

Intuitive Understanding of Quantum Computation and Post-Quantum Cryptography (Chapter 1)

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Abstract

Post-quantum cryptography is inevitable. National Institute of Standards and Technology (NIST) starts standardizing quantum-resistant public-key cryptography (aka post-quantum cryptography). The reason is that investment in quantum computing is blooming which poses significant threats to our currently deployed cryptographic algorithms. As a security engineer, to prepare for the apocalypse in advance, I've been watching the development of quantum computers and post-quantum cryptography closely. Never mind, I simply made up an excuse to study these fascinating scientific fields :) However, they are extremely hard to understand, at least to an amateur like me. This article shares with you my notes with the hope that you will have an *intuitive understanding* of the beautiful and mind-blowing quantum algorithms and post-quantum cryptography.

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[†]When asked, "How to say your name, Quan?", I answer, "It's the prefix of Quantum :)".

Introduction

To understand post-quantum cryptography, we have to understand quantum mechanics and quantum computers. Therefore, before studying post-quantum cryptography, I bought the classic book "Quantum Computation and Computation Information" by Michael Nielsen and Isaac Chuang [1]. The book was too advanced to me, so I took an excellent approachable quantum computation course by Umesh Vazirani [2]. In fact, I learned most quantum computation from Vazirani's course. Later on, I've realized that understanding quantum computers has nothing to do with understanding post-quantum cryptography. It was too late. I couldn't unlearn what I have learned. Therefore, I'll describe both quantum computers and post-quantum cryptography to make sure that you will make the same mistake as I did :) Joking aside, as a security engineer who is trained with "trust, but verify" mindset, I feel guilty to blind trust that Shor's quantum algorithms [3] break our current cryptographic protocols.

After "wasting" our time studying quantum algorithms, we'll study post-quantum cryptography. As Shor's algorithms solve factoring and discrete log problems in polynomial time, cryptographers had to find alternative cryptographic constructions that are presumably safe against quantum computers. The following cryptographic constructions are selected to advance to the 2nd round in NIST's post-quantum cryptography competition [4]: lattice-based cryptography, hash-based digital signature, code-based cryptography, multivariate public key cryptography and supersingular elliptic curve isogeny. I feel a headache just by reading these names :) They're independent of each other and each topic deserves its own research, so I'll describe them one-by-one in later chapters. They're all difficult to understand, but the most challenging obstacle is to overcome our fear in dealing with them. No worries, if you can't understand them, blame me for not explaining them well :)

1 Quantum Computation

Have you ever played computer games? American player Kyle Giersdorf won 3 million on the Fortnite game. Computer games are ruling the world, so I recommend you stop reading this article, instead go and play games :) Computer games are a strange world where games' creators invent rules and players follow with no questions asked. In the same spirit, we'll follow quantum computers' rules, play along and design quantum algorithms based on its rules. The rules are strange but they're not stranger than computer games' rules. Furthermore, we'll study quantum computation without saying a word about quantum physics. I don't even try to understand quantum physics because I don't want to be crazy :)

1.1 Quantum computers

To describe a classical computational system, we define its state, how to change its state and how to measure its state. For instance:

- State: n bits $x = x_1, x_2, \dots, x_n$ represented by n transistors.
- Classical logic gates such as NOT, AND, OR, NAND, etc are used to change x_1, x_2, \dots, x_n .
- Measurement: measure the transistors, based on transistors' voltages, we'll get n output bits.

In a similar way, to describe quantum computational system, we'll define quantum state, quantum gates and quantum measurement.

1.1.1 Quantum state

In classical computers, a bit is either 0 or 1 at any moment. In quantum computers, a quantum bit (aka qubit) can exist at both states 0 and 1 at the same time. In fact, a qubit is a superposition of states $|0\rangle$ and $|1\rangle$: $|q\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ where the amplitudes α_0, α_1 are complex numbers. On the one hand, the *ket* $| \rangle$ notation $|q\rangle, |0\rangle, |1\rangle$ just means quantum states, instead of classical ones. On the other hand, $|q\rangle$ denotes the state vector $|q\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \alpha_0 |0\rangle + \alpha_1 |1\rangle$.

Is it strange that α_0, α_1 are complex numbers instead of real numbers? It's even stranger to learn that we never have access to α_0, α_1 . As we'll see in the later section, when we measure $|q\rangle$, we'll get $|0\rangle$ with probability $|\alpha_0|^2$ and $|1\rangle$ with probability $|\alpha_1|^2$. I.e., we can observe these complex numbers' magnitudes (which are real numbers), but not the numbers themselves. Nature is mysterious!

Generalize the previous paragraphs, 2 qubits is a superposition of states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, for instance, $1/2(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. n qubits is a superposition of states $|00 \dots 0\rangle, \dots, |11 \dots 1\rangle$, i.e., $|q\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle$. It's amazing that n qubits hold information of 2^n states at the same time. This makes quantum computers more powerful than classical computers.

1.1.2 Quantum gates

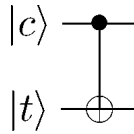
In classical computers, we use classical logic gates such as AND, NOT, OR, NAND to change bits' values. In quantum computers, we use quantum gates to change qubits. Recall that $|q\rangle$ denotes state vector $|q\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. To transform 2×1 column vectors, we will use 2×2 matrix. Therefore, we can describe quantum gates in the form of matrices.

Bit flip gate X Bit flip gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have $X|q\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix}$, i.e., it transforms $|0\rangle$ into $|1\rangle$ and $|1\rangle$ into $|0\rangle$.

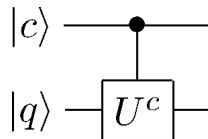
Phase flip gate Z Phase flip gate $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We have $Z|q\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ -\alpha_1 \end{pmatrix}$. If we denote $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|-\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$ then we have $Z|+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = |-\rangle$ and $Z|-\rangle = |+\rangle$.

The bit flip gate X and phase flip gate Z are basic and boring as they just transform back and forth between standard states $|0\rangle \leftrightarrow |1\rangle, |+\rangle \leftrightarrow |-\rangle$. The following 2 quantum gates are more interesting.

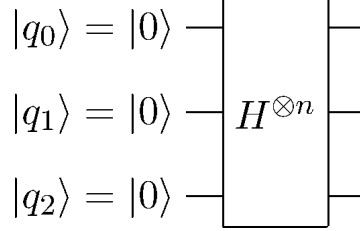
CNOT (Controlled-NOT) gate



CNOT (Controlled-NOT) gate: $|c\rangle |t\rangle \rightarrow |c\rangle |c \oplus t\rangle$ where c is the control bit, t is the target bit. What it means is that if c is 1 then it flips the target bit, otherwise it keeps the target bit as is. This can be generalized to implement if/else: if c is 1, execute gate $U^c = U$, else don't execute U (i.e. execute $U^c = U^0 = I$ identity).



Hadamard gate H Hadamard transform $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We have $H|0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, $H|1\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$. If we apply $H^{\otimes 2}$ (this notation just means we apply H to 1st and 2nd qubit independently) to 2 qubits $|00\rangle$, we have $H^{\otimes 2}|00\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)1/\sqrt{2}(|0\rangle + |1\rangle) = 1/2(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Observe that Hadamard gate transforms basis state into superposition of states, for instance, H transforms $|0\rangle$ into superposition of states $|0\rangle$, $|1\rangle$ and $H^{\otimes 2}$ transforms $|00\rangle$ into superposition of states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. We have $H^{\otimes n}|00\dots 0\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle$, i.e., we can create superposition of all basis states $|0\dots 0\rangle, \dots, |1\dots 1\rangle$ by applying $H^{\otimes n}$ gate to $|0\dots 0\rangle$. We'll use this trick over and over again in designing quantum algorithms.



If we take a closer look at Hadamard gate $H|0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, $H|1\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$, we'll see that $H|u\rangle = 1/\sqrt{2}(|0\rangle + (-1)^u|1\rangle)$. Therefore

$$\begin{aligned} H^{\otimes 2}|u_1u_2\rangle &= 1/\sqrt{2}(|0\rangle + (-1)^{u_1}|1\rangle)1/\sqrt{2}(|0\rangle + (-1)^{u_2}|1\rangle) \\ &= 1/2(|00\rangle + (-1)^{u_2}|01\rangle + (-1)^{u_1}|10\rangle + (-1)^{u_1u_2}|11\rangle) \\ &= 1/2((-1)^{(u_1,u_2)\cdot(0,0)}|00\rangle + (-1)^{(u_1,u_2)\cdot(0,1)}|01\rangle + (-1)^{(u_1,u_2)\cdot(1,0)}|10\rangle + (-1)^{(u_1,u_2)\cdot(1,1)}|11\rangle) \\ &= 1/2 \sum_x (-1)^{u\cdot x} |x\rangle \end{aligned}$$

For n qubits, we have $H^{\otimes n}|u_1u_2\dots u_n\rangle = \frac{1}{2^{n/2}} \sum_x (-1)^{u\cdot x} |x\rangle$ where $u\cdot x = u_1x_1 + u_2x_2 + \dots + u_nx_n$ is the inner product of u, x .

