - 1)$ with $R = \mathbb{Z}[p]$, every Jones polynomial $V_K(x)$ admits a unique finite expansion

$$V_K(x,p) = \sum_{q \geq 0} \big(a_q(K) + b_q(K)\, x\big)p^q,$$

where a_q, b_q are Vassiliev invariants of order $\leq q$, and $c_q := a_q + b_q$ matches the Birman-Lin h-layer up to a unitriangular change of variables [Lem. 3.4, Prop. 3.5, Lem. A.1]. Concretely, we read (a_q, b_q) directly from tabulated $V_K(t)$ by setting $t^{1/2} = x$, $t^{-1/2} = x - p$, and reducing via $x^2 = px + 1$; the p-adic order of $V_K - 1$ is then exactly the J_m-triviality index (the bandwidth) of K [Defs. 3.1 & 3.4].

2. **A comprehensive census up to 19 crossings.** Using the above pipeline, we computed (a_q, b_q) for all 352,152,252 prime knots with $\mathscr{N} \leq 19$ (Knot Atlas for $\mathscr{N} \leq 10$; Dartmouth for $11 \leq \mathscr{N} \leq 19$). Across this range we saw no counterexample to $m \leq \mathscr{N}(K)$ and consistently observed the stricter empirical cap

$$m \leq \left\lfloor \tfrac{\mathscr{N}(K)}{2} \right\rfloor + 1.$$

Figure 3 lists first occurrences by m, while Figure 4 plots the likelihood $P_m(\mathscr{N}) = \Pr(\mathscr{N}(K) \leq \mathscr{N} \mid K \text{ is } J_m\text{-trivial})$, making the uniform barrier visible as a sharp cutoff and suggesting an equipartition behavior of J_m-classes at large $\mathscr{N}$ [§4.1]. A heuristic for the halving phenomenon is that Jones-m-flatness and J_m-triviality are mirror-invariant, so flat subcubes occur in mirror pairs, effectively cutting the obstruction degree in half [§4].

3. **Symbol-skein checks and structural laws.** We verify computationally the symbol-level skein and the last-row fingerprint that power Part I. Appendix B constructs the symbol matrix on trefoil cubes and their residual cubes, reproducing the predicted $(-2)^r$ pattern along the bottom row after smoothing r crossings. We also record a clean min-law for the bandwidth under connected sum, $m(K_0 \# K_1) = \min\{m(K_0), m(K_1)\}$, which interacts well with the graded structure captured by c_m [§4.2]. These checks mirror the theoretical mechanisms of Part I: symbol recursion [cf. (20)/(21) there] and the shadow-level property (P) triggered by last-row data [Thm. 4.10 there].

Beyond confirming the barrier, the data suggest the conjectural refinement $m \leq \lfloor \mathscr{N}/2 \rfloor + 1$. In particular, for each $m > 1$ we exhibit prime J_m-trivial knots with $\mathscr{N}$ linear in m (via clasperian closures of Brunnian tangles), and, up to 19 crossings, the earliest examples align with the empirical cap [Prop. 4.1; Fig. 2]. The distribution functions $P_m(\mathscr{N})$ rise almost in parallel (log scale), indicating comparable prevalence of J_m-classes as $\mathscr{N}$ grows [Fig. 4].

Parts I and II. Readers primarily interested in the logic, symbol recursion $\Rightarrow$ last-row fingerprint $\Rightarrow$ property (P) $\Rightarrow$ barrier, will find the proofs in Part I, while the present paper shows that the same fingerprints are readily computable through the JVP and are seen ubiquitously in data. The two parts are thus complementary: Part I supplies the mechanism and theorems; Part II supplies the numerical landscape and practical tools for checking J_m-triviality on massive knot families.

1.1 Organization of this note

Section 2 recalls the finite-type/Boolean-cube formalism needed for computation. Section 3 defines the JVP, proves that its coefficients are finite type and explains how to extract them from $V_K(t)$;

it also records skein and product rules compatible with the Part I symbol calculus. Section 4 reports the main census: first occurrences by m, the empirical cap, distribution curves, and structural consequences (min-law, filtration point of view). Appendix A contains proofs and additional supporting results. Appendix B contains worked examples (including a trefoil symbol matrix) that make the last-row fingerprint and symbol-skein concrete. Together, these results supply the large-scale empirical substrate for the obstruction mechanism developed in Part I.

1.2 Conventions

Convention 1.1 (Conventions used throughout the text)**.**

- *A* knot shadow *(shadow for short) means the 4-regular plane graph obtained from a knot diagram by forgetting over/under information, and a crossing flip means changing the over/under choice at one vertex of the shadow.*
- *Over/under crossings are denoted as '+' and '−', and sometimes, when encoded by cube vertices, as* $+1$ *and* -1*.*
- *Smoothing a crossing always refers to* oriented *smoothing. Following the above crossing sign conventions, it is denoted as '0'.*
- $\mathscr{N}(K)$ *and* $\mathrm{cross}(K)$ *denote, respectively,* K*'s number of crossings and set of crossing indices. The number of crossings may always be that of a minimal representation of* K *in which case it is a knot invariant aka the crossing number.*
- *Let* S *be a fixed shadow with outside crossing set* $\mathrm{cross}(S)$*, and let* $D \in \mathcal{D}(S)$ *be a diagram carried by* S*. For a subset* $J \subseteq \mathrm{cross}(S)$*, we write* $D \setminus J$ *for the diagram obtained from* D *by performing the* oriented smoothing *at every crossing in* J *("0" smoothing). This is a concise form of Convention 3.1.*
- *A* C_k *tree (a degree-k tree clasper) is the embedded surface made of nodes, edges, and leaves. A* C_k *move is surgery along a connected* C_k *tree. Two links are* C_k*-equivalent if they are related by a sequence of* C_k *moves. A* C_k *move preserves every finite-type invariant of degree* $< k$ *[2, 3].*
- ***GH*** *refers to Goussarov-Habiro.*

2 Preliminaries

2.1 Finite-type invariants

Fix the category $\mathcal{K}$ of oriented links in S^3, considered up to ambient isotopy. Let $\mathcal{K}^{(d)}$ denote the set of (immersed) oriented singular links with at most d transverse double points and no other singularities. Let A be an abelian group (typically $A = \mathbb{Z}$ or $\mathbb{Z}[p]$).

Definition 2.1 (Vassiliev extension and order)**.** *A function* $v : \mathcal{K} \to A$ *is finite type of order* $\leq d$ *if it admits an extension* $\tilde{v} : \mathcal{K}^{(d)} \to A$ *determined inductively by the Vassiliev skein rule*

$$\tilde{v}\,(\textit{double point}) = \tilde{v}\,(\textit{positive}) - \tilde{v}\,(\textit{negative})$$

at each double point, and satisfies $\tilde{v}(L) = 0$ *whenever* L *has* $d{+}1$ *double points. The order (or type) of* v *is the least such* d*.*

Definition 2.2 (Symbol / weight system). *For v of order $\leq d$, its d-th symbol $\sigma_d(v)$ is the restriction of $\tilde{v}$ to $\mathcal{K}^{(d)}$ modulo the skein relations, equivalently a function on chord diagrams with d chords. The symbol factors through the 1T/4T relations, so $\sigma_d(v)$ defines a linear functional on the degree-d diagram space $\mathcal{A}_d$ (Bar-Natan).*

2.2 Pseudo-Boolean functions and finite-type invariants

Shadows and cubes. Fix a shadow S with outside crossing set $I = \mathrm{cross}(S)$, $|I| = n$. The associated *variation cube* is

$$\mathcal{Q}(S) \;=\; \{\varepsilon = (\varepsilon_i)_{i\in I} \mid \varepsilon_i \in \{\pm 1\}\} \;\cong\; \{\pm 1\}^n. \tag{1}$$

For each $\varepsilon \in \mathcal{Q}(S)$ let $D(\varepsilon) \in \mathcal{D}(S)$ denote the diagram carried by S with over/under choices encoded by ε (Convention 1.1).

Pseudo-Boolean functions on a fixed shadow. Let A be an abelian group (typically $\mathbb{Z}$ or $\mathbb{Z}[p]$). A *pseudo-Boolean function on the cube of S* is any map

$$f_S : \mathcal{Q}(S) \longrightarrow A, \qquad \varepsilon \longmapsto f_S(\varepsilon).$$

Given an ambient-isotopy invariant $v : \mathcal{K} \to A$, we obtain its restriction to the cube of S by evaluation at the vertices:

$$f_v^S(\varepsilon) \;:=\; v\big(D(\varepsilon)\big), \qquad \varepsilon \in \mathcal{Q}(S).$$

Thus, every knot/link invariant supplies, for each choice of shadow, a canonical pseudo-Boolean function on $\mathcal{Q}(S)$.

Faces and $\mathscr{U}$-subcubes. For $\mathscr{U} \subseteq I$ and a base state $\varepsilon^0 \in \mathcal{Q}(S)$ the *$\mathscr{U}$-subcube through ε^0* is

$$\mathcal{Q}_{\mathscr{U}}(\varepsilon^0) \;:=\; \{\varepsilon \in \mathcal{Q}(S) :\; \varepsilon_i = \varepsilon_i^0 \text{ for all } i \notin \mathscr{U}\}.$$

We freely regard alternating sums over such faces as *finite differences* (discrete derivatives).

