Cryptography

Fundamentals

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Mathematical Fundamentals

Modern cryptography relies on mathematical structures and methods.

We briefly discuss a number of fundamental topics from discrete mathematics, elementary number theory, computational complexity and probability theory.

Algebraic structures are covered in a separate chapter.

Sets

Sets are the most elementary mathematical structure. Finite sets play an important role in cryptography.

Example: $M = \{0, 1\}^{128}$ is the set of binary strings of length 128. Elements in M can be written in the form $b_1 b_2 \ldots b_{128}$ or

$$(b_1, b_2, \ldots, b_{128})$$

in vectorial notation. An element of M could, for example, represent one block of plaintext or ciphertext data. The cardinality of M is very large:

$$|M| = 2^{128} \approx 3.4 \cdot 10^{38}$$

Small and Large Numbers

It is important to help understand the difference between small, big and inaccessible numbers in practical computations. For example, one can easily store one terabyte (10^{12} bytes, i.e., around 2^{43} bits) of data. On the other hand, a large amount of resources are required to store one exabyte (one million terabytes) or 2^{63} bits and more than 2^{100} bits are out of reach.

The number of computing steps is also bounded: less than 2^{40} steps (say CPU clocks) are easily possible, 2^{60} operations require a lot of computing resources and take a significant amount of time, and more than 2^{100} operations are unfeasible. It is for example impossible to test 2^{128} different keys with conventional (non-quantum) computers.

Functions

Definition

A *function* or a *map*

$$f : X \to Y$$

consists of two sets (the *domain* X and the *codomain* Y) and a rule which assigns an output element (an *image*) $y = f(x) \in Y$ to each input element $x \in X$. The set of all $f(x)$ is a subset of Y called the *range* or *image* $im(f)$. Any $x \in X$ with $f(x) = y$ is called a *preimage* of y. Let $B \subset Y$, then we say that

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

is the *preimage* or *inverse image* of B under f.

Injective, Surjective and Bijective Maps

Definition

Let $f : X \to Y$ be a function.

- f is *injective* if different elements of the domain map to different elements of the range: for all $x_1,\ x_2 \in X$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. Equivalently, f is injective if for all $x_1,\ x_2 \in X$:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- f is *surjective* or *onto* if every element of the codomain Y is contained in the image of f, i.e., for every $y \in Y$ there exists an $x \in X$ with $f(x) = y$. In other words, f is surjective if $im(f) = Y$.
- f is *bijective* if it is both injective and surjective. Bijective functions are invertible and possess an inverse map $f^{-1} : Y \to X$ such that $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$.

Relations

Definition

A relation R on X is a subset of $X \times X$. R is called an *equivalence relation* if it satisfies the following conditions:

1. R is reflexive, i.e., $(x, x) \in R$ for all $x \in X$, and
2. R is symmetric, i.e., if $(x, y) \in R$ then $(y, x) \in R$, and
3. R is transitive, i.e., if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

If $(x, y) \in R$, then x and y are called *equivalent* and we write $x \sim y$. For $x \in X$, the subset $\overline{x} = \{y \in X \mid x \sim y\} \subset X$ is called the *equivalence class* of x. The set of equivalence classes of X gives the *quotient set*

$$X/\sim .$$

Residue Classes modulo n

Let $n \in \mathbb{N}$, $n \geq 2$. Define the following equivalence relation R_n on $\mathbb{Z}$:

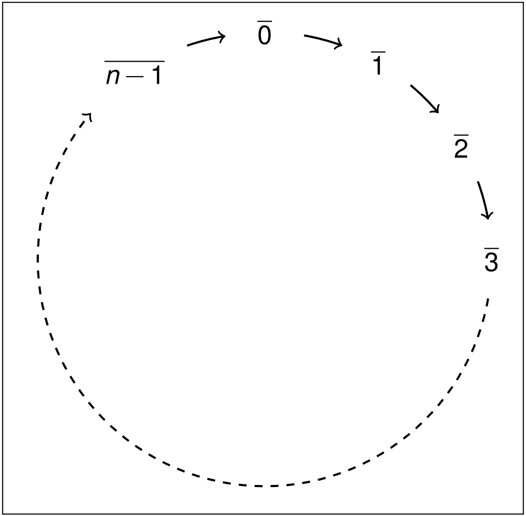
$$R_n = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \in n\mathbb{Z}\}$$

Note: $(x, y) \in R_n$ if the difference $x - y$ is divisible by n.
The equivalence class of $x \in \mathbb{Z}$ is the set

$$\overline{x} = \{\ldots, x - 2n, x - n, x, x + n, x + 2n, \ldots\}.$$

Now we have n *different* equivalence classes and the quotient set $\mathbb{Z}/\sim$ has n elements. We call this set the *residue classes modulo n* or *integers modulo n* and denote it by $\mathbb{Z}_n$ or $\mathbb{Z}/(n)$. Each residue class has a *standard representative* in the set $\{0, 1, \ldots, n - 1\}$ and elements in the same residue class are called *congruent modulo n*.

