Cryptography

Encryption Schemes and Definitions of Security

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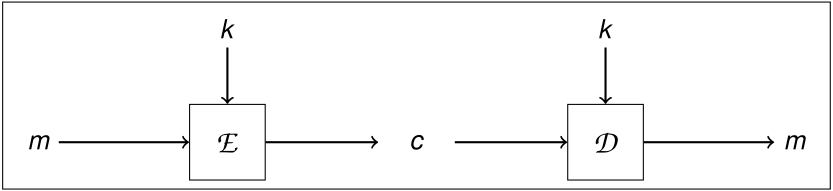
Encryption

An *encryption scheme* consists of algorithms which produce keys and transform plaintext into ciphertext and conversely.



Encryption and decryption algorithms.

The encryption algorithm is either *deterministic* or *randomized*, i.e., the ciphertext may depend on random input data.

Definition of Encryption

Definition

An *encryption scheme* or *cryptosystem* consists of

- A plaintext space $\mathcal{M}$, the set of plaintext or clear-text messages,
- A ciphertext space $\mathcal{C}$, the set of ciphertext messages,
- A key space $\mathcal{K}$, the set of keys,
- A randomized key generation algorithm $Gen(1^n)$ that takes the security parameter n as input and returns a key $k \in \mathcal{K}$,
- An encryption algorithm $\mathcal{E} = \{\mathcal{E}_k \mid k \in \mathcal{K}\}$, which is possibly randomized.
- A deterministic decryption algorithm $\mathcal{D} = \{\mathcal{D}_k \mid k \in \mathcal{K}\}$. An error symbol $\perp$ is returned if the ciphertext is invalid.

Definition of Encryption

We require that all algorithms (key generation, encryption, decryption) are *polynomial* with respect to the input size.

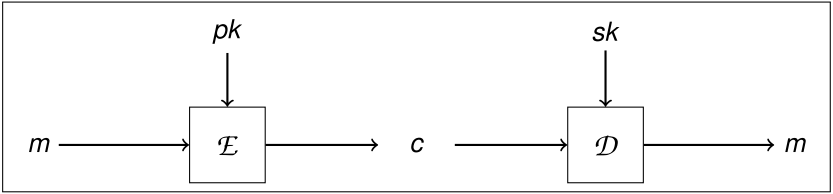
The scheme provides *correct decryption*, if for each key $k \in \mathcal{K}$ and all plaintexts $m \in \mathcal{M}$, one has

$$\mathcal{D}_k(\mathcal{E}_k(m)) = m.$$

Symmetric and Asymmetric Encryption Schemes

A scheme is said to be *symmetric-key* if encryption and decryption use the same secret key k. In contrast, *public-key* (*asymmetric-key*) encryption schemes use *key pairs* $k = (pk, sk)$, where pk is public and sk is private; encryption uses pk and decryption sk.

We discuss symmetric-key schemes first and later deal with public-key schemes.



Asymmetric encryption and decryption.

Kerkhoff's Principle

The security of a cryptographic scheme should be solely based on a *secret key*, not on the details of the system.

Definition (Kerkhoff's Principle)

A cryptosystem should be secure under the assumption that an attacker knows the encryption and decryption algorithms.

One-Time Pad

The *one-time pad* is an example of a simple but very powerful fixed-length symmetric encryption scheme. It uses the binary alphabet and the key length is equal to the message length. The security parameter n defines the length of plaintexts, ciphertexts and keys:

$$\mathcal{M} = \mathcal{C} = \mathcal{K} = \{0,1\}^n$$

The key generation algorithm $Gen(1^n)$ outputs a uniform random key $k \stackrel{\$}{\leftarrow} \{0,1\}^n$. A key k of length n is *used only for one message* $m \in \{0,1\}^n$. Encryption $\mathcal{E}_k$ and decryption $\mathcal{D}_k$ are identical and defined by a simple vectorial XOR operation:

$$c = \mathcal{E}_k(m) = m \oplus k \qquad m = \mathcal{D}_k(c) = c \oplus k$$

Vigenère Cipher

The *Vigenère cipher* of (key) length n is a classical example of a symmetric variable-length scheme over the alphabet of letters.

$$\mathcal{M} = \mathcal{C} = \Sigma^* \text{ and } \mathcal{K} = \Sigma^n, \text{ where } \Sigma = \{A, B, \ldots, Z\} \cong \mathbb{Z}_{26}.$$

$Gen(1^n)$ generates a uniform random key string $k \xleftarrow{\$} \Sigma^n$ of length n.

