

# A bound on the $m$ -triviality of knots

## Part I: Jones-Vassiliev unknot detection

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### Abstract

We prove a uniform barrier for Jones–Vassiliev truncations. If a knot  $K$  is  $J_m$ -trivial, that is, its Birman–Lin finite-type coefficients  $c_1, \dots, c_{m-1}$  vanish, and  $K$  is nontrivial, then  $m \leq \mathcal{N}(K)$ , the crossing number of  $K$ . For a fixed shadow  $S$ , the choices of over/under data at its crossings form a Boolean cube; a subcube is the face obtained by varying only a designated subset of crossings. We decorate  $S$  by degree- $m_j$  claspers so that flipping the  $j$ -th crossing is realized (modulo clasper calculus) by a  $C_{m_j}$ -move invisible to finite-type invariants below order  $m_j$ ; a weighted Goussarov–Habiro filtration packages these effects and produces an order- $< m$  flat family of knots all of whose members are  $m$ -trivial. On the Jones side we develop a symbol calculus for the Birman–Lin expansion in which alternating-sum symbols satisfy a skein recursion. When the Jones layers are constant through order  $m-1$  (equivalently, all order- $< m$  symbols vanish), the recursion forces a last-row fingerprint for the free term  $c_0$ : smoothing any  $m-1$  chosen crossings creates exactly  $m$  components. A shadow criterion then implies that any supercritical face (where  $m$  exceeds the number of varied crossings) is flat and carries only unknots. Applying this to a minimal shadow yields  $m \leq \mathcal{N}(K)$ . As a corollary, the Jones polynomial detects the unknot: if  $V_K(t) \equiv 1$  (so all  $c_n(K) = 0$ ), then  $K$  would be  $J_m$ -trivial for every  $m$ , contradicting the barrier for any  $m > \mathcal{N}(K)$ ; hence  $K$  is the unknot. Equivalently, for any diagram with  $C$  crossings, vanishing of  $c_1, \dots, c_C$  certifies the unknot, and every nontrivial knot satisfies  $\min\{n \geq 1 : c_n(K) \neq 0\} \leq \mathcal{N}(K)$ .

## 1 Introduction

Vassiliev’s program views finite-type data through alternating sums over faces of the skein cube of crossing changes [1], leading to the algebra of chord diagrams and the  $1T/4T$  relations and encompassing Kontsevich’s universal integral [2] and Bar-Natan’s graded algebra [3]; see also [4]. In this paper we pursue a complementary, more *topological* implementation built on *extended shadows* and *clasper-decorated crossings*.

Fix a shadow  $S$  with outside crossing set  $\text{cross}(S)$ . Choose a base diagram  $K_0$  carried by  $S$  and, for each crossing  $j$ , attach a degree- $m_j$  tree clasper  $C_{m_j}$  in a small ball meeting the diagram only near  $j$ . The resulting data

$$E = (K_0; \{C_{m_j}\}_{j \in \text{cross}(S)})$$

is an *extended shadow*. The key local move is that *flipping the  $j$ -th crossing is realized modulo clasper calculus by a  $C_{m_j}$ -move*; hence every finite-type invariant of degree  $< m_j$  is blind to that flip:

$$\partial_j f = f(+) - f(-) = 0 \quad \text{for } \deg f < m_j.$$

We therefore call the  $j$ -th crossing  $m_j$ -*invisible* (“crossing flip mod  $C_{m_j}$ ”). See Figure 1 for the local movie implementing this equivalence and §3 for the weighted GH filtration that packages multi-flip alternating sums. In particular, with  $m := \min_j m_j$ , the family  $\sigma \mapsto K_E(\sigma)$  carried by the decorated crossings is  $m$ -flat for all order  $< m$  finite-type data, i.e., any one of its members is  $m$ -trivial.

Specializing to the Jones polynomial, we use the Birman–Lin expansion

$$V_K\left(e^{h/2}\right) = \sum_{n \geq 0} c_n(K) h^n,$$

whose coefficients  $c_n$  are finite-type of order  $\leq n$ . On any  $m$ -flat family (hence, in particular, on the family carried by an extended shadow with  $\min m_j = m$ ), every Jones layer  $c_1, \dots, c_{m-1}$  is constant. When a single vertex in the family is  $J_m$ -trivial, these constants are forced to be zero, a situation we call *Jones-m-flatness*. A symbol-level skein recursion for the  $c_n$  then yields a *last-row fingerprint* for the free term  $c_0$ : smoothing any prescribed  $m-1$  crossings produces exactly  $m$  components. A shadow-level criterion converts this fingerprint into a structural statement: if  $m$  exceeds the number of crossings being varied (in particular, if  $m > \#\text{cross}(S)$ ), then the shadow must be a tree and every carried diagram is an unknot. Applying this to a minimal-crossing shadow of  $K$  gives the uniform barrier

$$m \leq \mathcal{N}(K).$$

As an immediate corollary, the Jones polynomial detects the unknot: if  $V_K(t) \equiv 1$  (equivalently,  $c_n(K) = 0$  for all  $n$ ), then  $K$  would be  $J_m$ -trivial for every  $m$ ; taking  $m > \mathcal{N}(K)$  contradicts the barrier, so  $K$  is the unknot. Equivalently, for any diagram with  $C$  crossings, vanishing of  $c_1, \dots, c_C$  certifies the unknot, and every nontrivial knot satisfies  $\min\{n \geq 1 : c_n(K) \neq 0\} \leq \mathcal{N}(K)$ .

## 1.1 Related work

The foundational insight of Birman and Lin is that the coefficients in the power-series expansions of the Jones, HOMFLY and Kauffman polynomials yield Vassiliev invariants [5]; their paper established the bridge between skein-type polynomials and finite-type theory and articulated early structural consequences and open directions for detection. Birman’s contemporaneous survey emphasized the breadth of the approach and new perspectives opened by finite-type methods [6].

Kontsevich constructed a universal integral whose weight systems classify finite-type invariants [2], while Bar-Natan developed the algebra of chord diagrams modulo 1T/4T, clarifying symbol calculus and functoriality [3]. These works underlie our symbol-level arguments: within the spectral framework, the top Walsh block encodes precisely the  $m$ -th graded piece  $\mathcal{A}_m$  of Bar-Natan’s chord-diagram algebra, and our cubical 1T/4T (Lemma A.1) is the pseudo-Boolean incarnation of these relations.

Goussarov’s  $n$ -equivalence and Habiro’s clasper calculus show that surgery on a connected  $C_m$  tree preserves all invariants of degree  $< m$  [7, 8]. We exploit this twice: (i) to produce  $m$ -flat families by decorating a fixed shadow with degree- $m_j$  trees, and (ii) via the *crossing flip mod  $C_{m_j}$*  gadget that renders the  $j$ -th crossing  $m_j$ -invisible. The weighted GH filtration then controls all multi-flip differences and underlies the GH–Taylor expansion we use for bookkeeping. Figure 1 illustrates the local movie realizing the flip modulo  $C_m$ .

A corollary of earlier work on polynomial/finite-type expansions states:

**Corollary 5.3 [9].** *If all Vassiliev invariants up to degree  $c$  vanish on a knot  $K$  of crossing number  $c$  then the knot has a trivial HOMFLY polynomial.*

Our barrier strengthens the Jones-side mechanism in a diagram-local fashion: from a length- $m$

initial Jones gap we build an  $m$ -flat family on an extended shadow via  $m$ -invisible flips, force the last-row fingerprint, and through a shadow criterion bound  $m$  by  $\mathcal{N}(K)$ , yielding a finite verification of Jones detection at minimal crossing number.

## 1.2 Organization of this work

**Section 2** lays out the preliminaries: finite-type invariants (Vassiliev order and symbols), the pseudo-Boolean variation cube attached to a shadow, and the Walsh-Fourier/spectral viewpoint that identifies order with polynomial degree and interprets facewise alternating sums as discrete derivatives. This section fixes notation used throughout (e.g., smoothing  $D \setminus J$ , residual cubes).

**Section 3** introduces extended shadows by decorating selected crossings of a fixed shadow with tree claspers of prescribed degrees  $m_j$ . We prove that the  $j$ -th crossing flip is  $m_j$ -invisible (a  $C_{m_j}$ -move) for invariants of degree  $< m_j$ , package multi-flip effects via a weighted Goussarov-Habiro filtration, and assemble a GH-Taylor expansion that keeps track of interactions among claspers. This yields  $m$ -flat families (constancy of all sub- $m$  layers) carried by the decorated shadow. Figure 1 depicts the clasper movie and the crossing flip modulo  $C_m$  used repeatedly later.

**Section 4** specializes to the Birman-Lin expansion of the Jones polynomial. We establish a symbol-level skein recursion for the Jones layers  $c_n$  and show that order-0 data  $c_0$  is a shadow invariant encoding component count. On any Jones- $m$ -flat subcube we prove the last-row fingerprint: smoothing any prescribed  $m-1$  crossings forces  $c_0 = (-2)^{m-1}$ , hence the component count rises by exactly one at each smoothing. This yields a degree cap for Jones-flatness on a subcube (Cor. 4.11). The symbol matrix in Figure 2 visualizes the skein constraints and the  $(-2)^r$  bottom-row pattern.

**Section 5** gives the main result, the uniform  $m$ -triviality barrier: if a nontrivial knot  $K$  is  $J_m$ -trivial, then  $m \leq \mathcal{N}(K)$  (Thm. 5.1). The proof transfers  $m$ -flatness from a decorated face of the extended cube back to the outside-crossing subcube, applies the last-row fingerprint with the Shadow Lemma to force global nugatoriness in the supercritical regime, and concludes the bound. Two corollaries follow: Jones detects the unknot (Thm. 5.3) and the Vassiliev version (Thm. 5.4).

**Appendix A** records a cubical  $1T/4T$  formulation that links the symbol calculus to Bar-Natan’s diagrammatics; **Appendix B** states and proves the Shadow Lemma, equating the component-increment property (P) with all crossings being nugatory (the shadow is a tree). These appendices supply the local-to-global bridge used in the proofs.

## 1.3 Part II: Computational evidence

The companion paper (Part II) [10] implements the symbol calculus developed here by packaging the Birman-Lin layers into a Jones-Vassiliev polynomial (JVP) in the rank-two module  $\mathbb{Z}[p][x]/(x^2 - px - 1)$ ; from tabulated Jones polynomials  $V_K(t)$  it reads off the finite-type layers via the substitutions  $t^{1/2} = x$ ,  $t^{-1/2} = x - p$ , followed by the reduction  $x^2 = px + 1$ . Using this extraction pipeline, Part II computes the layers for all 352,152,252 prime knots with  $\mathcal{N} \leq 19$  (Knot Atlas for  $\mathcal{N} \leq 10$ ; Dartmouth for  $11 \leq \mathcal{N} \leq 19$ ), finding no counterexample to the uniform barrier  $m \leq \mathcal{N}(K)$  proved in Theorem 5.1 here, and moreover uncovering a consistent empirical refinement  $m \leq \lfloor \mathcal{N}/2 \rfloor + 1$ ; see Fig. 3 (first  $J_m$ -trivial occurrences) and Fig. 4 (likelihood curves  $P_m(\mathcal{N})$ ) of Part II. Beyond the census, Part II reproduces our symbol-skein recursion and verifies the last-row fingerprint (Theorem 4.10/Corollary 4.11 here) on explicit model cubes and records a clean min-law under

connected sum for the JVP bandwidth (Proposition 4.2). Taken together, the large-scale data and the JVP calculus in Part II provide a practical, finite algorithmic mirror of the obstruction mechanism developed in Part I.

