A bound on the m-triviality of knots
Part I: Jones-Vassiliev unknot detection

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Abstract

We prove a uniform barrier for Jones–Vassiliev truncations. If a knot K is J_m-trivial, that is, its Birman-Lin finite-type coefficients $c_1, \ldots, c_{m-1}$ vanish, and K is nontrivial, then $m \leq \mathscr{N}(K)$, the crossing number of K. For a fixed shadow S, the choices of over/under data at its crossings form a Boolean cube; a subcube is the face obtained by varying only a designated subset of crossings. We decorate S by degree-m_j claspers so that flipping the j-th crossing is realized (modulo clasper calculus) by a C_{m_j}-move invisible to finite-type invariants below order m_j; a weighted Goussarov-Habiro filtration packages these effects and produces an order-$< m$ flat family of knots all of whose members are m-trivial. On the Jones side we develop a symbol calculus for the Birman–Lin expansion in which alternating-sum symbols satisfy a skein recursion. When the Jones layers are constant through order $m-1$ (equivalently, all order-$< m$ symbols vanish), the recursion forces a last-row fingerprint for the free term c_0: smoothing any $m-1$ chosen crossings creates exactly m components. A shadow criterion then implies that any supercritical face (where m exceeds the number of varied crossings) is flat and carries only unknots. Applying this to a minimal shadow yields $m \leq \mathscr{N}(K)$. As a corollary, the Jones polynomial detects the unknot: if $V_K(t) \equiv 1$ (so all $c_n(K) = 0$), then K would be J_m-trivial for every m, contradicting the barrier for any $m > \mathscr{N}(K)$; hence K is the unknot. Equivalently, for any diagram with C crossings, vanishing of $c_1, \ldots, c_C$ certifies the unknot, and every nontrivial knot satisfies $\min\{n \geq 1 : c_n(K) \neq 0\} \leq \mathscr{N}(K)$.

1 Introduction

Vassiliev's program views finite-type data through alternating sums over faces of the skein cube of crossing changes [1], leading to the algebra of chord diagrams and the $1T/4T$ relations and encompassing Kontsevich's universal integral [2] and Bar-Natan's graded algebra [3]; see also [4]. In this paper we pursue a complementary, more *topological* implementation built on *extended shadows* and *clasper-decorated crossings*.

Fix a shadow S with outside crossing set $\mathrm{cross}(S)$. Choose a base diagram K_0 carried by S and, for each crossing j, attach a degree-m_j tree clasper C_{m_j} in a small ball meeting the diagram only near j. The resulting data

$$E = (K_0; \{C_{m_j}\}_{j \in \mathrm{cross}(S)})$$

is an *extended shadow*. The key local move is that *flipping the j-th crossing is realized modulo clasper calculus by a C_{m_j}-move*; hence every finite-type invariant of degree $< m_j$ is blind to that flip:

$$\partial_j f \;=\; f(+) - f(-) \;=\; 0 \quad \text{for } \deg f < m_j.$$

We therefore call the j-th crossing m_j*-invisible* ("crossing flip mod C_{m_j}"). See Figure 1 for the local movie implementing this equivalence and §3 for the weighted GH filtration that packages multi-flip alternating sums. In particular, with $m := \min_j m_j$, the family $\sigma \mapsto K_E(\sigma)$ carried by the decorated crossings is m*-flat* for all order $< m$ finite-type data, i.e., any one of its members is m-trivial.

Specializing to the Jones polynomial, we use the Birman–Lin expansion

$$V_K\left(e^{h/2}\right) = \sum_{n\geq 0} c_n(K)\, h^n,$$

whose coefficients c_n are finite-type of order $\leq n$. On any m-flat family (hence, in particular, on the family carried by an extended shadow with $\min m_j = m$), every Jones layer $c_1, \ldots, c_{m-1}$ is constant. When a single vertex in the family is J_m-trivial, these constants are forced to be zero, a situation we call *Jones–m–flatness*. A symbol-level skein recursion for the c_n then yields a *last-row fingerprint* for the free term c_0: smoothing any prescribed $m-1$ crossings produces exactly m components. A shadow-level criterion converts this fingerprint into a structural statement: if m exceeds the number of crossings being varied (in particular, if $m > \#\mathrm{cross}(S)$), then the shadow must be a tree and every carried diagram is an unknot. Applying this to a minimal-crossing shadow of K gives the uniform barrier

$$m \leq \mathscr{N}(K).$$

As an immediate corollary, the Jones polynomial detects the unknot: if $V_K(t) \equiv 1$ (equivalently, $c_n(K) = 0$ for all n), then K would be J_m-trivial for every m; taking $m > \mathscr{N}(K)$ contradicts the barrier, so K is the unknot. Equivalently, for any diagram with C crossings, vanishing of $c_1, \ldots, c_C$ certifies the unknot, and every nontrivial knot satisfies $\min\{n \geq 1 : c_n(K) \neq 0\} \leq \mathscr{N}(K)$.

1.1 Related work

The foundational insight of Birman and Lin is that the coefficients in the power-series expansions of the Jones, HOMFLY and Kauffman polynomials yield Vassiliev invariants [5]; their paper established the bridge between skein-type polynomials and finite-type theory and articulated early structural consequences and open directions for detection. Birman's contemporaneous survey emphasized the breadth of the approach and new perspectives opened by finite-type methods [6].

Kontsevich constructed a universal integral whose weight systems classify finite-type invariants [2], while Bar-Natan developed the algebra of chord diagrams modulo 1T/4T, clarifying symbol calculus and functoriality [3]. These works underlie our symbol-level arguments: within the spectral framework, the top Walsh block encodes precisely the m-th graded piece $\mathcal{A}_m$ of Bar-Natan's chord-diagram algebra, and our cubical 1T/4T (Lemma A.1) is the pseudo-Boolean incarnation of these relations.

Goussarov's n-equivalence and Habiro's clasper calculus show that surgery on a connected C_m tree preserves all invariants of degree $< m$ [7, 8]. We exploit this twice: (i) to produce m-flat families by decorating a fixed shadow with degree-m_j trees, and (ii) via the *crossing flip mod* C_{m_j} gadget that renders the j-th crossing m_j-invisible. The weighted GH filtration then controls all multi-flip differences and underlies the GH–Taylor expansion we use for bookkeeping. Figure 1 illustrates the local movie realizing the flip modulo C_m.

A corollary of earlier work on polynomial/finite-type expansions states:

Corollary 5.3 [9]. *If all Vassiliev invariants up to degree c vanish on a knot K of crossing number c then the knot has a trivial HOMFLY polynomial.*

Our barrier strengthens the Jones-side mechanism in a diagram-local fashion: from a length-m

initial Jones gap we build an m-flat family on an extended shadow via m-invisible flips, force the last-row fingerprint, and through a shadow criterion bound m by $\mathcal{N}(K)$, yielding a finite verification of Jones detection at minimal crossing number.

1.2 Organization of this work

Section 2 lays out the preliminaries: finite-type invariants (Vassiliev order and symbols), the pseudo-Boolean variation cube attached to a shadow, and the Walsh-Fourier/spectral viewpoint that identifies order with polynomial degree and interprets facewise alternating sums as discrete derivatives. This section fixes notation used throughout (e.g., smoothing $D \setminus J$, residual cubes).

Section 3 introduces extended shadows by decorating selected crossings of a fixed shadow with tree claspers of prescribed degrees m_j. We prove that the j-th crossing flip is m_j-invisible (a C_{m_j}-move) for invariants of degree $< m_j$, package multi-flip effects via a weighted Goussarov-Habiro filtration, and assemble a GH-Taylor expansion that keeps track of interactions among claspers. This yields m-flat families (constancy of all sub-m layers) carried by the decorated shadow. Figure 1 depicts the clasper movie and the crossing flip modulo C_m used repeatedly later.

Section 4 specializes to the Birman-Lin expansion of the Jones polynomial. We establish a symbol-level skein recursion for the Jones layers c_n and show that order-0 data c_0 is a shadow invariant encoding component count. On any Jones-m-flat subcube we prove the last-row fingerprint: smoothing any prescribed $m-1$ crossings forces $c_0 = (-2)^{m-1}$, hence the component count rises by exactly one at each smoothing. This yields a degree cap for Jones-flatness on a subcube (Cor. 4.11). The symbol matrix in Figure 2 visualizes the skein constraints and the $(-2)^r$ bottom-row pattern.

Section 5 gives the main result, the uniform m-triviality barrier: if a nontrivial knot K is J_m-trivial, then $m \leq \mathcal{N}(K)$ (Thm. 5.1). The proof transfers m-flatness from a decorated face of the extended cube back to the outside-crossing subcube, applies the last-row fingerprint with the Shadow Lemma to force global nugatoriness in the supercritical regime, and concludes the bound. Two corollaries follow: Jones detects the unknot (Thm. 5.3) and the Vassiliev version (Thm. 5.4).

Appendix A records a cubical $1T/4T$ formulation that links the symbol calculus on cubes to Bar-Natan's diagrammatics; **Appendix B** states and proves the Shadow Lemma, equating the component-increment property (P) with all crossings being nugatory (the shadow is a tree). These appendices supply the local-to-global bridge used in the proofs.

1.3 Part II: Computational evidence

The companion paper (Part II) [10] implements the symbol calculus developed here by packaging the Birman-Lin layers into a Jones-Vassiliev polynomial (JVP) in the rank-two module $\mathbb{Z}[p][x]/(x^2-px-1)$; from tabulated Jones polynomials $V_K(t)$ it reads off the finite-type layers via the substitutions $t^{1/2} = x$, $t^{-1/2} = x - p$, followed by the reduction $x^2 = px + 1$. Using this extraction pipeline, Part II computes the layers for all 352,152,252 prime knots with $\mathcal{N} \leq 19$ (Knot Atlas for $\mathcal{N} \leq 10$; Dartmouth for $11 \leq \mathcal{N} \leq 19$), finding no counterexample to the uniform barrier $m \leq \mathcal{N}(K)$ proved in Theorem 5.1 here, and moreover uncovering a consistent empirical refinement $m \leq \lfloor \mathcal{N}/2 \rfloor + 1$; see Fig. 3 (first J_m-trivial occurrences) and Fig. 4 (likelihood curves $P_m(\mathcal{N})$) of Part II. Beyond the census, Part II reproduces our symbol-skein recursion and verifies the last-row fingerprint (Theorem 4.10/Corollary 4.11 here) on explicit model cubes and records a clean min-law under connected sum for the JVP bandwidth (Proposition 4.2). Taken together, the large-scale data and the JVP calculus in Part II provide a practical, finite algorithmic mirror of the obstruction mechanism developed in Part I.