For encryption and decryption, the message and the ciphertext is split into blocks of length n; the last block can be shorter. Encryption adds the key to each plaintext block, decryption subtracts the key.

$$c = \mathcal{E}_k(m) = \mathcal{E}_k(m_1 \| m_2 \| \ldots) = (m_1 + k \parallel m_2 + k \parallel \ldots) \mod 26$$
$$m = \mathcal{D}_k(c) = \mathcal{D}_k(c_1 \| c_2 \| \ldots) = (c_1 - k \parallel c_2 - k \parallel \ldots) \mod 26$$

Definition of Perfect Secrecy

One might want a cryptosystem that is *unbreakable* for any type of adversary, even if they have unlimited computing power. Claude Shannon defined the notion of *perfect secrecy*.

Definition

An encryption scheme is called *perfectly secret* if for all plaintexts $m_0,\ m_1 \in \mathcal{M}$ and all ciphertexts $c \in \mathcal{C}$:

$$Pr[\mathcal{E}_k(m_0) = c\,] = Pr[\mathcal{E}_k(m_1) = c\,]$$

The probabilities are computed over randomly generated keys $k \in \mathcal{K}$.

Perfect secrecy means that all plaintexts have the same probability for a given ciphertext. This is also called *perfect indistinguishability*.

One-Time Pad

Perfect secrecy is a very strong condition. However, it is achieved by the one-time-pad.

Theorem

The one-time pad is perfectly secret if the key is generated by a random bit generator and used only once.

In fact, given a ciphertext $c \in \{0, 1\}^n$, any plaintext m is possible and equally likely:

$$Pr[\mathcal{E}_k(m) = c] = \frac{1}{2^n}.$$

Example

A Vigenère cipher of key length 3 is perfectly secret for messages of length 3, if the key is only used once.

However, the same cipher is not perfectly secure for messages of length 4 (Exercise).

Computational Security

Perfect secrecy is too strong for most applications. Practical security should take the computing power of an adversary into account.

Definition

A scheme is (t, ε)-secure if any adversary running in time t (measured in CPU cycles) can break the scheme with probability of ε at most.

We consider a scheme to be *broken*, if an adversary can learn something about the plaintext from the ciphertext, i.e., if they obtain the output of any function of the plaintext, for example a plaintext bit or a sum of several plaintext bits.

Example of Computational Security

Assume the best-known attack against a scheme is exhaustive key search (brute force) and the key has length n. If testing a single key takes c CPU cycles and in total N CPU cycles are executed, then $\frac{N}{c}$ keys can be tested and the probability of success is approximately $\frac{N}{c2^n}$, if $\frac{N}{c} \ll 2^n$. Hence the scheme is $(N, \frac{N}{c2^n})$-secure.

Example: An adversary uses a computer with one 2 GHz CPU and performs a brute force attack against a scheme with 128-bit key length over the course of a year. Let's assume that $c = 1$. Then roughly 2^{55} keys can be tested and the scheme is $(2^{55}, 2^{-73})$-secure.

Note: an event with a probability of 2^{-73} will never occur in practice.

Asymptotic Definition

We give an asymptotic definition with respect to the security parameter.

Definition

An encryption scheme is called *computational secure*, if every probabilistic algorithm with polynomial running time can only break the scheme with *negligible probability* in the security parameter n.

Example: Suppose the best possible attack is a brute-force search of a key of length n and the running time of attackers is bounded by $N = p(n)$, where p is a polynomial.
Then the scheme is $(p(n), \frac{p(n)}{c \cdot 2^n})$-secure, where c is a constant. The probability $\frac{p(n)}{c \cdot 2^n}$ is *negligible*. The scheme is computationally secure.

Security under Different Types of Attacks

We now define security in well-defined experiments: suppose an adversary chooses two plaintexts and a challenger encrypts one of them. Can a polynomial-time adversary (who does not know the secret key) find the correct plaintext from the given ciphertext? *Indistinguishability* (IND) means that efficient adversaries are unable to find the correct plaintext out of two possibilities better than guessing at random.

We specify a *threat model* and consider attackers with certain capabilities:

- *Eavesdropping Attack* (EAV) Attack: the adversary gets the ciphertext.
- *Chosen Plaintext Attack* (CPA): the adversary can additionally choose plaintexts and gets the ciphertext.
- *Chosen Ciphertext Attack* (CCA): the adversary can additionally choose ciphertexts and gets the plaintext.

