Algebraic Topology

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Preface

Over the 2016–2017 academic year, I ran the standard algebraic topology sequence at MIT. The first semester deals with singular homology and cohomology, and Poicaré duality; the second builds up basic homotopy theory, spectral sequences, and characteristic classes.

I was lucky enough to have in the audience a student, Sanath Devalapurkar, who spontaneously decided to liveTEX the entire course. This resulted in a remarkably accurate record of what happened in the classroom – right down to random alarms ringing and embarassing mistakes on the blackboard. Sanath's TEX forms the basis of these notes.

My goal was to give a standard classical approach to these subjects. In the first semester, I tried to give an honest account of the relative cup products needed in the proof of Poincaré duality.

Contents

Part I

18.905: an introduction to algebraic topology

Chapter 1

Homology and CW-complexes

1 Introduction: singular simplices and chains

This is a course on algebraic topology. We'll discuss the following topics.

1. Singular homology
2. CW-complexes
3. Basics of category theory
4. Homological algebra
5. The Künneth theorem
6. UCT, cohomology
7. Cup and cap products, and
8. Poincaré duality.

The objects of study are of course topological spaces, and the machinery we develop in this course is designed to be applicable to a general space. But we are really interested in geometrically natural spaces. Here are some examples.

- The most basic example is *n-dimensional Euclidean space*, $\mathbf{R}^n$.
- The *n-sphere* $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$, topologized as a subspace of $\mathbf{R}^{n+1}$.
- Identifying antipodal points in S^n gives *real projective space* $\mathbf{RP}^n = S^n/(x \sim -x)$, i.e. the space of lines through the origin in $\mathbf{R}^{n+1}$.
- Call an ordered collection of k orthonormal vectors an *orthonormal k-frame*. The space of orthonormal k-frames in $\mathbf{R}^n$ forms the *Stiefel manifold* $V_k(\mathbf{R}^n)$, which is topologized as a subspace of $(S^{n-1})^k$.

- The *Grassmannian* $\mathrm{Gr}_k(\mathbf{R}^n)$ is the space of k-dimensional linear subspaces of $\mathbf{R}^n$. Forming the span gives us a surjection $V_k(\mathbf{R}^n) \to \mathrm{Gr}_k(\mathbf{R}^n)$, and the Grassmannian is given the quotient topology. For example, $\mathrm{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$.

All these examples are *manifolds*; that is, they are Hausdorff spaces locally homeomorphic to Euclidean space. Aside from $\mathbf{R}^n$ itself, the preceding examples are also compact. Such spaces exhibit a hidden symmetry, which is the culmination of 18.905: Poincaré duality.

As the name suggests, the central aim of algebraic topology is the usage of algebraic tools to study topological spaces. A common technique is to probe topological spaces via maps to them from simpler spaces. In different ways, this approach gives rise to singular homology and homotopy groups. We now detail the former; the latter takes stage in 18.906.

Definition 1.1. For $n \geq 0$, the *standard n-simplex* Δ^n is the convex hull of the standard basis $\{e_0, \ldots, e_n\}$ in $\mathbf{R}^{n+1}$:

$$\Delta^n = \left\{ \sum t_i e_i : \sum t_i = 1, t_i \geq 0 \right\} \subseteq \mathbf{R}^{n+1}.$$

The t_i are called *barycentric coordinates.*

The standard simplices are related by face inclusions $d^i \colon \Delta^{n-1} \to \Delta^n$ for $0 \leq i \leq n$, where d^i is the affine map that sends verticies to vertices, in order, and omits the vertex e_i.

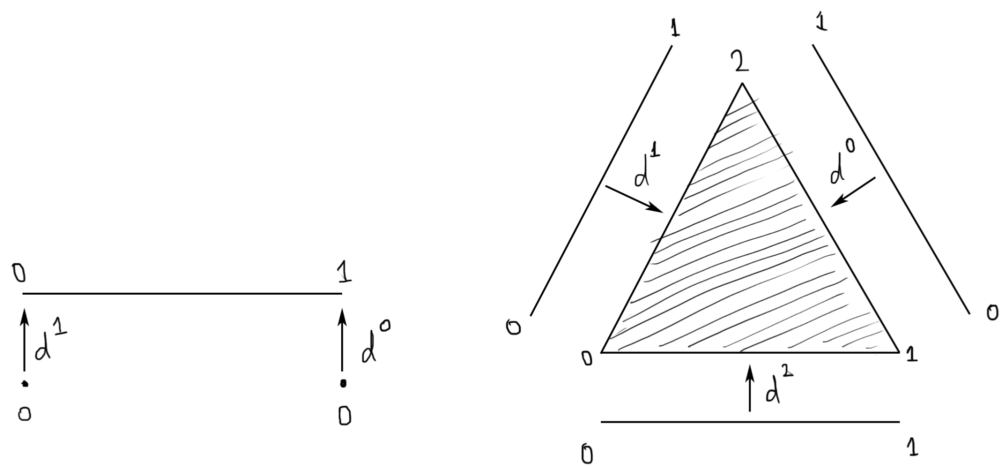


Figure 1.1: The face inclusions for $n = 1$ (left) and $n = 2$ (right).

Definition 1.2. Let X be any topological space. A *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$. Denote by $\mathrm{Sin}_n(X)$ the set of all n-simplices in X.

This seems like a rather bold construction to make, as $\mathrm{Sin}_n(X)$ is huge. But be patient!

For $0 \le i \le n$, precomposition by the face inclusion d^i produces a map $d_i\colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ sending $\sigma \mapsto \sigma \circ d^i$. This is the "$i$th face" of σ. This allows us to make sense of the "boundary" of a simplex, and we are particularly interested in simplices for which that boundary vanishes.

For example, if σ is a 1-simplex that forms a closed loop, then $d_1\sigma = d_0\sigma$. To express the condition that the boundary vanishes, we would like to write $d_0\sigma - d_1\sigma = 0$ – but this difference is no longer a simplex. To accommodate such formal sums, we will enlarge $\mathrm{Sin}_n(X)$ further by forming the free abelian group it generates.

Definition 1.3. The abelian group $S_n(X)$ of *singular n-chains* in X is the free abelian group generated by n-simplices

$$S_n(X) = \mathbf{Z}\mathrm{Sin}_n(X).$$

So an n-chain is a finite linear combination of simplices,

$$\sum_{i=0}^{k} a_i\sigma_i\,, \quad a_i \in \mathbf{Z}\,, \quad \sigma_i \in \mathrm{Sin}_n(X)\,.$$

If $n < 0$, $\mathrm{Sin}_n(X)$ is declared to be empty, so $S_n(X) = 0$.

We can now define the *boundary operator*

$$d\colon \mathrm{Sin}_n(X) \to S_{n-1}(X),$$

by

$$d\sigma = \sum_{i=0}^{n} (-1)^i d_i\sigma.$$

This extends to a homomorphism $d\colon S_n(X) \to S_{n-1}(X)$ by additivity.

We use this homomorphism to obtain something more tractable than the entirety of $S_n(X)$. First we restrict our attention to chains with vanishing boundary.

Definition 1.4. An *n-cycle* in X is an n-chain c with $dc = 0$. Denote $Z_n(X) = \ker(d : S_n(X) \to S_{n-1}(X))$.

For example, if σ is a 1-simplex forming a closed loop, then $\sigma \in Z_1(X)$ since $d\sigma = d_0\sigma - d_1\sigma = 0$.

It turns out that there's a cheap way to produce a cycle:

Theorem 1.5. *Any boundary is a cycle; that is, $d^2 = 0$.*

We'll leave the verification of this important result as a homework problem. What we have found, then, is that the singular chains form a "chain complex," as in the following definition.

Definition 1.6. A *graded abelian group* is a sequence of abelian groups, indexed by the integers. A *chain complex* is a graded abelian group $\{A_n\}$ together with homomorphisms $d : A_n \to A_{n-1}$ with the property that $d^2 = 0$.

The group of n-dimensional *boundaries* is

$$B_n(X) = \mathrm{im}(d : S_{n+1}(X) \to S_n(X)),$$

and the theorem tells us that this is a subgroup of the group of cycles: the "cheap" ones. If we quotient by them, what's left is the "interesting ones," captured in the following definition.

Definition 1.7. The *nth singular homology group* of X is:

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker(d : S_n(X) \to S_{n-1}(X))}{\mathrm{im}(d : S_{n+1}(X) \to S_n(X))}.$$

We use the same language for any chain complex: it has cycles, boundaries, and homology groups. The homology forms a graded abelian group.

Both $Z_n(X)$ and $B_n(X)$ are free abelian groups because they are subgroups of the free abelian group $S_n(X)$, but the quotient $H_n(X)$ isn't necessarily free. While $Z_n(X)$ and $B_n(X)$ are uncountably generated, $H_n(X)$ is finitely generated for the spaces we are interested in. If T is the torus, for example, then we will see that $H_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$ and σ as described previously is one of the two generators. We will learn to compute the homology groups of a wide variety of spaces. The n-sphere has the following homology groups:

$$H_q(S^n) = \begin{cases} \mathbf{Z} & \text{if } \; q = n > 0 \\ \mathbf{Z} & \text{if } \; q = 0, n > 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } \; q = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

2 More about homology

In the last lecture we introduced the standard n-simplex $\Delta^n \subseteq \mathbf{R}^{n+1}$. Singular simplices in a space X are maps $\sigma\colon \Delta^n \to X$ and constitute the set $\mathrm{Sin}_n(X)$. For example, $\mathrm{Sin}_0(X)$ consists of points of X. We also described the face inclusions $d^i : \Delta^{n-1} \to \Delta^n$, and the induced "face maps"

$$d_i : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X),\, 0 \le i \le n,$$

given by precomposing with face inclusions: $d_i\sigma = \sigma \circ d^i$. For homework you established some quadratic relations satisfied by these maps. A collection of sets $K_n, n \ge 0$, together with maps $d_i : K_n \to K_{n-1}$ related to each other in this way, is a *semi-simplicial set*. So we have assigned to any space X a semi-simplicial set $S_*(X)$.

To the semi-simplicial set $\{\mathrm{Sin}_n(X), d_i\}$ we then applied the free abelian group functor, obtaining a semi-simplicial abelian group. Using the d_is, we constructed a boundary map d which makes $S_*(X)$ a *chain complex* – that is, $d^2 = 0$. We capture this process in a diagram:

$$\begin{tikzcd}
\{\text{spaces}\} \arrow[r, "H_*"] \arrow[d, "\mathrm{Sin}_*"] & \{\text{graded abelian groups}\} \\
\{\text{semi-simplicial sets}\} \arrow[d, "\mathbf{Z}(-)"] & \\
\{\text{semi-simplicial abelian groups}\} \arrow[r] & \{\text{chain complexes}\} \arrow[uu, "\text{take homology}"']
\end{tikzcd}$$

Example 2.1. Suppose we have $\sigma\colon\Delta^1 \to X$. Define $\phi\colon\Delta^1 \to \Delta^1$ which sends $(t, 1-t) \mapsto (1-t, t)$. Precomposing σ with ϕ gives another singular simplex $\overline{\sigma}$ which reverses the orientation of σ. It is *not* true that $\overline{\sigma} = -\sigma$ in $S_1(X)$.

However, we claim that $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$, meaning there is a 2-chain in X whose boundary is $\overline{\sigma} + \sigma$. If $d_0\sigma = d_1\sigma$ so that $\sigma \in Z_1(X)$, then $\overline{\sigma}$ and $-\sigma$ are homologous: $[\overline{\sigma}] = -[\sigma]$ in $H_1(X)$.

To construct an appropriate boundary, consider the projection map $\pi : \Delta^2 \to \Delta^1$ that is the affine extension of the map sending e_0 and e_2 to e_0 and e_1 to e_1.

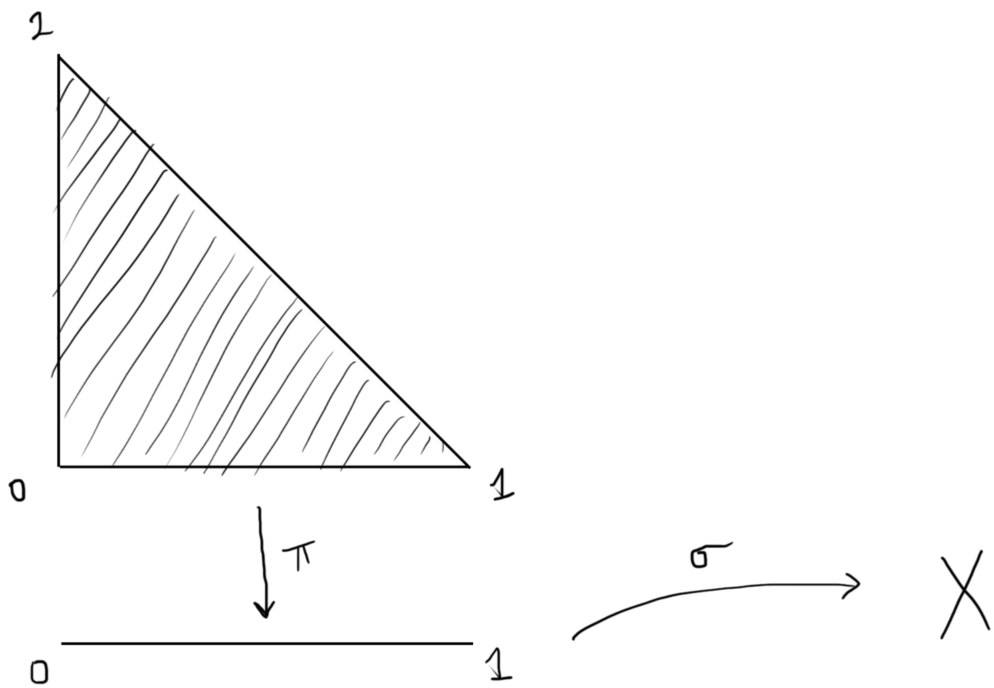


Figure 1.2: If σ is a 1-simplex in X, then precomposing by π gives a 2-simplex $\sigma \circ \pi$ in X.

We'll compute $d(\sigma \circ \pi)$. Some of the terms will be constant singular simplicies. Let's write $c_x^n : \Delta^n \to X$ for the constant map with value $x \in X$. Then

$$d(\sigma \circ \pi) = \sigma\pi d^0 - \sigma\pi d^1 + \sigma\pi d^2 = \overline{\sigma} - c^1_{\sigma(0)} + \sigma \,.$$

The constant simplex $c^1_{\sigma(0)}$ is an error term, and we wish to eliminate it. To achieve this we can use the constant 2-simplex $c^2_{\sigma(0)}$ at $\sigma(0)$; its boundary is

$$c^1_{\sigma(0)} - c^1_{\sigma(0)} + c^1_{\sigma(0)} = c^1_{\sigma(0)} \,.$$

So

$$\overline{\sigma} + \sigma = d(\sigma \circ \pi + c^2_{\sigma(0)}) \,,$$

and $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$ as claimed.

Some more language: two cycles that differ by a boundary dc are said to be *homologous*, and the chain c is a *homology* between them.

Let's compute the homology of the very simplest spaces, $\varnothing$ and $*$. For the first , $\mathrm{Sin}_n(\varnothing) = \varnothing$, so $S_*(\varnothing) = 0$. Hence $\cdots \to S_2 \to S_1 \to S_0$ is the zero chain complex. This means that $Z_*(\varnothing) = B_*(\varnothing) = 0$. The homology in all dimensions is therefore 0.

For $*$, we have $\mathrm{Sin}_n(*) = \{c^n_*\}$ for all $n \geq 0$. Consequently $S_n(*) = \mathbf{Z}$ for $n \geq 0$ and 0 for $n \leq 0$. For each i, $d_i c^n_* = c^{n-1}_*$, so the boundary maps $d\colon S_n(*) \to S_{n-1}(*)$ in the chain complex depend on the parity of n as follows:

$$d(c^n_*) = \sum_{i=0}^{n} (-1)^i c^{n-1}_* = \begin{cases} c^{n-1}_* & \text{for } n \text{ even, and} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This means that our chain complex is:

$$0 \leftarrow 0 \leftarrow \mathbf{Z} \leftarrow 1\mathbf{Z} \leftarrow 1\mathbf{Z} \leftarrow 1 \cdots \,.$$

The boundaries coincide with the cycles except in dimension zero, where $B_0(*) = 0$ while $Z_0(*) = \mathbf{Z}$. Therefore $H_0(*) = \mathbf{Z}$ and $H_i(*) = 0$ for $i \neq 0$.

We've defined homology groups for each space, but haven't considered what happens to maps between spaces. A continuous map $f\colon X \to Y$ induces a map $f_*\colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$ by composition:

$$f_* : \sigma \mapsto f \circ \sigma \,.$$

For f_* to be a map of semi-simplicial sets, it needs to commute with face maps. Explicitly, we need $f_* \circ d_i = d_i \circ f_*$. A diagram is said to be *commutative* if all composites with the same source and target are equal, so this is equivalent to commutativity of the below.

$$\begin{tikzcd} \mathrm{Sin}_n(X) \arrow[r, "f_*"] \arrow[d, "d_i"] & \mathrm{Sin}_n(Y) \arrow[d, "d_i"] \\ \mathrm{Sin}_{n-1}(X) \arrow[r, "f_*"] & \mathrm{Sin}_{n-1}(Y) \end{tikzcd}$$

We see that $d_i f_* \sigma = (f_*\sigma) \circ d^i = f \circ \sigma \circ d^i$, and $f_*(d_i\sigma) = f_*(\sigma \circ d^i) = f \circ \sigma \circ d^i$ as desired. The diagram remains commutative when we pass to the free abelian groups of chains.

If C_* and D_* are chain complexes, a *chain map* $f: C_* \to D_*$ is a collection of maps $f_n: C_n \to D_n$ such that the following diagram commutes for every n:

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "d_C"] & D_n \arrow[d, "d_D"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1} \end{tikzcd}$$

For example, if $f: X \to Y$ is a continuous map, then $f_*: S_*(X) \to S_*(Y)$ is a chain map as discussed above.

A chain map induces a map in homology $f_* : H_n(C) \to H_n(D)$. The method of proof is a so-called "diagram chase" and it will be the first of many. We check that we get a map $Z_n(C) \to Z_n(D)$. Let $c \in Z_n(C)$, so that $d_C c = 0$. Then $d_D f_n(c) = f_{n-1} d_C c = f_{n-1}(0) = 0$, because f is a chain map. This means that $f_n(c)$ is also an n-cycle, i.e., f gives a map $Z_n(C) \to Z_n(D)$.

Similarly, we also get a map $B_n(C) \to B_n(D)$. Let $c \in B_n(C)$, so that there exists $c' \in C_{n+1}$ such that $d_C c' = c$. Then $f_n(c) = f_n d_C c' = d_D f_{n+1}(c')$. Thus $f_n(c)$ is the boundary of $f_{n+1}(c')$, and f gives a map $B_n(C) \to B_n(D)$.

The two maps $Z_n(C) \to Z_n(D)$ and $B_n(C) \to B_n(D)$ give a map on homology $f_* : H_n(X) \to H_n(Y)$, as desired.

3 Categories, functors, natural transformations

From spaces and continuous maps, we constructed graded abelian groups and homomorphisms. We now cast this construction in the more general language of category theory.

Our discussion of category theory will be interspersed throughout the text, introducing new concepts as they are needed. Here we begin by introducing the basic definitions.

Definition 3.1. A *category* $\mathcal{C}$ consists of the following data.

- a class $\mathrm{ob}(\mathcal{C})$ of objects;
- for every pair of objects X and Y, a set of *morphisms* $\mathcal{C}(X, Y)$;
- for every object X a *identity morphism* $1_X \in \mathcal{C}(X, X)$; and
- for every triple of objects X, Y, Z, a *composition* map $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$, written $(f, g) \mapsto g \circ f$.

These data are required to satisfy the following:

- $1_Y \circ f = f$, and $f \circ 1_X = f$.
- Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

Note that we allow the collection of objects to be a class. This enables us to talk about a "category of all sets" for example. But we require each $\mathcal{C}(X,Y)$ to be set, and not merely a class. Some interesting categories have a *set* of objects; they are called *small categories.*

We will often write $X \in \mathcal{C}$ to mean $X \in \mathrm{ob}(\mathcal{C})$, and $f\colon X \to Y$ to mean $f \in \mathcal{C}(X,Y)$.

Definition 3.2. If $X,Y \in \mathcal{C}$, then $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon Y \to X$ with $f \circ g = 1_Y$ and $g \circ f = 1_X$. We may write

$$f : X \xrightarrow{\cong} Y$$

to indicate that f is an isomorphism.

Example 3.3. Many common mathematical structures can be arranged in categories.

- Sets and functions between them form a category Set.
- Abelian groups and homomorphisms form a category **Ab**.
- Topological spaces and continuous maps form a category **Top**.
- Simplicial sets and their maps form a category sSet.
- A monoid is the same as a category with one object, where the elements of the monoid are the morphisms in the category. It's a small category.
- The sets $[n] = \{0,\dots,n\}$ for $n \geq 0$ together with weakly order-preserving maps between them form the *simplex category* Δ, another small category. It contains as a subcategory the *semi-simplex category* Δ_{inj} with the same objects but only injective weakly order-preserving maps.
- A poset forms a category in which there is a morphism from x to y iff $x \leq y$. A small category with this property comes from a poset provided that the only isomorphisms are identities.

Categories may be related to each other by rules describing effect on both objects and morphisms.

Definition 3.4. Let $\mathcal{C},\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of the data of

- an assignment $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, and
- for all $x,y \in \mathrm{ob}(\mathcal{C})$, map $F : \mathcal{C}(x,y) \to \mathcal{C}(F(x),F(y))$.

These data are required to satisfy the following two properties:

- For all $X \in \mathrm{ob}\mathcal{C}$, $F(1_X) = 1_{F(X)} \in \mathcal{D}(F(X),F(X))$, and
- For all composable pairs of morphisms f,g in $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$.

We have defined quite a few functors already:

$$\mathrm{Sin}_n : \mathbf{Top} \to \mathbf{Set}\,, \quad S_n : \mathbf{Top} \to \mathbf{Ab}\,, \quad H_n : \mathbf{Top} \to \mathbf{Ab}\,,$$

for example. We also have defined, for each X, a morphism $d : S_n(X) \to S_{n-1}(X)$. This is a "morphism between functors." This property is captured by another definition.

Definition 3.5. Let $F, G{:}\,\mathcal{C} \to \mathcal{D}$. A *natural transformation* or *natural map* $\theta{:}\,F \to G$ consists of maps $\theta_X{:}\,F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$ such that for all $f{:}\,X \to Y$ the following diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

So for example the boundary map $d{:}\,S_n \to S_{n-1}$ is a natural transformation.

Natural transformations are so ... well, so *natural* that their occurance is indicated by a variety of terms: a *natural* or *canonical* map is precisely a natural transformation.

Example 3.6. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two categories, and assume that $\mathcal{C}$ is small. We may then form the *category of functors* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Its objects are the functors fron $\mathcal{C}$ to $\mathcal{D}$, and given two functors F, G, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})(F, G)$ is the set of natural transformations from F to G. We let the reader define the rest of the structure of this category, and check the axioms. We needed to assume that $\mathcal{C}$ is small in order to guarantee that there is no more than a set of natural transformations between functors.

For example, let G be a group (or a monoid) viewed as a one-object category. Any element $F \in \mathrm{Fun}(G, \mathbf{Ab})$ is simply a group action of G on $F(*) = A$, i.e., a representation of G in abelian groups. Given another $F' \in \mathrm{Fun}(G, \mathbf{Ab})$ with $F'(*) = A'$, then a natural transformation from $F \to F'$ is precisely a G-equivariant homomorphism $A \to A'$.

4 More about categories

Let Vect_k be the category of vector spaces over a field k, and linear transformations between them. Given a vector space V, you can consider the dual $V^* = \mathrm{Hom}(V, k)$. Does this give us a functor? If you have a linear transformation $f : V \to W$, you get a map $f^* : W^* \to V^*$, so this is like a functor, but the induced map goes the wrong way. This operation does preserve composition and identities, in an appropriate sense. This is an example of a *contravariant functor*.

We'll leave it to you to spell out the definition, but notice that there is a univeral example of a contravariant functor out of a category $\mathcal{C}$: $\mathcal{C} \to \mathcal{C}^{op}$, where $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$, but $\mathcal{C}^{op}(X, Y)$ is declared to be the set $\mathcal{C}(Y, X)$. The identity morphisms remain the same. To describe the composition in $\mathcal{C}^{op}$, we'll write f^{op} for $f \in \mathcal{C}(Y, X)$ regarded as an element of $\mathcal{C}^{op}(X, Y)$; then $f^{op} \circ g^{op} = (g \circ f)^{op}$.

Then a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same thing as a ("covariant") functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

Let $\mathcal{C}$ be a category, and let $Y \in \mathrm{ob}(\mathcal{C})$. We get a map $\mathcal{C}^{op} \to \mathrm{Set}$ that takes $X \mapsto \mathcal{C}(X,Y)$, and takes a map $X \to W$ to the map defined by composition $\mathcal{C}(W,Y) \to \mathcal{C}(X,Y)$. This is called the functor that is represented by Y. It is very important to note that $\mathcal{C}(-,Y)$ is contravariant, while, on the other hand, for any fixed X, $\mathcal{C}(Y,-)$ is a covariant functor.

Example 4.1. Recall that the simplex category Δ has objects the totally ordered sets $[n] = \{0,1,\dots,n\}$, with order preserving maps as morphisms. The "standard simplex" gives us a functor $\Delta\colon \Delta \to \mathbf{Top}$. Now fix a space X, and consider

$$[n] \mapsto \mathbf{Top}(\Delta^n, X)\,.$$

This gives us a contravariant functor $\Delta \to \mathbf{Top}$, or a covariant functor $\Delta^{op} \to \mathbf{Top}$. This functor carries in it all the face and degeneracy maps we discussed earlier, and their compositions. Let us make a definition.

Definition 4.2. Let $\mathcal{C}$ be any category. A *simplicial object* in $\mathcal{C}$ is a functor $K : \Delta^{op} \to \mathcal{C}$. Simplicial objects in $\mathcal{C}$ form a category with natural transformations as morphisms. Similarly, *semi-simplicial object* in $\mathcal{C}$ is a functor $\Delta_{inj}^{op} \to \mathcal{C}$,

So the singular functor Sin_* gives a functor from spaces to simplicial sets (and so, by restriction, to semi-simplicial sets).

In want to interject one more bit of categorical language that will often be useful to us.

Definition 4.3. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is a *split epimorphism* ("split epi" for short) if there exists $g : Y \to X$ (called a section or a splitting) such that $Y \xrightarrow{g} X \xrightarrow{f} Y$ is the identity.

Example 4.4. In the category of sets, a map $f : X \to Y$ is a split epimorphism exactly when, for every element of Y there exists some element of X whose image in Y is the original element. So f is surjective. Is every surjective map a split epimorphism? This reduces to the axiom of choice! because proving that if $y \in Y$, the map $g : Y \to X$ can be constructed by choosing some $x \in f^{-1}(y)$.

Every categorical definition is accompanied by a dual definition.

Definition 4.5. A map $g : Y \to X$ is a *split monomorphism* ("split mono" for short) if there is $f : X \to Y$ such that $f \circ g = 1_Y$.

Example 4.6. Again let $\mathcal{C} = \mathrm{Set}$. Any split monomorphism is an injection: If $y, y' \in Y$, and $g(y) = g(y')$, we want to show that $y = y'$. Apply f, to get $f(g(y)) = y = f(g(y')) = y'$. But the injection $\varnothing \to Y$ is a split monomorphism only if $Y = \varnothing$. So there's an assymetry in the category of sets.

Lemma 4.7. *A map is an isomorphism if and only if it is both a split epimorphism and a split monomorphism.*

Proof. Standard! □

The importance of these definitions is this: Functors will not in general respect "monomorphisms" or "epimorphisms," but:

Lemma 4.8. *If $f : X \to Y$ is a split epi or mono in $\mathcal{C}$, and you have a functor $F : \mathcal{C} \to \mathcal{D}$, then so is $F(f)$ in $\mathcal{D}$.*

Proof. Apply F to the diagram establishing f as a split epi or mono. □

Example 4.9. Suppose $\mathcal{C} = \mathbf{Ab}$, and you have a split epi $f : A \to B$. Let $g : B \to A$ be a section. We also have the inclusion $i : \ker f \to A$, and hence a map

$$[\, f \quad i\,] : B \oplus \ker f \to A\,.$$

We leave it to you to check that this map is an isomorphism, and to formulate a dual statement.

5 Homotopy, star-shaped regions

We've computed the homology of a point. Let's now compare the homology of a general space X to this example. There's always a unique map $X \to *$: $*$ is a "terminal object" in **Top**. We have an induced map

$$H_n(X) \to H_n(*) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{else} \end{cases}.$$

Any formal linear combination $c = \sum a_i x_i$ of points of X is a 0-cycle. The map to $*$ sends c to $\sum a_i \in \mathbf{Z}$. This defines an "augmentation" $\epsilon : H_*(X) \to H_*(*)$. If X is nonempty, the map $X \to *$ is split by any choice of point in X, so the augmentation is also split. The kernel of ϵ is the *reduced homology* $\widetilde{H}_*(X)$ of X, and we get a canonical splitting

$$H_*(X) \cong \widetilde{H}_*(X) \oplus \mathbf{Z}$$

Actually, it's useful to extend the definition to the empty space by the following device. Extend the singular chain complex for any space to include $\mathbf{Z}$ in dimension -1, with $d : S_0(X) \to S_{-1}(X)$ given by the augmentation ϵ. Let's write $\widetilde{S}_*(X)$ for this chain complex. When $X \neq \varnothing$, ϵ is surjective and you get the same answer as above. But

$$\widetilde{H}_q(\varnothing) = \begin{cases} \mathbf{Z} & \text{for } q = -1 \\ 0 & \text{for } q \neq -1\,. \end{cases}$$

What other spaces have trivial homology? A slightly non-obvious way to reframe the question is this:

When do two maps $X \to Y$ induce the same map in homology?

For example, when do $1_X : X \to X$ and $X \to * \to X$ induce the same map in homology? If they do, then $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism.

The key idea is that homology is a discrete invariant, so it should be unchanged by deformation. Here's the definition that makes "deformation" precise.

Definition 5.1. Let $f_0, f_1 : X \to Y$ be two maps. A *homotopy* from f_0 to f_1 is a map $h : X \times I \to Y$ (continuous, of course) such that $h(x,0) = f_0(x)$ and $f(x,1) = f_1(x)$. We say that f_0 and f_1 are *homotopic*, and that h is a *homotopy* between them. This relation is denoted by $f_0 \sim f_1$. It is an equivalence relation on maps from X to Y.

Transitivity follows from the gluing lemma of point set topology. We denote by $[X,Y]$ the set of *homotopy classes* of maps from X to Y. In the next lecture we'll prove the following result.

Theorem 5.2 (Homotopy invariance of homology)**.** *If $f_0 \sim f_1$, then $H_*(f_0) = H_*(f_1)$: homology cannot distinguish between homotopic maps.*

Suppose I have two maps $f_0, f_1 : X \to Y$ with a homotopy $h : f_0 \sim f_1$, and a map $g : Y \to Z$. Composing h with g gives a homotopy between $g \circ f_0$ and $g \circ f_1$. Precomposing also works: If $g : W \to X$ is a map and $f_0, f_1 : X \to Y$ are homotopic, then $f_0 g \sim f_1 g$. This lets us compose homotopy classes: we can complete the diagram:

$$\begin{tikzcd}
\mathbf{Top}(Y,Z) \times \mathbf{Top}(X,Y) \arrow[r] \arrow[d] & \mathbf{Top}(X,Z) \arrow[d] \\
{[Y,Z] \times [X,Y]} \arrow[r, dashed] & {[X,Z]}
\end{tikzcd}$$

Definition 5.3. The *homotopy category* (of topological spaces) $\mathrm{Ho}(\mathbf{Top})$ has the same objects as $\mathbf{Top}$, but $\mathrm{Ho}(\mathbf{Top})(X,Y) = [X,Y] = \mathbf{Top}(X,Y)/\sim$.

The first key property of homology is this:

Theorem 5.4 (Homotopy invariance of homology)**.** *The homology functor $H_* : \mathbf{Top} \to \mathbf{Ab}$ factors as $\mathbf{Top} \to \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Ab}$; it is a "homotopy functor."*

We will prove this in the next lecture, but let's stop now and think about some consequences.

Definition 5.5. A map $f : X \to Y$ is a *homotopy equivalence* if $[f] \in [X,Y]$ is an isomorphism in $\mathrm{Ho}(\mathbf{Top})$. In other words, there is a map $g : Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

Homotopy equivalences don't preserve most topological properties. For example, compactness is not a homotopy-invariant property: Consider the inclusion $S^{n-1} \subseteq \mathbf{R}^n - \{0\}$. A homotopy inverse $p : \mathbf{R}^n - \{0\} \to S^{n-1}$ can be obtained by dividing a (always nonzero!) vector by its length. Clearly $p \circ i = 1_{S^{n-1}}$. We have to find a homotopy $i \circ p \sim 1_{\mathbf{R}^n - \{0\}}$. This is t a map $(\mathbf{R}^n - \{0\}) \times I \to \mathbf{R}^n - \{0\}$, and we can use $(v, t) \mapsto tv + (1-t)\frac{v}{\|v\|}$.

On the other hand:

Corollary 5.6. *Homotopy equivalences induce isomorphisms in homology.*

Definition 5.7. A space X is *contractible* if the map $X \to *$ is a homotopy equivalence.

Corollary 5.8. *Let X be a contractible space. The augmentation $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism.*

Homotopy equivalences in general may be somewhat hard to visualize. A particularly simple and important class of homotopy equivalences is given by the following definition.

Definition 5.9. An inclusion $A \hookrightarrow X$ is a *deformation retract* provided that there is a map $h : X \times I \to X$ such that $h(x, 0) = x$ and $h(x, 1) \in A$ for all $x \in X$ and $h(a, t) = a$ for all $a \in A$ and $t \in I$.

For example, S^{n-1} is a deformation retract of $\mathbf{R}^n - \{0\}$.

We now set about constructing a proof of homotopy invariance of homology. The first step is to understand the analogue of homotopy on the level of chain complexes.

Definition 5.10. Let $C_\bullet, D_\bullet$ be chain complexes, and $f_0, f_1 : C_\bullet \to D_\bullet$ be chain maps. A chain homotopy $h : f_0 \sim f_1$ is a collection of homomorphisms $h : C_n \to D_{n+1}$ such that $dh + hd = f_1 - f_0$.

It's a really weird relation. Here's a picture.

$$\begin{tikzcd}
C_{n+2} \arrow[r, "f_1 - f_0"] \arrow[d, "d"] & D_{n+2} \arrow[d, "d"] \\
C_{n+1} \arrow[r, "f_1 - f_0"] \arrow[d, "d"] \arrow[ur, dashed, "h"] & D_{n+2} \arrow[d, "d"] \\
C_n \arrow[r, "f_1 - f_0"'] \arrow[ur, dashed, "h"] & D_n
\end{tikzcd}$$

Lemma 5.11. *If $f_0, f_1 : C_\bullet \to D_\bullet$ are chain homotopic, then $f_{0,*} = f_{1,*} : H(C) \to H(D)$.*

Proof. We want to show that for every $c \in Z_n(C_\bullet)$, the difference $f_1 c - f_0 c$ is a boundary. Well,

$$f_1 c - f_0 c = (dh + hd)c = dhc + hdc = dhc\,.$$

□

So homology invariance will follow from

Proposition 5.12. *Let $f_0, f_1 : X \to Y$ be homotopic. Then $f_{0*}, f_{1*} : S_\bullet(X) \to S_\bullet(Y)$ are chain homotopic.*

To prove this we will begin with a special case.

Definition 5.13. A subset $X \subseteq \mathbf{R}^n$ is *star-shaped* with respect to $b \in X$ if for every $x \in X$ the interval

$$\{tb + (1-t)x : t \in [0,1]\}$$

lies in X.

Any nonempty convex region is star shaped. Any star-shaped region X is contractible: A homotopy inverse to $X \to *$ is given by sending $* \mapsto b$. One composite is perforce the identity. A homotopy from the other composite to the identity 1_X is given by $(x,t) \mapsto tb + (1-t)x$.

So we should expect that $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism if X is star-shaped. In fact, using a piece of language that the reader can interpret:

Proposition 5.14. *$S_*(X) \to \mathbf{Z}$ is a chain homotopy equivalence.*

Proof. We have maps $S_*(X) \xrightarrow{\epsilon} \mathbf{Z} \xrightarrow{\eta} S_*(X)$ where $\eta(1) = c_0^0$. Clearly $\epsilon\eta = 1$, and the claim is that $\eta\epsilon \sim 1 : S_*(X) \to S_*(X)$. The chain map $\eta\epsilon$ concentrates everything at the origin: $\eta\epsilon\sigma = c_0^n$ for all $\sigma \in \mathrm{Sin}_n(X)$. Our chain homotopy $h : S_q(X) \to S_{q+1}(X)$ will actually send simplices to simplices. For $\sigma \in \mathrm{Sin}_q(X)$, define the chain homotopy evaluated on σ by means of the following "cone construction":

$$(b * \sigma)(t_0, \ldots, t_{q+1}) = t_0 b + (1-t_0)\sigma\left(\frac{(t_1, \ldots, t_{q+1})}{1-t_0}\right).$$

Explanation: The denominator $1-t_0$ makes the entries sum to 1, as they must if we are to apply σ to this vector. When $t_0 = 1$, this isn't defined, but it doesn't matter since we are multiplying by $1-t_0$. So $(b*\sigma)(0, t_1, \ldots, t_{q+1}) = b$; this is the vertex of the cone.

$\boxed{\text{picture needed}}$

Setting $t_0 = 0$, we find

$$d_0 b * \sigma = \sigma\,.$$

Setting $t_i = 0$ for $i > 0$, we find

$$d_i b * \sigma = h d_{i-1}\sigma\,.$$

Using the formula for the boundary operator, we find

$$db * \sigma = \sigma - b * d\sigma$$

... *unless* $q = 0$, when

$$db * \sigma = \sigma - c_b^0\,.$$

This can be assembled into the equation

$$db * + b * d = 1 - \eta\epsilon$$

which is what we wanted. □

6 Homotopy invariance of homology

We now know that the homology of a star-shaped region is trivial: in such a space, every cycle with augmentation 0 is a boundary. We will use that fact, which is a special case of homotopy invariance of homology, to prove the general result, which we state in somewhat stronger form:

Theorem 6.1. *A homotopy $h : f_0 \sim f_1 : X \to Y$ determines a natural chain homotopy $f_{0,*} \sim f_{1,*} : S_*(X) \to S_*(Y)$.*

The proof uses naturality (a lot). For a start, notice that if $k : g_0 \sim g_1 : C_* \to D_*$ is a chain homotopy, and $j : D_* \to E_*$ is another chain map, then the composites $j \circ k_n : C_n \to E_{n+1}$ give a chain homootpy $j \circ g_0 \sim j \circ g_1$. So if we can produce a chain homotopy homotopy between the chain maps induced by the two inclusions $i_0, i_1 : X \to X \times I$, we can get a chain homotopy k between $f_{0*} = h_* \circ i_{0*}$ and $f_{1*} = h_* \circ i_{1*}$ in the form $h_* \circ k$.

So now we want to produce a natural chain homotopy, with components $k_n : S_n(X) \to S_{n+1}(X \times I)$. The unit interval hosts a natural 1-simplex given by an identification $\Delta^1 \to I$, and we should imagine k as being given by "multiplying" by that 1-chain. This "multiplication" is a special case of a chain map

$$\times : S_*(X) \times S_*(Y) \to S_*(X \times Y)\,,$$

defined for any two spaces X and Y, with lots of good properties. It will ultimately be used to compute the homology of a product of two spaces in terms of the homology groups of the factors.

Here's the general result.

Theorem 6.2. *There exists a map $S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ that is:*

- *Natural, in the sense that if $f : X \to X'$ and $g : Y \to Y'$, and $a \in S_p(X)$ and $b \in S_p(Y)$ so that $a \times b \in S_{p+q}(X \times Y)$, then $f_*(a) \times g_*(b) = (f \times g)_*(a \times b)$.*
- *Bilinear, in the sense that $(a+a') \times b = (a \times b) + (a' \times b)$, and $a \times (b+b') = a \times b + a \times b'$.*
- *The Leibniz rule is satisfied, i.e., $d(a \times b) = (da) \times b + (-1)^p a \times db$.*
- *Normalized, in the following sense. Let $x \in X$ and $y \in Y$. Write $j_x : Y \to X \times Y$ sending $y \mapsto (x, y)$, and write $i_y : X \to X \times Y$ sending $x \mapsto (x, y)$. If $b \in S_q(Y)$, then $c_x^0 \times b = (j_x)_* b \in S_q(X \times Y)$, and if $a \in S_p(X)$, then $a \times c_y^0 = (i_y)_* a \in S_p(X \times Y)$.*

The Leibniz rule contains the first occurence of the "topologists sign rule"; we'll see these signs appearing often. Watch for when it appears in our proof.

Proof. We're going to use induction on $p+q$; the normalization axiom gives us the cases $p + q = 0, 1$. Let's assume that we've constructed the cross-product in total dimension $p+q-1$. We want to define $\sigma \times \tau$ for $\sigma \in S_p(X)$ and $\tau \in S_q(Y)$.

Note that there's a universal example of a p-simplex, namely the identity map $\iota_p : \Delta^p \to \Delta^p$. It's universal in the sense that given any p-simplex $\sigma : \Delta^p \to X$, you get

$\sigma = \sigma_*(\iota_p)$ where $\sigma_* : \mathrm{Sin}_p(\Delta^p) \to \mathrm{Sin}_p(X)$ is the map induced by σ. To define $\sigma \times \tau$ in general, then, it suffices to define $\iota_p \times \iota_q \in S_{p+q}(\Delta^p \times \Delta^q)$; we can (and must) then take $\sigma \times \tau = (\sigma \times \tau)_*(\iota_p \times \iota_q)$.

Our long list of axioms is useful in the induction. For one thing, if $p = 0$ or $q = 0$, normalization provides us with a choice. So now assume that both p and q are positive. We want the cross-product to satisfy the Leibnitz rule:

$$d(\iota_p \times \iota_q) = (d\iota_p) \times \iota_q + (-1)^p \iota_p \times d\iota_q \in S_{p+q-1}(\Delta^p \times \Delta^q)$$

Since $d^2 = 0$, a necessary condition for $\iota_p \times \iota_q$ to exist is that $d((d\iota_p)\times\iota_q + (-1)^p \iota_p \times d\iota_q) = 0$. Let's compute what this is, using the Leibnitz rule in dimension $p+q-1$ where we have it by the inductive assumption:

$$d((d\iota_p)\times\iota_q+(-1)^p\iota_p\times(d\iota_q)) = (d^2\iota_p)\times\iota_q+(-1)^{p-1}(d\iota_p)\times(d\iota_q)+(-1)^p(d\iota_p)\times d\iota_q+(-1)^q\iota_p\times(d^2\iota_q) = 0$$

because $d^2 = 0$. Note that this calculation would not have worked without the sign!

The subspace $\Delta^p \times \Delta^q \subseteq \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$ is convex, so by translation, it's homeomorphic to a star-shaped region. Therefore we know that $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ (remember, $p + q > 1$), which means that every cycle is a boundary. In other words, our necessary condition is also sufficient! So, choose any element with the right boundary and declare it to be $\iota_p \times \iota_q$.

The induction is now complete provided we can check that this choice satisfies naturality, bilinearity, and the Leibniz rule. We leave this as a relaxing exercise for the reader. □

The essential point here is that the space supporting the universal pair of simplices – $\Delta^p \times \Delta^q$ – has trivial homology. Naturality transports the result of that fact to the general situation.

The cross-product that this procedure constructs is not unique; it depends on a choice a choice of the chain $\iota_p \times \iota_q$ for each pair p, q with $p + q > 1$. The cone construction in the proof that star-shaped regions have vanishing homology provids us with a specific choice; but it turns out that any two choices are equivalent up to natural chain homotopy.

We return to homotopy invariance. To define our chain homotopy $h_X : S_n(X) \to S_{n+1}(X \times I)$, pick any 1-simplex $\iota : \Delta^1 \to I$ such that $d_0\iota = 1$ and $d_1\iota = 0$, and define

$$h_X\sigma = (-1)^n \sigma \times \iota\,.$$

Let's compute:

$$dh_X\sigma = (-1)^n d(\sigma \times \iota) = (-1)^n (d\sigma) \times \iota + \sigma \times (d\iota)$$

But $d\iota = c_1^0 - c_0^0 \in S_0(I)$, which means that we can continue (remembering that $|\partial\sigma| = n - 1$):

$$= -h_X d\sigma + (\sigma \times c_1^0 - \sigma \times c_0^0) = -h_X d\sigma + (\iota_{1*}\sigma - \iota_{0*}\sigma)\,,$$

using the normalization axiom of the cross-product. This is the result.

7 Homology cross product

In the last lecture we proved homotopy invariance of homology using the construction of a chain level bilinear cross-product

$$\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$$

that satisfied the Leibniz formula

$$d(a \times b) = (da) \times b + (-1)^p a \times (db)$$

What else does this map give us?

Let's abstract a little bit. Suppose we have three chain complexes A_*, B_*, and C_*, and suppose we have maps $\times : A_p \times B_q \to C_{p+q}$ that satisfy bilinearity and the Leibniz formula. What does this induce in homology?

Lemma 7.1. *This determines a bilinear map $H_p(A) \times H_q(B) \xrightarrow{\times} H_{p+q}(C)$.*

Proof. Let $a \in Z_p(A)$ and $b \in Z_q(A)$. We want to define $[a] \times [b] \in H_{p+q}(C)$. We hope that $[a] \times [b] = [a \times b]$. We need to check that $a \times b$ is a cycle. By Leibniz, $d(a \times b) = da \times b + (-1)^p a \times db$. Because a, b are boundaries, this is zero. Now we need to check that this thing is well-defined. Let's pick other cycles a' and b' in the same homology classes. We want $[a \times b] = [a' \times b']$. In other words, we need to show that $a \times b$ differs from $a' \times b'$ by a boundary. We can write $a' = a + d\overline{a}$ and $b' = b + d\overline{b}$, and compute, using bilinearity:

$$a' \times b' = (a + d\overline{a}) + (b + d\overline{b}) = a \times b + a \times d\overline{b} + (d\overline{a}) \times b + (d\overline{a}) \times (d\overline{b})$$

We need to deal with the last three terms here. But since $da = 0$,

$$d(a \times \overline{b}) = (-1)^p a \times (d\overline{b})\,.$$

Since $d\overline{b} = 0$,

$$d((\overline{a}) \times b) = (d\overline{a}) \times b\,.$$

And since $d^2\overline{b} = 0$,

$$d(a \times \overline{b}) = (d\overline{a}) \times (d\overline{b})\,.$$

This means that $a^d \times b^d$ and $a \times b$ differ by

$$d((-1)^p (a \times \overline{b}) + \overline{a} \times b + \overline{a} \times d\overline{b})\,,$$

and so are homologous.

The last step is to check bilinearity, which is left to the reader. $\square$

This gives the following result.

Theorem 7.2. *There is a map*

$$\times : H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y)$$

that is natural, bilinear, and normalized.

This map is also *uniquely defined* by these conditions, unlike the chain-level cross product.

I just want to mention an explicit choice of $\iota_p \times \iota_q$. This is called the Eilenberg-Zilber chain. You're highly encouraged to think about this yourself. It comes from a triangulation of the prism.

The simplices in this triangulation are indexed by order preserving injections

$$\omega : [p+q] \to [p] \times [q]$$

Injectivity forces $\omega(0) = (0,0)$ and $\omega(p+q) = (p,q)$. Each such map determines an affine map $\Delta^{p+q} \to \Delta^p \times \Delta^q$ of the same name. These will be the singular simplices making up $\iota_p \times \iota_q$. To specify the coefficients, think of ω as a staircase in the rectangle $[0,p] \times [0,q]$. Let $A(\omega)$ denote the area under that staircase. Then the Eilenberg-Zilber chain is given by

$$\iota_p \times \iota_q = \sum (-1)^{A(\omega)} \overline{\omega}$$

This description is in a paper by Eilenberg and Moore. It's very pretty, but it's combinatorially annoying to check that this satisfies the conditions of the theorem. It provides an explicit chain map

$$\beta_{X,Y} : S_*(X) \times S_*(Y) \to S_*(X \times Y)$$

that satisfies many good properties on the nose and not just up to chain homotopy. For example, it's *associative* –

$$\begin{tikzcd}
S_*(X) \times S_*(Y) \times S_*(Z) \arrow[r, "\beta_{X,Y} \times 1"] \arrow[d, "1 \times \beta Y{,}Z"] & S_*(X \times Y) \times S_*(Z) \arrow[d, "\beta_{X \times Y, Z}"] \\
S_*(X) \times S_*(Y \times Z) \arrow[r, "\beta_{X, Y \times Z}"] & S_*(X \times Y \times Z)
\end{tikzcd}$$

commutes – and *commutative* –

$$\begin{tikzcd}
S_*(X) \times S_*(Y) \arrow[r, "\beta_{X,Y}"] \arrow[d, "T"] & S_*(X \times Y) \arrow[d, "S_*(T)"] \\
S_*(Y) \times S_*(X) \arrow[r, "\beta_{Y,X}"] & S_*(X \times Y)
\end{tikzcd}$$

commutes, where on spaces $T(x,y) = (y,x)$, and on chain complexes $T(a,b) = (-1)^{pq}(b,a)$ when a has degree p and b has degree q.

We will see that these properties hold up to chain homotopy for any choice of chain-level cross product.

8 Relative homology

An ultimate goal of algebraic topology is to find means to compute the set of homotopy classes of maps from one space to another. This is important because many geometrical problems can be rephrased as such a computation. It's a lot more modest than wanting to characterize, somehow, all continuous maps from X to Y; but the very fact that it still contains a great deal of interesting information means that it is still very challenging to compute.

Homology is in a certain sense the best "additive" approximation to this problem; and its additivity makes it much more computable. To justify this, we want to describe the sense in which homology is "additive." Here are two related aspects of this claim.

1. If $A \subseteq X$ is a subspace, then $H_*(X)$ a combination of $H_*(A)$ and $H_*(X - A)$.

2. The homology $H_*(A \cup B)$ is like $H_*(A) + H_*(B) - H_*(A \cap B)$.

The first hope is captured by the long exact sequence of a pair, the second by the Meyer-Vietoris Theorem. Both facts show that homology behaves like a measure. The precise statement of both facts uses the machinery of exact sequences. We'll use the following language.

Definition 8.1. A *sequence* of abelian groups is a diagram of abelian groups of the form

$$\cdots \to C_{n+1} \xrightarrow{f_n} C_n \xrightarrow{f_{n-1}} C_{n-1} \to \cdots,$$

finite or infinite in either or direction or both directions, in which all composites are zero; that is, $\operatorname{im} f_n \subseteq f_{n-1}$ for all n. It is *exact* at C_n provided that this inequality is an equality.

Example 8.2. A sequence infinite in both directions is just a chain complex; it is exact at C_n if and only if $H_n(C_*) = 0$. So homology measures the failure of exactness.

Example 8.3. $0 \to A \xrightarrow{i} B$ is exact iff i is injective, and $B \xrightarrow{p} C \to 0$ is exact iff p is surjective.

Exactness is a key property in the development of algebraic topology, and "exact" is a great word for the concept. A foundational treatment of algebraic topology was published by Sammy Eilenberg and Norman Steenrod published in 1952. The story is that in the galleys for the book they left a blank space whenever the word representing this concept was used, and filled it in at the last minute.

Definition 8.4. An exact sequence that's infinite in both directions is a *long exact sequence.* A *short exact sequence* is an exact sequence of the form

$$0 \to A \xrightarrow{i} B \xRightarrow{p} C \to 0\,.$$

Any sequence of the form $0 \to A \to B \to C \to 0$ expands to a diagram

$$\begin{tikzcd}
\ker(p) \arrow[dr] & & \\
A \arrow[u] \arrow[r, "i"] & B \arrow[r, "p"] \arrow[dr] & C \\
& & \operatorname{coker}(i) \arrow[u]
\end{tikzcd}$$

It is short exact if and only if $A \xrightarrow{\cong} \ker p$ and $B \xrightarrow{\cong} \operatorname{coker}(i)$.

As suggested above, we will study the homology of a space X by comparing it to the homology of a subspace A and a complement or quotient construction. Note that $S_*(A)$ injects into $S_*(X)$. This suggests considering the quotient group

$$\frac{S_n(X)}{S_n(A)} .$$

This is the group of *relative n-chains* of the pair (X, A).

Let's formalize this a bit. Along with the category **Top** of spaces, we have the category $\mathbf{Top_2}$ of *pairs* of spaces. An object of $\mathbf{Top_2}$ is a space X together with a subspace A. A map $(X, A) \to (Y, B)$ is a continuous map $X \to Y$ that sends A into B.

There are four obvious functors relating **Top** and $\mathbf{Top_2}$:

$$X \mapsto (X, \varnothing) , \quad X \mapsto (X, X) ,$$

$$(X, A) \mapsto X , \quad (X, A) \mapsto A .$$

Do the relative chains form themselves into a chain complex?

Lemma 8.5. *Let A_* be a subcomplex of the chain complex B_*. There is a unique structure of chain complex on the quotient graded abelian group C_* with entries $C_n = B_n/A_n$ such that $B_* \to C_*$ is a chain map.*

Proof. To define $d : C_n \to C_{n-1}$, represent $c \in C_n$ by $b \in B_n$ and hope that $[db] \in B_{n-1}/A_{n-1}$ is well defined. If we replace b by $b + a$ for $a \in A_n$, we find

$$d(b + a) = db + da \equiv da \mod A_{n-1} ,$$

so our hope is justified. Then $d^2[b] = [d^2 b] = 0$. □

Definition 8.6. The *relative singular chain complex* of the pair (X, A) is

$$S_*(X, A) = \frac{S_*(X)}{S_*(A)} .$$

This is a functor from pairs of spaces to chain complexes. Of course

$$S_*(X, \varnothing) = S_*(X) , \quad S_*(X, X) = 0 .$$

Definition 8.7. The *relative singular homology* of the pair (X, A) is the homology of the relative singular chain complex:

$$H_n(X, A) = H_n(S_*(X, A)) .$$

One of the nice features of the absolute chain group $S_n(X)$ is that it is free as an abelian group. Is that also the case for its quotent $S_n(X, A)$? The map $\mathrm{Sin}_n(A) \to \mathrm{Sin}_n(X)$ is an injection. As long as $A \neq \varnothing$, this injection admits a splitting. (If $A = \varnothing$, then $S_n(X, A) = S_n(X)$ is indeed free.) So when we apply the free abelian group functor we obtain a split monomorphism. Then the induced map $S_*(X) \to S_*(A) \oplus S_*(X, A)$ is an isomorphism, so $S_*(X, A)$ is again free.

Example 8.8. Consider Δ^n, relative to its boundary $\partial\Delta^n := \bigcup \mathrm{im}\, d_i \cong S^{n-1}$. We have the identity map $\iota_n : \Delta^n \to \Delta^n$, the universal n-simplex, in $\mathrm{Sin}_n(\Delta^n) \subseteq S_n(\Delta^n)$. It is not a cycle; its boundary $d\iota_n \in S_{n-1}(\Delta^n)$ is the alternating sum of the faces of the n-simplex. Each of these singular simplices lies in $\partial\Delta^n$, so $d\iota_n \in S_{n-1}(\partial\Delta^n)$, and $[\iota_n] \in S_n(\Delta_n, \partial\Delta_n)$ *is* a *relative* cycle. We will see that the relative homology $H_n(\Delta^n, \partial\Delta^n)$ is infinite cyclic, with generator $[\iota_n]$.

9 The homology long exact sequence

A pair of spaces (X, A) gives rise to a short exact sequence of chain complexes:

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0 .$$

In homology, this will relate $H_*(A)$, $H_*(X)$, and $H_*(X, A)$.

To investigate what happens, let's suppse we have a general short exact sequence of chain complexes,

$$0 \to A_* \to B_* \to C_* \to 0 ,$$

and investigate what happens in homology.

Here is an expanded part of this short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "f"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g"] \arrow[d, "d"] & C_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "d"] & B_n \arrow[r, "g"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Clearly the composite $H(A_*) \to H_*(X) \to H_*(C)$ is trivial. Is this sequence exact? Let $[b] \in H_n(B)$ such that $g([b]) = 0$. It's determined by some $b \in B_n$ such that $d(b) = 0$. If $g([b]) = 0$, then there is some $\overline{c} \in C_{n+1}$ such that $d\overline{c} = gb$. Now, g is surjective, so there is some $\overline{b} \in B_{n+1}$ such that $g(\overline{b}) = \overline{c}$. Then we can consider $d\overline{b} \in B_n$, and $g(d(\overline{b})) = d(\overline{c}) \in C_n$. What is $b - d\overline{b}$? This maps to zero in C_n, so by exactness there is some $a \in A_n$ such that $f(a) = b - d\bar{b}$. Is a a cycle? Well, $f(da) = d(fa) = d(b - d\bar{b}) = db - d^2\bar{b} = db$, but we assumed that $db = 0$, so $f(da) = 0$. This means that da is zero because f is an injection by exactness. Therefore a is a cycle. What is $[a] \in H_n(A)$? Well, $f([a]) = [b - d\bar{b}] = [b]$ because $d\bar{b}$ is a cycle. Is the composite $H_n(A) \to H_n(B) \to H_n(C)$ zero? Yes, because the composite factors through zero. This proves exactness of $H_n(A) \to H_n(B) \to H_n(C)$.

On the other hand, $H_*(A) \to H_*(B)$ may fail to be injective, and $H_*(B) \to H_*(C)$ may fail to be surjective. Instead:

Theorem 9.1 (homology long exact sequence)**.** *Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. Then there is a natural homomorphism $\partial : H_n(C) \to H_{n-1}(A)$ such that the sequence*

$$\begin{tikzcd} & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll] \\ H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, "\partial"] \\ H_{n-1}(A) \arrow[r] & \cdots & \end{tikzcd}$$

is exact.

Proof. We'll construct ∂, and leave the rest as an exercise. Again:

$$\begin{tikzcd} 0 \arrow[r] & A_{n+1} \arrow[r, "f"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g"] \arrow[d, "d"] & C_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "d"] & B_n \arrow[r, "g"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

Let $c \in C_n$ such that $dc = 0$. The map g is surjective, so pick a $b \in B_n$ such that $g(b) = c$. Then consider $db \in B_{n-1}$. But $g(d(b)) = 0 = d(g(b)) = dc$. So by exactness, there is some $a \in A_{n-1}$ such that $f(a) = db$. How many choices are there of picking a? One, because a is injective. We need to check that a is a cycle. What is $d(a)$? Well, $d^2b = 0$, so da maps to 0 under f. But because f is injective, $da = 0$, i.e., a is a cycle. This means we can define $\partial[c] = [a]$.

To make sure that this is well-defined, let's make sure that this choice of homology class a didn't depend on the b that we chose. Pick some other b' such that $g(b') = c$. Then there is $a' \in A_{n-1}$ such that $f(a') = db'$. We want $a - a'$ to be a boundary, so that $[a] = [a']$. We want $\overline{a} \in A_n$ such that $d\overline{a} = a - a'$. Well, $g(b - b') = 0$, so by exactness, there is $\overline{a} \in A_n$ such that $f(\overline{a}) = b - b'$. What is $d\overline{a}$? Well, $d\overline{a} = d(b - b') = db - db'$. But $f(a - a') = b - b'$, so because f is injective, $d\overline{a} = a - a'$, i.e., $[a] = [a']$. What else do I have to check? It's an exercise to check that ∂ as defined here is a homomorphism. Also, left as an exercise to check that this doesn't depend on $c \in [c]$, and that ∂ actually makes the exact sequence above exact. □

Example 9.2. A pair of spaces (X, A) gives rise to a natural long exact sequence in homology:

$$\begin{tikzcd} & \cdots \arrow[r] & H_{n+1}(X,A) \arrow[dll, "\partial"'] \\ H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\partial"'] \\ H_{n-1}(A) \arrow[r] & \cdots & \end{tikzcd}$$

Example 9.3. Let's think again about the pair (D^n, S^{n-1}). By homotopy invariance we know that $H_q(D^n) = 0$ for $q > 0$, since D^n is contractible. So

$$\partial : H_q(D^n, S^{n-1}) \to H_{q-1}(S^{n-1})$$

is an isomorphism for $i > 1$. The bottom of the long exact sequence looks like this:

$$\begin{tikzcd} & 0 \arrow[r] & H_1(D^n, S^{n-1}) \arrow[dll] & \\ H_0(S^{n-1}) \arrow[r] & H_0(D^n) \arrow[r] & H_0(D^n, S^{n-1}) \arrow[r] & 0 \end{tikzcd}$$

When $n > 1$, both S^{n-1} and D^n are path-connected, so the map $H_0(S^{n-1}) \to H_0(D^n)$ is an isomorphism, and

$$H_1(D^n, S^{n-1}) = H_0(D^n, S^{n-1}) = 0\,.$$

When $n = 1$, we discover that

$$H_1(D^1, S^0) = \mathbf{Z} \quad \text{and} \quad H_0(D^1, S^0) = 0\,.$$

The generator of $H_1(D^1, S^0)$ is represented by any 1-simplex $\iota_1 : \Delta^1 \to D^1$ such that $d_0\iota = 1$ and $d_1\iota = 0$ (or vice versa). To go any further in this analysis, we'll need another tool, known as "excision."

We can set this up for reduced homology (as in Lecture 5) as well. Note that any map induces an isomorphism in $\widetilde{S}_{-1}$, so to a pair (X, A) we can associate a short exact sequence

$$0 \to \widetilde{S}_*(A) \to \widetilde{S}_*(X) \to S_*(X, A) \to 0$$

and hence a long exact sequence

$$\begin{tikzcd}
& & \cdots \arrow[r] & H_{n+1}(X,A) \arrow[dll, "\partial"'] \\
\widetilde{H}_n(A) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\partial"'] & \\
\widetilde{H}_{n-1}(A) \arrow[r] & \cdots & &
\end{tikzcd} .$$

The homology long exact sequence is often used in conjunction with an elementary fact about a map between exact sequences known as the *five lemma*. Suppose you have two exact sequences of abelian groups and a map between them – a "ladder":

$$\begin{tikzcd}
A_4 \arrow[r, "d"] \arrow[d, "f_4"] & A_3 \arrow[r, "d"] \arrow[d, "f_3"] & A_2 \arrow[r, "d"] \arrow[d, "f_2"] & A_1 \arrow[r, "d"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
B_4 \arrow[r, "d"] & B_3 \arrow[r, "d"] & B_2 \arrow[r, "d"] & B_1 \arrow[r, "d"] & B_0
\end{tikzcd}$$

When can we guarantee that f_2 is an isomorphism? We're going to "diagram chase." Just follow your nose, making assumptions as necessary.

Surjectivity: Let $b_2 \in B_2$. We want to show that there is something in A_2 mapping to b_2. We can consider $db_2 \in B_1$. Let's assume that f_1 is surjective. Then there's $a_1 \in A_1$ such that $f_1(a_1) = db_2$. What is da_1? Well, $f_0(da_1) = d(f_1(a_1)) = d(db) = 0$. So we want f_0 to be injective. Then da_1 is zero, so by exactness of the top sequence, there is some $a_2 \in A_2$ such that $da_2 = a_1$. What is $f_2(a_2)$? To answer this, begin by asking: What is $d(f_2(a_2))$? By commutativity, $d(f_2(a_2)) = f_1(d(a_2)) = f_1(a_1) = db_2$. Let's consider $b_2 - f_2(a_2)$. This maps to zero under d. So by exactness, there is $b_3 \in B_3$ such that $d(b_3) = b_2 - f_2(a_2)$. If we assume that f_3 is surjective, then there is $a_3 \in A_3$ such that $f_3(a_3) = b_3$. But now, $d(a_3) \in A_2$, and $f_2(d(a_3)) = d(f_3(a_3)) = b_2 - f_2(a_2)$. This means that $b_2 = f(a_2 + d(a_3))$, which guarantees surjectivity of f_2.

This proves the first half of the following important fact.

Proposition 9.4 (Five lemma). *In the map of exact sequences above,*

- *If f_0 is injective and f_1 and f_3 are surjective, then f_2 is surjective.*
- *If f_4 is surjective and f_3 and f_1 are injective, then f_2 is injective.*

Very commonly one knows that f_0, f_1, f_3, and f_4 are all isomorphisms, and concludes that f_2 is also an isomorphism. For example:

Corollary 9.5. *Let*

$$\begin{tikzcd}
0 \arrow[r] & A'_* \arrow[r] \arrow[d, "f"] & B'_* \arrow[r] \arrow[d, "g"] & C'_* \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_* \arrow[r] & B_* \arrow[r] & C_* \arrow[r] & 0
\end{tikzcd}$$

be a map of short exact sequences of chain complexes. If two of the three maps induced in homology by f, g, and h are isomorphisms, then so is the third.

Here's an application.

Proposition 9.6. *Let $(A, X) \to (B, Y)$ be a map of pairs, and assume that an two of $A \to B$, $X \to Y$, and $(X, A) \to (Y, B)$ induce isomorphims in homology. Then the third one does as well.*

Proof. Just apply the five lemma to the map between the two homology long exact sequences. □

10 Excision and applications

We have found two general properties of singular homology: homotopy invariance and the long exact sequence of a pair. We also claimed that $H_*(X, A)$ "depends only on $X - A$." You have to be careful about this. The following definition gives conditions that will capture the sense in which the relative homology of a pair (X, A) depends only on the complement of A in X.

Definition 10.1. A triple (X, A, U) where $U \subseteq A \subseteq X$, is *excisive* if $\overline{U} \subseteq \mathrm{Int}(A)$. The inclusion $(X - U, A - U) \subseteq (X, A)$ is then called an *excision*.

Theorem 10.2. *An excision induces an isomorphism in homology,*

$$H_*(X - U, A - U) \xrightarrow{\cong} H_*(X, A) .$$

So you can cut out the bits of the interior of A without changing the relative homology. The proof will take us a couple of days. Before we give applications, let me pose a different way to interpret the motto "$H_*(X, A)$ depends only on $X - A$." Collapsing the subspace A to a point gives us a map of pairs

$$(X, A) \to (X/A, *) .$$

When does this map induce an isomorphism in homology? Excision has the following consequence.

Corollary 10.3. *Assume that there is a subspace B of X such that* (1) $\overline{A} \subseteq \mathrm{Int}B$ *and* (2) $A \to B$ *is a deformation retract. Then*

$$H_*(X, A) \to H_*(X/A, *)$$

is an isomorphism.

Proof. The diagram of pairs

$$\begin{tikzcd}
(X,A) \arrow[r, "i"] \arrow[d] & (X,B) \arrow[d] & (X-A, B-A) \arrow[l, "j"'] \arrow[d, "k"] \\
(X/A, *) \arrow[r, "\bar\imath"] & (X/A, B/A) & (X/A - *, B/A - *) \arrow[l, "\bar\jmath"']
\end{tikzcd}$$

commutes. We want the left vertical to be a homology isomorphism, and will show that the rest of the perimeter consists of homology isomorphisms. The map k is a homeomorphism of pairs while j is an excision by assumption (1). The map i induces an isomorphism in homology by assumption (2), the long exact sequences, and the five-lemma. Since I is a compact Hausdorff space, the map $B \times I \to B/A \times I$ is again a quotient map, so the deformation $B \times I \to B$, that restricts to the constant deformation on A, descends to show that $* \to B/A$ is a deformation retract. So the map $\bar\imath$ is also a homology isomorphism. Finally, $\overline{*} \subseteq \operatorname{Int}(B/A)$ in X/A, by definition of the quotient topology, so $\bar\jmath$ induces an isomorphism by excision. □

Now what are some consequences? For a start, we'll finally get around to computing the homology of the sphere. It happens simultaneously with a computation of $H^*(D^n, S^{n-1})$. (Note that $S^{-1} = \varnothing$.) To describe generators, for each $n \geq 0$ pick a homeomorphism

$$(\Delta^n, \partial\Delta^n) \to (D^n, S^{n-1}),$$

and write

$$\iota_n \in S_n(D^n, S^{n-1})$$

for the corresponding relative n-chain.

Proposition 10.4. *Let $* \in S^{n-1}$ be any point (for $n > 0$).*

$$H_q(S^n) = \begin{cases} \mathbf{Z} = \langle[\partial\iota_{n+1}]\rangle & \text{if} \quad q = n > 0 \\ \mathbf{Z} = \langle[c_*^0]\rangle & \text{if} \quad q = 0, n > 0 \\ \mathbf{Z} \oplus \mathbf{Z} = \langle[c_*^0], [\partial\iota_1]\rangle & \text{if} \quad q = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_q(D^n, S^{n-1}) = \begin{cases} \mathbf{Z} = \langle[\iota_n]\rangle & \text{if} \quad q = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. The division into cases for $H_q(S^n)$ can be eased by employing reduced homology. Then the claim is merely that for $n \geq 0$

$$\widetilde{H}_q(S^{n-1}) = \begin{cases} \mathbf{Z} & \text{if} \quad q = n-1 \\ 0 & \text{if} \quad q \neq n-1 \end{cases}$$

and the map

$$\partial : H_q(D^n, S^{n-1}) \to \widetilde{H}_{q-1}(S^{n-1})$$

is an isomorphism. The second statement follows from the long exact sequence in reduced homology together with the fact that $\widetilde{H}_*(D^n) = 0$ since D^n is contractible. The first uses induction and pair of isomorphisms

$$\widetilde{H}_{q-1}(S^{n-1}) \xleftarrow{\cong} H_q(D^n, S^{n-1}) \xrightarrow{\cong} H_q(D^n/S^{n-1}, *)$$

since $D^n/S^{n-1} \cong S^n$. The right hand arrow is an isomorphism since S^{n-1} is a deformation retract of a neighborhood in D^n. □

Why should you care about this complicated homology calculation?

Corollary 10.5. *If $m \neq n$, then S^m and S^n are not homotopy equivalent.*

Proof. Their homology groups are not isomorphic. □

Corollary 10.6. *If $m \neq n$, then $\mathbf{R}^m$ and $\mathbf{R}^n$ are not homeomorphic.*

Proof. If m or n is zero, this is clear, so let $m, n > 0$. Assume we have a homeomorphism $f : \mathbf{R}^m \to \mathbf{R}^n$. This restricts to a homeomorphism $\mathbf{R}^m - \{0\} \to \mathbf{R}^n - \{0\}$. But these are homotopy equivalent to spheres, of different dimension. □

Theorem 10.7 (Brouwer fixed-point theorem)**.** *If $f : D^n \to D^n$ is continuous, then there is some point $x \in D^n$ such that $f(x) = x$.*

Proof. Suppose not. Then you can draw a ray from x through $f(x)$. It meets the boundary of D^n at a point $g(x) \in S^{n-1}$. Check that g is continuous. If x is on the boundary, then $x = g(x)$, so g is a null-homotopy of the identity map on S^{n-1}. This is inconsistent with our computation because the identity map induces the identity map on $H_{n-1}(S^{n-1}) \cong \mathbf{Z}$. □

picture needed

Our computation of the homology of a sphere also implies that there are many non-homotopic self-maps of S^n, for any $n \geq 1$. We will distinguish them by means of the "degree": A map $f : S^n \to S^n$ induces an endomorphism of the infinite cyclic group $H_n(S^n)$. Any endomorphism of an infinite cyclic group is given by multiplication by an integer. This integer is well defined (independent of a choice of basis), and any integer occurs. Thus $\mathrm{End}(\mathbf{Z}) = \mathbf{Z}_\times$, the monoid of integers under multiplication. The homotopy classes of self-maps of S^n also form a monoid, under composition, and:

Theorem 10.8. *Let $n \geq 1$. The degree map provides us with a surjective monoid homomorphism*

$$\deg : [S^n, S^n] \to \mathbf{Z}_\times \,.$$

Proof. Degree is multiplicative by functoriality of homology.

Construction of a map of degree k: If $n = 1$, this is just the winding number; an example is given by regardeing S^1 as unit complex numbers and sending z to z^k. The proof that this has degree k is an exercise.

Suppose we've constructed a map $f_k : S^{n-1} \to S^{n-1}$ of degree k. Extend it to a map $\overline{f}_k : D^n \to D^n$ by defining $\overline{f}_k(tx) = tf_k(x)$ for $t \in [0,1]$. We may then collapse the sphere to a point and identify the quotient with S^n. This gives us a new map $g_k : S^n \to S^n$ making the diagram below commute.

$$\begin{tikzcd}
H_{n-1}(S^{n-1}) \arrow[d, "f_{k*}"] & H_n(D^n, S^{n-1}) \arrow[l] \arrow[r] \arrow[d] & H_n(S^n) \arrow[d, "g_{k*}"] \\
H_{n-1}(S^{n-1}) & H_n(D^n, S^{n-1}) \arrow[l] \arrow[r] & H_n(S^n)
\end{tikzcd}$$

The horizontal maps are isomorphisms, so $\deg g_k = k$ as well. □

We will see (in 18.906) that this map is in fact an isomorphism.

11 The Eilenberg Steenrod axioms and the locality principle

Before we proceed to prove the excision theorem, let's review the properties of homology theory as we have developed them. They are captured by a set of axioms, due to Sammy Eilenberg and Norman Steenrod.

Definition 11.1. A *homology theory* (on **Top**) is:

- a sequence of functors $h_n : \mathbf{Top}_2 \to \mathbf{Ab}$ for all $n \in \mathbf{Z}$ and
- a sequence of natural transformations $\partial : h_n(X, A) \to h_{n-1}(A, \varnothing)$

such that:

- if $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $f_{0,*} = f_{1,*} : h_n(X, A) \to h_n(Y, B)$.
- excisions induce isomorphisms.
- for any pair (X, A), the sequence
$$\cdots \to h_{q+1}(X, A) \xrightarrow{\partial} h_q(A) \to h_q(X) \to h_q(X, A) \xrightarrow{\partial} \cdots$$
is exact, where we have written $h_q(X)$ for $h_q(X, \varnothing)$.
- (the dimension axiom): $h_n(*)$ is nonzero only in dimension zero.

