Lösningsförslag MMG300 del 1, 200320

1

(a) Felakt.

Motexempel: $x_k = k$.

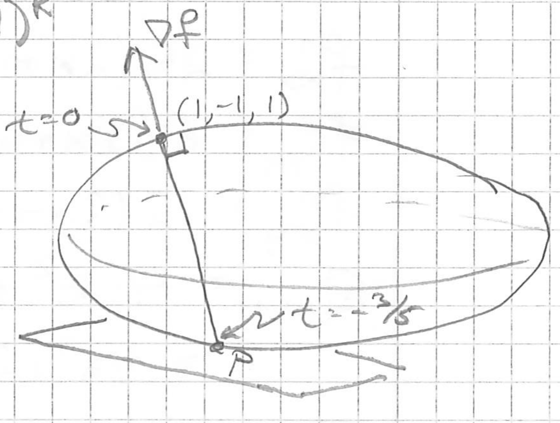
(b) Felakt.

Motexempel: $x_k = \frac{(-1)^k}{k}$

(c) Felakt.

Motexempel: $x_k = (-1)^k$

2



$f(x,y,z) = 2x^2+y^2+z^2$

normalvektor i $(1,-1,1)$:

$\nabla f(1,-1,1) =$

$(4x, 2y, 2z)\Big|_{(1,-1,1)}$

$= (4,-2,2)$

Normallinjens ekv. på parameterform:

$(x,y,z) = (1,-1,1) + t(4,-2,2)$

Skärningspunkt P:

$2\cdot(1+4t)^2 + (-1-2t)^2 + (1+2t)^2 = 4$

$16t + 32t^2 + 4t + 4t^2 + 4t + 4t^2 = 0$

$10t^2 + 6t = 0 \quad \Rightarrow t = 0$ el. $t = -\frac{3}{5}$

$\Rightarrow P = (1,-1,1) - \frac{3}{5}(4,-2,2)$

$= \left(-\frac{7}{5}, \frac{1}{5}, -\frac{1}{5}\right)$

Tangentplan i P:

$\nabla f(P) = \left(-4\cdot\frac{7}{5}, 2\left(\frac{1}{5}\right), 2\left(-\frac{1}{5}\right)\right)$

$= -\frac{2}{5}(14, -1, 1)$

Ekvation:

$14\left(x-\left(-\frac{7}{5}\right)\right) - 1\left(y-\frac{1}{5}\right) + 1\cdot\left(z-\left(-\frac{1}{5}\right)\right) = 0$

$\Leftrightarrow \quad 14x - y + z = \frac{-7\cdot 14 - 1 - 1}{5} = -\frac{100}{5} = -20$

Svar: $P = \left(-\frac{7}{5}, \frac{1}{5}, -\frac{1}{5}\right)$

Tang. plan: $14x - y + z = -20$

(3) Stationära punkter:

$f'_x = 6(2x+y)^2 + 4x = 0$ (1)

$f'_y = 3(2x+y)^2 + 2y = 0$ (2)

$(1) - 2\cdot(2) \Rightarrow x = y$

$\Rightarrow 3\cdot(3x)^2 + 2x = 0 \Rightarrow x\cdot(27x+2) = 0$

Två lösn.: $(0,0)$ och $\left(-\frac{2}{27}, -\frac{2}{27}\right)$

Kvadratiska former:

$f''_{xx} = 24(2x+y) + 4$

$f''_{yy} = 6(2x+y) + 2$

$f''_{xy} = 12(2x+y)$

$Q_{(0,0)} = 4\cdot h^2 + 2\cdot k^2 + 2\cdot 0\cdot h\cdot k$

$= 4h^2 + 2k^2$

Tecken ++ $\Rightarrow$ positivt definit $\Rightarrow$ lokalt min.

$Q_{\left(-\frac{2}{27}, -\frac{2}{27}\right)} = \Big/\ 2x+y = -\frac{2}{27}\cdot 3 = -\frac{2}{9},\ 6\cdot(2x+y) = -\frac{4}{3}\ \Big/$

$= -\frac{4}{3}h^2 + \frac{2}{3}k^2 + 2\cdot\left(-\frac{8}{3}\right)hk$

$= \frac{2}{3}\left(-2h^2 + \underline{k^2 - 8hk}\right)$

$= \frac{2}{3}\left((k-4h)^2 - 18h^2\right)$

Tecken +− $\Rightarrow$ indefinit $\Rightarrow$ sadelpunkt.

