CSE 533: The PCP Theorem and Hardness of Approximation (Autumn 2005)

Lecture 4: Proving PCP: Degree-Reduce and Expanderize

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1 The GAP-CG Problem and PCP Theorem

Definition 1.1. *A constraint graph (G, C) over an alphabet Σ is defined as an undirected graph G (V, E) with constraint c (c ϵC) specified for each edge (u, v) using alphabets in Σ.*

Note that the graph can have multi-edges(representing multiple constraints over the same pair of vertices) and self loops

Definition 1.2. $\text{MAX-CG}(\Sigma)_{c,s}$ $(0 < s \leq c \leq 1)$*: Given a constraint graph (G, C) find $\sigma : V \mapsto \Sigma$ such that number of edges (u, v) such that $(\sigma(u), \sigma(v))\epsilon C(u, v)$ is maximum.*

Definition 1.3. $\text{GAP-CG}(\Sigma)_{c,s}$ $(0 < s \leq c \leq 1)$*: Given a a constraint graph (G, C)*

- *output YES if $\exists\sigma$ such that fraction of edges satisfying constraints $\geq c$;*
- *output NO if $\forall\sigma$ the fraction of edges satisfying constraints $< s$;*
- *otherwise output anything.*

Theorem 1.4. *For any constant Σ_0 with cardinality > 1 and universal constant $k < k_0 \leq 1$, $\text{GAP-CG}(\Sigma)_{1,k}$ is NP-hard*

This along with the theorem stated below is sufficient to prove PCP theorem. We will concentrate on proving this theorem in the remaining lecture.

Theorem 1.5. *The following two statement are equivalent:*

1. *The PCP theorem;*
2. *There exists an universal constant $k < 1$ such that $\text{GAP-CG}_{1,k}$ is NP-hard.*

2 The proof of the PCP theorem: Proving $\text{GAP-CG}_{1,k}$ is NP-Hard

We try to reduce 3-coloring problem to $\text{GAP-CG}(\Sigma_0)_{1,k}$. The 3-coloring problem can be viewed alternatively as a GAP-CG problemwith a variable gap. Formally,

Definition 2.1. *3-coloring problem: Given a constrained graph* (G, C) *and* $m = \mid E \mid$

- *output YES if* $\exists\sigma$ *all edges satisfy the constraints;*
- *output NO if* $\forall\sigma$ *the fraction of edges satisfying constraints* $\leq m - 1 = (1 - \frac{1}{m})m$*;*

Thus the proof of PCP theorem will be complete if we can increase the gap(fraction of violating edges) from 1/m to some universal constant k. For this we will transform the given constrained graph in the 3-coloring problem using a series of sub-routines so as to obtain a constraint (G', C') with the following properties.

- $size(G', C') = O(size(G, C))$
- $gap(G', C') = \frac{m}{k} gap(G, C)$

The series of tansformations can be done in polynomial time. Thus if we can solve the GAP-CG problem over (G', C'), we can solve the 3-coloring problem over (G, C) in polynomial time.
The transformations would be done in steps such that in each step we amplify the gap by 2 without affecting the size of the constrained graph(upto some constant factor). Each step is composed of four sub-routines which have to be applied in a particular order to achieve the gap amplification. We shall discuss the first of these sub-routines in the next section.

3 Degree-reduce

Definition 3.1. *Transformation: Consider a constrained graph* (G, C) *with vertex set* V *and edge set* E*. Let* $deg(u)$ *denote the degree of any vertex* $u \epsilon V$*. Also let* d_0 *and* λ_0 *be any universal constant such that* $\lambda_0 < d_0$*. We will apply the folowing transformation to obtain a new constrained graph* (G^1, C^1) *with vertex set* V^1 *and edge set* E^1*.*

- *Replace each vertex* $u \epsilon V$ *by* $deg(u)$ *many vertices to get the new* V' *set. Call it* $cloud(u)$
- *Associate each edge* $(u, v) \epsilon E$ *to an edge from one of the new vertex in* $cloud(u)$ *(say* u_i*) to a new vertex in a* $cloud(v)$ *(say* v_j*) ensuring that each new vertex gets associated to exactly one edge.*
- *Put a* $deg(u, d_0, \lambda_0)$ *expander on each cloud with euality constraints on each edge.*

We can observe that in this process each new vertex u^1 has degree exactly equal to $d_0 + 1$. Also number of newly added edges is equal to $\sum_{u \epsilon V} \frac{deg(u) d_0}{2} = d_0 \sum_{u \epsilon V} \frac{deg(u)}{2} = d_0 \mid E \mid$. Thus $\mid E^1 \mid = (d_0 + 1) \mid E \mid$

Lemma 3.2. $gap^1 > \frac{gap}{0(1)}$

Proof. Let $\sigma^1 : V^1 \mapsto \Sigma$ be a best assignment for (G^1, C^1), we define $\sigma : V \mapsto \Sigma$ by taking the plurality vote from each cloud i.e take the character $\epsilon\Sigma$ which occurs maximum times among $\sigma^1(u^1)$ where $u^1 \epsilon cloud(u)$. We know that σ violates atleast $\gamma \mid E \mid$ constrains in (G, C). We define the set s^u as the set of vertices in $cloud(u)$ which disagree with $\sigma(u)$.