1.4 Conventions

Convention 1.1 (Conventions used throughout the text)**.**

- *A* knot shadow *(shadow for short) means the 4-regular plane graph obtained from a knot diagram by forgetting over/under information, and a crossing flip means changing the over/under choice at one vertex of the shadow.*
- *Over/under crossings are denoted as '+' and '−', and sometimes, when encoded by cube vertices, as* $+1$ *and* -1*.*
- *Smoothing a crossing always refers to* oriented *smoothing. Following the above crossing sign conventions, it is denoted as '0'.*
- $\mathscr{N}(K)$ *and* $\operatorname{cross}(K)$ *denote, respectively, K's number of crossings and set of crossing indices. The number of crossings may always be that of a minimal representation of* K *in which case it is a knot invariant aka the crossing number.*
- *Let* S *be a fixed shadow with outside crossing set* $\operatorname{cross}(S)$*, and let* $D \in \mathcal{D}(S)$ *be a diagram carried by* S*. For a subset* $J \subseteq \operatorname{cross}(S)$*, we write* $D \setminus J$ *for the diagram obtained from* D *by performing the* oriented smoothing *at every crossing in* J *("0" smoothing). This is a concise form of Convention 4.1.*
- *A* C_k *tree (a degree-k tree clasper) is the embedded surface made of nodes, edges, and leaves. A* C_k *move is surgery along a connected* C_k *tree. Two links are* C_k*-equivalent if they are related by a sequence of* C_k *moves. A* C_k *move preserves every finite-type invariant of degree* $< k$ *[7, 8].*
- ***GH*** *refers to Goussarov-Habiro.*

2 Preliminaries

2.1 Finite-type invariants

Fix the category $\mathcal{K}$ of oriented links in S^3, considered up to ambient isotopy. Let $\mathcal{K}^{(d)}$ denote the set of (immersed) oriented singular links with at most d transverse double points and no other singularities. Let A be an abelian group (typically $A = \mathbb{Z}$ or $\mathbb{Z}[p]$).

Definition 2.1 (Vassiliev extension and order)**.** *A function* $v : \mathcal{K} \to A$ *is finite type of order* $\leq d$ *if it admits an extension* $\tilde{v} : \mathcal{K}^{(d)} \to A$ *determined inductively by the Vassiliev skein rule*

$$\tilde{v}\,(\textit{double point}) = \tilde{v}\,(\textit{positive}) - \tilde{v}\,(\textit{negative})$$

at each double point, and satisfies $\tilde{v}(L) = 0$ *whenever* L *has* $d{+}1$ *double points. The order (or type) of* v *is the least such* d*.*

Definition 2.2 (Symbol / weight system). *For v of order $\le d$, its d-th symbol $\sigma_d(v)$ is the restriction of $\tilde{v}$ to $\mathcal{K}^{(d)}$ modulo the skein relations, equivalently a function on chord diagrams with d chords. The symbol factors through the 1T/4T relations, so $\sigma_d(v)$ defines a linear functional on the degree-d diagram space $\mathcal{A}_d$ (Bar-Natan).*

Remark 2.1 (Goussarov–Habiro invisibility). *A connected C_m move (surgery on a C_m clasper) preserves every invariant of order $< m$. We write "C_m is invisible to order $< m$" and use this repeatedly throughout this work.*

2.2 Pseudo-Boolean functions and finite-type invariants

Shadows and cubes. Fix a shadow S with outside crossing set $I = \mathrm{cross}(S)$, $|I| = n$. The associated *variation cube* is

$$\mathcal{Q}(S) \;=\; \{\varepsilon = (\varepsilon_i)_{i \in I} \mid \varepsilon_i \in \{\pm 1\}\} \;\cong\; \{\pm 1\}^n. \tag{1}$$

For each $\varepsilon \in \mathcal{Q}(S)$ let $D(\varepsilon) \in \mathcal{D}(S)$ denote the diagram carried by S with over/under choices encoded by ε (Convention 1.1).

Pseudo-Boolean functions on a fixed shadow. Let A be an abelian group (typically $\mathbb{Z}$ or $\mathbb{Z}[p]$). A *pseudo-Boolean function on the cube of S* is any map

$$f_S : \mathcal{Q}(S) \longrightarrow A, \qquad \varepsilon \longmapsto f_S(\varepsilon).$$

Given an ambient-isotopy invariant $v : \mathcal{K} \to A$, we obtain its restriction to the cube of S by evaluation at the vertices:

$$f_v^S(\varepsilon) \;:=\; v\big(D(\varepsilon)\big), \qquad \varepsilon \in \mathcal{Q}(S).$$

Thus, every knot/link invariant supplies, for each choice of shadow, a canonical pseudo-Boolean function on $\mathcal{Q}(S)$.

Faces and $\mathscr{U}$-subcubes. For $\mathscr{U} \subseteq I$ and a base state $\varepsilon^0 \in \mathcal{Q}(S)$ the *$\mathscr{U}$-subcube through ε^0* is

$$\mathcal{Q}_{\mathscr{U}}(\varepsilon^0) \;:=\; \{\varepsilon \in \mathcal{Q}(S) : \; \varepsilon_i = \varepsilon_i^0 \text{ for all } i \notin \mathscr{U}\}.$$

We freely regard alternating sums over such faces as *finite differences* (discrete derivatives).

Polynomial-valued invariants and coefficient layers. Let R be a commutative ring and let $J : \mathcal{K} \to R[p]$ be a polynomial-valued invariant,

$$J(K; p) \;=\; \sum_{m \ge 0} f_m(K)\, p^m, \qquad f_m : \mathcal{K} \to R.$$

On a fixed shadow S the coefficient functionals induce a family of pseudo-Boolean functions

$$f_{m,S}(\varepsilon) \;:=\; f_m\big(D(\varepsilon)\big), \qquad \varepsilon \in \mathcal{Q}(S),\; m \ge 0.$$

In particular, when J is the Jones–Vassiliev expansion (introduced in §4.1), we will write $c_{q,S}(\varepsilon) = c_q\big(D(\varepsilon)\big)$ for its q-th coefficient layer on $\mathcal{Q}(S)$.

Connected sums and product cubes. If a diagram $D(\varepsilon)$ splits as a connected sum $D_0(\varepsilon^0)\#D_1(\varepsilon^1)$ realized in disjoint disks, then the shadow cube factors as a Cartesian product $\mathcal{Q}(S) \cong \mathcal{Q}(S_0) \times \mathcal{Q}(S_1)$. When J is multiplicative under connected sum,

$$J_{K_0 \# K_1}(p) \; = \; J_{K_0}(p)\, J_{K_1}(p),$$

the coefficient layers satisfy the Cauchy rule at each vertex $(\varepsilon^0, \varepsilon^1)$:

$$f_{m,S}(\varepsilon^0, \varepsilon^1) \; = \; \sum_{i=0}^{m} f_{i,S_0}(\varepsilon^0)\, f_{m-i,S_1}(\varepsilon^1). \tag{2}$$

Residual cubes. For later use we smooth a specified set $J \subseteq I$ (oriented "0" smoothing from Convention 1.1) and denote by $\mathcal{Q}_{S\setminus J}$ the variation cube of the resulting shadow; see Definition 4.2. Order-0 constancy on residual cubes is recorded in Corollary 4.3.

2.3 Spectral representation

The Boolean cube-complex formalism enjoys a number of advantages. The Vassiliev skein relation is realized by a boundary operator, ∂_i, contracting/projecting along the i-th dimension. The type or order of the Vassiliev invariant becomes the *polynomial degree* of the pseudo-Boolean function. Beyond the observation that this may explain why polynomial invariants are native objects for encoding combinatorial finite-type data, we have tools from spectral analysis at our disposal.

Fourier basis and Walsh characters. Let $f : \{\pm 1\}^n \to \mathbb{C}$ be a pseudo-Boolean function. For every subset $S \subseteq [n]$, the Walsh character is given by,

$$\chi_S(\varepsilon) = \prod_{i \in S} \varepsilon_i \quad (\chi_\varnothing \equiv 1). \tag{3}$$

These characters form an orthonormal basis (with respect to the uniform measure on $\mathcal{Q}_n$), so

$$f(\varepsilon) = \sum_{S \subseteq [n]} \widehat{f}(S)\, \chi_S(\varepsilon), \qquad \widehat{f}(S) = 2^{-n} \sum_{\varepsilon} f(\varepsilon)\, \chi_S(\varepsilon). \tag{4}$$

is the Walsh-Fourier expansion of f. The polynomial degree, $\deg(f)$, matches the largest $|S|$ with $\widehat{f}(S) \neq 0$.

Discrete (finite) derivatives. For coordinate i we define the (normalized) derivative

$$(\partial_i f)(\varepsilon) \equiv \frac{1}{2}\big(f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)})\big), \tag{5}$$

where $\varepsilon^{(i,\pm)}$ is ε with the i-th entry set to ± 1. For a subset $T \subseteq [n]$,

$$\partial_T = \prod_{i \in T} \partial_i, \qquad (\partial_T f)(\varepsilon) = 2^{-|T|} \sum_{\sigma \in \{\pm 1\}^T} (-1)^{\#\{\sigma_i = -1\}} f(\varepsilon_{[n]\setminus T}, \sigma). \tag{6}$$

Equivalently, it is the alternating sum of f over the T-subcube through ε. For any fixed subset $T \subseteq [n] = \{1, \ldots, n\}$ the T-fold discrete derivative acts on Walsh characters by "dropping" the coordinates in T:

$$\partial_T\, \chi_S = \begin{cases} \chi_{S \setminus T}, & T \subseteq S, \\ 0, & T \not\subseteq S. \end{cases} \tag{7}$$

Because the characters $\{\chi_S\}_{S\subseteq[n]}$ form an orthonormal basis, we obtain the Fourier (Walsh) expansion of the derivative simply by applying this rule term-wise to the expansion of f:

$$(\partial_T f)(\varepsilon) = \sum_{[n]\supseteq U\supseteq T} \widehat{f}(U)\chi_{U\setminus T}(\varepsilon). \tag{8}$$

Thus, the Fourier coefficient of $\partial_T f$ at character χ_S (with $S\cap T=\varnothing$) is precisely the coefficient of f at the larger character $\chi_{S\cup T}$, namely, $\widehat{\partial_T f}(S)=\widehat{f}(S\cup T)$. This explicit form makes transparent how derivatives shift weight "down" the spectrum and why $\partial_T f$ vanishes whenever f has no Fourier support on sets containing T (the finite-type condition).

Remark 2.2 (Normalized vs. unnormalized derivatives)**.** *The standard Fourier/Walsh framework is formulated using the normalized difference operator* (5)*. The Vassiliev skein, on the other hand, adopts the unnormalized version, namely,*

$$(\partial_i f)(\varepsilon) \equiv f(\varepsilon^{(i,+)}) - f(\varepsilon^{(i,-)}). \tag{9}$$

At times, this convention alleviates the burden of dealing with factors such as 2^{-m} when taking derivatives over cube m-faces. For that reason, and for being consistent with the standard theory, unless stated otherwise, ∂_i means (9) *for the rest of this work.*

Connected sum and product rules. The variation cube encoding connected sums of knots, e.g. $K_0\#K_1$, is given by a cartesian product, $\mathcal{Q}\cong\mathcal{Q}_0\times\mathcal{Q}_1$. The set of coordinates S underlying $\mathcal{Q}$ can thus be partitioned into two subsets associated with the individual cubes, $S=S_0\cup S_1$. This facilitates composing the discrete derivatives of any pseudo-Boolean function as,

$$\partial_{T_0\cup T_1} = \partial_{T_0}\partial_{T1},\ \ \forall T_0\subseteq S_0,\, \forall T_1\subseteq S_1.$$

If, in addition, (2) holds, then

$$\partial_{T_0\cup T_1} f_m(\varepsilon_{K_0\#K_1}) = \sum_{i=0}^{m} \left[\partial_{T_0} f_i(\varepsilon_{K_0})\right]\left[\partial_{T_1} f_{m-i}(\varepsilon_{K_1})\right]. \tag{10}$$

Polynomial degree equals Vassiliev order. A pseudo-Boolean invariant f is said to be *finite type of (Vassiliev) order d* if

$$\partial_T f \equiv 0 \quad \text{for every } T\subseteq[n] \text{ with } |T|>d. \tag{11}$$

Intuitively, all $(d+1)$-dimensional faces evaluate to alternating sums of zero. The derivative representation immediately implies

$$\widehat{f}(S)=0 \quad \text{whenever } |S|>d. \tag{12}$$

Conversely, if $\widehat{f}(S)=0$ for $|S|>d$, then every $\partial_T f$ with $|T|>d$ vanishes.

$$\mathrm{Type}(f)=\deg(f)=\max\{|S| : \widehat{f}(S)\neq 0\}. \tag{13}$$

The derivative operator not only detects type but produces new finite-type functions of *singular* knots: Fix a face defined by coordinates T and take the alternating sum of f along that face. The result is $\partial_T f$ restricted to the complementary coordinates. If f is type d, then $\partial_T f$ is type $d-|T|$. Iterated extraction of "principal parts."