Unitary transformation If we take a closer look at X, Z, H we see that $I = X^2 = Z^2 = H^2$. This is not an accident. For every quantum transformation U , if we denote U^* as conjugate transpose of U then $UU^* = U^*U = I$. A matrix U satisfying the above equation is called a unitary matrix. In the above examples, it just happens that $X^* = X, Z^* = Z, H^* = H$ and hence $X^2 = XX^* = I, Z^2 = ZZ^* = I, H^2 = HH^* = I$.

1.1.3 Quantum measurement

In classical computers, measurement is a trivial operation because what you see is what you get. In quantum computers, we never have access to the qubit q 's complex amplitudes α_0, α_1 . When we measure $|q\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, we'll see $|0\rangle$ with probability $|\alpha_0|^2$ and see $|1\rangle$ with probability $|\alpha_1|^2$. Furthermore, the measured qubit forever collapses to state $|0\rangle$ or $|1\rangle$, i.e., you can't never put the measured qubit back to superposition of states $|0\rangle, |1\rangle$. To a certain extent, quantum measurement is a destructive operation. Finally, to make sure that the probabilities of measurement outcomes summing up to 1, α_0 and α_1 must satisfy the equation $|\alpha_0|^2 + |\alpha_1|^2 = 1$. This is the reason why you see $1/\sqrt{2}$ in state like $|+\rangle = 1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle$ because $(1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$. For n qubits $|q\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle$, we have $\sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1$.

The way quantum measurement works restricts quantum computers's computation capability. While n qubits $|q\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle$ holds information of 2^n basis states at the same time, every time we measure q , we only observe a tiny bit of information: a specific state $|i\rangle$ with probability $|\alpha_i|^2$ and even worst, $|q\rangle$ forever collapses to $|i\rangle$. In other words, most q 's information is hidden inside inaccessible $\alpha_j, j \in \{0, 1\}^n$. This is the reason why quantum computers is not 2^n more powerful than classical computers.

Partial measurement If we measure 2 qubits like $1/2|00\rangle + 1/2|01\rangle + 1/\sqrt{2}|10\rangle$, we'll see one of states $|00\rangle, |01\rangle, |10\rangle$. What would happen if we only measure the 1st qubit? If the measured 1st qubit is $|0\rangle$ then the 2 qubits become $1/\sqrt{2}(|00\rangle + |01\rangle)$, i.e., superposition of states where the 1st qubit is $|0\rangle$. Note that the amplitudes change from $1/2$ to $1/\sqrt{2}$ to make sure that the sum of new states' probabilities is 1. Similarly, if we measure the 1st qubit and we see $|1\rangle$ then the 2 qubits become $|10\rangle$ because among three states $|00\rangle, |01\rangle, |10\rangle$ only $|10\rangle$ has the 1st qubit as 1.

1.2 Reversible computation

The fact that any quantum transformation U is unitary has a profound impact on quantum computation. If we start with quantum state x and we apply quantum computation U to it then we can always get x back by applying U^* to $U|x\rangle$. The reason is that $U^*(U|x\rangle) = (U^*U)|x\rangle = I|x\rangle = |x\rangle$, i.e., all quantum computation is reversible. This contrasts with classical computers where there are many one way functions.

The question is if we are given a classical function $f(x)$, can we implement it using reversible quantum computation? The trick is to carry x to the output as well so that we have enough information to reverse the computation. There is quantum circuit U_f that implements $U_f|x\rangle|b\rangle = |x\rangle|b \oplus f(x)\rangle$.

1.3 Bernstein-Vazirani's algorithm

To demonstrate the power of quantum computation, we'll take a look at Bernstein-Vazirani's algorithm [5] that solves parity problems faster than any classical algorithm.

Given a function $f : \{0,1\}^n \rightarrow \{0,1\}$ as a "black box"¹ and $f(x) = u \cdot x \pmod 2$ for some hidden $u \in \{0,1\}^n$, find u .

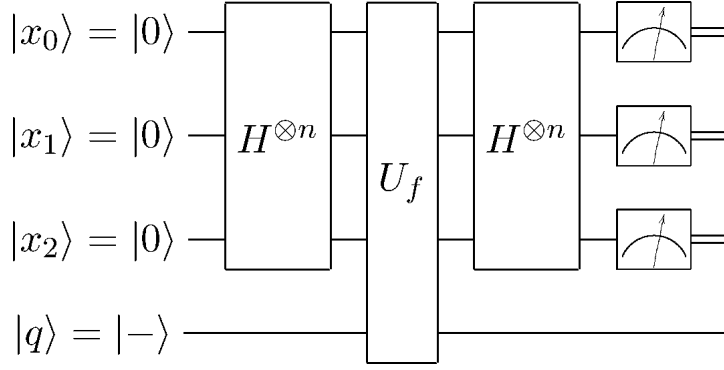
In classical computers, whenever we query the black box f we only get 1 bit of information $f(x)$. As $u \in \{0,1\}^n$ has n bits information, we need at least n queries to the black box f to find n -bit u . Furthermore, if we query f n times using the following x values: $10 \dots 0, 010 \dots 0, \dots, 00 \dots 1$ then we'll find u because i^{th} query reveals i^{th} bit of u . Therefore, the optimal classical algorithm requires n queries to f .

In quantum computers, if we denote U_f the quantum circuit that implements f then Bernstein-Vazirani's algorithm only uses U_f once. We'll describe the algorithm by working backwards. Let's recall what we've learned in the previous sections.

- When we measure n qubits $|q\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle$, we'll see state $|i\rangle, i \in \{0,1\}^n$ with probability $|\alpha_i|^2$. In a special case when $\alpha_u = 1, \alpha_{i \neq u} = 0$ or $|q\rangle = |u\rangle$, measuring $|q\rangle$ gives us $|u\rangle$.
- $H^{\otimes n} |u\rangle = H^{\otimes n} |u_1 u_2 \dots u_n\rangle = \frac{1}{2^{n/2}} \sum_x (-1)^{u \cdot x} |x\rangle$.
- $H^2 = I$. This implies $H^{\otimes n} H^{\otimes n} |x\rangle = I^{\otimes n} |x\rangle = |x\rangle$.

Based on the above facts, if we can set up quantum state $\frac{1}{2^{n/2}} \sum_x (-1)^{u \cdot x} |x\rangle = H^{\otimes n} |u\rangle$ then applying $H^{\otimes n}$ to it gives $|u\rangle$, so the final quantum measurement results in $|u\rangle$. Now, the question is how to set up the quantum state $\frac{1}{2^{n/2}} \sum_x (-1)^{u \cdot x} |x\rangle$. Note that we don't know u . However, we know $u \cdot x = f(x)$, which means that we can rewrite the above state as $\frac{1}{2^{n/2}} \sum_x (-1)^{f(x)} |x\rangle$. We make progress, but we're not done yet. Even though we don't know how to create state $\frac{1}{2^{n/2}} \sum_x (-1)^{f(x)} |x\rangle$ yet, we know how to create the state that looks similar to it using Hadamard transform $H^{\otimes n} |00 \dots 0\rangle = \frac{1}{2^{n/2}} \sum_x |x\rangle$. The last bit of the puzzle is how to create $(-1)^{f(x)}$. From "Reversible computation" section, we have $U_f |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$. Let's see what happens to $U_f |x\rangle |-\rangle = U_f |x\rangle (|0\rangle - |1\rangle) = U_f |x\rangle |0\rangle - U_f |x\rangle |1\rangle = |x\rangle |f(x)\rangle - |x\rangle |1 \oplus f(x)\rangle = |x\rangle (|f(x)\rangle - |1 \oplus f(x)\rangle) = |x\rangle (-1)^{f(x)} (|0\rangle - |1\rangle) = |x\rangle (-1)^{f(x)} |-\rangle$. Now, we have $(-1)^{f(x)}$.

¹We can input x to the "black box" and ask it to compute $f(x)$, but we don't have access to its internal computation process.



Putting together in the forward order, we have the algorithm as shown in the above figure:

1. $H^{\otimes n} |00\dots 0\rangle$ gives us $\frac{1}{2^{n/2}} \sum_x |x\rangle$.
2. Applying U_f to $\frac{1}{2^{n/2}} \sum_x |x\rangle |- \rangle$ gives us $\frac{1}{2^{n/2}} \sum_x (-1)^{f(x)} |x\rangle |- \rangle = \frac{1}{2^{n/2}} \sum_x (-1)^{u \cdot x} |x\rangle |- \rangle$
3. Apply $H^{\otimes n}$ to the first n qubits of $\frac{1}{2^{n/2}} \sum_x (-1)^{u \cdot x} |x\rangle |- \rangle$ gives us $|u\rangle |- \rangle$.
4. Measure the first n qubits of $|u\rangle |- \rangle$ gives us $|u\rangle$.

Now, we solved the problem but it's not clear why using U_f once gives us n bit information about u . If we look closer at the 2nd step, even though we use U_f once, we see that the sum $\frac{1}{2^{n/2}} \sum_x (-1)^{f(x)} |x\rangle |- \rangle$ have $f(x)$ for all $x \in \{0, 1\}^n$. How come? The reason is that U_f 's input can be a superposition of all basis states $\sum_x |x\rangle$ (this is in contrast with classical computers where the input is only 1 specific $|i\rangle$) and so the output contains all $f(x)$, $x \in \{0, 1\}^n$.

