Multiple Qubits, Gates, and Entangled States

In this section we will introduce you to multi-qubit states and operations. You will make your very own entangled state, known as a Bell state, and test some of the strangest properties of quantum physics.

The tutorials are broken down into the follow topics:

- CNOT and multi-qubit states
- T-gates (non-Clifford gates)
- Bell States
- GHZ states

Multiple Qubits

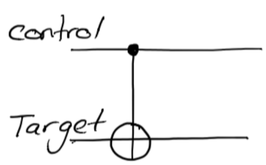
So far we have only considered a single qubit. The complex vector space of an n qubit system has a dimension equal to 2^n which we denote $\mathbb{C}^{2^n}$. The standard basis is the set of all binary strings for $k \in \{0, 2^n - 1\}$. For example, the basis for 2 qubits is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$; for 3 qubits, $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$; and for 4 qubits, $\{|0000\rangle, |0001\rangle, |0010\rangle, |0011\rangle, |0100\rangle, |0101\rangle, |0110\rangle, |0111\rangle, |1000\rangle, |1001\rangle, |1010\rangle, |1011\rangle, |1100\rangle, |1101\rangle, |1110\rangle, |1111\rangle\}$.

The number of terms increases exponentially. You can write them all out for five qubits (there will be 32 terms). Try writing them all out for 64 qubits! (This is like the famous wheat and chess board problem (https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem).) This exponential increase is one of the reasons why a quantum system is impossible to simulate on a conventional computer, but it is not simply because the number of terms grows exponentially.

A classical computer that has n-bits also has 2^n possible configurations. However, at any one point in time, it is in one and only one of these configurations. For example, a classical computer takes an n bit number, say 00000, and performs operations on it, mapping the input though an n-bit intermediate state such as 00001, which is then output as an n-bit number 10101. Interestingly, the quantum computer also takes in an n-bit number and outputs an n-bit number; but because of the superposition principle and the possibility of entanglement, the intermediate state is very different. To describe it requires 2^n complex numbers, giving a lot more room for maneuvering.

As an example, try running the "Random Classical Circuit" we have provided below. It takes the initial state $|0\rangle^{\otimes n}$ and it should produce the output $|10101\rangle$. By using X operations (NOTs) you can take the $|0\rangle^{\otimes n}$ to any classical state. Test it out!

To do interesting things in the quantum world, we need gates that perform conditional logic. The conditional gate we have provided is the Controlled NOT, or CNOT. It is represented by the element



The CNOT gate's action on classical basis states is to flip (apply a NOT or X gate to) the target qubit if the control qubit is $|1\rangle$; otherwise it does nothing. The CNOT plays the role of the classical XOR gate, but unlike the XOR, it is a two-output gate in order to be reversible (https://en.wikipedia.org/wiki/Reversible_computing)(as all quantum gates must be). It is represented by the matrix

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

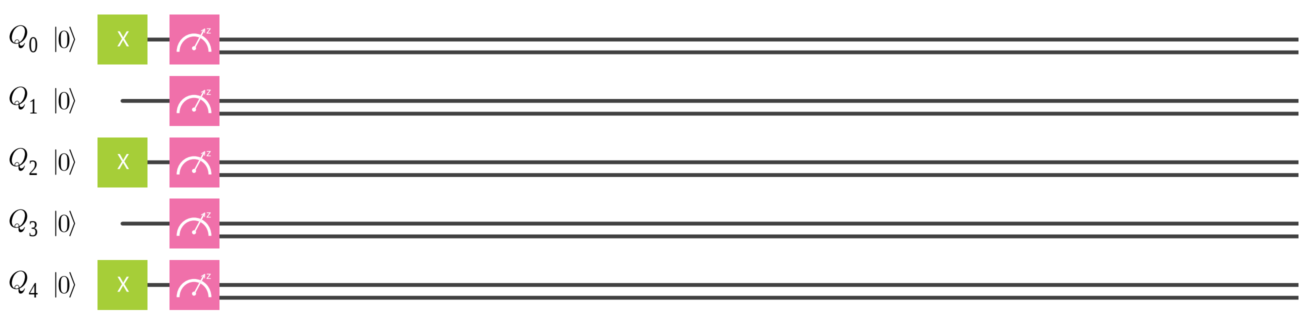
Try the "CNOT Circuits" below with different input states. Note that the X gates have prepared the qubits in a different configuration for each example. Here you can see the results we got when we ran these experiment on the processor:

CNOT test: May 3rd, 2016 9:21 pm

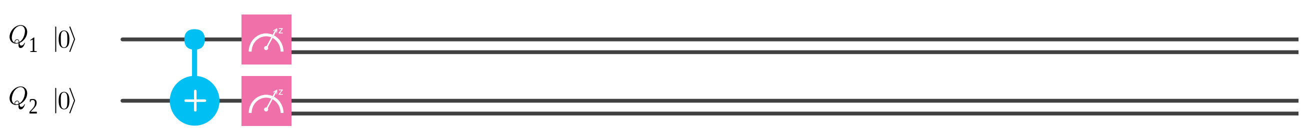
	00	01	10	11
00	0.965	0.014	0.007	0.014
01	0.040	0.943	0.009	0.008
10	0.010	0.026	0.038	0.926
11	0.026	0.010	0.947	0.017

Finally, many quantum algorithms (https://en.wikipedia.org/wiki/Quantum_algorithm) use the Hadamard gate as the first step because they map n qubits prepared in state $|0\rangle^{\otimes n}$ to a superposition of all 2^n orthogonal states with equal weight. Try out the five qubit version. You should see that it has made a quantum sphere that points in all directions with a small weight $1/(2^5)$. Try adding the CNOT gate and making your own new complicated quantum states. In the next sections we will show you examples of how quantum computers take advantage of strange states known as entangled states.

Random Classical Circuit



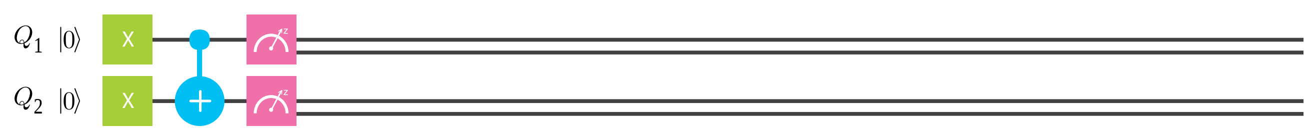
CNOT (with input 00)



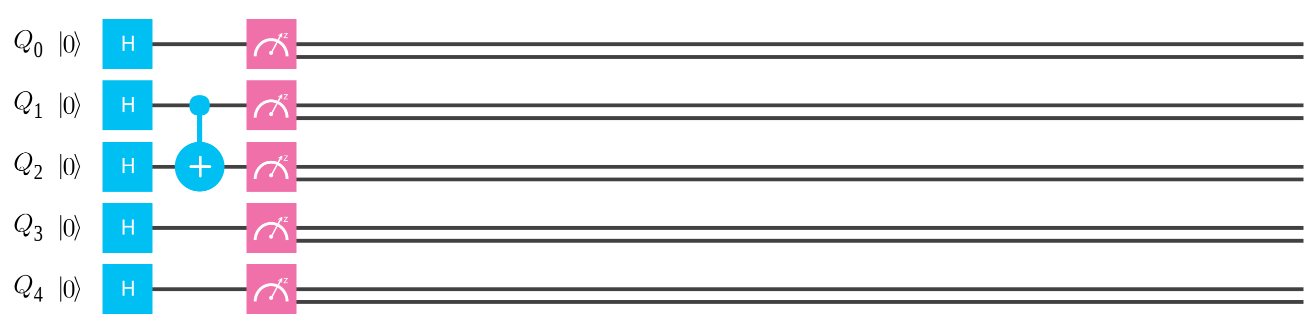
CNOT (with input 01)

CNOT (with input 10)

CNOT (with input 11)