## 1.4 Conventions

**Convention 1.1** (Conventions used throughout the text).

- A knot shadow (*shadow* for short) means the 4-regular plane graph obtained from a knot diagram by forgetting over/under information, and a crossing flip means changing the over/under choice at one vertex of the shadow.
- Over/under crossings are denoted as ‘+’ and ‘-’, and sometimes, when encoded by cube vertices, as +1 and -1.
- Smoothing a crossing always refers to oriented smoothing. Following the above crossing sign conventions, it is denoted as ‘0’.
- $\mathcal{N}(K)$  and  $\text{cross}(K)$  denote, respectively,  $K$ ’s number of crossings and set of crossing indices. The number of crossings may always be that of a minimal representation of  $K$  in which case it is a knot invariant aka the crossing number.
- Let  $S$  be a fixed shadow with outside crossing set  $\text{cross}(S)$ , and let  $D \in \mathcal{D}(S)$  be a diagram carried by  $S$ . For a subset  $J \subseteq \text{cross}(S)$ , we write  $D \setminus J$  for the diagram obtained from  $D$  by performing the oriented smoothing at every crossing in  $J$  (“0” smoothing). This is a concise form of Convention 4.1.
- A  $C_k$  tree (a degree- $k$  tree clasper) is the embedded surface made of nodes, edges, and leaves. A  $C_k$  move is surgery along a connected  $C_k$  tree. Two links are  $C_k$ -equivalent if they are related by a sequence of  $C_k$  moves. A  $C_k$  move preserves every finite-type invariant of degree  $< k$  [7, 8].
- **GH** refers to Goussarov-Habiro.

## 2 Preliminaries

### 2.1 Finite-type invariants

Fix the category  $\mathcal{K}$  of oriented links in  $S^3$ , considered up to ambient isotopy. Let  $\mathcal{K}^{(d)}$  denote the set of (immersed) oriented singular links with at most  $d$  transverse double points and no other singularities. Let  $A$  be an abelian group (typically  $A = \mathbb{Z}$  or  $\mathbb{Z}[p]$ ).

**Definition 2.1** (Vassiliev extension and order). *A function  $v : \mathcal{K} \rightarrow A$  is finite type of order  $\leq d$  if it admits an extension  $\tilde{v} : \mathcal{K}^{(d)} \rightarrow A$  determined inductively by the Vassiliev skein rule*

$$\tilde{v}(\text{double point}) = \tilde{v}(\text{positive}) - \tilde{v}(\text{negative})$$

at each double point, and satisfies  $\tilde{v}(L) = 0$  whenever  $L$  has  $d+1$  double points. The order (or type) of  $v$  is the least such  $d$ .

**Definition 2.2** (Symbol / weight system). *For  $v$  of order  $\leq d$ , its  $d$ -th symbol  $\sigma_d(v)$  is the restriction of  $\tilde{v}$  to  $\mathcal{K}^{(d)}$  modulo the skein relations, equivalently a function on chord diagrams with  $d$  chords. The symbol factors through the  $1T/4T$  relations, so  $\sigma_d(v)$  defines a linear functional on the degree- $d$  diagram space  $\mathcal{A}_d$  (Bar-Natan).*

**Remark 2.1** (Goussarov–Habiro invisibility). *A connected  $C_m$  move (surgery on a  $C_m$  clasper) preserves every invariant of order  $< m$ . We write “ $C_m$  is invisible to order  $< m$ ” and use this repeatedly throughout this work.*

## 2.2 Pseudo-Boolean functions and finite-type invariants

**Shadows and cubes.** Fix a shadow  $S$  with outside crossing set  $I = \text{cross}(S)$ ,  $|I| = n$ . The associated *variation cube* is

$$\mathcal{Q}(S) = \{\varepsilon = (\varepsilon_i)_{i \in I} \mid \varepsilon_i \in \{\pm 1\}\} \cong \{\pm 1\}^n. \quad (1)$$

For each  $\varepsilon \in \mathcal{Q}(S)$  let  $D(\varepsilon) \in \mathcal{D}(S)$  denote the diagram carried by  $S$  with over/under choices encoded by  $\varepsilon$  (Convention 1.1).

**Pseudo-Boolean functions on a fixed shadow.** Let  $A$  be an abelian group (typically  $\mathbb{Z}$  or  $\mathbb{Z}[p]$ ). A *pseudo-Boolean function on the cube of  $S$*  is any map

$$f_S : \mathcal{Q}(S) \longrightarrow A, \quad \varepsilon \longmapsto f_S(\varepsilon).$$

Given an ambient-isotopy invariant  $v : \mathcal{K} \rightarrow A$ , we obtain its restriction to the cube of  $S$  by evaluation at the vertices:

$$f_v^S(\varepsilon) := v(D(\varepsilon)), \quad \varepsilon \in \mathcal{Q}(S).$$

Thus, every knot/link invariant supplies, for each choice of shadow, a canonical pseudo-Boolean function on  $\mathcal{Q}(S)$ .

**Faces and  $\mathcal{U}$ -subcubes.** For  $\mathcal{U} \subseteq I$  and a base state  $\varepsilon^0 \in \mathcal{Q}(S)$  the  *$\mathcal{U}$ -subcube through  $\varepsilon^0$*  is

$$\mathcal{Q}_{\mathcal{U}}(\varepsilon^0) := \{\varepsilon \in \mathcal{Q}(S) : \varepsilon_i = \varepsilon_i^0 \text{ for all } i \notin \mathcal{U}\}.$$

We freely regard alternating sums over such faces as *finite differences* (discrete derivatives).

**Polynomial-valued invariants and coefficient layers.** Let  $R$  be a commutative ring and let  $J : \mathcal{K} \rightarrow R[p]$  be a polynomial-valued invariant,

$$J(K; p) = \sum_{m \geq 0} f_m(K) p^m, \quad f_m : \mathcal{K} \rightarrow R.$$

On a fixed shadow  $S$  the coefficient functionals induce a family of pseudo-Boolean functions

$$f_{m,S}(\varepsilon) := f_m(D(\varepsilon)), \quad \varepsilon \in \mathcal{Q}(S), m \geq 0.$$

In particular, when  $J$  is the Jones–Vassiliev expansion (introduced in §4.1), we will write  $c_{q,S}(\varepsilon) = c_q(D(\varepsilon))$  for its  $q$ -th coefficient layer on  $\mathcal{Q}(S)$ .

**Connected sums and product cubes.** If a diagram  $D(\varepsilon)$  splits as a connected sum  $D_0(\varepsilon^0) \# D_1(\varepsilon^1)$  realized in disjoint disks, then the shadow cube factors as a Cartesian product  $\mathcal{Q}(S) \cong \mathcal{Q}(S_0) \times \mathcal{Q}(S_1)$ . When  $J$  is multiplicative under connected sum,

$$J_{K_0 \# K_1}(p) = J_{K_0}(p) J_{K_1}(p),$$

the coefficient layers satisfy the Cauchy rule at each vertex  $(\varepsilon^0, \varepsilon^1)$ :

$$f_{m,S}(\varepsilon^0, \varepsilon^1) = \sum_{i=0}^m f_{i,S_0}(\varepsilon^0) f_{m-i,S_1}(\varepsilon^1). \quad (2)$$

**Residual cubes.** For later use we smooth a specified set  $J \subseteq I$  (oriented “0” smoothing from Convention 1.1) and denote by  $\mathcal{Q}_{S \setminus J}$  the variation cube of the resulting shadow; see Definition 4.2. Order-0 constancy on residual cubes is recorded in Corollary 4.3.

### 2.3 Spectral representation

The Boolean cube-complex formalism enjoys a number of advantages. The Vassiliev skein relation is realized by a boundary operator,  $\partial_i$ , contracting/projecting along the  $i$ -th dimension. The type or order of the Vassiliev invariant becomes the *polynomial degree* of the pseudo-Boolean function. Beyond the observation that this may explain why polynomial invariants are native objects for encoding combinatorial finite-type data, we have tools from spectral analysis at our disposal.

**Fourier basis and Walsh characters.** Let  $f : \{\pm 1\}^n \rightarrow \mathbb{C}$  be a pseudo-Boolean function. For every subset  $S \subseteq [n]$ , the Walsh character is given by,

$$\chi_S(\varepsilon) = \prod_{i \in S} \varepsilon_i \quad (\chi_\emptyset \equiv 1). \quad (3)$$

These characters form an orthonormal basis (with respect to the uniform measure on  $\mathcal{Q}_n$ ), so

$$f(\varepsilon) = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S(\varepsilon), \quad \widehat{f}(S) = 2^{-n} \sum_{\varepsilon} f(\varepsilon) \chi_S(\varepsilon). \quad (4)$$

is the Walsh-Fourier expansion of  $f$ . The polynomial degree,  $\deg(f)$ , matches the largest  $|S|$  with  $\widehat{f}(S) \neq 0$ .

**Discrete (finite) derivatives.** For coordinate  $i$  we define the (normalized) derivative

$$(\partial_i f)(\varepsilon) \equiv \frac{1}{2} (f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)})), \quad (5)$$

where  $\varepsilon^{(i,\pm)}$  is  $\varepsilon$  with the  $i$ -th entry set to  $\pm 1$ . For a subset  $T \subseteq [n]$ ,

$$\partial_T = \prod_{i \in T} \partial_i, \quad (\partial_T f)(\varepsilon) = 2^{-|T|} \sum_{\sigma \in \{\pm 1\}^T} (-1)^{\#\{\sigma_i = -1\}} f(\varepsilon_{[n] \setminus T}, \sigma). \quad (6)$$

Equivalently, it is the alternating sum of  $f$  over the  $T$ -subcube through  $\varepsilon$ . For any fixed subset  $T \subseteq [n] = \{1, \dots, n\}$  the  $T$ -fold discrete derivative acts on Walsh characters by “dropping” the coordinates in  $T$ :

$$\partial_T \chi_S = \begin{cases} \chi_{S \setminus T}, & T \subseteq S, \\ 0, & T \not\subseteq S. \end{cases} \quad (7)$$

Because the characters  $\{\chi_S\}_{S \subseteq [n]}$  form an orthonormal basis, we obtain the Fourier (Walsh) expansion of the derivative simply by applying this rule term-wise to the expansion of  $f$ :

$$(\partial_T f)(\varepsilon) = \sum_{[n] \supseteq U \supseteq T} \widehat{f}(U) \chi_{U \setminus T}(\varepsilon). \quad (8)$$

Thus, the Fourier coefficient of  $\partial_T f$  at character  $\chi_S$  (with  $S \cap T = \emptyset$ ) is precisely the coefficient of  $f$  at the larger character  $\chi_{S \cup T}$ , namely,  $\widehat{\partial_T f}(S) = \widehat{f}(S \cup T)$ . This explicit form makes transparent how derivatives shift weight “down” the spectrum and why  $\partial_T f$  vanishes whenever  $f$  has no Fourier support on sets containing  $T$  (the finite-type condition).

**Remark 2.2** (Normalized vs. unnormalized derivatives). *The standard Fourier/Walsh framework is formulated using the normalized difference operator (5). The Vassiliev skein, on the other hand, adopts the unnormalized version, namely,*

$$(\partial_i f)(\varepsilon) \equiv f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)}). \quad (9)$$

*At times, this convention alleviates the burden of dealing with factors such as  $2^{-m}$  when taking derivatives over cube  $m$ -faces. For that reason, and for being consistent with the standard theory, unless stated otherwise,  $\partial_i$  means (9) for the rest of this work.*

**Connected sum and product rules.** The variation cube encoding connected sums of knots, e.g.  $K_0 \# K_1$ , is given by a cartesian product,  $\mathcal{Q} \cong \mathcal{Q}_0 \times \mathcal{Q}_1$ . The set of coordinates  $S$  underlying  $\mathcal{Q}$  can thus be partitioned into two subsets associated with the individual cubes,  $S = S_0 \cup S_1$ . This facilitates composing the discrete derivatives of any pseudo-Boolean function as,

$$\partial_{T_0 \cup T_1} = \partial_{T_0} \partial_{T_1}, \quad \forall T_0 \subseteq S_0, \forall T_1 \subseteq S_1.$$

If, in addition, (2) holds, then

$$\partial_{T_0 \cup T_1} f_m(\varepsilon_{K_0 \# K_1}) = \sum_{i=0}^m [\partial_{T_0} f_i(\varepsilon_{K_0})] [\partial_{T_1} f_{m-i}(\varepsilon_{K_1})]. \quad (10)$$

**Polynomial degree equals Vassiliev order.** A pseudo-Boolean invariant  $f$  is said to be *finite type of (Vassiliev) order  $d$*  if

$$\partial_T f \equiv 0 \quad \text{for every } T \subseteq [n] \text{ with } |T| > d. \quad (11)$$

Intuitively, all  $(d+1)$ -dimensional faces evaluate to alternating sums of zero. The derivative representation immediately implies

$$\widehat{f}(S) = 0 \quad \text{whenever } |S| > d. \quad (12)$$

Conversely, if  $\widehat{f}(S) = 0$  for  $|S| > d$ , then every  $\partial_T f$  with  $|T| > d$  vanishes.

$$\text{Type}(f) = \deg(f) = \max\{|S| : \widehat{f}(S) \neq 0\}. \quad (13)$$

The derivative operator not only detects type but produces new finite-type functions of *singular* knots: Fix a face defined by coordinates  $T$  and take the alternating sum of  $f$  along that face. The result is  $\partial_T f$  restricted to the complementary coordinates. If  $f$  is type  $d$ , then  $\partial_T f$  is type  $d - |T|$ . Iterated extraction of “principal parts.”