Polynomial-valued invariants and coefficient layers. Let R be a commutative ring and let $J : \mathcal{K} \to R[p]$ be a polynomial-valued invariant,

$$J(K;p) \;=\; \sum_{m\geq 0} f_m(K)\, p^m, \qquad f_m : \mathcal{K} \to R.$$

On a fixed shadow S the coefficient functionals induce a family of pseudo-Boolean functions

$$f_{m,S}(\varepsilon) \;:=\; f_m\big(D(\varepsilon)\big), \qquad \varepsilon \in \mathcal{Q}(S),\; m \geq 0.$$

In particular, when J is the Jones–Vassiliev expansion (introduced in §3.1), we will write $c_{q,S}(\varepsilon) = c_q\big(D(\varepsilon)\big)$ for its q-th coefficient layer on $\mathcal{Q}(S)$.

Connected sums and product cubes. If a diagram $D(\varepsilon)$ splits as a connected sum $D_0(\varepsilon^0)\#D_1(\varepsilon^1)$ realized in disjoint disks, then the shadow cube factors as a Cartesian product $\mathcal{Q}(S) \cong \mathcal{Q}(S_0) \times \mathcal{Q}(S_1)$. When J is multiplicative under connected sum,

$$J_{K_0\#K_1}(p) \;=\; J_{K_0}(p)\, J_{K_1}(p),$$

the coefficient layers satisfy the Cauchy rule at each vertex $(\varepsilon^0, \varepsilon^1)$:

$$f_{m,S}(\varepsilon^0, \varepsilon^1) \;=\; \sum_{i=0}^{m} f_{i,S_0}(\varepsilon^0)\, f_{m-i,S_1}(\varepsilon^1). \tag{2}$$

2.3 Spectral representation

The Boolean cube-complex formalism enjoys a number of advantages. The Vassiliev skein relation is realized by a boundary operator, ∂_i, contracting/projecting along the i-th dimension. The type or order of the Vassiliev invariant becomes the *polynomial degree* of the pseudo-Boolean function. Beyond the observation that this may explain why polynomial invariants are native objects for encoding combinatorial finite-type data, we have tools from spectral analysis at our disposal.

Fourier basis and Walsh characters. Let $f : \{\pm 1\}^n \to \mathbb{C}$ be a pseudo-Boolean function. For every subset $S \subseteq [n]$, the Walsh character is given by,

$$\chi_S(\varepsilon) = \prod_{i \in S} \varepsilon_i \quad (\chi_\varnothing \equiv 1). \tag{3}$$

These characters form an orthonormal basis (with respect to the uniform measure on $\mathcal{Q}_n$), so

$$f(\varepsilon) = \sum_{S \subseteq [n]} \widehat{f}(S)\, \chi_S(\varepsilon), \qquad \widehat{f}(S) = 2^{-n} \sum_{\varepsilon} f(\varepsilon)\, \chi_S(\varepsilon). \tag{4}$$

is the Walsh-Fourier expansion of f. The polynomial degree, $\deg(f)$, matches the largest $|S|$ with $\widehat{f}(S) \neq 0$.

Discrete (finite) derivatives. For coordinate i we define the (normalized) derivative

$$(\partial_i f)(\varepsilon) \equiv \frac{1}{2}\big(f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)})\big), \tag{5}$$

where $\varepsilon^{(i,\pm)}$ is ε with the i-th entry set to ± 1. For a subset $T \subseteq [n]$,

$$\partial_T = \prod_{i \in T} \partial_i, \qquad (\partial_T f)(\varepsilon) = 2^{-|T|} \sum_{\sigma \in \{\pm 1\}^T} (-1)^{\#\{\sigma_i = -1\}}\, f(\varepsilon_{[n]\setminus T}, \sigma). \tag{6}$$

Equivalently, it is the alternating sum of f over the T-subcube through ε. For any fixed subset $T \subseteq [n] = \{1, \ldots, n\}$ the T-fold discrete derivative acts on Walsh characters by "dropping" the coordinates in T:

$$\partial_T\, \chi_S = \begin{cases} \chi_{S \setminus T}, & T \subseteq S, \\ 0, & T \nsubseteq S. \end{cases} \tag{7}$$

Because the characters $\{\chi_S\}_{S \subseteq [n]}$ form an orthonormal basis, we obtain the Fourier (Walsh) expansion of the derivative simply by applying this rule term-wise to the expansion of f:

$$(\partial_T f)(\varepsilon) = \sum_{[n] \supseteq U \supseteq T} \widehat{f}(U) \chi_{U \setminus T}(\varepsilon). \tag{8}$$

Thus, the Fourier coefficient of $\partial_T f$ at character χ_S (with $S \cap T = \varnothing$) is precisely the coefficient of f at the larger character $\chi_{S \cup T}$, namely, $\widehat{\partial_T f}(S) = \widehat{f}(S \cup T)$. This explicit form makes transparent how derivatives shift weight "down" the spectrum and why $\partial_T f$ vanishes whenever f has no Fourier support on sets containing T (the finite-type condition).

Remark 2.1 (Normalized vs. unnormalized derivatives). *The standard Fourier/Walsh framework is formulated using the normalized difference operator* (5)*. The Vassiliev skein, on the other hand, adopts the unnormalized version, namely,*

$$(\partial_i f)(\varepsilon) \equiv f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)}). \tag{9}$$

At times, this convention alleviates the burden of dealing with factors such as 2^{-m} *when taking derivatives over cube m-faces. For that reason, and for being consistent with the standard theory, unless stated otherwise,* ∂_i *means* (9) *for the rest of this note.*

Connected sum and product rules. The variation cube encoding connected sums of knots, e.g. $K_0 \# K_1$, is given by a cartesian product, $\mathcal{Q} \cong \mathcal{Q}_0 \times \mathcal{Q}_1$. The set of coordinates S underlying $\mathcal{Q}$ can thus be partitioned into two subsets associated with the individual cubes, $S = S_0 \cup S_1$. This facilitates composing the discrete derivatives of any pseudo-Boolean function as,

$$\partial_{T_0 \cup T_1} = \partial_{T_0} \partial_{T1}, \ \ \forall T_0 \subseteq S_0, \, \forall T_1 \subseteq S_1.$$

If, in addition, (2) holds, then

$$\partial_{T_0 \cup T_1} f_m(\varepsilon_{K_0 \# K_1}) = \sum_{i=0}^{m} \left[\partial_{T_0} f_i(\varepsilon_{K_0})\right] \left[\partial_{T_1} f_{m-i}(\varepsilon_{K_1})\right]. \tag{10}$$

Polynomial degree equals Vassiliev order. A pseudo-Boolean invariant f is said to be *finite type of (Vassiliev) order d* if

$$\partial_T f \equiv 0 \quad \text{for every } T \subseteq [n] \text{ with } |T| > d. \tag{11}$$

Intuitively, all $(d+1)$-dimensional faces evaluate to alternating sums of zero. The derivative representation immediately implies

$$\widehat{f}(S) = 0 \quad \text{whenever } |S| > d. \tag{12}$$

Conversely, if $\widehat{f}(S) = 0$ for $|S| > d$, then every $\partial_T f$ with $|T| > d$ vanishes.

$$\mathrm{Type}(f) = \deg(f) = \max\{|S| : \widehat{f}(S) \neq 0\}. \tag{13}$$

The derivative operator not only detects type but produces new finite-type functions of *singular* knots: Fix a face defined by coordinates T and take the alternating sum of f along that face. The result is $\partial_T f$ restricted to the complementary coordinates. If f is type d, then $\partial_T f$ is type $d - |T|$. Iterated extraction of "principal parts."

- Order-1 part: $g_1(\varepsilon) = \sum_{|S|=1} \widehat{f}(S) \chi_S(\varepsilon)$.
- Higher parts analogously via derivatives: the map $T \mapsto \partial_T f(1, \ldots, 1)$ lists exactly the Fourier coefficients of size $|T|$.

This is the discrete analogue of repeatedly differentiating a polynomial and evaluating at the origin to read off coefficients.

Concept	Description
Boolean variation cube	Encodes all simultaneous crossing variations; faces are natural domains for alternating-sum operators.
Walsh-Fourier expansion	Decomposes any pseudo-Boolean invariant into characters labelled by variation subsets.
Top Walsh-Fourier block	$\partial_{[m]}$ where m is the polynomial degree; the symbol.
Discrete derivatives	Are face-wise alternating sums; in Fourier space they simply drop indices.
Finite-type order d	Polynomial/Fourier degree $d \leftrightarrow$ vanishing of $(> d)$-fold derivatives.
Derivatives over subcubes	Isolate the homogeneous pieces and systematically generate lower-order invariants of singular knots.

Table 1: Pseudo-Boolean functions and finite-type invariants. Glossary of key concepts.

Symbols, cubes, and chords. In finite-type theory the m-th symbol is an invariant of exactly order m. That is, it is the restriction of the finite-type function to chord diagrams with m chords, or equivalently, is the m-th graded piece in the GH algebra (Definition 2.2). By definition, the symbol is independent of any particular variation of the singular crossings, i.e., it is fixed for a given cube. Over an m-dimensional variation cube it is extracted by the (unnormalized) m-fold discrete derivative and hence equals (up to a normalization factor) the top Walsh-Fourier block of the underlying pseudo-Boolean function,

$$\partial_{[m]} f = 2^m \hat{f}([m]).$$

The symbols factorize through the graded space $\mathcal{A}_m$ and hence depend only on the chord diagram class (modulo the 1T/4T relations) and not on the particular placement of singularities. The pseudo-Boolean framework offers concise descriptions of these local relations which consequently bridge the cubical definition of a symbol with the algebra of chord diagrams.

