

## Pseudo Random Number Generator

$$V_{i+1} = (aV_i + c) \bmod (m) \quad i = 0, 1, \dots,$$

where  $V_0$ : seed a: multiplier c: increment m: modulus

### Theorem : LCG has full period if and only if

1. If q is a prime that divides m, then q divides (a - 1).
2. The only positive integer that divides m and c is 1.
3. If 4 divides m, then 4 divides a - 1.

- exponential distribution inversion method:  $x = -\frac{1}{\lambda} \ln(1 - u)$

- generate poisson distribution:

1. Generate a random number  $U$

2.  $i = 0, p = e^{-\lambda}, F = p$ .

3. If  $U < F$ , set  $X = i$  and stop.

4.  $p = \frac{\lambda}{(i+1)} p, F = F + p, i = i + 1$ .

5. Goto Step 3.

- generate a random variable having density function

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if } 2 \leq x \leq 3 \\ \frac{2-x/3}{2} & \text{if } 3 \leq x \leq 6 \end{cases}$$

5.2

$$F(x) = \int_2^x \frac{x-2}{2} dx \quad (2 \leq x \leq 3), \quad \int_3^x \frac{2-\frac{x}{3}}{2} dx \quad (3 \leq x \leq 6)$$

$$F(x) = \begin{cases} \frac{(x-2)^2}{4} & (2 \leq x \leq 3) \Rightarrow 0 \sim 0.25 \\ 1 - \frac{(6-x)^2}{12} & (3 \leq x \leq 6) \Rightarrow 0.25 \sim 1 \end{cases}$$

1. Generate  $u_1 \sim U(0, 1)$

2. if  $u_1 \leq 0.25$ ,

$$u_1 = \frac{(x-2)^2}{4}, \quad x = 2 + 2\sqrt{u_1}$$

else

$$u_1 = \left(1 - \frac{(6-x)^2}{12}\right), \quad x = 6 - \sqrt{12(1-u_1)}$$

- Rejection Method

1. Discrete random variable

1. 도구 분포( $q_j$ ), 목표 분포( $p_j$ ) 설정. 도구분포는 목표 분포와 support set이 같아야 함.

- N(0, 1) support set:  $\{-\infty, x < \infty\}$
- 지수분포 support set:  $\{x > 0\}$
- 포아송 분포 support set:  $\{x = 0, 1, 2, \dots\}$
- 기하 분포 support set:  $\{x = 1, 2, \dots\}$
- 그 외 균등분포  $\lfloor a\mu \rfloor + 1$ 로부터 a까지의 정수 균등분포 생성 가능.

2. 모든  $p_j > 0$ 인 j에 대하여  $c = \max(\frac{p_j}{q_j})$ 인 상수 c 찾는다.

3. 다음을 반복한다.

1. 도구 분포  $q_j$ 로부터  $Y$ 를 생성한다.
2.  $U \sim Uniform(0, 1)$ 을 생성한다.

3.  $U \leq \frac{p_j}{cq_j}$  이면  $X = Y$ 를 반환하고 종료한다. 그렇지 않으면 Step 3.1로 돌아간다.

2. Continuous random variable

### Example 1: Beta Distribution

**Target Distribution:**  $f(x) = 20x(1-x)^3, \quad 0 < x < 1$  (Beta(2,4))

- 베타분포:  $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

- 평균:  $\frac{\alpha}{\alpha+\beta}$

**Instrument Distribution:**  $g(x) = 1, \quad 0 < x < 1$  (Uniform  $U(0, 1)$ )

Note that  $g(x)$  has the same support as  $f(x)$ .

**Calculating constant c:**

$$\frac{f(x)}{g(x)} = 20x(1-x)^3$$

To maximize this ratio, we take the derivative:

$$\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = 20[(1-x)^3 - 3x(1-x)^2] = 0$$

Solving for x:

$$(1-x) - 3x = 0 \implies 1 - 4x = 0 \implies x = \frac{1}{4}$$

Substitute  $x = 1/4$  back into the ratio:

$$c = \max \frac{f(x)}{g(x)} = 20 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^3 = \frac{135}{64} \approx 2.109$$

The acceptance condition is:

$$\frac{f(x)}{cg(x)} = \frac{256}{27} x(1-x)^3$$

1. Generate  $u_1 \sim U(0, 1), u_2 \sim U(0, 1)$ .

2. If  $u_2 \leq \frac{256}{27} u_1 (1-u_1)^3$ , set  $X = u_1$ .

3. Otherwise, return to Step 1.

### Numerical Example:

If  $u_1 = 0.37, u_2 = 0.68$ :

$$0.68 < \frac{256}{27} (0.37)(1-0.37)^3 \approx 0.877202$$

Condition is met. Stop.  $X = 0.37$ .

