Cryptography

Public-Key Encryption

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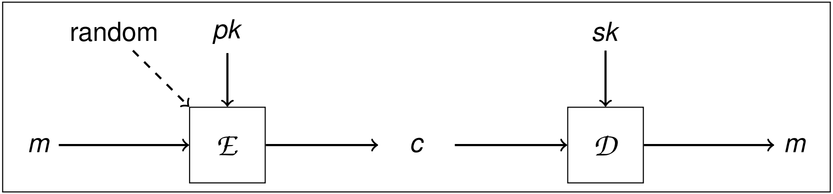
Public Keys

One of the main principles of *symmetric encryption* is that the encryption *and* decryption use a *secret key*. In contrast, public key schemes use a *public key* for encryption. Obviously, it is crucial that an adversary is not able to derive the private decryption key from the public encryption key.

Furthermore, the *authenticity* of public keys can represent a problem.



Public-key encryption and decryption.

Encryption scheme

Definition

A *public-key encryption scheme* (public-key cryptosystem) is given by:

- A plaintext space $\mathcal{M}$ and a ciphertext space $\mathcal{C}$,
- A key space $\mathcal{K} = \mathcal{K}_{pk} \times \mathcal{K}_{sk}$ (pairs of public and private keys),
- A randomized key generation algorithm $Gen(1^n)$, that takes a security parameter n as input and outputs a pair of keys (pk, sk),
- An encryption algorithm $\mathcal{E} = \{\mathcal{E}_{pk} \mid pk \in \mathcal{K}_{pk}\}$, which may be randomized. It takes a public key and a plaintext as input, and outputs the ciphertext or $\perp$.
- A deterministic decryption algorithm $\mathcal{D} = \{\mathcal{D}_{sk} \mid sk \in \mathcal{K}_{sk}\}$ that takes a private key and a ciphertext as input and outputs the plaintext or $\perp$.

Encryption and Decryption

Key generation, encryption and decryption algorithms must run in polynomial time.

The scheme provides *correct decryption* if

$$\mathcal{D}_{sk}(\mathcal{E}_{pk}(m)) = m$$

for all key pairs $(pk, sk) \in \mathcal{K}$ and all valid plaintexts $m \in \mathcal{M}$.

IND-CPA Security

Adversary		*Challenger*
	$\xleftarrow{pk,\ 1^n}$	$(pk, sk) \xleftarrow{\$} Gen(1^n)$ $b \xleftarrow{\$} \{0,1\}$
Choose m_0, m_1 $\lvert m_0 \rvert = \lvert m_1 \rvert$	$\xrightarrow{m_0,\ m_1}$	
	$\xleftarrow{c}$	$c \leftarrow \mathcal{E}_{pk}(m_b)$
Select m_0 $(b' = 0)$ or m_1 $(b' = 1)$	$\xrightarrow{b'}$	Compare b and b', output 1 or 0

Public-key EAV and CPA experiment. Since pk is public, an adversary can encrypt any chosen plaintext.

IND-CPA Security

The IND-CPA advantage of an adversary A is defined as

$$\mathrm{Adv}^{\text{ind-cpa}}(A) = |Pr[b' = b] - Pr[b' \neq b]|.$$

We use the same security definition as for symmetric schemes:

Definition

A public-key encryption scheme has *indistinguishable encryptions under a chosen plaintext attack* (IND-CPA secure or CPA-secure), if for every probabilistic polynomial time adversary A, the advantage $\mathrm{Adv}^{\text{ind-cpa}}(A)$ is negligible in n.

Since an adversary can encrypt m_0 and m_1 and compare the result with the challenge ciphertext c, it is obvious that public-key schemes with *deterministic* encryption cannot be IND-CPA secure.

IND-CCA2 Security

A more powerful adversary is able to perform an *adaptive chosen ciphertext attack* (CCA2).

In the CCA2 experiment, the adversary can additionally request the *decryption* of arbitrary ciphertexts (before and after choosing two plaintext messages), except that the challenge ciphertext c cannot be queried.

If the advantage of any probabilistic polynomial time adversary A is negligible in the CCA2 experiment, then a scheme is said to be *CCA2-secure*. Obviously, CCA2 security is stronger than CPA security.