3 Jones-m-triviality

3.1 Birman-Lin finite-type expansion of the Jones polynomial

We start from the Jones skein

$$x^{-2}V^{(+)} - x^2 V^{(-)} = (x - x^{-1})V^{(0)}. \tag{14}$$

and pass to the Birman–Lin substitution $x = e^{h/2}$. This reorganizes V_L into a power series

$$V_L(e^{h/2}) = \sum_{n \geq 0} c_n(L)\, h^n, \tag{15}$$

whose coefficients c_n are Vassiliev invariants of order $\leq n$ [4].

Definition 3.1 (Jones m-trivial, J_m-trivial)**.** *A knot K is J_m-trivial if its n-type coefficients $c_n(K) = 0$ for $1 \leq n \leq m-1$.*

Remark 3.1. *Note that if V_K has a h-gap of order m:*

$$V_K(e^{h/2}) = 1 + h^m V_K'(h), \tag{16}$$

then it is J_m-trivial.

Remark 3.2 (Every knot is J_m-trivial for some m)**.** *Every non-trivial knot is J_m-trivial for some $m \geq 2$ (as $c_1 = 0$ for knots). The unknot is J_m-trivial for any m.*

Remark 3.3. *Standard m-triviality (GH) implies J_m-triviality, because it forces every degree $\leq m-1$ finite-type invariant to agree with the unknot, hence the coefficients c_n vanish for $1 \leq n \leq m-1$.*

In the Walsh-Fourier viewpoint each coefficient layer becomes a pseudo-Boolean function on a fixed shadow cube, so facewise differences isolate orders and symbols. Two immediate consequences drive what follows: c_0 is a shadow invariant encoding component count, and the skein differentiates the layers, giving a symbol-level recursion.

Lemma 3.1 (Order-0 coefficient c_0)**.** *The order-0 coefficient in* (15) *encodes the number of linked components:*

$$c_0(L) = (-2)^{\ell(L)-1}.$$

Proof. This follows from $V_L(1) = (-2)^{\ell(L)-1}$. □

Lemma 3.2 (Finite-type coefficients skein relation)**.** *For every $n \geq 1$ one has*

$$\partial c_n = c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0) \; + \; R_{n-2} \,, \tag{17}$$

where the remainder R_{n-2} is an explicit linear combination of strictly lower Vassiliev orders ($\leq n-2$):

$$\begin{aligned} R_{n-2} = -\sum_{r\geq 1} \frac{1}{(2r)!}\, \partial c_{n-2r} \; + \sum_{r\geq 1} \frac{1}{(2r+1)!} \big(c_{n-2r-1}(+) + c_{n-2r-1}(-)\big) \\ + \sum_{r\geq 1} \frac{1}{2^{\,2r}(2r+1)!}\, c_{n-2r-1}(0). \end{aligned} \tag{18}$$

Convention 3.1 (Smoothing notation $D \setminus J$)**.** *Let S be a fixed shadow with outside crossing set* $\mathrm{cross}(S)$*, and let $D \in \mathcal{D}(S)$ be a diagram carried by S. For a subset $J \subseteq \mathrm{cross}(S)$, we write $D \setminus J$ for the diagram obtained from D by performing the* oriented smoothing *at every crossing in J (our "0" smoothing). Thus:*

- *$D \setminus \emptyset = D$, and for a singleton $\{i\}$ we have $D \setminus \{i\} = D(i = 0)$.*
- *Smoothing at distinct crossings is local and order–independent, so $D \setminus J$ is well defined (independent of any ordering of J).*
- *If $J \subseteq \mathrm{cross}(S)$ and $S \setminus J$ denotes the shadow with those vertices smoothed, then the* residual cube *$\mathcal{Q}_{S\setminus J}$ is the variation cube on the remaining crossings; functions of the form $D \mapsto f(D \setminus J)$ are naturally viewed on $\mathcal{Q}_{S\setminus J}$.*
- *We occasionally write $J_r \subseteq J$ to mean an arbitrarily chosen r-subset of J; by order independence, statements about $D \setminus J_r$ do not depend on the choice of J_r.*
- *In single-crossing skeins we retain $D(+), D(-), D(0)$; the set-notation agrees with this: $D \setminus \{i\} = D(i = 0)$.*

Definition 3.2. *For a shadow S and $J \subset \mathrm{cross}(S)$, the* ***residual cube*** *$\mathcal{Q}_{S\setminus J}$ is the variation cube of over/under choices on the remaining crossings after smoothing all J.*

Definition 3.3 (Set-indexed symbols). *For any $K \subset I$ and $J \subset I \setminus K$, define*

$$\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$$

i.e. first perform oriented smoothings at all crossings in K, then take the unnormalized alternating sum over the J-face (order $|J|$).

Convention 3.2 (Set/cardinality-indexed symbols). *From now on we write $\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$ to record both the smoothed subset K and the differenced subset J explicitly. If only the cardinalities matter, we write $\mathscr{S}_{r,n}$ for the class generated by all $\mathscr{S}_{K;J}$ with $|K| = r, |J| = n$.*

Lemma 3.3 (Symbol-skein recursion). *Fix a crossing i. For every $n \geq 1$,*

$$\mathscr{S}_{\varnothing;J} \;=\; 2\,\mathscr{S}_{\varnothing;J\setminus\{i\}} \;+\; \mathscr{S}_{\{i\};J\setminus\{i\}} \qquad (i \in J,\ |J| = n). \tag{19}$$

and more generally,

$$\mathscr{S}_{K;J} = 2\,\mathscr{S}_{K;J\setminus\{i\}} + \mathscr{S}_{K\cup\{i\};J\setminus\{i\}}. \tag{20}$$

3.2 Finite Jones expansions via ring isomorphism: The JVP

For the classification work of Section 4 we require a *finite* packaging of the Birman-Lin layers that is stable under basic operations and easy to compute from tabulated Jones polynomials. The Jones–Vassiliev polynomial (JVP) below does exactly this.

Lemma 3.4 (Jones–Vassiliev expansion (JVP)). *Let $R = \mathbb{Z}[p]$ and consider the R-algebra $R[x]/(x^2 - px - 1)$. For any oriented link K with Jones polynomial $V_K(x) \in \mathbb{Z}[x, x^{-1}]$, the following hold.*

1. *Under the homomorphism $p \mapsto x - x^{-1}$, the class of $V_K(x)$ in $R[x]/(x^2 - px - 1)$ admits a unique finite expansion*

$$V_K(x, p) = \sum_{q\geq 0} \big(a_q(K) + b_q(K)x\big)p^q, \qquad a_q(K), b_q(K) \in \mathbb{Z}.$$

   (Equivalently, $R[x]/(x^2 - px - 1)$ is a free R-module with basis $\{1, x\}$.)

2. *For each $q \geq 0$, the coefficient functionals $a_q, b_q : \mathcal{K} \to \mathbb{Z}$ are Vassiliev invariants of order $\leq q$; write $c_q := a_q + b_q$ likewise of order $\leq q$. In many degrees they are exactly of order q (see Proposition 3.6).*

The J_m-triviality index is unchanged from Birman-Lin (it is the p-adic order of $V_K - 1$), but now $K \in \mathcal{F}^m$ iff $V_K(x, p) \equiv 1 \pmod{p^m}$, giving an immediate *bandwidth filtration* and a clean min-law under connected sum. Computationally, the JVP lets us read (a_q, b_q) directly from standard $V_K(t)$ datasets by the substitutions $t^{1/2} = x$, $t^{-1/2} = x - p$, and the reduction $x^2 = px + 1$, which is what powers the large-scale experiments in Section 4.

Definition 3.4 (J_m-triviality; JVP coefficients). *A knot K is J_m-trivial if its q-type JVP coefficient $c_q(K) \equiv a_q(K) + b_q(K) = 0$ for $1 \leq q \leq m - 1$.*

Proposition 3.5 (Freeness and uniqueness for the JVP expansion). *Let $R = \mathbb{Z}[p]$ and consider the R-algebra $R[x]/(x^2 - px - 1)$. Then $R[x]/(x^2 - px - 1)$ is a free R-module of rank 2 with basis $\{1, x\}$. In particular, every Laurent polynomial $V(x) \in \mathbb{Z}[x, x^{-1}]$ admits a unique expansion*

$$V(x) = \sum_{q\geq 0} \big(a_q + b_q\, x\big)\, p^q \qquad (a_q, b_q \in \mathbb{Z}),$$

when viewed in $R[x]/(x^2 - px - 1)$ via the homomorphism $p \mapsto x - x^{-1}$.

Proposition 3.6 (Exact order of the JVP coefficients)**.** *In the Jones–Vassiliev expansion* $V_K(x,p) = \sum_{q\geq 0}(a_q(K) + b_q(K)x)p^q$ *of Lemma 3.4, each* a_q, b_q *is of order* $\leq q$*. Moreover:*

1. *Exactness transfers along* $h \leftrightarrow p$ *If the* h^q*–coefficient of* $V_K(e^h)$ *has nonzero order-q symbol for some knot* K*, then the order-q symbol* $\partial_{[q]} c_q(K)$ *is nonzero as well.*

2. *Existence for all* $q \geq 2$ *For every* $q \geq 2$ *there exists a knot* K_q *with* $\partial_{[q]} c_q(K_q) \neq 0$*; hence* c_q *(and therefore at least one of* a_q, b_q*) has exact order* q *on* K_q*.*

It turns out that the JVP free coefficients (type-0 Vassiliev invariants) encode the number of link components.

