~ Dealing with ∞

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$$\left\{\begin{array}{l} \dfrac{\infty}{\infty} \\ \\ \dfrac{f'(x)}{g'(x)} \to L \end{array}\right. \quad \text{as } x \to a$$

$$\Downarrow$$

$$\frac{f(x)}{g(x)} \to L \quad \text{as } x \to a$$

$a = \pm\infty$
$L = \pm\infty$ are OK.

Rates of growth:

Notation:

$f(x) \ll g(x)$ means $\dfrac{f(x)}{g(x)} \to 0$
$(x \to \infty)$ $(f_x, g_x > 0)$ $(x \to \infty)$

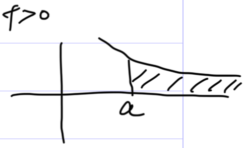
$\ln x \ll x^p \ll e^x \ll e^{x^2} \to \infty$ $(p > 0)$

Rates of decay:

$\dfrac{1}{\ln x} \gg \dfrac{1}{x^p} \gg \dfrac{1}{e^x} \gg \dfrac{1}{e^{x^2}}$

Relative growth.

Improper Integrals.



DEF: $\int_a^\infty f(x)dx = \lim_{N\to\infty} \int_a^N f(x)dx$,

the integral converges if limit exists. → area is finite.

diverges if not. → area is infinite.

Ex 1. $\int_0^\infty e^{-kx}dx \quad (k>0)$

$$\int_0^N e^{-Nx}dx = -\frac{1}{k}e^{-kx}\Big|_0^N = -\frac{1}{k}e^{-kN} - \left(-\frac{1}{k}\right), \text{ as } N\to\infty$$

$$-\frac{1}{k}e^{-kN} \to 0, \quad -\left(-\frac{1}{k}\right) \to -\frac{1}{k}$$

Short hand:

$$\int_0^\infty e^{-kx}dx = -\frac{1}{k}e^{-kx}\Big|_0^\infty = -\frac{1}{k}e^{-k\infty} - \left(-\frac{1}{k}\right) = 0 + \frac{1}{k}.$$

e.g. Radioactivity decay, in $0\le t \le T$

$\int_0^T Ae^{-kt}dt$: # particals decaied.

$\int_0^\infty Ae^{-kt}dt$ = total # of ...

$(T\to\infty)$

Ex 2 : $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$, key number in probability.

Ex 3 : $\int_1^\infty \frac{dx}{x}$ (border line case)

$$= \ln x\Big|_1^\infty = \ln\infty - \ln 1 = \infty - 0 = \infty$$

$$\int_1^\infty \frac{dx}{x}$$

$$= x^{-p+1}/_{-p+1}\Big|_1^\infty$$

$$= \frac{\infty^{-p+1}}{-p+1} - \frac{1}{-p+1} \begin{array}{l} \nearrow = \infty \text{ when } (-p+1)>0 \Rightarrow p<1 \\ \searrow = \frac{1}{p-1} \text{ when } (-p+1)<0 \Rightarrow p>1 \end{array}$$

$p\le 1$: diverge

$p>1$: converge

for $\int_1^\infty \frac{dx}{x^p}$.

$x\to\infty$, $x^{-p+1} \begin{array}{l}\nearrow x^{+n} \to \infty \\ \searrow x^{-n} \to \frac{1}{x^n} \to \frac{1}{\infty} \to 0\end{array}$

Converge
Diverge of improper inte $\int_a^\infty$

Limit Comparison. means $f(x)/g(x) \to 1$. as $x \to \infty$.

If $f(x) \sim g(x)$, as $x \to \infty$,

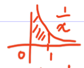
Then $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$: (a large a, tail) only interested in tail.

either: both converge;

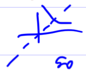
or: both diverge.

Ex. $\int_0^\infty \frac{dx}{\sqrt{x^2+10}} \sim \int_0^\infty \frac{dx}{\sqrt{x^2}} \sim \int_1^\infty \frac{dx}{x}$. diverges.

$\frac{\sqrt{x^2+10}}{\sqrt{x^2}} \to 1$



is infinite for other reasons.



symm. so also diverges.

Ex 2. $\int_{10}^\infty \frac{dx}{\sqrt{x^3+3}}$

$\sim \int_{10}^\infty \frac{dx}{x^{\frac{3}{2}}}$

$p = \frac{3}{2} > 1$, converges.

Ex 3. Check convergence of $\int_{-\infty}^\infty e^{-x^2}\,dx$

$= 2\int_0^\infty e^{-x^2}\,dx$

$\leq 2\int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x}\,dx$

definite — converge.

$e^{-x^2} \leq e^{-x}$ when $x \geq 1$

$\frac{f(x)}{g(x)} \to 1 \Rightarrow$ $\int_0^\infty f(x)$. $\int_0^\infty g(x)$. relative.

Improper Integral of 2nd Type.

$\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_0^1 \frac{dx}{x}$, $\int_0^1 \frac{dx}{x^2}$

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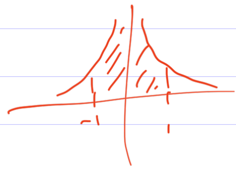
$\frac{1}{2}\int_{-1}^{1} \frac{dx}{x^2}$

$= x^{-1}\Big|_{-1}^{1}$

$= -2/2 = -1$

yet $\frac{1}{x^2} > 0$



be care when calculating on ∞ points.

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$\frac{1}{x}$ $\left(\frac{1}{x}\right)^2$ doubled height.

↑ diverges ↑ more diverges

$\int_0^1$ convergence.