# Matrix Decompositions

**Determinants** →

det (A) ⟶ maps A onto a scalar / real value num

$\forall A = n \times n$

"A square matrix that is not invertible is called singular or degenerate"
( $\det(A) \neq 0$ )

→ **Square Triangle Matrix.**

upper : → [ ] below diagonal is 0

lower : → [ ] above diagonal is 0

For $\Delta^r$ matrix, det is = product of diagonal elements

→ **Laplace Expansion:** OR COFACTOR EXPANSION

↳ method for computing determinant of matrix ($n \times n$)

3×3 $\qquad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\det(A) = a \det\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Applications : Matrix Inversion using conjugates
System of Equations [Cramer's Rule]

→ **Properties of Determinants:**

$\det(AB) = \det(A)\det(B)$

$\det(A) = \det(A^T)$

$\det(A^{-1}) = (\det(A))^{-1}$ if A is invertible

→ **Trace**

Sum of diagonal of square matrix

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

example
$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \text{Trace}(A) = a+e+i$

Trace Properties :
Linearity $\Rightarrow$ $tr(A+B) = tr(A) + tr(B)$
$tr(\alpha A) = \alpha\, tr(A)$ $\quad \alpha \in R$
$tr(I_n) = n$
$tr(AB) = tr(BA)$ $\quad A \in R^{n \times k}$
$B \in R^{k \times n}$

# Eigen Values & Eigen Vectors

Associated with square matrix

For a given Matrix A, an eigen value $\lambda$ is a scalar value such that their exists a non zero $v$ eigen vector satisfying the equation

$$\boxed{Av = \lambda v}$$

Eigen value equation $\Rightarrow$ $(A - \lambda I)v = 0$
$I$ is an identity matrix
$\lambda$ = eigen value
If $\det(A - \lambda I) = 0$

Characteristic Equation : $\Rightarrow$
Eigen value of [A] are found by solving characteristic equation.
$\rightarrow$ Roots of polynomial gives eigenvalues.

Application: used in SVD, PCA for dimensionality Reduction

Properties of Eigen Vectors & Values.

1. If $\lambda_1, \lambda_2, \dots \lambda_m$ are eigen values of $[A]_{m \times m}$ then corresponding eigen vectors $\vec{v_1}, \vec{v_2}, \dots \vec{v_m}$ are

linearly independent

2. Eigen values [A] = Eigen values [$A^T$] (not necc. same EV)
3. Similar matrix have same eigenvectors
4. Symmetric, +ve definite matrix always have +, real Eigenvalue.
5. det|A| = product of eigen values
6. Trace (A) = sum of eigen values
7. Eigen vectors for Symmetric Matrices ($A = A^T$)

eigen vectors are orthogonal.

$$\vec{v_i} \cdot \vec{v_j} = 0 \quad i \neq j$$

## Diagonalisation

Diagonal Matrix → 0 values everywhere on non diagonal place $\begin{bmatrix} c_1 & & \\ & c_2 & \\ & & \ddots & c_n \end{bmatrix}$

"A Matrix $A \in \mathbb{R}^{n \times n}$ is diagonalisable if it similar to a diagonal matrix. i.e. if there exits an invertible matrix $P \in \mathbb{R}^{n \times n}$ such that
$$D = P^{-1} A P$$"

Every symmetric matrix is diagonalisable i.e there exist an orthogonal matrix P such that

$$A = PDP^T$$

D = Diagonal matrix
P = orthogonal matrix containing eigen vectors

"linearly dependent vector is $v_1 = \left(\frac{1}{2}\right)$, $v_2 = \frac{1}{4}$ i.e $v_1 = \frac{1}{2}(v_2)$",
$v_1 = x v_2$

Eigen Decomposition: (Spectral Decomposition)

A square matrix $A \in \mathbb{R}^{n \times n}$ can be factored into

$$A = PDP^{-1}$$

$P \in \mathbb{R}^{n \times n}$

D= Diagonal matrix (diagonal values = eigen values of A)

**Application:** Process of breaking down square matrix into a set of its eigenvalues and eigenvectors

**Uses:** Simplifying matrix operations & solving matrix linear set of equations

PCA

## Steps for Eigen decomposition:

Find eigen values

↓

Find eigen vectors

↓

Form matrix P

↓

Find $P^{-1}$

↓

Form diagonal Matrix

example: $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det\left[\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] = 0$$

$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

$$\det\left|\begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}\right| = 0$$

**Eigen values:**

$$(4-\lambda)(3-\lambda) - 2 = 0$$

$$4(3-\lambda) - \lambda(3-\lambda) - 2 = 0$$

$$12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$$

$$10 - 7\lambda + \lambda^2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$\lambda^2 - (5+2)\lambda + 10 = 0$$

$$\lambda^2 - 5\lambda - 2\lambda + 10 = 0$$

$$\lambda(\lambda-5) - 2(\lambda-5) = 0$$

$$(\lambda-2)(\lambda-5)$$

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

**eigen vectors:**

$$(A - \lambda I)v = 0$$

For $\lambda_1 = 5$

$$\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

$$x_1 = x_2$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$

$$\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$2x_1 + \cancel{x}_2 = 0$

$2x_1 + x_2 = 0$

$2x_1 = -x_2$

$x_1 = 1$

$x_2 = -2$

$\therefore \; v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Construct matrices P

