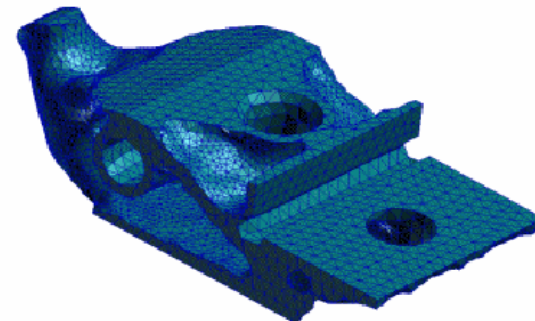
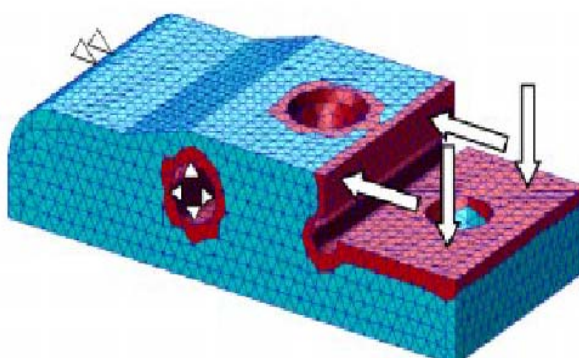
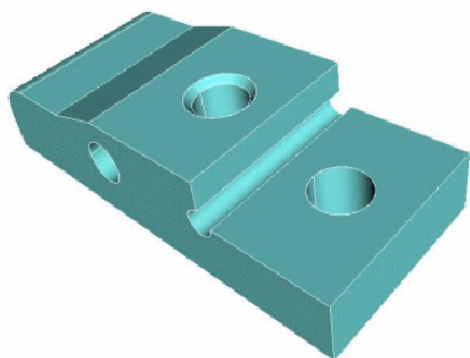


结构拓扑优化与材料设计



第7次课回顾

➤ 拓扑优化中的灵敏度分析

- 直接法
- 伴随法
- 柔顺性问题的灵敏度分析

第8次课主要内容

➤调用MMA优化子程序

➤结构应力拓扑优化

- 基本列式

- 应力约束优化问题中的奇异现象

- 放松方法

MMA程序介绍



- 移动渐近算法 (Method of Moving Asymptotes) 优化程序由瑞典皇家工学院Krister Svanberg教授于1987年开发。
- 后为使用于MBB梁问题及其他算例，对程序进行了重写。
- 2002年发布GCMMA，增强算法的全局收敛性。

[1] K. Svanberg, The method of moving asymptotes { a new method for structural optimization, International Journal for Numerical Methods in Engineering, 1987, 24, 359-373.

[2] K. Svanberg, A class of globally convergent optimization methods based on conservative convex separable approximations, SIAM Journal of Optimization, 2002, 12, 555-573.

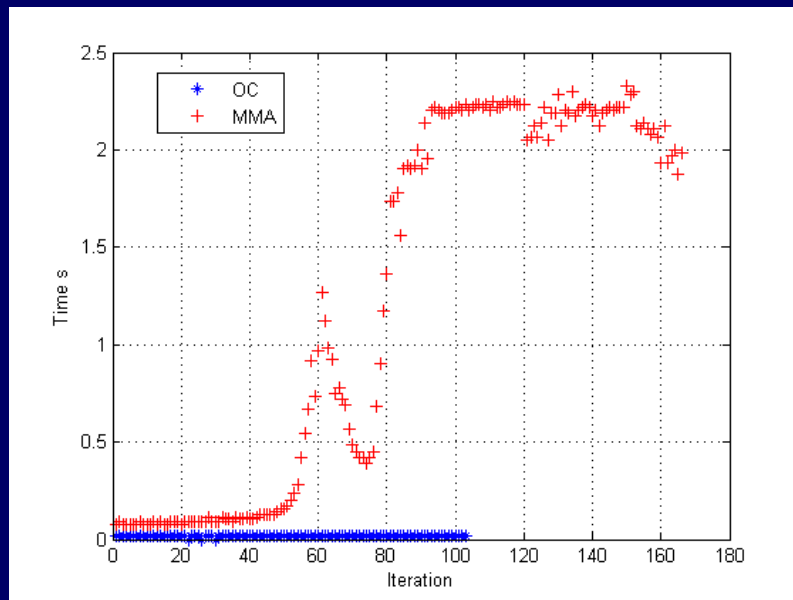
MMA程序介绍

- MMA的程序代码mma sub. m和subsolv. m可以免费用于教育及科研。可以通过向Svanberg教授发邮件的形式获取。
- <http://www.math.kth.se/~krille/>
- 求解相同的MBB梁问题，OC算法和MMA算法表现上有什么不同？
- 目标函数？ 计算时间？

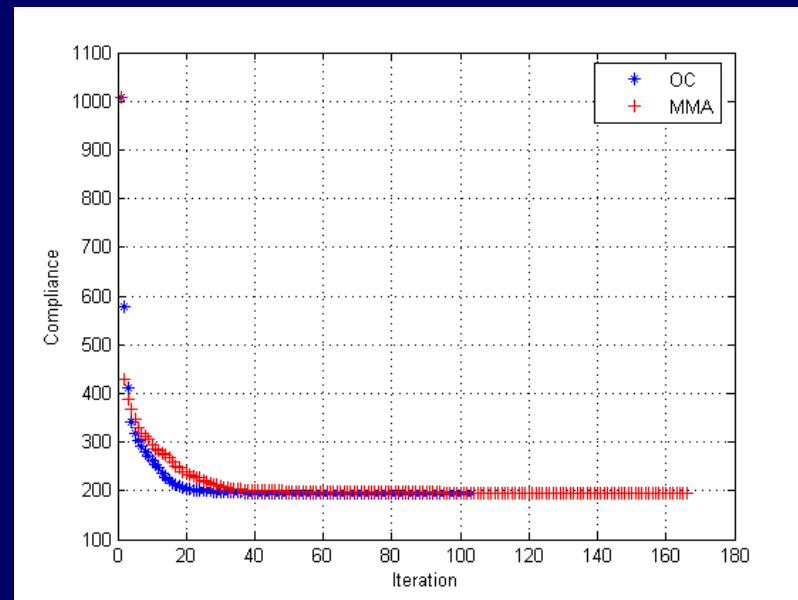
OC VS MMA

	Time s	Number of iterations
OC	18.86	41
MMA	96.36	97

➤ MBB problem 60x20 elements



Time pr. optimization iteration



➤ 对于相同的简单问题OC 计算效率高于 MMA.

