- Pseudo Random Number Generator

$$V_{i+1} = (aV_i + c) \quad \text{mod } (m) \quad i = 0, 1, \dots,$$

where V_0: seed a: multiplier c: increment m: modulus

Theorem : LCG has full period if and only if

1. If q is a prime that divides m, then q divides (a - 1).
2. The only positive integer that divides m and c is 1.
3. If 4 divides m, then 4 divides a - 1.

- exponential distribution inversion method: x = - $\frac{1}{\lambda} ln(1-u)$
- generate poissong distribution:

1. Generate a random number U
2. $i = 0, p = e^{-\lambda}, F = p.$
3. If $U < F$, set $X = i$ and **stop**.
4. $p = \frac{\lambda}{(i+1)} p, F = F + p, i = i + 1.$
5. Goto **Step 3**.

- Rejection Method

1. Discrete random variable
    1. 도구 분포(q_j), 목표 분포(p_j) 설정. 도구분포는 목표 분포와 support set이 같아야 함.
        - N(0, 1) support set: {-∞, x < ∞}
        - 지수분포 support set: {x > 0}
        - 포아송 분포 support set: {x = 0, 1, 2, …}
        - 기하 분포 support set: {x = 1, 2, …}
        - 그 외 균등분포 $\lfloor a\mu \rfloor + 1$로 1부터 a까지의 정수 균등분포 생성 가능.
    2. 모든 p_j > 0인 j에 대하여 $c = max(\frac{p_j}{q_j})$인 상수 c 찾는다.
    3. 다음을 반복한다.
        1. 도구 분포 q_j로부터 Y를 생성한다.
        2. $U \, Uniform(0, 1)$을 생성한다.
        3. $U \le \frac{p_Y}{cq_Y}$ 이면 $X = Y$를 반환하고 종료한다. 그렇지 않으면 **Step 3.1**로 돌아간다.
2. Continuous random variable

Example 1: Beta Distribution

Target Distribution: $f(x) = 20x(1-x)^3, \quad 0 < x < 1$ (Beta(2,4))

- 베타분포: $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
- 평균: $\frac{\alpha}{\alpha+\beta}$

Instrument Distribution: $g(x) = 1, \quad 0 < x < 1$ (Uniform $U(0, 1)$)

Note that $g(x)$ has the same support as $f(x)$.

Calculating constant c:

$$\frac{f(x)}{g(x)} = 20x(1-x)^3$$

To maximize this ratio, we take the derivative:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 20[(1-x)^3 - 3x(1-x)^2] = 0$$

Solving for x:

$$(1-x) - 3x = 0 \implies 1 - 4x = 0 \implies x = \frac{1}{4}$$

Substitute $x = 1/4$ back into the ratio:

$$c = \text{max } \frac{f(x)}{g(x)} = 20\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3 = \frac{135}{64} \approx 2.109$$

The acceptance condition is:

$$\frac{f(x)}{cg(x)} = \frac{256}{27} x(1-x)^3$$

1. Generate $u_1 \sim U(0, 1), u_2 \sim U(0, 1).$
2. If $u_2 \le \frac{256}{27} u_1(1-u_1)^3$, set $X = u_1.$
3. Otherwise, return to **Step 1**.

Numerical Example:

If $u_1 = 0.37, u_2 = 0.68$:

$$0.68 < \frac{256}{27}(0.37)(1-0.37)^3 \approx 0.877202$$

Condition is met. **Stop.** $X = 0.37$**.**

Example 2: Gamma Distribution

Target Distribution: Gamma(3/2, 1)

$$f(x) = Kx^{1/2}e^{-x}, \quad x > 0, \quad \text{where } K = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}$$

So, $f(x) = \frac{2}{\sqrt{\pi}}\sqrt{x}e^{-x}$.