Lemma 3.7. *Let* K *be a link with* ℓ_K *components. Then,* $c_0(K) \equiv a_0(K) + b_0(K) = (-2)^{\ell_K - 1}$*, and*

$$a_0(K) = \begin{cases} (-2)^{\ell_K-1}, & \ell_K \equiv 1 \mod 2 \\ 0, & \ell_K \equiv 0 \mod 2 \end{cases}, \quad b_0(K) = \begin{cases} 0, & \ell_K \equiv 1 \mod 2 \\ (-2)^{\ell_K-1}, & \ell_K \equiv 0 \mod 2 \end{cases} \tag{21}$$

Lemma 3.8 (Finite-type skein relations)**.** *For a knot/link dependent function,* $K \mapsto f(K)$*, let* $f^{(i=-)}$*,* $f^{(i=0)}$*, and* $f^{(i=+)}$ *be the function evaluated at a knot obtained from* K *by allowing the i-th crossing to vary as over, under, and complete annihilation via oriented smoothing, respectively. Define,* $c_q(K) \equiv a_q(K) + b_q(K)$ *(finite-type of order q). The Jones-Vassiliev skein relation induces the following three levels of recursive dependencies:*

$$\begin{aligned} &\textit{Polynomial level:} && \partial_i V = p\left[xV^{(i=-)} + V^{(i=0)} + (x-p)V^{(i=+)}\right], \\ &\textit{Finite-type level:} && \partial_i c_q = c_{q-1}^{(i=-)} + c_{q-1}^{(i=0)} + c_{q-1}^{(i=+)} - a_{q-2}^{(i=+)} + b_{q-2}^{(i=-)}, \\ &\textit{Symbol level:} && \partial_{[q]} c_q^{\varnothing} = 2\partial_{[q]\setminus\{i\}} c_{q-1}^{\varnothing} + \partial_{[q]\setminus\{i\}} c_{q-1}^{\{i\}}, \end{aligned} \tag{22}$$

where the unnormalized *partial derivative,* $\partial_i V \equiv V^{(i=+)} - V^{(i=-)}$*. The symbol after smoothing a crossing at* i*,* $\partial_{[q]\setminus\{i\}} c_{q-1}^{\{i\}}$*, occasionally differs from* $\partial_{[q]\setminus\{i\}} c_{q-1}^{\varnothing} \equiv \partial_{[q]\setminus\{i\}} c_{q-1}$*, hence the superscript notation.*

The JVP factorizes nicely over connected sums, $V_{K_0\#K_1} = V_{K_0}V_{K_1}$, and this carries over to the finite-type coefficients and the respective symbols.

Lemma 3.9. *Let* $c_q^r(K) \equiv a_q^r(K) + b_q^r(K)$ *(order-q finite-type invariant after* r *oriented smoothings). Then*

$$c_q^r(K_0\#K_1) = \sum_{i=0}^{q} c_i^{r_0}(K_0)c_{q-i}^{r_1}(K_1) + b_i^{r_0}(K_0)b_{q-1-i}^{r_1}(K_1), \tag{23}$$

where q *is bounded by the sum total of coefficients in either JVPs. Here,* $r = r_0 + r_1$*, and* r_0 *and* r_1 *are the number of smoothings associated with the underlying subcube components, i.e.,* $K_0\#K_1$ *is a vertex of* $\mathcal{Q}_0 \times \mathcal{Q}_1$*. Moreover, letting* $q = q_0 + q_1$*, where* q_0 *and* q_1 *are the symbol orders associated with the* K_0*-factor and* K_1*-factor (i.e., alternating sum over the underlying vertices in any individual subcube component), we have*

$$\partial_{[q_0+q_1]} c_{q_0+q_1}^{r_0+r_1}(K_0\#K_1) = \left[\partial_{[q_1]} c_{q_1}^{r_1}(K_1)\right]\left[\partial_{[q_0]} c_{q_0}^{r_0}(K_0)\right]. \tag{24}$$

4 Classifying J_m-trivial knots

We suspect that the bound is far from being strict. It follows immediately that all knots are at least J_2-trivial (as $a_1 = b_1 = c_1 = 0$ for knots) and indeed $\mathscr{N}(K) \geq 3$ for any non-trivial K. To stress test the uniform barrier, $m \leq \mathscr{N}(K)$, we have calculated the JVP coefficients of all knots up to 19 crossings (total of 352,152,252 knots). We used the Jones polynomial $\mathbb{Z}[t^{\pm 1/2}]$ datasets from Knot Atlas (0-10 crossings) and Dartmouth (11-19 crossings) to compute the JVP's q-order invariants, a_q and b_q, using the identification $t^{1/2} = x$, $t^{-1/2} = x - p$, and the relation $x^2 = px + 1$. This allowed us to determine m and classify knots by their J_m-triviality.

The first J_m-trivial knots for any crossing number from 4 to 19 are shown in Figure 3. Tables of the JVP coefficients of any knot up to 12 crossings may be found in the last pages of this note.The naming conventions are the ones used in the datasets.

The simplest J_2-trivial knot is of course the trefoil 3_1. The first J_3-trivial knot is 8_2 and the first J_4-trivial knot is 8_{14}. There are a couple of noteworthy observations that follow from this experiment. It became clear that the uniform barrier is much stricter than proven in Part I. Clearly, this does not affect any of the results therein. For knots with up to 19 crossings:

$$m \leq \left\lfloor \frac{\mathscr{N}(K)}{2} \right\rfloor + 1 < \mathscr{N}(K)$$

We conjecture that this is the case in general. The intuition behind it is that J_m-triviality and Jones-m-flatness are invariant to mirror symmetry. That means that m-flat subcubes come in pairs and this truncates the effective degree cap in the obstruction mechanism (Theorem 4.10, Corollary 4.11 in Part I) by a factor of two.

Proposition 4.1 (Prime J_m-trivial knots for any $m > 1$)**.** *For any $m > 1$ there exists at least one prime J_m-trivial knot with*

$$m \leq \mathscr{N}(K) \leq 8(m-1) \quad \textit{potentially} \quad m \leq \lfloor \mathscr{N}(K)/2 \rfloor + 1 \leq 4m - 3.$$

Proof. Figure 2 illustrates a basic local model: start from the unknot, insert a connected C_m tree whose surgery substitutes the trivial tangle $\mathscr{T}_m$ by the Brunnian tangle $\mathscr{B}_m$ (see Figure 1). Closing this tangle produces a knot that is C_m-equivalent to the unknot; in particular it is m-trivial in the GH sense and therefore J_m-trivial: its coefficients c_n agree with those of the unknot for $1 \leq n \leq m - 1$ (Theorem 4 in [5] refers to similar constructions). Moreover, there is a *distinguished* exterior crossing o whose flip "unlocks" the Brunnian component so that the knot becomes the unknot (the unlocking crossing is delineated in the last row in Figure 2). Because the only difference between the $\mathscr{B}_m$-closure and the $\mathscr{T}_m$-closure lives in the m-th graded GH piece, this flip is invisible to every degree $< m$ finite-type invariant (a crossing flip mod C_m). As the unknotting number of any such knot is 1 it is prime. Finally, $\mathscr{B}_m$ consists of $8(m-1)$ crossings, from which the bound on $\mathscr{N}(K)$ follows. □

4.1 Likelihood of J_m-triviality

Suppose that a J_m-trivial knot K is picked uniformly at random from all knots up to some crossing number, say 19 as in our experiment. What is the likelihood of its crossing number, $\mathscr{N}(K) \leq \mathscr{N}$ where $\mathscr{N} \leq 19$? The answer in this case is given by Figure 4 which shows the probability $P_m(\mathscr{N}) = \Pr(\mathscr{N}(K) \leq \mathscr{N} \mid K \text{ is } J_m\text{-trivial})$.

The stringent form of the bound is visible as $P_m(\mathscr{N}) = 0$ for $m > \lfloor \mathscr{N}/2 \rfloor + 1$ (no J_m-trivial knots in this regime). The growth of the probability with $\mathscr{N}$ is expected because J_m-trivial knots

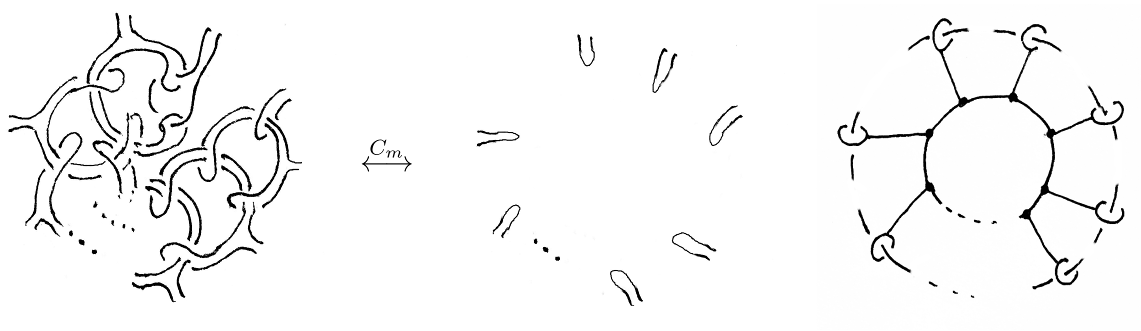


Figure 1: Brunnian tangle $\mathscr{B}_m$ (left), trivial tangle $\mathscr{T}_m$, and the associated clasper C_m tree (right). A C_m move realizes the local replacement of $\mathscr{T}_m$ by $\mathscr{B}_m$.

become more frequent with increasing knotedness. However, the growth rate, apart from being exponential (log scale of the ordinate axis), seems almost independent of m. This alludes to a form of an asymptotic equipartition property of J_m-trivial classes. In other words, when $\mathscr{N}$ is unbounded all J_m-trivial knots are equally likely.

