Algebraic Topology

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Preface

These are notes from the 18.905-906 sequence, taught by Haynes Miller in the academic year 2016-2017. The goal was to tell the story of classical algebraic topology, starting from the basics of homology and cohomology, all the way up to the Serre spectral sequence and characteristic classes. I tried to liveTEXthe entire course. As such, there are multiple typos and mathematical errors, which I am terribly ashamed of.

Haynes, a few other students from the class, and I are working to clean (and, in extreme cases, rewrite) the exposition in some chapters. Hopefully this project will be completed by the first month of 2018, in time for the second run of 18.906.

I'd appreciate any input on where these notes can be fixed. The github repository for the project is at

`https://github.com/sanathdevalapurkar/algtop-notes`.

Any comments can be sent to `sanathd@mit.edu`.

— Sanath

Things to fix

- As of January 19, 2018, all of Part II has been edited, with the exception of §3.3, §5.5, §5.6, §5.7, §5.8.1, and §5.9.
- Part I remains to be edited.
- The original version of the notes used `\cc` to denote both a script C and the complex numbers. Now, `\cc` denotes a script C and `\cC` denotes the complex numbers. This is a problem that needs to be fixed everywhere.

Contents

Part I

18.905: an introduction to algebraic topology

Chapter 1

Homology and CW-complexes

1.1 Introduction, simplices

In 18.905, which is the first half of this book, we will cover the following topics:

1. Singular homology
2. CW-complexes
3. Basics of category theory
4. Homological algebra
5. The Künneth theorem
6. UCT, cohomology
7. Cup and cap products, and
8. Poincaré duality.

Below are some examples of commonly encountered topological spaces.

- The most basic is *n-dimensional Euclidean space*, $\mathbf{R}^n$.
- The *n-sphere* $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$ is topologized as a subspace of $\mathbf{R}^{n+1}$.
- Identifying antipodal points in S^n gives *real projective space* $\mathbf{RP}^n = S^n/(x \sim -x)$, i.e. the space of lines through the origin in $\mathbf{R}^{n+1}$.

- Call an ordered collection of k orthonormal vectors an *orthonormal k-frame*. The space of orthonormal k-frames in $\mathbf{R}^n$ forms the *Stiefel manifold* $V_k(\mathbf{R}^n)$, which is topologized as a subspace of $(S^{n-1})^k$.

- Let $x \sim y$ if x and y are k-frames with the same span. The *Grassmannian* is the quotient $\mathrm{Gr}_k(\mathbf{R}^n) = V_k(\mathbf{R}^n)/\sim$. In particular, $\mathbf{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$.

The above are all *manifolds*, which are Hausdorff spaces locally homeomorphic to Euclidean space. Aside from $\mathbf{R}^n$ itself, the preceding examples are also compact. Such spaces exhibit a hidden symmetry, which is the culmination of 18.905: Poincaré duality.

As the name suggests, the central aim of algebraic topology is the usage of algebraic tools to study topological spaces. A common technique is to probe topological spaces via maps to them. In different ways, this approach gives rise to singular homology and homotopy groups. We now detail the former; the latter takes stage in 18.906.

Definition 1.1.1. For $n \geq 0$, the *standard n-simplex* Δ^n is the convex hull of the standard basis $\{e_0, \cdots, e_n\}$ in $\mathbf{R}^{n+1}$. More explicitly,

$$\Delta^n = \left\{ \sum t_i e_i : \sum t_i = 1, t_i \geq 0 \right\} \subseteq \mathbf{R}^{n+1}.$$

The t_i are called barycentric coordinates.

We will write i in lieu of e_i to refer to the vertices of Δ^n. The standard simplices are related by face inclusions $d^i \colon \Delta^{n-1} \to \Delta^n$ for $0 \leq i \leq n$, where d^i misses the vertex i.

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Definition 1.1.2. Let X be any topological space. A *singular n-simplex* in X is a continuous map $\Delta^n \to X$. Denote by $\mathrm{Sin}_n(X)$ the set of all n-simplices in X.

This seems like a rather bold construction to make, as $\mathrm{Sin}_n(X)$ is huge. Nonetheless, we will soon make it even larger.

Image me

(See drawing for a torus. The direction of the simplex is like an orientation given by ordering of the indices of Δ^n. The standard notation is $\sigma : \Delta^n \to X$.)

Depict this whole thing as an image, and remove the sentence

For $0 \leq i \leq n$, precomposition by the face inclusion d^i produces a map $d_i \colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ sending $\sigma \mapsto \sigma \circ d^i$, which is the ith face of σ. This allows us to make sense of the "boundary" of a simplex, and we are particularly interested in simplices for which that boundary vanishes.

For example, if σ is a 1-simplex that goes around the hole in a torus T, then $d_1\sigma = d_0\sigma$. To express that the boundary vanishes, we want to write $d_0\sigma - d_1\sigma = 0$, but this difference is no longer a simplex. To accommodate such formal sums, we will enlarge $\mathrm{Sin}_n(X)$ further by considering the free abelian group it generates.

Definition 1.1.3. The abelian group $S_n(X)$ of *singular n-chains* in X is the free abelian group generated by n-simplices

$$S_n(X) = \mathbf{Z}\mathrm{Sin}_n(X).$$

Its elements are finite linear combinations, i.e. formal sums $\sum_{i \in \text{finite set}} a_i\sigma$ $a_i \in \mathbf{Z}$. If $n < 0$, say that $S_n(X) = 0$. Now, define

idk what was meant by “display” in the comments

$$\partial \colon \mathrm{Sin}_n(X) \to S_{n-1}(X),$$

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i d_i\sigma.$$

This extends to a homomorphism $\partial \colon S_n(X) \to S_{n-1}(X)$ by additivity.

We use this homomorphism to obtain something more tractable than the entirety of $S_n(X)$. First we restrict our attention to chains with vanishing boundary.

Definition 1.1.4. An *n-cycle* in X is an n-chain c with $\partial c = 0$. Denote $Z_n(X) = \ker(S_n(X) \xrightarrow{\partial} S_{n-1}(X))$.

For example, with σ on the torus described before, $\sigma \in Z_1(X)$ since $\partial\sigma = d_0\sigma - d_1\sigma = 0$.

Theorem 1.1.5. *Any boundary is a cycle, i.e.,* $B_n(X) := \mathrm{Im}(\partial : S_{n+1}(X) \to S_n(X)) \subseteq Z_n(X)$.

Exercise 1.1.6. Deduce the above theorem by proving the following statements.

1. An order-preserving map $\phi : [n] \to [m]$ extends to an affine map $\phi^* : \Delta^n \to \Delta^m$. Give an explicit formula for ϕ^*.

2. Prove that any order-preserving map factors uniquely as the composite of an order-preserving surjection followed by an order-preserving injection.

3. Let $d^j : [n-1] \to [n]$ be the order-preserving injection omitting j as a value. Prove that an order-preserving injection $\phi : [n-k] \to [n]$ is uniquely a composition of the form $d^{j_k} d^{j_{k-1}} \cdots d^{j_1}$ with $0 \leq j_1 < j_2 < \cdots < j_k \leq n$. Define the j_i's in terms of ϕ. Verify the straightening rule

$$d^i d^j = d^{j+1} d^i \text{ if } i \leq j.$$

4. Let $s^i : [n+1] \to [n]$ be the order-preserving surjection repeating the value i. Show that any order-preserving surjection $\phi : [m] \to [n]$ is uniquely a composition of the form $(s^n)^{i_n}(s^{n-1})^{i_{n-1}} \cdots (s^0)^{i_0}$ by describing the i_j's in terms of ϕ. Verify a straightening rule for the composite $s^i s^j$.

5. Verify a straightening rule for $s^i d^j$.

6. Write down the straightening rules for the induced maps $d_i = (d^i)^* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ and $s_i = (s^i)^* : \mathrm{Sin}_m(X) \to \mathrm{Sin}_{m+1}(X)$. Use these to verify that $\partial^2 = 0 : S_n(X) \to S_{n-2}(X)$.

7. Let $f : X \to Y$ be a continuous map. This induces a map $f_* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$. Show that the f_* assemble to give a map of simplicial sets, i.e., show that the maps f_* commute with the maps induced by order-preserving maps $\phi : [m] \to [n]$.

With the preceding, we are prepared to define singular homology.

Definition 1.1.7. The *nth singular homology group* of X is:

$$H_n(X) = Z_n(X)/B_n(X) = \frac{\ker(\partial : S_n(X) \to S_{n-1}(X))}{\mathrm{im}(\partial : S_{n+1}(X) \to S_n(X))}.$$

Both $Z_n(X)$ and $B_n(X)$ are free abelian groups because they are subgroups of the free abelian group $S_n(X)$, but the quotient $H_n(X)$ isn't necessarily free. While $Z_n(X)$ and $B_n(X)$ are uncountably generated, $H_n(X)$ is finitely generated for the spaces we are interested in. If T is the torus for example, then $H_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$ and σ as described previously is one of the two generators.

1.2 Simplices, more about homology

Previously we introduced the standard n-simplex $\Delta^n \subseteq \mathbf{R}^{n+1}$. Singular simplices in a space X are maps $\sigma\colon \Delta^n \to X$ and constitute the set $\mathrm{Sin}_n(X)$. For example, $\mathrm{Sin}_0(X)$ consists of points of X. In addition to the *face maps* $d_i\colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ we described, there are also *degeneracy maps* $s_i\colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n+1}(X)$, and the collection $\{\mathrm{Sin}_n(X), d_i, s_i\}$ forms a *simplicial set*. Simplicial sets are combinatorial models for topological spaces. In the language of category theory, which we will discuss shortly, we have a functor **Top** $\to$ {simplicial sets}.

To the semi-simplicial set $\{\mathrm{Sin}_n(X), d_i\}$ we then applied the free abelian group functor, obtaining a semi-simplicial abelian group. Using the d_is, we constructed a boundary map ∂ which makes $S_*(X)$ a *chain complex* because $\partial^2 = 0$ (see Exercise 1.1.6). We capture this process in a diagram:

$$\begin{tikzcd}
\mathbf{Top} \arrow[r] \arrow[d] & \{\text{semi-simplicial sets}\} \arrow[r] & \{\text{semi-simplicial abelian groups}\} \arrow[d] \\
\{\text{simplicial sets}\} \arrow[ur] & & \{\text{chain complexes}\} \arrow[d, "\text{take homology}"] \\
& & \{\text{graded abelian group}\}
\end{tikzcd}$$

Given a chain complex $\partial\colon A_n \to A_{n-1}$, one can define its homology $H_n(A, \partial) = \ker \partial_n / \operatorname{im} \partial_n$.

Here's an example. Suppose we have $\sigma\colon \Delta^1 \to X$. Define $\phi\colon \Delta^1 \to \Delta^1$ which sends $(t, 1-t) \mapsto (1-t, t)$. Precomposing σ with ϕ gives another singular simplex $\overline{\sigma}$ which reverses the orientation of σ. It is *not* true that $\overline{\sigma} = -\sigma$ in $S_1(X)$.

However, we show that $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$, meaning there is a 2-chain in X whose boundary is $\overline{\sigma} + \sigma$. If $d_0\sigma = d_1\sigma$ so that $\sigma \in Z_1(X)$, then $\overline{\sigma}$ and $-\sigma$ are homologous: $[\overline{\sigma}] = -[\sigma]$ in $H_1(X)$.

Let π denote the projection map from $[0, 1, 2]$ to $[0, 1]$. Then, $\partial(\sigma \circ \pi) = \sigma\pi d^0 - \sigma\pi d^1 + \sigma\pi d^2 = \overline{\sigma} - c^1_{\sigma(0)} + \sigma$ where $c^1_{\sigma(0)}$ is the constant 1-simplex at $\sigma(0)$ (similarly for $c^n_{\sigma(0)}$). We wish to $c^1_{\sigma(0)}$ is an error term. To achieve this, consider the constant 2-simplex $c^2_{\sigma(0)}$ at $\sigma(0)$; its boundary is $c^1_{\sigma(0)} - c^1_{\sigma(0)} + c^1_{\sigma(0)} = c^1_{\sigma(0)}$. So $\overline{\sigma} + \sigma = \partial(\sigma \circ \pi + c^2_{\sigma(0)})$ and $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$ as claimed.

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To give the simplest explicit examples, let's compute the homologies of $\emptyset$ and $*$. For the former, $\mathrm{Sin}_n(\emptyset) = \emptyset$, so $S_*(\emptyset) = 0$. Hence $\cdots \to S_2 \to S_1 \to S_0$ is the zero chain complex. This means that $Z_*(\emptyset) = B_*(\emptyset) = 0$. The homology in all dimensions is therefore 0.

For $*$, we have $\mathrm{Sin}_n(*) = \{c_*^n\}$ for all $n \geq 0$. Consequently $S_n(*) = \mathbf{Z}$. The boundary maps $\partial\colon S_n(*) \to S_{n-1}(*)$ in the chain complex depend on the parity of n as follows:

$$\partial(c_*^n) = \sum_{i=0}^{n}(-1)^i c_*^{n-1} = \begin{cases} c_*^{n-1} & \text{for } n \text{ even, and} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This means that our chain complex is:

$$\cdots \to \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \to 0.$$

The boundaries coincide with the cycles except in dimension zero, where $B_0(*) = 0$ while $Z_0(*) = \mathbf{Z}$. Therefore $H_0(*) = \mathbf{Z}$ but $H_i(*) = 0$ for $i > 0$.

We've defined homology groups for each space, but haven't considered what happens to maps between spaces. A continuous map $f\colon X \to Y$ induces a map $f_*\colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$ by composition: $\sigma \mapsto f \circ \sigma =: f_*\sigma$. For f_* to be a map of semi-simplicial sets, it needs to commute with face maps. Explicitly, we need $f_* \circ d_i = d_i \circ f_*$. A diagram is said to be *commutative* if all composites with the same source and target are equal, so this is equivalent to commutativity of the below.

$$\begin{tikzcd} \mathrm{Sin}_n(X) \arrow[r, "f_*"] \arrow[d, "d_i"] & \mathrm{Sin}_n(Y) \arrow[d, "d_i"] \\ \mathrm{Sin}_{n-1}(X) \arrow[r, "f_*"] & \mathrm{Sin}_{n-1}(Y) \end{tikzcd}$$

We see that $d_i f_*\sigma = (f_*\sigma) \circ d^i = f \circ \sigma \circ d^i$, and $f_*(d_i\sigma) = f_*(\sigma \circ d^i) = f \circ \sigma \circ d^i$ as desired. The diagram remains commutative when we pass to the free abelian groups of chains.

If C_* and D_* are chain complexes, a *chain map* $f\colon C_* \to D_*$ is a collection of maps $f_n\colon C_n \to D_n$ such that the following diagram commutes for every n:

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial_C"] & D_n \arrow[d, "\partial_D"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1} \end{tikzcd}$$

For example, if $f\colon X \to Y$ is a continuous map, then $f_*\colon S_*(X) \to S_*(Y)$ is a chain map as discussed above.

A chain map induces a map in homology $f_* : H_n(C) \to H_n(D)$. The method of proof is a so-called "diagram chase" and it will be the first of many. We check that we get a map $Z_n(C) \to Z_n(D)$. Let $c \in Z_n(C)$, so that $\partial_C c = 0$. Then $\partial_D f_n(c) = f_{n-1}\partial_C c = f_{n-1}(0) = 0$, because f is a chain map. This means that $f_n(c)$ is also an n-cycle, i.e., f gives a map $Z_n(C) \to Z_n(D)$.

Similarly, we also get a map $B_n(C) \to B_n(D)$. Let $c \in B_n(C)$, so that there exists $c' \in C_{n+1}$ such that $\partial_C c' = c$. Then $f_n(c) = f_n \partial_C c' = \partial_D f_{n+1}(c')$. Thus $f_n(c)$ is the boundary of $f_{n+1}(c')$, and f gives a map $B_n(C) \to B_n(D)$.

The two maps $Z_n(C) \to Z_n(D)$ and $B_n(C) \to B_n(D)$ give a map on homology $f_* : H_n(X) \to H_n(Y)$, as desired.

1.3 Categories, functors, natural transformations

From spaces and continuous maps, we constructed graded abelian groups and homomorphisms. We now cast this construction in the more general language of category theory.

Our discussion of category theory will be interspersed throughout the text, introducing new concepts as they are needed. Here we begin by introducing the basic definitions.

Definition 1.3.1. A *category* $\mathcal{C}$ is a class $\mathrm{ob}(\mathcal{C})$ of objects, such that for every two objects X and Y there is a set of *morphisms* $\mathcal{C}(X, Y)$ (thought of as a set of maps from X to Y). We require that for all $X \in \mathrm{ob}(\mathcal{C})$, there exists an identity element $1_X \in \mathcal{C}(X, X)$, and for all $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, there is a composition $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ sending $(f, g) \mapsto g \circ f$. These in turn satisfy the following:

- $1_Y \circ f = f$, and $f \circ 1_X = f$.
- Composition is associative.

Note that, for set-theoretic reasons, we require the collection of objects to be a class. This enables us to talk about a "category of all sets" for example, but not a "category of all categories" because that is too large.

We will often write $X \in \mathcal{C}$ to mean $X \in \mathrm{ob}(\mathcal{C})$, and $f\colon X \to Y$ to mean $f \in \mathcal{C}(X, Y)$.

Definition 1.3.2. If $X, Y \in \mathcal{C}$, then $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon Y \to X$ with $f \circ g = 1_Y$ and $g \circ f = 1_X$, and we write $X \cong Y$.

Example 1.3.3. Many common mathematical structures can be arranged in categories.

- Sets and functions between them form a category Set.
- Abelian groups and homomorphisms form a category **Ab**.
- Topological spaces and continuous maps form a category **Top**.
- Simplicial sets and their maps form a category sSet.
- A monoid is the same as a category with one object, where the elements of the monoid are the morphisms in the category.
- The sets $[n] = \{0, \ldots, n\}$ for $n \geq 0$ together with weakly order-preserving maps between them form the simplex category Δ.
- A poset forms a category in which there is a morphism from x to y iff $x \leq y$. However, note that $x \leq y$ and $y \leq x$ imply $x \cong y$ rather than $x = y$. The latter holds if the only isomorphisms are identities.

A small category is one such that $\mathrm{ob}(\mathcal{C})$ is a set, not necessarily a class. While we cannot consider the category of all categories, it is sensible to define the category **Cat** of all small categories. But first, we must first define a "morphism of categories."

Definition 1.3.4. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, such that for all $x, y \in \mathrm{ob}(\mathcal{C})$, there is a map $\mathcal{C}(x, y) \to \mathcal{C}(F(x), F(y))$ that respects composition and the identity.

The diagram at the beginning of the previous section shows the functors we have built thus far (although explicit verification of functoriality is left to the reader).

We go a step further. Suppose we fix categories $\mathcal{C}$ and $\mathcal{D}$ and consider all functors $\mathcal{C} \to \mathcal{D}$. What is a "morphism of functors"?

Definition 1.3.5. Let $F, G\colon \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\theta\colon F \to G$ consists of maps $\theta(X)\colon F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$ such that the following diagram commutes for all $f\colon X \to Y$:

$$\begin{tikzcd}
F(X) \arrow[r, "\theta(X)"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\theta(Y)"] & G(Y)
\end{tikzcd}$$

Definition 1.3.6. If $\mathcal{C}, \mathcal{D}$ are categories, $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

Going further down the rabbit hole leads to higher category theory, which we will not delve into.

Natural transformations are central to algebraic topology. We will frequently describe certain maps as "natural," which is to say that they are natural transformations. The reader should determine the functors involved if they are not explicitly stated.

Example 1.3.7. The boundary map $\partial \colon S_n \to S_{n-1}$ is a natural transformation.

Let G be a group viewed as a one-point category. Any element $F \in \mathrm{Fun}(G, \mathbf{Ab})$ is simply a group action of G on $F(*) = A$, i.e., a representation of G in abelian groups. Given another $F' \in \mathrm{Fun}(G, \mathbf{Ab})$ with $F'(*) = A'$, then a natural transformation from $F \to F'$ is precisely a G-equivariant map $A \to A'$.

1.4 More about categories

Let $\mathrm{Vect}_{\mathbf{C}}$ be the category of $\mathbf{C}$-vector spaces with $\mathbf{C}$-linear transformations. Given such a vector space V, taking the dual gives another vector space $V^* = \mathrm{Hom}(V, \mathbf{C})$, and a linear transformation $f : V \to W$ dualizes to $f^* : W^* \to V^*$. This process resembles a functor $\mathrm{Vect}_{\mathbf{C}} \to \mathrm{Vect}_{\mathbf{C}}$, except that it reverses the direction of morphisms.

Definition 1.4.1. Let $\mathcal{C}$ be a category. Define its *opposite category* $\mathcal{C}^{op}$ to have the same objects as $\mathcal{C}$ but with morphisms reversed, so that for all $X, Y \in \mathrm{ob}(\mathcal{C})$, we have $\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$. Composition in $\mathcal{C}^{op}$ is the same as in $\mathcal{C}$.

If $\mathcal{D}$ is another category, then a *contravariant functor* $\mathcal{C} \to \mathcal{D}$ is an ordinary (or *covariant*) functor $\mathcal{C}^{op} \to \mathcal{D}$.

Let $\mathcal{C}$ be a category, and let $Y \in \mathrm{ob}(\mathcal{C})$. Consider the functor $\mathcal{C}^{op} \to$ Set that takes $X \in \mathrm{ob}(\mathcal{C})$ to the set $\mathcal{C}(X, Y)$, and takes a map $f \colon X \to W$ to the map $- \circ f \colon \mathcal{C}(W, Y) \to \mathcal{C}(X, Y)$ which is precomposition by f. This functor, denoted $\mathcal{C}(-, Y)$, is called the functor *represented by* Y. Similarly, there is a functor $\mathcal{C}(-, Y)$ which is called the functor *corepresented by* Y. Note that $\mathcal{C}(-, Y)$ is contravariant and $\mathcal{C}(Y, -)$ is covariant.

Recall that Δ has objects $[0], [1], [2], \cdots$ and there is a functor $\Delta \to$ **Top** that sends $[n] \mapsto \Delta^n$ (see Exercise 1.1.6). Let X be a space, and consider the functor $\mathbf{Top}^{op} \to \mathrm{Set}$ represented by X. Composing these functors gives a functor $\Delta^{op} \to \mathrm{Set}$ which sends $[n] \mapsto \mathbf{Top}(\Delta^n, X) =: \mathrm{Sin}_n(X)$. This is precisely the singular simplicial set of X.

Proposition 1.4.2. *Simplicial sets are precisely functors $\Delta^{op} \to \mathrm{Set}$, i.e., $s\mathrm{Set} = \mathbf{Fun}(\Delta^{op}, \mathrm{Set})$. More generally, simplicial objects in a category $\mathcal{C}$ are functors $\Delta^{op} \to \mathcal{C}$, i.e., $s\mathcal{C} = \mathbf{Fun}(\Delta^{op}, \mathcal{C})$.*

If Δ_{inj} is the subcategory of Δ whose morphisms are only the injective maps, then $ss\mathcal{C} := \mathbf{Fun}(\Delta_{inj}^{op}, \mathcal{C})$ is the category of semi-simplicial objects in $\mathcal{C}$ (which differ from simplicial objects in that they only have face maps).

Definition 1.4.3. Let $X, Y \in \mathcal{C}$. We say that a morphism $f\colon X \to Y$ is a *split epimorphism* if there exists $g : Y \to X$ (often called a section or a splitting) such that $Y \xrightarrow{g} X \xrightarrow{f} Y$ is the identity 1_Y.

We say that a morphism $g\colon Y \to X$ is a *split monomorphism* if there exists $f\colon X \to Y$ such that $Y \xrightarrow{g} X \xrightarrow{f} Y$ is the identity 1_Y.

Example 1.4.4. Let $\mathcal{C} = \mathrm{Set}$. If $f : X \to Y$ is a split epimorphism with $f \circ g = 1_Y$, then for every $y \in Y$, we have $f(g(y)) = y$ and thus f is surjective. Is every surjective map a split epimorphism? Constructing a splitting g amounts to picking $g(y) \in f^{-1}(y)$ for all $y \in Y$, so this reduces to the axiom of choice.

Now assume that $g : X \to Y$ is a split monomorphism. Suppose that $y, y' \in Y$ are such that $g(y) = g(y')$. Applying f gives $y = y'$, and thus split monomorphisms are injective. Conversely, if Y is nonempty, then every injection $g : Y \to X$ is a split monomorphism.

Example 1.4.5. A morphism is an isomorphism if and only if it is both a split epi and a split mono.

Lemma 1.4.6. *If $f : X \to Y$ is a split epi (resp. split mono) in $\mathcal{C}$, and $F : \mathcal{C} \to \mathcal{D}$, then $F(f)$ is a split epi (resp. split mono) in $\mathcal{D}$.*

Proof. If g splits f in $\mathcal{C}$, then $F(g)$ splits $F(f)$ in $\mathcal{D}$. □

Exercise 1.4.7. Suppose $A, B \in \mathcal{C} = \mathbf{Ab}$ and $f : A \to B$ is a split epi, so that $fg = 1$ for some $g\colon B \to A$. Let $i\colon \ker f \to A$ be the inclusion, and consider the sum $\ker f \oplus B \xrightarrow{[i,g]} A$. Show that $[i, g]$ is an isomorphism.

If $g : B \to A$ is a split mono, there exists $f : A \to B$ so that $fg = 1$. Let $p\colon A \to \operatorname{coker}(g)$ be the quotient map, and consider the map $A \xrightarrow{\begin{pmatrix} p \\ f \end{pmatrix}} \operatorname{coker}(g) \oplus B$. Show that this is an isomorphism.

We have to get into some topology, since it's on our agenda. In the category **Top**, the one-point space $*$ is terminal, meaning that for any space X there is a unique map $X \to *$. This induces a map

$$H_n(X) \to H_n(*) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

which is called the *augmentation map*.

Take a 0-cycle $\sum a_i x_i$ where the x_i are points in X. Consider its homology class $[\sum a_i x_i] \in H_0(X)$. Under the induced map above, this is sent to $[\sum a_i *] = (\sum a_i)\,[*]$.

Definition 1.4.8. A *pointed space* is a pair $(X, *)$, with $* \in X$ called the *basepoint*.

Let $* \to X \to *$ be the inclusion of the basepoint followed by the unique map to $*$. In homology we get a map $\mathbf{Z} \xrightarrow{\eta} H_*(X) \xrightarrow{\epsilon} \mathbf{Z}$, where the composition is the identity. The map ϵ is the augmentation map described above, so we see that it is a split epimorphism. This means that, by Exercise 1.4.7, $H_*(X) \cong \mathbf{Z} \oplus \operatorname{coker}\eta \cong \mathbf{Z} \oplus \ker \epsilon$. The *reduced homology* of $(X, *)$ is $H_*(X, *) = \operatorname{coker}\eta$. It's isomorphic to $H_*(X)$ in dimensions greater than 0, but differs by a factor of $\mathbf{Z}$ in dimension 0.

1.5 Homotopy, star-shaped regions

As homology is a functor $H_* : \mathbf{Top} \to \mathbf{Ab}$, it preserves isomorphisms: homeomorphic spaces have isomorphic homology. However, homology is not always able to distinguish between non-homeomorphic spaces. We introduce the looser notion of homotopy, which is a central concept of algebraic topology, and later we will show that homology is a homotopy invariant.

Definition 1.5.1. Let $f_0, f_1 : X \to Y$ be two maps. A *homotopy* from f_0 to f_1 is a map $h : X \times I \to Y$ such that $h(x, 0) = f_0(x)$ and $f(x, 1) = f_1(x)$. We say that f_0 and f_1 are *homotopic* and write $f_0 \sim f_1$. This notation is justified because it is indeed an equivalence relation (transitivity follows from the gluing lemma).

We denote by $[X, Y]$ the set $\mathbf{Top}(X, Y)/\sim$.

Suppose we have a map $g : Y \to Z$ and homotopic maps $f_0, f_1 : X \to Y$, with a homotopy $h : f_0 \sim f_1$. Then $g \circ h$ gives a homotopy $g \circ f_0 \sim g \circ f_1$. Similarly, if $g : W \to X$ is a map and $f_0, f_1 : X \to Y$ are homotopic, then $f_0 \circ g \sim f_1 \circ g$.

If $g_0 \sim g_1 : Y \to Z$ and $f_0 \sim f_1 : X \to Y$, then $g_0 \circ f_0 \sim g_0 \circ f_1 \sim g_1 \circ f_1$. Hence we are able to compose homotopy classes, giving the dotted arrow below.

$$\begin{tikzcd} \mathbf{Top}(Y,Z)\times\mathbf{Top}(X,Y) \arrow[r] \arrow[d] & \mathbf{Top}(X,Z) \arrow[d] \\ {[Y,Z]\times[X,Y]} \arrow[r, dashed] & {[X,Z]} \end{tikzcd}$$

Definition 1.5.2. The *homotopy category of topological spaces* is $\mathrm{Ho}(\mathbf{Top})$ whose objects are topological spaces and $\mathrm{Ho}(\mathbf{Top})(X, Y) = [X, Y] = \mathbf{Top}(X, Y)/\sim$.

Definition 1.5.3. A map $f : X \to Y$ is a *homotopy equivalence* if $[f] \in [X, Y]$ is an isomorphism in $\mathrm{Ho}(\mathbf{Top})$. In other words, there is $g : Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

This is an interesting category because it has *terrible* categorical properties.

This sentence seems a bit random atm; give examples of why it's terrible? or just remove?

In Section 1.6 we will prove:

Theorem 1.5.4 (Homotopy invariance of homology)**.** *If $f_0 \sim f_1$, then $H_*(f_0) = H_*(f_1)$.*

This theorem states that the homology functor $H_* : \mathbf{Top} \to \mathbf{Ab}$ factors as $\mathbf{Top} \to \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Ab}$. In particular, it cannot distinguish between homotopy equivalent spaces. (Caution: spaces with isomorphic homology need not be homotopy equivalent.)

Example 1.5.5. The inclusion $S^{n-1} \subseteq \mathbf{R}^n - \{0\}$ is a homotopy equivalence. The homotopy inverse $p : \mathbf{R}^n - \{0\} \to S^{n-1}$ can be obtained by dividing a (always nonzero!) vector by its length. Clearly $p \circ i = 1_{S^{n-1}}$. A homotopy $i \circ p \sim 1_{\mathbf{R}^n - \{0\}}$ is given by $(v, t) \mapsto tv + (1 - t)\frac{v}{||v||}$. This example shows that homotopy equivalence does not preserve compactness.

Definition 1.5.6. A space X is *contractible* if the map $X \to *$ is a homotopy equivalence.

Definition 1.5.7. A *star-shaped region* is a subspace X of $\mathbf{R}^n$ for some n such that $0 \in X$, such that for all $x \in X$, and for all $t \in [0,1]$, $tx \in X$.

For example, any convex region containing the origin is star-shaped. An argument similar to the one in Example 1.5.5 shows that the inclusion of $\{0\}$ into a star-shaped region is a homotopy equivalence. Thus star-shaped regions are contractible. Our goal now is to prove the following.

Theorem 1.5.8. *Let X be a star-shaped region. The augmentation map $\epsilon : H_*(X) \to \mathbf{Z}$ is an isomorphism, i.e., $H_0(X) \cong \mathbf{Z}$ and $H_i(X) \cong 0$ for $i > 0$.*

Before proving this, we will give a notion of homotopy in the category of chain complexes.

Definition 1.5.9. Let $C_\bullet, D_\bullet$ be chain complexes, and $f_0, f_1 : C_\bullet \to D_\bullet$ be chain maps. A *chain homotopy* $h : f_0 \sim f_1$ is a collection of homomorphisms $h : C_n \to D_{n+1}$ such that $\partial h + h\partial = f_1 - f_0$.

$$\begin{tikzcd}
C_{n+2} \arrow[r, "f_1-f_0"] \arrow[d, "\partial"] & D_{n+2} \arrow[d, "\partial"] \\
C_{n+1} \arrow[r, "f_1-f_0"] \arrow[d, "\partial"] \arrow[ur, dashed, "h"] & D_{n+2} \arrow[d, "\partial"] \\
C_n \arrow[r, "f_1-f_0"'] \arrow[ur, dashed, "h"] & D_n
\end{tikzcd}$$

The following lemma shows the significance of this condition.

Lemma 1.5.10. *If $f_0, f_1 : C_\bullet \to D_\bullet$ are chain homotopic, then $f_{0,*} = f_{1,*} : H(C) \to H(D)$.*

Proof. We show that $(f_1 - f_0)_* = 0$. Let $c \in Z_n(C_\bullet)(C)$, so that $\partial c = 0$. Then $(f_1 - f_0)_* c = (\partial h + h\partial)c = \partial hc + h\partial c = \partial hc$ is a boundary, which is zero in homology. $\square$

Proof of Theorem 1.5.8. The maps $\{0\} \to X$ and $X \to \{0\}$ induce $\mathbf{Z} \xrightarrow{\eta} S_*(X)$ and $S_*(X) \xrightarrow{\epsilon} \mathbf{Z}$ respectively. It is clear that $\epsilon\eta = 1 : \mathbf{Z} \to \mathbf{Z}$. We now proceed to show that $\eta\epsilon \sim 1 : S_*(X) \to S_*(X)$. Note that the composite $\eta\epsilon$ kills all chains in dimensions greater than zero, and on zero-chains we have $\eta\epsilon(\sum a_i x_i) = (\sum a_i)c_0$ where c_0 is the zero-simplex at the origin.

For $\sigma \in \mathrm{Sin}_q(X)$, define $h : \mathrm{Sin}_q(X) \to \mathrm{Sin}_{q+1}(X)$ as follows:

$$h\sigma(t_0, \cdots, t_{q+1}) = (1-t_0)\sigma\left(\frac{(t_0, \cdots, t_{q+1})}{1-t_0}\right).$$

This extends by linearity to a map $h : S_q(X) \to S_{q+1}(X)$. Observe that $d_0 h\sigma = \sigma$, and if $q \geq 1$, $d_i h\sigma = hd_{i-1}\sigma$. Then, if $\sigma \in \mathrm{Sin}_q(X)$ with $q \geq 1$,

$$\begin{aligned}
\partial h\sigma &= \sum_{i=0}^{q+1}(-1)^i d_i h\sigma \\
&= \sigma - h\sum_{i=0}^{q}(-1)^i d_i\sigma \\
&= \sigma - h\partial\sigma \\
\partial h\sigma + h\partial\sigma &= \sigma.
\end{aligned}$$

If $\sigma \in \mathrm{Sin}_0(X)$, then $d_1 h\sigma = c_0$ and we instead get $\partial h\sigma + h\partial\sigma = \sigma - c_0$. Thus $h : S_q(X) \to S_{q+1}(X)$ is a chain homotopy $\eta\epsilon \sim 1$ as desired. □

1.6 Homotopy invariance of homology

As a side product, we'll construct something called the cross product. Here's the theorem.

Theorem 1.6.1. *If $f_0, f_1 : X \to Y$ and $f_0 \sim f_1$, then $f_{0,*} \sim f_{1,*} : S_*(X) \to S_*(Y)$.*

Corollary 1.6.2. *With the same hypotheses, then $f_{0,*} = f_{1,*} : H_*(X) \to H_*(Y)$, because chain homotopic maps induce the same map on homology.*

The proof uses naturality (a lot). We'll produce a chain homotopy from the two inclusions $i_0, i_1 : X \to X \times I$, and then homotope f_0, f_1 through a map $X \times I \to Y$. Namely, we'll construct a chain homotopy $h_X : S_n(X) \to S_{n+1}(X \times I)$ which gives a homotopy $i_{0,*} \sim i_{1,*}$, and this'll be natural in X.

This gives the result we want. We can get a homotopy $h := g \circ h_* : S_n(X) \to S_{n+1}(X \times I) \to S_{n+1}(Y)$. Well:

$$\begin{aligned}
\partial h + h\partial &= \partial(g_* h_*) + g_* h_* \partial = g_* \partial h_* + g_* h_* \partial = g_*(\partial h_* + h_* \partial) \\
&= g_*(i_{1,*} - i_{0,*}) = g_* i_{1,*} - g_* i_{0,*} = (g \circ i_1)_* - (g \circ i_0)_* = f_{1,*} - f_{0,*}
\end{aligned}$$

(The last equality is because h_* is a chain homotopy between $i_{1,*}$ and $i_{0,*}$.) So $h = g_* \circ h_*$ is a chain homotopy. But how do we define $h_* : S_n(X) \to S_{n+1}(X \times I)$? You want to take $\sigma \mapsto$ "$\sigma \times I''$. More generally, we'll set up a cross product $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ that is natural, bilinear, satisfy the Leibniz rule, and are normalized.

Naturality is exactly what you'd expect it to be. If A, B, C are abelian groups, then $A \times B \to C$ is a bilinear map if $f(a + a', b) = f(a, b) + f(a', b)$ and similarly in the other variable (just substitute the $S_n(X)$ etc here). The Leibniz formula says:

$$\partial(a \times b) = (\partial a) \times b + (-1)^{|a|} a \times \partial b, \quad \text{where } |a| = p \text{ means that } a \in S_p(X).$$

The word normalized means that the following construction is correct. Suppose $q = 0$; then this is a map $S_p(X) \to S_0(Y) \to S_p(X \times Y)$, which (it suffices to define a map $\mathrm{Sin}_p(X) \times \mathrm{Sin}_0(Y) \to S_p(X \times Y)$ because $\left(\sum_i a_i\sigma_i, \sum_j b_j\tau_j\right) \mapsto \sum_{i,j} a_i b_i(\sigma_i \times \tau_j)$ by bilinearity) sending:

$$(\sigma, c_y^0) \mapsto \left(\begin{pmatrix} \sigma \\ c_y^p \end{pmatrix} : \Delta^p \to X \times Y\right)$$

This latter map is just the composition $\Delta^p \xrightarrow{\sigma} X \xrightarrow{\text{inclusion at } y \in Y} X \times Y$. When $p = 0$, we can send:

$$(c_x^0, \tau) \mapsto \left(\begin{pmatrix} c_0^p \\ \tau \end{pmatrix} : \Delta^p \to X \times Y\right)$$

This latter map is just the composition $\Delta^p \xrightarrow{\tau} Y \xrightarrow{\text{inclusion at } x \in X} X \times Y$. We have to check that this behaves correctly for the boundary map, namely, that Leibniz holds. We have:

$$\partial(\sigma \times c_y^0) = \partial\sigma \times c_y^0$$

and similarly. We're going to use induction to define this for $p+q$; we've only done this for $p + q = 0, 1$. Let's assume it's done for $p + q - 1$. First note that there's a universal example of a p-simplex, namely the map $\iota_p : \Delta^p \to \Delta^p$, because given any p-simplex $\sigma : \Delta^p \to X$, you get $\sigma = \sigma_*(\iota_p)$ where $\sigma_* : \mathrm{Sin}_p(\Delta^p) \to \mathrm{Sin}_p(X)$. It suffices to define $\iota_p \times \iota_q \in S_{p+q}(\Delta^p \times \Delta^q)$; you're supposed to think of the product of two simplices as a prism, which isn't a simplex itself - but you can triangulate it and look at it as the formal sum of two simplices. Then

$\sigma \times \tau = (\sigma \times \tau)_*(\iota_p \times \iota_q)$ where $(\sigma \times \tau)_* : S_{p+q}(\Delta^p \times \Delta^q) \to S_{p+q}(X \times Y)$. We need this to satisfy the Leibniz rule, so that:

$$S_{p+q-1}(\Delta^p \times \Delta^q) \ni \partial(\iota_p \times \iota_q) = (\partial \iota_p) \times \iota_q + (-1)^p \iota_p \times \partial \iota_q$$

A necessary condition for $\iota_p \times \iota_q$ to exist is that $\partial((\partial\iota_p) \times \iota_q + (-1)^p \iota_p \times \partial\iota_q) = 0$. Let's compute what this is.

$$\partial((\partial\iota_p)\times\iota_q+(-1)^p\iota_p\times\partial\iota_q) = \partial^2(\iota_p)\times\iota_q+(-1)^{p-1}(\partial\iota_p)\times(\partial\iota_q)+(-1)^p(\partial\iota_p)\times(\partial\iota_q)+(-$$

because $\partial^2 = 0$.

The subspace $\Delta^p \times \Delta^q \subseteq \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$ is convex, so by translation, it's homeomorphic to a star-shaped region. But we know that $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ because $p + q > 1$, which means that every cycle is a boundary. In other words, what we checked above is also sufficient! So, choose any element $\iota_p \times \iota_q$ with the right boundary. This means we're done if we check that this choice satisfies naturality, bilinearity, and the Leibniz rule (left to reader). We'll now define $h_X : S_n(X) \to S_{n+1}(X \times I)$ via $h_X c = c \times \iota$ where $\iota : \Delta^1 \to I$ is the obvious map. The cross product is in $S_{p+1}(X \times I)$. Let's compute:

$$\partial h_X = \partial(c \times \iota) = \partial c \times \iota + (-1)^{|c|} c \times \partial \iota$$

But now, $\partial\iota = c_1^0 - c_0^0 \in S_0(I)$, which means that this becomes $\partial c \times \iota + (-1)^{|c|}(\iota_{1,*} - \iota_{0,*})$. We'll do a little more next time.

1.7 Basically a recap of what we did last time

PSet 1 due today. Last time, we showed:

Theorem 1.7.1. *There exists a map $S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ that is:*

- *Natural, in the sense that if $f : X \to X'$ and $g : Y \to Y'$, and $a \in S_p(X)$ and $b \in S_p(Y)$ so that $a \times b \in S_{p+q}(X \times Y)$, then $f_*(a) \times g_*(b) = (f \times g)_*(a \times b)$.*

- *Bilinear, in the sense that $(a + a') \times b = (a \times b) + (a' \times b)$, and $a \times (b + b') = a \times b + a \times b'$.*

- *The Leibniz rule is satisfied, i.e., $\partial(a \times b) = (\partial a) \times b + (-1)^{|a|} a \times \partial b$.*

- *Normalized, in the following sense. Let $x \in X$ and $y \in Y$. Write $i_x : Y \to X \times Y$ sending $y \mapsto (x, y)$, and write $i_y : X \to X \times Y$ sending $x \mapsto (x, y)$. If $b \in S_q(Y)$, then $c^0_x \times b = (i_x)_* b \in S_q(X \times Y)$, and if $a \in S_p(X)$, then $a \times c^0_y = (i_y)_* a \in S_p(X \times Y)$.*

We were a little hasty in the end, so we're going to recall some things.

Proof sketch. There were two steps.

1. It's enough to define $\iota_p \times \iota_q \in S_{p+q}(\Delta^p \times \Delta^q)$ where $\iota_n : \Delta^n \to \Delta^n$ is the identity, because every other simplex is ι_n pushed forward, and the cross product is supposed to be natural.

2. Induction on $p+q$. The first thing we use is the Leibniz rule, namely $\partial(\iota_p \times \iota_q) = (\partial \iota_p) \times \iota_q + (-1)^p \iota_p \times \partial \iota_q$. Is this a boundary? A necessary thing for anything to be a boundary is that it's a cycle. But because $\Delta^p \times \Delta^q$ is homeomorphic to a star-shaped region, it's contractible - therefore $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$, a *sufficient* condition for something to be a boundary is that it's a cycle! We showed that $\partial(\iota_p \times \iota_q)$ is a cycle, and therefore a boundary. We just need to choose *some* class $[a]$ in $S_{p+q}(\Delta^p \times \Delta^q)$ such that $\partial([a]) = (\partial \iota_p) \times \iota_q + (-1)^p \iota_p \times \partial \iota_q$, and this works.

   a) Naturality is left to the reader.

   b) Let's check the Leibniz rule. Let $\sigma : \Delta^p \to X$ and $\tau : \Delta^q \to Y$. What's $\partial(\sigma \times \tau)$? We start by asking how we define $\sigma \times \tau$. Well, this is just $\sigma_* \iota_p \times \tau_* \iota_q$. Because of naturality, this is just $(\sigma \times \tau)_*(\iota_p \times \iota_q)$. This means that $\partial(\sigma \times \tau) = \partial((\sigma \times \tau)_*(\iota_p \times \iota_q))$. Now, we can use the naturality of the boundary map to see that this is just $(\sigma \times \tau)_* \partial(\iota_p \times \iota_q)$, which we can expand as:

$$\begin{aligned}
(\sigma \times \tau)_* \partial(\iota_p \times \iota_q) &= (\sigma \times \tau)_*((\partial \iota_p) \times \iota_q + (-1)^p \iota_p \times \partial \iota_q) \\
&= \sigma_*(\partial \iota_p) \times \tau_* \iota_q + (-1)^p(\sigma_* \iota_p \times \tau_*(\partial \iota_q)) \\
&= \partial(\sigma_* \iota_p) \times \tau_* \iota_q + (-1)^p(\sigma_* \iota_p \times \partial(\tau_* \iota_q)) \\
&= \partial \sigma \times \tau + (-1)^p \sigma \times \partial \tau
\end{aligned}$$

□

A key fact in this whole thing is that $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$. This method of proof, namely of reducing to things that have zero homology (aka acyclic spaces) is called the *method of acyclic models.*

What happens on the level of homology? Let's abstract a little bit. Suppose we have three chain complexes $A_\bullet$, $B_\bullet$, and $C_\bullet$, to be thought of as $S_*(X)$, $S_*(Y)$, and $S_*(X \times Y)$. Suppose we have maps $\times : A_p \times B_q \to C_{p+q}$ that satisfies bilinearity and the Leibniz formula. What does this induce in homology?

Lemma 1.7.2. *This determines a bilinear map $H_p(A) \times H_q(B) \xrightarrow{\times} H_{p+q}(C)$.*

Proof. Let $[a] \in H_p(A)$ where $a \in Z_p(A)$ such that $\partial a = 0$ (i.e., a is a cycle). Let $[b] \in H_q(B)$ where $b \in Z_q(A)$ such that $\partial b = 0$. We want to define $[a] \times [b] \in H_{p+q}(C)$. We hope that $[a] \times [b] = [a \times b]$. We need to check that $a \times b$ is a cycle; let's check. By Leibniz, $\partial(a \times b) = \partial a \times b + (-1)^p a \times \partial b$. Because a, b are boundaries, this is zero. We still need to check that this thing is well-defined. Let's pick another $[a'] = [a]$ and $[b'] = [b]$. We want $[a \times b] = [a' \times b']$. In other words, we need that $a \times b$ differs from $a' \times b'$ by a boundary. We can write $a' = a + \partial \overline{a}$ and $b' = b + \partial \overline{b}$. What's $a' \times b'$? It's:

$$a' \times b' = (a + \partial \overline{a}) + (b + \partial \overline{b}) = a \times b + \left(a \times \partial \overline{b} + (\partial \overline{a}) \times b + (\partial \overline{a}) \times (\partial \overline{b})\right)$$

But, well, $\partial(a \times \overline{b}) = \partial a \times \overline{b} + (-1)^p a \times \partial \overline{b} = (-1)^p a \times \partial \overline{b}$, and $\partial(\overline{a} \times b) = \partial \overline{a} \times b$, and $\partial(\overline{a} \times \partial \overline{b}) = \partial \overline{a} \times \partial \overline{b}$. This means that $a' \times b' = a \times b + \partial((-1)^{-p}(a \times \overline{b}) + \overline{a} \times b + \overline{a} \times \partial \overline{b})$. They differ by a boundary, so it's well-defined.

