

Discrete Mathematics #3

20192208 김형훈

8.3

Problem 29

$$f(n) = af\left(\frac{n}{b}\right) + cn^d$$

$$f\left(\frac{n}{b}\right) = af\left(\frac{n}{b^2}\right) + c\left(\frac{n}{b}\right)^d$$

$$f(n) = a\left(af\left(\frac{n}{b^2}\right) + c\left(\frac{n}{b}\right)^d\right) + cn^d$$

$$= a^2f\left(\frac{n}{b^2}\right) + ac\left(\frac{n}{b}\right)^d + cn^d$$

$$f\left(\frac{n}{b^2}\right) = af\left(\frac{n}{b^3}\right) + c\left(\frac{n}{b^2}\right)^d$$

$$f(n) = a^3f\left(\frac{n}{b^3}\right) + a^2c\left(\frac{n}{b^2}\right)^d + ac\left(\frac{n}{b}\right)^d + cn^d$$

$$\therefore f(n) = a^kf(1) + c\sum_{i=0}^{k-1}a^i\left(\frac{n}{b^i}\right)^d$$

$$a = b^d, a^i\left(\frac{n}{b^i}\right)^d = n^d$$

$$f(n) = a^kf(1) + cn^d\sum_{i=0}^{k-1}1$$

$$= a^kf(1) + cn^dk$$

$$n = b^k, k = \log_b n$$

$$a^k = a^{\log_b n} = n^{\log_b a}, \log_b a = d$$

$$f(n) = n^df(1) + cn^d\log_b n$$

Problem 30

$$f(n) = n^d f(1) + cn^d \log_b n$$

$$n^d f(1) = O(n^d)$$

$$cn^d \log_b n = cn^{d \frac{\log n}{\log b}} = \frac{c}{\log b} n^d \log n = O(n^d \log n)$$

n^d 보다 $n^d \log n$ 이 더 빠르게 증가하므로

$$\therefore f(n) = O(n^d \log n)$$

Problem 31

Problem 29와 마찬가지로

$$f(n) = a^k f(1) + c \sum_{i=0}^{k-1} a^i \left(\frac{n}{b^i}\right)^d$$

$$f(n) = a^k f(1) + cn^d \sum_{i=0}^{k-1} (ab^{-d})^i$$

$$= a^k f(1) + cn^d \frac{1-(ab^{-d})^k}{1-ab^{-d}}$$

$$n = b^k, k = \log_b n$$

$$f(n) = a^k f(1) + cn^d \frac{1-(ab^{-d})^{\log_b n}}{1-ab^{-d}}$$

$$= n^{\log_b a} f(1) + \frac{cn^d}{1-ab^{-d}} - \frac{cn^d (ab^{-d})^{\log_b n}}{1-ab^{-d}}$$

$$= n^{\log_b a} f(1) + n^d \frac{cb^d}{b^d - a} + \frac{cb^d n^d (ab^{-d})^{\log_b n}}{a - b^d}$$

$$= n^{\log_b a} f(1) + n^d \frac{cb^d}{b^d - a} + \frac{cb^d n^d n^{\log_b a} b^{-d}}{a - b^d}$$

$$= n^{\log_b a} f(1) + n^d \frac{cb^d}{b^d - a} + \frac{cb^d n^{d+\log_b a} b^{-d}}{a - b^d}$$

$$= n^{\log_b a} f(1) + n^d \frac{cb^d}{b^d - a} + \frac{cb^d n^{\log_b a}}{a - b^d}$$

$$= n^{\log_b a} \left(f(1) + \frac{cb^d}{a - b^d} \right) + n^d \left(\frac{cb^d}{b^d - a} \right)$$

$$C_1 = \frac{b^d c}{b^d - a}, C_2 = f(1) + \frac{b^d c}{a - b^d} \text{라고 할 때,}$$

$$f(n) = C_1 n^d + C_2 n^{\log_b a}$$

Problem 32

$$f(n) = C_1 n^d + C_2 n^{\log_b a}$$

$$a < b^d, \log_b a < d$$

n^d 이 $n^{\log_b a}$ 보다 빠르게 증가

$$\therefore f(n) = O(n^d)$$

Problem 33

Problem 32의 과정에서 $a > b^d$ 인 경우로 생각하면

$n^{\log_b a}$ 이 n^d 보다 빠르게 증가한다는 사실을 알 수 있다

$$\therefore f(n) = O(n^{\log_b a})$$