EAV Experiment

Definition (EAV Indistinguishability Experiment)

A challenger takes the security parameter 1^n as input, generates a key $k \in \mathcal{K}$ by running $Gen(1^n)$ and chooses a random bit $b \stackrel{\$}{\leftarrow} \{0, 1\}$. A probabilistic polynomial-time adversary A is given 1^n, but neither k nor b are not known to A. The adversary chooses two plaintexts m_0 and m_1 that are equal in length. The challenger returns the ciphertext $\mathcal{E}_k(m_b)$ of one of them. A tries to guess b and outputs a bit b'. The challenger outputs 1 (success) if $b = b'$, and 0 (failure) otherwise. The *EAV advantage* of A is defined as

$$\mathrm{Adv}^{\text{ind-eav}}(A) = \left| \, Pr[b' = b] - Pr[b' \neq b] \, \right| .$$

The probability is taken over all random variables in this experiment, i.e., the key k, bit b, encryption $\mathcal{E}_k$ and randomness of A.

EAV Experiment

Adversary		*Challenger*
	$\xleftarrow{\quad 1^n \quad}$	$k \xleftarrow{\$} Gen(1^n)$
		$b \xleftarrow{\$} \{0,1\}$
Choose m_0, m_1 $\lvert m_0 \rvert = \lvert m_1 \rvert$	$\xrightarrow{\quad m_0, m_1 \quad}$	
	$\xleftarrow{\quad c \quad}$	$c \leftarrow \mathcal{E}_k(m_b)$
Select m_0 $(b' = 0)$ or m_1 $(b' = 1)$	$\xrightarrow{\quad b' \quad}$	Compare b and b', output 1 or 0

Indistinguishability experiment in the presence of an eavesdropper.

EAV Security

Definition

An encryption scheme has *indistinguishable encryptions in the presence of an eavesdropper* (IND-EAV secure or EAV-secure), if for every probabilistic polynomial-time adversary A, the advantage $\mathrm{Adv}^{\text{ind-eav}}(A)$ is negligible in the security parameter n.

Example: Suppose a scheme does not encrypt the first bit. Then the scheme does not have EAV security: an adversary could choose two plaintexts that differ in their first bit. Then they can find the correct plaintext from the ciphertext and the EAV advantage is 1.

CPA Experiment

The CPA (Chosen Plaintext Attack) experiment gives an adversary more power than in the EAV experiment: they can freely choose plaintexts and get the corresponding ciphertexts from the challenger. This might help to decrypt a challenge ciphertext, at least if there are only two plaintext candidates m_0 and m_1. They may even ask the challenger to encrypt m_0 and m_1.

Implication: CPA security is stronger than EAV security and *deterministic* encryption schemes cannot be CPA-secure!

CPA Experiment

Adversary		*Challenger*
	$\xleftarrow{1^n}$	$k \xleftarrow{\$} Gen(1^n)$ $b \xleftarrow{\$} \{0,1\}$
Choose plaintext m'	$\xrightarrow{m'}$	
	$\xleftarrow{c'}$	$c' \leftarrow \mathcal{E}_k(m')$
Choose m_0, m_1	$\xrightarrow{m_0,\, m_1}$	
$\lvert m_0 \rvert = \lvert m_1 \rvert$	$\xleftarrow{c}$	$c \leftarrow \mathcal{E}_k(m_b)$
Choose plaintext m''	$\xrightarrow{m''}$	
	$\xleftarrow{c''}$	$c'' \leftarrow \mathcal{E}_k(m'')$
Select m_0 $(b' = 0)$ or m_1 $(b' = 1)$	$\xrightarrow{b'}$	Compare b and b', output 1 or 0

CPA Security

Definition

A scheme has *indistinguishable encryptions under chosen plaintext attack* (IND-CPA secure or CPA-secure), if for every probabilistic polynomial-time adversary A in the CPA experiment, the advantage

$$\mathrm{Adv}^{\text{ind-cpa}}(A) = |\ Pr[b' = b] - Pr[b' \neq b]\ |$$

is negligible in n.

CPA security is a key requirement for modern encryption schemes. However, it is hard to achieve! We will later see that secure block ciphers in certain operation modes can achieve this level of security.

CCA Security

The CCA2 (Chosen Ciphertext Attack) experiment is similar to the CPA experiment, but the adversary can also ask for the *decryption* of chosen ciphertexts (before and after receiving the challenge ciphertext, but except the challenge itself).

