# A bound on the $m$-triviality of knots
## Part I: Jones-Vassiliev unknot detection

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**Abstract**

We prove a uniform barrier for Jones–Vassiliev truncations. If a knot $K$ is $J_m$-trivial, that is, its Birman-Lin finite-type coefficients $c_1, \dots, c_{m-1}$ vanish, and $K$ is nontrivial, then $m \leq \mathscr{N}(K)$, the crossing number of $K$. For a fixed shadow $S$, the choices of over/under data at its crossings form a Boolean cube; a subcube is the face obtained by varying only a designated subset of crossings. We decorate $S$ by degree-$m_j$ claspers so that flipping the $j$-th crossing is realized (modulo clasper calculus) by a $C_{m_j}$-move invisible to finite-type invariants below order $m_j$; a weighted Goussarov-Habiro filtration packages these effects and produces an order-$< m$ flat family of knots all of whose members are $m$-trivial. On the Jones side we develop a symbol calculus for the Birman–Lin expansion in which alternating-sum symbols satisfy a skein recursion. When the Jones layers are constant through order $m-1$ (equivalently, all order-$< m$ symbols vanish), the recursion forces a last-row fingerprint for the free term $c_0$: smoothing any $m-1$ chosen crossings creates exactly $m$ components. A shadow criterion then implies that any supercritical face (where $m$ exceeds the number of varied crossings) is flat and carries only unknots. Applying this to a minimal shadow yields $m \leq \mathscr{N}(K)$. As a corollary, the Jones polynomial detects the unknot: if $V_K(t) \equiv 1$ (so all $c_n(K) = 0$), then $K$ would be $J_m$-trivial for every $m$, contradicting the barrier for any $m > \mathscr{N}(K)$; hence $K$ is the unknot. Equivalently, for any diagram with $C$ crossings, vanishing of $c_1, \dots, c_C$ certifies the unknot, and every nontrivial knot satisfies $\min\{n \geq 1 : c_n(K) \neq 0\} \leq \mathscr{N}(K)$.

# 1 Introduction

Vassiliev's program views finite-type data through alternating sums over faces of the skein cube of crossing changes [1], leading to the algebra of chord diagrams and the $1T/4T$ relations and encompassing Kontsevich's universal integral [2] and Bar-Natan's graded algebra [3]; see also [4]. In this paper we pursue a complementary, more *topological* implementation built on *extended shadows* and *clasper-decorated crossings*.

Fix a shadow $S$ with outside crossing set $\mathrm{cross}(S)$. Choose a base diagram $K_0$ carried by $S$ and, for each crossing $j$, attach a degree-$m_j$ tree clasper $C_{m_j}$ in a small ball meeting the diagram only near $j$. The resulting data

$$E = (K_0; \{C_{m_j}\}_{j \in \mathrm{cross}(S)})$$

is an *extended shadow*. The key local move is that *flipping the $j$-th crossing is realized modulo clasper calculus by a $C_{m_j}$-move*; hence every finite-type invariant of degree $< m_j$ is blind to that flip:

$$\partial_j f \;=\; f(+) - f(-) \;=\; 0 \quad \text{for } \deg f < m_j.$$

We therefore call the $j$-th crossing $m_j$-*invisible* ("crossing flip mod $C_{m_j}$"). See Figure 1 for the local movie implementing this equivalence and §3 for the weighted GH filtration that packages multi-flip alternating sums. In particular, with $m := \min_j m_j$, the family $\sigma \mapsto K_E(\sigma)$ carried by the decorated crossings is $m$-*flat* for all order $< m$ finite-type data, i.e., any one of its members is $m$-trivial.

Specializing to the Jones polynomial, we use the Birman–Lin expansion

$$V_K\left(e^{h/2}\right) = \sum_{n\geq 0} c_n(K)\, h^n,$$

whose coefficients $c_n$ are finite-type of order $\leq n$. On any $m$-flat family (hence, in particular, on the family carried by an extended shadow with $\min m_j = m$), every Jones layer $c_1, \dots, c_{m-1}$ is constant. When a single vertex in the family is $J_m$-trivial, these constants are forced to be zero, a situation we call *Jones–$m$–flatness*. A symbol-level skein recursion for the $c_n$ then yields a *last-row fingerprint* for the free term $c_0$: smoothing any prescribed $m-1$ crossings produces exactly $m$ components. A shadow-level criterion converts this fingerprint into a structural statement: if $m$ exceeds the number of crossings being varied (in particular, if $m > \#\text{cross}(S)$), then the shadow must be a tree and every carried diagram is an unknot. Applying this to a minimal-crossing shadow of $K$ gives the uniform barrier

$$m \leq \mathscr{N}(K).$$

As an immediate corollary, the Jones polynomial detects the unknot: if $V_K(t) \equiv 1$ (equivalently, $c_n(K) = 0$ for all $n$), then $K$ would be $J_m$-trivial for every $m$; taking $m > \mathscr{N}(K)$ contradicts the barrier, so $K$ is the unknot. Equivalently, for any diagram with $C$ crossings, vanishing of $c_1, \dots, c_C$ certifies the unknot, and every nontrivial knot satisfies $\min\{n \geq 1 : c_n(K) \neq 0\} \leq \mathscr{N}(K)$.

### 1.1 Related work

The foundational insight of Birman and Lin is that the coefficients in the power-series expansions of the Jones, HOMFLY and Kauffman polynomials yield Vassiliev invariants [5]; their paper established the bridge between skein-type polynomials and finite-type theory and articulated early structural consequences and open directions for detection. Birman's contemporaneous survey emphasized the breadth of the approach and new perspectives opened by finite-type methods [6].

Kontsevich constructed a universal integral whose weight systems classify finite-type invariants [2], while Bar-Natan developed the algebra of chord diagrams modulo 1T/4T, clarifying symbol calculus and functoriality [3]. These works underlie our symbol-level arguments: within the spectral framework, the top Walsh block encodes precisely the $m$-th graded piece $\mathcal{A}_m$ of Bar-Natan's chord-diagram algebra, and our cubical 1T/4T (Lemma A.1) is the pseudo-Boolean incarnation of these relations.

Goussarov's $n$-equivalence and Habiro's clasper calculus show that surgery on a connected $C_m$ tree preserves all invariants of degree $< m$ [7, 8]. We exploit this twice: (i) to produce $m$-flat families by decorating a fixed shadow with degree-$m_j$ trees, and (ii) via the *crossing flip mod $C_{m_j}$* gadget that renders the $j$-th crossing $m_j$-invisible. The weighted GH filtration then controls all multi-flip differences and underlies the GH–Taylor expansion we use for bookkeeping. Figure 1 illustrates the local movie realizing the flip modulo $C_m$.

A corollary of earlier work on polynomial/finite-type expansions states:

**Corollary 5.3 [9].** *If all Vassiliev invariants up to degree $c$ vanish on a knot $K$ of crossing number $c$ then the knot has a trivial HOMFLY polynomial.*

Our barrier strengthens the Jones-side mechanism in a diagram-local fashion: from a length-$m$

initial Jones gap we build an $m$-flat family on an extended shadow via $m$-invisible flips, force the last-row fingerprint, and through a shadow criterion bound $m$ by $\mathscr{N}(K)$, yielding a finite verification of Jones detection at minimal crossing number.

### 1.2 Organization of this work

**Section 2** lays out the preliminaries: finite-type invariants (Vassiliev order and symbols), the pseudo-Boolean variation cube attached to a shadow, and the Walsh-Fourier/spectral viewpoint that identifies order with polynomial degree and interprets facewise alternating sums as discrete derivatives. This section fixes notation used throughout (e.g., smoothing $D \setminus J$, residual cubes).

**Section 3** introduces extended shadows by decorating selected crossings of a fixed shadow with tree claspers of prescribed degrees $m_j$. We prove that the $j$-th crossing flip is $m_j$-invisible (a $C_{m_j}$-move) for invariants of degree $< m_j$, package multi-flip effects via a weighted Goussarov-Habiro filtration, and assemble a GH-Taylor expansion that keeps track of interactions among claspers. This yields $m$-flat families (constancy of all sub-$m$ layers) carried by the decorated shadow. Figure 1 depicts the clasper movie and the crossing flip modulo $C_m$ used repeatedly later.

**Section 4** specializes to the Birman-Lin expansion of the Jones polynomial. We establish a symbol-level skein recursion for the Jones layers $c_n$ and show that order-0 data $c_0$ is a shadow invariant encoding component count. On any Jones-$m$-flat subcube we prove the last-row fingerprint: smoothing any prescribed $m-1$ crossings forces $c_0 = (-2)^{m-1}$, hence the component count rises by exactly one at each smoothing. This yields a degree cap for Jones-flatness on a subcube (Cor. 4.11). The symbol matrix in Figure 2 visualizes the skein constraints and the $(-2)^r$ bottom-row pattern.

**Section 5** gives the main result, the uniform $m$-triviality barrier: if a nontrivial knot $K$ is $J_m$-trivial, then $m \leq \mathscr{N}(K)$ (Thm. 5.1). The proof transfers $m$-flatness from a decorated face of the extended cube back to the outside-crossing subcube, applies the last-row fingerprint with the Shadow Lemma to force global nugatoriness in the supercritical regime, and concludes the bound. Two corollaries follow: Jones detects the unknot (Thm. 5.3) and the Vassiliev version (Thm. 5.4).

**Appendix A** records a cubical $1T/4T$ formulation that links the symbol calculus on cubes to Bar-Natan's diagrammatics; **Appendix B** states and proves the Shadow Lemma, equating the component-increment property (P) with all crossings being nugatory (the shadow is a tree). These appendices supply the local-to-global bridge used in the proofs.

### 1.3 Part II: Computational evidence

The companion paper (Part II) [10] implements the symbol calculus developed here by packaging the Birman-Lin layers into a Jones-Vassiliev polynomial (JVP) in the rank-two module $\mathbb{Z}[p][x]/(x^2-px-1)$; from tabulated Jones polynomials $V_K(t)$ it reads off the finite-type layers via the substitutions $t^{1/2} = x$, $t^{-1/2} = x - p$, followed by the reduction $x^2 = px + 1$. Using this extraction pipeline, Part II computes the layers for all 352,152,252 prime knots with $\mathscr{N} \leq 19$ (Knot Atlas for $\mathscr{N} \leq 10$; Dartmouth for $11 \leq \mathscr{N} \leq 19$), finding no counterexample to the uniform barrier $m \leq \mathscr{N}(K)$ proved in Theorem 5.1 here, and moreover uncovering a consistent empirical refinement $m \leq \lfloor \mathscr{N}/2 \rfloor + 1$; see Fig. 3 (first $J_m$-trivial occurrences) and Fig. 4 (likelihood curves $P_m(\mathscr{N})$) of Part II. Beyond the census, Part II reproduces our symbol-skein recursion and verifies the last-row fingerprint (Theorem 4.10/Corollary 4.11 here) on explicit model cubes and records a clean min-law under connected sum for the JVP bandwidth (Proposition 4.2). Taken together, the large-scale data and the JVP calculus in Part II provide a practical, finite algorithmic mirror of the obstruction mechanism developed in Part I.

