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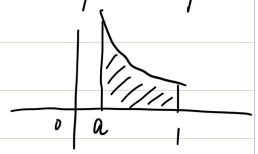
~ Improper Integrals (2nd kind)

have singularity at a finite place.

DEF. $\int_0^1 f(x)dx = \lim_{a\to 0^+} \int_a^1 f(x)dx$

converges if the limit exists

diverges if not.



Ex 1. $\int_0^1 \frac{dx}{\sqrt{x}}$

$= 2x^{1/2}\Big|_0^1$

$= 2-0$

$= 2$

Ex 2. $\int_0^1 \frac{dx}{x}$

$= \ln x\Big|_0^1$

$= \ln 1 - \ln 0^+$

$= 0 - (-\infty)$

$= \infty$

diverges.

In general. $\int_0^1 \frac{dx}{x^p}$

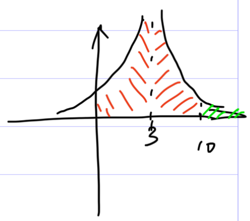
$= \frac{1}{1-p}$ $(p<1)$ | $=$ diverges $(p \geq 1)$

Contrast: div

$x \to 0^+$: $\frac{1}{x^{1/2}} \ll \boxed{\frac{1}{x}} \ll \boxed{\frac{1}{x^2}}$

$x \to \infty$: $\boxed{\frac{1}{x^{1/2}}} \gg \boxed{\frac{1}{x}} \gg \frac{1}{x^2}$

singularity.



$$\int_0^\infty \frac{dx}{(x-3)^2} = \int_0^5 \frac{dx}{(x-3)^2} + \int_5^\infty \frac{dx}{(x-3)^2}$$

$\int_0^5 \frac{dx}{(x-3)^2} \approx 2\cdot\int_0^1 \frac{dx}{x^2}$ ⇒ diverges.

$\int_5^\infty \frac{dx}{(x-3)^2} \sim \int_5^\infty \frac{dx}{x^2}$ ⇒ converges.

$\int_0^a$ converges

diverges

Infinite Series

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \cdots = 2$

Geometric Series:

$1+a+a^2+a^3+\cdots = \frac{1}{1-a}$ $\qquad \left(\frac{q^n-1}{q-1}\right)$

when $|a|<1$

Divergences:

$a=1$, $1+1+1\cdots = \frac{1}{1-1} = \frac{1}{0}$ diverge.

$a=-1$, $1-1+1-1\cdots = \frac{1}{1-(-1)} = \frac{1}{2}$ diverges.

$1 \quad 0 \quad 1 \quad 0$

$a=2$, $1+2+2^2+2^3+\cdots \overset{?}{=} \frac{1}{1-2} = -1$ diverges

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Notation:

$S_N = \sum_{n=0}^{N} a_n$ (partial sum)

$S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$ $\begin{cases} \text{limit exists: the series converges.} \\ \text{limit does not exist: the series diverges.} \end{cases}$

Ex 1. $\sum_{n=1}^{\infty} \frac{1}{n^2} \Leftrightarrow \int_1^{\infty} \frac{dx}{x^2}$ convergent.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ analogy $\int_1^{\infty} \frac{dx}{x^2} = 1$ (tail behavior)

Connection: use Riemann sum with $\Delta x = 1$.

Ex 2. $\sum_{n=1}^{\infty} \frac{1}{n^3} \Leftrightarrow \int_1^{\infty} \frac{dx}{x^3}$ convergent

$\sum_{n=1}^{\infty} \frac{1}{n^3} =$ irrational num. $\qquad \int_1^{\infty} \frac{dx}{x^3} = 1/2$

Ex 3: $\sum_{n=1}^{\infty} \frac{1}{n} \Leftrightarrow \int_1^{\infty} \frac{dx}{x}$ diverges.

① upper Riemann sum:

$\int_1^{\infty} \frac{dx}{x} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N-1} < ANS$ (S_N)

$= \ln N - 0$

$\Rightarrow \ln N < S_N$, $N\to\infty$

S_N divergence.

② lower Riemann sum:

$\ln N = \int_1^{\infty} \frac{dx}{x} > \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} = S_N - \frac{1}{1}$

$S_N < \ln N + 1$

$\therefore \ln N < S_N < \ln N + 1$

Infinite Series Converge Criteria & proof.

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Integral Comparison (btw Σ and $\int$)

If $f(x) \searrow$, $f(x) > 0$:

Then $\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx \right| < f(1)$;

and they converge or diverge together.

Limit Comparison:

If $f(n) \sim g(n)$ (i.e. $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$)

And $g(n) > 0$ for all n

Then $\Sigma f(n)$, $\Sigma g(n)$ converge or diverge together.

Ex $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}} \Leftrightarrow \sum_{1}^{\infty} \frac{1}{\sqrt{n^2}} = \Sigma \frac{1}{n}$, diverges.

Ex $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^5-n^2}} \Leftrightarrow \Sigma \frac{1}{n^{5/2}}$, converges.

$$\sum_{a}^{\infty} f(n) \text{ simplify to } \begin{cases} \int_a^{\infty} f(x)dx & (f = f) \\ \sum_{a}^{\infty} g(x) & (f \sim g) \end{cases}$$

$\to 0$