Concept	Description
<i>Boolean variation cube</i>	Encodes all simultaneous crossing variations; faces are natural domains for alternating-sum operators.
<i>Walsh-Fourier expansion</i>	Decomposes any pseudo-Boolean invariant into characters labelled by variation subsets.
<i>Top Walsh-Fourier block</i>	$\partial_{[m]}$ where $m$ is the polynomial degree; the symbol.
<i>Discrete derivatives</i>	Are face-wise alternating sums; in Fourier space they simply drop indices.
<i>Finite-type order <math>d</math></i>	Polynomial/Fourier degree $d \leftrightarrow$ vanishing of ( $> d$ )-fold derivatives.
<i>Derivatives over subcubes</i>	Isolate the homogeneous pieces and systematically generate lower-order invariants of singular knots.

Table 1: Pseudo-Boolean functions and finite-type invariants. Glossary of key concepts.

- Order-1 part:  $g_1(\varepsilon) = \sum_{|S|=1} \hat{f}(S) \chi_S(\varepsilon)$ .
- Higher parts analogously via derivatives: the map  $T \mapsto \partial_T f(1, \dots, 1)$  lists exactly the Fourier coefficients of size  $|T|$ .

This is the discrete analogue of repeatedly differentiating a polynomial and evaluating at the origin to read off coefficients.

**Symbols, cubes, and chords.** In finite-type theory the  $m$ -th symbol is an invariant of exactly order  $m$ . That is, it is the restriction of the finite-type function to chord diagrams with  $m$  chords, or equivalently, is the  $m$ -th graded piece in the GH algebra (Definition 2.2). By definition, the symbol is independent of any particular variation of the singular crossings, i.e., it is fixed for a given cube. Over an  $m$ -dimensional variation cube it is extracted by the (unnormalized)  $m$ -fold discrete derivative and hence equals (up to a normalization factor) the top Walsh-Fourier block of the underlying pseudo-Boolean function,

$$\partial_{[m]} f = 2^m \hat{f}([m]).$$

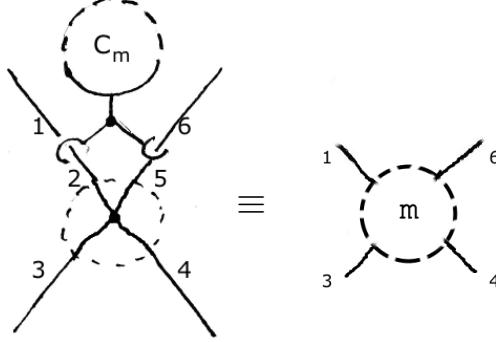
The symbols factorize through the graded space  $\mathcal{A}_m$  and hence depend only on the chord diagram class (modulo the 1T/4T relations) and not on the particular placement of singularities. The pseudo-Boolean framework offers concise descriptions of these local relations which consequently bridge the cubical definition of a symbol with the algebra of chord diagrams (see Lemma A.1 in the appendix).

### 3 Spectral $m$ -gaps

Let  $S$  be a knot shadow with  $N$  labeled double points  $1, \dots, N$ . Fix a choice of over/under data at those  $N$  points that yields a base diagram  $K_0$  (e.g., an unknot).

### 3.1 Extended shadows and $m$ -invisible crossings

For each  $j \in \{1, \dots, N\}$ , fix a simple tree clasper  $C_{m_j}$  for  $K_0$  of (Habiro) degree  $m_j \geq 1$ , embedded in a small ball meeting  $K_0$  only near the  $j$ -th double point, as shown below.



Assume the  $C_{m_j}$  are pairwise disjoint. All choices can be made within disjoint balls around the crossings. Call

$$E = (K_0; \{C_{m_j}\}_{j=1}^N)$$

the *extended shadow*. Below we prove that the clasper-decorated crossing  $m_j$  is invisible to degree  $< m_j$  invariants

$$\partial_j f = f^{(j=+)} - f^{(j=-)} = 0, \quad \deg(f) < m_j.$$

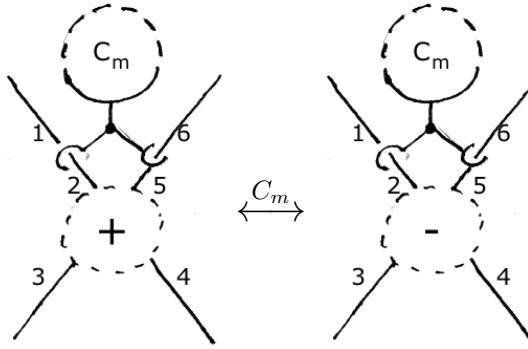
The diagrammatic representation



stands for a clasper-decorated crossing that

does not affect any degree- $< m$  invariant. Similarly, the net effect of flipping  $J$ -subset of crossings (alternating sums) is shifted to degrees  $\geq \sum_{j \in J} m_j$  of the GH filtration.

**Proposition 3.1** (Crossing flip modulo  $C_m$ ).



*Proof.* The proof is shown in Figure 1. □

### 3.2 The carried family

For any sign vector  $\sigma = (\sigma_1, \dots, \sigma_N) \in \{\pm 1\}^N$ , define  $K_E(\sigma)$  to be the knot obtained from  $K_0$  by resolving the  $j$ -th double point with the sign  $\sigma_j$  and, in addition, doing surgery along  $C_{m_j}$  iff

$\sigma_j = -1$ . Alternatively, we may perform all surgeries first and then fix the signs; disjointness of the  $C_{m_j}$  makes this well-defined up to higher GH degree.

A degree- $m_j$  tree realizes a  $C_{m_j}$ -move, hence, by Proposition 3.1,

$$K_E(\sigma) \equiv K_E(\sigma^{(j)}) \pmod{C_{m_j}},$$

so invariants of degree  $< m_j$  cannot see the  $j$ -th sign ( $j$ -th decorated crossing). Here  $\sigma^{(j)}$  is  $\sigma$  with the  $j$ -th entry flipped.

### 3.3 Weighted GH filtration and derivatives

Let  $F_n$  be the GH filtration on the free abelian group  $\mathbb{Z}\mathcal{K}$  on isotopy classes of knots (so  $F_n$  is generated by  $C_n$ -moves) [4]. Write  $\text{gr}_n^C := F_n/F_{n+1}$  and  $\text{gr}^C := \bigoplus_{n \geq 0} \text{gr}_n^C$ .

For  $J \subseteq \{1, \dots, N\}$  define the alternating multi-difference of the family  $K_E(\sigma)$  along the coordinates in  $J$ :

$$\Delta_J K_E := \sum_{S \subseteq J} (-1)^{|J|-|S|} K_E(\sigma^{(S)}),$$

where  $\sigma^{(S)}$  flips the entries of  $\sigma$  exactly on  $S$ . Because all  $C_{m_j}$  are disjoint and  $\deg C_{m_j} = m_j$ , standard clasper calculus implies  $\Delta_J K_E \in F_{w(J)}$ , where  $w(J) := \sum_{j \in J} m_j$  [8]. In particular, for  $J = \{j\}$  we have  $\Delta_{\{j\}} K_E \in F_{m_j}$ , i.e.,  $\partial_j f = 0$  for  $\deg f < m_j$ .

A corollary that packages the whole family:

**Corollary 3.2** (Weighted polynomiality). *Let  $f$  be a finite-type invariant of degree  $\leq d$ . Then for every  $J$ ,*

$$w(J) > d \implies f(\Delta_J K_E) = 0.$$

*Equivalently, the function  $\sigma \mapsto f(K_E(\sigma))$  is a polynomial in the  $N$  Boolean variables whose weighted degree (with  $\deg(\text{variable } j) = m_j$ ) is  $\leq d$ .*

### 3.4 The GH–Taylor expansion around $K_0$

Introduce formal commuting variables  $p_1, \dots, p_N$  with  $\deg p_j := m_j$ . Define the GH–Taylor series of the extended shadow by

$$\mathcal{T}_E(p_1, \dots, p_N) := K_0 + \sum_j p_j \Delta_{\{j\}} K_E + \sum_{|J| \geq 2} \prod_{j \in J} p_j \Delta_J K_E + \dots$$

The  $r$ -th summand is  $\sum_{|J|=r} (\prod_{j \in J} p_j) \Delta_J K_E$  and lies in filtration  $\geq \sum_{j \in J} m_j$ . Thus,  $\mathcal{T}_E$  is a well-defined element of the completed GH group  $\widehat{\mathbb{Z}\mathcal{K}}$ . For example,

$$\begin{aligned} \mathcal{T}_E(p_1, p_2, p_3) &:= \text{(diagram with 1 crossing)} + p_1 \text{(diagram with 1 crossing, m)} + p_2 \text{(diagram with 1 crossing, m)} + p_3 \text{(diagram with 1 crossing, m)} + \\ &\quad \text{(diagram with 2 crossings, mm)} + p_1 p_2 \text{(diagram with 2 crossings, mm)} + p_1 p_3 \text{(diagram with 2 crossings, mm)} + p_2 p_3 \text{(diagram with 2 crossings, mm)} + p_1 p_2 p_3 \text{(diagram with 3 crossings, mmm)} \\ &= \text{(diagram with 1 crossing)} + \mathcal{O}(h^m) \left( \text{(diagram with 1 crossing, m)} + \text{(diagram with 1 crossing, m)} + \text{(diagram with 1 crossing, m)} + \mathcal{O}(h^m) \left( \text{(diagram with 2 crossings, mm)} + \text{(diagram with 2 crossings, mm)} + \text{(diagram with 2 crossings, mm)} + \mathcal{O}(h^m) \left( \text{(diagram with 3 crossings, mmm)} \right) \right) \right) \end{aligned} \tag{14}$$

The shadow  $K_E(\sigma)$  modulo  $F_{>d}$  is recovered by substitution

$$K_E(\sigma) \equiv [\mathcal{T}_E]_{\leq d} \Big|_{p_j=(1-\sigma_j)/2}.$$

So for every finite-type invariant  $f$  of degree  $\leq d$ ,  $f(K_E(\sigma))$  equals the degree- $\leq d$  polynomial in the  $p_j$  obtained by applying  $f$  to  $\mathcal{T}_E$ .

Keeping only one-tree terms gives

$$\mathcal{T}_E \equiv K_0 + \sum_{j=1}^N p_j \underbrace{\Delta_{\{j\}} K_E}_{\in \text{gr}_{m_j}^C} \mod F_{>\min_{j \neq k}(m_j+m_k)}.$$

Identifying  $\Delta_{\{j\}} K_E$  with the class in  $\text{gr}_{m_j}^C$  represented by surgery on  $C_{m_j}$ , this is exactly

$$K_0 + \sum_{j=1}^N p_j C_{m_j} \implies V_{K_E}(h) = V_{K_0}(h) + \sum_{j=1}^N h^{m_j} V_{C_{m_j}}(h),$$

with  $p_j$  the grading variable and  $C_{m_j}$  the degree- $m_j$  tree class attached at the  $j$ -th crossing. Here,  $V_K(h)$  is the Taylor expansion of some polynomial invariant, e.g., Jones.