### 1.4 Conventions

**Convention 1.1** (Conventions used throughout the text)**.**

- *A* knot shadow *(shadow for short) means the 4-regular plane graph obtained from a knot diagram by forgetting over/under information, and a crossing flip means changing the over/under choice at one vertex of the shadow.*
- *Over/under crossings are denoted as '+' and '−', and sometimes, when encoded by cube vertices, as* $+1$ *and* $-1$*.*
- *Smoothing a crossing always refers to* oriented *smoothing. Following the above crossing sign conventions, it is denoted as '0'.*
- $\mathscr{N}(K)$ *and* $\mathrm{cross}(K)$ *denote, respectively, K's number of crossings and set of crossing indices. The number of crossings may always be that of a minimal representation of K in which case it is a knot invariant aka the crossing number.*
- *Let* $S$ *be a fixed shadow with outside crossing set* $\mathrm{cross}(S)$*, and let* $D \in \mathcal{D}(S)$ *be a diagram carried by* $S$*. For a subset* $J \subseteq \mathrm{cross}(S)$*, we write* $D \setminus J$ *for the diagram obtained from* $D$ *by performing the* oriented smoothing *at every crossing in* $J$ *("0" smoothing). This is a concise form of Convention 4.1.*
- *A* $C_k$ *tree (a degree-k tree clasper) is the embedded surface made of nodes, edges, and leaves. A* $C_k$ *move is surgery along a connected* $C_k$ *tree. Two links are* $C_k$*-equivalent if they are related by a sequence of* $C_k$ *moves. A* $C_k$ *move preserves every finite-type invariant of degree* $< k$ *[7, 8].*
- ***GH*** *refers to Goussarov-Habiro.*

## 2 Preliminaries

### 2.1 Finite-type invariants

Fix the category $\mathcal{K}$ of oriented links in $S^3$, considered up to ambient isotopy. Let $\mathcal{K}^{(d)}$ denote the set of (immersed) oriented singular links with at most $d$ transverse double points and no other singularities. Let $A$ be an abelian group (typically $A = \mathbb{Z}$ or $\mathbb{Z}[p]$).

**Definition 2.1** (Vassiliev extension and order)**.** *A function* $v : \mathcal{K} \to A$ *is finite type of order* $\leq d$ *if it admits an extension* $\tilde{v} : \mathcal{K}^{(d)} \to A$ *determined inductively by the Vassiliev skein rule*

$$\tilde{v}\,(\textit{double point}) = \tilde{v}\,(\textit{positive}) - \tilde{v}\,(\textit{negative})$$

*at each double point, and satisfies* $\tilde{v}(L) = 0$ *whenever* $L$ *has* $d{+}1$ *double points. The order (or type) of* $v$ *is the least such* $d$*.*

**Definition 2.2** (Symbol / weight system)**.** *For $v$ of order $\leq d$, its $d$-th symbol $\sigma_d(v)$ is the restriction of $\tilde{v}$ to $\mathcal{K}^{(d)}$ modulo the skein relations, equivalently a function on chord diagrams with $d$ chords. The symbol factors through the 1T/4T relations, so $\sigma_d(v)$ defines a linear functional on the degree-$d$ diagram space $\mathcal{A}_d$ (Bar-Natan).*

**Remark 2.1** (Goussarov–Habiro invisibility)**.** *A connected $C_m$ move (surgery on a $C_m$ clasper) preserves every invariant of order $< m$. We write "$C_m$ is invisible to order $< m$" and use this repeatedly throughout this work.*

## 2.2 Pseudo-Boolean functions and finite-type invariants

**Shadows and cubes.** Fix a shadow $S$ with outside crossing set $I = \mathrm{cross}(S)$, $|I| = n$. The associated *variation cube* is

$$\mathcal{Q}(S) \;=\; \{\varepsilon = (\varepsilon_i)_{i\in I} \mid \varepsilon_i \in \{\pm 1\}\} \;\cong\; \{\pm 1\}^n. \tag{1}$$

For each $\varepsilon \in \mathcal{Q}(S)$ let $D(\varepsilon) \in \mathcal{D}(S)$ denote the diagram carried by $S$ with over/under choices encoded by $\varepsilon$ (Convention 1.1).

**Pseudo-Boolean functions on a fixed shadow.** Let $A$ be an abelian group (typically $\mathbb{Z}$ or $\mathbb{Z}[p]$). A *pseudo-Boolean function on the cube of $S$* is any map

$$f_S : \mathcal{Q}(S) \longrightarrow A, \qquad \varepsilon \longmapsto f_S(\varepsilon).$$

Given an ambient-isotopy invariant $v : \mathcal{K} \to A$, we obtain its restriction to the cube of $S$ by evaluation at the vertices:

$$f_v^S(\varepsilon) \;:=\; v\big(D(\varepsilon)\big), \qquad \varepsilon \in \mathcal{Q}(S).$$

Thus, every knot/link invariant supplies, for each choice of shadow, a canonical pseudo-Boolean function on $\mathcal{Q}(S)$.

**Faces and $\mathscr{U}$-subcubes.** For $\mathscr{U} \subseteq I$ and a base state $\varepsilon^0 \in \mathcal{Q}(S)$ the *$\mathscr{U}$-subcube through $\varepsilon^0$* is

$$\mathcal{Q}_{\mathscr{U}}(\varepsilon^0) \;:=\; \{\varepsilon \in \mathcal{Q}(S) :\; \varepsilon_i = \varepsilon_i^0 \text{ for all } i \notin \mathscr{U}\}.$$

We freely regard alternating sums over such faces as *finite differences* (discrete derivatives).

**Polynomial-valued invariants and coefficient layers.** Let $R$ be a commutative ring and let $J : \mathcal{K} \to R[p]$ be a polynomial-valued invariant,

$$J(K;p) \;=\; \sum_{m\geq 0} f_m(K)\, p^m, \qquad f_m : \mathcal{K} \to R.$$

On a fixed shadow $S$ the coefficient functionals induce a family of pseudo-Boolean functions

$$f_{m,S}(\varepsilon) \;:=\; f_m\big(D(\varepsilon)\big), \qquad \varepsilon \in \mathcal{Q}(S),\; m \geq 0.$$

In particular, when $J$ is the Jones–Vassiliev expansion (introduced in §4.1), we will write $c_{q,S}(\varepsilon) = c_q\big(D(\varepsilon)\big)$ for its $q$-th coefficient layer on $\mathcal{Q}(S)$.

**Connected sums and product cubes.** If a diagram $D(\varepsilon)$ splits as a connected sum $D_0(\varepsilon^0)\#D_1(\varepsilon^1)$ realized in disjoint disks, then the shadow cube factors as a Cartesian product $\mathcal{Q}(S) \cong \mathcal{Q}(S_0) \times \mathcal{Q}(S_1)$. When $J$ is multiplicative under connected sum,

$$J_{K_0\#K_1}(p) \;=\; J_{K_0}(p)\, J_{K_1}(p),$$

the coefficient layers satisfy the Cauchy rule at each vertex $(\varepsilon^0, \varepsilon^1)$:

$$f_{m,S}(\varepsilon^0, \varepsilon^1) \;=\; \sum_{i=0}^{m} f_{i,S_0}(\varepsilon^0)\, f_{m-i,S_1}(\varepsilon^1). \tag{2}$$

**Residual cubes.** For later use we smooth a specified set $J \subseteq I$ (oriented “0” smoothing from Convention 1.1) and denote by $\mathcal{Q}_{S\setminus J}$ the variation cube of the resulting shadow; see Definition 4.2. Order-0 constancy on residual cubes is recorded in Corollary 4.3.

### 2.3 Spectral representation

The Boolean cube-complex formalism enjoys a number of advantages. The Vassiliev skein relation is realized by a boundary operator, $\partial_i$, contracting/projecting along the $i$-th dimension. The type or order of the Vassiliev invariant becomes the *polynomial degree* of the pseudo-Boolean function. Beyond the observation that this may explain why polynomial invariants are native objects for encoding combinatorial finite-type data, we have tools from spectral analysis at our disposal.

**Fourier basis and Walsh characters.** Let $f : \{\pm 1\}^n \to \mathbb{C}$ be a pseudo-Boolean function. For every subset $S \subseteq [n]$, the Walsh character is given by,

$$\chi_S(\varepsilon) = \prod_{i\in S} \varepsilon_i \quad (\chi_\varnothing \equiv 1). \tag{3}$$

These characters form an orthonormal basis (with respect to the uniform measure on $\mathcal{Q}_n$), so

$$f(\varepsilon) = \sum_{S\subseteq[n]} \widehat{f}(S)\,\chi_S(\varepsilon), \qquad \widehat{f}(S) = 2^{-n} \sum_{\varepsilon} f(\varepsilon)\,\chi_S(\varepsilon). \tag{4}$$

is the Walsh-Fourier expansion of $f$. The polynomial degree, $\deg(f)$, matches the largest $|S|$ with $\widehat{f}(S) \neq 0$.

**Discrete (finite) derivatives.** For coordinate $i$ we define the (normalized) derivative

$$(\partial_i f)(\varepsilon) \equiv \frac{1}{2}\big(f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)})\big), \tag{5}$$

where $\varepsilon^{(i,\pm)}$ is $\varepsilon$ with the $i$-th entry set to $\pm 1$. For a subset $T \subseteq [n]$,

$$\partial_T = \prod_{i\in T} \partial_i, \qquad (\partial_T f)(\varepsilon) = 2^{-|T|} \sum_{\sigma\in\{\pm1\}^T} (-1)^{\#\{\sigma_i=-1\}}\, f(\varepsilon_{[n]\setminus T}, \sigma). \tag{6}$$

Equivalently, it is the alternating sum of $f$ over the $T$-subcube through $\varepsilon$. For any fixed subset $T \subseteq [n] = \{1, \ldots, n\}$ the $T$-fold discrete derivative acts on Walsh characters by “dropping” the coordinates in $T$:

$$\partial_T\, \chi_S = \begin{cases} \chi_{S\setminus T}, & T \subseteq S, \\ 0, & T \not\subseteq S. \end{cases} \tag{7}$$

Because the characters $\{\chi_S\}_{S\subseteq[n]}$ form an orthonormal basis, we obtain the Fourier (Walsh) expansion of the derivative simply by applying this rule term-wise to the expansion of $f$:

$$(\partial_T f)(\varepsilon) = \sum_{[n]\supseteq U\supseteq T} \widehat{f}(U)\chi_{U\setminus T}(\varepsilon). \tag{8}$$

Thus, the Fourier coefficient of $\partial_T f$ at character $\chi_S$ (with $S\cap T=\varnothing$) is precisely the coefficient of $f$ at the larger character $\chi_{S\cup T}$, namely, $\widehat{\partial_T f}(S) = \widehat{f}(S\cup T)$. This explicit form makes transparent how derivatives shift weight "down" the spectrum and why $\partial_T f$ vanishes whenever $f$ has no Fourier support on sets containing $T$ (the finite-type condition).

