Vector Calculus

Partial Differentiation

Differentiating a multivariable function w.r.t one variable while treating other variables as constants.

$$f(x,y) = x^2y + 3xy^2$$

PD w.r.t x : $\frac{\partial f}{\partial x} = 2xy + 3y^2$

<u>Used in</u>

- Optimisation problems to find local extrema

PD w.r.t y : $\frac{\partial f}{\partial y} = x^2 + 6xy$

Gradient :

Vector that contains all the first partial derivatives of $f(x,y...)$
It point in direction of steepest ascent of function

Notation : for $f(x,y)$ $\boxed{\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \quad \text{gradient}}$

row vector
vector of partials

Example : $f(x,y) = x^2 + y^2$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, 2y)$$

Properties:

mag (∇f) = rate of increase
↑ mag ↑ ascent

Jacobians Matrix $\quad J_f = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_n}\right]$

matrix of partial derivatives

example: $F(x,y) = \begin{pmatrix} x^2y \\ e^x \cos y \end{pmatrix} \quad J(x,y) = \begin{pmatrix} 2xy & x^2 \\ e^x \cos y & -e^x \sin y \end{pmatrix}$

High Order Derivatives:

Derivatives of Partial derivatives.

2^{nd} order derivatives and higher

Example: $f(x,y) = x^2y + 3xy^2$

- Pure Second PD

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x^2y + 3xy^2)\right)$$

$$= \frac{\partial}{\partial x}\left(2xy + 3y^2\right)$$

$$\boxed{\frac{\partial f}{\partial x^2} = 2y}$$

- Mixed Second PD

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(x^2y + 3xy^2)\right)$$

$$= \frac{\partial}{\partial y}\left(2xy + 3y^2\right)$$

$$\boxed{\frac{\partial f}{\partial x\,\partial y} = 2x + 6y}$$

→ Used for studying curvature & surfaces

→ Optimization to study/classify critical points

Taylor Analysis.

Infinite series representation of smooth function as a sum of terms calculated from functions derivatives at a single point

"Approximate the function near that point by expressing it in polynomial series."

$f(x)$ → Infinitely differentiable at point a

Taylor series of f around a is:

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$$\cdots\cdots \frac{f^n(a)(x-a)^n}{n!}$$

Or in summation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

Uses:
- Approximatn using polynomials that are easier to compute
- → ML

- $f(a)$ – value of f at point a
- $\frac{f'(a)}{1!}(x-a)$ → linear approximation (slope of tangent line)
- $\frac{f''(a)(x-a)^2}{2!}$ → represent Quadratic curvature
- ↑ order term – capture increasingly finer details about function behaviour near a

example: $f(x) = e^x$

expand around $a = 0$

$$e^x = 1 + \frac{e^x(x-0)}{1!} + \frac{e^x(x-0)^2}{2!} + \frac{e^x(x-0)^3}{3!} + \cdots\cdots$$

$$\to e^x = 1 + \frac{e^x}{1!} + \frac{e^x}{2!} + \frac{e^x}{3!} + \cdots\cdots$$

Maclaurin series

special TS → $a = 0$

Multivariate Taylor Series:

extension of Taylor series to functions of more than one variable

" for a function $f(x_1, x_2, x_3 — x_n)$ that is smooth (infinitely differentiable at a point

$a = (a_1, a_2, a_3 \ldots a_n)$ the MTS expands as:

$$f(x) \approx f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f(a)}{\partial x_i \partial x_j} (x_i - a_i)(x_j - a_j) + \ldots$$

1st order MT Expansion,

(linear approximation to f near a) : $f(x) \approx f(a) + \sum_{i=1}^{n} \frac{\partial f(a)}{\partial x_i}(x_i - a_i)$

Tangent plane appx^n for function of multiple variables

example: Construct Taylor Series through second order for $f(x,y) = x^2 y + y^2$ at $(x,y) = (1,3)$

$f(1,3) = 3 + 9 = 12$

$\frac{\partial f}{\partial x} = 2xy + 0$ $\qquad \frac{\partial f}{\partial x}\Big|_{1,3} = 6$

$\frac{\partial f}{\partial y} = x^2 + 2y$ $\qquad \frac{\partial f}{\partial y}\Big|_{1,3} = 1 + 6 = 7$

$\frac{\partial^2 f}{\partial x^2} = 2y$ $\qquad \frac{\partial f}{\partial y}\Big|_{1,3} = 6$

$\frac{\partial^2 f}{\partial y^2} = 2$ $\qquad \frac{\partial^2 f}{\partial y^2}\Big|_{1,3} = 2$

$\frac{\partial^2 f}{\partial x \partial y} = 2x$ $\qquad \frac{\partial f}{\partial x \partial y}\Big|_{1,3} = 2$

$f(x,y) = \underset{f}{12} + \underset{f_x}{6}(x-1) + \underset{f_y}{7}(y-3) + \frac{1}{2!}\big(6(x-1)^2 + 2(y-3)^2$

$+ 2(x+1)(y-3))$

For a function $f(x,y) = x^2 + xy + y^2$, expand it around $(0,0)$

$f(0,0) = 0$

$\frac{\partial f}{\partial x} = 2x + y \Big|_{0,0} = 0$

$\frac{\partial f}{\partial y} = x + 2y \Big|_{0,0} = 0$ $\qquad \frac{\partial^2 f}{\partial y \partial x} = 1$

$\frac{\partial^2 f}{\partial x^2} = 2 \Big|_{0,0} = 2$

$\frac{\partial^2 f}{\partial y^2} = 2 \Big|_{0,0} = 2$

$\frac{\partial^2 f}{\partial x \partial y} = 1 \Big|_{0,0} = 1$

$f(x) = 0 + 0(x-0) + (y-0)^0 + \frac{1}{2!}\left(2(x-0)^2 + 2(y)^2 + xy\right) \ldots$
$+ xy$

$= \frac{2x^2 + 2y^2 + 2xy}{2!} \Rightarrow$

$\Rightarrow x^2 + y^2 + xy$

Other way using Hessen Matrix

$$\boxed{H(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$
at $(0,0)$

$\boxed{\text{2nd order taylor Series: General Formula}}$

$$\boxed{f(x) \simeq f(a) + \nabla f(a)^T (x-a) + \frac{1}{2}(x-a)^T H(a)(x-a)}$$

For example:

$$f(x,y) = 0 + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$