Trig. Integrals & Substitutions

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(Median 82. $A \ge 90$, $B \ge 75$, $C \ge 65$)

Basic:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left.\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned}\right\} \text{double angle formula.}$$

降角要升级

$$\left.\begin{aligned} \cos^2\theta &= \frac{1+\cos 2\theta}{2} \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \end{aligned}\right\} \text{half angle formula.}$$

降级要升角

Know:

$d\sin x = \cos x\,dx$: $\int \cos x\,dx = \sin x + C$

$d\cos x = -\sin x\,dx$: $\int \sin x\,dx = -\cos x + C$

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① $\int \sin^n x \cdot \cos^m x\,dx$ $(m, n = 0, 1, 2, \cdots)$

easy case: at least one exponent is odd.

(原型 ^{a}, 导数后 $^{a-1}$, $a+(a-1) = $ odd)

$\int A u u'\,du$

Ex $m=1$, $\int \sin^n x \cos x\,dx$

A: $u = \sin x$, $du = \cos x\,dx$ (substitution)

$$\begin{aligned} \text{ans} &= \int u^n \cdot du \\ &= u^{n+1} \cdot \frac{1}{n+1} + C \\ &= \sin^{n+1} x / n+1 + C \end{aligned}$$

Ex2: $\int \sin^3 x \cos^2 x\,dx$

odd = evens + 1 → form $\sin^1 x\,dx$ → du

use $\sin^2\theta + \cos^2\theta = 1$ to convert & unify to another trig.

$$\begin{aligned} \text{ans} &= \int (1-\cos^2 x)\sin x \cdot \cos^2 x\,dx \\ &= \int (1-\cos^2 x)\cos^2 x \cdot \sin x\,dx \\ &= \int (\cos^2 x - \cos^4 x) \cdot \sin x\,dx \end{aligned}$$

let $u = \cos x$, $du = -\sin x\,dx$

$$\begin{aligned} &= \int (u^2 - u^4) \cdot (-du) \\ &= \int (u^4 - u^2)\,du \\ &= u^5/5 - u^3/3 + C \\ &= \cos^5 x/5 - \cos^3 x/3 + C \end{aligned}$$

Ex $\int \sin^3 x \, dx$

$= \int (1-\cos^2 x) \cdot \sin x \, dx$

let $u = \cos x$, $du = -\sin x$

$= \int (1-u^2)(-du)$

$= \int (u^2-1) \, du$

$= u^3/3 - u + C$

$= \cos^3 x/3 - \cos x + C$

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② harder case: only even exps (m, n)

降级升角，直到奇数出现.

Ex 1: $\int \cos^2 x \, dx$

$= \int \frac{1+\cos 2x}{2} dx$

$= \int \frac{1}{2} dx + \frac{1}{2}\int \cos 2x \, dx$ < $u = \sin 2x$

$= \frac{1}{2}x + C_1 + \frac{1}{2} \cdot \frac{1}{2} u + C_2$

$= x/2 + \sin 2x/4 + C$

Ex 2: $\int \sin^2 x \cos^2 x \, dx$

$= \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} dx$

$= \frac{1}{4} \int (1-\cos^2 2x) \, dx$

$= \frac{1}{4}\int dx - \frac{1}{4} \int \cos^2 2x \cdot \frac{d2x}{2}$ Ex 1.

$= \frac{1}{4}x - \frac{1}{8}\left(2x/2 + \sin(2 \cdot 2x)/4\right) + C$

$= x/8 - \sin(4x)/32 + C$

or $= \int (\sin x \cos x)^2 dx$

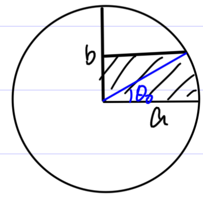
$= \int \left(\frac{\sin 2x}{2}\right)^2 dx$

$= \int \frac{\sin^2 2x}{4} dx$ 降级

$= \frac{1}{4}\int \frac{1-\cos 4x}{2} dx$

App & example of Trig. Substitution

Ex: $A = \int_0^b \sqrt{a^2 - y^2}\, dy$



$y = a\sin\theta$

$dy = a\cos\theta\, d\theta$

$y_1 = 0 = a\sin\theta_1 \Rightarrow \theta_1 = 0$

$y_2 = b = a\sin\theta_2 \Rightarrow \theta_2 = \sin^{(-1)}\frac{b}{a}$

$$A = \int_0^{\sin^{(-1)}\frac{b}{a}} \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\, d\theta$$

$$= \int_0^{\sin^{(-1)} b/a} a \cdot \cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 \int_0^{\sin^{(-1)} b/a} \underbrace{\cos^2\theta\, d\theta}_{}$$

$$= a^2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C \Big|_0^b \quad \text{Ex 1.}$$

indefinite inte.

$$\int = a^2\left(\frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4}\right) + C$$

$$= \frac{a^2 \arcsin(y/a)}{2} + \frac{a\sin\theta \cdot a\cos\theta}{2} + C$$

$$= \frac{a^2 \arcsin(y/a)}{2} + \frac{y \cdot \sqrt{a^2 - y^2}}{2} + C$$

$$A = \int_0^b = \left[\qquad + C\right]\Big|_0^b = \frac{a^2 \arcsin(b/a)}{2} + \frac{b \cdot \sqrt{a^2 - b^2}}{2}$$

$$= \underbrace{\frac{a^2\theta_0}{2}}_{\text{area of sector}} + \underbrace{\frac{b \cdot \sqrt{a^2 - b^2}}{2}}_{\text{area of triangle}}$$

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even: down grade order

odd: $(\square^{-1} dx)$ then $\int$