Residue Classes modulo n



Example: $\mathbb{Z}_2$

$\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ has only two elements. One has $\overline{-1} = \overline{1} = \overline{3}$ and $\overline{-2} = \overline{0} = \overline{2}$. The difference of two elements which are in the same class is divisible by 2 (i.e., their difference is even).

The standard representatives are 0, 1 and we have

$$\overline{0} = \{\dots, -4, -2, 0, 2, 4, \dots\},$$
$$\overline{1} = \{\dots, -3, -1, 1, 3, 5, \dots\}.$$

We may simple write 0 and 1 for the classes. Elements of $\mathbb{Z}_2$ can be added, and addition is the same as the XOR operation on bits: $0 \oplus 0 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$ and $1 \oplus 1 = 0$. Note that addition and subtraction is the same operation!

Example: $\mathbb{Z}_{26}$

$\mathbb{Z}_{26} = \{\overline{0}, \overline{1}, \ldots, \overline{25}\}$ has 26 elements. For example, one has $\overline{-14} = \overline{38}$, since $-14 - 38 = -52$ is a multiple of 26. The integers -14 and -38 are congruent modulo 26 and we write

$$-14 \equiv 38 \mod 26.$$

The standard representative of this residue class is 12 and

$$\overline{12} = \{\ldots, -14, 12, 38, 64, \ldots\}.$$

Computations with Residue Classes

Residue classes can be added, subtracted and multiplied. An arbitrary integer representative can be used, and it is reasonable to choose a small representative.

Examples: a) $79 - 180 \mod 26 \equiv 1 - 24 \equiv 1 + 2 = 3 \mod 26$.
b) $234577 \cdot 2328374 \cdot 2837289374 \mod 3 \equiv 1 \cdot 2 \cdot 2 \equiv 1 \mod 3$.

However, division is more tricky since rational numbers $\frac{b}{a}$ are not representatives of residue classes. We say that a is invertible modulo n if there exists $x \in \mathbb{Z}$ such that

$$ax \equiv 1 \mod n.$$

Then $x \equiv (a \mod n)^{-1}$.

Example: $(3 \mod 10)^{-1} \equiv 7$, since $3 \cdot 7 \equiv 1 \mod 10$.

Invertible Residue Classes

Proposition

An integer a is invertible modulo n if and only if $\gcd(a, n) = 1$*, i.e., if the greatest common divisor of a and n is* 1*.*

Example: 3 is invertible modulo 10, but 2 is not invertible modulo 10.

Definition

The invertible integers modulo n are called units mod n. The subset of units of $\mathbb{Z}_n$ is denoted by $\mathbb{Z}_n^*$.

Example: $\mathbb{Z}_{10}^* = \{\overline{1}, \overline{3}, \overline{7}, \overline{9}\}$.

Prime Numbers

Definition

An integer $p \geq 2$ is called a prime number if p is only divisible by ± 1 and $\pm p$.

If p is prime, then

$$\mathbb{Z}_p^* = \{\overline{1}, \ldots, \overline{p-1}\}.$$

Prime numbers play an important role in public-key cryptography. The Prime Number Theorem states that the density of primes in the first N integers is approximately

$$\frac{1}{\ln(N)}.$$

Extended Euclidean Algorithm

One of the key algorithms in elementary number theory is the *Extended Euclidean Algorithm*. The algorithm takes two nonzero integers a, b as input and computes $\gcd(a, b)$ as well as two integers $x, y \in \mathbb{Z}$ such that

$$\gcd(a, b) = ax + by.$$

The Extended Euclidean Algorithm is very efficient and can be used to compute the multiplicative inverse of $a \mod n$. If $\gcd(a, n) = 1$ then the algorithm outputs $x, y \in \mathbb{Z}$ such that

$$1 = ax + ny.$$

Then

$$1 \equiv ax \mod n$$

and thus $x \equiv (a \mod n)^{-1}$.