OC VS MMA

➤ MBB problem 60x20 elements



MMA优化结果



OC优化结果

MMA优化模型

$$\min_{x,y,z} : f_0(x) + a_0 z + \sum_{i=1}^m \left(c_i y_i + \frac{1}{2} d_i y_i^2 \right)$$

$$S.t. : f_i(x) - a_i z - y_i \leq 0, \quad i = 1, \dots, m$$

$$: x_j^{\min} \leq x_j \leq x_j^{\max}, \quad j = 1, \dots, n$$

$$: y_i \geq 0, \quad i = 1, \dots, m$$

$$: z \geq 0$$

➤ m, n 分别表示约束和设计变量的个数, x 表示设计变量向量, y 和 z 是正的优化变量。 f_0 表示目标函数, $f_1 \dots f_m$ 约束函数, a_i, c_i, d_i 为正常数。

MMA优化原模型

$$\min_{x,y,z} : f_0(x)$$

$$S.t. : f_i(x) \leq 0, \quad i = 1, \dots, m$$

$$: x_j^{\min} \leq x_j \leq x_j^{\max}, \quad j = 1, \dots, n$$

➤ 当优化变量y和z趋近于0的时候，MMA优化模型趋近于原始模型。Svanberg教授建议：

$$a_0 = 1, a_i = 0, c_i = 1000, d = 0$$

99行程序调用MMA

- 99行程序中调用MMA需要MMA的程序代码mma sub. m和subsol v. m
- 调用过程中需要提供灵敏度信息df0dx, df0dx2, dfdx, dfdx2。对应 $\frac{\partial f_0}{\partial x_j}, \frac{\partial^2 f_0}{\partial x_j^2}, \frac{\partial f_i}{\partial x_j}, \frac{\partial^2 f_i}{\partial x_j^2}$
- 由于拓扑优化问题的二阶导数很难获得, 调用MMA过程中仅需将df0dx2, dfdx2赋0值。

基于MMA程序的结构柔顺性优化程序实现

```
%%%% A 99 LINE TOPOLOGY OPTIMIZATION CODE BY OLE  
SIGMUND, JANUARY 2000 %%%  
%%%% CODE MODIFIED FOR INCREASED SPEED, September 2002,  
BY OLE SIGMUND %%%  
function top(nelx,nely,volfrac,penal,rmin);  
global time  
global cvec  
tic  
% INITIALIZE  
m = 1;  
n = nelx*nely;
```

初始化

```
x(1:nely, 1:nelx) = volfrac;  
xold1 = reshape(x, n, 1);  
xold2 = xold1;  
loop = 0;  
change = 1.;  
c = 1e6;  
xmin = 0.001;  
xmax = 1;  
dfdx2 = zeros(1, n);  
low = zeros(n, 1);  
upp = low;
```

初始化, 有限元分析

```
a0 = 1;  
a = 0;  
cmma = 1000;  
d = 0;  
move = 0.5;  
% START ITERATION  
while change > 0.01  
    loop = loop + 1;  
% FE-ANALYSIS  
    [U]=FE (nelx, nely, x, penal);
```

目标函数及敏度计算

% OBJECTIVE FUNCTION AND SENSITIVITY ANALYSIS

```
[KE] = 1k;
c = 0.;
dc = zeros(nely,nelx);
for j=1:1
    for ely = 1:nely
        for elx = 1:nelx
            n1 = (nely+1)*(elx-1)+ely;
            n2 = (nely+1)* elx    +ely;
            Ue = U([2*n1-1;2*n1; 2*n2-1;2*n2; 2*n2+1;2*n2+2;
2*n1+1;2*n1+2], j);
            c = c + x(ely,elx)^penal*Ue'*KE*Ue;
            dc(ely,elx) = dc(ely,elx) + -
penal*x(ely,elx)^(penal-1)*Ue'*KE*Ue;
        end
    end
end
end
```

最小柔顺性问题的伴随法灵敏度分析

$$\Phi = F^T U(\rho), \quad K(\rho)U = F$$

Adjoint method

$$\hat{\Phi} = \Phi + \lambda^T (KU - F)$$

$$\hat{\Phi}' = \frac{\partial \Phi}{\partial U} U' + \lambda^T (K'U + KU')$$

$$\left(\lambda^T K + \frac{\partial \hat{\Phi}}{\partial U} \right) U' = 0 \Rightarrow$$

$$K^T \lambda = - \left(\frac{\partial \hat{\Phi}}{\partial U} \right)^T \Rightarrow$$

$$\hat{\Phi}' = \lambda^T K'U$$

Compliance obj.

$$\left(\frac{\partial \hat{\Phi}}{\partial U} \right)^T = F \Rightarrow$$

$$K\lambda = -F \Rightarrow$$

$$\lambda = -U \Rightarrow$$

$$\hat{\Phi}' = -U^T K'U =$$

$$-\frac{p}{\rho_e} U_e^T K_e^0 U_e$$

$$K = \sum_e K_e$$
$$K_e = \rho_e^p K_e^0$$

滤波，MMA优化参数的准备

```
% FILTERING OF SENSITIVITIES
```

```
[dc] = check(nelx, nely, rmin, x, dc);
```

```
% MMA
```

```
x = reshape(x, n, 1);
```

```
xold1 = reshape(xold1, n, 1);
```

```
xold2 = reshape(xold2, n, 1);
```

```
dc = reshape(dc, n, 1);
```

```
df0dx2 = 0*dc;
```

```
fval = sum(x) / (volfrac*nelx*nely)-1;
```

```
dfdx = 1 / (volfrac*nelx*nely)*ones(1, n);
```

```
xmax = min(1, x+move);
```

```
xmin = max(0.001, x-move);
```