We add the following "Milnor axiom" to our definition. To state it, let I be a set and suppose that for each $i \in I$ we have a space X_i. We can form their disjoint union or *coproduct* $\coprod X_i$. The inclusion maps $X_i \to \coprod X_i$ induce maps $h_n(X_i) \to h_n(\coprod X_i)$, and these in turn induce a map from the direct sum, or coproduct.

$$\alpha : \bigoplus_i h_n(X_i) \to h_n\left(\coprod_{i \in I} X_i\right).$$

Then:

- The map α is an isomorphism for all n.

Ordinary singular homology satisfies these, with $h_0(*) = \mathbf{Z}$. We will soon add "coefficents" to homology, producing a homology theory whose value on a point is any prescribed abelian group. In later developments, it emerges that the dimension axiom is rather like the parallel postulate in Euclidean geometry: it's "obvious," but, as it turns out, the remaining axioms accomodate extremely interesting alternatives, in which $h_n(*)$ is nonzero for infinitely many values of n (both positive and negative).

Excision is a statement that homology is "localizable." To make this precise, we need some definitions.

Definition 11.2. Let X be a topological space. A family $\mathcal{A}$ of subsets of X is a *cover* if X is the union of the interiors of elements of $\mathcal{A}$.

Definition 11.3. Let $\mathcal{A}$ be a cover of X. An n-simplex σ is $\mathcal{A}$-small if there is $A \in \mathcal{A}$ such that the image of σ is entirely in A.

Notice that if $\sigma : \Delta^n \to X$ is $\mathcal{A}$-small, then so is $d^i\sigma$; in fact, for any simplicial operator ϕ, $\phi^*\sigma$ is again $\mathcal{A}$-small. Let's denote by $\mathrm{Sin}_*^{\mathcal{A}}(X)$ the graded set of $\mathcal{A}$-small simplices. This us a sub-simplicial set of $\mathrm{Sin}_*(X)$. Applying the free abelian group functor, we get the subcomplex

$$S_*^{\mathcal{A}}(X)$$

of $\mathcal{A}$-*small simplices.* Write $H_*^{\mathcal{A}}(X)$ for its homology.

Theorem 11.4. *The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ is a chain homotopy equivalence, so $H_*^{\mathcal{A}}(X) \to H_*(X)$ is an isomorphism.*

This will take a little time to prove. Let's see right now how it implies excision.

Suppose $X \supset A \supset U$ is excisive, so that $\overline{U} \subseteq \mathrm{Int}A$, or $\mathrm{Int}(X - U) \cup \mathrm{Int}A = X$. This if we let $B = X - U$, then $\mathcal{A} = \{A, B\}$ is a cover of X. Rewriting in terms of B,

$$(X - U, A - U) = (B, A \cap B)\,,$$

so we aim to show that

$$S_*(B, A \cap B) \to S_*(X, A)$$

induces an isomorphism in homology. We have the following diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r] \arrow[d, "="] & S_*^{\mathcal{A}}(X) \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X)/S_*(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

The middle vertical induces an isomorphism in homology by the locality principle, so the homology long exact sequences combine with the five-lemma to show that the right hand vertical is also a homology isomorphism. But

$$S_n^{\mathcal{A}}(X) = S_n(A) + S_n(B) \subseteq S_n(X)$$

and a simple result about abelian groups provides an isomorphism

$$\frac{S_n(B)}{S_n(A \cap B)} = \frac{S_n(B)}{S_n(A) \cap S_n(B)} \xrightarrow{\cong} \frac{S_n(A) + S_n(B)}{S_n(A)} = \frac{S_n^{\mathcal{A}}(X)}{S_n(A)},$$

so excision follows.

This case of a cover with two elements leads to another expression of excision, known as the "Mayer-Vietoris sequence." In describing it we will use the following notation for the various inclusion.

$$\begin{tikzcd} A \cap B \arrow[r, "j_1"] \arrow[d, "j_2"] & A \arrow[d, "i_1"] \\ B \arrow[r, "i_2"'] & X \end{tikzcd}$$

Theorem 11.5 (Mayer-Vietoris)**.** *Assume that $\mathcal{A} = \{A, B\}$ is a cover of X. There are natural maps $\partial : H_n(X) \to H_{n-1}(A \cap B)$ such that the sequence*

$$\begin{tikzcd} & \cdots \arrow[r, "\beta"] & H_{n+1}(X) \arrow[dll, "\partial"'] \\ H_n(A \cap B) \arrow[r, "\alpha"] & H_n(A) \oplus H_n(B) \arrow[r, "\beta"] & H_n(X) \arrow[dll, "\partial"'] \\ H_{n-1}(A \cap B) \arrow[r, "\alpha"] & \cdots & \end{tikzcd}$$

is exact, where

$$\alpha = \begin{bmatrix} j_{1*} \\ -j_{2*} \end{bmatrix}, \quad \beta = [\, i_{1*} \quad i_{2*} \,].$$

Proof. This is the homology long exact sequence associated to the short exact sequence of chain complexes

$$0 \to S_*(A \cap B) \xrightarrow{\alpha} S_*(A) \oplus S_*(B) \xrightarrow{\beta} S_*^{\mathcal{A}}(X) \to 0,$$

combined with the locality principle. □

12 Subdivision

We will begin the proof of the locality principle today, and finish it in the next lecture. The key is a process of subdivision of singular simplices. It will use the "cone construction" $b*$ from Lecture 5. The cone construction dealt with a region X in Euclidean space, star-shaped with respect to $b \in X$, and gave a chain-homotopy between the identity and the "constant map" on $S_*(X)$:

$$db * + b * d = 1 - \eta\epsilon$$

where $\epsilon : S_*(X) \to \mathbf{Z}$ is the augmentation and $\eta : \mathbf{Z} \to S_*(X)$ sends 1 to the constant 0-chain c_b^0.

Let's see how the cone construction can be used to "subdivide" an "affine simplex." An *affine simplex* is the convex hull of a finite set of points in Euclidean space. To make this non-degenerate, assume that the points $a_0, a_1, \ldots, a_n$, have the property that $\{a_1 - a_0, \ldots, a_n - a_0\}$ is linearly independent. The *barycenter* of this simplex is the center of mass of the vertices,

$$b = \frac{1}{n+1} \sum a_i \,.$$

Start with $n = 1$. To subdivide a 1-simplex, just cut it in half. For the 2-simplex, look at the subdivision of each face, and form the cone of them with the barycenter of the 2-simplex. This gives us a decomposition of the 2-simplex into six sub-simplices.

We want to formalize this process, and extend it to singular simplices (using naturality, of course). Define a natural transformation

$$\$: S_n(X) \to S_n(X)$$

by defining it on standard n-simplex, namely by specifying what $\$(\iota_n)$ is where $\iota_n : \Delta^n \to \Delta^n$ is the universal n-simplex, and then extending by naturality:

$$\$(\sigma) = \sigma_* \$(\iota_n) \,.$$

Here's the definition. When $n = 0$, define \$ to be the identity; i.e., $\$\iota_0 = \iota_0$. For $n > 0$, define

$$\$\iota_n := b_n * \$d\iota_n$$

where b_n is the barycenter of Δ^n. This makes a *lot* of sense if you draw out a picture, and it's a very clever definition that captures the geometry we described. Here's what we'll prove.

Proposition 12.1. *\$ is a natural chain map $S_*(X) \to S_*(X)$ that is naturally chain-homotopic to the identity.*

Proof. Let's try to prove that it's a chain map. We'll use induction on n. It's enough to show that $d\$\iota_n = \$d\iota_n$, because then, for any n-simplex σ,

$$d\$\sigma = d\$\sigma_*\iota_n = \sigma_* d\$\iota_n = \sigma_*\$d\iota_n = \$d\sigma_*\iota_n = \$d\sigma \,.$$

Dimension zero is easy: since $S_{-1} = 0$, $d\$\iota_0$ and $\$d\iota_0$ are both zero and hence equal.

For $n \geq 1$, we want to compute $d\$\iota_n$. This is:

$$\begin{aligned} d\$\iota_n &= d(b_n * \$d\iota_n) \\ &= (1 - \eta_b\epsilon - b_n * d)(\$d\iota_n) \end{aligned}$$

We'll use induction on n. What happens when $n = 1$? Well,

$$\eta_b\epsilon\$d\iota_1 = \eta_b\epsilon\$(c_1^0 - c_0^0) = \eta_b\epsilon(c_1^0 - c_0^0) = 0\,,$$

since ϵ takes sums of coefficients. So the $\eta_b\epsilon$ term drops out for any $n \geq 1$. Let's continue, using the inductive hypothesis:

$$\begin{aligned} d\$\iota_n &= (1 - b_n * d)(\$d\iota_n) \\ &= \$d\iota_n - b_n * d\$d\iota_n \\ &= \$d\iota_n - b_n\$d^2\iota_n \\ &= \$d\iota_n \end{aligned}$$

because $d^2 = 0$.

To define the chain homotopy T, we'll just write down a formula and not justify it. We just need to define $T\iota_n$ by naturality. So define:

$$T\iota_n = b_n * (\$\iota_n - \iota_n - Td\iota_n) \in S_{n+1}(\Delta^n)\,.$$

Once again, we're going to check that T is a chain homotopy by induction, and, again, we need to check only on the universal case.

When $n = 0$, the formula gives $T\iota_0 = 0$, so it's true that $dT\iota_0 - Td\iota_0 = \$\iota_0 - \iota_0$. Now let's assume that $dTc - Tdc = \$c - c$ for every $(n-1)$-chain. Let's start by computing $dT\iota_n$:

$$\begin{aligned} dT\iota_n &= d_n(b_n * (\$\iota_n - \iota_n - Td\iota_n)) \\ &= (1 - b_n * d)(\$\iota_n - \iota_n - Td\iota_n) \\ &= \$\iota_n - \iota_n - Td\iota_n - b_n * (d\$\iota_n - d\iota_n - dTd\iota_n) \end{aligned}$$

All we want now is that $b_n * (d\$\iota_n - d\iota_n - dTd\iota_n) = 0$. We can do this using the inductive hypothesis, because $d\iota_n$ is in dimension $n-1$.

$$\begin{aligned} dTd\iota_n &= -Td(d\iota_n) + \$d\iota_n - d\iota_n \\ &= \$d\iota_n - d\iota_n \\ &= d\$\iota_n - d\iota_n\,. \end{aligned}$$

This means that $d\$\iota_n - d\iota_n - dTd\iota_n = 0$, so T is indeed a chain homotopy. □

13 Proof of the Locality Principle

We have constructed the subdivision operator $\$: S_*(X) \to S_*(X)$, with the idea that it will shink chains and by iteration eventually render any chain $\mathcal{A}$-small. Does \$ succeed in making simplices smaller? Let's look first at the affine case. Recall that the "diameter" of a subset X of a metric space is given by

$$\operatorname{diam}(X) = \sup\{d(x,y) : x, y \in X\}\,.$$

Lemma 13.1. *Let σ be an affine n-simplex, and τ a simplex in $\$\sigma$. Then* $\operatorname{diam}(\tau) \le \frac{n}{n+1}\operatorname{diam}(\sigma)$.

Proof. Suppose that the vertices of σ are $v_0, v_1, \dots, v_n$. Let b be the barycenter of σ, and write the vertices of τ as $w_0 = b, w_1, \dots, w_n$. We compute

$$|b - v_i| = \left|\frac{v_0 + \cdots + v_n - (n+1)v_i}{n+1}\right| = \left|\frac{(v_0 - v_i) + (v_1 - v_i) + \cdots + (v_n - v_i)}{n+1}\right|\,.$$

One of the terms in the numerator is zero, so we can continue:

$$|b - v_i| \le \frac{n}{n+1}\max_{i,j}|v_i - v_j| = \frac{n}{n+1}\operatorname{diam}(\sigma)$$

Since $w_i \in \sigma$,

$$|b - w_i| \le \max_i |b - v_i| \le \frac{n}{n+1}\operatorname{diam}(\sigma)\,.$$

For the other cases, well, we use induction:

$$|w_i - w_j| \le \operatorname{diam}(\text{simplex in } \$d\sigma) \le \frac{n-1}{n}\operatorname{diam}(d\sigma) \le \frac{n}{n+1}\operatorname{diam}(\sigma)\,.$$

□

Now let's transfer this calculation to singular simplices in a space X equipped with a cover $\mathcal{A}$.

Lemma 13.2. *For any singular chain c, some iterate of the subdivision operator sends c to an $\mathcal{A}$-small chain.*

Proof. We may assume that c is a single simplex $\sigma : \Delta^n \to X$, because in generaly you just take the largest iterate of \$ needed to send each simplex in c to an $\mathcal{A}$-small chain. We now encounter another of the great virtues of singular homology: we pull $\mathcal{A}$ back to a cover of the standard simplex. Define an open cover $\sigma^{-1}\mathcal{A}$ of Δ^n defined by

$$\mathcal{U} := \{\sigma^{-1}(\operatorname{Int}(A)) : A \in \mathcal{A}\}\,.$$

The space Δ^n is a compact Hausdorff space, and so is subject to the Lebsegue covering lemma, which we apply to the open cover $\{\operatorname{Int}B : B \in \sigma^{-1}\mathcal{A}\}$.

Lemma 13.3 (Lebsegue covering lemma). *Let M be a compact metric space, and let $\mathfrak{U}$ be an open cover. Then there is $\epsilon > 0$ such that for all $x \in M$, $B_\epsilon(x) \subseteq U$ for some $U \in \mathfrak{U}$.*

□

To apply this, we will have to understand iterates of the subdivision operator.

Lemma 13.4. *For any $k \geq 1$, $\$^k \sim 1 : S_*(X) \to S_*(X)$.*

Proof. We construct T_k such that $dT_k + T_k d = \$^k - 1$. To begin, we take $T_1 = T$, since $dT + Td = \$ - 1$. Let's apply $\$$ to this equation. We get $\$dT + \$Td = \$^2 - \$$.Sum up these two equations to get

$$dT + Td + \$dT + \$Td = \$^2 - 1\,,$$

which simplifies to

$$d(\$ + 1)T + (\$ + 1)Td = \$^2 - 1$$

since $\$d = d\$$.

So define $T_2 = (\$ + 1)T$, and continuing, you see that we can define

$$T_k = (\$^{k-1} + \$^{k-2} + \cdots + 1)T\,.$$

□

We are now in position to prove the Locality Principle, which we recall:

Theorem 13.5. *Let $\mathcal{A}$ be a cover of a space X. The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ is a chain homotopy equivalence, so $H_*^{\mathcal{A}}(X) \to H_*(X)$ is an isomorphism.*

Proof. To prove surjectivity let c be an n-cycle in X. We want to find an $\mathcal{A}$-small n-cycle that is homologous to c. There's only one thing to do. Pick k such that $\$^k c$ is $\mathcal{A}$-small. This is a cycle because because $\$^k$ is a chain map. I want to compare this new cycle with c. That's what the chain homotopy T_k is designed for:

$$\$^k c - c = dT_k c + T_k dc = dT_k c$$

since c is a cycle. So $\$^k c$ and c are homologous.

Now for injectivity. Suppose c is a cycle in $S_n^{\mathcal{A}}(X)$ such that $c = db$ for some $b \in S_{n+1}(X)$. We want c to be a boundary of an $\mathcal{A}$-small chain. Use the chain homotopy T_k again: Suppose that k is such that $\$^k c$ is $\mathcal{A}$-small. Compute:

$$d\$^k b - c = d(\$^k - 1)b = d(dT_k + T_k d)b = dT_k c$$

so

$$c = d\$^k b - dT_k c = d(\$^k b - T_k c)\,.$$

Now, $\$^k b$ is $\mathcal{A}$-small, by choice of k. Is $T_k c$ also $\mathcal{A}$-small? I claim that it is. Why? It is enough to show that $T_k\sigma$ is $\mathcal{A}$-small if σ is. We know that $\sigma = \sigma_* \iota_n$. Because σ is $\mathcal{A}$-small, we know that $\sigma : \Delta^n \to X$ is the composition $i_*\overline{\sigma}$ where $\overline{\sigma} : \Delta^n \to A$ and $i : A \to X$ is the inclusion of some $A \in \mathcal{A}$. By naturality, then, $T_k\sigma = T_k i_*\overline{\sigma} = i_* T_k \overline{\sigma}$, which certainly is $\mathcal{A}$-small. □

This completes the proof of the Eilenberg Mac Lane axioms for singular homology. In the next chapter, we will develop a variety of practical tools, using these axioms to compute the singular homology of many spaces.

14 CW-complexes

There are various ways to model geometrically interesting spaces. Manifolds provide one important model, well suited to analysis. Another model, one we have not talked about, is given by simplicial complexes. It's very combinatorial, and constructing a simplicial complex model for a given space involves making a lot of choices that are combinatorial rather than topological in character. A more flexible model, one more closely reflecting topological information, is given by the theory of CW-complexes.

In building up a space as a CW-complex, we will successively "glue" cells onto what has been already built. It's a general construction.

Suppose we have a pair (B, A), and a map $f : A \to X$. Define a space $X \cup_f B$ (or $X \cup_A B$) in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ B \arrow[r] & X \cup_f B \end{tikzcd}$$

by

$$X \cup_f B = X \sqcup B / \sim$$

where the equivalence relation is generated by requiring that $a \sim f(a)$ for all $a \in A$. We say that we have "attached B to X along f (or along A)."

There are two kinds of equivalence classes in $X \cup_f B$: (1) singletons containing elements of $B - A$, and (2) $\{x\} \sqcup f^{-1}(x)$ for $x \in X$. The topology on $X \cup_f B$ is the quotient topology, and is characterized by a universal property: any solid-arrow commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, "j"] \arrow[ddr, "\bar{j}"] & \\ B \arrow[r] \arrow[drr, "\bar{g}"'] & X \cup_f B \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

can be filled in. It's a "push-out."

Example 14.1. If $X = *$, then $* \cup_f B = B/A$.

Example 14.2. If $A = \varnothing$, then $X \cup_f B$ is the coproduct $X \sqcup B$.

Example 14.3. If both,

$$B/\varnothing = * \cup_\varnothing B = * \sqcup B\,.$$

For example, $\varnothing/\varnothing = *$. This is creation from nothing. We won't get into the religious ramifications.

Example 14.4 (Attaching a cell)**.** A basic collection of pairs of spaces is given by the disks relative to their boundaries: (D^n, S^{n-1}). (Recall that $S^{-1} = \varnothing$.) In this context, D^n is called an "n-cell," and a map $f : S^{n-1} \to X$ allows us to attach an n-cell to X, to form

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ D^n \arrow[r] & X \cup_f D^n \end{tikzcd}$$

You might want to generalize this a little bit, and attach a bunch of n-cells all at once:

$$\begin{tikzcd} \coprod_{\alpha\in A} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ \coprod_{\alpha\in A} D^n_\alpha \arrow[r] & X \cup_f \coprod_{\alpha\in A} D^n_\alpha \end{tikzcd}$$

What are some examples? When $n = 0$, $(D^0, S^{-1}) = (*, \varnothing)$, so you are just adding a discrete set to X:

$$X \cup_f \coprod_{\alpha\in A} D^0 = X \sqcup A$$

More interesting:

$$\begin{tikzcd} S^0 \sqcup S^0 \arrow[r, "f"] \arrow[d, hook] & * \arrow[d] \\ D^1 \sqcup D^1 \arrow[r] & * \cup_f (D^1 \sqcup D^1) \end{tikzcd}$$

Again there's just one choice for f, and $* \cup_f (D^1 \sqcup D^1)$ is a figure 8, because you start with two 1-disks and identify the four boundary points together. Let me write $S^1 \vee S^1$ for this space. We can go on and attach a single 2-cell to manufacture a torus. Think of the figure 8 as the perimeter of a square with opposite sides identified. Then the inside of the square is a 2-cell, attached to the perimeter by a map I'll denote by $aba^{-1}b^{-1}$:

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, hook] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r] & (S^1 \vee S^1) \cup_f D^2 = T^2 . \end{tikzcd}$$

This example illuminates the following definition.

Definition 14.5. A *CW-complex* is a space X equipped with a sequence of subspaces

$$\varnothing = \mathrm{Sk}_{-1}X \subseteq \mathrm{Sk}_0 X \subseteq \mathrm{Sk}_1 X \subseteq \cdots \subseteq X$$

such that

- X is the union of the $\mathrm{Sk}_n X$'s, and
- for all n, there is a pushout diagram like this:

$$\begin{tikzcd} \coprod_{\alpha\in A_n} S^{n-1}_\alpha \arrow[r, "f_n"] \arrow[d, hook] & \mathrm{Sk}_{n-1}X \arrow[d] \\ \coprod_{\alpha\in A_n} D^n_\alpha \arrow[r, "g_n"] & \mathrm{Sk}_n X \end{tikzcd} .$$

The subspace $\mathrm{Sk}_n X$ is the *n-skeleton* of X. Sometimes it's convenent to use the alternate notation X_n for the n-skeleton. The first condition is intended topologically, so that a subset of X is open if and only if its intersection with each $\mathrm{Sk}_n X$ is open; or, equivalently, a map $f : X \to Y$ is continuous if and only if its restriction to each $\mathrm{Sk}_n X$ is continuous. The maps f_n are the *attaching maps* and the maps g_n are*characteristic maps.*

Example 14.6. We just constructed the torus as a CW complex with $\mathrm{Sk}_0 T^2 = *, \mathrm{Sk}_1 T^2 = S^1 \vee S^1$, and $\mathrm{Sk}_2 T^2 = T^2$.

Definition 14.7. A CW-complex is *finite-dimensional* if $\mathrm{Sk}_n X = X$ for some n; of *finite type* if each A_n is finite, i.e., finitely many cell in each dimension; and *finite* if it's finite-dimensional and of finite type.

The *dimension* of a CW complex is the largest n for which there are n cells. This is not obviously a topological invariant, but, have no fear, it turns out that it is.

In "CW," the "C" is for cell, and the "W" is for weak, because of the topology on a CW-complex. This definition is due to J. H. C. Whitehead. Here are a couple of important facts about them.

Theorem 14.8. *Any CW-complex is Hausdorff, and it's compact if and only if it's finite.*
Any compact smooth manifold admits a CW structure.

15 CW-complexes II

We have a few more general things to say about CW complexes.

Suppose X is a CW complex, with skeleton filtration $\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ and cell structure

$$\begin{tikzcd} \coprod_{\alpha\in A_n} S^{n-1}_\alpha \arrow[r, "f_n"] \arrow[d, hook] & X_{n-1} \arrow[d] \\ \coprod_{\alpha\in A_n} D^n_\alpha \arrow[r, "g_n"] & X_n \end{tikzcd} .$$

In each case, the boundary of a cell gets identified with part of the previous skeleton, but the "interior"

$$\mathrm{Int} D^n = \{x \in D^n : |x| < 1\}$$

does not. (Note that $\mathrm{Int}D^0 = D^0$.) Thus as sets – ignoring the topology –

$$X = \coprod_{n\geq 0}\coprod_{\alpha\in A_n} \mathrm{Int}(D^n_\alpha)\,.$$

The subsets $\mathrm{Int}D^n_\alpha$ are called "open n-cells," despite the fact that they not generally open in the topology on X, and (except when $n = 0$) they are not homeomorphic to compact disks.

Definition 15.1. Let X be a CW-complex with a cell structure $\{g_\alpha : D^n_\alpha \to X_n | \alpha \in A_n\}$. A *subcomplex* is a subspace $Y \subseteq X$ such that for all n, there is a subset B_n of A_n such that $Y_n = Y \cap X_n$ provides Y with a CW-structure with characteristic maps $\{g_\beta | \beta \in B_n\}$.

Example 15.2. $\mathrm{Sk}_n X \subseteq X$ is a subcomplex.

Proposition 15.3 (Bredon, p. 196)**.** *Let X be a CW-complex with a chosen cell structure. Any compact subspace of X lies in some finite subcomplex.*

Remark 15.4. For fixed cell structures, unions and intersections of subcomplexes are subcomplexes.

The n-sphere S^n (for $n > 0$) admits a very simple CW structure: Let $* = \mathrm{Sk}_0(S^n) = \mathrm{Sk}_1(S^n) = \cdots = \mathrm{Sk}_{n-1}(S^n)$, and attach an n-cell using the unique map $S^{n-1} \to *$. This is a minimal CW structure – you need at least two cells to build S^n.

This is great – much simpler than the simplest construction of S^n as a simplicial complex – but it is not ideal for all applications. Here's another CW-structure on S^n. Regard $S^n \subseteq \mathbf{R}^{n+1}$, filter the Euclidean space by leading subspaces

$$\mathbf{R}^k = \langle e_1, \ldots, e_k \rangle\,.$$

and define

$$\mathrm{Sk}_k S^n = S^n \cap \mathbf{R}^{k+1} = S^k\,.$$

picture needed

Now there are two k-cells for each k with $0 \leq k \leq n$, given by the two hemispheres of S^k. For each k there are two characteristic maps,

$$u, \ell : D^k \to S^k$$

defining the upper and lower hemispheres:

$$u(x) = (x, \sqrt{1-|x|^2})\,, \quad \ell(x) = (x, -\sqrt{1-|x|^2})\,.$$

Note that if $|x| = 1$ then $|u(x)| = |\ell(x)| = 1$, so each characteristic map restricts on the boundary to a map to S^{k-1}, and serves as an attaching map. This cell structure has the advantage that S^{n-1} is a subcomplex of S^n.

The case $n = \infty$ is allowed here. Then $\mathbf{R}^\infty$ denotes the countably infinite dimensional inner product space that is the topological union of the leading subspaces $\mathbf{R}^n$. The CW-complex S^∞ is of finite type but not finite dimensional. It has the following interesting property. We know that S^n is not contractible (because the identity map and a constant map have different behavior in homology), but:

Proposition 15.5. S^∞ *is contractible.*

Proof. This is an example of a "swindle," making use of infinite dimensionality. Let $T : \mathbf{R}^\infty \to \mathbf{R}^\infty$ send $(x_1, x_2, \ldots)$ to $(0, x_1, x_2, \ldots)$. This sends S^∞ to itself. The location of the leading nonzero entry is different for x and Tx, so the line segment joining x to Tx doesn't pass through the origin. Therefore

$$x \mapsto \frac{tx + (1-t)Tx}{|tx + (1-t)Tx|}$$

provides a homotopy $1 \sim T$. On the other hand, T is homotopic to the constant map with value $(1, 0, 0, \ldots)$, again by an affine homotopy. $\square$

This "inefficient" CW structure on S^n has a second advantage: it's "equivariant" with respect to the antipodal involution. This provides us with a CW structure on the orbit space for this action.

Recall that $\mathbf{RP}^k = S^k/\sim$ where $x \sim -x$. The quotient map $S^k \to \mathbf{RP}^k$ is a double cover, identifying upper and lower hemispheres. The inclusion of one sphere in the next is compatible with this equivalence relation, and gives us "linear" embeddings $\mathbf{RP}^{k-1} \subseteq \mathbf{RP}^k$. This suggests that

$$\varnothing \subseteq \mathbf{RP}^0 \subseteq \mathbf{RP}^1 \subseteq \cdots \subseteq \mathbf{RP}^n$$

might serve as a CW filtration. Indeed, for each k,

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & D^k \arrow[d, "u"] \\ \mathbf{RP}^{k-1} \arrow[r] & \mathbf{RP}^k \end{tikzcd}$$

is a pushout: A line in $\mathbf{R}^{k+1}$ either lies in $\mathbf{R}^k$ or is determined by a unique point in the upper hemisphere of S^k.

16 Homology of CW-complexes

The skeleton filtration of a CW complex leads to a long exact sequence in homology, showing that the relative homology $H_*(X_k, X_{k-1})$ controls how the homology changes when you pass from X_{k-1} to X_k. What is this relative homology? If we pick a set of attaching maps, we get the following diagram.

$$\begin{tikzcd} \coprod_\alpha S^{k-1} \arrow[r, hook] \arrow[d, "f"] & \coprod_\alpha D^k_\alpha \arrow[r] \arrow[d] & \bigvee_\alpha S^k_\alpha \arrow[d, dashed] \\ X_{k-1} \arrow[r, hook] & X_k \cup_f B \arrow[r] & X_k/X_{k-1} \end{tikzcd}$$

where $\bigvee$ is the wedge sum (disjoint union with all basepoints identified): $\bigvee_\alpha S^k_\alpha$ is a bouquet of spheres. The dotted map exists and is easily seen to be a homeomorphism.

Luckily, the inclusion $X_{k-1} \subseteq X_k$ satisfies what's needed to conclude that

$$H_q(X_k, X_{k-1}) \to H_q(X_k/X_{k-1}, *)$$

is an isomorphism. After all, X_{k-1} is a deformation retract of the space you get from X_k by deleting the center of each k-cell.

We know $H_q(X_k/X_{k-1}, *)$ very well:

$$\widetilde{H}_q(\bigvee_{\alpha\in A_k} S^k_\alpha) \cong \begin{cases} \mathbf{Z}[A_k] & q = k \\ 0 & q \neq k \end{cases}.$$

Lesson: The relative homology $H_q(X_k, X_{k-1})$ keeps track of the k-cells of X.

Definition 16.1. The group of *cellular n-chains* in a CW complex X is

$$C_k(X) := H_k(X_k, X_{k-1}) = \mathbf{Z}[A_k]\,.$$

If we put the fact that $H_q(X_k, X_{k-1}) = 0$ for $q \neq k, k+1$ into the homology long exact sequence of the pair, we find first that

$$H_q(X_{k-1}) \xrightarrow{\cong} H_q(X_k)\,\text{for}\,1 \neq k, k-1\,,$$

and then that there is a short exact sequence

$$0 \to H_k(X_k) \to C_k(X) \to H_{k-1}(X_{k-1}) \to 0\,.$$

So if we fix a dimension q, and watch how H_q varies as we move through the skelata of X, we find the following picture. Say $q > 0$. Since X_0 is discrete, $H_q(X_0) = 0$. Then $H_q(X_k)$ continues to be 0 till you get up to X_q. $H_q(X_q$ is a subgroup of the free abelian group $C_k(X)$ and hence is free abelian. Relations may get introduced into it when we pass to X_{q+1}; but thereafter all the maps

$$H_q(X_{q+1}) \to H_q(X_{q+2}) \to \cdots$$

are isomorphisms. All the q-dimensional homology of X is created on X_q, and all the relations in $H_q(X)$ occur by X_{q+1}.

This stable value of $H_q(X_k)$ maps isomorphically to $H_q(X)$, even if X is infinite dimensional. This is because the union of the images of any finite set of singular simplices in X is compact and so lies in a finite subcomplex and in particular lies in a finite skeleton. So any chain in X is the image of a chain in some skeleton. Since $H_q(X_k) \xrightarrow{\cong} H_q(X_{k+1})$ for $k > q$, we find that $H_q(X_q) \to H_q(X)$ is surjective. Similarly, if $c \in S_q(X_k)$ is a boundary in X, then it's a boundary in X_ℓ for some $\ell \geq k$. This shows that the map $H_q(X_{q+1}) \to H_q(X)$ is injective. We summarize:

Lemma 16.2. *Let $q \geq 0$. Then*

$$H_q(X_k) = 0 \text{ for } q < k$$

and

$$H_q(X_k) \xrightarrow{\cong} H_q(X) \text{ for } q > k\,.$$

In particular, $H_q(X) = 0$ if q exceeds the dimension of X.

We have defined the cellular n-chains of a CW complex X,

$$C_n(X) = H_n(X_n, X_{n-1}),$$

and found that it is the free abelian group on the set of n cells. We claim that these abelian groups are related to each other; they form the groups in a chain complex.

What should the boundary of an n-cell be? It is represented by a characteristic map $D^n \to X_n$ whose boundary is the attaching map

$$\alpha : S^{n-1} \to X_{n-1}$$

This is a lot of information, and hard to interpret because X_{n-1} is itself potentially a complicated space. But things get much simpler if I pinch out X_{n-2}. This suggests defining

$$d : C_n(X) = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_n) \to H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}(X).$$

The fact that $d^2 = 0$ is embedded in the following large diagram, in which the two columns and the central row are exact.

$$\begin{tikzcd}
C_{n+1}(X) = H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial_n"] \arrow[dr, "d"] & & 0 = H_{n-1}(X_{n-2}) \arrow[d] \\
H_n(X_n) \arrow[r, tail, "j_n"] \arrow[d] & C_n(X) = H_n(X_n, X_{n-1}) \arrow[r, "\partial_{n-1}"] \arrow[dr, "d"] & H_{n-1}(X_{n-1}) \arrow[d, "j_{n-1}"] \\
H_n(X_{n+1}) \arrow[d] & & C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \\
0 = H_n(X_{n+1}, X_n) & &
\end{tikzcd}$$

Now, $\partial_{n-1} \circ j_n = 0$. So the composite of the diagonals is zero, i.e., $d^2 = 0$, and we have a chain complex! This is the "cellular chain complex" of X.

We should compute the homology of this chain complex, $H_n(C_*(X)) = \ker d / \operatorname{im} d$. Now, $\ker d = \ker(j_{n-1} \circ \partial_{n-1})$. But j_{n-1} is injective, so

$$\ker d = \ker \partial_{n-1} = \operatorname{im} j_n = H_n(X_n).$$

On the other hand

$$\operatorname{im} d = j_n(\operatorname{im} \partial_n) = \operatorname{im} \partial_n \subseteq H_n(X_n).$$

So

$$H_n(C_*(X)) = H_n(X_n)/\operatorname{im} \partial_n = H_n(X_{n+1})$$

by exactness of the left column; but as we know this is exactly $H_n(X)$!

Theorem 16.3. *For a CW complex X, there is an isomorpphism*

$$H_*(C_*(X)) \cong H_*(X)$$

natural with respect to filtration-preserving maps between CW complexes.

This has an immediate and surprisingly useful corollary.

Corollary 16.4. *Suppose that the CW complex X has only even cells – that is, $X_{2k} \hookrightarrow X_{2k+1}$ is an isomorphism. Then $d = 0$ in the cellular chain complex of X, and*

$$H_*(X) \cong C_*(X) .$$

That is, $H_n(X) = 0$ for n odd, is free abelian for all n, and the rank of $H_n(X)$ for n even is the number of n-cells.

Example 16.5. Complex projective space $\mathbf{CP}^n$ has a CW structure in which

$$\mathrm{Sk}_{2k}\mathbf{CP}^n = \mathrm{Sk}_{2k+1}\mathbf{CP}^n = \mathbf{CP}^k .$$

The attaching $S^{2k-1} \to \mathbf{CP}^k$ sends $v \in S^{2k-1} \subseteq \mathbf{C}^n$ to the complex line through v. So

$$H_k(\mathbf{CP}^n) = \begin{cases} \mathbf{Z} & \text{for } 0 \le k \le 2n,\ k \text{ even} \\ 0 & \text{otherwise} . \end{cases}$$

Finally, notice that in our proof of Theorem 16.3 we used only properties contained in the Eilenberg-Steenrod axioms. As a result, any construction of a homology theory satisfying the Eilenberg-Steenrod axioms gives you the same values on CW complexes as singular homology.

17 Real projective space

Let's try to compute $H_*(\mathbf{RP}^n)$. This computation will invoke a second way to think of the cellular chain group $C_n(X)$. Each cell has a characteristic map $D^n \to X_n$, and we have the diagram

$$\begin{tikzcd} \coprod(D^n, S^{n-1}) \arrow[r] \arrow[dr] & (X_n, X_{n-1}) \arrow[d] \\ & (\bigvee S^{n-1}, *). \end{tikzcd}$$

We've shown that the vertical map induces an isomorphism in homology, and the diagonal does as well. (For example, $\coprod D^n$ has a CW structure in which the $(n-1)$-skeleton is $\coprod S^{n-1}$.) So

$$H_n(\coprod(D_n, S^{n-1})) \xrightarrow{\cong} C_n(X).$$

We have a CW structure on $\mathbf{RP}^n$ with $\mathrm{Sk}_k(\mathbf{RP}^n) = \mathbf{RP}^k$; there is one k-cell – which we'll denote by e_k – for each k between 0 and n. So the cellular chain complex looks like this:

$$\begin{tikzcd} 0 & C_0(\mathbf{RP}^n) \arrow[l] \arrow[d, equal] & C_1(\mathbf{RP}^n) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] & C_n(\mathbf{RP}^n) \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\ 0 & \mathbf{Z}\langle e^0\rangle \arrow[l] & \mathbf{Z}\langle e^1\rangle \arrow[l, "d=0"'] & \cdots \arrow[l] & \mathbf{Z}\langle e^n\rangle \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

The first differential is zero because we know what $H_0(\mathbf{RP}^n)$ is (it's $\mathbf{Z}$!). The differential in the cellular chain complex is given by the top row in the following commutative diagram.

$$\begin{tikzcd} C_n = H_n(\mathbf{RP}^n, \mathbf{RP}^{n-1}) \arrow[r, "\partial"] & H_{n-1}(\mathbf{RP}^{n-1}) \arrow[r] \arrow[dr] & H_{n-1}(\mathbf{RP}^{n-1}, \mathbf{RP}^{n-2}) = C_{n-1} \arrow[d, "\cong"'] \\ H_n(D^n, S^{n-1}) \arrow[u, "\cong"'] \arrow[r, "\partial", "\cong"'] & H_{n-1}(S^{n-1}) \arrow[u, "\pi_*"'] \arrow[r] & H_{n-1}(D^{n-1}/S^{n-2}, *)\,. \end{tikzcd}$$

The map $\pi : S^{n-1} \to \mathbf{RP}^{n-1}$ is the attaching map of the top cell of RP^n; that is, the double cover. The diagonal composite pinches the subspace $\mathbf{RP}^{n-1}$ to a point. The composite map $S^{n-1} \to D^{n-1}/S^{n-2}$ factors as follows:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{double cover}"] \arrow[dr] & \mathbf{RP}^{n-1} \arrow[r, "\text{pinch}"] & D^{n-1}/S^{n-2} \\ & S^{n-1}/S^{n-2} = S^{n-1} \vee S^{n-1} \arrow[ur] & \end{tikzcd}$$

One of the maps $S^{n-1} \to S^{n-1}$ from the wedge is the identity, and the other map is the antipodal map $\alpha : S^{n-1} \to S^{n-1}$. Write σ for a generator of $H_{n-1}(S^{n-1})$. Then in H_{n-1} we have $\sigma \mapsto (\sigma, \sigma) \mapsto \sigma + \alpha_*\sigma$. So we need to know the degree of the antipodal map on S^{n-1}. The antipodal map reverses all n coordinates in $\mathbf{R}^n$. Each reversal is a reflection, and acts on S^{n-1} by a map of degree -1. So

$$\deg \alpha = (-1)^n\,.$$

Therefore the cellular complex of $\mathbf{RP}^n$ is as follows:

$$\begin{array}{cccccccc} \text{dim} & -1 & 0 & 1 & \cdots & n & n+1 & \cdots \\ & 0 \xleftarrow{0} & \mathbf{Z} \xleftarrow{2} & \mathbf{Z} \xleftarrow{0} & \cdots \xleftarrow{2 \text{ or } 0} & \mathbf{Z} \longleftarrow & 0 \longleftarrow & \cdots \end{array}$$

The homology is then easy to read off.

Proposition 17.1. *The homology of real projective space is as follows.*

$$H_k(\mathbf{RP}^n) = \begin{cases} \mathbf{Z} & k = 0 \textit{ and } k = n \textit{ odd} \\ \mathbf{Z}/2\mathbf{Z} & k \textit{ odd, } 0 < k < n \\ 0 & \textit{otherwise}. \end{cases}$$

Here's a table. Missing entries are 0.

dim	0	1	2	3	4	5	⋯
$\mathbf{RP}^0$	$\mathbf{Z}$						
$\mathbf{RP}^1$	$\mathbf{Z}$	$\mathbf{Z}$					
$\mathbf{RP}^2$	$\mathbf{Z}$	$\mathbf{Z}/2$					
$\mathbf{RP}^3$	$\mathbf{Z}$	$\mathbf{Z}/2$	0	$\mathbf{Z}$			
$\mathbf{RP}^4$	$\mathbf{Z}$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$			
$\mathbf{RP}^5$	$\mathbf{Z}$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	0	$\mathbf{Z}$	

The moral: In real projective space, odd cells create new generators; even cells (except for the zero-cell) create torsion in the previous dimension.

This example illustrates the significance of cellular homology, and, therefore, of singular homology. A CW structure involves attaching maps

$$\coprod S^{n-1} \to \mathrm{Sk}_n X\,.$$

Knowing these, up to homotopy, determines the full homotopy type of the CW complex. Homology does not record all this information. Instead, it records only information about the composite obtaind by pinching out $\mathrm{Sk}_{n-1}X$.

$$\begin{tikzcd} \coprod_{a\in A_n} S^{n-1}_a \arrow[r] \arrow[dr] & \mathrm{Sk}_n X \arrow[d] \\ & \bigvee_{b\in A_{n-1}} S^{n-1}_b\,. \end{tikzcd}$$

In H_{n-1}, this can be identified with a map

$$\partial : \mathbf{Z}[A_n] \to \mathbf{Z}[A_{n-1}]$$

that is none other than the differential in the cellular chain complex.

The moral: homology picks off only the "first order" structure of a CW complex.

On the other hand, we'll see in the next lecture that it does a very good job of that.

18 Euler characteristic, and homology approximation

Theorem 18.1. *Let X be a finite CW-complex with a_n n-cells. Then*

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k a_k$$

depends only on the homotopy type of X; it is independent of the choice of CW structure.

This integer $\chi(X)$ is called the *Euler characteristic* of X. We will prove this theorem by showing that $\chi(X)$ equals a number computed from the homology groups of X, which are themselves homotopy invariants.

We'll need a little bit of information about the structure of finitely generated abelian groups.

Let A be an abelian group. The set of *torsion* elements of A,

$$\mathrm{Tors}(A)\{a \in A : na = 0 \text{ for some } n \neq 0\},$$

is a subgroup of A (since A is commutative). A group is *torsion free* if $\mathrm{Tors}(A) = 0$. For any A the quotient group $A/\mathrm{Tors}(A)$ is torsion free.

For a general abelian group, that's about all you can say. But now assume A is finitely generated. Then $\mathrm{Tors}(A)$ is a finite abelian group and $A/\mathrm{Tors}(A)$ is a finitely generated free abelian group, isomorphic to $\mathbf{Z}^r$ for some integer r called the *rank* of A. Pick elements of A that map to a set of generators of $A/\mathrm{Tors}(A)$, and use them to define a map $A/\mathrm{Tors}A \to A$ splitting the projection map. This shows that if A is finitely generated then

$$A \cong \mathrm{Tors}(A) \oplus \mathbf{Z}^r .$$

A finite abelian group A is necessarily of the form

$$\mathbf{Z}/n_1 \oplus \mathbf{Z}/n_2 \oplus \cdots \oplus \mathbf{Z}/n_t \text{ where } n_1|n_2|\cdots|n_t .$$

These are the "torsion coefficients" of A. They are well defined natural numbers.

Lemma 18.2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated abelian groups. Then*

$$\mathrm{rank}\, A - \mathrm{rank}\, B + \mathrm{rank}\, C = 0 .$$

Theorem 18.3. *Let X be a finite CW complex. Then*

$$\chi(X) = \sum_k (-1)^k \mathrm{rank}\, H_k(X) .$$

Proof. Pick a CW-structure with, say a_k k-cells for each k. We have the cellular chain complex C_*. Write H_*, Z_*, and B_* for the homology, the cycles, and the boundaries, in this chain complex. From the definitions, we have two families of short exact sequences:

$$0 \to Z_k \to C_k \to B_{k-1} \to 0$$

and

$$0 \to B_k \to Z_k \to H_k \to 0\,.$$

Let's use them and facts about rank rewrite the alternating sum is:

$$\begin{aligned}\sum_k (-1)^k a_k &= \sum_k (-1)^k \operatorname{rank}(C_k)\\ &= \sum_k (-1)^k (\operatorname{rank}(Z_k) + \operatorname{rank}(B_{k-1}))\\ &= \sum_k (-1)^k (\operatorname{rank}(B_k) + \operatorname{rank}(H_k) + \operatorname{rank}(B_{k-1}))\end{aligned}$$

The terms $\operatorname{rank} B_k + \operatorname{rank} B_{k-1}$ cancel because it's an alternating sum. This leaves $\sum_k (-1)^k \operatorname{rank} H_k$. But $H_k = H_k^{\text{sing}}(X)$. □

In the early part of the 20th century, "homology groups" were not discussed. It was Emmy Noether who first described things that way. Instead, people worked mainly with the sequence of ranks,

$$\beta_k = \operatorname{rank} H_k(X)\,,$$

which are known (following Poincaré) as the *Betti numbers* of X.

Given a CW-complex X of finite type, can we give a lower bound on the number of k-cells in terms of the homology of X? Let's see. $H_k(X)$ is finitely generated because $C_k(X) \hookleftarrow Z_k(X) \twoheadrightarrow H_k(X)$. Thus

$$H_k(X) = \bigoplus_{i=1}^{t(k)} \mathbf{Z}/n_i(k)\mathbf{Z} \oplus \mathbf{Z}^{r(k)}$$

where the $n_1(k)|\cdots|n_{t(k)}(k)$ are the torsion coefficients of $H_k(X)$ and $r(k)$ is the rank.

The minimal chain complex with $H_k = \mathbf{Z}^r$ and $H_q = 0$ for $q \neq k$ is just the chain complex with 0 everywhere else except for $\mathbf{Z}^r$ in the kth degree. The minimal chain complex *of free abelian groups* with $H_k = \mathbf{Z}/n\mathbf{Z}$ and $H_q = 0$ for $q \neq k$ is the chain complex with 0 everywhere else except in dimensions $k+1$ and k, where we have $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ These small complexes are called "elementary chain complexes."

This implies that a lower bound on the minimal number of k-cells is

$$r(k) + t(k) + t(k-1)\,.$$

The first two terms give generators for H_k, and the last gives relations for H_{k-1}.

These elementary chain complexes can be realized as the reduced cellular chains of CW complexes (at least if $k > 0$). A wedge of r copies of S^k has reduced cellular chains $\mathbf{Z}^r$ in dimension k and 0 in other dimensions. To construct a CW complex with reduced chains $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ in dimensions $k+1$ and k, start with S^k as k-skeleton and attach a $k+1$-cell by a map of degree n. For example, when $k = 1$ and $n = 2$, you have $\mathbf{RP}^2$. These CW complexes are called "Moore spaces."

This maximally efficient construction of a CW complex in a homotopy type can in fact be achieved:

Theorem 18.4 (Wall)**.** *Let X be a simply connected CW-complex of finite type. Then there exists a CW complex Y with $r(k)+t(k)+t(k-1)$ k-cells, for all k, and a homotopy equivalence $Y \to X$.*

We will prove this theorem in 18.906.

The construction of Moore spaces can be generalized:

Proposition 18.5. *For any graded abelian group A_* with $A_k = 0$ for $k \leq 0$, there exists a CW complex X with $\widetilde{H}_*(X) = A_*$.*

Proof. Let A be any abelian group. Pick generators for A. They determine a surjection from a free abelian group F_0. The kernel of that surjection is free, being a subgroup of a free abelian group. Write G_0 for minimal set of generators of F_0, and G_1 for a minimal set of generators for F_1.

Let $k \geq 1$. Define X_k to be the wedge of $|G_0|$ copies of S^k, so $H_k(X_k) = \mathbf{Z}^{|G_0|}$. Now define an attaching map

$$\alpha : \coprod_{b \in G_1} S^k_b \to X_k$$

by specifying it on each summand S^k_a. The generator $a \in G_1 \subseteq F_1$ is given by a linear combination of the generators of F_0, say

$$b = \sum_{i=1}^{s} n_i a_i \,.$$

Now map $S^k \to \bigvee^s S^k$ by pinching $(s-1)$ longitudes to points. Map the ith sphere in the wedge to $S^k_{a_i} \subseteq X_k$ by a map of degree n_i. The map on the summand S^k_a is then the composite of these two maps,

$$S^k_a \to \bigvee_i S^k_i \to \bigvee_a S^k_a$$

Altogether, we get a map α that realizes $F_1 \to F_0$ in H_k. So using it as an attaching map produces a CW complex X with $\widetilde{H}_q(X) = A$ for $q = k$ and 0 otherwise. Write $M(A,k)$ for a CW complex pruduced in this way.

Finally, given a graded abelian group A_*, for the wedge over k of the spaces $M(A,k)$. □

Such a space $M(A,k)$, with $\widetilde{H}_q(M(A,k)) = A$ for $q = k$ and 0 otherwise, is called a *Moore space of type* (A,k). The notation is a bit deceptive, since $M(A,k)$ cannot be made into a functor $\mathbf{Ab} \to \mathrm{Ho}\mathbf{Top}$.

19 Coefficients

As we have seen, abelian groups can be quite complicated, even finitely generated ones. Vector spaces over a field are so much simpler! A vector space is determined up to isomorphism by a single cardinality, its dimension. Wouldn't it be great to have a version of homology that took values in the category of vector spaces over a field?

We can do this, and more. Let R be any commutative ring at all. Instead of forming the free abelian group on $\mathrm{Sin}_*(X)$, we could just as form the free R-module:

$$S_*(X;R) = R\mathrm{Sin}_*(X)$$

This gives, first, a simplicial object in the category of R-modules. Forming the alternating sum of the face maps produces a chain complex *of R-modules*: $S_n(X;R)$ is an R-module for each n, and $d : S_n(X;R) \to S_{n-1}(X;R)$ is an R-module homomorphism. The homology groups are then again R-modules:

$$H_n(X;R) = \frac{\ker(d : S_n(X;R) \to S_{n-1}(X;R))}{\mathrm{im}(d : S_{n+1}(X;R) \to S_n(X;R))}.$$

This is the *singular homology of X with coefficients in the commutative ring R*. It satisfies all the Eilenberg-Steenrod axioms, but

$$H_n(*;R) = \begin{cases} R & \text{for} n = 0 \\ 0 & \text{otherwise}. \end{cases}$$

We could actually have replaced the ring R by any abelian group here, but this will become much clearer after we have the tensor product as a tool.

The rings that are most important in algebraic topology are simple ones: the integers and the prime fields $\mathbf{F}_p$ and $\mathbf{Q}$; typically, a PID.

As an experiment, let's compute $H_*(\mathbf{RP}^n;R)$ for various rings R. Let's start with $R = \mathbf{F}_2$, the field with 2 elements. This is a favorite among algebraic topologists, because using it for coefficients eliminates all sign issues. The cellular chain complex has $S_k = \mathbf{F}_2$ for $0 \le k \le n$, and the differential alternates between multiplication by 2 and by 0. But in $\mathbf{F}_2$, $2 = 0$: so $d = 0$, and the cellular chains coincide with the homology:

$$H_k(\mathbf{RP}^n;\mathbf{F}_2) = \begin{cases} \mathbf{F}_2 & \text{for } 0 \le k \le n \\ 0 & \text{otherwise}. \end{cases}$$

On the other hand, suppose that R is a ring in which 2 is invertible. The universal case is $\mathbf{Z}[1/2]$, but any subring of the rationals containing $1/2$ would do just as well, as would $\mathbf{F}_p$ for p odd. Now the cellular chain complex (in dimensions 0 through n) looks like

$$R \xleftarrow{0} R \xleftarrow{\cong} R \xleftarrow{0} R \xleftarrow{\cong} \cdots \xleftarrow{\cong} R$$

$$R \xleftarrow{0} R \xleftarrow{\cong} R \xleftarrow{0} R \xleftarrow{\cong} \cdots \xleftarrow{0} R$$

for n odd. Therefore

$$H_k(\mathbf{RP}^n;R) = \begin{cases} R & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

for n even, and

$$H_k(\mathbf{RP}^n; R) = \begin{cases} R & \text{for } k = 0 \\ R & \text{for } k = n \\ 0 & \text{otherwise}. \end{cases}$$

You get a much simpler result! Away from 2, even projective spaces look like a point, and odd projective spaces look like a sphere!

I'd like to generalize this process a little bit, and allow coefficients not just in a commutative ring, but more generally in a module M over a commutative ring; in particular, any abelian group. This is most cleanly done using the mechinism of the tensor product. That mechanism will also let us address the following natural question:

Question 19.1. Given $H_*(X;R)$, can we deduce $H_*(X;M)$ for an R-module M?

The answer is called the "universal coefficient theorem". I'll spend a few days developing what we need to talk about this.

20 Tensor product

The category of R-modules is what might be called a "categorical ring," in which addition corresponds to the direct sum, the zero element is the zero module, 1 is R itself, and multiplication is ... well the subject for today. We care about the tensor product for two reasons: First, it allows us to deal smoothly with bilinear maps such at the cross-product. Second, and perhaps more important, it will allow us relate homology with coefficients in an any abelian group to homology with coefficients in $\mathbf{Z}$.

Let's begin by recalling the definition of a bilinear map over a commutative ring R.

Definition 20.1. Given three R-modules, M, N, P, a *bilinear map* (or, to be explicit, *R-bilinear map*) is a function $\beta : M \times N \to P$ such that

$$\beta(x + x', y) = \beta(x, y) + \beta(x', y), \quad \beta(x, y + y') = \beta(x, y) + \beta(x, y'),$$

and

$$\beta(rx, y) = r\beta(x, y), \quad \beta(x, ry) = r\beta(x, y),$$

for $x, x' \in M$, $y, y' \in N$, and $r \in R$.

Example 20.2. $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by the dot product is an $\mathbf{R}$-bilinear map. The cross product $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is $\mathbf{R}$-bilinear. If R is a ring, the multiplication $R \times R \to R$ is R-bilinear, and the multiplication on an R-module M given by $R \times M \to M$ is R-bilinear. This enters into topology because the cross-product $H_m(X;R) \times H_n(Y;R) \xrightarrow{\times} H_{m+n}(X \times Y; R)$ is R-bilinear.

Wouldn't it be great to reduce stuff about bilinear maps to linear maps? We're going to do this by means of a universal property.

Definition 20.3. Let M, N be R-modules. A *tensor product* of M, N is a R-module P and a bilinear map $\beta_0 : M \times N \to P$ such that for every bilinear map $\beta : M \times N \to Q$ there is a unique factorization

$$\begin{tikzcd} M \times N \arrow[r, "\beta_0"] \arrow[dr, "\beta"'] & P \arrow[d, dashed, "f"] \\ & Q \end{tikzcd}$$

through an R-module homomorphism f.

We should have pointed out that the composition $f \circ \beta_0$ is indeed again R-bilinear; but this is easy to check.

So β_0 is a universal bilinear map out of $M \times N$. Instead of β_0 we're going to write $\otimes : M \times N \to P$. This means that $\beta(x, y) = f(x \otimes y)$ in the above diagram. There are lots of things to say about this. When you have something that is defined via a universal property, you know that it's unique . . . but you still have to check that it exists!

Construction 20.4. I want to construct a univeral R-bilinear map out of $M \times N$. Let $\beta : M \times N \to Q$ be any R-bilinear map. This β isn't linear. Maybe we should first extend it to a linear map. Consider $R\langle M \times N\rangle$, the free R-module generated by the set $M \times N$. Well, at least β is a map of sets, so there's a unique R-linear homomorphism $\overline{\beta} : R\langle M \times N\rangle \to Q$ extending it:

$$\begin{tikzcd} M \times N \arrow[rr, "\beta"] \arrow[dr, "{[-]}"'] & & Q \\ & R\langle M \times N\rangle \arrow[ur, "\overline{\beta}"'] & \end{tikzcd}$$

The map $[-]$ isn't bilinear. So we should quotient $R\langle M \times N\rangle$ by a submodule S of relations to make it bilinear. So S is the sub R-module generated by the four familes of elements (corresponding to the four relations in the definition of R-bilinearity):

1. $[(x + x', y)] - [(x, y)] - [(x' - y)]$

2. $[(x, y + y')] - [(x, y)] - [(x, y')]$

3. $[(rx, y)] - r[(x, y)]$

4. $[(x, ry)] - r[(x, y)]$

for $x, x' \in M$, $y, y' \in N$, and $r \in R$. Now the composite $M \times N \to R\langle M \times N\rangle/S$ *is* bilinear - we've quotiented out by all things that made it false! Now the map $R\langle M \times N\rangle \to Q$ factors through via $R\langle M \times N\rangle \to R\langle M \times N\rangle/S \xrightarrow{f} Q$ because β is bilinear, uniquely because the map to the quotient is surjective. This completes the construction.

You're never going to use this construction to compute anything. If you find yourself using this construction, stop and think about what you're doing. Here's an example: for any abelian group A,

$$A \times \mathbf{Z}/n\mathbf{Z} \to A/nA\,, (a,b) \mapsto ba \mod nA$$

is clearly bilinear, and is universal as such. Just look: If $\beta : A \times \mathbf{Z}/n\mathbf{Z} \to Q$ is bilinear then $\beta(na,b) = n\beta(a,b) = \beta(a,nb) = \beta(a,0) = 0$, so β factors through A/nA. And $A \times \mathbf{Z}/n\mathbf{Z} \to A/nA$ is surjective.

Remark 20.5. Note that the image of $M \times N$ in $R\langle M \times N\rangle/S$ generates it as an R-module. These $x \otimes y$ are called "decomposable tensors."

What are the properties of such a universal bilinear map?

Property 20.6 (Uniqueness)**.** Suppose $\beta_0 : M \times N \to P$ and $\beta_0' : M \times N \to P'$ are both universal. Then there's a linear map $f : P \to P'$ such that $\beta_0' = f\beta_0$ and a linear map $f' : P' \to P$ such that $\beta_0 = f'\beta_0'$. The composite $f'f : P \to P$ is a linear map such that $f'f\beta_0 = f'\beta_0' = \beta_0$. The identity map is another. But by universality, there's only one such linear map, so $f'f = 1_P$. An identical argument shows that $ff' = 1_{P'}$ as well, so they are inverse linear isomorphism. In brief:

> The target of a univeral R-bilinear map $\beta_0 : M \times N \to P$ is unique up to a unique linear isomorphism compatible with the map β_0.

This entitles us to speak of "the" universal bilinear map out of $M \times N$, and give the target a symbol: $M \otimes_R N$. If R is the ring of integers, or otherwise understood, we will drop it from the notation.

Property 20.7 (Functoriality)**.** Suppose $f : M \to M'$ and $: N \to N'$. Study the diagram

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[d, "f \times g"] \arrow[dr] & M \otimes N \arrow[d, dashed, "f \otimes g"] \\ M' \times N' \arrow[r, "\otimes"] & M' \otimes N' \end{tikzcd}$$

There is a unique R-linear map $f \otimes g$ because the diagonal map is R-bilinear and the map $M \times N \to M \otimes N$ is the universal R-bilinear map out of $M \times N$. You are invited to show that this construction is functorial.

Property 20.8 (Unitality, associativity, commutativity)**.** I said that this was going to be a "categorical ring," so we should check various properties of the tensor product. For example, $R \otimes_R M$ should be isomorphic to M. Let's think about this for a minute. I have an R-bilinear map $R \otimes M \to M$, given by multiplication. I just need to check the universal property. Suppose I have an R-bilinear map $\beta : R \times M \to P$. I have to construct a map $f : M \to P$ such that $\beta(r,x) = f(rx)$ and show it's unique. Our only choice is $f(x) = \beta(1,x)$, and that works.

Similarly, we should check that there's a unique isomorphism $L \otimes (M \otimes N) \xrightarrow{\cong} (L \otimes M) \otimes N$ that's compatible with $L \times (M \times N) \cong (L \times M) \times N$, and that there's a unique isomorphism $M \otimes N \to N \otimes M$ that's compatible with the switch map $M \times N \to N \times M$. There are a few other things to check, too: have fun.

Property 20.9 (Sums)**.** What happens with $M \otimes (\bigoplus_{\alpha \in A} N_\alpha)$? This might be a finite direct sum, or maybe an uncountable collection. How does this relate to $\bigoplus_{\alpha \in A}(M \otimes N_\alpha)$? Let's construct a map

$$f : \bigoplus_{\alpha \in A} (M \otimes N_\alpha) \to M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha \right).$$

We just need to define maps $M \otimes N_\alpha \to M \otimes (\bigoplus_{\alpha \in A} N_\alpha)$ because the direct sum is the coproduct. We can use $1 \otimes \mathrm{in}_\alpha$ where $\mathrm{in}_\alpha : N_\alpha \to \bigoplus_{\alpha \in A} N_\alpha$. These give you a map f.

What about a map the other way? We'll define a map out of the tensor product using the universal property. So we need to define a bilinear map out of $M \times (\bigoplus_{\alpha \in A} N_\alpha)$. Send (x, y) (where $y \in N_\beta$, say) to $x \otimes \mathrm{in}_\beta y$, where $\mathrm{in}_\beta : N_\beta \to \oplus N_\alpha$ is the incusion of a summand. It's up to you to check that these are inverses.

Property 20.10 (Distributivity)**.** Suppose $f : M' \to M$, $r \in R$, and $g_0, g_1 : N' \to N$. Then

$$f \otimes (g_0 + g_1) = f \otimes g_0 + f \otimes g_1 : M' \otimes N' \to M \otimes N$$

and

$$f \otimes r g_0 = r(f \otimes g_0) : M' \otimes N' \to M \otimes N\,.$$

We'll leave this to you to check.

Our immediate use of this construction is to give a clean definition of "homology with coefficients in M," where M is any abelian group. First, endow singular chains with coefficients in M like this:

$$S_*(X; M) = S_*(X) \otimes M$$

Then we define

$$H_n(X; M) = H_n(S_*(X; M))\,.$$

Since $S_n(X) = \mathbf{Z}\mathrm{Sin}_n(X)$, $S_n(X; M)$ is a direct sum of copies of M indexed by n-simplices in X. If M happens to be a ring, this coincides with the notation used in the last lecture. The boundary maps are just $d \otimes 1 : S_n(X) \otimes M \to S_{n-1}(X) \otimes M$.

As we have noted, the sequence

$$0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$$

is split short exact, and therefore applying $- \otimes M$ to it produces another split short exact sequence. So

$$S_n(X, A) \otimes M = S_n(A; M)/S_n(X; M)\,,$$

and it makes sense to use the notation $S_n(X,A;M)$ for this, and to define

$$H_n(X,A;M) = H_n(S_n(X,A;M)) .$$

Notice that

$$H_n(*;M) = \begin{cases} M & \text{for } n = 0 \\ 0 & \text{otherwise} . \end{cases}$$

The following result is immediate:

Proposition 20.11. *For any abelian group M, $(X,A) \mapsto H_*(X,A;M)$ provides a homology theory satisfying the Eilenberg-Steenrod axioms with $H_0(*;M) = M$.*

Suppose R is a commutative ring and A is an abelian group. Then $A \otimes R$ is naturally an R-module. So $S_*(X;R)$ is a chain complex of R-modules – *free* R-modules. We can go a little further: suppose that M is an R-module. Then $A \otimes M$ is an R-module; and $S_*(X;M)$ is a chain complex of R-modules. We can also write

$$S_*(X;M) = S_*(X;R) \otimes_R M .$$

This construction is natural in the R-module M; and, again using the fact that sums of exact sequences are exact, a short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0$$

leads to a short exact sequence of chain complexes

$$0 \to S_*(X;M') \to S_*(X;M) \to S_*(X;M'') \to 0$$

and hence to a long exact sequence in homology, a "coefficient long exact sequence":

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X;M'') \arrow[dll, "\partial"'] \\
H_n(X;M') \arrow[r] & H_n(X;M) \arrow[r] & H_n(X;M'') \arrow[dll, "\partial"'] \\
H_{n-1}(X;M') \arrow[r] & \cdots . &
\end{tikzcd}$$

A particularly important case is when R is a field; then $S_*(X;R)$ is a chain complex of vector spaces over R, and $H_*(X;R)$ is a graded vector space over R.

Question 20.12. A reasonable question is this: Suppose we know $H_*(X)$. Can we compute $H_*(X;M)$ for an abelian group M? More generally, suppose we know $H_*(X;R)$ and M is an R-module. Can we compute $H_*(X;M)$?

21 Tensor and Tor

We continue to study properties of the tensor product. Recall that

$$A \otimes \mathbf{Z}/n\mathbf{Z} = A/nA\,.$$

Consider the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0\,.$$

Let's tensor it with $\mathbf{Z}/2\mathbf{Z}$. We get

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0\,.$$

This cannot be a short exact sequence! This is a major tragedy: tensoring doesn't preserve exact sequences; one says that $\mathbf{Z}/n\mathbf{Z} \otimes -$ is not "exact." This is why we can't form homology with coefficients in A by simply tensoring homology with A.

Tensoring does respect certain exact sequences:

Proposition 21.1. *The functor $N \mapsto M \otimes_R N$ preserves cokernels; it is* right exact.

Proof. Suppose that $N \to N''$ is a surjection of R-modules, and M is any R-module. Then

$$\begin{tikzcd} M \otimes_R N \arrow[r] & M \otimes_R N'' \\ M \times N \arrow[u] \arrow[r, two heads] & M \times N''\,. \arrow[u] \end{tikzcd}$$

At least we know that $M \times -$ preserves surjections. But the image of $M \times N''$ generates $M \otimes_R N''$ as an R-module, so the image of $M \times N$ generates it as well. This implies that $M \otimes_R N \to M \otimes_R N''$ is surjective. □

How about this failure of exactness? What can we do about that? Failure of exactness is bad, so let's try to repair it. A key observation is that if M is *free*, then $M \otimes_R -$ is exact. If $M = R\langle S\rangle$, the free R-module on a set S, then $M \otimes_R N = \oplus_S N$, since tensoring distributes over direct sums. Then we remember:

Lemma 21.2. *If $M_i' \to M_i \to M_i''$ is exact for all $i \in I$, then so is*

$$\bigoplus M_i' \to \bigoplus M_i \to \bigoplus M_i''\,.$$

Proof. Clearly the composite is zero. Let $(x_i \in M_i, i \in I) \in \bigoplus M_i$ and suppose it maps to zero. That means that each x_i maps to zero in M_i'' and hence is in the image of some $x_i' \in M_i'$. Just make sure to take $x_i' = 0$ if $x_i = 0$. □

To exploit this observation, let's "resolve" M by free modules. This means: find a surjection from a free R-module, $F_0 \to M$. This amounts to specifying R-module generators. The kernel of $F_0 \to M$ won't generally be free. Let's make sure that it is by assuming that R is a PID, and write F_1 for the kernel. The failure of $M \otimes -$ to be exact is measured, at least partially, by the leftmost term (defined as a kernel) in the exact sequence

$$0 \to \mathrm{Tor}_1^R(M,N) \to F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0\,.$$

The notation suggests that this Tor term is independent of the resolution. This is indeed the case, as we shall show presently. But before we do, let's compute some Tor groups.

Example 21.3. For any PID R, if $M = F$ is free over R we can take $F_0 = F$ and $F_1 = 0$, and discover that then $\mathrm{Tor}_1^R(F,N) = 0$ for any N.

Example 21.4. Let $R = \mathbf{Z}$ and $M = \mathbf{Z}/n\mathbf{Z}$, and N any abelian group. When $R = \mathbf{Z}$ it is often omitted from the notation for Tor. There is a nice free resolution staring at us: $F_0 = F_1 = \mathbf{Z}$, and $F_1 \to F_0$ given by multiplication by n. The sequence looks like

$$0 \to \mathrm{Tor}_1(\mathbf{Z}/n\mathbf{Z}, N) \to \mathbf{Z} \otimes N \xrightarrow{n \otimes 1} \mathbf{Z} \otimes N \to \mathbf{Z}/n\mathbf{Z} \otimes N \to 0\,,$$

so

$$\mathbf{Z}/n\mathbf{Z} \otimes N = N/nN\,, \quad \mathrm{Tor}_1(\mathbf{Z}/n\mathbf{Z}, N) = \ker(n|N)\,.$$

The torsion in this case is the "n-torsion" in N. This accounts for the name.

Functors like Tor_1 can be usefully defined for any ring, and moving to that general case makes their significance clearer and illuminates the reason why Tor_1 is independent of choice of generators.

So let R be any ring and M a module over it. By picking R-module generators I can produce a surjection from a free R-module, $F_0 \to M$. Write K_0 for the kernel of this map. It is the module of relations among the generators. We can no longer guarantee that it's free, but we can at least find a set of module generators for it, and construct a surjection from a free R-module, $F_1 \to K_0$. Continuing in this way, we get a diagram like this –

$$\begin{tikzcd}
\cdots \arrow[rr] & & F_2 \arrow[rr, "d"] \arrow[dr] & & F_1 \arrow[rr, "d"] \arrow[dr] & & F_0 \arrow[dr] & & \\
& K_2 \arrow[ur] \arrow[dr] & & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & N \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

– in which the upside-down V subdiagrams are short exact sequences and F_s is free for all s. Splicing these exact sequences gives you an exact sequence in the top row. This is a *free resolution of* N. The top row, F_*, is a chain complex. It maps to the very short chain complex with N in degree 0 and 0 elsewhere, and this chain map is a homology isomorphism (or "quasi-isomorphism"). We have in effect replaced N with this chain complex of free modules. The module N may be very complicated, with generators, relations, relations between relations .... All this is laid out in front of us by the free resolution. Generators of F_0 map to generators for N, and generators for F_1 map to relations among those generators.

Now we can try to define higher Tor functors by tensoring F_* with N and taking homology. If R is a PID and the resolution is just $F_1 \to F_0$, forming homology is precisely taking cokernel and kernel, as we did above. In general, we define

$$\mathrm{Tor}_n^R(M,N) = H_n(M \otimes_R F_*)\,.$$

In the next lecture we will check that this is well-defined – independent of free resolution, and functorial in the arguments. For the moment, notice that

$$\mathrm{Tor}_n^R(M,F) = 0 \text{ for } n > 0 \quad \text{if } F \text{ is free}\,,$$

since I can take $F \xleftarrow{\cong} F \leftarrow 0 \leftarrow \cdots$ as a free resolution; and that

$$\mathrm{Tor}_0^R(M,N) = M \otimes_R N$$

since we know that $M \otimes_R -$ is right-exact.

22 The fundamental lemma of homological algebra

We will now show that the R-modules $\mathrm{Tor}_n^R(M,N)$ are well-defined and functorial. This will be an application of a very general principle.

Theorem 22.1 (Fundamental theorem of homological algebra)**.** *Let M and N be R-modules; let*

$$0 \leftarrow M \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots$$

be a sequence in which each E_n is free, and

$$0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$$

be an exact sequence; and let $f : M \to N$ be a homomorphism. Then we can lift f to a chain map $f_* : E_* \to F_*$, uniquely up to chain homotopy.

Proof. Let's try to construct f_0. Consider:

$$\begin{tikzcd}
0 \arrow[r] & K_0 = \ker(\epsilon_M) \arrow[r] \arrow[d, dashed, "g_0"] & E_0 \arrow[r, "\epsilon_M"] \arrow[d, dashed, "f_0"] & M \arrow[d, "f"] & \\
0 \arrow[r] & L_0 = \ker(\epsilon_N) \arrow[r] & F_0 \arrow[r, two heads, "\epsilon_N"] & N \arrow[r] & 0
\end{tikzcd}$$

We know that $E_0 = R\langle S\rangle$. What we do is push the generators of E into M via ϵ_M and then into F via f, and then lift them to F_0 via ϵ_N (which is possible because it's surjective). Then extend to a homomorphism, to get f_0. You can restrict it to kernels to get g_0.

Now the map $d : E_1 \to E_0$ satisifes $\epsilon_M \circ d = 0$, and so factors through a map to $K_0 = \ker \epsilon_M$. Similarly, $d : F_1 \to F_0$ factors through a map $F_1 \to L_1$, and this map must be surjective because the sequence $F_1 \to F_0 \to N$ is exact. We find ourselves in exactly the same situation:

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, dashed, "g_1"] & E_1 \arrow[r] \arrow[d, dashed, "f_1"] & K_0 \arrow[d, "g_0"] & \\
0 \arrow[r] & L_1 \arrow[r] & F_1 \arrow[r] & L_0 \arrow[r] & 0
\end{tikzcd}$$

So by we construct f_* by induction.

Now we need to prove the chain homotopy claim. So suppose I have $f_*, f'_* : E_* \to F_\bullet$, both lifting $f : M \to N$. Then $f'_n - f_n$ (which we'll rename ℓ_n) is a chain map lifting $0 : M \to N$. We want to consruct a chain null-homotopy of ℓ_*; that is, we want $h : E_n \to F_{n+1}$ such that $dh + hd = \ell$. At the bottom, $E_{-1} = 0$ so we want $h : E_0 \to F_1$ such that $dh = \ell_0$. This factorization happens in two steps.

$$\begin{tikzcd}
 & & E_0 \arrow[r] \arrow[d, "\ell_0"'] \arrow[dl] \arrow[dll, dashed, "h"'] & M \arrow[d, "0"] \\
F_1 \arrow[r, two heads] & L_0 \arrow[r] & F_0 \arrow[r] & N\,.
\end{tikzcd}$$

First, $d\ell_0 = 0$ implies that ℓ_0 factors through $L_1 = \ker \epsilon_N$. Next, $F_1 \to L_0$ is surjective, by exactness, and E_0 is free, so we can lift generators and extend R-linearly as indicated.

The next step is organized by the diagram

$$\begin{tikzcd}
 & & E_1 \arrow[r, "d"] \arrow[d, "\ell_1"] \arrow[dl, dashed] \arrow[dll, dashed, "h"'] & E_0 \arrow[d, "\ell_0"] \arrow[dl, "h"'] \\
F_2 \arrow[r, two heads] \arrow[rr, bend right, "d"'] & L_1 \arrow[r] & F_1 \arrow[r, "d"] & F_0
\end{tikzcd}$$

This diagram doesn't commute; while $dh = \ell_0$, we want to construct $h : E_1 \to F_2$ such that $dh = \ell_1 - hd$. Since

$$d(\ell_1 - hd) = \ell_0 d - dhd = (\ell_0 - dh)d = 0\,.$$

the map $\ell_1 - hd$ lifts to $L_1 = \ker d$. But then it lifts through F_2, since $F_2 \to L_1$ is surjective and E_1 is free.