Suppose $e = (u, v)$ is an edge that is violated by σ. Let e^1 be the corresponding edge in E^1. Then σ^1 either violates the edge e^1 or one of the endpoints of the edge belong to s^u or s^v. Thus

$$\gamma \mid E \mid \leq \text{number of edges violated by } \sigma^1 + \Sigma_u \mid s^u \mid.$$

$$\max(\text{number of edges violated by } \sigma^1, \Sigma_u \mid s^u \mid) \geq \frac{\gamma|E|}{2}.$$

We shall consider the following cases:

- number of edges violated by $\sigma^1 \geq \frac{\gamma|E|}{2} = \frac{\gamma|E^1|}{2d}$.
  Thus $gap^1 > \frac{gap}{0(1)}$

- $\Sigma_u \mid s^u \mid \geq \frac{\gamma|E|}{2}$: Let s^u_a represent the vertices in s^u which are mapped to the character a ϵ Σ by σ^1. We know

  $$\frac{|s^u_a|}{|s^u|} \leq \frac{1}{2}$$

  Hence due to the expander property of the cloud, number of edges out of s^u_a is atleast

  $$\phi(expander) \mid s^u_a \mid$$

  Each such edge will violate the equality constraint in the cloud.
  Therefore number of violated constraints $\geq \frac{1}{2O(1)} \Sigma_u \Sigma_a phi(expander) \mid s^u_a \mid$
  $\geq \frac{1}{O(1)} \Sigma_u \mid s^u \mid$
  $\geq \frac{1}{O(1)} \gamma \mid E \mid$
  $\geq \frac{1}{O(1)} \gamma \mid E^1 \mid$
  Tbus $gap^1 > \frac{gap}{0(1)}$ in both the cases

Hence Proved □

We can thus see that the degree-reduce subroutine transforms the constrained graph into a regular graph with degree $d_o + 1$ while reducing the gap atmost by a fixed constant. We shall now discuss another transformation which converts the graph into an expander.

4 Expanderize

Assume that the input constraint graph (G, C) is a regular graph. Apply the following transformations

- Let G' be a $(\mid V \mid, d_o, \lambda)$ expander with vertex set V (same as (G, C)) and the edge set E'
- Define dummy constraints C' over the edges in E' such that they are always satisfied
- Define (G^1, C^1) as the union of (G, C) and (G', C') in the following way
  - $V^1 = V = V'$
  - $E^1 = E \bigcup E'$

  We can observe the following effects of the transformation

- $gap(G, C) = 0 \Rightarrow gap^1(G^1, C^1) = 0$. This is easy to see since the new dummy constraints are always satisfied.
- $deg^1(G^1, C^1) = deg(G, C) + d_0 = d + d_0$
- The new constrained graph (G^1, C^1) is also an expander with $\lambda^1 \leq d_o + \lambda < d_0 + d$. This is based on the lemma on expanders which we saw in the previous class.
- If $d_0 = O(d)$ then $\mid E^1 \mid = O \mid E \mid$
- It follows that $gap^1 \geq \frac{gap}{O(1)}$

5 Before Stopping Random Walks and After Stopping Random Walks

For understaning the next subroutine we shall discuss special kinds of Random Walks on a regular graph and anlayze the probability of crossing a particular edge k times during the walks.

Definition 5.1. *After Stopping Random Walk over a regular graph $G(V, E)$ consists of the following steps*

1. *Pick a random vertex a ϵV*
2. *Take a random step e ϵE*
3. *Stop with probability $\frac{1}{t}$ else go to step 2*
4. *Call the final vertex b*

Definition 5.2. *Before Stopping Random Walk over a regular graph $G(V, E)$ starting from a vertex v consists of the following steps*

1. Stop with probability $\frac{1}{t}$ else go to step 3

2. Call the final vertex a

3. Take a random step e ϵE

Lemma 5.3. *Let k be a fixed constant ≥ 1 and (u, v) be an edge ϵE. If we do an ASRW conditioned on stepping on edge (u, v) exactly k times then*

1. The distribution on final vertex b is same as if we did a BSRW from v

2. The distribution on initial vertex a is same as if we did a ASRW from u

3. Conditionally, a and b are independent

Proof. 1. If we relax the restriction of stepping on (u, v) exactly k to the slightly looser restriction of stepping atleast k times then proof of 1 becomes immediate. This is because conditioned on the fact that we have to step on the edge atleast k times, once we reach the vertex v for the k^th time, there are no more additional restricitions. Thus the distribution on b(the final vertex) just becomes the same as the distribution of the final vertex if we were doing an BSRW from V.

Thus the probabilty $Pr[b = w \mid y \geq k]$ for any vertex $w\epsilon E$ is a fixed constant P_w independent of k. Here Y is random variable for the number of times we step on the edge (u, v)
P_w = $Pr[b = w \mid y \geq 1]$ = $\frac{Pr[(b=w) \wedge (Y \geq 1)]}{Pr(Y>=1)}$

$= \frac{Pr[(b=w) \wedge (Y=1)] \ + \ Pr[(b=w) \wedge (Y \geq 2)]}{Pr[Y=1] \ + \ Pr[Y \geq 2]}$

but P_w = $\frac{Pr[(b=w) \wedge (Y \geq 2)]}{Pr(Y>=2)}$

thus P_w = $Pr[(b = w) \mid (Y = 1)]$
Similarly P_w = $Pr[(b = w) \mid (Y = k)]$

Hence the distribution on final vertex is independent of k and is the same as the distribution we would get if we did a a BSRW from V.

2. If we reverse the random walk of part 1, we see that the distribution on 'a' conditioned on k u-v steps is same as the distibution on 'a' if we do a ASRW from u.

3. Given that we have a chain containing k u-v steps in the middle, it is easy to see that the path in the walk before the chain is conditionally independent from the path in the walk after the path.

□