The last step is to check bilinearity, which is left to the reader. ☐

This gives the following result.

Theorem 1.7.3. *There is a map $H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y)$ that's natural, bilinear, and normalized. This map is also* unique *(unlike the map $S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, which isn't unique because we there are uncountably many choices of $\iota_p \times \iota_q$, all differing by a boundary), because the map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$ is unique up to chain homotopy!*

Let's go back to homotopy invariance. Recall that if $f_0 \sim f_q : X \to Y$, then $f_{0,*} = f_{1,*} : S_*(X) \to S_*(Y)$. We proved this by showing that this reduces to showing that the two inclusions $i_0, i_1 : X \to X \times I$ induce the same map on $S_*(X) \to S_*(X \times I)$. The chain homotopy $h_X : i_{0,*} \sim i_{1,*}$ is defined as follows. Let $c \in S_p(X)$. We need to give an element of $S_{p+1}(X \times I)$, so if $\iota : \Delta^1 \to I$ is the obvious map, we just define $h_X(c) = (-1)^p c \times \iota$. Let's check that.

Let's compute $\partial h_X c$. This is $\partial((-1)^p c \times \iota) = (-1)^p \partial(c \times \iota)$, which, expanded out, is:

$$\partial((-1)^p c \times \iota) = (-1)^p \partial(c \times \iota) = (-1)^p (\partial c) \times \iota + (-1)^{2p} c \times \partial \iota$$

Well, $\partial \iota = c_1^0 - c_0^0 \in S_0(I)$, so this is equal to $(-1)^p(\partial c) \times \iota + c \times c_1^0 - c \times c_0^0$. But $c \times c_1^0 = (i_1)_* c - (i_0)_* c$. Therefore, this is $(-1)^p (\partial c) \times \iota + (i_1)_* c - (i_0)_* c$. On the other hand, what's $h_X \partial c = (-1)^{p-1}(\partial c) \times \iota$. Let's add them together:

$$\partial h_X c + h_X \partial c = (-1)^p (\partial c) \times \iota + (i_1)_* c - (i_0)_* c + (-1)^{p-1}(\partial c) \times \iota = (i_1)_* c - (i_0)_* c$$

So this is, by definition, a chain homotopy!

I just want to mention that there's an explicit choice of $\iota_p \times \iota_q$. This is called the Eilenberg-Zilber chain. You're highly encouraged to think about this yourself. We're going to consider $\Delta^{p+q} \to \Delta^p \times \Delta^q$. They're all affine maps, sending vertices to pairs of vertices. We're going to think of an ordered map $\omega : [p+q] \to [p] \times [q]$. We can complete the diagram to get:

$$\begin{tikzcd}
& {[p]} \\
{[p+q]} \arrow[ur, "pr_2\omega"] \arrow[r, "\omega"] \arrow[dr, "pr_1\omega"'] & {[p] \times [q]} \arrow[u, "pr_2"'] \arrow[d, "pr_1"] \\
& {[q]}
\end{tikzcd}$$

We also want ω to be injective (which requires that it takes $(0,0)$ to (p,q)). We can draw out a "staircase" in the $p \times q$ grid, and the area under the staircase defined by ω is denote $A(\omega)$. Define $\iota_p \times \iota_q = \sum (-1)^{A(\omega)} \overline{\omega}$ where $\overline{\omega}$ is the corresponding affine map $\Delta p + q \to \Delta^p \times \Delta^q$. It's combinatorially annoying to check that this satisfies the conditions of the theorem, but it's a good exercise to check it out. It's in a paper by Eilenberg-Moore.

1.8 Relative Homology

First, let's recall a little about homology. We showed that homology factors as a functor **Top** $\to$ h**Top** $\to$ **GrAb**. As a corollary:

Corollary 1.8.1. *If $f : X \to Y$ is a homotopy equivalence (i.e., an isomorphism in the homotopy category), the induced map on homology*

$H_*(f) : H_*(X) \to H_*(Y)$ is an isomorphism. This is the same thing as saying that if $f : X \to Y$ does not induce an isomorphism on homology, then f can't be a homotopy equivalence. Homology's therefore often used to distinguish spaces.

Example 1.8.2. If $X \times \mathbf{R}^n \to X$ is the projection map, this is a homotopy equivalence because $\mathbf{R}^n$ is contractible. Therefore, $H_*(X \times \mathbf{R}^n) \cong H_*(X)$.

Towards computing homology

Fix a space W, and consider the functor $X \mapsto [W, X]$. This is basically uncomputable - if you could, you'll win a Fields medal! But there's something more that homology has going for it is that it's "local". What do we mean by this? Homology is a bit like a measure:

1. If $A \subseteq X$ is a subspace, then $H_*(X)$ is related to $H_*(A) + H_*(X - A)$. Called the lexseq of a pair.
2. The homology $H_*(A \cup B)$ is like $H_*(A) + H_*(B) - H_*(A \cap B)$. Called the Mayer-Vietoris sequence.

The thing we use is the notion of exact sequences. Let me tell you about exact sequences.

Exact sequences

This is a story about abelian groups.

Definition 1.8.3. Let $A \xrightarrow{i} B \xrightarrow{p} C$ be a sequence of abelian groups and homomorphisms. We say that the sequence is exact (at B) if $\ker p = \operatorname{im} i$, i.e., $p \circ i = 0$ with no room for error.

Example 1.8.4. If you have a chain complex $\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$, it's exact at C_n if the homology $H_n(C)$ in dimension n is zero. So homology is the obstruction to exactness.

Example 1.8.5. $0 \to A \xrightarrow{i} B$ is exact iff i is injective, and $B \xrightarrow{p} C \to 0$ is exact iff p is surjective.

There's a beautiful book by Eilenberg and Steenrod, published in 1952, which was the founding of algebraic topology.

Example 1.8.6. If you have a sequence that's exact at every point, it's called a long exact sequence (henceforth called lexseq in these notes). If you have a sequence like $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ that's exact, then this is called a short exact sequence (henceforth called sexseq in these notes). This means that $p \circ i = 0$, i is injective, p is surjective. Also, this sequence factors like:

$$\begin{tikzcd}
\ker(p) \arrow[dr] & & \\
A \arrow[u] \arrow[r, "i"] & B \arrow[r, "p"] \arrow[dr] & C \\
& & \operatorname{coker}(i) \arrow[u]
\end{tikzcd}$$

So $A \cong \ker p$ and $B \cong \operatorname{coker}(i)$. These things are equivalence to short exactness.

Let's see how this appears in algebraic topology.

Definition 1.8.7. A pair of spaces is a space X together with subspaces $A \subseteq X$, denoted (X, A). We have a new category, called $\mathbf{Top}_2$ where morphisms $(X, A) \to (Y, B)$ are maps $f : X \to Y$ that take $A \to B$. There are two functors $\mathbf{Top} \to \mathbf{Top}_2$, sending $X \mapsto (X, \emptyset)$ and $X \mapsto (X, X)$. There are also two functors back to $\mathbf{Top}$, sending $(X, A) \mapsto A$ or $(X, A) \mapsto X$.

How does this behave on the level of chain complexes? If I have a pair (X, A), I get a map $\mathrm{Sin}_n(A) \to \mathrm{Sin}_n(X)$ that's clearly injective. Is this a split monomorphism? Yes, unless $A =$, because you can choose a point in A and send everything not in A to that point. Let's now apply the free abelian group functor to get $S_n(A) \to S_n(X)$. Is this a monomorphism? Yes, because monomorphisms are preserved by this functor. This is also split because being a split mono is a categorical property. This is split even when $A =$ because then $S_n() = 0$.

Definition 1.8.8. The relative n-chains is defined as $S_n(X, A) := S_n(X)/S_n(A)$. So we have a sexseq (ses) $0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$. Is $S_n(X, A)$ free abelian if $S_n(X)$ and $S_n(A)$ are? If you have a ses $0 \to A \to B \to C \to 0$ such that $A \to B$ is split, then it's homework to show that $B \cong A \oplus C$. So C must also be free abelian if B and A are; i.e., $S_n(X, A)$ is free abelian.

Example 1.8.9. Consider Δ^n, which contains its boundary $\partial\Delta^n := \bigcup \operatorname{im} d_i \simeq S^{n-1}$. We have the identity map $\iota_n : \Delta^n \to \Delta^n$, the universal n-simplex, which is in $\mathrm{Sin}_n(\Delta^n) \subseteq S_n(\Delta^n)$. Its boundary $\partial\iota_n \in S_{n-1}(\Delta^n)$, but it actually lands in $S_{n-1}(\partial\Delta^n)$. So $\partial\iota_n$ is *not* a boundary in $\partial\Delta^n$, as we'll see, but it certainly is a cycle. So it determines a homology class, $[\partial\iota_n]$, which, it turns out, generates $H_{n-1}(\partial\Delta^n) \simeq H_{n-1}(S^{n-1}) \cong \mathbf{Z}$.

I can think of $\iota_n \in S_n(\Delta^n, \partial\Delta^n)$, or rather the class of it mod $S_n(\partial\Delta^n)$. It's a relative chain. Is it a cycle? Let's branch off a bit.

Consider the ses $0 \to S_n(A) \to S_n(X) \to S_n(X,A) \to 0$. A $c \in S_n(X)$ determines a relative cycle if $\partial c \in S_{n-1}(A)$. I'm sorry, I've messed this up a little bit. There's so much to say here. I'm getting ahead of myself a little bit here. What I meant to say is, let's think of what ∂ does. We have a map of ses:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & S_n(A) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & S_n(X,A) \arrow[r] \arrow[d, dashed] & 0 \arrow[d] \\
0 \arrow[r] & S_{n-1}(A) \arrow[r] & S_{n-1}(X) \arrow[r] & S_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

Does the dotted map exist? We can pull $\bar{c} \in S_n(X,A)$ to some $c \in S_n(X)$, and then define $\partial\bar{c}$ to be the pushforward of ∂c. Is this well-defined? If c, c' both map to $\bar{c}$, then $c - c' = 0$, so there's some a in $S_n(A)$ that is sent to $c - c'$, and ∂ pushes this forward to say that ∂a maps to $\partial(c - c') = \partial c - \partial c'$. Since we're quotienting out by $S_{n-1}(A)$, this means that the pushforwards of ∂c and $\partial c'$ are the same. I'll leave it to you (although Professor Miller explained this in detail) to show that $\partial^2 = 0$.

Now let's continue. A class $c \in S_n(X)$ gives a relative cycle if and only if $\partial c \in S_{n-1}(A)$ because we want ∂c in $S_n(X,A)$ to be zero. So $\iota_n \in S_n(\Delta^n, \partial\Delta^n)$ is indeed a relative cycle since $\partial\iota_n \in S_{n-1}(\partial\Delta^n)$. Similarly, a class $c \in S_n(X)$ is a relative boundary if and only if there is a $b \in S_{n+1}(X)$ such that $\partial b = c \bmod S_n(A)$, i.e., $\partial b - c \in S_n(A)$. So $\iota_n \in S_n(\Delta^n, \partial\Delta^n)$ isn't a relative boundary. Therefore $H_n(\Delta^n, \partial\Delta^n) \cong \mathbf{Z} = \langle[\iota_n]\rangle$ where $H_n(X,A)$ denotes the relative homology. This stuff takes a little bit of time to get used to.

1.9 Long exact homology sequence, Excision, and Genealogy

WARNING: I've probably messed up typing some indices. Hopefully not.

5-lemma

Suppose you have two exact sequences of abelian groups.

$$\begin{tikzcd}
A_4 \arrow[r, "d"] \arrow[d, "f_4"] & A_3 \arrow[r, "d"] \arrow[d, "f_3"] & A_2 \arrow[r, "d"] \arrow[d, "f_2"] & A_1 \arrow[r, "r"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
B_4 \arrow[r, "d"] & B_3 \arrow[r, "d"] & B_2 \arrow[r, "d"] & B_1 \arrow[r, "r"] & B_0
\end{tikzcd}$$

When can we guarantee that f_2 is an isomorphism? We're going to diagram chase :-(. Just follow your nose.

Let $b_2 \in B_2$. We want to show that there is something in A_2 that can be pushed forward to b_2, i.e., prove surjectivity of f_2. We can consider $db_2 \in B_1$. Let's assume that f_1 is surjective. Then there's $a_1 \in A_1$ such that $f_1(a_1) = db_2$. What is da_1? Well, $f_0(da_1) = d(f_1(a_1)) = d(db) = 0$. So we want f_0 to be injective. Then da_1 is zero, so by exactness of the top sequence, there is some $a_2 \in A_2$ such that $da_2 = a_1$. What is $f_2(a_2)$? What is $d(f_2(a_2))$? By commutativity, $d(f_2(a_2)) = f_1(d(a_2)) = f_1(a_1) = db_2$. Let's consider $b_2 - f_2(a_2)$. This maps to zero under d. So by exactness, there is $b_3 \in B_3$ such that $d(b_3) = b_2 - f_2(a_2)$. If we assume that f_3 is surjective, then there is $a_3 \in A_3$ such that $f_3(a_3) = b_3$. But now, $d(a_3) \in A_2$, and $f_2(d(a_3)) = d(f_3(a_3)) = b_2 - f_2(a_2)$. But this means that $b_2 = f(a_2 + d(a_3))$, which guarantees surjectivity of f_2. This means that if f_1 is surjective, f_0 is injective, and f_3 is surjective, then f_2 is surjective.

A similar dual process says that f_2 is injective if f_1 is injective, f_3 is injective, and f_4 is surjective. If all of these conditions are satisfied, then f_2 is an isomorphism. This is the content of the five-lemma.

Relative homology

I guess I didn't really define this. Suppose you have a pair of spaces (X, A). Then you have an sexseq of chain complexes $0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$.

Definition 1.9.1. The relative homology of the pair $H_*(X, A) := H(S_*(X, A))$.

Example 1.9.2. $H_*(X, \emptyset) = H_*(X)$ because $S_*(\emptyset) = 0$. Another case is $H_*(X, X) = 0$ because $S_*(X, X) = S_*(X)/S_*(X) = 0$.

General study of homologies of sexseqs of chain complexes

Suppose I have three chain complexes $A_\bullet \to B_\bullet \to C_\bullet$. By the way, this is an important announcement. Henceforth, differentials in chain complexes will be denoted d, no longer ∂. For some reason. (It's so much easier for typing as well.) Assume that this is an exact sequence of chain complexes.

Is $H_*(A) \to H_*(B) \to H_*(C)$ exact? Let's push this a little further. Suppose I have a sexseq $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$. We can ask the same question as before. Let's write this out more explicitly.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "f"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g"] \arrow[d, "d"] & C_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "d"] & B_n \arrow[r, "g"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Let $[b] \in H_n(B)$ such that $g([b]) = 0$. It's determined by some $b \in B_n$ such that $d(b) = 0$. If $g([b]) = 0$, then there is some $\overline{c} \in C_{n+1}$ such that $d\overline{c} = gb$. Now, g is surjective, so there is some $\overline{b} \in B_{n+1}$ such that $g(\overline{b}) = \overline{c}$. Then we can consider $d\overline{b} \in B_n$, and $g(d(\overline{b})) = d(\overline{c}) \in C_n$. What is $b - d\overline{b}$? This maps to zero in C_n, so by exactness there is some $a \in A_n$ such that $f(a) = b - d\overline{b}$. Is a a cycle? Well, $f(da) = d(fa) = d(b - d\overline{b}) = db - d^2\overline{b} = db$, but we assumed that $db = 0$, so $f(da) = 0$. This means that da is zero because f is an injection by exactness. Therefore a is a cycle. What is $[a] \in H_n(A)$? Well, $f([a]) = [b - d\overline{b}] = [b]$ because $d\overline{b}$ is a cycle. Is the composite $H_n(A) \to H_n(B) \to H_n(C)$ zero? Yes, because the composite factors through zero. This proves exactness of $H_n(A) \to H_n(B) \to H_n(C)$.

Theorem 1.9.3 (lexseq in homology)**.** *Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a sexseq of chain complexes. Then there is a natural homomorphism*

$\partial : H_n(C) \to H_{n-1}(A)$ such that there's lexseq:

$$
\begin{tikzcd}
 & & {} \arrow[dll, "\partial"'] \\
H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, "\partial"'] \\
H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[dll, "\partial"'] \\
{} & &
\end{tikzcd}
$$

Proof. We'll construct ∂, and leave the rest as an exercise. We have our sexseq:

$$
\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "f"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g"] \arrow[d, "d"] & C_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "d"] & B_n \arrow[r, "g"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

Let $c \in C_n$ such that $dc = 0$. The map g is surjective, so pick a $b \in B_n$ such that $g(b) = c$. Then consider $db \in B_{n-1}$. But $g(d(b)) = 0 = d(g(b)) = dc$. So by exactness, there is some $a \in A_{n-1}$ such that $f(a) = db$. How many choices are there of picking a? One, because a is injective. We need to check that a is a cycle. What is $d(a)$? Well, $d^2 b = 0$, so da maps to 0 under f. But because f is injective, $da = 0$, i.e., a is a cycle. This means we can define $\partial[c] = [a]$.

To make sure that this is well-defined, let's make sure that this choice of homology class a didn't depend on the b that we chose. Pick some other b' such that $g(b') = c$. Then there is $a' \in A_{n-1}$ such that $f(a') = db'$. We want $a - a'$ to be a boundary, so that $[a] = [a']$. We want $\overline{a} \in A_n$ such that $d\overline{a} = a - a'$. Well, $g(b - b') = 0$, so by exactness, there is $\overline{a} \in A_n$ such that $f(\overline{a}) = b - b'$. What is $d\overline{a}$? Well, $d\overline{a} = d(b - b') = db - db'$. But $f(a - a') = b - b'$, so because f is injective, $d\overline{a} = a - a'$, i.e., $[a] = [a']$. What else do I have to check? It's an exercise to check that ∂ as defined here is a homomorphism. Also, left as an exercise to check that this doesn't depend on $c \in [c]$, and that ∂ actually makes the exact sequence above exact. $\square$

Mathematical Genealogy

I'm not typing in anything here. It's a rather big tree:

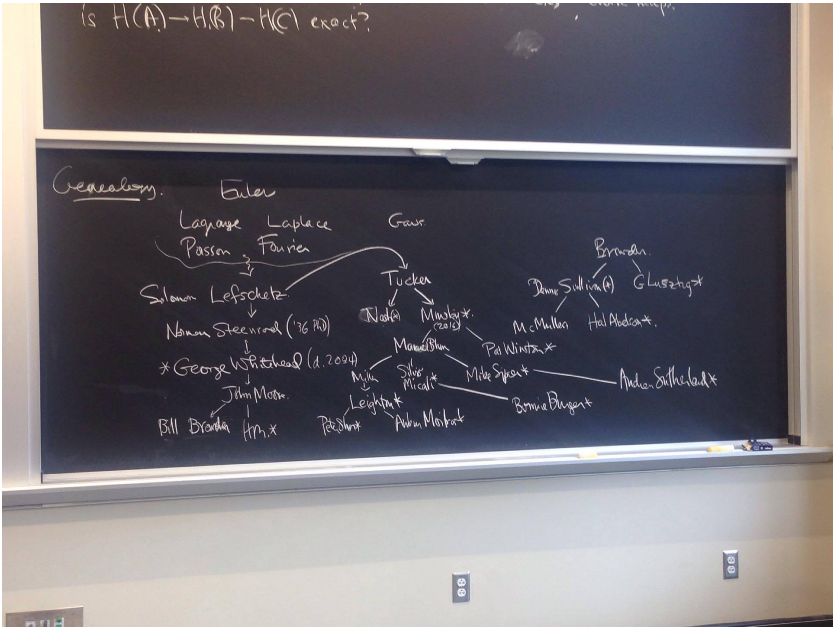

Figure 1.1: Mathematical genealogy, growing from Lefschetz, who was initially a chemist. The asterisks are meant to indicate that someone's at MIT.

1.10 Excision, and the Eilenberg-Steenrod axioms

We have homotopy invariance and the lexseq of a pair. We claimed that $H_*(X, A)$ "depends only on $X - A$". You have to be careful about this.

Definition 1.10.1. A triple (X, A, U) where $U \subseteq A \subseteq X$ is *excisive* if $\overline{U} \subseteq \text{Int}(A)$. This is a point-set definition. From an excisive triple you can get a pair $(X - U, A - U) \subseteq (X, A)$, and this is called an excision.

Theorem 1.10.2. *An excision induces an isomorphism in homology, i.e.,* $H_*(X - U, A - U) \cong H_*(X, A)$. *We might prove this on Wednesday.*

What are some consequences? We'll compute $H_*(S^n)$ and $H^*(D^n, S^{n-1})$. Here's the result. We'll use the homeomorphisms $D^n \simeq \Delta^n$ and $S^{n-1} \simeq \partial\Delta^n$. Let's write $S^0 = \{0, 1\}$, and $\iota_n : \Delta^n \to \Delta^n$.

Theorem 1.10.3. *1.*

$$H_q(S^n) = \begin{cases} \mathbf{Z} = \langle [c^0_*] \rangle & q = 0, n > 0 \\ \mathbf{Z} \oplus \mathbf{Z} = \langle [c^0_1], [\partial\iota_1] \rangle & q = n = 0 \\ \mathbf{Z} = \langle [\partial\iota_{n+1}] \rangle & q = n > 0 \\ 0 & \textit{else} \end{cases}$$

2.

$$H_q(D^n, S^{n-1}) = \begin{cases} \mathbf{Z} = \langle [\iota_n] \rangle & q = n \\ 0 & \textit{else} \end{cases}$$

If $n = 0$, then we say that $S^{-1} = \emptyset$. What are the generators of these groups?

Proof. We'll use the lexseq, homotopy invariance, and excision. We have the lexseq:

$$\begin{tikzcd} & & {} \arrow[dll, "\partial"'] \\ H_q(S^{n-1}) \arrow[r] & H_q(D^n) \arrow[r] & H_q(D^n, S^{n-1}) \arrow[dll, "\partial"'] \\ H_{1-1}(S^{n-1}) \arrow[r] & H_{q-1}(D^n) \arrow[r] & H_{q-1}(D^n, S^{n-1}) \arrow[dll, "\partial"'] \\ {} & & \end{tikzcd}$$

But we know that D^n is contractible, so $H_q(D^n) = \begin{cases} \mathbf{Z} & q = n \\ 0 & \text{else} \end{cases}$. This means that $\partial : H_q(D^n, S^{n-1}) \cong H_{q-1}(S^{n-1})$ for $q > 1$, but when $q = 1$, we get $0 \to H_1(D^n, S^{n-1}) \xrightarrow{\partial} H_0(S^{n-1}) \to H_0(D^n) \to H_0(D^n, S^{n-1}) \to 0$.

Let's think about the case $n > 1$. Then $H_0(S^{n-1}) = \mathbf{Z} = \langle [c^0_*] \rangle$, and $H_0(D^n) = \mathbf{Z} = \langle [c^0_*] \rangle$, so you have an isomorphism, which means that $H_0(D^n, S^{n-1}) = 0$ and $H_1(D^n, S^{n-1}) = 0$. Now, let's go to the case $n = 1$. Then $H_0(S^0) = \mathbf{Z} \oplus \mathbf{Z} = \langle [c^0_*], [\partial\iota_1] \rangle$ and $H_0(D^0) = \mathbf{Z}$. But $[\partial\iota_1]$ goes to zero, and $[c^0_*]$ goes to the generator of $\mathbf{Z} = H_0(D^0)$. This means that $H_1(D^1, S^0)$ is generated by $\langle [\iota_1] \rangle \cong \mathbf{Z}$ because the map $H_1(D^n, S^{n-1}) \to H_0(S^{n-1})$ is the boundary map, so $[\iota_1] \mapsto [\partial\iota_1]$.

Excision will come into play through the following statement.

Proposition 1.10.4. *If $n > 1, q > 1$, then $H_q(D^n, S^{n-1}) \to H_q(D^n/S^{n-1}, *) \cong H_q(S^n, *) \cong H_q(S^n)$, because $D^n/S^{n-1} \simeq S^n$. The claim is that this collapse map is an isomorphism.*

This provides the inductive step. Let's assume we've proved the proposition. Then $H_q(D^n, S^{n-1}) \cong H_{q-1}(S^{n-1})$. The proposition says that $H_q(S^n) \cong H_{q-1}(S^{n-1})$. But we also have the boundary map $\partial : H_{q+1}(D^n, S^{n-1}) \cong H_q(S^n)$, i.e., $H_{q+1}(D^n, S^{n-1}) \cong H_q(D^n, S^{n-1})$.

Now I want to prove the proposition.

Proof of proposition. We want to compare $H_q(D^n, S^{n-1})$ and $H_q(D^n/S^{n-1}, *)$. We'll use excision to do this. We have $D^n = \{x \in \mathbf{R}^{n+1} | |x| \leq 1\}$. Let $A = \{x | 1/3 \leq |x| \leq 1\}$ and $U = \{x | 2/3 < |x| \leq 1\}$, and $\{x | |x| = 1\} = S^{n-1} \subseteq U$. We need a preliminary step. $H_q(D^n, S^{n-1}) \to H_q(D^n, A)$. We claim that this an isomorphism, but this is true because of the lexseq and the 5-lemma. By excision, $H_q(D^n, A) \cong H_q(D^n - U, A - U)$. We can collapse the $(n-1)$-sphere, and $(D^n/S^{n-1} - U/S^{n-1}, A/S^{n-1} - U/S^{n-1}) = (D^n - U, A - U)$ because you're collapsing something from something that's already been collapsed! Now, we claim that $H_q(D^n/S^{n-1} - U/S^{n-1}, A/S^{n-1} - U/S^{n-1}) \cong H_q(D^n/S^{n-1}, A/S^{n-1})$, which is true by excision. THE FOLLOWING PART IS MOST DEFINITELY NOT RIGHT[1]. But also, $H_q(D^n/S^{n-1}, \text{disk}) = H_q(D^n/S^{n-1}, A/S^{n-1})$. Because the disk is contractible, using the lexseq and the 5-lemma completes the proof of the proposition. $\square$

$\square$

"This really turns me on, because I love homology." Why should you care about homology?

Corollary 1.10.5. *If $m \neq n$, then $S^m \not\simeq S^n$ because they have different homology groups.*

Corollary 1.10.6. *If $m \neq n$, then $\mathbf{R}^m \not\cong \mathbf{R}^n$, because they're not homeomorphic.*

[1] My own comment: this proof can be finished by noticing that $A \simeq S^{n-1}$ via $\mathbf{v} \mapsto \frac{\mathbf{v}}{||\mathbf{v}||}$, and that $D^n/S^{n-1} \simeq S^n$.

Proof. Let $m, n > 0$. Assume we have a homeomorphism $f : \mathbf{R}^m \to \mathbf{R}^n$. This restricts to $\mathbf{R}^m - \{0\} \to \mathbf{R}^n - \{0\}$, but each of these are homotopy equivalent to spheres, but we can't get a homotopy equivalence between two spheres of different dimension by the above corollary. $\square$

Eilenberg-Steenrod axioms

Definition 1.10.7. A homology theory (on **Top**) is:

- a functor $h_n : \mathbf{Top}_2 \to \mathbf{Ab}$ for all n. We'll write $h_n(X) = h_n(X, \emptyset)$
- natural transformations $\partial : h_n(X, A) \to h_{n-1}(A)$.

such that:

- if $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $f0,* \simeq f_{1,*} : h_n(X, A) \to h_n(Y, B)$.
- an excision induces isomorphisms.
- a lexseq:
$$\cdots \to h_{q+1}(X, A) \xrightarrow{\partial} h_q(A) \to h_q(X) \to h_q(X, A) \xrightarrow{\partial} \cdots$$
- (the dimension axiom): $h_n(*)$ is nonzero only in dimension zero. This is like the parallel postulate.

Example 1.10.8. Ordinary singular homology satisfies these.

Theorem 1.10.9 (Brouwer fixed-point theorem)**.** *If $f : D^n \to D^n$, then there is some point $x \in D^n$ such that $f(x) = x$.*

Proof. Suppose not. Then you can draw a ray from x to $f(x)$ to the boundary S^{n-1}, intersecting at a point $g(x)$. Left to you to check that g is continuous. If x was on the boundary, then $x = g(x)$. This is inconsistent by our computation because otherwise the identity on $H_{n-1}(S^{n-1})$ would be zero, contradiction! $\square$

1.11 Application of our previous calculation of $H_*(S^n)$ and the "locality principle"

Theorem 1.11.1. *Let $n \geq 1$. There is a surjective monoid homomorphism $[S^n, S^n] \to \mathbf{Z}_\times$, where $\mathbf{Z}_\times$ is the multiplicative monoid of $\mathbf{Z}$. $[S^n, S^n]$ is a monoid under composition. (This is basically the degree...)*

Proof. Given $f : S^n \to S^n$, take the homology, which is just a homomorphism $\mathbf{Z} \to \mathbf{Z}$, all of which are simply multiplication by an integer. The integer by which you're multiplying to get this homomorphism is the integer associated to f.

Construction. If $n = 1$, this is just the winding number. Suppose I've constructed this in dimension $n-1$. We have:

$$\begin{tikzcd}
H_{n-1}(S^{n-1}) \arrow[d, "n"] & H_n(D^n, S^{n-1}) \arrow[l] \arrow[r] \arrow[d] & H_n(S^n) \arrow[d] \\
H_{n-1}(S^{n-1}) & H_n(D^n, S^{n-1}) \arrow[l] \arrow[r] & H_n(S^n)
\end{tikzcd}$$

So we're basically suspending $f : S^{n-1} \to S^{n-1}$. More explicitly, if you have $f : S^{n-1} \to S^{n-1}$. We can extend to $\overline{f} : D^n \to D^n$ by sending $tx \mapsto tf(x)$ where tx denotes the ray connecting $x \in S^{n-1}$ to the origin, and we can then quotient out by S^{n-1} to get the map $S^n \to S^n$ as required. □

Addendum to the ES axioms

There's a further axiom, which isn't due to ES, but rather due to Milnor. It's this.

- Suppose I a set. For each $\alpha \in I$ there's have a space $X_\alpha \in \mathbf{Top}$. I can consider $\coprod_\alpha X_\alpha$. There are inclusion maps $X_\alpha \to \coprod_\alpha X_\alpha$. Then $\bigoplus_\alpha h_n(X_\alpha) \cong h_n\left(\coprod_\alpha X_\alpha\right)$.

This is known for ordinary singular homology.

Homological algebra

1. Suppose $A, B \subseteq C$ are abelian groups. Then $A + C \subseteq C$. You have a sexseq $0 \to A \cap B \to A \oplus B \to A + C \to 0$ where the map $c \mapsto (c, -c)$ is how the map $A \cap B \to A \oplus B$ is defined.

2. "Fundamental isomorphism for abelian groups" says the following. We have two sexseqs.

$$\begin{tikzcd}
0 \arrow[r] & A \cap B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \cap B \arrow[r] \arrow[d, dashed, "\cong"] & 0 \\
0 \arrow[r] & A \arrow[r] & A + B \arrow[r] & (A + B)/A \arrow[r] & 0
\end{tikzcd}$$

I'm not going to write out the diagram chase that we did.

3. "Snake lemma". Suppose I have[2]:

$$\begin{tikzcd}
& \ker f' \arrow[r] \arrow[d] & \ker f \arrow[r] \arrow[d] & \ker f'' \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker} f' \arrow[r] & \operatorname{coker} f \arrow[r] & \operatorname{coker} f'' &
\end{tikzcd}$$

Claim is that there's a map $\ker f'' \to \operatorname{coker} f$ so that $0 \to \ker f' \to \ker f \to \ker f'' \to \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f'' \to 0$. This is basically the lexseq in homology associated to the sexseqs of the following three chain complexes: $0 \to A' \to B' \to 0$, $0 \to A \to B \to 0$, and $0 \to A'' \to B'' \to 0$. Work this out yourself.

Locality

Definition 1.11.2. The (not necessarily open) cover of a topological space. Won't write this.

Definition 1.11.3. Let $\mathcal{A}$ be a cover of X. An n-simplex σ is $\mathcal{A}$-small if there is $A \in \mathcal{A}$ such that the image of σ is entirely in A.

Notice that if $\sigma : \Delta^n \to X$ is $\mathcal{A}$-small, then so is $d^i\sigma$. Let's denote by $\mathrm{Sin}_n^{\mathcal{A}}(X)$ the set of $\mathcal{A}$-small n-simplices. This means that we get a map $\mathrm{Sin}_n^{\mathcal{A}}(X) \to \mathrm{Sin}_{n-1}^{\mathcal{A}}(X)$. Let $S_n^{\mathcal{A}}(X) = \mathbf{Z}[\mathrm{Sin}_n^{\mathcal{A}}(X)]$. Then there's a subchain complex $S_*^{\mathcal{A}}(X)$.

Theorem 1.11.4. *The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ is a chain homotopy equivalence.*

Corollary 1.11.5. *If $H_*^{\mathcal{A}}(X) := H(S_*^{\mathcal{A}}(X))$, then $H_*^{\mathcal{A}}(X) \cong H_*(X)$.*

We'll do this on Monday.

---

[2]"It's my Turn", Jill Clayburgh

Example 1.11.6. If $\mathcal{A} = \{A, B\}$, then $\overline{X - B} = X - \mathrm{Int}(B) \subseteq \mathrm{Int}(A)$. Let $X - B = U$. Then $U \subseteq \overline{U} \subseteq \mathrm{Int}(A) \subseteq A \subseteq X$. This is an excision! So $U \subseteq A \subseteq X$ is an excision. But now, $(X - U, A - U) \to (X, A)$ is an excision, but $(X - U, A - U) = (B, A \cap B)$, so we have $(B, A \cap B) \to (X, A)$ is an excision. Now, also, $S_*^{\mathcal{A}}(X) = S_*(A) + S_*(B)$. Says he got off track, let me just write things out and explain in a moment.

$$\begin{tikzcd}
0 \arrow[r] & S_n(A) \cap S_n(B) = S_n(A \cap B) \arrow[r] \arrow[d] & S_n(B) \arrow[r] \arrow[d] & S_n(B, \arrow[d, dashed] \\
0 \arrow[r] & S_n(A) \arrow[r] & S_n(A) + S_n(B) = S_n^{\mathcal{A}}(X) \arrow[r] & S_n^{\mathcal{A}}(X)/
\end{tikzcd}$$

But we can consider $S_*(X)/S_*(A) = S_*(X, A)$. By the lexseq + 5 lemma, this thing is isomorphic to $S_n^{\mathcal{A}}(X)/S_n(A)$, so $S_*(B, A \cap B) \cong S_*(X, A)$ in homology. This is precisely the excision theorem. QED.

1.12 Mayer-Vietoris and Subdivision

(Is it Meyer-Vietoris or Mayer-Vietoris?) Today is the lecture with a lot of formulae.

Theorem 1.12.1. *Let $\mathcal{A}$ be a cover of X, so that $X = \bigcup_{A \in \mathcal{A}} \mathrm{Int}(A)$. Then the theorem we're going to prove is this. If $S_*^{\mathcal{A}}(X) = \sum_{A \in \mathcal{A}} S_*(A) \to S_*(X)$ induces an isomorphism in H_*. (This is a "quasi-isomorphism" of chain complexes.)*

Because this lecture's full of formulas, I'm going to stand around with a piece of paper in my hand.

Example 1.12.2. Let $\mathcal{A} = \{A, B\}$ of X, so that:

$$\begin{tikzcd}
A \cap B \arrow[r, "j_1"] \arrow[d, "j_2"] & A \arrow[d, "i_1"] \\
B \arrow[r, "i_2"'] & X
\end{tikzcd}$$

Then consider the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & S_*(A \cap B) \arrow[r, "\begin{pmatrix} j_{1*} \\ -j_{2*} \end{pmatrix}"] \arrow[d, equal] & S_*(A) \oplus S_*(B) \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S_*(A) \cap S_*(B) \arrow[r] & S_*(A) \oplus S_*(B) \arrow[r, "(i_{1*},\ i_{2*})"] \arrow[dr] & S_*(A) + S_*(B) \arrow[r] \arrow[d, hook] & 0 \\
 & & & S_*(X) &
\end{tikzcd}
$$

The map $S_*(A) + S_*(B) \hookrightarrow S_*(X)$ is a quasi-isomorphism, this is what locality says. Take the homology of this to get a lexseq:

$$
\begin{tikzcd}
\cdots \arrow[rr, "\begin{pmatrix} j_{1*} \\ -j_{2*} \end{pmatrix}"] & & H_{n+1}(X) \arrow[dll] \\
H_n(A \cap B) \arrow[r] & H_n(A) \oplus H_n(B) \arrow[r, "(i_{1*},\ i_{2*})"] & H_n(X) \arrow[dll] \\
H_{n-1}(A \cap B) \arrow[r] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}
$$

Voila, you have Mayer-Vietoris. (I have a different proof of this that I submitted in homework.)

The cone construction

Let $X \subseteq \mathbf{R}^N$ be a star-shaped region, and let $b \in \mathbf{R}^N$. Then we showed that the augmentation $S_*(X) \xrightarrow{\epsilon} \mathbf{Z}$ is a chain homotopy equivalence. There's another map going backwards $\mathbf{Z} \xrightarrow{\eta_b} S_*(X)$ sending $1 \mapsto c_b^0$. Clearly the composition $\epsilon \circ \eta_b$ is the identity. We want to show that η_b ad ϵ are chain homotopy inverses to each other. One direction is easy. The other map $S_*(X) \xrightarrow{\eta_b \epsilon} S_*(X)$ being homotopic to $1_{S_*(X)}$ is a little harder. This means that we want to construct a map $b* : S_n(X) \to S_{n+1}(X)$ such that $db * + b * d = 1 - \eta_b \epsilon$.

Consider some $\sigma : \Delta^1 \to X$. Then because X is star shaped, you can send σ to b. This gives a 2-simplex $b * \sigma$, called the *join*. We'll define this 2-simplex and label it so that the zero vertex is b itself, the

1 vertex is $d_1\sigma$, and the 2 vertex is $d_0\sigma$. Define $b * \sigma$ as follows (where $(t_0, \cdots, t_{n+1}) \in \Delta^{n+1}$):

$$b * \sigma(t_0, \cdots, t_n, t_{n+1}) = t_0 b + (1 - t_0)\sigma\left(\frac{t_1, \cdots, t_{n+1}}{1 - t_0}\right)$$

When $t_0 = 0$, then you recover exactly $\sigma(t_1, \cdots, t_{n+1})$ and when $t_0 = 1$, this is exactly b. (Why can you divide by $1 - t_0 = 0$?) This is a map $b* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n+1}(X)$, so we can extend linearly to get $S_n(X) \to S_{n+1}(X)$, also denoted $b*$. What is $d_i(b * \sigma)$? This is exactly:

$$d_i(b * \sigma) = \begin{cases} \sigma & i = 0 \\ c_b^0 & i = 1, n = 0 \\ b * d_{i-1}\sigma & i > 0, n > 0 \end{cases}$$

The latter thing seems true because in the case when $n = 1$, $d_2(b * \sigma)$ is the cone on $d_1\sigma$. The middle thing is true because when $n = 0$ you can't use the bottom thing (what is the boundary in that case?), and if you draw this out, noting our convention that when $t_0 = 1$ you have $d_1\sigma$ and when $t_0 = 1$ you have b, this automatically yields $d_1(b * \sigma) = b$ if $\sigma : \Delta^0 \to X$. We can rewrite this as follows. Here $c \in S_n(X)$.

$$d_i(b * c) = \begin{cases} c & i = 0 \\ b * d_0 c + \eta_b \epsilon c & i = 1 \\ b * d_{i-1}\sigma & i > 1 \end{cases}$$

Because d_0 of a 0-simplex is defined to be zero. This may seem confusing, but it's just a translation of what we wrote down above. We want to compute that this thing is actually a chain homotopy. Let's compute.

$$\begin{aligned} d(b * c) &= d_0(b * c) - d_1(b * c) + \sum_{i>1}(-1)^i d_i(b * c) \\ &= c - (b * d_0 c + \eta_b \epsilon c) + \sum_{i=2}^{n}(-1)^i b * d_{i-1} c \\ &= c - \eta_b \epsilon c - \sum_{j=0}^{n-1}(-1)^j b * d_j c \\ &= c - \eta_b \epsilon c - b * dc \end{aligned}$$

Here $j = i - 1$. The equality $\sum_{j=0}^{n-1}(-1)^j b * d_j c = b_* dc$ holds because $b*$ is linear (by definition on $S_n(X)$). This means that $b*$ is a chain homotopy, QED. This completes what we've claimed about the star shaped region. We want to use this cone construction to talk about subdivision.

Subdivide the standard simplex

Let's focus on the standard simplex. This is a nice thing about singular homology. For the 1-simplex, you just cut in half. For the 2-simplex, just look at the subdivision of each face, and look at the barycenter[3], and join the barycenter to the 1-simplex between each "half" 1-simplex. We want to formalize this process. Define a natural transformation $\$: S_n(X) \to S_n(X)$ by defining on standard n-simplex, namely by specifying what $\$(\iota_n)$ is where $\iota_n : \Delta^n \xrightarrow{\text{id}} \Delta^n$, and then extending by naturality (namely $\$(\sigma) = \sigma_* \$(\iota_n)$). Here's the definition. When $n = 0$, define $\$ = \text{id}$, i.e., $\$(\iota_0) = \iota_0$. For $n > 0$, define $\$\iota_n := b_n * \$d\iota_n$ where b_n is the barycenter of Δ^n. This makes a *lot* of sense if you draw out a picture, and it's a very clever definition that captures the geometry we described. Let me tell you what we'll prove about this, most likely on Wednesday.

Proposition 1.12.3. *$\$$ is a chain map $S_*(X) \to S_*(X)$, i.e., $\$d = d\$$. Also, $\$ \simeq 1$.*

Also, class is cancelled on Friday.

1.13 Locality (almost done!)

Theorem 1.13.1. *We discovered that $S_*^{\mathcal{A}}(X) \hookrightarrow S_*(X)$ is a subcomplex, and this induced an isomorphism in homology.*

We talked about subdivision and the cone construction, the latter of which dealt with a star-shaped region, relative to some point (which we can safely assume is the origin) b. If $\sigma : \Delta^n \to X$ is a map, then $b * \sigma : S_n(X) \to S_{n+1}(X)$ where $*$ is the join. We did all of this before. The property that this had is that it's a homotopy between 1 and $\eta_b \epsilon$, i.e., $db * +b * d = 1 - \eta_b \epsilon$. This is called equation $(*)$. Look above for the definition of η_b and ϵ. Hopefully you remember this story.

The subdivision operator $\$: S_*(X) \to S_*(X)$ for any space X is natural, so it's enough to say what $\$\iota_n$ and $\$\iota_0$ is. Define $\$\iota_0 = \iota_0$, and define $\$\iota_n = b_n * \$(d\iota_{n-1})$ where b_n is the barycenter of the n-simplex. The standard simplex is star-shaped relative to its barycenter, so by naturality, it suffices to do this for ι_n. The two key properties are the following.

[3]The barycenter of the n-simplex is $b_n := \frac{(1,\cdots,1)}{n+1}$.

Theorem 1.13.2. *1. \$ is a chain map.*

2. There is a chain homotopy $T : \$ \sim 1$.

Proof. Let's try to prove that it's a chain map. We'll use induction on n. It's enough to show that $d\$\iota_n = \$d\iota_n$, because:

$$d\$\sigma = d\$\sigma_*\iota_n = \sigma_* d\$\iota_n = \sigma_*\$d\iota_n = \$d\sigma_*\iota_n = \$d\sigma$$

We declared $d\$\iota_0 = d\iota_0 = 0$. But also $\$d\iota_0 = \$0 = 0$, so this works.

For $n \geq 1$, we want to compute $d\$\iota_n$. This is:

$$\begin{aligned} d\$\iota_n &= d(b_n * \$d\iota_n) \\ &= (1 - \eta_b\epsilon - b_n * d)(\$d\iota_n) && \text{by } (*) \end{aligned}$$

What happens when $n = 1$? Well:

$$\eta_b\epsilon\$d\iota_1 = \eta_b\epsilon\$(c_1^0 - c_0^0) = \eta_b\epsilon(c_1^0 - c^0 - 0) = 0$$

Because ϵ takes sums of coefficients, which is $1+(-1) = 0$. Let's continue.

$$\begin{aligned} d\$\iota_n &= ... \\ &= \$d\iota_n - b_n * d\$d\iota_n \\ &= \$d\iota_n - b_n\$d^2\iota_n \\ &= \$d\iota_n && \text{because } d^2 = 0 \end{aligned}$$

So we're done.

To define the chain homotopy T, we'll just write down a formula and not justify it. We just need to define $T\iota_n$ by naturality. So define:

$$T\iota_n = \begin{cases} 0 & n = 0 \\ b_n * (\$\iota_n - \iota_n - Td\iota_n) \in S_{n+1}(\Delta^n) & n > 0 \end{cases}$$

This is because $T : S_n(X) \to S_{n+1}(X)$ such that $dT + Td = \$ - 1$. I'm confused about this, so help me out. Hmm. The term $\$\iota_n - \iota_n - Td\iota_n$ is an n-chain. We're going to do this by induction. Again, we need to check only on the universal case.