Extended Euclidean Algorithm

Input: $a,\ b \in \mathbb{N}$

Output: $\gcd(a,b),\ x,\ y \in \mathbb{Z}$ such that $\gcd(a,b) = ax + by$

Initialisation: $x_0 = 1$, $x_1 = 0$, $y_0 = 0$, $y_1 = 1$, $sign = 1$

while $b \neq 0$ **do**
  $r = a \mod b$ // remainder of the integer division $a : b$
  $q = a/b$ // integer quotient
  $a = b$
  $b = r$
  $xx = x_1$
  $yy = y_1$
  $x_1 = q \cdot x_1 + x_0$
  $y_1 = q \cdot y_1 + y_0$
  $x_0 = xx$
  $y_0 = yy$
  $sign = -sign$
end while
$x = sign \cdot x0$
$y = -sign \cdot y0$
$gcd = a$
return $\mathrm{gcd},\ x,\ y$

Modular Exponentiation I

Modular exponentiation with a large basis, exponent and modulus plays an important role in cryptography. How can we efficiently compute

$$x^a \mod n\,?$$

If $a = 2^k$ then k-fold squaring modulo n gives the result:

$$x^a \mod n = ((((x^2 \mod n)^2 \mod n)^2 \mod n)^2 \ldots)^2 \mod n$$

For example, $x^{256} \mod n$ can be computed with only 8 squaring operations. After each squaring, reduce mod n in order to reduce the size of the result.

Modular Exponentiation II

If the exponent is not a power of 2, then it can still be written as *a sum of powers* of 2. This gives a product of factors of type $x^{(2^k)} \mod n$, and each factor can be computed by k modular squarings. We call this the *Fast Exponentiation Algorithm*.

Example: Compute $6^{41} \mod 59$. We have $41 = 2^5 + 2^3 + 2^0$ and first compute the following sequence of squares:

$$\begin{aligned}
6^2 &\equiv 36 \mod 59 \\
6^4 &\equiv 36^2 \equiv 57 \mod 59 \\
6^8 &\equiv 57^2 \equiv 4 \mod 59 \\
6^{16} &\equiv 4^2 \equiv 16 \mod 59 \\
6^{32} &\equiv 16^2 \equiv 20 \mod 59
\end{aligned}$$

Then $6^{41} = 6^{32} \cdot 6^8 \cdot 6 \equiv 20 \cdot 4 \cdot 6 \equiv 8 \mod 59$.

Cardinality

Proposition

Let X and Y be finite sets of cardinality $|X|$ and $|Y|$, respectively. Then:

1. $|X \times Y| = |X| \cdot |Y|$ *and* $|X^k| = |X|^k$ *for* $k \in \mathbb{N}$.
2. *Suppose* $|X| = n$ *and* $k \leq n$. *Then the number of subsets of X of cardinality k is given by the binomial coefficient* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example: There are $\binom{128}{2} = \frac{128 \cdot 127}{2} = 8128$ different binary words of length 128 with exactly two ones and 126 zeros.

Euler's φ-Function

Definition

Let $n \in \mathbb{N}$. Then Euler's φ-function is defined by the cardinality of the units mod n, i.e.,

$$\varphi(n) = |\mathbb{Z}_n^*|$$

Examples: a) $\varphi(10) = 4$.
b) If p is a prime number, then $\varphi(p) = p - 1$.
c) If p and q are different prime numbers, then $\varphi(pq) = (p-1)(q-1)$ (Why?).

Permutations

Definition

Let S be a finite set. A *permutation* of S is a bijective map $\sigma : S \to S$.

Proposition

Let S be a finite set and $|S| = n$. Then there are $n!$ permutations of S.

Note: the factorial increases very fast, for example

$$50! \approx 3.04 \cdot 10^{64}.$$

Permutations in Cryptography

Cryptographic operations often use permutations. A randomly chosen family of permutations of a set like $M = \{0,1\}^{128}$ would constitute an ideal block cipher. However, it is impossible to write down or store a general permutation since M has 2^{128} elements. Much simpler (and much less secure) are *bit permutations*, which permute only the *position* of the bits.

Example: $(5\ 7\ 1\ 2\ 8\ 6\ 3\ 4)$ defines a permutation on $X = \{0,1\}^8$: a byte $(b_1, b_2, \ldots, b_8)$ is mapped to $(b_5, b_7, b_1, b_2, b_8, b_6, b_3, b_4)$. There are $8!$ bit permutations of X (a small number), but $(2^8)!$ general permutations (a very large number).

Big-O Notation

We often need to analyze the computational complexity of algorithms, i.e., the resources (running time and space) as a function of the input size.

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$ be two functions on $\mathbb{N}$. Then we say that g is an *asymptotic upper bound* for f, if there exists a real number $C \in \mathbb{R}$ and an integer $n_0 \in \mathbb{N}$ such that

$$|f(n)| \leq C\,|g(n)| \text{ for all } n \geq n_0.$$

One writes $f = O(g)$ or $f \in O(g)$.