Definition

A scheme has *indistinguishable encryptions under chosen ciphertext attack* (IND-CCA2 secure or CCA2-secure), if for every probabilistic polynomial-time adversary A in the CCA2 experiment, the advantage

$$\mathrm{Adv}^{\text{ind-cca}}(A) = |\, Pr[b' = b] - Pr[b' \neq b]\, |$$

is negligible in n.

CCA2 security is significantly stronger than CPA security and can be achieved by combining a CPA-secure scheme and a secure MAC.

Constructing Secure Encryption Schemes

For the construction of secure encryption schemes, we use cryptographic *primitives* as building blocks.

- Pseudorandom generators (*prg*) take a seed (or key) as input and output a sequence of randomly looking bits. They are building block for stream ciphers, which use a pseudorandom keystream for encryption.
- Pseudorandom functions or permutations (*prf*, *prp*) are function families which are parametrized by a key. They transform input blocks into output blocks such that the mapping looks randomly. Block ciphers can be combined with an operation mode and define encryption schemes.

We will see that the security of encryption schemes can often be reduced to the security (pseudorandomness) of the underlying primitive.

Pseudorandom Generators

Randomness and pseudorandomness play a fundamental role in cryptography, especially for encryption, but also for other cryptographic operations. A pseudorandom string *looks like* a uniform binary string without being the output of a true random bit generator.

A *bit generator* G is a deterministic polynomial-time algorithm that takes an input seed $s \in \{0,1\}^n$ and outputs a string $G(s) \in \{0,1\}^{l(n)}$, where $l(.)$ is a polynomial and $l(n) > n$ for all $n \in \mathbb{N}$.

G is called a *pseudorandom generator* (prg), if the output cannot be distinguished from a uniform random sequence in polynomial time.

Pseudorandom Generator (prg) Experiment

The adversary has to distinguish between a true random sequence r and the output of the generator G.

Adversary		*Challenger*
	$\xleftarrow{\quad 1^n \quad}$	$s \xleftarrow{\$} \{0,1\}^n,\ b \xleftarrow{\$} \{0,1\}$
	$\xleftarrow{\quad c \quad}$	$r \xleftarrow{\$} \{0,1\}^{l(n)},\ c = \begin{cases} G(s) & \text{if } b = 1 \\ r & \text{if } b = 0 \end{cases}$
Distinguish	$\xrightarrow{\quad b' \quad}$	Compare b and b', output 1 or 0

Distinguishability experiment for a bit generator G.

Secure Pseudorandom Generators

Definition

A generator G is called a *pseudorandom generator*, if for all probabilistic polynomial-time distinguisher A in the prg experiment, the prg-advantage $\mathrm{Adv}^{\mathrm{prg}}(A) = |\, Pr[b' = b] - Pr[b' \neq b]\,|$ is negligible in the security parameter n.

The construction of a pseudorandom generator (prg) is a non-trivial task! The output of a *prg* can be used as *keystream*. Define a fixed-length encryption scheme by

$$c = \mathcal{E}_k(m) = m \oplus G(k) \quad \text{and} \quad m = \mathcal{D}_k(c) = c \oplus G(k).$$

Theorem

If G is a pseudorandom generator, then the associated encryption scheme is EAV-secure.

Pseudorandom Functions

Consider a family of functions

$$F: \ K_n \times D_n \to R_n,$$

where n is a security parameter and $K_n = \{0,1\}^n$ the set of keys. To simplify the notation, we assume that $D_n = R_n = \{0,1\}^l$ where $l = l(n)$ is polynomial in n. We suppose that F can be computed in *polynomial time*. Fixing $k \in K_n$ yields a function

$$F_k : D_n \to R_n.$$

F is called *pseudorandom* if the functions $F_k : D_n \to R_n$ *appear to be random* for a polynomial-time adversary who knows the input-output behavior of F_k, but is not given the secret key k. In other words, a polynomial-time adversary is not able to distinguish a pseudorandom function from a true random function.

Pseudorandom Function (prf) Experiment

Adversary		*Challenger*
		$k \xleftarrow{\$} \{0,1\}^n,\ b \xleftarrow{\$} \{0,1\}$
	$\xleftarrow{\quad 1^n \quad}$	$f = \begin{cases} F_k & \text{if } b = 1 \\ \text{random} & \text{if } b = 0 \end{cases}$
Choose m	$\xrightarrow{\quad m \quad}$	
	$\xleftarrow{\quad c \quad}$	$c = f(m)$
	$\vdots$	
Distinguish	$\xrightarrow{\quad b' \quad}$	Compare b and b', output 1 or 0

Distinguishability experiment for a function family.