When $n = 0$, $dT\iota_0 + Td\iota_0 = 0 + 0 = 0 = \$\iota_0 - \iota_0$ because $\$\iota_0 = \iota_0$. Now let's induct. For $n \geq 1$, let's start by computing $dT\iota_n$. This is:

$$\begin{aligned} dT\iota_n &= d_n(b_n * (\$\iota_n - \iota_n - Td\iota_n)) \\ &= (1 - b_n * d)(\$\iota_n - \iota_n - Td\iota_n) && \text{by } (*) \\ &= \$\iota_n - \iota_n - Td\iota_n - b_n * (d\$\iota_n - d\iota_n - dTd\iota_n) \end{aligned}$$

We can ignore the $\eta_b\epsilon$ part because we're in dimension ≥ 1. All we want now is that $b_n * (d\$\iota_n - d\iota_n - dTd\iota_n) = 0$. We can do this via induction, because $T(d\iota_n)$ is in dimension n (or is it $n-1$?):

$$\begin{aligned} dTd\iota_n &= -Td(d\iota_b) + \$d\iota_n - d\iota_n \\ &= \$d\iota_n - d\iota_n \\ &= d\$\iota_n - d\iota_n \end{aligned}$$

This means that $d\$\iota_n - d\iota_n - dTd\iota_n = 0$, so we're done. □

Corollary 1.13.3. $\$^k \sim 1 : S_*(X) \to S_*(X)$. *I.e., we're iterating subdivision. We want T_k such that $dT_k + T_k d = \$^k - 1$*

Proof. $dT + Td = \$ - 1$. Let's apply \$ to this. We get $\$dT + \$Td = \$^2 - \$$. Sum up these two things, so we get $dT + Td + \$dT + \$Td = \$^2 - 1$. But now, $\$d = d\$$, so the left hand side is $dT + d\$T + Td + \$Td = d(\$ + 1)T + (\$ + 1)Td$, i.e., $d(\$ + 1)T + (\$ + 1)Td = \$^2 - 1$. So define $T_2 = (\$ + 1)T$, and continuing, you see that $T_k = (\$^{k-1} + \$^{k-2} + \cdots + 1)T = \left(\sum_{i=0}^{k-1} \$^i\right) T$. □

Proposition 1.13.4 (Almost completes the proof of locality)**.** *Let $\mathcal{A}$ be a cover of X. For every chain $c \in S_n(X)$, there is a $k \geq 0$ such that $\$^k c \in S_n^{\mathcal{A}}(X)$. This is the geometric thing we have to prove.*

Proof. We may assume that $c : \sigma : \Delta^n \to X$, and this makes sense because you just take the max of the k of the terms of the sum. A great trick is the following: define an open cover $\mathfrak{U}$ of Δ^n defined by $\mathfrak{U} := \{\sigma^{-1}(\mathrm{Int}(A)) | A \in \mathcal{A}\}$. This is a cover, a basic result from topology. Then we use the Lesbegue covering lemma:

Lemma 1.13.5 (Lesbegue covering lemma)**.** *We'll pretend this is part of 18.901. Let M be a compact metric space (eg. Δ^n), and let $\mathfrak{U}$ be an open cover. Then there is $\epsilon > 0$ such that for all $x \in M$, there is $B_\epsilon(x) \subseteq U$ for some $U \in \mathfrak{U}$.*

Proof. Omitted, may be in 18.901. Or even 18.100B. □

Let's apply this to the cover we constructed. What we want is that for all $\epsilon > 0$, there is a k such that the diameter of the simplices in $\$^k\iota_n$ is less than ϵ. Let's do that.

Question 1.13.6. How small are these subdivided simplices in $\$^k\iota_n$?

For example, suppose $\sigma : \Delta^n \to \Delta^n$ is something in the subdivision. These are all affine simplices, i.e., it's determined by where the simplices of Δ^n go to in σ. We can write $\sigma = \langle v_, \cdots, v_n \rangle$. It could be in $\mathbf{R}^N$ if you wanted; maybe it's easier to think of it this way. The barycenter is $\frac{\sum_{i=0}^n v_i}{n+1}$. Let's compute:

$$\begin{aligned} |b - v_i| &= \left| \frac{v_0 + \cdots + v_n - (n+1)v_i}{n+1} \right| \\ &= \left| \frac{(v_0 - v_i) + (v_1 - v_i) + \cdots + (v_n - v_i)}{n+1} \right| \\ &\leq \frac{n}{n+1} \max_{i,j} |v_i - v_j| \\ &= \frac{n}{n+1} \operatorname{diam}(\operatorname{im} \sigma) \end{aligned}$$

The following lemma completes the proof because there's always a k such that $\left(\frac{n}{n+1}\right)^k < \epsilon$.

Lemma 1.13.7. *Let τ be a simplex in $\$^k\sigma$ where σ is an affine simplex. Then* $\operatorname{diam}(\tau) \leq \frac{n}{n+1}\operatorname{diam}(\sigma)$.

Proof. Let's write $\tau = \langle w_0 = b, \cdots, w_n \rangle$ and $\sigma = \langle v_0, \cdots, v_n \rangle$. We saw:

$$\begin{aligned} |b - w_i| &\leq \max_i |b - v_i| \\ &\leq \frac{n}{n+1} \operatorname{diam}(\sigma) \end{aligned}$$

For the other cases, well, we use induction:

$$\begin{aligned} |w_i - w_j| &\leq \operatorname{diam}(\text{simplex in } \$ d\sigma) \\ &\leq \frac{n-1}{n} \operatorname{diam}(d\sigma) \\ &\leq \frac{n}{n+1} \operatorname{diam}(\sigma) \end{aligned}$$

We're almost there. We'll finish the proof of locality on Friday. □

□

1.14 Concluding Locality! and CW-complexes

Again recall what $\mathcal{A}$-small covers, etc. are. We want to prove that:

Theorem 1.14.1. *$S_*^{\mathcal{A}}(X) \hookrightarrow S_*(X)$ is a quasi-isomorphism, i.e., an isomorphism on homology.*

We developed the subdivision operator $\$^k : S_*(X) \to S_*(X)$, and proved that it's a chain map. We showed that $T_k : \$^k \sim 1$.

Proof of locality. We want to prove surjectivity of $H_n(S_*^{\mathcal{A}}(X)) \to H_n(S_*(X)) = H_n(X)$. Let $c \in Z_n(C_\bullet)(X)$. We want to find an $\mathcal{A}$-small n-cycle that is homologous to c. There's only one thing to do. Pick k such that $\$^k c$ is $\mathcal{A}$-small. This is a cycle because $d\$^k c = \$^k dc = 0$ because $\$^k$ is a chain map. I want to compare this new cycle with c. Consider the chain homotopy T_k; then: $dT_k c + T_k dc = \$^k c - c$. But $dc = 0$, so $\$^k c - c = dT_k c$, so they differ by a boundary, and they're homologous.

Now for injectivity. Suppose $c \in S_n^{\mathcal{A}}(X)$ with $dc = 0$. Suppose that $c = db$ for some $b \in S_{n+1}(X)$, not necessarily $\mathcal{A}$-small. We want c to be a boundary of an $\mathcal{A}$-small chain. Well:

$$\begin{aligned}
&dT_k b + T_k db = \$^k b - b \\
\Rightarrow &dT_k b + T_k c = \$^k b - b \\
\Rightarrow &d(dT_k b + T_k c) = \$^k b - b = dT_k c = d\$^k b - c \\
\Rightarrow &c = d\$^k b - dT_k c = d(\$^k b - T_k c)
\end{aligned}$$

Now, $\$^k$ is $\mathcal{A}$-small. Is $T_k c$ also $\mathcal{A}$-small? I claim that it is. Why? It is enough to show that $T_k\sigma$ is $\mathcal{A}$-small if σ is. We know that $\sigma = \sigma_* \iota_n$. Because σ is $\mathcal{A}$-small, we know that $\sigma : \Delta^n \to X$ is the composition $i_*\overline{\sigma}$ where $\overline{\sigma} : \Delta^n \to A$ and $i : A \to X$ is the inclusion for some $A \in \mathcal{A}$. This means that $T_k\sigma = T_k i_* \overline{\sigma} = i_* T_k \overline{\sigma}$, which certainly is $\mathcal{A}$-small. $\square$

"Are you happy? You should be very happy, because we've finished our first portion of this course. We now have a whole package of homology."

CW-complexes

Simplicial complexes are rigid and combinatorial. But manifolds are smooth. In between, you have CW-complexes. (A lot of advertisement for this.) We want to "glue" things. This is the pushout construction.

Namely, if you have $i : A \hookrightarrow B$ and $f : A \to X$, then you define $X \cup_f B$ (or $X \cup_A B$) via:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ B \arrow[r] & X \cup_f B \end{tikzcd}$$

defined by $X \cup_f B = X \sqcup B/\sim$ where $\forall a \in A$, $f(a) \sim a$. This is X with B attached along f. There are two kinds of equivalence classes, namely elements of $B - A$, because anything not in A is just a singleton. The other is $\{x\} \cup f^{-1}(x)$ for $x \in X$, because anything that's not in $\operatorname{im} f$ is a singleton, but if something is in $\operatorname{im} f$, you identify it with its preimage. This is what it is as a set. It has a universal property. Suppose you have another space Y.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "j"] \arrow[ddr, "\overline{j}"] & \\ B \arrow[r] \arrow[drr, "\overline{g}"'] & X \cup_f B \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

such that $\overline{j} f = \overline{g} i$. The topology is right too because that's what the quotient topology does for you. As I wrote before, this is called a *pushout* of the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \\ B & \end{tikzcd}$$

Example 1.14.2. Let $X = *$. Then you have a pushout:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & * \arrow[d] \\ B \arrow[r] & * \cup_f B \end{tikzcd}$$

So then $* \cup_f B = B/A$.

Example 1.14.3.

$$\begin{tikzcd} \emptyset \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ B \arrow[r] & X \cup_f B \end{tikzcd}$$

It's then clear that this is exactly $X \sqcup B$.

Example 1.14.4. If both:

$$\begin{tikzcd} \emptyset \arrow[r, "f"] \arrow[d, hook, "i"'] & * \arrow[d] \\ B \arrow[r] & * \cup_f B \end{tikzcd}$$

So $B/\emptyset = * \sqcup B$. For example, $\emptyset/\emptyset = *$. This is "creation from nothing". "We won't get into the religious ramifications."

Example 1.14.5 (Attaching a cell, the most important)**.** Consider:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ D^n \arrow[r] & X \cup_f D^n \end{tikzcd}$$

This is called attaching a "cell". The D^n is what's called a cell. You're attaching a contractible space. You might want to generalize this a little bit:

$$\begin{tikzcd} \coprod_{\alpha \in A} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ \coprod_{\alpha \in A} D^n_\alpha \arrow[r] & X \cup_f \coprod_{\alpha \in A} D^n_\alpha \end{tikzcd}$$

What are some examples? When $n = 0$, the declaration is that $S^{-1} = \emptyset$, so this is:

$$\begin{tikzcd} \emptyset \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ \coprod_{\alpha \in A} * \arrow[r] & X \cup_f \coprod_{\alpha \in A} * \end{tikzcd}$$

You're just adding a bunch of points to X. This is a little more interesting. What about:

$$\begin{tikzcd} S^0 \sqcup S^0 \arrow[r, "f"] \arrow[d, hook, "i"'] & * \arrow[d] \\ D^1 \sqcup D^1 \arrow[r] & * \cup_f (D^1 \sqcup D^1) \end{tikzcd}$$

Then $* \cup_f (D^1 \sqcup D^1)$ is a figure 8, because you have two 1-disks, where you identify the four boundary points together. If we consider $(X, *), (Y, *)$, then $X \vee Y := X \sqcup Y/* \sim *$. So $* \cup_f (D^1 \sqcup D^1) = S^1 \vee S^1$. More interestingly:

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, hook, "i"'] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r] & (S^1 \vee S^1) \cup_f D^2 \end{tikzcd}$$

This is exactly the torus, i.e., $(S^1 \vee S^1) \cup_f D^2 = T^2$.

Definition 1.14.6. A *CW-complex* is a space X with a sequence of subspaces $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ (could be an infinite sequence) such that for all n, there is a pushout diagram like this:

$$\begin{tikzcd} \coprod_{\alpha \in A_n} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook, "i"'] & X_{n-1} \arrow[d] \\ \coprod_{\alpha \in A_n} D^n_\alpha \arrow[r] & X_n \end{tikzcd}$$

And $X = \bigcup X_n$, topologically (i.e. $A \subseteq X$ is open if and only if $A \cap X_n$ is open for all n). Often, $X_n = \mathrm{Sk}_n(X)$, called the n-skeleton, in honor of Halloween (coming right up!), of X.

Example 1.14.7. The torus is $\emptyset \subseteq T^2_0 \subseteq T^2_1 \subseteq T^2$. Here, $T^2_0 = *$ and $T^2_1 = S^1 \vee S^1$.

Definition 1.14.8. A CW-complex is *finite-dimensional* if $X_n = X$ for some n. Say that X is of *finite type* if each A_n is finite, i.e., finitely many cell in each dimension. Say that X is *finite* if it's finite-dimensional and of finite type.

In CW, the C is for cell, and the W is for weak, because of the topology on a CW-complex. This definition is due to J. H. C. Whitehead. Some people say that the "CW" comes from his name.

Theorem 1.14.9. *1. Any CW-complex is Hausdorff, and it's compact if and only if it's finite.*

2. Any compact smooth manifold admits a CW structure.

Proof. Not going to do this. □

Note that there could be multiple CW-structures on something.

1.15 CW-complexes and cellular homology

Recall:

Definition 1.15.1. A *CW-complex* is a space X with a sequence of subspaces $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ (could be an infinite sequence) such that for all n, there is a map $f : \coprod_{\alpha \in A_n} S_\alpha^{n-1} \to X_{n-1}$ (called the *attaching map*), such that there is a pushout diagram like this:

$$\begin{tikzcd} \coprod_{\alpha \in A_n} S_\alpha^{n-1} \arrow[r, "f"] \arrow[d, hook, "i"'] & X_{n-1} \arrow[d] \\ \coprod_{\alpha \in A_n} D_\alpha^n \arrow[r] & X_n \end{tikzcd}$$

And $X = \bigcup X_n$, topologically (i.e. $A \subseteq X$ is open if and only if $A \cap X_n$ is open for all n). Often, X_n is written $\mathrm{Sk}_n(X)$, and is called the n-skeleton of X.

Remark 1.15.2. This means that if you ignore the topology, i.e., as sets: $X = \coprod_{n \geq 0} \left(\coprod_{\alpha \in A_n} \mathrm{Int}(D_\alpha^n) \right)$ where $\mathrm{Int}(D^n) = \{x \in D^n : |x| < 1\}$ (the interior of D^n), so that $\mathrm{Int}(D^0) = D^0 = *$. The $\mathrm{Int}(D_\alpha^n)$ are called "open n-cells". Note that the open n-cells are not generally open in the topology on X.

Example 1.15.3. The n-sphere S^n. Let $\mathrm{Sk}_0(S^n) = * = \mathrm{Sk}_1(S^n) = \cdots = \mathrm{Sk}_{n-1}(S^n) \subseteq \mathrm{Sk}_n(S^n) = S^n$. We attach it by using the pushout:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d, hook] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

Here's another CW-structure. Let $\mathrm{Sk}_0(S^n) = S^0 = S^n \cap \mathbf{R}^1\langle \mathbf{e}_1 \rangle$, $\mathrm{Sk}_1(S^n) = S^1 = S^n \cap R\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$, $\mathrm{Sk}_2(S^n) = S^2 = S^n \cap \mathbf{R}\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$, etc. until $\mathrm{Sk}_n(S^n) = S^n$. I have to give maps $u, \ell : D^k \to S^k$ so that you have a pushout:

$$\begin{tikzcd} D^k \arrow[r] & S^k \\ S^{k-1} \arrow[u, hook] \arrow[r, equal] & \mathrm{Sk}_k(S^n) \arrow[u, hook] \end{tikzcd}$$

Let $u(x) = (x, \sqrt{1 - |x|^2})$ and $\ell(x)(x, -\sqrt{1 - |x|^2})$ where $x \in D^k$. Clearly u, ℓ take values in S^k.

There's another definition I have to make.

Definition 1.15.4. Let X be a CW-complex. A subcomplex of X is a subspace $Y \subseteq X$ such that $\emptyset \subseteq Y \cap X_0 \subseteq Y \cap X_1 \subseteq \cdots \subseteq Y \cap X_k \subseteq \cdots \subseteq Y$ is a CW-structure on Y.

Example 1.15.5. $X_n \subseteq X$ is a subcomplex of a CW-complex X. It's not that hard to see why. In particular, $S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq \bigcup_{n \geq 1} S^n =: S^\infty$. This of finite type, but isn't finite-dimensional.

Lemma 1.15.6. *S^∞ is contractible.*

Proof. S^0 itself is not contractible, but attaching two 1-cells makes this contractible. Similarly, S^1 isn't contractible, but attaching two 2-cells makes this contractible. This is the idea. You have $S^{k-1} \times I \to S^k$ by $(x, t) \mapsto u(tx + (1-t)\mathbf{e}_1)$ where u is the map we defined above. Therefore we get a map $S^\infty \times I \to S^\infty$ that's a contracting homotopy. □

Example 1.15.7. Recall $\mathbf{RP}^n = S^n/\sim$ where $x \sim -x$. There's a map from $S^n \to \mathbf{RP}^n$ that's a double cover. Let me propose a CW-decomposition. We have:

$$\begin{tikzcd} \cdots \arrow[r, hook] & S^{k-1} \arrow[r, hook] \arrow[d] & S^k \arrow[r] \arrow[d] & \cdots \arrow[r] & S^n \arrow[d] \\ \cdots \arrow[r, hook] & \mathbf{RP}^{k-1} \arrow[r, hook] & \mathbf{RP}^k \arrow[r, hook] & \cdots \arrow[r, hook] & \mathbf{RP}^n \end{tikzcd}$$

We claim that this is a CW-decomposition. We have a double cover $S^1 \to \mathbf{RP}^1 = S^1$. The maps $S^{k-1} \to \mathbf{RP}^{k-1}$ are not degree 2 maps

– they're different spaces! If we use the double cover $S^{k-1} \to \mathbf{RP}^{k-1}$, then we claim that there is a pushout:

$$\begin{tikzcd}
\mathbf{RP}^{k-1} \arrow[r, hook] & \mathbf{RP}^{k} \arrow[r, hook] & \cdots \arrow[r, hook] & \bigcup_{k\geq 0} \mathbf{RP}^k = \mathbf{RP}^\infty \\
S^{k-1} \arrow[u, "\text{double cover}"] \arrow[r] & D^k \arrow[u] & &
\end{tikzcd}$$

This is true if you notice that the preimage of any point of $\mathbf{RP}^k$ must be two points, one of which must be in the upper hemisphere, which is a disk, unless both points are on the equatorial sphere.

Homology of CW-complexes

Consider:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"] & B \arrow[r] \arrow[d] & B/A \arrow[d, dashed] \\
X \arrow[r, hook] & X \cup_f B \arrow[r] & (X \cup_f B)/X
\end{tikzcd}$$

By a diagram chase, the dotted arrow exists and is continuous. This is actually a pointed map. You can see that this is a homeomorphism. What if this is part of a CW-structure?

$$\begin{tikzcd}
\coprod_\alpha S^{k-1} \arrow[r, hook] \arrow[d, "f"] & \coprod_\alpha D^k_\alpha \arrow[r] \arrow[d] & \bigvee_\alpha S^k_\alpha \arrow[d, dashed] \\
X_{k-1} \arrow[r, hook] & X_k \cup_f B \arrow[r] & X_k/X_{k-1}
\end{tikzcd}$$

where $\bigvee$ is the wedge product (disjoint union with all basepoints identified). Then $\bigvee_\alpha S^k_\alpha$ is a bouquet of spheres. So $X_k/X_{k-1} \cong \bigvee_\alpha S^k_\alpha$. We know the homology of spheres very well by now, so let's exploit this.

Lemma 1.15.8. *$H_q(X_k, X_{k-1}) \to H_q(X_k/X_{k-1}, *)$ is an isomorphism.*

Proof. Later. □

But now, we know $H_q(X_k/X_{k-1}, *)$ very well! It's exactly $\widetilde{H}_q(\bigvee_{\alpha\in A_k} S^k_\alpha) \cong \begin{cases} \mathbf{Z}[A_k] & q = k \\ 0 & q \neq k \end{cases}$. Therefore the relative homology $H_q(X_k, X_{k-1})$ counts the number of k-cells of X.

Definition 1.15.9. Let $C_k(X) := H_k(X_k, X_{k-1})$. This is the "cellular k-chains" of X.

Corollary 1.15.10. *There's an exact sequence:*

$$\begin{tikzcd} & & H_{k+1}(X_k, X_{k-1}) = 0 \arrow[dll] \\ H_k(X_{k-1}) \arrow[r] & H_k(X_k) \arrow[r] & C_k(X) \arrow[dll] \\ H_{k-1}(X_{k-1}) \arrow[r] & H_{k-1}(X_k) \arrow[r] & H_{k-1}(X_k, X_{k-1}) = 0 \end{tikzcd}$$

And in other dimensions, $H_q(X_{k-1}) \cong H_q(X_k)$ for $q \neq k, k-1$. So:

1. *We have maps $H_q(X_k) \to H_q(X_{k+1}) \to \cdots$ that are all isomorphisms for $q < k$. (There was a lot of confusion here about what q is greater than or less than). All of these map to $H_q(X)$. There's another lemma that I will defer again:*

   Lemma 1.15.11. *This limit $H_q(X_k) \to H_q(X_{k+1}) \to \cdots \to H_q(X)$ is an isomorphism.*

   Proof. Deferred. □

2. *(There was a bit of confusion at this point, I'm not sure on what exactly.) $H_q(X_0) \to H_q(X_1) \to \cdots \to H_q(X_k)$ are all isomorphisms for $q > k$. Agreed? That's not what I wanted to say. I'll continue this on Wednesday. It's not supposed to be confusing.*

1.16 Homology of CW-complexes

Lemma 1.16.1. *There are isomorphisms:*

$$H_q(X_n, X_{n-1}) \xrightarrow{\cong} H_q(X_n/X_{n-1}, *) = H_q\left(\bigvee_{\alpha \in A_n} S^n_\alpha, *\right) = \begin{cases} 0 & q \neq n \\ \mathbf{Z}[A_n] & q = n \end{cases}$$

Let's talk about "characteristic maps". This is a map $\left(\coprod D^n_\alpha, \coprod S^{n-1}_\alpha\right) \to (X_n, X_{n-1})$. This is like a "relative homeomorphism" (I was drinking water, so this isn't exactly accurate). We have a map $H_q(X_n, X_{n-1}) \to H_q(X_n/X_{n-1}, *)$, to get a commutative diagram:

$$\begin{tikzcd}
H_q\left(\coprod D_\alpha^n, \coprod S_\alpha^{n-1}\right) \arrow[r] \arrow[d] & H_q\left(\bigvee S_\alpha^n, *\right) \arrow[d] \\
H_q(X_n, X_{n-1}) \arrow[r] & H_q(X_n/X_{n-1}, *)
\end{tikzcd}$$

The right arrow is an isomorphism. The top arrow is an isomorphism. The lemma says that the bottom map is an isomorphism, so that $H_q\left(\coprod D_\alpha^n, \coprod S_\alpha^{n-1}\right) \to H_q(X_n, X_{n-1})$ is an isomorphism. This is called the "cellular n-chains" on X.

Now, fix q. For $q = 0$, there is:

$$\begin{tikzcd}
H_1(X_1, X_0) \arrow[d] & H_0(X_1, X_0) = 0 & H_0(X_2, X_1) = 0 & \\
H_0(X_0) \arrow[r] \arrow[drrr] & H_0(X_1) \arrow[r] \arrow[u] \arrow[drr] & H_0(X_2) \arrow[r] \arrow[u] \arrow[dr] & \cdots \arrow[d] \\
 & H_1(X_2, X_1) = 0 \arrow[u] & & H_0(X)
\end{tikzcd}$$

We know that $H_0(X_1, X_0) = 0$, but $H_1(X_1, X_0)$ is not necessarily 0. This means that $H_0(X_0) \to H_0(X_1)$ is surjective, and $H_0(X_1) \cong H_0(X_2)$, and so on for higher dimensions. This makes sense because adding higher dimensional cells does not change path components.

Let's try this for $q > 0$. Then you have:

$$\begin{tikzcd}
 & H_{q+1}(X_q, X_{q+1}) = 0 \arrow[d] & H_{q+1}(X_{q+1}, X_q) \arrow[d] & & \\
\cdots \arrow[r] & H_q(X_{q-1}) \arrow[r] \arrow[dr] & H_q(X_q) \arrow[r] \arrow[d] & H_q(X_{q+1}) \arrow[r] \arrow[d] & {} \\
 & & H_q(X_q, X_{q-1}) & H_q(X_{q+1}, X_q) = 0 &
\end{tikzcd}$$

So the first maps ($H_q(X_0) \to H_q(X_1) \to \cdots$) are isomorphisms, the map $H_q(X_{q+1}) \to H_q(X_q)$ is an injection, and the map $H_q(X_q) \to H_q(X_{q+1})$ is surjective. But also, $H_q(X_{q+1}) \cong H_q(X_{q+2}) \cong \cdots$. But also, $H_q(X_0) \cong 0$, and we have:

Corollary 1.16.2. *$H_q(X) = 0$ for $q > \dim X = n$.*

Lemma 1.16.3. *$H_q(X_n) \cong H_q(X)$ for $n > 0$.*

I want you to have the following picture in mind. We have a diagram coming from the lexseq in the homology of a pair:

$$\begin{tikzcd}
C_{n+1}(X) = H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial"] \arrow[dr, "d"] & & 0 = H_{n-1} \arrow[d] \\
H_n(X_n) \arrow[r, "j"] \arrow[d] & C_n(X) = H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[dr, "d"] & H_{n-1}(\arrow[d] \\
H_n(X_{n+1}) \arrow[d] & & C_{n-1}(X) = H_{n-} \\
0 = H_n(X_{n+1}, X_n) & &
\end{tikzcd}$$

Now, $\partial \circ j = 0$. So the composite of the diagonals is zero, i.e., $d^2 = 0$, and we have a chain complex! More precisely, we get a chain complex, denoted $C_*(X)$. This is the "cellular chain complex" of X. We should compute the homology of this chain complex. Well, $H_n(C_*(X)) = \ker d / \operatorname{im} d$. Now, $\ker d = \ker(j \circ \partial)$. But j is injective, so $\ker d = \ker \partial$. Also, $\operatorname{im} d = \operatorname{im}(j \circ \partial) = j(\operatorname{im} \partial)$ because j is injective.

The kernel of ∂ is the image of j by exactness, but j is a monomorphism, so $\ker \partial \cong H_n(X)$. Now, $H_n(C_*(X)) \cong \frac{H_n(X)}{\operatorname{im}(\partial)}$. This is equal to $H_n(X_{n+1})$, again by exactness. But out lemma shows that $H_n(X_{n+1}) = H_n(X)$. In other words, we've proved:

Theorem 1.16.4. *If X is a CW-complex, then $H_*(C_*(X)) \cong H_*(X)$. I didn't use specific attaching maps at all, so this is natural in "skeletal" maps of CW-complexes.*

What is the differential? You have a relative cycle in dimension $(n+1)$, you're taking its boundary, and then working relative the $(n-1)$-skeleton. You'll see this better in the example we're going to do now, namely projective space.

Example 1.16.5. We'll try $H_*(\mathbf{RP}^n)$. We have: $\mathrm{sk}_k(\mathbf{RP}^n) = \mathbf{RP}^k$, which are just 1-dimensional subspaces of $\mathbf{R}^{k+1}$. Think of the inclusion $\mathbf{R}^{k+1} \to \mathbf{R}^{n+1}$ as the inclusion of the first $(k+1)$ basis vectors. This is a CW-complex because the map $S^{k-1} \to \mathbf{RP}^{k-1}$ is a double cover, and you have a pushout:

$$\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d, hook] & \mathbf{RP}^{k-1} \arrow[d, hook] \\
D^k \arrow[r] & \mathbf{RP}^k
\end{tikzcd}$$

The attaching maps are the double cover maps.

The notation is as follows. $\mathbf{RP}^n = \mathbf{RP}^{n-1} \cup_f D^n = \mathbf{RP}^{n-1} \cup_f e^n$. The e^n is the notation for an n-cell. In particular, $\mathbf{RP}^n = e_0 \cup_f e_1 \cup_f \cdots \cup_f e_n$. You have:

$$\begin{tikzcd}
0 & C_0(\mathbf{RP}^n) \arrow[l] \arrow[d] & C_1(\mathbf{RP}^n) \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & C_n(\mathbf{RP}^n) \arrow[l] \arrow[d] & 0 \\
& \mathbf{Z}\langle e^0 \rangle & \mathbf{Z}\langle e^1 \rangle \arrow[l, "d=0"'] & \cdots \arrow[l] & \mathbf{Z}\langle e^n \rangle \arrow[l] &
\end{tikzcd}$$

The first differential is zero because we know what $H_0(\mathbf{RP}^n)$ is (it's $\mathbf{Z}$!). I have $S^{n-1} \xrightarrow{f} \mathbf{RP}^{n-1} \to \mathbf{RP}^{n-1}/\mathbf{RP}^{n-2} = S^{n-1}$. Also recall the commutative diagram from before.

$$\begin{tikzcd}
H_n(D^n, S^{n-1}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(S^{n-1}, *) \arrow[d, "\cong"] \\
C_n = H_n(\mathbf{RP}^n, \mathbf{RP}^{n-1}) \arrow[r, "\partial"] & H_{n-1}(\mathbf{RP}^{n-1}) \arrow[r] & H_{n-1}(\mathbf{RP}^{n-1}, \mathbf{RP}^{n-2}) = C_{n-1}
\end{tikzcd}$$

The first map on the top is an isomorphism. The bottom composite is our differential. So the map $H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1}, *)$. Therefore, $S^{n-1} \xrightarrow{\text{double cover}} \mathbf{RP}^{n-1} \xrightarrow{\text{pinching}} S^{n-1}$.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\text{double cover}"] \arrow[dr] & \mathbf{RP}^{n-1} \arrow[r, "\text{pinching}"] & S^{n-1} \\
& S^{n-1}/S^{n-2} = S^{n-1} \vee S^{n-1} \arrow[ur] &
\end{tikzcd}$$

One of the maps $S^{n-1} \to S^{n-1}$ from the wedge is the identity, and the other map is the antipodal map, as can be seen by looking at a picture. If α is the antipodal map, then $S^{n-1} \vee S^{n-1} \to S^{n-1}$ is $[1, \alpha]$. If σ is a generator of $H_{n-1}(S^{n-1})$, we have $\sigma \mapsto (\sigma, \sigma) \mapsto \sigma + \alpha_* \sigma$. What is the degree of $\alpha_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$, so $\deg \alpha = (-1)^n$. Thus the composite, and hence the attaching map, is $(1 + (-1)^n)\sigma$. This means the cellular chain complex is:

$$0 \xleftarrow[0]{} \mathbf{Z} \xleftarrow[2]{} \mathbf{Z} \xleftarrow[0]{} \cdots \xleftarrow[2 \text{ or } 0]{} \mathbf{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

We'll continue next time[4].

1.17 Missing lemmas, $\mathbf{RP}^n$ again, and even CW-complexes

Lemma 1.17.1. *We want to show that $H_*(X_n, X_{n-1}) \cong H_*(X_n/X_{n-1}, *)$. We have the characteristic map $\left(\coprod_\alpha D^n, \coprod_\alpha S^{n-1}\right) \to (X_n, X_{n-1})$, where the map $\coprod_\alpha S^{n-1} \to X_{n-1}$ is the attaching map.*

$$\begin{tikzcd}
H_*(X_n, X_{n-1}) \arrow[d, "\cong"] & H_*\left(\coprod_\alpha D^n, \coprod_\alpha S^{n-1}\right) \arrow[l] \arrow[d, "{\cong, homework}"] \\
H_*(X_n/X_{n-1}, *) & H_*\left(\bigvee_\alpha S^n_\alpha, *\right) \arrow[l, "\cong"']
\end{tikzcd}$$

For preparation, we will talk about "strong deformation retracts". For example, $S^{n-1} \hookrightarrow D^n - \{0\}$. You just deform everything back radially.

Definition 1.17.2. A subspace of a space A inside X is a *strong deformation retract* if there is a homotopy $h : X \times I \to X$ such that $h(x, 0) = x$, $h(x, 1) \in A$, and $h(a, t) = a$ if $a \in A$.

Example 1.17.3. For example, for the map $S^{n-1} \hookrightarrow D^n - \{0\}$ can be defined as $h(x, t) = (1 - t)x + t\frac{x}{||x||}$.

A strong deformation retract is a homotopy equivalence, because we can just define the homotopy inverse to be $h(-, 1)$. Then $A \hookrightarrow X \xrightarrow{h(-,1)} A$ is the identity, and $X \xrightarrow{h(-,1)} A \hookrightarrow X$ is homotopic to the identity.

Example 1.17.4. The map $\coprod_\alpha S^{n-1}_\alpha \xrightarrow{\coprod} (D^{n-1}_\alpha - \{0\})$.

Terminology: if X is a CW-complex with filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$. A choice of characteristic maps is a "cell structure" for X. Note that this isn't specified in the CW-structure.

[4] Why don't we work in $\mathbf{Z}/2\mathbf{Z}$ coefficients? This is so much easier then. :P

Proof of the lemma. Let X be a CW-complex, with a choice of a cell structure, say with characteristic maps $g_\alpha : D^n_\alpha \to X_n$. Let $C_n = \{g_\alpha(0) | \alpha \in A_n\}$. We know that $X_{n-1} \hookrightarrow X_n - C_n$. We claim that this is a strong deformation retract. This follows from our observation that $\coprod_\alpha S^{n-1}_\alpha \xrightarrow{\coprod} (D^{n-1}_\alpha - \{0\})$ is a strong deformation retract. In particular, $X_{n-1} \hookrightarrow X_n - C_n$ is a homotopy equivalence.

For example, consider the torus. If you look at the fundamental polygon, and remove a hole, you can retract everything back to the boundary.

Now, we have:

$$\begin{tikzcd}
H_*\left(\coprod_\alpha D^n_\alpha, \coprod_\alpha S^{n-1}_\alpha\right) \arrow[r] \arrow[d] & H_*(X_n, X_{n-1}) \arrow[d] \\
H_*\left(\coprod_\alpha D^n_\alpha, \coprod_\alpha (D^n_\alpha - \{0\})\right) & H_*(X_n, X_n - C_n)
\end{tikzcd}$$

The downwards arrows are isomorphisms because of strong deformation retractions, homotopy invariance, lexseq, and the 5-lemma. Recall that if $U \subseteq A \subseteq X$, then $H_*(X - U) \cong H_*(X, A)$ if $\overline{U} \subseteq \mathrm{int}(A)$. Suppose we consider $X_{n-1} \subseteq X_n - C_n \subseteq X_n$. This is an excision because X_{n-1} is already closed, and $X_n - C_n$ is already open. Then excision tells us that $H_*(X_n - X_{n-1}, X_n - X_{n-1} - C_n)$. This means we can extend the diagram as follows.

$$\begin{tikzcd}
H_*\left(\coprod_\alpha D^n_\alpha, \coprod_\alpha S^{n-1}_\alpha\right) \arrow[r] \arrow[d] & H_*(X_n, X_{n-1}) \arrow[d] \\
H_*\left(\coprod_\alpha D^n_\alpha, \coprod_\alpha (D^n_\alpha - \{0\})\right) & H_*(X_n, X_n - C_n) \\
H_*(\coprod_\alpha (D^n_\alpha - S^{n-1}_\alpha), \coprod_\alpha (D^n_\alpha - S^{n-1}_\alpha - \{0\})) \arrow[r] \arrow[u, "\cong"] & H_*(X_n - X_{n-1}, X_n - X_{n-1} - C \arrow[u, "\cong"]
\end{tikzcd}$$

The left arrow on the second row is the excision from $\coprod_\alpha S^{n-1}_\alpha \subseteq \coprod_\alpha D^n_\alpha - \{0\} \subseteq \coprod_\alpha D^n_\alpha$. The bottom right arrow is an isomorphism because $\coprod_\alpha (D^n_\alpha - S^{n-1}_\alpha), \coprod_\alpha (D^n_\alpha - S^{n-1}_\alpha - \{0\}) \to (X_n - X_{n-1}, X_n - X_{n-1} - C_n)$ is a homeomorphism, and hence an isomorphism. This concludes the proof of the lemma. $\square$

Now for the second lemma

Lemma 1.17.5. *We have:*

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(X_{q-1}) \arrow[r] \arrow[d, equal] & H_q(X_q) \arrow[r, two heads] \arrow[d, "\cong"] \arrow[drr] & H_q(X_{q+1}) \arrow[r, "\cong"] \arrow[dr] & H_q(X_{q+2}) \arrow[d] \\
 & 0 & H_n(C_*(X_n)) \arrow[d, equal] & & H_n(X) \\
 & & \ker(C_n(X) \xrightarrow{d} C_{n-1}(X)) \arrow[d, hook] & & \\
 & & C_n(X_n) & &
\end{tikzcd}$$

So $H_q(X_q)$ is free abelian. The lemma is that $H_n(X_{n+1}) \to H_n(X)$ is an isomorphism.

For preparation, we'll talk about subcomplexes.

Definition 1.17.6. Let X be a CW-complex with a cell structure $\{g_\alpha : D^n_\alpha \to X_n | \alpha \in A_n\}$. A subcomplex is a subspace $Y \subseteq X$ such that for all n, there are $B_n \subseteq A_n$ such that $Y_n = Y \cap X_n$ is a CW-filtration for Y with characteristic maps $\{g_\beta | \beta \in B_n\}$.

Example 1.17.7. $X_n \subseteq X$ is a subcomplex.

Proposition 1.17.8 (Bredon, p. 196)**.** *Let X be a CW-complex with a chosen cell structure. Let $K \subseteq X$ be compact. Then K sits inside some finite subcomplex.*

Remark 1.17.9. For fixed cell structures, unions and intersections of subcomplexes are subcomplexes.

Proof of lemma 2. Let's do surjectivity. Pick $c \in Z_n(C_\bullet)(X)$. Well, $c = \sum c_i \sigma_i$ where $\sigma_i : \Delta^n \to X$. Since Δ^n is compact, $\sigma_i(\Delta^n)$ is compact, and thus $\bigcup \sigma_i(\Delta^n)$ is compact, and hence it lies in a finite subcomplex. Hence it sits in some X_N for some N, possibly very large. Thus $c \in S_n(X_N) \subseteq S_n(X)$. It's still a cycle because it was a cycle before. (This is a stronger result, we've proved that cycles come from cycles). This is more than enough.

Let's do injectivity. Let $c \in Z_n(C_\bullet)(X_{n+1})$. If i_* denotes the maps $H_n(X_q) \to H_n(X)$, then $i_*c \in Z_n(C_\bullet)(X)$. Suppose there was b such that $db = i_*(c)$, so that $i_*(c) = 0$ in $H_n(X)$. Well, $b = \sum b_i\tau_i$ where $\tau_i : \Delta^{n+1} \to X$. Then $\bigcup \tau_i(\Delta^{n+1})$ is compact, and thus sits inside X_M. So $b \in S_{n+1}(X_M)$, so the equation $db = i_*(c)$ is still true in X_M. So $[c] = 0$ in $H_n(X_M)$. It's not quite what I wanted.

This is good enough, because the maps $H_n(X_{n+1}) \to H_n(X_{n+2}) \to \cdots$ are all isomorphisms. □

We'll talk about real projective space next week.

Remark 1.17.10. Suppose X has only even cells. For example, $\mathbf{CP}^n$, namely complex lines in $\mathbf{C}^{n+1}$ through the origin, or $S^{2n+1}/v \sim \zeta z$ for any $\zeta \in \mathbf{C}$ such that $|\zeta| = 1$. I have a map $S^{2n-1} \to \mathbf{CP}^{n-1}$. We have:

$$\begin{tikzcd}
S^{2n-1} \arrow[r, hook] \arrow[d] & D^{2n} \arrow[d] \\
\mathbf{CP}^{n-1} \arrow[r, hook] & \mathbf{CP}^n
\end{tikzcd}$$

The same argument that we had before for $\mathbf{RP}^n$ show that the CW structure on $\mathbf{CP}^n$ is $\mathbf{CP}^0 \subseteq \mathbf{CP}^1 \subseteq \cdots \subseteq \mathbf{CP}^n$. So $\mathbf{CP}^n = D^0 \cup D^2 \cup \cdots \cup D^{2n}$.

Anyway, if you had X only with even cells, then $C_{\text{odd}}(X) = 0$, so $H_n(X) = \begin{cases} C_n(X) & n = 2k \\ 0 & n = 2k+1 \end{cases}$. We've shown that:

$$H_k(\mathbf{CP}^n) = \begin{cases} \mathbf{Z} & k = 2n \\ 0 & k = 2n+1 \end{cases}$$

1.18 Relative attaching maps, $\mathbf{RP}^n$, Euler characteristic, and homology approximation

Recall:

$$\begin{tikzcd}
H_n\left(\coprod_\alpha D^n_\alpha\right) \arrow[r, "\partial", "\cong"'] \arrow[d, "\text{char. map}"', "\cong"] & H_{n-1}\left(\coprod_\alpha S^{n-1}_\alpha\right) \arrow[d, "\text{attaching map}"'] & H_{n-1}\left(\coprod_\beta D^{n-1}_\beta, \coprod_\beta S^{n-2}_\beta\right) \arrow[r, "\cong"] & \widetilde{H} \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[d, equal] & H_{n-1}(X_{n-1}) \arrow[r, "j"] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "\cong"] \arrow[d, equal] \arrow[ur, dash, "\cong"] & {} \\
C_n(X) \arrow[rr, "d"] & & C_{n-1}(X) &
\end{tikzcd}$$

This boundary map d is the effect of $H_{n-1}(-)$ to:

$$\begin{tikzcd}
\coprod_\alpha S^{n-1}_\alpha \arrow[r, "f_{n-1}"] & X_{n-1} \arrow[r] \arrow[d] & X_{n-1}/X_{n-2} \cong \bigvee_\beta D^{n-1}_\beta / S^{n-2}_\beta \\
& X_n &
\end{tikzcd}$$

The composite in the top row of this diagram is called the "relative attaching map" because you're working relative to the $(n-2)$-skeleton.

Before, I coyly said before that there is a monoid homomorphism $\deg : [S^{n-1}, S^{n-1}] \to \mathbf{Z}_\times$ that sends $f \mapsto (H_{n-1}(f) : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1}))$. I said that this was surjective. This is actually an isomorphism. We won't prove injectivity here, but we'll do this in 18.906.

$\mathbf{RP}^m$

Recall the CW-structure $\mathbf{RP}^0 \subseteq \mathbf{RP}^1 \subseteq \cdots \subseteq \mathbf{RP}^{n-1} \subseteq \mathbf{RP}^n \subseteq \cdots \subseteq \mathbf{RP}^m$, where $\mathbf{RP}^{n-1}$ is the collection of lines in $\mathbf{R}^n$. The attaching map is the double cover $\pi : S^{n-1} \to \mathbf{RP}^{n-1}$ to get a pushout

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\text{double cover}"] \arrow[d] & \mathbf{RP}^{n-1} \arrow[d] \\
D^n \arrow[r] & \mathbf{RP}^n
\end{tikzcd}$$

The cellular chain complex C_* will look like:

$$0 \longleftarrow C_0 = \mathbf{Z} \longleftarrow C_1 = \mathbf{Z} \longleftarrow \cdots \longleftarrow C_{n-1} = \mathbf{Z} \longleftarrow C_n = \mathbf{Z} \longleftarrow \cdots \longleftarrow$$

The first map $C_1 \to C_0$ is easy because $\mathbf{RP}^m$ is connected. Thus $C_1 \to C_0$ is the zero map.

The relative attaching maps are: $S^{n-1} \xrightarrow{\pi} \mathbf{RP}^{n-1} \to \mathbf{RP}^{n-1}/\mathbf{RP}^{n-2} \cong S^{n-1}$. All we have to do is figure out the degree of this map. What happens when I collapse out $\mathbf{RP}^{n-2}$? This has the effect of collapsing out by the equator because you are collapsing all those points on the equator of S^{n-1} that go to $\mathbf{RP}^{n-1}$. So the composition $S^{n-1} \xrightarrow{\pi} \mathbf{RP}^{n-1} \to \mathbf{RP}^{n-1}/\mathbf{RP}^{n-2} \cong S^{n-1}$ splits as:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\pi"] \arrow[dr, "\text{pinching}"'] & \mathbf{RP}^{n-1} \arrow[r] & \mathbf{RP}^{n-1}/\mathbf{RP}^{n-2} \cong S^{n-1} \\
& S^{n-1}/S^{n-2} \arrow[ur] \arrow[r, equal] & S^{n-1}_u \vee S^{n-1}_\ell
\end{tikzcd}$$

The map $S_u^{n-1} \vee S_\ell^{n-1} \to S^{n-1}$ sends the top hemisphere to S^{n-1} itself via the identity, so the first factor is a homeomorphism. The lower hemisphere will be sent to S^{n-1} via the antipodal map (called α), which is also a homeomorphism, but I won't draw this in because I don't want to do that here. What does this do in homology? In $(n-1)$-dimensional homology, we choose a generator σ of $H_{n-1}(S^{n-1})$.

The pinch map sends σ to (σ, σ). The map from $S_u^{n-1} \vee S_\ell^{n-1}$ to S^{n-1} sends $(\sigma, \sigma) \mapsto \sigma + \alpha_* \sigma$. The degree of α_* is $(-1)^n$, as you saw in homework. The composite in homology for S^{n-1} is multiplication by $1 + (-1)^n$. Thus the differential $d : C_n(X) \to C_{n-1}(X)$ is multiplication by $1 + (-1)^n$. So the cellular chain complex now looks like, if n is even:

$$0 \longleftarrow C_0 = \mathbf{Z} \xleftarrow[0]{} C_1 = \mathbf{Z} \xleftarrow[2]{} C_2 = \mathbf{Z} \xleftarrow[0]{} \cdots \xleftarrow[0]{} C_{n-1} = \mathbf{Z} \xleftarrow[2]{} C_n = \mathbf{Z}$$

and if n is odd:

$$0 \longleftarrow C_0 = \mathbf{Z} \xleftarrow[0]{} C_1 = \mathbf{Z} \xleftarrow[2]{} C_2 = \mathbf{Z} \xleftarrow[0]{} \cdots \xleftarrow[2]{} C_{n-1} = \mathbf{Z} \xleftarrow[0]{} C_n = \mathbf{Z}$$

Thus:

$$H_k(\mathbf{RP}^n) = \begin{cases} \mathbf{Z} & k = 0 \text{ and } k = n \text{ odd} \\ \mathbf{Z}/2\mathbf{Z} & k \text{ odd}, 0 < k < n \\ 0 & \text{else} \end{cases}$$

This means that odd-dimensional real projective space is orientable, and even-dimensional real projective is non-orientable.

Euler char.

On Friday, I made the comment that things are simpler if you have (?). Here's a lemma.

