

## Final

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Do not turn this page until your instructor tells you to do so.
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- After the exam starts, write your name on every odd-numbered page. We reserve the right to deduct points if you do not, and you will not be allowed to do so after time is called.
- For short question, your answers must be written clearly inside the box region. Any answer outside the box will not be graded. For longer questions, if you run out of space, you must clearly mention in the space provided for the question if part of your answers is elsewhere.
- Try to answer all questions. Not all parts of a problem are weighted equally. Before you answer any question, read the problem carefully. Be precise and concise in your answers.
- You may consult at most *20 sheets of notes*. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- There are **14** pages on the exam (counting this one). Notify a proctor immediately if a page is missing.
- **You have 160 minutes: there are 6 questions on this exam worth a total of 100 points.**

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## 1 Multiple Choice Questions (18 points)

*Every question may have one or more choices that are correct. +3 if you mark all the correct choices and only those choices. Every other answer gets 0 points.*

1. Let  $\mathbb{G}$  be a group of order  $p$  and  $g$  be a random generator. Let  $e : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_T$  be an efficiently computable bilinear map. Assume that Computational Diffie-Hellman (CDH) problem is hard on  $\mathbb{G}$ . Which of the following is/are true?

- (a) The two distributions  $\{g, g^a, g^b, c, g^{abc}\}$  for  $a, b \leftarrow \mathbb{Z}_p$  and  $\{g, g^a, g^b, c, g^d\}$  for  $a, b, c, d \leftarrow \mathbb{Z}_p$  are indistinguishable.
- (b) Given  $g, g^a, g^b$  for  $a, b \leftarrow \mathbb{Z}_p$ , it is hard to compute  $g^{a/b}$ .
- (c) Given  $g, g^a$  for  $a \leftarrow \mathbb{Z}_p$ , it is easy to compute  $g^{a^2}$ .
- (d) Given  $g, g^a$  for  $a \leftarrow \mathbb{Z}_p$ , it is hard to compute  $g^{1/a}$ .

Your choices: **Solution: B,D**

2. Assume OWFs do not imply public-key encryption. Which of the following is/are implied by OWFs?

- (a) Digital Signatures.
- (b) Decisional Diffie-Hellman Assumption.
- (c) Secret Key Encryption.
- (d) Additively Homomorphic Secret Key Encryption.

Your choices: **Solution: A,C**

3. Which of the following implies a zero-knowledge proof for NP?

- (a) Pseudorandom Functions.
- (b)  $P = NP$ .
- (c) Decisional Diffie-Hellman Assumption.
- (d) Collision-Resistant Hash functions.

Your choices: **Solution: A,B,C,D**

4. Let  $\mathbb{G}$  be a DDH-hard group of order  $p$  and  $g$  be a random generator. Which of the following distributions are indistinguishable to  $(g, g^a, g^b, g^c)$  for randomly chosen  $a, b, c \leftarrow \mathbb{Z}_p$ ?

- (a)  $(g, g^a, g^{1/b}, g^{a/b})$  where  $a, b \leftarrow \mathbb{Z}_p$ .
- (b)  $(g, g^a, g^b, g^{a-b})$  where  $a, b \leftarrow \mathbb{Z}_p$ .

Name:

- (c)  $(g, g^a, g^b, g^{a/b})$  where  $a, b \leftarrow \mathbb{Z}_p$ .  
(d)  $(g, g^{b_1}, g^{ab_1}, g^{ab_2})$  where  $a, b_1, b_2 \leftarrow \mathbb{Z}_p$ .

Your choices: **Solution: A,C,D**

5. Which of the following primitives are implied by an Identity-Based Encryption Scheme?

- (a) Authenticated Secret key Encryption.  
(b) Digital Signatures.  
(c) Zero-Knowledge Proofs.  
(d) CCA-2 Public-Key Encryption.

Your choices: **Solution: A,B,C,D**

6. Which of the following is/are true?

- (a) CBC-MAC using random IV is secure.  
(b) Chained CBC mode is insecure.  
(c) One round SPN with key mixing step cannot be broken in  $2^{64}$  time.  
(d) Define the key space  $k = (a, b) \leftarrow \mathbb{Z}_p \times \mathbb{Z}_p$  where  $p$  is a prime. Define the encryption to take  $x \in \mathbb{Z}_p$  and output  $ax + b$ . This scheme is not perfectly secure.