4.2 Triviality calculus and grading

Definition 4.1 (Jones p-gap and bandwidth). *Let $m(K) \in \mathscr{N} \cup \{\infty\}$ be the least $m \geq 1$ with $c_m(K) \neq 0$ (set $m(\text{unknot}) = \infty$). Equivalently, $K \in \mathcal{F}^m$ if and only if $V_K(x,p) \equiv 1 \pmod{p^m}$, i.e. $a_q(K) = b_q(K) = 0$ for $1 \leq q < m$. We call $\mathcal{F}^m$ the* p-gap *(or Jones-bandwidth) filtration.*

Proposition 4.2 (Min-law for connected sum). *For all knots K_0, K_1,*

$$m(K_0\#K_1) = \min\{m(K_0), m(K_1)\}.$$

Equivalently, $\mathcal{F}^m$ is a multiplicative filtration: $K_0 \in \mathcal{F}^{m_0}, K_1 \in \mathcal{F}^{m_1} \Rightarrow K_0\#K_1 \in \mathcal{F}^{\min(m_0,m_1)}$.

Proof. Expand $V_{K_0}V_{K_1}$ in $\mathbb{Z}[p][x]/(x^2 - px - 1)$ and read off the p-orders; (32) shows all $p^{<m}$ layers vanish while the p^m layer adds. □

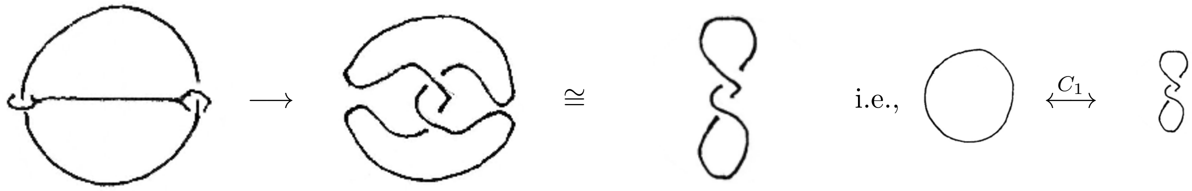
Remark 4.1 (The p-gap). *If one only assumes $c_q(K) = 0$ for $q < m$ (Definition 3.4), the term $\sum_{i=0}^{q-1} b_i(K_0)b_{q-1-i}(K_1)$ in* (38) *can prevent J_m-triviality from being closed under #; the p-gap hypothesis removes this obstruction.*

We may thus define equivalence classes of knots:

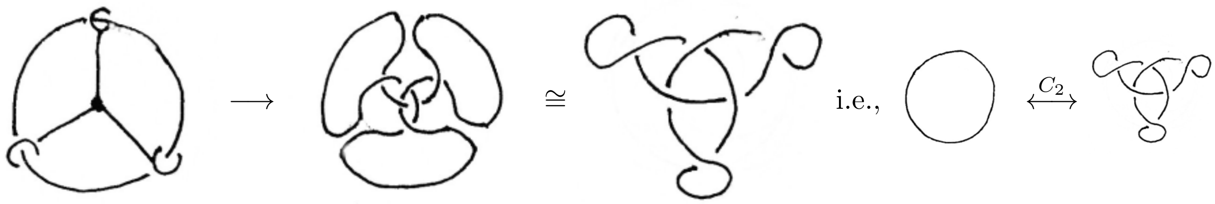
$$K \sim_m L \iff V_K(x,p) \equiv V_L(x,p) \pmod{p^m}.$$

Then $\sim_m$ respects connected sum, and on the associated graded $\mathrm{gr}^m := \mathcal{F}^m/\mathcal{F}^{m+1}$ we have an abelian monoid where the degree-m component is governed by the single layer c_m: the term $\sum_{i=0}^{q-1} b_i(K_0)b_{q-1-i}(K_1)$ in (38) vanishes and one may write

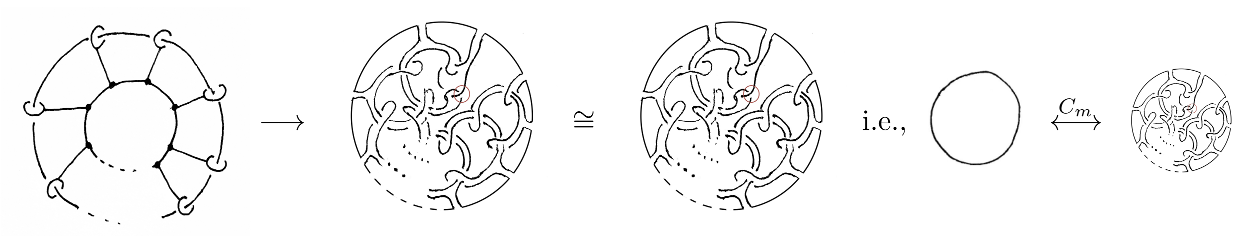
$$[c_m(K_0\#K_1)] = [c_m(K_0)] + [c_m(K_1)] \text{ in } \mathrm{gr}^m.$$



J_1-trivial



J_2-trivial



J_m-trivial

Figure 2: J_m-trivial knots as "clasperian" closures of the Brunnian tangle $\mathscr{B}_m$ (a C_m tree clasping the unknot). The clasper C_m-tree representation on the left column translates into a J_m-trivial (m-trivial) knot on the right. Equivalently, the unknot is C_m-equivalent to a J_m-trivial (m-trivial) knot. The unknotting number of these knots is 1 which follows from the fact that they all are clasperian closures of $\mathscr{B}_m$. Flipping the *unlocking crossing* in $\mathscr{B}_m$ turns the knot into the unknot (the unlocking crossing is delineated within the J_m-trivial knot). As both the original knot and the unknot agree on all Jones coefficients up to $m-1$, this single crossing flip does not change, $c_n = 0$, for $1 \leq n \leq m-1$. In fact, as a single C_m-move turns the unknot into an m-trivial (clasperian closure) knot, this particular crossing flip is invisible to all degree $< m$ finite-type invariants.

Knot	m	Knot	m
8_2	3	8_{14}	4
9_8	3		
10_{25}	3	10_{33}	4
K11a1	3	K11n30	4
K11a81	5		
K12n4	3	K12a17	4
K12a439	5	K12a214	6
13a_hyp_jones:71	3	13a_hyp_jones:754	4
13a_hyp_jones:757	5	13n_hyp_jones:4278	6
14a_hyp_jones:5	3	14a_hyp_jones:167	4
14a_hyp_jones:5988	6	14a_hyp_jones:19509	8
15a_hyp_jones:70	3	15a_hyp_jones:658	4
15a_hyp_jones:2437	5	15a_hyp_jones:49378	7
16a_hyp_jones:100	3	16a_hyp_jones:192	4
16a_hyp_jones:8741	5	16a_hyp_jones:24872	6
16a_hyp_jones:332201	8		
17a_hyp_jones:26	3	17a_hyp_jones:1014	4
17a_hyp_jones:6028	5	17a_hyp_jones:136317	6
17a_hyp_jones:330478	7	17a_hyp_jones:1743282	9
18a_hyp_jones:22	3	18a_hyp_jones:254	4
18a_hyp_jones:19404	5	18a_hyp_jones:24338	6
18a_hyp_jones:2189165	7	18a_hyp_jones:4068520	8
18n_hyp_jones:38726284, $m = 9$ (1st of 3 in 48,266,466)			
19n_hyp_jones:62	3	19n_hyp_jones:251	4
19a_hyp_jones:2276	5	19n_hyp_jones:124491	6
19a_hyp_jones:986383	7	19a_hyp_jones:34529444	8
19a_hyp_jones:28010794	9	19n_hyp_jones:199269550	9

Figure 3: The first occurring J_m-trivial knots in the Knot Atlas and Dartmouth Jones polynomial datasets up to 19 crossings (approximately 350 million knots). Note that $m \leq \lfloor \mathcal{N}/2 \rfloor + 1 < \mathcal{N}$.

Auxiliary lemmas and proofs

A Jones-Vassiliev polynomial

A.1 Proof of Lemma 3.4

Proof. The Jones polynomial lives in $\mathbb{Z}[x, x^{-1}]$. Its skein relation is given by

$$x^{-2}V^{(+)} - x^2V^{(-)} = (x - x^{-1})V^{(0)}, \tag{25}$$

where the superscripts designate the variation of a single crossing as over, under, and annihilation through oriented smoothing. There exists an isomorphism, $\mathbb{Z}[x, x^{-1}] \cong \mathbb{Z}[p][1, x] \cong \mathbb{Z}[p][x]/(x^2 - px - 1)$. In particular, letting $p = x - x^{-1}$, we obtain

$$x^2 = px + 1, \quad x^{-2} = p^2 - px + 1. \tag{26}$$

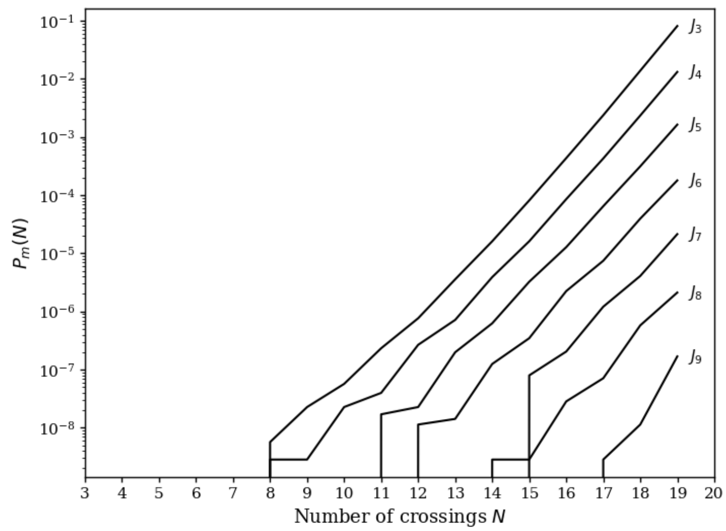


Figure 4: The likelihood of a J_m-trivial knot having not more than $\mathscr{N} \leq 19$ crossings (there are 352,152,252 such prime knots). The likelihood, $P_m(\mathscr{N})$, grows exponentially (log scale of the ordinate axis) with the number of crossings. The figure illustrates the uniform barrier, with $P_m(\mathscr{N}) = 0$ as soon as $m > \lfloor \mathscr{N}/2 \rfloor + 1$. In this range, J_m-trivial knots respect a stricter bound: $m \leq \lfloor \mathscr{N}/2 \rfloor + 1 < \mathscr{N}$.