Plain RSA

Definition

The plain (schoolbook) RSA encryption scheme is defined by:

- A polynomial-time key generation algorithm $Gen(1^n)$ that takes the security parameter 1^n as input, generates two random n-bit primes p and q and sets $N = pq$. Furthermore, two integers $e > 1$ and d with $ed \equiv 1 \mod (p-1)(q-1)$ are chosen. $Gen(1^n)$ outputs the public key $pk = (e, N)$ and the private key $sk = (d, N)$.
- The plaintext and the ciphertext space is $\mathbb{Z}_N^*$.
- The deterministic encryption algorithm takes a plaintext $m \in \mathbb{Z}_N^*$ and the public key pk as input and outputs

$$c = \mathcal{E}_{pk}(m) = m^e \mod N.$$

Definition

- The decryption algorithm takes a ciphertext $c \in \mathbb{Z}_N^*$ and the private key *sk* as input and outputs

$$m = \mathcal{D}_{sk}(c) = c^d \mod N.$$

For the correctness of the RSA scheme we have to show that

$$(m^e)^d \equiv m \mod N$$

for all $m \in \mathbb{Z}_N^*$. But this follows from Euler's Theorem: let $m \in \mathbb{Z}_N^*$, then we have

$$m^{\varphi(N)} \equiv 1 \mod N.$$

Since $ed = 1 + k\varphi(N)$ for some $k \in \mathbb{Z}$, we obtain

$$m^{ed} = m^1 (m^{\varphi(N)})^k \equiv m \cdot 1^k = m \mod N.$$

RSA Security

Factoring N breaks RSA, and so breaking RSA reduces to factoring, but the opposite direction is not known. The security of RSA is in fact based on the *RSA assumption*.

Definition

Consider the following experiment: run the RSA $Gen(1^n)$ algorithm. A uniform random ciphertext $c \xleftarrow{\$} \mathbb{Z}_N^*$ is chosen and an adversary obtains 1^n, e, N and c. The adversary has to find $m \in \mathbb{Z}_N^*$ such that

$$m^e \equiv c \mod N.$$

The RSA problem is hard relative to *Gen*, if for every probabilistic polynomial-time adversary, the probability of finding the correct plaintext m is negligible in n. The *RSA assumption* states that there is a key generation algorithm *Gen* such that the RSA problem is hard.

RSA Pitfalls

The plain RSA encryption scheme is *deterministic*. Therefore, it cannot be CPA-secure. But even if the plaintext messages are chosen uniformly at random from a large space, there are a number of pitfalls:

- Very small public exponents, e.g., $e = 3$, are insecure (low-exponent attack).
- Small decryption exponents, i.e., $d < \frac{1}{3}N^{1/4}$, are insecure (Wiener attack).
- Partially known plaintexts and keys can be attacked.
- Plain RSA encryption is *malleable*: a controlled modification of the ciphertext and the corresponding plaintext is possible.
- A *chosen ciphertext attack* against plain RSA is easy.

Primality Tests

RSA and other public-key schemes require *large prime numbers*. The asymptotic density of primes among the first N integer numbers is

$$\frac{1}{\ln(N)}.$$

Example: Let $N = 2^{2048}$. Then $\frac{1}{\ln(N)} \approx 0.0007$.

To generate a large prime, choose an *odd random number* of the required length and test its primality.

Fermat Test

The *Fermat test* is based on Euler's Theorem: if p is a prime and a is not divisible by p, then

$$a^{p-1} \equiv 1 \mod p.$$

To test whether n is prime, pick a number $1 < a < n$ and compute $a^{n-1} \mod n$. If the result is not 1, then n must be composite. If $a^{n-1} \equiv 1 \mod n$ then n could be a prime and one would repeat the test with a few other bases a.

However, there are infinitely many *composite numbers* n for which

$$a^{n-1} \equiv 1 \mod n$$

for all integers a with $\gcd(a, n) = 1$. They are called *Carmichael numbers*. The smallest Carmichael number is $n = 561$.

Miller-Rabin Test

Unlike the Fermat test, the probabilistic *Miller-Rabin test* can detect all composite numbers.

Suppose we want to test the primality of n. Let $n-1 = 2^s d$, where s is maximal. If n has the following property (COMP) then it must be *composite*:

(COMP) *There exists a base $a \in \mathbb{N}$ with $1 \leq a < n$, such that*

$$a^d \not\equiv \pm 1 \mod n \quad \text{and} \quad a^{2^r d} \not\equiv -1 \mod n \quad \text{for all } 1 \leq r \leq s-1.$$

If (COMP) is satisfied, then *n cannot be a prime* and a new number n is chosen. If n does not satisfy (COMP) for a given base a, then n *could be a prime*. The test is repeated several times (for different bases a) in order to increase the probability that n is indeed a prime. It can be shown that if n is composite then at most $\frac{1}{4}$ of the bases a do not satisfy (COMP).

