# Cryptography

## Hash Functions

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# Hash Functions

In general, a *cryptographic hash function* consists of a polynomial-time key generator (that takes a security parameter $1^n$ as input) and a hash algorithm. A *keyed* hash function

$$H_k : \{0,1\}^* \to \{0,1\}^n$$

takes a key and a binary string as input and outputs a hash value of length $n$.

In practice, hash functions are *unkeyed* or the key is fixed.

# Collisions and Collision Resistance

Since hash values are used as *message digests* or unique *identifiers*, their main requirement is *collision resistance*. A collision is given by two input values $x \neq x'$ with

$$H(x) = H(x').$$

## Definition

A function $H_k = H : D \to R$, where $H$, $k$, $D$ and $R$ depend on a security parameter $n$, is called *collision resistant*, if the probability that a probabilistic polynomial-time adversary finds a collision $H(x) = H(x')$, where $x,\ x' \in D$ and $x \neq x'$, is negligible in $n$.

# Weak Collision Resistance

There are two related requirements, which are weaker than collision resistance:

- *Second-preimage resistance* or *weak collision resistance* means that an adversary, who is given a uniform $x \in D$, is not able to find a second preimage $x' \in D$ with $x \neq x'$ such that $H(x) = H(x')$.
- *Preimage resistance* or *one-wayness* means that an adversary, who is given a uniform $y \in R$, is not able to find a preimage $x \in D$ such that $H(x) = y$.

# Random Oracle Model

An ideal unkeyed hash function is called a *random oracle*. The output of a random oracle is uniformly random, unless the same input is queried twice, in which case the oracle returns the same output.

A hash function that behaves as a random oracle is required in some security theorems and proofs. However, concrete implementations of a random oracle are impossible.

# Birthday Paradox

Hash values should not be too short. In fact, the *Birthday Paradox* demonstrates that collisions occur surprisingly often:

**Theorem**

*Let $k$ be the number of independent samples drawn from a uniform distribution on a set of size $N$. If $k \approx 1.2\sqrt{N}$, then the probability of a collision is around $50\%$.*

If we consider hash values of length $n$ and assume that they have a uniform distribution, then collisions occur after hashing around $\sqrt{2^n} = 2^{n/2}$ messages.

In order to minimize the risk of random collisions, hash values should be at least 200 bits long.

# Integrity Protection

Firstly, the hash value can be used as a short *identifier* of data.

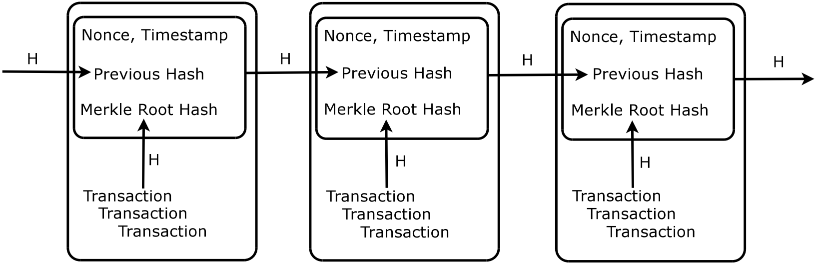
$$m \longrightarrow H(m)$$

The identifier is unique as long as the hash function is collision resistant. Hashes can be used to verify the *integrity* of messages. Note that the verifier needs access to the *authentic* message digest $H(m)$.

Hashes are used in the construction of *message authentication codes* (HMAC). *Signature schemes* first hash the message.

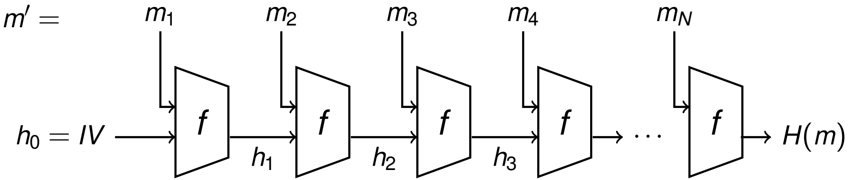
# Blockchain

A *blockchain* is a sequence of linked blocks. Each block contains the hash value of the previous block. A blockchain can be used as a *distributed ledger*, which records transactions in an efficient and verifiable way. Hash values protect the integrity of the blockchain: transactions in a block cannot be modified without changing all subsequent hash values.