The  $\Delta_J$  with  $|J| \geq 2$  package all interaction terms between different claspers, which first appear in filtration  $\sum_{j \in J} m_j$ . They are necessary for exactness beyond the linear (one-tree) regime.

### 3.5 $m$ -flat families

**Lemma 3.3** ( $m$ -flat family carried by an extended shadow). *Let  $m = \min_{1 \leq j \leq N}(m_j)$ . The knots carried by the shadow restricted to the decorated crossings,  $\sigma \mapsto K_E(\sigma)$ , share the following characterizing properties:*

1. *They are  $m$ -trivial (and hence also  $J_m$ -trivial, see Definition 4.1).*
2. *The order- $< m$  symbols (alternating sums of finite-type invariants) computed over subsets  $\{\sigma_j\}_{j \in J}$  of decorated crossings vanish.*

*Proof.* The GH-Taylor series shows that all knots  $\sigma \mapsto K_E(\sigma)$  are pairwise  $m$ -equivalent. Because the original shadow consists of at least one unknot it follows that each knot in  $\{K_E(\sigma)\}_{\sigma \in \{\pm 1\}^N}$  is  $m$ -trivial. As all finite-type sub- $m$  data is fixed over the cube coordinates  $\{\sigma_j\}_{j=1}^N$  it follows that,

$$f(\Delta_J K_E) = \partial_J f(K_E) = 0, \quad \deg(f) = |J| < m.$$

□

### 3.6 Uniform barrier and unknot detection

Extended shadows carry  $m$ -flat families of knots. Translating into a cube language, the extended cube  $\mathcal{Q}_E$  is a product of an  $m$ -flat base subcube  $\mathcal{Q}_S$  and the clasper decorations subcube  $\mathcal{Q}_C$ ,

$$E = S + p^m C \implies \mathcal{Q}_E = \mathcal{Q}_S \times \mathcal{Q}_C.$$

The extension cube's dimension depends on the clasper tree decorations. Normally, these scale linearly with the parameters  $m_j$ . Assuming  $m_j = \mathcal{O}(N)$ , the degree of the base shadow, and taking  $C_{m_j}$  as Brunnian claspers, we note that  $\dim \mathcal{Q}_E = \mathcal{O}(N^2)$ .

Applying the above construction using the finite-type expansion of the Jones polynomial leads to a bound on the size of  $m$ -flat families. We give here a proof sketch of the uniform barrier. The complete proof is provided later on following the formalization of the underlying concepts.

**Theorem** (Uniform barrier). *For any non-trivial  $K$  we have  $m \leq \mathcal{N}(K)$ .*

*Proof sketch.* Let  $S$  be the shadow of  $K$ . By Lemma 3.3 the subcube  $\mathcal{Q}_S$  is Jones- $m$ -flat /  $m$ -flat (all order- $< m$  symbols vanish). Theorem 4.10 and Corollary 4.11 show that if  $m > |\mathcal{Q}_S| = \mathcal{N}(K)$  the subcube  $\mathcal{Q}_S$  becomes completely flat, in which case all knots carried by  $S$  are trivial: this is a consequence of the Shadow lemma B.1 property P, and the Jones symbolic skein relation (21). Simply, smoothing crossings in a Jones- $m$ -flat cube results in  $m$  link components, indicating that the crossings are nugatory/cut-vertex. The clasper decorations occupy the higher degrees in the GH filtration and thus may be trivialized without affecting this conclusion. As  $K$  is non-trivial, we have here an obstruction mechanism that caps the size of flat knot families in extension cubes by limiting  $m$ : unless a knot is trivial, it cannot be ( $m$ -trivial)  $J_m$ -trivial with  $m > \mathcal{N}$ .  $\square$

The detection results follow directly from the barrier.

**Theorem.**  $V_K \equiv 1 \rightarrow K$  is the unknot.

*Proof.* This follows from the uniform barrier. If  $K$  were non-trivial and  $V_K = 1$ , i.e.,  $J_m$ -trivial for all  $m$ , then  $m \leq N(K)$  for all  $m$ . Contradiction.  $\square$

**Theorem** (Vassiliev). *A knot that is  $m$ -trivial for all  $m$  is the unknot.*

*Proof.* This follows from the uniform barrier. If  $K$  were non-trivial and  $m$ -trivial for all  $m$ , then  $m \leq N(K)$  for all  $m$ . Contradiction.  $\square$

## 4 $J_m$ -triviality and Jones- $m$ -flatness

### 4.1 Finite-type expansion of the Jones polynomial

We start from the Jones skein

$$x^{-2}V^{(+)} - x^2V^{(-)} = (x - x^{-1})V^{(0)}. \quad (15)$$

and pass to the Birman–Lin substitution  $x = e^{h/2}$ . This reorganizes  $V_L$  into a power series

$$V_L(e^{h/2}) = \sum_{n \geq 0} c_n(L) h^n, \quad (16)$$

whose coefficients  $c_n$  are Vassiliev invariants of order  $\leq n$  [5].

**Definition 4.1** (Jones  $m$ -trivial,  $J_m$ -trivial). *A knot  $K$  is  $J_m$ -trivial if its  $n$ -type coefficients  $c_n(K) = 0$  for  $1 \leq n \leq m - 1$ .*

**Remark 4.1.** *Note that if  $V_K$  has a  $h$ -gap of order  $m$ :*

$$V_K(e^{h/2}) = 1 + h^m V'_K(h), \quad (17)$$

*then it is  $J_m$ -trivial.*

**Remark 4.2** (Every knot is  $J_m$ -trivial for some  $m$ ). *Every non-trivial knot is  $J_m$ -trivial for some  $m \geq 2$  (as  $c_1 = 0$  for knots). The unknot is  $J_m$ -trivial for any  $m$ .*

**Remark 4.3.** *Standard  $m$ -triviality (GH) implies  $J_m$ -triviality, because it forces every degree  $\leq m - 1$  finite-type invariant to agree with the unknot, hence the coefficients  $c_n$  vanish for  $1 \leq n \leq m - 1$ .*

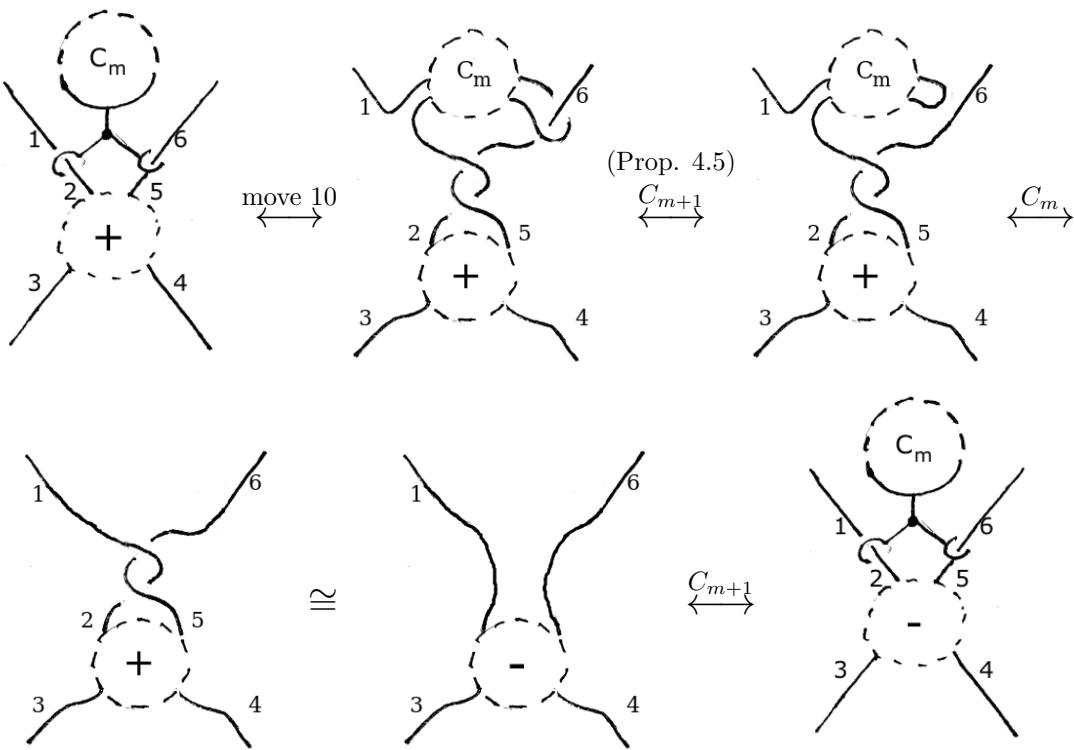


Figure 1: Proof of Proposition 3.1. The proof follows Habiro's clasper calculus. The  $C_m$ -tree confined to the upper dashed region is band-summed to a  $C_2$ -tree (a node). The movie consists of  $C_m$ ,  $C_{m+1}$  moves and isotopies. The two moves in the upper left corner are Habiro's move 10 (Figure 9 on p. 15 in [8]) and a  $C_{m+1}$  following his Proposition 4.5 (Figure 30 on p. 34 in [8]). The movie in its entirety preserves all degree- $< m$  invariants.

In the Walsh-Fourier viewpoint each coefficient layer becomes a pseudo-Boolean function on a fixed shadow cube, so facewise differences isolate orders and symbols. Two immediate consequences drive what follows:  $c_0$  is a shadow invariant encoding component count, and the skein differentiates the layers, giving a symbol-level recursion we use to set up the symbol calculus of §4.4.

**Lemma 4.1** (Order-0 coefficient  $c_0$ ). *The order-0 coefficient in (16) encodes the number of linked components:*

$$c_0(L) = (-2)^{\ell(L)-1}.$$

*Proof.* This follows from  $V_L(1) = (-2)^{\ell(L)-1}$ .  $\square$

**Lemma 4.2** (Finite-type coefficients skein relation). *For every  $n \geq 1$  one has*

$$\partial c_n = c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0) + R_{n-2}, \quad (18)$$

where the remainder  $R_{n-2}$  is an explicit linear combination of strictly lower Vassiliev orders ( $\leq n-2$ ):

$$\begin{aligned} R_{n-2} = & - \sum_{r \geq 1} \frac{1}{(2r)!} \partial c_{n-2r} + \sum_{r \geq 1} \frac{1}{(2r+1)!} (c_{n-2r-1}(+) + c_{n-2r-1}(-)) \\ & + \sum_{r \geq 1} \frac{1}{2^{2r}(2r+1)!} c_{n-2r-1}(0). \end{aligned} \quad (19)$$

*Proof.* Substitute  $x = e^{h/2}$  in the skein  $x^{-2}V^{(+)} - x^2V^{(-)} = (x - x^{-1})V^{(0)}$  and expand:

$$e^{-h} \sum_{n \geq 0} c_n(+) h^n - e^h \sum_{n \geq 0} c_n(-) h^n = (e^{h/2} - e^{-h/2}) \sum_{n \geq 0} c_n(0) h^n.$$

Equating the coefficient of  $h^n$  and separating even/odd exponential contributions yields

$$\partial c_n = - \sum_{m \geq 1} \frac{(-1)^m}{m!} c_{n-m}(+) + \sum_{m \geq 1} \frac{1}{m!} c_{n-m}(-) + \sum_{\substack{\alpha \geq 1 \\ \alpha \text{ odd}}} \frac{1}{2^{\alpha-1}\alpha!} c_{n-\alpha}(0).$$

Group the even/odd  $m$ 's on the left to obtain the displayed  $R_{n-2}$  and isolate the leading  $m = \alpha = 1$  terms to get (18). For fixed  $n$ , all sums are finite. Finally, each object in  $R_{n-2}$  has Vassiliev order at most  $n-2$  (because it uses indices  $< n-1$ ).  $\square$

**Convention 4.1** (Smoothing notation  $D \setminus J$ ). *Let  $S$  be a fixed shadow with outside crossing set  $\text{cross}(S)$ , and let  $D \in \mathcal{D}(S)$  be a diagram carried by  $S$ . For a subset  $J \subseteq \text{cross}(S)$ , we write  $D \setminus J$  for the diagram obtained from  $D$  by performing the oriented smoothing at every crossing in  $J$  (our “0” smoothing). Thus:*

- $D \setminus \emptyset = D$ , and for a singleton  $\{i\}$  we have  $D \setminus \{i\} = D(i=0)$ .
- Smoothing at distinct crossings is local and order-independent, so  $D \setminus J$  is well defined (independent of any ordering of  $J$ ).
- If  $J \subseteq \text{cross}(S)$  and  $S \setminus J$  denotes the shadow with those vertices smoothed, then the residual cube  $\mathcal{Q}_{S \setminus J}$  is the variation cube on the remaining crossings; functions of the form  $D \mapsto f(D \setminus J)$  are naturally viewed on  $\mathcal{Q}_{S \setminus J}$  (see Definition 4.2 and Corollary 4.3).