调用MMA

```
[xmma, ymma, zmma, lam, xsi, eta, mu, zet, s, low, upp] =  
mmasub(m, n, loop, x, xmin, xmax, xold1, xold2, ...  
c, dc, df0dx2, fval, dfdx, dfdx2, low, upp, a0, a, cmma, d);
```

```
change = max(max(abs(xmma-xold1)));  
disp([' It.: ' sprintf('%4i', loop) ' Obj.: '  
sprintf('%10.4f', c) ' Vol.: '  
sprintf('%6.3f', sum(sum(x))/(nelx*nely)) ' ch.: '  
sprintf('%6.3f', change )])
```

```
xold2 = xold1;  
xold1 = x;  
x = xmma;  
x = reshape(xmma, nely, nelx);  
% PLOT DENSITIES  
figure(1)  
colormap(gray); imagesc(-x); axis equal; axis tight;  
axis off; pause(1e-6);  
end
```

应力约束下的桁架拓扑优化

应力约束下桁架拓扑优化问题的提法

To Find $A_i, i = 1, 2, \dots, N$

$$\text{Min } W = \sum_{i=1}^N A_i \rho l_i$$

$$\text{St. } \sigma_{ij}^C \leq \sigma_{ij} \leq \sigma_{ij}^T, \quad A_i > 0,$$

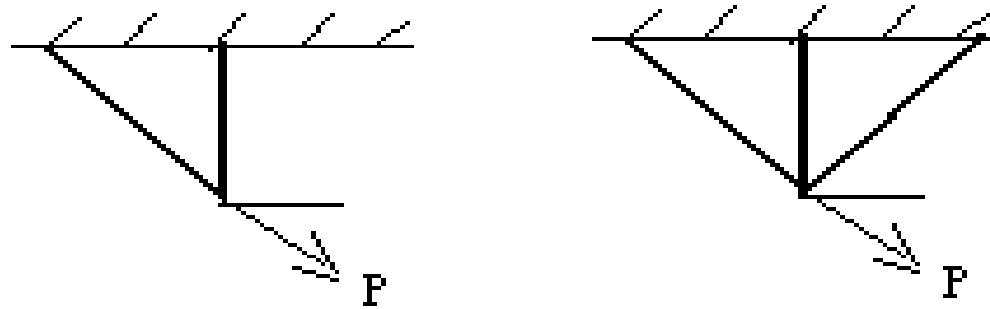
$$i = 1, 2, \dots, N; j = 1, 2, \dots, L$$

$$0 \leq A_i \leq A_i^U, \quad i = 1, 2, \dots, N$$

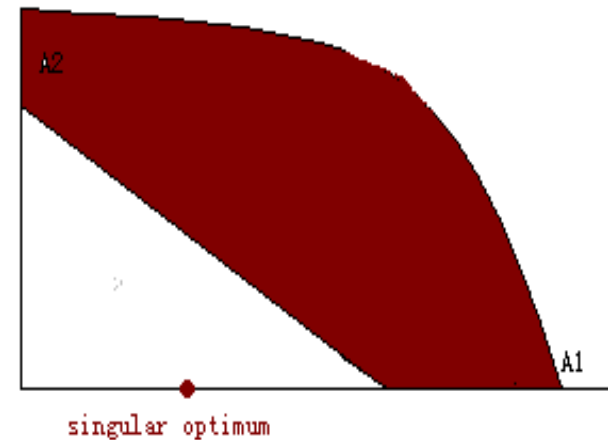
A_i 为杆件断面积, W 为桁架重量, ρ 为材料比重, l_i 为杆件长度, σ_{ij} 为杆件应力, 上标 C, T 分别为压缩和拉伸许用应力, U 为断面积上限