Your choices: **Solution: B**

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## 2 Bit Commitment (15 points)

1. Assuming a OWP  $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , construct an efficient algorithm  $\text{Com}(b; r)$ , where  $b \in \{0, 1\}$  and  $r \in \{0, 1\}^\ell$  (for any  $\ell \geq n$ ), such that:
  - (a) There does not exist  $r, r'$  such that  $\text{Com}(0; r) = \text{Com}(1; r')$ .
  - (b)  $\{\text{Com}(0; U_n)\} \approx_c \{\text{Com}(1; U_n)\}$  where  $U_n$  is the uniform distribution over  $\{0, 1\}^n$  and  $\approx_c$  denotes computational indistinguishability.

Prove that your construction satisfies both these properties.

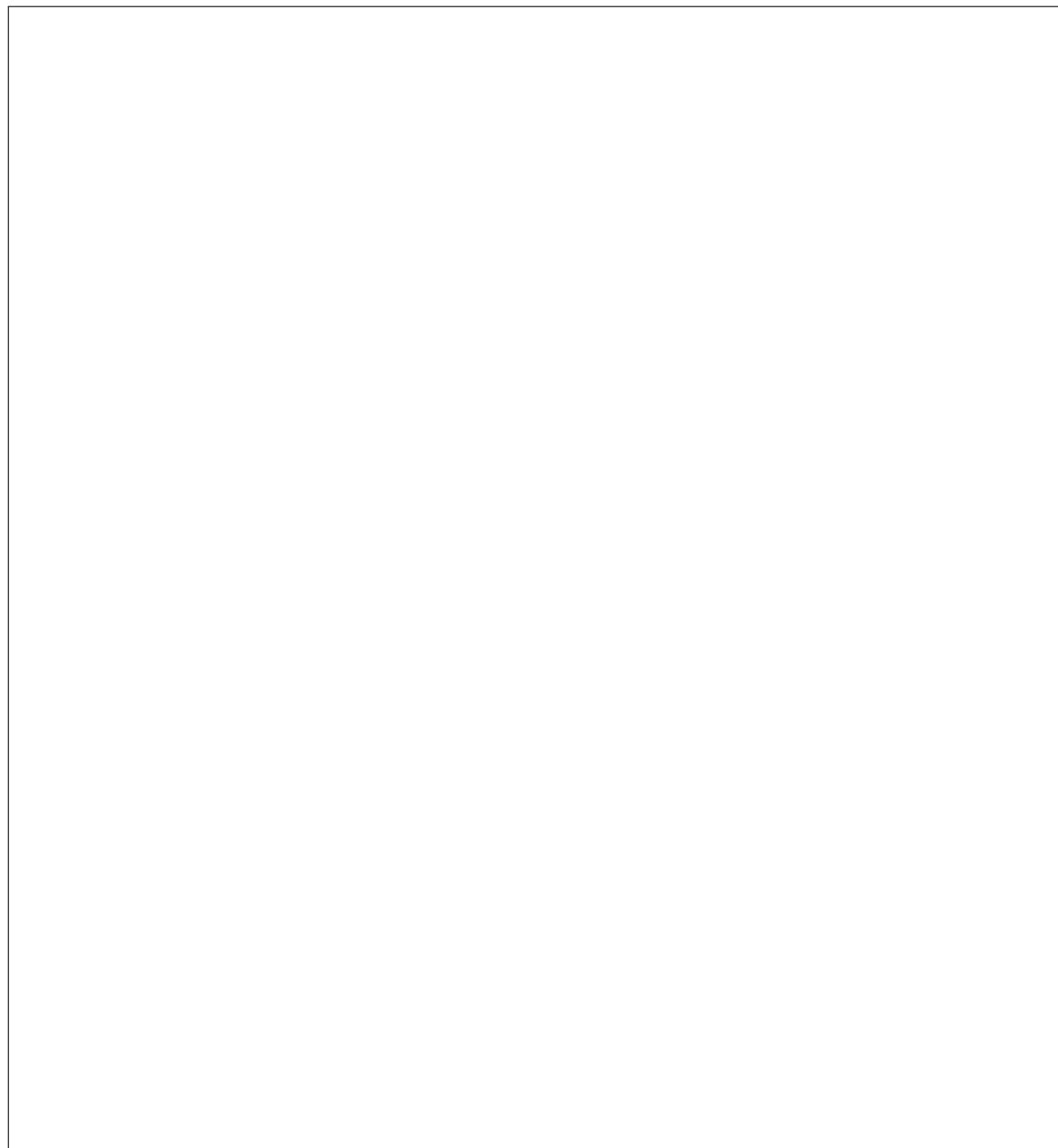
Name:

**Solution:**  $\text{Com}(b; r) = (f(r), h(r) \oplus b)$  where  $f$  is a one-way permutation and  $h$  is the hardcore bit. Binding follows since  $f$  is a one-way permutation and hiding follows from the hardcore bit.