- We occasionally write  $J_r \subseteq J$  to mean an arbitrarily chosen  $r$ -subset of  $J$ ; by order-independence, statements about  $D \setminus J_r$  do not depend on the choice of  $J_r$  (used, e.g., when discussing incremental smoothings and the last-row fingerprint).
- In single-crossing skeins we retain  $D(+), D(-), D(0)$ ; the set-notation agrees with this:  $D \setminus \{i\} = D(i=0)$ .

**Definition 4.2.** For a shadow  $S$  and  $J \subset \text{cross}(S)$ , the **residual cube**  $\mathcal{Q}_{S \setminus J}$  is the variation cube of over/under choices on the remaining crossings after smoothing all  $J$ .

**Corollary 4.3** (Order-0 constancy on residual cubes). Fix any shadow  $S$  of  $K$ . Because  $c_0$  is a type-0 (order-0) Vassiliev invariant, for any fixed subset  $J \subset \text{cross}(S)$  of crossings the map  $D \mapsto c_0(D \setminus J)$  is constant on the residual cube of over/under choices. Here  $\text{cross}(S)$  denotes the vertex set of the shadow (the common set of crossings for all  $D \in \mathcal{D}(S)$ ). In particular, under the Jones-flatness hypothesis of Lemma 4.7 below,  $c_0(D \setminus J) = (-2)^{|J|}$  for all  $D \in \mathcal{D}(S)$ .

*Proof.* Since  $c_0$  is type-0,  $\partial_T c_0 \equiv 0$  for all nonempty  $T$ , i.e., it is constant on every face of every residual cube  $\mathcal{Q}_{S \setminus J}$ .  $\square$

**Definition 4.3** (Set-indexed symbols). For any  $K \subset I$  and  $J \subset I \setminus K$ , define

$$\mathcal{S}_{K;J} := \partial_J c_{|J|}^K$$

i.e. first perform oriented smoothings at all crossings in  $K$ , then take the unnormalized alternating sum over the  $J$ -face (order  $|J|$ ).

**Convention 4.2** (Set/cardinality-indexed symbols). From now on we write  $\mathcal{S}_{K;J} := \partial_J c_{|J|}^K$  to record both the smoothed subset  $K$  and the differenced subset  $J$  explicitly. If only the cardinalities matter, we write  $\mathcal{S}_{r,n}$  for the class generated by all  $\mathcal{S}_{K;J}$  with  $|K|=r, |J|=n$ .

**Lemma 4.4** (Symbol-skein recursion). Fix a crossing  $i$ . For every  $n \geq 1$ ,

$$\mathcal{S}_{\emptyset;J} = 2\mathcal{S}_{\emptyset;J \setminus \{i\}} + \mathcal{S}_{\{i\};J \setminus \{i\}} \quad (i \in J, |J|=n). \quad (20)$$

and more generally,

$$\mathcal{S}_{K;J} = 2\mathcal{S}_{K;J \setminus \{i\}} + \mathcal{S}_{K \cup \{i\};J \setminus \{i\}}. \quad (21)$$

*Proof of Lemma 4.4 using state-sum for the order- $n$  symbol.* Classically, the symbol of  $c_n$  on a chord diagram  $D$  with chords  $\{j\}$  can be written as a state-sum over functions  $s : \{j\} \rightarrow \{1, 2\}$  (“double”/“delete” the chord) (see p. 71 in [4]):

$$\mathcal{S}_{K;J} = \sum_s \left( \prod_j s(j) \right) \cdot (-2)^{|\pi_0(s)|-1},$$

where  $|\pi_0(s)|$  is the number of components of the immersed curve obtained by applying the resolution prescribed by  $s$ . Fix a distinguished chord  $i$  and split the sum according to  $s(i)$ :

- If  $s(i) = 2$  (delete), the local multiplicative weight contributes a factor 2, and the resulting curve is obtained from  $D$  by forgetting the chord  $i$ . Summing over the remaining chords gives  $2\mathcal{S}_{K;J \setminus \{i\}}$ .
- If  $s(i) = 1$  (double), the local multiplicative weight contributes a factor 1, and the resulting curve is the oriented smoothing at  $i$ . Summing over the remaining chords gives  $\mathcal{S}_{K \cup \{i\};J \setminus \{i\}}$ .

Adding the two disjoint families of states yields (21).  $\square$

*Proof of Lemma 4.4 using Lemma 4.2.* Apply the  $(n - 1)$ -fold unnormalized face-difference  $\partial_{J \setminus \{i\}}$  (with  $i \in J$ ,  $|J| = n$ ) to the coefficient recursion (18):

$$\partial_J c_n = \partial_{J \setminus \{i\}}(c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0)) + \partial_{J \setminus \{i\}} R_{n-2}.$$

Since every term in  $R_{n-2}$  has Vassiliev order  $\leq n - 2$ , its  $(n - 1)$ -fold alternating sum vanishes identically:  $\partial_{J \setminus \{i\}} R_{n-2} \equiv 0$ . What remains are exactly the three order- $(n - 1)$  pieces on the right. Group the two  $(\pm)$  terms (they live on the same residual  $(n - 1)$ -face) and the smoothed term (which lives on the  $(n - 1)$ -face after oriented smoothing at  $i$ ), to obtain

$$\partial_J c_n = 2 \partial_{J \setminus \{i\}} c_{n-1} + \partial_{J \setminus \{i\}} c_{n-1}^{\{i\}},$$

which is (20).  $\square$

## 4.2 Order-0 constancy: $c_0$ is a shadow/cube invariant

The BL-expansion (16) free coefficient  $c_0$  is a 0-order finite-type invariant, which in turn makes it an invariant of variation cubes or shadows rather than of any single vertex (knot/link). In addition, we have seen it uniquely encodes the link component count. These two facts render it a powerful tool for analyzing the behavior of shadows under oriented smoothing.

**Corollary 4.5** (Shadow invariance of component counts). *Let  $S$  be a shadow and  $D(S)$  the set of all diagrams carried by  $S$ . Then:*

1. (**No smoothing**) *The number of link components  $\ell(D)$  is independent of  $D \in D(S)$ .*
2. (**Fixed smoothings**) *More generally, for any  $J \subseteq \text{cross}(S)$  (crossing set of  $S$ ), the component count of  $D \setminus J$  (link obtained from  $D$  by smoothing exactly the crossings in  $J$ ) is independent of  $D \in D(S)$ .*

*Proof.* By order-0 constancy (Corollary 4.3),  $D \mapsto c_0(D \setminus J)$  is constant on the residual cube determined by  $J$ . By Lemma 4.1,  $c_0(D \setminus J) = (-2)^{\ell(D \setminus J) - 1}$ , so  $\ell(D \setminus J)$  is constant as  $D$  varies. Taking  $J = \emptyset$  yields (1).  $\square$

**Corollary 4.6** ( $c_0^m$  fingerprint of property (P)). *Let  $D \in D(S)$  be a diagram (shadow  $S$ ). Fix a subset  $J \subseteq \text{cross}(D)$  of size  $|J| = m$ . If*

$$c_0(D \setminus J) = (-2)^m,$$

*then for every  $k$ -subset  $K \subset J$  (with  $0 \leq k \leq m$ ) we have*

$$c_0(D \setminus K) = (-2)^k \quad \text{and} \quad \#\text{components}(D \setminus K) = 1 + k. \quad (\mathbf{P})$$

*Equivalently, property (P) holds along the fixed set  $J$  for all  $k \leq m$ . If, moreover,  $m = |\text{cross}(D)|$  then by the Shadow Lemma (B.1),  $D$  is an unknot.*

*Proof.* By Lemma 4.1,  $c_0(L) = (-2)^{\ell(L) - 1}$ , so the hypothesis says  $D \setminus J$  has  $(m + 1)$  components. Smooth the  $m$  crossings in  $J$  in any order and write  $\delta_i \in \{0, 1\}$  for the increment in the component count at step  $i$ . Oriented smoothing changes the component count by at most 1, so  $\delta_i \leq 1$ . Since the final count is  $1 + \sum_{i=1}^m \delta_i = m + 1$ , each  $\delta_i = 1$ . Thus the first  $k$  smoothings (those in any prescribed  $k$ -subset  $K \subset J$ ) yield exactly  $1 + k$  components, hence  $c_0(D \setminus K) = (-2)^k$  (again by Lemma 4.1). Finally, order-0 constancy on residual cubes (Corollary 4.3) implies that for fixed  $J$  and  $K \subset J$  the value of  $c_0(D \setminus K)$  is independent of the over/under choices at the remaining crossings, so the same conclusion holds for every diagram carried by  $S$ .  $\square$

### 4.3 Jones-flat cubes

**Definition 4.4.** A variation cube all of whose vertices are unknots is said to be **Jones-flat**. Equivalently, every variation of the underlying shadow is an unknot.

**Lemma 4.7** (Jones-flatness). Denote by  $\delta_n$  the Kronecker delta  $\delta_{n0}$ . A necessary and sufficient condition for Jones-flatness is the vanishing of all non-trivial symbols,

$$\mathcal{S}_{0,n} = \partial_{[n]} c_n^0 = \delta_n. \quad (22)$$

*Proof.* Let  $\mathcal{S}_{r,n} = \partial_{[n]} c_n^r$  be the  $n$ -th symbol after  $r$  oriented smoothings. The symbol recursion (21) reads,

$$\mathcal{S}_{r,n-1} = \mathcal{S}_{r-1,n} - 2\mathcal{S}_{r-1,n-1},$$

where  $r$  and  $n$  are the cardinalities of the smoothed and differentiated crossing sets (Convention 4.2). As  $\mathcal{S}_{0,n} = \delta_n$ , it follows that,  $\mathcal{S}_{r,n} = (-2)^r \delta_n$ , and in particular, the free coefficient after  $r$  smoothings,  $c_0^r = (-2)^r$ . Fix a vertex  $K$  in the variation cube. Note that  $r = |\text{cross}(S)|$  is the number of crossings in any diagram obtained from the shadow. Therefore, by Corollary 4.6 (P) holds for all  $k \leq |\text{cross}(S)|$ , rendering  $K$  the unknot. As this holds for every vertex  $K$ , the cube is Jones-flat.

The converse direction, every variation is the unknot  $\rightarrow \mathcal{S}_{0,n} = \delta_n$ , follows immediately upon recognizing that all Fourier coefficients in all cubes vanish except for the trivial  $c_0 = 1$ , from which the symbols follow.  $\square$

### 4.4 Jones- $m$ -flat subcubes

**Definition 4.5** (Jones- $m$ -flat subcube). Let  $S$  be a shadow and  $\mathcal{U} \subseteq \text{cross}(S)$ . The  $\mathcal{U}$ -subcube  $\mathcal{Q}_{\mathcal{U}}$  is Jones- $m$ -flat if for every  $1 \leq n \leq m-1$  the coefficient  $c_n$  is constant on  $\mathcal{Q}_{\mathcal{U}}$  (equivalently, all facewise differences  $\partial_W c_n$  with  $\emptyset \neq W \subseteq \mathcal{U}$  vanish). If this holds for all  $m$ , call  $\mathcal{Q}_{\mathcal{U}}$  Jones- $\infty$ -flat or flat for short.