Concept	Description
Boolean variation cube	Encodes all simultaneous crossing variations; faces are natural domains for alternating-sum operators.
Walsh-Fourier expansion	Decomposes any pseudo-Boolean invariant into characters labelled by variation subsets.
Top Walsh-Fourier block	$\partial_{[m]}$ where m is the polynomial degree; the symbol.
Discrete derivatives	Are face-wise alternating sums; in Fourier space they simply drop indices.
Finite-type order d	Polynomial/Fourier degree $d \leftrightarrow$ vanishing of $(> d)$-fold derivatives.
Derivatives over subcubes	Isolate the homogeneous pieces and systematically generate lower-order invariants of singular knots.

Table 1: Pseudo-Boolean functions and finite-type invariants. Glossary of key concepts.

- Order-1 part: $g_1(\varepsilon) = \sum_{|S|=1} \widehat{f}(S)\chi_S(\varepsilon)$.
- Higher parts analogously via derivatives: the map $T \mapsto \partial_T f(1, \ldots, 1)$ lists exactly the Fourier coefficients of size $|T|$.

This is the discrete analogue of repeatedly differentiating a polynomial and evaluating at the origin to read off coefficients.

Symbols, cubes, and chords. In finite-type theory the m-th symbol is an invariant of exactly order m. That is, it is the restriction of the finite-type function to chord diagrams with m chords, or equivalently, is the m-th graded piece in the GH algebra (Definition 2.2). By definition, the symbol is independent of any particular variation of the singular crossings, i.e., it is fixed for a given cube. Over an m-dimensional variation cube it is extracted by the (unnormalized) m-fold discrete derivative and hence equals (up to a normalization factor) the top Walsh-Fourier block of the underlying pseudo-Boolean function,

$$\partial_{[m]} f = 2^m \hat{f}([m]).$$

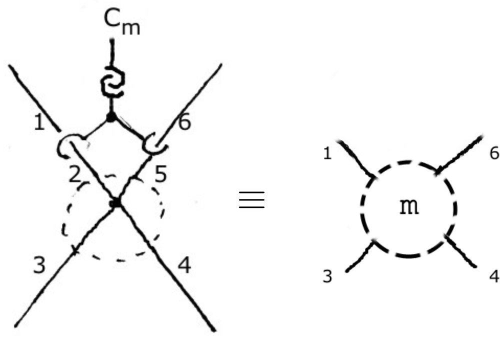
The symbols factorize through the graded space $\mathcal{A}_m$ and hence depend only on the chord diagram class (modulo the 1T/4T relations) and not on the particular placement of singularities. The pseudo-Boolean framework offers concise descriptions of these local relations which consequently bridge the cubical definition of a symbol with the algebra of chord diagrams (see Lemma A.1 in the appendix).

3 Spectral m-gaps

Let S be a knot shadow with N labeled double points $1, \ldots, N$. Fix a choice of over/under data at those N points that yields a base diagram K_0 (e.g., an unknot).

3.1 Extended shadows and m-invisible crossings

For each $j \in \{1, \dots, N\}$, fix a simple tree clasper C_{m_j} for K_0 of (Habiro) degree $m_j \geq 1$, embedded in a small ball meeting K_0 only near the j-th double point, as shown below.

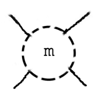


Assume the C_{m_j} are pairwise disjoint. All choices can be made within disjoint balls around the crossings. Call

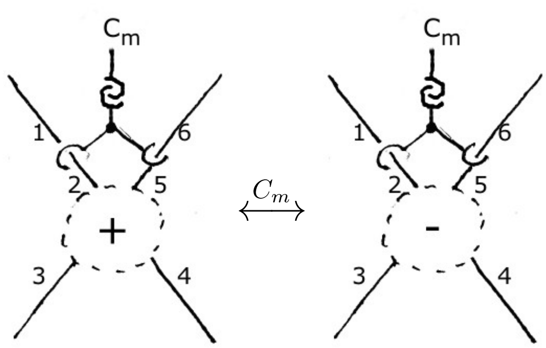
$$E = (K_0; \{C_{m_j}\}_{j=1}^{N})$$

the *extended shadow*. Below we prove that the clasper-decorated crossing m_j is invisible to degree $< m_j$ invariants

$$\partial_j f = f^{(j=+)} - f^{(j=-)} = 0, \quad \deg(f) < m_j.$$

The diagrammatic representation  stands for a clasper-decorated crossing that does not affect any degree-$< m$ invariant. Similarly, the net effect of flipping J-subset of crossings (alternating sums) is shifted to degrees $\geq \sum_{j \in J} m_j$ of the GH filtration.

Proposition 3.1 (Crossing flip modulo C_m)**.**



Proof. The proof is shown in Figure 1. $\square$

3.2 The carried family

For any sign vector $\sigma = (\sigma_1, \dots, \sigma_N) \in \{\pm 1\}^N$, define $K_E(\sigma)$ to be the knot obtained from K_0 by resolving the j-th double point with the sign σ_j and, in addition, doing surgery along C_{m_j} iff

$\sigma_j = -1$. Alternatively, we may perform all surgeries first and then fix the signs; disjointness of the C_{m_j} makes this well-defined up to higher GH degree.

A degree-m_j tree realizes a C_{m_j}-move, hence, by Proposition 3.1,

$$K_E(\sigma) \equiv K_E(\sigma^{(j)}) \mod C_{m_j},$$

so invariants of degree $< m_j$ cannot see the j-th sign (j-th decorated crossing). Here $\sigma^{(j)}$ is σ with the j-th entry flipped.

3.3 Weighted GH filtration and derivatives

Let F_n be the GH filtration on the free abelian group $\mathbb{Z}\mathcal{K}$ on isotopy classes of knots (so F_n is generated by C_n-moves) [4]. Write $\mathrm{gr}_n^C := F_n/F_{n+1}$ and $\mathrm{gr}^C := \bigoplus_{n\geq 0} \mathrm{gr}_n^C$.

For $J \subseteq \{1, \ldots, N\}$ define the alternating multi-difference of the family $K_E(\sigma)$ along the coordinates in J:

$$\Delta_J K_E \ := \ \sum_{S\subseteq J} (-1)^{|J|-|S|}\, K_E(\sigma^{(S)}),$$

where $\sigma^{(S)}$ flips the entries of σ exactly on S. Because all C_{m_j} are disjoint and $\deg C_{m_j} = m_j$, standard clasper calculus implies $\Delta_J K_E \in F_{w(J)}$, where $w(J) := \sum_{j\in J} m_j$ [8]. In particular, for $J = \{j\}$ we have $\Delta_{\{j\}} K_E \in F_{m_j}$, i.e., $\partial_j f = 0$ for $\deg f < m_j$.

A corollary that packages the whole family:

Corollary 3.2 (Weighted polynomiality)**.** *Let f be a finite-type invariant of degree $\leq d$. Then for every J,*

$$w(J) > d \implies f(\Delta_J K_E) = 0.$$

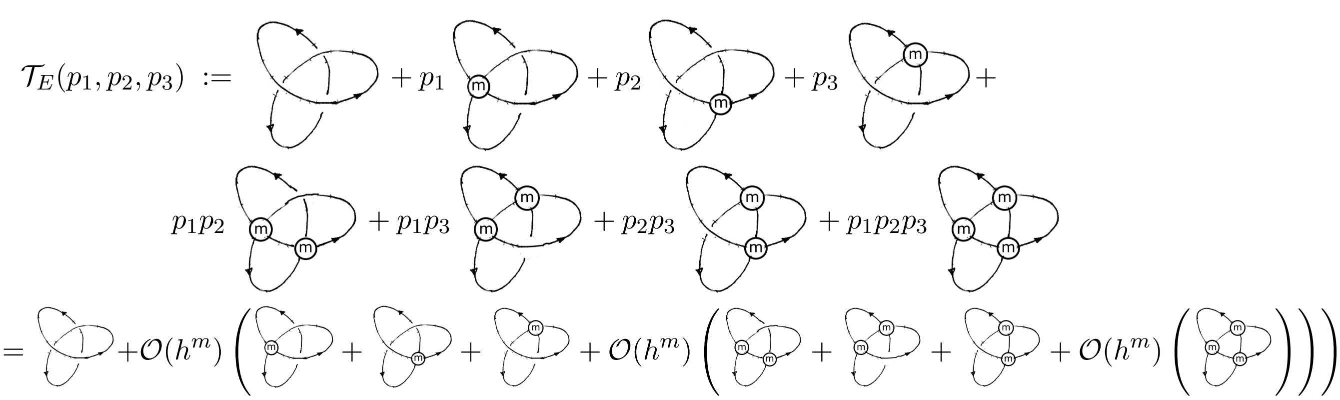
Equivalently, the function $\sigma \mapsto f\big(K_E(\sigma)\big)$ is a polynomial in the N Boolean variables whose weighted degree (with $\deg(\textit{variable } j) = m_j$*) is $\leq d$.*

3.4 The GH–Taylor expansion around K_0

Introduce formal commuting variables $p_1, \ldots, p_N$ with $\deg p_j := m_j$. Define the GH–Taylor series of the extended shadow by

$$\mathcal{T}_E(p_1, \ldots, p_N) \ := \ K_0 \ + \ \sum_j p_j\, \Delta_{\{j\}} K_E \ + \ \sum_{|J|\geq 2} \prod_{j\in J} p_j\, \Delta_J K_E \ + \ \cdots$$

The r-th summand is $\sum_{|J|=r}(\prod_{j\in J} p_j)\Delta_J K_E$ and lies in filtration $\geq \sum_{j\in J} m_j$. Thus, $\mathcal{T}_E$ is a well-defined element of the completed GH group $\widehat{\mathbb{Z}\mathcal{K}}$. For example,



(14)

The shadow $K_E(\sigma)$ modulo $F_{>d}$ is recovered by substitution

$$K_E(\sigma) \;\equiv\; \big[\mathcal{T}_E\big]_{\leq d}\Big|_{p_j=(1-\sigma_j)/2}.$$

So for every finite–type invariant f of degree $\leq d$, $f(K_E(\sigma))$ equals the degree-$\leq d$ polynomial in the p_j obtained by applying f to $\mathcal{T}_E$.