2. Assuming a PRG  $f : \{0, 1\}^n \rightarrow \{0, 1\}^{3n}$  (note that  $f$  may not be injective), construct an efficient algorithm  $\text{Com}(\sigma, b; r)$ , where  $\sigma \in \{0, 1\}^{3n}$ ,  $b \in \{0, 1\}$ , and  $r \in \{0, 1\}^\ell$  (for any  $\ell \geq n$ ), such that:

- (a) Let  $\text{Bad} = \{\sigma \mid \exists r, r' \text{ such that } \text{Com}(\sigma, 0; r) = \text{Com}(\sigma, 1; r')\}$ . Then,  $\mu(n) = \Pr_{\sigma \leftarrow U_{3n}}[\sigma \in \text{Bad}]$  is a negligible function.
- (b) We have that  $\{\sigma, \text{Com}(\sigma, 0; U_n)\} \approx_c \{\sigma, \text{Com}(\sigma, 1; U_n)\}$  for every  $\sigma \in \{0, 1\}^{3n}$ .

Prove that your construction satisfies both these properties.



**Solution:**  $\text{Com}(\sigma, b; r) = f(r)$  if  $b = 0$ ; else, it is equal to  $f(r) \oplus \sigma$  if  $b = 1$ . Hiding follows from pseudorandomness of  $f$ . Binding follows since  $|\{f(r) \oplus f(r') : r, r' \in \{0, 1\}^n\}| \leq 2^{2n}$  but  $\sigma$  is chosen uniformly from  $\{0, 1\}^{3n}$ .

### 3 Zero-Knowledge Proofs (15 points)

Let  $(P_0, V_0)$  and  $(P_1, V_1)$  be *public-coin*<sup>1</sup> honest verifier zero-knowledge proof systems for languages  $L_0$  and  $L_1$  respectively. For simplicity, let us assume that there are three messages in both protocols. The first message is from the prover to the verifier, denoted by  $\alpha_0$  (resp.  $\alpha_1$ ). The second message is just the verifier's random coins and this will be  $\beta_0$  (resp.  $\beta_1$ ) (we will assume that the length of the verifier's random coins is the same for both proof systems and this length will be  $n$ ). The final message is from the prover to the verifier, denoted by  $\gamma_0$  (resp.  $\gamma_1$ ). Let  $S_0$  and  $S_1$  be the honest verifier zero-knowledge simulators for the two proof systems. Specifically, for both  $c = 0$  and  $c = 1$ ,  $S_c(\beta_c)$  (for a randomly chosen  $\beta_c$ ) outputs  $(\alpha_c, \gamma_c)$  such that  $V_c(\alpha_c, \beta_c, \gamma_c) = 1$  and the distribution of  $(\alpha_c, \beta_c, \gamma_c)$  is identical to the real protocol.

Now consider the language  $L = \{(x_0, x_1) \mid x_0 \in L_0 \text{ or } x_1 \in L_1\}$ . Consider the following honest-verifier zero-knowledge protocol for  $L$ .<sup>2</sup> The protocol has a few missing parts and your task is to fill in the details. Also, fill in the details of the simulator  $S$  of the constructed honest-verifier, public-coin zero-knowledge protocol  $(P, V)$ .

