Vector Space

→ Set of vectors

For all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ (vector space) and all scalars $r, s \in R$, the following conditions apply:

These conditions are most important

1. Closure under Addition: $\vec{v} + \vec{w} \in V$
2. closure under scalar Multiplication: $r.\vec{v} \in V$
3. Must contain zero vector: $\vec{0} \in V$
4. Commutative: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
5. Associative: $(\vec{u}+\vec{v}) + \vec{w} = u + (\vec{v}+\vec{w})$
6. Scalar distributivity: $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

Basis: $c_1 v_1 + c_2 v_2 = 0 \Rightarrow$ Check for linear Independence
minimum no. of linear independent vectors that generate all vectors

Span: set of all linear combs of vectors $\{v_1, v_2 - v_n\}$

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vector: Inner product Conditions: for vectors $\vec{u}$ and $\vec{v}$, and scalar α

1. Symmetry: $\langle \vec{u}.\vec{v} \rangle = \langle \vec{v}.\vec{u} \rangle$
2. Positive Definiteness: $\langle \vec{u}.\vec{u} \rangle \geq 0$
   Inner product with itself is non-ve or 0 if $u=0$
3. Linearity:
   Additivity: $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, w \rangle$
   Homogeneity: $\langle \alpha\vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

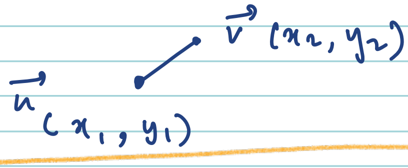
"Two vectors are orthogonal if $\langle \vec{u}.\vec{v} \rangle = 0$"

compute distance b/w two vectors

↳ Using Norms

dist b/w 2 point

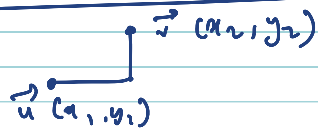
Euclidean Distance (L2 Norm)



$$d_e(\vec{u},\vec{v}) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

distance travelled in grid path
sum of absolute differences

Manhattan Distance (L1 Norm)



$$d_m(\vec{u},\vec{v}) = |x_2-x_1| + |y_2-y_1|$$

Angle b/w two vectors $\vec{x}$ and $\vec{y}$

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|u\|\,\|v\|}}$$

$\|u\|$ = Norm of $\vec{u}$ or $\sqrt{\langle \vec{x},\vec{x}\rangle}$

$\sqrt{(x_1)^2+(y_1)^2}$

$\|v\|$ = Norm of $\vec{v}$ $\sqrt{\langle \vec{y},\vec{y}\rangle}$

$\vec{u}\cdot\vec{v}$ = dot product

$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$\begin{bmatrix} x_1 & y_1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

example

$\vec{x} = (1, 2)$

$\vec{y} = (4, 6)$

$$= \left(\frac{\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix}}{\sqrt{\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}\sqrt{\begin{bmatrix}4 & 6\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix}}}\right)$$

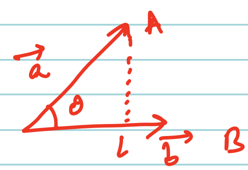
$$= \frac{4+12}{(\sqrt{1+4})(\sqrt{16+36})}$$

$$= \frac{16}{\sqrt{5}\sqrt{52}} = \frac{8}{\sqrt{65}}$$

$$\cos\theta = \frac{8}{\sqrt{65}}$$

Projections:

Mapping a vector onto another vector subspace



$|a|\cos\theta$.

projection of $\vec{a}$ on $\vec{b}$

$$\boxed{\text{proj}_b\, a = \frac{\langle a, b\rangle}{\|b\|^2}\, b}$$

example : $\vec{a} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\vec{u}$ on $\vec{b}$

$$\text{Proj}_{b}\, u = \frac{[-1 \quad 3]\begin{bmatrix}2\\-1\end{bmatrix}}{\left(\sqrt{(2)^2+(-1)^2}\right)^2}\begin{bmatrix}2\\-1\end{bmatrix}$$

$$= \frac{-2-3}{5}\begin{bmatrix}2\\-1\end{bmatrix} \Rightarrow \frac{-5}{5}\begin{bmatrix}2\\-1\end{bmatrix}$$

$$= -1\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

Project vector orthogonally to line

example:

$\vec{u}$ $\begin{bmatrix}-1\\-1\end{bmatrix}$, $y = 3x$ $\quad 3x - y = 0$, $x=1$, $y=3$ $\quad \vec{b}$ $\begin{bmatrix}1\\3\end{bmatrix}$

$$\text{proj}_{b}\, u = \frac{\langle u, b\rangle}{\|b\|^2}\, b$$

$$= \frac{(-1 \quad -1)(1 \quad 3)}{(\sqrt{1+9})^2}(1 \quad 3)$$

$$= \frac{(-1+(-3))}{(\sqrt{10})^2}(1 \quad 3) = \frac{-4}{(\sqrt{10})^2}(1 \quad 3)$$

$$= \frac{-4}{10}(1 \quad 3)$$

$$\left(\frac{-2}{5} \quad \frac{-\cancel{4}^{2}\times 3}{\cancel{10}\,5}\right) = \left(\frac{-2}{5} \quad \frac{-3}{5}\right)$$

- Data Compression
- computer Graphics (shadows)
- Data Analysis (Identify most imp features by projecting it onto principal component)

Gram Schmidt Process

Orthogonalisation Method

Orthogonalize set of vectors in an inner product space.

"Transform linearly independent (basis) vectors into orthogonal set of vectors"

<u>Steps:</u>

linearly Independent vectors

$\{v_1, v_2, v_3 \text{ ——— } v_n\}$

↓

Initialize 1st vector : $u_1 = v_1$

↓

Orthogonalize : $u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$

↓

Normalize vectors : $e_k = \frac{u_k}{\|u_k\|}$

QR Decomposition $[\,] = [\,]_O [\,]_{UTM}$

signal processing

why normalisation:

Simplify calculation

Numerical stability

example

$v_1 = (1, 1)$ $v_2 = (1, -1)$

$u_1 = v_1 = (1, 1)$

$u_2 = (1, -1) - \frac{(1,-1)(1,1)}{2}(1, 1)$

$u_2 = (1, -1) - \frac{(1-1)}{2}(1, 1)$

$u_2 = (1, -1) - 0$

$u_2 = (1, -1)$

normalize

$\|u_1\| \quad \sqrt{1+1} = \sqrt{2}$

$e_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$