Exactly the same process continues. □

This proof uses a property of freeness that is shared by a broader class of modules.

Definition 22.2. An R-module P is *projective* if any map out of P factors through any surjection:

$$\begin{tikzcd}
 & M \arrow[d] \\
P \arrow[r] \arrow[ur, dashed] & N
\end{tikzcd}$$

Every free module is projective; this is what we have been using; our proof of the fundamental lemma of homogolical algebra uses only that E_n is projective. Anything that's a direct summand in a projective is also projective. Any projective module is a direct summand of a free module. Over a PID, every projective is free, because any submodule of a free is free. But there are examples of nonfree projectives:

Example 22.3. Let k be a field and let R be the product ring $k \times k$. It acts on k in two ways, via $(a,b)c = ac$ and via $(a,b)c = bc$. This are both projective R-modules that are not free.

Now we will apply the Fundamental Lemma to verify that our proposed construction of Tor is independent of free (or projective!) resolution, and is functorial.

Suppose I have $f : N' \to N$. Pick arbitrary free resolutions $N' \leftarrow N'_*$ and $N \leftarrow N_*$, and pick any chain map $f_* F'_* \to F_*$ lifting f. We claim that the map induced in homology by $1 \otimes f_* : M \otimes_R F'_* \to M \otimes_R F_*$ is independent of the choice of lift. Suppose f'_* is another lift, and pick a chain homotopy $h : f \sim f'$. Since $M \otimes_R -$ is additive, the relation

$$1 \otimes h : 1 \otimes f \sim 1 \otimes f'$$

still holds. So $1 \otimes f$ and $1 \otimes f'$ induce the same map in homology.

For example, suppose that F_* and F'_* are two projective resolutions of N. Any two lifts of the identity map are chain-homotopic, and so induce the same map $H_*(M \otimes_R F_*) \to H_*(M \otimes_R F'_*)$. So if $f : F_* \to F'_*$ and $g : F'_* \to F_*$ are chain maps lifing the identity, then $f_* \circ g_*$ induces the same self-map of $H_*(M \otimes_R F'_*)$ as the identity self-map does, and so (by functoriality) is the identity. Similarly, $g_* \circ f_*$ induces the identity map on $H_*(M \otimes_R F_*)$. So they induce inverse isomorphisms.

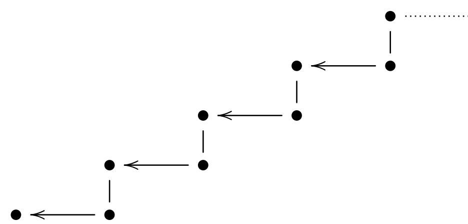
Putting all this together shows that any two projective resolutions of N induce canonically isomorphic modules $\mathrm{Tor}_n^R(M,N)$, and that a homomorphism $f : N' \to N$ induces a well defined map $\mathrm{Tor}_n^R(M,N') \to \mathrm{Tor}_n^R(M,N)$ that renders $\mathrm{Tor}_n^R(M,-)$ a functor.

Last comment about Tor is that there's a symmetry there. Of course, $M \otimes_R N \cong N \otimes_R M$. This uses the fact that R is commutative. This leads right on to saying that $\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(N,M)$. We've been computing Tor by taking a resolution of the second variable. But I could equally have taken a resolution of the first variable. This follows from the fundamental theorem of homological algebra.

Example 22.4. I want to give an example when you do have higher Tor modules. Let k be a field, and let $R = k[d]/(d^2)$. This is sometimes called the "dual numbers", or the exterior algebra over k. We're going to consider R-modules. Let's construct a projective resolution of k. What is an R-module M? It's just a k-vector space M with an operator d (given by multiplication by e) that satisfies $d^2 = 0$. Even though there's no grading around, I can still define the "homology" of M:

$$H(M;d) = \frac{\ker d}{\operatorname{im} d} .$$

This k-algebra is *augmented* by an algebra map $\epsilon : R \to k$ splitting the unit; $\epsilon(d) = 0$. Let's construct a free R-module resolution of this module. Here's a picture.



The vertical lines indicate multiplication by e. We could write this as

$$0 \leftarrow k \xleftarrow{\epsilon} R \xleftarrow{e} R \xleftarrow{e} R \leftarrow \cdots .$$

Now tensor this over R with an R-module M; so M is a vector space equipped with an operator d with $d^2 = 0$. Each copy of R gets replaced by a copy of M, and the differential gives multiplication by d on M. So taking homology gives

$$\operatorname{Tor}_n^R(k, M) = \begin{cases} k \otimes_R M = M/dM & \text{for } n = 0 \\ H(M; d) & \text{for } n > 0 \,. \end{cases}$$

So for example

$$\operatorname{Tor}_n^R(k, k) = k \text{ for } n \geq 0 \,.$$

23 Hom and Lim

We will now deveop more properties of the tensor product: its relationship to homomorphisms and to direct limits.

The tensor product arose in our study of bilinear maps. Even more natural are *linear maps*. Given a commutative ring R and two R-modules M and N, we can think about the collection of all R-linear maps from M to N. Not only does this set form an abelian group (under pointwise addition of homomorphisms); it forms an R-module, with

$$(rf)(y) = f(ry) = rf(y) \,, \quad r \in R,\, y \in M \,.$$

The check that this is again an R-module homomorphism uses commutativity of R. We will write $\underline{\operatorname{Hom}}_R(M, N)$, or just $\operatorname{Hom}(M, N)$, for this R-module.

Since $\operatorname{Hom}(M, N)$ is an R-module, we are entitled to think about what an R-module homomorphism into it is. Given

$$f : L \to \operatorname{Hom}(M, N)$$

we can define a new function

$$\hat{f} : L \times M \to N \,, \quad \hat{f}(x, y) = (f(x))(y) \in N \,.$$

You should check that this new function $\hat{f}$ is R-bilinear! So we get a natural map

$$\mathrm{Hom}(L, \mathrm{Hom}(M, N)) \to \mathrm{Hom}(L \otimes M, N) .$$

Conversely, given a map $\hat{f} : L \otimes M \to N$ and $x \in L$, we can define $f(x) : M \to N$ by the same formula. These are inverse operations, so:

Lemma 23.1. *The natural map* $\mathrm{Hom}(L, \mathrm{Hom}(M, N)) \to \mathrm{Hom}(L \otimes M, N)$ *is an isomorphism.*

One says that $\otimes$ and Hom are *adjoint*, a word suggested by Sammy Eilenberg to Dan Kan, who first formulated this relationship between functors.

This relationship makes a number of things obvious. For example, it's clear that

$$\mathrm{Hom}(\bigoplus_\alpha M_\alpha, N) = \prod_\alpha \mathrm{Hom}(M_\alpha, N)$$

and this implies that the tensor product distributes over arbitrary direct sums. We'll see another example in a minute.

The second thing we will discuss is a generalization of one perspective on how the rational numbers are constructed from the integers – by a limit process: there are compatible maps in the diagram

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "2"] \arrow[d, "1"] & \mathbf{Z} \arrow[r, "3"] \arrow[dl, "1/2"] & \mathbf{Z} \arrow[r, "4"] \arrow[dll, "1/3!"] & \mathbf{Z} \arrow[r, "5"] \arrow[dlll, "1/4!"] & \cdots \\
\mathbf{Q} & & & &
\end{tikzcd}$$

and $\mathbf{Q}$ is the "universal," or "initial," abelian group you can put in that position.

We will formalize this process, using partially ordered sets as indexing sets. Recall that a *partially ordered set*, or *poset*, is a small category $\mathcal{I}$ such that $\#\mathcal{I}(i, j) \le 1$ and the only isomorphisms are the identity maps. We will be interested in a particular class of posets.

Definition 23.2. A poset $(\mathcal{I}, \le)$ is *directed* if for every i, j, there exists a k such that $i \le k$ and $j \le k$.

Example 23.3. This is a very common condition. For example, the natural numbers $\mathbb{N}$ with inequality. Another example: if X is a space and I is the set of open subsets of X. It's directed by saying that $U \le V$ if $U \subseteq V$. This is because U, U' need not be comparable, but $U, U' \subseteq U \cup U'$. Another example is the positive natural numbers, with $i \le j$ if $i | j$. This is because $i, j | (ij)$.

Definition 23.4. Let $\mathcal{I}$ be a directed set. An $\mathcal{I}$-*directed diagram* in a category $\mathcal{C}$ is a functor $\mathcal{I} \to \mathcal{C}$. This means that for every $i \in \mathcal{I}$ we are given an object $X_i \in \mathcal{C}$, and for every $i \le j$ we are given a map $X_i \xrightarrow{f_{i,j}} X_j$, in such a way that $f_{i,i} = 1_{X_i}$ and if $i \le j \le k$ then $f_{i,k} = f_{j,k} \circ f_{i,j} : X_i \to X_k$.

Example 23.5. If $\mathcal{I} = (\mathbb{N}, \le)$, then you get a "linear system" $X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \to \cdots$.

Example 23.6. Suppose $\mathcal{I} = (\mathbb{N}_{>0}, |)$, i.e., the third example above. You can consider $\mathcal{I} \to \mathbf{Ab}$, say assigning to each i the integers $\mathbf{Z}$, and $f_{ij} : \mathbf{Z} \xrightarrow{j/i} \mathbf{Z}$. This seems like a more "choice-free" way to get at a construction of $\mathbf{Q}$ out of $\mathbf{Z}$.

These directed systems can be a little complicated. But there's a simple one, namely the constant one.

Example 23.7. Let $\mathcal{I}$ be any directed system. Any object $A \in \mathcal{C}$ determines an $\mathcal{I}$-directed set, namely the constant functor $c_A : \mathcal{I} \to \mathcal{C}$.

Not every directed system is constant, but we can try to find a best approximating constant system. To compare systems, we need morphisms. Of course, $\mathcal{I}$-directed systems in $\mathcal{C}$ are functors $\mathcal{I} \to \mathcal{C}$. They have natural transformations, and those are the morphisms in the category of $\mathcal{I}$-directed systems. That is to say, a morphism is a choice of map $g_i : X_i \to Y_i$, for each $i \in \mathcal{I}$, such that

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "g_i"] & X_j \arrow[d, "g_j"] \\ Y_i \arrow[r] & Y_j \end{tikzcd}$$

commutes for all $i \le j$.

Definition 23.8. Let $X : \mathcal{I} \to \mathcal{C}$ be a directed system. A *direct limit* is an object L and a map $X \to c_L$ that is initial among maps to constant systems. This means that given any other map to a constant system, say $X \to c_A$, there is a unique map $f : L \to A$ such that

$$\begin{tikzcd} & c_L \arrow[dd, "c_f"] \\ X \arrow[ur] \arrow[dr] & \\ & c_A \end{tikzcd}$$

commutes.

This is a universal property. So two different direct limits are canonically isomorphic; but a directed system may fail to have a direct limit. For example, the linear directed systems we used to creat the rational numbers exist in the category of finitely generated abelian groups; but $\mathbf{Q}$ is not finitely generated, and there's no finitely generated group that will serve as a direct limit in the category of finitely generated abelian groups.

Example 23.9. Suppose we have an increasing sequence of subspaces, $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$. This gives us a directed system of spaces, directed by the poset $(\mathbb{N}, \le)$. It's pretty clear that as a *set* the direct limit of this system is the union of the subspaces. Suppose that union is X. Then saying that X is the direct limit of this directed system of spaces is saying that the topology on X is determined by the topology on the subspaces; it's the "weak topology," characterized by the property that a map $f : X \to Y$ is continuous if and only if the restriction of f to each X_n is continuous. This is saying that a subset of X is open if and only if its intersection with each X_n is open in X.

Direct limits may be constructed from the material of coproducts and quotients. So suppose $X : \mathcal{I} \to \mathcal{C}$ is a directed system. To construct the direct limit, begin by forming the coproduct over the elements of $\mathcal{I}$,

$$\coprod_{i \in \mathcal{I}} X_i \,.$$

There are maps $\mathrm{in}_i : X_i \to \coprod X_i$, but they are not yet compatible with the order relation in $\mathcal{I}$. Form a quotient of the coproduct to enforce that compatibility:

$$\varinjlim_{i \in \mathcal{I}} X_i = \left(\coprod_{i \in \mathcal{I}} X_i \right) / \sim$$

where $\sim$ is the equivalence relation generated by requiring that for any $i \in \mathcal{I}$ and any $x \in X_i$,

$$\mathrm{in}_i x \sim \mathrm{in}_j f_{ij}(x) \,.$$

The process of forming the coproduct and the quotient will depend upon the category you are working in, and may not be possible. In sets, coproduct is disjoint union and the quotient just forms equivalence classes. In abelian groups, the coproduct is the direct sum and to form the quotient you divide by the subgroup generated by differences.

Proposition 23.10. *Let $\mathcal{I}$ be a direct set, and let $M : \mathcal{I} \to \mathbf{Mod}_R$ be a $\mathcal{I}$-directed system of R-modules. There is a natural isomorphism*

$$(\varinjlim_I M_i) \otimes_R N \cong \varinjlim_I (M_i \otimes_R N) \,.$$

Proof. Let's verify that both sides satisfy the same universal property. A map from $\varinjlim_I M_i) \otimes_R N$ to an R-module L is the same thing as a linear map $\varinjlim_I M_i \to \mathrm{Hom}_R(N, L)$. This is the same as a compatible family of maps $M_i \to \mathrm{Hom}_R(N, L)$, which in turn is the same as a compatible family of maps $M_i \otimes_R N \to L$, which is the same as a linear map $\varinjlim_I (M_i \otimes_R N) \to L$. □

Here's a lemma that lets us identify when a map to a constant functor is a direct limit.

Lemma 23.11. *Let $X : \mathcal{I} \to \mathbf{Ab}$ (or $\mathbf{Mod}_R$). A map $f : X \to c_L$ (given by $f_i : X_i \to L$ for $i \in \mathcal{I}$) is the direct limit if and only if:*

1. *For every $x \in L$, there exists an i and an $x_i \in X_i$ such that $f_i(x_i) = x$.*

2. *Let $x_i \in X_i$ be such that $f_i(x_i) = 0$ in L. Then there exists some $j \geq i$ such that $f_{ij}(x_i) = 0$ in X_j.*

Proof. Straightforward. □

This lemma generalizes the observation that $\mathbf{Q}$ is the colimit of the diagram we drew above for $\mathcal{I} = (\mathbb{N}_{>0}, |)$.

Proposition 23.12. *The direct limit functor* $\varinjlim_I : \mathrm{Fun}(\mathcal{I}, \mathbf{Ab}) \to \mathbf{Ab}$ *is exact. In other words, if* $X \xrightarrow{p} Y \xrightarrow{q} Z$ *is an exact sequence of* $\mathcal{I}$*-directed systems (meaning that at every degree we get an exact sequence of abelian groups), then* $\varinjlim_I X \to \varinjlim_I Y \to \varinjlim_I Z$ *is exact.*

Proof. First of all, $pi : X \to Z$ is zero, which is to say that it factors through the constant zero object, so $\varinjlim_I X \to \varinjlim_I Z$ is certainly the zero map. Let $y \in \varinjlim_I Y$, and suppose y maps to 0 in $\varinjlim_I Z$. By condition (1), there exists i such that $y = f_i(y_i)$ for some $y_i \in Y_i$. Then $0 = q(y) = f_i q(y_i)$ because q is a map of direct systems. This means that there is $j \geq i$ such that $f_{ij} q(y_i) = 0$ in Z_j. So $q f_{ij} y_i = 0$, again because q is a map of direct systems. We have an element in Y_j that maps to zero under q, so there is some $x_j \in X_j$ such that $p(x_j) = y_j$. Then $f_j(x_j) \in \varinjlim_I X$ maps to y. □

The exactness of the direct limit has many useful consequences. For example:

Corollary 23.13. *Let* $i \mapsto C(i)$ *be a directed system of chain complexes. Then there is a natural isomorphism*

$$\varinjlim_{i \in \mathcal{I}} H_*(C(i)) \to H_*(\varinjlim_{i \in \mathcal{I}} C(i)) \, .$$

Putting together things we have just said:

Corollary 23.14. $H_*(X; \mathbf{Q}) = H_*(X) \otimes \mathbf{Q}$.

So we can redefine the Betti numbers of a space X as

$$\beta_n = \dim_{\mathbf{Q}} H_n(X; \mathbf{Q})$$

and discuss the Euler characteristic entirely in terms of the rational vector spaces making up the rational homology of X.

24 Universal coefficient theorem

Suppose that we are given $H_*(X; \mathbf{Z})$. Can we compute $H_*(X; \mathbf{Z}/2\mathbf{Z})$? This is non-obvious. Consider the map $\mathbf{RP}^2 \to S^2$ that pinches $\mathbf{RP}^1$ to a point. In $H_2(\mathbf{RP}^2; \mathbf{Z}) = 0$, so in H_2 this map is zero. But in $\mathbf{Z}/2\mathbf{Z}$-coefficients, in dimension 2, this map gives an isomorphism. This shows that there's not a *functorial* relationship between $H_*(X; \mathbf{Z})$ and $H_*(X; \mathbf{Z}/2\mathbf{Z})$; the effect of a map in integral homology does not determine its effect in mod 2 homology. So how *do* we go between different coefficients? That's the mystery.

Let R be a commutative ring and M an R-module, and suppose we have a chain complex C_* of R-modules. It could be the singular complex of a space, but it doesn't have to be. Let's compare $H_n(C_*) \otimes M$ with $H_n(C_* \otimes M)$. (Here and below we'll just write $\otimes$ for $\otimes_R$.) The latter thing gives homology with coefficients in M. How can we compare these two? Let's investigate, and build up conditions on R and C_* as we go along.

First, there's a natural map

$$\alpha : H_n(C_*) \otimes M \to H_n(C_* \otimes M)\,,$$

sending $[z] \otimes m$ to $[z \otimes m]$. We propose to show that it is injective. The map α fits into a commutative diagram with exact columns like this:

$$\begin{tikzcd}
0 & 0 \\
H_n(C_*) \otimes M \arrow[u] \arrow[r, "\alpha"] & H_n(C_* \otimes M) \arrow[u] \\
Z_n(C_*) \otimes M \arrow[u] \arrow[r] & Z_n(C_* \otimes M) \arrow[u] \\
C_{n+1} \otimes M \arrow[u] \arrow[r, "="] & C_{n+1} \otimes M\,.
\end{tikzcd}$$

Now, $Z_n(C_* \otimes M)$ is a submodule of $C_n \otimes M$, but the map $Z_n(C) \otimes M \to C_n \otimes M$ need not be ... unless we impose more restrictions. If we can guarantee that it is, then a diagram chase shows that α is a monomorphism.

So let's assume that R is a PID and that C_n is a free R-module for all n. Then the submodule $B_{n-1}(C_*) \subseteq C_{n-1}$ is again free, so the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & Z_n(C_*) \arrow[r] & C_n \arrow[r] \arrow[dr] & B_{n-1}(C_*) \arrow[r] \arrow[d] & 0 \\
& & & C_{n-1} &
\end{tikzcd}$$

splits. So $Z_n(C_*) \to C_n$ is a split monomorphism, and hence $Z_n(C_*) \otimes M \to C_n \otimes M$ is too.

In fact, a little thought shows that this argument produces a splitting of the map α.

Now, α is not always an isomorphism. But it certainly is if $M = R$, and it's compatible with direct sums, so it certainly is if M is free. The idea is now to resolve M by frees, and see where that idea takes us.

So let

$$0 \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M. Again, we're using the assumption that R is a PID, to guarantee that $\ker(F_0 \to M)$ is free. Again using the assumption that each C_n is free, we get a short exact sequence of chain complexes

$$0 \to C_* \otimes F_1 \to C_* \otimes F_0 \to C_* \otimes M \to 0\,.$$

In homology, this gives a long exact sequence. Unsplicing it gives the left-hand column in the following diagram.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{coker}(H_n(C_*\otimes F_1)\to H_n(C_*\otimes F_0)) \arrow[r, "\cong"] \arrow[d] & \operatorname{coker}(H_n(C_*)\otimes F_1\to H_n(C_*)\otimes F_0)) \arrow[d] \\
H_n(C_*\otimes M) \arrow[r, "="] \arrow[d, "\partial"] & H_n(C_*\otimes M) \arrow[d] \\
\ker(H_{n-1}(C_*\otimes F_1)\to H_{n-1}(C_*\otimes F_0)) \arrow[r, "\cong"] \arrow[d] & \ker(H_{n-1}(C_*)\otimes F_1\to H_{n-1}(C_*)\otimes F_0) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The right hand column occurs because α is an isomorphism when the module involved is free. But

$$\operatorname{coker}(H_n(C_*)\otimes F_1\to H_n(C_*)\otimes F_0)) = H_n(C_*)\otimes M$$

and

$$\ker(H_{n-1}(C_*)\otimes F_1\to H_{n-1}(C_*)\otimes F_0) = \operatorname{Tor}_1^R(H_{n-1}(C_*), M)\,.$$

Theorem 24.1 (Universal Coefficient Theorem)**.** *Let R be a PID and C_* a chain complex of R-modules such that C_n is free for all n. Then there is a natural short exact sequence of R-modules*

$$0\to H_n(C_*)\otimes M \xrightarrow{\alpha} H_n(C_*\otimes M) \xrightarrow{\partial} \operatorname{Tor}_1^R(H_{n-1}(C_*), M)\to 0$$

that splits (but not naturally).

Example 24.2. The pinch map $\mathbf{RP}^2\to S^2$ induces the following map of universal coefficient short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H_2(\mathbf{RP}^2)\otimes\mathbf{Z}/2\mathbf{Z} \arrow[r] \arrow[d, "0"] & H_2(\mathbf{RP}^2;\mathbf{Z}/2\mathbf{Z}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \operatorname{Tor}_1(H_1(\mathbf{RP}^2),\mathbf{Z}/2\mathbf{Z}) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & H_2(S^2)\otimes\mathbf{Z}/2\mathbf{Z} \arrow[r, "\cong"] & H_2(S^2;\mathbf{Z}/2\mathbf{Z}) \arrow[r] & \operatorname{Tor}_1(H_1(S^2),\mathbf{Z}/2\mathbf{Z}) \arrow[r] & 0
\end{tikzcd}$$

This shows that the splitting of the universal coefficient short exact sequence cannot be made natural, and it explains the mystery that we began with.

Remark 24.3. The hypotheses are essential. Exercise: construct two counterexamples: one with $R=\mathbf{Z}$ but in which the groups in the chain complex are not free, and one in which $R=k[d]/d^2$ and the modules in C_* are free over R.

25 Künneth and Eilenberg-Zilber

We want to compute the homology of a product. Long ago, in Lecture 7, we constructed a bilinear map $S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, called the cross product. So we get a linear map $S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y)$, and it satisfies the Leibniz formula, i.e., $d(x \times y) = dx \times y + (-1)^p x \times dy$. The method we used was really an example of the fundamental theorem of homological algebra. It works with any coefficient ring, not just the integers.

Definition 25.1. Let C_*, D_* be two chain complexes. Their *tensor product* is the chain complex with

$$(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q \,.$$

The differential $(C_* \otimes D_*)_n \to (C_* \otimes D_*)_{n-1}$ sends $C_p \otimes D_q$ into the submodule $C_{p-1} \otimes D_q \bigoplus C_p \otimes D_{q-1}$ by

$$x \otimes y \mapsto dx \otimes y + (-1)^p x \otimes dy \,.$$

So the cross product is a map of chain complexes $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$. There are two questions:
(1) Is this map an isomorphism in homology?
(2) How is the homology of a tensor product of chain complexes related to the tensor product of their homologies?

It's easy to see what happens in dimension zero, because $\pi_0(X) \times \pi_0(Y) = \pi_0(X \times Y)$ implies that $H_0(X) \otimes H_0(Y) \xrightarrow{\cong} H_0(X \times Y)$.

Let's dispose of the purely algebraic question (2) first.

Theorem 25.2. *Let R be a PID and C_*, D_* be chain complexes of R-modules. Assume that C_n is a free R-module for all n. There is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C), H_q(D)) \to 0$$

natural in these data.

Proof. This is exactly the same as the proof for the UCT. It's a good idea to work through this on your own. □

Corollary 25.3. *Under these conditions, if $C_* \to R$ and $D_* \to R$ are homology isomorphisms then so is $C_* \otimes D_* \to R$.*

Our attack on question (1) is via the method of "acyclic models." This is really a special case of the fundamental lemma of homological algebra!

Definition 25.4. Let $\mathcal{C}$ be a category, and fix a set $\mathcal{M}$ of object in $\mathcal{C}$, to be called the "models." A functor $F : \mathcal{C} \to \mathbf{Ab}$ is *$\mathcal{M}$-free* if it is the free abelian group of a coproduct of corepresentable functors. That is, F is a direct sum of functors of the form $\mathbf{Z}\,\mathrm{Hom}_{\mathcal{C}}(M, -)$ where $M \in \mathcal{M}$.

Example 25.5. Since we are interested in the singular homology of a product of two spaces, it may be sensible to take as $\mathcal{C}$ the category of ordered pairs of spaces, $\mathcal{C} = \mathbf{Top}^2$, and for $\mathcal{M}$ the set of pairs of simplicies, $\mathcal{M} = \{(\Delta^p, \Delta^q) : p, q \geq 0\}$. Then

$$S_n(X \times Y) = \mathbf{Z}[\mathrm{Hom}_{\mathbf{Top}}(\Delta^n \times X) \times \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y) = \mathbf{Z}\,\mathrm{Hom}_{\mathbf{Top}^2}((\Delta^n, \Delta^n), (X, Y))\,.$$

is $\mathcal{M}$-free.

Example 25.6. With the same category and models,

$$(S_*(X) \otimes S_*(Y))_n = \bigoplus_{p+q=n} S_p(X) \otimes S_q(Y)\,,$$

is $\mathcal{M}$-free, since the tensor product has as free basis the set

$$\coprod_{p+q=n} \mathrm{Sin}_p(X) \times \mathrm{Sin}_q(Y) = \coprod_{p+q=n} \mathrm{Hom}_{\mathbf{Top}^2}((\Delta^p, \Delta^q), (X, Y))\,.$$

Definition 25.7. A natural transformation of functors $\theta : F \to G$ is an $\mathcal{M}$-*epimorphism* if $\theta_M : F(M) \to G(M)$ is a surjection of abelian groups for every $M \in \mathcal{M}$. A *sequence* of natural transformations is a composable pair $G' \to G \to G''$ with trivial composition. Let K be the objectwise kernel of $G \to G''$. There is a factorization $G' \to K$. The sequence is $\mathcal{M}$-*exact* if $G' \to K$ is a $\mathcal{M}$-epimorphism. Equivalently, $G'(M) \to G(M) \to G''(M)$ is exact for all $M \in \mathcal{M}$.

Example 25.8. We claim that

$$\cdots \to S_n(X \times Y) \to S_{n-1}(X \times Y) \to \cdots \to S_0(X \times Y) \to H_0(X \times Y) \to 0$$

is $\mathcal{M}$-exact. Just plug in (Δ^p, Δ^q): you get an exact sequence, since $\Delta^p \times \Delta^q$ is contractible.

Example 25.9. The sequence

$$\cdots \to (S_*(X) \otimes S_*(Y))_n \to (S_*(X) \otimes S_*(Y))_{n-1} \to \cdots \to S_0(X) \otimes S_0(Y) \to H_0(X) \otimes H_0(Y) \to 0\,.$$

is also $\mathcal{M}$-exact, by Corollary 25.3.

The terms "$\mathcal{M}$-free" and "$\mathcal{M}$-exact" relate to each other in the expected way:

Lemma 25.10. *Let $\mathcal{C}$ be a category with a set of models $\mathcal{M}$ and let $F, G, G' : \mathcal{C} \to \mathbf{Ab}$ be functors. Suppose that F is $\mathcal{M}$-free, let $G' \to G$ be a $\mathcal{M}$-epimorphism, and let $f : F \to G$ be any natural transformation. Then there is a lifting:*

$$\begin{tikzcd} & G' \arrow[d] \\ F \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"'] & G \end{tikzcd}$$

Proof. Clearly we may assume that $F(X) = \mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M, X)$. Suppose that $X = M \in \mathcal{M}$. We get:

$$\begin{tikzcd} & G'(M) \arrow[d] \\ \mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M,M) \arrow[ur, dashed, "\overline{f}_M"] \arrow[r, "f_M"] & G(M) \end{tikzcd}$$

Consider $1_M \in \mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M, M)$. Its image $f_M(1_M) \in G(M)$ is hit by some element in $c_M \in G'(M)$, since $G' \to G$ is an $\mathcal{M}$-epimorphism. Define $\overline{f}_M(1_M) - c_M$.

Now we exploit naturality! Any $\varphi : M \to X$ should produce a commutative diagram

$$\begin{tikzcd} \mathcal{C}(M,M) \arrow[r, "\overline{f}_M"] \arrow[d, "\varphi_*"] & G'(M) \arrow[d, "\varphi_*"] \\ \mathcal{C}(M,X) \arrow[r, "\overline{f}_X"] & G'(X) \end{tikzcd}$$

Chase 1_M around the diagram, to see what the value of $\overline{f}_X(\varphi)$ must be:

$$\overline{f}_X(\varphi) = \overline{f}_X(\varphi_*(1_M)) = \varphi_*(\overline{f}_M(1_M)) = \varphi_*(c_M)\,.$$

Now extend linearly. You should check that this does define a natural transformation. □

This is precisely the condition required to prove the Fundamental Lemma of Homological Algebra. So we have the

Theorem 25.11 (Acyclic Models)**.** *Let $\mathcal{M}$ be a set of models in a category $\mathcal{C}$. Let $\theta : F \to G$ be a natural transformation of functors fron $\mathcal{C}$ to* **Ab***. Let F_* and G_* be functors from $\mathcal{C}$ to chain complexes, with augmentations $F_0 \to F$ and $G_0 \to G$. Assume that F_n is $\mathcal{M}$-free for all n, and that $G_* \to G \to 0$ is an $\mathcal{M}$-exact sequence. Then there is a unique natural chain homotopy of chain maps $F_* \to G_*$ covering θ.*

Corollary 25.12. *Suppose furthermore that θ is a natural isomorphism. If each G_n is $\mathcal{M}$-free and $F_* \to F \to 0$ is an $\mathcal{M}$-exact sequence, then any natural chain map $F_* \to G_*$ covering θ is a natural chain homotopy equivalence.*

Applying this to our category $\mathbf{Top}^2$ with models as before, we get the following theorem that completes work we did in Lecture 7.

Theorem 25.13 (Eilenberg-Zilber theorem)**.** *There are unique chain homotopy classes of natural chain maps:*

$$S_*(X) \otimes S_*(Y) \leftrightarrows S_*(X \times Y)$$

covering the usual isomorphism

$$H_0(X) \otimes H_0(Y) \cong H_0(X \times Y)\,,$$

and they are natural chain homotopy inverses.

Corollary 25.14. *There is an isomorphism* $H(S_*(X) \otimes S_*(Y)) \cong H_*(X \times Y)$.

Combining this theorem with the algebraic Künneth theorem, we get:

Theorem 25.15 (Künneth theorem)**.** *Take coefficients in a PID R. There is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y)) \to 0$$

natural in X, Y. *It splits as R-modules, but not naturally.*

Example 25.16. If $R = k$ is a field, every module is already free, so the Tor term vanishes, and you get a Künneth isomorphism:

$$\times : H_*(X;k) \otimes_k H_*(Y;k) \xrightarrow{\cong} H_*(X \times Y;k)$$

This is rather spectacular. For example, what is $H_*(\mathbf{RP}^3 \times \mathbf{RP}^3;k)$, where k is a field? Well, if k has characteristic different from 2, $\mathbf{RP}^3$ has the same homology as S^3, so the product has the same homology as $S^3 \times S^3$: the dimensions are $1,0,0,2,0,0,1$. If $\mathrm{ch} k = 2$, on the other hand, the cohomology modules are either 0 or k, and we need to form the graded tensor product:

$$\begin{array}{cccc} k & k & k & k \\ k & k & k & k \\ k & k & k & k \\ k & k & k & k \end{array}$$

so the dimensions of the homology of the product are $1,2,3,4,3,2,1$.

The palindromic character of this sequence will be explained by Poincaré duality. Let's look also at what happens over the integers. Then we have the table of tensor products

	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$
$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$
$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}/2\mathbf{Z}$
0	0	0	0	0
$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$

There is only one nonzero Tor group, namely

$$\mathrm{Tor}_1^{\mathbf{Z}}(H_1(\mathbf{RP}^3), H_1(\mathbf{RP}^3)) = \mathbf{Z}/2\mathbf{Z}.$$

Putting this together, we get the groups

H_0	$\mathbf{Z}$
H_1	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
H_2	$\mathbf{Z}/2\mathbf{Z}$
H_3	$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
H_4	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
H_5	0
H_6	$\mathbf{Z}$

The failure of perfect symmetry here is interesting, and will also be explained by Poinaré duality.

Chapter 2

Cohomology and duality

26 Coproducts, cohomology

The next topic is cohomology. This is like homology, but it's a contravariant rather than covariant functor of spaces. There are three reasons why you might like a contravariant functor.
(1) Many geometric contructions *pull back*; that is, they behave contravariantly. For example, if I have some covering space $\widetilde{X} \to X$ and a map $f : Y \to X$, I get a pullback covering space $f^*\widetilde{X}$. A better example is vector bundles (that we'll talk about in 18.906) – they don't push out, they pullback. So if we want to study them by means of "natural" invariants, these invariants will have to lie in a (hopefully computable) group that also behaves contravariantly. This will lead to the theory of *characteristic classes*.
(2) The structure induced by the diagonal map from a space to its square induces stucture in contravariant functors that is more general and easier to study.
(3) Cohomology turns out to be the target of the Poincaré duality map.

Let's elaborate on point (2). Every space has a diagonal map

$$X \xrightarrow{\Delta} X \times X\,.$$

This induces a map $H_*(X;R) \to H_*(X \times X;R)$, for any coefficient group R. Now, if R is a ring, we get a cross product map

$$\times : H_*(X;R) \otimes_R H_*(X;R) \to H_*(X \times X;R)\,.$$

If R is a PID, the Künneth Theorem tells us that this map is a monomorphism. If the remaining term in the Künneth Theorem is zero, the cross product is an isomorphism. So if $H_*(X;R)$ is free over R (or even just flat over R), we get a "diagonal" or "coproduct"

$$\Delta : H_*(X;R) \to H_*(X;R) \otimes_R H_*(X;R)\,.$$

This map is universally defined, and natural in X, if R is a field.

This kind of structure is unfamiliar, and at first seems a bit strange. After all, the tensor product is defined by a universal property for maps *out* of it; maps *into* it just are what they are.

Still, it's often useful, and we pause to fill in some of its properties.

Definition 26.1. Let R be a ring. A (graded) coalgebra over R is a (graded) R-module M equipped with a comultiplication $\Delta : M \to M \otimes_R M$ and a counit map $\varepsilon : M \to R$ such that the following diagrams commute.

$$\begin{tikzcd} & M \arrow[dl, "="'] \arrow[d, "\Delta"] \arrow[dr, "="] & \\ R \otimes_R M & M \otimes_R M \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & M \otimes_R R \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & M \otimes_R M \arrow[d, "\Delta\otimes 1"] \\ M \otimes_R M \arrow[r, "1\otimes\Delta"] & M \otimes_R M \otimes_R M \end{tikzcd}$$

It is *commutative* if in addition

$$\begin{tikzcd} & M \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ M \otimes_R M \arrow[rr, "\tau"] & & M \otimes_R M \end{tikzcd}$$

commutes, where $\tau(x \otimes y) = (-1)^{|x|\cdot|y|} y \otimes x$ is the twist map.

Using acyclic models, we saw that the the Künneth map is coassociative and cocommutative: The diagrams

$$\begin{tikzcd} S_*(X) \otimes S_*(Y) \otimes S_*(Z) \arrow[r, "\times\otimes 1"] \arrow[d, "1\otimes\times"] & S_*(X \times Y) \otimes S_*(Z) \arrow[d, "\times"] \\ S_*(X) \otimes S_*(Y \times Z) \arrow[r, "\times"] & S_*(X \times Y \times Z) \end{tikzcd}$$

and

$$\begin{tikzcd} S_*(X) \otimes S_*(Y) \arrow[r, "\tau"] \arrow[d, "\times"] & S_*(Y) \otimes S_*(X) \arrow[d, "\times"] \\ S_*(X \times Y) \arrow[r, "T_*"] & S_*(Y \times X) \end{tikzcd}$$

commute up to natural chain homotopy, where τ is as defined above on the tensor product and $T X \times Y \to Y \times X$ is the swap map.

Corollary 26.2. *Suppose R is a PID and $H_*(X;R)$ is free over R. Then $H_*(X;R)$ has the natural structure of a commutative graded coalgebra over R.*

We could now just go on and talk about coalgebras. But they are less familiar, and available only if $H_*(X;R)$ is free over R. So instead we're going to dualize, talk about cohomology, and get an algebra structure. Some say that cohomology is better because you have algebras, but that's more of a sociological statement than a mathematical one.

Let's get on with it.

Definition 26.3. Let N be an abelian group. A *singular n-cochain* on X with values in N is a function $\mathrm{Sin}_n(X) \to N$.

If N is an R-module, then I can extend linearly to get an R-module homomorphism $S_n(X;R) \to N$.

Notation 26.4. Write

$$S^n(X;N) = \mathrm{Map}(\mathrm{Sin}_n(X), N) = \mathrm{Hom}_R(S_n(X;R), N)\,.$$

This is going to give us something contravariant, that's for sure. But we haven't quite finished dualizing. The differential $d : S_{n+1}(X;N) \to S_n(X;R)$ induces a "coboundary map"

$$d : S^n(X;N) \to S^{n+1}(X;N)$$

defined by

$$(df)(\sigma) = (-1)^{n+1} f(d\sigma)\,.$$

The sign is a little strange, and we'll see an explanation in a minute. Anyway, we get a "cochain complex," with a differential that *increases* degree by 1. We still have $d^2 = 0$, since

$$(d^2 f)(\sigma) = \pm d(f(d\sigma)) = \pm f(d^2\sigma) = \pm f(0) = 0\,,$$

so we can still take homology of this cochain complex.

Definition 26.5. The nth *singular cohomology group* of X with coefficients in an abelian group N is

$$H^n(X;N) = \frac{\ker(S^n(X;N) \to S^{n+1}(X;N))}{\mathrm{im}(S^{n-1}(X;N) \to S^n(X;N))}\,.$$

Let's first compute $H^0(X;N)$. A 0-cochain is a function $\mathrm{Sin}_0(X) \to N$; that is, a function (not required to be continuous!) $f : X \to N$. To compute df, take a 1-simplex $\sigma : \Delta^1 \to X$ and evaluate f on its boundary:

$$(df)(\sigma) = -f(d\sigma) = -f(\sigma(e_0) - \sigma(e_1)) = f(\sigma(e_1)) - f(\sigma(e_0))\,.$$

So f is a co*cycle* if it's constant on path components. That is to say:

Lemma 26.6. $H^0(X;N) = \mathrm{Map}(\pi_0(X), N)$.

Warning 26.7. $S^n(X;\mathbf{Z}) = \mathrm{Map}(\mathrm{Sin}_n(X);\mathbf{Z}) = \prod_{\mathrm{Sin}_n(X)} \mathbf{Z}$, which is probably an uncountable product. An awkward fact is that this is never free abelian.

The first thing a cohomology class does is to give a linear functional on homology, by "evaluation." Let's spin this out a bit.

We want to tensor together cochains and chains. But to do that we should make the differential in $S^*(X)$ go down, not up. Just as a notational matter, let's write

$$S^\vee_{-n}(X;N) = S^n(X;N)$$

and define a differential $d : S^\vee_{-n}(X) \to S^\vee_{-n-1}(X)$ to be the differential $d : S^n(X) \to S^{n+1}(X)$. Now $S^\vee_*(X)$ is a chain complex, albeit a negatively graded one. Form the graded tensor product, with

$$(S^\vee_*(X;N) \otimes S_*(X))_n = \bigoplus_{p+q=n} S^\vee_p(X;N) \otimes S_q(X)\,.$$

Now evaluation is a degree zero chain map

$$\langle -,-\rangle : S^\vee_*(X;N) \otimes S_*(X) \to N\,,$$

where N is regarded as a chain complex concentrated in degree 0. We would like this map to be a chain map. So let $f \in S^n(X;N)$ and $\sigma \in S_n(X)$, and compute

$$0 = d\langle f,\sigma\rangle = \langle df,\sigma\rangle + (-1)^n\langle f, d\sigma\rangle\,.$$

This forces

$$(df)(\sigma) = \langle df,\sigma\rangle = -(-1)^n f(d\sigma)\,.$$

Here's the payoff: There's a natural map

$$H_{-n}(S^\vee_*(X;N)) \otimes H_n(S_*(X)) \to H_0\left(S^\vee_*(X;N) \otimes S_*(X)\right) \to N$$

This gives us the *Kronecker pairing*

$$\langle -,-\rangle : H^n(X;N) \otimes H_n(X) \to N\,.$$

We can develop the properties of cohomology in analogy with properties of homology. For example: If $A \subseteq X$, there is a restriction map $S^n(X;N) \to S^n(A;N)$, induced by the injection $\mathrm{Sin}_n(A) \hookrightarrow \mathrm{Sin}_n(X)$. And as long as A is nonempty, we can split this injection, so any function $\mathrm{Sin}_n(A) \to N$ extends to $\mathrm{Sin}_n(X) \to N$. This means that $S^n(X;N) \to S^n(A;N)$ is surjective. (This is the case if $A = \varnothing$, as well!)

Definition 26.8. The *relative n-cochain group* with coefficients in N is

$$\ker\left(S^n(X;N) \to S^n(A;N)\right)\,.$$

This defines a sub cochain complex of $S^*(X;N)$, and we define

$$H^n(X,A;N) = H^n(S^*(X,A;N))\,.$$

The short exact sequence of cochain complexes

$$0 \to S^*(X,A;N) \to S^*(X;N) \to S^*(A;N) \to 0$$

induces the *long exact cohomology sequence*

$$\begin{tikzcd}
\cdots & & \\
H^1(X,A;N) \arrow[r] & H^1(X;N) \arrow[r] & H^1(A;N) \arrow[ull, "\delta"'] \\
H^0(X,A;N) \arrow[r] & H^0(X;N) \arrow[r] & H^0(A;N)\,. \arrow[ull, "\delta"']
\end{tikzcd}$$

27 Ext and UCT

Let R be a ring (probably a PID) and N an R-module. The singular cochains on X with values in N,

$$S^*(X;N) = \mathrm{Map}(\mathrm{Sin}_*(X), N)\,,$$

then forms a cochain complex of R-modules. It is contravariantly functorial in X and covariantly functorial in N. The Kronecker pairing defines a map

$$H^n(X;N) \otimes_R H_n(X;R) \to N$$

whose adjoint

$$\beta : H^n(X;N) \to \mathrm{Hom}_R(H_n(X;R), N)$$

gives us an estimate of the cohomology in terms of the homology of X. Here's how well it does:

Theorem 27.1 (Mixed variance Universal Coefficient Theorem)**.** *Let R be a PID and N an R-module, and let C_* be a chain-complex of free R-modules. Then there is a short exact sequence of R-modules,*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), N) \to H^n(\mathrm{Hom}_R(C_*, N)) \to \mathrm{Hom}_R(H_n(C_*), N) \to 0\,,$$

natural in C_* and N, that splits (but not naturally).

Taking $C_* = S_*(X;R)$, we the short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), N) \to H^n(X;N) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(X;R), N) \to 0$$

that splits, but not naturally. This also holds for relative cohomology.

What is this Ext?

The problem that arises is that $\mathrm{Hom}_R(-,N) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is not exact. Suppose I have an injection $M' \to M$. Is $\mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N)$ surjective? Does a map $M' \to N$ necessarily extend to a map $M \to N$? No! For example, $\mathbf{Z}/2\mathbf{Z} \hookrightarrow \mathbf{Z}/4\mathbf{Z}$ is an injection, but the identity map $\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ does not extend over $\mathbf{Z}/4\mathbf{Z}$.

On the other hand, if $M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is an exact sequence of R-modules then

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$$

is again exact. Check this statement!

Now homological algebra comes to the rescue to repair the failure of exactness! Pick a free resolution of M,

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0\,.$$

Apply Hom to get a chain complex

$$0 \to \mathrm{Hom}(F_0, N) \to \mathrm{Hom}(F_1, N) \to \mathrm{Hom}(F_2, N) \to \cdots\,.$$

Definition 27.2. $\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(F_*,N))$.

Remark 27.3. Ext is well-defined and functorial, by the fundamental lemma of homological algebra. If M is free (or projective) then $\mathrm{Ext}^n(M,-) = 0$ for $n > 0$, since we can take M as its own projective resolution. If R is a PID, then we can assume $F_1 = \ker(F_0 \to M)$ and $F_n = 0$ for $n > 1$, so $\mathrm{Ext}^n = 0$ if $n > 1$. If R is a field, then $\mathrm{Ext}^n = 0$ for $n > 0$.

Example 27.4. Let $R = \mathbf{Z}$ and take $M = \mathbf{Z}/k\mathbf{Z}$. This admits a simple free resolution: $0 \to \mathbf{Z} \xrightarrow{k} \mathbf{Z} \to \mathbf{Z}/k\mathbf{Z} \to 0$. Apply $\mathrm{Hom}(-,N)$ to it, and remember that $\mathrm{Hom}(\mathbf{Z},N) = N$, to get the very short cochain complex, with entries in dimensions 0 and 1:

$$0 \to N \xrightarrow{k} N \to 0\,.$$

Taking homology gives us

$$\mathrm{Hom}(\mathbf{Z}/k\mathbf{Z},N) = \ker(k|N) \qquad \mathrm{Ext}^1(\mathbf{Z}/k\mathbf{Z},N) = N/kN\,.$$

Proof. of Theorem 27.1 First of all, notice that

$$C_n/Z_n \cong B_{n-1}$$

is a submodule C_{n-1} and hence is free. Thus both of the following short exact sequences split:

$$0 \to Z_n \to C_n \to C_n/Z_n \to 0 \tag{2.1}$$

$$0 \to Z_n/B_n \to C_n/B_n \to C_n/Z_n \to 0\,. \tag{2.2}$$

Note that the second one can be rewritten as

$$0 \to H_n \to C_n/B_n \to B_{n-1} \to 0\,.$$

Start with the diagram

$$\begin{tikzcd}
0 \arrow[r] & B^n\,\mathrm{Hom}(C_*,N) \arrow[r] \arrow[d, dotted] & Z^n\,\mathrm{Hom}(C_*,N) \arrow[r] \arrow[d, "\cong"] & H^n(\mathrm{Hom}(C_*,N)) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1},N) \arrow[r] & \mathrm{Hom}(C_n/B_n,N) \arrow[r] & \mathrm{Hom}(H_n,N) \arrow[r] & 0
\end{tikzcd}$$

The bottom row arises from (2.2) and is exact because (2.2) splits. The middle arrow starts with $f : C_n \to N$ such that $C_{n+1} \xrightarrow{d} C_n \xrightarrow{f} N$ is zero. This condition is equivalent to requiring that f kill boundaries, and so it factors through a unique map $C_n/B_n \to N$.

We claim that the composite $B^n\,\mathrm{Hom}(C_*,N) \to \mathrm{Hom}(H_n,N)$ is trivial. So start with $f : C_n \to N$ such that $C_{n+1} \to C_n \to N$ is trivial. Then f kills B_n and so factors through C_n/B_n, giving an element of $\mathrm{Hom}(C_n/B_n,N)$; but it also kills the larger submodule Z_n, and hence factors through C_n/Z_n. This implies that the composite $H_n \to C_n/B_n \to N$ is trivial since $H_n \to C_n/B_n \to C_n/Z_n$ is.

So we can fill in the maps to get a map of short exact sequences. By the snake lemma, the right arrow is surjective and its kernel K fits into the short exact sequence at the top of the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & B^n \operatorname{Hom}(C_*, N) \arrow[r] \arrow[d, dotted, two heads] & \operatorname{Hom}(B_{n-1}, N) \arrow[r] \arrow[d, "="] & K \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & I \arrow[r] & \operatorname{Hom}(B_{n-1}, N) \arrow[r] & \operatorname{Ext}^1(H_{n-1}, N) \arrow[r] & 0
\end{tikzcd}$$

The bottom row of this diagram comes from the long exact sequence associated to the short exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1} \to 0\,,$$

so

$$I = \operatorname{im}\left(\operatorname{Hom}(Z_{n-1}, N) \to \operatorname{Hom}(B_{n-1}, N)\right)\,.$$

We claim that there is a surjection on the left as shown (making the square commutative). This completes the proof, since the snake lemma then implies that the right arrow is an isomorphism.

The left arrow occurs at the right of the diagram

$$\begin{tikzcd}
0 \arrow[r] & Z^{n-1} \operatorname{Hom}(C_*, N) \arrow[r] \arrow[d, dotted] & \operatorname{Hom}(C_{n-1}, N) \arrow[r] \arrow[d, two heads] & B^n \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \operatorname{Hom}(H_{n-1}, N) \arrow[r] & \operatorname{Hom}(Z_{n-1}, N) \arrow[r] & I \arrow[r] & 0
\end{tikzcd}$$

Here the middle surjection is induced from the split injection $Z_{n-1} \to C_{n-1}$. We need to construct the left arrow; this will finish the proof, since then the right arrow exists and is surjective again by the snake lemma.

So let $f : C_{n-1} \to N$ be such that $C_n \to C_{n-1} \to N$ is trivial. Its image under the vertical surjection is the composite $Z_{n-1} \to C_{n-1} \to N$. Now

$$\begin{tikzcd}
C_n \arrow[r] \arrow[d, two heads] & C_{n-1} \arrow[r] & N & \\
B_{n-1} \arrow[r] & Z_{n-1} \arrow[r] \arrow[u] \arrow[ur] & H_{n-1} \arrow[r] \arrow[u, dotted] & 0
\end{tikzcd}$$

The composite $B_{n-1} \to N$ is trivial since $C_n \to B_{n-1}$ is surjective, and the desired factorization through H_{n-1} follows. □

Remark 27.5. Question: Why is Ext called Ext?

Answer: It classifies extensions. Let R be a commutative ring, and let M, N be two R-modules. I can think about "extensions of M by N, that is, short exact sequences of the form

$$0 \to N \to L \to M \to 0\,.$$

For example, I have two extensions of $\mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$:

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

and

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0\,.$$

We'll say that two extensions are equivalent if there's a map of short exact sequences between them is the identity on N and M. The two extensions above aren't equivalent, for example.

Another definition of $\operatorname{Ext}^1_R(M,N)$ is the set of extensions like this modulo this notion of equivalence. The zero in the group is the split extension.

The universal coefficient theorem is useful in transferring properties of homology to cohomology. For example, if $f : X \to Y$ is a map that induces an isomorphism in $H_*(-;R)$, then it induces an isomorphism in $H^*(-;N)$ for any R-module N, at least provided that R is a PID. (This is true in general, however.)

Cohomology satisfies the appropriate analogues of the Eilenber Steenrod axioms.

Homotopy invariance: If $f_0 \sim f_1 : (X,A) \to (Y,B)$, then

$$f_0^* = f_1^* : H^*(Y,B;N) \to H^*(X,A;N)\,.$$

I can't use the UCT to address this because the UCT only tells you that things are isomorphic. But we did establish a chain homotopy $f_{0,*} \sim f_{1,*} : S_*(X,A) \to S_*(Y,B)$, and applying Hom converts chain homotopies to cochain homotopies.

Excision: If $U \subseteq A \subseteq X$ such that $\overline{U} \subseteq \operatorname{Int}(A)$, then $H^*(X,A;N) \to H^*(X-U, A-U;N)$ is an isomorphism. This follows from excision in homology and by the mixed variance UCT.

Mayer-Vietoris sequence: If $A, B \subseteq X$ are such that their interiors cover X, then there is a long exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
H^n(X;N) \arrow[r] & H^n(A;N) \oplus H^n(B;N) \arrow[r] & H^n(A \cap B;N) \arrow[dll] \\
H^{n+1}(X;N) \arrow[r] & \cdots
\end{tikzcd}$$

Milnor axiom: The inclusions induce an isomorphism

$$H^*(\coprod_\alpha X_\alpha;N) \to \prod_\alpha H^*(X_\alpha;N)\,.$$

28 Products in cohomology

We'll talk about the cohomology cross product first. Actually, the first step is to produce a map on chains that goes in the reverse direction from the cross product we constructed in Lecture 7.

Construction 28.1. For each pair of natural numbers p, q, we will define a natural homomorphism

$$\alpha : S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y)\,.$$

It suffices to define this on simplices, so let $\sigma : \Delta^{p+q} \to X \times Y$ be a singular $(p+q)$-simplex in the product. I can write $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ where $\sigma_1 : \Delta^{p+q} \to X$ and $\sigma_2 : \Delta^{p+q} \to Y$. I have to produce a p-simplex in X and a q-simplex in Y. First define two maps in the simplex category: the "front face" $\alpha_p : [p] \to [p+q]$, sending i to i for $0 \le i \le p$, and the "back face" $\omega_q : [q] \to [p+q]$, sending j to $j+p$ for $0 \le j \le q$. Use the same symbols for the affine extensions to maps $\Delta^p \to \Delta^{p+q}$ and $\Delta^q \to \Delta^{p+q}$. Now let

$$\alpha(\sigma) = \sigma_1 \circ \alpha_p \otimes \sigma_2 \circ \omega_q\,.$$

This seems like a very random construction; but it works! It's named after two great early algebraic topologists, Alexander and Whitney. For homework, you will show that these maps assemble into a chain map

$$\alpha : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)\,.$$

This works over any ring R. To get a map in cohomology, we should form

$$S^p(X;R) \otimes_R S^q(Y;R) \to \mathrm{Hom}_R(S_p(X;R) \otimes_R S_q(Y;R), R) \xrightarrow{\alpha^*} \mathrm{Hom}_R(S_{p+q}(X \times Y;R), R) = S^{p+q}(X \times Y)\,.$$

The first map goes like this: Given chain complexes C_* and D_*, we can consider the dual cochain complexes $\mathrm{Hom}_R(C_*, R)$ and $\mathrm{Hom}_R(D_*, R)$, and construct a chain map

$$\mathrm{Hom}_R(C_*, R) \otimes_R \mathrm{Hom}_R(D_*, R) \to \mathrm{Hom}_R(C_* \otimes_R D_*, R)$$

by

$$f \otimes g \mapsto \begin{cases} (x \otimes y \mapsto (-1)^{pq} f(x) g(y)) & |x| = |f| = p, |y| = |g| = q \\ 0 & \text{otherwise.} \end{cases}$$

Again, I leave it to you to check that this is a chain map.

Altogether, we have constructed a natural chain map

$$\times : S^p(X) \otimes S^q(Y) \to S^{p+q}(X \times Y)$$

From this, we get a homomorphism

$$H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y)\,.$$

I'm not quite done! As in the Künneth theorem, there is an evident natural map

$$H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) .$$

The composite

$$\times : H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y)$$

is the *cohomology cross product*.

It's not very easy to do computations with this, directly. We'll find indirect means. Let me make some points about this construction, though.

Definition 28.2. The *cup product* is the map obtained by taking $X = Y$ and composing with the map induced by the diagonal $\Delta : X \to X \times X$:

$$\cup : H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X), .$$

These definitions make good sense with any ring for coefficients.

Let's explore this definition in dimension zero. I claim that $H^0(X; R) \cong \mathrm{Map}(\pi_0(X), R)$ as rings. When $p = q = 0$, both α_0 and ω_0 are the identity maps, so we are just forming the pointwise product of functions.

There's a distinguished element in $H^0(X)$, namely the the function $\pi_0(X) \to R$ that takes on the value 1 on every path component. This is the identity for the cup product. This comes out because when $p = 0$ in our above story, then α_0 is just including the 0-simplex, and ω_q is the identity.

The cross product is also associative, even on the chain level.

Proposition 28.3. *Let $f \in S^p(X)$, $g \in S^q(Y)$, and $h \in S^r(Z)$, and let $\sigma : \Delta^{p+q+r} \to X \times Y \times Z$ be any simplex. Then*

$$((f \times g) \times h)(\sigma) = (f \times (g \times h))(\sigma) .$$

Proof. Write σ_{12} for the composite of σ with the projection map $X \times Y \times Z \to X \times Y$, and so on. Then

$$((f \times g) \times h)(\sigma) = (-1)^{(p+q)r}(f \times g)(\sigma_{12} \circ \alpha_{p+q})h(\sigma_3 \circ \omega_r) .$$

But

$$(f \times g)(\sigma_{12} \circ \alpha_{p+q}) = (-1)^{pq} f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \mu_q) ,$$

where μ_q is the "middle face," sending ℓ to $\ell + p$ for $0 \le \ell \le q$. In other words,

$$((f \times g) \times h)(\sigma) = (-1)^{pq+qr+rp} f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \mu_q) h(\sigma_3 \circ \omega_r) .$$

I've used associativity of the ring. But you get exactly the same thing when you expand $(f \times (g \times h))(\sigma)$, so the cross product is associative. □

Of course the diagonal map is "associative," too, and we find that the cup product is associative:

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)\,.$$

But this product is obviously not commutative on the level of cochains. It treats the two maps completely differently. But we have ways of dealing with this. You will show for homework that the method of acyclic models shows that

$$\alpha \cup \beta = (-1)^{|\alpha|\cdot|\beta|} \beta \cup \alpha\,.$$

So $H^*(X; R)$ forms a *commutative graded R-algebra.*

29 Cup product, continued

We have constructed an explicit map $S^p(X) \otimes S^q(Y) \xrightarrow{\times} S^{p+q}(Y)$ via:

$$(f \times g)(\sigma) = (-1)^{pq} f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \omega_q)$$

where $\alpha_p : \Delta^p \to \Delta^{p+q}$ takes $k \mapsto k$ for $0 \le k \le p$ and $\omega_q : \Delta^q \to \Delta^{p+q}$ sends $h \mapsto j + p$ for $0 \le j \le q$. This is a chain map, and it induces a "cross product" $H(S^p(X) \otimes S^q(Y)) \to H_{p+q}(X \times Y)$ and, by composing with the map induced by the diagonal embedding, a "cup product"

$$\cup : H^p(X) \otimes H^q(X) \to H^{p+q}(X)\,.$$

We formalize the structure that this product imposes on cohomology.

Definition 29.1. Let R be a commutative ring. A *graded R-algebra* is a graded R-module $\cdots, A_{-1}, A_0, A_1, A_2, \cdots$ equipped with maps $A_p \otimes_R A_q \to A_{p+q}$ and a map $\eta : R \to A_0$ that make the following diagram commute.

$$\begin{tikzcd}
A_p \otimes_R R \arrow[r, "1\otimes\eta"] \arrow[dr, "="'] & A_p \otimes_R A_0 \arrow[d] & A_0 \otimes_R A_q \arrow[d] & R \otimes_R A_q \arrow[l, "\eta\otimes 1"'] \arrow[dl, "="] \\
 & A_p & A_q &
\end{tikzcd}$$

$$\begin{tikzcd}
A_p \otimes_R (A_q \otimes_R A_r) \arrow[r] \arrow[d] & A_p \otimes_R A_{q+r} \arrow[d] \\
A_{p+q} \otimes_R A_r \arrow[r] & A_{p+q+r}
\end{tikzcd}$$

A graded R-algebra A is *commutative* if the following diagram commutes:

$$\begin{tikzcd}
A_p \otimes_R A_q \arrow[rr, "\tau"] \arrow[dr] & & A_q \otimes_R A_p \arrow[dl] \\
 & A_{p+q} &
\end{tikzcd}$$

where $\tau(x \otimes y) = (-1)^{pq} y \otimes x$.

We claim that $H^*(X;R)$ forms a commutative graded R-algebra under the cup product. This is nontrivial. On the cochain level, this is clearly not graded commutative. We're going to have to work hard – in fact, so hard that you're going to do some of it for homework. What needs to be checked is that the following diagram commutes up to natural chain homotopy.

$$\begin{tikzcd} S_*(X\times Y) \arrow[r, "T_*"] \arrow[d, "\alpha_{X,Y}"'] & S_*(Y\times X) \arrow[d, "\alpha_{Y,X}"] \\ S_*(X)\otimes_R S_*(Y) \arrow[r, "\tau"] & S_*(Y)\otimes_R S_*(X) \end{tikzcd}$$

Acyclic models helps us prove things like this.

You might hope that there is some way to produce a commutative product on a chain complex modeling $H^*(X)$. With coefficients in $\mathbf{Q}$, this is possible, by a construction due to Dennis Sullivan. With coefficients in a field of nonzero characteristic, it is not possible. Steenrod operations provide the obstruction.

My goal now is to compute the cohomology algebras of some spaces. Some spaces are easy! There is no choice for the product structure on $H^*(S^n)$, for example. (When $n=0$, we get a free module of rank 2 in dimension 0. This admits a variety of commutative algebra structures; but we have already seen that $H^0(S_0)=\mathbf{Z}\times\mathbf{Z}$ as an algebra.) Maybe the next thing to try is a product of spheres. More generally, we should ask whether there is an algebra structure on $H^*(X)\otimes H^*(Y)$ making the cross product an algebra map. If A and B are two graded algebras, there *is* a natural algebra structure on $A\otimes B$, given by $1=1\otimes 1$ and

$$(a'\otimes b')(a\otimes b)=(-1)^{|b'|\cdot|a|}a'a\otimes b'b\,.$$

If A and B are commutative, then so is $A\otimes B$ with this algebra structure.

Proposition 29.2. *The cohomology cross product*

$$\times : H^*(X)\otimes H^*(Y)\to H^*(X\times Y)$$

is an R-algebra homomorphism.

Proof. I have diagonal maps $\Delta_X : X\to X\times X$ and $\Delta_Y : Y\to Y\times Y$. The diagonal on $X\times Y$ factors as

$$\begin{tikzcd} X\times Y \arrow[rr, "\Delta_{X\times Y}"] \arrow[dr, "\Delta_X\times\Delta_Y"'] & & X\times Y\times X\times Y \\ & X\times X\times Y\times Y\,. \arrow[ur, "1\times T\times 1"'] & \end{tikzcd}$$

Let $\alpha_1, \alpha_2 \in H^*(X)$ and $\beta_1, \beta_2 \in H^*(Y)$. Then $\alpha_1 \times \beta_1, \alpha_2 \times \beta_2 \in H^*(X \times Y)$, and I want to calculate $(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2)$. Let's see:

$$\begin{aligned}(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) &= \Delta^*_{X\times Y}(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2)\\ &= (\Delta_X \times \Delta_Y)^*(1 \times T \times 1)^*(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2)\\ &= (\Delta_X \times \Delta_Y)^*(\alpha_1 \times T^*(\beta_1 \times \alpha_2) \times \beta_2)\\ &= (-1)^{|\alpha_2|\cdot|\beta_1|}(\Delta_X \times \Delta_Y)^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2)\end{aligned}$$

Now, I have a diagram:

$$\begin{tikzcd} H^*(X \times Y) & & H^*(X) \otimes_R H^*(Y) \arrow[ll, "\times_{X\times Y}"'] \\ H^*(X \times X \times Y \times Y) \arrow[u, "(\Delta_X \times \Delta_Y)^*"] & & H^*(X \times X) \otimes H^*(Y \times Y) \arrow[ll, "\times_{X\times X, Y\times Y}"'] \arrow[u, "\Delta_X^* \otimes \Delta_Y^*"'] \end{tikzcd}$$

This diagram commutes because the cross product is natural. We learn:

$$\begin{aligned}(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) &= (-1)^{|\alpha_2|\cdot|\beta_1|}(\Delta_X \times \Delta_Y)^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2)\\ &= (-1)^{|\alpha_2|\cdot|\beta_1|}(\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)\,.\end{aligned}$$

That's exactly what we wanted. □

Example 29.3. How about $H^*(S^p \times S^q)$? I'll assume that p and q are both positive, and leave the other cases to you. The Künneth theorem guarantees that $\times : H^*(S^p) \otimes H^*(S^q) \to H^*(S^p \times S^q)$ is an isomorphism. Write α for a generator of S^p and β for a generator of S^q; and use the same notations for the pullbacks of these elements to $S^p \times S^q$ under the projections. Then

$$H^*(S^p \times S^q) = \mathbf{Z}\langle 1, \alpha, \beta, \alpha\beta\rangle\,,$$

and

$$\alpha^2 = 0\,,\quad \beta^2 = 0\,,\quad \alpha\beta = (-1)^{pq}\beta\alpha\,.$$

This calculation is useful!

Corollary 29.4. *Let $p, q > 0$. Any map $S^{p+q} \to S^p \times S^q$ induces the zero map in $H^{p+q}(-)$.*

Proof. Let $f : S^{p+q} \to S^p \times S^q$ be such a map. It induces an algebra map $f^* : H^*(S^p \times S^q) \to H^*(S^{p+q})$. This map must kill α and β, for degree reasons. But then it also kills their product, since f^* is multiplicative. □

The space $S^p \vee S^q \vee S^{p+q}$ has the same cohomology groups as $S^p \times S^q$. Both are built as CW complexes with cells in dimensions $0, p, q$, and $p+q$. But they are not homotopy equivalent. We can see this now because there *is* a map $S^p \vee S^q \vee S^{p+q} \to S^{p+q}$ inducing an *isomorphism* in $H^{p+q}(-)$, namely, the map that pinches the other two factors to the basepoint.

30 Surfaces and nondegenerate symmetric bilinear forms

We are aiming towards a proof of a fundamental cohomological property of compact manifolds.

Definition 30.1. A (topological) manifold is a Hausdorff space such that every point has an open neighborhood that is homeomorphic to some (finite dimensional) Euclidean space.

If all these Euclidean space can be chosen to be $\mathbf{R}^n$, we have an n-manifold.

We will always make two additional assumptions about our manifolds.

Second countable: There is a countable neighborhood basis.

This is to avoid manifolds like an uncountable discrete set.

Good cover: There is a cover by Euclidean opens such that all finite intersections are either empty or again Euclidean.

This can be acheived in a smooth manifold by picking a metric and using convex neighborhoods.

In this lecture we will state a case of the Poincaré duality theorem and study some consequences of it, especially for compact 2-manifolds. This whole lecture will be happening with coefficients in $\mathbf{F}_2$.

Theorem 30.2. *Let M be a compact manifold of dimension n. There exists a unique class $[M] \in H_n(M)$, called the* fundamental class*, such that for every p,q with $p+q=n$ the pairing*

$$H^p(M) \otimes H^q(M) \xrightarrow{\cup} H^n(M) \xrightarrow{\langle -,[M]\rangle} \mathbf{F}_2$$

is perfect.

This means that the adjoint map

$$H^p(M) \to \mathrm{Hom}(H^q(M), \mathbf{F}_2)$$

is an isomorphism. Since cohomology vanishes in negative dimensions, one thing this implies is that $H^p(M) = 0$ for $p > n$. Since M is compact, $\pi_0(M)$ is finite, and

$$H^n(M) = \mathrm{Hom}(H^0(M), \mathbf{F}_2) = \mathrm{Hom}(\mathrm{Map}(\pi_0(M), \mathbf{F}_2), \mathbf{F}_2) = \mathbf{F}_2[\pi_0(M)]\,.$$

A vector space V admitting a perfect pairing $V \otimes W \to \mathbf{F}_2$ is necessarily finite dimensional; so $H^p(M)$ is in fact finite-dimensional for all p.

Combining this pairing with the universal coefficient theorem, we get isomorphisms

$$H^{n-p}(M) \xrightarrow{\cong} \mathrm{Hom}(H^p(M), \mathbf{F}_2) \xleftarrow{\cong} H_p(M)\,.$$

The homology and cohomology classes corresponding to each other under this isomorphism are said to be "Poincaré dual."

Using these isomorphisms, the cup product pairing can be rewritten as a homology pairing:

$$\begin{tikzcd} H_p(M) \otimes H_q(M) \arrow[r, "\pitchfork"] \arrow[d, "\cong"] & H_{n-p-q}(M) \arrow[d, "\cong"] \\ H^{n-p}(M) \otimes H^{n-q}(M) \arrow[r, "\cup"] & H^{2n-p-q}(M) . \end{tikzcd}$$

This is the *intersection pairing.* Here's how to think of this. Take homology classes $\alpha \in H_p(M)$ and $\beta \in H_q(M)$ and represent them (if possible!) as the image of the fundamental classes of submanifolds of M, of dimensions p and q. Move them if necessary to make them intersect "transversely." Then their intersection will be a submanifold of dimension $n-p-q$, and it will represent the homology class $\alpha \pitchfork \beta$.

This relationship between the cup product and the intersection pairing is the source of the symbol for the cup product.

Example 30.3. Let $M = T^2 = S^1 \times S^1$. We know that

$$H^1(M) = \mathbf{F}_2\langle \alpha, \beta \rangle$$

and $\alpha^2 = \beta^2 = 0$, while $\alpha\beta = \beta\alpha$ generates $H^2(M)$. The Poincaré duals of these classes are represented by cycles wrapping around one or the other of the two factor circles. They can be made to intersect in a single point. This reflects the fact that

$$\langle \alpha \cup \beta, [M] \rangle = 1 .$$

Similarly, the fact that $\alpha^2 = 0$ reflects the fact that its Poincaré dual cycle can be moved so as not to intersect itself.

This example exhibits a particularly interesting fragment of the statement of Poincaré duality: In an even dimensional manifold – say $n = 2k$ – the cup product pairing gives us a nondegenerate symmetric bilinear form on $H^k(M)$. As indicated above, this can equally well be considered a bilinear form on $H_k(M)$, and it is then to be thought of as describing the number of points (mod 2) two k-cycles intersect in, when put in general position relative to one another. It's called the *intersection form.* We'll denote it

$$\alpha \cdot \beta = \langle \alpha \cup \beta, [M] \rangle .$$

Example 30.4. In terms of the basis α, β, the intersection form has matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is a "hyperbolic form."

Let's discuss finite dimensional nondegenerate symmetric bilinear forms over $\mathbf{F}_2$ in general. A form on V restricts to a form on any subspace $W \subseteq V$, but the restricted form may be degenerate. Any subspace has an *orthogonal complement*

$$W^\perp = \{v \in V : v \cdot w = 0 \text{ for all } w \in W\} .$$

Lemma 30.5. *The restriction of a nondegenerate bilinear form on V to a subspace W is nondegenerate exactly when $W \cap W^\perp = 0$. In that case $W^\perp$ is also nondegenerate, and the splitting*

$$V \cong W \oplus W^\perp$$

respects the forms.

Using this easy lemma, we may inductively decompose a general (finite dimensional) symmetric bilinear form. First, if there is a vector $v \in V$ such that $v \cdot v = 1$, then it generates a nondegenerate subspace and

$$V = \langle v \rangle \oplus \langle v \rangle^\perp .$$

Continuing to split off one-dimensional subspaces brings us to the situation of a nondegenerate symmetric bilinear form such that $v \cdot v = 0$ for every vector. Unless $V = 0$ we can pick a nonzero vector. Since the form is nondegenerate, we may find another vector w such that $v \cdot w = 1$. The two together generate a 2-dimensional hyperbolic subspace. Split it off and continue. We conclude:

Proposition 30.6. *Any finite dimensional nondegenerate symmetric bilinear form splits as an orthogonal direct sum of forms with matrices* $[1]$ *and* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Let **Bil** be the set of isomorphism classes of finite dimensional nondegenerate symmetric bilinear forms over $\mathbf{F}_2$. I've just given a classification of these things. This is a commutative monoid under orthogonal direct sum. It can be regarded as the set of nonsingular symmetric matrices modulo the equivalence relation of "similarity": Two matrices M and N are *similar* if $N = AMA^T$ for some nonsingular A.

Claim 30.7.

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Proof. This is the same thing as saying that $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} = AA^T$ for some nonsingular A.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. □

It's easy to see that there are no further relations; **Bil** is the commutative monoid with two generators I and H, subject to the relation $I + H = 3I$.

Let's go back to topology. Let $n = 2$, $k = 1$ (so that $2k = n$). Then you get an intersection pairing on $H_1(M)$. Consider $\mathbf{RP}^2$. We know that $H_1(\mathbf{RP}^2) = \mathbf{F}_2$. This must be the form we labelled I. This says that anytime you have a nontrivial cycle on a projective plane, there's nothing I can do to remove its self interesections. You can see this. The projective plane is a Möbius band with a disk sown on along the boundary. The waist of the Möbius band serves as a generating cycle. The observation is that if this cycle is moved to intersect itself tranversely, it must intersect itself an odd number of times.

We can produce new surfaces from old by a process of "addition." Given two connected surfaces Σ_1 and Σ_2, cut a disk out of each one and sew them together along the resulting circles. This is the *connected sum* $\Sigma_1 \# \Sigma_2$.

Proposition 30.8. *There is an isomorphism*

$$H^1(\Sigma_1 \# \Sigma_2) \cong H^1(\Sigma_1) \oplus H^1(\Sigma_2)$$

compatible with the intersection forms.

Proof. Let's compute the cohomology of $\Sigma_1 \# \Sigma_2$ using Mayer-Vietoris. The two dimensional cohomology of $\Sigma_i - D^2)$ vanishes because the punctured surface retracts onto its 1-skeleton. The relevant fragment is

$$0 \to H^1(\Sigma_1 \# \Sigma_2) \to H^1(\Sigma_1 - D^2) \oplus H^1(\Sigma_2 - D^2) \to H^1(S^1) \xrightarrow{\delta} H^2(\Sigma_1 \# \Sigma_2) \to 0\,.$$

The boundary map must be an isomorphism, because the connected sum is a compact connected surface so has nontrivial H^2. □

The classification of surfaces may now be summarized as folows:

Theorem 30.9. *Formation of the intersection bilinear form gives isomorphism of commutative monoids* **Surf** $\to$ **Bil**.

This is a kind of model result of algebraic topology! – a complete algebraic classification of a class of geometric objects. The oriented surfaces correspond to the bilinear forms of type gH; g is the *genus*. But it's a little strange. We must have a relation corresponding to $H \oplus I = 3I$, namely

$$T^2 \# \mathbf{RP}^2 \cong (\mathbf{RP}^2)^{\#3}\,.$$

The two-fold connected sum $\mathbf{RP}^2 \# \mathbf{RP}^2$ is the Klein bottle K. In fact, more generally

Claim 30.10. If Σ is a nonoriented surface then $\Sigma \# T^2 \cong \Sigma \# K$.