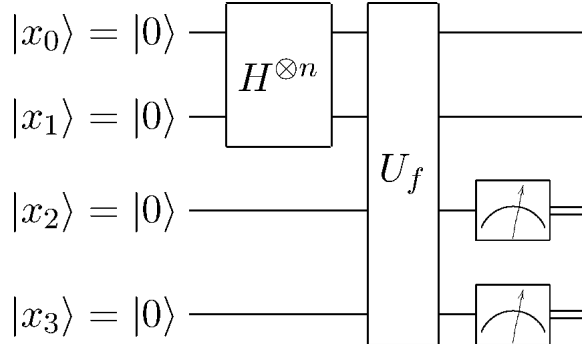
1.4 Simon's algorithm

Given a 2-to-1 function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ and $f(x) = f(x \oplus s)$ for some secret $s \in \{0, 1\}^n$, find s .

Designing classical algorithms is hard, let alone quantum algorithms. In this problem, the condition $f(x) = f(x \oplus s)$ is not even natural, so it's tough to even start the thinking process on how to solve the problem. We have no clue. One method that I found helpful is to play with our existing knowledge, analyze them and/or try to extend them with the hope that somehow it will lead us closer to the solution.

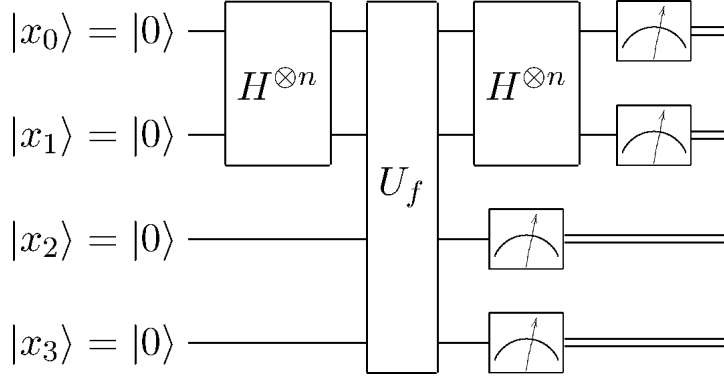
In the previous section, we've seen how to access all $f(x)$, $x \in \{0, 1\}^n$ using $H^{\otimes n}$, followed by U_f . Recall that $H^{\otimes n}$ gives us $\frac{1}{2^{n/2}} \sum_x |x\rangle$ and applying U_f to $\frac{1}{2^{n/2}} \sum_x |x\rangle |00\dots 0\rangle$ gives us $\frac{1}{2^{n/2}} \sum_x |x\rangle |f(x)\rangle$. Note that $\frac{1}{2^{n/2}} \sum_x |x\rangle |f(x)\rangle$ are $2n$ qubits as $f(x)$ is n -bit.

If we measure the 1st n qubits of $\frac{1}{2^{n/2}} \sum_x |x\rangle |f(x)\rangle$ and the 1st n qubits collapses to $|r\rangle$ for some r , then the $2n$ qubits collapses to $|r\rangle |f(r)\rangle$. We've made progress as we've just learned that this path leads nowhere :)



If we measure the 2nd n qubits of $\frac{1}{2^{n/2}} \sum_x |x\rangle |f(x)\rangle$ and we see $|f(r)\rangle$ for some r then the $2n$ qubit collapses to $|r\rangle |f(r)\rangle$. Wait a second, this wasn't correct. From the problem statement, we know $|f(r)\rangle = |f(r \oplus s)\rangle$, i.e., if we see $|f(r)\rangle$, then we see $|f(r \oplus s)\rangle$ as well because they both equal each other. In other words, the $2n$ qubits state collapses to $1/\sqrt{2}(|r\rangle |f(r)\rangle + |r \oplus s\rangle |f(r \oplus s)\rangle) = 1/\sqrt{2}(|r\rangle + |r \oplus s\rangle) |f(r)\rangle$. This is pretty cool as we've discovered a new state $1/\sqrt{2}(|r\rangle + |r \oplus s\rangle)$ that we've never seen before.

What are we going to do with $1/\sqrt{2}(|r\rangle + |r \oplus s\rangle)$? This is a specific state for some random r , so let's apply $H^{\otimes n}$ to make it in superposition of all basis states again. Are you sick of $H^{\otimes n}$ yet? :) I'm sorry that I have to make you real sick because we'll continue using $H^{\otimes n}$ over and over again. Applying $H^{\otimes n}$ to $|r\rangle$ gives us $\frac{1}{2^{n/2}} \sum_x (-1)^{r \cdot x} |x\rangle$ while applying $H^{\otimes n}$ to $|r \oplus s\rangle$ gives us $\frac{1}{2^{n/2}} \sum_x (-1)^{(r \oplus s) \cdot x} |x\rangle$, therefore applying $H^{\otimes n}$ to $1/\sqrt{2}(|r\rangle + |r \oplus s\rangle)$ gives us $\frac{1}{2^{(n+1)/2}} \sum_x ((-1)^{r \cdot x} + (-1)^{(r \oplus s) \cdot x}) |x\rangle$. The value $(-1)^{r \cdot x} + (-1)^{r \cdot x \oplus s \cdot x}$ is pretty special because if $s \cdot x = 1 \pmod 2$ then $(-1)^{r \cdot x} + (-1)^{r \cdot x \oplus 1} = (-1)^{r \cdot x} (1 + (-1)) = 0$ (i.e. we never see the corresponding $|x\rangle$), otherwise if $s \cdot x = 0$ then $(-1)^{r \cdot x} + (-1)^{r \cdot x \oplus 0} = 2(-1)^{r \cdot x} \neq 0$. What it means is after measurement, we only see $|x\rangle$ such that $s \cdot x = 0$ for some random x . The complete Simon algorithm [6] is shown in the below figure



All right, we get $s \cdot x = 0$ for some random x . This isn't enough to solve for s . What we can do is to repeat the algorithm so that we will receive a system of linear equations where each equation has the form $s \cdot x = 0$ with probably different random x . Roughly we need n equations of the form $s \cdot x = 0$ to find s using Gauss elimination algorithm.

1.5 Quantum Fourier Transform (QFT)

Denote ω N -root of unity: $\omega^N = 1$. Classical Fourier transform is defined by the following matrix:

$$F_N = 1/\sqrt{N} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{pmatrix}$$

Quantum Fourier Transform (QFT) is simply applying the same matrix to a quantum state:

$$F_N \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \end{pmatrix} \text{ or } F_N(\sum_j \alpha_j |j\rangle) = \sum_k \beta_k |k\rangle. \text{ In a special case, when } \alpha_0 = 1 \text{ and}$$

$\alpha_i = 0, i \neq 0$ we have $F_N |0\rangle = 1/\sqrt{N} \sum_k |k\rangle$, i.e., it transform $|0\rangle$ into superposition of all basis states. Did you notice that it looks similar to Hadamard transform $H^{\otimes n}$? The reason is that in fact, Hadamard transform is a special case of QFT.

I don't know about you but F_N looks scary as hell to me :) Fourier transform is beyond my depth, let alone QFT. However, QFT plays an important role in Shor's algorithms, so we have

to study it. One method to get away from dealing with deep math is to learn its properties without proving them. We can dig deeper into the proof once our math muscle is stronger. For now, let's see QFT's nice properties.

Efficient implementation QFT can be implemented in $O((\log N)^2)$. In other words, QFT can be efficiently implemented in polynomial time of N 's bit length.

Shift property $F_N \begin{pmatrix} \alpha_{N-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-2} \end{pmatrix} = \begin{pmatrix} 1\beta_0 \\ \omega\beta_1 \\ \omega^2\beta_2 \\ \vdots \\ \omega^{N-1}\beta_{N-1} \end{pmatrix}$ or $F_N(\alpha_{N-1}|0\rangle + \alpha_0|1\rangle + \alpha_1|2\rangle + \dots + \alpha_{N-2}|N-1\rangle =$

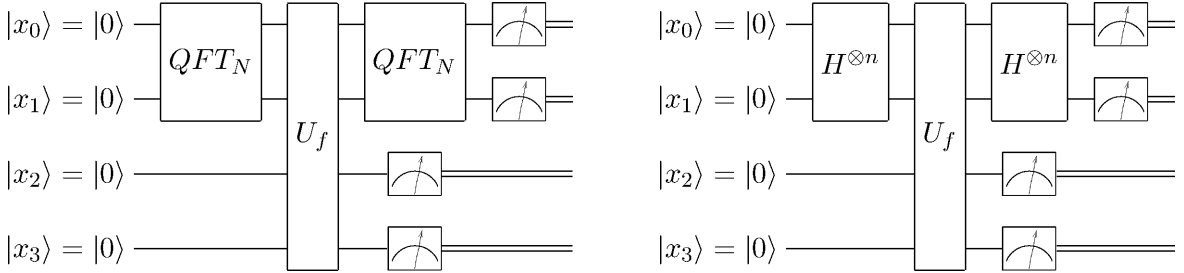
$\sum_k \omega^k \beta_k |k\rangle$, i.e., when we move the 1st amplitude α_0 to the 2nd position, the 2nd amplitude α_1 to the 3rd position, etc then applying QFT to the result corresponds to multiplying β_k with ω^k . When we measure $\sum_k \omega^k \beta_k |k\rangle$, we'll see specific state $|k\rangle$ with probability $|\omega^k \beta_k|^2 = |\omega^k|^2 |\beta_k|^2$. As the magnitude of ω^k is 1, we'll see specific state $|k\rangle$ with probability $|\beta_k|^2$. This is exactly the same as if we measure $\sum_k \beta_k |k\rangle$. What it means is that if we shift the input α_i , it doesn't change the measurement of the QFT's output.