Complete Superposition Circuit



Non-Clifford Gates

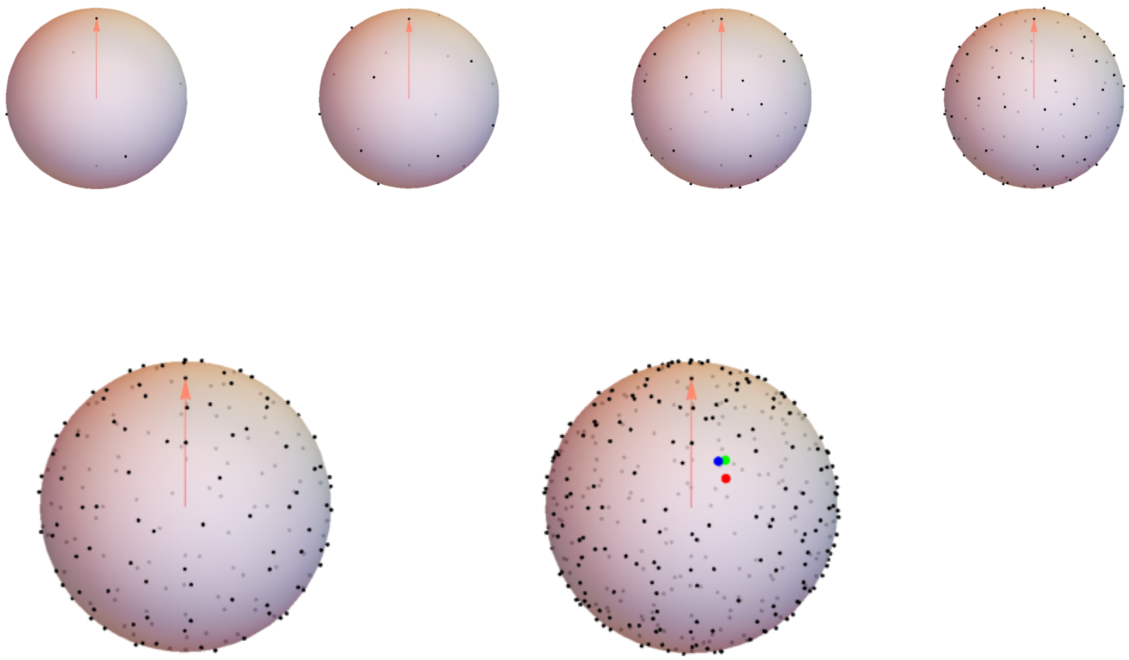
The most general operation that a quantum computer can perform is a unitary matrix in 2^n dimensions. A finite set of gates that can approximate any unitary matrix arbitrarily well is known as a universal gate set (https://en.wikipedia.org/wiki/Quantum_gate#Universal_quantum_gates). This is similar to how certain sets of classical logic gates, such as {AND, NOT}, are functionally complete (https://en.wikipedia.org/wiki/Functional_completeness) and can be used to build any Boolean function.

Up to this point, all the gates we have discussed ($X, Y, Z, H, S, S^\dagger$, and CNOT) are members of a special group of gates known as the Clifford group. These gates can be simulated efficiently on a classical computer (see the Gottesman-Knill (https://en.wikipedia.org/wiki/Gottesman%E2%80%93Knill_theorem) theorem). Therefore, the Clifford group is not universal. It cannot harness the full power of quantum computation; for that, we must include at least one non-Clifford gate in our circuits.

Any unitary matrix can be written as a combination of single- and two-qubit gates [*Barenco et al., 1995* (http://journals.aps.org/pra/abstract/10.1103/PhysRevA.52.3457?cm_mc_uid=43781767191014577577895&cm_mc_sid_50200000=1460741020)]. (This is unlike classical reversible computing, where 3-bit gates such as Toffoli (https://en.wikipedia.org/wiki/Toffoli_gate)are additionally required for functional completeness.) It turns out that adding almost any non-Clifford gate to single-qubit Clifford gates and CNOT gates is universal. There are several popular choices for non-Clifford gates, but we implement T as well as $T^\dagger$. These are given by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \qquad T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

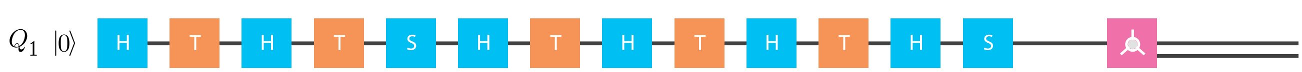
The T gate essentially makes it possible to reach all different points of the Bloch sphere. We can see that by increasing the number of T-gates in our circuit (the so-called T*-depth*) we start to cover the Bloch sphere more densely with states we can reach. The following figures depict the attainable states by increasing T-depth from 0, 1, ... up to 5. In the final Bloch sphere for T-depth 5, we have highlighted a few points in red, green, and blue, which correspond to the Clifford+T scores given below. Run these circuits to see if you end up at those points!