Secure Pseudorandom Functions

Definition

A keyed function family F as described above is called a *pseudorandom function* (prf) if for every probabilistic polynomial time adversary A in the prf-experiment, the prf-advantage $\mathrm{Adv}^{\mathrm{prf}}(A) = |\, Pr[b' = b] - Pr[b' \neq b] \,|$ is negligible in n.

A pseudorandom generator G can be derived from a pseudorandom function F by the following construction: choose a uniform random counter *ctr* and set

$$G(k) = F_k(ctr + 1) \parallel F_k(ctr + 2) \parallel F_k(ctr + 3) \parallel \ldots,$$

where *ctr* is viewed as an integer and addition is done modulo 2^l.

Pseudorandom Permutations

Pseudorandom permutations are an important special case of pseudorandom functions. They are given by a keyed family of *permutations*:

$$F : K_n \times D_n \rightarrow D_n$$

where $K_n = \{0, 1\}^n$ and $D_n = \{0, 1\}^l$. For any $k \in K_n$, the function $F_k : D_n \rightarrow D_n$ is a permutation, i.e., F_k is bijective.

A family F of permutations is called *pseudorandom*, if polynomial-time adversaries are not able to distinguish F from a truly random permutation.

Modern block ciphers (for example AES) are modeled to be pseudorandom permutations (prp). We will see that they are a main building block in the construction of secure encryption schemes.

Block Cipher

Definition

A family of permutations

$$E = E(n): \{0,1\}^n \times \{0,1\}^l \to \{0,1\}^l$$

is said to be a *block cipher*. n is the security parameter, $\{0,1\}^n$ is the key space, n is the *key length* and $l = l(n)$ is the *block length*.

Block ciphers can be thought of as concrete instances of families of *pseudorandom permutations*. Each key $k \in \{0,1\}^n$ yields a bijective function

$$E_k: \{0,1\}^l \to \{0,1\}^l.$$

Operation Modes

Block ciphers can only encrypt messages of block length l and an *operation mode* is needed to encrypt plaintexts of arbitrary length. The combination of a block cipher and an operation mode defines an encryption scheme with plaintexts and ciphertexts in $\{0,1\}^*$.

Here we discuss the following operation modes:

- ECB (Electronic Codebook Mode)
- CBC (Cipher Block Chaining Mode)
- CTR (Counter Mode)

Other operation modes exist, e.g., OFB, CFB and GCM, which will be dealt with later.

ECB Mode

Definition (ECB Mode)

Let E be a block cipher. We define an encryption scheme based on E. The key generation algorithm $Gen(1^n)$ outputs a uniform random key $k \xleftarrow{\$} \{0,1\}^n$ of length n. For encryption, set $m = m_1 \| m_2 \| \dots \| m_N$, where each m_i is a block of length l and the last message is padded by $10 \dots 0$. Define

$$c_i = E_k(m_i) \text{ for } i = 1, 2, \dots, N \text{ and } c = \mathcal{E}_k(m) = c_1 \| c_2 \| \dots \| c_N.$$

Decryption works in a similar way:

$$m_i = E_k^{-1}(c_i) \text{ for } i = 1, 2, \dots, N \text{ and } m = \mathcal{D}_k(c) = m_1 \| m_2 \| \dots \| m_N.$$

CBC Mode

Definition (Randomized CBC Mode)

The key generation and splitting of messages into blocks is the same as for the ECB mode. The CBC mode requires an initialization vector $IV \xleftarrow{\$} \{0,1\}^l$, which is chosen uniformly at random for each message. Then encrypt a message m by computing

$$c_0 = IV,\ c_i = E_k(m_i \oplus c_{i-1}) \text{ for } i = 1, 2, \ldots, N. \text{ Set}$$
$$c = \mathcal{E}_k(m) = c_0 \| c_1 \| c_2 \| \ldots \| c_N.$$