Periodic property If $f(x)$ is a periodic function with period r (i.e., $f(x) = f(x+r \bmod N)$) then $F_N f(x)$ is a periodic function with period N/r . Let's consider a special case $f(x) = 1/\sqrt{N/r} \sum_{j=0}^{N/r-1} |jr\rangle$, i.e., the amplitudes are $1/\sqrt{N/r}$ at $0, r, 2r, \dots, (N/r-1)r$ and are zeros everywhere else, hence $f(x)$ is a period function with period r . In this special case, $F_N(1/\sqrt{N/r} \sum_{j=0}^{N/r-1} |jr\rangle) = 1/\sqrt{r} \sum_{l=0}^{r-1} |lN/r\rangle$. Note that the right hand side's amplitudes are $1/\sqrt{r}$ at $0, N/r, 2(N/r), \dots, (r-1)(N/r)$ and are zeros everywhere else and hence it's a period function with period N/r .

1.6 Period finding

Assume $f: \{0, 1, \dots, N-1\} \rightarrow S$ (S is just some set) is a periodic function with hidden period r ($r|N$): $f(x) = f(x+r \bmod N)$. Find r .

The algorithm is shown in the left figure.



I guess you're disappointed with me because I didn't explain the thinking process in designing the solution, instead I jumped directly to the final algorithm as shown above. The reason is that if you look back Simon's algorithm (as shown in the right figure), you'll notice that it looks exactly the same as period finding's algorithm, except we replace $H^{\otimes n}$ in Simon's algorithm with QFT_N in period finding algorithm. Surprise!

The function f in period finding problem is different from Simon's algorithm and QFT_N is similar but different from $H^{\otimes n}$. Therefore, we have to analyze the period finding algorithm closely.

After applying QFT_N to $|0\rangle$, we get $1/\sqrt{N} \sum_x |x\rangle$. After U_f , we have $1/\sqrt{N} \sum_x |x\rangle |f(x)\rangle$. If we measure $|f(x)\rangle$ and get $f(k)$ for some specific k then as $f(k) = f(k+r) = \dots = f(k+r(N/r-1))$, we know that $1/\sqrt{N} \sum_x |x\rangle |f(x)\rangle$ collapses to $|k\rangle |f(k)\rangle + |k+r\rangle |f(k+r)\rangle + \dots + |k+r(N/r-1)\rangle |f(k+r(N/r-1))\rangle$. Rewrite the result as $\sum_{j=0}^{N/r-1} |jr+k\rangle |f(k)\rangle$. However,

as shifting doesn't affect QFT's measurement, we can simplify the 1st n qubits to $\sum_{j=0}^{N/r-1} |jr\rangle$. Now, using period property of QFT, we know that the last QFT_N will transform $\sum_{j=0}^{N/r-1} |jr\rangle$ into $\sum_{l=0}^{r-1} |lN/r\rangle$. After measuring the 1st n qubits, we'll get sN/r for some random s .

If we repeat the above algorithm a few times, we get s_1N/r , s_2N/r , s_3N/r , etc. It's easier to see that the greatest common divisor of them gives us N/r or r .

1.7 Shor's algorithms

In the previous section, we have built a general framework to find the period of an arbitrary periodic function. In this section, we'll describe Shor's algorithms [3] by transforming factoring problems and discrete log problems into period finding problems.

1.7.1 Shor's factoring algorithm

Given $n = p_1 * p_2$ where p_1, p_2 are prime numbers, find p_1, p_2 .

Shor's algorithm is based on the following classical observation. For a random number x , if we can find an even number r such that $x^r = 1 \pmod n$ then $\gcd(x^{r/2} - 1, n)$ or $\gcd(x^{r/2} + 1, n)$ gives us the factor of n . Let's see why. $x^r - 1 = (x^{r/2} - 1)(x^{r/2} + 1) = 0 \pmod n$ implies $(x^{r/2} - 1)(x^{r/2} + 1) = 0 \pmod{p_1}$, i.e., p_1 divides either $(x^{r/2} - 1)$ or $(x^{r/2} + 1)$. In other words, p_1 divides $\gcd(x^{r/2} - 1, n)$ or $\gcd(x^{r/2} + 1, n)$.

What does $x^r = 1 \pmod n$ mean? It means that if we fix the random number x and define $f(a) = x^a \pmod n$ then $f(a + r) = x^{a+r} = x^a x^r = x^a * 1 = f(a) \pmod n$, i.e., $f(a)$ is a periodic function with period r . Therefore, the algorithm is to generate random x , use the period finding algorithm to find period r of $f(a) = x^a \pmod n$ and finally compute $\gcd(x^{r/2} - 1, n)$ or $\gcd(x^{r/2} + 1, n)$. What if we find odd r , instead of even r ? We just repeat the algorithm with a different x .

1.7.2 Shor's discrete algorithm

Assuming p is a prime number and g is the generator of multiplicative group $\pmod p$, i.e., the set $\{g^0, g^1, \dots, g^{p-2}\}$ is the same as $\{1, 2, \dots, p-1\}$. Given $y = g^x \pmod p$, find x .

Let's take a look at the function $f(a, b) = g^a y^{-b} \pmod p$, we have $f(a + rx, b + r) = g^{a+rx} y^{-b-r} = g^{a+rx} (g^x)^{-b-r} = g^{a+rx-xb-rx} = g^{a-xb} = f(a, b) \pmod p$. I.e., $f(a, b)$ is a periodic function with tuple period (rx, r) . Using a period finding algorithm to find the period (r_1, r_2) of $f(a, b)$ and compute discrete log $x = r_1/r_2$.

2 Lattice-based cryptography

Lattice [7], [8] is a rare double-edged sword in cryptography. It can be used to build cryptographic protocols as well as to break them. Furthermore, lattice-based cryptography is assumed to be safe against quantum computers, while ECDH key exchange and RSA aren't. As quantum computers are more powerful than classical computers, can we deduce that lattice-based cryptography is safe against classical computers? No, we can't. It's because we start with a security assumption (not fact) which might turn out to be wrong. Like many other things in cryptography, lattice-based cryptography's security is an assumption, no one knows for sure. There is no evidence that lattice-based cryptography is safer than ECDH against classical computers. Therefore, it's better to deploy lattice-based key exchange in hybrid mode together with ECDH where the shared key is derived from both lattice and ECDH. Finally, lattice-based cryptography has extraordinary properties that no other cryptographic constructions have. It is used to build famous fully homomorphic encryption (FHE).²

Lattice-based cryptography often relies on the hardness of finding something small or solving equations with errors. Every time you see the words small and errors in the context of vectors, equations, polynomials, you know that you land in the realms of lattice-based cryptography.

²I copied some paragraphs from my previous article [9] because I'm lazy :)

Before studying lattice, let's warm up by solving Fermat's last equation $x^n + y^n = z^n$. I found a beautiful solution $z = \sqrt[n]{x^n + y^n}$. Sorry, I forgot that x, y, z must be integers :) All right, I found an integer solution as well $x = 0, z = y$:) What's wrong with me! I missed the conditions $x, y, z > 0$. The lessons learned are to pay attention to both the variables' domains and their constraints.

2.1 Lattice definitions

Definition Lattice ³ is a set of points $L = \{ \sum_{i=1}^{i=n} a_i v_i | a_i \in \mathbb{Z} \}$ where v_i are linearly independent vectors in \mathbb{R}^n and $B = \{v_1, \dots, v_n\}$ is a base of L . We often write v_i in the form of $n \times 1$ column vector and $B = (v_1, \dots, v_n)$ is a $n \times n$ matrix.

To understand this abstract definition, let's take a look at a concrete example where $v_1 = (2, 0), v_2 = (0, 2)$, i.e., v_1 and v_2 are twice unit vectors in x-axis and y-axis. The lattice L is the set of points $\{a_1 v_1 + a_2 v_2 = (2a_1, 2a_2) | a_1, a_2 \in \mathbb{Z}\}$, i.e., all even integer points in 2-dimensional space. This gives us a figure of lattice, but we haven't gone far in understanding the definition. One way to move forward is to question the details of the definition.

What if $\{v_1, \dots, v_n\}$ are dependent vectors, e.g., $v_1 = v_2 = (0, 2)$? In this case, a single vector v_1 is enough to generate L , i.e., the linearly independent condition is to remove redundancy in $\{v_1, \dots, v_n\}$.

What if a_i are real numbers, instead of integers? In this case, L is the whole 2-dimensional space regardless of the base $B = \{v_1, v_2\}$, i.e., L loses all its interesting math structure.