Keeping only one-tree terms gives

$$\mathcal{T}_E \equiv K_0 \;+\; \sum_{j=1}^{N} p_j \,\underbrace{\Delta_{\{j\}}K_E}_{\in\,\mathrm{gr}^C_{m_j}} \quad \bmod F_{>\min_{j\neq k}(m_j+m_k)}.$$

Identifying $\Delta_{\{j\}}K_E$ with the class in $\mathrm{gr}^C_{m_j}$ represented by surgery on C_{m_j}, this is exactly

$$K_0 + \sum_{j=1}^{N} p_j\, C_{m_j} \quad\Longrightarrow\quad V_{K_E}(h) = V_{K_0}(h) + \sum_{j=1}^{N} h^{m_j} V_{C_{m_j}}(h),$$

with p_j the grading variable and C_{m_j} the degree-m_j tree class attached at the j-th crossing. Here, $V_K(h)$ is the Taylor expansion of some polynomial invariant, e.g., Jones.

The Δ_J with $|J| \geq 2$ package all interaction terms between different claspers, which first appear in filtration $\sum_{j\in J} m_j$. They are necessary for exactness beyond the linear (one-tree) regime.

3.5 m-flat families

Lemma 3.3 (m-flat family carried by an extended shadow)**.** *Let $m = \min_{1\leq j\leq N}(m_j)$. The knots carried by the shadow restricted to the decorated crossings, $\sigma \mapsto K_E(\sigma)$, share the following characterizing properties:*

1. *They are m-trivial (and hence also J_m-trivial, see Definition 4.1).*
2. *The order-$<m$ symbols (alternating sums of finite-type invariants) computed over subsets $\{\sigma_j\}_{j\in J}$ of decorated crossings vanish.*

Proof. The GH-Taylor series shows that all knots $\sigma \mapsto K_E(\sigma)$ are pairwise m-equivalent. Because the original shadow consists of at least one unknot it follows that each knot in $\{K_E(\sigma)\}_{\sigma\in\{\pm1\}^N}$ is m-trivial. As all finite-type sub-m data is fixed over the cube coordinates $\{\sigma_j\}_{j=1}^N$ it follows that,

$$f(\Delta_J K_E) = \partial_J f(K_E) = 0, \quad \deg(f) = |J| < m.$$

□

3.6 Uniform barrier and unknot detection

Extended shadows carry m-flat families of knots. Translating into a cube language, the extended cube $\mathcal{Q}_E$ is a product of an m-flat base subcube $\mathcal{Q}_S$ and the clasper decorations subcube $\mathcal{Q}_C$,

$$E = S + p^m C \quad\Longrightarrow\quad \mathcal{Q}_E = \mathcal{Q}_S \times \mathcal{Q}_C.$$

The extension cube's dimension depends on the clasper tree decorations. Normally, these scale linearly with the parameters m_j. Assuming $m_j = \mathcal{O}(N)$, the degree of the base shadow, and taking C_{m_j} as Brunnian claspers, we note that $\dim Q_E = \mathcal{O}(N^2)$.

Applying the above construction using the finite-type expansion of the Jones polynomial leads to a bound on the size of m-flat families. We give here a proof sketch of the uniform barrier. The complete proof is provided later on following the formalization of the underlying concepts.

Theorem (Uniform barrier)**.** *For any non-trivial K we have $m \leq \mathscr{N}(K)$.*

Proof sketch. Let S be the shadow of K. By Lemma 3.3 the subcube $\mathcal{Q}_S$ is Jones-m-flat / m-flat (all order-$< m$ symbols vanish). Theorem 4.10 and Corollary 4.11 show that if $m > |\mathcal{Q}_S| = \mathscr{N}(K)$ the subcube $\mathcal{Q}_S$ becomes completely flat, in which case all knots carried by S are trivial: this is a consequence of the Shadow lemma B.1 property P, and the Jones symbolic skein relation (21). Simply, smoothing crossings in a Jones-m-flat cube results in m link components, indicating that the crossings are nugatory/cut-vertex. The clasper decorations occupy the higher degrees in the GH filtration and thus may be trivialized without affecting this conclusion. As K is non-trivial, we have here an obstruction mechanism that caps the size of flat knot families in extension cubes by limiting m: unless a knot is trivial, it cannot be (m-trivial) J_m-trivial with $m > \mathscr{N}$. □

The detection results follow directly from the barrier.

Theorem. $V_K \equiv 1 \longrightarrow K$ *is the unknot.*

Proof. This follows from the uniform barrier. If K were non-trivial and $V_K = 1$, i.e., J_m-trivial for all m, then $m \leq N(K)$ for all m. Contradiction. □

Theorem (Vassiliev)**.** *A knot that is m-trivial for all m is the unknot.*

Proof. This follows from the uniform barrier. If K were non-trivial and m-trivial for all m, then $m \leq N(K)$ for all m. Contradiction. □

4 J_m-triviality and Jones-m-flatness

4.1 Finite-type expansion of the Jones polynomial

We start from the Jones skein

$$x^{-2}V^{(+)} - x^2 V^{(-)} = (x - x^{-1})V^{(0)}. \tag{15}$$

and pass to the Birman–Lin substitution $x = e^{h/2}$. This reorganizes V_L into a power series

$$V_L(e^{h/2}) = \sum_{n \geq 0} c_n(L)\, h^n, \tag{16}$$

whose coefficients c_n are Vassiliev invariants of order $\leq n$ [5].

Definition 4.1 (Jones m-trivial, J_m-trivial)**.** *A knot K is J_m-trivial if its n-type coefficients $c_n(K) = 0$ for $1 \leq n \leq m - 1$.*

Remark 4.1. *Note that if V_K has a h-gap of order m:*

$$V_K(e^{h/2}) = 1 + h^m V'_K(h), \tag{17}$$

then it is J_m-trivial.

Remark 4.2 (Every knot is J_m-trivial for some m)**.** *Every non-trivial knot is J_m-trivial for some $m \geq 2$ (as $c_1 = 0$ for knots). The unknot is J_m-trivial for any m.*

Remark 4.3. *Standard m-triviality (GH) implies J_m-triviality, because it forces every degree $\leq m-1$ finite-type invariant to agree with the unknot, hence the coefficients c_n vanish for $1 \leq n \leq m - 1$.*

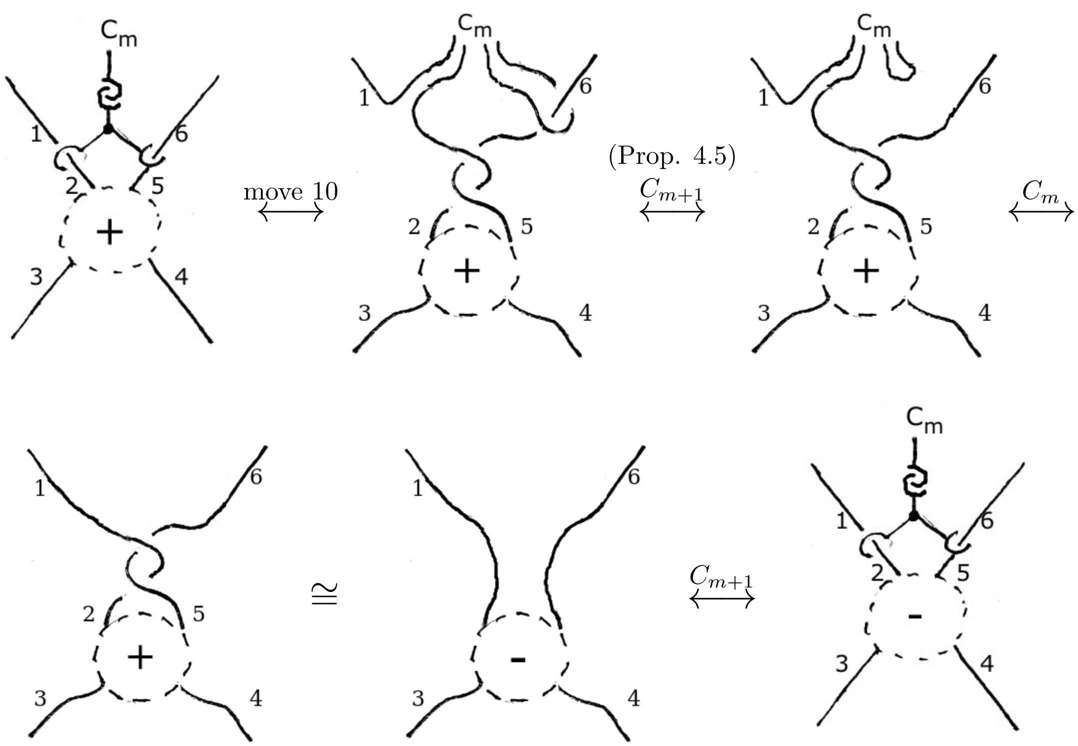


Figure 1: Proof of Proposition 3.1. The proof follows Habiro's clasper calculus. A C_{m+1}-tree (C_m-tree band-summed to a C_2 node) is attached to the branches of an outside crossing. The movie consists of C_m, C_{m+1} moves and isotopies. The two moves in the upper left corner are Habiro's move 10 (Figure 9 on p. 15 in [8]) and a C_{m+1} following his Proposition 4.5 (Figure 30 on p. 34 in [8]). The movie in its entirety preserves all degree-$< m$ invariants.

In the Walsh-Fourier viewpoint each coefficient layer becomes a pseudo-Boolean function on a fixed shadow cube, so facewise differences isolate orders and symbols. Two immediate consequences drive what follows: c_0 is a shadow invariant encoding component count, and the skein differentiates the layers, giving a symbol-level recursion we use to set up the symbol calculus of §4.4.

Lemma 4.1 (Order-0 coefficient c_0)**.** *The order-0 coefficient in* (16) *encodes the number of linked components:*

$$c_0(L) = (-2)^{\ell(L)-1}.$$

Proof. This follows from $V_L(1) = (-2)^{\ell(L)-1}$. □

Lemma 4.2 (Finite-type coefficients skein relation)**.** *For every $n \geq 1$ one has*

$$\partial c_n = c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0) \;+\; R_{n-2}\,, \tag{18}$$

where the remainder R_{n-2} is an explicit linear combination of strictly lower Vassiliev orders ($\leq n-2$):

$$\begin{aligned} R_{n-2} = -\sum_{r\geq 1} \frac{1}{(2r)!}\,\partial c_{n-2r} \;+\; \sum_{r\geq 1} \frac{1}{(2r+1)!}\big(c_{n-2r-1}(+) + c_{n-2r-1}(-)\big) \\ +\; \sum_{r\geq 1} \frac{1}{2^{\,2r}(2r+1)!}\, c_{n-2r-1}(0). \end{aligned} \tag{19}$$

Proof. Substitute $x = e^{h/2}$ in the skein $x^{-2}V^{(+)} - x^2 V^{(-)} = (x - x^{-1})V^{(0)}$ and expand:

$$e^{-h}\sum_{n\geq 0} c_n(+)h^n \;-\; e^{h}\sum_{n\geq 0} c_n(-)h^n \;=\; \big(e^{h/2} - e^{-h/2}\big)\sum_{n\geq 0} c_n(0)h^n.$$

Equating the coefficient of h^n and separating even/odd exponential contributions yields

$$\partial c_n = -\sum_{m\geq 1} \frac{(-1)^m}{m!}\, c_{n-m}(+) \;+\; \sum_{m\geq 1} \frac{1}{m!}\, c_{n-m}(-) \;+\; \sum_{\substack{\alpha\geq 1 \\ \alpha \text{ odd}}} \frac{1}{2^{\alpha-1}\alpha!}\, c_{n-\alpha}(0).$$