应力约束下的奇异最优解

奇异最优解最早是由Gino, Sved (1960) 在优化三杆桁架时发现

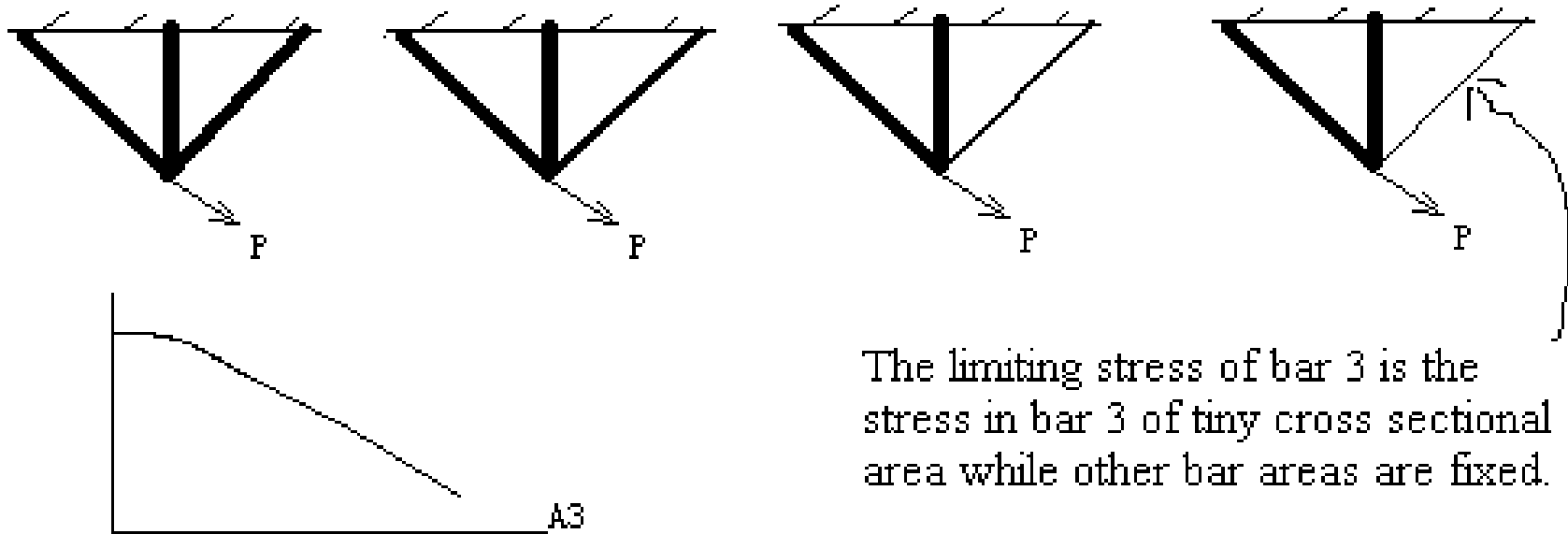


最小重量设计，三个荷载工况，全局最优解为 $W=12.0$ (8.0, 1.5, 0.0)，局部最优解为 $W=15.15$ (7.24, 2.135, 2.767)；从相当的初始解出发，用数学规划法无法得到全局最优解。



应力约束下的奇异最优解

奇异最优解产生的原因可用应力函数不连续，或应力约束是依赖于拓扑的约束来解释。



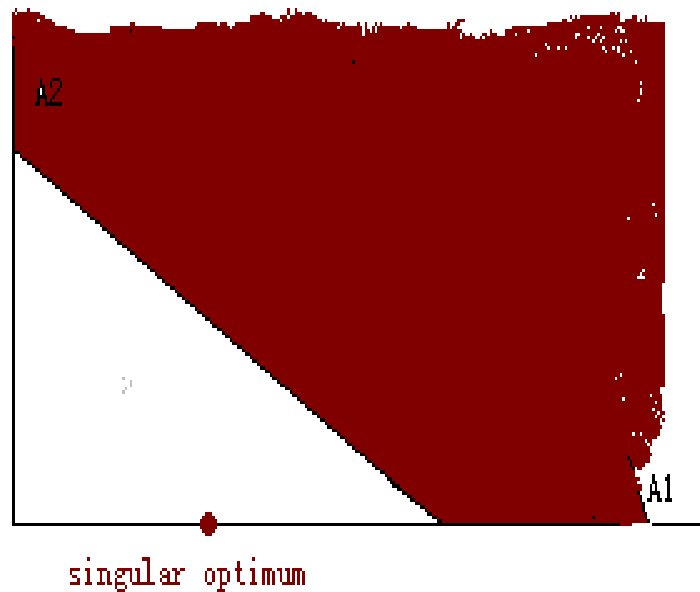
The limiting stress of bar 3 is the stress in bar 3 of tiny cross sectional area while other bar areas are fixed.