# Merkle-Damgård Construction

The *Merkle-Damgård construction* has found widespread use, including the MD-SHA family. The Merke-Damgård transform is based on a *compression function* $f : \{0,1\}^{n+l} \to \{0,1\}^n$, which maps $n+l$ input bits to $n$ output bits. The compression function is applied recursively. A message $m$ is padded and its length is appended, giving $m'$. In each step, $l$ bits of $m'$ are processed and the last output defines the hash value $H(m)$.



# SHA-1

SHA-1 is a Merkle-Damgård hash function based on a compression function

$$f : \{0,1\}^{160+512} \to \{0,1\}^{160}.$$

A 512-bit message block $m = W_0 \| W_1 \| \ldots \| W_{15}$ is subdivided into 16 words of length 32 bits. By XOR operations and a circular left shift by one position, 64 additional words $W_{16}, \ldots, W_{79}$ are generated:

$$W_j = (W_{j-16} \oplus W_{j-14} \oplus W_{j-8} \oplus W_{j-3}) \lll 1 \text{ for } 16 \leq j \leq 79$$

The 160-bit input vector $h = H_1 \| H_2 \| H_3 \| H_4 \| H_5$ is subdivided into five 32-bit words and copied to the initial status vector:

$$A \| B \| C \| D \| E \leftarrow H_1 \| H_2 \| H_3 \| H_4 \| H_5$$

# SHA-1 Compression Function

Then, 80 rounds of the SHA-1 compression function are performed, which update the status words $A||B||C||D||E$. In round $j$, the 32-bit message word $W_j$ is processed. A nonlinear bit-function $F$ (defined by AND, OR, NOT and XOR operations) and a constant $K$ are used. The function $F$ and the constant $K$ change every 20 rounds. The following function is used for the first 20 rounds:

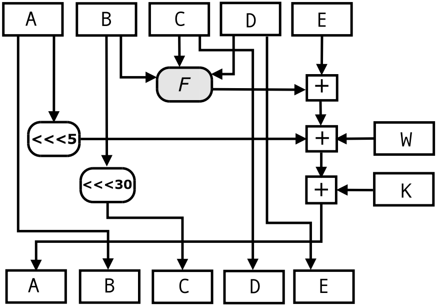
$$F(B,C,D) = (B \wedge C) \oplus (\neg B \wedge D)$$

After completing 80 rounds, the compression function outputs

$$f(h,m) = (A + H_1 \mid\mid B + H_2 \mid\mid C + H_3 \mid\mid D + H_4 \mid\mid E + H_5),$$

where $+$ denotes addition modulo $2^{32}$.

# SHA1 Compression Function




*One round of the SHA-1 compression function f which updates the 160-bit state. W is a 32-bit message chunk, F is a bit function (see above) and K a constant.* $\lll$ *denotes left-rotations and* $+$ *is addition modulo* $2^{32}$.

# SHA-1 Collision

In February 2017, a SHA-1 collision was found. The attack required $2^{63}$ SHA-1 calls, and took approximately 6500 CPU years and 100 GPU years. A prefix $P$ was chosen and two different 1024-bit messages $M^{(1)}$ and $M^{(2)}$ were found such that

$$H(P \| M^{(1)}) = H(P \| M^{(2)}).$$

Since $P$ is a valid preamble for PDF documents, the collision makes it possible to fabricate two different PDF files with the same SHA-1 hash value. Impressive examples have been published.

Here is the prefix $P$ in ASCII characters:

```
%PDF-1.3.%.......1 0 obj.<</Width 2 0 R/Height 3 0 R/Type 4 0 R/
Subtype 5 0 R/Filter 6 0 R/ColorSpace 7 0 R/Length 8 0 R/BitsPer
Component 8>>.stream......$SHA-1 is dead!!!!!./..#9u.9...<L.....
```

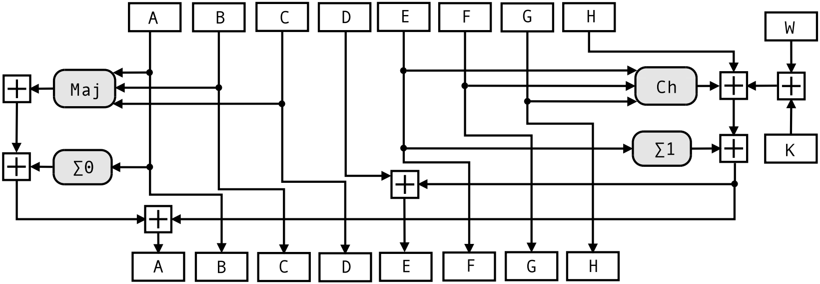
# SHA-2

The SHA-2 hash functions SHA-224, SHA-256, SHA-384 and SHA-512 are constructed in a similar way to SHA-1, but use an extended internal state and larger digests. It is assumed that SHA-2 offers better protection against collision-finding attacks. SHA-2 is now widely used in security protocols and applications.