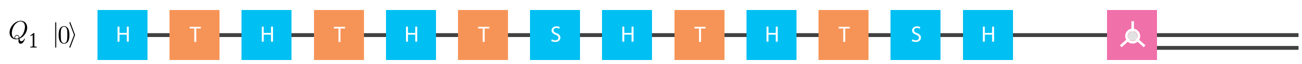
T-Depths are 0, 1, 2, 3, 4, and 5.

Red State

Green State



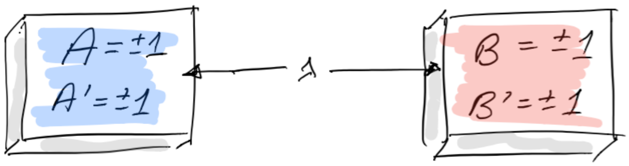
Blue State



Entanglement and Bell Tests

One of the infamous counterintuitive ideas of quantum mechanics is that two systems that appear too far apart to influence each other can nevertheless behave in ways that, though individually random, are too strongly correlated to be described by any classical local theory.

To understand this, we have outlined a simple Bell test experiment here. Imagine you have two systems (see blue and red systems below). Within each there are two measurements performed: A, A', B and B', that have outcomes 1 (or -1). Bell showed that if these measurements are chosen correctly for a given entangled state, the statistics can not be explained by any local hidden variable theory, and that there must be correlations that are beyond classical.



In 1969 John Clauser (https://en.wikipedia.org/wiki/John_Clauser), Michael Horne (https://en.wikipedia.org/w/index.php?title=Michael_Horne&action=edit&redlink=1), Abner Shimony (https://en.wikipedia.org/wiki/Abner_Shimony), and Richard Holt (https://en.wikipedia.org/w/index.php?title=Richard_Holt_%28physicist%29&action=edit&redlink=1) derived the following CHSH inequality $|\mathrm{C}| \leq 2$ where

$$C = \langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle$$

and the correlated expectation is given by

$$\langle AB \rangle = Pr(1,1) + P(0,0) - P(0,1) - P(0,1)$$

with 0 giving outcome $+1$ and 1 giving outcome -1. A correlation of 1 means both observables have even parity, and a correlation of -1 means both observables have odd parity.

It is simple to show that this inequality must be true if the theory obeys the following two assumptions, *locality* and *realism*:

Locality: No information can travel faster than the speed of light. There is a hidden variable λ that defines all the correlations so that

$$\langle AB \rangle = \sum_{\lambda} Pr(\lambda) A(\lambda) B(\lambda)$$

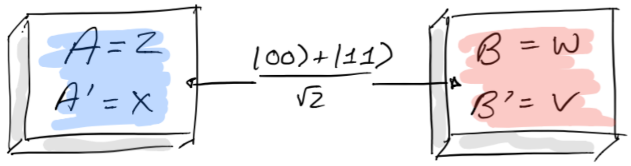
and C becomes

$$C = \sum_{\lambda} P(\lambda) \left[A(\lambda)\left(B(\lambda) - B'(\lambda)\right) + A'(\lambda)\left(B(\lambda) + B'(\lambda)\right) \right]$$

Realism: All observables have a definite value independent of the measurement (+1 or -1). This implies that either $|B(\lambda) + B'(\lambda)| = 2$ (or 0) while $|B(\lambda) - B'(\lambda)| = 0$ (or 2) respectively. That is, $|\mathrm{C}| = 2$, and noise will only make this smaller.

Perfectly reasonable, right? However, as you see, $|C| > 2$. How is this possible? The above assumptions must not be valid, and this is one of those astonishing counterintuitive ideas one must accept in the quantum world. Before you launch the scores below, let's try to understand what is happening and how each observable is measured and combined to give $|C|$.

The Bell experiment we have provided uses the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and the two measurements for system A are Z and X, while the two for B are $W = \frac{1}{\sqrt{2}}(Z + X)$ and $V = \frac{1}{\sqrt{2}}(Z - X)$.



For an ideal implementation the four correlated expectation values give,

$$\langle ZW\rangle = 1/\sqrt{2} \quad \langle ZV\rangle = 1/\sqrt{2} \quad \langle XW\rangle = 1/\sqrt{2} \quad \langle XV\rangle = -1/\sqrt{2}$$

which gives $|C| = 2\sqrt{2}$.