stress in bar 3

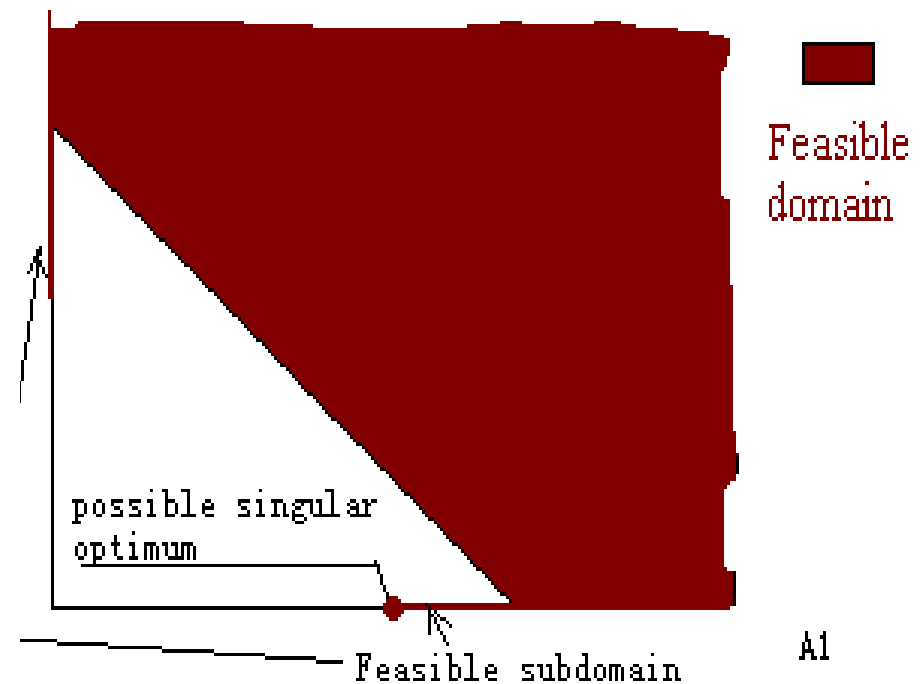
-

可行区的正确表示和错误表示

错误的图示



正确的图示



应力奇异最优解的 ε -松弛算法

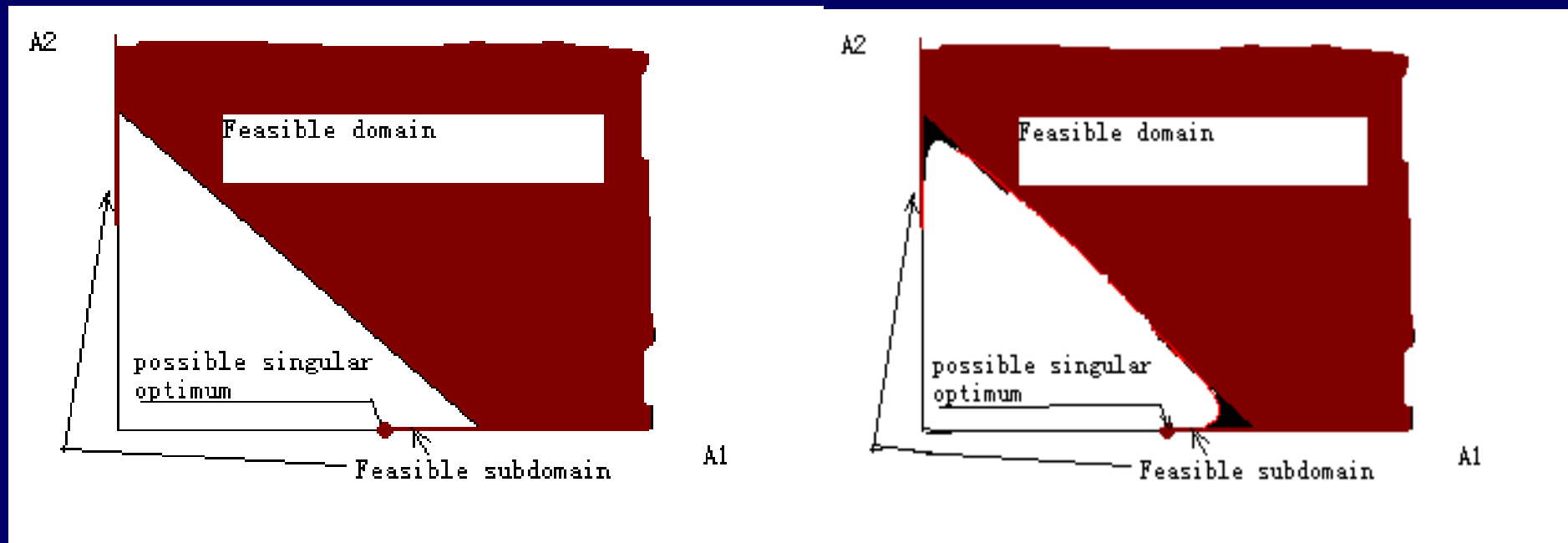
ε -relaxed approach(Cheng, Guo, 1997)

Cheng and Guo proposed a ε -relaxed approach in which the stress constraint is replaced by internal force constraints, and then internal force constraints were relaxed to eliminate possible singularities.

1. Replace $\sigma_i \leq \sigma_i^U$
by $A_i \sigma_i - A_i \sigma_i^U \leq 0$;
2. Relax $A_i \sigma_i - A_i \sigma_i^U \leq 0$
by $A_i \sigma_i - A_i \sigma_i^U \leq \varepsilon$

应力奇异最优解的 ε -松弛算法

ε -relaxed方法可使可行域的形状正则化



Before and after relaxation of constraints

应力奇异最优解的 ε -松弛算法

由点集映射原理，可以证明在如下两个结论：

1. 应力约束下的原始优化问题存在一个最优边界解 W_O^{opt}
2. 每一个迭代步中，可以获得

$$\{\varepsilon_k\}, \varepsilon_k > \varepsilon_{k+1} > 0 \text{ and } \lim_{k \rightarrow \infty} \varepsilon_k = 0$$

序列的变化对应的放松问题的全局最优解 $W_{\varepsilon_k}^{opt}$

这样：

$$\lim_{k \rightarrow \infty} W_{\varepsilon_k}^{opt} = W_O^{opt}$$

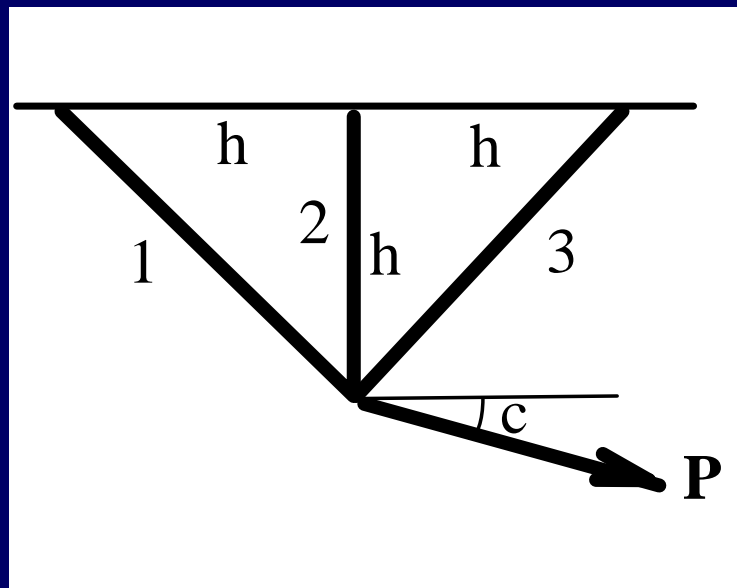
A three bar truss subjected to three load cases is optimized.

$E=1.0, h=1.0, \rho=1.0$.

The three loading cases are

(a) $P=40, c=45^\circ$; (b) $P=30, c=90^\circ$; (c) $P=20, c=135^\circ$.