- **Input.** The prover and verifier gets two statements  $x_0$  and  $x_1$  such that  $(x_0, x_1) \in L$ . The prover additionally obtains a witness  $(c, w)$  such that  $w$  is a valid witness for  $x_c \in L_c$ . Note that  $x_{1-c}$  may not be in  $L_{1-c}$ .
- $P \rightarrow V$ : In the first round, prover chooses  $\beta_{1-c} \leftarrow \{0, 1\}^n$  uniformly at random. It generates the first round message  $\alpha_c$  using the prover  $P_c$  on input  $x_c$  and the witness  $w$ . It generates  $\alpha_{1-c}$  as **Solution:  $[S_{1-c}(\beta_{1-c})]_1$** . It sends  $(\alpha_0, \alpha_1)$  to the verifier.
- $V \rightarrow P$ : In the second round, the verifier sends its random coins  $\beta \leftarrow \{0, 1\}^n$  to the prover.
- $P \rightarrow V$ : the last round, the prover generates  $\beta_c$  as **Solution:  $\beta_{1-c} \oplus \beta$** . It generates the third round message  $\gamma_c$  as **Solution:  $P_c(x_c, w, \alpha_c, \beta_c)$** . It generates  $\gamma_{1-c}$  as **Solution:  $[S_{1-c}(\beta_{1-c})]_2$** . It sends  $(\beta_0, \beta_1, \gamma_0, \gamma_1)$  to the verifier.
- The verifier accepts if  $\beta =$  **Solution:  $\beta_0 \oplus \beta_1$**  and  $V_0(\alpha_0, \beta_0, \gamma_0) = 1$  and  $V_1(\alpha_1, \beta_1, \gamma_1) = 1$ .

Give the description of the honest-verifier zero-knowledge simulator for the above protocol.

$S(\beta)$  outputs  $(\alpha = (\alpha_0, \alpha_1), \gamma = (\gamma_0, \gamma_1))$  where  $\beta_0 \leftarrow \{0, 1\}^n$  is chosen uniformly at random,

$\beta_1 =$  **Solution:  $\beta \oplus \beta_1$** ,

<sup>1</sup>By public-coin we mean that the verifier's messages in the protocol are outcomes of random coin tosses.

<sup>2</sup>Note that this implies that the verifier should not learn where  $x_0 \in L_0$  or  $x_1 \in L_1$ .

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$$\alpha_0 = \boxed{\text{Solution: } [\mathbf{S}_0(\beta_0)]_1},$$

$$\alpha_1 = \boxed{\text{Solution: } [\mathbf{S}_1(\beta_1)]_1},$$

$$\gamma_0 = \boxed{\text{Solution: } [\mathbf{S}_0(\beta_0)]_2},$$

$$\gamma_1 = \boxed{\text{Solution: } [\mathbf{S}_1(\beta_1)]_2}.$$



Name:

## 4 Pseudorandom Functions (25 points)

1. Assume pseudorandom functions exist. Construct a PRF such that any PPT attacker cannot distinguish between an oracle access to the PRF and an oracle access to a random function but given a single bit of the key, there exists an attacker that can distinguish between these two oracles with non-negligible advantage. Give proofs for all your claims.

**Solution:**  $f_{k,b}(x) = \begin{cases} g_k(x)_{1,\dots,n-1} \| b & \text{if } x = x^* \\ g_k(x) & \text{otherwise} \end{cases}$  where  $g$  is a PRF.

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2. A weak pseudorandom function  $F : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  is just like any other PRF except that it satisfies a weaker notion of security. Specifically, for any PPT attacker  $D$ , there exists a negligible function  $\text{negl}$  such that:

$$|\Pr[D^{O_k}(1^n) = 1] - \Pr[D^{O'}(1^n)]| \leq \text{negl}(n)$$

where  $k \leftarrow \{0, 1\}^n$  and  $O_k$  is an oracle that on any query, samples a random  $x \leftarrow \{0, 1\}^n$  and outputs  $F_k(x)$ .  $O'$  is another oracle such that on any query, samples  $x \leftarrow \{0, 1\}^n$  and outputs  $f(x)$  where  $f$  is a random function. Notice that the difference between weak PRF and a regular PRF is that in weak PRF, the attacker does not have the ability to choose inputs and learn the output of the function on the chosen inputs. Instead, each time the attacker queries the oracle, he gets the output of the function on randomly chosen inputs.

Construct a CPA-secure secret-key encryption from a weak PRF and prove its security.

**Solution:**  $\text{Enc}(k, m; r) = (r, F_k(r) \oplus m)$ .

Name:

3. Consider a PRF  $F : \mathcal{K} \times \mathcal{X} \rightarrow \mathcal{Y}$  where  $\mathcal{X}, \mathcal{Y}$  are groups with the group operation denoted by  $+$  and  $\cdot$  respectively. We say that  $F$  is input homomorphic if for any  $k \in \mathcal{K}$  and two inputs  $x_1, x_2 \in \mathcal{X}$

$$F(k, x_1) \cdot F(k, x_2) = F(k, x_1 + x_2)$$

- (a) Give a construction of collision-resistant hash functions from input homomorphic PRFs.  
(No need to prove collision-resistance)

**Solution:** For  $n > \log |\mathcal{Y}|$ , choose a matrix  $M$  randomly from  $\mathcal{X}^{2 \times n}$  and set the key of the hash function to be  $F_k(M)$ . On input  $x \in \{0, 1\}^n$ , compute  $\prod_i F_k(M_{x_i, i})$  using the homomorphic property.