**Remark 2.2** (Normalized vs. unnormalized derivatives)**.** *The standard Fourier/Walsh framework is formulated using the normalized difference operator* (5)*. The Vassiliev skein, on the other hand, adopts the unnormalized version, namely,*

$$(\partial_i f)(\varepsilon) \equiv f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)}). \tag{9}$$

*At times, this convention alleviates the burden of dealing with factors such as $2^{-m}$ when taking derivatives over cube $m$-faces. For that reason, and for being consistent with the standard theory, unless stated otherwise, $\partial_i$ means* (9) *for the rest of this work.*

**Connected sum and product rules.** The variation cube encoding connected sums of knots, e.g. $K_0\#K_1$, is given by a cartesian product, $\mathcal{Q}\cong\mathcal{Q}_0\times\mathcal{Q}_1$. The set of coordinates $S$ underlying $\mathcal{Q}$ can thus be partitioned into two subsets associated with the individual cubes, $S=S_0\cup S_1$. This facilitates composing the discrete derivatives of any pseudo-Boolean function as,

$$\partial_{T_0\cup T_1} = \partial_{T_0}\partial_{T1}, \quad \forall T_0\subseteq S_0, \forall T_1\subseteq S_1.$$

If, in addition, (2) holds, then

$$\partial_{T_0\cup T_1} f_m(\varepsilon_{K_0\#K_1}) = \sum_{i=0}^{m} \left[\partial_{T_0} f_i(\varepsilon_{K_0})\right]\left[\partial_{T_1} f_{m-i}(\varepsilon_{K_1})\right]. \tag{10}$$

**Polynomial degree equals Vassiliev order.** A pseudo-Boolean invariant $f$ is said to be *finite type of (Vassiliev) order $d$* if

$$\partial_T f \equiv 0 \quad \text{for every } T\subseteq[n] \text{ with } |T|>d. \tag{11}$$

Intuitively, all $(d+1)$-dimensional faces evaluate to alternating sums of zero. The derivative representation immediately implies

$$\widehat{f}(S) = 0 \quad \text{whenever } |S|>d. \tag{12}$$

Conversely, if $\widehat{f}(S)=0$ for $|S|>d$, then every $\partial_T f$ with $|T|>d$ vanishes.

$$\mathrm{Type}(f) = \deg(f) = \max\{|S| : \widehat{f}(S)\neq 0\}. \tag{13}$$

The derivative operator not only detects type but produces new finite-type functions of *singular* knots: Fix a face defined by coordinates $T$ and take the alternating sum of $f$ along that face. The result is $\partial_T f$ restricted to the complementary coordinates. If $f$ is type $d$, then $\partial_T f$ is type $d-|T|$. Iterated extraction of "principal parts."

| Concept | Description |
|---|---|
| *Boolean variation cube* | Encodes all simultaneous crossing variations; faces are natural domains for alternating-sum operators. |
| *Walsh-Fourier expansion* | Decomposes any pseudo-Boolean invariant into characters labelled by variation subsets. |
| *Top Walsh-Fourier block* | $\partial_{[m]}$ where $m$ is the polynomial degree; the symbol. |
| *Discrete derivatives* | Are face-wise alternating sums; in Fourier space they simply drop indices. |
| *Finite-type order d* | Polynomial/Fourier degree $d \leftrightarrow$ vanishing of ($> d$)-fold derivatives. |
| *Derivatives over subcubes* | Isolate the homogeneous pieces and systematically generate lower-order invariants of singular knots. |

Table 1: Pseudo-Boolean functions and finite-type invariants. Glossary of key concepts.

- Order-1 part: $g_1(\varepsilon) = \sum_{|S|=1} \widehat{f}(S)\chi_S(\varepsilon)$.
- Higher parts analogously via derivatives: the map $T \mapsto \partial_T f(1, \ldots, 1)$ lists exactly the Fourier coefficients of size $|T|$.

This is the discrete analogue of repeatedly differentiating a polynomial and evaluating at the origin to read off coefficients.

**Symbols, cubes, and chords.** In finite-type theory the $m$-th symbol is an invariant of exactly order $m$. That is, it is the restriction of the finite-type function to chord diagrams with $m$ chords, or equivalently, is the $m$-th graded piece in the GH algebra (Definition 2.2). By definition, the symbol is independent of any particular variation of the singular crossings, i.e., it is fixed for a given cube. Over an $m$-dimensional variation cube it is extracted by the (unnormalized) $m$-fold discrete derivative and hence equals (up to a normalization factor) the top Walsh-Fourier block of the underlying pseudo-Boolean function,

$$\partial_{[m]} f = 2^m \hat{f}([m]).$$

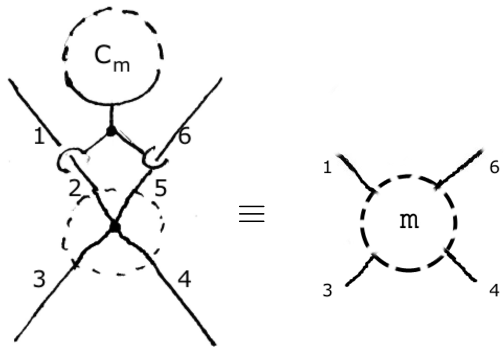
The symbols factorize through the graded space $\mathcal{A}_m$ and hence depend only on the chord diagram class (modulo the 1T/4T relations) and not on the particular placement of singularities. The pseudo-Boolean framework offers concise descriptions of these local relations which consequently bridge the cubical definition of a symbol with the algebra of chord diagrams (see Lemma A.1 in the appendix).

## 3 Spectral $m$-gaps

Let $S$ be a knot shadow with $N$ labeled double points $1, \ldots, N$. Fix a choice of over/under data at those $N$ points that yields a base diagram $K_0$ (e.g., an unknot).

### 3.1 Extended shadows and $m$-invisible crossings

For each $j \in \{1, \dots, N\}$, fix a simple tree clasper $C_{m_j}$ for $K_0$ of (Habiro) degree $m_j \geq 1$, embedded in a small ball meeting $K_0$ only near the $j$-th double point, as shown below.

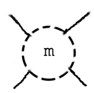



Assume the $C_{m_j}$ are pairwise disjoint. All choices can be made within disjoint balls around the crossings. Call

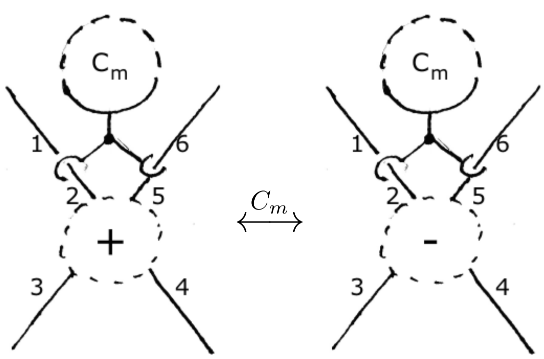
$$E = (K_0; \{C_{m_j}\}_{j=1}^{N})$$

the *extended shadow*. Below we prove that the clasper-decorated crossing $m_j$ is invisible to degree $< m_j$ invariants

$$\partial_j f = f^{(j=+)} - f^{(j=-)} = 0, \quad \deg(f) < m_j.$$

The diagrammatic representation  stands for a clasper-decorated crossing that does not affect any degree-$< m$ invariant. Similarly, the net effect of flipping $J$-subset of crossings (alternating sums) is shifted to degrees $\geq \sum_{j \in J} m_j$ of the GH filtration.

**Proposition 3.1** (Crossing flip modulo $C_m$)**.**




*Proof.* The proof is shown in Figure 1. □

### 3.2 The carried family

For any sign vector $\sigma = (\sigma_1, \dots, \sigma_N) \in \{\pm 1\}^N$, define $K_E(\sigma)$ to be the knot obtained from $K_0$ by resolving the $j$-th double point with the sign $\sigma_j$ and, in addition, doing surgery along $C_{m_j}$ iff

$\sigma_j = -1$. Alternatively, we may perform all surgeries first and then fix the signs; disjointness of the $C_{m_j}$ makes this well-defined up to higher GH degree.

A degree-$m_j$ tree realizes a $C_{m_j}$-move, hence, by Proposition 3.1,

$$K_E(\sigma) \equiv K_E(\sigma^{(j)}) \mod C_{m_j},$$

so invariants of degree $< m_j$ cannot see the $j$-th sign ($j$-th decorated crossing). Here $\sigma^{(j)}$ is $\sigma$ with the $j$-th entry flipped.

### 3.3 Weighted GH filtration and derivatives

Let $F_n$ be the GH filtration on the free abelian group $\mathbb{Z}\mathcal{K}$ on isotopy classes of knots (so $F_n$ is generated by $C_n$-moves) [4]. Write $\mathrm{gr}_n^C := F_n/F_{n+1}$ and $\mathrm{gr}^C := \bigoplus_{n\geq 0} \mathrm{gr}_n^C$.

For $J \subseteq \{1, \dots, N\}$ define the alternating multi-difference of the family $K_E(\sigma)$ along the coordinates in $J$:

$$\Delta_J K_E \ := \ \sum_{S \subseteq J} (-1)^{|J|-|S|} \, K_E(\sigma^{(S)}),$$

where $\sigma^{(S)}$ flips the entries of $\sigma$ exactly on $S$. Because all $C_{m_j}$ are disjoint and $\deg C_{m_j} = m_j$, standard clasper calculus implies $\Delta_J K_E \in F_{w(J)}$, where $w(J) := \sum_{j \in J} m_j$ [8]. In particular, for $J = \{j\}$ we have $\Delta_{\{j\}} K_E \in F_{m_j}$, i.e., $\partial_j f = 0$ for $\deg f < m_j$.

A corollary that packages the whole family:

**Corollary 3.2** (Weighted polynomiality)**.** *Let $f$ be a finite-type invariant of degree $\leq d$. Then for every $J$,*

$$w(J) > d \implies f(\Delta_J K_E) = 0.$$

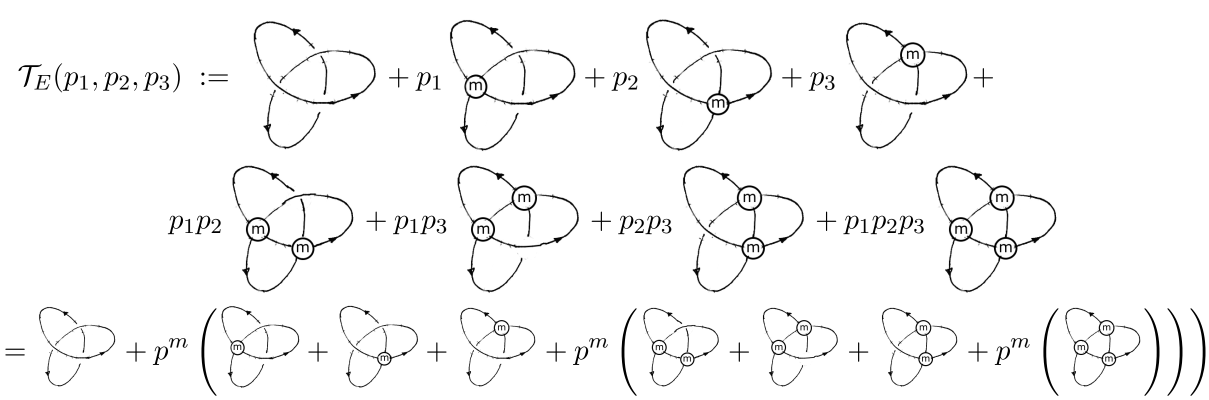
*Equivalently, the function $\sigma \mapsto f\big(K_E(\sigma)\big)$ is a polynomial in the $N$ Boolean variables whose weighted degree (with* $\deg($*variable* $j) = m_j$*) is $\leq d$.*

### 3.4 The GH–Taylor expansion around $K_0$

Introduce formal commuting variables $p_1, \dots, p_N$ with $\deg p_j := m_j$. Define the GH–Taylor series of the extended shadow by

$$\mathcal{T}_E(p_1, \dots, p_N) \ := \ K_0 \ + \ \sum_j p_j \, \Delta_{\{j\}} K_E \ + \ \sum_{|J|\geq 2} \prod_{j \in J} p_j \, \Delta_J K_E \ + \ \cdots$$

The $r$-th summand is $\sum_{|J|=r} (\prod_{j\in J} p_j) \Delta_J K_E$ and lies in filtration $\geq \sum_{j \in J} m_j$. Thus, $\mathcal{T}_E$ is a well-defined element of the completed GH group $\widehat{\mathbb{Z}\mathcal{K}}$. For example,




(14)

The shadow $K_E(\sigma)$ modulo $F_{>d}$ is recovered by substitution

$$K_E(\sigma) \;\equiv\; \big[\mathcal{T}_E\big]_{\le d}\Big|_{p_j=(1-\sigma_j)/2}.$$

So for every finite–type invariant $f$ of degree $\le d$, $f(K_E(\sigma))$ equals the degree-$\le d$ polynomial in the $p_j$ obtained by applying $f$ to $\mathcal{T}_E$.

