L'Hôpital's Rule

(or say L'Hospital)

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Convenient way to calculate limits (including new ones)

$x \ln x \quad x \to 0^+$

$x e^{-x} \quad x \to \infty$

$\frac{\ln x}{x} \quad x \to \infty$

Ex 1. $\lim_{x\to 1} \frac{x^{10}-1}{x^2-1}$ $\quad \frac{0}{0}$, indeterminate form.

$= \lim_{x\to 1} \frac{(x^{10}-1)/(x-1)}{(x^2-1)/(x-1)}$

$= \frac{10x^9}{2x}$

$= 5$

$f(x) = x^{10}-1$

$f(1) = 0$

$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{\Delta t\to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}$

$= f'(t)\big|_{t=0}$

$\to f'(x)\big|_{x=1}$

$= (x^{10}-1)'$

$= 10x^9\big|_1$

$= 10$

$\lim_{x\to a} \frac{f(x)}{g(x)}$ $\quad (f(a) = g(a) = 0)$

$= \frac{f(x)/(x-a)}{g(x)/(x-a)}$ $\quad$ $x \to a$, so $x \neq a$, $x-a \neq 0$; we can do $/(x-a)$ and it will help.

$= \frac{\frac{f(x)-0}{x-a}}{\frac{g(x)-0}{x-a}}$

$= \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(0)}{x-a}}$ $\quad$ now we see the form.

$= \frac{f'(a)}{g'(a)}$ $\quad$ (requires that $g'(a) \neq 0$, and works if so)

derive L'Hospital Rule

L'Hôpital's Rule (VI)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided 1. $f(x)=0,\ g(x)=0$ and (!!!)

2. the right-hand limit (right side of equation) exists. (you will calc it)

Ex 2 $\lim_{x\to 0}\frac{\sin 5x}{\sin 2x}$

A: $\because$ $\sin' 5x$ exists $= 5\cos 5x$

$\sin' 2x$ exists $= 2\cos 2x$

$\therefore$ ans $= \frac{\sin' 5x}{\sin' 2x}\lim_{x\to 0}$

$= \frac{5\cos 5x}{2\cos 2x}\lim_{x\to 0}$

$= \frac{5}{2}$

Ex 3. $\lim_{x\to 0}\frac{\cos x - 1}{x^2}$

$= \lim_{x\to 0}\frac{-\sin x}{2x}$ apply again

$= \lim_{x\to 0}\frac{-\cos x}{2}$

$= \frac{-\cos 0}{2}$

$= -\frac{1}{2}$

comparison with (linear) approx.

$\sin u \approx u \quad (u\to 0)$

Ex 2 $\frac{\sin 5x}{\sin 2x} \approx \frac{5x}{2x} \approx \frac{5}{2}$, when $x\to 0$

Ex 3 $\frac{\cos x - 1}{x^2} \approx \frac{(1-\frac{x^2}{2})-1}{x^2} = -\frac{1}{2}$

L'H def, Criteria, $\frac{0}{0}$

& usage.

Other cases:

$a = \pm\infty$ ok. (a is a in $f(a)$)

$f(a), g(a)$ $\pm\infty$ ok. $(\frac{\infty}{\infty})$

right hand part exists or $\pm\infty$ (then left $\leq$ right $= \pm\infty$)

Ex 4: $\lim_{x\to 0^+} x\ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ $\quad \frac{\infty}{\infty}$, appl.

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim_{x\to 0^+} -x$

$= 0$

Ex 5. $\lim_{x\to\infty} x e^{-px}$ $\quad (p>0)$

$= \lim_{x\to\infty} \frac{x}{e^{px}}$ $\quad \frac{\infty}{\infty}$

$\sim \frac{1}{pe^{px}}$

$= \frac{1}{\infty}$

$= 0$

Ex 5'. $\lim_{x\to\infty} \frac{e^{px}}{x^{100}}$

$= \lim_{x\to\infty} \left(\frac{e^{\frac{px}{100}}}{x}\right)^{100}$

$= \left(\lim_{x\to\infty} \frac{e^{\frac{px}{100}}}{x}\right)^{100}$

$\sim \left(\frac{\frac{p}{100} e^{px/100}}{1}\right)^{100}$

$= (\frac{\infty}{1})^{100}$

$= \infty$

e^{px} grows faster than any power of x $\quad (p>0)$

Ex 6. $\frac{\ln x}{x^{1/3}}$ $\quad (x\to\infty)$ $\quad (\frac{\infty}{\infty})$

$\sim \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$

$= \frac{3}{x^{1/3}} \Big|_{x\to\infty}$

$= \frac{3}{\infty}$

$= 0$

$\ln x$ grows slower than any power of x

Other conditions

$\frac{\infty}{\infty}$ ✓, 吕 $\nearrow \infty$, $\searrow -\infty$ ✓, $\frac{f(\infty)}{g(\infty)}$ ✓

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$(\frac{0}{0}, \frac{\infty}{\infty})$ 0^0 :

$$\lim_{x\to 0^+} x^x$$

$$x^x = e^{\ln x^x} = e^{x\ln x} \sim e^0 = 1.$$

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e.g. $\lim_{x\to 0^+} \frac{\sin x}{x^2} \frac{0}{0} \sim \frac{\cos x}{2x} \frac{\infty}{0} \sim \frac{-\sin x}{2} = 0$ $\times$

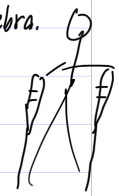
Look before L'HOP.

$$\frac{\sin x}{x^2} \approx \frac{x}{x^2} = \frac{1}{x}\Big|_{x\to 0^+} = \infty.$$

e.g. $\frac{x^5+2x^4+\cdots}{x^4+x^3+\cdots} \approx \frac{x^5}{x^4} = \infty$

Don't use L'Hospital as a crutch.

use common sense & primary algebra.



$$0^0 \Rightarrow 0\ln 0$$