Lemma 1.18.1. *If X is a CW-complex with only even cells (eg. $\mathbf{CP}^n, \mathbb{H}\mathbf{P}^n$), then $H_{odd}(X) = 0$, and $H_{even}(X) = C_{even}(X)$. Actually, I can just write $H_*(X) \cong C_*(X)$. Even homology groups are free abelian groups with rank given by the number of $(2q)$-cells.*

Proof. Trivial. □

Here's a result that'll improve this.

Theorem 1.18.2 ("Euler" because this is the generalization of the Euler characteristic)**.** *Let X be a finite CW-complex*[5]*. (We write A_n to index the n-cells.)*[6] *Then $\sum_{n=0}^{\infty}(-1)^n \# A_n =: \chi(X) =$ Euler characteristic is independent of the CW-structure on X.*

When n is even, the lemma is much stronger than this. I'm going to prove this theorem. Now I want to give a little reminder about the structure of finitely generated abelian groups.

Finitely generated abelian groups

If you have an abelian group A, you have a torsion subgroup $T(A)$, i.e., elements of finite order in A, i.e., $\{a \in A | \exists n \in \mathbf{Z}_{>0}, na = 0\}$. Then $A/T(A)$ is *torsion free.* For a general abelian group, that's all you can say. Assume A is finitely generated. Then $A/T(A)$ is also a finitely generated torsion free abelian group (take the image of the generators of A). This is actually a *free abelian group*, and so it's isomorphic to $\mathbf{Z}^r$. We say that r is the *rank* of A. It's an invariant of A.

Another fact is the following. Recall that any subgroup of A is finitely generated (nontrivial fact). This means that $T(A)$ is finitely generated. It is true that $T(A) = \mathbf{Z}/n_1 \oplus \mathbf{Z}/n_2 \oplus \cdots \oplus \mathbf{Z}/n_t$ where $n_1 | n_2 | \cdots | n_t$, where t is well-defined and is "the number of torsion generators". What this means for us is that $A \cong T(A) \oplus A/T(A) \cong \mathbf{Z}^r \oplus \mathbf{Z}/n_1 \oplus \mathbf{Z}/n_2 \oplus \cdots \oplus \mathbf{Z}/n_t$ where $n_1 | n_2 | \cdots | n_t$. If $0 \to A \to B \to C \to 0$ is a sexseq of finitely generated abelian groups, then $\operatorname{rank}(A) + \operatorname{rank}(B) = \operatorname{rank}(C)$.

1.19 "Euler's theorem", and "homology approximation" - CTC Wall. I) Singular homology, II) CW complexes, III) "homological algebra"

Theorem 1.19.1 ("Euler")**.** *Let X be a space which admits the structure of a finite CW complex. The sum $\sum_{h=0}^{\infty}(-1)^k \#(k\text{-cells})$ (generalizes $V - E + F$) is independent of that structure.*

Proof. Pick a CW-structure. We have $0 \to C_n \to \cdots \to C_2 \to C_1 \to C_0 \to 0$. We also have a sexseq $0 \to Z_k \to C_k \to B_{k-1} \to 0$, and another

[5] Some alarm starts ringing. "What are we supposed to do? It's just here to annoy us. It's ringing, but there's nothing to answer. Can we ignore it? (Turns off the light.) Let's just talk over it".

[6] Alarm ends, yay!

one $0 \to B_k \to Z_k \to H_k \to 0$. Let's use them and facts about rank that I talked about on Monday to compute what this alternating sum is. The Euler sum is the same as:

$$\begin{aligned}
\sum_{h=0}^{\infty}(-1)^k\#(k\text{-cells}) &= \sum_{k=0}^{\infty}(-1)^k \operatorname{Rank}(C_k) \\
&= \sum_{k=0}^{\infty}(-1)^k \operatorname{Rank}(Z_k) + \sum_{k=0}^{\infty}(-1)^k \operatorname{Rank}(B_{k-1}) \\
&= \sum_{k=0}^{\infty}(-1)^k(\operatorname{Rank}(H_k) + \operatorname{Rank}(B_k) + \operatorname{Rank}(B_{k-1}))
\end{aligned}$$

The terms $\operatorname{Rank} B_k + \operatorname{Rank} B_{k+1}$ telescope because it's an alternating sum, and hence vanish. The sum is $\sum_{k=0}^{\infty}(-1)^k \operatorname{Rank}(H_k)$. But $H_k(X) = H_k^{\text{sing}}(X)$ is an invariant of the space, independent of the CW-structure. □

Given $H_k(X)$, X a finite type CW-complex, what's a lower bound on the number of k-cells? Let's see. $H_k(X)$ is finitely generated because $C_k(X) \subseteq Z_k(X)$ is, and it surjects onto $H_k(X)$. Thus $H_k(X) = \bigoplus_{i=1}^{t(k)} \mathbf{Z}/n_i(k)\mathbf{Z} \oplus \mathbf{Z}^{r(k)}$ where the $n_1(k)|\cdots|n_{t(k)}(k)$ are the torsion indices.

The minimal chain complex with $H_k = \mathbf{Z}$ and $H_q = 0$ for $q \neq k$ is just the chain complex with 0 everywhere else except for $\mathbf{Z}$ in the kth degree. The minimal chain complex with $H_k = \mathbf{Z}/n\mathbf{Z}$ and $H_q = 0$ for $q \neq k$ is just the chain complex with 0 everywhere else except for $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ in dimension $k+1$ to k. These things are called elementary chain complexes.

A lower bound on the minimal number of k-cells is $r(k)+t(k)+t(k-1)$ where the last term comes for the "torsion generator in dimension $k-1$" (didn't catch that).

Theorem 1.19.2 (Wall). *Let X be a simply connected CW-complex of finite type. Then there exists a CW complex Y with $r(k)+t(k)+t(k-1)$ k-cells, for all k, and a homotopy equivalence $Y \to X$.*

I'm not going to prove this theorem. You can read Wall's theorem. You really can't ask for more. Oh, also here's a theorem.

Theorem 1.19.3. *Let X be connected and pointed $* \in X$. Then $\pi_1(X,*) \to H_1(X,*)$ exists, called the Hurewicz homomorphism, and it factors as $\pi_1(X,*) \to \pi_1(X,*)^{ab} \to H_1(X,*)$. The last map is an isomorphism.*

Some examples of Wall's theorem:

Example 1.19.4. We know that S^k has $\widetilde{H}_q(X) = \mathbf{Z}$ when $q = k$ and 0 else. Can you construct a space with $\widetilde{H}_q(X) = \mathbf{Z}/n\mathbf{Z}$ when $q = k$ and 0 else? We need to construct a space with the elementary chain complex with 0 everywhere else except for $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ in dimension $k+1$ to k. You need to have one 0-cell, do nothing until you get to dimension k, which is when you add a k-cell, and then use the attaching map $S^k \to S^k$ of degree n, i.e.:

$$\begin{tikzcd} S^k \arrow[r, "\text{degree } n"] \arrow[d] & S^k \arrow[d] \\ D^{k+1} \arrow[r] & X \end{tikzcd}$$

For example, when $k = 1$ and $n = 2$, you have $\mathbf{RP}^2$. This is called a "Moore space".

I brought up doing this in more generality with generators and relations, and Professor Miller built up on that:

Example 1.19.5. For more general abelian groups, you have a free abelian group F_0 sitting in a sexseq $0 \to F_1 \to F_0 \to M \to 0$ (this is an example of a *resolution of* M, which is what I'm going to start talking about). Then F_1 is also free. Pick some $k > 0$. You get a space whose homology is F_0, namely $\bigvee_\alpha S^k$, and a space whose homology is F_1, namely $\coprod S^k$. You can construct a map $\coprod S^k \to \bigvee_\alpha S^k$ such that the map $\alpha : F_1 \to F_0$ is what's induced on homology. Then you get:

$$\begin{tikzcd} \coprod S^k \arrow[r, "\text{gives } \alpha"] \arrow[d] & \bigvee_\alpha S^k \arrow[d] \\ \coprod D^{k+1} \arrow[r] & X \end{tikzcd}$$

Such an X is called a Moore space, and has homology M in dimension k and zero everywhere else. You can't make this into a functor, i.e., this can't be made into a functor $\mathbf{Ab} \to \mathbf{Top}$.

Homological algebra

You can put coefficients into homology. Let M an abelian group. You can talk about homology with coefficients in M. For example, $M =$

$\mathbf{Z}, \mathbf{Q}, \mathbf{Z}/n\mathbf{Z}, \cdots$. The $\mathbf{Z}/n\mathbf{Z}$ case when n is prime is pretty important because it's then a field.

Given X, you get a singular simplicial set $\mathrm{Sin}_*(X)$. Then we took the free abelian group S_* generated by $\mathrm{Sin}_*(X)$. I.e., $S_n = \mathbf{Z}[\mathrm{Sin}_n(X)] = \bigoplus_{\mathrm{Sin}_n(X)} \mathbf{Z}$. But I could replace $\mathbf{Z}$ with anything I wanted, and do the *exact* same construction. I can just as well as put any abelian group here.

Define the "singular chain complex with coefficients in M" as $S_n(X; M) = \bigoplus_{\mathrm{Sin}_n(X)} M$. There's a boundary map $d : S_n(X; M) \to S_{n-1}(X; M)$. Then the homology $H(S_*(X; M)) =: H_*(X; M)$. You can verify all the Eilenberg-Steenrod axioms yourself, except for one, namely the dimension axiom - $H_k(*; M) = \begin{cases} M & k = 0 \\ 0 & k \neq 0 \end{cases}$.

If you think about it, you'll realize that this whole unit in CW-complexes didn't use anything except for the Eilenberg-Steenrod axioms. This shows, by the way, that if you get some weird homology theory satisfying the Eilenberg-Steenrod axioms you get all the same results as if you used what we constructed before.

As an experiment, let's compute $H_*(\mathbf{RP}^n; \mathbf{Z}/2\mathbf{Z})$. The cellular chain complex is $0 \to \mathbf{Z}/2\mathbf{Z} \to \cdots \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$ where the maps are alternately multiplying by 2 and 0. But in this case, all the maps are 0 because $2 = 0$! So $H_k(\mathbf{RP}^n; \mathbf{Z}/2\mathbf{Z}) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & 0 \leq k \leq n \\ 0 & \text{else} \end{cases}$. How about $H_*(\mathbf{RP}^n; \mathbf{Q})$? Or $H_*(\mathbf{RP}^n; \mathbf{Z}[\frac{1}{p}])$ where $\mathbf{Z}[\frac{1}{p}] \subseteq \mathbf{Q}$? What about $\mathbf{Z}_{(p)} \subseteq \mathbf{Q}$ where you've localized at p?

Anyway, if I consider $H_*(\mathbf{RP}^n; \mathbf{Z}[\frac{1}{2}])$, then the cellular chain complex simplifies, but in a different way. You have $0 \to \mathbf{Z}[\frac{1}{2}] \to \cdots \to \mathbf{Z}[\frac{1}{2}] \to \mathbf{Z}[\frac{1}{2}] \to 0$. Multiplication by 2, however, is an isomorphism. So, $H_k(\mathbf{RP}^n; \mathbf{Z}[\frac{1}{2}]) = \begin{cases} \mathbf{Z}[\frac{1}{2}] & q = 0, n,\ q \text{ odd} \\ 0 & \text{else} \end{cases}$. You get a much simpler result. From this point of view, even projective spaces look like a point, and odd projective spaces look like a sphere!

It's a little awkward to go through this thing. I'd like to understand:

Question 1.19.6. How is $H_*(X; M)$ related to $H_*(X) = H_*(X; \mathbf{Z})$? This is a reasonable question.

The answer is called the "universal coefficient theorem". I'll spend a few days developing what we need to talk about this.

I want to talk about tensor products. Let me take a poll. Do you know tensor products? Working with a commutative ring instead of $\mathbf{Z}$? Actually, all of these examples $M = \mathbf{Z}, \mathbf{Q}, \mathbf{Z}/n\mathbf{Z}, \mathbf{Z}[\frac{1}{p}] \subseteq \mathbf{Q} \supseteq \mathbf{Z}_{(p)}, \cdots$ are rings. The boundary map $d : S_n(X; M) \to S_{n-1}(X; M)$ is a module homomorphism if $M = R$ is a (*always commutative*) ring.

This means that if R is a commutative ring, then $H_*(X; R)$ is an R-module. If R is a ring and M is an R-module, then $H_*(X; M)$ is an R-module. Just look at what you have here. The $\bigoplus_{\mathrm{Sin}_n(X)} M$ is an R-module, and d is an R-module homomorphism.

I'll admit, this is a little bit scary, because commutative rings are pretty complicated in general. I won't talk about some weirdo rings, though. I'll develop this more on Friday. Let me pass out homework.

1.20 $\otimes$

Welcome to algebraic topology! This is family weekend, so welcome. Today'll be more about algebra, and there'll be very little topology, I'm afraid. Today'll be about tensor products. I got your permission to talk about modules over a commutative ring. We're always going to let R be a commutative ring (they're going to be simple; for example, $\mathbf{Q}, \mathbf{F}_p, \mathbf{Z}, \mathbf{Z}/n\mathbf{Z}, \cdots$, PIDs).

I want to tell you that the category of R-modules is what's called a "categorical ring", where the addition corresponds to the direct sum, the zero element is the zero module, 1 is R itself, and multiplication is where you put a circle around a multiplication symbol.

The reason we do this is because of bilinear maps. Let me recall the definition of a bilinear map.

Definition 1.20.1. If I have M, N, P are R-modules, then a bilinear (or if you want to be annoying, R-bilinear) map is a map $\beta : M \times N \to P$ such that $\beta(x + x', y) = \beta(x, y) + \beta(x', y)$ and $\beta(x, y + y') = \beta(x, y) + \beta(x, y')$, and such that $\beta(rx, y) = r\beta(x, y)$ and $\beta(x, ry) = r\beta(x, y)$.

Example 1.20.2. $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by the dot product is a $\mathbf{R}$-bilinear map. The cross product $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ is $\mathbf{R}$-bilinear. More generally, if R is a ring then the multiplication $R \times R \to R$ is R-bilinear, and the multiplication on an R-module M given by $R \times M \to M$ is R-bilinear. This enters into topology because the map $H_n(X; R) \times H_n(Y; R) \xrightarrow{\times} H_{m+n}(X \times Y; R)$ is R-bilinear.

Wouldn't it be great to reduce stuff about bilinear maps to linear maps? We're going to do this by means of the universal property.

Definition 1.20.3. Let M, N be R-modules. A *tensor product* of M, N is a R-module P and a bilinear map $M \times N \xrightarrow{\beta_0} P$ such that for every bilinear map $M \times N \xrightarrow{\beta} Q$ there is a unique factorization.

$$\begin{tikzcd} M \times N \arrow[r, "\beta_0"] \arrow[dr, "\beta"'] & P \arrow[d, dashed, "f"] \\ & Q \end{tikzcd}$$

through an R-module homomorphism f. It's easy to check that $f \circ \beta_0$ is bilinear.

So β_0 is universal bilinear map out of $M \times N$. Instead of β_0 we're going to write $M \times N \xrightarrow{\otimes} P$. This means that $\beta(x, y) = f(x \otimes y)$ in the above diagram. There are lots of things to say about this. When you have something that is defined via a universal property, you first have to check that it exists!

Construction 1.20.4. I want to construct an R-bilinear map out of $M \times N$. I guess I should say it like this. Let $\beta : M \times N \to Q$ be any R-bilinear map. This β isn't linear. Maybe we should first extend it to a linear map. Consider $R\langle M \times N \rangle$, the free R-module generated by $M \times N$. Well, β is a map of sets, so there's a unique R-linear homomorphism $\overline{\beta} : R\langle M \times N \rangle \to Q$. Then I get a factorization:

$$\begin{tikzcd} M \times N \arrow[rr, "\beta"] \arrow[dr, "{[-]}"'] & & Q \\ & R\langle M \times N \rangle \arrow[ur, "\overline{\beta}"'] & \end{tikzcd}$$

The map $[-]$ isn't bilinear. So we should quotient $R\langle M \times N \rangle$ by a submodule S of relations. More precisely, S is the sub R-module generated by the relations needed to map $[-]$ a R-bilinear map, namely:

1. $[(x + x', y)] - [(x, y)] - [(x' - y)]$.
2. $[(x, y + y')] - [(x, y)] - [(x, y')]$.
3. $[(rx, y)] - r[(x, y)]$.
4. $[(x, ry)] - r[(x, y)]$

for all $x, x' \in M$ and $y, y' \in N$. Now, this map $[-]$ is bilinear - we've quotiented out by all things that made it false! Now the map $R\langle M \times N\rangle \to Q$ factors through via $R\langle M \times N\rangle \to R\langle M \times N\rangle/S \xrightarrow{f} Q$ because the map $\overline{\beta}$ is linear, and f is unique because the $\overline{\beta}$ is unique, so there's at most one factorization. We just checked that there was one, so we're done. We'll also write the composition $M \times N \xrightarrow{[-]} R\langle M \times N\rangle \to R\langle M \times N\rangle/S$ as $\otimes$.

You're never going to use this construction to compute anything. If you find yourself using this construction, stop and think about what you're doing.

Remark 1.20.5. Note that the image of (m, n) in $R\langle M \times N\rangle/S$ generates $R\langle M \times N\rangle/S$ as an R-module. The R-module $R\langle M \times N\rangle/S$ contains elements of the form $x \otimes y$ with $x \in M$ and $y \in N$ because they generate $R\langle M \times N\rangle$, and $R\langle M \times N\rangle/S$ is a quotient of that.

These $x \otimes y$ are called "decomposable tensors". (I've heard them called pure tensors.)

What are the properties of $R\langle M \times N\rangle/S =: P$?

1. How many maps are there that make the following diagram commute?

$$\begin{tikzcd} & P \arrow[dd, dashed] \\ M \otimes N \arrow[ur, "\otimes"] \arrow[dr, "\otimes"'] & \\ & P \end{tikzcd}$$

   By the uniqueness statement, there's only one map, namely the identity!

2. Suppose that we have two tensor products of M and N, say P and P'. We have

$$\begin{tikzcd} & P \arrow[dd, dashed, shift left, "b"] \\ M \otimes N \arrow[ur, "\otimes"] \arrow[dr, "\otimes"'] & \\ & P' \arrow[uu, dashed, shift right, "b'"'] \end{tikzcd}$$

And b, b' are unique. If you compose b and b', you'll see that you get the identity of P and P', depending on how you compose the maps. More precisely, you have:

$$\begin{tikzcd} & P \arrow[d, dashed, "b"] \\ M \otimes N \arrow[ur, "\otimes"] \arrow[dr, "\otimes"'] & P' \arrow[d, "b'"] \\ & P' \end{tikzcd}$$

and

$$\begin{tikzcd} & P' \arrow[d, dashed, "b'"] \\ M \otimes N \arrow[ur, "\otimes"] \arrow[dr, "\otimes"'] & P \arrow[d, "b"] \\ & P' \end{tikzcd}$$

Thus $bb' = 1$ and $b'b = 1$. So b, b' are isomorphisms, i.e., $P \cong P'$. We say that there is a canonical[7] isomorphism between any two constructions of a tensor products. The universal property defines the object up to canonical isomorphism. This is a general principle.

We can thus write the tensor product as if it just depended on just M and N. We write $M \otimes N$. A general element is a finite sum $\sum_i x_i \otimes y_i$. To be really honest, we'll write $M \otimes_R N$. If R is understood, we'll omit it. I'll usually forget to add the $\otimes_R$, and simply write $\otimes$.

3. Functoriality. If I have homomorphisms $M \times N \xrightarrow{f \times g} M' \times N'$. I have:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[d, "f \times g"] \arrow[dr] & M \otimes N \arrow[d, dashed] \\ M' \times N' \arrow[r, "\otimes"] & M' \otimes N' \end{tikzcd}$$

The dotted map exists because the diagonal map is R-bilinear. We write the map $M \otimes N \to M' \otimes N'$ as $f \otimes g$. We need to check stuff

[7]This means god given, but here it means that it's naturally constructed.

though.

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[d, "f\times g"] \arrow[dr] & M \otimes N \arrow[d, dashed, "f\otimes g"] \\
M' \times N' \arrow[r, "\otimes"] \arrow[d, "f'\times g'"] & M' \otimes N' \arrow[d, dashed, "f\otimes g"] \\
M'' \times N'' \arrow[r, "\otimes"] & M'' \otimes N''
\end{tikzcd}$$

And the composite matches up, i.e., $(f' \otimes g')(f \otimes g) = (f'f) \otimes g'g$.

4. I said that this was gonna be a categorical ring, so we need to check this. Well, $R \otimes_R M$ should be isomorphic to M. Let's think about this for a minute. I just need to check the universal property. Suppose I have an R-bilinear map $\beta : R \times M \to P$. We already have a universal R-bilinear map $\varphi : R \times M \to M$. I have to construct a universal factorization $f : M \to P$. Just let $f(x) = \beta(1, x)$. It's R-bilinear. We can check that this diagram commutes now because $f(\varphi(r, x)) = f(rx) = \beta(1, rx) = r\beta(1, x) = \beta(r, x)$. Well, this map $R \times M \to M$ is surjective, so there's at most one factorization. So we're done. There are other checks that are extremely boring, but they're part of the toolkit.

   I need to check that $L \otimes (M \otimes N) \cong (L \otimes M) \otimes N$ that's compatible with $L \times (M \times N) \cong (L \times M) \times N$. There's a canonical isomorphism. I don't know how to not say that this is trivial. Also, we need to check that $M \otimes N \cong N \otimes M$. (Just do this yourself. It's really easy.)

5. What happens with $M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$? It might be a finite direct sum, or maybe an uncountable collection. How does this relate to $\bigoplus_{\alpha \in A}(M \otimes N_\alpha)$? Let's construct a map $\displaystyle\bigoplus_{\alpha \in A}(M \otimes N_\alpha) \to M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$. We just need to define maps $M \otimes N_\alpha \to M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$ because direct sums are coproducts. Let this map be $1 \otimes \mathrm{in}_\alpha$ where $\mathrm{in}_\alpha : N_\alpha \to \bigoplus_{\alpha \in A} N_\alpha$. These give you a map $f : \bigoplus_{\alpha \in A}(M \otimes N_\alpha) \to M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$

   What about a map the other way? This is a bit trickier. An element of $M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right)$ is $x \otimes (y_\alpha)_{\alpha \in A}$, where you note that $y_\alpha = 0$ for all but finitely many $\alpha \in A$. Define $g : M \otimes \left(\bigoplus_{\alpha \in A} N_\alpha\right) \to$

$\bigoplus_{\alpha\in A}(M\otimes N_\alpha)$ via $x\otimes(y_\alpha)_{\alpha\in A}\mapsto(x\otimes y_\alpha)_{\alpha\in A}$. It's up to you to check that these are inverses and that you can extend to a general nondecomposable tensor by linearity.

We have not done any computations yet. I guess I should end with the statement that $S_*(X;M):=S_*(X)\otimes_R M$ if M is an R-module. We'll discuss on Monday the question we raised last time, namely:

Question 1.20.6. How is $H_*(X;M)$ related to $H_*(X)=H_*(X;\mathbf{Z})$? This is a reasonable question.

1.21 Tensor and Tor

Office hours are: for Hood, today from 1:30 to 3:30 in 2-390, and for me on Tuesday from 1 to 3 in 2-478. Point-set topology is the hardest part of this course, sorry for messing up the question on this week's pset. I like to emphasize the algebraic part.

Properties over $\otimes_R$

6. A ring is precisely specified by a map $R\otimes_{\mathbf{Z}}R\xrightarrow{\mu}R\xleftarrow{\eta}\mathbf{Z}$. You can define a ring purely diagrammatically. Associativity is commutativity of the following diagram:

$$\begin{tikzcd}
R\otimes R\otimes R \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & R\otimes R \arrow[d, "\mu"] \\
R\otimes R \arrow[r, "\mu"] & R
\end{tikzcd}$$

The identity map is commutativity of the following diagram.

$$\begin{tikzcd}
\mathbf{Z}\otimes R \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & R\otimes R \arrow[d, "\mu"] & R\otimes\mathbf{Z} \arrow[l, "1\otimes\eta"', "\cong"] \arrow[dl] \\
& R &
\end{tikzcd}$$

In fact, an R-module is an abelian group with a map $R\otimes M\xrightarrow{\varphi}M$ such that the following diagram commutes.

$$\begin{tikzcd}
R\otimes R\otimes M \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\eta"] & R\otimes M \arrow[d, "\varphi"] \\
R\otimes M \arrow[r, "\varphi"] & R
\end{tikzcd}$$

If A is an abelian group, then $R \otimes A$ is an R-module, where the multiplication is: $R \otimes (R \otimes A) \to (R \otimes R) \otimes A \xrightarrow{\mu \otimes 1} R \otimes A$. If A is an abelian group, then $A \to R \otimes A$ sending $a \mapsto 1 \otimes a$ is universal for maps from A to an R-module. We say that it's "initial". This means that if M is an R-module, there is a factorization:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & R \otimes A \arrow[dl, dashed] \\ M & \end{tikzcd}$$

Where the map $R \otimes A \to M$ is an R-module homomorphism and the map $A \to M$ is an abelian group homomorphism. Why is this true? We have a map $R \otimes A \xrightarrow{1 \otimes f} R \otimes M$, so the multiplication $\varphi : R \otimes M \to M$ is what we want. I.e., the extension is the composition:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & R \otimes A \arrow[d, "1 \otimes f"] \arrow[dl, "\varphi \circ (1 \otimes f)"] \\ M & R \otimes M \arrow[l, "\varphi"] \end{tikzcd}$$

Example 1.21.1. What if we let $A = \mathbf{Z}/n\mathbf{Z}$? Then if B is an abelian group (i.e., a $\mathbf{Z}$-module), $B \otimes \mathbf{Z}/n\mathbf{Z} \cong B/nB$.

7. Consider $0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$. Let's tensor with $\mathbf{Z}/2\mathbf{Z}$, to get $0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$. This cannot be a sexseq! But it's clear that the surjection $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ gives an isomorphism $\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$, i.e., $0 \to \mathbf{Z}/2\mathbf{Z} \xrightarrow{0} \mathbf{Z}/2\mathbf{Z} \xrightarrow{\cong} \mathbf{Z}/2\mathbf{Z} \to 0$. This is one of the major tragedies, that tensoring isn't exact. Exact means preserves exact sequences. The moral is that tensoring isn't generally exact, but preserves cokernels. More precisely:

Proposition 1.21.2. *The functor $N \otimes M \otimes_R N$ preserves cokernels. What do I mean? This means that this functor is* right exact*, i.e., if $N' \xrightarrow{i} N \xrightarrow{p} N'' \to 0$ is exact, then so is $M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$.*

Proof. We have:

$$\begin{tikzcd}
M \otimes_R N' \arrow[r, "1\otimes i"] & M \otimes_R N \arrow[r] \arrow[d] & M \otimes_R N'' \\
& M \otimes_R N/(\mathrm{im}(1 \otimes i)) = M \otimes_R N/I \arrow[ur, dashed, "\overline{\phi}"] &
\end{tikzcd}$$

At least we know that the composite $M \otimes_R N' \to M \otimes_R N \to M \otimes_R N''$ is zero. But this means that the dotted map exists, because the image I has to be sent to zero. The claim is that $\overline{\phi}$ is an isomorphism. We can construct an inverse to $\overline{\phi}$. It's easy to construct maps *out* of tensor products. This inverse will be a map $M \otimes_R N'' \xrightarrow{q} M \otimes_R N/I$. How do we construct maps out of a tensor product? Consider:

$$\begin{tikzcd}
M \otimes_R N'' \arrow[r, dashed, "\overline{q}"] & M \otimes_R N/I \\
M \times N'' \arrow[u] \arrow[ur, "q"] &
\end{tikzcd}$$

Where will $x \otimes y$ be sent? Let's pick $\overline{y} \in N$ such that $p\overline{y} = y$. I can do that because I supposed that p was surjective in the first place. Maybe I'm using the axiom of choice. (By the way, if I had a split exact sequence, tensoring will preserve split exact sequences, but not general exact sequences.) Anyway, map $x \otimes y \mapsto x \otimes \overline{y} + I$. That's the only thing I can think of doing, and so we pray and hope that it works. Let's check that this is well-defined first.

We know that $\overline{y}$ is only well-defined up to the image of something from N'. So consider $\overline{y}' = \overline{y} + iz$ for $z \in N'$. These are the only possible lifts. Then we get $x \otimes \overline{y}' = x \otimes (\overline{y} + iz) + x \otimes \overline{y} + x \otimes i(z) = x \otimes \overline{y} + (1 \otimes i)(x \otimes z) \in x \otimes \overline{y} + I$. Luckily, we divided out by I. There's *four* other things I have to check. I have to check that this is linear in each variable. This is just fussing around with the formula. Let's assume we've done that.

Pretty much by construction, $\overline{q}$ is the inverse for $\overline{p}$. This is because p takes $x \otimes \overline{y} + I$ to $x \otimes y$ because that's what $\overline{p}$ does – it just applies p to the second factor. $\square$

How about this failure of exactness? What can we do about that? Failure of exactness is bad, so let's try to repair it.

Think of a sexseq of chain complexes (that are bounded below by 0 and are nonnegatively graded) $0 \to N'_\bullet \to N_\bullet \to N''_\bullet \to 0$. We get an exact sequence $H_0 N' \to H_0 N \to H_0 N'' \to 0$. We already know that this isn't exact on the left because we have a lexseq in homology (because $H_1 N''$ need not be trivial). Let's imagine $M \otimes_R -$ as analogous to H_0. We already have an example of a functor that is right exact but not left exact (namely H_0), so this isn't unreasonable. I think I'll write down a theorem and finish the proof on Wednesday.

Theorem 1.21.3. *There are functors* $\mathrm{Tor}_n^R(M, -) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ *for* $n \geq 0$, *where I have a fixed ring* R *and a fixed* R*-module* M, *and natural transformations sending a sexseq* $0 \to N' \to N \to N'' \to 0$ *to* $\partial : \mathrm{Tor}_n^R(M, N'') \to \mathrm{Tor}_{n-1}^R(M, N')$ *such that you get a lexseq:*

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_n^R(M,N) \arrow[r] & \mathrm{Tor}_n^R(M,N'') \arrow[dll] \\
 \mathrm{Tor}_{n-1}^R(M,N') \arrow[r] & \mathrm{Tor}_{n-1}^R(M,N) \arrow[r] & \cdots &
\end{tikzcd}$$

such that $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$. *Basically,* Tor *fulfills the same role as homology.*

Some properties are as follows.

- $\mathrm{Tor}_q^R(M, N) = 0$ for $q > 1$ is R is a PID.
- $\mathrm{Tor}_q^R(M, F) = 0$ for $q > 0$ if F is a free R-module.

Let's explore what this gives us before we construct it.

Example 1.21.4. Let $R = \mathbf{Z}$, and consider the sexseq $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$. Because $\mathbf{Z}$ is free, you have:

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_n^{\mathbf{Z}}(C_\bullet)(M, \mathbf{Z}/n\mathbf{Z}) \arrow[dll] \\
 & \mathrm{Tor}_0^{\mathbf{Z}}(M,\mathbf{Z}) = M \otimes \mathbf{Z} \arrow[r, "\times n"] & \mathrm{Tor}_0^{\mathbf{Z}}(M,\mathbf{Z}) = M \otimes \mathbf{Z} \arrow[r] & \mathrm{Tor}_0^{\mathbf{Z}}(M, \mathbf{Z}/n\mathbf{Z}) = M/nM
\end{tikzcd}$$

So $\mathrm{Tor}_1^{\mathbf{Z}}(M, \mathbf{Z}/n\mathbf{Z}) = \ker(M \xrightarrow{n} M)$. This is the *n-torsion* of M. That's the origin of Tor. This is the key example to keep in mind. He said something like "In general, Tor isn't free, but it is here because $\mathbf{Z}$ is a PID."

Take a general R-module N. You can always take a free module F_0 that surjects onto N, i.e., $F_0 \to N \to 0$. For example, you can let F_0 be the free R-module on the underlying set of N. Form a sexseq $0 \to K_0 \to F_0 \to N \to 0$. You have an exact sequence for $n > 1$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_n^R(M, N) \arrow[dll] \\
\mathrm{Tor}_{n-1}^R(M, K_0) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

So for $n > 1$, $\mathrm{Tor}_n^R(M, N) \to \mathrm{Tor}_{n-1}^R(M, K_0)$ is an isomorphism. If $n = 1$, then:

$$\begin{tikzcd}
& \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_1^R(M, N) \arrow[dll] \\
M \otimes_R K_0 \arrow[r] & M \otimes_R F_0 \arrow[r] & M \otimes_R N \arrow[r] & 0
\end{tikzcd}$$

The maps between the tensors might be hard to compute, but you can compute this as a kernel.

But I've not constructed the functors yet. I just said to assume that it exists. This was so much fun, taking a free module and surjecting it onto N. What I'm trying to do is:

$$\begin{tikzcd}
\cdots \arrow[rr, dashed] & & F_2 \arrow[rr, dashed, "d"] \arrow[dr] & & F_1 \arrow[rr, dashed, "d"] \arrow[dr] & & F_0 \arrow[dr] & & \\
& K_2 \arrow[ur] \arrow[dr] & & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & N \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Where F_{i+1} surjects onto K_i and the F_i are free R-modules. Splicing these exact sequences gives you a exact sequence in the top row. This is what's called a *free resolution of* N. You can actually write this as:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] \arrow[d] & 0 \\
& & & N &
\end{tikzcd}$$

The F_0 are generators of N, the F_1 are relations, the F_2 are relations between relations, and so on. We say that these are syzygies. The singular term is syzygy.

1.22 More about Tor

Where is everybody? Looks like nobody wants to hear about Tor. You're the select few.

On Monday I gave "axioms" for Tor, basically by saying that $\operatorname{Tor}_n^R(M,-): \mathbf{Mod}_R \to \mathbf{Mod}_R$ is like a homology theory. Today I'm going to show a construction of Tor, and verify the axioms. Or at least the lexseq business. I also tried to show that it's a reasonable idea to study free resolutions, namely:

$$\begin{tikzcd}
\cdots \arrow[rr, dashed] & & F_2 \arrow[rr, dashed, "d"] \arrow[dr] & & F_1 \arrow[rr, dashed, "d"] \arrow[dr] & & F_0 \arrow[dr] & & \\
& K_2 \arrow[ur] \arrow[dr] & & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & N \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Where F_{i+1} surjects onto K_i and the F_i are free R-modules. Splicing these exact sequences gives you a exact sequence in the top row, which is a free resolution of N. Of course there are a lot of choices involved, so free resolutions aren't unique. The resolution $F_\bullet$ does *not* include N, and in the following diagram, the top row isn't exact but $\cdots \to F_0 \to N \to 0$ is exact.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \\
& & & N &
\end{tikzcd}$$

Then, note that:

$$H_q(F_\bullet) = \begin{cases} N & q = 0 \\ 0 & q > 0 \end{cases}$$

Construction 1.22.1. We construct $\operatorname{Tor}_n^R(M,N)$ via $H_n(M \otimes_R F_\bullet)$ where $F_\bullet$ is a free resolution of N.

I have to check that this is well-defined, that it's functorial, and that it satisfies the lexseq. Maybe I should also check what $H_0(M \otimes_R F_\bullet)$ is. I do get $M \otimes_R F_\bullet$, because tensoring with M is right exact, i.e., you have an exact sequence $M \otimes_R F_1 \xrightarrow{p} M \otimes_R F_0 \to M \otimes_R N \to 0$, and the zeroth homology is the cokernel of p, which is $M \otimes_R N$.

The check that it's well-defined goes like this. I call this the fundamental theorem of homological algebra.

Theorem 1.22.2 (Fundamental theorem of homological algebra)**.** *Let $f : M \to N$ be an R-module homomorphism. Let $\cdots \to E_1 \to E_0 \to M \to 0$ be such that each E_n is free, and $\cdots \to F_1 \to F_0 \to N \to 0$ is exact. The fundamental theorem says that I can lift the map $f : M \to N$ to a chain map $E_\bullet \to F_\bullet$ (i.e. they commute with the differentials and the augmentations $\epsilon_N : F_0 \to N$ and $\epsilon_M : E_0 \to M$), that is unique up to chain homotopy. I.e., they sit in the following commutative diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & E_2 \arrow[r] \arrow[d, dashed, "f_2"] & E_1 \arrow[r] \arrow[d, dashed, "f_1"] & E_0 \arrow[r, "\epsilon_M"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r, "\epsilon_N"] & N \arrow[r] & 0
\end{tikzcd}$$

I'm making a big deal about homological algebra in this course on algebraic topology, because this really is a homotopy-theoretic statement. It's part of the homotopy theory of chain complexes. I just want to mention something.

Definition 1.22.3. A projective R-module P is something such that there's a lift:

$$\begin{tikzcd}
 & M \arrow[d] \\
P \arrow[r] \arrow[ur, dashed] & N
\end{tikzcd}$$

Every free module is projective, clearly. Anything that's a direct summand in a projective is also projective. Any projective module is a direct summand of a free module.

Example 1.22.4. Let k be a field. Then $k \times k$ acts on k via $(a, b)c = ac$. This is an example of a projective that isn't free.

Remark 1.22.5. This proof uses only that E_n is projective. But if you have a PID, there's no difference between projective and free.

Proof of the fundamental theorem of homological algebra. Let's try to construct f_0. Consider:

$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d, dashed, "g_0"] & E_0 \arrow[r, "\epsilon_M"] \arrow[d, dashed, "f_0"] & M \arrow[d, "f"] & \\
0 \arrow[r] & L_0 = \ker(\epsilon_N) \arrow[r] & F_0 \arrow[r, two heads, "\epsilon_N"] & N \arrow[r] & 0
\end{tikzcd}$$

We know that $E_0 = R\langle S\rangle$. What we do is push forward the generators of E via ϵ_M, push it forward to f, and pull it back via ϵ_N which makes sense because it's surjective. This gives us f_0. You can restrict it to get g_0. Now I'm in exactly the same situation.

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, dashed, "g_1"] & E_1 \arrow[r, "\epsilon_M"] \arrow[d, dashed, "f_1"] & K_0 \arrow[d, "g_0"] & \\
0 \arrow[r] & L_1 \arrow[r] & F_1 \arrow[r] & L_0 \arrow[r] & 0
\end{tikzcd}$$

And g_1 exists. Now we need to prove the chain homotopy claim. Suppose I have $f_\bullet : E_\bullet \to F_\bullet$ and $f'_\bullet : E_\bullet \to F_\bullet$. Then $f'_n - f_n$ (which we'll rename ℓ_n) is a chain map lifting $0 : M \to N$. Let's rename things, so I have:

$$\begin{tikzcd}
\cdots \arrow[r] & E_2 \arrow[r] \arrow[d, "\ell_2"] & E_1 \arrow[r] \arrow[d, "\ell_1"] & E_0 \arrow[r, "\epsilon_M"] \arrow[d, "\ell_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r, "\epsilon_N"] & N \arrow[r] & 0
\end{tikzcd}$$

We want that $\ell_\bullet \simeq 0$. That is, we want $h : E_n \to F_{n+1}$ such that $dh + hd = \ell$. To begin with, we consider:

$$\begin{tikzcd}
 & E_0 \arrow[dl, dashed, "h"'] \arrow[d, "\ell_0"] \\
F_1 \arrow[r, "d"] & F_0
\end{tikzcd}$$

At the beginning, we want $dh = 0$. Well, we consider:

$$\begin{tikzcd}
 & & E_0 \arrow[dll, dashed] \arrow[dl] \arrow[d, "\ell_0"] \\
F_1 \arrow[r, two heads] & L_0 \arrow[r] & F_0
\end{tikzcd}$$

Because $F_1 \to L_0$ is a surjection, the lift exists, and we have $dh = \ell_0$. For the next step, we have:

$$\begin{tikzcd}
 & & E_1 \arrow[r] \arrow[dll, dashed] \arrow[d, "\ell_1"] & E_0 \arrow[d, "\ell_0"] \arrow[dl, "h"'] \\
F_2 \arrow[r, two heads] & L_1 \arrow[r] & F_1 \arrow[r, "d"] & F_0
\end{tikzcd}$$

So what do I want to do here? Ultimately what I want is that $dh = \ell_1 - hd$. Well, $d(\ell_1 - hd) = d\ell_1 - dhd = d\ell_1 - \ell_0 d = 0$ where the last equality comes because ℓ is a chain map. So now we can use exactness of $E_\bullet$ to define h. Exactly the same process continues. $\square$

That's it. We're going to use this several times. I'm glad to have mentioned the notion of projectivity, because we'll use it later. Now, apply $M \otimes_R -$ to that resolution (???). Suppose I have $f : N \to N'$, and get a map $f_\bullet : F_0 \to F'_\bullet$. Apply $M \otimes_R -$ to this, to get a chain map $M \otimes_R F_\bullet \to M \otimes_R F'_\bullet$ to get a map in homology $H_*(M \otimes_R F_\bullet) \to H_*(M \otimes_R F'_\bullet)$. How independent is this of the lifting that I chose? Suppose I have two chain maps $1 \otimes f_0, f \otimes f'_0 : M \otimes_R F_\bullet \to M \otimes_R F'_\bullet$. I can certainly form $1 \otimes h : M \otimes_R F_n \to M \otimes_R F'_{n+1}$. I know that $dh + hd = f - f'$. When I tensor, I get $1 \otimes (hd + dh) = 1 \otimes (f - f')$. But I want that $(1 \otimes h)(1 \otimes d) + (1 \otimes d)(1 \otimes h) = 1 \otimes f - 1 \otimes f'$. We use a further property of the tensor product:

8. If $f, f' : N \to N'$, then $1 \otimes (f + f') = 1 \otimes f + 1 \otimes f' : M \otimes_R N \to M \otimes_R N'$, and $1 \otimes (rf) = r(1 \otimes f)$. And $M \otimes_R -$ is an R-linear functor.

There's things called derived functors. In more general cases, you can't use chain complexes, but rather you use simplicial resolutions. There's non-additive homological algebra. Anyway, you check that $1 \otimes f$ and $1 \otimes f'$ are indeed chain homotopic, and so you're done.

I think I've verified that it's well-defined and functorial. What about the lexseq? Now, I start with a sexseq and want to get an lexseq. Suppose I have a sexseq $0 \to A \to B \to C \to 0$. I should first come up with an sexseq of resolutions. Consider:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
& F_0' \arrow[u] & & F_0'' \arrow[u] & \\
& F_1' \arrow[u] & & F_1'' \arrow[u] & \\
& \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

I want to get a free resolution in the middle. The only thing that I can think of doing is constructing:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & F_0' \arrow[u, "\epsilon_A"] \arrow[r] & F_0' \oplus F_0'' \arrow[u, dashed, "\epsilon_B"] \arrow[r] & F_0'' \arrow[u, "\epsilon_B"] \arrow[r] & 0 \\
& F_1' \arrow[u] & & F_1'' \arrow[u] & \\
& \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

In fact, it's the only choice because you have a free module and the sexseq splits. If I'm going to make this work, this is the only thing I can do. I need the augmentation, though. We can think of ϵ_B as a row vector. The first entry obviously has to be $i\epsilon$. And, well, there's a lift[8]

[8] There's some ambiguity here. I have to make a choice. It might seem that the boundary map is made up of the choices, but it *isn't*! I haven't proved that. It still needs to be proved.

because F_0'' is a free resolution:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & F_0' \arrow[r] \arrow[u, "\epsilon_A"] & F_0' \oplus F_0'' \arrow[r] \arrow[u, dashed, "\epsilon_B"] & F_0'' \arrow[r] \arrow[u, "\epsilon_B"'] \arrow[ul, dashed, "\overline{\epsilon}"] & 0 \\
 & F_1' \arrow[u] & & F_1'' \arrow[u] & \\
 & \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

So that $\epsilon_B = [i\epsilon, \overline{\epsilon}]$. This is surjective by the Snake lemma. Consider:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[u] & B \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F_0' \arrow[r] \arrow[u, "\epsilon_A"] & F_0' \oplus F_0'' \arrow[r] \arrow[u, dashed, "\epsilon_B"] & F_0'' \arrow[r] \arrow[u, "\epsilon_B"'] \arrow[ul, dashed, "\overline{\epsilon}"] & 0 \\
0 \arrow[r] & K_0' = \ker \epsilon_A \arrow[r] \arrow[u] & K_0 \arrow[r] \arrow[u] & K_0'' = \ker \epsilon_B \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The bottom row is exact by the 3×3-lemma. It's why I gave it to you for homework! Anyway, I now have a sexseq of free resolutions $0 \to F_\bullet' \to F_\bullet \to F_\bullet'' \to 0$. Now I want a sexseq $0 \to M \otimes_R F_\bullet' \to M \otimes_R F_\bullet \to M \otimes_R F_\bullet'' \to 0$, but I don't know this because $M \otimes_R -$ isn't exact. It's why we got into the business in the whole place. But $0 \to F_\bullet' \to F_\bullet \to F_\bullet'' \to 0$ is split. And applying any functor gives a splitting of the sexseq, i.e., $M \otimes_R -$ sends split sexseqs to (split) sexseq. This means that $0 \to M \otimes_R F_\bullet' \to M \otimes_R F_\bullet \to M \otimes_R F_\bullet'' \to 0$ also splits. Thus we're done proving the lexseq in Tor.

The key idea in homological algebra is that free modules are good.

1.23 Direct Limits

Goals are UCT (universal coefficient theorem), which is about $H_*(X; M)$ for varying M, the Künneth theorem (which is about $H_*(X \times Y)$), cohomology, and Poincaré duality.

Two more things about Tor

If R is a PID, then there's not so much to say about Tor, because any submodule of a free module is free. So that means that any R-module has a free resolution $0 \to F_1 = \ker(f) \to F_0 \xrightarrow{f}\to N \to 0$. It follows that $\mathrm{Tor}_n^R(M, N) = 0$ for $n > 1$. If you have a field, then tensoring is exact, so $\mathrm{Tor}_0^k(M, N) = 0$ for $n > 0$. That's why it's easy to work with fields. (There's also Prüfer rings, where every module is flat.) By the way, this means that if you have a sexseq $0 \to A \to B \to C \to 0$, then over a PID R, there's a six-term exact sequence $0 \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(M, B) \to \mathrm{Tor}_1^R(M, C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$.

Example 1.23.1. I want to give an example when you do have higher Tor. Let k be a field. Let $R = k[e]/(e^2)$. This is sometimes called the "dual numbers", or the exterior algebra over k. We're going to consider R-modules. Let's construct a projective resolution of k. Hmm. What is an R-module M? It's just a k-vector space M with an operator d that has action given by multiplication by e, and this satisfies $d^2 = 0$. And this is a chain complex! I guess it's not quite a chain complex because for us chain complexes are graded. So I guess it's an ungraded chain complex. I can consider $H(M; d) := \ker d/\operatorname{im} d$. Here's an example of an R-module. There's the augmentation $R \to k$ sending $e \mapsto 0$. This makes k an R-module, where $d = 0$. Let's construct a free resolution of k.