In the following, we describe the SHA-256 variant. The SHA-2 compression function $f$ takes as input a 256-bit status vector and a 512-bit message block and outputs an updated 256-bit status:

$$f : \{0,1\}^{256+512} \to \{0,1\}^{256}$$

# SHA-2 Compression Function




*One round of the SHA-2 compression function f. The functions Maj, Ch,* $\Sigma_0$ *and* $\Sigma_1$ *are defined in a similar way as the function F used in SHA-1.*

The 32-bit status words $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$ are updated. In each of the 64 rounds, a 32-bit chunk $W$ derived from the message block is processed. $K$ is a 32-bit constant that depends on the round number.

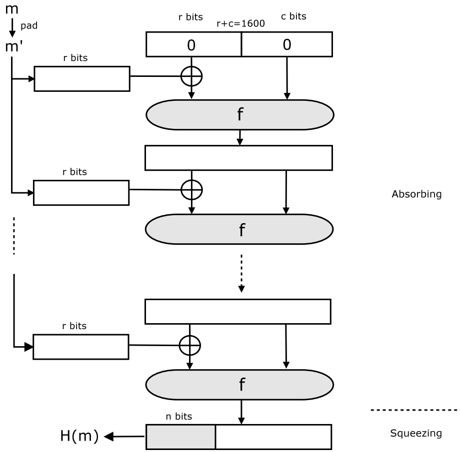
# SHA-3

After a competition to construct a new secure hash function called SHA-3, *Keccak* was selected as the winner. Keccak is not of Merkle-Damgård type, but rather based on a *sponge* construction.

In 2015, the Keccak variants SHA3-224, SHA3-256, SHA3-384, SHA3-512 with output lengths between 224 and 512 bits were standardized. The SHA-3 instance of Keccak uses a three-dimensional state array of $5 \times 5 \times 64 = 1600$ bits. The unkeyed Keccak-$f[1600]$ permutation operates on the 1600-bit state array, and it is assumed that $f$ behaves like a *random permutation*.

$$f : \{0,1\}^{1600} \rightarrow \{0,1\}^{1600}$$

# Keccak Operation




*Absorbing message blocks of length r and squeezing out the hash value.*

# SHA-3 Family

## Definition

The SHA-3 family of hash functions

$$H : \{0,1\}^* \to \{0,1\}^n$$

supports output lengths $n \in \{224, 256, 384, 512\}$. Depending on $n$, the rate $r$ and the capacity $c$ are fixed, where $r + c = 1600$. The input message $m$ is padded such that the length is a multiple of $r$. The padded message $m'$ is split into blocks $m_1, m_2, \ldots, m_N$ of length $r$:

$$m' = m \| 0110 \ldots 01 = m_1 \| m_2 \| \ldots \| m_N$$

The state $s = s_1 \| s_2$ is initialized by the zero vector $0^r \| 0^c$.

# SHA-3 Family

## Definition

During the *absorbing* phase, the state is recursively updated by the message blocks $m_i$:

$$s_1 \| s_2 \leftarrow f(s_1 \oplus m_i \parallel s_2) \text{ for } 1 \leq i \leq N.$$

Finally, the SHA-3 hash value is obtained by a single *squeezing* operation: $H(m)$ is given by the leftmost $n$ bits of the state.

# SHA-3 Family

An advantage of the sponge construction – in comparison to the Merkle-Damgård transform – is that the hash value does *not reveal the full state*, which prevents length extension attacks.

The SHA-3 standard also defines two *extendable-output functions* (XOF) called SHAKE128 and SHAKE256, with which the output can be extended to any desired length. In this case, the Keccak-$f$ function is applied multiple times during the squeezing phase in order to obtain the required number of output bits.

There are also keyed hash functions based on Keccak (KMAC128, KMAC256).