If we change $\{v_1, \dots, v_n\}$, will L change? Not always. L is defined as a set of points, i.e., the points are the essence of L , not vectors $\{v_1, \dots, v_n\}$ that are used to generate L . The vectors $\{v'_1 = (2, 0), v'_2 = (2, 2)\}$ ($v'_1 = v_1, v'_2 = v_1 + v_2$) also generate all even integer points in 2-dimensional space, i.e., the base $B = \{v_1, \dots, v_n\}$ of L is not unique. In fact, if U is an integer matrix with determinant ± 1 then $B' = BU$ is another base of L . In our example, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$\det(U) = 1 \text{ and } (v'_1, v'_2) = (v_1, v_1 + v_2) = (v_1, v_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We've made significant progress but we can go further by playing with the elements in the definition. If $a_i = 0$ then $0 = 0v_1 + \dots + 0v_n$ is in L . If $x = a_1 v_1 + \dots + a_n v_n$ is in L then $-x = (-a_1)v_1 + \dots + (-a_n)v_n$ is in L . If $x = a_1 v_1 + \dots + a_n v_n$ and $x' = a'_1 v_1 + \dots + a'_n v_n$ are in L then $x + x' = (a_1 + a'_1)v_1 + \dots + (a_n + a'_n)v_n$ is also in L . The above properties make L an additive subgroup of \mathbb{R}^n . The question is whether any additive subgroup of \mathbb{R}^n a lattice? When showing the above properties we haven't used the fact that a_i are integers at all. As mentioned above if a_i are real numbers then L contains all points in 2-dimensional space and the fact that a_i are integers make the lattice look discrete and not dense. Put everything together, we have an equivalent definition of lattice.

Definition Lattice is a discrete additive subgroup of \mathbb{R}^n .

You may wonder why I made an effort to introduce the 2nd definition. What's the point? In practical problems, no one is going to tell us that there is a lattice structure. We ourselves have to recognize the lattice structure hidden in the problem we're trying to solve. The 1st definition is not useful in this regard because we're not even sure yet there is a lattice, let alone know its basis vectors. The 2nd definition gives us signals to detect hidden lattice structure.

Let's look at the set of integer solutions of the following equation $2x_1 + 3x_2 = 0$. It's obvious that $(0, 0)$ is a solution. If (x_1, x_2) and (x'_1, x'_2) are solutions then $(-x_1, -x_2)$ and $(x_1 + x'_1, x_2 + x'_2)$ are solutions because $2(-x_1) + 3(-x_2) = -(2x_1 + 3x_2) = 0$ and $2(x_1 + x'_1) + 3(x_2 + x'_2) = (2x_1 + 3x_2) + (2x'_1 + 3x'_2) = 0 + 0 = 0$. The above properties make the set of solutions additive subgroup of \mathbb{R}^n and the integer condition makes it discrete. Therefore, the set of solutions is a lattice. In conclusion, we know that the set of solutions forms a lattice without solving the

³Lattice has a deep theory which is out of my depth. If you want to understand lattice in depth, I recommend studying it by excellent formal sources [7], [8].

equation, is it amazing? While this example is somewhat trivial, we'll use this observation in more complicated problems in later sections.

2.1.1 Successive minima

We briefly introduce a few concepts that we'll need in later sections.

$\lambda_1(L)$ denotes the length of a nonzero shortest vector in L . As 0 is always in L , in the definition of $\lambda_1(L)$, we have to add the nonzero condition to avoid trivial cases.

The i th successive minima $\lambda_i(L)$ is the smallest r such that L has i linearly independent vectors whose lengths are at most r .

2.2 Hard lattice computational problems

Public-key cryptography is based on hard computational problems. For instance, ECDH is based on hardness of discrete log problems while RSA is based on hardness of factoring problems. Similarly, lattice-based cryptography is based on hard lattice computational problems. Therefore, we briefly introduce the following hard lattice computational problems, mostly to introduce the terminologies.

Shortest Independent Vectors Problem (SIVP_γ): Given lattice $L(B)$ and small $\gamma > 0$, find n linearly independent lattice vectors whose length is at most $\gamma\lambda_n(L)$.

Bounded Distance Decoding Problems (BDD_γ): Given a lattice $L(B)$, small $\gamma > 0$ and a target point t that is guaranteed to be close to L (i.e. $\text{distance}(t, L) < d = \lambda_1(L)/\gamma$), find a unique lattice vector v such that $\|v - t\| < d$.

For the purpose of this article, it's fine to just remember the following: for lattice, it's difficult to find short lattice vectors or to find a lattice vector close to a given target point.

2.3 Short Integer Solution (SIS)

SIS Problem Given m random vector $a_i \in \mathbb{Z}_q^n$ (i.e. a_i is a n -dimensional vector where each coordinate is integer mod q), find nonzero solution $(z_1, \dots, z_m) \in \{-1, 0, 1\}$ of system of linear equations $z_1 a_1 + \dots + z_m a_m = 0$. In matrix form $Az = 0$ where $A = (a_1, \dots, a_m)$ is a $n \times m$ matrix.

Note that if there is no condition $z_i \in \{-1, 0, 1\}$ then we know how to solve this system of linear equations using Gaussian elimination.

Let's take a look at the set of solutions $\{(z_1, \dots, z_m)\}$. $(0, \dots, 0)$ is a trivial solution. If (z_1, \dots, z_m) and (z'_1, \dots, z'_m) are solutions then $(-z_1, \dots, -z_m)$ and $(z_1 + z'_1, \dots, z_m + z'_m)$ are solutions because $(-z_1)a_1 + \dots + (-z_m)a_m = -(z_1 a_1 + \dots + z_m a_m) = 0$ and $(z_1 + z'_1)a_1 + \dots + (z_m + z'_m)a_m = (z_1 a_1 + \dots + z_m a_m) + (z'_1 a_1 + \dots + z'_m a_m) = 0 + 0 = 0$. The above properties make the set of solutions $\{(z_1, \dots, z_m)\}$ an additive group and integer condition makes it discrete. Therefore, the set of integer solutions $\{(z_1, \dots, z_m)\}$ is a lattice. How's about the condition $z_i \in \{-1, 0, 1\}$? This condition forces the vector (z_1, \dots, z_m) to be small. Consequently, solving SIS problems is similar to finding small vectors in the lattice of integer solutions. In other words, solving SIS is a hard problem. It's pretty cool as whenever we see a hard problem, we have hope to use it to build cryptographic protocols. In the next section, we'll use the hardness of SIS problem to construct collision-resistant hash function.

2.3.1 Collision-resistant hash functions based on SIS

Given a random matrix $A \in \mathbb{Z}_q^{n \times m}$, the function $f_A(z) = Az$ where $z \in \{0, 1\}^m$ is a collision-resistant hash function.

We'll use proof by contradiction. If $f_A(z) = Az$ is not a collision-resistant hash function then we can find $z_1, z_2 \in \{0, 1\}^m$ such that $f_A(z_1) = f_A(z_2)$. This implies $Az_1 = Az_2$ or $A(z_1 - z_2) = 0$. As $z_1, z_2 \in \{0, 1\}^m$, we have $z_1 - z_2 \in \{-1, 0, 1\}^m$. It means that we found $z = z_1 - z_2 \in \{-1, 0, 1\}^m$ such that $Az = 0$. In other words, we can solve SIS problems (contradicts the fact that SIS problems are hard).

2.4 Learning With Errors (LWE)

Who came up with the name "learning with errors" [10]? Learning with accurate information is hard, let alone learning with misinformation and errors :) Maybe, it's the intention to make the attacker's life miserable.

LWE Problem Generate m random vectors $a_i \in Z_q^n$, small random errors e_i , a random secret $s \in Z_q^n$ and computes $b_i = \langle a_i, s \rangle + e_i \mod q$ ⁴. Given (a_i, b_i) , find s . In the matrix form, given $(A, b^t = s^t A + e^t \mod q)$ ⁵, find s where $A = (a_1 \cdots a_m)$ is an $n \times m$ matrix and $b = (b_1 \cdots b_m) \in Z_q^m$. A closely related problem is to not find s , but to distinguish $(A, b^t = s^t A + e^t \mod q)$ from random distribution.

Note that if there are no errors e_i then $b^t = s^t A$ is a system of linear equations which we know how to solve for s using Gaussian elimination.

Now, let's try to find our lattice in this LWE problem. If we remove errors e_i then if we fix matrix A , the set $L = \{s^t A \mod q\}$ forms a lattice. Why? $0 = 0A$ is in L . If $x_1 = s_1^t A$ and $x_2 = s_2^t A$ are in L then $-x_1$ and $x_1 + x_2$ are in L because $-x_1 = -s_1^t A = (-s_1^t)A$ and $x_1 + x_2 = s_1^t A + s_2^t A = (s_1^t + s_2^t)A$. The above properties make L additive group and the integer condition makes it discrete. Therefore, L is a lattice. In our LWE problem, $b^t = s^t A + e^t$ is a point close to the lattice point $s^t A$ because the difference between them is the small error vector e^t . Our task is given b^t , find the lattice point $s^t A$ close to it (note that knowing $s^t A$ is enough to find s^t using Gaussian elimination). This is the Bounded Distance Decoding (BDD_γ) problem which is hard to solve.

2.4.1 Ring-LWE

Define $R = Z_q[x]/(x^n + 1)$ where n is a power of 2, i.e., polynomials of degree n where coefficients are integers $\mod q$ and operations on polynomials are $\mod x^n + 1$ (i.e. we can replace x^n with -1). The definition of Ring-LWE is similar to LWE, except instead of using Z_q^n , we use R .

Ring-LWE Problem Generate m random polynomials $a_i \in R$, small random error polynomials e_i , a random secret $s \in R$, computes $b_i = a_i s + e_i$. Given (a_i, b_i) , find s .