Here we go. We're going to write $k = \bullet(= 1)$ and $R := (1 =)\bullet \xrightarrow{d} \bullet(= e)$. Well, we have $R \to k$ given by $(\bullet \xrightarrow{d} \bullet) \to \bullet$. The kernel is not free, so we get $(\bullet \xrightarrow{d} \bullet) \to (\bullet \xrightarrow{d} \bullet) \to \bullet$. And this continues, so we get a projective resolution $\cdots \xrightarrow{e} R \xrightarrow{e} R \xrightarrow{e} R \to k$. What is Tor? Well, $\mathrm{Tor}_*^R(M, k)$ is the homology of the following chain complex $\cdots \xrightarrow{d} M \xrightarrow{d} M \to 0$. What is that homology? Clearly $\mathrm{Tor}_0^R(M, k) = M \otimes_R k = M/dM = M/eM$. This is often called the module of indecomposables. And, $\mathrm{Tor}_n^R(M, k) = H(M; d)$.

Last comment about Tor is that there's a symmetry there. Of course, $M \otimes_R N \cong N \otimes_R M$. This uses the fact that R is commutative. This leads right on to saying that $\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(N,M)$. We've been computing Tor by taking a resolution of the second variable. But I could equally have taken a resolution of the first variable. This follows from the fundamental theorem of homological algebra.

Direct limits

A long time ago, I said what a poset was. Let me tell you a joke about posets. I was at a conference, and Quillen was giving a talk. He was a student of Raoul Bott. Quillen was giving his talk, and he used the term "poset". This word was invented by Garrett Birkhoff(?). Then Bott objected and said "What is this crazy word?", and Quillen responded "What are you talking about? Your colleague invented it!". Anyway, it's not a joke. I guess it's just a piece of MIT and Harvard rivalry.

Anyway, a poset is a small category $\mathcal{I}$ such that $\#\mathcal{I}(i,j) \leq 1$ and isomorphism implies identity. I want to talk about a *directed set.*

Definition 1.23.2. A poset $(\mathcal{I}, \leq)$ is *directed* if, for every i, j, there exists a k such that $i \leq k$ and $j \leq k$.

Example 1.23.3. For example, the natural numbers $\mathbf{Z}_{\geq 0}$ with equality. Another example: if X is a space and I is the set of open subsets of X. It's directed by saying that $U \leq V$ if $U \subseteq V$. This is because U, U' need not be comparable, but $U, U' \subseteq U \cup U'$. Another example is $\mathbf{Z}_{>0}$ where $i \leq j$ if $i|j$. This is because $i, j|(ij)$.

Definition 1.23.4. Let $\mathcal{I}$ be a directed set. An $\mathcal{I}$-directed diagram in $\mathcal{C}$ is a functor $\mathcal{I} \to \mathcal{C}$. This means that for every $i \in \mathcal{I}$, there is $X_i \in \mathcal{C}$, and for every $i \leq j$, there's a map $X_i \xrightarrow{f_{ij}} X_j$ (and similarly for composition).

Example 1.23.5. If $\mathcal{I} = (\mathbf{Z}_{\geq 0}, \leq)$, then you get $X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \to \cdots$. This is the most important.

Example 1.23.6. Suppose $\mathcal{I} = (\mathbf{Z}_{>0}, |)$, i.e., the third example above. You can consider $\mathcal{I} \to \mathbf{Ab}$, say assigning to each i the integers $\mathbf{Z}$, and $f_{ij} : \mathbf{Z} \xrightarrow{j/i} \mathbf{Z}$. You get the picture.

These directed systems are a little complicated. But there's a simple one, namely the constant one.

Example 1.23.7. Let $\mathcal{I}$ be any directed set. You have a constant functor $c_A : \mathcal{I} \to \mathcal{C}$ at some $A \in \mathcal{C}$.

Of course, $\mathcal{I}$-directed systems in $\mathcal{C}$ are functors $\mathcal{I} \to \mathcal{C}$. They have natural transformations, and those are the morphisms in the category of $\mathcal{I}$-directed systems. That just means that if I have two directed systems $X, Y : \mathcal{I} \to \mathcal{C}$, then a map from one to the other is a commuting diagram:

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "g_i"] & X_j \arrow[d, "g_j"] \\ Y_i \arrow[r] & Y_j \end{tikzcd}$$

for all $i \leq j$.

It'd be great if every directed system was constant, but this isn't true. This leads to direct limits.

Definition 1.23.8. A direct limit is an object L and a map $X \to c_L$, which is initial among maps to constant systems. This means that I have some other map $X \to c_A$, then there's a unique induced map $c_L \to c_A$ that is induced from a map $L \to A$. We write $\varinjlim_{i \in \mathcal{I}} X_i = \operatorname{colim}_{i \in I} X_i$. (My own note: it's also sometimes called an an inductive limit.)

This is a universal property. So two different direct limits are canonically isomorphic.

Example 1.23.9. Consider $\mathcal{I} = (\mathbf{Z}_{\geq 0}, \leq)$, then you get $X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \to \cdots$ in **Top**. What is the direct limit? It's going to be $\bigcup_i X_i$. But what's the topology? Give it the finest topology so that all of the maps to the union are open. This just means that a subset is open in $\bigcup_i X_i$ if the preimage is open.

Example 1.23.10. Recall our example where we let $\mathcal{I} = (\mathbf{Z}_{>0}, |)$, i.e., the third example above. You can consider $\mathcal{I} \to \mathbf{Ab}$, say assigning to each i the integers $\mathbf{Z}$, and $f_{ij} : \mathbf{Z} \xrightarrow{j/i} \mathbf{Z}$. The colimit is $\mathbf{Q}$, where you send $X_n \to \mathbf{Q}$ via $1 \mapsto \frac{1}{n}$.

Lemma 1.23.11. *Let $X : \mathcal{I} \to \mathbf{Ab}$ (or $\mathbf{Mod}_R$). A map $f : X \to c_L$ is the direct limit (we write $f_i : X_i \to L$) if and only if:*

1. *For every $x \in L$, there exists an i and an $x_i \in X_i$ such that $f_i(x_i) = x$.*

2. *Let $x_i \in X_i$ be such that $f_i(x_i) = 0$ in L. Then there exists some $j \geq i$ such that $f_{ij}(x_i) = 0$ in X_j.*

Proof. Straightforward. □

It generalizes the observation that $\mathbf{Q}$ is the colimit of the diagram we drew above for $\mathcal{I} = (\mathbf{Z}_{>0}, |)$.

Corollary 1.23.12. *The direct limit* $\varinjlim_I : \mathrm{Fun}(\mathcal{I}, \mathbf{Ab}) \to \mathbf{Ab}$ *is exact. In other words,* $X_\bullet \to Y_\bullet \xrightarrow{p} Z_\bullet$ *is an exact sequence of* $\mathcal{I}$*-directed systems (at every degree, we get an exact sequence of abelian groups), then* $\varinjlim_I X_\bullet \xrightarrow{i} \varinjlim_I Y_\bullet \xrightarrow{p} \varinjlim_I Z_\bullet$.

Proof. First of all, $X_\bullet \to Z_\bullet$ is zero. Thus it factors through the constant zero object, so that $\varinjlim_I X_\bullet \to \varinjlim_I Z_\bullet$ is zero. Let $y \in \varinjlim_I Y_\bullet$, and suppose y maps to 0 in $\varinjlim_I Z_\bullet$. By the first condition, there exists i such that $y = f_i(y_i)$ for some $y_i \in Y_i$. Then $p(y) = f_i p(y_i)$ because p is a map of systems. This is zero. This means that there is $j \geq i$ such that $f_{ij} p(y_i) = 0$. We have an element in Y_j that maps to zero, so there is some x_j that is the preimage of the element in Y_j. So we're done. □

This makes the world a nice place to live.

1.24 Universal coefficient theorem (and Hom, adjointness)

On Wednesday, we'll talk about the Künneth theorem, and later we'll talk about the Künneth theorem. We've been talking about tensor products of R-modules, but we can do something that's more natural in a way. That's the notion of $\mathrm{Hom}_R(M, N)$, which is the collection of R-linear homomorphisms between R-modules M and N. This is actually itself an R-module. It's an abelian group, first of all, because you can add morphisms. How does $r \in R$ act on $f \in \mathrm{Hom}_R(M, N)$? Just define $(rf)(x) = f \cdot f(x)$. You should check that this does actually define an R-module homomorphism. (This is trivial.) If R isn't commutative I guess there'd be an action of $Z(R)$ on $\mathrm{Hom}_R(M, N)$. In particular, $\mathrm{Hom}_R(M, -) : \mathbf{Mod}_R \to \mathbf{Mod}_R$.

Remark 1.24.1. You're supposed to technically write $\underline{\mathrm{Hom}}_R(M, N)$ to mean $\mathrm{Hom}_R(M, N)$ with the structure of an R-module. But in these notes, I will not do this.

I wanted to bring this up because it relates to tensor products in a beautiful way. Consider $\mathrm{Hom}_R(M \otimes_R N, L)$. This is the collection of R-bilinear maps $M \times N \to L$. I claim that $\mathrm{Hom}_R(M \otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$. The way this works is the following. Suppose $f : M \otimes_R N \to L$. Define $\widehat{f} : M \to \mathrm{Hom}_R(N, L)$ via $\widehat{f} : m \mapsto (n \mapsto f(m \otimes_R n))$. This is a special case of the notion of an adjoint functor, introduced by Dan Kan, who was actually here at MIT. This is a big part of category theory.

Proposition 1.24.2. *Let $\mathcal{I}$ be a direct set, and let $M : \mathcal{I} \to \mathbf{Mod}_R$ be a $\mathcal{I}$-directed system of R-modules. There is a natural isomorphism $(\varinjlim_I M_i) \otimes_R N \cong \varinjlim_I (M_i \otimes_R N)$. It's very technical but it might be very useful. Maybe in the homework for example* ;-)

Proof. Consider $\mathrm{Hom}_R((\varinjlim_I M_i) \otimes_R N, L) \cong \mathrm{Hom}_R(\varinjlim_I M_i, \mathrm{Hom}_R(N, L))$. That's cool, because this is the same thing as $\mathrm{Map}_{\mathrm{Fun}(\mathcal{I}, \mathbf{Mod}_R)}(\{M_i\}, c_{\mathrm{Hom}_R(N,L)})$ because $\{M_i\}$ is a $\mathcal{I}$-directed system. Now, this is the same as $\mathrm{Map}_{\mathrm{Fun}(\mathcal{I}, \mathbf{Mod}_R)}(\{M_i \otimes N\}, c_L)$ by the Hom-tensor adjunction. We can unspool this to see that this is $\mathrm{Hom}_R(\varinjlim_{\mathcal{I}} (M_i \otimes_R N), L)$. Now conclude via the Yoneda lemma (which we haven't discussed yet, but I think is a homework problem). Basically this states that $\mathrm{Map}_{\mathcal{C}}(X, -)$ determines X. $\square$

We'll talk a lot more about Hom and adjunctions, but not today.

Here's the question I want to talk about today. Suppose that I'm given $H_*(X; \mathbf{Z})$. Does it determine $H_*(X; \mathbf{Z}/2\mathbf{Z})$? Consider $\mathbf{RP}^2 \to S^2$. In homology with coefficients in $\mathbf{Z}$, in dimension 2, this map must induce 0. But in $\mathbf{Z}/2\mathbf{Z}$-coefficients, in dimension 2, this map gives an isomorphism. I could have considered reduced homology. This shows that there's not a functorial relationship between $H_*(X; \mathbf{Z})$ and $H_*(X; \mathbf{Z}/2\mathbf{Z})$. So how *do* we go between different coefficients? That's the mystery.

Let R be a commutative ring. Let M be a R-module. I want to think about some chain complex $C_\bullet$ of R-modules. It could be the singular complex of a space, but it doesn't have to be. I'm going to forget to write $\otimes_R$ now; I'll just write $\otimes$. I can consider $H_n(C_\bullet) \otimes M$, or $H_n(C_\bullet \otimes M)$. The latter thing gives homology with coefficients in M. How can we compare these two? I claim that I can construct a map $\alpha : H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M)$. Recall the exact sequence $0 \to B_n \to Z_n(C_\bullet) \to H_n \to 0$ that defines homology. So I get an exact sequence $B_n \otimes M \to Z_n(C_\bullet) \otimes M \to H_n(C_\bullet) \otimes M \to 0$. And there's a surjection $Z_n(C_\bullet)(C \otimes M) \to H_n(C_\bullet \otimes M)$. I have to tell you where $x \otimes m \in Z_n(C_\bullet) \otimes M$ goes. I'll send it to $x \otimes m \in Z_n(C_\bullet)(C_\bullet \otimes M)$. I claim that this is a cycle, because $d(x \otimes m) = (dx) \otimes m$. But $x \in \ker d$, so this is zero, and thus $x \otimes m \in Z_n(C_\bullet)(C_\bullet \otimes M)$. Does it descend to a map in homology? We want to check that $B_n \otimes M \to Z_n(C_\bullet)(C_\bullet \otimes M)$ is zero. Suppose $y \in C_{n+1}$. Then $dy \in C_n$. Where does $dy \otimes x$ go? Send it to $d(y \otimes x) \in B_n(C_\bullet \otimes M)$. And this maps to zero.

The problem is that α is not always an isomorphism. But it is if M is free, say $M = R\langle S\rangle$. That's because then $C_\bullet \otimes M \cong \bigoplus_S C_\bullet$. Now there's a little lemma that nobody tells you about, because it's obvious, but here it is anyway:

Lemma 1.24.3. *Suppose I have a collection of exact sequences of R-modules $A_i \to B_i \to C_i$. Then $\bigoplus A_i \to \bigoplus B_i \to \bigoplus C_i$ is short exact, i.e., $\bigoplus$ is an exact functor.*

Proof. The composition is obviously zero. If $(b_i) \in \bigoplus B_i$ maps to 0, then by exactness, there are a_i that map to b_i, and we assume that if some $b_i = 0$, then $a_i = 0$. ☐

This in particular implies that $H(\bigoplus C_i) \cong \bigoplus H(C_i)$.

Consider a free resolution of M. Assume R is a PID, so that M has a free resolution of the form $0 \to F_1 \to F_0 \to M \to 0$. Thus we get a chain complex $C_\bullet \otimes F_1 \to C_\bullet \otimes F_0 \to C_\bullet M \to 0$. Now I get a lexseq in Tor, namely $\cdots \to \operatorname{Tor}_1^R(C_\bullet, M) \to C_\bullet \otimes F_1 \to C_\bullet \otimes F_0 \to C_\bullet M \to 0$.

Now I'm going to make a second assumption. Suppose C_n is a free R-module for all n. At least that $\operatorname{Tor}_1^R(C_n, M) = 0$. This shouldn't bother you at all, because the chain complexes that we need satisfy this condition. In particular, we have a sexseq $0 \to C_\bullet \otimes F_1 \to C_\bullet \otimes F_0 \to C_\bullet M \to 0$. What happens then? We get a lexseq. I want to give an unspliced form of this (huge diagram coming up!).

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\operatorname{coker}(H_n(C_\bullet \otimes F_1) \to H_n(C_\bullet \otimes F_0)) \arrow[r, equal] \arrow[d] & H_n(C_\bullet) \otimes M \arrow[d, "\alpha"] \\
H_n(C_\bullet \otimes M) \arrow[r, equal] \arrow[d, "\partial"] & H_n(C_\bullet \otimes M) \arrow[d, "\partial"] \\
\ker(H_{n-1}(C_\bullet \otimes F_1) \to H_{n-1}(C_\bullet \otimes F_0)) \arrow[r, equal] \arrow[d] & \operatorname{Tor}_1^R(H_{n-1}(C_\bullet), M) \arrow[d] \\
0 \arrow[r, equal] & 0
\end{tikzcd}$$

Because $\ker(H_{n-1}(C_\bullet \otimes F_1) \to H_{n-1}(C_\bullet \otimes F_0)) = \ker(H_{n-1}(C_\bullet) \otimes F_1 \to H_{n-1}(C_\bullet) \otimes F_0)$ and $\operatorname{coker}(H_n(C_\bullet) \otimes F_1 \to H_n(C_\bullet) \otimes F_0)$. Also, $\ker(H_{n-1}(C_\bullet \otimes F_1) \to H_{n-1}(C_\bullet \otimes F_0)) = \operatorname{Tor}_1^R(H_{n-1}(C_\bullet), M)$ because of the lexseq $0 \to \operatorname{Tor}_1^R(H_{n-1}(C_\bullet), M) \to H_{n-1}(C_\bullet) \otimes F_1 \to H_{n-1}(C_\bullet) \otimes F_0 \to H_{n-1}(C_\bullet) \otimes M \to 0$.

Thus we have a sexseq, which gives the universal coefficient theorem:

Theorem 1.24.4 (Universal Coefficient Theorem)**.** *If R is a PID and C_n is free for all n, then there is a natural sexseq of R-modules:*

$$0 \to H_n(C_\bullet) \otimes M \xrightarrow{\alpha} H_n(C_\bullet \otimes M) \xrightarrow{\partial} \operatorname{Tor}_1^R(H_{n-1}(C_\bullet), M) \to 0$$

A further fact that we won't prove is that this splits as a sexseq of R-modules, but not naturally.

Example 1.24.5. Consider $H_2(\mathbf{RP}^2; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$, and we can consider $H_2(\mathbf{RP}^2; \mathbf{Z}) \otimes \mathbf{Z}/2\mathbf{Z} = 0$. So by the UCT, this must come from $\operatorname{Tor}_1^{\mathbf{Z}}(H_1(\mathbf{RP}^2; \mathbf{Z}), \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$. So ∂ is an isomorphism. This explains the mystery that we began with.

Remark 1.24.6. Suppose R is not a PID. For example, consider what we worked with before, e.g., $R = k[e]/(e^2)$, and let $M = k$ (so e acts as 0). Consider the chain complex $C_\bullet : \cdots \to R \xrightarrow{e} R \xrightarrow{e} R \xrightarrow{e} R \to 0$. This is actually the free resolution of k that we found before. In particular, $H_n(C_\bullet) = \begin{cases} k & n = 0 \\ 0 & n \neq 0 \end{cases}$. What is $C_\bullet \otimes_R k$? It's exactly $\cdots \to k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k \to 0$. So $H_n(C_\bullet \otimes_R k) = \begin{cases} k & n > 0 \\ 0 & n < 0 \end{cases} = \operatorname{Tor}_n^R(k, k)$. The two homologies are super different. The excess Tor's are accounted for via spectral sequences, which you'll see when you take 18.906.

The next step is to consider the homology of products.

1.25 Künneth and Eilenberg-Zilber

We want to compute the homology of a product. Long ago, we constructed a bilinear map $S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, called the cross product. So we get a linear map $S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y)$, and it satisfies the Leibniz formula, i.e., $d(x \times y) = dx \times y + (-1)^p x \times dy$. The method we used was really another example of the fundamental theorem of homological algebra.

Definition 1.25.1. Let $C_\bullet, D_\bullet$ be chain complexes that are bounded below (i.e., $C_i = 0$ and $D_i = 0$ for $i \ll 0$; we'll be looking at the case where they're zero if $i < 0$). Define $(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$. The boundedness says that it's a finite sum. The differential $(C_\bullet \otimes D_\bullet)_n \to (C_\bullet \otimes D_\bullet)_{n-1}$ sends $C_p \otimes D_q \to C_{p-1} \otimes D_q \bigoplus C_p \otimes D_{q-1}$ given by $x \otimes y \mapsto dx \otimes y + (-1)^p x \otimes dy$.

So the cross product is a map of chain complexes $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$. The Künneth theorem in dimension zero is really easy, because $\pi_0(X) \times \pi_0(Y) = \pi_0(X \times Y)$.

Acyclic models

Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \to \mathbf{Ab}$ be a functor. Fix a set of object in $\mathcal{C}$, and let $\mathcal{M}$ be the "models". If $\mathcal{C} = \mathbf{Top} \times \mathbf{Top}$, then $\mathcal{M}$ is the set of pairs of simplices.

Definition 1.25.2. We say that F is $\mathcal{M}$-free if it is a direct sum of the free abelian group of the corepresentable functors, i.e., F is a direct sum of $\mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M, -)$ where $M \in \mathcal{M}$.

Example 1.25.3. For example, $S_n(X \times Y) = \mathbf{Z}\operatorname{Hom}_{\mathbf{Top}}(\Delta^n \times Y) = \mathbf{Z}\operatorname{Hom}_{\mathbf{Top}\times\mathbf{Top}}((\Delta^n, \Delta^n), (X, Y))$. Another example is that $\bigoplus_{p+q=n} S_p(X) \otimes S_q(Y) = \bigoplus \mathbf{Z}\operatorname{Hom}_{\mathbf{Top}}(\Delta^p, X) \otimes \mathbf{Z}\operatorname{Hom}_{\mathbf{Top}}(\Delta^q, Y) = \bigoplus_{p+q=n} \mathbf{Z}\langle \operatorname{Hom}_{\mathbf{Top}}(\Delta^p, X) \times \operatorname{Hom}_{\mathbf{Top}}(\Delta^q, Y) \rangle = \bigoplus_{p+q=n} \mathbf{Z}\operatorname{Hom}_{\mathbf{Top}\times\mathbf{Top}}((\Delta^p, \Delta^q), (X, Y))$.

Definition 1.25.4. A natural transformation of functors $\theta : F \to G$ is a $\mathcal{M}$-epimorphism if $\theta_M : F(M) \to G(M)$ is a surjection of abelian groups for every $M \in \mathcal{M}$. Consider a composition of natural transformations of functors $G' \to G \to G''$ that is zero. Let K be the objectwise kernel of $G \to G''$. So there's a factorization $G' \to K$. Say that the sequence is $\mathcal{M}$-exact if $G' \to K$ is a $\mathcal{M}$-epi.

This means that $G'(M) \to G(M) \to G''(M)$ is exact for all $M \in \mathcal{M}$.

Example 1.25.5. We claim that $S_n(X \times Y) \to S_{n-1}(X \times Y) \to \cdots \to S_0(X \times Y) \to H_0(X \times Y) \to 0$ is $\mathcal{M}$-exact, because when I plug in (Δ^p, Δ^q), I get an exact sequence (it's contractible so all homology groups vanish).

Example 1.25.6. Consider the sequence $\cdots \to (S_*(X) \otimes S_*(Y))_1 \to S_0(X) \otimes S_0(Y) \to H_0(X) \otimes H_0(Y) \to 0$. Is this $\mathcal{M}$-exact? We don't know that yet, although it's true in each factor if you plug in a simplex. It turns out that it actually *is* $\mathcal{M}$-exact, but I'll come back to this in a few minutes.

I've been checking things for our chain complexes that'll come up in the Künneth theorem. Here's the key lemma that makes it all work.

Lemma 1.25.7. *Let $\mathcal{C}$ be a category with a set of models $\mathcal{M}$ and let $F, G, G' : \mathcal{C} \to \mathbf{Ab}$ be functors. Let F be $\mathcal{M}$-free, and let $G' \to G$ be a $\mathcal{M}$-epimorphism. Then there's a lifting:*

$$\begin{tikzcd} & G' \arrow[d] \\ F \arrow[r, "f"] \arrow[ur, dashed] & G \end{tikzcd}$$

Proof. Clearly we may assume that $F(X) = \mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M, X)$. Suppose $X = M$. We get:

$$\begin{tikzcd} & G'(M) \arrow[d] \\ \mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M, M) \arrow[r, "f_M"] \arrow[ur, dashed, "\overline{f}_M"] & G(M) \end{tikzcd}$$

Consider $1_M \in \mathbf{Z}\operatorname{Hom}_{\mathcal{C}}(M, M)$. This maps to $f_M(1_M)$. But because $G' \to G$ is a $\mathcal{M}$-epi, there is some c_M that maps to $f_M(1_M)$. This is going to be $\overline{f}_M(1_M)$, i.e., $\overline{f}_M(1_M) := c_M$.

Now we're done by naturality! Because given any $\varphi : M \to X$, we get a commutative diagram:

$$\begin{tikzcd} \mathcal{C}(M, X) \arrow[r, "\overline{f}_X"] \arrow[d] & G'(X) \arrow[d] \\ \mathcal{C}(M, M) \arrow[r, "\overline{f}_M"'] & G'(M) \end{tikzcd}$$

Now, $1_M \mapsto \varphi$, and so $\overline{f}_X(\varphi) = \varphi_*(c_M)$ by commutativity. Now extend linearly. $\square$

We already knew the following result, but now we can make this formal:

Theorem 1.25.8 (Eilenberg-Zilber theorem)**.** *There exists a natural chain map:*

$$\begin{tikzcd} S_*(X) \otimes S_*(Y) \arrow[r, "\times"] \arrow[d] & S_*(X \times Y) \arrow[d] \\ H_0(X) \otimes H_0(Y) \arrow[r, "\cong"] & H_0(X \times Y) \end{tikzcd}$$

That is unique up to natural chain homotopy (this is part we didn't show before). There's also a map $\alpha : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ *because* $H_0(X) \otimes H_0(Y) \to H_0(X \times Y)$ *is an isomorphism. These two maps* α *and* $\times$ *are naturally chain homotopy inverses.*

Corollary 1.25.9. *There is an isomorphism* $H(S_*(X) \otimes S_*(Y)) \cong H_*(X \times Y)$*.*

We will show the following:

Theorem 1.25.10. *Let* $C_\bullet, D_\bullet$ *be chain complexes, bounded below, where* C_n *is a free R-module for all n. Then we have a generalization of the universal coefficient theorem. There is a sexseq:*

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \arrow[r, equal] & (H_*(C) \otimes H_*(D))_n \arrow[dll] \\ H_n(C_\bullet \otimes D_\bullet) \arrow[r] & \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C), H_q(D)) \arrow[r] & 0 \end{tikzcd}$$

This is true over any PID, too.

Proof. This is exactly the same as the proof for the UCT. It's a good idea to work through this on your own. □

So, by combining this theorem and previous corollary, we get:

Theorem 1.25.11 (Künneth theorem)**.** *If R is a PID, there is a natural sexseq (that splits, but not naturally):*

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \arrow[r, equal] & (H_*(X) \otimes H_*(X))_n \arrow[dll] \\ H_n(X \times Y) \arrow[r] & \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(X), H_q(Y)) \arrow[r] & 0 \end{tikzcd}$$

Example 1.25.12. If $R = k$ is a field, every module is already free, so the Tor term vanishes, and you get a Künneth isomorphism:

$$H_*(X;k) \otimes_k H_*(Y;k) \cong H_*(X \times Y;k)$$

This is rather spectacular. For example, what is $H_*(\mathbf{RP}^2 \times \mathbf{RP}^3; \mathbf{Z}/2\mathbf{Z})$? Well, directly, we see that there is 1 cell in dimensions 0 and 5, 2 cells in dimensions 1 and 4, 3 cells in dimensions 2 and 3. Note the symmetry. This isn't an accident, it's Poincaré duality, which we'll get to soon. By Künneth, it's $H_*(\mathbf{RP}^2; \mathbf{Z}/2\mathbf{Z}) \otimes_{\mathbf{Z}/2\mathbf{Z}} H_*(\mathbf{RP}^3)$, i.e.:

$$H_n(\mathbf{RP}^2 \times \mathbf{RP}^3; \mathbf{Z}/2\mathbf{Z}) = \bigoplus_{p+q=n} H_p(\mathbf{RP}^2; \mathbf{Z}/2\mathbf{Z}) \otimes H_q(\mathbf{RP}^3; \mathbf{Z}/2\mathbf{Z}) = \begin{cases} \text{work it out yo} \end{cases}$$

Chapter 2

Cohomology and duality

2.1 A few more things about coefficients, cohomology

Problem Set 5 now due Friday! Recall our example where we let $\mathcal{I} = (\mathbf{Z}_{>0}, |)$. You can consider $\mathcal{I} \to \mathbf{Ab}$, say assigning to each i the integers $\mathbf{Z}$, and $f_{ij} : \mathbf{Z} \xrightarrow{j/i} \mathbf{Z}$. The colimit is $\mathbf{Q}$, where you send $X_n \to \mathbf{Q}$ via $1 \mapsto \frac{1}{n}$. It's rather natural. But it's not the simplest way to define $\mathbf{Q}$. You just could have defined $\mathbf{Q} = \varinjlim(\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{4} \mathbf{Z} \to \cdots)$.

We saw that $\varinjlim$ is exact, that it commutes with tensor products, and that homology commutes with direct limits. So if I had a chain complex $C_\bullet$ of abelian groups, then the process of computing homology involves exact sequences. If I want to compute:

$$H(C_\bullet \otimes \mathbf{Q}) = H(C_\bullet \otimes \varinjlim \mathbf{Z}) = H(\varinjlim(C_\bullet \otimes \mathbf{Z})) = H(\varinjlim C_\bullet) = \varinjlim H(C_\bullet) = H(C_\bullet) \otimes \varinjlim$$

Thus $- \otimes \mathbf{Q}$ is exact, i.e., $\mathbf{Q}$ is *flat* over $\mathbf{Z}$, so $\mathrm{Tor}_1^{\mathbf{Z}}(M, \mathbf{Q}) = 0$. By UCT, this means that $H_*(X) \otimes \mathbf{Q} \simeq H_*(X; \mathbf{Q})$.

Definition 2.1.1. The nth Betti number of X is defined to be $\beta_n := \dim_{\mathbf{Q}} H_n(X; \mathbf{Q})$.

The Euler characteristic can just as well have been defined as $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \beta_n$.

Oh, let me ask if 11 am works for those of you who are going to take 18.906. OK? It seems like it's too early.

Anyway, every space has a diagonal map $X \xrightarrow{\Delta} X \times X$. This induces a map $H_*(X; R) \to H_*(X \times X; R)$, where R is a PID. But now, we have a Künneth map $\alpha : H_*(X;R) \otimes_R H_*(X;R) \to H_*(X \times X;R)$. We can get some kind of comultiplication if α is an isomorphism. And, well, the Künneth theorem says that $\mathrm{Tor}_1^R(H_*(X;R), H_*(X;R)) = 0$ if and only if α is an isomorphism. This condition is satisfied, for example, if each $H_n(X;R)$ is flat over R for all n (for example, free over R). A special case is when R is a field. So you get a comultiplication $\Delta : H_*(X;R) \to H_*(X;R) \otimes_R H_*(X;R)$ if this condition is satisfied.

This diagonal map has many properties.

Definition 2.1.2. Let R be a ring. A (graded, bounded below) coalgebra over R is a (graded) R-module M with a multiplication $\Delta : M \to M \otimes_R M$ and an augmentation map $\varepsilon : M \to R$ such that all of the following diagrams commute:

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[d] \arrow[dr] & \\
R \otimes_R M & M \otimes_R M \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & M \otimes_R R
\end{tikzcd}$$

Where the diagonal maps are the canonical isomorphisms. And you have coassociativity:

$$\begin{tikzcd}
M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & M \otimes_R M \arrow[d, "\Delta\otimes 1"] \\
M \otimes_R M \arrow[r, "1\otimes\Delta"'] & M \otimes_R M \otimes_R M
\end{tikzcd}$$

And it's cocommutative (he'll just say commutative because there's no need to say "co" if we know we're working with coalgebras, but I want to write it anyway – I'll probably abuse it though) if:

$$\begin{tikzcd}
 & M \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] & \\
M \otimes_R M \arrow[rr, "\tau"] & & M \otimes_R M
\end{tikzcd}$$

Where $\tau(x \otimes y) = (-1)^{|x|\cdot|y|} y \otimes x$ is the twisting map.

Example 2.1.3. The Künneth map is coassociative and cocommutative.

Consider:

$$\begin{tikzcd}
S_*(X) \otimes S_*(Y) \arrow[r, "\tau"] \arrow[d, "\times"] & S_*(Y) \otimes S_*(X) \arrow[d, "\times"] \\
S_*(X \times Y) \arrow[r, "\tau_*"] & S_*(Y \times X)
\end{tikzcd}$$

Where τ is as defined above on the tensor product and τ is also used to denote the twisting map $X \times Y \to Y \times X$. And this diagram commutes up to chain homotopy.

Corollary 2.1.4. *Let k be a field. Then $H_*(X;k)$ has the natural structure of a cocommutative graded coalgebraic structure.*

I could just talk about coalgebras. But one of my friends told me that nobody in France knew what coalgebras were. So we're going to talk about cohomology, and get an algebra structure. Some say that cohomology is better because you have algebras, but that's more of a sociological statement than a mathematical one.

Slogan 2.1.5. Cohomology gives algebras. It's a contravariant functor on spaces.

A better reason for looking a cohomology is that there are many geometric constructions that pull back. For example, if I have some covering space $\widetilde{X} \to X$, and I have a map $f : Y \to X$, I get a pullback covering space $f^*\widetilde{X}$. A better example is vector bundles (that we'll talk about in 18.906) – they don't push out, they pullback. This'll give the theory of *characteristic classes*. Another even better reason is that cohomology is the target of the Poincaré duality map.

Definition 2.1.6. Let N be an abelian group. A singular n-cochain on X with coefficients in N is a function $\mathrm{Sin}_n(X) \to N$.

If N is an R-module, then I can extend linearly to get a map $S_n(X;R) \to N$.

Notation 2.1.7. Write $S^n(X;N) := \mathrm{Map}(\mathrm{Sin}_n(X);N) = \mathrm{Hom}_R(S_n(X;R);N)$.

This is going to give me something contravariant, that's for sure. But I want to get a cochain complex $(S^*(X;N),d)$. Cochain means that the differential increases the degree.

I have a map $(-,-) : S^n(X;N) \otimes S_n(X;R) \to N$ given by evaluation. I want d on $S^*(X;N)$ that makes the map $S^n(X;N) \otimes S_n(X;R) \to N$ is a chain map where we regard N as a chain complex with zeros everywhere except in dimension 0. But the way I've said it doesn't quite make sense because $S^*(X;N)$ isn't a chain complex! So let $S^*(X;N)$ be a chain complex with $S^n(X;N)$ in dimension $(-n)$. But now it's not bounded below. So, to be honest, I should say that we're going to make a decision. If $C_\bullet, D_\bullet$ aren't necessarily bounded below chain complex, then $(C_\bullet \otimes D_\bullet)_n := \bigoplus_{i+j=n} C_i \otimes D_j$.

Let $f \in S^n(X;R)$ and $\sigma \in S_n(X;R)$. This is a chain map if $d(f,\sigma) = 0$. And $d(f,\sigma) = (df,\sigma) + (-1)^{|f|}(f, d\sigma)$. Let me erase this. This doesn't make sense. Scratch that. Let's start over.

Let $f \in S^{n+1}(X;R)$ and $\sigma \in S_n(X;R)$. Then $(df,\sigma)+(-1)^{|f|}(f,d\sigma) = d(f,\sigma) = 0$. So $(df)(\sigma) = (-1)^{|f|+1} f(d\sigma)$. That's what the differential is. This makes $S^*(X;N)$ a cochain complex. So you can consider its homology.

Definition 2.1.8. The nth cohomology of X with coefficients in N is $H^n(X;N) := H^n(S^*(X;N))$. This is a contravariant functor on **Top**.

If you fix X, I get a covariant functor $H^n(X;-)$ on $\mathbf{Mod}_R$.

Construction 2.1.9. I have a chain map $(,) : S^*(X;N) \otimes_R S_*(X;R) \to N$. I can apply homology to this, to get $\alpha : H^*(X;N) \otimes_R H_*(X;R) \to H(S^*(X;N) \otimes_R S_*(X;R))$. In particular, you get a natural pairing $(,) : H^*(X;N) \otimes_R H_*(X;R) \xrightarrow{\alpha} H(S^*(X;N) \otimes_R S_*(X;R)) \xrightarrow{(,)} N$. This "evaluation map" is called the Kronecker pairing.

Warning 2.1.10. $S^n(X;\mathbf{Z}) = \mathrm{Map}(\mathrm{Sin}_n(X);\mathbf{Z}) = \prod_{\mathrm{Sin}_n(X)} \mathbf{Z}$, which is probably an uncountable product. An awkward fact is that this is never free abelian.

Construction 2.1.11. If $A \subseteq X$, there is a restriction map $S^n(X;N) \to S^n(A;N)$. There is an injection $\mathrm{Sin}_n(A) \hookrightarrow \mathrm{Sin}_n(X)$. And as long as A is empty, I can split this. So any function $\mathrm{Sin}_n(A) \to N$ extends to $\mathrm{Sin}_n(X) \to N$. This means that $S^n(X;N) \to S^n(A;N)$ is surjective. You can define the kernel as $S^n(X,A;N)$, which sits in $0 \to S^n(X,A;N) \to S^n(X;N) \to S^n(A;N) \to 0$. This gives the *relative cochains*.

I can take the homology of this sexseq to get a lexseq:

$$
\begin{tikzcd}
 & \cdots & \\
H^1(X,A;N) \arrow[r] & H^1(X;N) \arrow[r] & H^1(A) \arrow[ull, "\delta"'] \\
H^0(X,A;N) \arrow[r] & H^0(X;N) \arrow[r] & H^0(A) \arrow[ull, "\delta"']
\end{tikzcd}
$$

And $H^0(X;N)$ sits in the cokernel of $\mathrm{Map}(\mathrm{Sin}_0(X),N) \to \mathrm{Map}(\mathrm{Sin}_1(X),N)$, so that $H^0(X;N) = \mathrm{Map}(\pi_0(X),N)$.

2.2 Cohomology, Ext, cup product

Let R be a ring, probably a PID. It's often a field, but it could be $\mathbf{Z}$. Let N be an R-module. Define $S^n(X;N) = \mathrm{Map}(\mathrm{Sin}_n(X),N)$. There's a boundary map $d : S^n(X;N) \to S^{n+1}(X;N)$ that takes a cochain f to the map df defined by $df(\sigma) = (-1)^{n+1} f(d\sigma)$ where $\sigma \in \mathrm{Sin}_{n+1}(X)$. Now, $d^2 = 0$, so $H^n(X;N) := H^n(S^*(X;N))$. This is a contravariant functor from **Top** to **Ab** and it's covariant in the coefficients.

When $n = 0$, you have $0 \to S^0(X;N) \xrightarrow{d} S^1(X;N)$. Thus $H^0(X;N) = \ker d$. Well, $S^0(X;N) = \mathrm{Map}(X,N)$, and d sends a 0-cochain f to $\sigma \mapsto \pm f(d\sigma) = \pm(f(\sigma(0)) - f(\sigma(1)))$. So a function is in the kernel of d if its values on the ends of any path is the same. Thus $H^0(X;N) = \mathrm{Map}(\pi_0(X),N)$.

We also talked about the Kronecker pairing. This gave an evaluation $H^n(X;N) \otimes_R H_n(X;R) \to N$. Taking the adjoint gives a map $H^n(X;N) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(X;R),N)$. We can try to understand cohomology in terms of homology. β'll often be an isomorphism, but not always.

Theorem 2.2.1 (UCT for cohomology). *There is a natural sexseq:*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R),N) \to H^n(X;N) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(X;R),N) \to 0$$

that splits, but not naturally. This also holds for relative cohomology.

I will tell you what Ext means now. I will prove this on Friday.

The problem that arises is that $\mathrm{Hom}_R(-,N) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is not exact. More precisely, it preserves right exact sequences, but not left exact sequences. Consider $M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ an exact sequence of R-modules. This gives a sequence $0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$. If I have $f : M'' \to N$, and I compose with p to get a zero map, then is f zero? Well, yes, because p is surjective. Now suppose I have $g : M \to N$, such that $i \circ g = 0$. So it facts through the cokernel, and you get a unique factorization $M'' \to N$ (unique because $M \to M''$ is surjective).

Suppose I have an injection $0 \to M' \to M$. Is $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N)$ surjective? If I have some map $M' \to N$ and $M' \hookrightarrow N$, then does this extend to a map $M \to N$? No! For example, if you have $1 : \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z} \hookrightarrow \mathbf{Z}/4\mathbf{Z}$, then you can't lift to a map $\mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$. This works, though, if the sexseq splits.

Homological algebra now comes to the rescue! Pick a free resolution of M given by $\cdots \to F_2 \to F_1 \to M \to F_0 \to 0$. If I apply Hom, I get a chain complex $0 \to \mathrm{Hom}(F_0, N) \to \mathrm{Hom}(F_1, N) \to \mathrm{Hom}(F_2, N) \to \cdots$.

Definition 2.2.2. Define $\mathrm{Ext}^n_R(M, N) = H^n(\mathrm{Hom}_R(F_\bullet, N))$ as the n-dimensional homology of this chain complex.

Remark 2.2.3. If R is a PID, then $\mathrm{Ext}^n = 0$ if $n > 1$. If R is a field, then $\mathrm{Ext}^n = 0$ for $n > 0$. Also, Ext is well-defined and functorial (by the fundamental theorem of homological algebra). Another important point is that $\mathrm{Hom}_R(-, N)$ takes chain homotopies to chain homotopies. This is a pretty important thing, and is something to think about for a minute. This is because if I have $M' \to M$, I get an induced map $\mathrm{Hom}_R(M', N) \leftarrow \mathrm{Hom}_R(M, N)$, and this is actually an R-module map, and in particular, additive. And this means that the $dh - hd = f_1 - f_0$ is preserved. Lastly, if M is free or projective, then $\mathrm{Ext}^n(M, -) = 0$ for $n > 0$. In addition, $\mathrm{Ext}^0(M, N) = \mathrm{Hom}_R(M, N)$.

Recall the trick that if I have a sexseq $0 \to A \to B \to C \to 0$, and I have free resolutions $F'_\bullet \to A$ and $F''_\bullet \to 0$, I can get a free resolution $F_\bullet \to B$ to get a sexseq $0 \to F'_\bullet \to F_\bullet \to F''_\bullet \to 0$. I can now apply $\mathrm{Hom}_R(-, N)$ to get a sexseq of cochain complexes, because this sequence splits in any given degree. Thus, I get a lexseq:

$$
\begin{tikzcd}
 & & 0 \arrow[dll] \\
\mathrm{Hom}_R(M'', N) \arrow[r] & \mathrm{Hom}_R(M, N) \arrow[r] & \mathrm{Hom}_R(M', N) \arrow[dll] \\
\mathrm{Ext}^1_R(M'', N) \arrow[r] & \mathrm{Ext}^1_R(M, N) \arrow[r] & \mathrm{Ext}^1_R(M', N) \arrow[dll] \\
\dots & &
\end{tikzcd}
$$

In this sense, Ext is like a cohomology theory for R-modules.

Let us use this to make a calculation.

Example 2.2.4. Let $R = \mathbf{Z}$, and look at the sexseq $0 \to \mathbf{Z} \xrightarrow{k} \mathbf{Z} \to \mathbf{Z}/k\mathbf{Z} \to 0$. Here N is some abelian group. So, our lexseq will look like:

$$
\begin{tikzcd}
 & & 0 \arrow[dll] \\
\mathrm{Hom}_R(\mathbf{Z}/k\mathbf{Z}, N) \arrow[r] & \mathrm{Hom}_R(\mathbf{Z}, N) = N \arrow[r] & \mathrm{Hom}_R(\mathbf{Z}, N) = N \arrow[dll] \\
\mathrm{Ext}^1_R(\mathbf{Z}/k\mathbf{Z}, N) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\dots & &
\end{tikzcd}
$$

The map $N \to N$ in this lexseq is multiplication by k. Thus $\mathrm{Hom}(\mathbf{Z}/k\mathbf{Z}, N) = \ker(N \xrightarrow{k} N)$. And, well, $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/k\mathbf{Z}, N) = N/kN$.

Let's get some consequences of cohomology from the UCT. Even independent of that, we get some properties.

Properties of cohomology

1. It's homotopy invariant. This means that if $f_0 \sim f_1 : (X, A) \to (Y, B)$, then $H^*(X, A; N) \xleftarrow{f_0^*, f_1^*} H^*(Y, B; N)$ are equal. I can't

use the UCT to address this because the UCT only tells you that things are isomorphic (use the 5-lemma). But we did establish a chain homotopy $f_{0,*} \sim f_{1,*} : S_*(X, A) \to S_*(Y, B)$, and applying Hom still retains this chain homotopy, and hence you get the same map on cohomology.

2. Excision. If $U \subseteq A \subseteq X$ such that $\overline{U} \subseteq \operatorname{Int}(A)$, then $H^*(X, A; N) \leftarrow H^*(X - U, A - U; N)$ is an isomorphism. This follows from the UCT (in the relative form, which is also true).

3. Mayer-Vietoris sequence. If I have $A, B \subseteq X$ such that their interiors cover X, then I have an lexseq:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
H^n(X;N) \arrow[r] & H^n(A;N) \oplus H^n(B;N) \arrow[r] & H^n(A \cap B;N) \arrow[dll] \\
H^{n+1}(X;N) \arrow[r] & \cdots &
\end{tikzcd}$$

2.3 Universal coefficient theorem, and products in H^*

We talked about cohomology $H^*(X, A; N)$, which was contravariant in (X, A). We can repeat some of the arguments for homology in the cohomological context. We can also relate cohomology to homology. This is the purpose of the universal coefficient theorem for cohomology. I won't actually prove this here, because I've put up notes on the website that covers this.

Theorem 2.3.1 (Mixed variance UCT). *Let R be a PID, let N be a R-module, and let $C_\bullet$ be a chain complex of free R-modules. We've decided that $\operatorname{Hom}_R(C_\bullet, N)$ is a cochain complex. I'm always a little confused on how to write the homology of a cochain complex? Should I write H^n or H_n? Maybe this is a personal problem, and I should keep it personal? We'll just write H^n. (Some ridiculous notation with the n sitting on the line in H was suggested, but this'd be* impossible *to TeX!)*

Anyway, we had a map $H^n \operatorname{Hom}_R(C_\bullet, N) \to \operatorname{Hom}_R(H_n(C_\bullet), N)$. The theorem is that this is surjective, which has kernel $\operatorname{Ext}^1_R(H_{n-1}(C_\bullet), N)$.

I.e, there is a natural sexseq:

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C_\bullet), N) \to H^n \operatorname{Hom}_R(C_\bullet, N) \to \operatorname{Hom}_R(H_n(C_\bullet), N) \to 0$$

that splits, but not naturally.