There's more to be said about this. Away from characteristic 2, symmetric bilinear forms and quadratic forms are interchangeable. But over $\mathbf{F}_2$ you can ask for a quadratic form q such that

$$q(x+y) = q(x) + q(y) + x \cdot y\,.$$

This is a "quadratic refinement" of the symmetric bilinear form. Of course it implies that $x \cdot x = 0$ for all x, so this will correspond to some further structure on an oriented surface. This structure is a "framing," a trivialization of the normal bundle of an embedding into a high dimensional Euclidean space. There are then further invariants of this framing; this is the story of the Kervaire invariant.

31 A plethora of products

Recall that we have the Kronecker pairing

$$\langle -, - \rangle : H^p(X : R) \otimes H_p(X : R) \to R\,.$$

It's obviously not "natural," because H^p is contravariant while homology is covariant. But given $f : X \to Y$, $b \in H^p(Y)$, and $x \in H_p(X)$, we can ask: How does $\langle f^* b, x\rangle$ relate to $\langle b, f_* x\rangle$?

Claim 31.1. $\langle f^* b, x\rangle = \langle b, f_* x\rangle$.

Proof. This is easy! I find it useful to write out diagrams of where things are. We're going to work on the chain level.

$$\begin{tikzcd}
\mathrm{Hom}(S_p(Y), R) \otimes S_p(X) \arrow[r, "1\otimes f_*"] \arrow[d, "f^*\otimes 1"] & \mathrm{Hom}(S_p(Y), R) \otimes S_p(Y) \arrow[d, "\langle -,- \rangle"] \\
\mathrm{Hom}(S_p(X), R) \otimes S_p(X) \arrow[r, "\langle -,- \rangle"] & R
\end{tikzcd}$$

We want this diagram to commute. Suppose $[\beta] = b$ and $[\xi] = x$. Then from the to left, going to the right and then down gives

$$\beta \otimes \xi \mapsto \beta \otimes f_*(\xi) \mapsto \beta(f_*\xi)\,.$$

The other way gives

$$\beta \otimes \xi \mapsto f^*(\beta) \otimes \xi = (\beta \circ f_*) \otimes \xi \mapsto (\beta \circ f_*)(\xi)\,.$$

This is exactly $\beta(f_*\xi)$. □

There's actually another product around:

$$\mu : H(C_*) \otimes H(D_*) \to H(C_* \otimes D_*)$$

given by $[c] \otimes [d] \mapsto [c \otimes d]$. I used it to pass from the chain level computation we did to the homology statement.

We also have the two cross products:

$$\times : H_p(X) \otimes H_q(Y) \to H_{p+q}(X \times Y)$$

and

$$\times : H^p(X) \otimes H^q(Y) \to H^{p+q}(X \times Y)\,.$$

You should think of this as fishy because both maps are in the same direction. This is OK because we used different things to make these constructions: the chain-level cross product (or Eilenberg-Zilber map) for homology and the Alexander-Whitney map for cohomology. Still, they're related:

Lemma 31.2. *Let $a \in H^p(X), b \in H^q(Y), x \in H_p(X), y \in H_q(Y)$. Then:*

$$\langle a \times b, x \times y \rangle = (-1)^{|x| \cdot |b|} \langle a, x \rangle \langle b, y \rangle$$

Proof. Look at the chain-level cross product and the Alexander-Whitney map

$$\times : S_*(X) \otimes S_*(Y) \leftrightarrows S_*(X \times Y) : \alpha$$

Both of them are the identity in dimension 0, and both sides are projective resolutions with respect to the models (Δ^p, Δ^q); so by acyclic models they are natural chain homotopy inverses.

Say $[f] = a, [g] = b, [\xi] = x, [\eta] = y$. Write fg for the composite

$$S_p(X) \otimes S_q(Y) \xrightarrow{\times} S_{p+q}(X \times Y) \xrightarrow{f \otimes g} R \otimes R \to R\,.$$

Then:

$$(f \times g)(\xi \times \eta) = (fg)\alpha(\xi \times \eta) \sim (fg)(\xi \otimes \eta) = (-q)^{pq} f(\xi) g(\eta)\,.$$

□

We can use this to prove a restricted form of the Künneth theorem in cohomology.

Theorem 31.3. *Let R be a PID. Assume that $H_p(X)$ is a finitely generated free R-module for all p. Then*

$$\times : H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$$

is an isomorphism.

Proof. Write $M^\vee$ for the linear dual of an R-module M. By our assumption about $H_p(X)$, the map

$$H_p(X)^\vee \otimes H_q(Y)^\vee \to (H_p(X) \otimes H_q(Y))^\vee\,,$$

sending $f \otimes g$ to $(x \otimes y \mapsto (-1)^{pq} f(x) g(y))$, is an isomorphism. The homology Künneth theorem guarantees that the bottom map in the following diagram is an isomorphism.

$$\begin{tikzcd}
\bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) \arrow[rr, "\times"] \arrow[d, "\cong"] & & H^n(X \times Y) \arrow[d, "\cong"] \\
\bigoplus_{p+q=n} H_p(X)^\vee \otimes H_q(Y)^\vee \arrow[r, "\cong"] & \left(\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y)\right)^\vee & H_n(X \times Y)^\vee \arrow[l, "\cong"']
\end{tikzcd}$$

Commutativity of this diagram is exactly the content of Lemma 31.2. □

We saw before that $\times$ is an algebra map, so under the conditions of the theorem it is an isomorphism of algebras. You do need some finiteness assumption, even if you are working over a field. For example let T be an infinite set, regarded as a space with the discrete topology. Then $H^0(T; R) = \mathrm{Map}(T, R)$. But

$$\mathrm{Map}(T, R) \otimes \mathrm{Map}(T, R) \to \mathrm{Map}(T \times T, R)$$

sending $f \otimes g$ to $(s,t) \to f(s)g(t)$ is not surjective; the characteristic function of the diagonal is not in the image, for example.

There are more products around. For example, there is a map

$$H^p(Y) \otimes H^q(X,A) \to H^{p+q}(Y \times X, Y \times A)\,.$$

Constructing this is on your homework. Suppose $Y = X$. Then I get

$$\cup : H^*(X) \otimes H^*(X,A) \to H^*(X \times X, X \times A) \xrightarrow{\Delta^*} H^*(X,A)$$

where $\Delta : (X,A) \to (X \times X, X \times A)$ is the "relative diagonal." This "relative cup product" makes $H^*(X,A)$ into a module over the graded algebra $H^*(X)$. The relative cohomology is *not* a ring – it doesn't have a unit, for example – but it is a module. And the long exact sequence of the pair is a sequence of $H^*(X)$-modules.

I want to introduce you to one more product, which will enter into our expression of Poincaré duality. This is the *cap product*. What can I do with $S^p(X) \otimes S_n(X)$? Well, I can form the composite:

$$S^p(X) \otimes S_n(X) \xrightarrow{1 \times (\alpha_{X,X} \circ \Delta_*)} S^p(X) \otimes S_p(X) \otimes S_{n-p}(X) \xrightarrow{\langle -,- \rangle \otimes 1} S_{n-p}(X)$$

Using our explicit formula for α, we can write:

$$\cap : \beta \otimes \sigma \mapsto \beta \otimes (\sigma \circ \alpha_p) \otimes (\sigma \circ \omega_q) \mapsto (\beta(\sigma \circ \alpha_p))\,(\sigma \circ \omega_q)$$

We are evaluating the cochain on *part of* the chain, leaving a lower dimensional chain left over.

This composite is a chain map, and so induces a map in homology:

$$\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X)\,.$$

Here are some properties of the cap product.

Lemma 31.4. $(\alpha \cup \beta) \cap x = \alpha \cap (\beta \cap x)$ *and* $1 \cap x = x$.

Proof. Easy to check from the definition. □

This makes $H_*(X)$ into a module over $H^*(X)$. These are not hard things to check. There's a lot of structure, and the fact that $H^*(X)$ forms an algebra is a good thing. Notice how the dimensions work. Long ago a bad choice was made: we should index cohomology with negative numbers, so that the grading in $\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X)$ makes sense. A cochain complex with positive grading is the same as a chain complex with negative grading.

There are also two slant products. Maybe we won't talk about them. We will check a few things about cap products, and then we'll get into the machinery of Poincaré duality.

32 Cap product and "Čech" cohomology

We have a few more things to say about the cap product, and will then use it to give a statement of Poincaré duality.

Let R be a commutative ring of coefficients. The cap product

$$\cap : H^p(X) \otimes H_n(X) \to H_q(X)\,, \quad p+q=n\,,$$

comes from a chain level composite

$$S^p(X) \otimes S_n(X) \xrightarrow{1\otimes\alpha} S^p(X) \otimes S_p(X) \otimes S_q(X) \xrightarrow{\langle -,-\rangle\otimes 1} R \otimes S_q(X) \cong S_q(X)\,.$$

Using the Alexander-Whitney map this can be written as follows.

$$\cap : \beta \otimes \sigma \mapsto \beta \otimes (\sigma \circ \alpha_p) \otimes (\sigma \circ \omega_q) \mapsto (\beta(\sigma \circ \alpha_p))\,(\sigma \circ \omega_q)$$

Proposition 32.1. *The cap product enjoys the following properties.*
(1) $(a \cup b) \cap x = a \cap (b \cap x)$ *and* $1 \cap x = x$*:* $H_*(X)$ *is a module for* $H^*(X)$*.*
(2) Given a map $f : X \to Y$*,* $b \in H^p(Y)$*, and* $x \in H_n(X)$*,*

$$f_*(f^*(b) \cap x) = b \cap f_*(x)\,.$$

(3) Let $\epsilon : H_*(X) \to R$ *be the augmentation. Then*

$$\varepsilon(b \cap x) = \langle b, x\rangle\,.$$

(4) Cap and cup are adjoint:

$$\langle a \cap b, x\rangle = \langle a, b \cap x\rangle$$

Proof. (1) We proved this in the last lecture.
(2) Let β be a cocycle representing b, and σ an n-simplex in X. Then

$$\begin{aligned} f_*(f^*(\beta) \cap \sigma) &= f_*((f^*(\beta)(\sigma \circ \alpha_p)) \cdot (\sigma \circ \omega_q)) \\ &= f_*(\beta(f \circ \sigma \circ \alpha_p) \cdot (\sigma \circ \omega)) \\ &= \beta(f \circ \sigma \circ \alpha_p) \cdot f_*(\sigma \circ \omega_q) \\ &= \beta(f \circ \sigma \circ \alpha_p) \cdot (f \circ \sigma \circ \omega_q) \\ &= \beta \cap f_*(\sigma) \end{aligned}$$

This formula goes by many names: the "projection formula," or "Frobenius reciprocity."
(3) We get zero unless $p = n$. Again let $\sigma \in \mathrm{Sin}_n(X)$, and compute:

$$\varepsilon(\beta \cap \sigma) = \varepsilon(\beta(\sigma) \cdot c^0_{\sigma(n)}) = \beta(\sigma)\varepsilon(c^0_{\sigma(n)}) = \beta(\sigma) = \langle \beta, \sigma\rangle\,.$$

$\square$

Here now is a statement of Poincaré duality. It deals with the homological structure of compact topological manifolds. The statement will use the notion of orientability, which we will have more to say about. Since M is locally Euclidean, excision guarantees that for any $p \in M$ there is an isomorphism

$$j_a : H_n(M, M - a; R) \leftarrow H_n(\mathbf{R}^n, \mathbf{R}^n - 0; R) = R\,.$$

An "orientation" is a choice of generators for these free R-modules that varies continuously with a. With coefficients in $\mathbf{F}_2$, any manifold is uniquely oriented.

Theorem 32.2 (Poincaré duality). *Let M be a topological n-manifold that is compact and oriented with respect to a PID R. Then there is a unique class $[M] \in H_n(M; R)$ that restricts to the orientation class in $H_n(M, M - a; R)$ for every $a \in M$. It has the property that*

$$- \cap [M] : H^p(M; R) \to H_q(M; R)\,, \quad p + q = n\,,$$

is an isomorphism for all p.

You might want to go back to Lecture 25 and verify that $\mathbf{RP}^3 \times \mathbf{RP}^3$ satisfies this theorem.

Our proof of Poincaré duality will be by induction. In order to make the induction go we will prove a substantially more general theorem, one that involves relative homology and cohomology. So we begin by understanding how the cup product behaves in relative homology.

Suppose $A \subseteq X$ is a subspace. We have:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
S^p(X) \otimes S_n(A) \arrow[r, "i^* \otimes 1"] \arrow[d, "1 \otimes i_*"] & S^p(A) \otimes S_n(A) \arrow[r, "\cap"] & S_q(A) \arrow[d, "i_*"] \\
S^p(X) \otimes S_n(X) \arrow[rr, "\cap"] \arrow[d] & & S_q(X) \arrow[d] \\
S^p(X) \otimes S_n(X, A) \arrow[rr, dashed] \arrow[d] & & S_q(X, A) \arrow[d] \\
0 & & 0
\end{tikzcd}$$

The left sequence is exact because $0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$ splits and tensoring with $S^p(X)$ (which is not free!) therefore leaves it exact. The solid arrow diagram commutes precisely by the chain-level projection formula. There is therefore a uniquely defined map on cokernels.

This chain map yields the *relative cap product*

$$\cap : H^p(X) \otimes H_n(X, A) \to H_q(X, A)$$

It renders $H_*(X,A)$ a module for $H^*(X)$.

I want to come back to an old question, about the significance of relative homology. Suppose that $K \subset X$ is a subspace, and consider the relative homology $H_*(X, X-K)$. Since the complement of $X-K$ in X is K, these groups should be regarded as giving information about K. If I enlarge K, I make $X-K$ smaller, $K \subseteq L \subseteq$ induces $H_*(X, X-L) \to H_*(X-K)$: the relative homology is *contravariant* in the variable K (regarded as an element of the poset of subspaces of X, at least).

Excision gives insight into how $H_*(X, X-K)$ depends on K. Suppose $K \subseteq U \subseteq X$ such that $\overline{K} \subseteq \mathrm{Int}(U)$. To simplify things, let's just suppose that K is closed and U is open. Then $X-U$ is closed, $X-K$ is open, and $X-U \subseteq X-K$, so excision asserts that the inclusion map

$$H_*(U, U-K) \to H_*(X, X-K)$$

is an isomorphism.

The cap product puts some structure on $H_*(X, X-K)$: it's a module over $H^*(X)$. But we can do better! We just decided that $H_*(X, X-K) = H_*(U, U-K)$, so the $H^*(X)$ action factors through an action by $H^(U)$, for any open set U containing K. How does this refined action change when I decrease U?

Lemma 32.3. *Let $U \supseteq V \supseteq K$. Then:*

$$\begin{tikzcd}
H^p(U) \otimes H_n(X, X-K) \arrow[dd, "i^*\otimes 1"'] \arrow[dr, "\cap"] & \\
& H_q(X, X-K) \\
H^p(V) \otimes H_n(X, X-K) \arrow[ur, "\cap"'] &
\end{tikzcd}$$

commutes.

Proof. Hint: use projection formula again. □

Let $\mathcal{U}_K$ be the set of open neighborhoods of K in X. It is partially ordered by reverse inclusion. This poset is directed, since an intersection of two opens is open. By the lemma, $H^p : \mathcal{U}_K \to \mathbf{Ab}$ is a directed system.

Definition 32.4. The *Čech cohomology* of K is

$$\check{H}^p(K) := \varinjlim_{U \in \mathcal{U}_K} H^p(U) .$$

I apologize for this bad notation; its apparent dependence on the way K is sitting in X is not recorded.

Since tensor product commutes with direct limits, we now get a cap product pairing

$$\cap : \check{H}^p(K) \otimes H_n(X, X-K) \to H_q(X, X-K)$$

satifying the expected properties. This is the best you can do. It's the natural structure that this relative homology has: $H_*(X, X-K)$ is a module over $\check{H}^*(K)$.

There are compatible restricton maps $H_p(U) \to H^p(K)$, so there is a natural map

$$\check{H}^*(K) \to H^p(K) .$$

This map is often an isomorphism. Suppose $K \subseteq X$ satisfies the condition (a "regular neighborhood" condition) that for every open $U \supseteq K$, there exists an open V such that $U \supseteq V \supseteq K$ such that $K \hookrightarrow V$ is a homotopy equivalence (or actually just a homology isomorphism).

Lemma 32.5. *Under these conditions, $\check{H}^*(K) \to H^*(K)$ is an isomorphism.*

Proof. We will check that the map to $H^p(K)$ satisfies the conditions we established in Lecture 23 to be a direct limit.

So let $x \in H^p(K)$. Let U be a neighborood of K in X such that $H^p(U) \to H^p(K)$ is an isomorphism. Then indeed x is in the image of $H^p(U)$.

Then let U be a neighborhood of K and let $x \in H^p(U)$ restrict to 0 in $H^p(K)$. Let V be a sub-neighborood such that $H^p(V) \to H^p(K)$ is an isomorphism. Then x restricts to 0 in $H^p(V)$. □

On the other hand, here's an example that distinguishes $\check{H}^*$ from H^*. This is a famous example – the "topologist's sine curve." The topologist's sine curve is the subspace of $\mathbf{R}^2$ defined as follows. It is union of two subsets, A and B. A is the graph of $\sin(\pi/x)$ where $0 < x \leq 1$. B is a continuous curve from $(0,-1)$ to $(1,0)$ and not meeting A. This is a counterexample for a lot of things; you've probably seen it in 18.901.

What is the singular homology of the topologist sine curve? Use Mayer-Vietoris! I can choose V to be some connected portion of the continuous curve from $(0,-1)$ to $(1,0)$, and U to contain the rest of the space in a way that intersects V in two open intervals. Then V is contractible, and U is made up of two contractible connected components. (This space is not locally connected, and one of these path components is not closed.)

The Mayer-Vietoris sequence looks like

$$0 \to H_1(X) \xrightarrow{\partial} H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0 .$$

The two path components of $U \cap V$ do not become connected in U, so $\partial = 0$ and we find that $\varepsilon : H_*(X) \xrightarrow{\cong} H_*(*)$ and hence $H^*(X) \cong H^*(*)$.

How about $\check{H}^*$? Let $X \subset U$ be an open neighborhood. The interval is contained in some ϵ-neighborhood that's contained in U. This implies that there exists a neighborhood $X \subseteq V \subseteq U$ such that $V \sim S^1$. This implies that

$$\varinjlim_{U \in \mathcal{U}_X} H^*(U) \cong H^*(S^1)$$

by an argument we will detail later. So $\check{H}^*(X) \neq H^*(X)$.

33 $\check{H}^*$ as a cohomology theory, and the fully relative $\cap$ product

Let X be any space, and let $K \subseteq X$ be a closed subspace. We've defined the Čech cohomology of K as the direct limit of $H^*(U)$ as U ranges over the poset $\mathcal{U}_K$ of open neighborhoods of K. This often coincides with $H^*(K)$ but will not be the same in general. Nevertheless it behaves like a cohomology theory. To expand on this claim, we should begin by defining a relative version.

Suppose $L \subseteq K$ is a pair of closed subsets of a space X. Let (U, V) be a "neighborhood pair" for (K, L):

$$\begin{array}{ccc} L & \subseteq & K \\ \cap\!\!\!| & & \cap\!\!\!| \\ V & \subseteq & U \end{array}$$

These again form a directed set $\mathcal{U}_{K,L}$, with partial order given by reverse inclusion of pairs. Then define

$$\check{H}^p(K, L) = \varinjlim_{(U,V) \in \mathcal{U}_{K,L}} H^p(U, V) .$$

Theorem 33.1. *Let (K, L) be a closed pair in X. There is a long exact sequence*

$$\cdots \to \check{H}^p(K, L) \to \check{H}^p(K) \to \check{H}^p(L) \xrightarrow{\delta} \check{H}^{p+1}(K, L) \to \cdots$$

that is natural in the pair.

Also, a form of excision holds:

Theorem 33.2 (Excision)**.** *Suppose $A, B \subseteq X$ are closed. Then the inclusion induces isomorphisms*

$$\check{H}^p(A \cup B, A) \xrightarrow{\cong} \check{H}^p(B, A \cap B) .$$

So Čech cohomology is better suited to closed subsets than singular cohomology is.

Čech cohomology appeared as the natural algebra acting on $H^*(X, X - K)$, where K is a closed subspace of X:

$$\cap : \check{H}^p(K) \otimes H_n(X, X - K) \to H_q(X, X - K) , \quad p + q = n .$$

If we fix $x_K \in H_n(X, X - K)$, then capping with x_K gives a map

$$\cap x_K : \check{H}^p(K) \to H_q(X, X - K) , \quad p + q = n .$$

We will be very interested in showing that this map is an isomorphism under certain conditions. This is a kind of duality result, comparing cohomology and relative homology and reversing the dimensions. We'll try to show that such a map is an isomorphism by embedding it in a map of long exact sequences and using the five-lemma.

For a start, let's think about how these maps vary as I change K. So let L be a closed subset of K, so $X - K \subseteq X - L$ and I get a "restriction map"

$$i_* : H_n(X, X - K) \to H_n(X, X - L)\,.$$

Define x_L as the image of x_K. The diagram

$$\begin{tikzcd}
\check{H}^p(K) \arrow[r] \arrow[d, "- \cap x_K"'] & \check{H}^p(L) \arrow[d, "- \cap x_L"'] \\
H_q(X, X - K) \arrow[r] & H_q(X, X - L)
\end{tikzcd}$$

commutes by the projection formula. This embeds in a bigger diagram:

Theorem 33.3. *There is a "fully relative" cap product*

$$\cap : \check{H}^p(K, L) \otimes H_n(X, X - K) \to H_q(X - L, X - K)\,, \quad p + q = n\,,$$

such that for any $x_K \in H_n(X, X - K)$ the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K, L) \arrow[r] \arrow[d, "- \cap x_K"'] & \check{H}^p(K) \arrow[r] \arrow[d, "- \cap x_K"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "- \cap x_L"'] & \check{H}^{p+1}(K, L) \arrow[r] \arrow[d, "- \cap x_K"'] & \cdots \\
\cdots \arrow[r] & H_q(X - L, X - K) \arrow[r] & H_q(X, X - K) \arrow[r] & H_q(X, X - L) \arrow[r, "\partial"] & H_{q-1}(X - L, X - K) \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Here x_L is x_K restricted to $H_n(X, X - L)$.

What I have to do is define a cap product of the following form (bottom row):

$$\begin{tikzcd}
\check{H}^p(K) \otimes H_n(X, X - K) \arrow[r, "\cap"] & H_q(X, X - K) \\
\check{H}^p(K, L) \otimes H_n(X, X - K) \arrow[r, "\cap"] \arrow[u] & H_q(X - L, X - K)
\end{tikzcd}$$

(where $p + q = n$)

Our map $\check{H}^p(K) \otimes H_n(X, X-K) \to H_q(X, X-K)$ came from $S^p(U) \otimes S_n(U, U-K) \to S_q(U, U - K)$ where $U \supseteq K$, defined via $\beta \otimes \sigma \mapsto \beta(\sigma \circ \alpha_p) \cdot (\sigma \circ \omega_q)$. I'm hoping to get:

$$\begin{tikzcd}
S^p(U) \otimes S_n(U, U - K) \arrow[r] & S_q(U, U - K) \\
S^p(U, V) \otimes S_n(U - L)/S_n(U - K) \arrow[r] \arrow[u] & S_q(U - L)/S_q(U - K) \arrow[u]
\end{tikzcd}$$

where again we have inclusions (U, V open and K, L closed):

$$\begin{tikzcd}
K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\
U & V \arrow[l, hook']
\end{tikzcd}$$

The bottom map $S^p(U,V)\otimes S_n(U-L)/S_n(U-K)\to S_q(U-L)/S_q(U-K)$ makes sense. We can evaluate a cochain that kills everything on V. This means that we can add in $S_n(V)$ to get $S^p(U,V)\otimes(S_n(U-L)+S_n(V))/S_n(U-K)\to S_q(U-L)/S_q(U-K)$ by sending $\beta\otimes\tau\mapsto 0$ where $\tau:\Delta^n\to V$. This means that the diagram:

$$\begin{tikzcd}
S^p(U)\otimes S_n(U,U-K) \arrow[r] & S_q(U,U-K) \\
S^p(U,V)\otimes(S_n(U-L)+S_n(V))/S_n(U-K) \arrow[u] \arrow[r] & S_q(U-L)/S_q(U-K) \arrow[u]
\end{tikzcd}$$

commutes. It's not that far off from where we want to go.

Now, $(U-L)\cup V=U$. I have this covering of U by two open sets. In $S_n(U-L)+S_n(V)$ we're taking the sum of n-chains. We have a map $S_*(U-L)+S_*(V)\to S_*(U)$. We have already worked through this – the locality principle! This tells us that $S_*(U-L)+S_*(V)\to S_*(U)$ is a homotopy equivalence. Hence we can extend our diagram:

$$\begin{tikzcd}
S^p(U)\otimes S_n(U,U-K) \arrow[r] & S_q(U,U-K) \\
S^p(U,V)\otimes(S_n(U-L)+S_n(V))/S_n(U-K) \arrow[u] \arrow[r] \arrow[d, "\simeq"] & S_q(U-L)/S_q(U-K) \arrow[u] \\
S^p(U,V)\otimes S_n(U)/S_n(U-K) &
\end{tikzcd}$$

We want the homology of $S_n(U)/S_n(U-K)$ to approximate $H_n(X,X-K)$.

Claim 33.4. There is an isomorphism $H_n(S_*(U)/S_*(U-K))=H_n(U,U-K)\to H_n(X,X-K)$.

Proof. This is exactly excision! Remember our recasting of excision in the previous lecture. □

This means that what we've constructed really *is* what we want! We now have our large lexseq:

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K,L) \arrow[r] \arrow[d, dashed, "-\cap x_K"'] & \check{H}^p(K) \arrow[r] \arrow[d, "-\cap x_k"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap x_L"'] & \check{H}^{p+1}(K,L) \arrow[r] \arrow[d, "-\cap x_K"'] & \cdots \\
\cdots \arrow[r] & H_q(X-L,X-K) \arrow[r] & H_q(X,X-K) \arrow[r] & H_q(X,X-L) \arrow[r, "\partial"] & H_{q-1}(X-L,X-K) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_q(U-L,U-K) \arrow[r] \arrow[u, "{\cong,\ \text{five-lemma}}"] & H_q(U,U-K) \arrow[r] \arrow[u, "{\cong,\ \text{locality}}"] & H_q(U,U-L) \arrow[rr] \arrow[u, "{\cong,\ \text{locality}}"] & & \cdots
\end{tikzcd}$$

As desired.

The diagram:

$$\begin{tikzcd}
\check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap x_L"] & \check{H}^{p+q}(K,L) \arrow[d, "-\cap x_K"] \\
H_q(X, X-L) \arrow[r, "\partial"] & H_{q-1}(X-L, X-K)
\end{tikzcd}$$

says that:

$$(\delta b) \cap x_k = \partial(b \cap x_L)$$

It's rather wonderful! You have a decreasing sequence below and an increasing one above.

I want to reformulate all of this in a more useful fashion, from Mayer-Vietoris. We had two different proofs, one from locality, and another one that we'll remind you of:

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d, "\cong"] & C_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

then you get a lexseq:

$$\cdots \to C_{n+1} \to C'_{n+1} \oplus A_n \to A'_n \xrightarrow{\partial} C_n \to \cdots$$

You can use this to prove Mayer-Vietoris – I will do this in a special case. (This is exactly what I did in a homework assignment[1]!) We have a ladder of lexseqs:

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(X, X-A\cup B) \arrow[r] \arrow[d] & H_q(X, X-A) \arrow[r] \arrow[d] & H_{q-1}(X-A, X-A\cup B) \arrow[r] \arrow[d, "\cong\text{, excision}"] & \cdots \\
\cdots \arrow[r] & H_q(X, X-B) \arrow[r] & H_q(X, X-A\cap B) \arrow[r] & H_{q-1}(X-A\cap B, X-B) \arrow[r] & \cdots
\end{tikzcd}$$

This means that (using the lexseq of the ladder) you have a lexseq:

$$\cdots \to H_q(X, X-A\cup B) \to H_q(X, X-A) \oplus H_q(X, X-B) \to H_q(X, X-A\cap B) \to H_{q-1}(X, X-A\cup B) \to \cdots$$

---

[1] Suppose $A \subseteq X$ is a subspace of X. Then there is a lexseq in reduced homology $\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to H_n(X,A) \to \widetilde{H}_{n-1}(A) \to \cdots$ that can be obtained by using the lexseq in homology of the sexseq $0 \to \widetilde{S}_*(A) \to \widetilde{S}_*(X) \to S_*(X,A) \to 0$.

Now suppose $X = A \cup B$. Consider the ladder:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(A, A\cap B) \arrow[r] \arrow[d] & \widetilde{H}_n(A\cap B) \arrow[r] \arrow[d] & \widetilde{H}_n(A) \arrow[r] \arrow[d] & H_n(A, A\cap B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X, B) \arrow[r] & \widetilde{H}_n(B) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & H_n(X, B) \arrow[r] & \cdots
\end{tikzcd}$$

The first and fourth maps as shown are isomorphisms because of excision. The lexseq from the ladder (see above) therefore yields the Mayer-Vietoris sequence $\cdots \to \widetilde{H}_n(A\cap B) \to \widetilde{H}_n(B) \oplus \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(A\cap B) \to \cdots$.

This can be used to give a lexseq for Čech cohomology:

$$\cdots \to \check{H}^p(A \cup B) \to \check{H}^p(A) \oplus \check{H}^p(B) \to \check{H}^p(A \cap B) \to \check{H}^{p+q}(A \cup B) \to \cdots$$

so that we're going to get a commutative Mayer-Vietoris ladder:

Theorem 33.5. *There's a "Mayer-Vietoris" ladder:*

$$\begin{tikzcd}
\to \check{H}^p(A \cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \check{H}^p(A) \oplus \check{H}^p(B) \arrow[r] \arrow[d, "(-\cap x_A)\oplus -\cap x_B"] & \check{H}^p(A \cap B) \arrow[r] \arrow[d] & \check{H}^{p+q}(A \cup B) \arrow[r] \arrow[d] & {} \\
\to H_q(X, X - A \cup B) \arrow[r] & H_q(X, X-A) \oplus H_q(X, X-B) \arrow[r] & H_q(X, X - A \cap B) \arrow[r] & H_{q-1}(X, X - A \cup B) &
\end{tikzcd}$$

where I have four cohomology classes $x_{A\cup B}, x_A, x_B, x_{A\cap B}$ that commute in:

$$\begin{tikzcd}
& H_n(X, X-A) \arrow[dr] & \\
H_n(X, X - A \cap B) \arrow[ur] \arrow[dr] & & H_n(X, X - A \cap B) \\
& H_n(X, X-B) \arrow[ur] &
\end{tikzcd}$$

This is the most complicated blackboard for the rest of the course. Also xymatrix is not compiling properly because the diagram is too big!

34 $\check{H}^*$ as a cohomology theory

Office hours: today, Hood in 4-390 from 1:30 to 3:30 and Miller in 4-478 from 1-3 on Tuesday. Note that pset 6 is due Wednesday. Also, Wednesday we'll have a lightning review of π_1 and covering spaces.

We're coming to the end of the course, and there are going to be oral exams. I have some questions that I'd like to ask you. They won't be super advanced, detailed questions – they'll be basic things. I'll post a list of examples of questions. I won't select questions from that list, that's cruel and isn't the point. The oral will be 40 minutes. It'll be fun – better than a written exam. It's much better than grading a written exam!

PLEASE DRAW A PICTURE WHEN READING THIS IF YOU DIDN'T COME TO CLASS!

Cofinality

Let $\mathcal{I}$ be a directed set. Let $A : \mathcal{I} \to \mathbf{Ab}$ be a functor. If I have a functor $f : \mathcal{K} \to \mathcal{I}$, then I get $Af : \mathcal{K} \to \mathbf{Ab}$, i.e., $(Af)_j = A_{f(j)}$.

I can form $\varinjlim_{\mathcal{K}} Af$ and $\varinjlim_I A$. I claim you have a map $\varinjlim_{\mathcal{K}} Af \to \varinjlim_{\mathcal{I}} A$. All I have to do is the following:

$$\begin{tikzcd}
\varinjlim_J Af \arrow[r] & \varinjlim_I A \\
A_{f(j)} \arrow[u, "\mathrm{in}_j"]
\end{tikzcd}$$

So I have to give you maps $A_{f(j)} \to \varinjlim_I A$ for various j. I know what to do, because I have $\mathrm{in}_{f(j)} : A_{f(j)} \to \varinjlim_I A$. Are they compatible when I change j? Suppose I have $j' \le j$. Then I get a map $f(j') \to f(j)$, so I have a map $A_{f(j')} \to A_{f(j)}$, and thus the maps are compatible. Hence I get:

$$\begin{tikzcd}
\varinjlim_J Af \arrow[r] & \varinjlim_I A \\
(Af)_j = A_{f(j)} \arrow[u, "\mathrm{in}_j"] \arrow[ur, dashed, "\mathrm{in}_{f(j)}"]
\end{tikzcd}$$

Example 34.1. Suppose $K \supseteq L$ be closed, then I get a map $\check{H}^*(K) \to \check{H}^*(L)$. Is this a homomorphism? Well, $\check{H}^*(K) = \varinjlim_{U \in \mathcal{U}_K} H^*(U)$ and $\check{H}^*(L) = \varinjlim_{V \in \mathcal{U}_L} H^*(V)$. This is an example of a $\mathcal{I}$ and $\mathcal{K}$ that I care about. Well, $\mathcal{U}_K \subseteq \mathcal{U}_L$, and thus I get a map $\check{H}^*(K) \to \check{H}^*(L)$, which is what I wanted.

I can do something for relative cohomology. Suppose:

$$\begin{tikzcd}
K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\
K' & L' \arrow[l, hook']
\end{tikzcd}$$

I get a homomorphism $\check{H}^*(K, L) \to \check{H}^*(K', L')$ because I have $\mathcal{U}_{K,L} \to \mathcal{U}_{K',L'}$.

This isn't exactly what we need:

Question 34.2. When does $f : \mathcal{K} \to \mathcal{I}$ induce an isomorphism $\varinjlim_J Af \to \varinjlim_I A$?

This is a lot like taking a sequence and a subsequence and asking when they have the same limit. There's a cofinality condition in analysis, that has a similar expression here.

Definition 34.3. $f : \mathcal{K} \to \mathcal{I}$ is cofinal if for all $i \in \mathcal{I}$, there exists $j \in \mathcal{K}$ such that $i \le f(j)$.

Example 34.4. If f is surjective.

Lemma 34.5. *If f is cofinal, then $\varinjlim_J Af \to \varinjlim_I A$ is an isomorphism.*

Proof. Check that $\{A_{f(j)} \to \varinjlim_I A\}$ satisfies the necessary and sufficient conditions:

1. For all $a \in \varinjlim_I A$, there exists j and $a_j \in A_{f(j)}$ such that $a_j \mapsto a$. We know that there exists some i and $a_i \in A$ such that $a_i \mapsto a$. Pick j such that $f(j) \geq i$, so we get a map $a_i \to a_{f(j)}$, and by compatibility, we get $a_{f(j)} \mapsto a$.

2. The other condition is also just as easy.

□

This is a very convenient condition.

Example 34.6. I had a perverse way of constructing $\mathbf{Q}$ by using the divisibility directed system. A much simpler (linear!) directed system is $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{4} \mathbf{Z} \to \cdots$. This has the same colimit as the divisibility directed system because $n|n!$, so we have a cofinal map between directed systems.

How about the direct limits in the Čech cohomology case?

Example 34.7. Do I have a map $\check{H}^*(K,L) \to \check{H}^*(K)$? Suppose:

$$\begin{tikzcd} K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\ U & V \arrow[l, hook'] \end{tikzcd}$$

Then $\check{H}^p(K,L) = \varinjlim_{(U,V)\in\mathcal{U}_{K,L}} H^p(U,V)$ and $\check{H}^p(K) = \varinjlim_{U\in\mathcal{U}_K} H^p(U)$. I have a map of directed sets $\mathcal{U}_{K,L} \to \mathcal{U}_K$ by sending $(U,V) \mapsto U$. I didn't have to use cofinality. I want a long exact sequence, though, and I'm going to do this by saying that it's a directed limit of a long exact sequence. I'm going to have to have all of these various Čech cohomologies as being the directed limit over the *same* indexing set.

I'd really like to say that $\check{H}^p(K) = \varinjlim_{U\in\mathcal{U}_K} H^p(U) \cong \varinjlim_{(U,V)\in\mathcal{U}_{K,L}} H^p(U)$. Thus I need to show that $\mathcal{U}_{K,L} \to \mathcal{U}_K$ where $(U,V) \mapsto U$ is cofinal. This is easy, because if $U \in \mathcal{U}_K$, just pick (U,U), i.e., $\mathcal{U}_{K,L} \to \mathcal{U}_K$ is cofinal. How about $\mathcal{U}_{K,L} \to \mathcal{U}_L$ by $(U,V) \mapsto V$; is it cofinal? Yes! For $V \in \mathcal{U}_L$, pick (X,V)! This means that $\cdots\check{H}^{p-1}(L) \to \check{H}^p(K,L) \to \check{H}^p(K) \to \check{H}^p(L) \to \check{H}^{p+1}(K,L)$ is $\varinjlim_{\mathcal{U}_{K,L}}(\cdots \to H^p(U,V) \to \cdots)$, and hence exact.

How about excision? I need this to get to Mayer-Vietoris!

Lemma 34.8. *Assume X is normal and A,B are closed subsets. Then $\check{H}^p(A\cup B, B) \to \check{H}^p(A, A\cap B)$ is an isomorphism.*

Proof. Well, $\check{H}^p(A\cup B,B)$ is $\varinjlim$ over $\mathcal{U}_{A\cup B,B}$ and $\check{H}^p(A,A\cap B)$ is $\varinjlim$ over $\mathcal{U}_{A,A\cap B}$. Let $W \supseteq A$ and $Y \supseteq B$ are neighborhoods. I claim that $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A\cup B,B}$ sending $(W,Y) \mapsto (W\cup Y, Y)$ and $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A,A\cap B}$ sending $(W,Y) \mapsto (W, W\cap Y)$ are cofinal.

If I give you $(U,V) \in \mathcal{U}_{A\cup B,B}$, define $(W,V) \in \mathcal{U}_A \times \mathcal{U}_B$ where $W = U$ and $Y = V$, so $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A\cup B,B}$ is surjective, hence cofinal. The latter is trickier. Let $U \supseteq A$

and $V \supseteq A \cap B$. Here's where normality comes into play. Separate $B - V$ from A. Let $T \supseteq B - V$. Shit. *Shit!*

Maybe I'll leave this to you. I'll put this on the board on Wednesday. Anyway, I'll use normality to show that $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A,A\cap B}$ is cofinal, and thus this verifies excision – so you actually have excision. □

35 Finish off the proof of $\check{H}^p$ excision, topological manifolds, fundamental classes

The end of the proof

Let's finish off the proof from last time. Suppose A, B are closed in normal X. Excision for $\check{H}^p$:

$$\begin{tikzcd}
\varinjlim_{(W,Y)\in\mathcal{U}_A\times\mathcal{U}_B} H^p(W\cup Y, Y) \arrow[r, "{\cong,\text{ ordinary excision}}"] \arrow[d, "{\cong,\text{ cofinality/surjectivity}}"] & \varinjlim_{\mathcal{U}_A\times\mathcal{U}_B} H^p(W, W\cap Y) \arrow[d, "{\cong,\text{ cofinality, see below}}"] \\
\varinjlim_{(U,V)\in\mathcal{U}_{A\cup B,B}} H^p(U,V) \arrow[r] \arrow[d, equal] & \varinjlim_{(U,V)\in\mathcal{U}_{A,A\cap B}} H^p(U,V) \arrow[d, equal] \\
\check{H}^p(A\cup B, B) \arrow[r] & \check{H}^p(A, A\cap B)
\end{tikzcd}$$

$\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A,A\cap B}$ is cofinal since: start with $(U,V) \supseteq (A, A\cap B)$. Using normality, separate $B \cap (X - V) \subseteq T$ and $A \subseteq S$. Take $W = U \cap S$ and $Y = V \cup T$. Then $A \subseteq W \subseteq U$ and $A \cap B \subseteq W \cap Y = S \cap V \subseteq V$.

This means that $\check{H}^p$ satisfies excision, hence Mayer-Vietoris. Let's put this in the drawer for now.

Topological manifolds + Poincaré duality

yayyyyyyyyyyyyy finally

Fundamental class and orientation local system

Definition 35.1. A *topological manifold* is a Hausdorff space M such that for every $x \in M$, there exists a neighborhood $U \ni x$ that is homeomorphic to some Euclidean space $\mathbf{R}^n$. It's called an n-manifold if all U are homeomorphic to $\mathbf{R}^n$ for the *same* n.

Example 35.2. $\mathbf{R}^n$, duh. $\varnothing$ is an n-manifold for every n. The sphere S^n. The Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$, introduced in the beginning of the course. I don't know exactly what the dimension of this is, but you can figure it out. Also, $V_k(\mathbf{R}^n)$, and surfaces.

These things are the most interesting things to look at.

Warning 35.3. We assume the following.

1. There exists a countable basis.
2. There exists a good cover, i.e., all nonempty intersections are Euclidean as well (always true for differentiable manifolds because you can take geodesic neighborhoods, and in particular for the manifolds we listed above).

This is the context in which duality works.

Definition 35.4. Let X be any space, and let $a \in X$. The local homology of X at a is the homology $H_*(X, X - a)$. We're always working over a commutative ring.

For example, $H_q(\mathbf{R}^n, \mathbf{R}^n - 0) = \begin{cases} \text{free of rank 1} & q = n \\ 0 & q \neq n \end{cases}$. This means that local homology is picking out the characteristic feature of Euclidean space. Therefore we also have $H_q(M, M - a) = \begin{cases} \text{free of rank 1} & q = n \\ 0 & q \neq n \end{cases}$ for n-manifolds.

Notation 35.5. Let $j_a : (M, \varnothing) \to (M, M - a)$ be the inclusion.

Definition 35.6. A fundamental class for M (an n-manifold) is $[M] \in H_n(M)$ such that for every $a \in M$, the image of $[M]$ under $j_{a,*} : H_n(M) \to H_n(M, M - a)$ is a generator of $H_n(M, M - a)$.

This is somehow trying to say that this class $[M]$ covers the whole manifold.

Example 35.7. When does a space have a fundamental class?

	$\mathbf{R}^2$	$\mathbf{RP}^2$	T^2
$R = \mathbf{Z}$	no!	no!	yes! you did this for homework
$R = \mathbf{Z}/2\mathbf{Z}$	no!	yes!	yes!

Something about orientability and compactness seem to be involved.

What do we have?

Definition 35.8. $o_M = \coprod_{a \in M} H_n(M, M - a)$ as a set. This has a map $p : o_M \to M$.

Construction 35.9. This can be topologized in Euclidean neighborhoods. Let $U \cong \mathbf{R}^n$ be an Euclidean neighborhood of a. I can always arrange so that a corresponds to 0. We have the open disk sitting inside the closed disk: $\widetilde{D^n} \subseteq D^n \subseteq \mathbf{R}^n$ that corresponds to some open $V \subseteq \overline{V} \subseteq U$. Let $x \in V$. I have a diagram:

$$\begin{tikzcd}
H_n(M, M - \overline{V}) \arrow[d] & H_n(U, U - \overline{V}) \arrow[l, "\cong"', "\text{excision of } M - U"] \arrow[d, "\cong"] \arrow[r, equal] & H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \arrow[d, "{\cong, \text{ homotopy equivalence}}"] \\
H_n(M, M - x) & H_n(U, U - x) \arrow[l] \arrow[r, equal] & H_n(\mathbf{R}^n, \mathbf{R}^n - 0)
\end{tikzcd}$$

Hence $H_n(M, M-\overline{V}) \cong H_n(M, M-x)$. Thus I can collect points in o_M together when they come from the same class in $H_n(M, M-\overline{V})$, so they form "sheets".

I have a map $V \times H_n(M, M-\overline{V}) \to o_M|_V = p^{-1}(V)$ by sending $(x, c) \mapsto (j_x)_*(c) \in H_n(M, M-x)$, and this map is bijective (that's what comes from excision). This LHS has a nice topology by letting $H_n(M, M-\overline{V})$ be discrete. I'm topologizing o_M as the weakest topology these generate.

"Have I been sufficiently obscure enough? This is not supposed to be a complicated point". This $o_M \to M$ is called the *orientation local system*, and is a covering space.

Definition 35.10. A continuous map $p: E \to B$ is a covering space if:

1. $p^{-1}(b)$ is discrete for all $b \in B$.

2. For every b there's a neighborhood V and a map $p^{-1}(V) \to p^{-1}(b)$ such that $p^{-1}(V) \xrightarrow{\cong} V \times p^{-1}(b)$ is a homeomorphism.

That's exactly the way we topologized o_M. There's more structure though because $H_n(M, M-\overline{V})$ is an R-module!

Definition 35.11. A local system (of R-modules) $p: E \to B$ is a covering space together with structure maps $E \times_B E := \{(e, e') | pe = pe'\} \xrightarrow{+} E$ and $z: B \to E$ such that:

$$\begin{tikzcd}
E \times_B E \arrow[rr, "+"] \arrow[dr] & & E \arrow[dl] & R \times E \arrow[l] \\
& B \arrow[r, equal] & B \arrow[u, "z"'] &
\end{tikzcd}$$

making $p^{-1}(b)$ a R-module.

We have $H_n(M) \xrightarrow{j_x} H_n(M, M-x)$, which gives a *section* of o_M. If I have a covering space $p: E \to B$, a section is a continuous map $s: B \to E$ such that $ps = 1_B$. Write $\Gamma(E)$ to be the set of sections. If E is a local system, this is an R-module. Hence $H_n(M) \xrightarrow{j_x} H_n(M, M-x)$ gives a map $j: H_n(M) \to \Gamma(o_M)$. This is pretty cool because it's telling you about this high-dimensional homology of M into something "discrete".

Theorem 35.12. *If M is compact then $j: H_n(M) \to \Gamma(o_M)$ is an isomorphism, and $H_q(M) = 0$ for $q > n$.*

This is case of Poincaré duality actually because $\Gamma(o_M)$ is somewhat like zero-dimensional cohomology. If this is trivial, like it is for a torus, so if the manifold is connected, then $\Gamma(o_M)$ is just R.

36 Fundamental class

Note that if M is a compact manifold, then $H_q(M;R) = 0$ for $q \gg 0$, and if R is a PID, then for all q, $H_q(M;R)$ is finitely-generated. This follows from:

Claim 36.1. Suppose X admits an open cover $\{U_i\}_{i=1}^n$ such that all intersections are either empty or contractible (this is what you get for a good cover on a manifold). Then $H_q(X;R) = 0$ for $q \geq n$, and if R is a PID, then for all q, $H_q(X;R)$ is finitely-generated.

Proof. Induct. Certainly true for $n = 1$. Let $Y = \bigcup_{i=1}^{n-1} U_i$, then this statement is true by induction – and similarly for $Y \cap U_n$. Now use Mayer-Vietoris. You have $\cdots \to H_q(Y \cap U_n) \to H_q(Y) \oplus H_q(U_n) \to H_q(X) \to H_{q-1}(Y \cap U_n) \to \cdots$. When $q = n - 1$, $H_q(Y \cap U_n)$ could be nonzero, and so you might get something nontrivial (???). Also, you'll get a sexseq by unsplicing the lexseq: $0 \to H_q(Y) \oplus H_q(U_n)/\text{something} \to H_q(X) \to$ submodule of $H_{q-1}(Y \cap U_n) \to 0$, where you use R being a PID to conclude that submodule of $H_{q-1}(Y \cap U_n)$ is finitely generated. □

Let M be an n-manifold. We had a map $j : H_n(M) \to \Gamma(M; o_M)$. Here $\Gamma(M; o_M)$ is the collection of compatible elements of $H_n(M, M - x)$ for $x \in M$. This map $j : H_n(M) \to \Gamma(M; o_M)$ sends $c \mapsto (x \mapsto j_x c)$ where $j_x : H_n(M, \varnothing) \to H_n(M, M - x)$. I want to make two refinements.

You can't expect j to be surjective, except maybe when M is compact. Here's why. Let $c \in Z_n(M)$. It's a sum of simplices, and each simplex is compact, and so the union of the images is compact, and hence there's a compact subset $K \subseteq M$ such that $c \in Z_n(K)$. Now if I take $x \notin K$, then the map $H_n(K) \to H_n(M)$ splits as $H_n(K) \to H_n(M - x) \to H_n(M)$. In the relative homology, $H_n(M, M - x)$, the map $H_n(K) \to H_n(M) \to H_n(M, M - x)$ sends c to zero.

Definition 36.2. Let σ be a section of $p : E \to B$ (local system). Then the support of σ is defined as $\operatorname{supp}(\sigma) = \overline{\{x \in B | \sigma(x) \neq 0\}}$. The collection of all sections with compact support is $\Gamma_c(B; E)$, and it's a submodule of $\Gamma(B; E)$.

The first refinement is that $j : H_n(M) \to \Gamma(M; o_M)$ lands in $\Gamma_c(M; o_M)$, because homology is compactly supported.

The second refinement seems a little artificial but is part of the inductive process. Let $A \subseteq M$ be closed. Then you have a restriction map $H_n(M, M - A) \xrightarrow{j_x} H_n(M, M - x)$ for $x \in A$. Thus you get a map $j : H_n(M, M - A) \to \Gamma_c(A; o_M|_A)$, the latter of which we'll just denote $\Gamma_c(A; o_M)$.

Theorem 36.3. *The map $j : H_n(M, M - A) \to \Gamma_c(A; o_M|_A)$ is an isomorphism and $H_q(M, M - A) = 0$ for $q > n$. (If $A = M$ then $j : H_n(M) \to \Gamma_c(M; o_M)$ is an isomorphism.)*

Proof. For $X = \mathbf{R}^n$ and $A = D^n$. Well, $o_{\mathbf{R}^n} = \mathbf{R}^n \times H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$ is trivial (i.e., a product projection), so $\Gamma(D^n; o_{\mathbf{R}^n}) = \operatorname{Hom}_{\mathbf{Top}}(D^n, H_n(\mathbf{R}^n, \mathbf{R}^n - 0))$ where $H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$ is discrete, and this is therefore just a map from π_0 into this, and thus $\Gamma(D^n; o_{\mathbf{R}^n}) = R$ (your coefficient). But also, $H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \cong R$, so you have that j gives $H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \to \Gamma_c(D^n; o_{\mathbf{R}^n}|_{D^n})$.

Say that this is true for $A, B, A\cap B$ – we'll prove this for $A\cup B$. Obviously, use Mayer-Vietoris. I have a restriction $\Gamma_c(A\cup B; o_M) \to \Gamma_c(A; o_M) \oplus \Gamma_c(B; o_M)$ that sits in an exact sequence $0 \to \Gamma_c(A\cup B; o_M) \xrightarrow{\text{inclusion, determined by } A,B} \Gamma_c(A; o_M) \oplus \Gamma_c(B; o_M) \to \Gamma_c(A \cap B; o_M)$. This is a gluing lemma. We also have a relative Mayer-Vietoris $H_n(M, M - A \cup B) \to H_n(M, M - A) \oplus H_n(M, M - B) \to H_n(M, M - A \cap B)$, so we have:

$$\begin{tikzcd}
 & 0 \arrow[r] & \Gamma_c(A\cup B; o_M) \arrow[r, "\text{inclusion } A,B\to A\cup B"] & \Gamma_c(A; o_M) \oplus \Gamma_c(B; o_M) \arrow[r] & \Gamma_c(A\cap B; o_M) \\
H_{n+1}(M, M - A\cap B) = 0 \arrow[r, "0"] & H_n(M, M - A\cup B) \arrow[u] \arrow[r] & H_n(M, M-A) \oplus H_n(M, M-B) \arrow[u, "j_*", "\cong"'] \arrow[r] & H_n(M, M - A\cap B) \arrow[u, "j_*", "\cong"']
\end{tikzcd}$$

This is a "local-to-global" argument. "I don't feel like going through the point-set topology – the rest of the proof is just annoyance." See Bredon's book for the conclusion of the proof. ∎

Corollary 36.4. $j : H_n(M) \to \Gamma_c(M; o_M)$ *is an isomorphism.*

Definition 36.5. An R-orientation for M is a section σ of $\Gamma(M; o_M^\times)$ where $o_M^\times$ is the covering space of M given by the generators (as R-modules) of the fibers of o_M.

If M is compact, then $j : H_n(M) \to \Gamma(M; o_M)$, and you get $[M] \leftrightarrow \sigma$. When does that exist?

Over $\mathbf{Z}$: $o_M^\times \to M$ is a double cover of M (over every element you have two possible elements given by the two possible orientations (± 1)). If M is an n-manifold and $f : N \to M$ is a covering space, then N is also locally Euclidean. I have the orientation local system to get a pullback local system:

$$\begin{tikzcd}
f^* o_M = N \times_M o_M \arrow[r] \arrow[d] & o_M \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

Because $N \to M$ is a covering space, the fibers of $f^* o_M$ are the same as the fibers of o_N, so actually, $f^* o_M \cong o_N$. For example, suppose $N = o_M^\times$. What happens if I consider:

$$\begin{tikzcd}
o_N = N \times_M N \arrow[r] \arrow[d] & N \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

But now, I have the identity $N \to N$ that sits compatibly as:

$$\begin{tikzcd}
N \arrow[r, "\text{id}"] \arrow[d, "\text{id}"] & N \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

And hence you get $N \to o_N^\times$, which is a section of $o_N^\times \to N$. The conclusion is that $N = o_M^\times$ is canonically oriented (even if M is not oriented!). If M is oriented, then the local system is trivial and you have the trivial double cover.

The overarching conclusion is: if M is an n-manifold, then:

1. $H_q(M) = 0$ for $q > n$.
2. If M is compact, then $H_n(M) \xrightarrow{\cong} \Gamma(M, o_M)$.
3. If M is connected and compact, then:
    a) if M is oriented with respect to R, then $H_n(M) \cong \Gamma(M, o_M) \cong R$.
    b) (I have no idea what was happening here, we didn't reach to a conclusion for a while.) if M is not orientable, then $o_M^\times$ is nontrivial. If $o_M^\times$ has a section, then it's trivial (and so is o_M) because if it has a section $\sigma : M \to o_M^\times$, define $M \times R^\times \xrightarrow{\cong} o_M^\times$ by sending $(x, r) \mapsto r\sigma(x) \in o_M^\times$ (and the same thing $M \times R \xrightarrow{\cong} o_M$ for the orientation local system itself). I don't see an argument to conclude that if M is nonorientable, then there aren't any section of o_M. In particular, if $R = \mathbf{Z}$, then $H_n(M; \mathbf{Z}) = 0$. I'm going to leave this as a statement without proof, unless any of you can help me.

If a section $\sigma(x) = 0$ for some x, then $\sigma = 0$.

Remark 36.6. Prof. Miller talked with me about this after class. If I recall correctly, one way to think about this is as follows. If you have a local system $p : E \to B$, this can be viewed as a representation of $\pi_1(B) \to R^\times$, and the $\Gamma(B; E) = (E_x)^{\pi_1(B)}$ where $\pi_1(B)$ acts on the fibers by multiplication. Thus $(E_x)^{\pi_1(B)} = R^{\pi_1(B)}$. If $R = \mathbf{Z}$, then $R^\times = \{\pm 1\}$, so $R^{\pi_1(B)} = \{r | ar = r, a \in \pi_1(B)\}$, so that $\mathbf{Z}^{\pi_1(B)} = 0$. Hence there are no sections of o_M, as desired. For a ring R, $o_{M,R} = o_{M,\mathbf{Z}} \otimes R$. Something else for $\mathbf{Z}/2\mathbf{Z}$. A higher homotopy theoretic perspective is that if you have a fibration $E \to B$, then $E = PB \times_{\Omega B} F$ where F is the fiber of the fibration, so that $\Gamma(B; E) = \mathrm{Map}_{\Omega B}(PB, F) = F^{h\Omega B}$. In the case of a covering space you recover what you have above since $\pi_0(\Omega B) = \pi_1(B)$.

37 Covering spaces and Poincaré duality

Miller's office hours are tomorrow, from 1-3 in 2-478. The first half of this lecture was just explaining the remark above by using less technology.

On the website, there are notes on $\pi_1(X, *)$. I'm assuming people have seen this thing. Assume X is path-connected, and let $* \in X$. There's another technical assumption: semi-locally simply connected (SLSC), which means that for every $b \in X$ and neighborhood $b \in U$, there exists a smaller neighborhood $b \in V \subseteq U$ such that $\pi_1(V, b) \to \pi_1(X, b)$ is trivial. This is a very very weak condition.

Theorem 37.1. *Let X be a path-connected, SLSC space with $* \in X$. Then there is an equivalence of categories between covering spaces over X and sets with an action of*

$\pi_1(X,*)$. The way this functor goes is by sending $p : E \to X$ to $p^{-1}(*)$, which has an action of $\pi_1(X,*)$ in the obvious way by path-lifting.

Example 37.2 (Stupidest possible case)**.** Suppose $\mathrm{id} : X \to X$ is sent to $*$ with the trivial action. This is the terminal covering space over X.

We've been interested in $\Gamma(E;X)$, which is the same thing as $\mathrm{Map}_X(X \to X, E \to X) \cong \mathrm{Map}_{\pi_1(X)}(*, E_*) = (E_*)^{\pi_1(B)}$, the fixed points of the action. We also thought about the case of E being a local system of R-modules, and the same functor gives an equivalence between local systems of R-modules and $R[\pi_1(X)]$-modules, i.e., representations of $\pi_1(X)$.

Recall that o_M is the orientation local system, but now *over* $\mathbf{Z}$. Thus, over a general ring, $o_{M,R} = o_M \otimes R$. We were thinking about what happens with a closed path-connected SLSC subset $* \in A \subseteq M$ of an n-manifold M, and then considering $\Gamma(A, o_M \otimes R)$, which we now see to be $(o_M \otimes R)_*^{\pi_1(A,*)}$. How many options do we have here?

That is to say, this local system o_M is the same thing as the free abelian group $H_n(M, M-*)$ with an action of $\pi_1(X,*)$. There aren't many options for this action. In other words, this is a homomorphism $\pi_1(M,*) \to \mathrm{Aut}(H_n(M,M-*))$. I haven't chosen a generator for $H_n(M,M-*)$, and there's only two automorphisms, i.e., we get a homomorphism $w_1 : \pi_1(M,*) \to \mathbf{Z}/2\mathbf{Z}$. This homomorphism is called the "first Stiefel-Whitney class". 18.906 will describe all the Stiefel-Whitney classes. With R-coefficients, I get a map $\pi_1(M,*) \to \mathrm{Aut}(H_n(M,M-*;R)) \cong R^\times$. This is a natural construction, so this homomorphism $\pi_1(M,*) \to R^\times$ factors through $\pi_1(M,*) \to \mathbf{Z}/2\mathbf{Z}$. This lets us get a good handle on what the sections are: $\Gamma(A; o_M \otimes R) = H_n(M, M-*;R)^{\pi_1(X,*)}$, but our analysis shows that:

$$\Gamma(A;o_M\otimes R) = H_n(M,M-*;R)^{\pi_1(X,*)}$$
$$= \begin{cases} H_n(M,M-*;R)\cong R & \text{if } w_1 = 1\text{, well-defined up to sign; the orientable case} \\ \ker(R \xrightarrow{2} R) & \text{if } w_1 \neq 1\text{, and this is a canonical identification} \end{cases}$$

where we get the latter thing because then $a = -a$, i.e., $2a = 0$. In particular, if $R = \mathbf{Z}/2\mathbf{Z}$, since $\mathrm{Aut}_{\mathbf{Z}/2\mathbf{Z}}(\mathbf{Z}/2\mathbf{Z}) = 1$, you always have a unique orientation. If $R = \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}, \mathbf{Q}$, then $\ker(R \xrightarrow{2} R) = 0$.

We had a general theorem:

Theorem 37.3. *$H_n(M, M-A;R) \xrightarrow{j,\cong} \Gamma_c(A;o_M\otimes R)$ and $H_q(M,M-A;R) = 0$ for $q > n$.*

Corollary 37.4. *If M is connected and $A = M$, and if M is not compact, then $H_n(M;R) = 0$. If M is compact, then the work we just did shows that $H_n(M;R) = \begin{cases} R & \text{oriented} \\ \ker(R \xrightarrow{2} R) & \text{nonorientable} \end{cases}$.*

Poincaré duality, finally

Assume M is R-oriented. Let $K \subseteq M$ be compact. Then $H_n(M, M-K) \xrightarrow{\cong} \Gamma(K; o_M \otimes R)$. Picking an orientation picks an isomorphism $\Gamma(K; o_M \otimes R) \cong R$. This gives some $[M]_K$, which is called the fundamental class along K. If $K = M$, then $[M]_M =: [M] \in H_n(M; R)$.

Suppose $K \subseteq L$ are compact subsets. We now combine all of our results above:

Theorem 37.5 (Fully relative Poincaré duality). *If $p+q=n$, then $\check{H}^p(K, L; R) \xrightarrow{\cap[M]_K} H_q(M-L, M-K; R)$ is an isomorphism.*

Proof. "It's, like, not hard at this point." One thing we did was set up an LES for $\check{H}$ of a pair, which implies that we may assume that $L = \varnothing$. We want to prove that $\check{H}^p(K; R) \xrightarrow{\cap[M]_K} H_q(M, M-K; R)$ is an isomorphism. Now there's a standard local-to-global process.

In the local case, if $M = \mathbf{R}^n$ and $K = D^n$, then this is saying that $\check{H}^p(D^n; R) \cong H^p(D^n; R) \xrightarrow{\cap[\mathbf{R}^n]_{D^n}} H_q(\mathbf{R}^n, \mathbf{R}^n - D^n; R)$ where the first isomorphism comes from analysis we did earlier about Čech and ordinary cohomology coinciding. If $p \neq 0$, then both sides are zero. When $p=0$, we are asking that $H^0(D^n; R) \xrightarrow{\cap[\mathbf{R}^n]_{D^n}} H_n(\mathbf{R}^n, \mathbf{R}^n - D^n; R)$. They're both equal to R, and we are just capping along $[\mathbf{R}^n]_{D^n}$, because we found that $1 \cap [\mathbf{R}^n]_{D^n} = [\mathbf{R}^n]_{D^n}$, as desired.

We carefully set up the Mayer-Vietoris sequence ladder (Theorem 33.5) that allows us to put this all together. "We're not going to go through the details because there's point set topology that I don't like there." Note that normality is not needed for K, L compact because compact sets in Hausdorff spaces can always be separated, normal or not. I just reversed the order in which things are usually taught in books. □

We have time for one beginning application.

Corollary 37.6 (Relative Poincaré duality). *Suppose $K = M$ and M is compact and R-oriented. Then $\check{H}^p(M, L; R) \xrightarrow{\cap[M]} H_{n-p}(M-L, R)$ is an isomorphism.*

Corollary 37.7 (Poincaré duality, corollary of corollary). *Let M be compact and R-oriented, then $H^p(M; R) \xrightarrow{\cap[M]} H_{n-p}(M; R)$ is an isomorphism.*

Proof. Follows from the above corollary since $\check{H}^p(M; R)$ is literally equal to $H^p(M; R)$ □

That's the most beautiful form of all. If you do have an L, you have this ladder, where all vertical maps are isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(M,L) \arrow[r] \arrow[d, dashed, "-\cap[M]"'] & \check{H}^p(M) \arrow[r] \arrow[d, "-\cap[M]"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap[M]_L"'] & \check{H}^{p+1}(M,L) \arrow[r] \arrow[d, "-\cap[M]"'] & \cdots \\
\cdots \arrow[r] & H_q(L) \arrow[r] & H_q(M) \arrow[r] & H_q(M, M-L) \arrow[r, "\partial"] & H_{q-1}(L) \arrow[r] & \cdots
\end{tikzcd}$$

This is a consistency statement for Poincaré duality. On Wednesday, we'll specialize even further, and prove the Jordan curve theorem as well as study the cohomology rings of things we haven't worked through before.

38 Applications

Please check exam schedule! Also, a sample exam is posted. This is the payoff day. All this stuff about Poincaré duality has got to be good for something. Recall:

Theorem 38.1 (Fully relative duality)**.** *Let M be a R-oriented n-manifold. Let $L \subseteq K \subseteq M$ be compact (M need not be compact). Then $[M]_K \in H_n(M, M-K)$, and capping gives an isomorphism:*

$$\check{H}^p(K, L; R) \xrightarrow{\cap [M]_k, \cong} H_{n-p}(M-L, M-K; R)$$

Today we'll think about the case $L = \varnothing$, so this is saying:

$$\check{H}^p(K; R) \xrightarrow{\cap [M]_k, \cong} H_{n-p}(M, M-K; R)$$

Corollary 38.2. *$\check{H}^q(K; R) = 0$ for $q > n$.*

We can contrast this with singular (co)homology. Here's an example:

Example 38.3 (Barratt-Milnor)**.** A two-dimensional version K of the Hawaiian earring, i.e., nested spheres all tangent to a point whose radii are going to zero. What they proved is that $H_q(K; \mathbf{Q})$ is uncountable for every $q > 1$. But if you look at the Čech cohomology, stuff vanishes.

That's nice.

How about an even more special subcase? Suppose $M = \mathbf{R}^n$. The result is called Alexander duality. This says:

Theorem 38.4 (Alexander duality)**.** *If $\varnothing \neq K \subseteq \mathbf{R}^n$ be compact. Then $\check{H}^{n-q}(K; R) \xrightarrow{\cong} \widetilde{H}_{q-1}(\mathbf{R}^n - K; R)$*

Proof. We have the LES of a pair, which gives an isomorphism $\partial : H_q(\mathbf{R}^n, \mathbf{R}^n - K; R) \xrightarrow{\cong} \widetilde{H}_{q-1}(\mathbf{R}^n - K; R)$, so the composition $\partial \circ (- \cap [M]_K)$ is an isomorphism by Poincaré duality. □

For most purposes, this is the most useful duality theorem.

Example 38.5 (Jordan curve theorem)**.** $q = 1$ and $R = \mathbf{Z}$. Then this is saying that $\check{H}^{n-1}(K) \xrightarrow{\cong} \widetilde{H}_0(\mathbf{R}^n - K)$. But $\widetilde{H}_0(\mathbf{R}^n - K)$ is free on $\#\pi_0(\mathbf{R}^n - K) - 1$ generators. If $n = 2$, for example, and $K \cong S^1$, then $\check{H}^{n-1}(K) = H^{n-1}(K) \cong H^{n-1}(S^1)$, so $H^1(S^1) \cong \widetilde{H}_0(\mathbf{R}^2 - K)$. Hence there are *two* components in the complement of K. This could also be the topologist's sine curve as well. This is the Jordan curve theorem.

Consider the UCT, which states that there's a sexseq $0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z}) \to H^q(X) \to \mathrm{Hom}(H_q(X), \mathbf{Z}) \to 0$ that splits, but not naturally. First, note that $\mathrm{Hom}(H_q(X), \mathbf{Z})$ is always torsion-free. If I assume that $H_{q-1}(X)$ is finitely generated, then $\mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z})$ is a finite abelian group, but in particular it's torsion.

The UCT is making the decomposition of $H^q(X)$ into its torsion-free and torsion parts. I can divide by torsion, so that $H^q(X)/\mathrm{tors} \cong \mathrm{Hom}(H_q(X), \mathbf{Z})$. But there's also an isomorphism $\mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z}) \to \mathrm{Hom}(H_q(X), \mathbf{Z})$ because $\mathbf{Z}$ is torsion-free. Therefore I get an isomorphism $\alpha : H^q(X)/\mathrm{tors} \to \mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z})$. I.e.:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z}) \arrow[r] & H^q(X) \arrow[r] \arrow[d] & \mathrm{Hom}(H_q(X), \mathbf{Z}) \arrow[r] & 0 \\
& & H^q(X)/\mathrm{tors} \arrow[r, "\alpha"] \arrow[ur, "\cong"] & \mathrm{Hom}(H_q(X)/\mathrm{tors}/\mathbf{Z}) \arrow[u, "\cong"'] &
\end{tikzcd}$$

Or I could say it like this: the Kronecker pairing can be quotiented by torsion, and you get an induced map $H^q(X)/\mathrm{tors} \otimes H_q(X)/\mathrm{tors} \to \mathbf{Z}$ is a perfect pairing, which means that the adjoint map $H^q(X)/\mathrm{tors} \xrightarrow{\cong} \mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z})$. Let's combine this with Poincaré duality.

Let $X = M$ be a compact oriented n-manifold. Then $H^{n-q}(X) \xrightarrow{-\cap[M], \cong} H_q(M)$, and so we get a perfect pairing $H^q(X)/\mathrm{tors} \otimes H^{n-q}(X)/\mathrm{tors} \to \mathbf{Z}$. And what is that pairing? It's the cup product! We have:

$$\begin{tikzcd}
H^q(M) \otimes H^{n-q}(M) \arrow[r] \arrow[d, "1\otimes(-\cap[M])"'] & \mathbf{Z} \\
H^q(M) \otimes H_q(M) \arrow[ur, "\langle,\rangle"'] &
\end{tikzcd}$$

And, well:

$$\langle a, b \cap [M] \rangle = \langle a \cup b, [M] \rangle$$

Thus the map $H^q(M) \otimes H^{n-q}(M) \to \mathbf{Z}$ is $a \otimes b \mapsto \langle a \cup b, [M] \rangle$, and it's a perfect pairing. This is a purely cohomological version, and is the most useful statement.

Example 38.6. Suppose $M = \mathbf{CP}^2 = D^0 \cup D^2 \cup D^4$, and its homology is $\mathbf{Z}\,0\,\mathbf{Z}\,0\,\mathbf{Z}$, and so its cohomology is the same. Let $a \in H^2(\mathbf{CP}^2)$. Then we have $H^2(\mathbf{CP}^2) \otimes H^2(\mathbf{CP}^2) \to \mathbf{Z}$, and so $a \cup a$ is a generator of $H^4(\mathbf{CP}^2)$, and hence specifies an orientation for $\mathbf{CP}^2$. The conclusion is that $H^*(\mathbf{CP}^2) = \mathbf{Z}[a]/(a^3)$ where $|a| = 2$.

How about $\mathbf{CP}^3$? It just adds a 6-cell, so its homology is $\mathbf{Z}\,0\,\mathbf{Z}\,0\,\mathbf{Z}\,0\,\mathbf{Z}$, and so its cohomology is the same. But then $a^3 = a \cup a \cup a$ is a generator of $H^6(\mathbf{CP}^2)$, and etc. Thus in general, we have:

$$H^*(\mathbf{CP}^n) = \mathbf{Z}[a]/(a^{n+1})$$

These things are finite CW-complexes, so you find:

$$H^*(\mathbf{CP}^\infty) = \mathbf{Z}[a] \tag{2.3}$$

Example 38.7. Suppose I look at maps $f : S^m \to S^n$. One of the most interesting things is that there are lots of non null-homotopic maps $S^m \to S^n$ if $m > 2$. For example, $\eta : S^3 \to S^2$ that's the attaching map for the 4-cell in $\mathbf{CP}^2$. This is called the Hopf fibration. It's essential. Why is it nullhomotopic? If η was null homotopic, then $\mathbf{CP}^2 \simeq S^2 \wedge S^4$. That's compatible with the cohomology in each dimension, but not into the cohomology ring! There's a map $S^2 \wedge S^4 \to S^2$ that collapses S^4, and the generator in $H^*(S^2)$ has $a^2 = 0$, so $a^2 = 0$ in $H^*(S^2 \wedge S^4)$. But this is not compatible with our computation that $H^*(\mathbf{CP}^2) = \mathbf{Z}[a]/(a^3)$ where $|a| = 2$.

With coefficients in a field k, then the torsion is zero, so you find that if M is compact k-oriented, then if the characteristic of $k = 2$, there's no condition for M to be oriented, and if the characteristic of k is not 2, then M is $\mathbf{Z}$-oriented. Thus we get that $H^q(M;k) \otimes_k H^{n-q}(M;k) \to k$ is a perfect pairing.

Example 38.8. Exactly the same argument as for complex projective space shows that:

$$H^*(\mathbf{RP}^n; \mathbf{F}_2) = \mathbf{F}_2[a]/(a^{n+1})$$

where $|a| = 1$. So:

$$H^*(\mathbf{RP}^\infty; \mathbf{F}_2) = \mathbf{F}_2[a] \tag{2.4}$$

where $|a| = 1$.

I'll end with the following application.

Theorem 38.9. *Suppose $f : \mathbf{R}^{m+1} \supseteq S^m \to S^n \subseteq \mathbf{R}^{n+1}$ that is equivariant with respect to the antipodal action, i.e., $f(-x) = -f(x)$. Then $m \le n$.*

So there are *no* equivariant maps from $S^m \to S^n$ if $m > n$!

Proof. Suppose I have a map like that: the map on spheres induces a map $\overline{f} : \mathbf{RP}^m \to \mathbf{RP}^n$. We claim that $H_1(\overline{f})$ is an isomorphism. Let $\pi : S^n \to \mathbf{RP}^n$ denote the map. Let $\sigma : I \to S^m$ be defined via $\sigma(0) = v$ and $\sigma(1) = -v$. So this gives a 1-cycle $\sigma : I \to S^m \to \mathbf{RP}^m$, and $H_1(\mathbf{RP}^n) = [\pi\sigma]$ is generated by this thing. When I map this thing to $\mathbf{RP}^n$, we send $\pi\sigma$ to a generator. What we've actually proved, therefore, is that $H_1(\mathbf{RP}^m) \cong H_1(\mathbf{RP}^n)$. This is also true with mod 2 coefficients, i.e., $H_1(\overline{f}, \mathbf{F}_2) \neq 0$.