- 감마분포: $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$}
- 감마분포의 평균: $\frac{\alpha}{\beta} = \frac{3/2}{1} = 3/2$}

Instrument Distribution: Exponential with mean 3/2

$$g(x) = \frac{2}{3} e^{-\frac{2}{3}x}, \quad x > 0$$

Calculating constant c:

Ratio:

$$\frac{f(x)}{g(x)} = \frac{\frac{2}{\sqrt{\pi}} x^{1/2} e^{-x}}{\frac{2}{3} e^{-2x/3}} = \frac{3}{\sqrt{\pi}} x^{1/2} e^{-x/3}$$

To find max, differentiate and set to 0:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 0 \implies \frac{1}{2}x^{-1/2}e^{-x/3} - \frac{1}{3}x^{1/2}e^{-x/3} = 0$$

$$\frac{1}{2} = \frac{1}{3}x \implies x = \frac{3}{2}$$

Substitute $x = 3/2$:

$$c = \text{max } \frac{f(x)}{g(x)} = \frac{3}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{1/2} e^{-1/2} = \frac{3\sqrt{3}}{\sqrt{2\pi e}} \approx 1.257$$

The acceptance condition simplifies to:

$$\frac{f(x)}{cg(x)} = \left(\frac{2e}{3}\right)^{1/2} x^{1/2} e^{-x/3}$$

1. Generate $u_1 \sim U(0, 1), u_2 \sim U(0, 1)$. Set $Y = -\frac{3}{2}\ln(u_1)$
2. If $u_2 \le \left(\frac{2eY}{3}\right)^{1/2} e^{-Y/3}$, set $X = Y$.
3. Otherwise, return to **Step 1**.

- Composition Method

Suppose that we have efficient method to generate two discrete random variables with distribution as $\{p_j^1, j \ge 0\}$ and $\{p_j^2, j \ge 0\}$. And we want to simulate random variable X having distribution

$$Pr[X = j] = \alpha p_j^1 + (1-\alpha)p_j^2, \quad j \ge 0$$

where $0 < \alpha < 1$.

To generate X, if random variables X_1 and X_2 have distributions

$$Pr[X_1 = j] = p_j^1$$

$$Pr[X_2 = j] = p_j^2$$

Then, the random variable X defined as

$$X = \begin{cases} X_1 & \text{with probability } \alpha \\ X_2 & \text{with probability } 1-\alpha \end{cases}$$

has distribution

$$Pr[X = j] = \alpha p_j^1 + (1-\alpha)p_j^2$$

Thus, X can be generated by the following steps.

1. Generate $u \sim U(0, 1)$
2. If $u < \alpha$, generate X_1, else generate X_2.

Example

$$p_j = Pr[X = j] = \begin{cases} 0.05 & \text{for } j = 1, 2, 3, 4, 5 \\ 0.15 & \text{for } j = 6, 7, 8, 9, 10 \end{cases}$$

Note that $p_j = 0.5p_j^1 + 0.5p_j^2$ where

$$p_j^1 = 0.1 \quad j = 1, 2, \dots, 10$$

$$p_j^2 = \begin{cases} 0 & j = 1, 2, 3, 4, 5 \\ 0.2 & j = 6, 7, 8, 9, 10 \end{cases}$$

1. Generate $u_1 \sim U(0, 1)$ and $u_2 \sim U(0, 1)$
2. if $u_1 < 0.5$, set $X = \lfloor 10u_2 \rfloor + 1$, otherwise $X = \lfloor 5u_2 \rfloor + 6$.

- **Alias Method**

n - 1개의 합으로

1. 제일 작은 i, 제일 큰 j 찾기
2. i row 1 col 제외 전부 0
3. 1 col j, i row 제외 전부 0
    - 전부 0인 row가 나오면 양변에 $\frac{n-1}{n-2}$ 곱함.