Running Time

RSA encryption and decryption both require one exponentiation modulo N. Let $n = \text{size}\,(N)$. The running time is

$$O(n^3),$$

which is polynomial, but not very fast. Therefore, a large number of such exponentiations should be avoided.

Often, one chooses the constant encryption exponent $e = 2^{16} + 1$, so that the running-time of encryption is only $O(n^2)$. Furthermore, the decryption $c^d \mod N$ can be accelerated by a factor of around 4 by using the *Chinese Remainder Theorem* (CRT). Recall that the CRT gives a decomposition of rings

$$\mathbb{Z}_N \cong \mathbb{Z}_p \times \mathbb{Z}_q \ \text{ for } p \neq q$$

So c^d can be computed modulo p and modulo q, and both have size $n/2$. Then exponentiations are around $2^3 = 8$ times faster.

RSA-OAEP

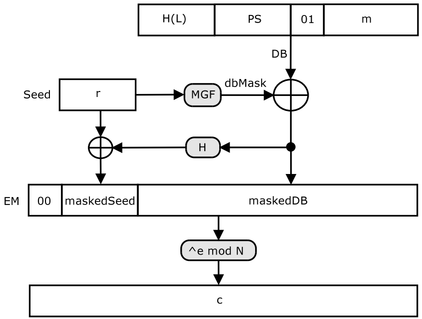
Plain RSA is not CPA-secure, even if the parameters p, q, e, d are appropriately chosen, because the scheme is *deterministic*.

Furthermore, the ciphertext is *malleable* and chosen ciphertext attacks are possible. In fact, an improved RSA scheme called *Optimal Asymmetric Encryption Padding* (OAEP) is standardized as PKCS #1 version 2.2 and RFC 8017.

The message m is transformed into an encoded and randomized message EM (see below), which is subsequently encrypted using plain RSA.

Decryption first computes EM, verifies the integrity of the result and then recovers m (or outputs $\perp$).

RSA-OAEP



Encryption of a plaintext m using RSA-OAEP. L is a label or empty. PS is a padding string consisting of zero bytes. r is a random seed of the same length as the output of the hash function. MGF is a mask generating function based on the hash function, but with larger output length. Note that the maximum length of m is a little shorter than the length of N.

Security of RSA-OAEP

Theorem

RSA-OAEP is secure against adaptive chosen ciphertexts attacks *(CCA2-secure) under the RSA assumption and in the random oracle model.*

Factoring Methods

Factoring is assumed to be a hard problem, at least on conventional computers. Obviously, if the factoring assumption turns out to be wrong, then RSA is broken.

Trial division is an elementary factoring method. It suffices to test numbers $\leq \sqrt{N}$. A list of small primes is useful (sieve method), and otherwise all odd numbers (or perhaps all numbers not divisible by 2, 3 or 5) need to be tested. The worst-case complexity is $O(\sqrt{N})$ and the running time is exponential in size (N).

Pollard's ρ

Pollard's ρ *algorithm* searches for an integer x such that $\gcd(x, N)$ is either p or q, e.g., $x \equiv 0 \mod p$, but $x \not\equiv 0 \mod q$. The idea is to generate a pseudorandom sequence $x_i = f(x_{i-1})$ of integers modulo N and to find a collision $x_k \equiv x_{2k}$ modulo p or q using *Floyd's cycle finding algorithm*. Note that

$$x_k \equiv x_{2k} \mod p \Longleftrightarrow x_k - x_{2k} \equiv 0 \mod p.$$

The algorithm computes pairs (x_k, x_{2k}) of integers modulo N and checks whether $\gcd(x_k - x_{2k}, N) > 1$.

By the birthday paradox, around $O(\sqrt{p}) \approx O(N^{1/4})$ iterations should be sufficient to find a collision modulo p.

Fermat Factorization

Fermat factorization uses a representation of N as a difference of squares:

$$N = x^2 - y^2 = (x+y)(x-y)$$

To find x and y, begin with the integer $x = \lceil \sqrt{N} \rceil$ and increment x by 1 until $x^2 - N$ is square, say y^2, so that $x^2 - N = y^2$.