That means that $H^1(\overline{f}; \mathbf{F}_2) \neq 0$ by UCT. But what is this? This is a map $H^1(f; \mathbf{F}_2) : H^*(\mathbf{RP}^n; \mathbf{F}_2) \to H^*(\mathbf{RP}^n; \mathbf{F}_2)$, i.e., a map $\mathbf{F}_2[a]/(a^{n+1}) \to \mathbf{F}_2[a] \to (a^{m+1})$. Thus $a \mapsto a$. There's not a lot of ways to do this if $m > n$. Thus what we've shown that $m \le n$. ∎

This is the Borsuk-Ulam theorem from the '20s, I think. This is an example of how you can use the cohomology ring structure for projective space.

Please check the website for details about your finals. I will ask you to sign a form, to make sure that you don't share the questions or that you haven't heard the questions beforehand. I have a fixed set of questions that'll guide the conversation.

Part II

18.906 – homotopy theory

Introduction

Here is an overview of this part of the book.

1. **General homotopy theory.** This includes category theory; because it started as a part of algebraic topology, we'll speak freely about it here. We'll also cover the general theory of homotopy groups, long exact sequences, and obstruction theory.

2. **Bundles.** One of the major themes of this part of the book is the use of bundles to understand spaces. This will include the theory of classifying spaces; later, we will touch upon connections with cohomology.

3. **Spectral sequences.** It is impossible to describe everything about spectral sequences in the duration of a single course, so we will focus on a special (and important) example: the Serre spectral sequence. As a consequence, we will derive some homotopy-theoretic applications. For instance, we will relate homotopy and homology (via the Hurewicz theorem, Whitehead's theorem, and "local" versions like Serre's mod C theory).

4. **Characteristic classes.** This relates the geometric theory of bundles to algebraic constructions like cohomology described earlier in the book. We will discuss many examples of characteristic classes, including the Thom, Euler, Chern, and Stiefel-Whitney classes. This will allow us to apply a lot of the theory we built up to geometry.

Chapter 3

Homotopy groups

Insert an outline of the content of each lecture.

39 Limits, colimits, and adjunctions

Limits and colimits

We will freely use the theory developed in the first part of this book (see §3). Suppose $\mathcal{I}$ is a small category (so that it has a *set* of objects), and let $\mathcal{C}$ be another category.

Definition 39.1. Let $X : \mathcal{I} \to \mathcal{C}$ be a functor. A *cone under* X is a natural transformation η from X to a constant functor; explicitly, this means that for every object i of $\mathcal{I}$, we must have a map $\eta_i : X_i \to Y$, such that for every $f : i \to j$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{tikzcd} X_i \arrow[d, "f_*"] \arrow[dr, "\eta_i"] & \\ X_j \arrow[r, "\eta_j"] & Y. \end{tikzcd}$$

A *colimit* of X is an initial cone (L, τ_i) under X; explicitly, this means that for all cones (Y, η_i) under X, there exists a unique natural transformation $h : L \to Y$ such that $h \circ \tau_i = \eta_i$.

As always for category theoretic concepts, some examples are in order.

Example 39.2. If $\mathcal{I}$ is a discrete category (i.e., only a set, with identity maps), the colimit of any functor $\mathcal{I} \to \mathcal{C}$ is the coproduct. This already illustrates an important point about colimits: they need not exist in general (since, for example, coproducts need not exist in a general category). Examples of categories $\mathcal{C}$ where the colimit of a functor $\mathcal{I} \to \mathcal{C}$ exists: if $\mathcal{C}$ is sets, or spaces, the colimit is the disjoint union. If $\mathcal{C} = \mathbf{Ab}$, a candidate for the colimit would be the product: but this only works if $\mathcal{I}$ is finite; in general, the correct thing is to take the (possibly infinite) direct sum.

Example 39.3. Let $\mathcal{I} = \mathbf{N}$, considered as a category via its natural poset structure; then a functor $\mathcal{I} \to \mathcal{C}$ is simply a linear system of objects and morphisms in $\mathcal{C}$. As a specific example, suppose $\mathcal{C} = \mathbf{Ab}$, and consider the diagram $X : \mathcal{I} \to \mathcal{C}$ defined by the system

$$\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \to \cdots$$

The colimit of this diagram is $\mathbf{Q}$, where the maps are:

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "2"] \arrow[dr, "1"'] & \mathbf{Z} \arrow[r, "3"] \arrow[d, "1/2"] & \mathbf{Z} \arrow[r, "4"] \arrow[dl, "1/3!"] & \cdots \\
& \mathbf{Q} & &
\end{tikzcd}$$

Example 39.4. Let G be a group; we can view this as a category with one object, where the morphisms are the elements of the group (composition is given by the group structure). If $\mathcal{C} = \mathbf{Top}$ is the category of topological spaces, a functor $G \to \mathcal{C}$ is simply a group action on a topological space X. The colimit of this functor is the orbit space of the G-action on X.

Example 39.5. Let $\mathcal{I}$ be the category whose objects and morphisms are determined by the following graph:

$$\begin{tikzcd}
& a \arrow[dl] \arrow[dr] & \\
b & & c.
\end{tikzcd}$$

The colimit of a diagram $\mathcal{I} \to \mathcal{C}$ is called a *pushout*.

If $\mathcal{C} = \mathbf{Top}$, again, a functor $\mathcal{I} \to \mathcal{C}$ is determined by a diagram of spaces:

$$\begin{tikzcd}
& A \arrow[dl, "f"] \arrow[dr, "g"] & \\
B & & C.
\end{tikzcd}$$

The colimit of such a functor is just the pushout $B \cup_A C := B \sqcup C/\sim$, where $f(a) \sim g(a)$ for all $a \in A$. We have already seen this in action before: the same construction appears in the process of attaching cells to CW-complexes.

If $\mathcal{C}$ is the category of groups, instead, the colimit of such a functor is the free product quotiented out by a certain relation (the same as for topological spaces); this is called the *amalgamated free product*.

Example 39.6. Suppose $\mathcal{I}$ is the category defined by the following graph:

$$a \rightrightarrows b.$$

The colimit of a diagram $\mathcal{I} \to \mathcal{C}$ is called the *coequalizer* of the diagram.

One can also consider cones *over* a diagram $X : \mathcal{I} \to \mathcal{C}$: this is simply a cone in the opposite category.

Definition 39.7. With notation as above, the *limit* of a diagram $X : \mathcal{I} \to \mathcal{C}$ is a terminal object in cones over X.

For instance, products are limits, just like in Example 39.2. (This example also shows that abelian groups satisfy an interesting property: finite products are the same as finite coproducts!)

Exercise 39.8. Revisit the examples provided above: what is the limit of each diagram? For instance, the limit of the diagram described in Example 39.4 is just the fixed points!

Adjoint functors

Adjoint functors are very useful — and very natural — objects. We already have an example: let $\mathcal{C}^{\mathcal{I}}$ be the functor category $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$. (We've been working in this category this whole time!) Let's make an additional assumption on $\mathcal{C}$, namely that all $\mathcal{I}$-indexed colimits exist. All examples considered above satisfy this assumption.

There is a functor $\mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, given by sending any object to the constant functor taking that value. The process of taking the colimit of a diagram supplies us with a functor $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$. We can characterize this functor via a formula[1]:

$$\mathcal{C}(\mathrm{colim}_{i \in \mathcal{I}} X_i, Y) = \mathcal{C}^{\mathcal{I}}(X, \mathrm{const}_Y),$$

where X is some functor from $\mathcal{I}$ to $\mathcal{C}$. This formula is reminiscent of the adjunction operator in linear algebra, and is in fact our first example of an adjunction.

Definition 39.9. Let $\mathcal{C}, \mathcal{D}$ be categories, with specified functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. An *adjunction between F and G* is an isomorphism:

$$\mathcal{D}(FX, Y) = \mathcal{C}(X, GY),$$

which is natural in X and Y. In this situation, we say that F is a *left adjoint* of G and G is a *right adjoint* of X.

This notion was invented by Dan Kan, who worked in the MIT mathematics department until he passed away in 2013.

We've already seen an example above, but here is another one:

Definition 39.10 (Free groups)**.** There is a forgetful functor $u : \mathrm{Grp} \to \mathrm{Set}$. Any set X gives rise to a group FX, namely the free group on X elements. This is determined by a universal property: set maps $X \to u\Gamma$ are the same as group maps $FX \to \Gamma$, where Γ is any group. This is exactly saying that the free group functor the left adjoint to the forgetful functor u.

[1] There is an analogous formula for the limit of a diagram:

$$\mathcal{C}(W, \lim_{i \in \mathcal{I}} X_i) = \mathcal{C}^{\mathcal{I}}(\mathrm{const}_W, X).$$

In general, "free objects" come from left adjoints to forgetful functors.

Definition 39.11. A category $\mathcal{C}$ is said to be *cocomplete* if all (small) colimits exist in $\mathcal{C}$. Similarly, one says that $\mathcal{C}$ is *complete* if all (small) limits exist in $\mathcal{C}$.

The Yoneda lemma

One of the many important concepts in category theory is that an object is determined by the collection of all maps out of it. The Yoneda lemma is a way of making this precise. An important reason to even bother thinking about objects in this fashion comes from our discussion of colimits. Namely, how do we even know that the notion is well-defined?

The colimit of an object is characterized by maps out of it; precisely:

$$\mathcal{C}(\operatorname{colim}_{j\in\mathcal{J}} X_j, Y) = \mathcal{C}^{\mathcal{J}}(X_\bullet, \operatorname{const}_Y).$$

The two sides are naturally isomorphic, but if the colimit exists, how do we know that it is unique? This is solved by Yoneda lemma[2]:

Theorem 39.12 (Yoneda lemma)**.** *Consider the functor* $\mathcal{C}(X,-) : \mathcal{C} \to \mathrm{Set}$. *Suppose* $G : \mathcal{C} \to \mathrm{Set}$ *is another functor. It turns out that:*

$$\operatorname{nt}(\mathcal{C}(X,-), G) \simeq G(X).$$

Proof. Let $x \in G(X)$. Define a natural transformation that sends a map $f : X \to Y$ to $f_*(x) \in G(Y)$. On the other hand, we can send a natural transformation $\theta : C(X,-) \to G$ to $\theta_X(1_X)$. Proving that these are inverses is left as an exercise — largely in notation — to the reader. □

In particular, if $G = \mathcal{C}(Y,-)$ — these are called *corepresentable* functors — then $\operatorname{nt}(\mathcal{C}(X,-),\mathcal{C}(Y,-)) \simeq \mathcal{C}(Y,X)$. Simply put, natural isomorphisms $\mathcal{C}(X,-) \to \mathcal{C}(Y,-)$ are the same as isomorphisms $Y \to X$. As a consequence, the object that a corepresentable functor corepresents is unique (at least up to isomorphism).

From the Yoneda lemma, we can obtain some pretty miraculous conclusions. For instance, functors with left and/or right adjoints are very well-behaved (the "constant functor" functor is an example where both adjoints exist), as the following theorem tells us.

Theorem 39.13. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. If* F *admits a right adjoint, it preserves colimits. Dually, if* F *admits a left adjoint, it preserves limits.*

Proof. We'll prove the first statement, and leave the other as an (easy) exercise. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that admits a right adjoint G, and let $X : \mathcal{I} \to \mathcal{C}$ be a small $\mathcal{I}$-indexed diagram in $\mathcal{C}$. For any object Y of $\mathcal{C}$, there is an isomorphism

$$\operatorname{Hom}(\operatorname{colim}_{\mathcal{I}} X, Y) \simeq \lim_{\mathcal{I}} \operatorname{Hom}(X, Y).$$

[2] Sometimes "you-need-a-lemma"!

This follows easily from the definition of a colimit. Let Y be any object of $\mathcal{D}$; then, we have:

$$\begin{aligned}\mathcal{D}(F(\operatorname{colim}_{\mathcal{I}} X), Y) &\simeq \mathcal{C}(\operatorname{colim}_{\mathcal{I}} X, G(Y))\\ &\simeq \lim_{\mathcal{I}} \mathcal{C}(X, G(Y))\\ &\simeq \lim_{\mathcal{I}} \mathcal{D}(F(X), Y)\\ &\simeq \mathcal{D}(\operatorname{colim}_{\mathcal{I}} F(X), Y).\end{aligned}$$

The Yoneda lemma now finishes the job. □

40 Compactly generated spaces

A lot of homotopy theory is about loop spaces and mapping spaces. Standard topology doesn't do very well with mapping spaces, so we will narrate the story of *compactly generated spaces*. One nice consequence of working with compactly generated spaces is that the category is Cartesian-closed (a concept to be defined below).

CGHW spaces

Some constructions commute for "categorical reasons". For instance, limits commute with limits. Here is an exercise to convince you of a special case of this.

Exercise 40.1. Let X be an object of a category $\mathcal{C}$. The *overcategory* (or the *slice category*) $\mathcal{C}_{/X}$ has objects given by morphisms $p : Y \to X$ in $\mathcal{C}$, and morphisms given by the obvious commutativity condition.

1. Assume that $\mathcal{C}$ has finite products. What is the left adjoint to the functor $X \times - : \mathcal{C} \to \mathcal{C}_{/X}$ that sends Y to the object $X \times Y \xrightarrow{\mathrm{pr}_1} X$?

2. As a consequence of Theorem 39.13, we find that $X \times - : \mathcal{C} \to \mathcal{C}_{/X}$ preserves limits. The composite $\mathcal{C} \to \mathcal{C}_{/X} \to \mathcal{C}$, however, probably does not.

   - What is the limit of a diagram in $\mathcal{C}_{/X}$?
   - Let $Y : \mathcal{I} \to \mathcal{C}$ be any diagram. Show that

   $$\lim_{i \in \mathcal{I}}{}^{\mathcal{C}_{/X}} (X \times Y_i) \simeq X \times \lim_{i \in \mathcal{I}}{}^{\mathcal{C}} Y_i.$$

   What happens if $\mathcal{I}$ only has two objects and only identity morphisms?

However, colimits and limits need not commute! An example comes from algebra. The coproduct in the category of commutative rings is the tensor product (exercise!). But $\left(\lim \mathbf{Z}/p^k\mathbf{Z}\right) \otimes \mathbf{Q} \simeq \mathbf{Z}_p \otimes \mathbf{Q} \simeq \mathbf{Q}_p$ is clearly not $\lim \left(\mathbf{Z}/p^k\mathbf{Z} \otimes \mathbf{Q}\right) \simeq \lim 0 \simeq 0$!

We also need not have an isomorphism between $X \times \operatorname{colim}_{j \in \mathcal{J}} Y_j$ and $\operatorname{colim}_{j \in \mathcal{J}} (X \times Y_j)$. One example comes a quotient map $Y \to Z$: in general, the induced map $X \times Y \to X \times Z$ is not necessarily another quotient map. A theorem of Whitehead's says that this problem is rectified if we assume that X is a compact Hausdorff space. Unfortunately, a lot of interesting maps are built up from more "elementary" maps by such a procedure, so we would like to repair this problem.

We cannot simply do this by restricting ourselves to compact Hausdorff spaces: that's a pretty restrictive condition to place. Instead (motivated partially by the Yoneda lemma), we will look at topologies detected by maps from compact Hausdorff spaces.

Definition 40.2. Let X be a space. A subspace $F \subseteq X$ is said to be *compactly closed* if, for any map $k : K \to X$ from a compact Hausdorff space K, the preimage $k^{-1}(F) \subseteq K$ is closed.

It is clear that any closed subset is compactly closed, but there might be compactly closed sets which are not closed in the topology on X. This motivates the definition of a k-space:

Definition 40.3. A topological space X is said to be a *k-space* if every compactly closed set is closed.

The k comes either from "kompact" and/or Kelly, who was an early topologist who worked on such foundational topics.

It's clear that X is a k-space if and only if the following statement is true: a map $X \to Y$ is continuous if and only if, for every compact Hausdorff space K and map $k : K \to X$, the composite $K \to X \to Y$ is continuous. For instance, compact Hausdorff spaces are k-spaces. First countable (so metric spaces) and CW-complexes are also k-spaces.

In general, a topological space X need not be a k-space. However, it can be "k-ified" to obtain another k-space denoted kX. The procedure is simple: endow X with the topology consisting of all compactly closed sets. The reader should check that this is indeed a topology on X; the resulting topological space is denoted kX. This construction immediately implies, for instance, that the identity $kX \to X$ is continuous.

Let $k\mathbf{Top}$ be the category of k-spaces. This is a subcategory of the category of topological spaces, via a functor $i : k\mathbf{Top} \hookrightarrow \mathbf{Top}$. The process of k-ification gives a functor $\mathbf{Top} \to k\mathbf{Top}$, which has the property that:

$$k\mathbf{Top}(X, kY) = \mathbf{Top}(iX, Y).$$

Notice that this is another example of an adjunction! We can conclude from this that $k(iX \times iY) = X \times^{k\mathbf{Top}} Y$, where X and Y are k-spaces. One can also check that $kiX \simeq X$.

The takeaway is that $k\mathbf{Top}$ has good categorical properties inherited from $\mathbf{Top}$: it is a complete and cocomplete category. As we will now explain, this category has more categorical niceness, that does not exist in $\mathbf{Top}$.

Mapping spaces

Let X and Y be topological spaces. The set $\mathbf{Top}(X, Y)$ of continuous maps from X to Y admits a topology, namely the compact-open topology. If X and Y are k-spaces, we can make a slight modification: define a topology on $k\mathbf{Top}(X, Y)$ generated by the sets

$$W(k : K \to X, \text{ open } U \subseteq Y) := \{f : X \to Y : f(k(K)) \subseteq U\}.$$

We write Y^X for the k-ification of $k\mathbf{Top}(X, Y)$.

Proposition 40.4. *1. The functor* $(k\mathbf{Top})^{op} \times k\mathbf{Top} \to k\mathbf{Top}$ *given by* $(X, Y) \to Y^X$ *is a functor of both variables.*

2. $e : X \times Z^X \to Z$ *given by* $(x, f) \mapsto f(x)$ *and* $i : Y \to (X \times Y)^X$ *given by* $y \mapsto (x \mapsto (x, y))$ *are continuous.*

Proof. The first statement is left as an exercise to the reader. For the second statement, see [Str09, Proposition 2.11]. □

As a consequence of this result, we can obtain a very nice adjunction. Define two maps:

- $k\mathbf{Top}(X \times Y, Z) \to k\mathbf{Top}(Y, Z^X)$ via

$$(f : X \times Y \to Z) \mapsto (Y \xrightarrow{i} (X \times Y)^X \to Z^X).$$

- $k\mathbf{Top}(Y, Z^X) \to k\mathbf{Top}(X \times Y, Z)$ via

$$(f : Y \to Z^X) \mapsto (X \times Y \to X \times Z^X \xrightarrow{e} X).$$

By [Str09, Proposition 2.12], these two maps are continuous inverses, so there is a natural homeomorphism

$$k\mathbf{Top}(X \times Y, Z) \simeq k\mathbf{Top}(Y, Z^X).$$

This motivates the definition of a Cartesian closed category.

Definition 40.5. A category $\mathcal{C}$ with finite products is said to be *Cartesian closed* if, for any object X of $\mathcal{C}$, the functor $X \times - : \mathcal{C} \to \mathcal{C}$ has a right adjoint.

Our discussion above proves that $k\mathbf{Top}$ is Cartesian closed, while this is not satisfied by $\mathbf{Top}$. As we will see below, this has very important ramifications for algebraic topology.

Exercise 40.6. Insert Exercise 2 from 18.906.

41 "Cartesian closed", Hausdorff, Basepoints

Pushouts are colimits, so the quotient space $X/A = X \cup_A *$ is an example of a colimit. Let Y be a topological space, and consider the functor $Y \times - : \mathbf{Top} \to \mathbf{Top}$. Applying this to the pushout square, we find that $(Y \times X) \cup_{Y\times A} * \simeq (Y \times X)/(Y \times A)$. As we discussed in §40, this product is *not* the same as $Y \times (X/A)$! There is a bijective map $Y\times X/Y\times A \to Y\times(X/A)$, but it is not, in general, a homeomorphism. From a categorical point of view (see Theorem 39.13), the reason for this failure stems from $Y \times -$ not being a left adjoint.

The discussion in §40 implies that, when working with k-spaces, that functor is indeed a left adjoint (in fancy language, the category $k\mathbf{Top}$ is Cartesian closed), which means that — in $k\mathbf{Top}$ — there is a homeomorphism $Y \times X/Y \times A \to Y \times (X/A)$. This addresses the issues raised in §40. The ancients had come up with a good definition of a topology — but k-spaces are better! Sometimes, though, we can be greedy and ask for even more: for instance, we can demand that points be closed. This leads to a further refinement of k-spaces.

I don't like point-set topology, so I'll return to editing this lecture at the end.

"Hausdorff"

Definition 41.1. A space is "weakly Hausdorff" if the image of every map $K \to X$ from a compact Hausdorff space K is closed.

Another way to say this is that the map itself if closed. Clearly Hausdorff implies weakly Hausdorff. Another thing this means is that every point in X is closed (eg $K = *$).

Proposition 41.2. *Let X be a k-space.*

1. *X is weakly Hausdorff iff $\Delta : X \to X \times^k X$ is closed. In algebraic geometry such a condition is called separated.*

2. *Let $R \subseteq X \times X$ be an equivalence relation. If R is closed, then X/R is weakly Hausdorff.*

Definition 41.3. A space is compactly generated if it's a weakly Hausdorff k-space. The category of such spaces is called $\mathbf{CG}$.

We have a pair of adjoint functors $(i, k) : \mathbf{Top} \to k\mathbf{Top}$. It's possible to define a functor $k\mathbf{Top} \to \mathbf{CG}$ given by $X \mapsto X/\bigcap$all closed equivalence relations. It is easy to check that if Z is weakly Hausdorff, then Z^X is weakly Hausdorff (where X is a k-space). What this implies is that $\mathbf{CG}$ is also Cartesian closed!

I'm getting a little tired of point set stuff. Let's start talking about homotopy and all that stuff today for a bit. You know what a homotopy is. I will not worry about point-set topology anymore. So when I say $\mathbf{Top}$, I probably mean $\mathbf{CG}$. A homotopy between $f, g : X \to Y$ is a map $h : I \times X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[dr, "i_0"'] \arrow[drr, "f"] & & \\ & I \times X \arrow[r, "h"] & Y \\ X \arrow[ur, "i_1"] \arrow[urr, "g"'] & & \end{tikzcd}$$

We write $f \sim g$. We define $[X, Y] = \mathbf{Top}(X, Y)/\sim$. Well, a map $I \times X \to Y$ is the same as a map $X \to Y^I$ but also $I \to Y^X$. The latter is my favorite! It's a path of maps from f to g. So $[X, Y] = \pi_0 Y^X$.

To talk about higher homotopy groups and induct etc. we need to talk about basepoints.

Basepoints

A pointed space is $(X, *)$ with $* \in X$. This gives a category $\mathbf{Top}_*$ where the morphisms respect the basepoint. This has products because $(X, *) \times (Y, *) = (X \times Y, (*, *))$. How about coproducts? It has coproducts as well. This is the wedge product, defined as $X \sqcup Y/*_X \sim *_Y =: X \vee Y$. This is `\vee`, not `\wedge`. Is this category also Cartesian closed?

Define the space of pointed maps $Z_*^X \subseteq Z^X$ topologized as a subspace. Does the functor $Z \mapsto Z_*^X$ have a left adjoint? Well $\mathbf{Top}(W, Z^X) = \mathbf{Top}(X \times W, Z)$. What about $\mathbf{Top}(W, Z_*^X)$? This is $\{f : X \times W \to Z : f(*, w) = * \forall w \in W\}$. That's not quite what I wanted either! Thus $\mathbf{Top}_*(W, Z_*^X) = \{f : X \times W \to Z : f(*, w) = * = f(x, *) \forall x \in X, w \in W\}$. These send both "axes" to the basepoint. Thus, $\mathbf{Top}_*(W, Z_*^X) = \mathbf{Top}_*(X \wedge W, Z)$ where $X \wedge W = X \times W/X \vee W$ because $X \vee W$ are the "axes".

So $\mathbf{Top}_*$ is not Cartesian closed, but admits something called the smash product[3]. What properties would you like? Here's a good property: $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ are bijective in pointed spaces. If you work in $k\mathbf{Top}$ or $\mathbf{CG}$, then they are homeomorphic! It also has a unit.

Oh yeah, some more things about basepoints! So there's a canonical forgetful functor $i : \mathbf{Top}_* \to \mathbf{Top}$. Let's see. If I have $\mathbf{Top}(X, iY) = \mathbf{Top}_*(??, Y)$? This is $X_+ = X \sqcup *$. Thus we have a left adjoint $(-)_+$. It is clear that $(X \sqcup Y)_* = X_+ \vee Y_+$. The unit for the smash product is $*_+ = S^0$.

On Friday I'll talk about fibrations and fiber bundles.

42 Fiber bundles, fibrations, cofibrations

Having set up the requisite technical background, we can finally launch ourselves from point-set topology to the world of homotopy theory.

[3] Remark by Sanath: this is like the tensor product.

Fiber bundles

Definition 42.1. A fiber[4] bundle is a map $p : E \to B$, such that for every $b \in B$, there exists:

- an open subset $U \subseteq B$ that contains b, and
- a map $p^{-1}(U) \to p^{-1}(b)$ such that $p^{-1}(U) \to U \times p^{-1}(b)$ is a homeomorphism.

If $p : E \to B$ is a fiber bundle, E is called the *total space*, B is called the *base space*, p is called a *projection*, and F (sometimes denoted $p^{-1}(b)$) is called the *fiber over b*.

In simpler terms: the preimage over every point in B looks like a product, i.e., the map $p : E \to B$ is "locally trivial" in the base.

Here is an equivalent way of stating Definition 42.1: there is an open cover $\mathcal{U}$ (called the *trivializing cover*) of B, such that for every $U \subseteq \mathcal{U}$, there is a space F, and a homeomorphism $p^{-1}(U) \simeq U \times F$ that is compatible with the projections down to U. (So, for instance, a trivial example of a fiber bundle is just the projection map $B \times F \xrightarrow{\mathrm{pr}_1} B$.)

Fiber bundles are naturally occurring objects. For instance, a covering space $E \to B$ is a fiber bundle with discrete fibers.

Example 42.2 (The Hopf fibration)**.** The Hopf fibration is an extremely important example of a fiber bundle. Let $S^3 \subset \mathbf{C}^2$ be the 3-sphere. There is a map $S^3 \to \mathbf{CP}^1 \simeq S^2$ that is given by sending a vector v to the complex line through v and the origin. This is a non-nullhomotopic map, and is a fiber bundle whose fiber is S^1.

Here is another way of thinking of the Hopf fibration. Recall that $S^3 = SU(2)$; this contains as a subgroup the collection of matrices $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$. This subgroup is simply S^1, which acts on S^3 by translation; the orbit space is S^2.

The Hopf fibration is a map between smooth manifolds. A theorem of Ehresmann's says that it is not too hard to construct fiber bundles over smooth manifolds:

Theorem 42.3 (Ehresmann)**.** *Suppose E and B are smooth manifolds, and let $p : E \to B$ be a smooth (i.e., C^∞) map. Then p is a fiber bundle if:*

1. *p is a* submersion*, i.e., $dp : T_e E \to T_{p(e)} B$ is a surjection, and*
2. *p is* proper*, i.e., preimages of compact sets are compact.*

The purpose of this part of the book is to understand fiber bundles through algebraic methods like cohomology and homotopy. This means that we will usually need a "niceness" condition on the fiber bundles that we will be studying; this condition is made precise in the following definition (see [May99]).

[4] Or "fibre", if you're British.

Definition 42.4. Let X be a space. An open cover $\mathcal{U}$ of X is said to be *numerable* if there exists a subordinate partition of unity, i.e., for each $U \in \mathcal{U}$, there is a function $f_U : X \to [0,1] = I$ such that $f^{-1}((0,1]) = U$, and any $x \in X$ belongs to only finitely many $U \in \mathcal{U}$. The space X is said to be *paracompact* if any open cover admits a numerable refinement.

This isn't too restrictive for us algebraic topologists since CW-complexes are paracompact.

Definition 42.5. A fiber bundle is said to be *numerable* if it admits a numerable trivializing cover.

Fibrations and path liftings

For our purposes, though, fiber bundles are still too narrow. Fibrations capture the essence of fiber bundles, although it is not at all immediate from their definition that this is the case!

Definition 42.6. A map $p : E \to B$ is called a *(Hurewicz[5]) fibration* if it satisfies the *homotopy lifting property* (commonly abbreviated as HLP): suppose $h : I \times W \to B$ is a homotopy; then there exists a lift[6] (given by the dotted arrow) that makes the diagram commute:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ I \times W \arrow[r, "h"'] \arrow[ur, dashed, "\overline{h}"] & B, \end{tikzcd} \tag{3.1}$$

At first sight, this seems like an extremely alarming definition, since the HLP has to be checked for *all* spaces, *all* maps, and *all* homotopies! The HLP is not impossible to check, though.

Exercise 42.7. Check that the projection $\mathrm{pr}_1 : B \times F \to B$ is a fibration.

Exercise 42.8. Check the following statements.

- Fibrations are closed under pullbacks. In other words, if $p : E \to B$ is a fibration and $X \to B$ is any map, then the induced map $E \times_B X \to X$ is a fibration.

- Fibrations are closed under exponentiation and products. In other words, if $p : E \to B$ is a fibration, then $E^A \to B^A$ is another fibration.

- Fibrations are closed under composition.

[5]Named after Witold Hurewicz, who was one of the first algebraic topologists at MIT.
[6]Note that we place no restriction on the *uniqueness* of this lift.

Exercise 42.9. Let $p : E_0 \to B_0$ be a fibration, and let $f : B \to B_0$ be a homotopy equivalence. Prove that the induced map $B \times_{B_0} E_0 \to E_0$ is a homotopy equivalence. (Warning: this exercise has a lot of technical details! The end of this chapter describes an alternative[7] solution to this exercise, when E_0 and $B \times_{B_0} E_0$ are CW-complexes.)

n't forget to do
s!

There is a simple geometric interpretation of what it means for a map to be a fibration, in terms of "path liftings". To understand this description, we will reformulate the diagram (3.1). Given that we are working in the category of CGWH spaces, one of the first things we can attempt to do is adjoint the I; this gives the following diagram.

$$\begin{tikzcd} E \arrow[r, "p"] & B \\ W \arrow[u, "f"] \arrow[r, "\widehat{h}"'] & B^I \arrow[u, "\mathrm{ev}_0"'] \end{tikzcd} \tag{3.2}$$

By the definition of the pullback of a diagram, the data of this diagram is equivalent to a map $W \to B^I \times_B E$. Explicitly,

$$B^I \times_B E = \{(\omega, e) \in B^I \times E \text{ such that } \omega(0) = p(e)\}.$$

Suppose the desired dotted map exists (i.e., $p : E \to B$ satisfied the HLP). This would beget (again, by adjointness) a lifted homotopy $\widehat{\overline{h}} : W \to E^I$. Since we already have a map[8] $\widetilde{p} : E^I \to B^I \times_B E$ given by $\omega \mapsto (p\omega, \omega(0))$, the existence of the lift $\overline{h}$ in the diagram (3.1) is equivalent to the existence of a lift in the following diagram.

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ W \arrow[ur, dashed, "\widehat{\overline{h}}"] \arrow[r] & B^I \times_B E \end{tikzcd}$$

Obviously the universal example of a space W that makes the diagram (3.2) commute is $B^I \times_B E$ itself. If p is a fibration, we can make the lift in the following diagram.

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ B^I \times_B E \arrow[ur, dashed, "\lambda"] \arrow[r, "1"] & B^I \times_B E \end{tikzcd}$$

The map λ is called a *lifting function*. To understand why, suppose $(\omega, e) \in B^I \times_B E$, so that $\omega(0) = p(e)$. In this case, $\lambda(\omega, e)$ defines a path in E such that

$$p \circ \lambda(\omega, e) = \omega, \text{ and } \lambda(\omega, e)(0) = e.$$

[7] "Alternative" in the sense that the proof uses statements not covered yet in this book.

[8] Clearly $(p\omega)(0) = p(\omega(0))$, so this map is well-defined (i.e., the image lands in $B^I \times_B E$).

Taking a step back and assessing the situation, we find that the lifting function λ starts with a path ω in B, and some point in E mapping down to $\omega(0)$, and produces a "lifted" path in E which lives over ω. In other words, the map λ is a path lifting: it's a continuous way to lift paths in the base space B to the total space E.

The following result is a "consistency check".

Theorem 42.10 (Dold)**.** *Let $p : E \to B$ be a map. Assume there's a numerable cover of B, say $\mathcal{U}$, such that for every $U \in \mathcal{U}$, the restriction $p|_{p^{-1}(U)} : p^{-1}U \to U$ is a fibration. (In other words, p is* locally *a fibration over the base). Then p itself is a fibration.*

In particular, one consequence of this theorem is that every numerable fiber bundle is a fibration. Our discussion above tells us that numerable fiber bundles satisfy the homotopy (and hence path) lifting property. This is great news, as we will see shortly.

43 Fibrations and cofibrations

Comparing fibers over different points

Let $p : E \to B$ be a fibration. Above, we saw that this implies that paths in B "lift" to paths in E. Let us consider a path $\omega : I \to B$ with $\omega(0) = a$ and $\omega(1) = b$. Denote by F_a the fiber over a. If the world plays fairly, the path lifting property of fibrations should beget a (unique[9]) map $F_a \to F_b$. The goal of this subsection is to construct such a map.

Consider the diagram:

$$\begin{tikzcd}
F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed, "h"] & I \arrow[r, "\omega"'] & B.
\end{tikzcd}$$

This commutes since $\omega(0) = a$. Utilizing the homotopy lifting property, there is a dotted arrow that makes the entire diagram commute. If $x \in F_a$, the image $h(1,x)$ is in F_b, and $h(0,x) = x$. This supplies us with a map $f : F_a \to F_b$, given by $f(x) = h(1,x)$.

We're now faced with a natural question: is f unique up to homotopy? Namely: if we have two homotopic paths ω_0, ω_1 with $\omega_0(0) = \omega_1(0) = a$, and $\omega_0(1) = \omega_1(1) = b$, along with a given homotopy $g : I \times I \to B$ between ω_0 and ω_1, such that $f_0, f_1 : F_a \to F_b$ are the associated maps (defined by $h_0(1,x)$ and $h_1(1,x)$), respectively, are f_0 and f_1 homotopic?

We have a diagram of the form:

$$\begin{tikzcd}
((\partial I \times I) \cup (I \times \{0\})) \times F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed] & I \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

To get a homotopy between f_0 and f_1, we need the dotted arrow to exist.

[9] At least up to homotopy.

It's an easy exercise to recognize that our diagram is equivalent to the following.

$$\begin{tikzcd}
I \times F_a \arrow[r, "\simeq"] & ((\partial I \times I) \cup (I \times 0)) \times F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times I \times F_a \arrow[r, "\simeq"'] & I \times I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed] & I \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

Letting $W = I \times F_a$ in the definition of a fibration (Definition 42.6) thus gives us the desired lift, i.e., a homotopy $f_0 \simeq f_1$.

We can express the uniqueness (up to homotopy) of lifts of homotopic paths in a functorial fashion. To do so, we must introduce the fundamental groupoid of a space.

Definition 43.1. Let X be a topological space. The *fundamental groupoid* $\Pi_1(X)$ of X is a category (in fact, groupoid), whose objects are the points of X, and maps are homotopy classes of paths in X. The composition of compatible paths σ and ω is defined by:

$$(\sigma \cdot \omega)(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \sigma(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The results of the previous sections can be succinctly summarized in the following neat statement.

Proposition 43.2. *Any fibration $p : E \to B$ gives a functor $\Pi_1(B) \to$ **Top**.*

This is the beginning of a beautiful story involving fibrations. (The interested reader should look up "Grothendieck construction".)

Cofibrations

Let $i : A \to X$ be a map of spaces. If Y is another topological space, when is the induced map $Y^X \to Y^A$ a fibration? This is asking for the map i to be "dual" to a fibration.

By the definition of a fibration, we want a lifting:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\mathrm{in}_0"'] & Y^X \arrow[d] \\
I \times W \arrow[r] \arrow[ur, dashed] & Y^A.
\end{tikzcd}$$

Adjointing over, we get:

$$\begin{tikzcd}
A \times W \arrow[r, "i \times 1"] \arrow[d, "1 \times \mathrm{in}_0"'] & X \times W \arrow[d] \arrow[ddr] & \\
A \times W \times I \arrow[r] \arrow[drr] & X \times I \times W \arrow[dr, dashed] & \\
& & Y.
\end{tikzcd}$$

Again adjointing over, this diagram transforms to:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\
A \times I \arrow[r] \arrow[drr] & X \times I \arrow[dr, dashed] & \\
& & Y^W.
\end{tikzcd}$$

This discussion motivates the following definition of a "cofibration": as mentioned above, this is "dual" to the notion of fibration.

Definition 43.3. A map $i : A \to X$ of spaces is said to be a *cofibration* if it satisfies the *homotopy extension property* (sometimes abbreviated as "HEP"): for any space Y, there is a dotted map in the following diagram that makes it commute:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\
A \times I \arrow[r] \arrow[drr] & X \times I \arrow[dr, dashed] & \\
& & Y.
\end{tikzcd}$$

Again, using the definition of a pushout, the universal example of such a space Y is the pushout $X \cup_A (A \times I)$. Equivalently, we are therefore asking for the existence of a dotted arrow in the following diagram.

$$\begin{tikzcd}
X \cup_A (A \times I) \arrow[r] \arrow[dr] & X \times I \arrow[d, dashed] \\
& Z,
\end{tikzcd}$$

for any Z. Using the universal property of a pushout, this is equivalent to the existence of a dotted arrow in the following diagram.

$$\begin{tikzcd}
X \cup_A (A \times I) \arrow[r] \arrow[dr] & X \times I \arrow[d, dashed] \\
& X \cup_A (A \times I) \arrow[d] \\
& Z,
\end{tikzcd}$$

which is, in turn, equivalent to asking $X \cup_A (A \times I)$ to be a retract of $X \times I$.

Example 43.4. $S^{n-1} \hookrightarrow D^n$ is a cofibration.

Properly draw out this figure!

Figure 3.1: Drawing by John Ni.

In particular, setting $n = 1$ in this example, $\{0, 1\} \hookrightarrow I$ is a cofibration.

Here are some properties of the class of cofibrations of CGWH spaces.

- It's closed under cobase change: if $A \to X$ is a cofibration, and $A \to B$ is any map, the pushout $B \to X \cup_A B$ is also cofibration. (Exercise!)
- It's closed under finite products. (This is surprising.)
- It's closed under composition. (Exercise!)
- Any cofibration is a closed inclusion[10].

is is not obvious; uld we include a oof?

44 Homotopy fibers

An important, but easy, fact about fibrations is that the canonical map $X \to *$ from any space X is a fibration [11]. This is because the dotted lift in the diagram below can be taken to the map $(t, w) \mapsto f(w)$:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ I \times W \arrow[r] \arrow[ur, dashed] & *. \end{tikzcd}$$

However:

[10]Note that the dual statement for fibrations would state: any fibration $p : E \to B$ is a quotient map. This is definitely not true: fibrations do not have to be surjective! For instance, the trivial map $\varnothing \to B$ is a fibration. (Fibrations are surjective on path components though, because of path lifting.)

[11]Model category theorists get excited about this, because this says that all objects in the associated model structure on topological spaces is fibrant.

Exercise 44.1. The inclusion $* \hookrightarrow X$ is not always a cofibration; if it is, say that $*$ is a *nondegenerate basepoint* of X. Give an example of a compactly generated space X for which this is true.

If $*$ has a neighborhood in X that contracts to $*$, the inclusion $* \hookrightarrow X$ is a cofibration. Note that if $*$ is a nondegenerate basepoint, the canonical map $X^A \xrightarrow{\text{ev}} X$ is a fibration, where A is a pointed subspace of X (with basepoint given by $*$). The fiber of ev is exactly the space of pointed maps $A \to X$.

Remark 44.2. In Example 43.4, we saw that $\{0,1\} \hookrightarrow I$ is a cofibration; this implies that the map $Y^I \to Y \times Y$ (given by $\omega \mapsto (\omega(0), \omega(1))$) is a fibration.

"Fibrant replacements"

The purpose of this subsection is to provide a proof of the following result, which says that every map can be "replaced" (up to homotopy) by a fibration.

Theorem 44.3. *For any map $f : X \to Y$, there is a space $T(f)$, along with a fibration $p : T(f) \to Y$ and a homotopy equivalence $X \xrightarrow{\simeq} T(f)$, such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] \\ & Y. \end{tikzcd}$$

Proof. Consider the map $Y^I \xrightarrow{\binom{\text{ev}_0}{\text{ev}_1}} Y \times Y$. Let $T(f)$ be the pullback of the following diagram:

$$\begin{tikzcd} T(f) \arrow[r] \arrow[d] & Y^I \arrow[d, "\binom{\text{ev}_0}{\text{ev}_1}"] \\ X \times Y \arrow[r, "f \times 1"'] & Y \times Y. \end{tikzcd}$$

So, as a set, we can write

$$T(f) = \{(x, \omega) \in X \times Y^I | f(x) = \omega(0)\}.$$

Let us check that the canonical map $T(f) \to Y$, given by $(x, \omega) \mapsto \omega(1)$, is a fibration. The projection map $\text{pr} : X \times Y \to Y$ is a fibration, so it suffices to show that the map $T(f) \to X \times Y$ is also a fibration. Since fibrations are closed under pullbacks, we are reduced to checking that the map $Y^I \to Y \times Y$ is a fibration; but this is exactly saying that the inclusion $\{0,1\} \hookrightarrow I$ is a cofibration, which it is (Example 43.4).

To prove that X is homotopy equivalent to $T(f)$, we need to produce a map $X \to T(f)$. This is equivalent to giving maps $X \to X \times Y$ and $X \to Y^I$ that have compatible images in $Y \times Y$. The first map can be chosen to be $X \xrightarrow{\binom{1}{f}} X \times Y$. Define the map $X \to Y^I$ by sending $x \in X$ to the constant loop at $f(x)$. It is clear that both composites

$X \to X \times Y \to Y \times Y$ and $X \to Y^I \to Y \times Y$ are the same; this defines a map $X \to T(f)$, denoted g. As one can easily check, the composite $X \to T(f) \xrightarrow{p} Y$ is the map $f : X \to Y$ that we started off with. It remains to check that this map $X \xrightarrow{g} T(f)$ is a homotopy equivalence. We will construct a homotopy inverse to this map.

The composite $X \to T(f) \to X \times Y \to X$ is the identity, so one candidate for a homotopy inverse to g is the composite

$$T(f) \to X \times Y \xrightarrow{pr_1} X.$$

To prove that this map is indeed a homotopy inverse to g, we need to consider the composite $T(f) \to X \xrightarrow{g} T(f)$, which sends $(x, \omega) \mapsto x \mapsto (x, c_{f(x)})$ where, recall, $c_{f(x)}$ is the constant path at x. We need to produce a homotopy between this composite and the identity on $T(f)$.

Let $s \in I$. Given $\omega \in Y^I$, define a new loop ω_s by $\omega_s(t) = \omega(st)$. For instance, $\omega_1 = \omega$, and $\omega_0 = c_{\omega(0)}$ — so, the loop ω_s "sucks in" the point $\omega(1)$. This is precisely what we need to produce a homotopy between the composite $T(f) \to X \xrightarrow{g} T(f)$ and $\mathrm{id}_{T(f)}$, since the only constraint on $(x, \omega) \in T(f)$ is on $\omega(0)$. The following map provides the desired homotopy equivalence $X \simeq T(f)$.

$$\begin{aligned} H : I \times T(f) &\to T(f) \\ (s, (x, \omega)) &\mapsto (x, \omega_s). \end{aligned}$$

$\square$

Example 44.4 (Path-loop fibration)**.** This is a silly, but important, example. If $X = *$, the space $T(f)$ consists of paths ω in Y such that $\omega(0) = *$. In other words, $T(f) = Y^I_*$; this is called the *(based) path space* of Y, and is denoted by $P(Y, *)$, or simply by PY. The fiber of the fibration $T(f) = PY \to Y$ consists of paths that begin and end at $*$, i.e., loops on Y based at $*$. This is denoted ΩY, and is called the *(based) loop space* of Y. The resulting fibration $PY \to Y$ is called the *path-loop fibration.*

Exercise 44.5 ("Cofibrant replacements")**.** In this exercise, you will prove the analogue of Theorem 44.3 for cofibrations. Let $f : X \to Y$ be any map. Show that f factors (functorially) as a composite $X \to M \to Y$, where $X \to M$ is a cofibration and $M \to Y$ is a homotopy equivalence.

Solution 44.6. Define Mf via the pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & Y \arrow[d, "g"] \\ I \times X \arrow[r] & Mf. \end{tikzcd}$$

Define $r : Mf \to Y$ via $r(y) = y$ on Y and $r(x, s) = f(x)$ on $X \times I$. Then, clearly, $rg = \mathrm{id}_Y$. There is a homotopy $\mathrm{id}_{Mf} \simeq gr$ given by the map $h : Mf \times I \to Mf$, defined by the formulae

$$h(y, t) = y, \text{ and } h((x, s), t) = (x, (1 - t)s).$$

We now have to check that $X \to Mf$ is a cofibration, i.e., that $Mf \times I$ retracts onto $Mf \times \{0\} \cup_X (X \times I)$. This can be done by "pushing" $Y \times I$ to $Y \times \{0\}$ and $X \times I \times I$ down to $X \times I$, while fixing $X \times \{0\}$.

It is easy to see that this factorization is functorial: if $f : X \to Y$ is sent to $g : W \to Z$ via $p : X \to W$ and $q : Y \to Z$, then $Mf \to Mg$ can be defined as the dotted map in the following diagram (which exists, by the universal property of the pushout):

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"] \arrow[dd, "\mathrm{in}_0"'] & & Y \arrow[dr, "q"] \arrow[dd] & \\
& W \arrow[rr, "g"] \arrow[dd] & & Z \arrow[dd] \\
X \times I \arrow[rr] \arrow[dr, "p\times\mathrm{id}"'] & & Mf \arrow[dr, dashed] & \\
& W \times I \arrow[rr] & & Mg.
\end{tikzcd}$$

Fix the overlapping arrows here, I don't know how to do this...

Homotopy fibers

One way to define the fiber (over a basepoint) of a map $f : X \to Y$ is via the pullback

$$\begin{tikzcd}
f^{-1}(*) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
* \arrow[r] & Y.
\end{tikzcd}$$

If $g : W \to X$ is another map such that the composite $W \xrightarrow{g} X \xrightarrow{f} Y$ is trivial, the map g factors through $f^{-1}(*)$. In homotopy theory, maps are generally not trivial "on the nose"; instead, we usually have a nullhomotopy of a map. Nullhomotopies of composite maps do not factor through this "strict" fiber; this leads to the notion of a homotopy fiber.

Definition 44.7 (Homotopy fiber)**.** The *homotopy fiber* of a map $f : X \to Y$ is the pullback:

$$\begin{tikzcd}
F(f,*) \arrow[r] \arrow[d] & T(f) \arrow[r, "\simeq"] \arrow[d, "p"] & X \arrow[dl, "f"] \\
* \arrow[r] & Y &
\end{tikzcd}$$

As a set, we have

$$F(f,*) = \{(x,\omega) \in X \times Y^I | f(x) = \omega(0), \omega(1) = *\}. \tag{3.3}$$

A nullhomotopic composite $W \to X \xrightarrow{f} Y$ factors as $W \to F(f,*) \to X \xrightarrow{f} Y$.

Warning 44.8. The ordinary fiber and the homotopy fiber of a map are generally not the same! There is a canonical map $p^{-1}(*) \to F(p,*)$, but it is generally not a homotopy equivalence.

Proposition 44.9. *Suppose $p : X \to Y$ is a fibration. Then the canonical map $p^{-1}(*) \to F(p,*)$ is a homotopy equivalence.*

You will prove this in a series of exercises.

Exercise 44.10. Prove Proposition 44.9 by working through the following statements.

1. Let $p : E \to B$ be a fibration. Suppose $g : X \to B$ lifts across p up to homotopy, i.e., there exists a map $f : X \to E$ such that $p \circ f \simeq g$. Prove that there exists a map $f' : X \to E$ that is homotopic to f, such that $p \circ f' = g$ (on the nose).

2. Show that if $p : E \to B$ and $p' : E' \to B$ are fibrations, and $f : E \to E'$ such that $p' \circ f = p$, the map f is a *fiber homotopy equivalence*: there is a homotopy inverse $g : E' \to E$ such that g, and the two homotopies $fg \simeq \mathrm{id}_{E'}$ and $gf \simeq \mathrm{id}_E$ are all fiber preserving (e.g., $p \circ g = p'$).

3. Conclude Proposition 44.9.

Before we proceed, recall that we constructed the homotopy fiber by replacing $f : X \to Y$ by a fibration. In doing so, we implicitly made a choice: we could have replaced the map $* \to Y$ by a fibration. Are the resulting pullbacks the same?

By replacing $* \to Y$ by a fibration (namely, the path-loop fibration), we end up with the following pullback diagram:

$$\begin{tikzcd}
F'(f,*) \arrow[r] \arrow[d] & P(Y,*) \arrow[r, "\simeq"] \arrow[d] & * \arrow[dl] \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

As a set, we have

$$F'(f,*) = \{(x,\omega) \in X \times Y^I \text{ such that } \omega(0) = * \text{ and } \omega(1) = f(x)\}.$$

Our description of $F(f,*)$ in (3.3) is almost exactly the same — except that the directions of the paths are reversed. Thus there's a homeomorphism $F'(f,*) \simeq F(f,*)$ given by reversing directions of paths.

Remark 44.11. One could also replace both $f : X \to Y$ and $* \to Y$ by fibrations, and the resulting pullback is also homeomorphic to $F(f,*)$. (Prove this, if the statement is not immediate.)

45 Barratt-Puppe sequence

Fiber sequences

Recall, from the previous section, that we have a pullback diagram:

$$\begin{tikzcd}
& F(f,*) \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "p"] \arrow[dr, "\simeq"] & \\
f^{-1}(*) \arrow[r] \arrow[ur] & X \arrow[r, "f"'] & Y & * \arrow[l]
\end{tikzcd}$$

Consider a pointed map[12] $f : X \to Y$ (so that $f(*) = *$). Then, we will write Ff for the homotopy fiber $F(f,*)$.

Since we're exploring the homotopy fiber Ff, we can ask the following, seemingly silly, question: what is the fiber of the canonical map $p : Ff \to X$ (over the basepoint of X)? This is precisely the space of loops in Y! Since p is a fibration (recall that fibrations are closed under pullbacks), the homotopy fiber of p is also the "strict" fiber! Our expanded diagram is now:

$$\begin{tikzcd}
& \Omega Y = p^{-1}(*) \arrow[d] & & \\
& F(f,*) \arrow[r] \arrow[d, "p"] & PY \arrow[d, "p"] \arrow[dr, "\simeq"] & \\
f^{-1}(*) \arrow[r] \arrow[ur] & X \arrow[r, "f"'] & Y & *. \arrow[l]
\end{tikzcd}$$

It's easy to see that the composite $Ff \xrightarrow{p} X \xrightarrow{f} Y$ sends $(x, \omega) \mapsto f(x)$; this is a pointed nonconstant map. (Note that the basepoint we're choosing for Ff is the image of the basepoint in $f^{-1}(*)$ under the canonical map $f^{-1}(*) \hookrightarrow Ff$.)

While the composite $fp : Ff \to Y$ is not zero "on the nose", it is nullhomotopic, for instance via the homotopy $h : Ff \times I \to Y$, defined by

$$h(t, (x, \omega)) = \omega(t).$$

Exercise 45.1. Let $f : X \to Y$ and $g : W \to X$ be pointed maps. Establish a homeomorphism between the space of pointed maps $W \xrightarrow{p} Ff$ such that $fp = g$ and the space of pointed nullhomotopies of the composite fg.

[12]Some people call such a map "based", but this makes it sound like we're doing chemistry, so we won't use it.

This exercise proves that the homotopy fiber is the "kernel" in the homotopy category of pointed spaces and pointed maps between them.

Define $[W, X]_* = \pi_0(X_*^W)$; this consists of the pointed homotopy classes of maps $W \to X$. We may view this as a pointed set, whose basepoint is the constant map. Fixing W, this is a contravariant functor in X, so there are maps $[W, Ff]_* \to [W, X]_* \to [W, Y]_*$. This composite is not just nullhomotopic: it is "exact"! Since we are working with pointed sets, we need to describe what exactness means in this context: the preimage of the basepoint in $[W, Y]_*$ is exactly the image of $[W, Ff]_* \to [W, X]_*$. (This is exactly a reformulation of Exercise 45.1.) We say that $Ff \to X \xrightarrow{f} Y$ is a *fiber sequence.*

Remark 45.2. Let $f : X \to Y$ be a map of spaces, and suppose we have a homotopy commutative diagram:

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, "\Omega g"'] & Ff \arrow[r] \arrow[d, dashed] & X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\
\Omega Y' \arrow[r] & Ff' \arrow[r] & X' \arrow[r, "f'"'] & Y.
\end{tikzcd}$$

Then the dotted map exists, but it depends on the homotopy $f'h \simeq gf$.

Iterating fiber sequences

Let $f : X \to Y$ be a pointed map, as before. As observed above, we have a composite map $Ff \xrightarrow{p} X \xrightarrow{f} Y$, and the strict fiber (homotopy equivalent to the homotopy fiber) of p is ΩY. This begets a map $i(f) : \Omega Y \to Ff$; iterating the procedure of taking fibers gives:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_3 \arrow[r, "p_4"] & Fp_2 \arrow[r, "p_3"] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
& \Omega Fp_0 \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[ur, "i(p_2)"] & \Omega X \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[ur, "i(p_1)"] & \Omega Y \arrow[u, "\simeq"] \arrow[ur, "i(f)"] & & &
\end{tikzcd}$$

All the p_i in the above diagram are fibrations. Each of the dotted maps in the above diagram can be filled in up to homotopy. The most obvious guess for what these dotted maps are is simply $\Omega X \xrightarrow{\Omega f} \Omega Y$. But *that is the wrong map*!

The right map turns out to be $\Omega X \xrightarrow{\overline{\Omega f}} \Omega Y$:

Lemma 45.3. *The following diagram commutes to homotopy:*

$$\begin{tikzcd}
& Fp \\
\Omega X \arrow[ur, "i(p)"] \arrow[r, "\overline{\Omega f}"'] & \Omega Y; \arrow[u]
\end{tikzcd}$$

here, $\overline{\Omega f}$ is the diagonal in the following diagram:

$$\begin{tikzcd}
\Omega X \arrow[r, "-"] \arrow[d, "\Omega f"'] \arrow[dr, "\overline{\Omega f}"] & \Omega X \arrow[d, "\Omega f"] \\
\Omega Y \arrow[r, "-"'] & \Omega Y,
\end{tikzcd}$$

where the map $- : \Omega X \to \Omega X$ *sends* $\omega \mapsto \overline{\omega}$.

Proof.

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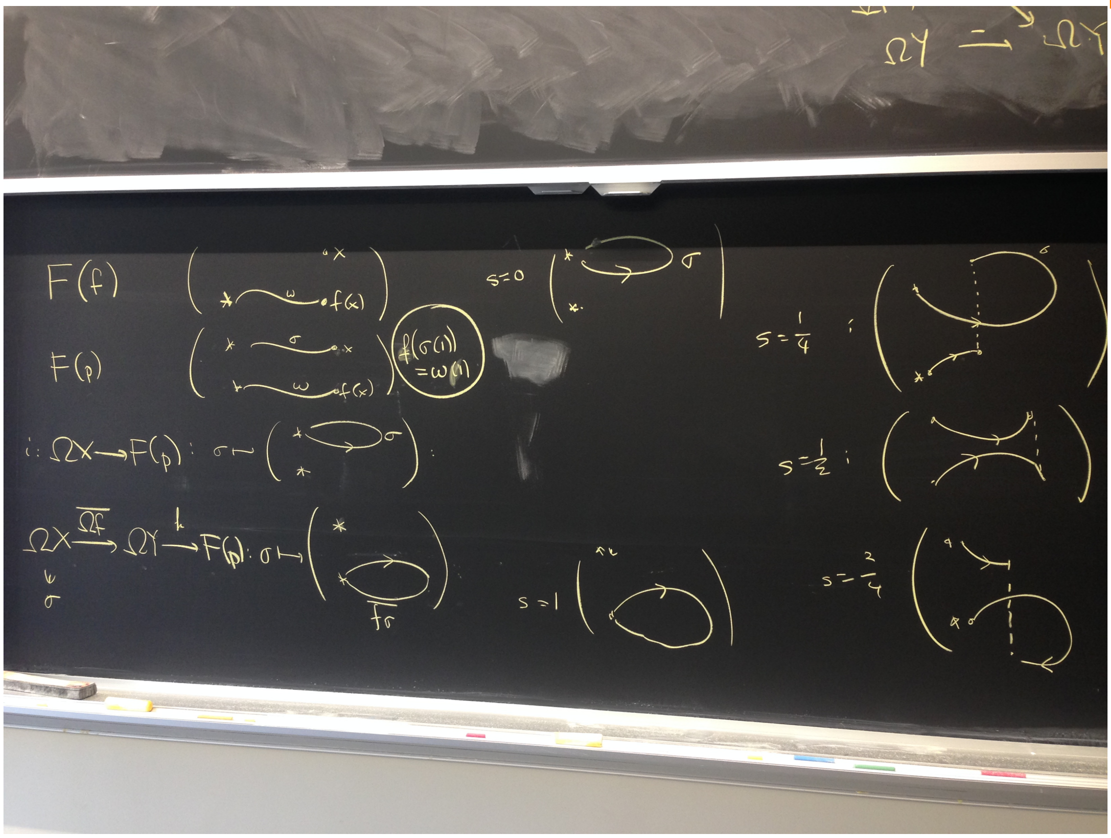

Figure 3.2: A proof of this lemma.

$\square$

By the above lemma, we can extend our diagram to:

$$
\begin{tikzcd}
\cdots \arrow[r] & Fp_4 \arrow[r] & Fp_3 \arrow[r] & Fp_2 \arrow[r] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
\cdots \arrow[r] & \Omega Fp_1 \arrow[u, "\simeq"'] \arrow[r, "\overline{\Omega p_2}"] & \Omega Ff \arrow[u, "\simeq"'] \arrow[ur, "i(p_2)"] \arrow[r, "\overline{\Omega p}"] & \Omega X \arrow[u, "\simeq"'] \arrow[ur, "i(p_1)"] \arrow[r, "\overline{\Omega f}"] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "i(f)"] & & & \\
\Omega^2 X \arrow[u, "\simeq"'] \arrow[r, "\Omega^2 f"'] & \Omega^2 Y \arrow[u, "\simeq"'] \arrow[ur, "\overline{\Omega i(p_0)}"'] & & & & & &
\end{tikzcd}
$$

We have a special name for the sequence of spaces sneaking along the bottom of this diagram:

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to \Omega Ff \to \Omega X \to \Omega Y \to Ff \to X \xrightarrow{f} Y;$$

this is called the *Barratt-Puppe sequence*. Applying $[W, -]_*$ to the Barratt-Puppe sequence of a map $f : X \to Y$ gives a long exact sequence.

The most important case of this long exact sequence comes from setting $W = S^0 = \{\pm 1\}$; in this case, we get terms like $\pi_0(\Omega^n X)$. We can identify $\pi_0(\Omega^n X)$ with $[S^n, X]_*$: to see this for $n = 2$, recall that $\Omega^2 X = (\Omega X)^{S^1}$; because $(S^1)^{\wedge n} = S^n$ (see below for a proof of this fact), we find that

$$(\Omega X)^{S^1} \simeq (X_*^{S^1})_*^{S^1} = X_*^{S^1 \wedge S^1} = X_*^{S^2}, \tag{3.4}$$

as desired.

The space ΩX is a group in the homotopy category; this implies that $\pi_0 \Omega X = \pi_1 X$ is a group! For $n > 1$, we know that

$$\pi_n(X) = [(D^n, S^{n-1}), (X, *)] = [(I^n, \partial I^n), (X, *)].$$

Exercise 45.4. Prove that $\pi_n(X)$ is an abelian group for $n > 2$.

Applying π_0 to the Barratt-Puppe sequence (see Equation 3.4) therefore gives a long exact sequence (of groups when the homotopy groups are in degrees greater than 0, and of pointed sets in degree 0):

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 Ff \to \pi_1 X \to \pi_1 Y \to \pi_0 Ff \to \pi_0 X \to \pi_0 X.$$

46 Relative homotopy groups

Spheres and homotopy groups

The functor Ω (sending a space to its based loop space) admits a left adjoint. To see this, recall that $\Omega X = X_*^{S^1}$, so that

$$\mathbf{Top}_*(W, \Omega X) = \mathbf{Top}_*(S^1 \wedge W, X).$$

Definition 46.1. The *reduced suspension* ΣW is $S^1 \wedge W$.

If $A \subseteq X$, then

$$X/A \wedge Y/B = (X \times Y)/((A \times Y) \cup_{A\times B} (X \times B)).$$

Since $S^1 = I/\partial I$, this tells us that $\Sigma X = S^1 \wedge X$ can be identified with $I \times X/(\partial I \times X \cup I \times *)$: in other words, we collapse the top and bottom of a cylinder to a point, as well as the line along a basepoint.

The same argument says that $\Sigma^n X$ (defined inductively as $\Sigma(\Sigma^{n-1}X)$) is the left adjoint of the n-fold loop space functor $X \mapsto \Omega^n X$. In other words, $\Sigma^n X = (S^1)^{\wedge n} \wedge X$. We claim that $S^1 \wedge S^n \simeq S^{n+1}$. To see this, note that

$$S^1 \wedge S^n = I/\partial I \wedge I^n \wedge \partial I^n = (I \times I^n)/(\partial I \times I^n \cup I \times \partial I^n).$$

The denominator is exactly ∂I^{n+1}, so $S^1 \wedge S^n \simeq S^{n+k}$. It's now easy to see that $S^k \wedge S^n \simeq S^{k+n}$.

Definition 46.2. The *nth homotopy group* of X is $\pi_n X = \pi_0(\Omega^n X)$.

This is, as we noted in the previous section, $[S^0, \Omega^n X]_* = [S^n, X]_* = [(I^n, \partial I^n), (X, *)]$.

The homotopy category

Define the *homotopy category of spaces* Ho(**Top**) to be the category whose objects are spaces, and whose hom-sets are given by taking π_0 of the mapping space. To check that this is indeed a category, we need to check that if $f_0, f_1 : X \to Y$ and $g : Y \to Z$, then $gf_0 \simeq gf_1$ — but this is clear. Similarly, we'd need to check that $f_0 h \simeq f_1 h$ for any $h : W \to X$. We can also think about the homotopy category of pointed spaces (and pointed homotopies) Ho($\mathbf{Top}_*$); this is the category we have been spending most of our time in. Both Ho(**Top**) and Ho($\mathbf{Top}_*$) have products and coproducts, but very few other limits or colimits. From a category-theoretic standpoint, these are absolutely terrible.

Let W be a pointed space. We would like the assignment $X \mapsto X_*^W$ to be a homotopy functor. It clearly defines a functor $\mathbf{Top}_* \to \mathbf{Top}_*$, so this desire is equivalent to providing a dotted arrow in the following diagram:

$$\begin{tikzcd}
\mathbf{Top}_* \arrow[r, "X \mapsto X_*^W"] \arrow[d] & \mathbf{Top}_* \arrow[d] \\
\mathrm{Ho}(\mathbf{Top}_*) \arrow[r, dashed] & \mathrm{Ho}(\mathbf{Top}_*).
\end{tikzcd}$$

Before we can prove this, we will check that a homotopy $f_0 \sim f_1 : X \to Y$ is the same as a map $I_+ \wedge X \to Y$. There is a nullhomotopy if the basepoint of I is one of the endpoints, so a homotopy is the same as a map $I \times X/I \times * \to Y$. The source is just $I_+ \wedge X$, as desired.

A homotopy $f_0 \simeq f_1 : X \to Y$ begets a map $(I_+ \wedge X)^W \to Y_*^W$. For the assignment $X \mapsto X_*^W$ to be a homotopy functor, we need a natural transformation $I_+ \wedge X_*^W \to Y_*^W$, so this map is not quite what's necessary. Instead, we can attempt to construct a map $I_+ \wedge X_*^W \to (I_+ \wedge X)_*^W$.

We can construct a general map $A \wedge X_*^W \to (A \wedge X)_*^W$: there is a map $A \wedge X_*^W \to A_*^W \wedge X_*^W$, given by sending $a \mapsto c_a$; then the exponential law gives a homotopy $A_*^W \wedge X_*^W \to (A \wedge X)_*^W$. This, in turn, gives a map $I_+ \wedge X_*^W \to (I_+ \wedge X)_*^W \to Y_*^W$, thus making $X \mapsto X_*^W$ a homotopy functor.

Motivated by our discussion of homotopy fibers, we can study composites which "behave" like short exact sequences.

Definition 46.3. A *fiber sequence* in $\mathrm{Ho}(\mathbf{Top}_*)$ is a composite $X \to Y \to Z$ that is isomorphic, in $\mathrm{Ho}(\mathbf{Top}_*)$, to some composite $Ff \xrightarrow{p} E \xrightarrow{f} B$; in other words, there exist (possibly zig-zags of) maps that are homotopy equivalences, that make the following diagram commute:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
Ff \arrow[r, "p"'] & E \arrow[r, "f"'] & B.
\end{tikzcd}$$

Let us remark here that if $A' \xrightarrow{\sim} A$ is a homotopy equivalence, and $A \to B \to C$ is a fiber sequence, so is the composite $A' \xrightarrow{\sim} A \to B \to C$.

Exercise 46.4. Prove the following statements.

- Ω takes fiber sequences to fiber sequences.
- $\Omega F f \simeq F \Omega f$. Check this!

We've seen examples of fiber sequences in our elaborate study of the Barratt-Puppe sequence.

Example 46.5. Recall our diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_4 \arrow[r] & Fp_3 \arrow[r] & Fp_2 \arrow[r] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
\cdots \arrow[r] \arrow[u, phantom] & \Omega Fp_1 \arrow[r, "-\overline{\Omega p_2}"] \arrow[u, "\simeq"'] & \Omega Ff \arrow[r, "-\overline{\Omega p}"] \arrow[u, "\simeq"'] \arrow[ur, "i(p_2)"] & \Omega X \arrow[r, "-\overline{\Omega f}"] \arrow[u, "\simeq"'] \arrow[ur, "i(p_1)"] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "i(f)"] & & & \\
\Omega^2 X \arrow[r, "\Omega f"'] \arrow[u, "\simeq"'] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "\overline{\Omega i(f)}"'] & & & & & &
\end{tikzcd}$$

The composite $Ff \to X \xrightarrow{f} Y$ is canonically a fiber sequence. The above diagram shows that $\Omega Y \to F \xrightarrow{p} X$ is another fiber sequence: it is isomorphic to $Fp \to F \to X$ in

$\mathrm{Ho}(\mathbf{Top}_*)$. Similarly, the composite $\Omega X \xrightarrow{\overline{\Omega f}} \Omega Y \to F$ is another fiber sequence; this implies that $\Omega X \xrightarrow{\Omega f} \Omega Y \to F$ is also an example of a fiber sequence (because these two fiber sequences differ by an automorphism of ΩX)

Applying Ω again, we get $\Omega F \xrightarrow{\Omega p} \Omega X \xrightarrow{\Omega f} \Omega Y$. Since this is a looping of a fiber sequence, and taking loops takes fiber sequences to fiber sequences (Exercise 46.4), this is another fiber sequence. Looping again gives another fiber sequence $\Omega^2 Y \xrightarrow{\Omega i} \Omega F \xrightarrow{\Omega p} \Omega X$. (For the category-theoretically–minded folks, this is an unstable version of a triangulated category.)

The long exact sequence of a fiber sequence

As discussed at the end of §45, applying $\pi_0 = [S^0, -]_*$ to the Barratt-Puppe sequence associated to a map $f : X \to Y$ gives a long exact sequence:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \pi_2 Y \arrow[dll] \\
 & \pi_1 F \arrow[r] & \pi_1 X \arrow[r] & \pi_1 Y \arrow[dll] \\
 & \pi_0 F \arrow[r] & \pi_0 X. &
\end{tikzcd}$$

of pointed sets. The space $\Omega^2 X$ is an *abelian* group object in $\mathrm{Ho}(\mathbf{Top})$ (in other words, the multiplication on $\Omega^2 X$ is commutative up to homotopy). This implies $\pi_1(X)$ is a group, and that $\pi_k(X)$ is abelian for $k \geq 2$; hence, in our diagram above, all maps (except on π_0) are group homomorphisms.

Consider the case when $X \to Y$ is the inclusion $i : A \hookrightarrow X$ of a subspace. In this case,

$$Fi = \{(a, \omega) \in A \times X_*^I | \omega(1) = a\};$$

this is just the collection of all paths that begin at $* \in A$ and end in A. This motivates the definition of *relative homotopy groups*:

Definition 46.6. Define:

$$\pi_n(X, A, *) = \pi_n(X, A) \coloneqq \pi_{n-1} Fi = [(I^n, \partial I^n, (\partial I^n \times I) \cup (I^{n-1} \times 0)), (X, A, *)].$$

We have a sequence of inclusions

$$\partial I^n \times I \cup I^{n-1} \times 0 \subset \partial I^n \subset I^n.$$

One can check that

$$\pi_{n-1} Fi = [(I^n, \partial I^n, (\partial I^n \times I) \cup (I^{n-1} \times 0)), (X, A, *)].$$

This gives a long exact sequence on homotopy, analogous to the long exact sequence in relative homology:

$$\begin{tikzcd} & & \cdots \arrow[r] & \pi_2(X,A) \arrow[dll] \\ \pi_1 A \arrow[r] & \pi_1 X \arrow[r] & \pi_1(X,A) \arrow[dll] \\ \pi_0 A \arrow[r] & \pi_0 X \end{tikzcd} \tag{3.5}$$

47 Action of π_1, simple spaces, and the Hurewicz theorem

In the previous section, we constructed a long exact sequence of homotopy groups:

$$\begin{tikzcd} & & \cdots \arrow[r] & \pi_2(X,A) \arrow[dll] \\ \pi_1 A \arrow[r] & \pi_1 X \arrow[r] & \pi_1(X,A) \arrow[dll] \\ \pi_0 A \arrow[r] & \pi_0 X, \end{tikzcd}$$

which looks suspiciously similar to the long exact sequence in homology. The goal of this section is to describe a relationship between homotopy groups and homology groups.

Before we proceed, we will need the following lemma.

Lemma 47.1 (Excision)**.** *If $A \subseteq X$ is a cofibration, there is an isomorphism*

$$H_*(X,A) \xrightarrow{\cong} \widetilde{H}_*(X/A).$$

Under this hypothesis,

$$X/A \simeq \textit{Mapping cone of } i : A \to X;$$

here, the mapping cone is the homotopy pushout in the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "in_1"] & X \arrow[d] \\ CA \arrow[r] & X \cup_A CA, \end{tikzcd}$$

where CA is the cone on A, defined by

$$CA = A \times I / A \times 0.$$

This lemma is dual to the statement that the homotopy fiber is homotopy equivalent to the strict fiber for fibrations.

Unfortunately, $\pi_*(X, A)$ is definitely not $\pi_*(X/A)$! For instance, there is a cofibration sequence

$$S^1 \to D^2 \to S^2.$$

We know that $\pi_* S^1$ is just $\mathbf{Z}$ in dimension 1, and is zero in other dimensions. On the other hand, we do not, and probably will never, know the homotopy groups of S^2. (A theorem of Edgar Brown in [Bro57] says that these groups are computable, but this is super-exponential.)

π_1-action

There is more structure in the long exact sequence in homotopy groups that we constructed last time, coming from an action of $\pi_1(X)$. There is an action of $\pi_1(X)$ on $\pi_n(X)$: if x, y are points in X, and $\omega : I \to X$ is a path with $\omega(0) = x$ and $\omega(1) = y$, we have a map $f_\omega : \pi_n(X, x) \to \pi_n(X, y)$; this, in particular, implies that $\pi_1(X, *)$ acts on $\pi_n(X, *)$. When $n = 1$, the action $\pi_1(X)$ on itself is by conjugation.

In fact, one can also see that $\pi_1(A)$ acts on $\pi_n(X, A, *)$. It follows (by construction) that all maps in the long exact sequence of Equation (3.5) are equivariant for this action of $\pi_1(A)$. Moreover:

Proposition 47.2 (Peiffer identity). *Let $\alpha, \beta \in \pi_2(X, A)$. Then $(\partial\alpha) \cdot \beta = \alpha\beta\alpha^{-1}$.*

Definition 47.3. A topological space X is said to be *simply connected* if it is path connected, and $\pi_1(X, *) = 1$.

Let $p : E \to B$ be a covering space with E and B connected. Then, the fibers are discrete, hence do not have any higher homotopy. Using the long exact sequence in homotopy groups, we learn that $\pi_n(E) \to \pi_n(B)$ is an isomorphism for $n > 1$, and that $\pi_1(E)$ is a subgroup of $\pi_1(B)$ that classifies the covering space. In general, we know from Exercise 47.7 that ΩB acts on the homotopy fiber Fp. Since Ff is discrete, this action factors through $\pi_0(\Omega B) \simeq \pi_1(B)$.

Definition 47.4. A space X is said to be *n-connected* if $\pi_i(X) = 0$ for $i \leq n$.

Note that this is a well-defined condition, although we did not specify the basepoint: 0-connected implies path connected. Suppose $E \to B$ is a covering space, with the total space E being n-connected. Then, Hopf showed that the group $\pi_1(B)$ determines the homology $H_i(B)$ in dimensions $i < n$.

Sometimes, there are interesting spaces which are not simply connected, for which the π_1-action is nontrivial.