The allowable stress are : $\sigma_{ia} = \pm 5.0, i=1,3$ $\sigma_{2a} = \pm 20.0$

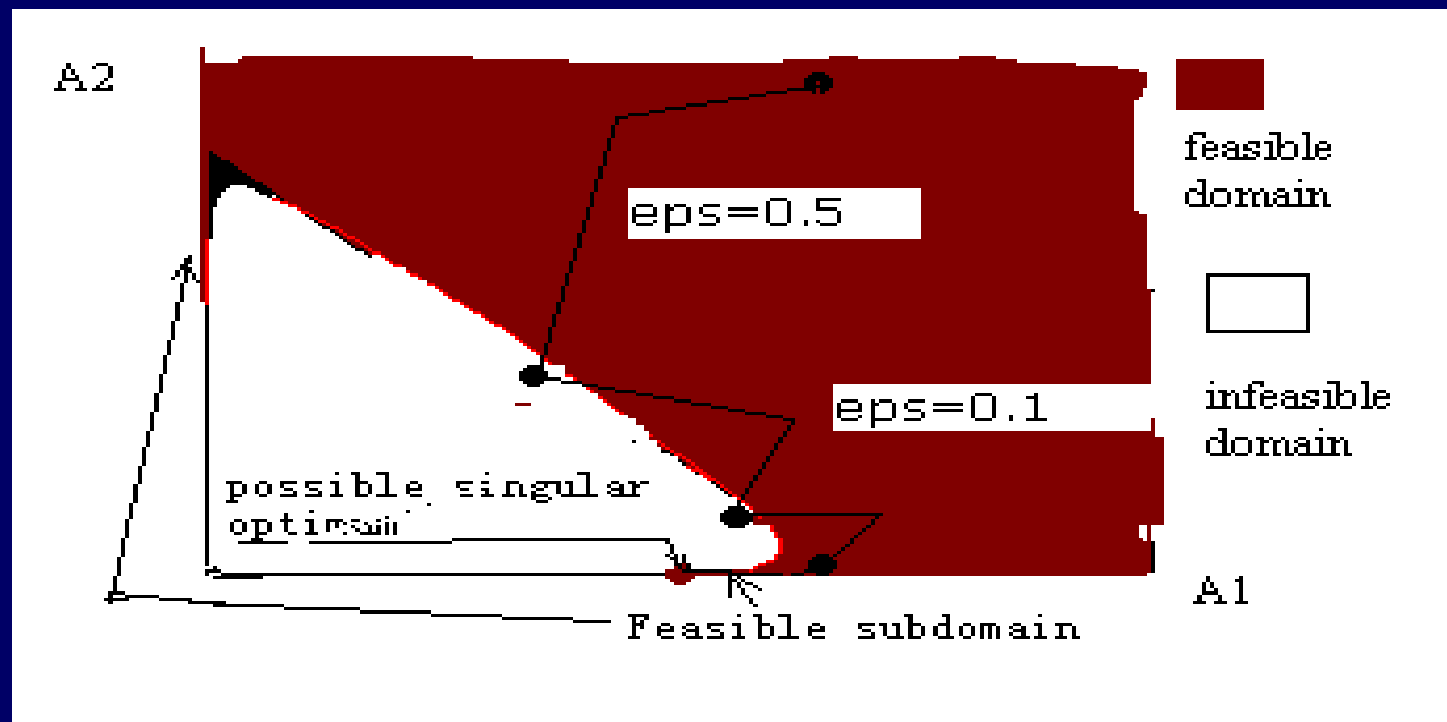


初始设计与不同优化列式所得优化结果的影响

Ini. Design			Relaxed Formulation				Stress
A1	A2	A3	A1	A2	A3	W	
8.0	2.4	3.2	7.024	2.135	2.767	15.97	W=15.97
0.0	12.0	18.0	0.000	2.828	8.000	14.14	W=15.97
0.7	4.0	10.0	0.000	2.828	8.000	14.14	W=15.97
1.0	1.0	1.0	7.024	2.135	2.767	15.97	W=15.97
10.0	10.0	10.0	7.024	2.135	2.767	15.97	W=15.97
1.0	1.0	0.1	8.00	1.50	0.00	12.81	W=15.97
10.0	10.0	0.1	8.00	1.50	0.00	12.81	W=15.97
Gl. Opt			8.00	1.50	0.00	12.81	

应力奇异最优解的 ε -松弛算法

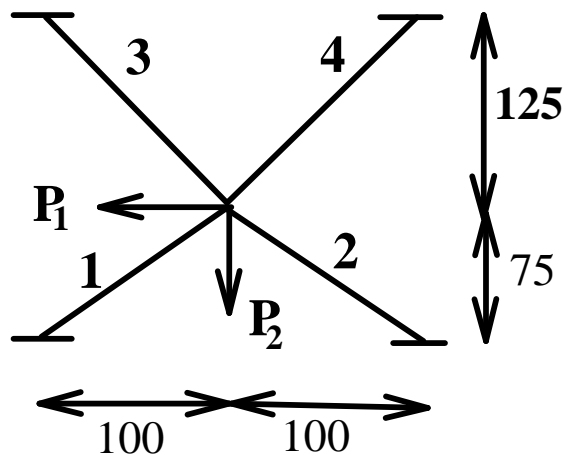
- ε -relaxed方法对原问题有较好的近似
- 但获得全局最优解仍然是困难的
- 初始解的选择对于获得全局最优解非常重要
- 为了拓展对全局最优解的吸引，进一步使用连续化策略改善 ε -relaxed算法。



Iteration history by Continuation method

应力奇异最优解的 ε -松弛算法

- 通过 ε -松弛延拓算法，可以有效解决应力约束下桁架拓扑优化奇异性困难，以桁架构件截面积为设计变量，将拓扑优化与尺寸优化统一实现。
- ε -松弛延拓算法可以显著的提高基于梯度的优化算法，从任意给定的均匀解收敛到全局最优解的概率。