Keeping only one-tree terms gives

$$\mathcal{T}_E \equiv K_0 \;+\; \sum_{j=1}^{N} p_j \underbrace{\Delta_{\{j\}}K_E}_{\in\, \mathrm{gr}^C_{m_j}} \quad \mathrm{mod}\ F_{>\min_{j\neq k}(m_j+m_k)}.$$

Identifying $\Delta_{\{j\}}K_E$ with the class in $\mathrm{gr}^C_{m_j}$ represented by surgery on $C_{m_j}$, this is exactly

$$K_0 + \sum_{j=1}^{N} p_j\, C_{m_j} \quad\Longrightarrow\quad V_{K_E}(h) = V_{K_0}(h) + \sum_{j=1}^{N} h^{m_j} V_{C_{m_j}}(h),$$

with $p_j$ the grading variable and $C_{m_j}$ the degree-$m_j$ tree class attached at the $j$-th crossing. Here, $V_K(h)$ is the Taylor expansion of some polynomial invariant, e.g., Jones.

The $\Delta_J$ with $|J| \ge 2$ package all interaction terms between different claspers, which first appear in filtration $\sum_{j\in J} m_j$. They are necessary for exactness beyond the linear (one-tree) regime.

### 3.5 $m$-flat families

**Lemma 3.3** ($m$-flat family carried by an extended shadow)**.** *Let $m = \min_{1\le j\le N}(m_j)$. The knots carried by the shadow restricted to the decorated crossings, $\sigma \mapsto K_E(\sigma)$, share the following characterizing properties:*

1. *They are $m$-trivial (and hence also $J_m$-trivial, see Definition 4.1).*
2. *The order-$<m$ symbols (alternating sums of finite-type invariants) computed over subsets $\{\sigma_j\}_{j\in J}$ of decorated crossings vanish.*

*Proof.* The GH-Taylor series shows that all knots $\sigma \mapsto K_E(\sigma)$ are pairwise $m$-equivalent. Because the original shadow consists of at least one unknot it follows that each knot in $\{K_E(\sigma)\}_{\sigma\in\{\pm1\}^N}$ is $m$-trivial. As all finite-type sub-$m$ data is fixed over the cube coordinates $\{\sigma_j\}_{j=1}^{N}$ it follows that,

$$f(\Delta_J K_E) = \partial_J f(K_E) = 0, \quad \deg(f) = |J| < m.$$

□

### 3.6 Uniform barrier and unknot detection

Extended shadows carry $m$-flat families of knots. Translating into a cube language, the extended cube $\mathcal{Q}_E$ is a product of an $m$-flat base subcube $\mathcal{Q}_S$ and the clasper decorations subcube $\mathcal{Q}_C$,

$$E = S + p^m C \quad\Longrightarrow\quad \mathcal{Q}_E = \mathcal{Q}_S \times \mathcal{Q}_C.$$

The extension cube's dimension depends on the clasper tree decorations. Normally, these scale linearly with the parameters $m_j$. Assuming $m_j = \mathcal{O}(N)$, the degree of the base shadow, and taking $C_{m_j}$ as Brunnian claspers, we note that $\dim Q_E = \mathcal{O}(N^2)$.

Applying the above construction using the finite-type expansion of the Jones polynomial leads to a bound on the size of $m$-flat families. We give here a proof sketch of the uniform barrier. The complete proof is provided later on following the formalization of the underlying concepts.

**Theorem** (Uniform barrier)**.** *For any non-trivial $K$ we have $m \leq \mathscr{N}(K)$.*

*Proof sketch.* Let $S$ be the shadow of $K$. By Lemma 3.3 the subcube $\mathcal{Q}_S$ is Jones-$m$-flat / $m$-flat (all order-$< m$ symbols vanish). Theorem 4.10 and Corollary 4.11 show that if $m > |\mathcal{Q}_S| = \mathscr{N}(K)$ the subcube $\mathcal{Q}_S$ becomes completely flat, in which case all knots carried by $S$ are trivial: this is a consequence of the Shadow lemma B.1 property P, and the Jones symbolic skein relation (21). Simply, smoothing crossings in a Jones-$m$-flat cube results in $m$ link components, indicating that the crossings are nugatory/cut-vertex. The clasper decorations occupy the higher degrees in the GH filtration and thus may be trivialized without affecting this conclusion. As $K$ is non-trivial, we have here an obstruction mechanism that caps the size of flat knot families in extension cubes by limiting $m$: unless a knot is trivial, it cannot be ($m$-trivial) $J_m$-trivial with $m > \mathscr{N}$. □

The detection results follow directly from the barrier.

**Theorem.** *$V_K \equiv 1 \longrightarrow K$ is the unknot.*

*Proof.* This follows from the uniform barrier. If $K$ were non-trivial and $V_K = 1$, i.e., $J_m$-trivial for all $m$, then $m \leq N(K)$ for all $m$. Contradiction. □

**Theorem** (Vassiliev)**.** *A knot that is $m$-trivial for all $m$ is the unknot.*

*Proof.* This follows from the uniform barrier. If $K$ were non-trivial and $m$-trivial for all $m$, then $m \leq N(K)$ for all $m$. Contradiction. □

# 4 $J_m$-triviality and Jones-$m$-flatness

## 4.1 Finite-type expansion of the Jones polynomial

We start from the Jones skein

$$x^{-2}V^{(+)} - x^2V^{(-)} = (x - x^{-1})V^{(0)}. \tag{15}$$

and pass to the Birman–Lin substitution $x = e^{h/2}$. This reorganizes $V_L$ into a power series

$$V_L(e^{h/2}) = \sum_{n\geq 0} c_n(L)\, h^n, \tag{16}$$

whose coefficients $c_n$ are Vassiliev invariants of order $\leq n$ [5].

**Definition 4.1** (Jones $m$-trivial, $J_m$-trivial)**.** *A knot $K$ is $J_m$-trivial if its $n$-type coefficients $c_n(K) = 0$ for $1 \leq n \leq m - 1$.*

**Remark 4.1.** *Note that if $V_K$ has a $h$-gap of order $m$:*

$$V_K(e^{h/2}) = 1 + h^m V_K'(h), \tag{17}$$

*then it is $J_m$-trivial.*

**Remark 4.2** (Every knot is $J_m$-trivial for some $m$)**.** *Every non-trivial knot is $J_m$-trivial for some $m \geq 2$ (as $c_1 = 0$ for knots). The unknot is $J_m$-trivial for any $m$.*

**Remark 4.3.** *Standard $m$-triviality (GH) implies $J_m$-triviality, because it forces every degree $\leq m-1$ finite-type invariant to agree with the unknot, hence the coefficients $c_n$ vanish for $1 \leq n \leq m-1$.*

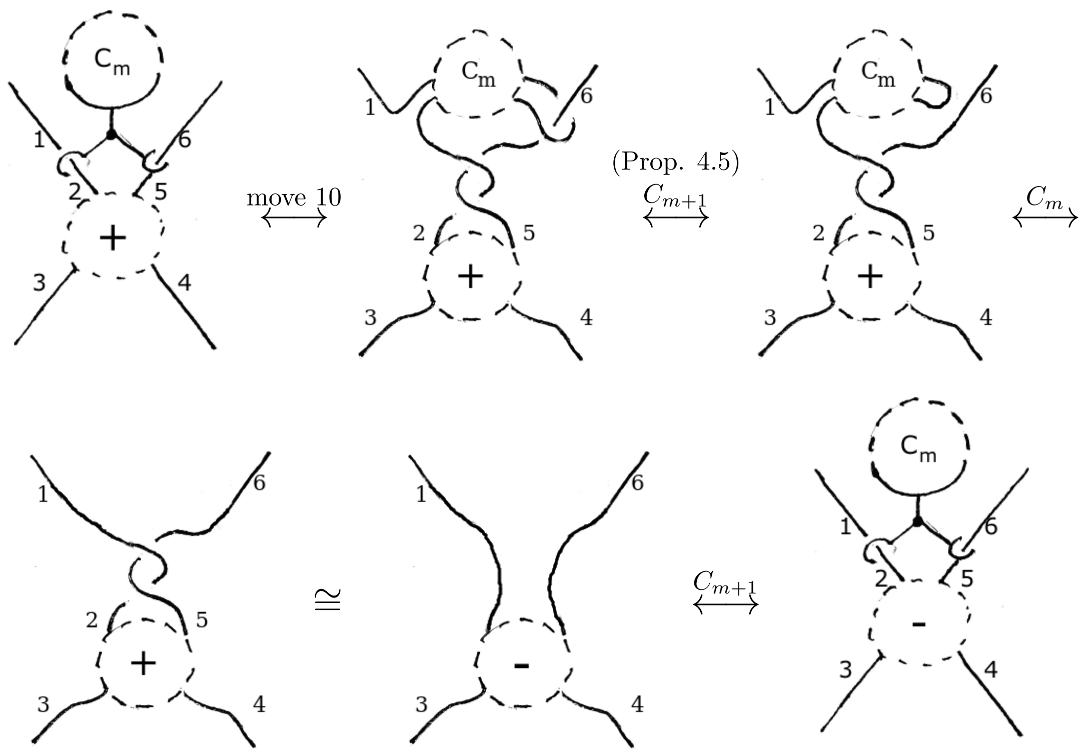



Figure 1: Proof of Proposition 3.1. The proof follows Habiro's clasper calculus. The $C_m$-tree confined to the upper dashed region is band-summed to a $C_2$-tree (a node). The movie consists of $C_m$, $C_{m+1}$ moves and isotopies. The two moves in the upper left corner are Habiro's move 10 (Figure 9 on p. 15 in [8]) and a $C_{m+1}$ following his Proposition 4.5 (Figure 30 on p. 34 in [8]). The movie in its entirety preserves all degree-$< m$ invariants.

In the Walsh-Fourier viewpoint each coefficient layer becomes a pseudo-Boolean function on a fixed shadow cube, so facewise differences isolate orders and symbols. Two immediate consequences drive what follows: $c_0$ is a shadow invariant encoding component count, and the skein differentiates the layers, giving a symbol-level recursion we use to set up the symbol calculus of §4.4.

**Lemma 4.1** (Order-0 coefficient $c_0$)**.** *The order-0 coefficient in* (16) *encodes the number of linked components:*

$$c_0(L) = (-2)^{\ell(L)-1}.$$

*Proof.* This follows from $V_L(1) = (-2)^{\ell(L)-1}$. □

**Lemma 4.2** (Finite-type coefficients skein relation)**.** *For every $n \geq 1$ one has*

$$\partial c_n = c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0) \; + \; R_{n-2} \,, \tag{18}$$

*where the remainder $R_{n-2}$ is an explicit linear combination of strictly lower Vassiliev orders ($\leq n-2$):*

$$\begin{aligned} R_{n-2} = -\sum_{r\geq 1} \frac{1}{(2r)!}\, \partial c_{n-2r} \; + \sum_{r\geq 1} \frac{1}{(2r+1)!} \big(c_{n-2r-1}(+) + c_{n-2r-1}(-)\big) \\ + \sum_{r\geq 1} \frac{1}{2^{\,2r}(2r+1)!}\, c_{n-2r-1}(0). \end{aligned} \tag{19}$$

*Proof.* Substitute $x = e^{h/2}$ in the skein $x^{-2}V^{(+)} - x^2 V^{(-)} = (x - x^{-1})V^{(0)}$ and expand:

$$e^{-h} \sum_{n\geq 0} c_n(+)h^n \; - \; e^{h} \sum_{n\geq 0} c_n(-)h^n \; = \; \big(e^{h/2} - e^{-h/2}\big) \sum_{n\geq 0} c_n(0)h^n.$$

Equating the coefficient of $h^n$ and separating even/odd exponential contributions yields

$$\partial c_n = -\sum_{m\geq 1} \frac{(-1)^m}{m!}\, c_{n-m}(+) \; + \sum_{m\geq 1} \frac{1}{m!}\, c_{n-m}(-) \; + \sum_{\substack{\alpha\geq 1 \\ \alpha \text{ odd}}} \frac{1}{2^{\alpha-1}\alpha!}\, c_{n-\alpha}(0).$$

