

$$V_{i+1} = (aV_i + c) \mod (m) \quad i = 0, 1, \dots,$$

where V_0 : seed a : multiplier c : increment m : modulus

Theorem : LCG has full period if and only if

1. If q is a prime that divides m , then q divides $(a - 1)$.
2. The only positive integer that divides m and c is 1.
3. If 4 divides m , then 4 divides $a - 1$.

- exponential distribution inversion method: $x = -\frac{1}{\lambda} \ln(1 - u)$
- generate poisson distribution:

1. Generate a random number U
2. $i = 0, p = e^{-\lambda}, F = p$.
3. If $U < F$, set $X = i$ and **stop**.
4. $p = \frac{\lambda}{(i+1)} p, F = F + p, i = i + 1$.
5. Goto **Step 3**.

- generate a random variable having density function

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if } 2 \leq x \leq 3 \\ \frac{2-x/3}{2} & \text{if } 3 \leq x \leq 6 \end{cases}$$

5.2

$$F(x) = \int_2^x \frac{x-2}{2} dx \quad (2 \leq x \leq 3), \quad \int_3^x \frac{2-x/3}{2} dx \quad (3 \leq x \leq 6)$$

$$F(x) = \begin{cases} \frac{(x-2)^2}{4} & (2 \leq x \leq 3) \Rightarrow 0 \sim 0.25 \\ 1 - \frac{(6-x)^2}{12} & (3 \leq x \leq 6) \Rightarrow 0.25 \sim 1 \end{cases}$$

1. Generate $u_1 \sim U(0, 1)$

2. If $u_1 \leq 0.25$,

$$u_1 = \frac{(x-2)^2}{4}, \quad x = 2 + 2\sqrt{u_1}$$

else

$$u_1 = \left(1 - \frac{(6-x)^2}{12}\right), \quad x = 6 - \sqrt{12(1 - u_1)}$$

- Rejection Method

1. Discrete random variable

1. 도구 분포(q_j), 목표 분포(p_j) 설정. 도구분포는 목표 분포와 support set이 같아야 함.
 - $N(0, 1)$ support set: $\{-\infty, x < \infty\}$
 - 지수분포 support set: $\{x > 0\}$
 - 포아송 분포 support set: $\{x = 0, 1, 2, \dots\}$
 - 기하 분포 support set: $\{x = 1, 2, \dots\}$
 - 그 외 균등분포 $[a, \mu] + 1$ 로 1부터 a 까지의 정수 균등분포 생성 가능.
2. 모든 $p_j > 0$ 인 j 에 대하여 $c = \max(\frac{p_j}{q_j})$ 인 상수 c 찾는다.
3. 다음을 반복한다.
 1. 도구 분포 q_j 로부터 Y 를 생성한다.
 2. $U \sim \text{Uniform}(0, 1)$ 을 생성한다.
 3. $U \leq \frac{p_Y}{c q_Y}$ 이면 $X = Y$ 를 반환하고 종료한다. 그렇지 않으면 **Step 3.1**로 돌아간다.

2. Continuous random variable

Example 1: Beta Distribution

Target Distribution: $f(x) = 20x(1-x)^3, \quad 0 < x < 1$ (Beta(2,4))

- 베타분포: $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$
- 평균: $\frac{\alpha}{\alpha+\beta}$

Instrument Distribution: $g(x) = 1, \quad 0 < x < 1$ (Uniform $U(0, 1)$)

Note that $g(x)$ has the same support as $f(x)$.

Calculating constant c :

$$\frac{f(x)}{g(x)} = 20x(1-x)^3$$

To maximize this ratio, we take the derivative:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = 20[(1-x)^3 - 3x(1-x)^2] = 0$$

Solving for x :

$$(1-x) - 3x = 0 \Rightarrow 1 - 4x = 0 \Rightarrow x = \frac{1}{4}$$

Substitute $x = 1/4$ back into the ratio:

$$c = \max \frac{f(x)}{g(x)} = 20 \left(\frac{1}{4} \right) \left(\frac{3}{4} \right)^3 = \frac{135}{64} \approx 2.109$$

The acceptance condition is:

$$\frac{f(x)}{cg(x)} = \frac{256}{27} x(1-x)^3$$

1. Generate $u_1 \sim U(0, 1), u_2 \sim U(0, 1)$.
2. If $u_2 \leq \frac{256}{27} u_1(1-u_1)^3$, set $X = u_1$.
3. Otherwise, return to **Step 1**.