In $R = \mathbb{Z}[p][x]/(x^2 - px - 1)$ we reduce every product via $x^2 = px + 1$; all identities below are taken in R. Substituting (26) into the skein relation (25) and rearranging yield the Vassiliev skein relation,

$$\partial V \equiv V^{(+)} - V^{(-)} = p\left[xV^{(-)} + V^{(0)} + (x-p)V^{(+)}\right]. \tag{27}$$

The above definitions imply that the zero modulo in the underlying ring is $x^2 - px - 1$, while p may be of any degree. Because $x^2 - px - 1 = 0$ is monic, $\mathbb{Z}[p][x]/(x^2 - px - 1)$ is free of rank 2 over $\mathbb{Z}[p]$ with basis $\{1, x\}$, and thus may be written as

$$V_K(x,p) = \sum_{q \geq 0} \left[a_q(K) + b_q(K)x\right] p^q. \tag{28}$$

where the coefficients $a_q(K), b_q(K)$ are well defined and unique. These coefficients are q-type Vassiliev invariants. This follows from the fact that repeated application of the Vassiliev skein relation (27) increases the lowest degree of p. In particular,

$$\partial^n V_K \equiv 0 \mod p^n, \tag{29}$$

i.e., the p^q coefficients of $\partial^n V_K$ vanish for every $q < n$. □

A.2 Proof of Proposition 3.5

Proof. Because $x^2 - px - 1$ is monic of degree 2, reduction modulo this polynomial shows $R[x]/(x^2 - px - 1)$ is generated over R by $\{1, x\}$. Suppose $F(p) + G(p)\,x = 0$ in $R[x]/(x^2 - px - 1)$. Passing to the fraction field $K = \mathbb{Q}(p)$, the image of x in $K[x]/(x^2 - px - 1)$ has minimal polynomial $t^2 - pt - 1$, hence $\{1, x\}$ is K-linearly independent. Therefore $F = G = 0$ in $K[p]$, and since the module is torsion-free over R, it follows $F, G \in R$ vanish. Uniqueness of the coefficients $\{a_q, b_q\}$ follows. □

A.3 Relation to Birman-Lin expansion

Lemma A.1 (Birman–Lin change with explicit unitriangularity)**.** *Put $x = e^h$, so $p = x - x^{-1} = 2\sinh h = 2h + \frac{1}{3}h^3 + \frac{1}{60}h^5 + \frac{1}{2520}h^7 + \cdots$. Its compositional inverse is*

$$h = \tfrac{1}{2}p - \tfrac{1}{48}p^3 + \tfrac{3}{1280}p^5 - \tfrac{5}{14336}p^7 + \tfrac{35}{589{,}824}p^9 + \cdots .$$

Let $V(h) = \sum_{j\geq 0} u_j h^j$ be the (Birman–Lin) h–expansion. Substituting $h = h(p)$ and collecting coefficients gives a unitriangular transformation

$$V(h(p)) = \sum_{q\geq 0} \Big(\sum_{j\leq q} T_{qj}\, u_j \Big)\, p^q, \qquad T_{qq} = 2^{-q}.$$

In particular, the p^q–coefficient depends only on $\{u_j\}_{j\leq q}$, so order cannot increase *under $h \mapsto p$. Moreover, the first rows are*

$$\begin{aligned}
[p^1]:&\quad \tfrac{1}{2}u_1,\\
[p^2]:&\quad \tfrac{1}{4}u_2,\\
[p^3]:&\quad -\tfrac{1}{48}u_1 + \tfrac{1}{8}u_3,\\
[p^4]:&\quad -\tfrac{1}{48}u_2 + \tfrac{1}{16}u_4,\\
[p^5]:&\quad \tfrac{3}{1280}u_1 - \tfrac{1}{64}u_3 + \tfrac{1}{32}u_5.
\end{aligned}$$

If the h^j–coefficient is of Vassiliev order $\leq j$ (Birman–Lin), then the Jones–Vassiliev coefficients a_q, b_q (and $c_q = a_q + b_q$) are of order $\leq q$.

Proof. The series $p(h)$ has only odd powers with unit leading term after rescaling by 2; its compositional inverse $h(p)$ is likewise odd and of the form $h = \frac{1}{2}p + \mathcal{O}(p^3)$. The coefficient extraction maps $\{h^j\} \leftrightarrow \{p^q\}$ via a unitriangular matrix, so order cannot increase under the change of variables. Combine with Lemma 3.4's skein-modulo-p argument (29). □

A.4 Proof of Proposition 3.6

Proof. (1) The change of variables $p = x - x^{-1} = 2\sinh h$ is unitriangular: $p = 2h + O(h^3)$ and $h = \frac{1}{2}p + O(p^3)$ (Lemma A.1). Thus extracting the p^q–layer is a unitriangular linear combination of the h^j–layers with $j \leq q$, and the top line comes from h^q. Therefore nonvanishing of the order-q symbol at h^q forces $\partial_{[q]} c_q \neq 0$.

(2) Appendix B shows $\partial_{[2]} c_2(\text{trefoil}) = -3 \neq 0$ and $\partial_{[3]} c_3(\text{trefoil}) = -12 \neq 0$. By the symbol product rule (Lemma 3.9, (24)), for connected sums

$$\partial_{[q_0+q_1]} c_{q_0+q_1}(K_0 \# K_1) = \big(\partial_{[q_0]} c_{q_0}(K_0)\big) \cdot \big(\partial_{[q_1]} c_{q_1}(K_1)\big).$$

Hence symbols multiply and orders add. Because every integer $q \geq 2$ is a sum of 2's and 3's, taking connected sums of trefoils realizes nonzero $\partial_{[q]} c_q$ for all $q \geq 2$. (For knots, $c_1 \equiv 0$.) This proves existence of exact order q for all $q \geq 2$. □

A.5 Proof of Lemma 3.7

Proof. The Jones polynomial of a two-component unlink is $V_{\bigcirc\bigcirc} = -x - x^{-1}$, and the corresponding JVP is $V_{\bigcirc\bigcirc} = p - 2x$. It is well known that evaluating the Jones polynomial at $x = 1$ yields $V_{\bigcirc\bigcirc}(x = 1)^{l_K - 1} = (-2)^{l_K - 1}$, where l_K is the number of link components. As $p = x - x^{-1}$ we have $p = 0$ once $x = 1$. Therefore, evaluating the Jones polynomial at 1 amounts to evaluating the

JVP at $x = 1$, $p = 0$, namely, the number of link components are encoded exclusively by the free coefficient. Note, however, that p vanishes also for $x = -1$, in which case $V_{\bigcirc\bigcirc}(x = -1, p = 0) = 2$, from which we conclude that $V_{\bigcirc\bigcirc}(x = \pm 1, p = 0) = \mp 2$, and,

$$V_K(x = \pm 1, p = 0) = V_{\bigcirc\bigcirc}(x = \pm 1, p = 0)^{l_K - 1} = (\mp 2)^{l_K - 1}. \tag{30}$$

Therefore, $V_K(x = \pm 1, p = 0) = a_0(K) \pm b_0(K) = (\mp 2)^{l_K - 1}$. Adding and subtracting the two identities for $x = 1$ and $x = -1$ yields the stated result. □

A.6 Proof of Lemma 3.8

Proof. The JVP skein relation is obtained in the proof of the preceding lemma, namely, (27). The skein relations at the finite-type level is obtained by substituting (28) into (27). Doing so and equating coefficients of p^q on both sides while noting that $x^2 = px + 1$, yields

$$\begin{aligned}
\partial_i a_q &= a_q^{(i=+)} - a_q^{(i=-)} = b_{q-1}^{(i=-)} + a_{q-1}^{(i=0)} + b_{q-1}^{(i=+)} - a_{q-2}^{(i=+)}, \\
\partial_i b_q &= b_q^{(i=+)} - b_q^{(i=-)} = a_{q-1}^{(i=-)} + b_{q-1}^{(i=0)} + a_{q-1}^{(i=+)} + b_{q-2}^{(i=-)}.
\end{aligned} \tag{31}$$

Letting $c_q \equiv a_q + b_q$ it now follows that,

$$\partial_i c_q = c_q^{(i=+)} - c_q^{(i=-)} = c_{q-1}^{(i=-)} + c_{q-1}^{(i=0)} + c_{q-1}^{(i=+)} - a_{q-2}^{(i=+)} + b_{q-2}^{(i=-)} \tag{32}$$