To run this experiment with our hardware we need the following quantum score and 4 measurements.

In the first part of the experiment the qubits are initially prepared in the ground state $|00\rangle$. The H takes the first qubit to the equal superposition $\frac{|00\rangle + |10\rangle}{\sqrt{2}}$ and the CNOT gate flips the second qubit if the first is excited, making the state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$. This is the entangled state (commonly called a *Bell state*) required for this test. In the first experiment the measurements are of the observable Z and $W = \frac{1}{\sqrt{2}}(X + Z)$. To rotate the measurement basis to the W axis, use the sequence of gates S-H-T-H and then perform a standard measurement. The correlator $\langle ZW\rangle$ should be close to $1/\sqrt{2}$ and is found using the above equation.

In the second experiment the two observables are Z and $V = \frac{1}{\sqrt{2}}(-X + Z)$. To rotate to this basis we use the sequence of gates S-H-$T^\dagger$-H and then perform a standard measurement. The correlator $\langle ZV\rangle$ is found in a similar way as before and should be close to $1/\sqrt{2}$.

Finally, in the third and fourth experiment the correlators $\langle XW\rangle$ and $\langle XV\rangle$ are measured and should be close to $1/\sqrt{2}$ and $-1/\sqrt{2}$ respectively. The W and V measurement is performed the same way as above and the X via a Hadamard gate before a standard measurement.

Here you can see the results we got when we ran this experiment on the processor:

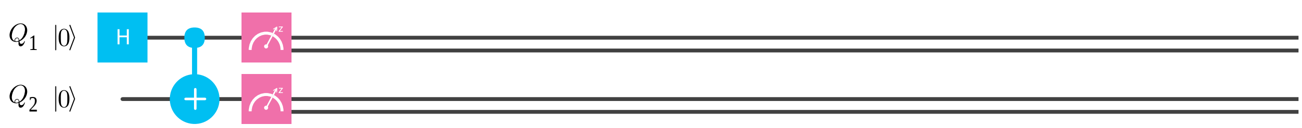
Belltest: 8192 shots May 2nd 11:44pm

	P(00)	P(11)	P(01)	P(10)	⟨AB⟩
ZW	0.434	0.380	0.070	0.116	0.629
ZV	0.409	0.415	0.100	0.076	0.648
XW	0.452	0.375	0.090	0.083	0.654
XV	0.110	0.077	0.451	0.36	-0.626

$|C| = 2.56 \pm 0.03$

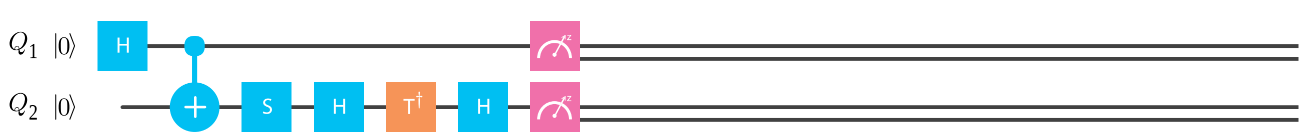
Try it out for yourself! Compare what we got with the simulations (with both ideal and realistic parameters).