- **Chi-square Test**

$$X^2 = \sum_{j=1}^{k} \frac{(f_j - e_j)^2}{e_j} \sim \chi^2_{k-1-l} \text{ distribution}$$

- l: 추정한 모수의 개수
- 기대 빈도가 5 이상이여야 함

- **Kolmogorov-Smirnov Test**

Example

Observed samples are $0.1, 0.2, 0.3, 0.9$ with $n = 4$

Empirical CDF values:

$$F(0.1) = 0.25, \quad F(0.2) = 0.5, \quad F(0.3) = 0.75, \quad F(0.9) = 1.0$$

Theoretical CDF (Target Distribution):

$$\tilde{F}(x) = x$$

Test Statistics Calculation:

$$D^+ = \max\{0.15, 0.3, 0.45, 0.1\} = 0.45$$
$$D^- = \max\{0.1, -0.05, -0.2, 0.15\} = 0.15$$

For $\alpha = 0.01, n = 4$ $D_{\alpha,n} = 0.733$. Therefore, do not reject H_0.

Instead of $D_{\alpha,n}$, use the following table of $c_{1-\alpha}$.

1 - α	0.85	0.90	0.95	0.975	0.990
$c_{1-\alpha}$	1.138	1.224	1.358	1.480	1.628

$$\text{If } (\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}})D > c_{1-\alpha}, \quad \text{then reject } H_0$$

Example If $\alpha = 0.05$,

$$(\sqrt{4} + 0.12 + \frac{0.11}{\sqrt{4}})0.45 = 0.97875 < c_{1-\alpha} \quad \text{Thus, do not reject } H_0$$

- **Monte Carlo Approach**

Let $g(x)$ be a function of variable x. Computing

$$\theta = \int_0^1 g(x)dx$$

If $U \sim U(0,1)$, then

$$\theta = E[g(U)]$$

Monte Carlo Approach to compute θ: If $u_1, \ldots, u_k$ iid $U(0,1)$, then

$$\sum_{i=1}^{k} \frac{g(u_i)}{k} \to E[g(U)] = \theta \text{ as } k \to \infty$$

- Example 1

For $\theta = \int_a^b g(x)dx$

Use substitution technique

$$y = \frac{x-a}{b-a}, \quad x = a + (b-a)y, \quad dx = (b-a)dy$$
$$\int_a^b g(x)dx = \int_0^1 g(a + (b-a)y)(b-a)dy$$
$$= \int_0^1 h(y)dy$$

1. Generate $u_1, u_2, \ldots$, iid $U(0,1)$

$$\theta = \text{average of } g(a + (b-a)u_i)(b-a) = h(u_i)$$

For $\theta = \int_0^\infty g(x)dx$

$$y = \frac{1}{x+1} \quad x = \frac{1}{y} - 1 \quad dx = -\frac{1}{y^2}dy$$
$$\int_0^\infty g(x)dx = \int_1^0 g(\frac{1}{y} - 1)(-\frac{1}{y^2})dy$$
$$= \int_0^1 g(\frac{1}{y} - 1)\frac{1}{y^2}dy$$

- **Example: Estimation of π**

$X, Y \sim U(-1,1)$ Define

$$I = \begin{cases} 1 & \text{if } X^2 + Y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[I] = Pr[X^2 + Y^2 \leq 1] = \frac{\pi}{4}$$

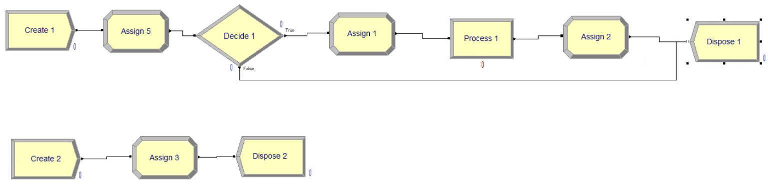

Figure 1: Sspolicy

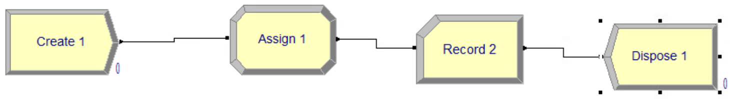

Figure 2: newsboy