Let $J \subseteq \text{cross}(\mathcal{Q})$ be a q-subset of crossing indices, $|J| = q$. The symbol skein relation follows by taking derivative, $\partial_{J\setminus\{i\}}$, on both sides of (32),

$$\begin{aligned}
\partial_J c_q = \partial_{J\setminus\{i\}} c_q^{(i=+)} &- \partial_{J\setminus\{i\}} c_q^{(i=-)} \\
&= \partial_{J\setminus\{i\}} c_{q-1}^{(i=-)} + \partial_{J\setminus\{i\}} c_{q-1}^{(i=0)} + \partial_{J\setminus\{i\}} c_{q-1}^{(i=+)} - \partial_{J\setminus\{i\}} a_{q-2}^{(i=+)} + \partial_{J\setminus\{i\}} b_{q-2}^{(i=-)}
\end{aligned} \tag{33}$$

Invoking the type condition, $\partial_T f = \text{const}$, for any order-$|T|$ invariant f,

$$\partial_{J\setminus\{i\}} c_{q-1}^{(i=+)} = \partial_{J\setminus\{i\}} c_{q-1}^{(i=-)} = \partial_{J\setminus\{i\}} c_{q-1}.$$

By the same argument, $\partial_{J\setminus\{i\}} a_{q-2}^{(i=+)} = \partial_{J\setminus\{i\}} b_{q-2}^{(i=-)} = 0$. Note, however, that $\partial_{J\setminus\{i\}} c_{q-1}^{(i=0)}$ generally differs from its non-smoothed counterpart, $\partial_{J\setminus\{i\}} c_{q-1}$. To make this difference explicit, we shall record the subset $K \subseteq J$ of smoothed crossings using the explicit index-set notation of the main text, $\mathscr{S}_{K;J} := \partial_J c_{|J|}^K$ (first perform oriented smoothings at all crossings in K, then take the unnormalized alternating sum over the J-face).

$$\partial_J c_q = 2\partial_{J\setminus\{i\}} c_{q-1} + \partial_{J\setminus\{i\}} c_{q-1}^{(i=0)} \quad \longrightarrow \quad \mathscr{S}_{\varnothing;J} = 2\mathscr{S}_{\varnothing;J\setminus\{i\}} + \mathscr{S}_{\{i\};J\setminus\{i\}} \tag{34}$$

Clearly, the skein holds for any subset K of smoothings, hence

$$\mathscr{S}_{K;J} = 2\mathscr{S}_{K;J\setminus\{i\}} + \mathscr{S}_{K\cup\{i\};J\setminus\{i\}} \tag{35}$$

An operatoric form of the symbol skein may be obtained by defining the pair of linear maps:

$$v_i(\mathscr{S}_{K;J}) = \mathscr{S}_{K;J\setminus\{i\}}, \quad h_i(\mathscr{S}_{K;J}) = \mathscr{S}_{K\cup\{i\};J\setminus\{i\}}$$

Using these we may write (35) as

$$\text{id} \circ \mathscr{S}_{K;J} = (2v_i + h_i) \circ \mathscr{S}_{K;J},$$

hence, $2v_i + h_i$ acts as an identity on symbols, $\text{id} = 2v_i + h_i$. □

A.7 Proof of Lemma 3.9

Proof. Given a connected sum of two knots the JVP factorizes as $V_{K_0\#K_1} = V_{K_0}V_{K_1}$. Therefore,

$$\begin{aligned} V_{K_0\#K_1} &= \\ &\sum_{i,j} \left[a_i(K_0) + b_i(K_0)x\right]\left[a_j(K_1) + b_j(K_1)x\right] p^{i+j} = \sum_{i,j} \left[a_i(K_0)a_j(K_1) + b_i(K_0)b_j(K_1)\right] p^{i+j} \\ &\qquad + \left[a_i(K_0)b_j(K_1) + b_i(K_0)a_j(K_1)\right] xp^{i+j} + b_i(K_0)b_j(K_1)xp^{i+j+1}, \end{aligned} \tag{36}$$

where the relation, $x^2 = px + 1$, has been used. The order-m finite-type coefficient of $V_{K_0\#K_1}$ is thus given as,

$$\begin{aligned} a_m(K_0\#K_1) &= \textstyle\sum_{i=0}^m a_i(K_0)a_{m-i}(K_1) + b_i(K_0)b_{m-i}(K_1), \\ b_m(K_0\#K_1) &= \textstyle\sum_{i=0}^m a_i(K_0)b_{m-i}(K_1) + b_i(K_0)a_{m-i}(K_1) + b_i(K_0)b_{m-1-i}(K_1). \end{aligned} \tag{37}$$

Summing up both these identities yields,

$$c_m(K_0\#K_1) = \sum_{i=0}^m c_i(K_0)c_{m-i}(K_1) + b_i(K_0)b_{m-1-i}(K_1). \tag{38}$$

Taking $m = m_0 + m_1$, where m_i is the order-m_i symbol of the i-th subcube, Q^i, of which K_i is a vertex, the symbols factorize accordingly,

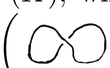
$$\begin{aligned} \partial_{[m]}c_m(K_0\#K_1) &= \sum_{i=0}^m \partial_{[m_0]}c_i(K_0)\cdot\partial_{[m_1]}c_{m-i}(K_1) + \partial_{[m_0]}b_i(K_0)\cdot\partial_{[m_1]}b_{m-1-i}(K_1) \\ &= \sum_{i=m_0}^m \partial_{[m_0]}c_i(K_0)\cdot\partial_{[m_1]}c_{m-i}(K_1) + \partial_{[m_0]}b_i(K_0)\cdot\partial_{[m_1]}b_{m-1-i}(K_1) \\ &= \partial_{[m_0]}c_{m_0}(K_0)\cdot\partial_{[m_1]}c_{m_1}(K_1). \end{aligned} \tag{39}$$

The second line, of which the running index starts from m_0, follows from the type condition, $\partial_{[m_0]}c_i(K_0) = \partial_{[m_0]}b_i(K_0) = 0$, $\forall i < m_0$. The third line follows by the same argument, as $\partial_{[m_1]}c_{m-i}(K_1) = 0$, $\forall i > m_0$, and $\partial_{[m_1]}b_{m-1-i}(K_1) = 0$, $\forall i \geq m_0$.

The homogeneous smoothing of crossings across individual subcube components, that is, smoothing the same crossings in any vertex of $\mathcal{Q}^i$, only changes the smoothing index of the finite-type coefficient, $c_{m_i}^{n_i}(K_i)$, $i = 0, 1$. In other words, the above analysis applies to variation cubes whose vertices are knots/links obtained after any number of successive smoothings. $\square$

B Example computations

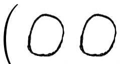
B.1 JVP of basic knots and links

To simplify notation we shall denote (K), where K is an inline diagram, as the JVP of K. By common normalization, the unknot, $\Big($  $\Big)$ = $\Big($  $\Big)$ = $\Big($  $\Big)$ = 1. The invariants of other basic knots and links may be obtained by iterating the polynomial skein (22). As we are working in a quotient rank-2 module ($x^2 = px + 1$), the following identity is useful

$$(px+1)(p^2 - px + 1) = p^3x - p^2x^2 + px + p^2 - px + 1 = p^3x + p^2 - p^2(px+1) + 1 = 1,$$

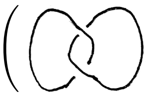
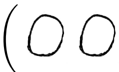
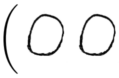
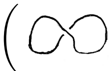
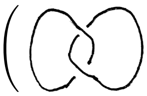
hence, $(p^2 - px + 1)^{-1} = px + 1$. Note also that the JVP transforms under mirror symmetry as: $p \mapsto -p$ and $x \mapsto x - p$.

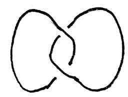
1. Unlink.

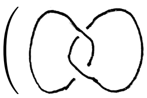
$0 = \Big($  $\Big) - \Big($  $\Big) = p\Big[x\Big($  $\Big) + \Big($  $\Big) + (x-p)\Big($  $\Big)\Big]$

$\Big($  $\Big) = p - 2x$

2. Hopf link.

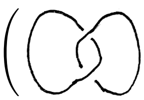
$\Big($  $\Big) - \Big($  $\Big) = p\Big[x\Big($  $\Big) + \Big($  $\Big) + (x-p)\Big($  $\Big)\Big]$

$(p^2 - px + 1)\Big($  $\Big) - (p - 2x) = px(p - 2x) + p$

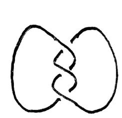
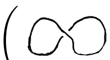
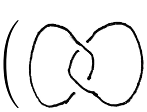
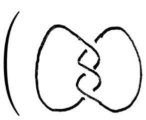
$\Big($  $\Big) = (px + 1)\left[p^2x - 2p(px + 1) + 2p - 2x\right] = (px + 1)(-p^2x - 2x)$

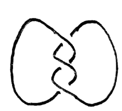
$= (-p^3 - 2p)(px + 1) - p^2x - 2x = -p^4x - p^3 - 3p^2x - 2p - 2x$

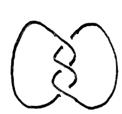
The mirrored Hopf's JVP is, accordingly,

$\Big($  $\Big) = p^5 - p^4x + 4p^3 - 3p^2x + 4p - 2x$

3. Trefoil.

$\Big($  $\Big) - \Big($  $\Big) = p\Big[x\Big($  $\Big) + \Big($  $\Big) + (x-p)\Big($  $\Big)\Big]$

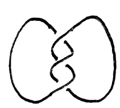
$(p^2 - px + 1)\Big($  $\Big) - 1 = px + p(-p^4x - p^3 - 3p^2x - 2p - 2x)$