Numerical Example:

If $u_1 = 0.37, u_2 = 0.68$:

$$0.68 < \frac{256}{27} (0.37)(1-0.37)^3 \approx 0.877202$$

Condition is met. **Stop.** $X = 0.37$.

Example 2: Gamma Distribution

Target Distribution: Gamma(3/2, 1)

$$f(x) = Kx^{1/2}e^{-x}, \quad x > 0, \quad \text{where } K = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}$$

$$\text{So, } f(x) = \frac{2}{\sqrt{\pi}} \sqrt{x} e^{-x}.$$

- 감마분포: $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$
- 감마분포의 평균: $\frac{\alpha}{\beta} = \frac{3/2}{1} = 3/2$
- 감마함수: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha-1)!$
- 부분 적분: $\int_0^\infty f(x)g'(x)dx = [f(x)g(x)]_0^\infty - \int_0^\infty f'(x)g(x)dx$

Instrument Distribution: Exponential with mean 3/2

$$g(x) = \frac{2}{3} e^{-\frac{2}{3}x}, \quad x > 0$$

Calculating constant c :

Ratio:

$$\frac{f(x)}{g(x)} = \frac{\frac{2}{\sqrt{\pi}} x^{1/2} e^{-x}}{\frac{2}{3} e^{-2x/3}} = \frac{3}{\sqrt{\pi}} x^{1/2} e^{-x/3}$$

To find max, differentiate and set to 0:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = 0 \Rightarrow \frac{1}{2} x^{-1/2} e^{-x/3} - \frac{1}{3} x^{1/2} e^{-x/3} = 0$$

$$\frac{1}{2} = \frac{1}{3} x \Rightarrow x = \frac{3}{2}$$

Substitute $x = 3/2$:

$$c = \max \frac{f(x)}{g(x)} = \frac{3}{\sqrt{\pi}} \left(\frac{3}{2} \right)^{1/2} e^{-1/2} = \frac{3\sqrt{3}}{\sqrt{2\pi e}} \approx 1.257$$

The acceptance condition simplifies to:

$$\frac{f(x)}{cg(x)} = \left(\frac{2e}{3} \right)^{1/2} x^{1/2} e^{-x/3}$$

1. Generate $u_1 \sim U(0, 1), u_2 \sim U(0, 1)$. Set $Y = -\frac{3}{2} \ln(u_1)$
2. If $u_2 \leq \left(\frac{2eY}{3} \right)^{1/2} e^{-Y/3}$, set $X = Y$.
3. Otherwise, return to **Step 1**.

- Composition Method

Suppose that we have efficient method to generate two discrete random variables with distribution as $\{p_j^1, j \geq 0\}$ and $\{p_j^2, j \geq 0\}$. And we want to simulate random variable X having distribution

$$Pr[X = j] = \alpha p_j^1 + (1 - \alpha) p_j^2, \quad j \geq 0$$

where $0 < \alpha < 1$.

To generate X , if random variables X_1 and X_2 have distributions

$$Pr[X_1 = j] = p_j^1$$

$$Pr[X_2 = j] = p_j^2$$

Then, the random variable X defined as

$$X = \begin{cases} X_1 & \text{with probability } \alpha \\ X_2 & \text{with probability } 1 - \alpha \end{cases}$$

has distribution

$$Pr[X = j] = \alpha p_j^1 + (1 - \alpha) p_j^2$$

Thus, X can be generated by the following steps.

1. Generate $u \sim U(0, 1)$
2. If $u < \alpha$, generate X_1 , else generate X_2 .

Example

$$p_j = Pr[X = j] = \begin{cases} 0.05 & \text{for } j = 1, 2, 3, 4, 5 \\ 0.15 & \text{for } j = 6, 7, 8, 9, 10 \end{cases}$$

Note that $p_j = 0.5p_j^1 + 0.5p_j^2$ where

$$p_j^1 = 0.1 \quad j = 1, 2, \dots, 10$$

$$p_j^2 = \begin{cases} 0 & j = 1, 2, 3, 4, 5 \\ 0.2 & j = 6, 7, 8, 9, 10 \end{cases}$$

1. Generate $u_1 \sim U(0, 1)$ and $u_2 \sim U(0, 1)$
2. if $u_1 < 0.5$, set $X = \lfloor 10u_2 \rfloor + 1$, otherwise $X = \lfloor 5u_2 \rfloor + 6$.