Fermat factorization always works, since $N = pq$ can be written in the following way as a difference of two squares:

$$pq = \left(\frac{1}{2}(p+q)\right)^2 - \left(\frac{1}{2}(p-q)\right)^2 = x^2 - y^2$$

However, Fermat's method is only efficient if the prime factors are close to one another, i.e., if y is small. In general, the running time is $O(\sqrt{N})$ and hence exponential w.r.t. the size of N.

Example: Fermat Factorization

Let $N = 59987$. Then $x = \lceil \sqrt{N} \rceil = 245$. Compute $x^2 - N = 38$ which is not a square number.

Increment x and set $x = 246$. Then $x^2 - N = 529 = 23^2$. Set $y = 23$ and obtain

$$N = x^2 - y^2 = (x+y)(x-y) = (246-23)(246+23) = 223 \cdot 269.$$

Quadratic Sieve

The *quadratic sieve* generalizes the Fermat factorization and is currently the fastest algorithm for numbers with less than around 100 decimal digits. One looks for integers x and y such that $x^2 \equiv y^2 \mod N$, but $x \not\equiv \pm y \mod N$. This implies

$$N \text{ divides } x^2 - y^2 = (x+y)(x-y),$$

but N divides neither $x+y$ nor $x-y$. Hence $gcd(x-y, N)$ must be a non-trivial divisor of N and equals either p or q.

The idea is to multiply several (non-quadratic) numbers $x^2 - N$ with small prime factors (they are called *smooth over a factor base*), so that their product is a square. The difficult task is to find smooth numbers.

The running time of the quadratic sieve is *sub-exponential*:

$$O(e^{(1+o(1))\sqrt{\ln(N)\ \ln(\ln(N))}})$$

Quadratic Sieve Example

Let $N = 10441$, then $\sqrt{N} \approx 102.2$. We compute $x^2 - N$ for a couple of integers $x \geq 103$ and look for small prime factors.

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x=103  103^2-10441 = 168 = 2^3 * 3 * 7
x=104  104^2-10441 = 375 = 3 * 5^3
x=105  105^2-10441 = 584 = 2^3 * 73
x=106  106^2-10441 = 795 = 3 * 5 * 53
x=107  107^2-10441 = 1008 = 2^4 * 3^2 * 7
x=108  108^2-10441 = 1223 = 1223
x=109  109^2-10441 = 1440 = 2^5 * 3^2 * 5
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$$(103^2 - N) \cdot (104^2 - N) \cdot (107^2 - N) \cdot (109^2 - N) = 2^{12} \cdot 3^6 \cdot 5^4 \cdot 7^2$$

Now we set $x = 103 \cdot 104 \cdot 107 \cdot 109$ and $y = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7$, and obtain $x^2 \equiv y^2 \mod N$. We have $x \equiv 7491$, $y \equiv 10052$ modulo N and get $\gcd(7491 - 10052, 10441) = 197$, which is indeed a divisor of 10441.

Number Field Sieve

The *general number field sieve* (a generalization of the quadratic sieve) is currently the most efficient algorithm for factoring large integers. With massive computing resources, numbers with more than 200 digits, for example the 768-bit *RSA Challenge*, could be factored using this method.

The heuristic complexity of the number field sieve is sub-exponential:

$$O(e^{(c+o(1))\ln(N)^{1/3}\ \ln(\ln(N))^{2/3}}),$$

where $c = \sqrt[3]{\frac{64}{9}} \approx 1.92$. The relative success of the number field sieve is a reason why a secure RSA modulus should have *more than* 2000 *bits*.

Other Methods

Pollard's $p-1$ method can be applied if $p-1$ or $q-1$ decompose into a product of small primes. In this case, one can guess multiples k of $p-1$. We have

$$(p-1) \mid k \Rightarrow a^k \equiv 1 \mod p.$$

Then $\gcd(a^k - 1, N)$ is either p (method successful) or N (failure).

The *Elliptic Curve Factorization Method* (ECM) is another interesting factoring method with sub-exponential running time. ECM is suitable for finding prime factors with up to 80 decimal digits. For larger primes, the quadratic sieve or the number field sieve are more efficient.

Furthermore, *Shor's algorithm* can factor large numbers in polynomial time on a *quantum computer*. However, sufficiently large and stable quantum computers are not yet available.