Strategy of the proof. I have a sexseq $0 \to Z_n(C_\bullet) \to C_n \to C_n/Z_n(C_\bullet) \to 0$ where $Z_n(C_\bullet) = \ker d_n$ (where $C_\bullet$ is a chain complex). But $C_n/Z_n(C_\bullet) = B_{n-1} \hookrightarrow C_{n-1}$ where $B_n(C_\bullet) = \operatorname{im} d_{n-1}$ (this last thing probably has wrong indexing). We assumed C_{n-1} is a free R-module, and that R is a PID, so that B_{n-1} is also free. Thus the sexseq $0 \to Z_n(C_\bullet) \to C_n \to B_{n-1} \to 0$ splits.

Another sexseq that's important is $0 \to Z_n(C_\bullet)/B_n(C_\bullet) \to C_n/B_n(C_\bullet) \to C_n/Z_n(C_\bullet) \to 0$. If you like, you can think of this as follows:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Z_n(C_\bullet)/B_n(C_\bullet) \arrow[r] \arrow[u] & C_n/B_n(C_\bullet) \arrow[r] \arrow[u] & C_n/Z_n(C_\bullet) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z_n(C_\bullet) \arrow[r] \arrow[u] & C_n \arrow[r] \arrow[u] & C_n/Z_n(C_\bullet) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B_n(C_\bullet) \arrow[r] \arrow[u] & B_n(C_\bullet) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Thus $0 \to Z_n(C_\bullet)/B_n(C_\bullet) \to C_n/B_n(C_\bullet) \to C_n/Z_n(C_\bullet) \to 0$ splits. But $Z_n(C_\bullet)/B_n(C_\bullet) = H_{n-1}$, so this gives: $0 \to H_{n-1} \to C_n/B_n(C_\bullet) \to B_{n-1} \to 0$.

Now, we have a sexseq $0 \to B^n \operatorname{Hom}_R(C_\bullet, N) \to Z^n \operatorname{Hom}_R(C_\bullet, N) \to H^n \operatorname{Hom}_R(C_\bullet, N) \to 0$. We want to compare this to $\operatorname{Hom}_R(H_n(C_\bullet), N)$. But, now, $0 \to H_{n-1} \to C_n/B_n(C_\bullet) \to B_{n-1} \to 0$ splits, so we get a sexseq $0 \to \operatorname{Hom}(B_{n-1}, N) \to \operatorname{Hom}(C_n/B_n(C_\bullet), N) \to \operatorname{Hom}(H_{n-1}(C_\bullet), N) \to 0$. Let me write this out:

$$\begin{tikzcd}
0 \arrow[r] & B^n \operatorname{Hom}_R(C_\bullet, N) \arrow[r] \arrow[d, dashed] & Z^n \operatorname{Hom}_R(C_\bullet, N) \arrow[r] \arrow[d, dashed] & H^n \operatorname{Hom}_R(C_\bullet, N) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \operatorname{Hom}(B_{n-1}, N) \arrow[r] & \operatorname{Hom}(C_n/B_n(C_\bullet), N) \arrow[r] & \operatorname{Hom}(H_{n-1}(C_\bullet), N) \arrow[r] & {}
\end{tikzcd}$$

An element of $Z^n \operatorname{Hom}_R(C_\bullet, N)$ is a map $f : C_n \to N$ such that $f \circ d = 0$. So this map f factors as $C_n/B_n(C_\bullet) \to N$. Thus we have the middle dotted map, and it's actually an isomorphism. You can then check compatibility, to get the left dotted map.

Is the map $H^n \operatorname{Hom}_R(C_\bullet, N) \to \operatorname{Hom}(H_{n-1}(C_\bullet), N)$ surjective? Well, use the snake lemma. I can then use diagram chasing to see that the map is indeed surjective, with kernel given by the kernel of $B^n \operatorname{Hom}_R(C_\bullet, N) \to \operatorname{Hom}(B_{n-1}, N)$. And the rest of the proof amounts to showing that this kernel is $\operatorname{Ext}^1_R(H_{n-1}(C_\bullet), N)$. And for splitting we can construct a splitting map, see the notes. $\square$

Remark 2.3.2. Miguel: Why is Ext called Ext?

Miller: It deals with extensions. Let R be a commutative ring, and let M, N be two R-modules. I can think about extensions $0 \to N \to L \to M \to 0$. Well, for example, I have two extensions $0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$, and $0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$. Say that two extensions are equivalent if there's a map of sexseqs between them is the identity on N and M. The two extensions above aren't equivalent, for example.

Another definition of $\operatorname{Ext}^1_R(M, N)$ is the set of extensions like this modulo this notion of equivalence. The zero in the group is the split extension.

Also, Ext is contravariant in the first variable, but not in the second variable. If you want to find the Ext groups, you can use an injective resolution of the second variable, or a projective resolution of the first variable. These are what are known as derived functors. Tor is a left derived functor because it uses a projective resolution that goes off to the left, but Ext is a right derived functor because it uses an injective resolution that goes off to the right.

Products

We'll talk about the cohomology cross product.

Construction 2.3.3. Define $S^p(X) \otimes S^q(Y) \xrightarrow{\times} S^{p+q}(X \times Y)$ as follows. Let σ be a $(p+q)$-simplex in $X \times Y$. Let $f \otimes g \in S^p(X) \otimes S^q(Y)$. We'll define $f \times g \in S^{p+q}(X \times Y)$. Then $f : S_p(X) \to R$ and $g : S_q(Y) \to R$. I can write $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ where $\sigma_1 : \Delta^{p+q} \to X$ and $\sigma_2 : \Delta^{p+q} \to Y$. Define:

$$(f \times g)(\sigma) = f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \omega_q)$$

where $\alpha_p : \Delta^p \to \Delta^{p+q}$ takes $k \mapsto k$ where $k \in [p]$. And $\omega_q : \Delta^q \to \Delta^{p+q}$ sends $\ell \mapsto \ell + p$ where $\ell \in [q]$. It is an extremely (idiotic I think was the word used) construction that I have never gotten used to.

Remark 2.3.4. It is *incredibly* stupid.

We get a map on cochains. We need to check that this is a chain map $S^*(X) \otimes S^*(Y) \to S^*(X \times Y)$. This is your homework! What this means is that you get a map $H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y)$. I guess I have coefficients in R. I'm not quite done, but I have a map $H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y))$. The composition $H^*(X) \otimes H^*(Y) \to H^*(S^*(X) \otimes S^*(Y)) \to H^*(X \times Y)$ is the cross product.

It's not very easy to do computations with this. It's just hard to deal with. Let me make some points about this construction, though.

Definition 2.3.5. Now take $X = Y$. Then I get $H^p(X) \otimes H^q(X) \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X)$ where $\Delta : X \to X \times X$ is the diagonal. This composition is called the cup product. Some people write $\smile$, and others write $\cup$. (I'll TeX it as $\cup$.)

Some properties of the cup product are the following. I claim that $H^0(X) \cong \mathrm{Map}(\pi_0(X), R)$ as rings. This α and ω stuff collapses if $p = q = 0$. There's nothing to do ... they're both the identity maps. So this isomorphism is clear. Also, inside $H^0(X)$, we pick the element that maps to $(c \mapsto 1) \in \mathrm{Map}(\pi_0(X), R)$. This is the identity for the cup product. This comes out because when $p = 0$ in our above story, then α_0 is just including the 0-simplex, and ω is the identity, so this is completely clear from that description.

Proposition 2.3.6. *If $f \in S^p(X)$ and $g \in S^q(Y)$ and $h \in S^r(Z)$, then $((f \times g) \times h)(\sigma) = (f \times (g \times h))(\sigma)$ where $\sigma : \Delta^{p+q+r} \to X \times Y \times Z$.*

Proof. Well, $((f \times g) \times h)(\sigma) = (f \times g)(\sigma_{12} \circ \alpha_{p+q}) h(\sigma_3 \circ \omega_r)$ where $\sigma_{12} : \Delta^{p+q+r} \to X \times Y$ and $\sigma_3 : \Delta^{p+q+r} \to Z$. But $(f \times g)(\sigma_{12} \circ \alpha_{p+q}) = f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \mu_q)$ where μ_q is the "middle portion" that sends $\ell \mapsto \ell + p$ where $\ell \in [q]$. In other words, $((f \times g) \times h)(\sigma) = f(\sigma_1 \circ \alpha_p) g(\sigma_2 \circ \mu_q) h(\sigma_3 \circ \omega_r)$. I've used associativity of the ring. But this thing is exactly the same as $(f \times (g \times h))(\sigma)$, so this is associative. $\square$

Therefore, $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.

But this product is obviously not commutative. It treats the two maps completely differently. But we have ways of dealing with this. This comes from the story of acyclic models, where I'll show that $\alpha \cup \beta = (-1)^{|\alpha|\cdot|\beta|}\beta \cup \alpha$. Thus $H^*(X)$ forms a *graded commutative ring.*

2.4 Cup product, continued

We can construct an explicit map $S^p(X) \otimes S^q(Y) \xrightarrow{\times} S^{p+q}(Y)$ via:

$$(f \times g)(\sigma) = f(\sigma_1 \circ \alpha_p)g(\sigma_2 \circ \omega_q)$$

where $\alpha_p : \Delta^p \to \Delta^{p+q}$ takes $k \mapsto k$ where $k \in [p]$, and $\omega_q : \Delta^q \to \Delta^{p+q}$ sends $\ell \mapsto \ell + p$ where $\ell \in [q]$.

I would like to modify this definition a bit, and correct what I said. I think it's more natural to put a sign so that:

$$(f \times g)(\sigma) = (-1)^{pq} f(\sigma_1 \circ \alpha_p)g(\sigma_2 \circ \omega_q)$$

You'll see why we're adding the sign soon. The claim is that $\times$ is a chain map, so we get a map $H(S^p(X) \otimes S^q(Y)) \to H_{p+q}(X \times Y)$. There is a map $H_p(X) \otimes H_q(Y) \xrightarrow{\mu} H(S^p(X) \otimes S^q(Y))$ defined as follows. Given chain complexes $C_\bullet$ and $D_\bullet$, define $\mu : H(C_\bullet) \otimes H(D_\bullet) \to H(C_\bullet \otimes D_\bullet)$. We've done this before, but let me just say: $\mu : [x] \otimes [y] \mapsto [x \otimes y]$. For some reason, he said that we'll not call this μ. But I don't want to delete the μ so I'll just leave it there.

We get the cup product as follows. Given $\Delta : X \to X \times X$, we get $\cup : H_p(X) \otimes H_q(X) \to H_{p+q}(X \times X) \xrightarrow{\Delta^*} H_{p+q}(X)$. We proved that there was a class $1 \in H_0(X) = \mathrm{Map}(\pi_0(X), R) \ni (\alpha \mapsto 1)$. This acts as a unit for the cup product. Also, the cup product is strictly associative.

Definition 2.4.1. Let R be a commutative ring. A *graded R-algebra* is a graded R-module $\cdots, A_{-1}, A_0, A_1, A_2, \cdots$ (integer graded sequence) (some people take a direct sum, but I find no reason for doing that) with maps $A_p \otimes_R A_q \to A_{p+q}$ and a map $R \to A_0$ (that determines the unit), that makes the following diagram commute.

$$\begin{tikzcd}
A_p \otimes_R (A_q \otimes_R A_r) \arrow[r] \arrow[d] & A_p \otimes_R A_{q+r} \arrow[d] \\
(A_{p+q}) \otimes_R A_r \arrow[r] & A_{p+q+r}
\end{tikzcd}$$

Definition 2.4.2. A graded R-algebra A is said to be (graded) commutative id the following diagram commutes:

$$x \otimes y \longmapsto (-1)^{|x|\cdot|y|} y \otimes x$$

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr] & & A \otimes A \arrow[dl] \\ & A & \end{tikzcd}$$

We claim that $H_*(X)$ forms a graded commutative ring under the cup product. This is nontrivial. On the cochain level, this is clearly not graded commutative. We're going to have to work hard – in fact, so hard that you're going to do some of it for homework.

We'll do a chain level construction. I say there's a map $\alpha : S_n(X \times Y) \to \bigoplus_{p+q=n} S_p(X) \otimes S_n(Y)$. First I'll tell you what happens to n-simplices $\sigma : \Delta^n \to X \times Y$. Let $\sigma_1 : \Delta^n \to X \times Y \to X$, and similarly for σ_2. Define $S_n(X \times Y) \xrightarrow{\alpha} \bigoplus_{p+q=n} S_p(X) \otimes S_n(Y)$ by sending:

$$\sigma \mapsto \sum_{p+q=n} (\sigma_1 \circ \alpha_p) \otimes (\sigma_2 \otimes \omega_q)$$

We claim that this is a chain map, and that this induces the cross product on cochains.

Remark 2.4.3. This map α is called the Alexander-Whitney map.

If I have two chain complexes, I want to consider a map $\mu : \mathrm{Hom}(C_\bullet, R) \otimes \mathrm{Hom}(D_\bullet, R) \to \mathrm{Hom}(C_\bullet \otimes D_\bullet, R)$ given by $f \otimes g \mapsto (x \otimes y \mapsto (-1)^{pq} f(x) g(y))$ where $|f| = |x| = p$ and $|g| = |y| = q$. I haven't quite done the right thing here, have I? I have to write:

$$f \otimes g \mapsto \begin{cases} (x \otimes y \mapsto (-1)^{pq} f(x) g(y)) & |x| = |f| = p, |y| = |g| = q \\ 0 & \text{else} \end{cases}$$

You should check that this is a chain map.

Recall that we have: $S^p(X) \otimes S^q(Y) = \mathrm{Hom}(S_p(X), R) \otimes_R \mathrm{Hom}(S_q(Y), R)$. The map μ we constructed just now gives a map $\mathrm{Hom}(S_p(X), R) \otimes_R \mathrm{Hom}(S_q(Y), R) \xrightarrow{\mu} \mathrm{Hom}(S_p(X) \otimes S_q(Y), R) \xrightarrow{\alpha} \mathrm{Hom}(S_{p+q}(X \times Y), R) = S^{p+q}(X \times Y)$. This is exactly the cross product $\times : S^p(X) \otimes S^q(Y) \to S^{p+q}(X \times Y)$.

Now, α is a natural transformation. Acyclic models comes into play. For homework, you're going to check that the following diagram commutes.

$$\begin{tikzcd}
S_*(X \times Y) \arrow[r, "T_*"] \arrow[d, "\alpha_{X,Y}"'] & S_*(Y \times X) \arrow[d, "\alpha_{Y,X}"] \\
S_*(X) \otimes_R S_*(Y) \arrow[r, "\tau"] & S_*(Y) \otimes_R S_*(X)
\end{tikzcd}$$

Acyclic models helps us prove things like this.

All of this implies that $H_*(X; R)$ is graded commutative. It's a theorem that you can't find a commutative multiplication. This is where Steenrod operations come from. They're called cohomology operations, and we'll talk more about this in 18.906.

My goal is to compute the cohomology of some space.

Proposition 2.4.4. *$H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X \times Y)$ is a R-algebra homomorphism.*

If A and B are graded R-algebras, then $(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes_R B_q$, and $(a \otimes b)(a' \otimes b') = (-1)^{|a'|\cdot|b|} aa' \otimes bb'$. This tensor product is graded commutative if A and B are.

Proof. I have $\Delta_X : X \to X \times X$ and $\Delta_Y : Y \to Y \times Y$. I also have $\Delta_{X \times Y} : X \times Y \to X \times Y \times X \times Y$, which factors as $(1 \times T \times 1) \circ (\Delta_X \times \Delta_Y)$. Let $\alpha_1, \alpha_2 \in H^*(X)$ and $\beta_1, \beta_2 \in H^*(Y)$. Then $\alpha_1 \times \beta_1, \alpha_2 \times \beta_2 \in H^*(X \times Y)$. I want to calculate what $(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2)$ is. Let's see:

$$\begin{aligned}
(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) &= \Delta^*_{X \times Y}(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\
&= (\Delta_X \times \Delta_Y)^*(1 \times T \times 1)^*(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\
&= (\Delta_X \times \Delta_Y)^*(\alpha_1 \times T^*(\beta_1 \times \alpha_2) \times \beta_2) \\
&= (-1)^{|\alpha_2|\cdot|\beta_1|}(\Delta_X \times \Delta_Y)^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2)
\end{aligned}$$

Now, I have a diagram:

$$\begin{tikzcd}
H^*(X \times Y) & H^*(X) \otimes_R H^*(Y) \arrow[l, "\times_{X \times Y}"'] \\
H^*(X \times X \times Y \times Y) \arrow[u, "(\Delta_X \times \Delta_Y)^*"] & H^*(X \times X) \otimes H^*(Y \times Y) \arrow[l, "\times_{X \times X, Y \times Y}"'] \arrow[u, "\Delta_X^{\times} \otimes \Delta_Y^*"']
\end{tikzcd}$$

This diagram commutes because the cross product is natural. This is exactly what commutativity of the diagram means. This means that:

$$\begin{aligned}(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) &= (-1)^{|\alpha_2|\cdot|\beta_1|}(\Delta_X \times \Delta_Y)^*(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2) \\ &= (-1)^{|\alpha_2|\cdot|\beta_1|}(\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)\end{aligned}$$

That's exactly what we wanted. $\square$

Example 2.4.5. How about $H^*(S^p)$? Let $p > 0$. This is: $H^k(S_p) = \begin{cases}\mathbf{Z} & k = 0, p \\ 0 & \text{else}\end{cases}$. Say that σ_p generates $H^p(S^p)$. We now have $H^*(S^p) \otimes H^*(S^q) \to H^*(S^p \times S^q)$. From the Künneth theorem, we know that $H^k(S^p \times S^q) = \begin{cases}\mathbf{Z} & k = 0, p, q, p+q \\ 0 & \text{else}\end{cases}$. This structure $H^*(S^p) \otimes H^*(S^q) \cong \mathbf{Z}[\sigma_p, \sigma_q]/(\sigma_p^2, \sigma_q^2) \cong H^*(S^p \times S^q)$ where $\sigma_p\sigma_q = (-1)^{|p|\cdot|q|}\sigma_q\sigma_p$ where by $\mathbf{Z}[\sigma_p, \sigma_q]$ I mean the free $\mathbf{Z}$-algebra on σ_p, σ_q. When $p = q = 1$, this gives an exterior algebra.

This is to be contrasted with $X = S^p \vee S^q \vee S^{p+q}$. This has the same homology as the product of two spheres. But in cohomology, the product of the generators must be zero. There is a diagram:

$$\begin{tikzcd} X \arrow[r] & S^p \vee S^q \\ S^p \arrow[u, hook] & \end{tikzcd}$$

On cohomology, notice that $\sigma_p\sigma_q = 0$ in $H^*(S^p \vee S^q)$ because there's no $(p+q)$-dimensional cohomology. Therefore we find that $S^p \vee S^q \vee S^{p+q} \not\simeq S^p \times S^q$.

It's very interesting to think about what the attaching map is:

$$\begin{tikzcd} S^{p+q-1} \arrow[r] \arrow[d] & S^p \vee S^q \arrow[d] \\ D^{p+q} \arrow[r] & S^p \times S^q \end{tikzcd}$$

On Wednesday we'll have a relaxed talk about surfaces. BTW I might not TeX that.

2.5 Surfaces and symmetric nondegenerate bilinear forms

Never mind, I decided to TeX this.

I think that this should be in an appendix...

Case of Poincaré duality

Let M be a compact manifold of dimension n (this is the kind of thing most of the world cares about). This whole lecture will be happening with coefficients in $\mathbf{F}_2$. There is a cup product pairing $H^p(M) \otimes H^q(M) \to H^{p+q}(M)$.

Theorem 2.5.1. *There exists a unique class $[M] \in H_n(M)$, called the fundamental class, such that there's a pairing $H^p(M) \otimes H^q(M) \xrightarrow{\cup} H^n(M) \xleftarrow{\langle -,[M]\rangle} \mathbf{F}_2$ when $p+q=n$. This pairing is a nonsingular (aka perfect) pairing, i.e., the map $H^p(M) \to \mathrm{Hom}(H^q(M), \mathbf{F}_2)$ defined by $x \mapsto (y \mapsto \langle x \cup y, [M]\rangle)$ is an isomorphism.*

One thing I should say is that $H_*(M)$ is zero in dimensions above n, and $H_i(M)$ are finitely generated.

Corollary 2.5.2. *From the UCT, it follows that $H^p(M) \xrightarrow{\cong} \mathrm{Hom}(H^q(M), \mathbf{F}_2) = H_q(M)$. The corresponding classes under this association are said to be Poincaré dual to each other.*

Remark 2.5.3. This means I get a pairing $H_q(M) \otimes H_q(M) \to \mathbf{F}_2$ in homology when $p+q=n$. This is called the *intersection pairing*. If you have one p-cycle and one q-cycle, this means that I can make them intersect transversely, i.e., they intersect in points – or not at all. I count the number of points modulo 2, and this is the element in $\mathbf{F}_2$.

Example 2.5.4. Let $M = T^2 = S^1 \times S^1$. We know that $H^1(M) = \mathbf{F}_2 \times \mathbf{F}_2 = \langle x, y\rangle$ where $x^2 = 0$ and $y^2 = 0$, and $xy \neq 0$ (this is the generator of $H^2(M)$). In terms of homology, this means that I can pick cycles on the torus so that they intersect on one point. This makes sense. Also, $x^2 = 0 = y^2$ because the second copy of x and y can just be moved so that they don't intersect.

Example 2.5.5 (Especially interesting case)**.** If $n = 2k$, I get a symmetric bilinear form on $H_k(M)$ via $H_k(M) \otimes H_k(M) \to \mathbf{F}_2$ that is nondegenerate.

Symmetric bilinear forms

I've really screwed up when TeXing this.

If $W \subseteq V$ is a subspace, then a bilinear form on V gives a bilinear form on W, but this need not be nonsingular even if the bilinear form on V is. Observe though that the induced bilinear form on W is nonsingular if and only if $W \cap W^\perp = 0$. This result should be obvious, but I don't want to say why.

I have a map $V \xrightarrow{\cong} V^*$ by sending $v \mapsto (v' \mapsto v \cdot v')$. Inside V, I have $W^\perp$, and I have a quotient $V^* \xrightarrow{\text{Res}} W^*$, with kernel $\ker \text{Res}$. I want to claim that $V/W^\perp \cong W^*$. We know that $\dim W + \dim W^\perp = \dim V =: n$. If W is nonsingular, then $V = W \oplus W^\perp$. This is true as vector spaces, but also as quadratic forms.

To see this, pick a basis $e_1, \cdots, e_k$ for W. The quadratic form is represented by some matrix $\big(e_i \cdot e_j\big)$. It's nonsingular exactly when its determinant is nonzero, i.e., 1. Then I can also pick a basis for $W^\perp$. Then I have nondegenerate quadratic form on V given by $\begin{pmatrix} (e_i \cdot e_j) & 0 \\ 0 & (...) \end{pmatrix}$ where the bottom left thing is that of $W^\perp$. Since the quadratic form on V has determinant 1, the matrix for $W^\perp$ also has determinant 1, and hence $V = W \oplus W^\perp$ as quadratic forms.

Now:

1. Suppose $v \in V$ has $v \cdot v = 1$. Then $V = \langle v \rangle \oplus \langle v \rangle^\perp$. This is an orthogonal splitting into nonsingular forms.

2. Assume $v \cdot v = 0$ for all[1] $v \in V$. Anyway, in our general situation, v might be zero – so pick $v \neq 0$. But the form is nonsingular, so there exists w such that $v \cdot w \neq 0$, i.e., $v \cdot w = 1$. Then $\langle v, w \rangle$ is a 2-dimensional subspace sitting inside V, so it splits off. Thus we get a hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (or something like this). (Example: torus)

Thus we conclude:

[1] Your intuition about dot products isn't very useful here, for example the torus. This is because the matrix for the quadratic form on the torus is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Also, $(x+y) \cdot (x+y) = 0$. This means that every cycle on the torus can be moved to have no self-intersection, at least modulo 2.

Proposition 2.5.6. *Any nonsingular symmetric bilinear form on a finite dimensional vector space over* $\mathbf{F}_2$ *is isomorphic to:*

$$\begin{pmatrix} \begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix} & 0 \\ 0 & \begin{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} & \\ & \end{pmatrix}\end{pmatrix}$$

where the bottom left thing has some number of the hyperbolic $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Let **Bil** be the set of nonsingular symmetric bilinear forms that are finite dimensional over $\mathbf{F}_2$ modulo isomorphisms. I've just given a classification of these things. This is a commutative monoid under $\bigoplus$. This **Bil** is {nonsingular matrices}/similarity where similarity means $M \sim N$ if $N = AMA^T$ for some nonsingular A.

Claim 2.5.7.

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1\end{pmatrix} \sim \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}$$

This is the same thing as saying that $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1\end{pmatrix} = AA^T$ for some nonsingular A.

Proof. Let $A = \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$. □

So, **Bil** is the commutative monoid generated by (1), $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ with relation $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1\end{pmatrix} \sim \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}$.

Connected surfaces

Let's go back to topology. Let $n = 2$, $k = 1$ (so that $2k = n$). Then you get an intersection pairing on $H_1(M)$. Suppose I consider $\mathbf{RP}^2$. We know that $H_1(\mathbf{RP}^2) = \mathbf{F}_2$. This must be that "(1)"-form. This says that anytime you have a cycle on a projective plane, there's nothing I can do to remove its self interesections. This is also seen on a Möbius band that I will not draw because I don't know how to draw fundamental polygons in LaTeX. But this can be seen by looking at the projection from the Möbius band to the circle.

We also had the notion of connected sums. Let's compute the homology of $\Sigma_1 \# \Sigma_2$ using Mayer-Vietoris, like you did in the exercise. You get $H_2(\Sigma_1 \# \Sigma_2) \to H_1(S^1) \to H_1(\Sigma_1 - D^2) \oplus H_1(\Sigma_2 - D^2) \to H_1(\Sigma_1 \# \Sigma_2)$. When you pierce a surface, it collapses down into its 1-skeleton, so $H_2(\Sigma_1 - D^2) \oplus H_2(\Sigma_2 - D^2) \cong 0$. Let's restrict ourselves to connected surfaces. The map $H_1(S^1) \to H_1(\Sigma_1 - D^2) \oplus H_1(\Sigma_2 - D^2)$ is the zero map when you think of it as a boundary. Actually, that argument needs to use something, because we know that with integer coefficients, this isn't right – you need to find an oriented chain in that case – but this orientation issue goes away in $\mathbf{F}_2$ because there's no issues with $\pm$. And actually, the map $H_1(\Sigma_1 \# \Sigma_2) \to H_0(S^1)$ is zero as well, so $H_1(\Sigma_1 - D^2) \oplus H_1(\Sigma_2 - D^2) \cong H_1(\Sigma_1) \oplus H_1(\Sigma_2) \cong H_1(\Sigma_1 \# \Sigma_2)$ that respect the intersection pairing.

Let **Surf** be the commutative monoid of connected compact surfaces modulo homeomorphism. What we've proved is that there's a map **Surf** $\to$ **Bil** that takes connected sums to direct sums of orthogonal bilinear forms. And this map is an isomorphism. I.e.:

Theorem 2.5.8. *There is an isomorphism of commutative monoids* **Surf** $\to$ **Bil**.

This is a little strange. What is our claim $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ saying in this context? It's saying that $T^2 \# \mathbf{RP}^2 \cong (\mathbf{RP}^2)^{\#3}$.

Claim 2.5.9. If Σ is nonoriented, then $\Sigma \# T^2 \cong \Sigma \# K$ (where K is the Klein bottle $\mathbf{RP}^2 \# \mathbf{RP}^2$).

If it's nonoriented, then a Möbius strip M sits into Σ. Uh I guess you're dragging your attachment of the torus along M?

There's more to be said about this. You can think of quadratic forms if you're working over the reals or something. You can't do this modulo 2. Under the story we talked about, the oriented surfaces are the ones where the dot product is always zero. A better thing to ask over $\mathbf{F}_2$ are quadratic forms such that $q(x+y) = q(x) + q(y) + x \cdot y$. This leads to the Kervaire invariant, etc.

Happy Thanksgiving! See you on Monday.

2.6 A plethora of products

Remark 2.6.1. Please note that these notes might be absolute bullshit, because I came back today from my flight at 8 am.

Recall that we have the Kronecker pairing $\langle , \rangle : H^p(X) \otimes H_p(X) \to R$, which obviously isn't natural because H^p is contravariant while homology isn't.

Consider the following. Given $f : X \to Y$, and $b \in H^p(Y)$ and $x \in H_p(X)$, how does $\langle f^* b, x\rangle$ relate to $\langle b, f_* x\rangle$?

Claim 2.6.2.

$$\langle f^* b, x\rangle = \langle b, f_* x\rangle$$

Proof. Easy! I find it useful to write out diagrams of where things are. We're gonna work on the chain model.

$$\begin{tikzcd}
\mathrm{Hom}(S_p(X), R) \otimes S_p(X) \arrow[rr, "\langle , \rangle"] & & R \\
\mathrm{Hom}(S_p(Y), R) \otimes S_p(X) \arrow[u, "f^* \otimes 1"] \arrow[r, "1 \otimes f_*"] & \mathrm{Hom}(S_p(Y), R) \otimes S_p(Y) \arrow[ur, "\langle , \rangle"'] &
\end{tikzcd}$$

The top line is what gives the Kronecker pairing. Note that f^* means contravariant and f_* means covariant. This diagram commuting *is* the statement we want to prove. Let's see. Suppose $[\beta] = b$ and $[\xi] = x$. Then from the bottom left, going to the right and then the top is $\beta \otimes \xi \mapsto \beta \otimes f_*(\xi) \mapsto \beta(f_* \xi)$. The other way is $\beta \otimes \xi \mapsto f^*(\beta) \otimes \xi = (\beta \circ f) \otimes \xi \mapsto (\beta \circ f)(\xi)$. This is exactly $\beta(f_* \xi)$, because, oh, that's just how composition works. $\square$

There's another product around. We called this μ I think, where $\mu : H(C_\bullet) \otimes H(D_\bullet) \to H(C_\bullet \otimes D_\bullet)$ given by $[c] \otimes [d] \mapsto [c \otimes d]$. I'm secretly using this in the above proof.

We also have the cross product(s!) $\times : H_p(X) \otimes H_q(Y) \to H_{p+q}(X \times Y)$ and $\times : H^p(X) \otimes H^q(Y) \to H^{p+q}(X \times Y)$. You should think of this as fishy because both maps are in the same direction – but this is OK because we're using different things to make these constructions. Still, they're related:

Theorem 2.6.3. *Let $a \in H^p(X), b \in H^p(Y), x \in H_p(X), y \in H_q(Y)$. Then:*

$$\langle a \times b, x \times y\rangle = (-1)^{|x| \cdot |b|} \langle a, x\rangle \langle b, y\rangle$$

This isn't just idle – although idle things are a great thing to do!

Proof. Say $[\alpha] = a, [\beta] = b, [\xi] = x, [\eta] = y$. Then recall that $\langle a \times b, x \times y\rangle$ comes from $(\alpha \times \beta)(\sigma) = (-1)^{pq}\alpha(\sigma_1 \circ \alpha_p)\beta(\sigma_2 \circ \omega_q)$ where $\sigma : \Delta^{p+q} \to X \times Y$. There's two uses of the symbol α, and there'll be a third one in a minute, but they'll all have subscripts, so I hope you'll forgive me.

Recall also the ((p, q)th component of the) Alexander-Whitney map $\alpha_{X,Y} : S_{p+q}(X \times Y) \to S_p(X) \otimes S_q(Y)$. We have a big diagram:

$$\begin{tikzcd}
S_p(X) \otimes S_q(Y) \arrow[r, "\times"] \arrow[dr, "1\sim"] & S_{p+q}(X \times Y) \arrow[d, "\alpha_{X,Y}"] \arrow[dr, "\alpha\times\beta"] & \\
 & S_p(X) \otimes S_p(Y) \arrow[r, "\alpha\cdot\beta"'] & R
\end{tikzcd}$$

Where $\alpha \cdot \beta : (\xi \otimes \eta) \mapsto (-1)^{pq}\alpha(\xi) \cdot \beta(\xi)$. Now, the diagonal arrow $S_p(X) \otimes S_q(Y) \to S_p(X) \otimes S_p(Y)$ is unique up to chain homotopy, and is homotopic to the identity – this is what the method of acyclic models tells me. (See the statement of the Eilenberg-Zilber theorem above.)

Thus we find that $\alpha_{X,Y}(\xi \times \eta) = \xi \otimes \eta + (dh + hd)(\xi \otimes \eta)$ for some chain homotopy h. I think we're really down now, because ξ and η are both cycles, and hence d will kill them (wait it seems like it only kills η???). So $\alpha_{X,Y}(\xi \times \eta) = \xi \otimes \eta$. Now, $\alpha \cdot \beta$ is a cocycle (check!). When I apply α it kills the dh factor (what???), therefore $(\alpha \cdot \beta)\alpha_{X,Y}(\xi \times \eta) = (\alpha \cdot \beta)(\xi \otimes \eta)$. $\square$

All because of the magic of acyclic models.

Let's now try to prove a Künneth theorem for H^*. Let $R = k$ be a field (eg $\mathbf{F}_p, \mathbf{Q}$) that's our coefficient. Then we have $H_*(X) \otimes_k H_*(Y) \cong H_*(X \times Y)$. Also, this map $H^p(X) \otimes H_p(X) \to k$ has an adjoint $H^p(X) \to \mathrm{Hom}_k(H_p(X), k) =: H_p(X)^\vee$, which is an isomorphism because Ext vanishes over a field.

Theorem 2.6.4. *Let k be a field. Assume that $H_p(X)$ is finite-dimensional for all p. Then $H^*(X) \otimes H^*(Y) \cong H^*(X \times Y)$.*

Proof. We have:

$$\begin{tikzcd}
H^*(X)\otimes H^*(Y) \arrow[r, "\times"] \arrow[d, "\cong"] & H^*(X\times Y) \arrow[d, "\cong"] \\
H_*(X)\otimes H_*(Y)^\vee \arrow[d, "\zeta"] & H_*(X\times Y)^\vee \arrow[dl, "\cong"] \\
(H_*(X)\otimes H_*(Y))^\vee &
\end{tikzcd}$$

Where $\zeta : \alpha\otimes\beta \mapsto (x\otimes y \mapsto \pm\alpha(x)\beta(y))$. The theorem we proved above implies that this diagram commutes.

In general, I might have two (graded) vector spaces U, V, and consider $U^\vee\otimes V^\vee \to (U\otimes V)^\vee$ by the above formula. Well, $(U\otimes V)^\vee = \mathrm{Hom}_k(U\otimes V, k) = \mathrm{Hom}_k(U, V^\vee)$. Thus I get a map $U^\vee \otimes V^\vee \to \mathrm{Hom}_k(U, V^\vee)$. This map is an isomorphism when U or V is finite dimensional. I also have $\widehat{\alpha} : U^\vee \otimes W \to \mathrm{Hom}(U, W)$ via $\alpha\otimes w \mapsto (w \mapsto \alpha(u)w)$. And that's the map we have in mind. The image of $\widehat{\alpha}$ consists of finite rank homomorphisms because a general tensor is a finite sum. This is therefore an isomorphism if U or W is finite dimensional.

This shows that the map ζ above is an isomorphism if $H_*(X)$ or $H_*(Y)$ is finite-dimensional. We're done by commutativity. □

We saw before that $\times$ is an algebra map! So this is an isomorphism of algebras.

There are more products around. There is a map $H^p(Y)\otimes H^q(X,A) \to H^{p+q}(Y\times X, Y\times A)$. Constructing this is on your homework. You can see how this comes about. This comes from the map on the chain level, and it comes from looking at the cochains, and you're going to get a map (???). Anyway. Suppose $Y = X$. Then I get $\cup : H^*(X)\otimes H^*(X,A) \to H^*(X\times X, X\times A) \xrightarrow{\Delta^*} H^*(X,A)$ where $\Delta : (X,A) \to (X\times X, X\times A)$. This "relative cup product" makes $H^*(X,A)$ into a graded module over $H^*(X)$. This is *not* a ring – it doesn't have a unit, for example – but it is a module. Also the lexseq is a sequence of $H^*(X)$-modules. I'm just making statements here.

I want to introduce you to *one more* product, which we'll talk more about, and forms the foundation of Poincaré duality. This is the cap product. What can I do with $S^p(X)\otimes S^n(X)$? Well, I get big map:

$$S^p(X)\otimes S_n(X) \xrightarrow{1\times(\alpha_{X,X}\circ\Delta_*)} S^p(X)\otimes S_p(X)\otimes S_{n-p}(X) \xrightarrow{\langle -,-\rangle\otimes 1} S_{n-p}(X)$$

This composite participates in a chain map. This induces a map in homology $\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X)$ that comes from μ. This is a pretty interesting map.

Lemma 2.6.5. *$(\alpha \cup \beta) \cap x = \alpha \cap (\beta \cap x)$ and $1 \cap x = x$.*

Proof. Easy to check from the definition. □

This makes $H_*(X)$ into a module over $H^*(X)$. These are not hard things to check. There's a lot of structure, and the fact that $H^*(X)$ forms an algebra is a good thing. Notice how the dimensions work. People made a mistake before, and they should have indexed cohomology with negative numbers, so that $\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X)$ makes sense. A cochain complex with positive grading is the same as a chain complex with negative grading.

There's also slant products (two of them!). Maybe we won't talk about them. We will check a few things about cap products, and then we'll go into Poincaré duality. We can prove nice theorems then.

2.7 Cap product and "Čech" cohomology

Let R be a commutative ring with coefficients. The cap product is a map $\cap : H^p(X) \otimes H_n(X) \to H_q(X)$ where $p + q = n$. This comes from a chain level map $S^p(X) \otimes S_n(X) \xrightarrow{1 \otimes \alpha} S^p(X) \otimes S_p(X) \otimes S_q(X) \xrightarrow{\langle -, - \rangle \otimes 1} R \otimes S_q(X) \cong S_q(X)$. Using our explicit formula for α, we can write:

$$\cap : \beta \otimes \sigma \mapsto \beta \otimes (\sigma \circ \alpha_p) \otimes (\sigma \circ \omega_q) \mapsto (\beta(\sigma \circ \alpha_p)) \cdot (\sigma \circ \omega_q)$$

There's many things to say:

1. $H_*(X)$ is a module for $H^*(X)$.
2. The only reasonable thing to ask for in terms of naturality is the following. Suppose $f : X \to Y$, and let $b \in H^p(Y)$ and $x \in H_n(X)$. We then have $f_*(f^*(b) \cap x) = b \cap f_*(x)$, where $f^*(b) \cap x \in H_q(X)$ and $b \cap f_*(x) \in H_n(X)$. This is called a projection formula. To see this, let $[\beta] = b$. Then:

$$\begin{aligned} f_*(f^*(\beta) \cap \sigma) &= f_*((f^*(\beta)(\sigma \circ \alpha_p)) \cdot (\sigma \circ \omega_q)) \\ &= f_*(\beta(f \circ \sigma \circ \alpha_p) \cdot (\sigma \circ \omega)) \\ &= \beta(f \circ \sigma \circ \alpha_p) \cdot f_*(\sigma \circ \omega_q) \\ &= \beta(f \circ \sigma \circ \alpha_p) \cdot (f \circ \sigma \circ \omega_q) \\ &= \beta \cap f_*(\sigma) \end{aligned}$$

So we're done.

3. There's a relation between the cap and Kronecker product. Any space has an augmentation $\varepsilon : X \to *$, so I get $\varepsilon_* : H_*(X) \to R$. Maybe we should compute $\varepsilon_*(\beta \cap \sigma)$. I will get zero unless $p = n$ and $q = 0$. What does our formula say? This just says that $\varepsilon_*(b \cap x) = \varepsilon_*(\beta(\sigma) \cdot c^0_{\sigma(n)}) = \beta(\sigma)\varepsilon_*(c^0_{\sigma(n)}) = \beta(\sigma) = \langle \beta, \sigma \rangle$ because ε_* counts the number of points, i.e., it's 1. Hence $\varepsilon_*(b \cap x) = \langle b, x \rangle$.

4. What is $\langle a \cup b, x \rangle$? This is $\varepsilon_*((a \cup b) \cap x) = \varepsilon_*(a \cap (b \cap x))$ by an assertion in the previous lecture (namely that $(\alpha \cup \beta) \cap x = \alpha \cap (\beta \cap x)$ and $1 \cap x = x$), which becomes $\langle a, b \cap x \rangle$. In other words, $\langle a \cup b, x \rangle = \langle a, b \cap x \rangle$. So the cup product is adjoint to the cap product.

Relative $\cap$

There's a lot of structure, but we want more. We want to now try to understand the relative cap product. Suppose $A \subseteq X$ is a subspace. We have:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
S^p(X) \otimes S_n(A) \arrow[r, "i^* \otimes 1"] \arrow[d, "1 \otimes i_*"] & S^p(A) \otimes S_n(A) \arrow[r, "\cap"] & S_q(A) \arrow[d] \\
S^p(X) \otimes S_n(X) \arrow[rr, "\cap"] \arrow[d] & & S_q(X) \arrow[d] \\
S^p(X) \otimes S_n(X, A) \arrow[rr, dashed] \arrow[d] & & S_q(X, A) \arrow[d] \\
0 & & 0
\end{tikzcd}$$

The left sequence is exact because $0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$ splits and tensoring with $S^p(X)$ still leaves it exact. We have to check that this diagram commutes.

Let $\beta \otimes \sigma \in S^p(X) \otimes S_n(A)$. We then get:

$$\beta \otimes \sigma \xrightarrow{i^* \otimes 1} i^*\beta \otimes \sigma \to i^*(\beta) \cap \sigma \xrightarrow{i_*} i_*(i^*(\beta) \cap \sigma)$$

And:

$$\beta \otimes \sigma \xrightarrow{1\otimes i_*} \beta \otimes i_*\sigma \to \beta \cap i_*(\sigma)$$

So they're equal by the projection formula. Hence you get $\cap : H^p(X) \otimes H_n(X, A) \to H_q(X, A)$ that makes $H_*(X, A)$ a $H^*(X)$-module.

A different perspective on excision

Recall what excision is. We know that $H_*(X - U, A - U) \cong H_*(X, A)$. There's another perspective on this. Suppose $K \subseteq U \subseteq X$ such that $\overline{K} \subseteq \mathrm{Int}(U)$. To simplify things, suppose K is closed and U is open. Let $A = X - K \supseteq X - U = V$. Then excision says that $H_*(X - V, A - V) = H_*(X - (X - U), (X - K) - (X - U)) \cong H_*(X, A) = H_*(X, X - K)$. There's a simpler expression: $H_*(X - (X - U), (X - K) - (X - U)) = H_*(U, U - K)$, so $H_*(U, U - K) \cong H_*(X, X - K)$, i.e., it depends only on an open neighborhood of K. A question that we now have is: how does this depend on $H_*(K)$? $H^*(K)$? This is really what Poincaré duality wants to understand.

Example 2.7.1. We'll eventually be talking, for example, about $X = S^3$ and $K =$ knot.

We want to understand $H_*(X, X - K)$ better. We have a cap product $H^p(X) \otimes H_n(X, X - K) \to H_q(X, X - K)$. We just decided that $H_n(X, X - K) \cong H_n(U, U - K)$, and so I have the cap product $H^p(U) \otimes H_n(U, U - K) \to H_q(U - U - K)$. Hence I get the cap product map like $H^p(U) \otimes H_n(X, X - K) \to H_q(X, X - K)$. But this seems to depend upon a choice of U. What if I make U smaller?

Lemma 2.7.2. *Let $U \supseteq V \supseteq K$. Then:*

$$\begin{tikzcd}
H^p(U) \otimes H_n(X, X - K) \arrow[d, "i^* \otimes 1"'] \arrow[dr, "\cap"] & \\
 & H_q(X, X - K) \\
H^p(V) \otimes H_n(X, X - K) \arrow[ur, "\cap"'] &
\end{tikzcd}$$

Proof. Hint: use projection formula again. $\square$

Let $\mathcal{U}_K$ be the set of open neighborhoods of K in X. This is a poset (actually a directed set because you can take intersections), under reverse-inclusion as the ordering. This lemma says that $H^p : \mathcal{U}_K \to \mathbf{Ab}$.

Definition 2.7.3. $\check{H}^p(K) := \varinjlim_{U \in \mathcal{U}_K} H^p(U)$.

This is bad notation because it depends on the way K is sitting in X.

You therefore get $\check{H}^p(K) \otimes H_n(X, X - K) \xrightarrow{\cap} H_q(X, X - K)$. This is the best you can do. It's the natural structure that this relative homology has, i.e., $H_*(X, X - K)$ is a module over $\check{H}^*(K)$.

Sometimes, $\check{H}^*(K)$ will just be $H^*(K)$. Suppose $K \subseteq X$ satisfies the condition (called the "regularity" condition) that for every open $U \supseteq K$, there exists an open V such that $U \supseteq V \supseteq K$ such that $K \to V$ is a homotopy equivalence (or actually just a homology isomorphism). (for example, a smooth knot in S^3) Then:

Lemma 2.7.4. *Suppose $\mathcal{I}$ is a directed set (nonempty). Let $F : \mathcal{I} \to$ $\mathbf{Ab}$, and suppose I have a natural transformation $\theta : F \to c_A$ (for example, a map from F to its direct limit). This expresses A as $\varinjlim_{\mathcal{I}} F$ provided that for all i, there is $j \geq i$ such that $F(i) \to A$ factors through $F(j) \to A$, which should be an isomorphism, i.e.:*

$$\begin{tikzcd}
F(i) \arrow[rr] \arrow[dr] & & A \\
& F(j) \arrow[ur, "\cong"] &
\end{tikzcd}$$

Proof. Given $a \in A$, it has to come from somewhere to be a direct limit. This is obviously true. Also, for any i and $a_i \in F(i)$ such that $a_i \mapsto 0$, then there exists $j \geq i$ such that $a_i \mapsto 0 \in F(j)$. This is also obvious. $\square$

Remark 2.7.5. This is a really strong condition by the way. It is a really stupid way.

This works in the case that K is regular in X. Thus, under this condition, $\check{H}^p(K) \cong H^p(K)$. One other comment is that more generally, if X is an Euclidean neighborhood retract (a retract of a neighborhood in some $\mathbf{R}^n$), and K is locally compact, then $\check{H}^p(K)$ depends only on K, and it is isomorphic to Čech cohomology (which is a different type of cohomology theory).

2.8 $\check{H}^*$ as a cohomology theory, and the fully relative $\cap$ product

pset 6 is now doe December 7. If you read any book about Poincaré duality, there'll be an incomprehensible smear of stuff about cap products. I want to give you a cleaner explanation. The motivation is that it's the main step in the proof of Poincaré duality.