Group the even/odd m's on the left to obtain the displayed R_{n-2} and isolate the leading $m = \alpha = 1$ terms to get (18). For fixed n, all sums are finite. Finally, each object in R_{n-2} has Vassiliev order at most $n-2$ (because it uses indices $< n-1$). □

Convention 4.1 (Smoothing notation $D \setminus J$)**.** *Let S be a fixed shadow with outside crossing set* $\mathrm{cross}(S)$*, and let $D \in \mathcal{D}(S)$ be a diagram carried by S. For a subset $J \subseteq$* $\mathrm{cross}(S)$*, we write $D \setminus J$ for the diagram obtained from D by performing the* oriented smoothing *at every crossing in J (our "0" smoothing). Thus:*

- *$D \setminus \emptyset = D$, and for a singleton $\{i\}$ we have $D \setminus \{i\} = D(i = 0)$.*
- *Smoothing at distinct crossings is local and order–independent, so $D \setminus J$ is well defined (independent of any ordering of J).*
- *If $J \subseteq$* $\mathrm{cross}(S)$ *and $S \setminus J$ denotes the shadow with those vertices smoothed, then the* residual cube *$\mathcal{Q}_{S\setminus J}$ is the variation cube on the remaining crossings; functions of the form $D \mapsto f(D \setminus J)$ are naturally viewed on $\mathcal{Q}_{S\setminus J}$ (see Definition 4.2 and Corollary 4.3).*

- *We occasionally write $J_r \subseteq J$ to mean an arbitrarily chosen r-subset of J; by order–independence, statements about $D \setminus J_r$ do not depend on the choice of J_r (used, e.g., when discussing incremental smoothings and the last–row fingerprint).*
- *In single-crossing skeins we retain $D(+), D(-), D(0)$; the set-notation agrees with this: $D \setminus \{i\} = D(i = 0)$.*

Definition 4.2. *For a shadow S and $J \subset \operatorname{cross}(S)$, the* ***residual cube*** *$\mathcal{Q}_{S\setminus J}$ is the variation cube of over/under choices on the remaining crossings after smoothing all J.*

Corollary 4.3 (Order-0 constancy on residual cubes)**.** *Fix any shadow S of K. Because c_0 is a type-0 (order-0) Vassiliev invariant, for any fixed subset $J \subset \operatorname{cross}(S)$ of crossings the map $D \mapsto c_0(D \setminus J)$ is constant on the residual cube of over/under choices. Here $\operatorname{cross}(S)$ denotes the vertex set of the shadow (the common set of crossings for all $D \in \mathcal{D}(S)$). In particular, under the Jones-flatness hypothesis of Lemma 4.7 below, $c_0(D \setminus J) = (-2)^{|J|}$ for all $D \in \mathcal{D}(S)$.*

Proof. Since c_0 is type-0, $\partial_T c_0 \equiv 0$ for all nonempty T, i.e., it is constant on every face of every residual cube $\mathcal{Q}_{S\setminus J}$. □

Definition 4.3 (Set-indexed symbols)**.** *For any $K \subset I$ and $J \subset I \setminus K$, define*

$$\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$$

i.e. first perform oriented smoothings at all crossings in K, then take the unnormalized alternating sum over the J-face (order $|J|$).

Convention 4.2 (Set/cardinality-indexed symbols)**.** *From now on we write $\mathscr{S}_{K;J} := \partial_J c^K_{|J|}$ to record both the smoothed subset K and the differenced subset J explicitly. If only the cardinalities matter, we write $\mathscr{S}_{r,n}$ for the class generated by all $\mathscr{S}_{K;J}$ with $|K| = r, |J| = n$.*

Lemma 4.4 (Symbol-skein recursion)**.** *Fix a crossing i. For every $n \geq 1$,*

$$\mathscr{S}_{\varnothing;J} \;=\; 2\,\mathscr{S}_{\varnothing;J\setminus\{i\}} \;+\; \mathscr{S}_{\{i\};J\setminus\{i\}} \qquad (i \in J,\ |J| = n). \tag{20}$$

and more generally,

$$\mathscr{S}_{K;J} = 2\,\mathscr{S}_{K;J\setminus\{i\}} + \mathscr{S}_{K\cup\{i\};J\setminus\{i\}}. \tag{21}$$

Proof of Lemma 4.4 using state-sum for the order-n symbol. Classically, the symbol of c_n on a chord diagram D with chords $\{j\}$ can be written as a state-sum over functions $s : \{j\} \to \{1, 2\}$ ("double"/"delete" the chord) (see p. 71 in [4]):

$$\mathscr{S}_{K;J} = \sum_s \Big(\prod_j s(j)\Big) \cdot (-2)^{\,|\pi_0(s)|-1},$$

where $|\pi_0(s)|$ is the number of components of the immersed curve obtained by applying the resolution prescribed by s. Fix a distinguished chord i and split the sum according to $s(i)$:

- If $s(i) = 2$ (delete), the local multiplicative weight contributes a factor 2, and the resulting curve is obtained from D by forgetting the chord i. Summing over the remaining chords gives $2\,\mathscr{S}_{K;J\setminus\{i\}}$.
- If $s(i) = 1$ (double), the local multiplicative weight contributes a factor 1, and the resulting curve is the oriented smoothing at i. Summing over the remaining chords gives $\mathscr{S}_{K\cup\{i\};J\setminus\{i\}}$.

Adding the two disjoint families of states yields (21). ☐

Proof of Lemma 4.4 using Lemma 4.2. Apply the $(n-1)$-fold unnormalized face-difference $\partial_{J\setminus\{i\}}$ (with $i \in J$, $|J| = n$) to the coefficient recursion (18):

$$\partial_J c_n = \partial_{J\setminus\{i\}}\big(c_{n-1}(+) + c_{n-1}(-) + c_{n-1}(0)\big) \;+\; \partial_{J\setminus\{i\}} R_{n-2}.$$

Since every term in R_{n-2} has Vassiliev order $\leq n-2$, its $(n-1)$-fold alternating sum vanishes identically: $\partial_{J\setminus\{i\}} R_{n-2} \equiv 0$. What remains are exactly the three order-$(n-1)$ pieces on the right. Group the two $(\pm)$ terms (they live on the same residual $(n-1)$-face) and the smoothed term (which lives on the $(n-1)$-face after oriented smoothing at i), to obtain

$$\partial_J c_n = 2\,\partial_{J\setminus\{i\}} c_{n-1} \;+\; \partial_{J\setminus\{i\}} c_{n-1}^{\{i\}},$$

which is (20). ☐

4.2 Order-0 constancy: c_0 is a shadow/cube invariant

The BL-expansion (16) free coefficient c_0 is a 0-order finite-type invariant, which in turn makes it an invariant of variation cubes or shadows rather than of any single vertex (knot/link). In addition, we have seen it uniquely encodes the link component count. These two facts render it a powerful tool for analyzing the behavior of shadows under oriented smoothing.

Corollary 4.5 (Shadow invariance of component counts)**.** *Let S be a shadow and $D(S)$ the set of all diagrams carried by S. Then:*

1. *(**No smoothing**) The number of link components $\ell(D)$ is independent of $D \in D(S)$.*
2. *(**Fixed smoothings**) More generally, for any $J \subseteq \mathrm{cross}(S)$ (crossing set of S), the component count of $D \setminus J$ (link obtained from D by smoothing exactly the crossings in J) is independent of $D \in D(S)$.*

Proof. By order-0 constancy (Corollary 4.3), $D \mapsto c_0(D \setminus J)$ is constant on the residual cube determined by J. By Lemma 4.1, $c_0(D \setminus J) = (-2)^{\ell(D\setminus J)-1}$, so $\ell(D \setminus J)$ is constant as D varies. Taking $J = \varnothing$ yields (1). ☐

Corollary 4.6 (c_0^m fingerprint of property (P))**.** *Let $D \in \mathcal{D}(S)$ be a diagram (shadow S). Fix a subset $J \subseteq \mathrm{cross}(D)$ of size $|J| = m$. If*

$$c_0(D \setminus J) = (-2)^m,$$

then for every k-subset $K \subset J$ (with $0 \leq k \leq m$) we have

$$c_0(D \setminus K) = (-2)^k \quad \textit{and} \quad \#\textit{components}(D \setminus K) = 1 + k. \tag{\textbf{P}}$$

Equivalently, property (P) holds along the fixed set J for all $k \leq m$. If, moreover, $m = |\mathrm{cross}(D)|$ then by the Shadow Lemma (B.1), D is an unknot.

Proof. By Lemma 4.1, $c_0(L) = (-2)^{\ell(L)-1}$, so the hypothesis says $D \setminus J$ has $(m+1)$ components. Smooth the m crossings in J in any order and write $\delta_i \in \{0,1\}$ for the increment in the component count at step i. Oriented smoothing changes the component count by at most 1, so $\delta_i \leq 1$. Since the final count is $1 + \sum_{i=1}^{m} \delta_i = m+1$, each $\delta_i = 1$. Thus the first k smoothings (those in any prescribed k-subset $K \subset J$) yield exactly $1+k$ components, hence $c_0(D \setminus K) = (-2)^k$ (again by Lemma 4.1). Finally, order-0 constancy on residual cubes (Corollary 4.3) implies that for fixed J and $K \subset J$ the value of $c_0(D \setminus K)$ is independent of the over/under choices at the remaining crossings, so the same conclusion holds for every diagram carried by S. ☐

4.3 Jones-flat cubes

Definition 4.4. *A variation cube all of whose vertices are unknots is said to be **Jones-flat**. Equivalently, every variation of the underlying shadow is an unknot.*

Lemma 4.7 (Jones-flatness)**.** *Denote by δ_n the Kronecker delta δ_{n0}. A necessary and sufficient condition for Jones-flatness is the vanishing of all non-trivial symbols,*

$$\mathscr{S}_{0,n} = \partial_{[n]} c_n^0 = \delta_n. \tag{22}$$

Proof. Let $\mathscr{S}_{r,n} = \partial_{[n]} c_n^r$ be the n-th symbol after r oriented smoothings. The symbol recursion (21) reads,

$$\mathscr{S}_{r,n-1} = \mathscr{S}_{r-1,n} - 2\mathscr{S}_{r-1,n-1},$$

where r and n are the cardinalities of the smoothed and differentiated crossing sets (Convention 4.2). As $\mathscr{S}_{0,n} = \delta_n$, it follows that, $\mathscr{S}_{r,n} = (-2)^r \delta_n$, and in particular, the free coefficient after r smoothings, $c_0^r = (-2)^r$. Fix a vertex K in the variation cube. Note that $r = |\mathrm{cross}(S)|$ is the number of crossings in any diagram obtained from the shadow. Therefore, by Corollary 4.6 (P) holds for all $k \leq |\mathrm{cross}(S)|$, rendering K the unknot. As this holds for every vertex K, the cube is Jones-flat.

