Linear Algebra

Matrices : Rectangular array of numbers in rows & columns

vectors : both size & direction

$\vec{v}$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Uses: To solve linear equations
- Linear Transformations
- Eigen values & Eigen Vectors
- PCA
- Neural networks
- Linear Regression.

Linear Equations:

Represent straight line when plotted

1st power of variable is 1

$$Ax + By = C$$

$$2x + 3y = 8$$
$$y = 5x - 1$$
$$x = 4$$

Guassian elimination:

Used to solve system of linear Equations

Example:

$$x - 2y + z = 0$$
$$2x + 2y - 3z = 5$$
$$4x - 7y + z = -1$$

$$Ax = b$$

STEP 1: Augmented Matrix →

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

STEP 2: 1st Goal → 0 below 1st entry in 1st colum using Row operations

$$\begin{bmatrix} -1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{bmatrix} \xrightarrow[-4r_1 + r_3]{-2r_1 + r_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

STEP 3: 2nd goal: 0 below 2nd entry in second column

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -5 & 5 \end{bmatrix} \xrightarrow[r_2]{-5r_2 +} \underbrace{\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 10 & 10 \end{bmatrix}}_{\text{echelon form}}$$

STEP 4: Use 3rd row to find the third unknown (z) & find 2nd & 1st similarly.

$10z = 10$

$\boxed{z = 1}$

solve for 2nd unknown (y)

$y - 3z = -1$

$y = -1 + 3z$

$y = -1 + 3$

$\boxed{y = 2}$

solve for 1st unknown (x)

$x - 2y + 1 = 0$

$x = 2y - 1$

$x = 4 - 1$

$x = 3$

$\boxed{(x, y, z) \rightarrow (3, 2, 1)}$

solution of system

Cases of Guassian elimination Method:

Case 1

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & 10 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

t_{nn} c_{nn}

when $t_{nn} \neq 0$

Unique Solution

Consistent system

Case 2

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & 10 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

t_{nn} c_{nn}

when $t_{nn} = 0$, $c_{nn} \neq 0$

No solution

Inconsistent contradiction

Case 3

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & 10 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

t_{nn} c_{nn}

when $t_{nn} = 0$, $c_{nn} = 0$

"Infinitely many solutions"

Consistent

REF and RREF

REF

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

0's below pivots

all non zero rows above all zero rows

example: $\begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

after REF in linear equation

- Guassian elimination substitution can be done to get values.
- REF not for inverse matrix since it is not fully reduced.

RREF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1s at leading places (pivots) and pivot column each has only 1 non zero value

example: $\begin{bmatrix} 1 & 7 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

"Fully simplified"

- easier to interpret results.
- "Used to find inverse of matrix

Inverse of Matrices:

Use elementary operation to Transform matrix into Reduced REF

Conditions for Inverse:

- Square matrix $n \times n$
- Full rank (linearly independent)
- $\det(\text{Matrix}) \neq 0$

If A_{RREF} = full rank
Inverse Exists

If last row of A_{RREF} are all zero then it is not full rank as $^{-1}$ doesn't occur.

If A is full rank then,

$A^{-1} = E_r E_{r-1} \cdots E_1$
(all elementary)

for RREF)

Inverse of 2x2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \boxed{A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \quad 2\times2$$

exist only if $ad - bc \neq 0$

Inverse for 3x3

$$\boxed{A^{-1} = \frac{1}{|\det(A)|} \operatorname{adj}(A)} \quad 3\times3$$

exists if $\det(A) \neq 0$

Steps (3x3):
1. Find $\det(A)$ if $\neq 0$, continue
2. Find cofactor of C_{ij} each and make cofactor matrix
3. Make adjugate matrix: Transpose of cofactor Matrix
4. Divide adj (A) with det (A)

Rank Nullity Theoram

Rank : no. of linearly independent rows or cols vectors of matrix

(Non zero eigen values of matrix)

Nullity: how many independent solution exist to homo. eq $Ax = 0$

Example:

$$A \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 1$, $N(A) = 2$, $n = 3$

$1 + 2 = 3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 4$

$N(A)$ 2

$n = 6$

For any matrix A of size $m \times n$

$[A]_{m\times n}$

Rank Nollity Theoram states that:

$$\boxed{\text{Rank}(A) + \text{Nollity}(A) = n}$$

(no of columns)