Group the even/odd $m$'s on the left to obtain the displayed $R_{n-2}$ and isolate the leading $m = \alpha = 1$ terms to get (18). For fixed $n$, all sums are finite. Finally, each object in $R_{n-2}$ has Vassiliev order at most $n-2$ (because it uses indices $< n-1$). □

**Convention 4.1** (Smoothing notation $D \setminus J$)**.** *Let $S$ be a fixed shadow with outside crossing set* cross($S$)*, and let $D \in \mathcal{D}(S)$ be a diagram carried by $S$. For a subset $J \subseteq$* cross($S$)*, we write $D \setminus J$ for the diagram obtained from $D$ by performing the* oriented smoothing *at every crossing in $J$ (our "0" smoothing). Thus:*

- *$D \setminus \emptyset = D$, and for a singleton $\{i\}$ we have $D \setminus \{i\} = D(i = 0)$.*
- *Smoothing at distinct crossings is local and order–independent, so $D \setminus J$ is well defined (independent of any ordering of $J$).*
- *If $J \subseteq$* cross($S$) *and $S \setminus J$ denotes the shadow with those vertices smoothed, then the* residual cube *$\mathcal{Q}_{S\setminus J}$ is the variation cube on the remaining crossings; functions of the form $D \mapsto f(D \setminus J)$ are naturally viewed on $\mathcal{Q}_{S\setminus J}$ (see Definition 4.2 and Corollary 4.3).*

- *We occasionally write $J_r \subseteq J$ to mean an arbitrarily chosen $r$-subset of $J$; by order–independence, statements about $D \setminus J_r$ do not depend on the choice of $J_r$ (used, e.g., when discussing incremental smoothings and the last–row fingerprint).*

- *In single-crossing skeins we retain $D(+), D(-), D(0)$; the set-notation agrees with this: $D \setminus \{i\} = D(i = 0)$.*

**Definition 4.2.** *For a shadow $S$ and $J \subset \mathrm{cross}(S)$, the* ***residual cube*** *$\mathcal{Q}_{S\setminus J}$ is the variation cube of over/under choices on the remaining crossings after smoothing all $J$.*

**Corollary 4.3** (Order-0 constancy on residual cubes)**.** *Fix any shadow $S$ of $K$. Because $c_0$ is a type-0 (order-0) Vassiliev invariant, for any fixed subset $J \subset \mathrm{cross}(S)$ of crossings the map $D \mapsto c_0(D \setminus J)$ is constant on the residual cube of over/under choices. Here $\mathrm{cross}(S)$ denotes the vertex set of the shadow (the common set of crossings for all $D \in \mathcal{D}(S)$). In particular, under the Jones-flatness hypothesis of Lemma 4.7 below, $c_0(D \setminus J) = (-2)^{|J|}$ for all $D \in \mathcal{D}(S)$.*

*Proof.* Since $c_0$ is type-0, $\partial_T c_0 \equiv 0$ for all nonempty $T$, i.e., it is constant on every face of every residual cube $\mathcal{Q}_{S\setminus J}$. □

**Definition 4.3** (Set-indexed symbols)**.** *For any $K \subset I$ and $J \subset I \setminus K$, define*

$$\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$$

*i.e. first perform oriented smoothings at all crossings in $K$, then take the unnormalized alternating sum over the $J$-face (order $|J|$).*

**Convention 4.2** (Set/cardinality-indexed symbols)**.** *From now on we write $\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$ to record both the smoothed subset $K$ and the differenced subset $J$ explicitly. If only the cardinalities matter, we write $\mathscr{S}_{r,n}$ for the class generated by all $\mathscr{S}_{K;J}$ with $|K| = r, |J| = n$.*

**Lemma 4.4** (Symbol-skein recursion)**.** *Fix a crossing $i$. For every $n \geq 1$,*

$$\mathscr{S}_{\varnothing;J} \;=\; 2\,\mathscr{S}_{\varnothing;J\setminus\{i\}} \;+\; \mathscr{S}_{\{i\};J\setminus\{i\}} \qquad (i \in J,\ |J| = n). \tag{20}$$

*and more generally,*

$$\mathscr{S}_{K;J} = 2\,\mathscr{S}_{K;J\setminus\{i\}} + \mathscr{S}_{K\cup\{i\};J\setminus\{i\}}. \tag{21}$$

*Proof of Lemma 4.4 using state-sum for the order-$n$ symbol.* Classically, the symbol of $c_n$ on a chord diagram $D$ with chords $\{j\}$ can be written as a state-sum over functions $s : \{j\} \to \{1, 2\}$ ("double"/"delete" the chord) (see p. 71 in [4]):

$$\mathscr{S}_{K;J} = \sum_s \Big( \prod_j s(j) \Big) \cdot (-2)^{\,|\pi_0(s)|-1},$$

where $|\pi_0(s)|$ is the number of components of the immersed curve obtained by applying the resolution prescribed by $s$. Fix a distinguished chord $i$ and split the sum according to $s(i)$:

- If $s(i) = 2$ (delete), the local multiplicative weight contributes a factor 2, and the resulting curve is obtained from $D$ by forgetting the chord $i$. Summing over the remaining chords gives $2\,\mathscr{S}_{K;J\setminus\{i\}}$.

- If $s(i) = 1$ (double), the local multiplicative weight contributes a factor 1, and the resulting curve is the oriented smoothing at $i$. Summing over the remaining chords gives $\mathscr{S}_{K\cup\{i\};J\setminus\{i\}}$.

Adding the two disjoint families of states yields (21). □

*Proof of Lemma 4.4 using Lemma 4.2.* Apply the $(n-1)$-fold unnormalized face-difference $\partial_{J\setminus\{i\}}$ (with $i \in J$, $|J| = n$) to the coefficient recursion (18):

$$\partial_J c_n = \partial_{J\setminus\{i\}}\big(c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0)\big) \; + \; \partial_{J\setminus\{i\}} R_{n-2}.$$

Since every term in $R_{n-2}$ has Vassiliev order $\leq n-2$, its $(n-1)$-fold alternating sum vanishes identically: $\partial_{J\setminus\{i\}} R_{n-2} \equiv 0$. What remains are exactly the three order-$(n-1)$ pieces on the right. Group the two $(\pm)$ terms (they live on the same residual $(n-1)$-face) and the smoothed term (which lives on the $(n-1)$-face after oriented smoothing at $i$), to obtain

$$\partial_J c_n = 2\,\partial_{J\setminus\{i\}} c_{n-1} \; + \; \partial_{J\setminus\{i\}} c_{n-1}^{\{i\}},$$

which is (20). □

## 4.2 Order-0 constancy: $c_0$ is a shadow/cube invariant

The BL-expansion (16) free coefficient $c_0$ is a 0-order finite-type invariant, which in turn makes it an invariant of variation cubes or shadows rather than of any single vertex (knot/link). In addition, we have seen it uniquely encodes the link component count. These two facts render it a powerful tool for analyzing the behavior of shadows under oriented smoothing.

**Corollary 4.5** (Shadow invariance of component counts)**.** *Let $S$ be a shadow and $D(S)$ the set of all diagrams carried by $S$. Then:*

1. *(**No smoothing**) The number of link components $\ell(D)$ is independent of $D \in D(S)$.*
2. *(**Fixed smoothings**) More generally, for any $J \subseteq \mathrm{cross}(S)$ (crossing set of $S$), the component count of $D \setminus J$ (link obtained from $D$ by smoothing exactly the crossings in $J$) is independent of $D \in D(S)$.*

*Proof.* By order-0 constancy (Corollary 4.3), $D \mapsto c_0(D \setminus J)$ is constant on the residual cube determined by $J$. By Lemma 4.1, $c_0(D \setminus J) = (-2)^{\ell(D\setminus J)-1}$, so $\ell(D \setminus J)$ is constant as $D$ varies. Taking $J = \varnothing$ yields (1). □

**Corollary 4.6** ($c_0^m$ fingerprint of property (P))**.** *Let $D \in \mathcal{D}(S)$ be a diagram (shadow $S$). Fix a subset $J \subseteq \mathrm{cross}(D)$ of size $|J| = m$. If*

$$c_0(D \setminus J) = (-2)^m,$$

*then for every $k$-subset $K \subset J$ (with $0 \leq k \leq m$) we have*

$$c_0(D \setminus K) = (-2)^k \quad \textit{and} \quad \#\textit{components}(D \setminus K) = 1 + k. \tag{\textbf{P}}$$

*Equivalently, property (P) holds along the fixed set $J$ for all $k \leq m$. If, moreover, $m = |\mathrm{cross}(D)|$ then by the Shadow Lemma (B.1), $D$ is an unknot.*

*Proof.* By Lemma 4.1, $c_0(L) = (-2)^{\ell(L)-1}$, so the hypothesis says $D \setminus J$ has $(m+1)$ components. Smooth the $m$ crossings in $J$ in any order and write $\delta_i \in \{0, 1\}$ for the increment in the component count at step $i$. Oriented smoothing changes the component count by at most 1, so $\delta_i \leq 1$. Since the final count is $1 + \sum_{i=1}^m \delta_i = m + 1$, each $\delta_i = 1$. Thus the first $k$ smoothings (those in any prescribed $k$-subset $K \subset J$) yield exactly $1 + k$ components, hence $c_0(D \setminus K) = (-2)^k$ (again by Lemma 4.1). Finally, order-0 constancy on residual cubes (Corollary 4.3) implies that for fixed $J$ and $K \subset J$ the value of $c_0(D \setminus K)$ is independent of the over/under choices at the remaining crossings, so the same conclusion holds for every diagram carried by $S$. □

### 4.3 Jones-flat cubes

**Definition 4.4.** *A variation cube all of whose vertices are unknots is said to be* ***Jones-flat****. Equivalently, every variation of the underlying shadow is an unknot.*

**Lemma 4.7** (Jones-flatness)**.** *Denote by $\delta_n$ the Kronecker delta $\delta_{n0}$. A necessary and sufficient condition for Jones-flatness is the vanishing of all non-trivial symbols,*

$$\mathscr{S}_{0,n} = \partial_{[n]} c_n^0 = \delta_n. \tag{22}$$

*Proof.* Let $\mathscr{S}_{r,n} = \partial_{[n]} c_n^r$ be the $n$-th symbol after $r$ oriented smoothings. The symbol recursion (21) reads,

$$\mathscr{S}_{r,n-1} = \mathscr{S}_{r-1,n} - 2\mathscr{S}_{r-1,n-1},$$

where $r$ and $n$ are the cardinalities of the smoothed and differentiated crossing sets (Convention 4.2). As $\mathscr{S}_{0,n} = \delta_n$, it follows that, $\mathscr{S}_{r,n} = (-2)^r \delta_n$, and in particular, the free coefficient after $r$ smoothings, $c_0^r = (-2)^r$. Fix a vertex $K$ in the variation cube. Note that $r = |\mathrm{cross}(S)|$ is the number of crossings in any diagram obtained from the shadow. Therefore, by Corollary 4.6 (P) holds for all $k \leq |\mathrm{cross}(S)|$, rendering $K$ the unknot. As this holds for every vertex $K$, the cube is Jones-flat.