**Remark 4.4.** “Flat” here means all finite-type layers below  $|\mathcal{U}|$  are constant on the  $\mathcal{U}$ -subcube; only when  $\mathcal{U}$  is the full crossing set does the Shadow Lemma B.1 implies all vertices are unknots.

**Remark 4.5** (Flat subcube constants). On a flat subcube every positive-order ( $n \geq 1$ )  $c_n$  is constant. Whenever a shadow subcube contains unknot vertices it pins these constants to 0. The analogous anchor for Jones- $m$ -flat subcubes is that orders  $1 \leq n \leq m-1$  vanish identically by definition. Think of  $J_m$ -trivial vertices as playing the same role for Jones- $m$ -flat subcubes that the unknot plays for flat subcubes.

**Lemma 4.8** (Symbols vanish on Jones- $m$ -flat subcubes through order  $m-1$ ). If  $\mathcal{Q}_{\mathcal{U}}$  is Jones- $m$ -flat, then  $\mathcal{S}_{\emptyset;J} \equiv 0$  for every nonempty  $J \subseteq \mathcal{U}$  with  $|J| \leq m-1$ .

*Proof.* For each  $1 \leq n \leq m-1$ , the function  $c_n$  is constant on  $\mathcal{Q}_{\mathcal{U}}$  (by definition of Jones- $m$ -flatness). Therefore, all face derivatives vanish,  $\partial_{[n]} c_n \equiv 0$ .  $\square$

For  $K \subset I$  and  $J \subset I \setminus K$  set  $\mathcal{S}_{K;J} := \partial_J c_{|J|}^K$ . When only the cardinalities  $r = |K|$  and  $q = |J|$  matter we write  $\mathcal{S}_{r,q}$  for the rank-1 line generated by all  $\mathcal{S}_{K;J}$  with  $|K| = r, |J| = q$ ; in this quotient the specific choice of  $J$  is forgotten.

Define the pair of operators:

$$v_i : \mathcal{S}_{r,q} \rightarrow \mathcal{S}_{r,q-1}, \quad h_i : \mathcal{S}_{r,q} \rightarrow \mathcal{S}_{r+1,q-1}, \quad (23)$$

that is,  $v_i$  lowers symbol order by one by taking the partial difference at  $i$ ,  $h_i$  lowers symbol order and increases the smoothing index by one after oriented smoothing at  $i$ ; concretely  $v_i(\mathcal{S}_{K;J}) = \mathcal{S}_{K;J \setminus \{i\}}$ ,  $h_i(\mathcal{S}_{r,q}) = \mathcal{S}_{K \cup \{i\}; J \setminus \{i\}}$ . The symbol skein may thus be expressed as

$$\mathcal{S}_{K;J} = 2v_i(\mathcal{S}_{K;J}) + h_i(\mathcal{S}_{K;J}), \quad (24)$$

i.e.,  $2v_i + h_i$  acts as an identity on  $\mathcal{S}_{K;J}$ .

#### 4.4.1 Symbol module and operator calculus

Fix a shadow  $S$  with outside crossing set  $I = \text{cross}(S)$ . For  $r, n \geq 0$ , let  $\mathbb{Z}^{\mathcal{D}(S)}$  denote the free abelian group on diagrams carried by  $S$ , and let  $c_n^r : \mathcal{D}(S) \rightarrow \mathbb{Q}$  be the order- $(\leq n)$  Vassiliev functional after  $r$  oriented smoothings. Define the symbol extraction map  $E_{r,n} : \mathbb{Q}^{\mathcal{D}(S)} \rightarrow \mathcal{S}_{r,n}$  by

$$E_{r,n}(F) := \partial_{[n]} F^r \quad (\text{unnormalized alternating sum on } n\text{-faces}),$$

and set  $\mathcal{S}_{r,n} := \text{im } E_{r,n}$  (the symbol module of order  $n$  after  $r$  smoothings). Elements of  $\mathcal{S}_{r,n}$  are written  $\mathcal{S}_{r,n}$  in the sequel (Figure 2 visualizes the grid  $(r, n)$ ).

Define linear maps

$$\begin{aligned} v_i : \mathcal{S}_{r,n} &\rightarrow \mathcal{S}_{r,n-1}, & v_i(\mathcal{S}_{K;J}) &:= \mathcal{S}_{K;J \setminus \{i\}}, \\ h_i : \mathcal{S}_{r,n} &\rightarrow \mathcal{S}_{r+1,n-1}, & h_i(\mathcal{S}_{K;J}) &:= \mathcal{S}_{K \cup \{i\}; J \setminus \{i\}}, \end{aligned}$$

with  $|K| = r$ ,  $|J| = n$ . These are well-defined on  $\mathcal{S}_{r,n}$  because the assignments are linear in  $c_n^r$  and depend only on the image  $E_{r,n}(c_n^r)$ . (Linearity: the constructions are formed from alternating sums. Well-definedness follows a posteriori from the operator identity below.)

**Corollary 4.9** (path independence of smoothing at the symbol level). *For any subset  $J \subset I$  with  $|J| = k$ , the operator that smooths the crossings in  $J$  (in any order) acts on symbols by*

$$h_J := \prod_{i \in J} h_i = \prod_{i \in J} (\text{id} - 2v_i) = \sum_{L \subseteq J} (-2)^{|L|} v_L, \quad (25)$$

where  $v_L := \prod_{i \in L} v_i$  (the products are order-independent). In particular, path independence means the result depends only on the set  $J$ , not the order in which its elements are smoothed. (This follows from the one-crossing skein  $h_i = \text{id} - 2v_i$  and commutativity of the discrete differences.)

**Remark 4.6.** In general  $v_L \mathcal{S}_{\emptyset;J}$  depends on the set  $L$ , not only on its size. On Jones- $m$ -flat subcubes the dependence disappears when acting on  $\mathcal{S}_{\emptyset;J}$  with  $|J| = m-1$ , because all intermediate top-row symbols vanish,  $v_L \mathcal{S}_{\emptyset;J} = \mathcal{S}_{\emptyset;J \setminus L} = 0$ , for  $0 \leq |L| < m-1$  (Lemma 4.8); this is the only specialization used in the proof of Theorem 4.10.

#### 4.4.2 Last-row fingerprint and degree cap

**Theorem 4.10** (Last-row fingerprint of Jones- $m$ -flat subcubes). *Let  $\mathcal{Q}_{\mathcal{U}}$  be Jones- $m$ -flat. For any prescribed set  $J \subseteq \mathcal{U}$  of  $m-1$  crossings and any diagram  $D$  carried by  $S$ , the free (order-0) coefficient satisfies the **last-row fingerprint***

$$c_0(D \setminus J) = (-2)^{m-1}.$$

Equivalently, for every such  $J$ ,

$$\mathcal{S}_{J;\emptyset} = (-2)^{m-1} \mathcal{S}_{\emptyset;\emptyset} = (-2)^{m-1}.$$

As this holds for every  $J \subseteq \mathcal{U}$  with  $|J| = m - 1$  we write  $c_0^{m-1} = (-2)^{m-1}$ . By Corollary 4.6, this last-row equality implies property (P) for every  $k \leq m - 1$  (uniformly over all diagrams carried by the same shadow).

*Proof.* Fix  $J \subseteq \mathcal{U}$  with  $|J| = m - 1$ . Apply (25),

$$\mathcal{S}_{J;\emptyset} = h_J(\mathcal{S}_{\emptyset;J}) = \sum_{L \subseteq J} (-2)^{|L|} v_L \mathcal{S}_{\emptyset;J}.$$

Each term satisfies  $v_L \mathcal{S}_{\emptyset;J} = \mathcal{S}_{\emptyset;J \setminus L}$ . On a Jones- $m$ -flat subcube we have exactly the vanishing

$$\mathcal{S}_{\emptyset;J \setminus L} = 0 \quad \text{for } 1 \leq |J \setminus L| \leq m - 1$$

(Lemma 4.8), so all terms with  $0 \leq |L| < m - 1$  die after restriction to the  $\mathcal{U}$ -slice. The only surviving term is,  $L = J$ :  $v_J \mathcal{S}_{\emptyset;J} = \mathcal{S}_{\emptyset;\emptyset} = c_0$ . Hence

$$\mathcal{S}_{J;\emptyset} = (-2)^{m-1} \mathcal{S}_{\emptyset;\emptyset} = (-2)^{m-1},$$

i.e.  $c_0(D \setminus J) = (-2)^{m-1}$ . This is the last-row fingerprint.

*From last-row fingerprint to property (P).* By Lemma 4.1,  $c_0(L) = (-2)^{\ell(L)-1}$ . Thus  $\mathcal{S}_{J;\emptyset} = (-2)^{m-1}$  means that after smoothing the prescribed  $|J| = m - 1$  crossings, the component count is  $m$ . Smoothing in any order can raise the component count by at most 1 at each step; hitting the final value  $m$  forces every intermediate increment to be exactly +1. Hence for every  $K \subseteq J$ ,  $c_0(D \setminus K) = (-2)^{|K|}$ , i.e. property (P) holds along  $J$  (Corollary 4.6).  $\square$

**Corollary 4.11** (Degree cap of  $m$ -flatness). *Let  $S$  be a shadow,  $\mathcal{U} \subset \text{cross}(S)$  a set of  $d := |\mathcal{U}|$  crossings, and  $\mathcal{Q}_{\mathcal{U}}$  the  $\mathcal{U}$ -subcube. Assume  $\mathcal{Q}_{\mathcal{U}}$  is Jones- $m$ -flat, i.e. for every  $1 \leq n < m$  the coefficient  $c_n$  is constant on  $\mathcal{Q}_{\mathcal{U}}$ . Then:*

(i) Supercritical case yields (P) along  $\mathcal{U}$ . If  $m > d$ , then property (P) holds along  $\mathcal{U}$ : for every  $K \subseteq \mathcal{U}$ , smoothing the crossings in  $K$  increases the component count by  $|K|$ , and

$$\mathcal{S}_{K;\emptyset} = c_0(D \setminus K) = (-2)^{|K|} \quad \text{for all } D \in \mathcal{D}(S).$$

(ii) Global nugatory cap. If moreover  $\mathcal{U} = \text{cross}(S)$  (so  $d$  is the total number of crossings of  $S$ ), then the shadow is a tree (every crossing nugatory), hence all diagrams carried by  $S$  are unknots; in particular the complete cube is isotopically constant.

(iii) Degree cap (contrapositive). In the global situation  $\mathcal{U} = \text{cross}(S)$ , if the complete cube is not isotopically constant, then necessarily  $m \leq d$ .

*Proof.*

(i) Set  $d = |\mathcal{U}|$  and suppose  $m > d$ . Then  $m' := d + 1 \leq m$ . Jones- $m$ -flatness implies Jones- $m'$ -flatness (constancy of  $c_n$  for  $n < m'$ ), so we may apply Theorem 4.10 (the last-row fingerprint) with the prescribed  $(m' - 1)$ -set  $J = \mathcal{U}$ . This gives

$$c_0(D \setminus \mathcal{U}) = (-2)^d \quad \text{for every } D \in \mathcal{D}(S).$$

By Corollary 4.6, the equality  $c_0(D \setminus \mathcal{U}) = (-2)^d$  forces the component count to increase by exactly one at each of the  $d$  smoothings, hence for every  $K \subseteq \mathcal{U}$  we have  $c_0(D \setminus K) = (-2)^{|K|}$  and  $\#\text{components}(D \setminus K) = 1 + |K|$ . This is property (P) along  $\mathcal{U}$ .

(ii) If additionally  $\mathcal{U} = \text{cross}(S)$ , then (P) holds on the full crossing set. By the Shadow Lemma B.1,  $S$  is a tree and every diagram carried by  $S$  is an unknot; consequently the complete cube is isotopically constant.