• Alias Method

n - 1개의 합으로

1. 제일 작은 i, 제일 큰 j 찾기
2. i row 1 col 제외 전부 0
3. 1 col j, i row 제외 전부 0

- 전부 0인 row가 나오면 양변에 $\frac{n-1}{n-2}$ 곱함.

• Chi-square Test

$$X^2 = \sum_{j=1}^k \frac{(f_j - e_j)^2}{e_j} \sim \chi_{k-1-l}^2 \text{ distribution}$$

- l: 추정한 모수의 개수
- 기대 빈도가 5 이상이어야 함

• Kolmogorov-Smirnov Test

Example

Observed samples are 0.1, 0.2, 0.3, 0.9 with $n = 4$

Empirical CDF values:

$$F(0.1) = 0.25, \quad F(0.2) = 0.5, \quad F(0.3) = 0.75, \quad F(0.9) = 1.0$$

Theoretical CDF (Target Distribution):

$$\tilde{F}(x) = x$$

Test Statistics Calculation:

$$D^+ = \max\{0.15, 0.3, 0.45, 0.1\} = 0.45$$

$$D^- = \max\{0.1, -0.05, -0.2, 0.15\} = 0.15$$

For $\alpha = 0.01, n = 4$ $D_{\alpha, n} = 0.733$. Therefore, do not reject H_0 .

Instead of $D_{\alpha, n}$, use the following table of $c_{1-\alpha}$.

$1 - \alpha$	0.85	0.90	0.95	0.975	0.990
$c_{1-\alpha}$	1.138	1.224	1.358	1.480	1.628

$$\text{If } (\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}})D > c_{1-\alpha}, \quad \text{then reject } H_0$$

Example If $\alpha = 0.05$,

$$(\sqrt{4} + 0.12 + \frac{0.11}{\sqrt{4}})0.45 = 0.97875 < c_{1-\alpha} \quad \text{Thus, do not reject } H_0$$

• Monte Carlo Approach

Let $g(x)$ be a function of variable x . Computing

$$\theta = \int_0^1 g(x)dx$$

If $U \sim U(0, 1)$, then

$$\theta = E[g(U)]$$

Monte Carlo Approach to compute θ : If u_1, \dots, u_k iid $U(0, 1)$, then

$$\sum_{i=1}^k \frac{g(u_i)}{k} \rightarrow E[g(U)] = \theta \text{ as } k \rightarrow \infty$$

• Example 1

For $\theta = \int_a^b g(x)dx$

Use substitution technique

$$y = \frac{x-a}{b-a}, \quad x = a + (b-a)y, \quad dx = (b-a)dy$$

$$\begin{aligned} \int_a^b g(x)dx &= \int_0^1 g(a + (b-a)y)(b-a)dy \\ &= \int_0^1 h(y)dy \end{aligned}$$

1. Generate u_1, u_2, \dots , iid $U(0, 1)$

$$\theta = \text{average of } g(a + (b-a)u_i)(b-a) = h(u_i)$$

For $\theta = \int_0^\infty g(x)dx$

$$y = \frac{1}{x+1} \quad x = \frac{1}{y} - 1 \quad dx = -\frac{1}{y^2}dy$$

$$\begin{aligned} \int_0^\infty g(x)dx &= \int_1^0 g\left(\frac{1}{y} - 1\right)\left(-\frac{1}{y^2}\right)dy \\ &= \int_0^1 g\left(\frac{1}{y} - 1\right)\frac{1}{y^2}dy \end{aligned}$$

• Example: Estimation of π

$X, Y \sim U(-1, 1)$ Define

$$I = \begin{cases} 1 & \text{if } X^2 + Y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[I] = Pr[X^2 + Y^2 \leq 1] = \frac{\pi}{4}$$

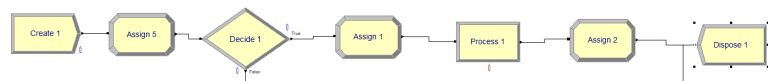


Figure 1: Sspolicy

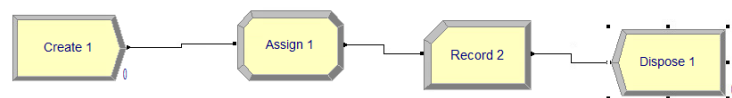


Figure 2: newsboy