The converse direction, every variation is the unknot $\to$ $\mathscr{S}_{0,n} = \delta_n$, follows immediately upon recognizing that all Fourier coefficients in all cubes vanish except for the trivial $c_0 = 1$, from which the symbols follow. □

### 4.4 Jones-$m$-flat subcubes

**Definition 4.5** (Jones–$m$–flat subcube)**.** *Let $S$ be a shadow and $\mathscr{U} \subseteq \mathrm{cross}(S)$. The $\mathscr{U}$-subcube $\mathcal{Q}_{\mathscr{U}}$ is Jones–$m$–flat if for every $1 \leq n \leq m-1$ the coefficient $c_n$ is constant on $\mathcal{Q}_{\mathscr{U}}$ (equivalently, all facewise differences $\partial_W c_n$ with $\emptyset \neq W \subseteq \mathscr{U}$ vanish). If this holds for all $m$, call $\mathcal{Q}_{\mathscr{U}}$ Jones-$\infty$-flat or flat for short.*

**Remark 4.4.** *"Flat" here means all finite-type layers below $|\mathscr{U}|$ are constant on the $\mathscr{U}$-subcube; only when $\mathscr{U}$ is the full crossing set does the Shadow Lemma B.1 implies all vertices are unknots.*

**Remark 4.5** (Flat subcube constants)**.** *On a flat subcube every positive-order ($n \geq 1$) $c_n$ is constant. Whenever a shadow subcube contains unknot vertices it pins these constants to* 0*. The analogous anchor for Jones-m-flat subcubes is that orders $1 \leq n \leq m-1$ vanish identically by definition. Think of $J_m$-trivial vertices as playing the same role for Jones-m-flat subcubes that the unknot plays for flat subcubes.*

**Lemma 4.8** (Symbols vanish on Jones-$m$-flat subcubes through order $m-1$)**.** *If $\mathcal{Q}_{\mathscr{U}}$ is Jones-$m$-flat, then $\mathscr{S}_{\varnothing;J} \equiv 0$ for every nonempty $J \subseteq \mathscr{U}$ with $|J| \leq m-1$.*

*Proof.* For each $1 \leq n \leq m-1$, the function $c_n$ is constant on $\mathcal{Q}_{\mathscr{U}}$ (by definition of Jones-$m$-flatness). Therefore, all face derivatives vanish, $\partial_{[n]} c_n \equiv 0$. □

For $K \subset I$ and $J \subset I \setminus K$ set $\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$. When only the cardinalities $r = |K|$ and $q = |J|$ matter we write $\mathscr{S}_{r,q}$ for the rank-1 line generated by all $\mathscr{S}_{K;J}$ with $|K| = r, |J| = q$; in this quotient the specific choice of $J$ is forgotten.

Define the pair of operators:

$$v_i : \mathscr{S}_{r,q} \to \mathscr{S}_{r,q-1}, \quad h_i : \mathscr{S}_{r,q} \to \mathscr{S}_{r+1,q-1}, \tag{23}$$

that is, $v_i$ lowers symbol order by one by taking the partial difference at $i$, $h_i$ lowers symbol order and increases the smoothing index by one after oriented smoothing at $i$; concretely $v_i(\mathscr{S}_{K;J}) = \mathscr{S}_{K;J\setminus\{i\}}$, $h_i(\mathscr{S}_{r,q}) = \mathscr{S}_{K\cup\{i\};J\setminus\{i\}}$. The symbol skein may thus be expressed as

$$\mathscr{S}_{K;J} = 2v_i(\mathscr{S}_{K;J}) + h_i(\mathscr{S}_{K;J}), \tag{24}$$

i.e., $2v_i + h_i$ acts as an identity on $\mathscr{S}_{K;J}$.

#### 4.4.1 Symbol module and operator calculus

Fix a shadow $S$ with outside crossing set $I = \mathrm{cross}(S)$. For $r, n \geq 0$, let $\mathbb{Z}^{\mathcal{D}(S)}$ denote the free abelian group on diagrams carried by $S$, and let $c_n^r : \mathcal{D}(S) \to \mathbb{Q}$ be the order-($\leq n$) Vassiliev functional after $r$ oriented smoothings. Define the symbol extraction map $E_{r,n} : \mathbb{Q}^{\mathcal{D}(S)} \to \mathcal{S}_{r,n}$ by

$$E_{r,n}(F) := \partial_{[n]} F^r \qquad \text{(unnormalized alternating sum on $n$-faces)},$$

and set $\mathcal{S}_{r,n} := \mathrm{im}\, E_{r,n}$ (the symbol module of order $n$ after $r$ smoothings). Elements of $\mathcal{S}_{r,n}$ are written $\mathscr{S}_{r,n}$ in the sequel (Figure 2 visualizes the grid $(r, n)$).

Define linear maps

$$v_i : \mathcal{S}_{r,n} \to \mathcal{S}_{r,n-1}, \qquad v_i\big(\mathscr{S}_{K;J}\big) := \mathscr{S}_{K;J\setminus\{i\}},$$

$$h_i : \mathcal{S}_{r,n} \to \mathcal{S}_{r+1,n-1}, \qquad h_i\big(\mathscr{S}_{K;J}\big) := \mathscr{S}_{K\cup\{i\};J\setminus\{i\}},$$

with $|K| = r,\ |J| = n$. These are well-defined on $\mathcal{S}_{r,n}$ because the assignments are linear in $c_n^r$ and depend only on the image $E_{r,n}(c_n^r)$. (Linearity: the constructions are formed from alternating sums. Well-definedness follows a posteriori from the operator identity below.)

**Corollary 4.9** (path independence of smoothing at the symbol level)**.** *For any subset $J \subset I$ with $|J| = k$, the operator that smooths the crossings in $J$ (in any order) acts on symbols by*

$$h_J := \prod_{i\in J} h_i = \prod_{i\in J}(\mathrm{id} - 2\, v_i) = \sum_{L\subseteq J} (-2)^{|L|}\, v_L, \tag{25}$$

*where $v_L := \prod_{i\in L} v_i$ (the products are order-independent). In particular, path independence means the result depends only on the set $J$, not the order in which its elements are smoothed. (This follows from the one-crossing skein $h_i = \mathrm{id} - 2v_i$ and commutativity of the discrete differences.)*

**Remark 4.6.** *In general $v_L \mathscr{S}_{\varnothing;J}$ depends on the set $L$, not only on its size. On Jones-$m$-flat subcubes the dependence disappears when acting on $\mathscr{S}_{\varnothing;J}$ with $|J| = m-1$, because all intermediate top-row symbols vanish, $v_L \mathscr{S}_{\varnothing;J} = \mathscr{S}_{\varnothing;J\setminus L} = 0$, for $0 \leq |L| < m-1$ (Lemma 4.8); this is the only specialization used in the proof of Theorem 4.10.*

#### 4.4.2 Last-row fingerprint and degree cap

**Theorem 4.10** (Last-row fingerprint of Jones-$m$-flat subcubes)**.** *Let $\mathcal{Q}_{\mathscr{U}}$ be Jones-$m$-flat. For any prescribed set $J \subseteq \mathscr{U}$ of $m-1$ crossings and any diagram $D$ carried by $S$, the free (order-0) coefficient satisfies the **last-row fingerprint***

$$c_0(D \setminus J) = (-2)^{m-1}.$$

*Equivalently, for every such $J$,*

$$\mathscr{S}_{J;\varnothing} = (-2)^{m-1} \mathscr{S}_{\varnothing;\varnothing} = (-2)^{m-1}.$$

*As this holds for every $J \subseteq \mathscr{U}$ with $|J| = m-1$ we write $c_0^{m-1} = (-2)^{m-1}$. By Corollary 4.6, this last-row equality implies property (P) for every $k \le m-1$ (uniformly over all diagrams carried by the same shadow).*

*Proof.* Fix $J \subseteq \mathscr{U}$ with $|J| = m-1$. Apply (25),

$$\mathscr{S}_{J;\varnothing} = h_J\big(\mathscr{S}_{\varnothing;J}\big) = \sum_{L\subseteq J} (-2)^{|L|}\, v_L\, \mathscr{S}_{\varnothing;J}.$$

Each term satisfies $v_L\, \mathscr{S}_{\varnothing;J} = \mathscr{S}_{\varnothing;J\setminus L}$. On a Jones-$m$-flat subcube we have exactly the vanishing

$$\mathscr{S}_{\varnothing;J\setminus L} = 0 \quad \text{for } 1 \le |J \setminus L| \le m-1$$

(Lemma 4.8), so all terms with $0 \le |L| < m-1$ die after restriction to the $\mathscr{U}$-slice. The only surviving term is, $L = J$: $v_J \mathscr{S}_{\varnothing;J} = \mathscr{S}_{\varnothing;\varnothing} = c_0$. Hence

$$\mathscr{S}_{J;\varnothing} = (-2)^{m-1}\, \mathscr{S}_{\varnothing;\varnothing} = (-2)^{m-1},$$

i.e. $c_0(D \setminus J) = (-2)^{m-1}$. This is the last-row fingerprint.

*From last-row fingerprint to property (P).* By Lemma 4.1, $c_0(L) = (-2)^{\ell(L)-1}$. Thus $\mathscr{S}_{J;\varnothing} = (-2)^{m-1}$ means that after smoothing the prescribed $|J| = m-1$ crossings, the component count is $m$. Smoothing in any order can raise the component count by at most 1 at each step; hitting the final value $m$ forces every intermediate increment to be exactly $+1$. Hence for every $K \subseteq J$, $c_0(D \setminus K) = (-2)^{|K|}$, i.e. property (P) holds along $J$ (Corollary 4.6). □

**Corollary 4.11** (Degree cap of $m$-flatness)**.** *Let $S$ be a shadow, $\mathscr{U} \subset \operatorname{cross}(S)$ a set of $d := |\mathscr{U}|$ crossings, and $\mathcal{Q}_{\mathscr{U}}$ the $\mathscr{U}$-subcube. Assume $\mathcal{Q}_{\mathscr{U}}$ is Jones-$m$-flat, i.e. for every $1 \le n < m$ the coefficient $c_n$ is constant on $\mathcal{Q}_{\mathscr{U}}$. Then:*

(i) Supercritical case yields (P) along $\mathscr{U}$. *If $m > d$, then property* (P) *holds along $\mathscr{U}$: for every $K \subseteq \mathscr{U}$, smoothing the crossings in $K$ increases the component count by $|K|$, and*

$$\mathscr{S}_{K;\varnothing} = c_0(D\setminus K) = (-2)^{|K|} \quad \textit{for all } D \in \mathcal{D}(S).$$

(ii) Global nugatory cap. *If moreover $\mathscr{U} = \operatorname{cross}(S)$ (so $d$ is the total number of crossings of $S$), then the shadow is a tree (every crossing nugatory), hence all diagrams carried by $S$ are unknots; in particular the complete cube is isotopically constant.*
(iii) Degree cap (contrapositive). *In the global situation $\mathscr{U} = \operatorname{cross}(S)$, if the complete cube is not isotopically constant, then necessarily $m \le d$.*

*Proof.*

(i) Set $d = |\mathscr{U}|$ and suppose $m > d$. Then $m' := d+1 \le m$. Jones-$m$-flatness implies Jones-$m'$-flatness (constancy of $c_n$ for $n < m'$), so we may apply Theorem 4.10 (the last-row fingerprint) with the prescribed $(m'-1)$-set $J = \mathscr{U}$. This gives

$$c_0(D\setminus\mathscr{U}) = (-2)^d \quad \text{for every } D \in \mathcal{D}(S).$$

By Corollary 4.6, the equality $c_0(D\setminus\mathscr{U}) = (-2)^d$ forces the component count to increase by exactly one at each of the $d$ smoothings, hence for every $K \subseteq \mathscr{U}$ we have $c_0(D \setminus K) = (-2)^{|K|}$ and #components$(D\setminus K) = 1 + |K|$. This is property (P) along $\mathscr{U}$.

(ii) If additionally $\mathscr{U} = \operatorname{cross}(S)$, then (P) holds on the full crossing set. By the Shadow Lemma B.1, $S$ is a tree and every diagram carried by $S$ is an unknot; consequently the complete cube is isotopically constant.

