# Pseudorandom Functions

## CS 276: Introduction to Cryptography

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# Overview

1. Pseudorandom Functions
   - Definitions
   - Construction of PRF from PRG
2. PRFs from DDH: Naor-Reingold
3. Summary

# The Problem with Random Functions

## Random Functions

Consider the set of all functions $f : \{0,1\}^n \to \{0,1\}^n$:

- Total number: $(2^n)^{2^n} = 2^{n \cdot 2^n}$
- To describe a random function: need $n \cdot 2^n$ bits
- This is **exponential** in $n$!

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## Goal

Can we have a function that:

- Looks random to any PPT adversary
- But can be described with only **polynomial** (in $n$) bits?
- And can be evaluated efficiently?

# Pseudorandom Functions: Intuition

## Key Idea

A pseudorandom function (PRF) is a function that:

- Can be described with a short **key** $k$ (polynomial in $n$)
- Given key $k$, can evaluate $F_k(x)$ efficiently
- Cannot be distinguished from a truly random function by any PPT adversary
- Adversary can only query the function polynomially many times

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## Applications

- Symmetric encryption
- Message authentication codes (MACs)
- Key derivation
- Many other cryptographic protocols

# Function Ensemble

**Definition 1 (Function Ensemble)**

A **function ensemble** is a sequence of random variables $F_1, F_2, \ldots, F_n, \ldots$ denoted as $\{F_n\}_{n \in \mathbb{N}}$ such that $F_n$ assumes values in the set of functions mapping $n$-bit input to $n$-bit output.

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## Notation

- We write $\{F_n\}_n$ or simply $F_n$ when clear from context
- Each $F_n$ is a random variable over functions
- Can generalize to functions mapping $n$-bit inputs to $m$-bit outputs

# Random Function Ensemble

**Definition 2 (Random Function Ensemble)**

We denote a random function ensemble by $\{R_n\}_{n\in\mathbb{N}}$, where $R_n$ is uniformly distributed over all functions from $\{0,1\}^n$ to $\{0,1\}^n$.

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## Key Properties

- A sampling of $R_n$ requires $n \cdot 2^n$ bits to describe
- Truly random: each input-output pair is independent
- This is our "ideal" benchmark for PRFs

# Efficiently Computable Function Ensemble

**Definition 3 (Efficiently Computable Function Ensemble)**

A function ensemble $\{F_n\}_n$ is **efficiently computable** if:

1. **Succinct**: $\exists$ PPT algorithm $I$ and mapping $\phi$ such that $\phi(I(1^n))$ and $F_n$ are identically distributed
2. **Efficient**: $\exists$ poly-time machine $V$ such that $V(i, x) = f_i(x)$ for every $x \in \{0,1\}^n$, where $i$ is in the range of $I(1^n)$ and $f_i = \phi(i)$

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## Key Insight

- A sample from $F_n$ can be generated by sampling a key $k \in \{0,1\}^n$
- The key $k$ is the description of the function
- Only $n$ bits needed (vs $n \cdot 2^n$ for random functions)!

# Pseudorandom Function Ensemble

**Definition 4 (Pseudorandom Function Ensemble)**

A function ensemble $F = \{F_n\}_{n \in \mathbb{N}}$ is **pseudorandom** if for every non-uniform PPT oracle adversary $\mathcal{A}$, there exists a negligible function $\epsilon(n)$ such that:

$$\left| \Pr[\mathcal{A}^{F_n}(1^n) = 1] - \Pr[\mathcal{A}^{R_n}(1^n) = 1] \right| \leq \epsilon(n)$$

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## Key Points

- Adversary $\mathcal{A}$ has **oracle access** to the function
- Can query the function polynomially many times
- Each oracle call costs 1 unit of time
- Cannot distinguish PRF from truly random function

# PRF from PRG: The GGM Construction

**Goal**

Construct a PRF from a length-doubling PRG $G : \{0,1\}^n \to \{0,1\}^{2n}$.

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Construct a PRF from a length-doubling PRG $G : \{0,1\}^n \to \{0,1\}^{2n}$.

## Key Idea: Binary Tree

- View the PRF evaluation as traversing a binary tree
- Root: the key $K$
- Each level: use PRG to generate two children
- Leaf: the output

# GGM Construction: Setup

## Notation

Given PRG $G : \{0,1\}^n \to \{0,1\}^{2n}$:

- $G_0(x)$: first $n$ bits of $G(x)$
- $G_1(x)$: last $n$ bits of $G(x)$

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## Construction

For key $K \in \{0,1\}^n$ and input $x = x_1 x_2 \cdots x_n \in \{0,1\}^n$:

$$F_n^{(K)}(x_1 x_2 \cdots x_n) := G_{x_n}(G_{x_{n-1}}(\cdots(G_{x_1}(K))\cdots))$$