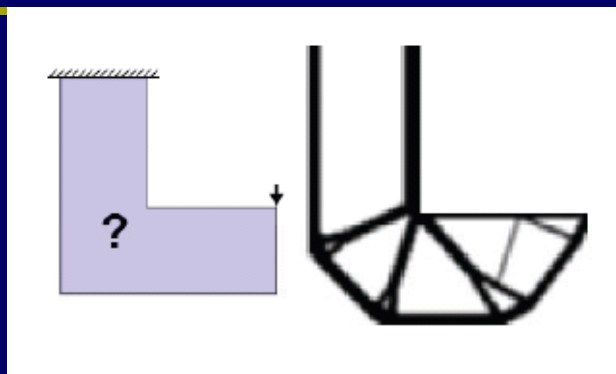


应用 ε -松弛和延拓算法在不同约束处理下求解四杆桁架的优化结果

SF-stress constraint; IF-internal force constraint;
uniform initial design

ε	A1	A2	A3	A4	W
2.0	0.0529	0.1451	0.1521	0.1592	79.39
1.5	0.0481	0.2083	0.2586	0.2021	105.79
1.0	0.0372	0.2757	0.3361	0.2483	132.66
0.5	0.0210	0.3480	0.4109	0.2969	159.42
0.1	0.0046	0.4115	0.4653	0.3373	180.48
0.01	0.0000	0.4274	0.4757	0.3471	185.15
0.001	0.0000	0.4286	0.4773	0.3476	185.62
Hoback	0.0000	0.5600	0.3980	0.3620	191.00
SF	1.0157	1.0115	0.4939	0.4886	410.69
IF	0.0000	1.2500	0.0000	2.1479	500.08

考虑应力约束的拓扑优化问题—连续体



$$\begin{aligned} \min_{\rho} : & \sum_{e=1}^N v_e \rho_e \\ \text{s.t.} : & (\sigma_e)_{VM} \leq \rho_e^p \sigma^*, \quad \text{if } \rho_e > 0, \quad e = 1, \dots, N \\ & : 0 \leq \rho \leq 1 \\ & : \mathbf{K}(\rho) \mathbf{U} = \mathbf{F} \end{aligned}$$

应力约束的 ε -relaxation方法

Original Constraint $(\sigma_e)_{VM} \leq \rho_e^p \sigma^*$ if $\rho_e > 0$

Rewritten $\left(\frac{(\sigma_e)_{VM}}{\rho_e^p \sigma^*} - 1 \right) \leq 0$ if $\rho_e > 0$

Reformulation $\rho_e \left(\frac{(\sigma_e)_{VM}}{\rho_e^p \sigma^*} - 1 \right) \leq 0$

应力约束的 ε -relaxation方法

Original Constraint $(\sigma_e)_{VM} \leq \rho_e^p \sigma^*$ if $\rho_e > 0$

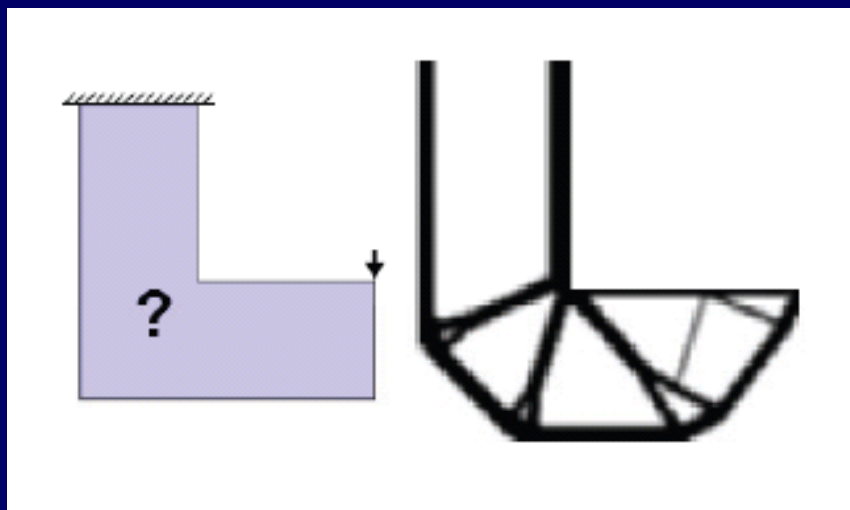
Rewritten $\left(\frac{(\sigma_e)_{VM}}{\rho_e^p \sigma^*} - 1 \right) \leq 0$ if $\rho_e > 0$

Reformulation $\rho_e \left(\frac{(\sigma_e)_{VM}}{\rho_e^p \sigma^*} - 1 \right) \leq 0$

Relaxed constraint, Cheng and Guo (1997):

$$\rho_e \left(\frac{(\sigma_e)_{VM}}{\rho_e^p \sigma^*} - 1 \right) \leq \varepsilon(1 - \rho_e), \quad \varepsilon^2 = \rho_{min} \leq \rho_e$$

最大应力最小的拓扑优化问题—连续体



$$\begin{aligned} \min_{\rho} : & \max_e(\sigma_e)_{VM} \\ \text{s.t.} : & \sum_{e=1}^N v_e \rho_e \leq V^* \\ & 0 \leq \rho \leq 1 \\ & \mathbf{K}(\rho) \mathbf{U} = \mathbf{F} \end{aligned}$$