The converse direction, every variation is the unknot $\rightarrow$ $\mathscr{S}_{0,n} = \delta_n$, follows immediately upon recognizing that all Fourier coefficients in all cubes vanish except for the trivial $c_0 = 1$, from which the symbols follow. □

4.4 Jones-m-flat subcubes

Definition 4.5 (Jones–m–flat subcube)**.** *Let S be a shadow and $\mathscr{U} \subseteq \mathrm{cross}(S)$. The $\mathscr{U}$-subcube $\mathcal{Q}_{\mathscr{U}}$ is Jones–m–flat if for every $1 \leq n \leq m-1$ the coefficient c_n is constant on $\mathcal{Q}_{\mathscr{U}}$ (equivalently, all facewise differences $\partial_W c_n$ with $\emptyset \neq W \subseteq \mathscr{U}$ vanish). If this holds for all m, call $\mathcal{Q}_{\mathscr{U}}$ Jones-∞-flat or flat for short.*

Remark 4.4. *"Flat" here means all finite-type layers below $|\mathscr{U}|$ are constant on the $\mathscr{U}$-subcube; only when $\mathscr{U}$ is the full crossing set does the Shadow Lemma B.1 implies all vertices are unknots.*

Remark 4.5 (Flat subcube constants)**.** *On a flat subcube every positive-order ($n \geq 1$) c_n is constant. Whenever a shadow subcube contains unknot vertices it pins these constants to* 0*. The analogous anchor for Jones-m-flat subcubes is that orders $1 \leq n \leq m-1$ vanish identically by definition. Think of J_m-trivial vertices as playing the same role for Jones-m-flat subcubes that the unknot plays for flat subcubes.*

Lemma 4.8 (Symbols vanish on Jones-m-flat subcubes through order $m-1$)**.** *If $\mathcal{Q}_{\mathscr{U}}$ is Jones-m-flat, then $\mathscr{S}_{\varnothing;J} \equiv 0$ for every nonempty $J \subseteq \mathscr{U}$ with $|J| \leq m-1$.*

Proof. For each $1 \leq n \leq m-1$, the function c_n is constant on $\mathcal{Q}_{\mathscr{U}}$ (by definition of Jones-m-flatness). Therefore, all face derivatives vanish, $\partial_{[n]} c_n \equiv 0$. □

For $K \subset I$ and $J \subset I \setminus K$ set $\mathscr{S}_{K;J} := \partial_J c_{|J|}^K$. When only the cardinalities $r = |K|$ and $q = |J|$ matter we write $\mathscr{S}_{r,q}$ for the rank-1 line generated by all $\mathscr{S}_{K;J}$ with $|K| = r, |J| = q$; in this quotient the specific choice of J is forgotten.

Define the pair of operators:

$$v_i : \mathscr{S}_{r,q} \rightarrow \mathscr{S}_{r,q-1}, \quad h_i : \mathscr{S}_{r,q} \rightarrow \mathscr{S}_{r+1,q-1}, \tag{23}$$

that is, v_i lowers symbol order by one by taking the partial difference at i, h_i lowers symbol order and increases the smoothing index by one after oriented smoothing at i; concretely $v_i(\mathscr{S}_{K;J}) = \mathscr{S}_{K;J\setminus\{i\}}$, $h_i(\mathscr{S}_{r,q}) = \mathscr{S}_{K\cup\{i\};J\setminus\{i\}}$. The symbol skein may thus be expressed as

$$\mathscr{S}_{K;J} = 2v_i(\mathscr{S}_{K;J}) + h_i(\mathscr{S}_{K;J}), \tag{24}$$

i.e., $2v_i + h_i$ acts as an identity on $\mathscr{S}_{K;J}$.

4.4.1 Symbol module and operator calculus

Fix a shadow S with outside crossing set $I = \mathrm{cross}(S)$. For $r, n \geq 0$, let $\mathbb{Z}^{\mathcal{D}(S)}$ denote the free abelian group on diagrams carried by S, and let $c_n^r : \mathcal{D}(S) \to \mathbb{Q}$ be the order-($\leq n$) Vassiliev functional after r oriented smoothings. Define the symbol extraction map $E_{r,n} : \mathbb{Q}^{\mathcal{D}(S)} \to \mathcal{S}_{r,n}$ by

$$E_{r,n}(F) := \partial_{[n]} F^r \qquad \text{(unnormalized alternating sum on } n\text{-faces)},$$

and set $\mathcal{S}_{r,n} := \operatorname{im} E_{r,n}$ (the symbol module of order n after r smoothings). Elements of $\mathcal{S}_{r,n}$ are written $\mathscr{S}_{r,n}$ in the sequel (Figure 2 visualizes the grid (r, n)).

Define linear maps

$$v_i : \mathcal{S}_{r,n} \to \mathcal{S}_{r,n-1}, \qquad v_i\big(\mathscr{S}_{K;J}\big) := \mathscr{S}_{K;J\setminus\{i\}},$$
$$h_i : \mathcal{S}_{r,n} \to \mathcal{S}_{r+1,n-1}, \qquad h_i\big(\mathscr{S}_{K;J}\big) := \mathscr{S}_{K\cup\{i\};J\setminus\{i\}},$$

with $|K| = r,\ |J| = n$. These are well-defined on $\mathcal{S}_{r,n}$ because the assignments are linear in c_n^r and depend only on the image $E_{r,n}(c_n^r)$. (Linearity: the constructions are formed from alternating sums. Well-definedness follows a posteriori from the operator identity below.)

Corollary 4.9 (path independence of smoothing at the symbol level)**.** *For any subset $J \subset I$ with $|J| = k$, the operator that smooths the crossings in J (in any order) acts on symbols by*

$$h_J := \prod_{i\in J} h_i = \prod_{i\in J}(\mathrm{id} - 2\,v_i) = \sum_{L\subseteq J}(-2)^{|L|}\,v_L, \tag{25}$$

where $v_L := \prod_{i\in L} v_i$ (the products are order-independent). In particular, path independence means the result depends only on the set J, not the order in which its elements are smoothed. (This follows from the one-crossing skein $h_i = \mathrm{id} - 2v_i$ and commutativity of the discrete differences.)

Remark 4.6. *In general $v_L\mathscr{S}_{\varnothing;J}$ depends on the set L, not only on its size. On Jones-m-flat subcubes the dependence disappears when acting on $\mathscr{S}_{\varnothing;J}$ with $|J| = m-1$, because all intermediate top-row symbols vanish, $v_L\mathscr{S}_{\varnothing;J} = \mathscr{S}_{\varnothing;J\setminus L} = 0$, for $0 \leq |L| < m-1$ (Lemma 4.8); this is the only specialization used in the proof of Theorem 4.10.*

4.4.2 Last-row fingerprint and degree cap

Theorem 4.10 (Last-row fingerprint of Jones-m-flat subcubes)**.** *Let $\mathcal{Q}_{\mathscr{U}}$ be Jones-m-flat. For any prescribed set $J \subseteq \mathscr{U}$ of $m-1$ crossings and any diagram D carried by S, the free (order-0) coefficient satisfies the **last-row fingerprint***

$$c_0(D \setminus J) = (-2)^{m-1}.$$

Equivalently, for every such J,

$$\mathscr{S}_{J;\varnothing} = (-2)^{m-1}\mathscr{S}_{\varnothing;\varnothing} = (-2)^{m-1}.$$

As this holds for every $J \subseteq \mathscr{U}$ with $|J| = m-1$ we write $c_0^{m-1} = (-2)^{m-1}$. By Corollary 4.6, this last-row equality implies property (P) for every $k \le m-1$ (uniformly over all diagrams carried by the same shadow).

Proof. Fix $J \subseteq \mathscr{U}$ with $|J| = m-1$. Apply (25),

$$\mathscr{S}_{J;\varnothing} = h_J\big(\mathscr{S}_{\varnothing;J}\big) = \sum_{L\subseteq J} (-2)^{|L|}\, v_L\, \mathscr{S}_{\varnothing;J}.$$

Each term satisfies $v_L\, \mathscr{S}_{\varnothing;J} = \mathscr{S}_{\varnothing;J\setminus L}$. On a Jones-$m$-flat subcube we have exactly the vanishing

$$\mathscr{S}_{\varnothing;J\setminus L} = 0 \quad \text{for } 1 \le |J \setminus L| \le m-1$$

(Lemma 4.8), so all terms with $0 \le |L| < m-1$ die after restriction to the $\mathscr{U}$-slice. The only surviving term is, $L = J$: $v_J \mathscr{S}_{\varnothing;J} = \mathscr{S}_{\varnothing;\varnothing} = c_0$. Hence

$$\mathscr{S}_{J;\varnothing} = (-2)^{m-1}\, \mathscr{S}_{\varnothing;\varnothing} = (-2)^{m-1},$$

i.e. $c_0(D \setminus J) = (-2)^{m-1}$. This is the last-row fingerprint.

From last-row fingerprint to property (P). By Lemma 4.1, $c_0(L) = (-2)^{\ell(L)-1}$. Thus $\mathscr{S}_{J;\varnothing} = (-2)^{m-1}$ means that after smoothing the prescribed $|J| = m-1$ crossings, the component count is m. Smoothing in any order can raise the component count by at most 1 at each step; hitting the final value m forces every intermediate increment to be exactly $+1$. Hence for every $K \subseteq J$, $c_0(D \setminus K) = (-2)^{|K|}$, i.e. property (P) holds along J (Corollary 4.6). $\square$

Corollary 4.11 (Degree cap of m-flatness)**.** *Let S be a shadow, $\mathscr{U} \subset \operatorname{cross}(S)$ a set of $d := |\mathscr{U}|$ crossings, and $\mathcal{Q}_{\mathscr{U}}$ the $\mathscr{U}$-subcube. Assume $\mathcal{Q}_{\mathscr{U}}$ is Jones-m-flat, i.e. for every $1 \le n < m$ the coefficient c_n is constant on $\mathcal{Q}_{\mathscr{U}}$. Then:*

(i) Supercritical case yields (P) along $\mathscr{U}$. *If $m > d$, then property* (P) *holds along $\mathscr{U}$: for every $K \subseteq \mathscr{U}$, smoothing the crossings in K increases the component count by $|K|$, and*

$$\mathscr{S}_{K;\varnothing} = c_0(D\setminus K) = (-2)^{|K|} \quad \textit{for all } D \in \mathcal{D}(S).$$

(ii) Global nugatory cap. *If moreover $\mathscr{U} = \operatorname{cross}(S)$ (so d is the total number of crossings of S), then the shadow is a tree (every crossing nugatory), hence all diagrams carried by S are unknots; in particular the complete cube is isotopically constant.*

(iii) Degree cap (contrapositive). *In the global situation $\mathscr{U} = \operatorname{cross}(S)$, if the complete cube is not isotopically constant, then necessarily $m \le d$.*

Proof.

(i) Set $d = |\mathscr{U}|$ and suppose $m > d$. Then $m' := d+1 \le m$. Jones-m-flatness implies Jones-m'-flatness (constancy of c_n for $n < m'$), so we may apply Theorem 4.10 (the last-row fingerprint) with the prescribed $(m'-1)$-set $J = \mathscr{U}$. This gives

$$c_0(D\setminus\mathscr{U}) = (-2)^d \quad \text{for every } D \in \mathcal{D}(S).$$

By Corollary 4.6, the equality $c_0(D\setminus\mathscr{U}) = (-2)^d$ forces the component count to increase by exactly one at each of the d smoothings, hence for every $K \subseteq \mathscr{U}$ we have $c_0(D \setminus K) = (-2)^{|K|}$ and #components$(D\setminus K) = 1 + |K|$. This is property (P) along $\mathscr{U}$.

(ii) If additionally $\mathscr{U} = \operatorname{cross}(S)$, then (P) holds on the full crossing set. By the Shadow Lemma B.1, S is a tree and every diagram carried by S is an unknot; consequently the complete cube is isotopically constant.