(iii) The last assertion is the contrapositive of (ii): if the complete cube is not isotopically constant, then (when  $\mathcal{U} = \text{cross}(S)$ ) the hypothesis  $m > d$  cannot hold, so  $m \leq d$ .  $\square$

**Remark 4.7** (Symbol matrix illustration). *The last-row fingerprint behavior of symbols from Theorem 4.10 and Corollary 4.11 is depicted in Figure 2.*

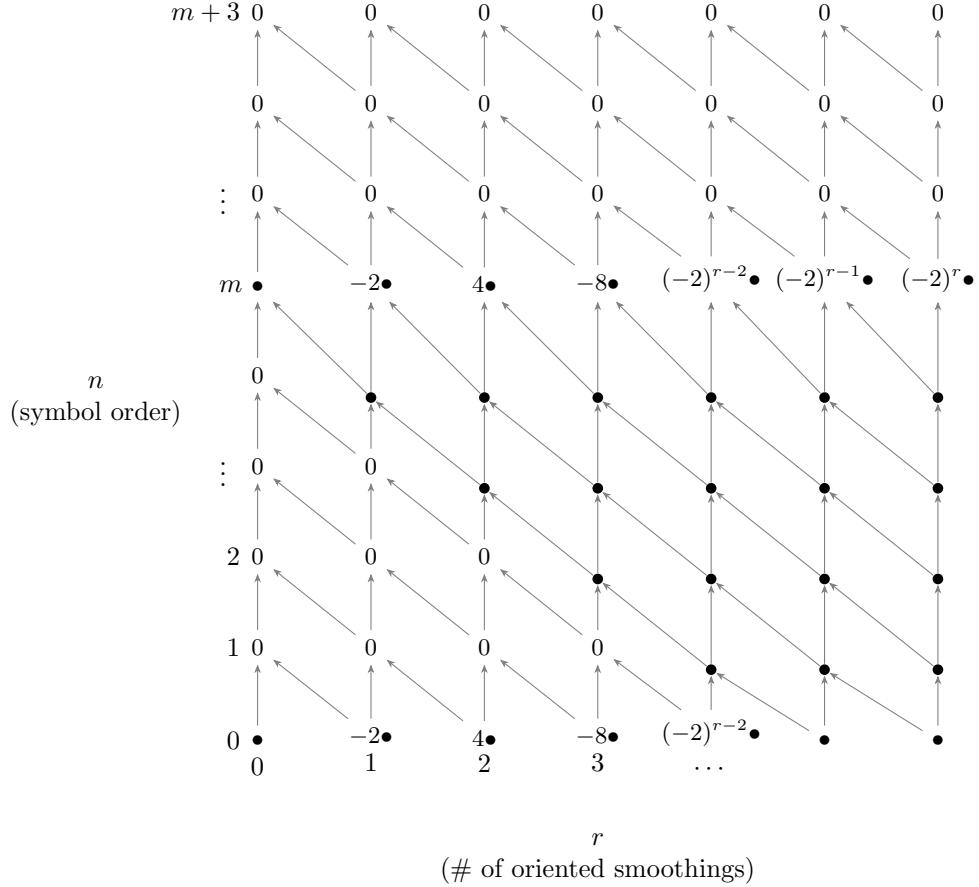


Figure 2: The symbols matrix  $[\mathcal{S}_{r,n}]$  of a Jones- $m$ -flat subcube ( $\mathcal{S}_{0,n} = 0$  for  $1 \leq n \leq m-1$ ) exhibiting the last-row fingerprint from Theorem 4.10. The skein operator  $2v + h$ , where  $v$  and  $h$  are the pair in (23) constrains the triad,  $\mathcal{S}_{r,n}$ ,  $\mathcal{S}_{r,n-1}$ , and  $\mathcal{S}_{r+1,n-1}$ , which ultimately leads to the last-row fingerprint. The distinctive  $(-2)^r$  pattern is evident in both the order- $m$  symbol and the order-0 (free) coefficients,  $\mathcal{S}_{r,0} = c_0^r$ .

## 5 Uniform $m$ -triviality barrier

**Theorem 5.1** (Uniform  $m$ -triviality barrier). *Let  $K$  be  $J_m$ -trivial. If  $K$  is nontrivial, then*

$$m \leq \mathcal{N}(K).$$

*Proof of Theorem 5.1.* Let  $S$  be the shadow of  $K$ . Fix an extended shadow  $E = (K_0; \{C_{m_j}\})$  as in §3.1 with  $\min_j m_j = m$ . The clasper subcube consists of all clasper states, including the inactive (trivial) tangles. Let  $T$  be the set of cube coordinates underlying all inactive clasper decorations, and let  $\mathcal{Q}_{S_T} := \{T\} \times \mathcal{Q}_S$  be the corresponding  $T$ -face of the extended cube. The following lemma shows that  $m$ -flatness carries over from  $\mathcal{Q}_{S_T}$  to  $\mathcal{Q}_S$ .

**Lemma 5.2** (Decorated-to-outside transfer). *Let  $f$  be any finite-type invariant of degree  $< m$ . Then for every nonempty  $J \subseteq \text{cross}(S)$  with  $|J| \leq m - 1$ ,*

$$\partial_J(f \text{ on } \{T\} \times \mathcal{Q}_S) = 0.$$

*In particular, the face  $\{T\} \times \mathcal{Q}_S$  is  $m$ -flat, so every Jones layer  $c_n$  with  $1 \leq n \leq m - 1$  is constant on  $\{T\} \times \mathcal{Q}_S$ .*

*Proof of Lemma 5.2.* Fix a vertex  $\varepsilon \in \mathcal{Q}_S$  and write  $\sigma(\varepsilon) \in \{\pm 1\}^N$  for the same sign vector. By construction (Figure 1), the knot  $K_T(\varepsilon)$  at the  $T$ -face and the decorated knot  $K_E(\sigma(\varepsilon))$  differ by surgeries along a disjoint family of  $C_m$  trees, one for each index with  $\sigma_j(\varepsilon) = -1$ . Thus  $K_T(\varepsilon)$  and  $K_E(\sigma(\varepsilon))$  are  $C_m$ -equivalent, hence  $f(K_T(\varepsilon)) = f(K_E(\sigma(\varepsilon)))$  by GH invisibility (order  $< m$  invariants are preserved by  $C_m$  moves). Taking the alternating sum over the  $J$ -face,

$$\partial_J f|_{\{T\} \times \mathcal{Q}_S} = \sum_{S \subseteq J} (-1)^{|J|-|S|} f(K_T(\varepsilon^S)) = \sum_{S \subseteq J} (-1)^{|J|-|S|} f(K_E(\sigma(\varepsilon^S))) = \partial_J(f \circ K_E).$$

By Lemma 3.3 and Corollary 3.2,  $\partial_J(f \circ K_E) = 0$  for  $|J| \leq m - 1$  (the decorated  $J$ -differences land in filtration  $\geq \sum_{j \in J} m_j \geq m$ , so degree  $< m$  invariants vanish on them). Hence  $\partial_J(f \text{ on } \{T\} \times \mathcal{Q}_S) = 0$  for  $|J| \leq m - 1$ .  $\square$

By Lemma 5.2 the subcube  $\mathcal{Q}_{S_T}$  is Jones- $m$ -flat /  $m$ -flat (all order- $< m$  symbols vanish), i.e., the Jones layers  $c_n \equiv 0$ , for  $1 \leq n \leq m - 1$  on  $\mathcal{Q}_{S_T}$ . In other words, each knot carried by  $\{T\} \times S$  is  $J_m$ -trivial as  $K$  itself is  $J_m$ -trivial. If  $m > \dim \mathcal{Q}_{S_T}$ , Theorem 4.10 and Corollary 4.6 give (P) on the full outside set  $\mathcal{U} = \text{cross}(S)$  (as  $T$  consists of trivial tangle states). Corollary 4.11(ii) (together with Lemma B.1) implies the  $T$ -resolved extended shadow  $\{T\} \times S$  is a tree and the face/subcube  $\mathcal{Q}_{S_T}$  is isotopically constant. As  $K$  is isotopic to  $\{T\} \times K$ , it follows that it is the unknot, a contradiction. Thus  $m \leq \dim \mathcal{Q}_{S_T} = \dim \mathcal{Q}_S$ , and by choosing  $S$  from a minimal diagram of  $K$ , this equals  $\mathcal{N}(K)$ .  $\square$

Figuratively, one may say there is a limit to the degree to which a nontrivial knot  $K$  may pretend to be ‘trivial’. This degree is set by the order  $m$  of its first non-trivial finite-type coefficient. The price to pay for this aspiration comes in the form of increased knottedness; its number of crossings scales as

$$\mathcal{N}(K) \geq m.$$

In a sense, the unknot is the only knot that can afford being  $\infty$ ly trivial.

**Theorem 5.3** (Jones detects the unknot; finite check). *With the Jones normalization of (15), write the expansion (16),  $V_K(e^{h/2}) = \sum_{n \geq 0} c_n(K)h^n$ . If  $V_K \equiv 1$  then  $K$  is the unknot. Equivalently, on a minimal diagram with  $\mathcal{N}(K)$  crossings, if  $c_n(K) = 0$  for  $1 \leq n \leq \mathcal{N}(K)$  then  $V_K \equiv 1$ .*

*Proof.*  $V_K \equiv 1 \Rightarrow c_n \equiv 0$  for  $n \geq 1$ . If  $K$  were non-trivial, Theorem 5.1 would give  $n \leq \mathcal{N}(K)$  for all  $n$ , a contradiction.  $\square$

**Theorem 5.4** (Vassiliev unknot conjecture). *A knot that is  $m$ -trivial for all  $m$  is the unknot.*

*Proof.* Suppose that  $K$  were non-trivial. The statement implies  $K$  is  $J_m$ -trivial for all  $m$ . By Theorem 5.1,  $m \leq \mathcal{N}(K)$  for all  $m$ . Contradiction.  $\square$

## 6 Conclusions and outlook

Birman–Lin showed that coefficients in the series expansions of skein polynomials (Jones, HOMFLY, Kauffman) define Vassiliev invariants, suggesting that polynomial–finite-type bridges might ultimately settle unknot detection via control of low-order layers [5, 6]. Kontsevich’s integral and Bar-Natan’s diagrammatics organize those layers via weight systems and the  $1T/4T$  relations [2, 3]. On the geometric side, Goussarov–Habiro theory provides local moves that are invisible below degree  $m$  (connected  $C_m$  claspers) [7, 8].

What has been missing is a *diagram-local* mechanism that: (i) packages finite-type layers on a fixed shadow; (ii) turns their local vanishing into a rigid, shadow-level constraint; and (iii) transports this flatness across many outside crossings at once. Our contribution is precisely this triptych, expressed entirely in terms of extended shadows and clasper decorations:

**(A)  $m$ -invisible crossings on extended shadows.** Fix a shadow and, near each outside crossing  $j$ , attach a degree- $m_j$  tree clasper. Crossing flips are then realized *modulo clasper calculus* by  $C_{m_j}$  moves, hence are invisible to all invariants of degree  $< m_j$ . With  $m := \min_j m_j$ , the resulting carried family is  $m$ -flat for order  $< m$  finite-type data (weighted GH filtration).

**(B) Jones layers and a symbol-level skein.** For the Birman–Lin expansion  $V_K(e^{h/2}) = \sum_n c_n(K)h^n$ , each  $c_n$  has order  $\leq n$ . On any  $m$ -flat family the layers  $c_1, \dots, c_{m-1}$  are constant, and whenever a single vertex is  $J_m$ -trivial these constants pin to zero (Jones– $m$ -flatness). A symbol-level skein recursion then forces a *last-row fingerprint* for the free term  $c_0$ : smoothing any  $m-1$  chosen crossings produces exactly  $m$  components (property (P) along that set).