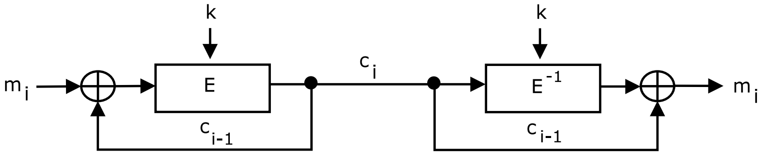
Decryption of a ciphertext c is defined by

$$m_i = E_k^{-1}(c_i) \oplus c_{i-1} \text{ for } i = 1, 2, \ldots, N. \text{ Then}$$
$$m = \mathcal{D}_k(c) = m_1 \| m_2 \| \ldots \| m_N.$$

CBC Mode

We can easily verify that the CBC mode has correct decryption:

$$E_k^{-1}(c_i) \oplus c_{i-1} = E_k^{-1}(E_k(m_i \oplus c_{i-1})) \oplus c_{i-1} = m_i \oplus c_{i-1} \oplus c_{i-1} = m_i$$



Encryption and decryption in CBC mode.

CTR Mode

Definition (Randomized CTR Mode)

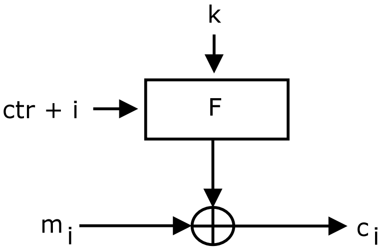
Let F be family of functions or permutations. We define an encryption scheme based on F. The key generation algorithm $Gen(1^n)$ outputs a uniform random key $k \xleftarrow{\$} \{0,1\}^n$ of length n. A plaintext message m is split into blocks of length l, i.e., $m = m_1 \| m_2 \| \dots \| m_N$; the last block can be shorter. A uniform random counter $ctr \xleftarrow{\$} \{0,1\}^l$ is chosen and incremented for each block. The encrypted counter gives keystream:

$$c_i = F_k(ctr + i) \oplus m_i \text{ for } i = 1, 2, \dots, N \text{ and}$$
$$c = \mathcal{E}_k(m) = ctr \| c_1 \| c_2 \| \dots \| c_N.$$

Decryption uses the same keystream:

$$m_i = F_k(ctr + i) \oplus c_i \text{ for } i = 1, 2, \dots, N \text{ and}$$
$$m = \mathcal{D}_k(c) = m_1 \| m_2 \| \dots \| m_N.$$

CTR Mode



Encryption in CTR mode. The keystream is XORed with the plaintext. Decryption is almost identical, with plaintext and ciphertext swapped.

Security of the CTR Mode

Theorem

If F is a pseudorandom function (prf or prp), then the randomized CTR mode has indistinguishable encryptions under a chosen-plaintext attack, i.e., the encryption scheme is IND-CPA secure.

Proof (idea): We have to show that any adversary A in the CPA experiment achieves only a negligible advantage. Since F is a *prf*, A cannot distinguish between the keystream $F_k(crt+1), F_k(crt+2), \dots$ and a true random sequence, if the counter values are fresh. Then CTR mode encryption is essentially a *one-time-pad* and A's advantage is zero. However, A can ask for the encryption of chosen plaintexts. If counter values of A's queries overlap with counters used for the challenge ciphertext, then A can decrypt a block of the challenge ciphertext and win the experiment. But a careful analysis shows that the probability of an overlap and hence $\mathrm{Adv}^{\text{ind-cpa}}(A)$ is negligible.

Security of the CBC Mode

The CBC mode provides a similar level of security as the CTR mode.

Theorem

If E is a pseudorandom permutation (prp), then the randomized CBC mode is IND-CPA secure.

This type of security guarantee can easily be misunderstood.

- CBC and CTR mode security is not unconditional, but rather depends on the underlying *prp* or *prf*.
- Only certain type of attacks are considered, since we made assumptions about the setup and the adversary (uniform random secret keys, a polynomial-time adversary who is only looking at plaintexts and ciphertexts, e.g., no side-channel attacks).
- The security guarantee is only asymptotic, i.e., for large n.

CCA Security and Malleability

It is easy to see that neither CBC nor CTR mode yields a CCA2-secure encryption scheme. Security against chosen ciphertext attacks requires *non-malleability*; a controlled modification of the ciphertext should be impossible and the result of decrypting a modified ciphertext should not be related to the original plaintext.

Example: Consider the CTR mode. If a single bit of the ciphertext is flipped, then only the corresponding plaintext bit changes. Hence the CTR mode is clearly malleable.

CCA2 security can be achieved by adding a MAC (*Message Authentication Code*). This is, for example, implemented by the Galois Counter Mode (GCM) which we shall deal with later.