Let X be any space, and let $K \subseteq X$ be a closed subspace. We'll write $\check{H}^p(K) = \varinjlim_{U \in \mathcal{U}_K} H^p(U)$ (and call it the Čech cohomology – but there's a different definition, although in certain cases they are isomorphic). We should write down a relative version of this as well. Suppose $L \subseteq K$ be closed. Define $\check{H}^p(K, L) = \varinjlim_{(U,V) \in \mathcal{U}_{K,L}} H^p(U, V)$ where $K \supseteq L$ and $K \subseteq U$, and $L \subseteq V$ such that $V \subseteq U$. Here $\mathcal{U}_{K,L}$ is the directed set:

$$\mathcal{U}_{K,L} \text{ is the set of } \begin{tikzcd} K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\ U & V \arrow[l, hook'] \end{tikzcd}$$

such that U and V are open.

Theorem 2.8.1. *There is a lexseq:*

$$\cdots \to \check{H}^p(K, L) \to \check{H}^p(K) \to \check{H}^p(L) \xrightarrow{\delta} \check{H}^{p+1}(K, L) \to \cdots$$

All the maps in the lexseq are mysterious.

Also, excision holds:

Theorem 2.8.2 (Excision). *Suppose $A, B \subseteq X$ are closed. Then $\check{H}^p(A \cup B, A) \cong \check{H}^p(B, A \cap B)$. This is a form of excision that doesn't hold for ordinary cohomology.*

So Čech cohomology is better suited for talking about closed sets.

We defined $\cap : \check{H}^p(K) \otimes H_n(X, X - K) \to H_q(X, X - K)$, such that $p + q = n$. Fix $x_K \in H_n(X, X - K)$. Then capping with x_K gives a map $\cap x_k : \check{H}^p(K) \to H_q(X, X - K)$.

Now, if $K \supseteq L$, then $X - K \subseteq X - L$, so I get $H_n(X, X - K) \xrightarrow{i_*} H_n(X, X - L)$. Then $x_K \mapsto x_L$. We can now extend our lexseq:

$$\begin{tikzcd} \cdots \arrow[r] & \check{H}^p(K, L) \arrow[r] & \check{H}^p(K) \arrow[r] \arrow[d, "- \cap x_k"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "- \cap x_L"'] & \\ & H_q(X - L, X - K) \arrow[r] & H_q(X, X - K) \arrow[r] & H_q(X, X - L) \arrow[r, "\partial"] & H_{q-1} \end{tikzcd}$$

Where the bottom thing comes from the lexseq of the triple (see exercise 9 on your pset) $X - K \subseteq X - L \subseteq X$ (I think). Then we can extend our theorem on the lexseq:

Theorem 2.8.3. *There is a lexseq and a ladder:*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K,L) \arrow[r] \arrow[d, dashed, "-\cap x_K"'] & \check{H}^p(K) \arrow[r] \arrow[d, "-\cap x_k"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap x_L"'] & {} \\
\cdots \arrow[r] & H_q(X-L, X-K) \arrow[r] & H_q(X, X-K) \arrow[r] & H_q(X, X-L) \arrow[r, "\partial"] & H_{q-1}
\end{tikzcd}$$

What I have to do is define a cap product of the following form (bottom row):

$$\begin{tikzcd}
\check{H}^p(K) \otimes H_n(X, X-K) \arrow[r, "\cap"] & H_q(X, X-K) \\
\check{H}^p(K,L) \otimes H_n(X, X-K) \arrow[u] \arrow[r, "\cap"] & H_q(X-L, X-K)
\end{tikzcd}$$

(where $p + q = n$)

I want to define this fully relative cap product $\check{H}^p(K,L) \otimes H_n(X, X-K) \to H_q(X-L, X-K)$. We'll use this in the inductive proof of (some) important theorem.

Our map $\check{H}^p(K) \otimes H_n(X, X-K) \to H_q(X, X-K)$ came from $S^p(U) \otimes S_n(U, U-K) \to S_q(U, U-K)$ where $U \supseteq K$, defined via $\beta \otimes \sigma \mapsto \beta(\sigma \circ \alpha_p) \cdot (\sigma \circ \omega_q)$. I'm hoping to get:

$$\begin{tikzcd}
S^p(U) \otimes S_n(U, U-K) \arrow[r] & S_q(U, U-K) \\
S^p(U,V) \otimes S_n(U-L)/S_n(U-K) \arrow[u] \arrow[r] & S_q(U-L)/S_q(U-K) \arrow[u]
\end{tikzcd}$$

where again we have inclusions (U, V open and K, L closed):

$$\begin{tikzcd}
K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\
U & V \arrow[l, hook']
\end{tikzcd}$$

The bottom map $S^p(U,V) \otimes S_n(U-L)/S_n(U-K) \to S_q(U-L)/S_q(U-K)$ makes sense. We can evaluate a cochain that kills everything on V.

This means that we can add in $S_n(V)$ to get $S^p(U,V) \otimes (S_n(U-L) + S_n(V))/S_n(U-K) \to S_q(U-L)/S_q(U-K)$ by sending $\beta \otimes \tau \mapsto 0$ where $\tau : \Delta^n \to V$. This means that the diagram:

$$\begin{tikzcd}
S^p(U) \otimes S_n(U, U-K) \arrow[r] & S_q(U, U-K) \\
S^p(U,V) \otimes (S_n(U-L) + S_n(V))/S_n(U-K) \arrow[u] \arrow[r] & S_q(U-L)/S_q(U-K) \arrow[u]
\end{tikzcd}$$

commutes. It's not that far off from where we want to go.

Now, $(U-L) \cup V = U$. I have this covering of U by two open sets. In $S_n(U-L) + S_n(V)$ we're taking the sum of n-chains. We have a map $S_*(U-L) + S_*(V) \to S_*(U)$. We have already worked through this – the locality principle! This tells us that $S_*(U-L) + S_*(V) \to S_*(U)$ is a homotopy equivalence. Hence we can extend our diagram:

$$\begin{tikzcd}
S^p(U) \otimes S_n(U, U-K) \arrow[r] & S_q(U, U-K) \\
S^p(U,V) \otimes (S_n(U-L) + S_n(V))/S_n(U-K) \arrow[u] \arrow[r] \arrow[d, "\simeq"] & S_q(U-L)/S_q(U-K) \arrow[u] \\
S^p(U,V) \otimes S_n(U)/S_n(U-K) &
\end{tikzcd}$$

We want the homology of $S_n(U)/S_n(U-K)$ to approximate $H_n(X, X-K)$.

Claim 2.8.4. There is an isomorphism $H_n(S_*(U)/S_*(U-K)) = H_n(U, U-K) \to H_n(X, X-K)$.

Proof. This is exactly excision! Remember our recasting of excision in the previous lecture. $\square$

This means that what we've constructed really *is* what we want! We now have our large lexseq:

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K,L) \arrow[r] \arrow[d, dashed, "-\cap x_K"'] & \check{H}^p(K) \arrow[r] \arrow[d, "-\cap x_k"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap x_L"'] & {} \\
\cdots \arrow[r] & H_q(X-L, X-K) \arrow[r] & H_q(X, X-K) \arrow[r] & H_q(X, X-L) \arrow[r, "\partial"] & H_{q-1} \\
\cdots \arrow[r] & H_q(U-L, U-K) \arrow[r] \arrow[u, "{\cong, \text{five-lemma}}"] & H_q(U, U-K) \arrow[r] \arrow[u, "{\cong, \text{locality}}"] & H_q(U, U-L) \arrow[r] \arrow[u, "{\cong, \text{locality}}"] & {}
\end{tikzcd}$$

As desired.

The diagram:

$$\begin{tikzcd}
\check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap x_L"] & \check{H}^{p+q}(K,L) \arrow[d, "-\cap x_K"] \\
H_q(X, X-L) \arrow[r, "\partial"] & H_{q-1}(X-L, X-K)
\end{tikzcd}$$

says that:

$$(\delta b) \cap x_k = \partial(b \cap x_L)$$

It's rather wonderful! You have a decreasing sequence below and an increasing one above.

I want to reformulate all of this in a more useful fashion, from Mayer-Vietoris. We had two different proofs, one from locality, and another one that we'll remind you of:

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d, "\cong"] & C_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

then you get a lexseq:

$$\cdots \to C_{n+1} \to C'_{n+1} \oplus A_n \to A'_n \xrightarrow{\partial} C_n \to \cdots$$

You can use this to prove Mayer-Vietoris – I will do this in a special case. (This is exactly what I did in a homework assignment[2]!) We have

[2] Suppose $A \subseteq X$ is a subspace of X. Then there is a lexseq in reduced homology

a ladder of lexseqs:

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(X, X-A\cup B) \arrow[r] \arrow[d] & H_q(X, X-A) \arrow[r] \arrow[d] & H_{q-1}(X-A, X-A\cup \arrow[d, "\cong\text{, excision}"] \\
\cdots \arrow[r] & H_q(X, X-B) \arrow[r] & H_q(X, X-A\cap B) \arrow[r] & H_{q-1}(X-A\cap B, X-
\end{tikzcd}$$

This means that (using the lexseq of the ladder) you have a lexseq:

$$\cdots \to H_q(X, X{-}A{\cup}B) \to H_q(X, X{-}A){\oplus}H_q(X, X{-}B) \to H_q(X, X{-}A{\cap}B) \to H_{q-1}$$

This can be used to give a lexseq for Čech cohomology:

$$\cdots \to \check{H}^p(A{\cup}B) \to \check{H}^p(A){\oplus}\check{H}^p(B) \to \check{H}^p(A{\cap}B) \to \check{H}^{p+q}(A{\cup}B) \to \cdots$$

so that we're going to get a commutative Mayer-Vietoris ladder:

Theorem 2.8.5. *There's a "Mayer-Vietoris" ladder:*

$$\begin{tikzcd}
\to \check{H}^p(A\cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \check{H}^p(A) \oplus \check{H}^p(B) \arrow[r] \arrow[d, "(-\cap x_A)\oplus-\cap x_B"] & \check{H}^p(A\cap B) - \arrow[d] \\
\to H_q(X, X-A\cup B) \arrow[r] & H_q(X, X-A) \oplus H_q(X, X-B) \arrow[r] & H_q(X, X-A\cap
\end{tikzcd}$$

where I have four cohomology classes $x_{A\cup B}, x_A, x_B, x_{A\cap B}$ that commute

---

$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to H_n(X,A) \to \widetilde{H}_{n-1}(A) \to \cdots$ that can be obtained by using the lexseq in homology of the sexseq $0 \to \widetilde{S}_*(A) \to \widetilde{S}_*(X) \to S_*(X,A) \to 0$.

Now suppose $X = A \cup B$. Consider the ladder:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(A, A\cap B) \arrow[r] \arrow[d] & \widetilde{H}_n(A\cap B) \arrow[r] \arrow[d] & \widetilde{H}_n(A) \arrow[r] \arrow[d] & H_n(A, A\cap B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X, B) \arrow[r] & \widetilde{H}_n(B) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & H_n(X, B) \arrow[r] & \cdots
\end{tikzcd}$$

The first and fourth maps as shown are isomorphisms because of excision. The lexseq from the ladder (see above) therefore yields the Mayer-Vietoris sequence $\cdots \to \widetilde{H}_n(A\cap B) \to \widetilde{H}_n(B) \oplus \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(A\cap B) \to \cdots$.

in:

$$
\begin{tikzcd}
 & H_n(X, X - A) \arrow[dr] & \\
H_n(X, X - A \cap B) \arrow[ur] \arrow[dr] & & H_n(X, X - A \cap B) \\
 & H_n(X, X - B) \arrow[ur] &
\end{tikzcd}
$$

This is the most complicated blackboard for the rest of the course. Also xymatrix is not compiling properly because the diagram is too big!

2.9 $\check{H}^*$ as a cohomology theory

Office hours: today, Hood in 4-390 from 1:30 to 3:30 and Miller in 4-478 from 1-3 on Tuesday. Note that pset 6 is due Wednesday. Also, Wednesday we'll have a lightning review of π_1 and covering spaces.

We're coming to the end of the course, and there are going to be oral exams. I have some questions that I'd like to ask you. They won't be super advanced, detailed questions – they'll be basic things. I'll post a list of examples of questions. I won't select questions from that list, that's cruel and isn't the point. The oral will be 40 minutes. It'll be fun – better than a written exam. It's much better than grading a written exam!

PLEASE DRAW A PICTURE WHEN READING THIS IF YOU DIDN'T COME TO CLASS!

$\check{H}^*$ versus H^*

Let's come up with an example that distinguishes $\check{H}^*$ and H^*. This is a famous example – the topologist's sine curve. The topologist's sine curve is defined as follows. Consider the graph of $\sin(\pi/x)$ where $0 < x \leq 1$, where we draw a continuous curve from $(0, -1)$ to $(1, 0)$. This is a counterexample for a lot of things, you've probably seen it in 18.901.

What is H_* of the topologist sine curve? Use Mayer-Vietoris! I can choose the bottom half to be some connected portion of the continuous curve from $(0, -1)$ to $(1, 0)$, and the top half to be the rest of the space that intersects the bottom half (in two spots). The top is the union of two path components, each contractible.

To see this, suppose I have $\sigma : I \to$ top such that $\sigma(0) = (0, b)$ for some $-3/2 < b \leq 1$ (the $-3/2$ is arbitrary, just choose something ≤ 0). If $b < 1$, pick $\epsilon > 0$ such that if we write $\sigma = (\sigma_1, \sigma_2)$, then $\sigma_2(t) < 1$ for all $t \in [0, \epsilon)$, where we use continuity. Then $\sigma|_{[0,\epsilon)}$ can't be on the sine curve. If $b = 1$, pick $\epsilon > 0$ such that $\sigma_2(t) > -1$ for all $t \in [0, \epsilon)$. Similarly, it can't be on the sine curve.

Thus, H_1 is simple: it fits into $0 \to H_1(X) \to H_0(\text{top} \cap \text{bottom}) \to H_0(\text{top}) \oplus H_0(\text{top}) \to H_0(X) \to 0$. This is basically $0 \to H_1(X) \xrightarrow{\partial} \to \mathbf{Z} \oplus \mathbf{Z} \hookrightarrow \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0$, so $\partial = 0$. This means that $H_*(X) \cong H_*(*)$ and this implies that $H^*(X) \cong H^*(*)$.

How about $\check{H}^*$? Let $X \subset U$ be an open neighborhood. The interval is contained in some ϵ-neighborhood that's contained in U. This implies that there exists a neighborhood $X \subseteq V \subseteq U$ such that $V \sim S^1$. Therefore, $\varinjlim_{U \in \mathcal{U}_X} H^*(U) \cong H^*(S^1)$ by "cofinality". So $\check{H}^*$ and H^* differ.

Cofinality

Let $\mathcal{I}$ be a directed set. Let $A : \mathcal{I} \to \mathbf{Ab}$ be a functor. If I have a functor $f : \mathcal{K} \to \mathcal{I}$, then I get $Af : \mathcal{K} \to \mathbf{Ab}$, i.e., $(Af)_j = A_{f(j)}$.

I can form $\varinjlim_{\mathcal{K}} Af$ and $\varinjlim_{I} A$. I claim you have a map $\varinjlim_{\mathcal{K}} Af \to \varinjlim_{\mathcal{I}} A$. All I have to do is the following:

$$\begin{tikzcd} \varinjlim_J Af \arrow[r] & \varinjlim_I A \\ A_{f(j)} \arrow[u, "\mathrm{in}_j"] & \end{tikzcd}$$

So I have to give you maps $A_{f(j)} \to \varinjlim_I A$ for various j. I know what to do, because I have $\mathrm{in}_{f(j)} : A_{f(j)} \to \varinjlim_I A$. Are they compatible when I change j? Suppose I have $j' \leq j$. Then I get a map $f(j') \to f(j)$, so I have a map $A_{f(j')} \to A_{f(j)}$, and thus the maps are compatible. Hence I get:

$$\begin{tikzcd} \varinjlim_J Af \arrow[r] & \varinjlim_I A \\ (Af)_j = A_{f(j)} \arrow[u, "\mathrm{in}_j"] \arrow[ur, dashed, "\mathrm{in}_{f(j)}"'] & \end{tikzcd}$$

Example 2.9.1. Suppose $K \supseteq L$ be closed, then I get a map $\check{H}^*(K) \to \check{H}^*(L)$. Is this a homomorphism? Well, $\check{H}^*(K) = \varinjlim_{U \in \mathcal{U}_K} H^*(U)$ and

$\check{H}^*(L) = \varinjlim_{V \in \mathcal{U}_L} H^*(V)$. This is an example of a $\mathcal{I}$ and $\mathcal{K}$ that I care about. Well, $\mathcal{U}_K \subseteq \mathcal{U}_L$, and thus I get a map $\check{H}^*(K) \to \check{H}^*(L)$, which is what I wanted.

I can do something for relative cohomology. Suppose:

$$\begin{tikzcd} K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\ K' & L' \arrow[l, hook'] \end{tikzcd}$$

I get a homomorphism $\check{H}^*(K, L) \to \check{H}^*(K', L')$ because I have $\mathcal{U}_{K,L} \to \mathcal{U}_{K',L'}$.

This isn't exactly what we need:

Question 2.9.2. When does $f : \mathcal{K} \to \mathcal{I}$ induce an isomorphism $\varinjlim_J Af \to \varinjlim_I A$?

This is a lot like taking a sequence and a subsequence and asking when they have the same limit. There's a cofinality condition in analysis, that has a similar expression here.

Definition 2.9.3. $f : \mathcal{K} \to \mathcal{I}$ is cofinal if for all $i \in \mathcal{I}$, there exists $j \in \mathcal{K}$ such that $i \leq f(j)$.

Example 2.9.4. If f is surjective.

Lemma 2.9.5. *If f is cofinal, then $\varinjlim_J Af \to \varinjlim_I A$ is an isomorphism.*

Proof. Check that $\{A_{f(j)} \to \varinjlim_I A\}$ satisfies the necessary and sufficient conditions:

1. For all $a \in \varinjlim_I A$, there exists j and $a_j \in A_{f(j)}$ such that $a_j \mapsto a$. We know that there exists some i and $a_i \in A$ such that $a_i \mapsto a$. Pick j such that $f(j) \geq i$, so we get a map $a_i \to a_{f(j)}$, and by compatibility, we get $a_{f(j)} \mapsto a$.

2. The other condition is also just as easy.

$\square$

This is a very convenient condition.

Example 2.9.6. I had a perverse way of constructing $\mathbf{Q}$ by using the divisibility directed system. A much simpler (linear!) directed system is $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{4} \mathbf{Z} \to \cdots$. This has the same colimit as the divisibility directed system because $n|n!$, so we have a cofinal map between directed systems.

How about the direct limits in the Čech cohomology case?

Example 2.9.7. Do I have a map $\check{H}^*(K,L) \to \check{H}^*(K)$? Suppose:

$$\begin{tikzcd} K \arrow[d, hook] & L \arrow[l, hook'] \arrow[d, hook] \\ U & V \arrow[l, hook'] \end{tikzcd}$$

Then $\check{H}^p(K,L) = \varinjlim_{(U,V)\in\mathcal{U}_{K,L}} H^p(U,V)$ and $\check{H}^p(K) = \varinjlim_{U\in\mathcal{U}_K} H^p(U)$. I have a map of directed sets $\mathcal{U}_{K,L} \to \mathcal{U}_K$ by sending $(U,V) \mapsto U$. I didn't have to use cofinality. I want a long exact sequence, though, and I'm going to do this by saying that it's a directed limit of a long exact sequence. I'm going to have to have all of these various Čech cohomologies as being the directed limit over the *same* indexing set.

I'd really like to say that $\check{H}^p(K) = \varinjlim_{U\in\mathcal{U}_K} H^p(U) \cong \varinjlim_{(U,V)\in\mathcal{U}_{K,L}} H^p(U)$. Thus I need to show that $\mathcal{U}_{K,L} \to \mathcal{U}_K$ where $(U,V) \mapsto U$ is cofinal. This is easy, because if $U \in \mathcal{U}_K$, just pick (U,U), i.e., $\mathcal{U}_{K,L} \to \mathcal{U}_K$ is cofinal. How about $\mathcal{U}_{K,L} \to \mathcal{U}_L$ by $(U,V) \mapsto V$; is it cofinal? Yes! For $V \in \mathcal{U}_L$, pick (X,V)! This means that $\cdots \check{H}^{p-1}(L) \to \check{H}^p(K,L) \to \check{H}^p(K) \to \check{H}^p(L) \to \check{H}^{p+1}(K,L)$ is $\varinjlim_{\mathcal{U}_{K,L}} (\cdots \to H^p(U,V) \to \cdots)$, and hence exact.

How about excision? I need this to get to Mayer-Vietoris!

Lemma 2.9.8. *Assume X is normal and A, B are closed subsets. Then $\check{H}^p(A\cup B, B) \to \check{H}^p(A, A\cap B)$ is an isomorphism.*

Proof. Well, $\check{H}^p(A\cup B, B)$ is $\varinjlim$ over $\mathcal{U}_{A\cup B,B}$ and $\check{H}^p(A, A\cap B)$ is $\varinjlim$ over $\mathcal{U}_{A,A\cap B}$. Let $W \supseteq A$ and $Y \supseteq B$ are neighborhoods. I claim that $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A\cup B,B}$ sending $(W,Y) \mapsto (W\cup Y, Y)$ and $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A,A\cap B}$ sending $(W,Y) \mapsto (W, W\cap Y)$ are cofinal.

If I give you $(U,V) \in \mathcal{U}_{A\cup B,B}$, define $(W,V) \in \mathcal{U}_A \times \mathcal{U}_B$ where $W = U$ and $Y = V$, so $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A\cup B,B}$ is surjective, hence cofinal. The latter is trickier. Let $U \supseteq A$ and $V \supseteq A \cap B$. Here's where normality comes into play. Separate $B - V$ from A. Let $T \supseteq B - V$. Shit. *Shit!*

Maybe I'll leave this to you. I'll put this on the board on Wednesday. Anyway, I'll use normality to show that $\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A,A\cap B}$ is cofinal, and thus this verifies excision – so you actually have excision. □

2.10 Finish off the proof of $\check{H}^p$ excision, topological manifolds, fundamental classes

The end of the proof

Let's finish off the proof from last time. Suppose A, B are closed in normal X. Excision for $\check{H}^p$:

$$\begin{tikzcd}
\varinjlim_{(W,Y)\in\mathcal{U}_A\times\mathcal{U}_B} H^p(W\cup Y, Y) \arrow[r, "\cong\text{, ordinary excision}"] \arrow[d, "\cong\text{, cofinality/surjectivity}"] & \varinjlim_{\mathcal{U}_A\times\mathcal{U}_B} H^p(W, W\cap Y) \arrow[d, "\cong\text{, cofinality, see below}"] \\
\varinjlim_{(U,V)\in\mathcal{U}_{A\cup B,B}} H^p(U,V) \arrow[r] \arrow[d, equal] & \varinjlim_{(U,V)\in\mathcal{U}_{A,A\cap B}} H^p(U,V) \arrow[d, equal] \\
\check{H}^p(A\cup B, B) \arrow[r] & \check{H}^p(A, A\cap B)
\end{tikzcd}$$

$\mathcal{U}_A \times \mathcal{U}_B \to \mathcal{U}_{A,A\cap B}$ is cofinal since: start with $(U,V) \supseteq (A, A\cap B)$. Using normality, separate $B\cap(X-V) \subseteq T$ and $A \subseteq S$. Take $W = U\cap S$ and $Y = V\cup T$. Then $A \subseteq W \subseteq U$ and $A\cap B \subseteq W\cap Y = S\cap V \subseteq V$.

This means that $\check{H}^p$ satisfies excision, hence Mayer-Vietoris. Let's put this in the drawer for now.

Topological manifolds + Poincaré duality

yayyyyyyyyyyyy finally

Fundamental class and orientation local system

Definition 2.10.1. A *topological manifold* is a Hausdorff space M such that for every $x \in M$, there exists a neighborhood $U \ni x$ that is homeomorphic to some Euclidean space $\mathbf{R}^n$. It's called an n-manifold if all U are homeomorphic to $\mathbf{R}^n$ for the *same* n.

Example 2.10.2. $\mathbf{R}^n$, duh. $\emptyset$ is an n-manifold for every n. The sphere S^n. The Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$, introduced in the beginning of the course. I don't know exactly what the dimension of this is, but you can figure it out. Also, $V_k(\mathbf{R}^n)$, and surfaces.

These things are the most interesting things to look at.

Warning 2.10.3. We assume the following.

1. There exists a countable basis.
2. There exists a good cover, i.e., all nonempty intersections are Euclidean as well (always true for differentiable manifolds because you can take geodesic neighborhoods, and in particular for the manifolds we listed above).

This is the context in which duality works.

Definition 2.10.4. Let X be any space, and let $a \in X$. The local homology of X at a is the homology $H_*(X, X-a)$. We're always working over a commutative ring.

For example, $H_q(\mathbf{R}^n, \mathbf{R}^n - 0) = \begin{cases} \text{free of rank 1} & q = n \\ 0 & q \neq n \end{cases}$. This means that local homology is picking out the characteristic feature of Euclidean space. Therefore we also have $H_q(M, M-a) = \begin{cases} \text{free of rank 1} & q = n \\ 0 & q \neq n \end{cases}$ for n-manifolds.

Notation 2.10.5. Let $j_a : (M, \emptyset) \to (M, M-a)$ be the inclusion.

Definition 2.10.6. A fundamental class for M (an n-manifold) is $[M] \in H_n(M)$ such that for every $a \in M$, the image of $[M]$ under $j_{a,*} : H_n(M) \to H_n(M, M-a)$ is a generator of $H_n(M, M-a)$.

This is somehow trying to say that this class $[M]$ covers the whole manifold.

Example 2.10.7. When does a space have a fundamental class?

	$\mathbf{R}^2$	$\mathbf{RP}^2$	T^2
$R = \mathbf{Z}$	no!	no!	yes! you did this for homework
$R = \mathbf{Z}/2\mathbf{Z}$	no!	yes!	yes!

Something about orientability and compactness seem to be involved.

What do we have?

Definition 2.10.8. $o_M = \coprod_{a \in M} H_n(M, M-a)$ as a set. This has a map $p : o_M \to M$.

Construction 2.10.9. This can be topologized in Euclidean neighborhoods. Let $U \cong \mathbf{R}^n$ be an Euclidean neighborhood of a. I can always arrange so that a corresponds to 0. We have the open disk sitting inside the closed disk: $\widetilde{D^n} \subseteq D^n \subseteq \mathbf{R}^n$ that corresponds to some open $V \subseteq \overline{V} \subseteq U$. Let $x \in V$. I have a diagram:

$$\begin{tikzcd}
H_n(M, M - \overline{V}) \arrow[d] & H_n(U, U - \overline{V}) \arrow[l, "\cong"', "\text{excision of } M-U"] \arrow[r, equal] \arrow[d, "\cong"] & H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \arrow[d, "{\cong, \text{ homotopy equivalence}}"] \\
H_n(M, M - x) & H_n(U, U - x) \arrow[l] \arrow[r, equal] & H_n(\mathbf{R}^n, \mathbf{R}^n - 0)
\end{tikzcd}$$

Hence $H_n(M, M - \overline{V}) \cong H_n(M, M - x)$. Thus I can collect points in o_M together when they come from the same class in $H_n(M, M - \overline{V})$, so they form "sheets".

I have a map $V \times H_n(M, M - \overline{V}) \to o_M|_V = p^{-1}(V)$ by sending $(x, c) \mapsto (j_x)_*(c) \in H_n(M, M - x)$, and this map is bijective (that's what comes from excision). This LHS has a nice topology by letting $H_n(M, M - \overline{V})$ be discrete. I'm topologizing o_M as the weakest topology these generate.

"Have I been sufficiently obscure enough? This is not supposed to be a complicated point". This $o_M \to M$ is called the *orientation local system*, and is a covering space.

Definition 2.10.10. A continuous map $p : E \to B$ is a covering space if:

1. $p^{-1}(b)$ is discrete for all $b \in B$.
2. For every b there's a neighborhood V and a map $p^{-1}(V) \to p^{-1}(b)$ such that $p^{-1}(V) \xrightarrow{\cong} V \times p^{-1}(b)$ is a homeomorphism.

That's exactly the way we topologized o_M. There's more structure though because $H_n(M, M - \overline{V})$ is an R-module!

Definition 2.10.11. A local system (of R-modules) $p : E \to B$ is a covering space together with structure maps $E \times_B E := \{(e, e')|pe = pe'\} \xrightarrow{+} E$ and $z : B \to E$ such that:

$$\begin{tikzcd}
E \times_B E \arrow[rr, "+"] \arrow[dr] & & E \arrow[dl, leftrightarrow] & R \times E \arrow[l] \\
& B \arrow[r, equal] & B \arrow[u, "z"] &
\end{tikzcd}$$

making $p^{-1}(b)$ a R-module.

We have $H_n(M) \xrightarrow{j_x} H_n(M, M-x)$, which gives a *section* of o_M. If I have a covering space $p : E \to B$, a section is a continuous map $s : B \to E$ such that $ps = 1_B$. Write $\Gamma(E)$ to be the set of sections. If E is a local system, this is an R-module. Hence $H_n(M) \xrightarrow{j_x} H_n(M, M-x)$ gives a map $j : H_n(M) \to \Gamma(o_M)$. This is pretty cool because it's telling you about this high-dimensional homology of M into something "discrete".

Theorem 2.10.12. *If M is compact then $j : H_n(M) \to \Gamma(o_M)$ is an isomorphism, and $H_q(M) = 0$ for $q > n$.*

This is case of Poincaré duality actually because $\Gamma(o_M)$ is somewhat like zero-dimensional cohomology. If this is trivial, like it is for a torus, so if the manifold is connected, then $\Gamma(o_M)$ is just R.

2.11 Fundamental class

Note that if M is a compact manifold, then $H_q(M; R) = 0$ for $q \gg 0$, and if R is a PID, then for all q, $H_q(M; R)$ is finitely-generated. This follows from:

Claim 2.11.1. Suppose X admits an open cover $\{U_i\}_{i=1}^n$ such that all intersections are either empty or contractible (this is what you get for a good cover on a manifold). Then $H_q(X; R) = 0$ for $q \geq n$, and if R is a PID, then for all q, $H_q(X; R)$ is finitely-generated.

Proof. Induct. Certainly true for $n = 1$. Let $Y = \bigcup_{i=1}^{n-1} U_i$, then this statement is true by induction – and similarly for $Y \cap U_n$. Now use Mayer-Vietoris. You have $\cdots \to H_q(Y \cap U_n) \to H_q(Y) \oplus H_q(U_n) \to H_q(X) \to H_{q-1}(Y \cap U_n) \to \cdots$. When $q = n-1$, $H_q(Y \cap U_n)$ could be nonzero, and so you might get something nontrivial (???). Also, you'll get a sexseq by unsplicing the lexseq: $0 \to H_q(Y) \oplus H_q(U_n)/\text{something} \to H_q(X) \to$ submodule of $H_{q-1}(Y \cap U_n) \to 0$, where you use R being a PID to conclude that submodule of $H_{q-1}(Y \cap U_n)$ is finitely generated. □

Let M be an n-manifold. We had a map $j : H_n(M) \to \Gamma(M; o_M)$. Here $\Gamma(M; o_M)$ is the collection of compatible elements of $H_n(M, M-x)$ for $x \in M$. This map $j : H_n(M) \to \Gamma(M; o_M)$ sends $c \mapsto (x \mapsto j_x c)$ where $j_x : H_n(M, \emptyset) \to H_n(M, M-x)$. I want to make two refinements.

You can't expect j to be surjective, except maybe when M is compact. Here's why. Let $c \in Z_n(M)$. It's a sum of simplices, and each simplex is compact, and so the union of the images is compact, and hence there's a compact subset $K \subseteq M$ such that $c \in Z_n(K)$. Now if I take $x \notin K$, then the map $H_n(K) \to H_n(M)$ splits as $H_n(K) \to H_n(M-x) \to H_n(M)$. In the relative homology, $H_n(M, M-x)$, the map $H_n(K) \to H_n(M) \to H_n(M, M-x)$ sends c to zero.

Definition 2.11.2. Let σ be a section of $p : E \to B$ (local system). Then the support of σ is defined as $\mathrm{supp}(\sigma) = \overline{\{x \in B | \sigma(x) \neq 0\}}$. The collection of all sections with compact support is $\Gamma_c(B; E)$, and it's a submodule of $\Gamma(B; E)$.

The first refinement is that $j : H_n(M) \to \Gamma(M; o_M)$ lands in $\Gamma_c(M; o_M)$, because homology is compactly supported.

The second refinement seems a little artificial but is part of the inductive process. Let $A \subseteq M$ be closed. Then you have a restriction map $H_n(M, M-A) \xrightarrow{j_x} H_n(M, M-x)$ for $x \in A$. Thus you get a map $j : H_n(M, M-A) \to \Gamma_c(A; o_M|_A)$, the latter of which we'll just denote $\Gamma_c(A; o_M)$.

Theorem 2.11.3. *The map $j : H_n(M, M-A) \to \Gamma_c(A; o_M|_A)$ is an isomorphism and $H_q(M, M-A) = 0$ for $q > n$. (If $A = M$ then $j : H_n(M) \to \Gamma_c(M; o_M)$ is an isomorphism.)*

Proof. For $X = \mathbf{R}^n$ and $A = D^n$. Well, $o_{\mathbf{R}^n} = \mathbf{R}^n \times H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$ is trivial (i.e., a product projection), so $\Gamma(D^n; o_{\mathbf{R}^n}) = \mathrm{Hom}_{\mathbf{Top}}(D^n, H_n(\mathbf{R}^n, \mathbf{R}^n - 0))$ where $H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$ is discrete, and this is therefore just a map from π_0 into this, and thus $\Gamma(D^n; o_{\mathbf{R}^n}) = R$ (your coefficient). But also, $H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \cong R$, so you have that j gives $H_n(\mathbf{R}^n, \mathbf{R}^n - D^n) \to \Gamma_c(D^n; o_{\mathbf{R}^n}|_{D^n})$.

Say that this is true for $A, B, A \cap B$ – we'll prove this for $A \cup B$. Obviously, use Mayer-Vietoris. I have a restriction $\Gamma_c(A \cup B; o_M) \to \Gamma_c(A; o_M) \oplus \Gamma_c(B; o_M)$ that sits in an exact sequence $0 \to \Gamma_c(A \cup B; o_M) \xrightarrow{\text{inclusion, determined by } A,B} \Gamma_c(A; o_M) \oplus \Gamma_c(B; o_M) \to \Gamma_c(A \cap B; o_M)$. This is a gluing lemma. We also have a relative Mayer-Vietoris $H_n(M, M - A \cup B) \to H_n(M, M-A) \oplus H_n(M, M-B) \to H_n(M, M - A \cap B)$, so we have:

$$\begin{tikzcd}
& 0 \arrow[r] & \Gamma_c(A\cup B;o_M) \arrow[r, "\text{inclusion } A,B\to A\cup B"] & \Gamma_c(A;o_M)\oplus\Gamma_c(B;o_M) \arrow[r] & \cdots \\
H_{n+1}(M,M-A\cap B)=0 \arrow[r, "0"] & H_n(M,M-A\cup B) \arrow[ur] \arrow[r] & H_n(M,M-A)\oplus H_n(M,M-B) \arrow[u, "j_*"', "\cong"] & &
\end{tikzcd}$$

This is a "local-to-global" argument. "I don't feel like going through the point-set topology – the rest of the proof is just annoyance." See Bredon's book for the conclusion of the proof. □

Corollary 2.11.4. $j : H_n(M) \to \Gamma_c(M; o_M)$ *is an isomorphism.*

Definition 2.11.5. An R-orientation for M is a section σ of $\Gamma(M; o_M^\times)$ where $o_M^\times$ is the covering space of M given by the generators (as R-modules) of the fibers of o_M.

If M is compact, then $j : H_n(M) \to \Gamma(M; o_M)$, and you get $[M] \leftrightarrow \sigma$. When does that exist?

Over **Z**: $o_M^\times \to M$ is a double cover of M (over every element you have two possible elements given by the two possible orientations (± 1)). If M is an n-manifold and $f : N \to M$ is a covering space, then N is also locally Euclidean. I have the orientation local system to get a pullback local system:

$$\begin{tikzcd}
f^*o_M = N\times_M o_M \arrow[r] \arrow[d] & o_M \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

Because $N \to M$ is a covering space, the fibers of f^*o_M are the same as the fibers of o_N, so actually, $f^*o_M \cong o_N$. For example, suppose $N = o_M^\times$. What happens if I consider:

$$\begin{tikzcd}
o_N = N\times_M N \arrow[r] \arrow[d] & N \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

But now, I have the identity $N \to N$ that sits compatibly as:

$$\begin{tikzcd}
N \arrow[r, "\text{id}"] \arrow[d, "\text{id}"] & N \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

And hence you get $N \to o_N^\times$, which is a section of $o_N^\times \to N$. The conclusion is that $N = o_M^\times$ is canonically oriented (even if M is not oriented!). If M is oriented, then the local system is trivial and you have the trivial double cover.

The overarching conclusion is: if M is an n-manifold, then:

1. $H_q(M) = 0$ for $q > n$.

2. If M is compact, then $H_n(M) \xrightarrow{\cong} \Gamma(M, o_M)$.

3. If M is connected and compact, then:

   a) if M is oriented with respect to R, then $H_n(M) \cong \Gamma(M, o_M) \cong R$.

   b) (I have no idea what was happening here, we didn't reach to a conclusion for a while.) if M is not orientable, then $o_M^\times$ is nontrivial. If $o_M^\times$ has a section, then it's trivial (and so is o_M) because if it has a section $\sigma : M \to o_M^\times$, define $M \times R^\times \xrightarrow{\cong} o_M^\times$ by sending $(x, r) \mapsto r\sigma(x) \in o_M^\times$ (and the same thing $M \times R \xrightarrow{\cong} o_M$ for the orientation local system itself). I don't see an argument to conclude that if M is nonorientable, then there aren't any section of o_M. In particular, if $R = \mathbf{Z}$, then $H_n(M; \mathbf{Z}) = 0$. I'm going to leave this as a statement without proof, unless any of you can help me.

If a section $\sigma(x) = 0$ for some x, then $\sigma = 0$.

Remark 2.11.6. Prof. Miller talked with me about this after class. If I recall correctly, one way to think about this is as follows. If you have a local system $p : E \to B$, this can be viewed as a representation of $\pi_1(B) \to R^\times$, and the $\Gamma(B; E) = (E_x)^{\pi_1(B)}$ where $\pi_1(B)$ acts on the fibers by multiplication. Thus $(E_x)^{\pi_1(B)} = R^{\pi_1(B)}$. If $R = \mathbf{Z}$, then $R^\times = \{\pm 1\}$, so $R^{\pi_1(B)} = \{r | ar = r, a \in \pi_1(B)\}$, so that $\mathbf{Z}^{\pi_1(B)} = 0$. Hence there are no sections of o_M, as desired. For a ring R, $o_{M,R} = o_{M,\mathbf{Z}} \otimes R$. Something else for $\mathbf{Z}/2\mathbf{Z}$. A higher homotopy theoretic perspective is that if you have a fibration $E \to B$, then $E = PB \times_{\Omega B} F$ where F is the fiber of the fibration, so that $\Gamma(B; E) = \mathrm{Map}_{\Omega B}(PB, F) = F^{h\Omega B}$. In the case of a covering space you recover what you have above since $\pi_0(\Omega B) = \pi_1(B)$.

2.12 Covering spaces and Poincaré duality

Miller's office hours are tomorrow, from 1-3 in 2-478. The first half of this lecture was just explaining the remark above by using less technology.

On the website, there are notes on $\pi_1(X, *)$. I'm assuming people have seen this thing. Assume X is path-connected, and let $* \in X$. There's another technical assumption: semi-locally simply connected (SLSC), which means that for every $b \in X$ and neighborhood $b \in U$, there exists a smaller neighborhood $b \in V \subseteq U$ such that $\pi_1(V, b) \to \pi_1(X, b)$ is trivial. This is a very very weak condition.

Theorem 2.12.1. *Let X be a path-connected, SLSC space with $* \in X$. Then there is an equivalence of categories between covering spaces over X and sets with an action of $\pi_1(X, *)$. The way this functor goes is by sending $p : E \to X$ to $p^{-1}(*)$, which has an action of $\pi_1(X, *)$ in the obvious way by path-lifting.*

Example 2.12.2 (Stupidest possible case)**.** Suppose $\mathrm{id} : X \to X$ is sent to $*$ with the trivial action. This is the terminal covering space over X.

We've been interested in $\Gamma(E; X)$, which is the same thing as $\mathrm{Map}_X(X \to X, E \to X) \cong \mathrm{Map}_{\pi_1(X)}(*, E_*) = (E_*)^{\pi_1(B)}$, the fixed points of the action. We also thought about the case of E being a local system of R-modules, and the same functor gives an equivalence between local systems of R-modules and $R[\pi_1(X)]$-modules, i.e., representations of $\pi_1(X)$.

Recall that o_M is the orientation local system, but now *over* $\mathbf{Z}$. Thus, over a general ring, $o_{M,R} = o_M \otimes R$. We were thinking about what happens with a closed path-connected SLSC subset $* \in A \subseteq M$ of an n-manifold M, and then considering $\Gamma(A, o_M \otimes R)$, which we now see to be $(o_M \otimes R)_*^{\pi_1(A,*)}$. How many options do we have here?

That is to say, this local system o_M is the same thing as the free abelian group $H_n(M, M - *)$ with an action of $\pi_1(X, *)$. There aren't many options for this action. In other words, this is a homomorphism $\pi_1(M, *) \to \mathrm{Aut}(H_n(M, M - *))$. I haven't chosen a generator for $H_n(M, M - *)$, and there's only two automorphisms, i.e., we get a homomorphism $w_1 : \pi_1(M, *) \to \mathbf{Z}/2\mathbf{Z}$. This homomorphism is called the "first Stiefel-Whitney class". 18.906 will describe all the Stiefel-Whitney classes. With R-coefficients, I get a map $\pi_1(M, *) \to \mathrm{Aut}(H_n(M, M - *; R)) \cong R^\times$. This is a natural construction, so this homomorphism $\pi_1(M, *) \to R^\times$ factors through $\pi_1(M, *) \to \mathbf{Z}/2\mathbf{Z}$. This lets us get a good handle on what the sections are: $\Gamma(A; o_M \otimes R) = H_n(M, M -$

$*; R)^{\pi_1(X,*)}$, but our analysis shows that:

$$\Gamma(A; o_M \otimes R) = H_n(M, M - *; R)^{\pi_1(X,*)}$$
$$= \begin{cases} H_n(M, M - *; R) \cong R & \text{if } w_1 = 1\text{, well-defined up to sign; the ori} \\ \ker(R \xrightarrow{2} R) & \text{if } w_1 \neq 1\text{, and this is a canonical identific} \end{cases}$$

where we get the latter thing because then $a = -a$, i.e., $2a = 0$. In particular, if $R = \mathbf{Z}/2\mathbf{Z}$, since $\mathrm{Aut}_{\mathbf{Z}/2\mathbf{Z}}(\mathbf{Z}/2\mathbf{Z}) = 1$, you always have a unique orientation. If $R = \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}, \mathbf{Q}$, then $\ker(R \xrightarrow{2} R) = 0$.

We had a general theorem:

Theorem 2.12.3. *$H_n(M, M-A; R) \xrightarrow{j,\cong} \Gamma_c(A; o_M \otimes R)$ and $H_q(M, M-A; R) = 0$ for $q > n$.*

Corollary 2.12.4. *If M is connected and $A = M$, and if M is not compact, then $H_n(M; R) = 0$. If M is compact, then the work we just did shows that $H_n(M; R) = \begin{cases} R & \textit{oriented} \\ \ker(R \xrightarrow{2} R) & \textit{nonorientable} \end{cases}$.*

Poincaré duality, finally

Assume M is R-oriented. Let $K \subseteq M$ be compact. Then $H_n(M, M - K) \xrightarrow{\cong} \Gamma(K; o_M \otimes R)$. Picking an orientation picks an isomorphism $\Gamma(K; o_M \otimes R) \cong R$. This gives some $[M]_K$, which is called the fundamental class along K. If $K = M$, then $[M]_M =: [M] \in H_n(M; R)$.

Suppose $K \subseteq L$ are compact subsets. We now combine all of our results above:

Theorem 2.12.5 (Fully relative Poincaré duality)**.** *If $p + q = n$, then $\check{H}^p(K, L; R) \xrightarrow{\cap [M]_K} H_q(M - L, M - K; R)$ is an isomorphism.*

Proof. "It's, like, not hard at this point." One thing we did was set up an LES for $\check{H}$ of a pair, which implies that we may assume that $L = \emptyset$. We want to prove that $\check{H}^p(K; R) \xrightarrow{\cap [M]_K} H_q(M, M - K; R)$ is an isomorphism. Now there's a standard local-to-global process.

In the local case, if $M = \mathbf{R}^n$ and $K = D^n$, then this is saying that $\check{H}^p(D^n; R) \cong H^p(D^n; R) \xrightarrow{\cap [\mathbf{R}^n]_{D^n}} H_q(\mathbf{R}^n, \mathbf{R}^n - D^n; R)$ where the first isomorphism comes from analysis we did earlier about Čech and ordinary cohomology coinciding. If $p \neq 0$, then both sides are zero. When $p = 0$, we are asking that $H^0(D^n;R) \xrightarrow{\cap[\mathbf{R}^n]_{D^n}} H_n(\mathbf{R}^n, \mathbf{R}^n - D^n; R)$. They're both equal to R, and we are just capping along $[\mathbf{R}^n]_{D^n}$, because we found that $1 \cap [\mathbf{R}^n]_{D^n} = [\mathbf{R}^n]_{D^n}$, as desired.

We carefully set up the Mayer-Vietoris sequence ladder (Theorem 33.5) that allows us to put this all together. "We're not going to go through the details because there's point set topology that I don't like there." Note that normality is not needed for K, L compact because compact sets in Hausdorff spaces can always be separated, normal or not. I just reversed the order in which things are usually taught in books. $\square$

We have time for one beginning application.