(iii) The last assertion is the contrapositive of (ii): if the complete cube is not isotopically constant, then (when $\mathscr{U} = \operatorname{cross}(S)$) the hypothesis $m > d$ cannot hold, so $m \le d$. □

**Remark 4.7** (Symbol matrix illustration)**.** *The last-row fingerprint behavior of symbols from Theorem 4.10 and Corollary 4.11 is depicted in Figure 2.*

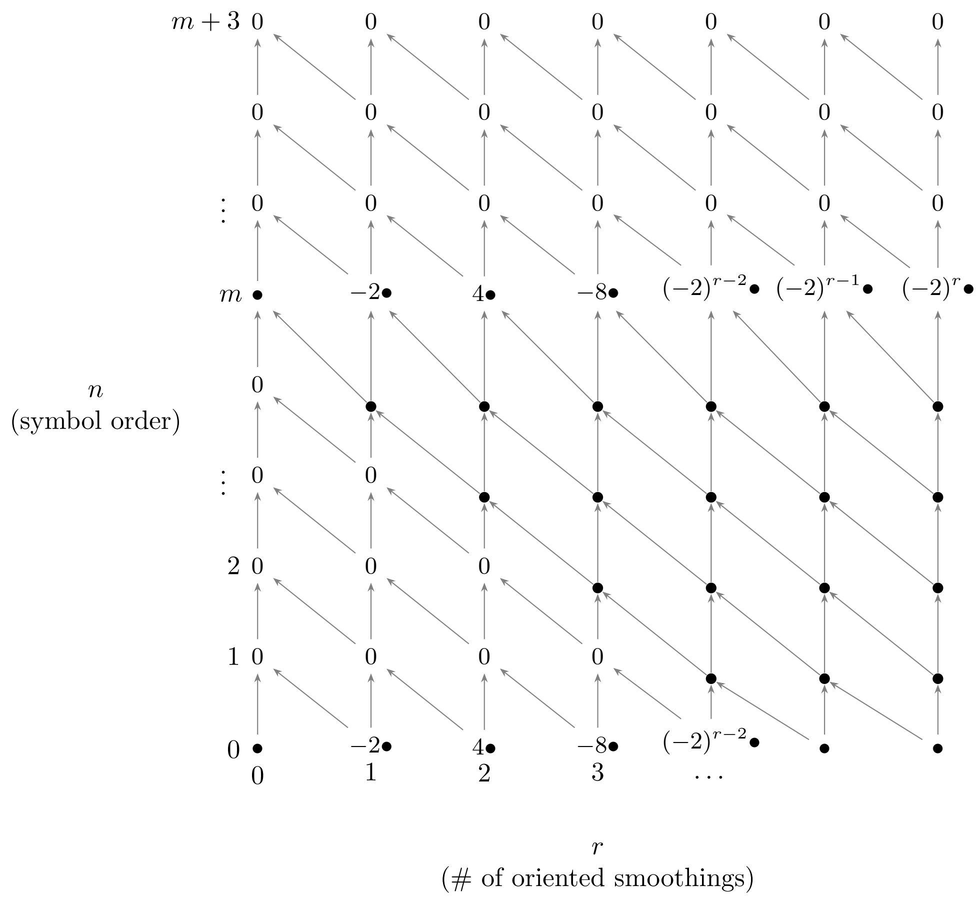



Figure 2: The symbols matrix $[\mathscr{S}_{r,n}]$ of a Jones-$m$-flat subcube ($\mathscr{S}_{0,n} = 0$ for $1 \le n \le m-1$) exhibiting the last-row fingerprint from Theorem 4.10. The skein operator $2v + h$, where $v$ and $h$ are the pair in (23) constrains the triad, $\mathscr{S}_{r,n}, \mathscr{S}_{r,n-1}$, and $\mathscr{S}_{r+1,n-1}$, which ultimately leads to the last-row fingerprint. The distinctive $(-2)^r$ pattern is evident in both the order-$m$ symbol and the order-0 (free) coefficients, $\mathscr{S}_{r,0} = c_0^r$.

# 5 Uniform $m$-triviality barrier

**Theorem 5.1** (Uniform $m$-triviality barrier)**.** *Let $K$ be $J_m$-trivial. If $K$ is nontrivial, then*

$$m \le \mathscr{N}(K).$$

*Proof of Theorem 5.1.* Let $S$ be the shadow of $K$. Fix an extended shadow $E = (K_0; \{C_{m_j}\})$ as in §3.1 with $\min_j m_j = m$. The clasper subcube consists of all clasper states, including the inactive (trivial) tangles. Let $T$ be the set of cube coordinates underlying all inactive clasper decorations, and let $\mathcal{Q}_{S_T} := \{T\} \times \mathcal{Q}_S$ be the corresponding $T$-face of the extended cube. The following lemma shows that $m$-flatness carries over from $\mathcal{Q}_{S_T}$ to $\mathcal{Q}_S$.

**Lemma 5.2** (Decorated-to-outside transfer)**.** *Let $f$ be any finite-type invariant of degree $< m$. Then for every nonempty $J \subseteq \operatorname{cross}(S)$ with $|J| \le m-1$,*

$$\partial_J\big(f \text{ on } \{T\} \times \mathcal{Q}_S\big) \;=\; 0.$$

*In particular, the face $\{T\} \times \mathcal{Q}_S$ is $m$-flat, so every Jones layer $c_n$ with $1 \le n \le m-1$ is constant on $\{T\} \times \mathcal{Q}_S$.*

*Proof of Lemma 5.2.* Fix a vertex $\varepsilon \in \mathcal{Q}_S$ and write $\sigma(\varepsilon) \in \{\pm 1\}^N$ for the same sign vector. By construction (Figure 1), the knot $K_T(\varepsilon)$ at the $T$-face and the decorated knot $K_E(\sigma(\varepsilon))$ differ by surgeries along a disjoint family of $C_m$ trees, one for each index with $\sigma_j(\varepsilon) = -1$. Thus $K_T(\varepsilon)$ and $K_E(\sigma(\varepsilon))$ are $C_m$-equivalent, hence $f\big(K_T(\varepsilon)\big) = f\big(K_E(\sigma(\varepsilon))\big)$ by GH invisibility (order $< m$ invariants are preserved by $C_m$ moves). Taking the alternating sum over the $J$-face,

$$\partial_J f\big|_{\{T\}\times Q_S} = \sum_{S\subseteq J}(-1)^{|J|-|S|} f\big(K_T(\varepsilon^S)\big) = \sum_{S\subseteq J}(-1)^{|J|-|S|} f\big(K_E(\sigma(\varepsilon^S))\big) = \partial_J\big(f\circ K_E\big).$$

By Lemma 3.3 and Corollary 3.2, $\partial_J(f \circ K_E) = 0$ for $|J| \le m-1$ (the decorated $J$-differences land in filtration $\ge \sum_{j\in J} m_j \ge m$, so degree $< m$ invariants vanish on them). Hence $\partial_J\big(f \text{ on } \{T\}\times Q_S\big) = 0$ for $|J| \le m-1$. □

By Lemma 5.2 the subcube $\mathcal{Q}_{S_T}$ is Jones-$m$-flat / $m$-flat (all order-$< m$ symbols vanish), i.e., the Jones layers $c_n \equiv 0$, for $1 \le n \le m-1$ on $\mathcal{Q}_{S_T}$. In other words, each knot carried by $\{T\} \times S$ is $J_m$-trivial as $K$ itself is $J_m$-trivial. If $m > \dim \mathcal{Q}_{S_T}$, Theorem 4.10 and Corollary 4.6 give (P) on the full outside set $\mathscr{U} = \operatorname{cross}(S)$ (as $T$ consists of trivial tangle states). Corollary 4.11(ii) (together with Lemma B.1) implies the $T$-resolved extended shadow $\{T\} \times S$ is a tree and the face/subcube $\mathcal{Q}_{S_T}$ is isotopically constant. As $K$ is isotopic to $\{T\} \times K$, it follows that it is the unknot, a contradiction. Thus $m \le \dim \mathcal{Q}_{S_T} = \dim \mathcal{Q}_S$, and by choosing $S$ from a minimal diagram of $K$, this equals $\mathscr{N}(K)$. □

Figuratively, one may say there is a limit to the degree to which a nontrivial knot $K$ may pretend to be 'trivial'. This degree is set by the order $m$ of its first non-trivial finite-type coefficient. The price to pay for this aspiration comes in the form of increased knottedness; its number of crossings scales as

$$\mathscr{N}(K) \ge m.$$

In a sense, the unknot is the only knot that can afford being $\infty$ly trivial.

**Theorem 5.3** (Jones detects the unknot; finite check)**.** *With the Jones normalization of* (15)*, write the expansion* (16)*, $V_K(e^{h/2}) = \sum_{n\ge 0} c_n(K)h^n$. If $V_K \equiv 1$ then $K$ is the unknot. Equivalently, on a minimal diagram with $\mathscr{N}(K)$ crossings, if $c_n(K) = 0$ for $1 \le n \le \mathscr{N}(K)$ then $V_K \equiv 1$.*

*Proof.* $V_K \equiv 1 \Rightarrow c_n \equiv 0$ for $n \ge 1$. If $K$ were non-trivial, Theorem 5.1 would give $n \le \mathscr{N}(K)$ for all $n$, a contradiction. □

**Theorem 5.4** (Vassiliev unknot conjecture)**.** *A knot that is $m$-trivial for all $m$ is the unknot.*

*Proof.* Suppose that $K$ were non-trivial. The statement implies $K$ is $J_m$-trivial for all $m$. By Theorem 5.1, $m \le \mathscr{N}(K)$ for all $m$. Contradiction. □

# 6 Conclusions and outlook

Birman–Lin showed that coefficients in the series expansions of skein polynomials (Jones, HOMFLY, Kauffman) define Vassiliev invariants, suggesting that polynomial–finite-type bridges might ultimately settle unknot detection via control of low-order layers [5, 6]. Kontsevich's integral and Bar-Natan's diagrammatics organize those layers via weight systems and the $1T/4T$ relations [2, 3]. On the geometric side, Goussarov–Habiro theory provides local moves that are invisible below degree $m$ (connected $C_m$ claspers) [7, 8].

What has been missing is a *diagram-local* mechanism that: (i) packages finite-type layers on a fixed shadow; (ii) turns their local vanishing into a rigid, shadow-level constraint; and (iii) transports this flatness across many outside crossings at once. Our contribution is precisely this triptych, expressed entirely in terms of extended shadows and clasper decorations:

**(A) $m$-invisible crossings on extended shadows.** Fix a shadow and, near each outside crossing $j$, attach a degree-$m_j$ tree clasper. Crossing flips are then realized *modulo clasper calculus* by $C_{m_j}$ moves, hence are invisible to all invariants of degree $< m_j$. With $m := \min_j m_j$, the resulting carried family is $m$-flat for order $< m$ finite-type data (weighted GH filtration).

**(B) Jones layers and a symbol-level skein.** For the Birman–Lin expansion $V_K(e^{h/2}) = \sum_n c_n(K)h^n$, each $c_n$ has order $\leq n$. On any $m$-flat family the layers $c_1, \ldots, c_{m-1}$ are constant, and whenever a single vertex is $J_m$-trivial these constants pin to zero (Jones–$m$–flatness). A symbol-level skein recursion then forces a *last-row fingerprint* for the free term $c_0$: smoothing any $m-1$ chosen crossings produces exactly $m$ components (property (P) along that set).