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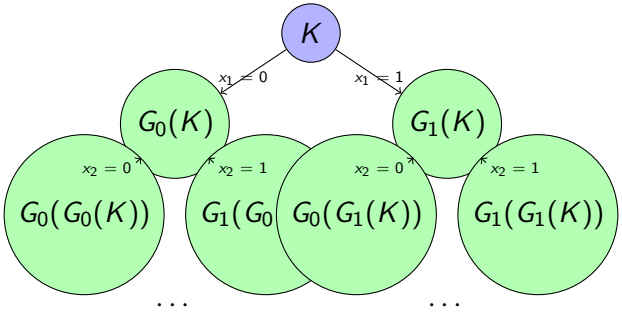
## Algorithm

1. Set $y \leftarrow K$
2. For $i = 1$ to $n$: update $y \leftarrow G_{x_i}(y)$
3. Output $y$

# GGM Construction: Binary Tree View



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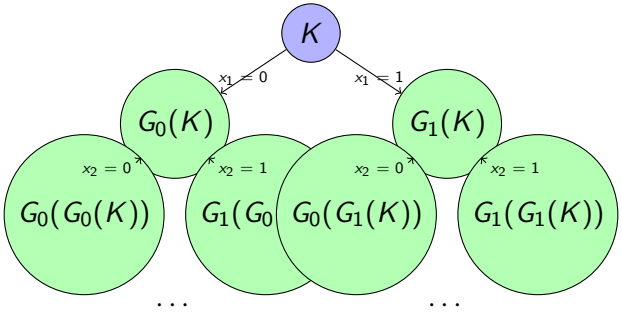


## Intuition

- To evaluate $F_K(x_1 \cdots x_n)$, follow path $x_1, x_2, \ldots, x_n$
- At each level, use $G_0$ if bit is 0, $G_1$ if bit is 1
- Final node is the output

# GGM Construction: Theorem

**Theorem 5 (GGM)**

*The function ensemble $\{F_n\}_{n\in\mathbb{N}}$ constructed above is pseudorandom.*

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**Proof Strategy**

- Assume for contradiction that $\{F_n\}$ is not a PRF
- Use hybrid argument to show this breaks the PRG
- Key challenge: adversary can make multiple queries
- Need sub-hybrids to handle this

# GGM Proof: Hybrids

## Hybrid $H_i$

For $i \in \{0, 1, \ldots, n\}$, define hybrid $H_i$:

$$H_i^{(K_i)}(x_1 x_2 \ldots x_n) := G_{x_n}(G_{x_{n-1}}(\cdots(G_{x_{i+1}}(K_i(x_1 \ldots x_i)))\cdots))$$

where $K_i$ is a random function from $\{0,1\}^i$ to $\{0,1\}^n$.

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## Key Observations

- $H_0$: PRF (all levels use PRG)
- $H_n$: Random function (all levels random)
- $H_i$: Levels 0 to $i$ random, levels $i+1$ to $n$ use PRG

# GGM Proof: The Challenge

**Problem**

We cannot directly reduce $H_i$ vs $H_{i+1}$ to PRG security because:

- Adversary can make multiple oracle queries
- Different queries may share prefixes
- Need to handle queries carefully

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## Solution: Sub-Hybrids

- Define sub-hybrids $H_{i,j}$ for $j \in \{0, \ldots, q\}$ where $q$ is number of queries
- $H_{i,0} = H_i$ and $H_{i,q} = H_{i+1}$
- Each sub-hybrid handles queries one at a time

# GGM Proof: Sub-Hybrids

## Sub-Hybrid $H_{i,j}$

Let $R_i : \{0,1\}^i \to \{0,1\}^n$ and $S_i : \{0,1\}^{i+1} \to \{0,1\}^n$ be random functions.
For query $x = x_1 \dots x_n$:

1. Initialize list $L$ of $i$-bit prefixes seen
2. If $|L| < j$ or $(x_1 \dots x_i) \in L$:
   - Set $y \leftarrow S_i(x_1 \dots x_{i+1})$ (random)
   - Append $(x_1 \dots x_i)$ to $L$
3. Else:
   - Set $y \leftarrow R_i(x_1 \dots x_i)$ (random)
4. For $a = i + 2$ to $n$: update $y \leftarrow G_{x_a}(y)$
5. Output $y$

# GGM Proof: Outer Adversary

## Construction of $\mathcal{B}$

$\mathcal{B}$ on input $z \in \{0,1\}^{2n}$ (either from $U_{2n}$ or $G(U_n)$):

1. Parse $z$ as $z_0 \| z_1$ where $z_0, z_1 \in \{0,1\}^n$
2. For queries except the $(j^* + 1)$-th: respond as $H_{i^*,j^*}$
3. For the $(j^* + 1)$-th query $(x_1 \dots x_n)$:
   - Set $y \leftarrow z_{x_{i^*+1}}$
   - For $a = i^* + 2$ to $n$: update $y \leftarrow G_{x_a}(y)$
   - Respond with $y$
4. Output whatever $\mathcal{A}$ outputs

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## Analysis

- If $z \leftarrow G(U_n)$: output is from $H_{i^*,j^*}$
- If $z \xleftarrow{\$} U_{2n}$: output is from $H_{i^*,j^*+1}$
- $\mathcal{B}$ breaks PRG security

# Naor-Reingold PRF: Motivation

**Question**

Can we construct PRFs from number-theoretic assumptions like DDH?