What's the relationship between element of R with our familiar vector Z_q^n ? Let's write down a polynomial $p(x)$ in R

$$p(x) = a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

The coefficients $(a_{n-1}, \cdots, a_1, a_0)$ is a vector in Z_q^n . On one hand, to a certain extent, we can cast Ring-LWE problems into LWE problems. On the other hand, polynomial $\mod (x^n + 1)$ has more structure than Z_q^n . Let's briefly take a look at $p(x)x \mod x^n + 1$

$$\begin{aligned} p(x)x &= (a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0)x \\ &= a_{n-1}x^n + a_{n-2}x^{n-1} + \cdots + a_1x^2 + a_0x \\ &= a_{n-1}(-1) + a_{n-2}x^{n-1} + \cdots + a_1x^2 + a_0x \\ &= a_{n-2}x^{n-1} + \cdots + a_1x^2 + a_0x - a_{n-1} \end{aligned}$$

I.e. multiply $p(x)$ with x corresponds to transform from $(a_{n-1}, \cdots, a_1, a_0)$ to $(a_{n-2}, \cdots, a_1, a_0, -a_{n-1})$. Properties like this make implementation of Ring-LWE more efficient than LWE. However, whenever you have extra math structure, it might later come back and help cryptanalysis. There is an on-going debate about the trade-off between efficiency of Ring-LWE and its extra math structure to security.

⁴The notation $\langle x, y \rangle$ denotes the inner product of 2 vectors $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$, i.e., $x_1y_1 + \cdots + x_ny_n$

⁵The notation s^t means the transpose of s

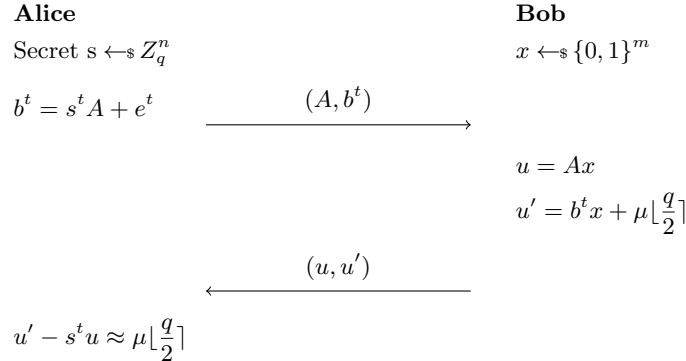
2.5 Regev's LWE public key cryptosystem

In this section, we'll take a look at Regev's public key cryptosystem [10] that is based on the hardness of LWE problem. The protocol only encrypts a single bit μ but it can be generalized to encrypt arbitrary data. Note that Regev's encryption is only semantically secure, i.e., the ciphertext is indistinguishable from random distribution and it's safe against eavesdropper. It's not safe against active adversary, i.e., it's not safe against chosen ciphertext attack.

To build any public key cryptosystem, the first step is to construct private/public key pair. LWE's description gives us a hint on how to do it: Alice's private key is s and her public key is $(A, b^t = s^t A + e^t)$. As $(A, b^t = s^t A + e^t)$ is indistinguishable from random distribution, the public key (A, b^t) doesn't leak any information about the private key s . The question is how to use the public key (A, b^t) for encryption?

As semantic security concerns indistinguishability from random, we'll try to generate random alike traffic to confuse eavesdroppers. We need another math property: if $x \in \{0, 1\}^m$ then (A, Ax) is indistinguishable from random distribution. Therefore, using Alice's public key $(A, b^t = s^t A + e^t)$, Bob can generate $x \in \{0, 1\}^m$ and send $(Ax, b^t x)$ to Alice. This is safe because $(Ax, b^t x)$ is indistinguishable from random distribution. What can Alice do? Note that $b^t x = (s^t A + e^t)x = s^t Ax + e^t x$ and Alice knows the private key s^t , so she can compute $b^t x - s^t Ax = e^t x$. I.e., Alice can compute the noise $e^t x$ (e^t and x are small), but the eavesdropper can't. Awesome, Bob can generate random-alike traffic $(Ax, b^t x)$ and Alice can denoise it. We're happy, we make the eavesdropper confused, except we confuse ourselves as well because we haven't encrypted anything yet :)

We're close to the final protocol. The last piece of the puzzle is how to include the bit μ into the traffic. What will happen if Bob sends $(Ax, b^t x + \mu)$? Alice uses the same computation to get $b^t x - s^t Ax = e^t x + \mu$. However, Alice doesn't know x , so while she can compute the noise $e^t x + \mu$, she can't extract μ from it. The idea to solve this puzzle is to encode the "meaning" of μ , instead of μ itself into the traffic: if $\mu = 0$, we doesn't include it into the traffic, but if $\mu = 1$ then we include large number $\mu \lfloor \frac{q}{2} \rfloor$ into traffic. When Alice receives it, she computes $b^t x - s^t Ax$ and if the result is small, she knows that μ was 0, otherwise if $b^t x - s^t Ax$ is large, she knows that μ was 1. The complete protocol is shown in the below figure.



To encrypt a bit μ , choose random $x \in \{0, 1\}^m$, compute ciphertext as follow ($u = Ax, u' = b^t x + \mu \lfloor \frac{q}{2} \rfloor$).

To decrypt, compute $u' - s^t u = b^t x + \mu \lfloor \frac{q}{2} \rfloor - s^t Ax = (s^t A + e^t)x + \mu \lfloor \frac{q}{2} \rfloor - s^t Ax = e^t x + \mu \lfloor \frac{q}{2} \rfloor \approx \mu \lfloor \frac{q}{2} \rfloor$ and test whether it is closer to 0 ($\mu = 0$ case) or $\lfloor \frac{q}{2} \rfloor \bmod q$ ($\mu = 1$ case).

2.6 Lattice cryptanalysis of subset sum problem

LLL algorithm [11] is a celebrated algorithm, there is a dedicated book just for LLL algorithm and its applications. For the purpose of the article, we'll use only one property of the LLL algorithm. Given a lattice $L(B)$ in \mathbb{R}^n where B is a base of L , LLL algorithm will find a relatively short vector b : $\|b\| < \beta^n \lambda_1(L)$ where $1 < \beta < 2$.

Using the above property, to solve certain equations, we'll prove the following:

- The solutions belong to a lattice.
- The solutions are short vectors.

After that we'll let LLL algorithm do its job to find short vectors (solutions) in the above lattice for us. Let's apply the described method to solve the following subset sum problem [12]: given n positive numbers a_i and a positive number M , find $x_i \in \{0, 1\}$ such that $a_1x_1 + \dots + a_nx_n = M$.

Let's consider the following vectors

$$\begin{aligned} b_1 &= (1, 0, \dots, 0, -a_1) \\ b_2 &= (0, 1, \dots, 0, -a_2) \\ &\vdots \\ b_n &= (0, 0, \dots, 1, -a_n) \\ b_{n+1} &= (0, 0, \dots, 0, M) \end{aligned}$$

We have

$$\begin{aligned} x_1b_1 + x_2b_2 + \dots + x_nb_n + 1b_{n+1} \\ &= (x_1, 0, \dots, 0, -a_1x_1) \\ &+ (0, x_2, \dots, 0, -a_2x_2) \\ &\vdots \\ &+ (0, 0, \dots, x_n, -a_nx_n) \\ &+ (0, 0, \dots, 0, M) \\ &= (x_1, x_2, \dots, x_n, -a_1x_1 - a_2x_2 + \dots - a_nx_n + M) \\ &= (x_1, x_2, \dots, x_n, 0) \end{aligned}$$

I.e., $(x_1, x_2, \dots, x_n, 0)$ is a vector in the lattice with basis vectors b_1, \dots, b_{n+1} . Furthermore, $(x_1, x_2, \dots, x_n, 0)$ is a short vector because $x_i \in \{0, 1\}$. Therefore, LLL algorithm will help find the short vector (solution) $(x_1, x_2, \dots, x_n, 0)$ for us.

2.7 Lattice-based homomorphic encryption

Let's assume we have data that we store in a cloud. To protect our data, we encrypt them and keep the key to ourselves. On the other hand, we want to take advantage of cloud's computing power, so we want the cloud to compute on our ciphertexts without knowing what our plaintexts are. Homomorphic encryption is a special type of encryption that achieves the previous goal. In this section, we'll describe a simple lattice-based homomorphic encryption [13].

We'll use $\mathbb{Z}_q[x]/(x^{2^k} + 1)$ (q is a prime number), i.e., polynomials whose coefficients are in \mathbb{Z}_q and all operations are $\text{mod } x^{2^k} + 1$. We also use a small modulus t that is much smaller than q . Note that everything in this section including secret key, message, ciphertext are polynomials.

The secret key is a polynomial s in $\mathbb{Z}_q[x]/(x^{2^k} + 1)$.

To encrypt a message polynomial $m \in \mathbb{Z}_t[x]/(x^{2^k} + 1)$, we randomly generate polynomial a , small error polynomial e and the ciphertext is simply $c = \text{Enc}(m) = (c_0, c_1) = (-a, as + m + et)$.

To decrypt a ciphertext $c = (c_0, c_1)$, we compute $c_1 + c_0s \text{ mod } t = as + m + et - as \text{ mod } t = m + et \text{ mod } t = m$.