**(C) Shadow criterion and the uniform barrier.** A shadow-level criterion converts (P) into global nugatoriness: when  $m$  exceeds the number of varied crossings, the shadow collapses to a tree and every carried diagram is an unknot. Applied to a minimal shadow, this yields the uniform bound

$$m \leq \mathcal{N}(K).$$

Equivalently: a nontrivial knot cannot be  $J_m$ -trivial beyond its crossing number. As a corollary, the Jones polynomial detects the unknot: if  $V_K(t) \equiv 1$  then  $K$  would be  $J_m$ -trivial for every  $m$ , contradicting the bound for  $m > \mathcal{N}(K)$ ; hence  $K$  is the unknot. A practical form is: for any diagram with  $C$  crossings, vanishing of  $c_1, \dots, c_C$  certifies the unknot.

## Auxiliary lemmas and proofs

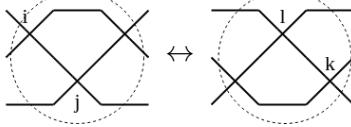
### A Cubical 1T/4T

**Lemma A.1** (Cubical 1T/4T). *Let  $S, S'$  be shadows with variation cubes  $\mathcal{Q}(S), \mathcal{Q}(S')$ . Let  $f : \mathcal{Q}(S) \rightarrow \mathbb{C}$  and  $f' : \mathcal{Q}(S') \rightarrow \mathbb{C}$  be pseudo-Boolean functions encoding finite-type invariants on these cubes. Write  $I = \text{cross}(S)$  and  $I' = \text{cross}(S')$ .*

(1T) *If  $S' = S \# \mathcal{O}$  (adding a singular twist), then  $\dim \mathcal{Q}(S') = \dim \mathcal{Q}(S) + 1$  and  $\mathcal{Q}(S') \cong \mathcal{Q}(S) \times \{\pm 1\}$  with  $f'(\varepsilon, \eta) = f(\varepsilon)$ . If  $m = \dim \mathcal{Q}(S) = |I|$ , then*

$$\partial_{I \cup \{\text{new}\}} f' \equiv 0, \quad \partial_I f' = \partial_I f.$$

(4T) *Suppose  $S$  and  $S'$  differ by the usual local 4T replacement supported on two crossings  $(i, j) \in I$  vs.  $(k, l) \in I'$ . Assume  $|I| = |I'| =: m$  and fix the natural bijection  $I \setminus \{i, j\} \cong I' \setminus \{k, l\}$  induced by the local move:*



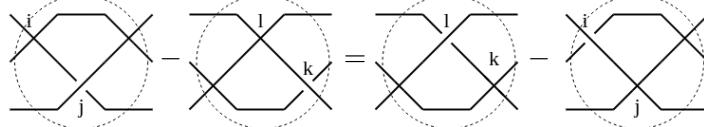
*Then the order- $(m - 1)$  Walsh coefficients satisfy*

$$\hat{f}(I \setminus \{j\}) - \hat{f}(I' \setminus \{k\}) = \hat{f}'(I' \setminus \{l\}) - \hat{f}(I \setminus \{i\}). \quad (26)$$

*and the symbols agree (up to the natural identification of coordinates),*

$$\partial_I f = \partial_{I'} f'.$$

**Remark A.1.** *The relation (26) is the pseudo-Boolean incarnation of the 4T, commonly represented by*



*Proof.* (1T). The cubical 1T relation follows immediately from the fact that  $f(\varepsilon^{(i,+)})) = f(\varepsilon^{(i,-)})$  for a nugatory crossing whose index is  $i$ . Therefore,  $\partial_i f = 0$ , from which it follows that the symbol  $\partial_I f = 0$ .

(4T). The shadows  $S$  and  $S'$  differ in two crossings,  $\{i, j\}$  vs.  $\{k, l\}$ . Taking the  $(m - 2)$ -fold discrete derivative over the residual cube defined by the crossing set  $I \setminus \{i, j\}$  (equivalently,  $I' \setminus \{k, l\}$ ), yields an order-2 invariant of knots with  $(m - 2)$  singular crossings. Denote,  $g_2 := \partial_{I \setminus \{i, j\}} f$  and  $g'_2 := \partial_{I' \setminus \{k, l\}} f'$ , the respective order-2 invariants over  $\mathcal{Q}(S)/\mathcal{Q}_2$  and  $\mathcal{Q}(S')/\mathcal{Q}'_2$ . As both invariants are defined over 2-dimensional cubes  $\{\pm 1\}^2$ , their Walsh-Fourier expansion is

$$\begin{aligned} g_2(\varepsilon_i, \varepsilon_j) &= \hat{g}_2(\emptyset) + \hat{g}_2(i)\varepsilon_i + \hat{g}_2(j)\varepsilon_j + \hat{g}_2(ij)\varepsilon_i\varepsilon_j, \\ g'_2(\varepsilon_k, \varepsilon_l) &= \hat{g}'_2(\emptyset) + \hat{g}'_2(k)\varepsilon_k + \hat{g}'_2(l)\varepsilon_l + \hat{g}'_2(kl)\varepsilon_k\varepsilon_l. \end{aligned} \quad (27)$$

Note that both pseudo-Boolean functions share the same free coefficient  $\hat{g}_2(\emptyset)$ . This follows from the derivative rule (8) which shows that  $\hat{g}_2(\emptyset)$  is the order- $(m - 2)$  Fourier coefficient  $\hat{f}(I \setminus \{i, j\})$  whose

Walsh character  $\chi_{I \setminus \{i,j\}}(\varepsilon)$  is independent of  $\varepsilon_i, \varepsilon_j$ . This coefficient is the same as  $\hat{f}'(I' \setminus \{k,l\})$  as they are obtained using the same set of  $m - 2$  singular crossings.

By Reidemeister-III we have the following identities:

$$g_2 \left( \begin{array}{c} \text{Diagram 1} \\ \text{with } i, j, k \end{array} \right) = g'_2 \left( \begin{array}{c} \text{Diagram 2} \\ \text{with } l, k \end{array} \right) \quad g_2 \left( \begin{array}{c} \text{Diagram 3} \\ \text{with } i, j, k \end{array} \right) = g'_2 \left( \begin{array}{c} \text{Diagram 4} \\ \text{with } l, k \end{array} \right) \quad (28)$$

which translate to

$$g_2(1, 1) = g'_2(1, 1), \quad g_2(-1, -1) = g'_2(-1, -1). \quad (29)$$

These together with (27) yield,

$$\pm [\hat{g}_2(i) + \hat{g}_2(j) - \hat{g}'_2(k) - \hat{g}'_2(l)] = \hat{g}'_2(kl) - \hat{g}_2(ij), \quad (30)$$

implying,  $\hat{g}'_2(kl) - \hat{g}_2(ij) = 0$ , and  $\hat{g}_2(i) - \hat{g}'_2(k) = \hat{g}'_2(l) - \hat{g}_2(j)$ . The lemma follows upon recognizing that (see (8)),  $\hat{g}_2(i) = \hat{f}(I \setminus j)$ ,  $\hat{g}_2(j) = \hat{f}(I \setminus i)$ ,  $\hat{g}'_2(k) = \hat{f}'(I' \setminus l)$ ,  $\hat{g}'_2(l) = \hat{f}'(I' \setminus k)$ , and the symbols,  $\hat{f}(I) = \hat{g}_2(ij) = \hat{g}'_2(kl) = \hat{f}'(I')$   $\rightarrow \partial_I f = 2^{|I|} \hat{f}(I) = 2^{|I'|} \hat{f}'(I') = \partial_{I'} f'$ .  $\square$

## B Shadow Lemma

**Definition B.1** (Nugatory crossing; cut-vertex). *A crossing is nugatory iff there exists a Jordan curve in the projection plane meeting the diagram only at that crossing; at the shadow level, this is equivalent to the vertex being a cut-vertex (removing it disconnects the shadow). [11, 12]*

**Lemma B.1** (Shadow). *Let  $S$  be a connected shadow with  $n > 0$  vertices and let*

$$\mathcal{D}(S) = \{\text{all } 2^n \text{ diagrams carried by } S\}.$$

*For a diagram  $D \in \mathcal{D}(S)$  write  $c(D)$  for its set of crossings and, for  $K \subseteq c(D)$ , let  $D \setminus K$  be the link obtained from  $D$  after smoothing every crossing in  $K$ .*

*The following statements are equivalent:*

1. **Component-increment property (P).** *For every  $D \in \mathcal{D}(S)$  and every subset  $K \subseteq c(D)$  of size  $k$ ,*

$$\#\text{components of } (D \setminus K) = 1 + k. \quad (\text{P})$$

2. *Every vertex of  $S$  is a cut-vertex (equivalently, every crossing is nugatory in Tait's sense). Hence  $S$  is the planar embedding of a tree in which each edge appears exactly once.*

3. *All diagrams in  $\mathcal{D}(S)$  are diagrams of the unknot (or of a split unlink if  $S$  is disconnected).*

*Consequently, any diagram satisfying (P)—and every diagram obtained from it by crossing flips—represents the unknot.*

*Proof.*

(1)  $\Rightarrow$  (2) Work with oriented diagrams and fix a vertex  $x$  of the connected shadow  $S$ . By Corollary 4.5, for a fixed set of smoothings the number of components of  $D \setminus J$  is independent of the over/under choices in  $D \in \mathcal{D}(S)$  (the component count after smoothing a fixed set is a shadow invariant); thus we may argue at the level of the shadow. Property (P) with  $|K| = 1$  says that for every diagram  $D$  the oriented smoothing at  $x$  yields two components  $D \setminus \{x\} = L_A \sqcup L_B$ .

Suppose  $x$  were not a cut-vertex. Then the plane graph  $S \setminus \{x\}$  is connected; hence there exists a path in  $S \setminus \{x\}$  from an arc of  $L_A$  to an arc of  $L_B$ , and along that path choose the first vertex  $y$  where the two sides meet. In the diagram  $D \setminus \{x\}$  the crossing  $y$  is a crossing between different components  $L_A$  and  $L_B$ . Now smooth  $y$  as well. Oriented smoothing at a crossing that connects distinct components does not increase the number of components (it either keeps it the same or reduces it by one). Therefore

$$\#\pi_0(D \setminus \{x, y\}) \leq 2,$$

contradicting Property (P) for  $|K| = 2$ , which requires  $1+2=3$  components. Hence our assumption was false:  $S \setminus \{x\}$  must be disconnected, i.e.  $x$  is a cut-vertex (nugatory) [11].

**(2)  $\Rightarrow$  (3)** Theorem 3 in [13]: *A shadow  $S$  has only unknot diagrams iff every vertex of  $S$  is a cut-vertex.* Assumption (2) therefore forces every diagram in  $\mathcal{D}(S)$  to be an unknot diagram.

**(3)  $\Rightarrow$  (1)** Seifert's genus formula.

Fix a diagram  $D \in \mathcal{D}(S)$  and orient it. Apply *Seifert's algorithm* [14]: smoothing all  $n$  crossings produces  $s$  Seifert circles and a surface with Euler characteristic  $\chi = s - n$ . Because  $D$  is an unknot, its genus must be zero. For a surface with one boundary component,

$$\chi = 1 - 2g \implies 0 = g = \frac{1 - (s - n)}{2},$$

hence

$$s = n + 1. \tag{1}$$

Now smooth an *arbitrary* subset  $K \subseteq c(D)$  of size  $k$ . In the Seifert-surface picture, each smoothed crossing removes exactly one band, thereby increasing the number of boundary components (link components) by one. Starting from one component and using induction on  $|K|$  with (1), we obtain exactly  $1 + k$  components—property (P).

**(2)  $\Leftrightarrow$  (1)** (direct argument)

- **(2)  $\Rightarrow$  (1)** If every crossing is nugatory, smoothing one crossing splits one component into two. Repeating the argument shows the component count always increases by one.
- **(1)  $\Rightarrow$  (2)** The single-crossing case  $k = 1$  recovers the nugatory condition used in the first implication.

Thus all three statements are equivalent.  $\square$

**Corollary B.2** (“Tree diagrams”). *A connected knot diagram satisfies property (P) iff its shadow is a tree. Such a diagram is sometimes called a descending or standard diagram; it can be reduced to a round circle by  $n$  Reidemeister I moves (one per nugatory crossing) followed by  $n$  paired Reidemeister II moves along the tree edges.*

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