### Example 2: Gamma Distribution

**Target Distribution:** Gamma(3/2, 1)

$$f(x) = Kx^{1/2}e^{-x}, \quad x > 0, \quad \text{where } K = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}$$

So,  $f(x) = \frac{2}{\sqrt{\pi}} \sqrt{x} e^{-x}$ .

- 감마분포:  $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

- 감마분포의 평균:  $\frac{\alpha}{\beta} = \frac{3/2}{1} = 3/2$

- 감마함수:  $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha-1)!$

- 부분 적분:  $\int_0^\infty f(x)g'(x)dx = [f(x)g(x)]_0^\infty - \int_0^\infty f'(x)g(x)dx$

**Instrument Distribution:** Exponential with mean 3/2

$$g(x) = \frac{2}{3} e^{-\frac{2}{3}x}, \quad x > 0$$

**Calculating constant c:**

Ratio:

$$\frac{f(x)}{g(x)} = \frac{\frac{2}{\sqrt{\pi}} x^{1/2} e^{-x}}{\frac{2}{3} e^{-2x/3}} = \frac{3}{\sqrt{\pi}} x^{1/2} e^{-x/3}$$

To find max, differentiate and set to 0:

$$\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = 0 \implies \frac{1}{2} x^{-1/2} e^{-x/3} - \frac{1}{3} x^{1/2} e^{-x/3} = 0$$

$$\frac{1}{2} = \frac{1}{3} x \implies x = \frac{3}{2}$$

Substitute  $x = 3/2$ :

$$c = \max \frac{f(x)}{g(x)} = \frac{3}{\sqrt{\pi}} \left(\frac{3}{2}\right)^{1/2} e^{-1/2} = \frac{3\sqrt{3}}{\sqrt{2\pi e}} \approx 1.257$$

The acceptance condition simplifies to:

$$\frac{f(x)}{cg(x)} = \left(\frac{2e}{3}\right)^{1/2} x^{1/2} e^{-x/3}$$

1. Generate  $u_1 \sim U(0, 1), u_2 \sim U(0, 1)$ . Set  $Y = -\frac{3}{2} \ln(u_1)$

2. If  $u_2 \leq \left(\frac{2eY}{3}\right)^{1/2} e^{-Y/3}$ , set  $X = Y$ .

3. Otherwise, return to Step 1.

- Composition Method

Suppose that we have efficient method to generate two discrete random variables with distribution as  $\{p_j^1, j \geq 0\}$  and  $\{p_j^2, j \geq 0\}$ . And we want to simulate random variable X having distribution

$$Pr[X = j] = \alpha p_j^1 + (1-\alpha)p_j^2, \quad j \geq 0$$

where  $0 < \alpha < 1$ .

To generate X, if random variables  $X_1$  and  $X_2$  have distributions

$$Pr[X_1 = j] = p_j^1$$

$$Pr[X_2 = j] = p_j^2$$

Then, the random variable X defined as

$$X = \begin{cases} X_1 & \text{with probability } \alpha \\ X_2 & \text{with probability } 1 - \alpha \end{cases}$$

has distribution

$$Pr[X = j] = \alpha p_j^1 + (1-\alpha)p_j^2$$

Thus, X can be generated by the following steps.

1. Generate  $u \sim U(0, 1)$
2. If  $u < \alpha$ , generate  $X_1$ , else generate  $X_2$ .

## Example

$$p_j = Pr[X = j] = \begin{cases} 0.05 & \text{for } j = 1, 2, 3, 4, 5 \\ 0.15 & \text{for } j = 6, 7, 8, 9, 10 \end{cases}$$

Note that  $p_j = 0.5p_j^1 + 0.5p_j^2$  where

$$p_j^1 = 0.1 \quad j = 1, 2, \dots, 10$$

$$p_j^2 = \begin{cases} 0 & j = 1, 2, 3, 4, 5 \\ 0.2 & j = 6, 7, 8, 9, 10 \end{cases}$$

1. Generate  $u_1 \sim U(0, 1)$  and  $u_2 \sim U(0, 1)$
2. if  $u_1 < 0.5$ , set  $X = \lfloor 10u_2 \rfloor + 1$ , otherwise  $X = \lfloor 5u_2 \rfloor + 6$ .

- Alias Method

n - 1개의 합으로

1. 제일 작은 i, 제일 큰 j 찾기
2. i row 1 col 제외 전부 0
3. 1 col j, i row 제외 전부 0
  - 전부 0인 row가 나오면 양변에  $\frac{n-1}{n-2}$  곱함.