Example 47.5. Consider the space $S^1 \vee S^2$. The universal cover is just $\mathbf{R}$, with a 2-sphere S^2 stuck on at every integer point. This space is simply connected, so the Hurewicz theorem says that $\pi_2(E) \simeq H_2(E)$. Since the real line is contractible, we can collapse it to a point: this gives a countable bouquet of 2-spheres. As a consequence, $\pi_2(E) \simeq H_2(E) = \bigoplus_{i=0}^{\infty} \mathbf{Z}$.

There is an action of $\pi_1(S^1 \vee S^2)$ on E: the action does is shift the 2-spheres on the integer points of $\mathbf{R}$ (on E) to the right by 1 (note that $\pi_1(S^1 \vee S^2) = \mathbf{Z}$). This tells us that $\pi_2(E) \simeq \mathbf{Z}[\pi_1(B)]$ as a $\mathbf{Z}[\pi_1(B)]$-module; this is the same action of $\pi_1(E)$ on $\pi_2(E)$. We should be horrified: $S^1 \vee S^2$ is a very simple 3-complex, but its homotopy is huge!

Simply-connectedness can sometimes be a restrictive condition; instead, to simplify the long exact sequence, we define:

Definition 47.6. A topological space X is said to be *simple* if it is path-connected, and $\pi_1(X)$ acts trivially on $\pi_n(X)$ for $n \geq 1$.

Note, in particular, that $\pi_1(X)$ is abelian for a simple space.

Being simple is independent of the choice of basepoint. If $\omega : x \mapsto x'$ is a path in X, then $\omega_\sharp : \pi_n(X,x) \to \pi_n(X,x')$ is a group isomorphism. There is a (trivial) action of $\pi_1(X,x)$ on $\pi_n(X,x)$, and another (potentially nontrivial) action of $\pi_1(X,x')$ on $\pi_n(X,x')$. Both actions are compatible: hence, if $\pi_1(X,x)$ acts trivially, so does $\pi_1(X,x')$.

If X is path-connected, there is a map $\pi_n(X,*) \to [S^n, X]$. It is clear that this map is surjective, so one might expect a factorization:

$$\begin{tikzcd}
\pi_n(X,*) \arrow[rr, two heads] \arrow[dr] & & {[S^n, X]} \\
& \pi_1(X,*)\backslash\pi_n(X,*) \arrow[ur] &
\end{tikzcd}$$

Exercise 47.7. Prove that $\pi_1(X,*)\backslash\pi_n(X,*) \simeq [S^n, X]$. To do this, work through the following exercises.

Let $f : X \to Y$ be a map of spaces, and let $* \in Y$ be a fixed basepoint of Y. Denote by Ff the homotopy fiber of f; this admits a natural fibration $p : Ff \to X$, given by $(x,\sigma) \mapsto x$. If ΩY denotes the (based) loop space of Y, we get an action $\Omega Y \times Ff \to Ff$, given by

$$(\omega, (x,\sigma)) \mapsto (x, \sigma \cdot \omega),$$

where $\sigma \cdot \omega$ is the concatenation of σ and ω, defined, as usual, by

$$\sigma \cdot \omega(t) = \begin{cases} \omega(2t) & 0 \leq t \leq 1/2 \\ \sigma(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

(Note that when X is the point, this defines a "multiplication" $\Omega Y \times \Omega Y \to \Omega Y$; this is associative and unital up to homotopy.) On connected components, we therefore get an action of $\pi_0 \Omega Y \simeq \pi_1 Y$ on $\pi_0 Ff$.

There is a canonical map

$$Ff \times \Omega Y \to Ff \times_X Ff,$$

given by $((x,\sigma),\omega) \mapsto ((x,\sigma),(x,\sigma)\cdot\omega)$. Prove that this map is a homotopy equivalence (so that the action of ΩY on Ff is "free"), and conclude that two elements in $\pi_0 Ff$ map to the same element of $\pi_0 X$ if and only if they are in the same orbit under the action of $\pi_1 Y$.

Let X be path connected, with basepoint $* \in X$. Conclude that $\pi_1(X,*)\backslash\pi_n(X,*) \simeq [S^n, X]$ by proving that the surjection $\pi_n(X,*) \to [S^n, X]$ can be identified with the orbit projection for the action of $\pi_1(X,*)$ on $\pi_n(X,*)$.

If X is simple, then the quotient $\pi_1(X,*)\backslash\pi_n(X,*)$ is simply $\pi_n(X,*)$, so Exercise 47.7 implies that $\pi_n(X,*) \cong [S^n, X]$ — independently of the basepoint; in other words, these groups are canonically the same, i.e., two paths $\omega, \omega' : x \to y$ give the same map $\omega_\sharp = \omega'_\sharp : \pi_n(X,x) \to \pi_n(X,y)$.

Exercise 47.8. A *H-space* is a pointed space X, along with a pointed map $\mu : X \times X \to X$, such that the maps $x \mapsto \mu(x,*)$ and $x \mapsto \mu(*,x)$ are both pointed homotopic to the identity. In this exercise, you will prove that path connected H-spaces are simple.

Denote by $\mathcal{C}$ the category of pairs (G,H), where G is a group that acts on the group H (on the left); the morphisms in $\mathcal{C}$ are pairs of homomorphisms which are compatible with the group actions. This category has finite products. Explain what it means for an object of $\mathcal{C}$ to have a "unital multiplication", and prove that any object (G,H) of $\mathcal{C}$ with a unital multiplication has G and H abelian, and that the G-action on H is trivial. Conclude from this that path connected H-spaces are simple.

Hurewicz theorem

Definition 47.9. Let X be a path-connected space. The Hurewicz map $h : \pi_n(X,*) \to H_n(X)$is defined as follows: an element in $\pi_n(X,*)$ is represented by $\alpha : S^n \to X$; pick a generator $\sigma \in H_n(S^n)$, and send

$$\alpha \mapsto \alpha_*(\sigma) \in H_n(X).$$

We will see below that h is in fact a homomorphism.

This is easy in dimension 0: a point is a 0-cycle! In fact, we have an isomorphism $H_0(X) \simeq \mathbf{Z}[\pi_0(X)]$. (This isomorphism is an example of the Hurewicz theorem.)

When $n = 1$, we have $h : \pi_1(X,*) \to H_1(X)$. Since $H_1(X)$ is abelian, this factors as $\pi_1(X,*) \to \pi_1(X,*)^{ab} \to H_1(X)$. The Hurewicz theorem says that the map $\pi_1(X,*)^{ab} \to H_1(X)$ is an isomorphism. We will not prove this here; see [Hat15, Theorem 2A.1] for a proof.

The Hurewicz theorem generalizes these results to higher dimensions:

Theorem 47.10 (Hurewicz)**.** *Suppose X is a space for which $\pi_i(X) = 0$ for $i < n$, where $n \geq 2$. Then the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism.*

Before the word "isomorphism" can make sense, we need to prove that h is a homomorphism. Let $\alpha, \beta : S^n \to X$ be pointed maps. The product $\alpha\beta \in \pi_n(X)$ is the composite:

$$\alpha\beta : S^n \xrightarrow{\delta,\text{ pinching along the equator}} S^n \vee S^n \xrightarrow{\beta\vee\alpha} X \vee X \xrightarrow{\nabla} X,$$

where $\nabla : X \vee X \to X$ is the fold map, defined by:

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "1"] & \\ X \vee X \arrow[r, "\nabla"] & X \\ X \arrow[u] \arrow[ur, "1"'] & \end{tikzcd}$$

To show that h is a homomorphism, it suffices to prove that for two maps $\alpha, \beta : S^n \to X$, the induced maps on homology satisfy $(\alpha+\beta)_* = \alpha_* + \beta_*$ — then,

$$h(\alpha+\beta) = (\alpha+\beta)_*(\sigma) = \alpha_*(\sigma) + \beta_*(\sigma) = h(\alpha) + h(\beta).$$

To prove this, we will use the pinch map $\delta : S^n \to S^n \vee S^n$, and the quotient maps $q_1, q_2 : S^n \vee S^n \to S^n$; the induced map $H_n(S^n) \to H_n(S^n) \oplus H_n(S^n)$ is given by the diagonal map $a \mapsto (a,a)$. It follows from the equalities

$$(f \vee g)\iota_1 = f, \ (f \vee g)\iota_2 = g,$$

where $\iota_1, \iota_2 : S^n \hookrightarrow S^n \vee S^n$ are the inclusions of the two wedge summands, that the map $(f \vee g)_*((\iota_1)_* + (\iota_2)_*)$ sends $(x,0)$ to $f_*(x)$, and $(0,x)$ to $g_*(x)$. In particular,

$$(x,x) \mapsto f_*(x) + g_*(x),$$

so the composite $H_n(S^n) \to H_n(X)$ sends $x \mapsto (x,x) \mapsto f_*(x) + g_*(x)$. This composite is just $(f+g)_*(x)$, since the composite $(f \vee g)\delta$ induces the map $(f+g)_*$ on homology.

It is possible to give an elementary proof of the Hurewicz theorem, but we won't do that here: instead, we will prove this as a consequence of the Serre spectral sequence.

Example 47.11. Since $\pi_i(S^n) = 0$ for $i < n$, the Hurewicz theorem tells us that $\pi_n(S^n) \simeq H_n(S^n) \simeq \mathbf{Z}$.

Example 47.12. Recall the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. The long exact sequence on homotopy groups tells us that $\pi_i(S^3) \xrightarrow{\simeq} \pi_i(S^2)$ for $i > 2$, where the map is given by $\alpha \mapsto \eta\alpha$. As we saw above, $\pi_3(S^3) = \mathbf{Z}$, so $\pi_3(S^2) \simeq \mathbf{Z}$, generated by η.

One can show that $\pi_{4n-1}(S^{2n}) \otimes \mathbf{Q} \simeq \mathbf{Q}$. A theorem of Serre's says that, other than $\pi_n(S^n)$, these are the only non-torsion homotopy groups of spheres.

48 Examples of CW-complexes

Bringing you up-to-speed on CW-complexes

Definition 48.1. A *relative CW-complex* is a pair (X, A), together with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X,$$

such that for all n, the space X_n sits in a pushout square:

$$\begin{tikzcd}
\coprod_{\alpha \in \Sigma_n} S^{n-1} \arrow[r] \arrow[d, "\text{attaching maps}"'] & \coprod_{\alpha \in \Sigma_n} D^n \arrow[d, "\text{characteristic maps}"] \\
X_{n-1} \arrow[r] & X_n,
\end{tikzcd}$$

and $X = \varinjlim X_n$.

If $A = \varnothing$, this is just the definition of a CW-complex. In this case, X is also compactly generated. (This is one of the reasons for defining compactly generated spaces.) Often, X will be a CW-complex, and A will be a subcomplex. If A is Hausdorff, then so is X.

If X and Y are both CW-complexes, define

$$(X \times^k Y)_n = \bigcup_{i+j=n} X_i \times Y_j;$$

this gives a CW-structure on the product $X \times^k Y$. Any closed smooth manifold admits a CW-structure.

Example 48.2 (Complex projective space). The complex projective n-space $\mathbf{CP}^n$ is a CW-complex, with skeleta $\mathbf{CP}^0 \subseteq \mathbf{CP}^1 \subseteq \cdots \subseteq \mathbf{CP}^n$. Indeed, any complex line through the origin meets the hemisphere defined by $\begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}$ with $\|z\| = 1$, $\Im(z_n) = 0$, and $\Re(z_n) \geq 0$. Such a line meets this hemisphere (which is just D^{2n}) at one point — unless it's on the equator; this gives the desired pushout diagram:

$$\begin{tikzcd}
S^{2n-1} \arrow[r] \arrow[d] & D^{2n} \arrow[d] \\
\mathbf{CP}^{n-1} \arrow[r] & \mathbf{CP}^n.
\end{tikzcd}$$

Example 48.3 (Grassmannians). Let $V = \mathbf{R}^n$ or $\mathbf{C}^n$ or $\mathbf{H}^n$, for some fixed n. Define the Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$ to be the collection of k-dimensional subspaces of V. This is equivalent to specifying a $k \times n$ rank k matrix.

For instance, $\mathrm{Gr}_2(\mathbf{R}^4)$ is, as a set, the disjoint union of:

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 1 & * & * \end{pmatrix}.$$

Motivated by this, define:

Definition 48.4. The j-skeleton of $\mathrm{Gr}(V)$ is

$$\mathrm{sk}_j \mathrm{Gr}_k(V) = \{A : \text{row echelon representation with at most } j \text{ free entries}\}.$$

For a proof that this is indeed a CW-structure, see [MS74, §6].

The top-dimensional cell tells us that

$$\dim \mathrm{Gr}_k(\mathbf{R}^n) = k(n-k).$$

The complex Grassmannian has cells in only even dimensions. We know the homology of Grassmannians: Poincaré duality is visible if we count the number of cells. (Consider, for instance, in $\mathrm{Gr}_2(\mathbf{R}^4)$).

49 Relative Hurewicz and J. H. C. Whitehead

Here is an "alternative definition" of connectedness:

Definition 49.1. Let $n \geq 0$. The space X is said to be $(n-1)$-*connected* if, for all $0 \leq k \leq n$, any map $f : S^{k-1} \to X$ extends:

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & X \\ D^k \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

When $n = 0$, we know that $S^{-1} = \varnothing$, and $D^0 = *$. Thus being (-1)-connected is equivalent to being nonempty. When $n = 1$, this is equivalent to path connectedness. You can check that this is exactly the same as what we said before, using homotopy groups.

As is usual in homotopy theory, there is a relative version of this definition.

Definition 49.2. Let $n \geq 0$. Say that a pair (X, A) is n-*connected* if, for all $0 \leq k \leq n$, any map $f : (D^k, S^{k-1}) \to (X, A)$ extends:

$$\begin{tikzcd} (D^k, S^{k-1}) \arrow[r, "f"] \arrow[d, dashed] & (X, A) \\ (A, A) \arrow[ur] & \end{tikzcd}$$

up to homotopy. In other words, there is a homotopy between f and a map with image in A, such that $f|_{S^{k-1}}$ remains unchanged.

0-connectedness implies that A meets every path component of X. Equivalently:

Definition 49.3. (X, A) is n-connected if:

- when $n = 0$, the map $\pi_0(A) \to \pi_0(X)$ surjects.
- when $n > 0$, the canonical map $\pi_0(A) \xrightarrow{\simeq} \pi_0(X)$ is an isomorphism, and for all $a \in A$, the group $\pi_k(X, A, a)$ vanishes for $1 \leq k \leq n$. (Equivalently, $\pi_0(A) \xrightarrow{\simeq} \pi_0(X)$ and $\pi_k(A, a) \to \pi_k(X, A)$ is an isomorphism for $1 \leq k < n$ and is onto for $k = n$.)

The relative Hurewicz theorem

Assume that $\pi_0(A) = * = \pi_0(X)$, and pick $a \in A$. Then, we have a comparison of long exact sequences, arising from the classical (i.e., non-relative) Hurewicz map:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(A) \arrow[r] \arrow[d, "h"] & \pi_1(X) \arrow[r] \arrow[d, "h"] & \pi_1(X,A) \arrow[r] \arrow[d, "h"] & \pi_0(A) \arrow[r] \arrow[d, "h"] & \pi_0(X) \arrow[d, "h"] & \\
\cdots \arrow[r] & H_1(A) \arrow[r] & H_1(X) \arrow[r] & H_1(X,A) \arrow[r] & H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(X,A)
\end{tikzcd}$$

To define the relative Hurewicz map, let $\alpha \in \pi_n(X, A)$, so that $\alpha : (D^n, S^{n-1}) \to (X, A)$; pick a generator of $H_n(D^n, S^{n-1})$, and send it to an element of $H_n(X, A)$ via the induced map $\alpha_* : H_n(D^n, S^{n-1}) \to H_n(X, A)$.

Because $H_n(X, A)$ is abelian, the group $\pi_1(A)$ acts trivially on $H_n(X, A)$; in other words, $h(\omega(\alpha)) = h(\alpha)$. Consequently, the relative Hurewicz map factors through the group $\pi_n^\dagger(X, A)$, defined to be the quotient of $\pi_n(X, A)$ by the normal subgroup generated by $(\omega\alpha)\alpha^{-1}$, where $\omega \in \pi_1(A)$ and $\alpha \in \pi_n(X, A)$. This begets a map $\pi_n^\dagger(X, A) \to H_n(X, A)$.

Theorem 49.4 (Relative Hurewicz)**.** *Let $n \geq 1$, and assume (X, A) is n-connected. Then $H_k(X, A) = 0$ for $0 \leq k \leq n$, and the map $\pi_{n+1}^\dagger(X, A) \to H_{n+1}(X, A)$ constructed above is an isomorphism.*

We will prove this later using the Serre spectral sequence.

The Whitehead theorems

J. H. C. Whitehead was a rather interesting character. He raised pigs.

Whitehead was interested in determining when a continuous map $f : X \to Y$ that is an isomorphism in homology or homotopy is a homotopy equivalence.

Definition 49.5. Let $f : X \to Y$ and $n \geq 0$. Say that f is a *n-equivalence*[13] if, for every $* \in Y$, the homotopy fiber $F(f, *)$ is $(n-1)$-connected.

For instance, f being a 0-equivalence simply means that $\pi_0(X)$ surjects onto $\pi_0(Y)$ via f. For $n > 0$, this says that $f : \pi_0(X) \to \pi_0(Y)$ is a bijection, and that for every $* \in X$:

$$\pi_k(X, *) \to \pi_k(Y, f(*)) \text{ is } \begin{cases} \text{an isomorphism} & 1 \leq k < n \\ \text{onto} & k = n. \end{cases}$$

[13]Some sources sometimes use "n-connected".

Using the "mapping cylinder" construction (see Exercise 44.5), we can always assume $f : X \to Y$ is a cofibration; in particular, that $X \hookrightarrow Y$ is a closed inclusion. Then, $f : X \to Y$ is an n-equivalence if and only if (Y, X) is n-connected.

Theorem 49.6 (Whitehead). *Suppose $n \geq 0$, and $f : X \to Y$ is n-connected. Then:*

$$H_k(X) \xrightarrow{f} H_k(Y) \text{ is } \begin{cases} \text{an isomorphism} & 1 \leq k < n \\ \text{onto} & k = n. \end{cases}$$

Proof. When $n = 0$, because $\pi_0(X) \to \pi_0(Y)$ is surjective, we learn that $H_0(X) \simeq \mathbf{Z}[\pi_0(X)] \to \mathbf{Z}[\pi_0(Y)] \simeq H_0(Y)$ is surjective. To conclude, use the relative Hurewicz theorem. (Note that the relative Hurewicz dealt with $\pi_n^\dagger(X, A)$, but the map $\pi_n(X, A) \to \pi_n^\dagger(X, A)$ is surjective.) □

The case $n = \infty$ is special.

Definition 49.7. f is a *weak equivalence* (or an ∞-equivalence, to make it sound more impressive) if it's an n-equivalence for all n, i.e., it's a π_*-isomorphism.

Putting everything together, we obtain:

Corollary 49.8. *A weak equivalence induces an isomorphism in integral homology.*

How about the converse?

If $H_0(X) \to H_0(Y)$ surjects, then the map $\pi_0(X) \to \pi_0(Y)$ also surjects. Now, assume X and Y path connected, and that $H_1(X)$ surjects onto $H_1(Y)$. We would like to conclude that $\pi_1(X) \to \pi_1(Y)$ surjects. Unfortunately, this is hard, because $H_1(X)$ is the abelianization of $\pi_1(X)$. To forge onward, we will simply give up, and assume that $\pi_1(X) \to \pi_1(Y)$ is surjective.

Suppose $H_2(X) \to H_2(Y)$ surjects, and that $f_* : H_1(X) \xrightarrow{\simeq} H_1(Y)$. We know that $H_2(Y, X) = 0$. On the level of the Hurewicz maps, we are still stuck, because we only obtain information about $\pi_2^\dagger$. Let us assume that $\pi_1(X)$ is trivial[14]. Under this assumption, we find that $\pi_1(Y) = 0$. This implies $\pi_2(Y, X)$ is trivial. Arguing similarly, we can go up the ladder.

Theorem 49.9 (Whitehead). *Let $n \geq 2$, and assume that $\pi_1(X) = 0 = \pi_1(Y)$. Suppose $f : X \to Y$ such that:*

$$H_k(X) \to H_k(Y) \text{ is } \begin{cases} \text{an isomorphism} & 1 \leq k < n \\ \text{onto} & k = n; \end{cases}$$

then f is an n-equivalence.

Setting $n = \infty$, we obtain:

[14] This is a pretty radical assumption; for the following argument to work, it would technically be enough to ask that $\pi_1(X)$ acts trivially on $\pi_2(Y, X)$: but this is basically impossible to check.

Corollary 49.10. *Let X and Y be simply-connected. If f induces an isomorphism in homology, then f is a weak equivalence.*

This is incredibly useful, since homology is actually computable! To wrap up the story, we will state the following result, which we will prove in a later section.

Theorem 49.11. *Let Y be a CW-complex. Then a weak equivalence $f : X \to Y$ is in fact a homotopy equivalence.*

50 Cellular approximation, cellular homology, obstruction theory

In previous sections, we saw that homotopy groups play well with (maps between) CW-complexes. Here, we will study maps between CW-complexes themselves, and prove that they are, in some sense, "cellular" themselves.

Cellular approximation

Definition 50.1. Let X and Y be CW-complexes, and let $A \subseteq X$ be a subcomplex. Suppose $f : X \to Y$ is a continuous map. We say that $f|_A$ is skeletal[15] if $f(\Sigma_n) \subseteq Y_n$.

Note that a skeletal map might not take cells in A to cells in Y, but it takes n-skeleta to n-skeleta.

Theorem 50.2 (Cellular approximation)**.** *In the setup of Definition 50.1, the map f is homotopic to some other continuous map $f' : X \to Y$, relative to A, such that f' is skeletal on all of X.*

To prove this, we need the following lemma.

Lemma 50.3 (Key lemma)**.** *Any map $(D^n, S^{n-1}) \to (Y, Y_{n-1})$ factors as:*

$$\begin{tikzcd} (D^n, S^{n-1}) \arrow[r] \arrow[dr, dashed] & (Y, Y_{n-1}) \\ & (Y_n, Y_{n-1}) \arrow[u] \end{tikzcd}$$

"Proof." Since D^n is compact, we know that $f(D^n)$ must lie in some finite subcomplex K of Y. The map $D^n \to K$ might hit some top-dimensional cell $e^m \subseteq K$, which does not have anything attached to it; hence, we can homotope this map to miss a point, so that it contracts onto a lower-dimensional cell. Iterating this process gives the desired result. □

Using this lemma, we can conclude the cellular approximation theorem.

[15] Some would say cellular.

"Proof" of Theorem 50.2. We will construct the homotopy $f \simeq f'$ one cell at a time. Note that we can replace the space A by the subspace to which we have extended the homotopy.

Consider a single cell attachment $A \to A \cup D^m$; then, we have

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{skeletal}"'] & A \cup D^m \arrow[dl, "\text{may not be skeletal}"] \\ Y & \end{tikzcd}$$

Using the "compression lemma" from above, the rightmost map factors (up to homotopy) as the composite $A \cup D^m \to Y_m \to Y$. Unfortunately, we have not extended this map to the whole of X, although we could do this if we knew that the inclusion of a subcomplex is a cofibration. But this is true: there is a cofibration $S^{n-1} \to D^n$, and so any pushout of these maps is a cofibration! This allows us to extend; we now win by iterating this procedure. $\square$

As a corollary, we find:

Exercise 50.4. The pair (X, X_n) is n-connected.

Cellular homology

vide a link!

Let (X, A) be a relative CW-complex with $A \subseteq X_{n-1} \subseteq X_n \subseteq \cdots \subseteq X$. In the previous part that $H_*(X_n, X_{n-1}) \simeq \widetilde{H}_*(X_n/X_{n-1})$. More generally, if $B \to Y$ is a cofibration, there is an isomorphism (see [Bre93, p. 433]):

$$H_*(Y, B) \simeq \widetilde{H}_*(Y/B).$$

Since $X_n/X_{n-1} = \bigvee_{\alpha \in \Sigma_n} S^n_\alpha$, we find that

$$H_*(X_n, X_{n-1}) \simeq \mathbf{Z}[\Sigma_n] = C_n.$$

The composite $S^{n-1} \to X_{n-1} \to X_{n-1}/X_{n-2}$ is called a relative attaching map.

There is a boundary map $d : C_n \to C_{n-1}$, defined by

$$d : C_n = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}.$$

Exercise 50.5. Check that $d^2 = 0$.

Using the resulting chain complex, denoted $C_*(X, A)$, one can prove that there is an isomorphism

$$H_n(X, A) \simeq H_n(C_*(X, A)).$$

vide a link!

(In the previous part, we proved this for CW-pairs, but not for relative CW-complexes.) The incredibly useful cellular approximation theorem therefore tells us that the effect of maps on homology can be computed.

Of course, the same story runs for cohomology: one gets a chain complex which, in dimension n, is given by

$$C^n(X, A; \pi) = \mathrm{Hom}(C_n(X, A), \pi) = \mathrm{Map}(\Sigma_n, \pi),$$

where π is any abelian group.

Obstruction theory

Using the tools developed above, we can attempt to answer some concrete, and useful, questions.

Question 50.6. Let $f : A \to Y$ be a map from a space A to Y. Suppose (X, A) is a relative CW-complex. When can we find an extension in the diagram below?

$$\begin{tikzcd}
X \arrow[dr, dashed] & \\
A \arrow[u, hook] \arrow[r, "f"'] & Y
\end{tikzcd}$$

The lower level obstructions can be worked out easily:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X_0 \arrow[r, hook] \arrow[dl, dashed] & X_1 \arrow[dll, dashed] \\
\varnothing \neq Y & &
\end{tikzcd}$$

Thus, for instance, if two points in X_0 are connected in X_1, we only have to check that they are also connected in Y.

For $n \geq 2$, we can form the diagram:

$$\begin{tikzcd}
\coprod_{\alpha \in \Sigma_n} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook] & X_{n-1} \arrow[r, "g"] \arrow[d] & Y \\
\coprod_{\Sigma_n} D^n_\alpha \arrow[r] & X_n \arrow[ur, dashed] &
\end{tikzcd}$$

The desired extension exists if the composite $S^{n-1}_\alpha \xrightarrow{f_\alpha} X_{n-1} \to Y$ is nullhomotopic.

Clearly, $g \circ f_\alpha \in [S^{n-1}, Y]$. To simplify the discussion, let us assume that Y is simple; then, Exercise 47.7 says that $[S^{n-1}, Y] = \pi_{n-1}(Y)$. This procedure begets a map $\Sigma_n \xrightarrow{\theta} \pi_{n-1}(Y)$, which is a n-cochain, i.e., an element of $C^n(X, A; \pi_{n-1}(Y))$. It is clear that $\theta = 0$ if and only if the map g extends to $X_n \to Y$.

Proposition 50.7. *θ is a cocycle in $C^n(X, A; \pi_{n-1}(Y))$, called the "obstruction cocycle".*

Proof. θ gives a map $H_n(X_n, X_{n-1}) \to \pi_{n-1}(Y)$. We would like to show that the composite

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n) \to H_n(X_n, X_{n-1}) \xrightarrow{\theta} \pi_{n-1}(Y)$$

is trivial. We have the long exact sequence in homotopy of a pair (see Equation (3.5)):

$$\begin{tikzcd}
\pi_{n+1}(X_{n+1}, X_n) \arrow[r, two heads] \arrow[d] & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial"] \\
\pi_n(X_n) \arrow[r] \arrow[d] & H_n(X_n) \arrow[d] \\
\pi_n(X_n, X_{n-1}) \arrow[r, two heads] \arrow[d, "\partial"'] & H_n(X_n, X_{n-1}) \arrow[d, "\theta"] \\
\pi_{n-1}(X_{n-1}) \arrow[r, "g_*"'] & \pi_{n-1}(Y)
\end{tikzcd}$$

This diagram commutes, so θ is indeed a cocycle. $\square$

Our discussion above allows us to conclude:

Theorem 50.8. *Let (X, A) be a relative CW-complex and Y a simple space. Let $g : X_{n-1} \to Y$ be a map from the $(n-1)$-skeleton of X. Then $g|_{X_{n-2}}$ extends to X_n if and only if $[\theta(g)] \in H^n(X, A; \pi_{n-1}(Y))$ is zero.*

Corollary 50.9. *If $H^n(X, A; \pi_{n-1}(Y)) = 0$ for all $n > 2$, then any map $A \to Y$ extends to a map $X \to Y$ (up to homotopy[16]); in other words, there is a dotted lift in the following diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \\
X \arrow[ur, dashed] &
\end{tikzcd}$$

For instance, every map $A \to Y$ factors through the cone if $H^n(CA, A; \pi_{n-1}(Y)) \simeq \widetilde{H}^{n-1}(A; \pi_{n-1}(Y)) = 0$.

51 Conclusions from obstruction theory

The main result of obstruction theory, as discussed in the previous section, is the following.

Theorem 51.1 (Obstruction theory)**.** *Let (X, A) be a relative CW-complex, and Y a simple space. The map $[X, Y] \to [A, Y]$ is:*

1. *is onto if $H^n(X, A; \pi_{n-1}(Y)) = 0$ for all $n \geq 2$.*

2. *is one-to-one if $H^n(X, A; \pi_n(Y)) = 0$ for all $n \geq 1$.*

[16]In fact, this condition is unnecessary, since the inclusion of a subcomplex is a cofibration.

Remark 51.2. The first statement implies the second. Indeed, suppose we have two maps $g_0, g_1 : X \to Y$ and a homotopy $h : g_0|_A \simeq g_0|_A$. Assume the first statement. Consider the relative CW-complex $(X \times I, A \times I \cup X \times \partial I)$. Because (X, A) is a relative CW-complex, the map $A \hookrightarrow X$ is a cofibration; this implies that the map $A \times I \cup X \times \partial I \to X \times I$ is also a cofibration.

$$\begin{aligned} H^n(X \times I, A \times I \cup X \times \partial I; \pi) &\simeq \widetilde{H}^n(X \times I/(A \times I \cup X \times \partial I); \pi) \\ &= H^n(\Sigma X/A; \pi) \simeq \widetilde{H}^{n-1}(X/A; \pi). \end{aligned}$$

We proved the following statement in the previous section.

Proposition 51.3. *Suppose $g : X_{n-1} \to Y$ is a map from the $(n-1)$-skeleton of X to Y. Then $g|_{X_{n-2}}$ extends to $X_n \to Y$ iff $[\theta(g)] = 0$ in $H^n(X, A; \pi_{n-1}(Y))$.*

An immediate consequence is the following.

Theorem 51.4 (CW-approximation)**.** *Any space admits a weak equivalence from a CW-complex.*

This tells us that studying CW-complexes is not very restrictive, if we work up to weak equivalence.

It is easy to see that if W is a CW-complex and $f : X \to Y$ is a weak equivalence, then $[W, X] \xrightarrow{\simeq} [W, Y]$. We can now finally conclude the result of Theorem 49.11:

Corollary 51.5. *Let X and Y be CW-complexes. Then a weak equivalence $f : X \to Y$ is a homotopy equivalence.*

Postnikov and Whitehead towers

Let X be path connected. There is a space $X_{\leq n}$, and a map $X \to X_{\leq n}$ such that $\pi_i(X_{\geq n}) = 0$ for $i > n$, and $\pi_i(X) \xrightarrow{\simeq} \pi_i(X_{\leq n})$ for $i \leq n$. This pair $(X, X_{\leq n})$ is essentially unique up to homotopy; the space $X_{\leq n}$ is called the *nth Postnikov section* of X. Since Postnikov sections have "simpler" homotopy groups, we can try to understand X by studying each of its Postnikov sections individually, and then gluing all the data together.

Suppose A is some abelian group. We saw, in the first partthat there is a space provide a link
$M(A, n)$ with homology given by:

$$\widetilde{H}_i(M(A, n)) = \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

This space was constructed from a free resolution $0 \to F_1 \to F_0 \to A \to 0$ of A. We can construct a map $\bigvee S^n \to \bigvee S^n$ which realizes the first two maps; coning this off gets $M(A, n)$. By Hurewicz, we have:

$$\pi_i(M(A, n)) = \begin{cases} 0 & i < n \\ A & i = n \\ ?? & i > n \end{cases}$$

It follows that, when we look at the nth Postnikov section of $M(A,n)$, we have:

$$\pi_i(M(A,n)_{\le n}) = \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

In some sense, therefore, this Postnikov section is a "designer homotopy type". It deserves a special name: $M(A,n)_{\le n}$ is called an *Eilenberg-MacLane space*, and is denoted $K(A,n)$. By the fiber sequence $\Omega X \to PX \to X$ with $PX \simeq *$, we find that $\Omega K(\pi,n) \simeq K(\pi,n-1)$. Eilenberg-MacLane spaces are unique up to homotopy.

Note that $n = 1$, A does not have to be abelian, but you can still construct $K(A,1)$. This is called the *classifying space* of G; such spaces will be discussed in more detail in the next chapter. Examples are in abundance: if Σ is a closed surface that is not S^2 or $\mathbf{R}^2$, then $\Sigma \simeq K(\pi_1(\Sigma),1)$. Perhaps simpler is the identification $S^1 \simeq K(\mathbf{Z},1)$.

Example 51.6. We can identify $K(\mathbf{Z},2)$ as $\mathbf{CP}^\infty$. To see this, observe that we have a fiber sequence $S^1 \to S^{2n+1} \to \mathbf{CP}^n$. The long exact sequence in homotopy tells us that the homotopy groups of $\mathbf{CP}^n$ are the same as the homotopy groups of S^1, until $\pi_* S^{2n+1}$ starts to interfere. As n grows, we obtain a fibration $S^1 \to S^\infty \to \mathbf{CP}^\infty$. Since S^∞ is weakly contractible (it has no nonzero homotopy groups), we get the desired result.

Example 51.7. Similarly, we can identify $K(\mathbf{Z}/2\mathbf{Z},1)$ as $\mathbf{RP}^\infty$.

Since $\pi_1(K(A,n)) = 0$ for $n > 1$, it follows that $K(A,n)$ is automatically a simple space. This means that

$$[S^k, K(A,n)] = \pi_k(K(A,n)) = H^n(S^k, A).$$

In fact, a more general result is true:

Theorem 51.8 (Brown representability)**.** *If X is a CW-complex, then* $[X, K(A,n)] = H^n(X;A)$.

We will not prove this here, but one can show this simply by showing that the functor $[-, K(A,n)]$ satisfies the Eilenberg-Steenrod axioms. Somehow, these Eilenberg-MacLane spaces $K(A,n)$ completely capture cohomology in dimension n.

If X is a CW-complex, then we may assume that $X_{\le n}$ is also a CW-complex. (Otherwise, we can use cellular approximation and then kill homotopy groups.) Let us assume that X is path connected; then $X_{\le 1} = K(\pi_1(X),1)$. We may then form a (commuting) tower:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \cdots \arrow[l] \\
 & X_{\le 3} \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
 & X_{\le 2} \arrow[d] & K(\pi_2(X),2) \arrow[l] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & X_{\le 1} \arrow[r, equal] & K(\pi_1(X),1),
\end{tikzcd}$$

since $K(\pi_n(X),n) \to X_{\le n} \to X_{\le n-1}$ is a fiber sequence. This decomposition of X is called the *Postnikov tower* of X.

Denote by $X_{>n}$ the fiber of the map $X \to X_{\le n}$ (for instance, $X_{>1}$ is the universal cover of X); then, we have

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, equal] & \vdots \arrow[d] & \cdots \arrow[l] \\
X_{>3} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\le 3} \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
X_{>2} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\le 2} \arrow[d] & K(\pi_2(X),2) \arrow[l] \\
X_{>1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\le 1} \arrow[d] \arrow[r, equal] & K(\pi_1(X),1) \\
X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}$$

The leftmost tower is called the *Whitehead tower* of X, named after George Whitehead.

I can take the fiber of $X_{>1} \to X$, and I get $K(\pi_1(X),0)$; more generally, the fiber of

$X_{>n} \to X_{>n-1}$ is $K(\pi_n(X), n-1)$. This yields the following diagram:

$$\begin{tikzcd}
\cdots & \vdots \arrow[d] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \cdots \\
K(\pi_3(X),2) \arrow[r] & X_{>3} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 3} \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
K(\pi_2(X),1) \arrow[r] & X_{>2} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 2} \arrow[d] & K(\pi_2(X),2) \arrow[l] \\
K(\pi_1(X),0) \arrow[r] & X_{>1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 1} \arrow[d] \arrow[r, equal] & K(\pi_1(X),1) \\
 & X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}$$

We can construct Eilenberg-MacLane spaces as cellular complexes by attaching cells to the sphere to kill its higher homotopy groups. The complexity of homotopy groups, though, shows us that attaching cells to compute the cohomology of Eilenberg-MacLane spaces is not feasible.

Chapter 4

Vector bundles

52 Vector bundles, principal bundles

Let X be a topological space. A point in X can be viewed as a map $* \to X$; this is a cross section of the canonical map $X \to *$. Motivated by this, we will define a vector space over B to be a space $E \to B$ over B with the following extra data:

- a multiplication $\mu : E \times_B E \to E$, compatible with the maps down to B;
- a "zero" section $s : B \to E$ such that the composite $B \xrightarrow{s} E \to B$ is the identity;
- an inverse $\chi : E \to E$, compatible with the map down to B; and
- an action of $\mathbf{R}$:

$$\begin{tikzcd} \mathbf{R} \times E \arrow[r, equal] \arrow[dr, "p\circ \mathrm{pr}_2"'] & (B \times \mathbf{R}) \times_B E \arrow[r] \arrow[d] & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Because $\mathbf{R}$ is a field, the last piece of data shows that $p^{-1}(b)$ is a $\mathbf{R}$-vector space for any point $b \in B$.

Example 52.1. A rather silly example of a vector space over B is the projection $B \times V \to B$ where V is a (real) vector space, which we will always assume to be finite-dimensional.

Example 52.2. Consider the map

$$\mathbf{R} \times \mathbf{R} \xrightarrow{(s,t) \mapsto (s,st)} \mathbf{R} \times \mathbf{R},$$

over $\mathbf{R}$ (the structure maps are given by projecting onto the first factor). It is an isomorphism on all fibers, but is zero everywhere else. The kernel is therefore 0 everywhere, except over the point $0 \in \mathbf{R}$. This the "skyscraper" vector bundle over B.

Sheaf theory accommodates examples like this.

One can only go so far you can go with this simplistic notion of a "vector space" over B. Most interesting and naturally arising examples have a little more structure, which is exemplified in the following definition.

Definition 52.3. A *vector bundle* over B is a vector space over B that is locally trivial (in the sense of Definition 42.1).

Remark 52.4. We will always assume that the space B admits a numerable open cover (see Definition 42.4) which trivializes the vector bundle. Moreover, the dimension of the fiber will always be finite.

If $p : E \to B$ is a vector bundle, then E is called the *total space*, p is called the *projection map*, and B is called the *base space*. We will always use a Greek letter like ξ or ζ to denote a vector bundle, and $E(\xi) \to B(\xi)$ denotes the actual projection map from the total space to the base space. The phrase "ξ is a vector bundle over B" will also be shortened to $\xi \downarrow B$.

Example 52.5.
1. Following Example 52.1, one example of a vector bundle is the trivial bundle $B \times \mathbf{R}^n \to B$, denoted by $n\epsilon$.

2. In contrast to this silly example, one gets extremely interesting examples from the Grassmannians $\mathrm{Gr}_k(\mathbf{R}^n)$, $\mathrm{Gr}_k(\mathbf{C}^n)$, and $\mathrm{Gr}_k(\mathbb{H}^n)$. For simplicity, let K denote $\mathbf{R}, \mathbf{C}$, or $\mathbb{H}$. Over $\mathrm{Gr}_k(K^n)$ lies the *tautological bundle* γ. This is a sub-bundle of $n\epsilon$ (i.e., the fiber over any point $x \in \mathrm{Gr}_k(K^n)$ is a subspace of the fiber of $n\epsilon$ over x). The total space of γ is defined as:
$$E(\gamma) = \{(V, x) \in \mathrm{Gr}_k(K^n) \times K^n : x \in V\}$$
This projection map down to $\mathrm{Gr}_k(K^n)$ is the literal projection map
$$(V, x) \mapsto V.$$
Exercise 52.6. Prove that γ, as defined above, is locally trivial; so γ defines a vector bundle over $\mathrm{Gr}_k(K^n)$.

   For instance, when $k = 1$, we have $\mathrm{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$. In this case, γ is one-dimensional (i.e., the fibers are all of dimension 1); this is called a *line bundle*. In fact, it is the "canonical line bundle" over $\mathbf{RP}^{n-1}$.

3. Let M be a smooth manifold. Define τ_M to be the tangent bundle $TM \to M$ over M. For example, if $M = S^{n-1}$, then
$$TS^{n-1} = \{(x, v) \in S^{n-1} \times \mathbf{R}^n : v \cdot x = 0\}.$$

Constructions with vector bundles

One cannot take the kernels of a map of vector bundles; but just about anything which can be done for vector spaces can also be done for vector bundles:

1. Pullbacks are legal: if $p' : E' \to B'$, then the leftmost map in the diagram below is also a vector bundle.
$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$
For instance, if $B = *$, the pullback is just the fiber of E' over the point $* \to B'$. If ξ is the bundle $E' \to B'$, we denote the pullback $E \to B$ as $f^*\xi$.

2. If $p : E \to B$ and $p' : E' \to B'$, then we can take the product $E \times E' \xrightarrow{p\times p'} B \times B'$.

3. If $B = B'$, we can form the pullback:
$$\begin{tikzcd} E \oplus E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \arrow[r, "\Delta"'] & B \times B \end{tikzcd}$$
The bundle $E \oplus E'$ is called the *Whitney sum*. For instance, it is an easy exercise to see that
$$n\epsilon = \epsilon \oplus \cdots \oplus \epsilon.$$

4. If $E, E' \to B$ are two vector bundles over B, we can form another vector bundle $E \otimes_{\mathbf{R}} E' \to B$ by taking the fiberwise tensor product. Likewise, taking the fiberwise Hom begets a vector bundle $\mathrm{Hom}_{\mathbf{R}}(E, E') \to B$.

Example 52.7. Recall from Example 52.5(2) that the tautological bundle γ lives over $\mathbf{RP}^{n-1}$; we will write $L = E(\gamma)$. The tangent bundle $\tau_{\mathbf{RP}^{n-1}}$ also lives over $\mathbf{RP}^{n-1}$. As this is the first explicit pair of vector bundles over the same space, it is natural to wonder what is the relationship between these two bundles.

At first glance, one might guess that $\tau_{\mathbf{RP}^{n-1}} = \gamma^\perp$; but this is false! Instead,
$$\tau_{\mathbf{RP}^{n-1}} = \mathrm{Hom}(\gamma, \gamma^\perp).$$
To see this, note that we have a 2-fold covering map $S^{n-1} \to \mathbf{RP}^{n-1}$; therefore, $T_x(\mathbf{RP}^{n-1})$ is a quotient of $T(S^n)$ by the map sending $(x, v) \mapsto (-x, -v)$, where $v \in T_x(S^n)$. Therefore,
$$T_x\mathbf{RP}^{n-1} = \{(x, v) \in S^{n-1} \times \mathbf{R}^n : v \cdot x = 0\}/((x, v) \sim (-x, -v)).$$
This is exactly the fiber of $\mathrm{Hom}(\gamma, \gamma^\perp)$ over $x \in \mathbf{RP}^{n-1}$, since the line through x can be mapped to the line through $\pm v$.

Exercise 52.8. Prove that if γ is the tautological vector bundle over $\mathrm{Gr}_k(K^n)$, for $K = \mathbf{R}, \mathbf{C}, \mathbb{H}$, then
$$\tau_{\mathrm{Gr}_k(K^n)} = \mathrm{Hom}(\gamma, \gamma^\perp).$$

Metrics and splitting exact sequences

A *metric* on a vector bundle is a continuous choice of inner products on fibers.

Lemma 52.9. *Any vector bundle ξ over X admits a metric.*

Intuitively speaking, this is true because if g, g' are both inner products on V, then $tg + (1-t)g'$ is another. Said differently, the space of metrics forms a real affine space.

Proof. Pick a trivializing open cover of X, and a subordinate partition of unity. This means that we have a map $\phi_U : U \to [0,1]$, such that the preimage of the complement of 0 is U. Moreover,
$$\sum_{x\in U} \phi_U(x) = 1.$$
Over each one of these trivial pieces, pick a metric g_U on $E|_U$. Let
$$g \coloneqq \sum_U \phi_U g_U;$$
this is the desired metric on ξ. $\square$

We remark that, in general, one cannot pick metrics for vector bundles. For instance, this is the case for vector bundles which arise in algebraic geometry.

Definition 52.10. Suppose $E, E' \to B$ are vector bundles over B. An *isomorphism* is a map $\alpha : E \to E'$ over B that is a linear isomorphism on each fiber.

In particular, the map α admits an inverse (over B).

Corollary 52.11. *Any exact*[1] *sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles (over the same base) splits.*

Proof sketch. Pick a metric for E. Consider the composite
$$E'^\perp \subseteq E \to E''.$$
This is an isomorphism: the dimensions of the fibers are the same. It follows that
$$E \cong E' \oplus E'^\perp \cong E' \oplus E'',$$
as desired. $\square$

Note that this splitting is not natural.

[1] This is the obvious definition.

53 Principal bundles, associated bundles

I-invariance

We will denote by $\mathrm{Vect}(B)$ the set of isomorphism classes of vector bundles over B. (Justify the use of the word "set"!)

Consider a vector bundle $\xi \downarrow B$. If $f : B' \to B$, taking the pullback gives a vector bundle denoted $f^*\xi$. This operation descends to a map $f^* : \mathrm{Vect}(B) \to \mathrm{Vect}(B')$; we therefore obtain a functor $\mathrm{Vect} : \mathbf{Top}^{op} \to \mathrm{Set}$. One might expect this functor to give some interesting invariants of topological spaces.

Theorem 53.1. *Let $I = \Delta^1$. Then* Vect *is I-invariant. In other words, the projection $X \times I \to X$ induces an isomorphism* $\mathrm{Vect}(X) \to \mathrm{Vect}(X \times I)$.

One important corollary of this result is:

Corollary 53.2. Vect *is a homotopy functor.*

Proof. Consider two homotopic maps $f, g : B \to B'$, so there exists a homotopy $H : B' \times I \to B$. If $\xi \downarrow B$, we need to prove that $f_0^*\xi \simeq f_1^*\xi$. This is far from obvious.

Consider the following diagram.

$$\begin{tikzcd} B' \times I \arrow[r, "H"] \arrow[d, "\mathrm{pr}"'] & B \\ B' \end{tikzcd}$$

The leftmost map is an isomorphism under Vect, by Theorem 53.1. Let $\eta \downarrow B$ be a vector bundle such that $\mathrm{pr}^*\eta \simeq f^*\xi$. For any $t \in I$, define a map ϵ_t: $B' \to B' \times I$ sends $x \mapsto (x, t)$. We then have isomorphisms:

$$f_t^*\xi \simeq \epsilon_t^* \, f^*\xi \simeq \epsilon_t^* \, \mathrm{pr}^*\eta \simeq (\mathrm{pr} \circ \epsilon_t)^*\eta \simeq \eta,$$

as desired. □

It is easy to see that $\mathrm{Vect}(X) \to \mathrm{Vect}(X \times I)$ is injective. In the next lecture, we will prove surjectivity, allowing us to conclude Theorem 53.1.

Principal bundles

Definition 53.3. Let G be a topological group[2]. A *principal G-bundle* is a right action of G on P such that:

- G acts freely.
- The orbit projection $P \to P/G$ is a fiber bundle.

[2]We will only care about discrete groups and Lie groups.

These are not unfamiliar objects, as the next example shows.

Example 53.4. Suppose G is discrete. Then the fibers of the orbit projection $P \to P/G$ are all discrete. Therefore, the condition that $P \to P/G$ is a fiber bundle is simply that it's a covering projection (the action is "properly discontinuous").

As a special case, let X be a space with universal cover $\widetilde{X} \downarrow X$. Then $\pi_1(X)$ acts freely on $\widetilde{X}$, and $\widetilde{X} \downarrow X$ is the orbit projection. It follows from our discussion above that this is a principal bundle. Explicit examples include the principal $\mathbf{Z}/2$-bundle $S^{n-1} \downarrow \mathbf{RP}^{n-1}$, and the Hopf fibration $S^{2n-1} \downarrow \mathbf{CP}^{n-1}$, whcih is a principle S^1-bundle.

By looking at the universal cover, we can classify covering spaces of X. Remember how that goes: if F is a set with left $\pi_1(X)$-action, the dotted map in the diagram below is the desired covering space.

$$\begin{tikzcd} \widetilde{X} \times F \arrow[r] \arrow[d, "p\circ \mathrm{pr}_1"'] & \widetilde{X} \times F/\sim \arrow[dl, "q"] \\ X & \end{tikzcd}$$

Here, we say that $(y, gz) \sim (yg, z)$, for elements $y \in \widetilde{X}$, $z \in F$, and $g \in \pi_1(X)$.

Fix $y_0 \in \widetilde{X}$ over $* \in X$. Then it is easy to see that $F \xrightarrow{\sim} q^{-1}(*)$, via the map $z \mapsto (y_0, z)$. This is all neatly summarized in the following theorem from point-set topology.

Theorem 53.5 (Covering space theory)**.** *There is an equivalence of categories:*

$$\{\text{Left } \pi_1(X)\text{-sets}\} \xrightarrow{\simeq} \{\text{Covering spaces of } X\},$$

with inverse functor given by taking the fiber over the basepoint and lifting a loop in X to get a map from the fiber to itself.

Example 53.4 shows that covering spaces are special examples of principal bundles. The above theorem therefore motivates finding a more general picture.

Construction 53.6. Let $P \downarrow B$ is a principal G-bundle. If F is a left G-space, we can define a new fiber bundle, exactly as above:

$$\begin{tikzcd} P \times F \arrow[r] \arrow[d] & P \times F/\sim \arrow[dl, "q"] \\ B & \end{tikzcd}$$

This is called an *associated bundle*, and is denoted $P \times_G F$.

We must still justify that the resulting space over B is indeed a new fiber bundle with fiber F. Let $x \in B$, and let $y \in P$ over x. As above, we have a map $F \to q^{-1}(*)$ via the map $z \mapsto [y, z]$. We claim that this is a homeomorphism. Indeed, define a map $q^{-1}(*) \to F$ via

$$[y', z'] = [y, gz'] \mapsto gz',$$

where $y' = yg$ for some g (which is necessarily unique).

Exercise 53.7. Check that these two maps are inverse homeomorphisms.

Definition 53.8. A vector bundle $\xi \downarrow B$ is said to be an n-plane bundle if the dimensions of all the fibers are n.

Let $\xi \downarrow B$ be an n-plane bundle. Construct a principal $\mathrm{GL}_n(\mathbf{R})$-bundle $P(\xi)$ by defining

$$P(\xi)_b = \{\text{bases for } E(\xi)_b = \mathrm{Iso}(\mathbf{R}^n, E(\xi)_b)\}.$$

To define the topology, note that (topologically) we have

$$P(B \times \mathbf{R}^n) = B \times \mathrm{Iso}(\mathbf{R}^n, \mathbf{R}^n),$$

where $\mathrm{Iso}(\mathbf{R}^n, \mathbf{R}^n) = \mathrm{GL}_n(\mathbf{R})$ is given the usual topology as a subspace of $\mathbf{R}^{n^2}$.

There is a right action of $\mathrm{GL}_n(\mathbf{R})$ on $P(\xi) \downarrow B$, given by precomposition. It is easy to see that this action is free and simply transitive. One therefore has a principal action of $\mathrm{GL}_n(\mathbf{R})$ on $P(\xi)$. The bundle $P(\xi)$ is called the *principalization* of ξ.

Given the principalization $P(\xi)$, we can recover the total space $E(\xi)$. Consider the associated bundle $P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n$ with fiber $F = \mathbf{R}^n$, with $\mathrm{GL}_n(\mathbf{R})$ acting on $\mathbf{R}^n$ from the left. Because this is a linear action, $P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n$ is a vector bundle. One can show that

$$P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n \simeq E(\xi).$$

Fix a topological group G. Define $\mathrm{Bun}_G(B)$ as the set of isomorphism classes of G-bundles over B. An isomorphism is a G-equivariant homeomorphism over the base. Again, arguing as above, this begets a functor $\mathrm{Bun}_G : \mathbf{Top} \to \mathrm{Set}$. The above discussion gives a natural isomorphism of functors:

$$\mathrm{Bun}_{\mathrm{GL}_n(\mathbf{R})}(B) \simeq \mathrm{Vect}(B).$$

The I-invariance theorem will therefore follow immediately from:

Theorem 53.9. Bun_G *is I-invariant.*

Remark 53.10. Principal bundles allow a description of "geometric structures on ξ". Suppose, for instance, that we have a metric on ξ. Instead of looking at all ordered bases, we can attempt to understand all ordered orthonormal bases in each fiber. This give the *frame bundle*

$$\mathrm{Fr}(B) = \{\text{ordered orthonormal bases of } E(\xi)_b\};$$

these are isometric isomorphisms $\mathbf{R}^n \to E(\xi)_b$. Again, there is an action of the orthogonal group on $\mathrm{Fr}(B)$: in fact, this begets a principal $O(n)$-bundle. Such examples are in abundance: consistent orientations give an $SO(n)$-bundle. Trivializations of the vector bundle also give principal bundles. This is called "reduction of the structure group".

One useful fact about principal G-bundles (which should not be too surprising) is the following statement.

Theorem 53.11. *Every morphism of principal G-bundles is an isomorphism.*

Proof. Let $p : P \to B$ and $p' : P' \to B$ be two principal G-bundles over B, and let $f : P \to P'$ be a morphism of principal G-bundles. For surjectivity of f, let $y \in P'$. Consider $x \in P$ such that $p(x) = p'(y)$. Since $p(x) = p'f(x)$ we conclude that $y = f(x)g$ for some $g \in G$. But $f(x)g = f(xg)$, so xg maps to y, as desired. To see that f is injective, suppose $f(x) = f(y)$. Now $p(x) = p'f(x) = p(y)$, so there is some $g \in G$ such that $xg = y$. But $f(y) = f(xg) = f(x)g$, so $g = 1$, as desired. We will leave the continuity of f^{-1} as an exercise to the reader. $\square$

Theorem 53.11 says that if we view $\mathrm{Bun}_G(B)$ as a category where the morphisms are given by morphisms of principal G-bundles, then it is a groupoid.

54 I-invariance of Bun_G, and G-CW-complexes

Let G be a topological group. We need to show that the functor $\mathrm{Bun}_G : \mathbf{Top}^{op} \to \mathrm{Set}$ is I-invariant, i.e., the projection $X \times I \xrightarrow{\mathrm{pr}} X$ induces an isomorphism $\mathrm{Bun}_G(X) \xrightarrow{\cong} \mathrm{Bun}_G(X \times I)$. Injectivity is easy: the composite $X \xrightarrow{\mathrm{in}_0} X \times I \xrightarrow{\mathrm{pr}} X$ gives you a splitting $\mathrm{Bun}_G(X) \xrightarrow{\mathrm{pr}_*} \mathrm{Bun}_G(X \times I) \xrightarrow{\mathrm{in}_0} \mathrm{Bun}_G(X)$ whose composite is the identity.

The rest of this lecture is devoted to proving surjectivity. We will prove this when X is a CW-complex (Husemoller does the general case; see [Hus94, §4.9]). We begin with a small digression.

G-CW-complexes

We would like to define CW-complexes with an action of the group G. The naïve definition (of a space with an action of the group G) will not be sufficient; rather, we will require that each cell have an action of G.

In other words, we will build G-CW-complexes out of "G-cells". This is supposed to be something of the form $D^n \times H\backslash G$, where H is a closed subgroup of G. Here, the space $H\backslash G$ is the orbit space, viewed as a right G-space. The boundary of the G-cell $D^n \times H\backslash G$ is just $\partial D^n \times H\backslash G$. More precisely:

Definition 54.1. A G-CW-complex is a (right) G-space X with a filtration $0 = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X$ such that for all n, there exists a pushout square:

$$\begin{tikzcd} \coprod \partial D^n_\alpha \times H_\alpha\backslash G \arrow[r] \arrow[d] & \coprod D^n_\alpha \times H_\alpha\backslash G \arrow[d] \\ X_{n-1} \arrow[r] & X_n, \end{tikzcd}$$

and X has the direct limit topology.

Notice that a CW-complex is a G-CW-complex for the trivial group G.

Theorem 54.2. *If G is a compact Lie group and M a compact smooth G-manifold, then M admits a G-CW-structure.*

This is the analogue of the classical result that a compact smooth manifold is homotopy equivalent to a CW-complex, but it is much harder to prove the equivariant statement.

Note that if G acts principally (Definition 53.3) on P, then every G-CW-structure on P is "free", i.e., $H_\alpha = 0$.

1. If X is a G-CW-complex, then X/G inherits a CW-structure whose n-skeleton is given by $(X/G)_n = X_n/G$.

2. If $P \to X$ is a principal G-bundle, then a CW-structure on X lifts to a G-CW-structure on P.

Proof of I-invariance

Recall that our goal is to prove that every G-bundle over $X \times I$ is pulled back from some vector bundle over X.

As a baby case of Theorem 53.1 we will prove that if X is contractible, then any principal G-bundle over X is trivial, i.e., $P \simeq X \times G$ as G-bundles.

Let us first prove the following: if $P \downarrow X$ has a section, then it's trivial. Indeed, suppose we have a section $s : X \to P$. Since P has an action of the group on it, we may extend this to a map $X \times G \to P$ by sending $(x, g) \mapsto gs(x)$. As this is a map of G-bundles over X, it is an isomorphism by Theorem 53.11, as desired.

To prove the statement about triviality of any principal G-bundle over a contractible space, it therefore suffices to construct a section for any principal G-bundle. Consider the constant map $X \to P$. Then the following diagram commutes up to homotopy, and hence (by Exercise 44.10(1)) there is an *actual* section of $P \to X$, as desired.

$$\begin{tikzcd} & P \arrow[d] \\ X \arrow[ur, "\text{const}"] \arrow[r] & X \end{tikzcd}$$

For the general case, we will assume X is a CW-complex. For notational convenience, let us write $Y = X \times I$. We will use descending induction to construct the desired principal G-bundle over X.

To do this, we will filter Y by subcomplexes. Let $Y_0 = X \times 0$; in general, we define

$$Y_n = X \times 0 \cup X_{n-1} \times I.$$

It follows that we may construct Y_n out of Y_{n-1} via a pushout:

$$\begin{tikzcd} \coprod_{\alpha \in \Sigma_{n-1}} (\partial D^{n-1} \times I \cup D^{n-1}_\alpha \times 0) \arrow[r] \arrow[d, "\coprod_{\alpha \in \Sigma_{n-1}} f_\alpha \times 1_I \cup \phi_\alpha \times 0"'] & \coprod_\alpha (D^{n-1}_\alpha \times I) \arrow[d] \\ Y_{n-1} \arrow[r] & Y_n, \end{tikzcd}$$

where the maps f_α and ϕ_α are defined as:

$$\begin{tikzcd}
\partial D_\alpha^{n-1} \arrow[r, "f_\alpha"] \arrow[d] & X_{n-2} \arrow[d] \\
D_\alpha^{n-1} \arrow[r, "\phi_\alpha"'] & X_{n-1}
\end{tikzcd}$$

In other words, the f_α are the attaching maps and the ϕ_α are the characteristic maps.

Consider a principal G-bundle $P \xrightarrow{p} Y = X \times I$. Define $P_n = p^{-1}(Y_n)$; then we can build P_n from P_{n-1} in a similar way:

$$\begin{tikzcd}
\coprod_\alpha (\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \times G \arrow[r] \arrow[d] & \coprod_\alpha (D_\alpha^{n-1} \times I) \times G \arrow[d] \\
P_{n-1} \arrow[r] & P_n
\end{tikzcd}$$

Note that this isn't *quite* a G-CW-structure. Recall that we are attempting to fill in a dotted map:

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d] & P_0 \arrow[d] \\
Y \arrow[r, "\mathrm{pr}"'] & Y_0 = X
\end{tikzcd}$$

sh this...

I'm constructing this inductively– we have $P_{n-1} \to P_0$. So I want to define $\coprod_\alpha (D_\alpha^{n-1} \times I) \times G \to P_0$ that's equivariant. That's the same thing as a map $\coprod_\alpha (D_\alpha^{n-1} \times I) \to P_0$ that's compatible with the map from $\coprod (\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0)$. Namely, I want to fill in:

$$\begin{tikzcd}
\coprod_\alpha (\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \arrow[r] \arrow[d] & \coprod_\alpha (D_\alpha^{n-1} \times I) \arrow[d] \arrow[ddr, dashed] & \\
\coprod_\alpha (\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \times G \arrow[r] \arrow[d] & \coprod_\alpha (D_\alpha^{n-1} \times I) \times G \arrow[d] \arrow[dr, dashed] & \\
P_{n-1} \arrow[r] \arrow[drr, "\text{induction}"] & P_n \arrow[dr] & \\
 & & P_0 \arrow[d] \\
 & & X
\end{tikzcd} \tag{4.1}$$

Now, I know that $(D^{n-1} \times I, \partial D^{n-1} \times I \cup D^{n-1} \times 0) \simeq (D^{n-1} \times I, D^{n-1} \times 0)$. So what I have is:

$$\begin{tikzcd} D^{n-1}\times 0 \arrow[r, "\text{induction}"] \arrow[d] & P_0 \arrow[d] \\ D^{n-1}\times I \arrow[r, "\phi\circ pr"'] \arrow[ur, dashed] & X \end{tikzcd}$$

So the dotted map exists, since $P_0 \to X$ is a fibration!

OK, so note that I haven't checked that the outer diagram in Equation 4.1 commutes, because otherwise we wouldn't get $P_n \to P_0$.

Exercise 54.3. Check my question above.

Turns out this is easy, because you have a factorization:

$$\begin{tikzcd} D^{n-1}\times 0 \arrow[r] \arrow[d] & P_{n-1} \arrow[r, "\text{induction}"] & P_0 \arrow[d] \\ D^{n-1}\times I \arrow[rr, "\phi\circ pr"'] \arrow[urr, dashed] & & X \end{tikzcd}$$

Oh my god, look what time it is! Oh well, at least we got the proof done.

55 Classifying spaces: the Grassmann model

We will now shift our focus somewhat and talk about classifying spaces for principal bundles and for vector bundles. We will do this in two ways: the first will be via the Grassmann model and the second via simplicial methods.

Lemma 55.1. *Over a compact Hausdorff space, any n-plane bundle embeds in a trivial bundle.*

Proof. Let $\mathcal{U}$ be a trivializing open cover of the base B; since B is compact, we may assume that $\mathcal{U}$ is finite with k elements. There is no issue with numerability, so there is a subordinate partition of unity ϕ_i. Consider an n-plane bundle $E \to B$. By trivialization, there is a fiberwise isomorphism $p^{-1}(U_i) \xrightarrow{f_i} \mathbf{R}^n$ where the $U_i \in \mathcal{U}$. A map to a trivial bundle is the same thing as a bundle map in the following diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbf{R}^N \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

We therefore define $E \to (\mathbf{R}^n)^k$ via

$$e \mapsto (\phi_i(p(e)) f_i(e))_{i=1,\cdots,k}.$$

This is a fiberwise linear embedding, generally called a "Gauss map". Indeed, observe that this map has no kernel on every fiber, so it is an embedding. $\square$

The trivial bundle has a metric on it, so choosing the orthogonal complement of the embedding of Lemma 55.1, we obtain:

Corollary 55.2. *Over a compact Hausdorff space, any n-plane bundle has a complement (i.e. a $\xi^\perp$ such that $\xi \oplus \xi^\perp$ is trivial).*

Another way to say this is that if B is a compact Hausdorff space with an n-plane bundle ξ, there is a map $f : X \to \mathrm{Gr}_n(\mathbf{R}^{kn})$; this is exactly the Gauss map. It has the property that taking the pullback $f^*\gamma^n$ of the tautologous bundle over $\mathrm{Gr}_n(\mathbf{R}^{kn})$ gives back ξ.

In general, we do not have control over the number k. There is an easy fix to this problem: consider the tautologous bundle γ^n over $\mathrm{Gr}_n(\mathbf{R}^\infty)$, defined as the union of $\mathrm{Gr}_n(\mathbf{R}^m)$ and given the limit topology. This is a CW-complex of finite type (i.e. finitely many cells in each dimension). Note that $\mathrm{Gr}_n(\mathbf{R}^m)$ are not the m-skeleta of $\mathrm{Gr}_n(\mathbf{R}^\infty)$!

The space $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is "more universal":

Lemma 55.3. *Any (numerable) n-plane bundle is pulled back from $\gamma^n \downarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ via the Gauss map.*

Lemma 55.3 is a little bit tricky, since the covering can be wildly uncountable; but this is remedied by the following bit of point-set topology.

Lemma 55.4. *Let $\mathcal{U}$ be a numerable cover of X. Then there's another numerable cover $\mathcal{U}'$ such that:*

1. *the number of open sets in $\mathcal{U}'$ is countable, and*

2. *each element of $\mathcal{U}'$ is a disjoint union of elements of $\mathcal{U}$.*

If $\mathcal{U}$ is a trivializing cover, then $\mathcal{U}'$ is also a trivializing cover.

Proof. See [Hus94, Proposition 3.5.4]. □

It is now an exercise to deduce Lemma 55.3. The main result of this section is the following.

Theorem 55.5. *The map $[X, \mathrm{Gr}_n(\mathbf{R}^\infty)] \to \mathrm{Vect}_n(X)$ defined by $[f] \mapsto [f^*\gamma^n]$ is bijective, where $[f]$ is the homotopy class of f and $[f^*\gamma^n]$ is the isomorphism class of the bundle $f^*\gamma^n$.*

This is why $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is also called the *classifying space* for n-plane bundles. The Grassmannian provides a very explicit geometric description for the classifying space of n-plane bundles. There is a more abstract way to produce a classifying space for principal G-bundles, which we will describe in the next section; the Grassmannian is the special case when $G = \mathrm{GL}_n(\mathbf{R})$.

Proof. We have already shown surjectivity, so it remains to prove injectivity. Suppose $f_0, f_1 : X \to \mathrm{Gr}_n(\mathbf{R}^\infty)$ such that $f_0^*\gamma^n$ and $f_1^*\gamma^n$ are isomorphic over X. We need to construct a homotopy $f_0 \simeq f_1$. For ease of notation, let us identify $f_0^*\gamma^n$ and $f_1^*\gamma_n$ with each other; call it $\xi : E \downarrow X$.

The maps f_i are the same thing as Gauss maps $g_i : E \to \mathbf{R}^\infty$, i.e., maps which are fiberwise linear embeddings. The homotopy $f_0 \simeq f_1$ is created by saying that we have a homotopy from g_0 to g_1 through Gauss maps, i.e., through other fiberwise linear embeddings.

In fact, we will prove a much stronger statement: *any two* Gauss maps $g_0, g_1 : E \to \mathbf{R}^\infty$ are homotopic through Gauss maps. This is very far from true if I didn't have a $\mathbf{R}^\infty$ on the RHS there.

Let us attempt (and fail!) to construct an affine homotopy between g_0 and g_1. Consider the map $tg_0 + (1-t)g_1$ for $0 \le t \le 1$. In order for these maps to define a homotopy via Gauss maps, we need the following statement to be true: for all t, if $tg_0(v) + (1-t)g_1(v) = 0 \in \mathbf{R}^\infty$, then $v = 0$. In other words, we need $tg_0 + (1-t)g_1$ to be injective. Of course, this is not guaranteed from the injectivity of g_0 and g_1!

Instead, we will construct a composite of affine homotopies between g_0 and g_1 using the fact that $\mathbf{R}^\infty$ is an infinite-dimensional Euclidean space. Consider the following two linear isometries:

$$\begin{tikzcd}
 & \mathbf{R}^\infty = \langle e_0, e_1, \cdots \rangle \arrow[dl, "e_i \mapsto e_{2i}"', "\alpha"] \arrow[dr, "e_i \mapsto e_{2i+1}", "\beta"'] & \\
\mathbf{R}^\infty & & \mathbf{R}^\infty
\end{tikzcd}$$

Then, we have four Gauss maps: g_0, $\alpha \circ g_0$, $\beta \circ g_1$, and g_1. There are affine homotopies through Gauss maps:

$$g_0 \simeq \alpha \circ g_0 \simeq \beta \circ g_1 \simeq g_1.$$

We will only show that there is an affine homotopy through Gauss maps $g_0 \simeq \alpha \circ g_0$; the others are left as an exercise. Let t and v be such that $tg_0(v) + (1-t)\alpha g_0(v) = 0$. Since g_0 and αg_0 are Gauss maps, we may suppose that $0 < t < 1$. Since $\alpha g_0(v)_i$ has only even coordinates, it follows by definition of the map α that $g_0(v)$ only had nonzero coordinates only in dimensions congruent to 0 mod 4. Repeating this argument proves the desired result. $\square$

56 Simplicial sets

In order to discuss the simplicial model for classifying spaces of G-bundles, we will embark on a long digression on simplicial sets (which will last for three sections). We begin with a brief review of some of the theory of simplicial objects (see also Part 1).

Review

We denote by $[n]$ the set $\{0, 1, \cdots, n\}$, viewed as a totally ordered set. Define a category Δ whose objects are the sets $[n]$ for $n \ge 0$, with morphisms order preserving maps. There are maps $d^i : [n] \to [n+1]$ given by omitting i (called coface maps) and codegeneracy maps $s^i : [n] \to [n-1]$ that's the surjection which repeats i. As discussed in Exercise **??**, any order-preserving map can be written as the composite of these maps, and there are famous relations that these things satisfy. They generate the category Δ.

There is a functor $\Delta : \Delta \to \mathbf{Top}$ defined by sending $[n] \mapsto \Delta^n$, the standard n-simplex. To see that this is a functor, we need to show that maps $\phi : [n] \to [m]$ induce maps $\Delta^n \to \Delta^m$. The vertices of Δ^n are indexed by elements of $[n]$, so we may just extend ϕ as an affine map to a map $\Delta^n \to \Delta^m$.

Let X be a space. The set of singular n-simplices $\mathbf{Top}(\Delta^n, X)$ defines the singular simplicial set $\mathrm{Sin} : \Delta^{op} \to \mathrm{Set}$.

Definition 56.1. Let $\mathcal{C}$ be a category. Denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$, i.e., the category $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$. We write $X_n = X([n])$, called the n-simplices.

Explicitly, this gives an object $X_n \in \mathcal{C}$ for every $n \geq 0$, as well as maps $d_i : X_{n+1} \to X_n$ and $s_i : X_{n-1} \to X_n$ given by the face and degeneracy maps.

Example 56.2. Suppose $\mathcal{C}$ is a small category, for instance, a group. Notice that $[n]$ is a small category, with:

$$[n](i,j) = \begin{cases} \{\leq\} & \text{if } i \leq j \\ \varnothing & \text{else.} \end{cases}$$

We are therefore entitled to think about $\mathrm{Fun}([n], \mathcal{C})$. This begets a simplicial set $N\mathcal{C}$, called the *nerve of* $\mathcal{C}$, whose n-simplices are $(N\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$. Explicitly, an n-simplex in the nerve is $(n+1)$-objects in $\mathcal{C}$ (possibly with repetitions) and a chain of n composable morphisms. The face maps are given by composition (or truncation, at the end of the chain of morphisms). The degeneracy maps just compose with the identity at that vertex.

For example, if G is a group regarded as a category, then $(NG)_n = G^n$.

Realization

The functor Sin transported us from spaces to simplicial sets. Milnor described a way to go the other way.

Let X be a simplicial set. We define the realization $|X|$ as follows:

$$|X| = \left(\coprod_{n \geq 0} \Delta^n \times X_n \right) / \sim,$$

where $\sim$ is the equivalence relation defined as:

$$\Delta^m \times X_m \ni (v, \phi^* x) \sim (\phi_* v, x) \in \Delta^n \times X$$

for all maps $\phi : [m] \to [n]$ where $v \in \Delta^m$ and $x \in X_n$.

Example 56.3. The equivalence relation is telling us to glue together simplices as dictated by the simplicial structure on X. To see this in action, let us look at $\phi^* = d_i : X_{n+1} \to X_n$ and $\phi_* = d^i : \Delta^n \to \Delta^{n+1}$. In this case, the equivalence relation then says that $(v, d_i x) \in \Delta^n \times X_n$ is equivalent to $(d^i v, x) \in \Delta^{n+1} \times X_{n+1}$. In other words: the n-simplex indexed by $d_i x$ is identified with the ith face of the $(n+1)$-simplex indexed by x.

There's a similar picture for the degeneracies s^i, where the equivalence relation dictates that every element of the form $(v, s_i x)$ is already represented by a simplex of lower dimension.

Example 56.4. Let $n \geq 0$, and consider the simplicial set $\mathrm{Hom}_\Delta(-, [n])$. This is called the "simplicial n-simplex", and is commonly denoted Δ^n for good reason: we have a homeomorphism $|\Delta^n| \simeq \Delta^n$. It is a good exercise to prove this using the explicit definition.

For any simplicial set X, the realization $|X|$ is naturally a CW-complex, with

$$\mathrm{sk}_n|X| = \left(\coprod_{k \leq n} \Delta^k \times X_k\right) / \sim .$$

The face maps give the attaching maps; for more details, see [GJ99, Proposition I.2.3]. This is a very combinatorial way to produce CW-complexes.

The geometric realization functor and the singular simplicial set give two functors going back and forth between spaces and simplicial sets. It is natural to ask: do they form an adjoint pair? The answer is yes:

$$\begin{tikzcd} s\mathrm{Set} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\mathrm{Sin}"] \end{tikzcd}$$

For instance, let X be a space. There is a continuous map $\Delta^n \times \mathrm{Sin}_n(X) \to X$ given by $(v, \sigma) \mapsto \sigma(v)$. The equivalence relation defining $|\mathrm{Sin}(X)|$ says that the map factors through the dotted map in the following diagram:

$$\begin{tikzcd} {|\mathrm{Sin}(X)|} \arrow[r, dashed] & X \\ \coprod \Delta^n \times \mathrm{Sin}_n(X) \arrow[u, two heads] \arrow[ur] & \end{tikzcd}$$

The resulting map is the counit of the adjunction.

