~ Partial Fractions.

20210926 23:30

methods always work. But maybe slowly.

Step 0: long division.

$$\frac{R(x)}{Q(x)} = \text{quotient} + \frac{P(x)}{Q(x)}$$

deg $P <$ deg $Q$.

1: factor denomitor $Q$. (hard)

2: setup

3: cover-up. $\left( \frac{A}{\cancel{(\ )^3}} + \frac{B}{(\ )^2} + \frac{C}{(\ )^1} + \dots, \quad \frac{Ax+B}{(x^2+\cdot)} \right)$

$x=0$ etc.

4: integrate. (ln, sub, trig, ...)

num: 1 deg less then den.

den: $(\ )^4$ all patterns 4 3 2 1 should appear.

e.g. $\frac{1}{(x^2+4)^2} = \frac{Ax^1+B}{(x^2+4)^2} + \frac{Cx^1+D}{(x^2+4)^1}$

e.g. $\int \frac{dx}{(x^2+4)^3}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$= \int \frac{2\sec^2 \, d\theta}{4^3 \sec^6\theta}$

$\frac{2}{4^3}\int \frac{d\theta}{\sec^4\theta}$

$\frac{1}{?}\int \cos^4\theta \, d\theta$

$\frac{1}{?}\int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$\frac{1}{?}\int C + 2\cos 2\theta + \cos^2 2\theta$ ✓

$\int \cos 2\theta + \int \frac{1+\cos 4\theta}{2}$ ✓ ✓

$= -\sin 2\theta$ ✓ $\quad = -\sin 4\theta$

e.g. $\int \frac{dx}{x^2+2x+3} = \int \frac{A}{i} + \frac{B}{i}$

Or $= \int \frac{dx}{(x+1)^2+2}$ (completing the square.)

$(\sqrt{2}\tan\theta)^2+2$

$= \cdots = \tan^{-1}\frac{x+1}{\sqrt{2}} \,/\, \ln(x^2+2x+3) \cdots$

Computer will do it for you.

Do not be intimidated by them.

# Integration by Parts.

$$(uv)' = u'v + uv'$$

$$uv' = (uv)' - u'v$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

if it's hard to beat him, beat his brother, on the supervision of their parents.

also $\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$

Ex 1: $\int \ln x\,dx$

$= \int \ln x \cdot \underset{v'}{1}\,dx$

$= \ln x \cdot x - \int \frac{1}{x} \cdot \underset{v}{x}\,dx$

$= x\ln x - x + C$

if treat $\ln x$ as $\square'$: we don't know the original $f$. so we can only treat $\ln x$ as original $f$, then 1 is the differenciated.

Ex 2: $\int (\ln x)^2\,dx$

$= \int \ln x \cdot \underset{v'}{\ln x}\,dx$

$= \ln x(x\ln x - x + C_1) - \int \frac{1}{x} \cdot (x\ln x - x + C_2)\,dx$

$= [\,] - \int (\ln x - 1 + \frac{C_2}{x})\,dx$

$= \int (\ln x)^2 \cdot 1\,dx$

$= x \cdot (\ln x)^2 - \int 2\ln x \cdot \frac{1}{x} \cdot x\,dx$

$= x(\ln x)^2 - 2\int \ln x\,dx$ Ex 1.

$= x(\ln x)^2 - 2(x\ln x - x) + C$

— better.

$u = (\ln x)^2$, $u' = 2\ln x \cdot \frac{1}{x}$.

$v = x$, $v' = 1$.

Ex 3. (Reduction formula)

$$\int (\ln x)^n dx$$

$u = (\ln x)^n$ , $u' = n(\ln x)^{n-1} \cdot \frac{1}{x}$

$v = x$ , $v' = 1$

$$= x \cdot (\ln x)^n - n\int (\ln x)^{n-1} dx$$

$F_n(x) = \int (\ln x)^n dx$

$F_n(x) = x(\ln x)^n - nF_{n-1}(x)$

$F_0 = x \cdot 1 - 0 = x \quad +C$

$F_1 = x(\ln x) - F_0 = x\ln x - x \quad +C$

$F_2 = x(\ln x)^2 - 2F_1 = x(\ln x)^2 - 2(x\ln x - x) + C$

Ex 4. $\int x^n e^x dx$ (also work for $x^n \sin x$, $x^n \cos x$)

$u = x^n$ , $u' = nx^{n-1}$ easier. so formula may exist.

$v = e^x$ , $v' = e^x$ don't get worse.

$$= x^n e^x - \int n x^{n-1} e^x dx$$

$F_n = x^n e^x - nF_{n-1} \quad (+C)$

So $\int xe^x dx = xe^x - e^x \quad (+C)$

(p.s. $\int \ln x \, dx = x\ln x - x \quad +C$ )

Advanced guessing.

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$\frac{1}{7} \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$\frac{1}{7} \int C + 2\cos 2\theta + \cos^2 2\theta$

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