(iii) The last assertion is the contrapositive of (ii): if the complete cube is not isotopically constant, then (when $\mathscr{U} = \operatorname{cross}(S)$) the hypothesis $m > d$ cannot hold, so $m \le d$. $\square$

Remark 4.7 (Symbol matrix illustration)**.** *The last-row fingerprint behavior of symbols from Theorem 4.10 and Corollary 4.11 is depicted in Figure 2.*

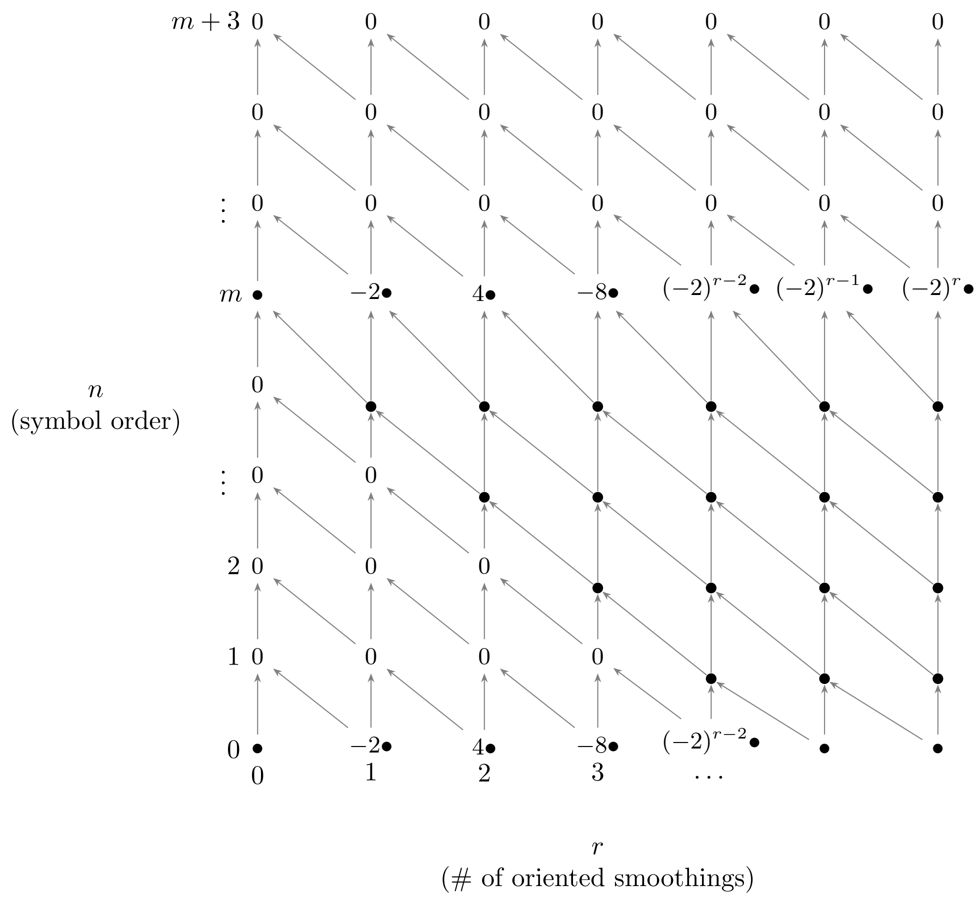


Figure 2: The symbols matrix $[\mathscr{S}_{r,n}]$ of a Jones-m-flat subcube ($\mathscr{S}_{0,n} = 0$ for $1 \leq n \leq m-1$) exhibiting the last-row fingerprint from Theorem 4.10. The skein operator $2v + h$, where v and h are the pair in (23) constrains the triad, $\mathscr{S}_{r,n}, \mathscr{S}_{r,n-1}$, and $\mathscr{S}_{r+1,n-1}$, which ultimately leads to the last-row fingerprint. The distinctive $(-2)^r$ pattern is evident in both the order-m symbol and the order-0 (free) coefficients, $\mathscr{S}_{r,0} = c_0^r$.

5 Uniform m-triviality barrier

Theorem 5.1 (Uniform m-triviality barrier)**.** *Let K be J_m-trivial. If K is nontrivial, then*

$$m \leq \mathscr{N}(K).$$

Proof of Theorem 5.1. Let S be the shadow of K. Fix an extended shadow $E = (K_0; \{C_{m_j}\})$ as in §3.1 with $\min_j m_j = m$. The clasper subcube consists of all clasper states, including the inactive (trivial) tangles. Let T be the set of cube coordinates underlying all inactive clasper decorations, and let $\mathcal{Q}_{S_T} := \{T\} \times \mathcal{Q}_S$ be the corresponding T-face of the extended cube. The following lemma shows that m-flatness carries over from $\mathcal{Q}_{S_T}$ to $\mathcal{Q}_S$.

Lemma 5.2 (Decorated-to-outside transfer)**.** *Let f be any finite-type invariant of degree $< m$. Then for every nonempty $J \subseteq \operatorname{cross}(S)$ with $|J| \le m-1$,*

$$\partial_J\big(f \text{ on } \{T\} \times \mathcal{Q}_S\big) \;=\; 0.$$

In particular, the face $\{T\} \times \mathcal{Q}_S$ is m-flat, so every Jones layer c_n with $1 \le n \le m-1$ is constant on $\{T\} \times \mathcal{Q}_S$.

Proof of Lemma 5.2. Fix a vertex $\varepsilon \in \mathcal{Q}_S$ and write $\sigma(\varepsilon) \in \{\pm 1\}^N$ for the same sign vector. By construction (Figure 1), the knot $K_T(\varepsilon)$ at the T-face and the decorated knot $K_E(\sigma(\varepsilon))$ differ by surgeries along a disjoint family of C_m trees, one for each index with $\sigma_j(\varepsilon) = -1$. Thus $K_T(\varepsilon)$ and $K_E(\sigma(\varepsilon))$ are C_m-equivalent, hence $f\big(K_T(\varepsilon)\big) = f\big(K_E(\sigma(\varepsilon))\big)$ by GH invisibility (order $< m$ invariants are preserved by C_m moves). Taking the alternating sum over the J-face,

$$\partial_J f\big|_{\{T\}\times Q_S} = \sum_{S\subseteq J} (-1)^{|J|-|S|} f\big(K_T(\varepsilon^S)\big) = \sum_{S\subseteq J} (-1)^{|J|-|S|} f\big(K_E(\sigma(\varepsilon^S))\big) = \partial_J\big(f \circ K_E\big).$$

By Lemma 3.3 and Corollary 3.2, $\partial_J(f \circ K_E) = 0$ for $|J| \le m-1$ (the decorated J-differences land in filtration $\ge \sum_{j\in J} m_j \ge m$, so degree $< m$ invariants vanish on them). Hence $\partial_J\big(f \text{ on } \{T\} \times Q_S\big) = 0$ for $|J| \le m-1$. $\square$

By Lemma 5.2 the subcube $\mathcal{Q}_{S_T}$ is Jones-m-flat / m-flat (all order-$< m$ symbols vanish), i.e., the Jones layers $c_n \equiv 0$, for $1 \le n \le m-1$ on $\mathcal{Q}_{S_T}$. In other words, each knot carried by $\{T\} \times S$ is J_m-trivial as K itself is J_m-trivial. If $m > \dim \mathcal{Q}_{S_T}$, Theorem 4.10 and Corollary 4.6 give (P) on the full outside set $\mathscr{U} = \operatorname{cross}(S)$ (as T consists of trivial tangle states). Corollary 4.11(ii) (together with Lemma B.1) implies the T-resolved extended shadow $\{T\} \times S$ is a tree and the face/subcube $\mathcal{Q}_{S_T}$ is isotopically constant. As K is isotopic to $\{T\} \times K$, it follows that it is the unknot, a contradiction. Thus $m \le \dim \mathcal{Q}_{S_T} = \dim \mathcal{Q}_S$, and by choosing S from a minimal diagram of K, this equals $\mathscr{N}(K)$. $\square$

Figuratively, one may say there is a limit to the degree to which a nontrivial knot K may pretend to be 'trivial'. This degree is set by the order m of its first non-trivial finite-type coefficient. The price to pay for this aspiration comes in the form of increased knottedness; its number of crossings scales as

$$\mathscr{N}(K) \ge m.$$

In a sense, the unknot is the only knot that can afford being ∞ly trivial.

Theorem 5.3 (Jones detects the unknot; finite check)**.** *With the Jones normalization of* (15)*, write the expansion* (16)*, $V_K(e^{h/2}) = \sum_{n\ge 0} c_n(K) h^n$. If $V_K \equiv 1$ then K is the unknot. Equivalently, on a minimal diagram with $\mathscr{N}(K)$ crossings, if $c_n(K) = 0$ for $1 \le n \le \mathscr{N}(K)$ then $V_K \equiv 1$.*

Proof. $V_K \equiv 1 \Rightarrow c_n \equiv 0$ for $n \ge 1$. If K were non-trivial, Theorem 5.1 would give $n \le \mathscr{N}(K)$ for all n, a contradiction. $\square$

Theorem 5.4 (Vassiliev unknot conjecture)**.** *A knot that is m-trivial for all m is the unknot.*

Proof. Suppose that K were non-trivial. The statement implies K is J_m-trivial for all m. By Theorem 5.1, $m \le \mathscr{N}(K)$ for all m. Contradiction. $\square$

6 Conclusions and outlook

Birman–Lin showed that coefficients in the series expansions of skein polynomials (Jones, HOMFLY, Kauffman) define Vassiliev invariants, suggesting that polynomial–finite-type bridges might ultimately settle unknot detection via control of low-order layers [5, 6]. Kontsevich's integral and Bar-Natan's diagrammatics organize those layers via weight systems and the $1T/4T$ relations [2, 3]. On the geometric side, Goussarov–Habiro theory provides local moves that are invisible below degree m (connected C_m claspers) [7, 8].

What has been missing is a *diagram-local* mechanism that: (i) packages finite-type layers on a fixed shadow; (ii) turns their local vanishing into a rigid, shadow-level constraint; and (iii) transports this flatness across many outside crossings at once. Our contribution is precisely this triptych, expressed entirely in terms of extended shadows and clasper decorations:

(A) m-invisible crossings on extended shadows. Fix a shadow and, near each outside crossing j, attach a degree-m_j tree clasper. Crossing flips are then realized *modulo clasper calculus* by C_{m_j} moves, hence are invisible to all invariants of degree $< m_j$. With $m := \min_j m_j$, the resulting carried family is m-flat for order $< m$ finite-type data (weighted GH filtration).

(B) Jones layers and a symbol-level skein. For the Birman–Lin expansion $V_K(e^{h/2}) = \sum_n c_n(K)h^n$, each c_n has order $\leq n$. On any m-flat family the layers $c_1, \ldots, c_{m-1}$ are constant, and whenever a single vertex is J_m-trivial these constants pin to zero (Jones–m–flatness). A symbol-level skein recursion then forces a *last-row fingerprint* for the free term c_0: smoothing any $m-1$ chosen crossings produces exactly m components (property (P) along that set).