Likewise, we can write down the unit of the adjunction: if $K \in s\mathrm{Set}$, the map $K \to \mathrm{Sin}|K|$ sends $x \in K_n$ to the map $\Delta^n \to |K|$ defined via $v \mapsto [(v, x)]$.

This is the beginning of a long philosophy in semi-classical homotopy theory, of taking any homotopy-theoretic question and reformulating it in simplicial sets. For instance, one can define homotopy groups in simplicial sets. For more details, see [GJ99].

We will close this section with a definition that we will discuss in the next section. Let $\mathcal{C}$ be a category. From our discussion above, we conclude that the realization $|N\mathcal{C}|$ of its nerve is a CW-complex, called the *classifying space* $B\mathcal{C}$ of $\mathcal{C}$; the relation to the notion of classifying space introduced in §55 will be elucidated upon in a later section.

57 Properties of the classifying space

One important result in the story of geometric realization introduced in the last section is the following theorem of Milnor's.

Theorem 57.1 (Milnor)**.** *Let X be a space. The map $|\mathrm{Sin}(X)| \to X$ is a weak equivalence.*

As a consequence, this begets a functorial CW-approximation to X. Unforunately, it's rather large.

In the last section, we saw that $|-|$ was a left adjoint. Therefore, it preserves colimits (Theorem 39.13). Surprisingly, it also preserves products:

Exercise 57.2 (Hard)**.** Let X and Y be simplicial sets. Their product $X \times Y$ is defined to be the simplicial set such that $(X \times Y)_n = X_n \times Y_n$. Under this notion of product, there is a homeomorphism

$$|X \times Y| \xrightarrow{\simeq} |X| \times |Y|.$$

It is important that this product is taken in the category of k-spaces.

Last time, we defined the classifying space $B\mathcal{C}$ of $\mathcal{C}$ to be $|N\mathcal{C}|$.

Theorem 57.3. *The natural map $B(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} B\mathcal{C} \times B\mathcal{D}$ is a homeomorphism*[3].

Proof. It is clear that $N(\mathcal{C} \times \mathcal{D}) \simeq N\mathcal{C} \times N\mathcal{D}$. Since $B\mathcal{C} = |N\mathcal{C}|$, the desired result follows from Exercise 57.2. □

In light of Theorem 57.3, it is natural to ask how natural transformations behave under the classifying space functor. To discuss this, we need some categorical preliminaries.

The category **Cat** is Cartesian closed (Definition 40.5). Indeed, the right adjoint to the product is given by the functor $\mathcal{D} \mapsto \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, as can be directly verified.

Consider the category $[1]$. This is particularly simple: a functor $[1] \to \mathcal{C}$ is just an arrow in $\mathcal{C}$. It follows that a functor $[1] \to \mathcal{D}^{\mathcal{C}}$ is a natural transformation between two functors f_0 and f_1 from $\mathcal{C}$ to $\mathcal{D}$. By our discussion above, this is the same as a functor $\mathcal{C} \times [1] \to \mathcal{D}$.

By Theorem 57.3, we have a homeomorphism $B([1] \times \mathcal{C}) \simeq B[1] \times B\mathcal{C}$. One can show that $B[1] = \Delta^1$, so a natural transformation between f_0 and f_1 begets a map $\Delta^1 \times B\mathcal{C} \to B\mathcal{D}$ between Bf_0 and Bf_1. Concretely:

[3] Recall that if $\mathcal{C}$ and $\mathcal{D}$ are categories, the product $\mathcal{C} \times \mathcal{D}$ is the category whose objects are pairs of objects of $\mathcal{C}$ and $\mathcal{D}$, and whose morphisms are pairs of morphisms in $\mathcal{C}$ and $\mathcal{D}$.

Lemma 57.4. *A natural transformation $\theta : f_0 \to f_1$ where $f_0, f_1 : \mathcal{C} \to \mathcal{D}$ induces a homotopy $Bf_0 \sim Bf_1 : B\mathcal{C} \to B\mathcal{D}$.*

An interesting comment is in order. The notion of a homotopy is "reversible", but that is definitely not true for natural transformations! The functor B therefore "forgets the polarity in **Cat**".

Lemma 57.4 is quite powerful: consider an adjunction $L \dashv R$ where $L : \mathcal{C} \to \mathcal{D}$; then we have natural transformations given by the unit $1_{\mathcal{C}} \to RL$ and the counit $LR \to 1_{\mathcal{D}}$. By Lemma 57.4 we get a homotopy equivalence between $B\mathcal{C}$ and $B\mathcal{D}$. In other words, two categories that are related by any adjoint pair are homotopy equivalent.

A special case of the above discussion yields a rather surprising result. Consider the category $[0]$. Let $\mathcal{D}$ be another category such that there is an adjoint pair $L \dashv R$ where $L : [0] \to \mathcal{D}$. Then L determines an object $*$ of $\mathcal{D}$. Let d be any object of $\mathcal{D}$. We have the counit $LR(d) \to d$; but $LR(d) = *$, so there is a unique morphism $* \to X$. (To see uniqueness, note that the adjunction $L \dashv R$ gives an identification $\mathcal{D}(*, X) = \mathcal{C}(0,0) = 0$.) In other words, such a category $\mathcal{D}$ is simply a category with an initial object.

Arguing similarly, any category $\mathcal{D}$ with adjunction $L \dashv R$ where $L : \mathcal{D} \to [0]$ is simply a category with a terminal object. From our discussion above, we conclude that if $\mathcal{D}$ is any category with a terminal (or initial) object, then $B\mathcal{D}$ is contractible.

58 Classifying spaces of groups

The constructions of the previous sections can be summarized in a single diagram:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, "\text{nerve}"] & s\mathrm{Set} \arrow[d, "|-|"] \\ \mathbf{Gp} \arrow[u, hook] \arrow[r, "B"'] & \mathbf{Top} \end{tikzcd}$$

The bottom functor is defined as the composite along the outer edge of the diagram. The space BG for a group G is called the *classifying space of* G. At this point, it is far from clear what BG is classifying. The goal of the next few sections is to demystify this definition.

Lemma 58.1. *Let G be a group, and $g \in G$. Let $c_g : G \to G$ via $x \mapsto gxg^{-1}$. Then the map $Bc_g : BG \to BG$ is homotopic to the identity.*

Proof. The homomorphism c_g is a functor from G to itself. It suffices to prove that there is a natural transformation θ from the identity to c_g. This is rather easy to define: it sends the only object to the only object: we define $\theta_* : * \to *$ to be the map given by $* \xrightarrow{g} *$ specified by $g \in \mathrm{Hom}_G(*, *) = G$. In order for θ to be a natural transformation, we need the following diagram to commute, which it obviously does:

$$\begin{tikzcd} * \arrow[r, "g"] \arrow[d, "1"'] & * \arrow[d, "gxg^{-1}"] \\ * \arrow[r, "g"'] & *. \end{tikzcd}$$

□

Groups are famous for acting on objects. Viewing groups as categories allows for an abstract definition a group action on a set: it is a functor $G \to \mathrm{Set}$. More generally, if $\mathcal{C}$ is a category, an action of $\mathcal{C}$ is a functor $\mathcal{C} \xrightarrow{X} \mathrm{Set}$. We write $X_c = X(c)$ for an object c of $\mathcal{C}$.

Definition 58.2. The "translation" category $X\mathcal{C}$ has objects given by

$$\mathrm{ob}(X\mathcal{C}) = \coprod_{c \in \mathcal{C}} X_c,$$

and morphisms defined via $\mathrm{Hom}_{X\mathcal{C}}(x \in X_c, y \in X_d) = \{f : c \to d : f_*(x) = y\}$.

There is a projection $X\mathcal{C} \to \mathcal{C}$. (For those in the know: this is a special case of the Grothendieck construction.)

Example 58.3. The group G acts on itself by left translation. We will write $\widetilde{G}$ for this G-set. The translation category $\widetilde{G}G$ has objects as G, and maps $x \to y$ are elements yx^{-1}. This category is "unicursal", in the sense that there is exactly one map from one object to another object. Every object is therefore initial and terminal, so the classifying space of this category is trivial by the discussion at the end of §57. We will denote by EG the classifying space $B(\widetilde{G}G)$. The map $\widetilde{G}G \to G$ begets a canonical map $EG \to BG$.

The G also acts on itself by right translation. Because of associativity, the right and left actions commute with each other. It follows that the right action is equivariant with respect to the left action, so we get a right action of G on EG.

Claim 58.4. This action of G on EG is a principal action, and the orbit projection is $EG \to BG$.

To prove this, let us contemplate the set $N(\widetilde{G}G)_n$. An element is a chain of composable morphisms. In this case, it is actually just a sequence of $n+1$ elements in G, i.e., $N(\widetilde{G}G)_n = G^{n+1}$. The right action of G is simply the diagonal action. We claim that this is a free action. More precisely:

Lemma 58.5 (Shearing)**.** *If G is a group and X is a G-set, and if $X \times^{\Delta} G$ has the diagonal G-action and $X \times G$ has G acting on the second factor by right translation, then $X \times^{\Delta} G \simeq X \times G$ as G-sets.*

Proof. Define a bijection $X \times^{\Delta} G \mapsto X \times G$ via $(x,g) \mapsto (xg^{-1}, g)$. This map is equivariant since $(x,g)\cdot h = (xh, gh)$, while $(xg^{-1}, g)\cdot h = (xg^{-1}, gh)$. The element (xh, gh) is sent to $(xh(gh)^{-1}, gh)$, as desired. The inverse map $X \times G \to X \times^{\Delta} G$ is given by $(x,g) \mapsto (xg, g)$. $\square$

We know that G acts freely on $N(\widetilde{G}G)_n$, soo a nonidentity group element is always going to send a simplex to another simplex. It follows that G acts freely on EG.

To prove the claim, we need to understand the orbit space. The shearing lemma shows that quotienting out by the action of G simply cancels out one copy of G from the product $N(\widetilde{G}G) = G^n$. In symbols:

$$N(\widetilde{G}G)/G \simeq G^n \simeq (NG)_n.$$

Of course, it remains to check the compatibility with the face and degeneracy maps. We will not do this here; but one can verify that everything works out: the realization is just BG!

We need to be careful: the arguments above establish that $EG/G \simeq BG$ when G is a finite group. The case when G is a topological group is more complicated. To describe this generalization, we need a preliminary categorical definition.

Let $\mathcal{C}$ be a category, with objects $\mathcal{C}_0$ and morphisms $\mathcal{C}_1$. Then we have maps $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{\text{compose}} \mathcal{C}_1$ and two maps (source and target) $\mathcal{C}_1 \to \mathcal{C}_0$, and the identity $\mathcal{C}_0 \to \mathcal{C}_1$. One can specify the same data in any category $\mathcal{D}$ with pullbacks. Our interest will be in the case $\mathcal{D} = \mathbf{Top}$; in this case, we call $\mathcal{C}$ a "category in **Top**".

Let G be a topological group acting on a space X. We can again define XG, although it is now a category in **Top**. Explicitly, $(XG)_0 = X$ and $(XG)_1 = G \times X$ as spaces. The nerve of a topological category begets a simplicial space. In general, we will have

$$(N\mathcal{C})_n = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times \cdots \times_{\mathcal{C}_0} \mathcal{C}_1.$$

The geometric realization functor works in exactly the same way, so the realization of a simplicial space gets a topological space. The above discussion passes through with some mild topological conditions on G (namely, if G is an absolute neighborhood retract of a Lie group); we conclude:

Theorem 58.6. *Let G be an absolute neighborhood retract of a Lie group. Then EG is contractible, and G acts from the right principally. Moreover, the map $EG \to BG$ is the orbit projection.*

A generalization of this result is:

Exercise 58.7. Let X be a G-set. Show that

$$EG \times_G X \simeq B(XG).$$

59 Classifying spaces and bundles

Let $\pi : Y \to X$ be a map of spaces. This defines a "descent category" $\check{C}(\pi)$ whose objects are the points of Y, whose morphisms are points of $Y \times_X Y$, and whose structure morphisms are the obvious maps. Let cX denote the category whose objects and morphisms are both given by points of X, so that the nerve NcX is the constant simplicial object with value X. There is a functor $\check{C}(\pi) \to cX$ specified by the map π.

Let $\mathcal{U}$ be a cover of X. Let $\check{C}(\mathcal{U})$ denote the descent category associated to the obvious map $\epsilon : \coprod_{U\in\mathcal{U}} U \to X$. It is easy to see that $\epsilon : B\check{C}(\mathcal{U}) \simeq X$ if $\mathcal{U}$ is numerable. The morphism determined by $x \in U \cap V$ is denoted $x_{U,V}$. Suppose $p : P \to X$ is a principal G-bundle. Then $\mathcal{U}$ trivializes p if there are homeomorphisms $t_U : p^{-1}(U) \xrightarrow{\simeq} U \times G$ over U. Specifying such homeomorphisms is the same as a trivialization of the pullback bundle $\epsilon^* P$.

This, in turn, is the same as a functor $\theta_P : \check{C}(\mathcal{U}) \to G$. To see this, we note that the G-equivariant composite $t_V \circ t_U^{-1} : (U \cap V) \times G \to (U \cap V) \times G$ is determined by the value of $(x, 1) \in (U \cap V) \times G$. The map $U \cap V \to G$ is denoted $f_{U,V}$. Then, the functor $\theta_P : \check{C}(\mathcal{U}) \to G$ sends every object of $\check{C}(\mathcal{U})$ to the point, and $x_{U,V}$ to $f_{U,V}(x)$.

On classifying spaces, we therefore get a map $X \xleftarrow{\simeq} B\check{C}(\mathcal{U}) \xrightarrow{\theta_P} BG$, where the map on the left is given by ϵ.

Exercise 59.1. Prove that $\theta_P^* EG \simeq \epsilon^* P$.

This suggests that BG is a classifying space for principal G-bundles (in the sense of §55). To make this precise, we need to prove that two principal G-bundles are isomorphic if and only if the associated maps $X \to BG$ are homotopic.

To prove this, we will need to vary the open cover. Say that $\mathcal{V}$ *refines* $\mathcal{U}$ if for any $V \in \mathcal{U}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A *refinement* is a function $p : \mathcal{V} \to \mathcal{U}$ such that $V \subseteq p(V)$. A refinement p defines a map $\coprod_{V\in\mathcal{V}} V \to \coprod_{U\in\mathcal{U}} U$, denoted ρ.

As both $\coprod_{V\in\mathcal{V}} V$ and $\coprod_{U\in\mathcal{U}} U$ cover X, we get a map $\check{C}(\mathcal{V}) \to \check{C}(\mathcal{U})$ over cX. Taking classifying spaces begets a diagram:

$$\begin{tikzcd} B\check{C}(\mathcal{V}) \arrow[r] \arrow[dr] & B\check{C}(\mathcal{U}) \arrow[d] \\ & X \end{tikzcd}$$

Let t be trivialization of P for the open cover $\mathcal{U}$. The construction described above begets a functor $B\check{C}(\mathcal{U}) \to BG$, so we get a trivialization s for $\mathcal{V}$. This is a homeomorphism $s_V : p^{-1}(V) \to V \times G$ which fits into the following diagram:

$$\begin{tikzcd} p^{-1}(V) \arrow[r, "s_V", "\sim"'] \arrow[d] & V \times G \arrow[d] \\ p^{-1}(\rho(V)) \arrow[r, "\sim", "t_{\rho(V)}"'] & \rho(V) \times G \end{tikzcd}$$

By construction, there is a large commutative diagram:

$$\begin{tikzcd}
B\check{C}(\mathcal{V}) \arrow[r] \arrow[rr, bend left] \arrow[dr, "\sim"'] & B\check{C}(\mathcal{U}) \arrow[r] \arrow[d, "\sim"] & BG \\
& X. &
\end{tikzcd}$$

This justifies dropping the symbol $\mathcal{U}$ in the notation for the map θ_P.

Consider two principal G-bundles over X:

$$\begin{tikzcd}
P \arrow[r, "\simeq"] \arrow[dr] & Q \arrow[d] \\
& X,
\end{tikzcd}$$

and suppose I have trivializations $(\mathcal{U}, t)$ of P and $(\mathcal{W}, s)$ of Q. Let $\mathcal{V}$ be a common refinement, so that there is a diagram:

$$\begin{tikzcd}
& \check{C}(\mathcal{U}) \arrow[dr, "\theta_P^{\mathcal{U}}"] & \\
\check{C}(\mathcal{V}) \arrow[ur] \arrow[rr, bend left=10, "\theta_P^{\mathcal{V}}"] \arrow[rr, bend right=10, "\theta_Q^{\mathcal{V}}"'] \arrow[dr] & & G \\
& \check{C}(\mathcal{W}) \arrow[ur, "\theta_Q^{\mathcal{W}}"'] &
\end{tikzcd}$$

Included in the diagram is a mysterious natural transformation $\beta : \theta_P^{\mathcal{V}} \to \theta_Q^{\mathcal{V}}$, whose construction is left as an exercise to the reader. Its existence combined with Lemma 57.4 implies that the two maps $\theta_P, \theta_Q : B\check{C}(\mathcal{V}) \simeq X \to BG$ are homotopic, as desired.

Should we describe this? It's rather technical...

Topological properties of BG

Before proceeding, let us summarize the constructions discussed so far. Let G be some topological group (assumed to be an absolute neighborhood retract of a Lie group). We constructed EG, which is a contractible space with G acting freely on the right (this works for any topological group). There is an orbit projection $EG \to BG$, which is a principal G-bundle under our assumption on G. The space BG is universal, in the sense that there is a bijection

$$\mathrm{Bun}_G(X) \xleftarrow{\simeq} [X, BG]$$

given by $f \mapsto [f^* EG]$.

Let E be a space such that G acts on E from the left. If $P \to B$ is any principal G-bundle, then $P \times E \to P \times_G E$ is another principal G-bundle. In the case $P = EG$, it follows that if E is a contractible space on which G acts, then the quotient $EG \times_G E$ is a model for BG. Recall that EG is contractible. Therefore, if E is a contractible space on which

which G acts freely, then the quotient $G\backslash E$ is a model for BG. Of course, one can run the same argument in the case that G acts on E from the right. Although the construction with simplicial sets provided us with a very concrete description of the classifying space of a group G, we could have chosen any principal action on a contractible space in order to obtain a model for BG.

Suppose X is a pointed path connected space. Remember that X has a contractible path space $PX = X^I_*$. The canonical map $PX \to X$ is a fibration, with fiber ΩX.

Consider the case when $X = BG$. Then, we can compare the above fibration with the fiber bundle $EG \to BG$:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \Omega BG \arrow[d] \\ * \simeq EG \arrow[r, dashed] \arrow[d] & PBG \simeq * \arrow[d] \\ BG \arrow[r, equal] & BG \end{tikzcd}$$

The map $EG \to BG$ is nullhomotopic; a choice of a nullhomotopy is exactly a lift into the path space. Therefore, the dotted map $EG \to PBG$ exists in the above diagram. As EG and PBG are both contractible, we conclude that ΩBG is weakly equivalent to G. In fact, this weak equivalence is a H-map, i.e., it commutes up to homotopy with the multiplication on both sides.

Remark 59.2 (Milnor)**.** If X is a countable CW-complex, then ΩX is not a CW-complex, but it is *homotopy* equivalent (not just weakly equivalent) to one. Moreover, ΩX is weakly equivalent to a topological group GX such that $BGX \simeq X$.

Examples

We claim that $BU(n) \simeq \mathrm{Gr}_n(\mathbf{C}^\infty)$. To see this, let $V_n(\mathbf{C}^\infty)$ is the contractible space of complex n-frames in $\mathbf{C}^\infty$, i.e., isometric embeddings of $\mathcal{C}^n$ into $\mathcal{C}^\infty$. The Lie group $U(n)$ acts principally on $V_n(\mathbf{C}^\infty)$ by precomposition, and the quotient $V_n(\mathbf{C}^\infty)/U(n)$ is exactly the Grassmannian $\mathrm{Gr}_n(\mathbf{C}^\infty)$. As $\mathrm{Gr}_n(\mathbf{C}^\infty)$ is the quotient of a principal action of $U(n)$ on a contractible space, our discussion in the previous section implies the desired claim.

Let G be a compact Lie group (eg finite).

Theorem 59.3 (Peter-Weyl)**.** *There exists an embedding $G \hookrightarrow U(n)$ for some n.*

Since $U(n)$ acts principally on $V_n(\mathbf{C}^\infty)$, it follows G also acts principally on $V_n(\mathcal{C}^\infty)$. Therefore $V_n(\mathcal{C}^\infty)/G$ is a model for BG. It is not necessarily that this the most economic description of BG.

For instance, in the case of the symmetric group Σ_n, we have a much nicer geometric description of the classifying space. Let $\mathrm{Conf}_n(\mathbf{R}^k)$ denote embeddings of $\{1, \cdots, n\} \to \mathbf{R}^k$ (ordered distinct n-tuples). This space is definitely *not* contractible! However, the classifying space $\mathrm{Conf}_n(\mathbf{R}^\infty)$ is contractible. The symmetric group obviously acts freely on this (for finite groups, a principal action is the same as a free action). It follows that $B\Sigma_n$ is the space of *un*ordered configurations of n distinct points in $\mathbf{R}^\infty$. Using Cayley's theorem from classical group theory, we find that if G is finite, a model for BG is the quotient $\mathrm{Conf}_n(\mathbf{R}^\infty)/G$.

We conclude this chapter with a construction of Eilenberg-Maclane spaces via classifying spaces. If A is a topological abelian group, then the multiplication $\mu : A \times A \to A$ is a homomorphism. Applying the classifying space functor begets a map $m : BA \times BA \to BA$. If G is a finite group, then $BA = K(A, 1)$. The map m above gives a topological abelian group model for $K(A, 1)$. There is nothing preventing us from iterating this construction: the space $B^2 A$ sits in a fibration

$$BA \to EBA \simeq * \to B^2 A.$$

It follows from the long exact sequence in homotopy that the homotopy groups of $B^2 A$ are the same as that of BA, but shifted up by one. Repeating this procedure multiple times gives us an explicit model for $K(A, n)$:

$$B^n A = K(A, n).$$

Chapter 5

Spectral sequences

> Spectral sequences are one of those things for which anybody who is anybody must suffer through. Once you've done that, it's like linear algebra. You stop thinking so much about the 'inner workings' later.
>
> – Haynes Miller

60 The spectral sequence of a filtered complex

Our goal will be to describe a method for computing the homology of a chain complex. We will approach this problem by assuming that our chain complex is equipped with a filtration; then we will discuss how to compute the associated graded of an induced filtration on the homology, given the homology of the associated graded of the filtration on our chain complex.

We will start off with a definition.

Definition 60.1. A *filtered chain complex* is a chain complex C_* along with a sequence of subcomplexes F_sC_* such that the group C_n has a filtration by

$$F_0C_n \subset F_1C_n \subseteq \cdots,$$

such that $\bigcup F_sC_n = C_n$.

The differential on C_* begets the structure of a chain complex on the associated graded $\mathrm{gr}_s C_n = F_sC_n/F_{s-1}C_n$; in other words, the differential on C_* respects the filtration, hence begets a differential $d : \mathrm{gr}_s C_n \to \mathrm{gr}_s C_{n-1}$.

The canonical example of a filtered chain complex to keep in mind is the homology of a filtered space (such as a CW-complex). Let X be a filtered space, i.e., a space equipped with a filtration $X_0 \subseteq X_1 \subseteq \cdots$ such that $\bigcup X_n = X$. We then have a filtration of the chain complex $C_*(X)$ by the subcomplexes $C_*(X_n)$.

For ease of notation, let us write

$$E^0_{s,t} = \mathrm{gr}_s C_{s+t} = F_sC_{s+t}/F_{s-1}C_{s+t},$$

so the differential on C_* gives a differential $d^0 : E^0_{s,t} \to E^0_{s,t-1}$. A first approximation to the homology of C_* might therefore be the homology $H_{s+t}(\mathrm{gr}_s C_*)$. We will denote this group by $E^1_{s,t}$. This is the homology of the associated graded of the filtration F_*C_*.

We can get an even better approximation to H_*C_* by noticing that there is a differential even on $E^1_{s,t}$. By construction, there is a short exact sequence of chain complexes

$$0 \to F_{s-1}C_* \to F_sC_* \to \mathrm{gr}_s C_* \to 0,$$

so we get a long exact sequence in homology. The differential on $E^1_{s,t}$ is the composite of the boundary map in this long exact sequence with the natural map $H_*(F_{s-1}C_*) \to H_*(\mathrm{gr}_{s-1}C_*)$; more precisely, it is the composite

$$d^1 : E^1_{s,t} = H_{s+t}(\mathrm{gr}_s C_*) \xrightarrow{\partial} H_{s+t-1}(F_{s-1}C_*) \to H_{s+t-1}(\mathrm{gr}_{s-1}C_*) = E^1_{s-1,t}.$$

It is easy to check that $(d^1)^2 = 0$.

This construction is already familiar from cellular chains: in this case, $E^1_{s,t}$ is exactly $H_{s+t}(X_s, X_{s-1})$, which is exactly the cellular s-chains when $t = 0$ (and is 0 if $t \neq 0$). The d^1 differential is constructed in exactly the same way as the differential on cellular chains.

In light of this, we define $E^2_{s,t}$ to be the homology of the chain complex $(E^1_{*,*}, d^1)$; explicitly, we let

$$E^2_{s,t} = \ker(d^1 : E^1_{s,t} \to E^1_{s-1,t}) / \operatorname{im}(d^1 : E^1_{s+1,t} \to E^1_{s,t}).$$

Does this also have a differential d^2? The answer is yes. We will inductively define $E^r_{s,t}$ via a similar formula: if $E^{r-1}_{*,*}$ and the differential $d^{r-1} : E^{r-1}_{s,t} \to E^{r-1}_{s-r+1,t+r-2}$ are both defined, we set

$$E^r_{s,t} = \ker(d^{r-1} : E^{r-1}_{s,t} \to E^{r-1}_{s-r+1,t+r-2}) / \operatorname{im}(d^{r-1} : E^{r-1}_{s+r-1,t-r+2} \to E^{r-1}_{s,t}).$$

The differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ is defined as follows. Let $[x] \in E^r_{s,t}$ be represented by an element of $x \in E^1_{s,t}$, i.e., an element of $H_{s+t}(\mathrm{gr}_s C_*)$. As above, the boundary map induces natural maps $\partial : H_{s+t}(\mathrm{gr}_s C_*) \to H_{s+t-1}(F_{s-1}C_*)$ and $\partial : H_{s+t-1}(F_{s-r}C_*) \to H_{s+t-1}(\mathrm{gr}_{s-r}C_*)$. The element $\partial x \in H_{s+t-1}(F_{s-1}C_*)$ in fact lifts to an element of $H_{s+t-1}(F_{s-r}C_*)$. The image of this element under ∂ inside $H_{s+t-1}(\mathrm{gr}_{s-r}C_*) = E^1_{s-r,t+r-1}$ begets a class in $E^r_{s-r,t+r-1}$; this is the desired differential.

Exercise 60.2. Fill in the missing details in this construction of d^r, and show that $(d^r)^2 = 0$.

We have proven most of the statements in the following theorem.

Theorem-Definition 60.3. *Let F_*C be a filtered complex. Then there exist natural*

1. *bigraded groups $(E^r_{s,t})_{s\geq 0, t\in \mathbf{Z}}$ for any $r \geq 0$, and*
2. *differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ for any $r \geq 0$.*

such that $E^{r+1}_{s,t}$ is the homology of $(E^r_{*,*}, d^r)$, and (E^0, d^0) and (E^1, d^1) are as above. If F_*C is bounded below, then this spectral sequence converges to $\mathrm{gr}_* H_*(C)$*, in the sense that there is an isomorphism:*

$$E^\infty_{s,t} \simeq \mathrm{gr}_s H_{s+t}(C). \tag{5.1}$$

This is called a *homology spectral sequence*. One should think of each $E^r_{*,*}$ as a "page", with lattice points $E^r_{s,t}$. We still need to describe the symbols used in the formula (5.1).

There is a filtration $F_s H_n(C) := \mathrm{im}(H_n(F_s C) \to H_n(C))$, and $\mathrm{gr}_s H_*(C)$ is the associated graded of this filtration. Taking formula (5.1) literally, we only obtain information about the associated graded of the homology of C_*. Over vector spaces, this is sufficient to determine the homology of C_*, but in general, one needs to solve an extension problem.

To define the notation E^∞ used above, let us assume that the filtration F_*C is bounded below (so $F_{-1}C = 0$). It follows that $E^0_{s,t} = F_s C_{s+t}/F_{s-1}C_{s+t} = 0$ for $s < 0$, so the spectral sequence of Theorem-Definition 60.3 is a "right half plane" spectral sequence. It follows that in our example, the differentials from the group in position (s,t) must have vanishing d^{s+1} differential.

In turn, this implies that there is a surjection $E^{s+1}_{s,t} \to E^{s+2}_{s,t}$. This continues: we get surjections

$$E^{s+1}_{s,t} \to E^{s+2}_{s,t} \to E^{s+3}_{s,t} \to \cdots,$$

and the direct limit of this directed system is defined to be $E^\infty_{s,t}$.

For instance, in the case of cellular chains, we argued above that $E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$, so that $E^1_{s,t} = 0$ if $t \neq 0$, and the d^1 differential is just the differential in the cellular chain complex. It follows that $E^2_{s,t} = H^{cell}_s(X)$ if $t = 0$, and is 0 if $t \neq 0$. All higher differentials are therefore zero (because either the target or the source is zero!), so $E^r_{s,t} = E^2_{s,t}$ for every $r \geq 2$. In particular $E^\infty_{s,t} = H^{cell}_s(X)$ when $t = 0$, and is 0 if $t \neq 0$. There are no extension problems either: the filtration on X is bounded below, so Theorem-Definition 60.3 implies that $\mathrm{gr}_s H_{s+0}(X) = H_s(X) \simeq H^{cell}_s(X) = E^\infty_{s,t}$.

In a very precise sense, the datum of the spectral sequence of a filtered complex F_*C_* determines the homology of C_*:

Corollary 60.4. *Let $C \xrightarrow{f} D$ be a map of filtered complexes. Assume that the filtration on C and D are bounded below and exhaustive. Assume also that $E^r(f)$ is an isomorphism for some r. Then $f_* : H_*(C) \to H_*(D)$ is an isomorphism.*

Proof. The map $E^r(f)$ is an isomorphism which is also also a chain map, i.e., it is compatible with the differential d^r. It follows that $E^{r+1}(f)$ is an isomorphism. By induction, we conclude that $E^\infty_{s,t}(f)$ is an isomorphism for all s,t. Theorem-Definitino 60.3 implies that the map $\mathrm{gr}_s(f_*) : \mathrm{gr}_s H_*(C) \to \mathrm{gr}_s H(D)$ is an isomorphism.

We argue by induction using the short exact sequence:

$$0 \to F_s H_*(C) \to F_{s+1} H_*(C) \to \mathrm{gr}_{s+1} H_*(C) \to 0.$$

We have $\mathrm{gr}_0 H_n(C) = F_0 H_n(C) = \mathrm{im}(H_n(F_0 C) \to H_n(C))$, so the base case follows from the five lemma. In general, f induces an isomorphism an isomorphism on the groups on the left (by the inductive hypothesis) and right (by the above discussion), so it follows that $F_s f_*$ is an isomorphism by the five lemma. Since the filtration $F_* C_*$ was exhaustive, it follows that f_* is an isomorphism. □

Serre spectral sequence

In this book, we will give two constructions of the Serre spectral sequence. The second will appear later. Fix a fibration $E \xrightarrow{p} B$, with B a CW-complex. We obtain a filtration on E by taking the preimage of the s-skeleton of B, i.e., $E_s = p^{-1}\mathrm{sk}_s B$. It follows that there is a filtration on $S_*(E)$ given by

$$F_s S_*(E) = \mathrm{im}(S_*(p^{-1}\mathrm{sk}_s(B)) \to S_* E).$$

This filtration is bounded below and exhaustive. The resulting spectral sequence of Theorem-Definition 60.3 is the Serre spectral sequence.

Let us be more explicit. We have a pushout square:

$$\begin{tikzcd}
E_{s-1} \arrow[r] \arrow[d] & E_s \arrow[d] \\
B_{s-1} \arrow[r] & B_s \\
\coprod_{\alpha \in \Sigma_s} S^{s-1}_\alpha \arrow[r] \arrow[u] & \coprod_{\alpha \in \Sigma_s} D^s_\alpha \arrow[u]
\end{tikzcd}$$

Let F_α be the preimage of the center of α cell. In particular, we have a pushout:

$$\begin{tikzcd}
E_{s-1} \arrow[r] & E_s \\
\coprod_{\alpha \in \Sigma_s} S^{s-1}_\alpha \times F_\alpha \arrow[r] \arrow[u] & \coprod_{\alpha \in \Sigma_s} D^s_\alpha \times F_\alpha \arrow[u]
\end{tikzcd}$$

We know that

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) = \bigoplus_{\alpha \in \Sigma_s} H_{s+t}(D^s_\alpha \times F_\alpha, S^{s-1}_\alpha \times F_\alpha).$$

We can suggestively view this as $\bigoplus_{\alpha \in \Sigma_s} H_{s+1}((D^s_\alpha, S^{s-1}_\alpha) \times F_\alpha)$. By the Künneth formula (at least, if our coefficients are in a field), this is exactly $\bigoplus_{\alpha \in \Sigma_s} H_t(F_\alpha)$. In analogy with our discussion above regarding the spectral sequence coming from the cellular chain complex, one would like to think of this as "$C_s(B; H_t(F_\alpha))$". Sadly, there are many things wrong with writing this.

For instance, suppose B isn't connected. The fibers F_α could have completely different homotopy types, so the symbol $C_s(B; H_t(F_\alpha))$ does not make any sense. Even if

B was path-connected, there would still be no canonical way to identify the fibers over different points. Instead, we obtain a functor $H_t(p^{-1}(-)) : \Pi_1(B) \to \mathbf{Ab}$, i.e., a "local coefficient system" on B. So, the right thing to say is "$E^2_{s,t} = H_s(B; \underline{H_t(\text{fiber})})$".

To define precisely what $H_s(B; \underline{H_t(\text{fiber})})$ means, let us pick a basepoint in B, and build the universal cover $\widetilde{B} \to B$. This has an action of $\pi_1(B,*)$, so we obtain an action of $\pi_1(B,*)$ on the chain complex $S_*(\widetilde{B})$. Said differently, $S_*(\widetilde{B})$ is a chain complex of right modules over $\mathbf{Z}[\pi_1(B)]$. If B is connected, a local coefficient system on B is the same thing as a (left) action of $\pi_1(B)$ on $H_t(p^{-1}(*))$. Then, we define a chain complex:

$$S_*(B; \underline{H_t(p^{-1}(*))}) = S_*(\widetilde{B}) \otimes_{\mathbf{Z}[\pi_1(B)]} H_t(p^{-1}(*));$$

the differential is induced by the $\mathbf{Z}[\pi_1(B)]$-equivariant differential on $S_*(\widetilde{B})$. Our discussion above implies that the homology of this chain complex is the E^2-page.

We will always be in the case where that local system is trivial, so that $H_*(B; \underline{H_*(p^{-1}(*))})$ is just $H_*(B; H_*(p^{-1}(*)))$. For instance, this is the case if $\pi_1(B)$ acts trivially on the fiber. In particular, this is the case if B is simply connected.

61 Exact couples

Let us begin with a conceptual discussion of exact couples. As a special case, we will recover the construction of the spectral sequence associated to a filtered chain complex (Theorem-Definition 60.3).

Definition 61.1. An *exact couple* is a diagram of (possilby (bi)graded) abelian groups

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which is exact at each joint.

As $jkjk = 0$, the map $E \xrightarrow{jk} E$ is a differential, denoted d. An exact couple determines a "derived couple":

$$\begin{tikzcd} A' \arrow[rr, "i'=i|_{\mathrm{im}\, i}"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd} \tag{5.2}$$

where $A' = \mathrm{im}(i)$ and $E' = H_*(E,d)$. Iterating this procedure, we get exact sequences

$$\begin{tikzcd} A^r \arrow[rr, "i_r"] & & A^r \arrow[dl, "j_r"] \\ & E^r \arrow[ul, "k_r"] & \end{tikzcd}$$

where the next exact couple is the derived couple of the preceding exact couple.

It remains to define the maps in the above diagram. Define $j'(ia) = ja$. *A priori*, it is not clear that this well-defined. For one, we need $[ja] \in E'$; for this, we must check that $dja = 0$, but $d = jk$, and $jkja = 0$ so this follows. We also need to check that j' is well-defined modulo boundaries. To see this, suppose $ia = 0$. We then need to know that ja is a boundary. But if $ia = 0$, then $a = ke$ for some e, so $ja = jke = de$, as desired.

Define $k' : H(E,d) \to \operatorname{im} i$ via $k'([e]) \mapsto ke$. As before, we need to check that this is well-defined. For instance, we have to check that $ke \in \operatorname{im} i$. Since $de = 0$ and $d = jk$, we learn that $jke = 0$. Thus ke is killed by j, and therefore, by exactness, is in the image of i. We also need to check that k' is independent of the choice of representative of the homology class. Say $e = de'$. Then $kd = kde' = kjke' = 0$.

Exercise 61.2. Check that these maps indeed make diagram (5.2) into an exact couple.

It follows that we obtain a spectral sequence, in the sense of Theorem-Definition 60.3.

Exercise 61.3. By construction,

$$A^r = \operatorname{im}(i^r|_A) = i^r A.$$

Show, by induction, that

$$E^r = \frac{k^{-1}(i^r A)}{j(\ker i^r)}$$

and that

$$i_r(a) = ia, \;\; j_r(i^r a) = [ja], \;\; k_r(e) = ke.$$

Intuitively: an element of E^1 will survive to E^r if its image in A^1 can be pulled back under i^{r-1}. The differential d^r is obtained by the homology class of the pushforward of this preimage via j to E^1.

Remark 61.4. In general, the groups in consideration will be bigraded. It is clear by construction that $\deg(i') = \deg(i)$, $\deg(k') = \deg(k)$, and $\deg(j') = \deg(j) - \deg(i)$. It follows by an easy inductive argument that

$$\deg(d^r) = \deg(j) + \deg(k) - (r-1)\deg(i).$$

The canonical example of an exact couple is that of a filtered complex; the resulting spectral sequence is precisely the spectral sequence of Theorem-Definition 60.3. If C_* is a filtered chain complex, we let $A_{s,t} = H_{s+t}(F_s C_*)$, and $E^1_{s,t} = E_{s,t} = H_{s+t}(\mathrm{gr}_s C_*)$. The exact couple is precisely that which arises from the long exact sequence in homology associated to the short exact sequence of chain complexes

$$0 \to F_{s-1} C_* \to F_s C_* \to \mathrm{gr}_s C_* \to 0.$$

Note that in this case, the exact couple is one of bigraded groups, so Remark 61.4 dictates the bidegrees of the differentials.

We will conclude this section with a brief discussion of the convergence of the spectral sequence constructed above. Assume that $i : A \to A$ satisfies the property that

$$\ker(i) \cap \bigcap i^r A = 0.$$

Let $\widetilde{A}$ be the colimit of the directed system

$$A \xrightarrow{i} A \xrightarrow{i} A \to \cdots$$

There is a natural filtration on $\widetilde{A}$. Let I denote the image of the map $A \to \widetilde{A}$; the kernel of this map is $\bigcup \ker(i^r)$. The groups $i^r I$ give an exhaustive filtration of $\widetilde{A}$, and the quotients $i^r I/i^{r+1} I$ are all isomorphic to I/iI (since i is an isomorphism on $\widetilde{A}$). Then we have an isomorphism

$$E^\infty \simeq I/iI. \tag{5.3}$$

Indeed, we know from Exercise 61.3 that

$$E^\infty \simeq \frac{k^{-1}\left(\bigcap i^r A\right)}{j\left(\bigcup \ker i^r\right)};$$

by our assumption on i, this is

$$\frac{\ker(k)}{j\left(\bigcup \ker i^r\right)} \simeq \frac{j(A)}{j\left(\bigcup \ker i^r\right)}.$$

But there is an isomorphism $A/iA \to j(A)$ which clearly sends $iA + \bigcup \ker i^r$ to $j\left(\bigcup \ker i^r\right)$. By our discussion above, $A/\bigcup \ker i^r \simeq I$, and $iA/\bigcup \ker i^r \simeq iI$. Modding out by iI on both sides, we get (5.3).

62 The homology of ΩS^n, and the Serre exact sequence

The goal of this section is to describe a computation of the homology of ΩS^n via the Serre spectral sequence, as well as describe a "degenerate" case of the Serre spectral sequence.

The homology of ΩS^n

Let us first consider the case $n = 1$. The space ΩS^1 is the base of a fibration $\Omega S^1 \to PS^1 \to S^1$. Comparing this to the fibration $\mathbf{Z} \to \mathbf{R} \to S^1$, we find that $\Omega S^1 \simeq \mathbf{Z}$. Equivalently, this follows from the discussion in §59 and the observation that $S^1 \simeq K(\mathbf{Z}, 1)$.

Having settled that case, let us now consider the case $n > 1$. Again, there is a fibration $\Omega S^n \to PS^n \to S^n$. In general, if $F \to E \to B$ is a fibration and the space F has torsion-free homology, we can (via the universal coefficients theorem) rewrite the E^2-page:

$$E^2_{s,t} = H_s(B; H_t(F)) \simeq H_s(B) \otimes H_t(F).$$

Since S^n has torsion-free homology, the Serre spectral sequence (see §60) runs:

$$E^2_{s,t} = H_s(S^n) \otimes H_t(\Omega S^n) \Rightarrow H_*(PS^n) = \mathbf{Z}.$$

Since $H_s(S^n)$ is concentrated in degrees 0 and n, we learn that E^2-page is concentrated in columns $s = 0, n$. For instance, if $n = 4$, then the E^2-page (without the differentials drawn in) looks like:

$$
H_*(\Omega S^4)\quad
\begin{array}{c|cccccc}
0 & H_5(\Omega S^n) & & & & H_5(\Omega S^n) & \\
0 & H_4(\Omega S^n) & & & & H_4(\Omega S^n) & \\
0 & H_3(\Omega S^n) & & & & H_3(\Omega S^n) & \\
0 & H_2(\Omega S^n) & & & & H_2(\Omega S^n) & \\
0 & H_1(\Omega S^n) & & & & H_1(\Omega S^n) & \\
0 & H_0(\Omega S^n) & & & & H_0(\Omega S^n) & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

$$H_*(S^4)$$

We know that $H_0(\Omega S^n) = \mathbf{Z}$. Since the target has homology concentrated in degree 0, we know that $E^2_{n,0}$ has to be killed. The only possibility is that it is hit by a differential, or that it supports a nonzero differential.

There are not very many possibilities for differentials in this spectral sequence. In fact, up until the E^n-page, there are no differentials (either the target or source of the differential is zero), so $E^2 \simeq E^3 \simeq \cdots \simeq E^n$. On the E^n-page, there is only one possibility for a differential: $d^n : E^2_{n,0} \to E^n_{0,n-1}$. This differential has to be a monomorphism because if it had anything in its kernel, that will be left over in the position. In our example above (with $n = 4$), we have

$$
\begin{tikzcd}
0 & H_5(\Omega S^n) & & & & H_5(\Omega S^n) & \\
0 & H_4(\Omega S^n) & & & & H_4(\Omega S^n) & \\
0 & H_3(\Omega S^n) & & & & H_3(\Omega S^n) & \\
0 & H_2(\Omega S^n) & & & & H_2(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
0 & H_1(\Omega S^n) & & & & H_1(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
0 & H_0(\Omega S^n) & & & & H_0(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{tikzcd}
$$

$H_*(\Omega S^4)$

$H_*(S^4)$

However, we still do not know the group $E^n_{0,n-1}$. If it is bigger than $\mathbf{Z}$, then d^n is not surjective. There can be no other differentials on the E^r-page for $r \geq n+1$ (because of sparsity), so the d^n differential is our last hope in killing everything in degree $(0, n-1)$. This means that d^n is an epimorphism. We find that $E^n_{0,n-1} = H_{n-1}(\Omega S^n) \simeq \mathbf{Z}$, and that d^n is an isomorphism.

We have now discovered that $H_{n-1}(\Omega S^n) \simeq \mathbf{Z}$ — but there is a lot more left in the E^2-page! For instance, we still have a $\mathbf{Z}$ in $E^n_{n,n-1}$. Because $H^*(PS^n)$ is concentrated in degree 0, this, too, must die! We are in exactly the same situation as before, so the same arguments show that the differential $d^n : E^n_{n,n-1} \to E^n_{0,2(n-1)}$ has to be an isomorphism. Iterating this argument, we find:

$$H_q(\Omega S^n) \simeq \begin{cases} \mathbf{Z} & \text{if } (n-1) | q \geq 0 \\ 0 & \text{else} \end{cases}$$

This is a great example of how useful spectral sequences can be.

Remark 62.1. The loops ΩX is an associative H-space. Thus, as is the case for any H-space, the homology $H_*(\Omega X; R)$ is a graded associative algebra. Recall that the suspension functor Σ is the left adjoint to the loops functor Ω, so there is a unit map $A \to \Omega\Sigma A$. This in turn begets a map $\widetilde{H}_*(A) \to H_*(\Omega\Sigma A)$.

Recall that the universal tensor algebra $\mathrm{Tens}(\widetilde{H}_*(A))$ is the free associative algebra on $\widetilde{H}_*(A)$. Explicitly:

$$\mathrm{Tens}(\widetilde{H}_*(A)) = \bigoplus_{n \geq 0} \widetilde{H}_*(A)^{\otimes n}.$$

In particular, by the universal property of $\mathrm{Tens}(\widetilde{H}_*(A))$, we get a map $\alpha : \mathrm{Tens}(\widetilde{H}_*(A)) \to H_*(\Omega\Sigma A)$.

Theorem 62.2 (Bott-Samelson)**.** *The map α is an isomorphism if R is a PID and $H_*(A)$ is torsion-free.*

For instance, if $A = S^{n-1}$ then $\Omega S^n = \Omega\Sigma A$. Theorem 62.2 then shows that

$$H_*(\Omega S^n) = \mathrm{Tens}(\widetilde{H}_*(S^{n-1})) = \langle 1, x, x^2, x^3, \cdots \rangle,$$

where $|x| = n-1$. It is a mistake to call this "polynomial", since if n is even, x is an odd class (in particular, x squares to zero by the Koszul sign rule).

Theorem 62.2 suggests thinking of $\Omega\Sigma A$ as the "free associative algebra" on A. Let us make this idea more precise.

Remark 62.3. The space ΩA is homotopy equivalent to a topological monoid $\Omega_M A$, called the *Moore loops* on A. This means that $\Omega_M A$ has a *strict* unit and is *strictly* associative (i.e., not just up to homotopy). Concretely,

$$\Omega_M A := \{(\ell, \omega) : \ell \in \mathbf{R}_{\geq 0}, \omega : [0, \ell] \to A, \omega(0) = * = \omega(\ell)\},$$

topologized as a subspace of the product. There is an identity class $1 \in \Omega_M A$, given by $1 = (0, c_*)$ where c_* is the constant loop at the basepoint $*$. The addition on this space is just given by concatenatation. In particular, the lengths get added; this overcomes the obstruction to ΩA not being strictly associative, so the Moore loops $\Omega_M A$ are indeed strictly associative. If the basepoint is nondegenerate, it is not hard to see that the inclusion $\Omega A \hookrightarrow \Omega_M A$ is a homotopy equivalence.

Given the space A, we can form the free monoid $\mathrm{FreeMon}(A)$. The elements of this space are just formal sequences of elements of A (with topology coming from the product topology), and the multiplication is given by juxtaposition. Let us adjoin the element $1 = *$. As with all free constructions, there is a map $A \to \mathrm{FreeMon}(A)$ which is universal in the sense that any map $A \to M$ to a monoid factors through $\mathrm{FreeMon}(A)$.

The unit $A \to \Omega\Sigma A$ is a map from A to a monoid, so we get a monoid map $\beta : \mathrm{FreeMon}(A) \to \Omega\Sigma A$.

Theorem 62.4 (James)**.** *The map $\beta : \mathrm{FreeMon}(A) \to \Omega\Sigma A$ is a weak equivalence if A is path-connected.*

The free monoid looks very much like the tensor product, as the following theorem of James shows.

Theorem 62.5 (James)**.** *Let $J(A) = \mathrm{FreeMon}(A)$. There is a splitting:*

$$\Sigma J(A) \simeq_w \Sigma \left(\bigvee_{n\geq 0} A^{\wedge n} \right).$$

Applying homology to the splitting of Theorem 62.5 shows that:

$$\widetilde{H}_*(J(A)) \simeq \bigoplus_{n\geq 0} \widetilde{H}_*(A^{\wedge n}).$$

Assume that our coefficients are in a PID, and that $\widetilde{H}_*(A)$ is torsion-free; then this is just $\bigoplus_{n\geq 0} \widetilde{H}_*(A)^{\otimes n}$. In particular, we recover our computation of $H_*(\Omega S^n)$ from these general facts.

The Serre exact sequence

Suppose $\pi : E \to B$ is a fibration over a path-connected base. Assume that $\widetilde{H}_s(B) = 0$ for $s < p$ where $p \geq 1$. Let $* \in B$ be a chosen basepoint. Denote by F the fiber $\pi^{-1}(*)$. Assume $\widetilde{H}_t(F) = 0$ for $t < q$, where $q \geq 1$. We would like to use the Serre spectral sequence to understand $H_*(E)$. As always, we will assume that $\pi_1(B)$ acts trivially on $H_*(F)$.

Recall that the Serre spectral sequence runs

$$E^2_{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E).$$

Our assumptions imply that $E^2_{0,0} = \mathbf{Z}$, and $E^2_{0,t} = 0$ for $t < q$. Moreover, $E^2_{s,0} = 0$ for $s < p$. In particular, $E^2_{0,q+t} = H_{q+t}(F)$ and $E^2_{p+k,0} = H_{p+k}(B)$ — the rest of the spectral sequence is mysterious.

By sparsity, the first possible differential is $d^p : H_p(B) \to H_{p-1}(F)$, and $d^{p+q} : H_{p+1}(B) \to H_p(F)$. In the mysterious zone, there are differentials that hit $E^2_{p,q}$.

Again by sparsity, the only differential is $d^s : E^s_{s,0} \to E^s_{0,s-1}$ for $s < p + q - 1$. This is called a *transgression*. It is the last possible differential which has a chance at being nonzero. This means that the cokernel of d^s is $E^\infty_{0,s-1}$. There is also a map $E^\infty_{s,0} \to E^s_{s,0}$. We obtain a mysterious composite

$$0 \to E^\infty_{s,0} \to E^s_{s,0} \simeq H_s(B) \xrightarrow{d^s} E^s_{0,s-1} \simeq H_{s-1}(F) \to E^\infty_{0,s-1} \to 0. \tag{5.4}$$

Let $n < p + q - 1$. Recall that $F_s H_n(E) = \mathrm{im}(H_*(\pi^{-1}(\mathrm{sk}_s(B))) \to H_*(E))$, so $F_0 H_n(E) = E^\infty_{0,n}$. Here, we are using the fact that $F_{-1}H_*(E) = 0$. In particular, there is a map $E^\infty_{0,n} \to H_n(E)$. By our hypotheses, there is only one other potentially nonzero filtration in this range of dimensions, so we have a short exact sequence:

$$0 \to F_0 H_n(E) = E^\infty_{0,n} \to H_n(E) \to E^\infty_{n,0} \to 0 \tag{5.5}$$

Splicing the short exact sequences (5.4) and (5.5), we obtain a long exact sequence:

$$H_{p+q-1}(F) \to \cdots \to H_n(F) \to H_n(E) \to H_n(B) \xrightarrow{\text{transgression}} H_{n-1}(F) \to H_{n-1}(E) \to \cdots$$

This is called the *Serre exact sequence*. In this range of dimensions, homology behaves like homotopy.

63 Edge homomorphisms, transgression

Recall the Serre spectral sequence for a fibration $F \to E \to B$ has E^2-page given by

$$E^2_{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E).$$

If B is path-connected, $\widetilde{H}_t(F) = 0$ for $t < q$, $\widetilde{H}_s(B) = 0$ for $s < p$, and $\pi_1(B)$ acts trivially on $H_*(F)$, we showed that there is a long exact sequence (the Serre exact sequence)

$$H_{p+q-1}(F) \xrightarrow{\bullet} H_{p+q-1}(E) \to H_{p+q-1}(B) \to H_{p+q-2}(F) \to \cdots \tag{5.6}$$

Let us attempt to describe the arrow marked by $\bullet$.

Let $(E^r_{p,q}, d^r)$ be any spectral sequence such that $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$; such a spectral sequence is called a *first quadrant* spectral sequence. The Serre spectral sequence is a first quadrant spectral sequence. In a first quadrant spectral sequence, the d^2-differential $d^2 : E^2_{0,t} \to E^2_{-2,t+1}$ is zero, since $E^2_{s,t}$ vanishes for $s < 0$. This means that $H_t(F) = H_0(B; H_t(F)) = E^2_{0,t}$ surjects onto $E^3_{0,t}$. Arguing similarly, this surjects onto $E^4_{0,t}$. Eventually, we find that $E^r_{0,t} \simeq E^{t+2}_{0,t}$ for $r \geq t+2$. In particular,

$$E^{t+2}_{0,t} \simeq E^{\infty}_{0,t} \simeq \mathrm{gr}_0 H_t(E) \simeq F_0 H_t(E),$$

which sits inside $H_t(E)$. The composite

$$E^2_{0,t} = H_t(F) \to E^3_{0,t} \to \cdots \to E^{t+2}_{0,t} \subseteq F_0 H_t(E) \to H_t(E)$$

is precisely the map $\bullet$! Such a map is known as an *edge homomorphism*.

The map $F \to E$ is the inclusion of the fiber; it induces a map $H_t(F) \to H_t(E)$ on homology. We claim that this agrees with $\bullet$. Recall that $F_0 H_t(E)$ is defined to be $\mathrm{im}(H_t(F_0 E) \to H_t(E))$. In the construction of the Serre spectral sequence, we declared that $F_0 E$ is exactly the preimage of the zero skeleton. Since B is simply connected, we find that $F_0 E$ is exactly the fiber F.

To conclude the proof of the claim, consider the following diagram:

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F \arrow[d] \\
F \arrow[r] \arrow[d] & E \arrow[d] \\
* \arrow[r, hook] & B
\end{tikzcd}$$

The naturality of the Serre spectral sequence implies that there is an induced map of spectral sequences. Tracing through the symbols, we find that this observation proves our claim.

The long exact sequence (5.6) also contains a map $H_s(E) \to H_s(B)$. The group $F_s H_s(E) = H_s(E)$ maps onto $\mathrm{gr}_s H_s(E) \simeq E^{\infty}_{s,0}$. If F is connected, then $H_s(B) = H_s(B; H_0(F)) = E^2_{s,0}$. Again, the d^2-differential $d^2 : E^2_{s+2,-1} \to E^2_{s,0}$ is trivial (since the source is zero). Since $E^3 = \ker d^2$, we have an injection $E^3_{s,0} \to E^2_{s,0}$. Repeating the same argument, we get injections

$$E^{\infty}_{s,0} = E^{s+1}_{s,0} \to \cdots \to E^2_{s,0} \to E^2_{s,0} = H_s(B).$$

Composing with the map $H_s(E) \to E^{\infty}_{s,0}$ gives the desired map $H_s(E) \to H_s(B)$ in the Serre exact sequence. This composite is also known as an edge homomorphism.

As above, this edge homomorphism is the map induced by $E \to B$. This can be proved by looking at the induced map of spectral sequences coming from the following map of fiber sequences:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d] \\ E \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

The topologically mysterious map is the boundary map $\partial : H_{p+q-1}(B) \to H_{p+q-2}(F)$. Such a map is called a *transgression*. Again, let $(E^r_{s,t}, d^r)$ be a first quadrant spectral sequence. In our case, $E^2_{n,0} = H_n(B)$, at least F is connected. As above, we have injections

$$i : E^n_{n,0} \to \cdots \to E^3_{n,0} \to E^2_{n,0} = H_n(B).$$

Similarly, we have surjections

$$s : E^2_{0,n-1} \to E^3_{0,n-1} \to \cdots \to E^n_{0,n-1}.$$

There is a differential $d^n : E^n_{n,0} \to E^n_{0,n-1}$. The transgression is defined as the *linear relation* (not a function!) $E^2_{n,0} \to E^2_{0,n-1}$ given by

$$x \mapsto i^{-1} d^n s^{-1}(x).$$

However, the reader should check that in our case, the transgression is indeed a well-defined function.

Topologically, what is the origin of the transgression? There is a map $H_n(E,F) \xrightarrow{\pi_*} H_n(B,*)$, as well as a boundary map $\partial : H_n(E,F) \to H_{n-1}(F)$. We claim that:

$$\operatorname{im}\pi_* = \operatorname{im}(E^n_{n,0} \to H_n(B) = E^2_{n,0}), \quad \partial \ker \pi_* = \ker(H_{n-1}(F) = E^2_{0,n-1} \to E^n_{0,n-1}).$$

Proof sketch. Let $x \in H_n(B)$. Represent it by a cycle $c \in Z_n(B)$. Lift it to a chain in the total space E. In general, this chain will not be a cycle (consider the Hopf fibration). The differentials record this boundary; let us recall the geometric construction of the differential. Saying that the class x survives to the E^n-page is the same as saying that we can find a lift to a chain σ in E, with $d\sigma \in S_{n-1}(F)$. Then $d^n(x)$ is represented by the class $[dc] \in H_{n-1}(F)$. This is precisely the trangression.

Informally, we lift something from $H_n(B)$ to $S_n(E)$; this is well-defined up to something in F. In particular, we get an element in $H_n(E,F)$. We send it, via ∂, to an element of $H_{n-1}(F)$ — and this is precisely the transgression. □

An example

We would like to compare the Serre exact sequence (5.6) with the homotopy exact sequence:

$$* \to \pi_{p+q-1}(F) \to \pi_{p+q-1}(E) \to \pi_{p+q-1}(B) \xrightarrow{\partial} \pi_{p+q-2}(F) \to \cdots$$

There are Hurewicz maps $\pi_{p+q-1}(X) \to H_{p+q-1}(X)$. We claim that there is a map of exact sequences between these two long exact sequences.

$$\begin{tikzcd}
H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[r, "\partial"'] & H_{p+q-2}(F) \arrow[r] & \cdots \\
\pi_{p+q-1}(E) \arrow[u, "h"'] \arrow[r, "\pi_*"'] & \pi_{p+q-1}(B) \arrow[u, "h"] \arrow[r] & \pi_{p+q-2}(F) \arrow[u, "h"] \arrow[r] & \cdots
\end{tikzcd}$$

The leftmost square commutes by naturality of Hurewicz. The commutativity of the righmost square is not immediately obvious. For this, let us draw in the explicit maps in the above diagram:

$$\begin{tikzcd}
 & & H_{p+q-1}(E,F) \arrow[dl] \arrow[dr] & & \\
H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[rr, "\partial"'] & & H_{p+q-2}(F) \arrow[r] & \cdots \\
\pi_{p+q-1}(E) \arrow[u, "h"'] \arrow[r, "\pi_*"'] \arrow[dr] & \pi_{p+q-1}(B) \arrow[u, "h"] \arrow[rr] & & \pi_{p+q-2}(F) \arrow[u, "h"] \arrow[r] & \cdots \\
 & \pi_{p+q-1}(E,F) \arrow[u, "s", "\cong"'] \arrow[uuur] \arrow[urr] & & &
\end{tikzcd}$$

The map marked s is an isomorphism (and provides the long arrow in the above diagram, which makes the square commute), since

$$\pi_n(E,F) = \pi_{n-1}(\mathrm{hofib}(F \to E)) = \pi_{n-1}(\Omega B) = \pi_n(B).$$

Let us now specialize to the case of the fibration

$$\Omega X \to PX \to X.$$

Assume that X is connected, and $* \in X$ is a chosen basepoint. Let $p \geq 2$, and suppose that $\widetilde{H}_s(X) = 0$ for $s < p$. Arguing as in §62, we learn that the Serre spectral sequence we know that the homology of ΩX begins in dimension $p-1$ since $PX \simeq *$, so $q = p-1$. Likewise, if we knew $\widetilde{H}_n(\Omega X) = 0$ for $n < p-1$, then the same argument shows that $\widetilde{H}_n(X) = 0$ for $n < p$.

A surprise gust: the Hurewicz theorem

The discussion above gives a proof of the Hurewicz theorem; this argument is due to Serre.

Theorem 63.1 (Hurewicz, Serre's proof)**.** *Let $p \geq 1$. Suppose X is a pointed space with $\pi_i(X) = 0$ for $i < p$. Then $\widetilde{H}_i(X) = 0$ for $i < p$ and $\pi_p(X)^{ab} \to H_p(X)$ is an isomorphism.*

Proof. Let us assume the case $p = 1$. This is classical: it is Poincaré's theorem. We will only use this result when X is a loop space, in which case the fundamental group is already abelian.

Let us prove this by induction, using the loop space fibration. By assumption, $\pi_i(\Omega X) = 0$ for $i < p-1$. By our inductive hypothesis, $\widetilde{H}_i(\Omega X) = 0$ for $i < p-1$, and $\pi_{p-1}(\Omega X) \xrightarrow{\simeq} H_{p-1}(\Omega X)$. By our discussion above, we learn that $\widetilde{H}_i(X) = 0$ for $i < p$. The Hurewicz map $\pi_p(X) \xrightarrow{h} H_p(X)$ fits into a commutative diagram:

$$\begin{tikzcd}
\pi_{p-1}(\Omega X) \arrow[r, "\simeq"] & H_{p-1}(\Omega X) \\
\pi_p(X) \arrow[u, "\simeq"] \arrow[r, "h"'] & H_p(X) \arrow[u, "\simeq"', "\text{transgression}"]
\end{tikzcd}$$

It follows from the Serre exact sequence that the transgression is an isomorphism. □

64 Serre classes

Definition 64.1. A class $\mathbf{C}$ of abelian groups is a *Serre class* if:

1. $0 \in \mathbf{C}$.

2. if I have a short exact sequence $0 \to A \to B \to C \to 0$, then $A \& C \in \mathbf{C}$ if and only if $B \in \mathbf{C}$.

Some consequences of this definition: a Serre class is closed under isomorphisms (easy). A Serre class is closed under subobjects and quotients, because there is a short exact equence

$$0 \to A \hookrightarrow B \to B/A \to 0.$$

Consider an exact sequence $A \to B \to C$ (not necessarily a *short* exact sequence). If $A, C \in \mathbf{C}$, then $B \in \mathbf{C}$ because we have a short exact sequence:

$$\begin{tikzcd}
 & & & \operatorname{coker} i \arrow[r] \arrow[d] & 0 \\
 & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[ur] & C & 0 \\
0 \arrow[r] & \ker p \arrow[ur] & & &
\end{tikzcd}$$

Some examples are in order.

Example 64.2. 1. $\mathbf{C} = \{0\}$, and $\mathbf{C}$ the class of all abelian groups.

2. Let $\mathbf{C}$ be the class of all torsion abelian groups. We need to check that $\mathbf{C}$ satisfies the second condition of Definition **??**. Consider a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

We need to show that B is torsion if A and C are torsion. To see this, let $b \in B$. Then $p(b)$ is killed by some integer n, so there exists $a \in A$ such that $i(a) = nb$. SInce A is torsion, it follows that b is torsion, too.

3. Let $\mathcal{P}$ be a set of primes. Define:

$$\mathbf{C}_{\mathcal{P}} = \{A : \text{if } p \notin \mathcal{P}, \text{ then } p : A \xrightarrow{\simeq} A, \text{ i.e., } A \text{ is a } \mathbf{Z}[1/p]\text{-module}\}$$

Let $\mathbf{Z}_{(\mathcal{P})} = \mathbf{Z}[1/p : p \notin \mathcal{P}] \subseteq \mathbf{Q}$.

For instance, if $\mathcal{P}$ is the set of all primes, then $\mathbf{C}_{\mathcal{P}}$ is the Serre class of all abelian groups. If $\mathcal{P}$ is the set of all primes other than ℓ, then $\mathbf{C}_{\mathcal{P}}$ is the Serre class consisting of all $\mathbf{Z}[1/\ell]$-modules. If $\mathcal{P} = \{\ell\}$, then $\mathbf{C}_{\{\ell\}} =: \mathbf{C}_\ell$ is the Serre class of all $\mathbf{Z}_{(\ell)}$-modules. If $\mathcal{P} = \varnothing$, then $\mathbf{C}_\varnothing$ is all rational vector spaces.

4. If $\mathbf{C}$ and $\mathbf{C}'$ are Serre classes, then so is $\mathbf{C} \cap \mathbf{C}'$. For instance, $\mathbf{C}_{\text{tors}} \cap \mathbf{C}_{\text{fg}}$ is the Serre class $\mathbf{C}_{\text{finite}}$. Likewise, $\mathbf{C}_p \cap \mathbf{C}_{\text{tors}}$ is the Serre class of all p-torsion abelian groups.

Here are some straightforward consequences of the definition:

1. If $C_\bullet$ is a chain complex, and $C_n \in \mathbf{C}$, then $H_n(C_\bullet) \in \mathbf{C}$.

2. Suppose F_*A is a filtration on an abelian group. If $A \in \mathbf{C}$, then $\text{gr}_n A \in \mathbf{C}$ for all n. If F_*A is finite and $\text{gr}_n A \in \mathbf{C}$ for all n, then $A \in \mathbf{C}$.

3. Suppose we have a spectral sequence $\{E_r\}$. If $E^2_{s,t} \in \mathbf{C}$, then $E^r_{s,t} \in \mathbf{C}$ for $r \geq 2$. It follows that if $\{E^r\}$ is a right half-plane spectral sequence, then $E^{s+1}_{s,t} \twoheadrightarrow E^{s+2}_{s,t} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{s,t} \in \mathbf{C}$.

Thus, if the spectral sequence comes from a filtered complex (which is bounded below, such that for all n there exists an s such that $F_s H_n(C) = H_n(C)$, i.e., the homology of the filtration stabilizes), then $E^\infty_{s,t} = \text{gr}_s H_{s+t}(C)$. This means that if the $E^2_{s,t} \in \mathbf{C}$ for all $s + t = n$, then $H_n(C) \in \mathbf{C}$.

To apply this to the Serre spectral sequence, we need an additional axiom for Definition 64.1:

2. if $A, B \in \mathbf{C}$, then so are $A \otimes B$ and $\text{Tor}_1(A, B)$.

All of the examples given above satisfy this additional axiom.

Terminology 64.3. $f : A \to B$ is said to be a $\mathbf{C}$-epimorphism if $\operatorname{coker} f \in \mathbf{C}$, a $\mathbf{C}$-monomorphism if $\ker f \in \mathbf{C}$, and a $\mathbf{C}$-isomorphism if it is a $\mathbf{C}$-epimorphism and a $\mathbf{C}$-monomorphism.

Proposition 64.4. *Let $\pi : E \to B$ be a fibration and B path connected, such that the fiber $F = \pi^{-1}(*)$ is path connected. Suppose $\pi_1(B)$ acts trivially on $H_*(F)$.*

Let $\mathbf{C}$ be a Serre class satisfiying Axiom 2. Let $s \geq 3$, and assume that $H_n(E) \in \mathbf{C}$ where $1 \leq n < s-1$ and $H_t(B) \in \mathbf{C}$ for $1 \leq t < s$. Then $H_t(F) \in \mathbf{C}$ for $1 \leq t < s-1$.

Proof. We will do the case $s = 3$, for starters. We're gonna want to relate the low-dimension homology of these groups. What can I say? We know that $H_0(E) = \mathbf{Z}$ since it's connected. I have $H_1(E) \to H_1(B)$, via π. This is one of the edge homomorphisms, and thus it surjects (no possibility for a differential coming in). I now have a map $H_1(F) \to H_1(E)$. But I have a possible $d^2 : H_2(B) \to H_1(F)$, which is a transgression that gives:

$$H_2(B) \xrightarrow{\partial} H_1(F) \to H_1(E) \to H_1(B) \to 0$$

Let me take a step back and say something general. You might be interested in knowing when something in $H_n(F)$ maps to zero in $H_n(E)$. I.e., what's the kernel of $H_n(F) \to H_n(E)$. The sseq gives an obstruction to being an isomorphism. The only way that something can be killed by $H_n(F) \to H_n(E)$ is described by:

$$\ker(H_n(F) \to H_n(E)) = \bigcup \left(\text{im of } d^r \text{ hitting } E^r_{0,n}\right)$$

You can also say what the cokernel is: it's whatever's left in $E^\infty_{s,t}$ with $s + t = n$. These obstruct $H_n(F) \to H_n(E)$ from being surjective.

In the same way, I can do this for the base. If I have a class in $H_n(E)$, that maps to $H_n(B)$, the question is: what's the image? Well, the only obstruction is the possibility is that the element in $H_n(B)$ supports a nonzero differential. Thus:

$$\mathrm{im}(H_n(E) \xrightarrow{\pi_*} H_n(B)) = \bigcap \left(\ker(d^r : E^r_{r,0} \to \cdots)\right)$$

Again, you can think of the sseq as giving obstructions. And also, the obstruction to that map being a monomorphism that might occur in lower filtration along the same total degree line.

Back to our argument. We had the low-dimensional exact sequence:

$$H_2(B) \xrightarrow{\partial} H_1(F) \to H_1(E) \to H_1(B) \to 0$$

Here $p = 3$, so we have $H_2(B) \in \mathbf{C}$ and $H_1(E) \in \mathbf{C}$. Thus $H_1(F) \in \mathbf{C}$. That's the only thing to check when $p = 3$.

Let's do one more case of this induction. What does this say? Now I'll do $p = 4$. We're interested in knowing if $E^2_{0,3} \in \mathbf{C}$. There are now two possible differentials! I have $H_2(F) = E^2_{0,2} \twoheadrightarrow E^3_{0,2}$. This quotient comes from $d^2 : E^2_{2,1} \to E^2_{0,2}$. Now, $d^3 : E^3_{3,0} \to E^3_{0,2}$ which gives a surjection $E^3_{0,2} \twoheadrightarrow E^4_{0,2} \simeq E^\infty_{0,2} \hookrightarrow H_2(E)$. Now, our assumptions were that $E^2_{2,1}, E^3_{3,0}, H_2(E) \in \mathbf{C}$. Thus $E^3_{0,2} \in \mathbf{C}$ and so $E^2_{0,2} = H_2(F) \in \mathbf{C}$. Ta-da! □

We're close to doing actual calculations, but I have to talk about the multiplicative structure on the Serre sseq first.

65 Mod C Hurewicz, Whitehead, cohomology spectral sequence

We had $\mathbf{C}_{fg}$ and $\mathbf{C}_{tors}$, and

$$\mathbf{C}_{\mathcal{P}} = \{A | \ell : A \xrightarrow{\simeq} A, \ell \notin \mathcal{P}\}, \quad \mathbf{C}_p = \mathbf{C}_{\{p\}}, \quad \mathbf{C}_{p'} = \mathbf{C}_{\text{not } p}$$

Another one is $\mathbf{C}_{p'} \cap \mathbf{C}_{tors}$, which consists of torsion groups such that p is an isomorphism on A. There is therefore no p-torsion, and it has only prime-to-p torsion. This is the same thing as saying that $A \otimes \mathbf{Z}_{(p)} = 0$.

Theorem 65.1 (Mod $\mathbf{C}$ Hurewicz). *Let X be simply connected and $\mathbf{C}$ a Serre class such that $A, B \in \mathbf{C}$ implies that $A \otimes B, \mathrm{Tor}_1(A, B) \in \mathbf{C}$ (this is axiom 2). Assume also that if $A \in \mathbf{C}$, then $H_j(K(A,1)) = H_j(BA) \in \mathbf{C}$ for all $j > 0$. (This is valid for all our examples, and is what is called Axiom 3.)*

Let $n \geq 1$. Then $\pi_i(X) \in \mathbf{C}$ for any $1 < i < n$ if and only if $H_i(X) \in \mathbf{C}$ for any $1 < i < n$, and $\pi_n(X) \to H_n(X)$ is a mod $\mathbf{C}$ isomorphism.

Example 65.2. For $1 < i < n$, the group $\widetilde{H}_i(X)$ is:

1. torsion;
2. finitely generated;
3. finite;
4. $- \otimes \mathbf{Z}_{(p)} = 0$

if and only if $\pi_i(X)$ for $1 < i < n$.

Proof. Look at $\Omega X \to PX \to X$. Then $\pi_1 \Omega X \in \mathbf{C}$. Look at Davis+Kirk. □

There's a Whitehead theorem that comes out of this, that I want to state for you.

Theorem 65.3 (Mod $\mathbf{C}$ Whitehead theorem). *Let $\mathbf{C}$ be a Serre class satisfying axioms 1, 2, 3, and:*

(2′) $A \in \mathbf{C}$ implies that $A \otimes B \in \mathbf{C}$ for any B.

This is satisfied for all our examples except $\mathbf{C}_{fg}$.

Suppose I have $f : X \to Y$ where X, Y are simply connected. Suppose $\pi_2(X) \to \pi_2(Y)$ is onto. Let $n \geq 2$. Then $\pi_i(X) \to \pi_i(Y)$ is a $\mathbf{C}$-isomorphism for $2 \leq i \leq n$ and is a $\mathbf{C}$-epimorphism for $i = n$, with the same statement for H_i.

These kind of theorems help us work locally at a prime, and that's super. You'll see this in the next assignment, which is mostly up on the web. You'll also see this in calculations which we'll start doing in a day or two.

Change of subject here. Today I'm going to say a lot of things for which I won't give a proof. I want to talk about cohomology sseq.