<u>Svar</u>: lok. min i $(0,0)$

sadel i $\left(-\frac{2}{27}, -\frac{2}{27}\right)$

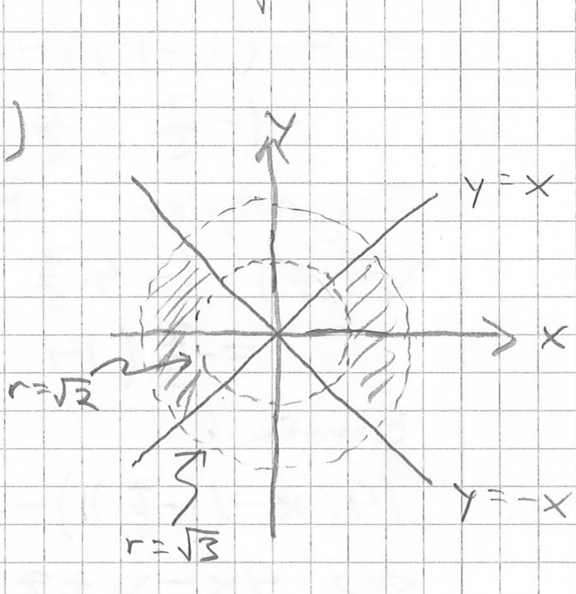
(4) (a) $y^2 = x^2 \Leftrightarrow |y| = |x|$

$\Leftrightarrow y = \pm x$

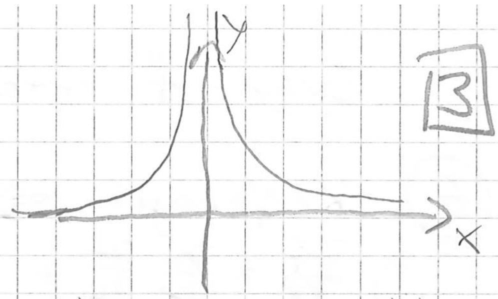
ej öppen

ej sluten

ej sammanhängande.



(b) $x^2y=1 \Leftrightarrow y=\frac{1}{x^2}$



3

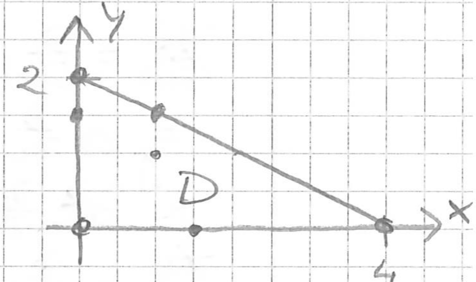
sluten

ej öppen

ej sammanhängande.

D_2 utgörs av punkter på de två kurvorna.

5 D är kompakt och sammanhängande, och f är kontinuerlig, så f antar max M och min m på D och värdemängden är $f(D)=[m,M]$.



Kandidatjakt!

1. Inre punkter

$f'_x = 2x+y-3=0$

$f'_y = x+2y-3=0$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Randkurvor:

$\underline{y=0}$: $f(x,0)=x^2-3x=(x-\frac{3}{2})^2-\frac{9}{4}$

$\underline{x=0}$: $f(0,y)=y^2-3y=(x-\frac{3}{2})^2-\frac{9}{4}$

$\underline{x+2y=4}$: $f(4-2y,y)=(4-2y)^2+(4-2y)y+y^2$
$-3(4-2y)-3y=3y^2-9y+4$
$=3(y-\frac{3}{2})^2+4-\frac{27}{4}$

3. Hörnpunkter $(0,0)$, $(4,0)$ och $(0,2)$

Jämförelse:

$f(1,1)=-3$ min

$f(3/2,0)=\frac{9}{4}-\frac{9}{2}=-\frac{9}{4}$

$f(0,3/2)=-9/4$

$f(1,3/2)=\frac{9}{4}-3-2=-\frac{11}{4}$

$f(0,0)=0$

$f(4,0)=4$ max

$f(0,2)=-2$

Svar: f antar värdena $[-3, 4]$

(6) Kedjeregeln $\Rightarrow$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x}\dfrac{\partial f}{\partial u} + \dfrac{\partial v}{\partial x}\dfrac{\partial f}{\partial v} = a\dfrac{\partial f}{\partial u} + b\dfrac{\partial f}{\partial v} \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y}\dfrac{\partial f}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial u} + \dfrac{\partial f}{\partial v} \end{cases}$$

Transformering av andra derivator:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right)$$
$$= a\frac{\partial}{\partial u}\left(a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right) + b\frac{\partial}{\partial v}\left(a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right)$$
$$= a^2\frac{\partial^2 f}{\partial u^2} + 2ab\frac{\partial^2 f}{\partial u\partial v} + b^2\frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$
$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$
$$= \frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u\partial v} + \frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$
$$= a\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + b\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$
$$= a\frac{\partial^2 f}{\partial u^2} + (a+b)\frac{\partial^2 f}{\partial u\partial v} + b\frac{\partial^2 f}{\partial v^2}$$

PDEn $\Leftrightarrow$

$$3\left(a^2 f''_{uu} + 2ab f''_{uv} + b^2 f''_{vv}\right)$$
$$= \left(f''_{uu} + 2f''_{uv} + f''_{vv}\right) + 2\left(a f''_{uu} + (a+b)f''_{uv} + b f''_{vv}\right)$$

För att detta ska vara $\Leftrightarrow$ $f''_{uv} = 0$ måste

$$\begin{cases} 3a^2 = 1 + 2a \\ 3b^2 = 1 + 2b \\ 6ab \neq 2 + 2(a+b) \end{cases}$$

$$3a^2 - 2a = 1$$
$$3\left(a - \tfrac{1}{3}\right)^2 = 1 + \tfrac{1}{3} = \tfrac{4}{3}$$
$$a - \tfrac{1}{3} = \pm\tfrac{2}{3} \qquad a = 1 \text{ el. } -\tfrac{1}{3}$$

Vi får inte välja $a = b = 1$ eller $a = b = -\frac{1}{3}$ ty variabelbytet är då inte bijektivt.

/Kontroll: $6\cdot 1\cdot(-\frac{1}{3}) \neq 2+2(1-\frac{1}{3})$ /

Svar: De två möjligheterna är

$(a,b)=(1,-\frac{1}{3})$ eller $(a,b)=(-\frac{1}{3},1)$

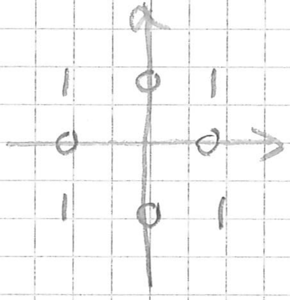
7 (a) Felakt.

Motexempel: $n=1$, $D=\mathbb{R}$, $f(x)=|x|$, $\bar{a}=0$.

(b) Felakt.

Motexempel: $n=2$, $D=\mathbb{R}^2$, $\bar{a}=(0,0)$

$$f(x,y)=\begin{cases} 1 & ;\ x\neq 0 \text{ och } y\neq 0 \\ 0 & ;\ x=0 \text{ eller } y=0 \end{cases}$$



(f är ej ens kontinuerlig i $(0,0)$)

(c) Felakt.

Motexempel: $n=1$ $D=\mathbb{R}$, $\bar{a}=0$

$$f(x)=\begin{cases} x^2\cdot\sin(\frac{1}{x}) & ,\ x\neq 0 \\ 0 & ,\ x=0 \end{cases}$$

$f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$ existerar för $x\neq 0$

$f'(0)=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\to 0} h\sin\frac{1}{h}=0$ existerar.

Men $\lim_{x\to 0} f'(x)$ existerar ej då $\cos\frac{1}{x}$ saknar gränsvärde.

Så f är ej C^1.

Obs: $n=1$, så differentierbar $\Leftrightarrow$ deriverbar.