Bell State ZZ Measurement



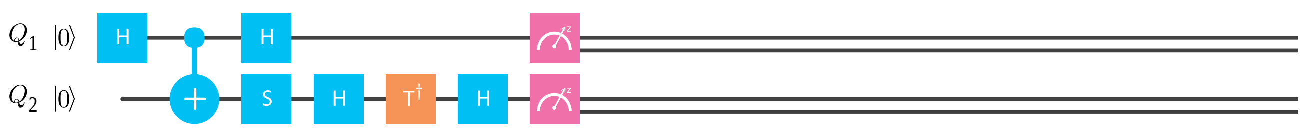
Bell State ZW Measurement

Bell State ZV Measurement



Bell State XW Measurement

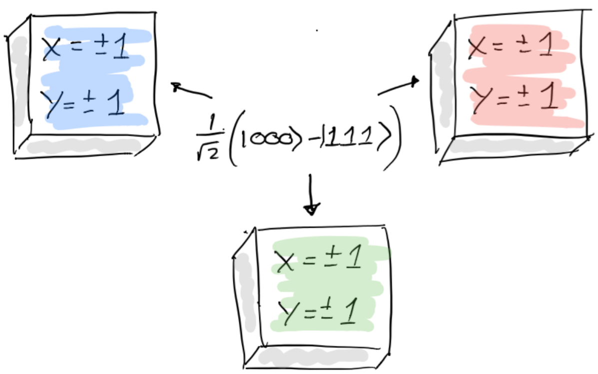
Bell State XV Measurement



GHZ States

Perhaps even stranger than Bell states are their three-qubit generalization, the *GHZ states.* An example of one of these states is $\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$. The measured results should be half $|000\rangle$ and half $|111\rangle$. GHZ states are named after Greenberger, Horne, and Zeilinger, who were the first to study them in 1997. GHZ states are also known as "Schroedinger cat states" or just "cat states."

In the 1990 paper by N. David Mermin, *What's wrong with these elements of reality?*, the GHZ states demonstrate an even stronger violation of local reality than Bell's inequality. Instead of a *probabilistic* violation of an inequality, the GHZ states lead to a *deterministic* violation of an equality.



Imagine you have three independent systems which we denote by a blue, red, and green box. You are asked to solve the following problem: in each box there are two questions, labeled X and Y, that have only two possible outcomes, $+1$ or -1. You must come up with a solution to the following set of identities.

$XYY = 1.$

$YXY = 1.$

$YYX = 1.$

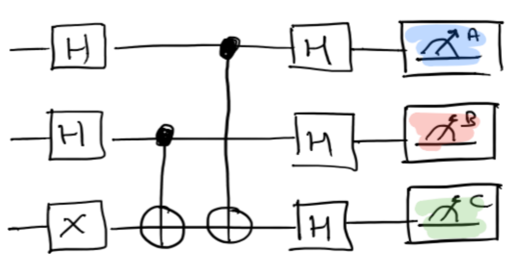
$XXX = -1.$

Try it!

After a while you will realize this is not possible. The simple way to show this is the following: if we multiply the first three equations together, we can simplify squared quantities and obtain $XXX = 1$, which contradicts the fourth identity.

Amazingly enough, a GHZ state can provide a solution to this problem. Then we have to accept what quantum mechanics teaches us: there are not local hidden elements of reality associated with each qubit which predetermine the outcomes of measurements in the X and Y bases. So, as Mermin pointed out, the GHZ test described above contradicts the possibility of physics being described by local reality! As opposed to the Bell test, which provides only a statistical evidence for the contradiction, the GHZ test can rule out the local reality description with certainty after a single run of the experiment (not including the effects of noise and imperfections in our system).

To make this state we use the following circuit, which is slightly different to the standard way of creating a GHZ (in our hardware the CNOT gates are constrained in their orientation). In the first part of the circuit, the ground state is taken to a superposition $\frac{1}{2}(|001\rangle + |011\rangle + |101\rangle + |111\rangle)$. The two CNOTs now entangle



all the qubits into the state $\frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$. The final three Hadamard gates map this to the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$.

To make the measurements in the X and Y basis we again rotate the measurement using the circuits you have seen before. For example, consider the XXX measurement. Note that flipping all three qubits of the GHZ state gives the same state with the minus sign. In other words, the GHZ state is a -1 eigenvector of a three-qubit Pauli operator XXX. This implies

$$XXX = -1$$

for each realization of the experiment. Next consider the Pauli operator XYY. One can check that the GHZ state is a $+1$ eigenevector of XYY. Therefore,

$$XYY = 1$$

for each realization of the experiment. Likewise,

$YXY = 1$, and $YYX = 1$.

One can verify this by running the experiments using the circuits provided below.

Here you can see the results we got when we ran this experiment on the processor:

GHZ test: 8192 shots May 3rd 1:20 AM

	P(000)	P(001)	P(010)	P(011)	P(100)	P(101)	P(110)	P(111)
YYX	0.201	0.038	0.016	0.114	0.030	0.258	0.258	0.086
XXX	0.046	0.226	0.125	0.007	0.250	0.025	0.094	0.228
YXY	0.222	0.042	0.012	0.111	0.034	0.254	0.251	0.074
XYY	0.227	0.039	0.013	0.128	0.033	0.247	0.229	0.084

$\langle YYX\rangle = 0.661$ $\quad\langle XXX\rangle = -0.657$ $\quad\langle YXY\rangle = 0.675$ $\quad\langle XYY\rangle = 0.661$