$\Big($  $\Big) = (px + 1)\left[-p^5x - p^4 - 3p^3x - 2p^2 - px + 1\right]$

$= (-p^6 - 3p^4 - p^2)x^2 - 2p^5x - p^4 - 5p^3x - 2p^2 + 1$

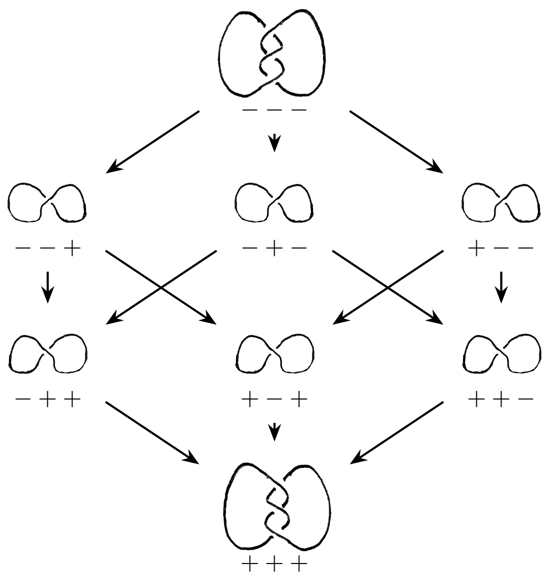
$= -p^7x - p^6 - 5p^5x - 4p^4 - 6p^3x - 3p^2 + 1$

The mirrored trefoil's JVP is

$\Big($  $\Big) = -p^8 + p^7x - 6p^6 + 5p^5x - 10p^4 + 6p^3x - 3p^2 + 1$

B.2 Variation cubes and symbols

Take the shadow of the trefoil knot above and consider its variation cube $\mathcal{Q}_3 = \{\pm\}^3$. In this cube, the antipodal vertices/variations, $---$ and $+++$, encode a trefoil and its mirror symmetric counterpart while the six remaining vertices are unknots. The JVP's finite-type coefficients, $c_q = a_q + b_q$, associated with each vertex may be summarized in a table.



Variation (ε)	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$---$	1	0	−3	6	−10	5	−6	1	−1
$--+$	1	0	0	0	0	0	0	0	0
$-+-$	1	0	0	0	0	0	0	0	0
$+--$	1	0	0	0	0	0	0	0	0
$-++$	1	0	0	0	0	0	0	0	0
$+-+$	1	0	0	0	0	0	0	0	0
$++-$	1	0	0	0	0	0	0	0	0
$+++$	1	0	−3	−6	−4	−5	−1	−1	0

To compute the symbols we shall define a family of pseudo-Boolean functions of increasing cube dimensions. Formally, let $c_q : \{\pm\}^q \to \mathbb{Z}$ whose Walsh-Fourier expansion is,

$$c_q(\varepsilon) = \sum_{S \subseteq [q]} \hat{c}_q(S)\chi_S(\varepsilon).$$

Computing the Fourier coefficients, $\hat{c}_q(S)$, for the above variations is immediate as the finite-type coefficients in all but two vertices vanish. The Fourier coefficients are computed as

$$\hat{c}_q(S) = 2^{-q} \sum_{\varepsilon \in \{\pm 1\}^q} c_q(\varepsilon)\chi_S(\varepsilon).$$

Computing the coefficients of interest based on the values in the above table,

$$\begin{aligned}
&\hat{c}_0(\emptyset) = c_0 = 1 \\
&\hat{c}_1(\emptyset) = c_1 = 0, \quad \hat{c}_1([1]) = 0 \\
&\hat{c}_2(\emptyset) = -3/4, \quad \hat{c}_2([1]) = 0, \quad \hat{c}_2([2]) = -3/4 \\
&\hat{c}_3(\emptyset) = 0, \quad \hat{c}_3([1]) = -12/8, \quad \hat{c}_3([2]) = 0, \quad \hat{c}_3([3]) = -12/8.
\end{aligned}$$

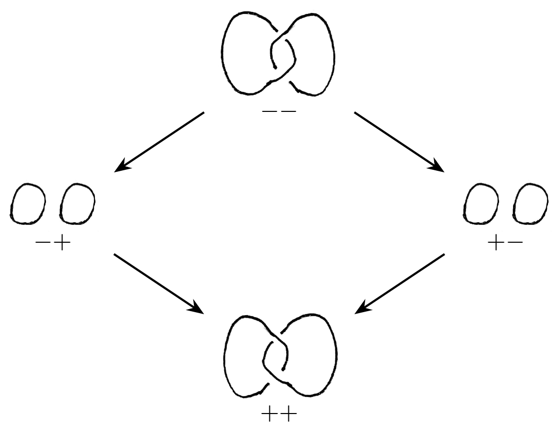
The symbols may then be extracted as $\partial_{[q]} c_q = 2^q \hat{c}_q([q])$. Hence, $\partial_{[q]} c_q = 1, 0, -3, -12$ for $q = 0, 1, 2, 3$.

B.2.1 Validating the symbol matrix

We have obtained the first column in a small symbol matrix and we may proceed completing the rest of its columns using the symbol skein (22). Once this is done we may validate the symbols in the rest of the matrix by reiterating the preceding steps for reduced cubes obtained via smoothing crossings in the shadow of the trefoil.

$$\begin{array}{r|rrrr} 3 & -12 & \bullet & \bullet & \bullet \\ 2 & -3 & -6 & \bullet & \bullet \\ 1 & 0 & -3 & 0 & \bullet \\ 0 & 1 & -2 & 1 & -2 \\ \hline {}^{q\uparrow}\!\diagup_{r\rightarrow} & 0 & 1 & 2 & 3 \end{array} \qquad (\star\star)$$

The symbols at $r = 1$, i.e., $\partial_{[q]} c_q^1$, may be validated by reiterating the above computation for a variation cube after smoothing one crossing in the trefoil shadow.



Variation (ε)	c_0	c_1	c_2	c_3	c_4	c_5
$--$	-2	4	-3	4	-1	1
$-+$	-2	1	0	0	0	0
$+-$	-2	1	0	0	0	0
$++$	-2	-2	-3	-1	-1	0

Computing the coefficients of interest based on the values in the above table,

$$\begin{aligned} &\hat{c}_0^1(\emptyset) = c_0 = -2 \\ &\hat{c}_1^1(\emptyset) = -1/2, \qquad \hat{c}_1^1([1]) = -3/2 \\ &\hat{c}_2^1(\emptyset) = -6/4, \qquad \hat{c}_2^1([1]) = 0, \qquad \hat{c}_2^1([2]) = -6/4. \end{aligned}$$

The symbols may then be extracted as $\partial_{[q]} c_q^r = 2^q \hat{c}_q^r([q])$. Hence, $\partial_{[q]} c_q^1 = -2, -3, -6$ for $q = 0, 1, 2$. These values agree with those in the $r = 1$ column of the symbol matrix. Reiterating this process further yields the remaining symbol values in $(\star\star)$.

The bullets are placeholders that are filled in any of the following cases: the introduction of nugatory crossings (underlying Jones flat-cubes), Markov moves applied homogeneously across the cube, or connected sums. Extending our original cube as, $\mathcal{Q}_3 \mapsto \mathcal{Q}_3 \times \mathcal{Q}_n$, where $\mathcal{Q}_n = \{\pm\}^n$ is a flat cube all of whose vertices are n-fold twists, we end up with

$q \uparrow$ \ $r \rightarrow$	0	1	2	3
4	0	0	0	0
3	−12	24	−48	96
2	−3	−6	36	−120
1	0	−3	0	36
0	1	−2	1	−2

whose third row displays the last-row fingerprint of the flat subcube, $-12 \cdot (-2)^r$.

C J_5- and J_6-trivial knots up to 12 crossings and their JVP

J_6-trivial knots up to 12 crossings (4 knots; $4/2978 \approx 0.13\%$)

$K \setminus {a_q \atop b_q}$	0	6	7	8	9	10	11	12	13	14	15	16
$K12a1248$	1	−12	0	−52	0	−67	0	−38	0	−10	0	−1
	0	0	24	0	44	0	30	0	9	0	1	0
$K12a1249$	1	12	0	16	0	7	0	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
$K12a214$	1	12	0	16	0	7	0	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
$K12a358$	1	−12	0	−28	0	−23	0	−8	0	−1	0	0
	0	0	−24	0	−44	0	−30	0	−9	0	−1	0

J_5-trivial knots up to 12 crossings (8 knots; $8/2978 \approx 0.26\%$)

$K \setminus {a_q \atop b_q}$	0	5	6	7	8	9	10	11	12	13	14	15	16
$K11a267$	1	0	−18	0	−57	0	−68	0	−38	0	−10	0	−1
	0	6	0	29	0	45	0	30	0	9	0	1	0
$K11a282$	1	0	24	0	44	0	30	0	9	0	1	0	0
	0	−12	0	−28	0	−23	0	−8	0	−1	0	0	0
$K11a81$	1	0	12	0	16	0	7	0	1	0	0	0	0
	0	12	0	28	0	23	0	8	0	1	0	0	0
$K11a91$	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	12	0	16	0	7	0	1	0	0	0	0	0
$K11n125$	1	0	−18	0	−39	0	−29	0	−9	0	−1	0	0
	0	−6	0	−41	0	−61	0	−37	0	−10	0	−1	0
$K11n157$	1	0	12	0	16	0	7	0	1	0	0	0	0
	0	−6	0	−11	0	−6	0	−1	0	0	0	0	0
$K12a439$	1	0	12	0	16	0	7	0	1	0	0	0	0
	0	−6	0	−5	0	−1	0	0	0	0	0	0	0
$K12n482$	1	0	6	0	5	0	1	0	0	0	0	0	0
	0	6	0	11	0	6	0	1	0	0	0	0	0

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