To see how this encryption is additive homomorphic, let's take a look at two encryptions of m and m' : $c = \text{Enc}(m) = (c_0, c_1) = (-a, as + m + et)$ and $c' = \text{Enc}(m') = (c'_0, c'_1) = (-a', a's + m' + e't)$. If we add (c_0, c_1) and (c'_0, c'_1) together, we have:

$$\begin{aligned}
(c_0, c_1) + (c'_0, c'_1) &= (c_0 + c'_0, c_1 + c'_1) \\
&= (-a - a', as + m + et + a's + m' + e't) \\
&= (-(a + a'), (a + a')s + (m + m') + (e + e')t)
\end{aligned}$$

If we denote $a'' = a + a'$, $m'' = m + m'$, $e'' = e + e'$, then we see that $c'' = (c''_0, c''_1) = (c_0, c_1) + (c'_0, c'_1)$ is the encryption of $m'' = m + m'$ with error $e'' = e + e'$. To recap, what we've done is to add 2 ciphertexts together without knowing the messages, but the result corresponds to the sum of the messages. It's pretty cool, right?

Another nice property is that if you multiply a polynomial p to the encryption of m then the result corresponds to encryption of pm . To see why it's the case, let's take a look at polynomial p and encryption of m : $(c_0, c_1) = (-a, as + m + et)$. We have:

$$\begin{aligned}
p(c_0, c_1) &= (pc_0, pc_1) \\
&= (p(-a), p(as + m + et)) \\
&= (-pa, pas + pm + pet)
\end{aligned}$$

If we denote $a' = -pa$, $m' = pm$, $e' = pe$, then we see that $(c'_0, c'_1) = p(c_0, c_1)$ is the encryption of $m' = pm$ with error $e' = pe$.

The final note is that the error increases in both cases. For lattice-based cryptography to work, the error must be small. Therefore, various techniques have been designed to reduce the error over time. We won't discuss error reduction techniques here, instead, we'll take a look at an awesome application of the previous homomorphic encryption in the next section.

2.8 Lattice-based private information retrieval

Let's say a server has a public database (e.g. movies, songs, lyrics, stories, books) with n items x_1, \dots, x_n . A user wants to see a single item x_i at index i from the database without revealing to the server what item has been downloaded. This is to protect the user's privacy. An obvious solution is the user downloads all n items from the database. This has perfect privacy, but it costs significant bandwidth and user's local storage. We'll trade CPU with bandwidth and storage using the above homomorphic encryption [14]. The basic protocol works as follows.

The user forms a sequence of 0 and 1 where only at index i , it's 1 while the remaining numbers are 0: $0, \dots, 0, \underbrace{1}_{\text{index } i}, 0, \dots, 0$. The user uses homomorphic encryption to encrypt the above sequence, i.e., $c_1 = \text{Enc}(0), \dots, c_{i-1} = \text{Enc}(0), c_i = \text{Enc}(1), c_{i+1} = \text{Enc}(0), \dots, c_n = \text{Enc}(0)$. The user sends c_1, \dots, c_n to the server.

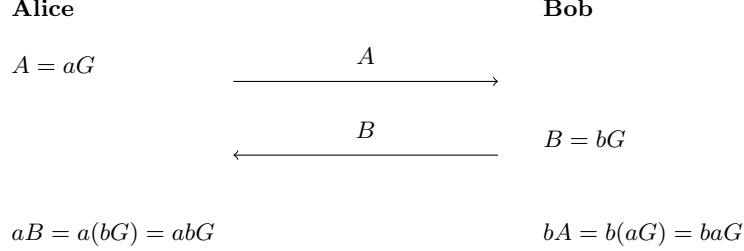
The server computes $x = x_1c_1 + \dots + x_nc_n$ without knowing what i is and sends x to the user.

The user decrypts x and the result is x_i . Why's that? By homomorphic property, $x = x_1c_1 + \dots + x_nc_n$ corresponds to the encryption of $x_1.0 + \dots + x_{i-1}.0 + x_i.1 + x_{i+1}.0 + \dots + x_n.1 = 0 + \dots + 0 + x_i + 0 + \dots + 0 = x_i$.

3 Supersingular Isogeny Diffie-Hellman(SIDH)

SIDH [15][16][17][18] is the hardest to understand post-quantum cryptography construction. If you read my previous article "Intuitive Advanced Cryptography" [9], you would know that ECDH uses 1 elliptic curve, then pairing increases the difficulty significantly by using 2 elliptic curves. SIDH is the next difficulty level where it uses a graph (set) of elliptic curves. Therefore, this chapter requires prerequisite: you must watch the movie Inception by Christopher Nolan.

I'm serious! Furthermore, we'll give up on the ambitious goal of analyzing SIDH's security. Instead, we'll set a humble (not-so-humble) goal to understand how SIDH works. If we can't achieve our goal, change it to an easier one :)



Let's recall a few terminologies in elliptic curve and ECDH protocol. In this article, we'll use supersingular⁶ elliptic curve E defined over a finite field $F_q = F_{p^2}$. An extraordinary property of elliptic curve is that we can define addition operation $+$ between its points, i.e., given 2 points $P, Q \in E$, the operation $P + Q$ makes sense and the result is another point $R \in E$. How points addition operation $+$ is defined is not our concern, what we care is that $(E, +)$ forms a group. The order of a point P is the smallest number n for which $nG = 0$ where 0 denotes the identity of E . The ECDH protocol works as follows. Alice generates a random private key a , computes her public key $A = aG$ and sends A to Bob. Bob generates a random private key b , computes his public key $B = bG$ and sends B to Alice. Now Alice computes $aB = a(bG) = abG$ and Bob computes $bA = b(aG) = baG$. We notice that Alice and Bob compute the same result abG and it's their shared key that can be used for encryption.

Let's look at ECDH from a different perspective using graph terminology. Let consider points in the elliptic curve E as nodes in a graph and 2 points are connected by an edge if their difference is G . Using graph terminology, Alice computes her public key aG by travelling through the following path in the graph: $P_1 = G \xrightarrow{G} P_2 = 2G \xrightarrow{G} P_3 = 3G \cdots \xrightarrow{G} P_a = aG$. The security relies on the fact that knowing the destination point $P_a = aG$ doesn't help the adversary compute how many edges \xrightarrow{G} that Alice has gone through. In terms of practical computation, Alice does not compute $P_a = aG$ linearly as shown above, instead she uses a double-and-add algorithm. In special case when $a = 2^k$, Alice computes $P_1 = G, P_2 = 2P_1, P_4 = 2P_2, \cdots, P_{2^k} = 2P_{2^{k-1}}$. In other words, in ECDH protocol, each node is a point and we travel from points to points within 1 elliptic curve. In SIDH protocol, a node is an elliptic curve⁷ and we travel from elliptic curves to elliptic curves and if we look inside 1 node/elliptic curve we'll see multiple points inside it. It's similar to dreams in the movie Inception. Inside each node (elliptic curve) in SIDH protocol, there are other nodes (points) within that elliptic curve. From another perspective, it's similar to abstraction layers. At the SIDH layer, each node in a graph is an elliptic curve and we abstract away the detail in the lower layer of how points within that elliptic curve look like. We talked about nodes, but we haven't defined edges in SIDH protocol. I.e. what connects 1 node (1 elliptic curve) to another?

The connection from 1 node (1 elliptic curve) to another in SIDH protocol is an isogeny. Formally, isogeny is a non-constant rational map $E_1 \xrightarrow{\phi} E_2$ which is also a group homomorphism from $E_1(F_q)$ to $E_2(F_q)$. What rational map means is not our concern but we'll pay attention to what group homomorphism is. In this context, group homomorphism means that $\phi(\alpha P + \beta Q) = \alpha\phi(P) + \beta\phi(Q)$ where $P, Q \in E_1, \alpha, \beta \in \mathbb{Z}$. I.e., if we're given $\phi(P), \phi(Q)$ then we can compute ϕ of any linear combination $\alpha P + \beta Q$ of P and Q . In the next paragraphs, we'll introduce a few new concepts and isogeny's properties.

The kernel of $E_1 \xrightarrow{\phi} E_2$ is the set of points $P \in E_1$ such as $\phi(P) = 0$.

⁶Let's ignore what supersingular means. For advanced readers, the word supersingular probably triggers your negative reaction because using supersingular curves in ECDH is not safe. However, SIDH uses a graph of elliptic curves, not points within 1 elliptic curve like ECDH, so the computational hard problem is different.

⁷For strict readers, a node is a set of isomorphic elliptic curves that have the same j-invariants.

The degree of isogeny $E_1 \xrightarrow{\phi} E_2$, denoted as $\deg \phi$, is the number of elements in the kernel $\ker(\phi)$.

The multiplication-by- n map $E \xrightarrow{[n]} E$ maps a point P to nP . The n -torsion group of E , denoted $E[n]$ is the set of all points $P \in E$ such that nP is the identity. In other words, n -torsion group is the kernel of a multiplication-by- n map.