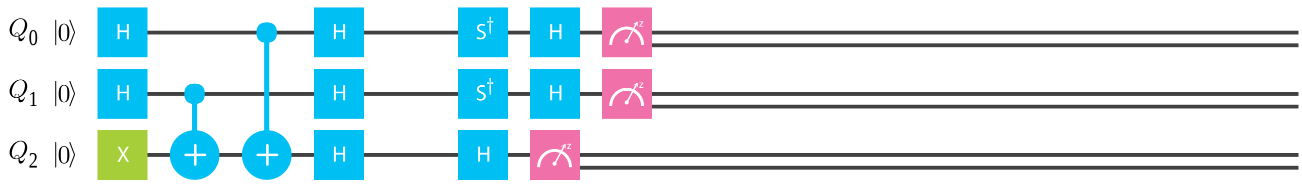
$$m = \langle YYX\rangle\langle XXX\rangle\langle YXY\rangle\langle XYY\rangle = -0.194 \pm 0.005$$

EXAMPLE CIRCUITS:

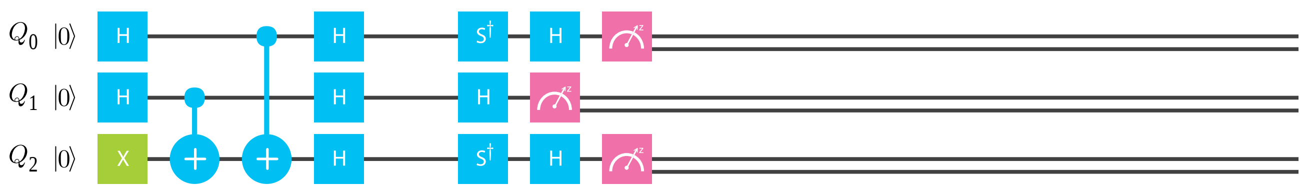
The first circuit shown below creates a GHZ state and then measures all qubits in the standard basis. The measured results should be half 000 and half 111. The remaining four circuits describe the GHZ test. Each circuit prepares the GHZ state and then measures the three qubits by choosing the measurement bases according to YYX, YXY, XYY, and XXX respectively.

GHZ State

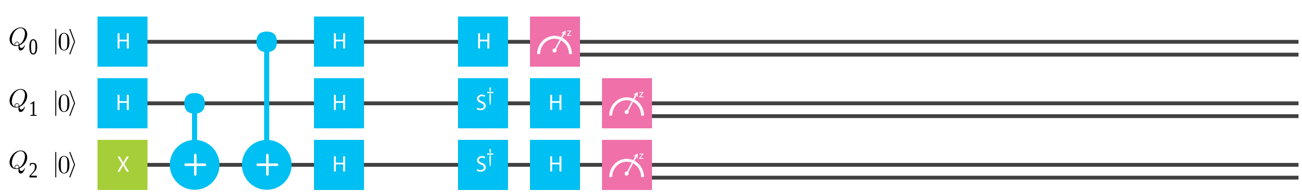
GHZ State YYX Measurement



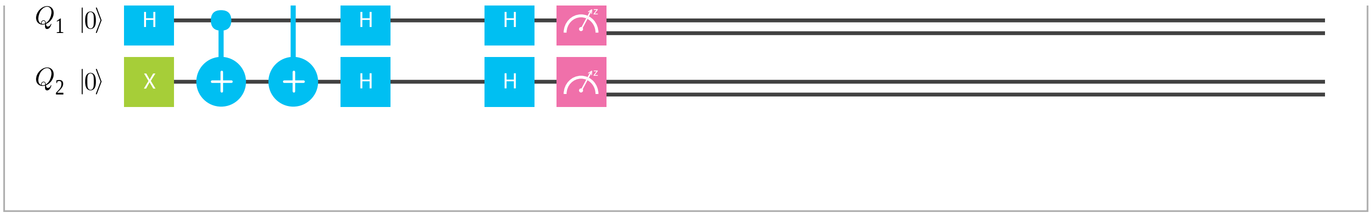
GHZ State YXY Measurement



GHZ State XYY Measurement



GHZ State XXX Measurement



Q_1 $|0\rangle$
Q_2 $|0\rangle$
H
X
H
H
H
H