$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ sign

$D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

$$P^{-1} = \frac{1}{-2-1}\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$D = PVP^{-1}$$

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## Eigen Decomposition :

For Square Matrix $A \in R^{n \times n}$, A can be factored into

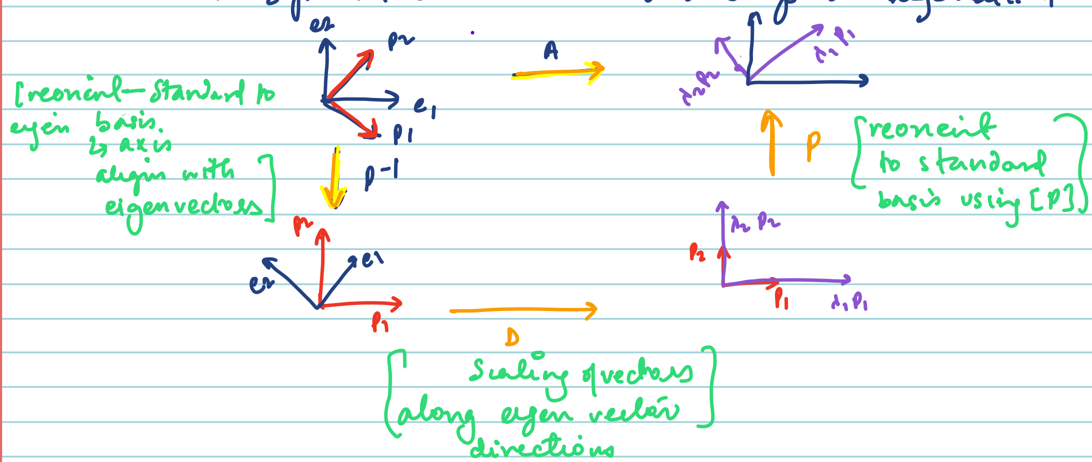
$$A = PDP^{-1}$$

where

$P = R^{n \times n}$

D = Digonal matrix

" A symmetric matrix can always be diagonalized "



## Eigen Decomposition with power

$$A^K = (PDP^{-1})^K$$

$$= (PD\underbrace{P^{-1}P}_{I}DP^{-1}PDP^{-1} \_\_\_\_ PDP^{-1})^K$$

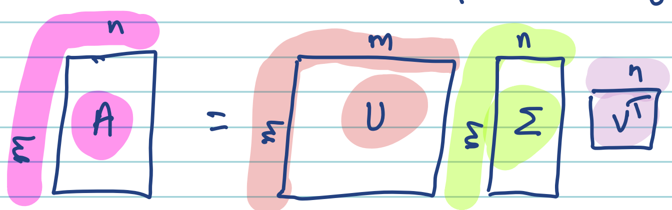
$$\therefore \quad = PD^KP^{-1}$$

$$\boxed{\therefore \; A^K = PD^KP^{-1}}$$

## Singular Value Decomposition :-

let $A \in R^{m \times n}$ be rectangular [] of rank $r \in [0, \min(m,n)]$

the SVD is a decomposition of the form




$U$ = Orthogonal matrix $U \in R^{m \times m}$
square with colvectors

$V$ = Orthogonal matrix $V \in R^{n \times n}$
with column vectors

$\Sigma$ = $m \times n$ matrix with

square root of eigen values of $A^TA$ matrix

$\Sigma_{ii} = \sigma_i \geq 0$ singular values
$i = 1 \ldots r$

$\sigma_1 \geq \sigma_2 \geq \sigma_3 — \sigma_r \geq 0$
desc order

$\Sigma_{ij} = 0 \quad i \neq j$

$u_i$ = left singular vector
$v_i$ = right singular vectors

Singular values : Square root of eigen values

## Steps to find SVD:

For matrix A

↓

Multiply with $A^T$ to make $\Rightarrow A^T A$ (it symmetric [square [] )

↓

Calculate eigen values of $A^T A$ & eigenvectors (divide it with unit eigenvector

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \sqrt{1^2+1^2} \quad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

↓

$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$ always in descending order

same size as A

PC corresponds to top K diagonal elements which capture most variance value & important comp.

↓

$u_1 = \frac{1}{\sigma_1} A v_1$

$u_2 = \frac{1}{\sigma_2} A v_2$

$U = [u_1, u_2]$

left singular vector are image of right singular vectors normalised by division

" for symmetric matrix, $P^{-1} = P^T$ $\therefore A = PDP^T$ "

| | Eigen decomposition | SVD |
|---|---|---|
| Matrix | Square matrix | Any matrix |
| Decomposition | $A = PDP^{-1}$ | $A = U \Sigma V^T$ |
| Components | P-eigenvect[], D-(eigenvalues diagonal []) | U = left singular vector<br>$\Sigma$ = diagonal (sing = $\sqrt{ev}$)<br>$V^T$ = right singular vector |
| Conditions | Matrix must be diagonalisable | No condition |
| Orthogonality | No | U and V are orthogonal [] |
| Application | PCA | Data reduction, noise reduction, PCA |
| | → for non symmetric matrix, | → SV are always non- |

additional

EV u can be real & complex

v & v are orthogonal.