最大应力最小的可微化处理

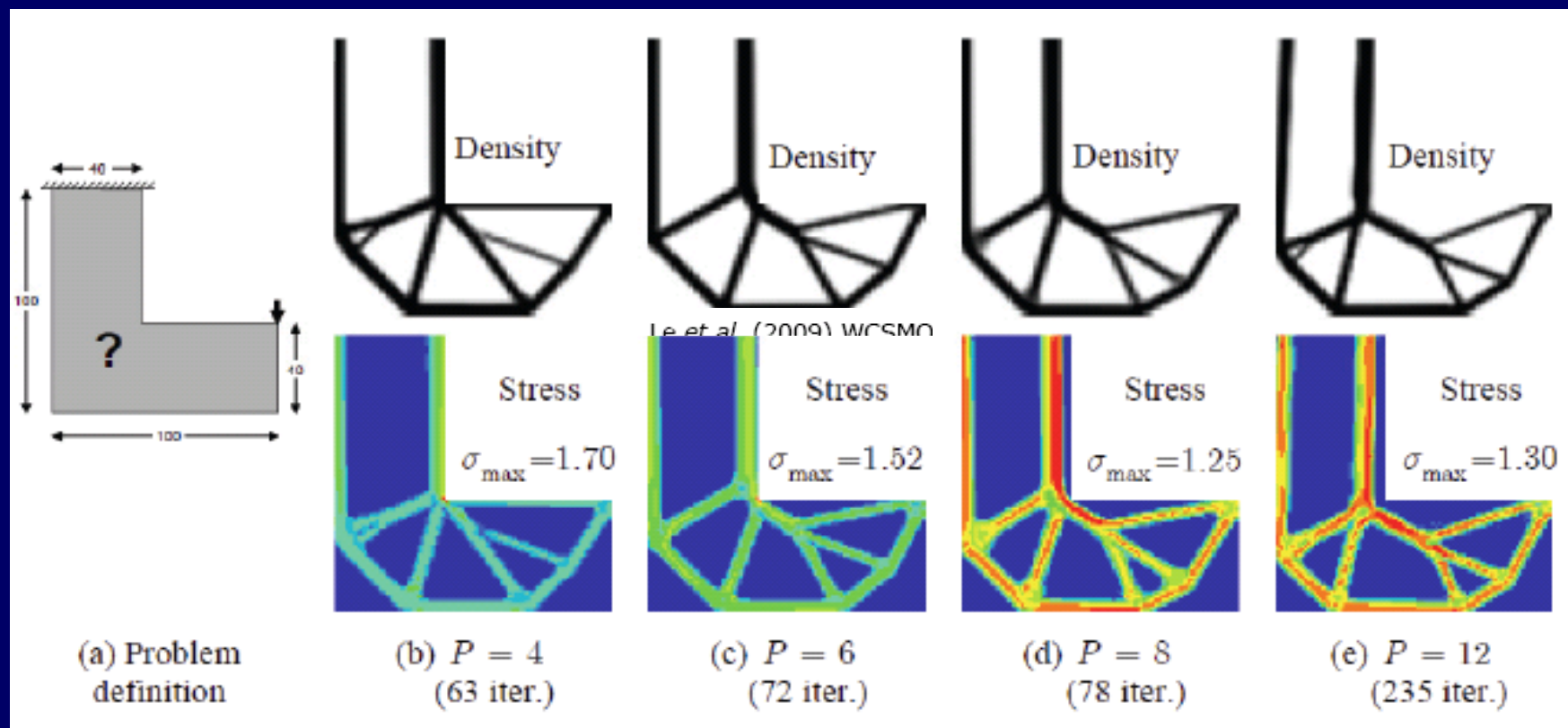
$$\begin{aligned} \min_{\rho} : & \max_e (\sigma_e)_{VM} \\ \text{s.t.} : & \sum_{e=1}^N v_e \rho_e \leq V^* \\ & : 0 \leq \rho \leq 1 \\ & : \mathbf{K}(\rho) \mathbf{U} = \mathbf{F} \end{aligned}$$

Multi objectives=>
multiple constraints

$$\begin{aligned} \min_{\rho} : & \left(\sum_{e=1}^N \sigma_e^q \right)^{1/q} \\ \text{s.t.} : & \sum_{e=1}^N v_e \rho_e \leq V^* \\ & : 0 \leq \rho \leq 1 \\ & : \mathbf{K}(\rho) \mathbf{U} = \mathbf{F} \end{aligned}$$

One objective

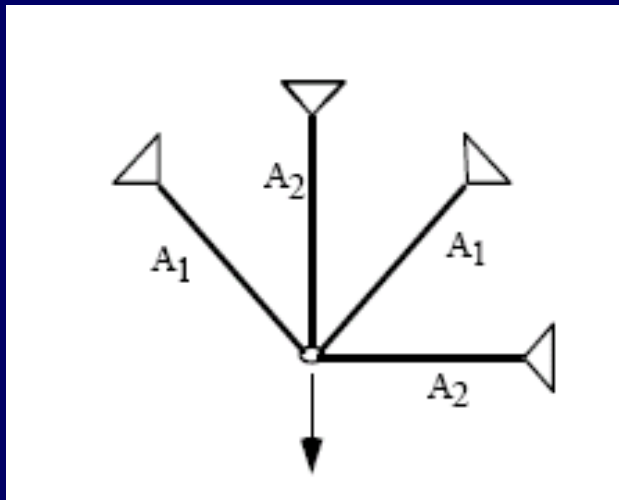
应力优化结果



Le et al. (2009) WCSMO

THE END

应力约束的奇异最优解



$$\min 4A_1 + 2A_2$$

$$\text{st. : } A_1 + A_2 - 5\sqrt{2} \geq 0 \quad \text{if } A_1 \neq 0$$

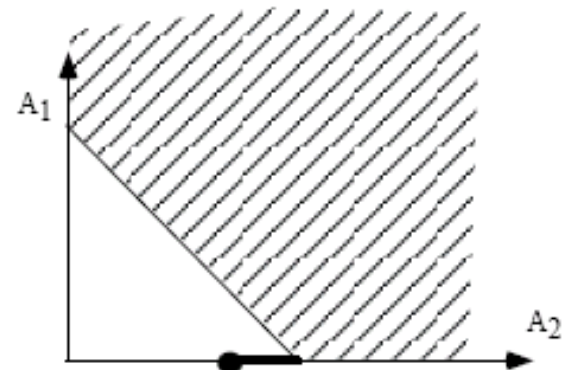
$$A_1 + A_2 - 5 \geq 0 \quad \text{if } A_2 \neq 0$$

$$A_1 \geq 0, \quad A_2 \geq 0$$

$$E = 1, \quad P = 1$$

$$\rho_1 = 2, \quad \rho_2 = 1$$

$$\sigma_1^Y = \frac{1}{5\sqrt{2}}, \quad \sigma_2^Y = \frac{1}{5}$$



应力约束的 ε -relaxation 方法