- Chi-square Test

$$X^2 = \sum_{j=1}^k \frac{(f_j - e_j)^2}{e_j} \sim \chi^2_{k-1-l} \text{ distribution}$$

- l: 추정한 모수의 개수
- 기대 빈도가 5 이상이여야 함
- Kolmogorov-Smirnov Test

## Example

Observed samples are 0.1, 0.2, 0.3, 0.9 with  $n = 4$

**Empirical CDF values:**

$$F(0.1) = 0.25, \quad F(0.2) = 0.5, \quad F(0.3) = 0.75, \quad F(0.9) = 1.0$$

**Theoretical CDF (Target Distribution):**

$$\tilde{F}(x) = x$$

**Test Statistics Calculation:**

$$D^+ = \max\{0.15, 0.3, 0.45, 0.1\} = 0.45$$

$$D^- = \max\{0.1, -0.05, -0.2, 0.15\} = 0.15$$

For  $\alpha = 0.01, n = 4$   $D_{\alpha,n} = 0.733$ . Therefore, do not reject  $H_0$ .

Instead of  $D_{\alpha,n}$ , use the following table of  $c_{1-\alpha}$ .

$1 - \alpha$	0.85	0.90	0.95	0.975	0.990
$c_{1-\alpha}$	1.138	1.224	1.358	1.480	1.628

$$\text{If } (\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}})D > c_{1-\alpha}, \quad \text{then reject } H_0$$

**Example** If  $\alpha = 0.05$ ,

$$(\sqrt{4} + 0.12 + \frac{0.11}{\sqrt{4}})0.45 = 0.97875 < c_{1-\alpha} \quad \text{Thus, do not reject } H_0$$

- Monte Carlo Approach

Let  $g(x)$  be a function of variable  $x$ . Computing

$$\theta = \int_0^1 g(x)dx$$

If  $U \sim U(0, 1)$ , then

$$\theta = E[g(U)]$$

**Monte Carlo Approach to compute  $\theta$ :** If  $u_1, \dots, u_k \sim U(0, 1)$ , then

$$\sum_{i=1}^k \frac{g(u_i)}{k} \rightarrow E[g(U)] = \theta \text{ as } k \rightarrow \infty$$

- Example 1

$$\text{For } \theta = \int_a^b g(x)dx$$

**Use substitution technique**

$$\begin{aligned} y &= \frac{x-a}{b-a}, \quad x = a + (b-a)y, \quad dx = (b-a)dy \\ \int_a^b g(x)dx &= \int_0^1 g(a + (b-a)y)(b-a)dy \\ &= \int_0^1 h(y)dy \end{aligned}$$

1. Generate  $u_1, u_2, \dots, \sim U(0, 1)$

$$\theta = \text{average of } g(a + (b-a)u_i)(b-a) = h(u_i)$$

$$\text{For } \theta = \int_0^\infty g(x)dx$$

$$\begin{aligned} y &= \frac{1}{x+1} \quad x = \frac{1}{y} - 1 \quad dx = -\frac{1}{y^2}dy \\ \int_0^\infty g(x)dx &= \int_1^0 g\left(\frac{1}{y} - 1\right)\left(-\frac{1}{y^2}\right)dy \\ &= \int_0^1 g\left(\frac{1}{y} - 1\right)\frac{1}{y^2}dy \end{aligned}$$

- Example: Estimation of  $\pi$

$$X, Y \sim U(-1, 1) \text{ Define}$$

$$I = \begin{cases} 1 & \text{if } X^2 + Y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[I] = Pr[X^2 + Y^2 \leq 1] = \frac{\pi}{4}$$

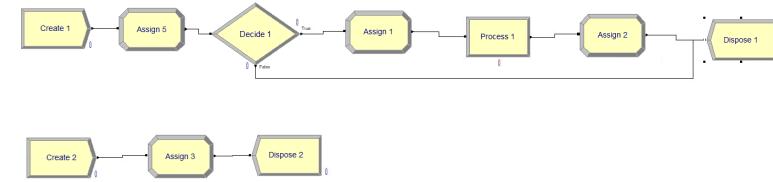


Figure 1: Sspolicy

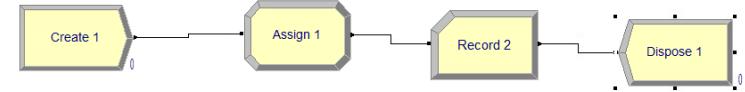


Figure 2: newsboy