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Can we construct PRFs from number-theoretic assumptions like DDH?

**Naor-Reingold PRF**

- Based on DDH assumption
- Output is a group element (not a bit string)
- More efficient in some settings
- Key is longer: $(n+1)$ group elements

# Naor-Reingold PRF: Construction

## Setup

- Group ensemble $\{\mathbb{G}_n\}$ where DDH is hard
- Generator $g$ of $\mathbb{G}_n$
- Key space: $\mathcal{K}$

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## Key Generation

Key $K = (h, u_1, u_2, \ldots, u_n)$ where:

- $u, u_1, \ldots, u_n \xleftarrow{\$} \{0, \ldots, |\mathbb{G}_n| - 1\}$
- $h = g^u$

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## Function Evaluation

For input $x = x_1 x_2 \cdots x_n \in \{0,1\}^n$:

$$F_n(K, x) = h^{\prod_{i=1}^n u_i^{x_i}} = g^{u \cdot \prod_{i=1}^n u_i^{x_i}}$$

# Naor-Reingold PRF: Properties

## Key Differences from GGM

- **Key length**: $(n+1)$ group elements vs $n$ bits
- **Output**: Group element vs bit string
- **Assumption**: DDH vs PRG (which needs OWP)
- **Structure**: Algebraic vs tree-based

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## Advantages

- Can be more efficient in practice
- Natural for group-based cryptography
- Useful for certain applications

# Naor-Reingold PRF: Security

**Lemma 6**

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**Proof Strategy**

- Similar to GGM proof: hybrid argument
- Key difference: nodes in tree are not independent
- Must handle DDH relations carefully
- Use DDH challenge to embed in reduction

# Naor-Reingold Proof: Hybrids

## Hybrid $H_n^j$

For $j \in \{0, \ldots, n\}$, let $S_n^j : \{0,1\}^j \to \{0, \ldots, |\mathbb{G}_n| - 1\}$ be a random function.

$$H_n^j((u, u_{j+1} \ldots u_n), x) = \left(g^{S_n^j(x_1 \ldots x_j)}\right)^{\prod_{i=j+1}^{n} u_i^{x_i}}$$

where $S_n^0(\cdot) = u$ (constant function).

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## Observations

- $H_n^0$: Naor-Reingold PRF
- $H_n^n$: Random function (uniform group element)
- $H_n^j$: First $j$ bits use random function, rest use key

# Naor-Reingold Proof: Sub-Hybrids

## Sub-Hybrid $H_n^{j,k}$

Similar to GGM, define sub-hybrids $H_n^{j,k}$ for $k \in \{0, \ldots, q\}$:

- Handle queries one at a time
- First $k$ queries with prefix $(x_1 \ldots x_j)$: use random function $D_n^j$
- Rest: use random function $C_n^j$
- $H_n^{j,0} = H_n^j$ and $H_n^{j,q} = H_n^{j+1}$

# Naor-Reingold Proof: Outer Adversary

**Challenge**

$\mathcal{B}$ receives DDH challenge $(g, A = g^a, B = g^b, C)$ where $C$ is either $g^{ab}$ (DDH tuple) or $g^c$ (random).

# Naor-Reingold Proof: Outer Adversary

## Challenge

$\mathcal{B}$ receives DDH challenge $(g, A = g^a, B = g^b, C)$ where $C$ is either $g^{ab}$ (DDH tuple) or $g^c$ (random).

## Construction of $\mathcal{B}$

1. Sample $u, u_{j^*+1}, \ldots, u_n$ uniformly
2. For first $k^*$ queries: respond as $H_n^{j^*,k^*}$
3. For $(k^*+1)$-th query $(x_1 \ldots x_n)$:
   - If $x_{j^*+1} = 0$: set $y \leftarrow A$, else $y \leftarrow C$
   - For $i = j^* + 2$ to $n$: update $y \leftarrow y \cdot u_i^{x_i}$
   - Output $g^y$
4. For remaining queries: use appropriate DDH relations
5. Output whatever $\mathcal{A}$ outputs

# Naor-Reingold Proof: Key Insight

## DDH Relation

- If $C = g^{ab}$: responses match $H_n^{j^*,k^*}$
- If $C = g^c$: responses match $H_n^{j^*,k^*+1}$
- $\mathcal{B}$ can distinguish DDH tuples from random
- This contradicts DDH assumption

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## Complexity

- Unlike GGM, nodes are not independent
- Must maintain DDH relations across all queries
- More careful handling needed

# Key Takeaways

1. **PRFs**: Functions that look random but have short descriptions (keys)
2. **GGM Construction**: Build PRF from PRG using binary tree structure
3. **Hybrid Arguments**: Essential proof technique with sub-hybrids for multiple queries
4. **Naor-Reingold**: PRF from DDH assumption with group element outputs
5. **Applications**: Foundation for symmetric encryption, MACs, and more

# Next Steps

- PRFs are powerful building blocks
- Next: Applications of PRFs
  - Symmetric encryption schemes
  - Message authentication codes
  - More advanced protocols
- Questions?