Cohomology sseq

We're building up this powerful tool using spectral sequences. We saw how powerful the cup product was, and that is what cohomology is good for. In cohomology, things get turned upside down:

Definition 65.4. A *decreasing filtration* of an object A is

$$A \supseteq \cdots \supseteq F^{-1}A \supseteq F^0 A \supseteq F^1 A \supseteq F^2 A \supseteq \cdots \supseteq 0$$

This is called "bounded above" if $F^0 A = A$. Write $\mathrm{gr}^s A = F^s A / F^{s+1} A$.

Example 65.5. Suppose X is a filtered space. So there's an increasing filtration $\varnothing = F_{-1}X \subseteq F_0 X \subseteq \cdots$. Let R be a commutative ring of coefficients. Then I have $S^*(X)$, where the differential goes up one degree. Define

$$F^s S^*(X) = \ker(S^*(X) \to S^*(F_{s-1}X))$$

For instance, $F^0 S^*(X) = S^*(X)$. Thus this is a bounded above decreasing filtration . My computer will run out of juice so TEXthis up later!

Example 65.6. Let $X = E \xrightarrow{\pi} B =$ CW-complex with $\pi_1(B)$ acting trivially on $H_t(F)$. Then $F_s E = \pi^{-1}(\mathrm{sk}_s B)$. Thus I get a filtration on $S^*(E)$, and

$$F^s H^*(X) = \ker(H^*(X) \to H^*(F_{s-1}X))$$

Doing everything the same as before, we get a *cohomology spectral sequence*. Here are some facts.

1. First, you have $E_r^{s,t}$ (note that indices got reversed). There's a differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$, so that the total degree of the differential is 1.

2. You discover that
$$E_2^{s,t} \simeq H^s(B; H^t(F))$$

3. and $E_\infty^{s,t} \simeq \mathrm{gr}^s H^{s+t}(E)$.

4.

66 A few examples, double complexes, Dress sseq

Way back in 905 I remember computing the cohomology ring of $\mathbf{CP}^n$ using Poincaré duality. Let's do it fresh using the fiber sequence

$$S^1 \to S^{2n+1} \to \mathbf{CP}^n$$

where S^1 acts on S^{2n+1}. Here we know the cohomology of the fiber and the total space, but not the cohomology of the base. Let's look at the cohomology sseq for this. Then

$$E_2^{s,t} = H^s(\mathbf{CP}^n; H^t(S^1)) \simeq H^s(\mathbf{CP}^n) \otimes H^t(S^1) Rightarrow H^{s+t}(S^{2n+1})$$

The isomorphism $H^s(\mathbf{CP}^n; H^t(S^1)) \simeq H^s(\mathbf{CP}^n) \otimes H^t(S^1)$ follows from the UCT.

We know at least that $\mathbf{CP}^n$ is simply connected by the lexseq of homotopy groups. I don't have to worry about local coefficients. Let's work with the case S^5. We know that $\mathbf{CP}^n$ is simply connected, so the one-dimensional cohomology is 0. The only way to kill $E_2^{0,1}$ is by sending it via d_2 to $E_2^{2,0}$. Is this map surjective? Yes, it's an isomorphism.

Now I'm going to give names to the generators of these things; see the below diagram. $E_2^{2,1}$ is in total degree 3 and so we have to get rid of it. I will compute d_2 on this via Leibniz:

$$d_2(xy) = (d_2 x)y - x d_2 y = (d_2 x)y = y^2$$

which gives (iterating the same computation):

$$\begin{tikzcd}
0 & \mathbf{Z}x \arrow[drr, "d_2"'] & 0 & \mathbf{Z}xy \arrow[drr, "d_2"'] & 0 & \mathbf{Z}xy^2 \\
0 & \mathbf{Z} & 0 & \mathbf{Z}y & 0 & \mathbf{Z}y^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

This continues until the end where you reach $\mathbf{Z}xy^{??}$ which is a permanent cycle since it lasts until the E_∞-page.

Another example: let C_m be the cyclic group of order m sitting inside S^1. How can we analyse $S^{2n+1}/C_m =: L$? This is the lens space. We have a map $S^{2n+1}/C_m \to S^{2n+1}/S^1 = \mathbf{CP}^n$. This is a fiber bundle whose fiber is S^1/C_m. The spectral sequence now runs:

$$E^2_{s,t} = H_s(\mathbf{CP}^n) \otimes H_t(S^1/C_m) \Rightarrow H_{s+t}(L)$$

We know the whole E^2 term now:

$$\begin{tikzcd}
0 & \mathbf{Z} & 0 & \mathbf{Z} & 0 & \mathbf{Z} \\
0 & \mathbf{Z} & 0 & \mathbf{Z} \arrow[ull, "m"'] & 0 & \mathbf{Z} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

In cohomology, we have something dual:

$$\begin{tikzcd}
0 & u \arrow[drr, "m"'] & 0 & uy \arrow[drr, "m"'] & 0 & uy^2 \\
0 & 1 & 0 & y & 0 & y^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

What's the ring structure? We get that $H^*(L) = \mathbf{Z}[y,v]/(my, y^{n+1}, yv, v^2)$ where $|v| = 2n+1$ and $|y| = 2$. By the way, when $m = 1$, this is $\mathbf{RP}^{2n+1}$. This is a computation of the cohomology of odd real projective spaces. Remember that odd projective spaces are orientable and you're seeing that here because you're picking up a free abelian group in the top dimension.

Double complexes

$A_{s,t}$ is a bigraded abelian group with $d_h : A_{s,t} \to A_{s-1,t}$ and $d_v : A_{s,t} \to A_{s,t-1}$ such that $d_v d_h = d_h d_v$. Assume that $\{(smt) : s+t = n, A_{s,t} \neq 0\}$ is finite for any n. Then

$$(tA)_n = \bigoplus_{s+t=n} A_{s,t}$$

Under this assumption, there's only finitely many nonzero terms. I like this personally because otherwise I'd have to decide between the direct sum and the direct product, so we're avoiding that here. It's supposed to be a chain complex. Here's the differential:

$$d(a_{s,t}) = d_h a_{s,t} + (-1)^s d_v a_{s,t}$$

Then $d^2 = 0$, as you can check.

Question 66.1. What is $H_*(tA_*)$?

Define a filtration as follows:

$$F_p(tA)_n = \bigoplus_{s+t=n, s\le p} A_{s,t} \subseteq (tA)_n$$

This kinda obviously gives a filtered complex. Let's compute the low pages of the sseq. What is $\mathrm{gr}_s(tA)$? Well

$$\mathrm{gr}_s(tA)_{s+t} = (F_s/F_{s-1})_{s+t} = A_{s,t}$$

This associated graded object has its own differential $\mathrm{gr}_s(tA)_{s+t} = A_{s,t} \xrightarrow{d_v} A_{s,t-1} = \mathrm{gr}_s(tA)_{s+t-1}$. Let $E^0_{s,t} = \mathrm{gr}_s(tA)_{s+t} = A_{s+t}$, so that $d^0 = d_v$. Then $E^1 = H(E^0_{s,t}, d^0) = H(A_{s,t}; d_v) =: H^v_{s,t}(A)$. So computing E^1 is ez. Well, what's d^1 then?

To compute d^1 I take a vertical cycle that and the differential decreases the ... by 1, so that d^1 is induced by d_h. This means that I can write $E^2_{s,t} = H^h_{s,t}(H^v(A))$.

Question 66.2. You can also do $'E^2_{s,t} = H^v_{s,t}(H^h(A))$, right?

Rather than do that, you can define the transposed double complex $A^\top_{t,s} = A_{s,t}$, and $d^\top_h(a_{s,t}) = (-1)^s d_v(a_{s,t})$ and $d^\top_v(a_{s,t}) = (-1)^t d_h a_{s,t}$. When I set the signs up like that, then

$$tA^\top \simeq tA$$

as complexes and not just as groups (because of those signs). Thus, you get a spectral sequence

$${}^\top E^2_{s,t} = H^v_{s,t}(H^h(A))$$

converging to the same thing. I'll reserve telling you about Dress' construction until Monday because I want to give a double complex example. It's not ... it's just a very clear piece of homological algebra.

Example 66.3 (UCT)**.** For this, suppose I have a (not necessarily commutative) ring R. Let C_* be a chain complex, bounded below of right R-modules, and let M be a left R-module. Then I get a new chain complex of abelian groups via $C_* \otimes_R M$. What is $H(C_* \otimes_R M)$? I'm thinking of M as some kind of coefficient. Let's assume that each C_n is projective, or at least flat, for all n.

Shall we do this?

Let $M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ be a projective resolution of M as a left R-module. Then $H_*(P_*) \xrightarrow{\simeq} M$. Form $C_* \otimes_R P_*$: you know how to do this! I'll define $A_{s,t}$ to be $C_s \otimes_R P_t$. It's got two differentials, and it's a double complex. Let's work out the two sseqs.

Firstly, let's take it like it stands and take homology wrt P first. I'm organizing it so that C is along the base and P is along the fiber. What is the vertical homology $H^v(A_{*,*})$? If the C are projective then tensoring with them is exact, so that $H^v(A_{s,*}) = C_s \otimes_R H_*(P_*)$, so that $E^1_{s,t} = H^v_{s,t}(A_{*,*}) = C_s \otimes M$ if $t = 0$ and 0 otherwise. The spectral sequence is concentrated in one row. Thus,

$$E^2_{s,t} = \begin{cases} H_s(C_* \otimes_R M) & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$$

This is canonically the same thing as $E^\infty_{s,0} \simeq H_s(tA)$.

Let me go just one step further here. The game is to look at the *other* spectral sequence, where I do horizontal homology first. Then $H^h(A_{*,*}) = H_t(C_*) \otimes P_s$ again because the P_* are projective. Thus,

$$E^2_{s,t} = H^v(H^h(A_{*,*})) = \mathrm{Tor}^R_s(H_t(C), M) \Rightarrow H_{s+t}(C_* \otimes_R M)$$

That's the *universal coefficients spectral sequence.*

What happens if R is a PID? Only two columns are nonzero, and $E^2_{0,n} = H_n(C) \otimes_R M$ and $E^2_{1,n-1} = \mathrm{Tor}_1(H_{n-1}(C), M)$. This exactly gives the universal coefficient exact sequence.

Later we'll use this stuff to talk about cohomology of classifying spaces and Grassmannians and Thom isomorphisms and so on.

67 Dress spectral sequence, Leray-Hirsch

I think I have to be doing something tomorrow, so no office hours then. The new pset is up, and there'll be one more problem up. There are two more things about spectral sequences, and specifically the multiplicative structure, that I have to tell you about. The construction of the Serre sseq isn't the one that we gave. He did stuff with simplicial homology, but as you painfully figured out, $\Delta^s \times \Delta^t$ isn't another simplex. Serre's solution was to not use simplices, but to use cubes. He defined a new kind of homology using the n-cube. It's more complicated and unpleasant, but he worked it out.

Dress' sseq

Dress made the following variation on this idea, which I think is rather beautiful. We have a trivial fiber bundle $\Delta^t \to \Delta^s \times \Delta^t \to \Delta^s$. Let's do with this what we did with homology in the first place. Dress started with some map $\pi : E \to B$ (not necessarily a fibration), and he thought about the set of maps from $\Delta^s \times \Delta^t \to \Delta^s$ to $\pi : E \to B$. This set is denoted $\mathrm{Sin}_{s,t}(\pi)$. This forgets down to $S_s(B)$. Altogether, this $\mathrm{Sin}_{*,*}(\pi)$ is a functor $\Delta^{op} \times \Delta^{op} \to \mathrm{Set}$, forming a "bisimplicial set".

The next thing we did was to take the free R-module, to get a bisimplicial R-module $R\mathrm{Sin}_{*,*}(\pi)$. We then passed to chain complexes by forming the alternating sum. We can do this in two directions here! (The s is horizontal and t is vertical.) This gives us a double complex. We now get a spectral sequence! I hope it doesn't come as a surprise that you can compute the horizontal – you can compute the vertical differential first, and then taking the horizontal differential gives the homology of B with coefficients in something. Oh actually, the totalization $tR\mathrm{Sin}_{*,*}(\pi) \simeq R\mathrm{Sin}_*(E) = S_*(E)$. We'll have

$$E^2_{s,t} = H_s(B; \text{crazy generalized coefficients}) \Rightarrow H_{s+t}(E)$$

These coefficients may not even be local since I didn't put any assumptions on π! This is like the "Leray" sseq, set up without sheaf theory. If π is a fibration, then those crazy generalized coefficients is the local system given by the homology of the fibers. This gives the Serre sseq.

This has the virtue of being completely natural. Another virtue is that I can form $\mathrm{Hom}(-, R)$, and this gives rise to a multiplicative double complex. Remember that the cochains on a space form a DGA, and that's where the cup product comes from. The same story puts a bigraded multiplication on this double complex, and that's true *on the nose*. That gives rise a multiplicative cohomology sseq.

This is very nice, but the only drawback is that the paper is in German. That was item one in my agenda.

Leray-Hirsch

This tells you condition under which you can compute the cohomology of a total space. Anyway. We'll see.

Let's suppose I have a fibration $\pi : E \to B$. For simplicity suppose that B is path connected, so that gives meaning to the fiber F which we'll also assume to be path-connected. All cohomology is with coefficients in a ring R. I have a sseq

$$E_2^{s,t} = H^s(B; \underline{H^t F}) \Rightarrow H^{s+t}(E)$$

If you want assume that $\pi_1(B)$ acts trivially so that that cohomology in local coefficients is just cohomology with coefficients in $H^* F$. I have an algebra map $\pi^* : H^*(B) \to H^*(E)$, making $H^*(E)$ into a module over $H^*(B)$. We have $E_2^{*,t} = H^*(B; H^t(F))$, and this is a $H^*(B)$-module. That's part of the multiplicative structure, since $E_2^{*,0} = H^* B$. This row acts on every other row by that module structure.

Everything in the bottom row is a permanent cycle, i.e., survives to the E_∞-page. In other words

$$H^*(B) = E_2^{*,0} \twoheadrightarrow E_3^{*,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{*,0}$$

Each one of these surjections is an algebra map.

What the multiplicative structure is telling us is that $E_r^{*,0}$ is a graded algebra acting on $E_r^{*,t}$. Thus, $E_\infty^{*,t}$ is a module for $H^*(B)$.

Really I should be saying that it's a module for $H^*(B; \underline{H^0(F)})$. Can I guarantee that the $\pi_1(B)$-action on F is trivial. We know that $F \to *$ induces an iso on H^0 (that's part of being path-connected). So if you have a fibration whose fiber is a point, there's no possibility for an action. This fibration looks the same as far as H^0 of the fiber is concerned. Thus the $\pi_1(B)$-action is trivial on $H^0(F)$, so saying that it's a $H^*(B)$-module is fine.

Where were we? We have module structures all over the place. In particular, we know that $H^*(E)$ is a module over $H^*(B)$ as we saw, and also $E_\infty^{*,t}$ is a $H^*(B)$-module. These better be compatible!

Define an increasing filtration on $H^*(E)$ via $F_t H^n(E) = F^{n-t}H^n(E)$. For instance, $F_0 H^n(E) = F^n H^n(E)$. What is that? In our picture, we have the associated quotients along the diagonal on $E_\infty^{s,t}$ given by $s+t = n$. In the end, since we know that $F^{n+1}H^n(E) = 0$, it follows that

$$F_0 H^n(E) = F^n H^n(E) = E_\infty^{n,0} = \operatorname{im}(\pi^* : H^n(B) \to H^n(E))$$

With respect to this filtration, we have

$$\operatorname{gr}_t H^*(E) = E_\infty^{*,t}$$

I learnt this idea from Dan Quillen. It's a great idea. This increasing filtration $F_* H^*(E)$ is a filtration by $H^*(B)$-modules, and $\operatorname{gr}_t H^*(E) = E_\infty^{*,t}$ is true as H^*B-modules. It's exhaustive and bounded below.

This is a great perspective. Let's use it for something. Let me give you the Leray-Hirsch theorem.

Theorem 67.1 (Leray-Hirsch). *Let $\pi : E \to B$.*

1. *Suppose B and F are path-connected.*

2. *Suppose that $H^t(F)$ is free[1] of finite rank as a R-module.*

3. *Also suppose that $H^*(E) \twoheadrightarrow H^*(F)$. That's a big assumption; it's dual is saying that the homology of the fiber injects into the homology of E. This is called "totally non-homologous to zero" – this is a great phrase, I don't know who invented it.*

Pick an R-linear surjection $\sigma : H^*(F) \to H^*(E)$; this defines a map $\overline{\sigma} : H^*(B) \otimes_R H^*(F) \to H^*(E)$ via $\overline{\sigma}(x \otimes y) = \pi^*(x) \cup \sigma(y)$. This is the $H^*(B)$-linear extension. Then $\overline{\sigma}$ is an isomorphism.

[1] Everything is coefficients in R

Remark 67.2. It's not natural since it depends on the choice of σ. It tells you that $H^*(E)$ is free as a $H^*(B)$-module. That's a good thing.

Proof. I'm going to use our Serre sseq

$$E_2^{s,t} = H^s(B; \underline{H^t F}) \Rightarrow H^{s+t}(E)$$

Our map $H^*(E) \to H^*(F)$ is an edge homomorphism in the sseq, which means that it factors as $H^*(E) \to E_2^{0,*} = H^0(B; \underline{H^*(F)}) \subseteq H^*(F)$. Since $H^*(E) \to H^*(F)$, we have $H^0(B; \underline{H^*(F)}) \simeq H^*(F)$. Thus the $\pi_1(B)$-action on F is trivial.

Question 67.3. What's this arrow $H^*(E) \to E_2^{0,*}$? We have a map $H^*(E) \to H^*(E)/F^1 = E_\infty^{0,*}$. This includes into $E_2^{0,*}$.

Now you know that the E_2-term is $H^s(B; H^t(F))$. By our assumption on $H^*(F)$, this is $H^s(B) \otimes_R H^t(F)$, as algebras. What do the differentials look like? I can't have differentials coming off of the fiber, because if I did then the restriction map to the fiber wouldn't be surjective, i.e., that $d_r|_{E_r^{0,\infty}} = 0$. The differentials on the base are of course zero. This proves that d_r is zero on every page by the algebra structure! This means that $E_\infty = E_2$, i.e., $E_\infty^{*,t} = H^*(B) \otimes H^t(F)$.

Now I can appeal to the filtration stuff that I was talking about, so that $E_\infty^{*,t} = \mathrm{gr}_t H^*(E)$. Let's filter $H^*(B) \otimes H^*(F)$ by the degree in $H^*(F)$, i.e., $F_q = \bigoplus_{t \le q} H^*(B) \otimes H^t(F)$. The map $\overline{\sigma} : H^*(B) \otimes H^*(F) \to H^*(E)$ is filtration preserving, and it's an isomorphism on the associated graded. This is the identification $H^*(B) \otimes H^t(F) = E_\infty^{*,t} = \mathrm{gr}_t H^*(E)$. Since the filtrations are exhaustive and bounded below, we conclude that $\overline{\sigma}$ itself is an isomorphism. □

68 Integration, Gysin, Euler, Thom

Today there's a talk by
 the one
 the only
 JEAN-PIERRE SERRE
 OK let's begin.

Umkehr

Let $\pi : E \to B$ be a fibration and suppose B is path-connected. Suppose the fiber has no cohomology above some dimension d. The Serre sseq has nothing above row d.

Let's look at $H^n(E)$. This happens along total degree n. We have this neat increasing filtration that I was talking about on Monday whose associated quotients are the rows in this thing. So I can divide out by it (i.e I divide out by $F_{d-1}H^n(E)$). Then I get

$$H^n(E) \twoheadrightarrow H^n(E)/F_{d-1}H^n(E) = E_\infty^{n-d,d} \rightarrowtail E_2^{n-d,d} = H^{n-d}(B; \underline{H^d(F)})$$

That's because on the E_2 page, at that spot, there's nothing hitting it, but there might be a differential hitting it. There it is; here's another edge homomorphism.

Remark 68.1. This is a *wrong-way map*, also known as an "umkehr" map. It's also called a *pushforward map*, or the *Gysin map*.

We know from the incomprehensible discussion that I was giving on Monday that this was a filtration of modules over $H^*(B)$, so that this map $H^n(E) \to H^{n-d}(B; \underline{H^d(F)})$ is a $H^*(B)$-module map.

Example 68.2. F is a compact connected d-manifold with a given R-orientation. Thus $H^d(F) \simeq R$, given by $x \mapsto \langle x, [F] \rangle$. There might some local cohomology there, but I do get a map $H^n(E; R) \to H^{n-d}(B; \overline{R})$. This is such a map, and it has a name: it's written $\pi_!$ or π_*. I'll write π_*.

Of course, if $\pi_1(B)$ fixes $[F] \in H_d(F; R)$, then $\underline{R}$-cohomology is R-cohomology. Thus our map is now $H^n(E; R) \to H^{n-d}(B; R)$. Sometimes it's also called a pushforward map. Note that we also get a projection formula

$$\pi_*(\pi^*(b) \cup e) = b \cup \pi_*(e)$$

where π^* is the pushforward, $e \in H^n(E)$ and $b \in H^s(B)$. Others call this Frobenius reciprocity.

Gysin

Suppose $H^*(F) = H^*(S^{n-1})$. In practice, $F \cong S^{n-1}$, or even $F \simeq S^{n-1}$. In that case, $\pi : E \to B$ is called a *spherical fibration* Then the spectral sequence is *even simpler*! It has only two nonzero rows!

Let's pick an orientation for S^{n-1}, to get an isomorphism $H^{n-1}(S^{n-1})$. Well the spectral sequence degenerates, and you get a long exact sequence

$$\cdots \to H^s(B) \xrightarrow{\pi^*} H^s(E) \xrightarrow{\pi_*} H^{s-n+1}(B; \underline{R}) \xrightarrow{d_n} H^{s+1}(B) \xrightarrow{\pi^*} H^{s+1}(E) \to \cdots$$

That's called the *Gysin sequence*[2]. Because everything is a module over $H^*(B)$, this is a lexseq of $H^*(B)$-modules.

Let me be a little more explicit. Suppose we have an orientation. We now have a differential $H^0(B) \to H^n(B)$. We have the constant function $1 \in H^0(B)$, and this maps to something in B. This is called the *Euler class*, and is denoted e.

Since d_n is a module homomorphism, we have $d_n(x) = d_n(1 \cdot x) = d_n(1) \cdot x = e \cdot x$ where x is in the cohomology of B. Thus our lexseq is of the form

$$\cdots \to H^s(B) \xrightarrow{\pi^*} H^s(E) \xrightarrow{\pi_*} H^{s-n+1}(B; \underline{R}) \xrightarrow{e \cdot -} H^{s+1}(B) \xrightarrow{\pi^*} H^{s+1}(E) \to \cdots$$

Some facts about the Euler class

Suppose $E \to B$ has a section $\sigma : B \to E$ (so that $\pi\sigma = 1_B$). So, if it came from a vector bundle, I'm asking that there's a nowhere vanishing cross-section of that vector bundle.

[2]pronounced Gee-sin

Let's apply cohomology, so that you get $\sigma^*\pi^* = 1_{H^*(B)}$. Thus π^* is monomorphic. In terms of the Gysin sequence, this means that $H^{s-n}(B) \xrightarrow{e\cdot -} H^s(B)$ is zero. But this implies that

$$\boxed{e = 0}$$

Thus, if you don't have a nonzero Euler class then you cannot have a section! If your Euler class is zero sometimes you can conclude that your bundle has a section, but that's a different story.

The Euler class of the tangent bundle of a manifold when paired with the fundamental class is the Euler characteristic. More precisely, if M is oriented connected compact n-manifold, then

$$\langle e(\tau_M), [M]\rangle = \chi(M)$$

That's why it's called the Euler class. (He didn't know about spectral sequences or cohomology.)

Time for Thom

This was done by Rene Thom. Let ξ be a n-plane bundle over X. I can look at $H^*(E(\xi), E(\xi) - \text{section})$. If I pick a metric, this is $H^*(D(\xi), S(\xi))$, where $D(\xi)$ is the disk bundle[3] and $S(\xi)$ is the sphere bundle. If there's no point-set annoyance, this is $\widetilde{H}^*(D(\xi)/S(\xi))$.

If X is a compact Hausdorff space, then ... The open disk bundle $D^0(\xi) \simeq E(\xi)$. This quotient $D(\xi)/S(\xi) = E(\xi)^+$ since you get the one-point compactification by embedding into a compact Hausdorff space ($D(\xi)$ here) and then quotienting by the complement (which is $S(\xi)$ here). This is called the *Thom space* of ξ. There are two notations: some people write $\mathrm{Th}(\xi)$, and some people (Atiyah started this) write X^ξ.

Example 68.3 (Dumb)**.** Suppose ξ is the zero vector bundle. Then your fibration is $\pi : X \to X$. What's the Thom space? The disk bundle is X, and the boundary of a disk is empty, so $\mathrm{Th}(0) = X^0 = X \sqcup *$.

The Thom space is a pointed space (corresponding to ∞ or the point which $S(\xi)$ is collapsed to).

I'd like to study its cohomology, because it's interesting. There's no other justification. Maybe I'll think of it as the relative cohomology.

So, guess what? We've developed sseqs and done cohomology. Anything else we'd like to do to groups and functors and things?

Let's make the spectral sequence relative!

[3] $D(\xi) = \{v \in E(\xi) : \|v\| \le 1\}$.

I have a path connected B, and I'll study:

$$\begin{tikzcd} F_0 \arrow[r, hook] \arrow[d] & F \arrow[d] \\ E_0 \arrow[r, hook] \arrow[d] & E \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Then if you sit patiently and work through things, we get

$$E_2^{s,t} = H^s(B; H^t(F, F_0)) \Rightarrow_s H^{s+t}(E, E_0)$$

Note that $\Rightarrow_s$ means that s determines the filtration.

Let's do this with the Thom space. We have $D(\xi) \xrightarrow{\simeq} X$. That isn't very interesting. In our case, we have an incredibly simple spectral sequence, where everything on the E_2-page is concentrated in row n. Thus the E_2 page *is* the cohomology of

$$\widetilde{H}^{s+n}(\mathrm{Th}(\xi)) = H^{s+n}(D(\xi), S(\xi)) \simeq H^s(B; \underline{R})$$

where $\underline{R} = \underline{H^n(D^n, S^{n-1})}$. This is a canonical isomorphism of $H^*(B)$-modules.

Suppose your vector bundle ξ is oriented, so that $\underline{R} = R$. Now, if $s = 0$, then I have $1 \in H^0(B)$. This gives $u \in H^n(\mathrm{Th}(\xi))$, which is called the *Thom class*.

The cohomology of B is a free module of rank one over $H^*(B)$, so that $H^*(\mathrm{Th}(\xi))$ is also a $H^*(B)$-module that is free of rank 1, generated by u.

Let me finish by saying one more thing. This is why the Thom space is interesting. Notice one more thing: there's a lexseq of a pair

$$\cdots \to \widetilde{H}^s(\mathrm{Th}(\xi)) \to H^s(D(\xi)) \to H^s(S(\xi)) \to \widetilde{H}^{s+1}(\mathrm{Th}(\xi)) \to \cdots$$

We have synonyms for these things:

$$\cdots \to H^{s-n}(X) \to H^s(X) \to H^s(S(\xi)) \to H^{s-n+1}(X) \to \cdots$$

And aha, this is is exactly the same form as the Gysin sequence. Except, oh my god, what have I done here?

Yeah, right! In the Gysin sequence, the map $H^{s-n}(X) \to H^s(X)$ was multiplication by the Euler class. The Thom class u maps to some $e' \in H^n(X)$ via $\widetilde{H}^n(D(\xi), S(\xi)) \to H^n(D(\xi)) \simeq H^n(X)$. And the map $H^{s-n}(X) \to H^s(X)$ is multiplication by e'. Guess what? This is the Gysin sequence.

You'll explore more in homework.

I'll talk about characteristic classes on Friday.

Chapter 6

Characteristic classes

69 Grothendieck's construction of Chern classes

Generalities on characteristic classes

We would like to apply algebraic techniques to study G-bundles on a space. Let A be an abelian group, and $n \geq 0$ an integer.

Definition 69.1. A *characteristic class* for principal G-bundles (with values in $H^n(-;A)$) is a natural transformation of functors **Top** → **Ab**:

$$\mathrm{Bun}_G(X) \xrightarrow{c} H^n(X;A)$$

Concretely: if $P \to Y$ is a principal G-bundle over a space X, and $f: X \to Y$ is a continuous map of spaces, then

$$c(f^*P) = f^*c(P).$$

The motivation behind this definition is that $\mathrm{Bun}_G(X)$ is still rather mysterious, but we have techniques (developed in the last section) to compute the cohomology groups $H^n(X;A)$. It follows by construction that if two bundles over X have two different characteristic classes, then they cannot be isomorphic. Often, we can use characteristic classes to distinguish a given bundle from the trivial bundle.

Example 69.2. The Euler class takes an oriented real n-plane vector bundle (with a chosen orientation) and produces an n-dimensional cohomology class $e: \mathrm{Vect}_n^{or}(X) = \mathrm{Bun}_{SO(n)}(X) \to H^n(X;\mathbf{Z})$. This is a characteristic class. To see this, we need to argue that if $\xi \downarrow X$ is a principal G-bundle, we can pull the Euler class back via $f: X \to Y$. The bundle $f^*\xi \downarrow Y$ has a orientation if ξ does, so it makes sense to even talk about the Euler class of $f^*\xi$. Since all of our constructions were natural, it follows that $e(f^*\xi) = f^*e(\xi)$.

Similarly, the mod 2 Euler class is $e_2: \mathrm{Vect}_n(X) = \mathrm{Bun}_{O(n)}(X) \to H^n(X;\mathbf{Z}/2\mathbf{Z})$ is another Euler class. Since everything has an orientation with respect to $\mathbf{Z}/2\mathbf{Z}$, the mod 2 Euler class is well-defined.

By our discussion in §58, we know that $\mathrm{Bun}_G(X) = [X, BG]$. Moreover, as we stated in Theorem 51.8, we know that $H^n(X; A) = [X, K(A,n)]$ (at least if X is a CW-complex). One moral reason for cohomology to be easier to compute is that the spaces $K(A,n)$ are infinite loop spaces (i.e., they can be delooped infinitely many times). It follows from the Yoneda lemma that characteristic classes are simply maps $BG \to K(A,n)$, i.e., elements of $H^n(BG; A)$.

Example 69.3. The Euler class e lives in $H^n(BSO(n); \mathbf{Z})$; in fact, it is $e(\xi)$, the Euler class of the universal oriented n-plane bundle over $BSO(n)$. A similar statement holds for $e_2 \in H^n(BO(n); \mathbf{Z}/2\mathbf{Z})$. For instance, if $n = 2$, then $SO(2) = S^1$. It follows that

$$BSO(2) = BS^1 = \mathbf{CP}^\infty.$$

We know that $H^*(\mathbf{CP}^\infty; \mathbf{Z}) = \mathbf{Z}[e]$ — it's the polynomial algebra on the "universal" Euler class! Similarly, $O(1) = \mathbf{Z}/2\mathbf{Z}$, so

$$BO(1) = B\mathbf{Z}/2 = \mathbf{RP}^\infty.$$

We know that $H^*(\mathbf{RP}^\infty; \mathbf{F}_2) = \mathbf{F}_2[e_2]$ — as above, it is the polynomial algebra over $\mathbf{Z}/2\mathbf{Z}$ on the "universal" mod 2 Euler class.

Chern classes

These are one of the most fundamental example of characteristic classes.

Theorem 69.4 (Chern classes)**.** *There is a unique family of characteristic classes for complex vector bundles that assigns to a complex n-plane bundle ξ over X the nth* Chern class *$c_k^{(n)}(\xi) \in H^{2k}(X; \mathbf{Z})$, such that:*

1. *$c_0^{(n)}(\xi) = 1$.*
2. *If ξ is a line bundle, then $c_1^{(1)}(\xi) = -e(\xi)$.*
3. *The* Whitney sum formula *holds: if ξ is a p-plane bundle and η is a q-plane bundle (and if $\xi \oplus \eta$ denotes the fiberwise direct sum), then*

$$c_k^{(p+q)}(\xi \oplus \eta) = \sum_{i+j=k} c_i^{(p)}(\xi) \cup c_j^{(q)}(\eta) \in H^{2k}(X; \mathbf{Z}).$$

Moreover, if ξ_n is the universal n-plane bundle, then

$$H^*(BU(n); \mathbf{Z}) \simeq \mathbf{Z}[c_1^{(n)}, \cdots, c_n^{(n)}],$$

where $c_k^{(n)} = c_k^{(n)}(\xi_n)$.

This result says that all characteristic classes for complex vector bundles are given by polynomials in the Chern classes because the cohomology of $BU(n)$ gives all the characteristic classes. It also says that there are no universal algebraic relations among the Chern classes: you can specify them independently.

Remark 69.5. The $(p+q)$-plane bundle $\xi_p \times \xi_q = \mathrm{pr}_1^*\xi_p \oplus \mathrm{pr}_2^*\xi_q$ over $BU(p)\times BU(q)$ is classified by a map $BU(p)\times BU(q) \xrightarrow{\mu} BU(p+q)$. The Whitney sum formula computes the effect of μ on cohomology:

$$\mu^*(c_k^{(n)}) = \sum_{i+j=k} c_i^{(p)} \times c_j^{(q)} \in H^{2k}(BU(p)\times BU(q)),$$

where, you'll recall,

$$x \times y := \mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y.$$

The Chern classes are "stable", in the following sense. Let ϵ be the trivial one-dimensional complex vector bundle, and let ξ be an n-dimensional vector bundle. What is $c_k^{(n+q)}(\xi \oplus \epsilon^q)$? For this, the Whitney sum formula is valuable.

The trivial bundle is characterized by the pullback:

$$\begin{tikzcd} X \times \mathbf{C}^n = n\epsilon \arrow[r] \arrow[d] & \mathbf{C}^n \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

By naturality, we find that if $k > 0$, then $c_k^{(n)}(n\epsilon) = 0$. The Whitney sum formula therefore implies that

$$c_k^{(n+q)}(\xi \oplus \epsilon^q) = c_k^{(n)}(\xi).$$

This phenomenon is called stability: the Chern class only depends on the "stable equivalence class" of the vector bundle (really, they are only defined on "K-theory", for those in the know). For this reason, we will drop the superscript on $c_k^{(n)}(\xi)$, and simply write $c_k(\xi)$.

Grothendieck's construction

Let ξ be an n-plane bundle. We can consider the vector bundle $\pi : \mathbf{P}(\xi) \to X$, the projectivization of ξ: an element of the fiber of $\mathbf{P}(\xi)$ over $x \in X$ is a line inside ξ_x, so the fibers are therefore all isomorphic to $\mathbf{CP}^{n-1}$.

Let us compute the cohomology of $\mathbf{P}(\xi)$. For this, the Serre spectral sequence will come in handy:

$$E_2^{s,t} = H^s(X; H^t(\mathbf{CP}^{n-1})) \Rightarrow H^{s+t}(\mathbf{P}(\xi)).$$

Remark 69.6. Why is the local coefficient system constant? The space X need not be simply connected, but $BU(n)$ is simply connected since $U(n)$ is simply connected. Consider the projectivization of the universal bundle $\xi_n \downarrow BU(n)$; pulling back via $f : X \to BU(n)$ gives the bundle $\pi : \mathbf{P}(\xi) \to X$. The map on fibers $H^*(\mathbf{P}(\xi_n)_{f(x)}) \to H^*(\mathbf{P}(\xi_n)_x)$ is an isomorphism which is equivariant with respect to the action of the fundamental group of $\pi_1(X)$ via the map $\pi_1(X) \to \pi_1(BU(n)) = 0$.

Because $H^*(\mathbf{CP}^{n-1})$ is torsion-free and finitely generated in each dimension, we know that

$$E_2^{s,t} \simeq H^s(X) \otimes H^t(\mathbf{CP}^{n-1}).$$

The spectral sequence collapses at E_2, i.e., that $E_2 \simeq E_\infty$, i.e., there are no differentials. We know that the E_2-page is generated as an algebra by elements in the cohomology of the fiber and elements in the cohomology of the base. Thus, it suffices to check that elements in the cohomology of the fiber survive to E_∞. We know that

$$E_2^{0,2t} = \mathbf{Z}\langle x^t\rangle, \text{ and } E_2^{0,2t+1} = 0,$$

where $x = e(\lambda)$ is the Euler class of the canonical line bundle $\lambda \downarrow \mathbf{CP}^{n-1}$.

In order for the Euler class to survive the spectral sequence, it suffices to come up with a two dimensional cohomology class in $\mathbf{P}(\xi)$ that restricts to the Euler class over $\mathbf{CP}^{n-1}$. We know that λ itself is the restriction of the tautologous line bundle over $\mathbf{CP}^\infty$. There is a tautologous line bundle $\lambda_\xi \downarrow \mathbf{P}(\xi)$, given by the tautologous line bundle on each fiber. Explicitly:

$$E(\lambda_\xi) = \{(\ell, y) \in \mathbf{P}(\xi) \times_X E(\xi) | y \in \ell \subseteq \xi_x\}.$$

Thus, x is the restriction $e(\lambda_\xi)|_{\text{fiber}}$ of the Euler class to the fiber. It follows that the class x survives to the E_∞-page.

Using the Leray-Hirsch theorem (Theorem 67.1), we conclude that

$$H^*(\mathbf{P}(\xi)) = H^*(X)\langle 1, e(\lambda_\xi), e(\lambda_\xi)^2, \cdots, e(\lambda_\xi)^{n-1}\rangle.$$

For simplicity, let us write $e = e(\lambda_\xi)$. Unforunately, we don't know what e^n is, although we do know that it is a linear combination of the e^k for $k < n$. In other words, we have a relation

$$e^n + c_1 e^{n-1} + \cdots + c_{n-1}e + c_n = 0,$$

where the c_k are elements of $H^{2k}(X)$. These are the Chern classes of ξ. By construction, they are unique!

To prove Theorem 69.4(2), note that when $n = 1$ the above equation reads

$$e + c_1 = 0,$$

as desired.

70 $H^*(BU(n))$, splitting principle

Theorem 69.4 claimed that the Chern classes, which we constructed in the previous section, generate the cohomology of BU as a polynomial algebra. Our goal in this section is to prove this result.

The cohomology of $BU(n)$

Recall that $BU(n)$ supports the universal principal $U(n)$-bundle $EU(n) \to BU(n)$. Given any left action of $U(n)$ on some space, we can form the associated fiber bundle. For instance, the associated bundle of the $U(n)$-action on $\mathbf{C}^n$ yields the universal line bundle ξ_n.

Likewise, the associated bundle of the action of $U(n)$ on $S^{2n-1} \subseteq \mathcal{C}^n$ is the unit sphere bundle $S(\xi_n)$, the unit sphere bundle. By construction, the fiber of the map $EU(n) \times_{U(n)} S^{2n-1} \to BU(n)$ is S^{2n-1}. Since

$$S^{2n-1} = U(n)/(1 \times U(n-1)),$$

we can write

$$EU(n) \times_{U(n)} S^{2n-1} \simeq EU(n) \times_{U(n)} (U(n)/U(n-1)) \simeq EU(n)/U(n-1) = BU(n-1).$$

In other words, $BU(n-1)$ is the unit sphere bundle of the tautologous line bundle over $BU(n)$. This begets a fiber bundle:

$$S^{2n-1} \to BU(n-1) \to BU(n),$$

which provides an inductive tool (via the Serre spectral sequence) for computing the homology of $BU(n)$. In §68, we observed that the Serre spectral sequence for a spherical fibration was completely described bythe Gysin sequence.

Recall that if B is oriented and $S^{2n-1} \to E \xrightarrow{\pi} B$ is a spherical bundle over B, then the Gysin sequence was a long exact sequence

$$\cdots \to H^{q-1}(E) \xrightarrow{\pi_*} H^{q-2n}(B) \xrightarrow{e\cdot} H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\pi_*} \cdots$$

Let us assume that the cohomology ring of E is polynomial and concentrated in even dimensions. For the base case of the induction, both these assumptions are satisfied (since $BU(0) = *$ and $BU(1) = \mathbf{CP}^\infty$).

These assumptions imply that if q is even, then the map π_* is zero. In particular, multiplication by $e|_{H^{\text{even}}(B)}$ (which we will also denote by e) is injective, i.e., e is a nonzero divisor. Similarly, if q is odd, then $e \cdot H^{q-2n}(B) = H^q(B)$. But if $q = 1$, then $H^{q-2n}(B) = 0$; by induction on q, we find that $H^{\text{odd}}(B) = 0$. Therefore, if q is even, then $H^{q-2n+1}(B) = 0$. This implies that there is a short exact sequence

$$0 \to H^*(B) \xrightarrow{e\cdot} H^*(B) \to H^*(E) \to 0. \tag{6.1}$$

In particular, the cohomology of E is the cohomology of B quotiented by the ideal generated by the nonzero divisor e.

For instance, when $n = 1$, then $B = \mathbf{CP}^\infty$ and $E \simeq *$. We have the canonical generator $e \in H^2(\mathbf{CP}^\infty)$; these deductions tell us the well-known fact that $H^*(\mathbf{CP}^\infty) \simeq \mathbf{Z}[e]$.

Consider the surjection $H^*(B) \xrightarrow{\pi^*} H^*(E)$. Since $H^*(E)$ is polynomial, we can lift the generators of $H^*(E)$ to elements of $H^*(B)$. This begets a splitting $s : H^*(E) \to$

$H^*(B)$. The existence of the Euler class $e \in H^*(B)$ therefore gives a map $H^*(E)[e] \xrightarrow{\overline{s}} H^*(B)$. We claim that this map is an isomorphism.

This is a standard algebraic argument. Filter both sides by powers of e, i.e., take the e-adic filtration on $H^*(E)[e]$ and $H^*(B)$. Clearly, the associated graded of $H^*(E)[e]$ just consists of an infinite direct sum of the cohomology of E. The associated graded of $H^*(B)$ is the same, thanks to the short exact sequence (6.1). Thus the induced map on the associated graded $\mathrm{gr}^*(\overline{s})$ is an isomorphism. In this particular case (but not in general), we can conclude that $\overline{s}$ is an isomorphism: in any single dimension, the filtration is finite. Thus, using the five lemma over and over again, we see that the map $\overline{s}$ an isomorphism on each filtered piece. This implies that $\overline{s}$ itself is an isomorphism, as desired.

This argument proves that

$$H^*(BU(n-1)) = \mathbf{Z}[c_1, \cdots, c_{n-1}].$$

In particular, there is a map $\pi^* : H^*(BU(n)) \to H^*(BU(n-1))$ which an isomorphism in dimensions at most $2n$. Thus, the generators c_i have *unique* lifts to $H^*(BU(n))$. We therefore get:

Theorem 70.1. *There exist classes $c_i \in H^{2i}(BU(n))$ for $1 \leq i \leq n$ such that:*

- *the canonical map $H^*(BU(n)) \xrightarrow{\pi_*} H^*(BU(n-1))$ sends*

$$c_i \mapsto \begin{cases} c_i & i < n \\ 0 & i = n, \text{ and} \end{cases}$$

- $c_n := (-1)^n e \in H^{2n}(BU(n))$.

Moreover,

$$\boxed{H^*(BU(n)) \simeq \mathbf{Z}[c_1, \cdots, c_n]}.$$

The splitting principle

Theorem 70.2. *Let $\xi \downarrow X$ be an n-plane bundle. Then there exists a space $\mathrm{Fl}(\xi) \xrightarrow{\pi} X$ such that:*

1. *$\pi^*\xi = \lambda_1 \oplus \cdots \lambda_n$, where the λ_i are line bundles on Y, and*
2. *the map $\pi^* : H^*(X) \to H^*(\mathrm{Fl}(\xi))$ is monic.*

Proof. We have already (somewhat) studied this space. Recall that there is a vector bundle $\pi : \mathbf{P}(\xi) \to X$ such that

$$H^*(\mathbf{P}(\xi)) = H^*(X)\langle 1, e, \cdots, e^{n-1}\rangle.$$

Moreover, in §69, we proved that there is a complex line bundle over $\mathbf{P}(\xi)$ which is a subbundle of $\pi^*\xi$. In other words, $\pi^*\xi$ splits as a sum of a line bundle and some other bundle (by Corollary 52.11). Iterating this construction proves the existence of $\mathrm{Fl}(\xi)$. □

This proof does not give much insight into the structure of $\mathrm{Fl}(\xi)$. Remember that the *frame bundle* $\mathrm{Fr}(\xi)$ of ξ: an element of $\mathrm{Fr}(\xi)$ is a linear, inner-product preserving map $\mathbf{C}^n \to E(\xi)$. This satisfies various properties; for instance:

$$E(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} \mathbf{C}^n.$$

Moreover,

$$\mathbf{P}(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} U(n)/(1 \times U(n-1)).$$

The *flag bundle* $\mathrm{Fl}(\xi)$ is defined to be

$$\mathrm{Fl}(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} U(n)/(U(1) \times \cdots \times U(1)).$$

The product $U(1)\times\cdots\times U(1)$ is usually denoted T^n, since it is the maximal torus in $U(n)$. For the universal bundle $\xi_n \downarrow BU(n)$, the frame bundle is exactly $EU(n)$; therefore, $\mathrm{Fl}(\xi_n)$ is just the bundle given by $BT^n \to BU(n)$. By construction, the fiber of this bundle is $U(n)/T^n$. In particular, there is a monomorphism $H^*(BU(n)) \hookrightarrow H^*(BT^n)$. The cohomology of BT^n is extremely simple — it is the cohomology of a product of $\mathbf{CP}^\infty$'s, so

$$H^*(BT^n) \simeq \mathbf{Z}[t_1, \cdots, t_n],$$

where $|t_k| = 2$. The t_i are the Euler classes of $\pi_i^* \lambda_i$, under the projection map $\pi_i : BT^n \to \mathbf{CP}^\infty$.

71 The Whitney sum formula

As we saw in the previous section, there is an injection $H^*(BU(n)) \hookrightarrow H^*(BT^n)$. What is the image of this map?

The symmetric group sits inside of $U(n)$, so it acts by conjugation on $U(n)$. This action stabilizes this subgroup T^n. By naturality, Σ_n acts on the classifying space BT^n. Since Σ_n acts by conjugation on $U(n)$, it acts on $BU(n)$ in a way that is homotopic to the identity (Lemma 58.1). However, each element $\sigma \in \Sigma_n$ simply permutes the factors in $BT^n = (\mathbf{CP}^\infty)^n$; we conclude that $H^*(BU(n); R)$ actually sits inside the invariants $H^*(BT^n; R)^{\Sigma_n}$.

Recall the following theorem from algebra:

Theorem 71.1. *Let Σ_n act on the polynomial algebra $R[t_1, \cdots, t_n]$ by permuting the generators. Then*

$$R[t_1, \cdots, t_n]^{\Sigma_n} = R[\sigma_1^{(n)}, \cdots, \sigma_n^{(n)}],$$

where the σ_i are the elementary symmetric polynomials, defined via

$$\prod_{i=1}^n (x - t_i) = \sum_{j=0}^n \sigma_i^{(n)} x^{n-i}.$$

For instance,

$$\sigma_1^{(n)} = -\sum t_i,\ \sigma_n^{(n)} = (-1)^n \prod t_i.$$

If we impose a grading on $R[t_1, \cdots, t_n]$ such that $|t_i| = 2$, then $|\sigma_i^{(n)}| = 2i$. It follows from our discussion in §70 that the ring $H^*(BT^n)^{\Sigma_n}$ has the same size as $H^*(BU(n))$.

Consider an injection of finitely generated abelian groups $M \hookrightarrow N$, with quotient Q. Suppose that, after tensoring with any field, the map $M \to N$ an isomorphism. If $Q \otimes k = 0$, then $Q = 0$. Indeed, if $Q \otimes \mathbf{Q} = 0$ then Q is torsion. Similarly, if $Q \otimes \mathbf{F}_p = 0$, then Q has no p-component. In particular, $M \simeq N$. Applying this to the map $H^*(BU(n) \to H^*(BT^n)^{\Sigma_n}$, we find that

$$H^*(BU(n); R) \xrightarrow{\simeq} H^*(BT^n; R)^{\Sigma_n} = R[\sigma_1^{(n)}, \cdots, \sigma_n^{(n)}].$$

What happens as n varies? There is a map $R[t_1, \cdots, t_n] \to R[t_1, \cdots, t_{n-1}]$ given by sending $t_n \mapsto 0$ and $t_i \mapsto t_i$ for $i \neq n$. Of course, we cannot say that this map is equivariant with respect to the action of Σ_n. However, it *is* equivariant with respect to the action of Σ_{n-1} on $R[t_1, \cdots, t_n]$ via the inclusion of $\Sigma_{n-1} \hookrightarrow \Sigma_n$ as the stabilizer of $n \in \{1, \cdots, n\}$. Therefore, the Σ_n-invariants sit inside the Σ_{n-1}-invariants, giving a map

$$R[t_1, \cdots, t_n]^{\Sigma_n} \to R[t_1, \cdots, t_n]^{\Sigma_{n-1}} \to R[t_1, \cdots, t_{n-1}]^{\Sigma_{n-1}}.$$

We also find that for $i < n$, we have $\sigma_i^{(n)} \mapsto \sigma_i^{(n-1)}$ and $\sigma_n^{(n)} \mapsto 0$.

Where do the Chern classes go?

To answer this question, we will need to understand the multiplicativity of the Chern class. We begin with a discussion about the Euler class. Suppose $\xi^p \downarrow X, \eta^q \downarrow Y$ are oriented real vector bundles; then, we can consider the bundle $\xi \times \eta \downarrow X \times Y$, which is another oriented real vector bundle. The orientation is given by picking oriented bases for ξ and η. We claim that

$$e(\xi \times \eta) = e(\xi) \times e(\eta) \in H^{p+q}(X \times Y).$$

Since $D(\xi \times \eta)$ is homeomorphic to $D(\xi) \times D(\eta)$, and $S(\xi \times \eta) = D(\xi) \times S(\eta) \cup S(\xi) \times D(\eta)$, we learn from the relative Künneth formula that

$$H^*(D(\xi \times \eta), S(\xi \times \eta)) \leftarrow H^*(D(\xi), S(\xi)) \otimes H^*(D(\eta), S(\eta)).$$

It follows that

$$u_{\xi \times \eta} = u_\xi \times u_\eta \in H^{p+q}(\mathrm{Th}(\xi) \times \mathrm{Th}(\eta));$$

this proves the desired result since the Euler class is the image of the Thom class under the map $H^n(\mathrm{Th}(\xi)) \to H^n(D(\xi)) \simeq H^n(B)$.

Consider the diagonal map $\Delta : X \to X \times X$. The cross product in cohomology then pulls back to the cup product, and the direct product of fiber bundles pulls back to the Whitney sum. It follows that

$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta).$$

If $\xi^n \downarrow X$ is an n-dimensional complex vector bundle, then we defined[1]

$$c_n(\xi) = (-1)^n e(\xi_{\mathbf{R}}).$$

We need to describe the image of $c_n(\xi_n)$ under the map $H^{2n}(BU(n)) \to H^{2n}(BT^n)^{\Sigma_n}$.

Let $f : BT^n \to BU(n)$ denote the map induced by the inclusion of the maximal torus. Then, by construction, we have a splitting

$$f^*\xi_n = \lambda_1 \oplus \cdots \oplus \lambda_n.$$

Thus,

$$(-1)^n e(\xi) \mapsto (-1)^n e(\lambda_1 \oplus \cdots \oplus \lambda_n) = (-1)^n e(\lambda_1) \cup \cdots \cup e(\lambda_n).$$

The discussion above implies that f^* sends the right hand side to $(-1)^n t_1 \cdots t_n = \sigma_n^{(n)}$. In other words, the top Chern class maps to $\sigma_n^{(n)}$ under the map f^*.

Our discussion in the previous sections gives a commuting diagram:

$$\begin{tikzcd}
H^*(BU(n)) \arrow[r] \arrow[d] & H^*(BT^n)^{\Sigma_n} \arrow[d] \\
H^*(BU(n-1)) \arrow[r] & H^*(BT^{n-1})^{\Sigma_{n-1}}
\end{tikzcd}$$

Arguing inductively, we find that going from the top left corner to the bottom left corner to the bottom right corner sends

$$c_i \mapsto c_i \mapsto \sigma_i^{(n-1)} \text{ for } i < n.$$

Likewise, going from the top left corner to the top right corner to the bottom right corner sends

$$c_i \mapsto \sigma_i^{(n)} \mapsto \sigma_i^{(n-1)} \text{ for } i < n.$$

We conclude that the map f^* sends $c_i^{(i)} \mapsto \sigma_i^{(i)}$.

Proving the Whitney sum formula

By our discussion above, the Whitney sum formula of Theorem 69.4 reduces to proving the following identity:

$$\sigma_k^{(p+q)} = \sum_{i+j=k} \sigma_i^{(p)} \cdot \sigma_j^{(q)} \tag{6.2}$$

inside $\mathbf{Z}[t_1, \cdots, t_p, t_{p+1}, \cdots, t_{p+q}]$. Here, $\sigma_i^{(p)}$ is thought of as a polynomial in $t_1, \cdots, t_p$, while $\sigma_i^{(q)}$ is thought of as a polynomial in $t_{p+1}, \cdots, t_{p+q}$. To derive Equation (6.2), simply

[1] There's a slight technical snag here: a complex bundle doesn't have an orientation. However, its underlying oriented real vector bundle does.

compare coefficients in the following:

$$\begin{aligned}
\sum_{k=0}^{p+q} \sigma_k^{(p+q)} x^{p+q-k} &= \prod_{i=1}^{p+q}(x - t_i) \\
&= \prod_{i=1}^{p}(x - t_i) \cdot \prod_{j=p+1}^{p+q}(x - t_j) \\
&= \left(\sum_{i=0}^{p} \sigma_i^{(p)} x^{p-i}\right)\left(\sum_{j=0}^{q} \sigma_j^{(p)} x^{q-j}\right) \\
&= \sum_{k=0}^{p+q}\left(\sum_{i+j=k} \sigma_i^{(p)} \sigma_j^{(q)}\right) x^{p+q-k}.
\end{aligned}$$

72 Stiefel-Whitney classes, immersions, cobordisms

There is a result analogous to Theorem 69.4 for all vector bundles (not necessarily oriented):

Theorem 72.1. *There exist a unique family of characteristic classes $w_i : \mathrm{Vect}_n(X) \to H^n(X; \mathbf{F}_2)$ such that for $0 \le i$ and $i > n$, we have $w_i = 0$, and:*

1. $w_0 = 1$;
2. $w_1(\lambda) = e(\lambda)$; *and*
3. *the Whitney sum formula holds:*

$$w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \cup w_j(\eta)$$

Moreover:

$$H^*(BO(n); \mathbf{F}_2) = \mathbf{F}_2[w_1, \cdots, w_n],$$

where $w_n = e_2$.

Remark 72.2. We can express the Whitney sum formula simply by defining the *total Steifel-Whitney class*

$$1 + w_1 + w_2 + \cdots =: w.$$

Then the Whitney sum formula is just

$$w(\xi \oplus \eta) = w(\xi) \cdot w(\eta).$$

Likewise, the Whitney sum formula can be stated by defining the total Chern class.

Remark 72.3. Again, the Steifel-Whitney classes are stable:

$$w(\xi \oplus k\epsilon) = w(\xi).$$

Again, Grothendieck's definition works since the splitting principle holds. There is an injection $H^*(BO(n)) \hookrightarrow H^*(B(\mathbf{Z}/2\mathbf{Z})^n)$. To compute $H^*(BO(n))$, our argument for computing $H^*(BU(n))$ does not immediately go through, although there is a fiber sequence

$$S^{n-1} \to EO(n) \times_{O(n)} O(n)/O(n-1) \to BO(n);$$

the problem is that $n-1$ can be even or odd. We still have a Gysin sequence, though:

$$\cdots \to H^{q-n}(BO(n)) \xrightarrow{e\cdot} H^q(BO(n)) \xrightarrow{\pi^*} H^q(BO(n-1)) \to H^{q-n+1}(BO(n)) \to \cdots$$

In order to apply our argument for computing $H^*(BU(n))$ to this case, we only need to know that e is a nonzero divisor. The splitting principle gave a monomorphism $H^*(BO(n)) \hookrightarrow H^*((\mathbf{RP}^\infty)^n)$. The fact that e is a nonzero divisor follows from the observation that under this map,

$$e_2 = w_n \mapsto e_2(\lambda_1 \oplus \cdots \oplus \lambda_n) = t_1 \cdots t_n,$$

using the same argument as in §71; however, $t_1 \cdots t_n$ is a nonzero divisor, since $H^*((\mathbf{RP}^\infty)^n)$ is an integral domain.

Immersions of manifolds

The theory developed above has some interesting applications to differential geometry.

Definition 72.4. Let M^n be a smooth closed manifold. An *immersion* is a smooth map from M^n to $\mathbf{R}^{n+k}$, denoted $f : M^n \looparrowright \mathbf{R}^{n+k}$, such that $(\tau_{M^n})_x \hookrightarrow (\tau_{\mathbf{R}^{n+k}})_{f(x)}$ for $x \in M$.

Informally: crossings are allows, but not cusps.

Example 72.5. There is an immersion $\mathbf{RP}^2 \looparrowright \mathbf{R}^3$, known as *Boy's surface*.

Question 72.6. When can a manifold admit an immersion into an Euclidean space?

Assume we had an immersion $i : M^n \looparrowright \mathbf{R}^{n+k}$. Then we have an embedding $f : \tau_M \to i^*\tau_{\mathbf{R}^{n+k}}$ into a trivial bundle over M, so τ_M has a k-dimensional complement, called ξ such that

$$\tau_M \oplus \xi = (n+k)\epsilon.$$

Apply the total Steifel-Whitney class, we have

$$w(\tau)w(\xi) = 1,$$

since there's no higher Steifel-Whitney class of a trivial bundle. In particular,

$$w(\xi) = w(\tau)^{-1}.$$

Example 72.7. Let $M = \mathbf{RP}^n \looparrowright \mathbf{R}^{n+k}$. Then, we know that

$$\tau_{\mathbf{RP}^n} \oplus \epsilon \simeq (n+1)\lambda^* \simeq (n+1)\lambda,$$

where $\lambda \downarrow \mathbf{RP}^n$ is the canonical line bundle. By Remark 72.3, we have

$$w(\tau_{\mathbf{RP}^n}) = w(\tau_{\mathbf{RP}^n} \oplus \eta) = w((n+1)\lambda) = w(\lambda)^{n+1}.$$

It remains to compute $w(\lambda)$. Only the first Steifel-Whitney class is nonzero. Writing $H^*(\mathbf{RP}^n) = \mathbf{F}_2[x]/x^{n+1}$, we therefore have $w(\lambda) = x$. In particular,

$$w(\tau_{\mathbf{RP}^n}) = (1+x)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} x^i.$$

It follows that

$$w_i(\tau_{\mathbf{RP}^n}) = \binom{n+1}{i} x^i.$$

The total Steifel-Whitney class of the complement of the tangent bundle is:

$$w(\xi) = (1+x)^{-n-1}.$$

The most interesting case is when n is a power of 2, i.e., $n = 2^s$ for some integer s. In this case, since taking powers of 2 is linear in characteristic 2, we have

$$w(\xi) = (1+x)^{-1-2^s} = (1+x)^{-1}(1+x)^{-2^s} = (1+x)^{-1}(1+x^{2^s})^{-1}.$$

As all terms of degree greater than 2^s are zero, we conclude that So

$$w(\xi) = 1 + x + x^2 + \cdots + x^{2^s-1} + 2x^s = 1 + x + x^2 + \cdots + x^{2^s-1}.$$

As $x^{2^s-1} \neq 0$, this means that $k = \dim \xi \geq 2^s - 1$. We conclude:

Theorem 72.8. *There is no immersion* $\mathbf{RP}^{2^s} \looparrowright \mathbf{R}^{2\cdot 2^s-2}$.

The following result applied to $\mathbf{RP}^{2^s}$ shows that the above result is sharp:

Theorem 72.9 (Whitney). *Any smooth compact closed manifold* $M^n \looparrowright \mathbf{R}^{2n-1}$.

However, Whitney's result is *not* sharp for a general smooth compact closed manifold. Rather, we have:

Theorem 72.10 (Brown–Peterson, Cohen). *A closed compact smooth n-manifold* $M^n \looparrowright \mathbf{R}^{2n-\alpha(n)}$, *where $\alpha(n)$ is the number of 1s in the dyadic expansion of n.*

This result is sharp, since if $n = \sum 2^{d_i}$ for the dyadic expansion, then $M = \prod_i \mathbf{RP}^{2^{d_i}} \not\looparrowright \mathbf{R}^{2n-\alpha(n)-1}$.

Cobordism, characteristic numbers

If we have a smooth closed compact n-manifold, then it embeds in $\mathbf{R}^{n+k}$ for some $k \gg 0$. The normal bundle then satisfies

$$\tau_M \oplus \nu_M = (n+k)\epsilon.$$

A piece of differential topology tells us that if k is large, then $\nu_M \oplus N\epsilon$ is independent of the bundle for some N.

This example, combined with Remark **??**, shows that $w(\nu_M)$ is independent of k. We are therefore motivated to think of Stiefel-Whitney classes as coming from $H^*(BO;\mathbf{F}_2) = \mathbf{F}_2[w_1, w_2, \cdots]$, where $BO = \varinjlim BO(n)$. Similarly, Chern classes should be thought of as coming from $H^*(BU;\mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots]$. This exa

Definition 72.11. The characteristic number of a smooth closed compact n-manifold M is defined to be $\langle w(\nu_M), [M]\rangle$.

Note that the fundamental class $[M]$ exists, since our coefficients are in $\mathbf{F}_2$, where everything is orientable.

This definition is very useful when thinking about cobordisms.

Definition 72.12. Two (smooth closed compact) n-manifolds M, N are *(co)bordant* if there is an $(n+1)$-dimensional manifold W^{n+1} with boundary such that

$$\partial W \simeq M \sqcup N.$$

For instance, when $n = 0$, the manifold $* \sqcup *$ is *not* cobordant to $*$, but it is cobordant to the empty set. However, $* \sqcup * \sqcup *$ is cobordant to $*$. Any manifold is cobordant to itself, since $\partial(M \times I) = M \sqcup M$. In fact, cobordism forms an equivalence relation on manifolds.

Example 72.13. A classic example of a cobordism is the "pair of pants"; this is the following cobordism between S^1 and $S^1 \sqcup S^1$: add image

Let us define

$$\Omega_n^O = \{\text{cobordism classes of } n\text{-manifolds}\}.$$

This forms a group: the addition is given by disjoint union. Note that every element is its own inverse. Moreover, $\bigoplus_n \Omega_n^O = \Omega_*^O$ forms a graded ring, where the product is given by the Cartesian product of manifolds. Our discussion following Definition 72.12 shows that $\Omega_0^O = \mathbf{F}_2$.

Exercise 72.14. Every 1-manifold is nullbordant, i.e., cobordant to the point.

Thom made the following observation. Suppose an n-manifold M is embedded into Euclidean space, and that M is nullbordant via some $(n+1)$-manifold W, so that $\nu_W|_M = \nu_M$. In particular,

$$\langle w(\nu_M), [M]\rangle = \langle w(\nu_W)|_M, [M]\rangle.$$

On the other hand, the boundary map $H_{n+1}(W,M) \xrightarrow{\partial} H_n(M)$ sends the relative fundamental class $[W,M]$ to $[M]$. Thus

$$\langle w(\nu_M),[M]\rangle = \langle w(\nu_M),\partial[W,M]\rangle = \langle \delta w(\nu_M),[W,M]\rangle.$$

However, we have an exact sequence

$$H^n(W) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(W,M).$$

Since $w(\nu_M)$ is in the image of i^*, it follows that $\delta w(\nu_M) = 0$. In particular, the characteristic number of a nullbordant manifold is zero. Thus, we find that "Stiefel-Whitney numbers tell all":

Proposition 72.15. *Characteristic numbers are cobordism invariants. In other words, characteristic numbers give a map*

$$\Omega_n^O \to \mathrm{Hom}(H^n(BO),\mathbf{F}_2) \simeq H_n(BO).$$

More is true:

Theorem 72.16 (Thom, 1954)**.** *The map of graded rings $\Omega_*^O \to H_*(BO)$ defined by the characteristic number is an inclusion. Concretely, if $w(M^n) = w(N^n)$ for all $w \in H^n(BO)$, then M^n and N^n are cobordant.*

The way that Thom proved this was by expressing Ω_*^O is the graded homotopy ring of some space, which he showed is the product of mod 2 Eilenberg-MacLane spaces. Along the way, he also showed that:

$$\Omega_*^O = \mathbf{F}_2[x_i : i \neq 2^s - 1] = \mathbf{F}_2[x_2, x_4, x_5, x_6, x_8, \cdots]$$

This recovers the result of Exercise 72.14 (and so much more!).

73 Oriented bundles, Pontryagin classes, Signature theorem

We have a pullback diagram

$$\begin{tikzcd} BSO(n) \arrow[r] \arrow[d, "\text{double cover}"] & S^\infty \arrow[d] \\ BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z} \end{tikzcd}$$

The bottom map is exactly the element $w_1 \in H^1(BO(n);\mathbf{F}_2)$. It follows that a vector bundle $\xi \downarrow X$ represented by a map $f : X \to BO(n)$ is orientable iff $w_1(\xi) = f^*(w_1) = 0$, since this is equivalent to the existence of a factorization:

$$\begin{tikzcd} & BSO(n) \arrow[r] \arrow[d] & S^\infty \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "\xi"'] & BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z} \end{tikzcd}$$

The fiber sequence $BSO(n) \to BO(n) \to \mathbf{RP}^\infty$ comes from a fiber sequence $SO(n) \to O(n) \to \mathbf{Z}/2\mathbf{Z}$ of groups. For $n \geq 3$, we can kill $\pi_1(SO(n)) = \mathbf{Z}/2\mathbf{Z}$, to get a double cover $\mathrm{Spin}(n) \to SO(n)$. The group $\mathrm{Spin}(n)$ is called the *spin group*. We have a cofiber sequence

$$B\mathrm{Spin}(n) \to BSO(n) \xrightarrow{w_2} K(\mathbf{Z}/2\mathbf{Z}, 2).$$

If $w_2(\xi) = 0$, we get a further lift in the above diagram, begetting a *spin structure* on ξ.

Bott computed that $\pi_2(\mathrm{Spin}(n)) = 0$. However, $\pi_3(\mathrm{Spin}(n)) = \mathbf{Z}$; killing this gives the *string group* $\mathrm{String}(n)$. Unlike $\mathrm{Spin}(n)$, $SO(n)$, and $O(n)$, this is not a finite-dimensional Lie group (since we have an infinite dimensional summand $K(\mathbf{Z}, 2)$). However, it can be realized as a topological group. The resulting maps

$$\mathrm{String}(n) \to \mathrm{Spin}(n) \to SO(n) \to O(n)$$

are just the maps in the Whitehead tower for $O(n)$. Taking classifying spaces, we get

$$\begin{tikzcd} & B\mathrm{String}(n) \arrow[d] & \\ & B\mathrm{Spin}(n) \arrow[r, "p_1/2"] \arrow[d] & K(\mathbf{Z}, 4) \\ & BSO(n) \arrow[r] \arrow[d] & K(\mathbf{Z}/2\mathbf{Z}, 2) \\ X \arrow[r, "\xi"'] \arrow[ur, dashed] \arrow[uur, dashed] \arrow[uuur, dashed] & BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z} \end{tikzcd}$$

Computing the (mod 2) cohomology of $BSO(n)$ is easy. We have a double cover $BSO(n) \to BO(n)$ with fiber S^0. Consequently, there is a Gysin sequence:

$$0 \to H^q(BO(n)) \xrightarrow{w_1} H^{q+1}(BO(n)) \xrightarrow{\pi^*} H^{q+1}(BSO(n)) \to 0$$

since w_1 is a nonzero divisor. The standard argument shows that

$$H^*(BSO(n)) = \mathbf{F}_2[w_2, \cdots, w_n].$$

However, it is *not* easy to compute $H^*(B\mathrm{Spin}(n))$ and $H^*(B\mathrm{String}(n))$; these are extremely complicated (and only become more complicated for higher connective covers of $BO(n)$). However, we will remark that they are concentrated in even degrees.

To define integral characteristic classes for oriented bundles, we will need to study Chern classes a little more. Let ξ be a complex n-plane bundle, and let $\overline{\xi}$ denote the conjugate bundle. What is the total Chern class $c(\overline{\xi})$? Recall that the Chern classes $c_k(\overline{\xi})$ occur as coefficients in the identity

$$\sum c_i(\overline{\xi})e(\lambda_{\overline{\xi}})^{n-i} = 0,$$

where $\lambda_{\overline{\xi}} \downarrow \mathbf{P}(\overline{\xi})$. Note that $\mathbf{P}(\overline{\xi}) = \mathbf{P}(\xi)$. By construction, $\lambda_{\overline{\xi}} = \overline{\lambda_\xi}$. In particular, we find that

$$e(\lambda_{\overline{\xi}}) = -e(\lambda_\xi).$$

It follows that

$$0 = \sum_{i=0}^n c_i(\overline{\xi})e(\overline{\lambda_\xi})^{n-i} = \sum_{i=0}^n c_i(\overline{\xi})(-1)^{n-i}e(\lambda_\xi)^{n-i} = (-1)^n e(\lambda_\xi)^n + \cdots$$

This is *not* monic, and hence doesn't define the Chern classes of $\overline{\xi}$. We do, however, get a monic polynomial by multiplying this identity by $(-1)^n$:

$$\sum_{i=0}^n (-1)^i c_i(\overline{\xi})e(\lambda_\xi)^{n-i} = 0.$$

It follows that

$$\boxed{c_i(\overline{\xi}) = (-1)^i c_i(\xi).}$$

If ξ is a real vector bundle, then

$$c_i(\xi\otimes\mathbf{C}) = c_i(\overline{\xi\otimes\mathbf{C}}) = (-1)^i c_i(\xi\otimes\mathbf{C}).$$

If i is odd, then $2c_i(\xi\otimes\mathbf{C}) = 0$. If R is a $\mathbf{Z}[1/2]$-algebra, we therefore define:

Definition 73.1. Let ξ be a real n-plane vector bundle. Then the kth Pontryagin class of ξ is defined to be

$$p_k(\xi) = (-1)^k c_{2k}(\xi\otimes\mathcal{C}) \in H^{4k}(X;R).$$

Notice that this is 0 if $2k > n$, since $\xi\otimes\mathcal{C}$ is of complex dimension n. The Whitney sum formula now says that:

$$(-1)^k p_k(\xi\oplus\eta) = \sum_{i+j=k}(-1)^i p_i(\xi)(-1)^j p_j(\eta) = (-1)^k \sum_{i+j=k} p_i(\xi)p_j(\eta).$$

If ξ is an oriented real $2k$-plane bundle, one can calculate that

$$p_k(\xi) = e(\xi)^2 \in H^{4k}(X;R).$$

We can therefore write down the cohomology of $BSO(n)$ with coefficients in a $\mathbf{Z}[1/2]$-algebra:

$* =$	2	4	6	8	10	12
$H^*(BSO(2))$	e_2	(e_2^2)				
$H^*(BSO(3))$		p_1				
$H^*(BSO(4))$		p_1, e_4		(e_4^2)		
$H^*(BSO(5))$		p_1		p_2		
$H^*(BSO(6))$		p_1	e_6	p_2		(e_6^2)
$H^*(BSO(7))$		p_1		p_2		p_3

Here, $p_k \mapsto e_{2k}^2$. In the limiting case (i.e., for $BSO = BSO(\infty)$), we get a polynomial algebra on the p_i.

Applications

We will not prove any of the statements in this section; it only serves as an outlook. The first application is the following analogue of Theorem 72.16:

Theorem 73.2 (Wall)**.** *Let M^n, N^n be oriented manifolds. If all Stiefel-Whitney numbers and Pontryagin numbers coincide, then M is oriented cobordant to N, i.e., there is an $(n+1)$-manifold W^{n+1} such that*

$$\partial W^{n+1} = M \sqcup -N.$$

The most exciting application of Pontryagin classes is to Hirzebruch's "signature theorem". Let M^{4k} be an oriented $4k$-manifold. Then, the formula

$$x \otimes y \mapsto \langle x \cup y, [M] \rangle$$

defines a pairing

$$H^{2k}(M)/\text{torsion} \otimes H^{2k}(M)/\text{tors} \to \mathbf{Z}.$$

Poincaré duality implies that this is a perfect pairing, i.e., there is a nonsingular symmetric bilinear form on $H^{2k}(M)/\text{torsion} \otimes \mathbf{R}$. Every symmetric bilinear form on a real vector space can be diagonalized, so that the associated matrix is diagonal, and the only nonzero entries are ± 1. The number of 1s minus the number of -1s is called the *signature* of the bilinear form. When the bilinear form comes from a $4k$-manifold as above, this is called the signature of the manifold.

Lemma 73.3 (Thom)**.** *The signature is an oriented bordism invariant.*

This is an easy thing to prove using Lefschetz duality, which is a deep theorem. Hirzebruch's signature theorem says:

Theorem 73.4 (Hirzebruch signature theorem)**.** *There exists an explicit rational polynomial $L_k(p_1, \cdots, p_k)$ of degree $4k$ such that*

$$\langle L(p_1(\tau_M), \cdots, p_1(\tau_M)), [M] \rangle = \text{signature}(M).$$

The reason the signature theorem is so interesting is that the polynomial $L(p_1(\tau_M), \cdots, p_1(\tau_M))$ is defined only in terms of the tangent bundle of the manifold, while the signature is defined only in terms of the topology of the manifold. This result was vastly generalized by Atiyah and Singer to the Atiyah-Singer index theorem.

Example 73.5. One can show that

$$L_1(p_1) = p_1/3.$$

The Hirzebruch signature theorem implies that $\langle p_1(\tau), [M^4] \rangle$ is divisible by 3.

Example 73.6. From Hirzebruch's characterization of the L-polynomial, we have

$$L_2(p_1, p_2) = (7p_2 - p_1^2)/45.$$

This imposes very interesting divisibility constraints on the characteristic classes of a tangent bundle of an 8-manifold. This particular polynomial was used by Milnor to produce "exotic spheres", i.e., manifolds which are homeomorphic to S^7 but not diffeomorphic to it.

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