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- (b) Give a construction of CPA secure public key encryption from input homomorphic PRFs. Prove its security.

**Solution:**  $(r, F_k(r) \cdot m)$  is a secret key additively homomorphic encryption. This implies

Name:

an CPA-secure PKE.

## 5 Private Information Retrieval from DDH (15 points)

In the problem of private information retrieval, there is a server who holds a large database  $D \in \{0,1\}^n$ . The client wishes to learn the bit in the  $i$ -th position of the database, i.e., client wants to learn  $D_i$ . The client additionally wishes the server not to learn the position  $i$ . One simple way to ensure is to ask the server to send the entire database for every client query. But this has huge communication cost when  $n$  is of the order of several thousand GB. We are interested in solutions where the communication cost is substantially lower than the size of the database.

In this problem, we will construct a private information retrieval from DDH where the length of the message from client to server is large (of the order of  $n$ ) but the communication cost from the server to client is small and independent of the length of the database.

1. Let  $(\mathbb{G}, g, p)$  be a DDH-hard group. On input a location  $L \in [n]$ , the client does the following. It chooses

$$P = \begin{pmatrix} g_{1,0} & g_{2,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & \cdots & g_{n,1} \end{pmatrix}$$

where each  $g_{i,b}$  is chosen uniformly from  $\mathbb{G}$ . It then chooses a random  $r \leftarrow \mathbb{Z}_p$  and computes

$$Q = \begin{pmatrix} g_{1,0}^r & g_{2,0}^r & \cdots & g_{L-1,0}^r & Q_{L,0} & g_{L+1,0}^r & \cdots & g_{n,0}^r \\ g_{1,1}^r & g_{2,1}^r & \cdots & g_{L-1,1}^r & Q_{L,1} & g_{L+1,1}^r & \cdots & g_{n,1}^r \end{pmatrix}$$

where  $Q_{L,0} =$  **Solution:**  $g_{L,0}^r$

and  $Q_{L,1} =$  **Solution:**  $g \cdot g_{L,1}^r$ .

It sends  $P, Q$  to the server.

2. The server on receiving  $P$  and  $Q$  computes,

$$h = \prod_{i=1}^n P_{i,D_i}$$

and

$$k = \text{Solution: } \prod_{i=1}^n Q_{i,D_i}$$

and sends  $(h, k)$  to the client.

3. The client retrieves  $D_i$  as (and argue why it is correct):

Name:

**Solution:** Check if  $k = h^r$  and if yes, output 0. Else, output 1.

4. Prove that  $P, Q$  hides the location  $L$  using the DDH assumption.

**Solution:** Under the DDH assumption,  $(g, g^r, g_{L,1}, g \cdot g_{L,1}^r)$  is indistinguishable to  $(g, g^r, g_{L,1}, g^s)$  for a randomly chosen  $s$ .

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## 6 Witness Encryption (12 points)

1. A witness encryption scheme for a language  $L \in \text{NP}$  is defined as follows:
  - Let  $x$  be some statement. The encryption algorithm takes  $x$  and a message  $m$  and outputs a ciphertext  $c$ .
  - The decryption algorithm takes the ciphertext  $c$  and a witness  $w$  such that  $(x, w)$  satisfies the NP relation for  $L$  and outputs the message  $m$ .

We require two properties:

- **Correctness.** For every  $w$  such that  $(x, w)$  that satisfies the NP relation for  $L$ , we require the decryption to output the message  $m$  with probability 1.
- **Security.** If  $x \notin L$ , for every two messages  $m_0, m_1$ ,

$$\text{Enc}(x, m_0) \approx_c \text{Enc}(x, m_1)$$

Give a construction of witness encryption scheme for any language  $L \in \text{NP}$  using an indistinguishability obfuscator. Prove correctness and security.

**Solution:**  $\text{Enc}(x, m)$ : Output  $iO(C_{x,m})$  where  $C_{x,m}$  takes a witness  $w$  and it outputs  $m$  if and only if  $(x, w)$  satisfies the NP relation.