**(C) Shadow criterion and the uniform barrier.** A shadow-level criterion converts (P) into global nugatoriness: when $m$ exceeds the number of varied crossings, the shadow collapses to a tree and every carried diagram is an unknot. Applied to a minimal shadow, this yields the uniform bound

$$m \leq \mathscr{N}(K).$$

Equivalently: a nontrivial knot cannot be $J_m$-trivial beyond its crossing number. As a corollary, the Jones polynomial detects the unknot: if $V_K(t) \equiv 1$ then $K$ would be $J_m$-trivial for every $m$, contradicting the bound for $m > \mathscr{N}(K)$; hence $K$ is the unknot. A practical form is: for any diagram with $C$ crossings, vanishing of $c_1, \ldots, c_C$ certifies the unknot.

# Auxiliary lemmas and proofs

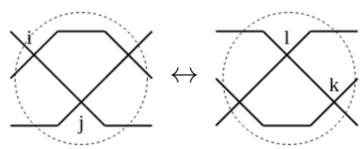
## A Cubical 1T/4T

**Lemma A.1** (Cubical 1T/4T)**.** *Let $S, S'$ be shadows with variation cubes $\mathcal{Q}(S), \mathcal{Q}(S')$. Let $f : \mathcal{Q}(S) \to \mathbb{C}$ and $f' : \mathcal{Q}(S') \to \mathbb{C}$ be pseudo-Boolean functions encoding finite-type invariants on these cubes. Write $I = \operatorname{cross}(S)$ and $I' = \operatorname{cross}(S')$.*

***(1T)*** *If $S' = S\#\mathcal{O}$ (adding a singular twist), then* $\dim \mathcal{Q}(S') = \dim \mathcal{Q}(S) + 1$ *and* $\mathcal{Q}(S') \cong \mathcal{Q}(S) \times \{\pm 1\}$ *with $f'(\varepsilon, \eta) = f(\varepsilon)$. If $m = \dim \mathcal{Q}(S) = |I|$, then*

$$\partial_{I \cup \{\text{new}\}} f' \equiv 0, \qquad \partial_I f' = \partial_I f.$$

***(4T)*** *Suppose $S$ and $S'$ differ by the usual local 4T replacement supported on two crossings $(i, j) \in I$ vs. $(k, l) \in I'$. Assume $|I| = |I'| =: m$ and fix the natural bijection $I \setminus \{i, j\} \cong I' \setminus \{k, l\}$ induced by the local move:*



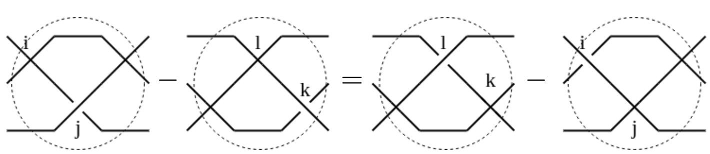

*Then the order-$(m-1)$ Walsh coefficients satisfy*

$$\widehat{f}(I \backslash \{j\}) - \widehat{f}'(I' \backslash \{k\}) \;=\; \widehat{f}'(I' \backslash \{l\}) - \widehat{f}(I \backslash \{i\}). \tag{26}$$

*and the symbols agree (up to the natural identification of coordinates),*

$$\partial_I f = \partial_{I'} f'.$$

**Remark A.1.** *The relation* (26) *is the pseudo-Boolean incarnation of the 4T, commonly represented by*




*Proof. (1T).* The cubical 1T relation follows immediately from the fact that $f(\varepsilon^{(i,+)}) = f(\varepsilon^{(i,-)})$ for a nugatory crossing whose index is $i$. Therefore, $\partial_i f = 0$, from which it follows that the symbol $\partial_I f = 0$.

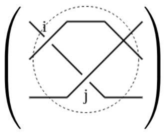
*(4T).* The shadows $S$ and $S'$ differ in two crossings, $\{i, j\}$ vs. $\{k, l\}$. Taking the $(m-2)$-fold discrete derivative over the residual cube defined by the crossing set $I \setminus \{i, j\}$ (equivalently, $I' \setminus \{k, l\}$), yields an order-2 invariant of knots with $(m-2)$ singular crossings. Denote, $g_2 := \partial_{I \setminus \{i,j\}} f$ and $g_2' := \partial_{I' \setminus \{l,k\}} f'$, the respective order-2 invariants over $\mathcal{Q}(S)/\mathcal{Q}_2$ and $\mathcal{Q}(S')/\mathcal{Q}_2'$. As both invariants are defined over 2-dimensional cubes $\{\pm 1\}^2$, their Walsh-Fourier expansion is

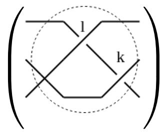
$$\begin{aligned} g_2(\varepsilon_i, \varepsilon_j) &= \hat{g}_2(\emptyset) + \hat{g}_2(i)\varepsilon_i + \hat{g}_2(j)\varepsilon_j + \hat{g}_2(ij)\varepsilon_i\varepsilon_j, \\ g_2'(\varepsilon_k, \varepsilon_l) &= \hat{g}_2(\emptyset) + \hat{g}_2'(k)\varepsilon_k + \hat{g}_2'(l)\varepsilon_l + \hat{g}_2'(kl)\varepsilon_k\varepsilon_l. \end{aligned} \tag{27}$$

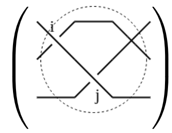
Note that both pseudo-Boolean functions share the same free coefficient $\hat{g}_2(\emptyset)$. This follows from the derivative rule (8) which shows that $\hat{g}_2(\emptyset)$ is the order-$(m-2)$ Fourier coefficient $\hat{f}(I \setminus \{i, j\})$ whose

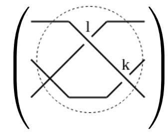
Walsh character $\chi_{I\setminus\{i,j\}}(\varepsilon)$ is independent of $\varepsilon_i, \varepsilon_j$. This coefficient is the same as $\hat{f}'(I' \setminus \{k,l\})$ as they are obtained using the same set of $m-2$ singular crossings.

By Reidemreister-III we have the following identities:

$g_2$ 

$= g_2'$ 

$g_2$ 

$= g_2'$ 

(28)

which translate to

$$g_2(1,1) = g_2'(1,1), \quad g_2(-1,-1) = g_2'(-1,-1). \tag{29}$$

These together with (27) yield,

$$\pm\left[\hat{g}_2(i) + \hat{g}_2(j) - \hat{g}_2'(k) - \hat{g}_2'(l)\right] = \hat{g}_2'(kl) - \hat{g}_2(ij), \tag{30}$$

implying, $\hat{g}_2'(kl) - \hat{g}_2(ij) = 0$, and $\hat{g}_2(i) - \hat{g}_2'(k) = \hat{g}_2'(l) - \hat{g}_2(j)$. The lemma follows upon recognizing that (see (8)), $\hat{g}_2(i) = \hat{f}(I \setminus j)$, $\hat{g}_2(j) = \hat{f}(I \setminus i)$, $\hat{g}_2'(k) = \hat{f}'(I' \setminus l)$, $\hat{g}_2'(l) = \hat{f}'(I' \setminus k)$, and the symbols, $\hat{f}(I) = \hat{g}_2(ij) = \hat{g}_2'(kl) = \hat{f}'(I') \longrightarrow \partial_I f = 2^{|I|}\hat{f}(I) = 2^{|I'|}\hat{f}'(I') = \partial_{I'} f'$. □

# B Shadow Lemma

**Definition B.1** (Nugatory crossing; cut-vertex)**.** *A crossing is nugatory iff there exists a Jordan curve in the projection plane meeting the diagram only at that crossing; at the shadow level, this is equivalent to the vertex being a cut-vertex (removing it disconnects the shadow). [11, 12]*

**Lemma B.1** (Shadow)**.** *Let $S$ be a connected shadow with $n > 0$ vertices and let*

$$\mathcal{D}(S) = \{\textit{all } 2^n \textit{ diagrams carried by } S\}.$$

*For a diagram $D \in \mathcal{D}(S)$ write $c(D)$ for its set of crossings and, for $K \subseteq c(D)$, let $D \setminus K$ be the link obtained from $D$ after* smoothing *every crossing in $K$.*
*The following statements are equivalent:*

1. ***Component-increment property (P)**. For every $D \in \mathcal{D}(S)$ and every subset $K \subseteq c(D)$ of size $k$,*
$$\#\textit{components of } (D \setminus K) = 1 + k. \tag{P}$$

2. *Every vertex of $S$ is a* cut-vertex *(equivalently, every crossing is* nugatory *in Tait's sense). Hence $S$ is the planar embedding of a tree in which each edge appears exactly once.*

3. *All diagrams in $\mathcal{D}(S)$ are diagrams of the unknot (or of a split unlink if $S$ is disconnected).*

*Consequently, any diagram satisfying (P)—and every diagram obtained from it by crossing flips—represents the unknot.*

*Proof.*
**(1) ⇒ (2)** Work with oriented diagrams and fix a vertex $x$ of the connected shadow $S$. By Corollary 4.5, for a fixed set of smoothings the number of components of $D \setminus J$ is independent of the over/under choices in $D \in \mathcal{D}(S)$ (the component count after smoothing a fixed set is a shadow invariant); thus we may argue at the level of the shadow. Property $(P)$ with $|K| = 1$ says that for every diagram $D$ the oriented smoothing at $x$ yields two components $D \setminus \{x\} = L_A \sqcup L_B$.

Suppose $x$ were not a cut-vertex. Then the plane graph $S \setminus \{x\}$ is connected; hence there exists a path in $S \setminus \{x\}$ from an arc of $L_A$ to an arc of $L_B$, and along that path choose the first vertex $y$ where the two sides meet. In the diagram $D \setminus \{x\}$ the crossing $y$ is a crossing between different components $L_A$ and $L_B$. Now smooth $y$ as well. Oriented smoothing at a crossing that connects distinct components does not increase the number of components (it either keeps it the same or reduces it by one). Therefore

$$\#\pi_0\big(D \setminus \{x, y\}\big) \ \le\ 2,$$

contradicting Property $(P)$ for $|K| = 2$, which requires $1+2 = 3$ components. Hence our assumption was false: $S \setminus \{x\}$ must be disconnected, i.e. $x$ is a cut-vertex (nugatory) [11].

**(2) ⇒ (3)** Theorem 3 in [13]: *A shadow $S$ has only unknot diagrams iff every vertex of $S$ is a cut-vertex.* Assumption (2) therefore forces every diagram in $\mathcal{D}(S)$ to be an unknot diagram.

**(3) ⇒ (1)** Seifert's genus formula.

Fix a diagram $D \in \mathcal{D}(S)$ and orient it. Apply *Seifert's algorithm* [14]: smoothing all $n$ crossings produces $s$ Seifert circles and a surface with Euler characteristic $\chi = s - n$. Because $D$ is an unknot, its genus must be zero. For a surface with one boundary component,

$$\chi = 1 - 2g \quad \Longrightarrow \quad 0 = g = \frac{1 - (s - n)}{2},$$

hence

$$s = n + 1. \tag{1}$$

Now smooth an *arbitrary* subset $K \subseteq c(D)$ of size $k$. In the Seifert-surface picture, each smoothed crossing removes exactly one band, thereby increasing the number of boundary components (link components) by one. Starting from one component and using induction on $|K|$ with (1), we obtain exactly $1 + k$ components—property (P).

**(2) ⇔ (1)** (direct argument)

- **(2) ⇒ (1)** If every crossing is nugatory, smoothing one crossing splits one component into two. Repeating the argument shows the component count always increases by one.
- **(1) ⇒ (2)** The single-crossing case $k = 1$ recovers the nugatory condition used in the first implication.

Thus all three statements are equivalent. □

**Corollary B.2** ("Tree diagrams"). *A connected knot diagram satisfies property (P)* iff *its shadow is a tree. Such a diagram is sometimes called a* descending *or* standard *diagram; it can be reduced to a round circle by $n$ Reidemeister I moves (one per nugatory crossing) followed by $n$ paired Reidemeister II moves along the tree edges.*

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