Asymptotic Complexity: Examples

1. $f(n) = 2n^3 + n^2 + 7n + 2$. Since $n^2 \leq n^3$, $n \leq n^3$ and $1 \leq n^3$ for $n \geq 1$, one has $f(n) \leq (2+1+7+2)n^3$. Set $C = 12$ and $n_0 = 1$. Thus $f = O(n^3)$ and so f has cubic growth in n.
2. $f(n) = 100 + \frac{20}{n+1}$. Set $C = 101$ and $n_0 = 19$. Since $\frac{20}{n+1} \leq 1$ for $n \geq 19$, we have $f = O(1)$. Hence f is asymptotically bounded by a constant.
3. $f(n) = 5\sqrt{2^{n+3} + n^2 - 2n}$. Then $f = O(2^{n/2})$, and so f grows exponentially in n.

Complexity of Algorithms

Definition

If the running time of an algorithm is $f(n)$, where f is a *polynomial* and n is the input *size*, then the algorithm has *polynomial running time* and belongs to the complexity class **P**.

Examples: a) The functions in the above examples (1) and (2) are polynomial.
b) The running time of the Extended Euclidean Algorithm on input $a, b \in \mathbb{N}$ is $O(\text{size }(a) \text{size }(b))$, so the algorithm is polynomial on the maximal input size.

Polynomial-time algorithms are usually regarded as *efficient*. In computer science, one is usually interested in the *worst-case* complexity of algorithms. However, when looking at the complexity of attacks against cryptographic schemes, their *average-case* complexity is much more important.

Negligible Functions

We need the notion of a *negligible* function in the context of the probability of successful attacks.

Definition

Let $f : \mathbb{N} \to \mathbb{R}$ be a function. We say that f is *negligible* in n, if $f = O(\frac{1}{q(n)})$ for all polynomials q, or equivalently, if $f = O(\frac{1}{n^c})$ for all $c > 0$.

Negligible functions are eventually smaller than any inverse polynomial. This means that $f(n)$ approaches zero faster than any of the functions $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \ldots$

Example: $f(n) = 10e^{-n}$ and $2^{-\sqrt{n}}$ are negligible in n.
$f(n) = \frac{1}{n^2+3n}$ is not negligible, since $f(n) = O(\frac{1}{n^2})$, but $f \neq O(\frac{1}{n^3})$.

Probability

We refer to textbooks on probability theory. We only consider *discrete probability spaces* and need the following notions:

- Probability space $(\Omega, \mathcal{S}, Pr)$, where Ω is a sample space, $\mathcal{S} = \mathcal{P}(\Omega)$ the set of events and $Pr : \mathcal{S} \to [0, 1]$ a probability distribution.
- Independent events A, B, i.e., $P(A \cap B) = P(A) \cdot P(B)$, and mutually independent events $A_1, \ldots, A_n$.
- The conditional probability $P[A|B] = \frac{P(A \cap B)}{P(B)}$ of events A, B.
- Random variables $X : \Omega \to \mathbb{R}$, their expectation $E[X]$ and variance $V[X]$.
- Probability mass function (pmf) $p_X(x) = Pr[X = x]$ of a random variable X.

Uniform Distribution and Random Bits

Definition

Pr has a uniform distribution if all elementary events have equal probability: $Pr[\{\omega\}] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

Random bits (or random numbers) are quite important in cryptography (but difficult to generate).

Definition

A random bit generator (RBG) outputs a sequence of bits such that the corresponding random variables $X_1, X_2, X_3, \ldots$ satisfy

1. $Pr[X_n = 0] = Pr[X_n = 1] = \frac{1}{2}$ for all $n \in \mathbb{N}$ (uniform distribution), and
2. $X_1, X_2, \ldots, X_n$ are mutually independent for all $n \in \mathbb{N}$.

Birthday Paradox

Let $x_1, x_2, \ldots, x_n$ be a sequence in a sample space Ω. We say that there is a *collision* if at least two elements in the sequence are identical.

Proposition

Let Pr be a uniform distribution on a set Ω of cardinality n. If we draw $k = \left\lceil \sqrt{2 \ln(2) n} \right\rceil \approx 1.2 \sqrt{n}$ independent samples from Ω, then the probability of a collision is around 50%.

This fact is called *birthday paradox*: only $k = 23$ random birthdays ($n = 365$) are on average sufficient for a collision.

For $|\Omega| = 2^n$, around $2^{n/2}$ independent samples probably give a collision.