The composition of isogenies $E_0 \xrightarrow{\phi_0} E_1$ and $E_1 \xrightarrow{\phi_1} E_2$ is an isogeny $\phi_1 \circ \phi_0 : E_0 \rightarrow E_2$ ($E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2$) and $\deg(\phi_1 \circ \phi_0) = \deg \phi_1 * \deg \phi_0$. This composition property is important as it allows us to combine edges to create paths such as $E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2$. Let's look at a path with more than 2 edges. If we have a series of k edges (isogenies), each have degree 2 as follows: $E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} E_3 \cdots \xrightarrow{\phi_{k-1}} E_k$ then the composition $\phi_k \circ \phi_{k-1} \cdots \phi_0 : E_0 \rightarrow E_k$ has order 2^k . Do you notice the similarity with ECDH?

$$\begin{array}{lllll} \text{ECDH:} & P_1 = G & P_2 = 2P_1 & P_4 = 2P_2 \cdots & P_{2^k} = 2P_{2^{k-1}} \\ \text{SIDH:} & E_1 = \phi_0(E_0) & E_2 = \phi_1(E_1) & E_3 = \phi_2(E_2) \cdots & E_k = \phi_{k-1}(E_{k-1}) \end{array}$$

The previous paragraph assumes that we know how to construct isogeny ϕ_i . It's time to discuss isogeny's construction. Given a finite subgroup $G \in E_1$, Vélu formula⁸ constructs an isogeny $\phi : E_1 \rightarrow E_2$ that has G as kernel and we write $E_2 = \phi(E_1) = E_1/\langle G \rangle$. What it means is that if we start at E_1 , choose a subgroup G as kernel, then we not only can construct an edge (isogeny) from E_1 but we also know how to compute the other end E_2 of the constructed edge. In other words, an edge (isogeny) is determined by its kernel (subgroup of starting node). We'll come back to this property in later paragraphs.

We're not ready to describe the protocol yet. The elliptic curves in SIDH have special structure, so we have to look at them first. We will work with extension field $F_q = F_{p^2}$ where p is a prime of the form $p = l_A^{e_A} l_B^{e_B} - 1$, $l_A = 2, l_B = 3$. Fix a supersingular curve E_0 over F_q . We have $E_0[2^{e_A}] \cong Z_{2^{e_A}} \times Z_{2^{e_A}}$. In layman's term, it means that there exist 2 points P_A, Q_A of order 2^{e_A} and they're the basis of 2^{e_A} -torsion group of E_0 (aka kernel of multiplication-by- 2^{e_A} map). We write $E_0[2^{e_A}] = \langle P_A, Q_A \rangle$. Similarly, $\langle P_B, Q_B \rangle = E_0[2^{e_B}]$. All E_0, P_A, Q_A, P_B, Q_B are public parameters.

Let's combine all the knowledge in the previous paragraphs and see how far we can go. If we fix a point $R_0 = [m_A]P_A + [n_A]Q_A$, $m_A, n_A \in \mathbb{Z}$ of order 2^{e_A} then we can efficiently construct an isogeny $\phi_A : E_0 \rightarrow E_A$ having $\langle R_0 \rangle$ as kernel. Wait a second, we knew how to compute ϕ_A from kernel $\langle R_0 \rangle$ using Vélu's formula, so what's new? The key word is "efficient". Note that the kernel $\langle R_0 \rangle$ is a large group with 2^{e_A} elements and Vélu's formula requires us to explicitly list all the elements in the kernel. We'll use composition of isogenies property to decompose ϕ_A into smaller isogenies. Recall that the composition of isogenies of degree 2 is the isogeny $\phi_k \circ \phi_{k-1} \cdots \phi_0 : E_0 \rightarrow E_k$ of order 2^k : $E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} E_3 \cdots \xrightarrow{\phi_{k-1}} E_k$. The idea to construct $\phi_A : E_0 \rightarrow E_A$ is to find $\phi_i, i = 0, \dots, e_A - 1$ (each has small order 2 and hence is efficiently computable) such that $\phi_A = \phi_{e_A} \cdots \phi_1 \circ \phi_0$ and $E_k = E_A$ has $\langle K_A \rangle$ as kernel. Let's see how we can construct isogeny of order 2: $E_0 \xrightarrow{\phi_0} E_1$. To do that, we need to find a point of order 2 that generates the kernel of ϕ_0 (recall that isogeny is determined by its kernel). The point that we're looking for is $2^{e_A-1}R_0$. Why's that? We know that R_0 has order 2^{e_A} , i.e., $2^{e_A}R_0 = 0$ or $2(2^{e_A-1}R_0) = 0$. In other words, $2^{e_A-1}R_0$ is a point of order 2 and it's the point we're looking for. Denote $R_1 = \phi_0(R_0)$ then R_1 has order 2^{e_A-1} in E_1 . Using a similar argument, we see that $2^{(e_A-1)-1}R_1$ is the kernel of isogeny $\phi_1 : E_1 \rightarrow E_2$. In general, the following algorithm will construct $E_0, E_1, \dots, E_{e_A-1}$ and isogenies $\phi_0, \phi_1, \dots, \phi_{e_A-1}$

$$E_{i+1} = E_i / \langle 2^{e_A-i-1}R_i \rangle, \phi_i : E_i \rightarrow E_{i+1}, R_{i+1} = \phi_i(R_i)$$

Finally, after all the hard work, we're ready to describe SIDH protocol.

⁸The explicit formula is complicated, so we'll skip it.

Alice

$$E_A = E_0 / \langle [m_A]P_A + [n_A]Q_A \rangle = E_0 / \langle K_A \rangle$$

$$E_0 \xrightarrow{\phi_A} E_A$$

Bob

$$E_B = E_0 / \langle [m_B]P_B + [n_B]Q_B \rangle = E_0 / \langle K_B \rangle$$

$$E_0 \xrightarrow{\phi_B} E_B$$

$$\phi_A(P_B),$$

$$\phi_A(Q_B),$$

$$E_A$$

$$\longrightarrow$$

$$\phi_B(P_A),$$

$$\phi_B(Q_A),$$

$$E_B$$

$$\longleftarrow$$

$$E_B \xrightarrow{\phi'_A} E_{AB}$$

$$E_{AB} =$$

$$E_B / \langle [m_A]\phi_B(P_A) + [n_A]\phi_B(Q_A) \rangle$$

$$E_A \xrightarrow{\phi'_B} E_{BA}$$

$$E_{BA} =$$

$$E_A / \langle [m_B]\phi_A(P_B) + [n_B]\phi_A(Q_B) \rangle$$

The starting node (elliptic curve) for both Alice and Bob is E_0 . Alice chooses her secret isogeny (path) $\phi_A : E_0 \rightarrow E_A$ by choosing a secret kernel $K_A = \langle [m_A]P_A + [n_A]Q_A \rangle$ where m_A, n_A are randomly generated in $\{0, \dots, 2^{e_A-1}\}$. We know how to compute ϕ_A efficiently from the previous paragraphs. Alice then computes $\phi_A(P_B), \phi_A(Q_B)$ and sends her public key $\{E_A, \phi_A(P_B), \phi_A(Q_B)\}$ to Bob. Bob chooses his secret isogeny (path) $\phi_B : E_0 \rightarrow E_B$ by choosing a secret kernel $K_B = \langle [m_B]P_B + [n_B]Q_B \rangle$ where m_B, n_B are randomly generated in $\{0, \dots, 3^{e_B-1}\}$. Bob computes $\phi_B(P_A), \phi_B(Q_A)$ and sends his public key $\{E_B, \phi_B(P_A), \phi_B(Q_A)\}$ to Alice.

After receiving Bob's public key, Alice uses $E_B, \phi_B(P_A), \phi_B(Q_A)$ to construct isogeny $\phi'_A : E_B \rightarrow E_{AB}$ with kernel $\langle [m_A]\phi_B(P_A) + [n_A]\phi_B(Q_A) \rangle$. Similarly, after receiving Bob's public key, Bob uses $E_A, \phi_A(P_B), \phi_A(Q_B)$ to construct isogeny $\phi'_B : E_A \rightarrow E_{BA}$ with kernel $\langle [m_B]\phi_A(P_B) + [n_B]\phi_A(Q_B) \rangle$.

At this point, I guess you have many questions. What are $\phi_B(P_A), \phi_B(Q_A)$ for? What does complicated formula $E_0 / \langle [m_A]\phi_B(P_A) + [n_A]\phi_B(Q_A) \rangle$ mean? As isogeny is group homomorphism, we have $\phi_B([m_A]P_A + [n_A]Q_A) = [m_A]\phi_B(P_A) + [n_A]\phi_B(Q_A)$. Furthermore, we have $K_A = [m_A]P_A + [n_A]Q_A$, so $\phi_B(K_A) = [m_A]\phi_B(P_A) + [n_A]\phi_B(Q_A)$. This means that $\phi_B(P_A), \phi_B(Q_A)$ allows Alice to compute $\phi_B(K_A)$ although Alice does not know Bob's secret isogeny ϕ_B . This discussion shed some light on the cumbersome formula $E_B / \langle [m_A]\phi_B(P_A) + [n_A]\phi_B(Q_A) \rangle$, it is just $E_B / \langle \phi_B(K_A) \rangle$. Now, let's revisit $E_{AB} : E_0 \xrightarrow{\phi_B} E_B \xrightarrow{\phi'_A} E_{AB}$, we have $E_{AB} = \phi'_A(\phi_B(E_0)) = E_0 / \langle K_A, K_B \rangle = \phi'_B(\phi_A(E_0)) = E_{BA}$. I.e., Alice and Bob compute the same shared secret.

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