(C) Shadow criterion and the uniform barrier. A shadow-level criterion converts (P) into global nugatoriness: when m exceeds the number of varied crossings, the shadow collapses to a tree and every carried diagram is an unknot. Applied to a minimal shadow, this yields the uniform bound

$$m \leq \mathscr{N}(K).$$

Equivalently: a nontrivial knot cannot be J_m-trivial beyond its crossing number. As a corollary, the Jones polynomial detects the unknot: if $V_K(t) \equiv 1$ then K would be J_m-trivial for every m, contradicting the bound for $m > \mathscr{N}(K)$; hence K is the unknot. A practical form is: for any diagram with C crossings, vanishing of $c_1, \ldots, c_C$ certifies the unknot.

Auxiliary lemmas and proofs

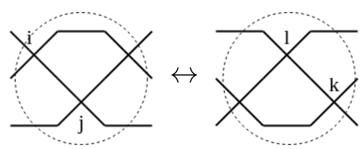
A Cubical 1T/4T

Lemma A.1 (Cubical 1T/4T)**.** *Let S, S' be shadows with variation cubes $\mathcal{Q}(S), \mathcal{Q}(S')$. Let $f : \mathcal{Q}(S) \to \mathbb{C}$ and $f' : \mathcal{Q}(S') \to \mathbb{C}$ be pseudo-Boolean functions encoding finite-type invariants on these cubes. Write $I = \mathrm{cross}(S)$ and $I' = \mathrm{cross}(S')$.*

(1T) *If $S' = S\#\mathcal{O}$ (adding a singular twist), then $\dim \mathcal{Q}(S') = \dim \mathcal{Q}(S) + 1$ and $\mathcal{Q}(S') \cong \mathcal{Q}(S) \times \{\pm 1\}$ with $f'(\varepsilon, \eta) = f(\varepsilon)$. If $m = \dim \mathcal{Q}(S) = |I|$, then*

$$\partial_{I \cup \{\mathrm{new}\}} f' \equiv 0, \qquad \partial_I f' = \partial_I f.$$

(4T) *Suppose S and S' differ by the usual local 4T replacement supported on two crossings $(i, j) \in I$ vs. $(k, l) \in I'$. Assume $|I| = |I'| =: m$ and fix the natural bijection $I \setminus \{i, j\} \cong I' \setminus \{k, l\}$ induced by the local move:*



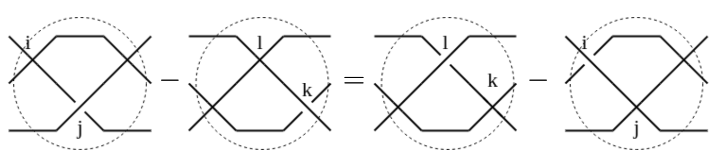
Then the order-$(m-1)$ Walsh coefficients satisfy

$$\widehat{f}(I \backslash \{j\}) - \widehat{f}'(I' \backslash \{k\}) \;=\; \widehat{f}'(I' \backslash \{l\}) - \widehat{f}(I \backslash \{i\}). \tag{26}$$

and the symbols agree (up to the natural identification of coordinates),

$$\partial_I f = \partial_{I'} f'.$$

Remark A.1. *The relation* (26) *is the pseudo-Boolean incarnation of the 4T, commonly represented by*



Proof. (1T). The cubical 1T relation follows immediately from the fact that $f(\varepsilon^{(i,+)}) = f(\varepsilon^{(i,-)})$ for a nugatory crossing whose index is i. Therefore, $\partial_i f = 0$, from which it follows that the symbol $\partial_I f = 0$.

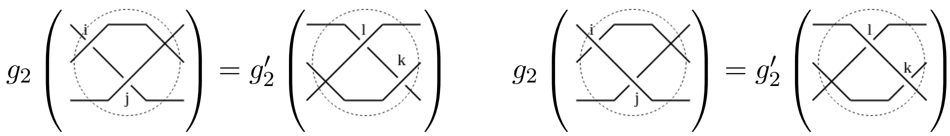
(4T). The shadows S and S' differ in two crossings, $\{i, j\}$ vs. $\{k, l\}$. Taking the $(m-2)$-fold discrete derivative over the residual cube defined by the crossing set $I \setminus \{i, j\}$ (equivalently, $I' \setminus \{k, l\}$), yields an order-2 invariant of knots with $(m-2)$ singular crossings. Denote, $g_2 := \partial_{I \setminus \{i,j\}} f$ and $g_2' := \partial_{I' \setminus \{l,k\}} f'$, the respective order-2 invariants over $\mathcal{Q}(S)/\mathcal{Q}_2$ and $\mathcal{Q}(S')/\mathcal{Q}_2'$. As both invariants are defined over 2-dimensional cubes $\{\pm 1\}^2$, their Walsh-Fourier expansion is

$$\begin{aligned} g_2(\varepsilon_i, \varepsilon_j) &= \hat{g}_2(\emptyset) + \hat{g}_2(i)\varepsilon_i + \hat{g}_2(j)\varepsilon_j + \hat{g}_2(ij)\varepsilon_i\varepsilon_j, \\ g_2'(\varepsilon_k, \varepsilon_l) &= \hat{g}_2(\emptyset) + \hat{g}_2'(k)\varepsilon_k + \hat{g}_2'(l)\varepsilon_l + \hat{g}_2'(kl)\varepsilon_k\varepsilon_l. \end{aligned} \tag{27}$$

Note that both pseudo-Boolean functions share the same free coefficient $\hat{g}_2(\emptyset)$. This follows from the derivative rule (8) which shows that $\hat{g}_2(\emptyset)$ is the order-$(m-2)$ Fourier coefficient $\hat{f}(I \setminus \{i, j\})$ whose

Walsh character $\chi_{I\setminus\{i,j\}}(\varepsilon)$ is independent of $\varepsilon_i, \varepsilon_j$. This coefficient is the same as $\hat{f}'(I' \setminus \{k,l\})$ as they are obtained using the same set of $m-2$ singular crossings.

By Reidemreister-III we have the following identities:



(28)

which translate to

$$g_2(1,1) = g_2'(1,1), \quad g_2(-1,-1) = g_2'(-1,-1). \tag{29}$$

These together with (27) yield,

$$\pm\left[\hat{g}_2(i) + \hat{g}_2(j) - \hat{g}_2'(k) - \hat{g}_2'(l)\right] = \hat{g}_2'(kl) - \hat{g}_2(ij), \tag{30}$$

implying, $\hat{g}_2'(kl) - \hat{g}_2(ij) = 0$, and $\hat{g}_2(i) - \hat{g}_2'(k) = \hat{g}_2'(l) - \hat{g}_2(j)$. The lemma follows upon recognizing that (see (8)), $\hat{g}_2(i) = \hat{f}(I \setminus j)$, $\hat{g}_2(j) = \hat{f}(I \setminus i)$, $\hat{g}_2'(k) = \hat{f}'(I' \setminus l)$, $\hat{g}_2'(l) = \hat{f}'(I' \setminus k)$, and the symbols, $\hat{f}(I) = \hat{g}_2(ij) = \hat{g}_2'(kl) = \hat{f}'(I') \longrightarrow \partial_I f = 2^{|I|}\hat{f}(I) = 2^{|I'|}\hat{f}'(I') = \partial_{I'} f'$. $\square$

B Shadow Lemma

Definition B.1 (Nugatory crossing; cut-vertex). *A crossing is nugatory iff there exists a Jordan curve in the projection plane meeting the diagram only at that crossing; at the shadow level, this is equivalent to the vertex being a cut-vertex (removing it disconnects the shadow). [11, 12]*

Lemma B.1 (Shadow). *Let S be a connected shadow with $n > 0$ vertices and let*

$$\mathcal{D}(S) = \{\textit{all } 2^n \textit{ diagrams carried by } S\}.$$

For a diagram $D \in \mathcal{D}(S)$ write $c(D)$ for its set of crossings and, for $K \subseteq c(D)$, let $D \setminus K$ be the link obtained from D after smoothing *every crossing in K.*

The following statements are equivalent:

1. ***Component-increment property (P)**. For every $D \in \mathcal{D}(S)$ and every subset $K \subseteq c(D)$ of size k,*
$$\#\textit{components of } (D \setminus K) = 1 + k. \tag{P}$$
2. *Every vertex of S is a* cut-vertex *(equivalently, every crossing is* nugatory *in Tait's sense). Hence S is the planar embedding of a tree in which each edge appears exactly once.*
3. *All diagrams in $\mathcal{D}(S)$ are diagrams of the unknot (or of a split unlink if S is disconnected).*

Consequently, any diagram satisfying (P)—and every diagram obtained from it by crossing flips—represents the unknot.

Proof.
(1) ⇒ (2) Work with oriented diagrams and fix a vertex x of the connected shadow S. By Corollary 4.5, for a fixed set of smoothings the number of components of $D \setminus J$ is independent of the over/under choices in $D \in \mathcal{D}(S)$ (the component count after smoothing a fixed set is a shadow invariant); thus we may argue at the level of the shadow. Property (P) with $|K| = 1$ says that for every diagram D the oriented smoothing at x yields two components $D \setminus \{x\} = L_A \sqcup L_B$.

Suppose x were not a cut-vertex. Then the plane graph $S \setminus \{x\}$ is connected; hence there exists a path in $S \setminus \{x\}$ from an arc of L_A to an arc of L_B, and along that path choose the first vertex y where the two sides meet. In the diagram $D \setminus \{x\}$ the crossing y is a crossing between different components L_A and L_B. Now smooth y as well. Oriented smoothing at a crossing that connects distinct components does not increase the number of components (it either keeps it the same or reduces it by one). Therefore

$$\#\pi_0\big(D \setminus \{x, y\}\big) \ \le\ 2,$$

contradicting Property (P) for $|K| = 2$, which requires $1+2 = 3$ components. Hence our assumption was false: $S \setminus \{x\}$ must be disconnected, i.e. x is a cut-vertex (nugatory) [11].

(2) ⇒ (3) Theorem 3 in [13]: *A shadow S has only unknot diagrams iff every vertex of S is a cut-vertex.* Assumption (2) therefore forces every diagram in $\mathcal{D}(S)$ to be an unknot diagram.

(3) ⇒ (1) Seifert's genus formula.

Fix a diagram $D \in \mathcal{D}(S)$ and orient it. Apply *Seifert's algorithm* [14]: smoothing all n crossings produces s Seifert circles and a surface with Euler characteristic $\chi = s-n$. Because D is an unknot, its genus must be zero. For a surface with one boundary component,

$$\chi = 1 - 2g \quad\Longrightarrow\quad 0 = g = \frac{1-(s-n)}{2},$$

hence

$$s = n + 1. \tag{1}$$

Now smooth an *arbitrary* subset $K \subseteq c(D)$ of size k. In the Seifert-surface picture, each smoothed crossing removes exactly one band, thereby increasing the number of boundary components (link components) by one. Starting from one component and using induction on $|K|$ with (1), we obtain exactly $1 + k$ components—property (P).

(2) ⇔ (1) (direct argument)

- **(2) ⇒ (1)** If every crossing is nugatory, smoothing one crossing splits one component into two. Repeating the argument shows the component count always increases by one.
- **(1) ⇒ (2)** The single-crossing case $k = 1$ recovers the nugatory condition used in the first implication.

Thus all three statements are equivalent. □

Corollary B.2 ("Tree diagrams")**.** *A connected knot diagram satisfies property (P)* iff *its shadow is a tree. Such a diagram is sometimes called a* descending *or* standard *diagram; it can be reduced to a round circle by n Reidemeister I moves (one per nugatory crossing) followed by n paired Reidemeister II moves along the tree edges.*

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