Corollary 2.12.6 (Relative Poincaré duality)**.** *Suppose $K = M$ and M is compact and R-oriented. Then $\check{H}^p(M, L; R) \xrightarrow{\cap[M]} H_{n-p}(M - L, R)$ is an isomorphism.*

Corollary 2.12.7 (Poincaré duality, corollary of corollary)**.** *Let M be compact and R-oriented, then $H^p(M;R) \xrightarrow{\cap[M]} H_{n-p}(M;R)$ is an isomorphism.*

Proof. Follows from the above corollary since $\check{H}^p(M;R)$ is literally equal to $H^p(M;R)$ $\square$

That's the most beautiful form of all. If you do have an L, you have this ladder, where all vertical maps are isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(M,L) \arrow[r] \arrow[d, dashed, "-\cap[M]"'] & \check{H}^p(M) \arrow[r] \arrow[d, "-\cap[M]"'] & \check{H}^p(L) \arrow[r, "\delta"] \arrow[d, "-\cap[M]_L"'] & \check{H}^{p+1}(M,L) \arrow[r] \arrow[d, "-\cap[M]"'] & \cdots \\
\cdots \arrow[r] & H_q(L) \arrow[r] & H_q(M) \arrow[r] & H_q(M, M-L) \arrow[r, "\partial"] & H_{q-1}(L) \arrow[r] & \cdots
\end{tikzcd}$$

This is a consistency statement for Poincaré duality. On Wednesday, we'll specialize even further, and prove the Jordan curve theorem as well as study the cohomology rings of things we haven't worked through before.

2.13 Applications

Please check exam schedule! Also, a sample exam is posted. This is the payoff day. All this stuff about Poincaré duality has got to be good for something. Recall:

Theorem 2.13.1 (Fully relative duality). *Let M be a R-oriented n-manifold. Let $L \subseteq K \subseteq M$ be compact (M need not be compact). Then $[M]_K \in H_n(M, M-K)$, and capping gives an isomorphism:*

$$\check{H}^p(K, L; R) \xrightarrow{\cap[M]_k, \cong} H_{n-p}(M-L, M-K; R)$$

Today we'll think about the case $L = \emptyset$, so this is saying:

$$\check{H}^p(K; R) \xrightarrow{\cap[M]_k, \cong} H_{n-p}(M, M-K; R)$$

Corollary 2.13.2. *$\check{H}^q(K; R) = 0$ for $q > n$.*

We can contrast this with singular (co)homology. Here's an example:

Example 2.13.3 (Barratt-Milnor)**.** A two-dimensional version K of the Hawaiian earring, i.e., nested spheres all tangent to a point whose radii are going to zero. What they proved is that $H_q(K; \mathbf{Q})$ is uncountable for every $q > 1$. But if you look at the Čech cohomology, stuff vanishes.

That's nice.

How about an even more special subcase? Suppose $M = \mathbf{R}^n$. The result is called Alexander duality. This says:

Theorem 2.13.4 (Alexander duality)**.** *If $\emptyset \neq K \subseteq \mathbf{R}^n$ be compact. Then $\check{H}^{n-q}(K; R) \xrightarrow{\cong} \widetilde{H}_{q-1}(\mathbf{R}^n - K; R)$*

Proof. We have the LES of a pair, which gives an isomorphism $\partial : H_q(\mathbf{R}^n, \mathbf{R}^n - K; R) \xrightarrow{\cong} \widetilde{H}_{q-1}(\mathbf{R}^n - K; R)$, so the composition $\partial \circ (- \cap [M]_K)$ is an isomorphism by Poincaré duality. $\square$

For most purposes, this is the most useful duality theorem.

Example 2.13.5 (Jordan curve theorem)**.** $q = 1$ and $R = \mathbf{Z}$. Then this is saying that $\check{H}^{n-1}(K) \xrightarrow{\cong} \widetilde{H}_0(\mathbf{R}^n - K)$. But $\widetilde{H}_0(\mathbf{R}^n - K)$ is free on $\#\pi_0(\mathbf{R}^n - K) - 1$ generators. If $n = 2$, for example, and $K \cong S^1$, then $\check{H}^{n-1}(K) = H^{n-1}(K) \cong H^{n-1}(S^1)$, so $H^1(S^1) \cong \widetilde{H}_0(\mathbf{R}^2 - K)$. Hence there are *two* components in the complement of K. This could also be the topologist's sine curve as well. This is the Jordan curve theorem.

Consider the UCT, which states that there's a sexseq $0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z}) \to H^q(X) \to \mathrm{Hom}(H_q(X), \mathbf{Z}) \to 0$ that splits, but not naturally. First, note that $\mathrm{Hom}(H_q(X), \mathbf{Z})$ is always torsion-free. If I assume that $H_{q-1}(X)$ is finitely generated, then $\mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z})$ is a finite abelian group, but in particular it's torsion.

The UCT is making the decomposition of $H^q(X)$ into its torsion-free and torsion parts. I can divide by torsion, so that $H^q(X)/\mathrm{tors} \cong \mathrm{Hom}(H_q(X), \mathbf{Z})$. But there's also an isomorphism $\mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z}) \to \mathrm{Hom}(H_q(X), \mathbf{Z})$ because $\mathbf{Z}$ is torsion-free. Therefore I get an isomorphism $\alpha : H^q(X)/\mathrm{tors} \to \mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z})$. I.e.:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbf{Z}}(H_{q-1}(X), \mathbf{Z}) \arrow[r] & H^q(X) \arrow[r] \arrow[d] & \mathrm{Hom}(H_q(X), \mathbf{Z}) \arrow[r] & 0 \\
& & H^q(X)/\mathrm{tors} \arrow[r, "\alpha", "\cong"'] \arrow[ur, "\cong"] & \mathrm{Hom}(H_q(X)/\mathrm{tors}/\mathbf{Z}) \arrow[u, "\cong"'] &
\end{tikzcd}$$

Or I could say it like this: the Kronecker pairing can be quotiented by torsion, and you get an induced map $H^q(X)/\mathrm{tors} \otimes H_q(X)/\mathrm{tors} \to \mathbf{Z}$ is a perfect pairing, which means that the adjoint map $H^q(X)/\mathrm{tors} \xrightarrow{\cong} \mathrm{Hom}(H_q(X)/\mathrm{tors}, \mathbf{Z})$. Let's combine this with Poincaré duality.

Let $X = M$ be a compact oriented n-manifold. Then $H^{n-q}(X) \xrightarrow{-\cap[M],\cong} H_q(M)$, and so we get a perfect pairing $H^q(X)/\mathrm{tors} \otimes H^{n-q}(X)/\mathrm{tors} \to \mathbf{Z}$. And what is that pairing? It's the cup product! We have:

$$\begin{tikzcd}
H^q(M) \otimes H^{n-q}(M) \arrow[r] \arrow[d, "1\otimes(-\cap[M])"'] & \mathbf{Z} \\
H^q(M) \otimes H_q(M) \arrow[ur, "\langle , \rangle"'] &
\end{tikzcd}$$

And, well:

$$\langle a, b \cap [M] \rangle = \langle a \cup b, [M] \rangle$$

Thus the map $H^q(M) \otimes H^{n-q}(M) \to \mathbf{Z}$ is $a \otimes b \mapsto \langle a \cup b, [M] \rangle$, and it's a perfect pairing. This is a purely cohomological version, and is the most useful statement.

Example 2.13.6. Suppose $M = \mathbf{CP}^2 = D^0 \cup D^2 \cup D^4$, and its homology is $\mathbf{Z}\,0\,\mathbf{Z}\,0\,\mathbf{Z}$, and so its cohomology is the same. Let $a \in H^2(\mathbf{CP}^2)$. Then we have $H^2(\mathbf{CP}^2) \otimes H^2(\mathbf{CP}^2) \to \mathbf{Z}$, and so $a \cup a$ is a generator of $H^4(\mathbf{CP}^2)$, and hence specifies an orientation for $\mathbf{CP}^2$. The conclusion is that $H^*(\mathbf{CP}^2) = \mathbf{Z}[a]/(a^3)$ where $|a| = 2$.

How about $\mathbf{CP}^3$? It just adds a 6-cell, so its homology is $\mathbf{Z}\,0\,\mathbf{Z}\,0\,\mathbf{Z}\,0\,\mathbf{Z}$, and so its cohomology is the same. But then $a^3 = a \cup a \cup a$ is a generator of $H^6(\mathbf{CP}^2)$, and etc. Thus in general, we have:

$$H^*(\mathbf{CP}^n) = \mathbf{Z}[a]/(a^{n+1})$$

These things are finite CW-complexes, so you find:

$$H^*(\mathbf{CP}^\infty) = \mathbf{Z}[a] \tag{2.1}$$

Example 2.13.7. Suppose I look at maps $f : S^m \to S^n$. One of the most interesting things is that there are lots of non null-homotopic maps $S^m \to S^n$ if $m > 2$. For example, $\eta : S^3 \to S^2$ that's the attaching map for the 4-cell in $\mathbf{CP}^2$. This is called the Hopf fibration. It's essential. Why is it nullhomotopic? If η was null homotopic, then $\mathbf{CP}^2 \simeq S^2 \wedge S^4$. That's compatible with the cohomology in each dimension, but not into the cohomology ring! There's a map $S^2 \wedge S^4 \to S^2$ that collapses S^4, and the generator in $H^*(S^2)$ has $a^2 = 0$, so $a^2 = 0$ in $H^*(S^2 \wedge S^4)$. But this is not compatible with our computation that $H^*(\mathbf{CP}^2) = \mathbf{Z}[a]/(a^3)$ where $|a| = 2$.

With coefficients in a field k, then the torsion is zero, so you find that if M is compact k-oriented, then if the characteristic of $k = 2$, there's no condition for M to be oriented, and if the characteristic of k is not 2, then M is $\mathbf{Z}$-oriented. Thus we get that $H^q(M;k) \otimes_k H^{n-q}(M;k) \to k$ is a perfect pairing.

Example 2.13.8. Exactly the same argument as for complex projective space shows that:

$$H^*(\mathbf{RP}^n; \mathbf{F}_2) = \mathbf{F}_2[a]/(a^{n+1})$$

where $|a| = 1$. So:

$$H^*(\mathbf{RP}^\infty; \mathbf{F}_2) = \mathbf{F}_2[a] \tag{2.2}$$

where $|a| = 1$.

I'll end with the following application.

Theorem 2.13.9. *Suppose $f : \mathbf{R}^{m+1} \supseteq S^m \to S^n \subseteq \mathbf{R}^{n+1}$ that is equivariant with respect to the antipodal action, i.e., $f(-x) = -f(x)$. Then $m \leq n$.*

So there are *no* equivariant maps from $S^m \to S^n$ if $m > n$!

Proof. Suppose I have a map like that: the map on spheres induces a map $\overline{f} : \mathbf{RP}^m \to \mathbf{RP}^n$. We claim that $H_1(\overline{f})$ is an isomorphism. Let $\pi : S^n \to \mathbf{RP}^n$ denote the map. Let $\sigma : I \to S^m$ be defined via $\sigma(0) = v$ and $\sigma(1) = -v$. So this gives a 1-cycle $\sigma : I \to S^m \to \mathbf{RP}^m$, and $H_1(\mathbf{RP}^n) = [\pi\sigma]$ is generated by this thing. When I map this thing to $\mathbf{RP}^n$, we send $\pi\sigma$ to a generator. What we've actually proved, therefore, is that $H_1(\mathbf{RP}^m) \cong H_1(\mathbf{RP}^n)$. This is also true with mod 2 coefficients, i.e., $H_1(\overline{f}, \mathbf{F}_2) \neq 0$.

That means that $H^1(\overline{f}; \mathbf{F}_2) \neq 0$ by UCT. But what is this? This is a map $H^1(f; \mathbf{F}_2) : H^*(\mathbf{RP}^n; \mathbf{F}_2) \to H^*(\mathbf{RP}^n; \mathbf{F}_2)$, i.e., a map $\mathbf{F}_2[a]/(a^{n+1}) \to \mathbf{F}_2[a] \to (a^{m+1})$. Thus $a \mapsto a$. There's not a lot of ways to do this if $m > n$. Thus what we've shown that $m \leq n$. $\square$

This is the Borsuk-Ulam theorem from the '20s, I think. This is an example of how you can use the cohomology ring structure for projective space.

Please check the website for details about your finals. I will ask you to sign a form, to make sure that you don't share the questions or that you haven't heard the questions beforehand. I have a fixed set of questions that'll guide the conversation.

Part II

18.906 – homotopy theory

Introduction

Here is an overview of this part of the book.

1. **General homotopy theory.** This includes category theory; because it started as a part of algebraic topology, we'll speak freely about it here. We'll also cover the general theory of homotopy groups, long exact sequences, and obstruction theory.

2. **Bundles.** One of the major themes of this part of the book is the use of bundles to understand spaces. This will include the theory of classifying spaces; later, we will touch upon connections with cohomology.

3. **Spectral sequences.** It is impossible to describe everything about spectral sequences in the duration of a single course, so we will focus on a special (and important) example: the Serre spectral sequence. As a consequence, we will derive some homotopy-theoretic applications. For instance, we will relate homotopy and homology (via the Hurewicz theorem, Whitehead's theorem, and "local" versions like Serre's mod C theory).

4. **Characteristic classes.** This relates the geometric theory of bundles to algebraic constructions like cohomology described earlier in the book. We will discuss many examples of characteristic classes, including the Thom, Euler, Chern, and Stiefel-Whitney classes. This will allow us to apply a lot of the theory we built up to geometry.

Chapter 3

Homotopy groups

Insert an outline of the content of each lecture.

3.1 Limits, colimits, and adjunctions

Limits and colimits

We will freely use the theory developed in the first part of this book (see §**??**). Suppose $\mathcal{I}$ is a small category (so that it has a *set* of objects), and let $\mathcal{C}$ be another category.

Definition 3.1.1. Let $X : \mathcal{I} \to \mathcal{C}$ be a functor. A *cone under* X is a natural transformation η from X to a constant functor; explicitly, this means that for every object i of $\mathcal{I}$, we must have a map $\eta_i : X_i \to Y$, such that for every $f : i \to j$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{tikzcd} X_i \arrow[d, "f_*"] \arrow[dr, "\eta_i"] & \\ X_j \arrow[r, "\eta_j"] & Y. \end{tikzcd}$$

A *colimit* of X is an initial cone (L, τ_i) under X; explicitly, this means that for all cones (Y, η_i) under X, there exists a unique natural transformation $h : L \to Y$ such that $h \circ \tau_i = \eta_i$.

As always for category theoretic concepts, some examples are in order.

Example 3.1.2. If $\mathcal{I}$ is a discrete category (i.e., only a set, with identity maps), the colimit of any functor $\mathcal{I} \to \mathcal{C}$ is the coproduct. This already illustrates an important point about colimits: they need not exist in general (since, for example, coproducts need not exist in a general category). Examples of categories $\mathcal{C}$ where the colimit of a functor $\mathcal{I} \to \mathcal{C}$ exists: if $\mathcal{C}$ is sets, or spaces, the colimit is the disjoint union. If $\mathcal{C} = \mathbf{Ab}$, a candidate for the colimit would be the product: but this only works if $\mathcal{I}$ is finite; in general, the correct thing is to take the (possibly infinite) direct sum.

Example 3.1.3. Let $\mathcal{I} = \mathbf{N}$, considered as a category via its natural poset structure; then a functor $\mathcal{I} \to \mathcal{C}$ is simply a linear system of objects and morphisms in $\mathcal{C}$. As a specific example, suppose $\mathcal{C} = \mathbf{Ab}$, and consider the diagram $X : \mathcal{I} \to \mathcal{C}$ defined by the system

$$\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \to \cdots$$

The colimit of this diagram is $\mathbf{Q}$, where the maps are:

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "2"] \arrow[dr, "1"] & \mathbf{Z} \arrow[r, "3"] \arrow[d, "1/2"] & \mathbf{Z} \arrow[r, "4"] \arrow[dl, "1/3!"] & \cdots \\
& \mathbf{Q} & &
\end{tikzcd}$$

Example 3.1.4. Let G be a group; we can view this as a category with one object, where the morphisms are the elements of the group (composition is given by the group structure). If $\mathcal{C} = \mathbf{Top}$ is the category of topological spaces, a functor $G \to \mathcal{C}$ is simply a group action on a topological space X. The colimit of this functor is the orbit space of the G-action on X.

Example 3.1.5. Let $\mathcal{I}$ be the category whose objects and morphisms are determined by the following graph:

$$\begin{tikzcd}
& a \arrow[dl] \arrow[dr] & \\
b & & c.
\end{tikzcd}$$

The colimit of a diagram $\mathcal{I} \to \mathcal{C}$ is called a *pushout*.

If $\mathcal{C} = \mathbf{Top}$, again, a functor $\mathcal{I} \to \mathcal{C}$ is determined by a diagram of spaces:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B & & C. \end{tikzcd}$$

The colimit of such a functor is just the pushout $B \cup_A C := B \sqcup C/\sim$, where $f(a) \sim g(a)$ for all $a \in A$. We have already seen this in action before: the same construction appears in the process of attaching cells to CW-complexes.

If $\mathcal{C}$ is the category of groups, instead, the colimit of such a functor is the free product quotiented out by a certain relation (the same as for topological spaces); this is called the *amalgamated free product.*

Example 3.1.6. Suppose $\mathcal{I}$ is the category defined by the following graph:

$$a \rightrightarrows b.$$

The colimit of a diagram $\mathcal{I} \to \mathcal{C}$ is called the *coequalizer* of the diagram.

One can also consider cones *over* a diagram $X : \mathcal{I} \to \mathcal{C}$: this is simply a cone in the opposite category.

Definition 3.1.7. With notation as above, the *limit* of a diagram $X : \mathcal{I} \to \mathcal{C}$ is a terminal object in cones over X.

For instance, products are limits, just like in Example 3.1.2. (This example also shows that abelian groups satisfy an interesting property: finite products are the same as finite coproducts!)

Exercise 3.1.8. Revisit the examples provided above: what is the limit of each diagram? For instance, the limit of the diagram described in Example 3.1.4 is just the fixed points!

Adjoint functors

Adjoint functors are very useful — and very natural — objects. We already have an example: let $\mathcal{C}^{\mathcal{I}}$ be the functor category $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$. (We've been working in this category this whole time!) Let's make an additional assumption on $\mathcal{C}$, namely that all $\mathcal{I}$-indexed colimits exist. All examples considered above satisfy this assumption.

There is a functor $\mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, given by sending any object to the constant functor taking that value. The process of taking the colimit of a diagram supplies us with a functor $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$. We can characterize this functor via a formula[1]:

$$\mathcal{C}(\operatorname{colim}_{i\in\mathcal{I}} X_i, Y) = \mathcal{C}^{\mathcal{I}}(X, \mathrm{const}_Y),$$

where X is some functor from $\mathcal{I}$ to $\mathcal{C}$. This formula is reminiscent of the adjunction operator in linear algebra, and is in fact our first example of an adjunction.

Definition 3.1.9. Let $\mathcal{C}, \mathcal{D}$ be categories, with specified functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. An *adjunction between F and G* is an isomorphism:

$$\mathcal{D}(FX, Y) = \mathcal{C}(X, GY),$$

which is natural in X and Y. In this situation, we say that F is a *left adjoint* of G and G is a *right adjoint* of X.

This notion was invented by Dan Kan, who worked in the MIT mathematics department until he passed away in 2013.

We've already seen an example above, but here is another one:

Definition 3.1.10 (Free groups)**.** There is a forgetful functor $u : \mathrm{Grp} \to \mathrm{Set}$. Any set X gives rise to a group FX, namely the free group on X elements. This is determined by a universal property: set maps $X \to u\Gamma$ are the same as group maps $FX \to \Gamma$, where Γ is any group. This is exactly saying that the free group functor the left adjoint to the forgetful functor u.

In general, "free objects" come from left adjoints to forgetful functors.

Definition 3.1.11. A category $\mathcal{C}$ is said to be *cocomplete* if all (small) colimits exist in $\mathcal{C}$. Similarly, one says that $\mathcal{C}$ is *complete* if all (small) limits exist in $\mathcal{C}$.

The Yoneda lemma

One of the many important concepts in category theory is that an object is determined by the collection of all maps out of it. The Yoneda lemma is a way of making this precise. An important reason to even bother

[1] There is an analogous formula for the limit of a diagram:

$$\mathcal{C}(W, \lim_{i\in\mathcal{I}} X_i) = \mathcal{C}^{\mathcal{I}}(\mathrm{const}_W, X).$$

thinking about objects in this fashion comes from our discussion of colimits. Namely, how do we even know that the notion is well-defined?

The colimit of an object is characterized by maps out of it; precisely:

$$\mathcal{C}(\operatorname{colim}_{j\in\mathcal{J}} X_j, Y) = \mathcal{C}^{\mathcal{J}}(X_\bullet, \mathrm{const}_Y).$$

The two sides are naturally isomorphic, but if the colimit exists, how do we know that it is unique? This is solved by Yoneda lemma[2]:

Theorem 3.1.12 (Yoneda lemma)**.** *Consider the functor* $\mathcal{C}(X,-) : \mathcal{C} \to \mathrm{Set}$*. Suppose* $G : \mathcal{C} \to \mathrm{Set}$ *is another functor. It turns out that:*

$$\mathrm{nt}(\mathcal{C}(X,-), G) \simeq G(X).$$

Proof. Let $x \in G(X)$. Define a natural transformation that sends a map $f : X \to Y$ to $f_*(x) \in G(Y)$. On the other hand, we can send a natural transformation $\theta : C(X,-) \to G$ to $\theta_X(1_X)$. Proving that these are inverses is left as an exercise — largely in notation — to the reader. □

In particular, if $G = \mathcal{C}(Y,-)$ — these are called *corepresentable* functors — then $\mathrm{nt}(\mathcal{C}(X,-), \mathcal{C}(Y,-)) \simeq \mathcal{C}(Y,X)$. Simply put, natural isomorphisms $\mathcal{C}(X,-) \to \mathcal{C}(Y,-)$ are the same as isomorphisms $Y \to X$. As a consequence, the object that a corepresentable functor corepresents is unique (at least up to isomorphism).

From the Yoneda lemma, we can obtain some pretty miraculous conclusions. For instance, functors with left and/or right adjoints are very well-behaved (the "constant functor" functor is an example where both adjoints exist), as the following theorem tells us.

Theorem 3.1.13. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. If* F *admits a right adjoint, it preserves colimits. Dually, if* F *admits a left adjoint, it preserves limits.*

Proof. We'll prove the first statement, and leave the other as an (easy) exercise. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that admits a right adjoint G, and let $X : \mathcal{I} \to \mathcal{C}$ be a small $\mathcal{I}$-indexed diagram in $\mathcal{C}$. For any object Y of $\mathcal{C}$, there is an isomorphism

$$\mathrm{Hom}(\operatorname{colim}_{\mathcal{I}} X, Y) \simeq \lim_{\mathcal{I}} \mathrm{Hom}(X, Y).$$

[2]Sometimes "you-need-a-lemma"!

This follows easily from the definition of a colimit. Let Y be any object of $\mathcal{D}$; then, we have:

$$\begin{aligned}\mathcal{D}(F(\operatorname{colim}_{\mathcal{I}} X), Y) &\simeq \mathcal{C}(\operatorname{colim}_{\mathcal{I}} X, G(Y)) \\ &\simeq \lim_{\mathcal{I}} \mathcal{C}(X, G(Y)) \\ &\simeq \lim_{\mathcal{I}} \mathcal{C}(F(X), Y) \\ &\simeq \mathcal{C}(\operatorname{colim}_{\mathcal{I}} F(X), Y).\end{aligned}$$

The Yoneda lemma now finishes the job. $\square$

3.2 Compactly generated spaces

A lot of homotopy theory is about loop spaces and mapping spaces. Standard topology doesn't do very well with mapping spaces, so we will narrate the story of *compactly generated spaces*. One nice consequence of working with compactly generated spaces is that the category is Cartesian-closed (a concept to be defined below).

CGHW spaces

Some constructions commute for "categorical reasons". For instance, limits commute with limits. Here is an exercise to convince you of a special case of this.

Exercise 3.2.1. Let X be an object of a category $\mathcal{C}$. The *overcategory* (or the *slice category*) $\mathcal{C}_{/X}$ has objects given by morphisms $p : Y \to X$ in $\mathcal{C}$, and morphisms given by the obvious commutativity condition.

1. Assume that $\mathcal{C}$ has finite products. What is the left adjoint to the functor $X \times - : \mathcal{C} \to \mathcal{C}_{/X}$ that sends Y to the object $X \times Y \xrightarrow{\mathrm{pr}_1} X$?

2. As a consequence of Theorem 3.1.13, we find that $X \times - : \mathcal{C} \to \mathcal{C}_{/X}$ preserves limits. The composite $\mathcal{C} \to \mathcal{C}_{/X} \to \mathcal{C}$, however, probably does not.

   - What is the limit of a diagram in $\mathcal{C}_{/X}$?
   - Let $Y : \mathcal{I} \to \mathcal{C}$ be any diagram. Show that

     $$\lim_{i \in \mathcal{I}}{}^{\mathcal{C}_{/X}} (X \times Y_i) \simeq X \times \lim_{i \in \mathcal{I}}{}^{\mathcal{C}} Y_i.$$

     What happens if $\mathcal{I}$ only has two objects and only identity morphisms?

However, colimits and limits need not commute! An example comes from algebra. The coproduct in the category of commutative rings is the tensor product (exercise!). But $\left(\lim \mathbf{Z}/p^k\mathbf{Z}\right) \otimes \mathbf{Q} \simeq \mathbf{Z}_p \otimes \mathbf{Q} \simeq \mathbf{Q}_p$ is clearly not $\lim \left(\mathbf{Z}/p^k\mathbf{Z} \otimes \mathbf{Q}\right) \simeq \lim 0 \simeq 0$!

We also need not have an isomorphism between $X \times \mathrm{colim}_{j\in\mathcal{J}} Y_j$ and $\mathrm{colim}_{j\in\mathcal{J}}(X \times Y_j)$. One example comes a quotient map $Y \to Z$: in general, the induced map $X \times Y \to X \times Z$ is not necessarily another quotient map. A theorem of Whitehead's says that this problem is rectified if we assume that X is a compact Hausdorff space. Unfortunately, a lot of interesting maps are built up from more "elementary" maps by such a procedure, so we would like to repair this problem.

We cannot simply do this by restricting ourselves to compact Hausdorff spaces: that's a pretty restrictive condition to place. Instead (motivated partially by the Yoneda lemma), we will look at topologies detected by maps from compact Hausdorff spaces.

Definition 3.2.2. Let X be a space. A subspace $F \subseteq X$ is said to be *compactly closed* if, for any map $k : K \to X$ from a compact Hausdorff space K, the preimage $k^{-1}(F) \subseteq K$ is closed.

It is clear that any closed subset is compactly closed, but there might be compactly closed sets which are not closed in the topology on X. This motivates the definition of a k-space:

Definition 3.2.3. A topological space X is said to be a *k-space* if every compactly closed set is closed.

The k comes either from "kompact" and/or Kelly, who was an early topologist who worked on such foundational topics.

It's clear that X is a k-space if and only if the following statement is true: a map $X \to Y$ is continuous if and only if, for every compact Hausdorff space K and map $k : K \to X$, the composite $K \to X \to Y$ is continuous. For instance, compact Hausdorff spaces are k-spaces. First countable (so metric spaces) and CW-complexes are also k-spaces.

In general, a topological space X need not be a k-space. However, it can be "k-ified" to obtain another k-space denoted kX. The procedure is simple: endow X with the topology consisting of all compactly closed sets. The reader should check that this is indeed a topology on X; the resulting topological space is denoted kX. This construction immediately implies, for instance, that the identity $kX \to X$ is continuous.

Let $k\mathbf{Top}$ be the category of k-spaces. This is a subcategory of the category of topological spaces, via a functor $i : k\mathbf{Top} \hookrightarrow \mathbf{Top}$. The process of k-ification gives a functor $\mathbf{Top} \to k\mathbf{Top}$, which has the property that:

$$k\mathbf{Top}(X, kY) = \mathbf{Top}(iX, Y).$$

Notice that this is another example of an adjunction! We can conclude from this that $k(iX \times iY) = X \times^{k\mathbf{Top}} Y$, where X and Y are k-spaces. One can also check that $kiX \simeq X$.

The takeaway is that $k\mathbf{Top}$ has good categorical properties inherited from $\mathbf{Top}$: it is a complete and cocomplete category. As we will now explain, this category has more categorical niceness, that does not exist in $\mathbf{Top}$.

Mapping spaces

Let X and Y be topological spaces. The set $\mathbf{Top}(X, Y)$ of continuous maps from X to Y admits a topology, namely the compact-open topology. If X and Y are k-spaces, we can make a slight modification: define a topology on $k\mathbf{Top}(X, Y)$ generated by the sets

$$W(k : K \to X, \text{ open } U \subseteq Y) := \{f : X \to Y : f(k(K)) \subseteq U\}.$$

We write Y^X for the k-ification of $k\mathbf{Top}(X, Y)$.

Proposition 3.2.4. *1. The functor $(k\mathbf{Top})^{op} \times k\mathbf{Top} \to k\mathbf{Top}$ given by $(X, Y) \to Y^X$ is a functor of both variables.*

2. $e : X \times Z^X \to Z$ given by $(x, f) \mapsto f(x)$ and $i : Y \to (X \times Y)^X$ given by $y \mapsto (x \mapsto (x, y))$ are continuous.

Proof. The first statement is left as an exercise to the reader. For the second statement, see [Str09, Proposition 2.11]. □

As a consequence of this result, we can obtain a very nice adjunction. Define two maps:

- $k\mathbf{Top}(X \times Y, Z) \to k\mathbf{Top}(Y, Z^X)$ via

$$(f : X \times Y \to Z) \mapsto (Y \xrightarrow{i} (X \times Y)^X \to Z^X).$$

- $k\mathbf{Top}(Y, Z^X) \to k\mathbf{Top}(X \times Y, Z)$ via

$$(f : Y \to Z^X) \mapsto (X \times Y \to X \times Z^X \xrightarrow{e} X).$$

By [Str09, Proposition 2.12], these two maps are continuous inverses, so there is a natural homeomorphism

$$k\mathbf{Top}(X \times Y, Z) \simeq k\mathbf{Top}(Y, Z^X).$$

This motivates the definition of a Cartesian closed category.

Definition 3.2.5. A category $\mathcal{C}$ with finite products is said to be *Cartesian closed* if, for any object X of $\mathcal{C}$, the functor $X \times - : \mathcal{C} \to \mathcal{C}$ has a right adjoint.

Our discussion above proves that $k\mathbf{Top}$ is Cartesian closed, while this is not satisfied by $\mathbf{Top}$. As we will see below, this has very important ramifications for algebraic topology.

Exercise 3.2.6.

Insert Exercise 2 from 18.906.

3.3 "Cartesian closed", Hausdorff, Basepoints

Pushouts are colimits, so the quotient space $X/A = X \cup_A *$ is an example of a colimit. Let Y be a topological space, and consider the functor $Y \times - : \mathbf{Top} \to \mathbf{Top}$. Applying this to the pushout square, we find that $(Y \times X) \cup_{Y \times A} * \simeq (Y \times X)/(Y \times A)$. As we discussed in §3.2, this product is *not* the same as $Y \times (X/A)$! There is a bijective map $Y \times X/Y \times A \to Y \times (X/A)$, but it is not, in general, a homeomorphism. From a categorical point of view (see Theorem 3.1.13), the reason for this failure stems from $Y \times -$ not being a left adjoint.

The discussion in §3.2 implies that, when working with k-spaces, that functor is indeed a left adjoint (in fancy language, the category $k\mathbf{Top}$ is Cartesian closed), which means that — in $k\mathbf{Top}$ — there is a homeomorphism $Y \times X/Y \times A \to Y \times (X/A)$. This addresses the issues raised in §3.2. The ancients had come up with a good definition of a topology — but k-spaces are better! Sometimes, though, we can be greedy and ask for even more: for instance, we can demand that points be closed. This leads to a further refinement of k-spaces.

I don't like point-set topology, so I'll return to editing this lecture at the end.

"Hausdorff"

Definition 3.3.1. A space is "weakly Hausdorff" if the image of every map $K \to X$ from a compact Hausdorff space K is closed.

Another way to say this is that the map itself if closed. Clearly Hausdorff implies weakly Hausdorff. Another thing this means is that every point in X is closed (eg $K = *$).

Proposition 3.3.2. *Let X be a k-space.*

1. *X is weakly Hausdorff iff $\Delta : X \to X \times^k X$ is closed. In algebraic geometry such a condition is called separated.*

2. *Let $R \subseteq X \times X$ be an equivalence relation. If R is closed, then X/R is weakly Hausdorff.*

Definition 3.3.3. A space is compactly generated if it's a weakly Hausdorff k-space. The category of such spaces is called **CG**.

We have a pair of adjoint functors $(i, k) : \mathbf{Top} \to k\mathbf{Top}$. It's possible to define a functor $k\mathbf{Top} \to \mathbf{CG}$ given by $X \mapsto X/\bigcap$ all closed equivalence relations. It is easy to check that if Z is weakly Hausdorff, then Z^X is weakly Hausdorff (where X is a k-space). What this implies is that **CG** is also Cartesian closed!

I'm getting a little tired of point set stuff. Let's start talking about homotopy and all that stuff today for a bit. You know what a homotopy is. I will not worry about point-set topology anymore. So when I say **Top**, I probably mean **CG**. A homotopy between $f, g : X \to Y$ is a map $h : I \times X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "i_0"'] \arrow[drr, "f"] & & \\
& I \times X \arrow[r, "h"] & Y \\
X \arrow[ur, "i_1"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

We write $f \sim g$. We define $[X, Y] = \mathbf{Top}(X, Y)/\sim$. Well, a map $I \times X \to Y$ is the same as a map $X \to Y^I$ but also $I \to Y^X$. The latter is my favorite! It's a path of maps from f to g. So $[X, Y] = \pi_0 Y^X$.

To talk about higher homotopy groups and induct etc. we need to talk about basepoints.

Basepoints

A pointed space is $(X, *)$ with $* \in X$. This gives a category $\mathbf{Top}_*$ where the morphisms respect the basepoint. This has products because $(X, *) \times$

$(Y, *) = (X \times Y, (*, *))$. How about coproducts? It has coproducts as well. This is the wedge product, defined as $X \sqcup Y / *_X \sim *_Y =: X \vee Y$. This is `\vee`, not `\wedge`. Is this category also Cartesian closed?

Define the space of pointed maps $Z^X_* \subseteq Z^X$ topologized as a subspace. Does the functor $Z \mapsto Z^X_*$ have a left adjoint? Well $\mathbf{Top}(W, Z^X) = \mathbf{Top}(X \times W, Z)$. What about $\mathbf{Top}(W, Z^X_*)$? This is $\{f : X \times W \to Z : f(*, w) = * \forall w \in W\}$. That's not quite what I wanted either! Thus $\mathbf{Top}_*(W, Z^X_*) = \{f : X \times W \to Z : f(*, w) = * = f(x, *) \forall x \in X, w \in W\}$. These send both "axes" to the basepoint. Thus, $\mathbf{Top}_*(W, Z^X_*) = \mathbf{Top}_*(X \wedge W, Z)$ where $X \wedge W = X \times W / X \vee W$ because $X \vee W$ are the "axes".

So $\mathbf{Top}_*$ is not Cartesian closed, but admits something called the smash product[3]. What properties would you like? Here's a good property: $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ are bijective in pointed spaces. If you work in $k\mathbf{Top}$ or $\mathbf{CG}$, then they are homeomorphic! It also has a unit.

Oh yeah, some more things about basepoints! So there's a canonical forgetful functor $i : \mathbf{Top}_* \to \mathbf{Top}$. Let's see. If I have $\mathbf{Top}(X, iY) = \mathbf{Top}_*(??, Y)$? This is $X_+ = X \sqcup *$. Thus we have a left adjoint $(-)_+$. It is clear that $(X \sqcup Y)_* = X_+ \vee Y_+$. The unit for the smash product is $*_+ = S^0$.

On Friday I'll talk about fibrations and fiber bundles.

3.4 Fiber bundles, fibrations, cofibrations

Having set up the requisite technical background, we can finally launch ourselves from point-set topology to the world of homotopy theory.

Fiber bundles

Definition 3.4.1. A fiber[4] bundle is a map $p : E \to B$, such that for every $b \in B$, there exists:

- an open subset $U \subseteq B$ that contains b, and
- a map $p^{-1}(U) \to p^{-1}(b)$ such that $p^{-1}(U) \to U \times p^{-1}(b)$ is a homeomorphism.

[3] Remark by Sanath: this is like the tensor product.
[4] Or "fibre", if you're British.

If $p : E \to B$ is a fiber bundle, E is called the *total space*, B is called the *base space*, p is called a *projection*, and F (sometimes denoted $p^{-1}(b)$) is called the *fiber over b*.

In simpler terms: the preimage over every point in B looks like a product, i.e., the map $p : E \to B$ is "locally trivial" in the base.

Here is an equivalent way of stating Definition 3.4.1: there is an open cover $\mathcal{U}$ (called the *trivializing cover*) of B, such that for every $U \subseteq \mathcal{U}$, there is a space F, and a homeomorphism $p^{-1}(U) \simeq U \times F$ that is compatible with the projections down to U. (So, for instance, a trivial example of a fiber bundle is just the projection map $B \times F \xrightarrow{\mathrm{pr}_1} B$.)

Fiber bundles are naturally occurring objects. For instance, a covering space $E \to B$ is a fiber bundle with discrete fibers.

Example 3.4.2 (The Hopf fibration)**.** The Hopf fibration is an extremely important example of a fiber bundle. Let $S^3 \subset \mathbf{C}^2$ be the 3-sphere. There is a map $S^3 \to \mathbf{CP}^1 \simeq S^2$ that is given by sending a vector v to the complex line through v and the origin. This is a non-nullhomotopic map, and is a fiber bundle whose fiber is S^1.

Here is another way of thinking of the Hopf fibration. Recall that $S^3 = SU(2)$; this contains as a subgroup the collection of matrices $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$. This subgroup is simply S^1, which acts on S^3 by translation; the orbit space is S^2.

The Hopf fibration is a map between smooth manifolds. A theorem of Ehresmann's says that it is not too hard to construct fiber bundles over smooth manifolds:

Theorem 3.4.3 (Ehresmann)**.** *Suppose E and B are smooth manifolds, and let $p : E \to B$ be a smooth (i.e., C^∞) map. Then p is a fiber bundle if:*

1. *p is a* submersion*, i.e., $dp : T_eE \to T_{p(e)}B$ is a surjection, and*

2. *p is* proper*, i.e., preimages of compact sets are compact.*

The purpose of this part of the book is to understand fiber bundles through algebraic methods like cohomology and homotopy. This means that we will usually need a "niceness" condition on the fiber bundles that we will be studying; this condition is made precise in the following definition (see [May99]).

Definition 3.4.4. Let X be a space. An open cover $\mathcal{U}$ of X is said to be *numerable* if there exists a subordinate partition of unity, i.e., for each $U \in \mathcal{U}$, there is a function $f_U : X \to [0,1] = I$ such that $f^{-1}((0,1]) = U$, and any $x \in X$ belongs to only finitely many $U \in \mathcal{U}$. The space X is said to be *paracompact* if any open cover admits a numerable refinement.

This isn't too restrictive for us algebraic topologists since CW-complexes are paracompact.

Definition 3.4.5. A fiber bundle is said to be *numerable* if it admits a numerable trivializing cover.

Fibrations and path liftings

For our purposes, though, fiber bundles are still too narrow. Fibrations capture the essence of fiber bundles, although it is not at all immediate from their definition that this is the case!

Definition 3.4.6. A map $p : E \to B$ is called a *(Hurewicz[5]) fibration* if it satisfies the *homotopy lifting property* (commonly abbreviated as HLP): suppose $h : I \times W \to B$ is a homotopy; then there exists a lift[6] (given by the dotted arrow) that makes the diagram commute:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ I \times W \arrow[r, "h"'] \arrow[ur, dashed, "\overline{h}"] & B, \end{tikzcd} \tag{3.1}$$

At first sight, this seems like an extremely alarming definition, since the HLP has to be checked for *all* spaces, *all* maps, and *all* homotopies! The HLP is not impossible to check, though.

Exercise 3.4.7. Check that the projection $\mathrm{pr}_1 : B \times F \to B$ is a fibration.

Exercise 3.4.8. Check the following statements.

- Fibrations are closed under pullbacks. In other words, if $p : E \to B$ is a fibration and $X \to B$ is any map, then the induced map $E \times_B X \to X$ is a fibration.

[5] Named after Witold Hurewicz, who was one of the first algebraic topologists at MIT.

[6] Note that we place no restriction on the *uniqueness* of this lift.

- Fibrations are closed under exponentiation and products. In other words, if $p : E \to B$ is a fibration, then $E^A \to B^A$ is another fibration.

- Fibrations are closed under composition.

Exercise 3.4.9. Let $p : E_0 \to B_0$ be a fibration, and let $f : B \to B_0$ be a homotopy equivalence. Prove that the induced map $B \times_{B_0} E_0 \to E_0$ is a homotopy equivalence. (Warning: this exercise has a lot of technical details! The end of this chapter describes an alternative[7] solution to this exercise, when E_0 and $B \times_{B_0} E_0$ are CW-complexes.)

Don't forget to do this!

There is a simple geometric interpretation of what it means for a map to be a fibration, in terms of "path liftings". To understand this description, we will reformulate the diagram (3.1). Given that we are working in the category of CGWH spaces, one of the first things we can attempt to do is adjoint the I; this gives the following diagram.

$$\begin{tikzcd} E \arrow[r, "p"] & B \\ W \arrow[u, "f"] \arrow[r, "\widehat{h}"'] & B^I \arrow[u, "\mathrm{ev}_0"'] \end{tikzcd} \tag{3.2}$$

By the definition of the pullback of a diagram, the data of this diagram is equivalent to a map $W \to B^I \times_B E$. Explicitly,

$$B^I \times_B E = \{(\omega, e) \in B^I \times E \text{ such that } \omega(0) = p(e)\}.$$

Suppose the desired dotted map exists (i.e., $p : E \to B$ satisfied the HLP). This would beget (again, by adjointness) a lifted homotopy $\widehat{\widehat{h}} : W \to E^I$. Since we already have a map[8] $\widetilde{p} : E^I \to B^I \times_B E$ given by $\omega \mapsto (p\omega, \omega(0))$, the existence of the lift $\overline{h}$ in the diagram (3.1) is equivalent to the existence of a lift in the following diagram.

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ W \arrow[ur, dashed, "\widehat{\widehat{h}}"] \arrow[r] & B^I \times_B E \end{tikzcd}$$

[7] "Alternative" in the sense that the proof uses statements not covered yet in this book.

[8] Clearly $(p\omega)(0) = p(\omega(0))$, so this map is well-defined (i.e., the image lands in $B^I \times_B E$).

Obviously the universal example of a space W that makes the diagram (3.2) commute is $B^I \times_B E$ itself. If p is a fibration, we can make the lift in the following diagram.

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ B^I \times_B E \arrow[r, "1"] \arrow[ur, dashed, "\lambda"] & B^I \times_B E \end{tikzcd}$$

The map λ is called a *lifting function.* To understand why, suppose $(\omega, e) \in B^I \times_B E$, so that $\omega(0) = p(e)$. In this case, $\lambda(\omega, e)$ defines a path in E such that

$$p \circ \lambda(\omega, e) = \omega, \text{ and } \lambda(\omega, e)(0) = e.$$

Taking a step back and assessing the situation, we find that the lifting function λ starts with a path ω in B, and some point in E mapping down to $\omega(0)$, and produces a "lifted" path in E which lives over ω. In other words, the map λ is a path lifting: it's a continuous way to lift paths in the base space B to the total space E.

The following result is a "consistency check".

Theorem 3.4.10 (Dold)**.** *Let $p : E \to B$ be a map. Assume there's a numerable cover of B, say $\mathcal{U}$, such that for every $U \in \mathcal{U}$, the restriction $p|_{p^{-1}(U)} : p^{-1}U \to U$ is a fibration. (In other words, p is* locally *a fibration over the base). Then p itself is a fibration.*

In particular, one consequence of this theorem is that every numerable fiber bundle is a fibration. Our discussion above tells us that numerable fiber bundles satisfy the homotopy (and hence path) lifting property. This is great news, as we will see shortly.

3.5 Fibrations and cofibrations

Comparing fibers over different points

Let $p : E \to B$ be a fibration. Above, we saw that this implies that paths in B "lift" to paths in E. Let us consider a path $\omega : I \to B$ with $\omega(0) = a$ and $\omega(1) = b$. Denote by F_a the fiber over a. If the world plays fairly, the path lifting property of fibrations should beget a (unique[9]) map $F_a \to F_b$. The goal of this subsection is to construct such a map.

[9] At least up to homotopy.

Consider the diagram:

$$\begin{tikzcd}
F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed, "h"] & I \arrow[r, "\omega"'] & B.
\end{tikzcd}$$

This commutes since $\omega(0) = a$. Utilizing the homotopy lifting property, there is a dotted arrow that makes the entire diagram commute. If $x \in F_a$, the image $h(1, x)$ is in F_b, and $h(0, x) = x$. This supplies us with a map $f : F_a \to F_b$, given by $f(x) = h(1, x)$.

We're now faced with a natural question: is f unique up to homotopy? Namely: if we have two homotopic paths ω_0, ω_1 with $\omega_0(0) = \omega_1(0) = a$, and $\omega_0(1) = \omega_1(1) = b$, along with a given homotopy $g : I \times I \to B$ between ω_0 and ω_1, such that $f_0, f_1 : F_a \to F_b$ are the associated maps (defined by $h_0(1, x)$ and $h_1(1, x)$), respectively, are f_0 and f_1 homotopic?

We have a diagram of the form:

$$\begin{tikzcd}
((\partial I \times I) \cup (I \times \{0\})) \times F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed] & I \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

To get a homotopy between f_0 and f_1, we need the dotted arrow to exist.

It's an easy exercise to recognize that our diagram is equivalent to the following.

$$\begin{tikzcd}
I \times F_a \arrow[r, "\simeq"] & ((\partial I \times I) \cup (I \times 0)) \times F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times I \times F_a \arrow[r, "\simeq"'] & I \times I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed] & I \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

Letting $W = I \times F_a$ in the definition of a fibration (Definition 3.4.6) thus gives us the desired lift, i.e., a homotopy $f_0 \simeq f_1$.

We can express the uniqueness (up to homotopy) of lifts of homotopic paths in a functorial fashion. To do so, we must introduce the fundamental groupoid of a space.

Definition 3.5.1. Let X be a topological space. The *fundamental groupoid* $\Pi_1(X)$ of X is a category (in fact, groupoid), whose objects are the points of X, and maps are homotopy classes of paths in X. The composition of compatible paths σ and ω is defined by:

$$(\sigma \cdot \omega)(t) = \begin{cases} \omega(2t) & 0 \leq t \leq 1/2 \\ \sigma(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

The results of the previous sections can be succinctly summarized in the following neat statement.

Proposition 3.5.2. *Any fibration $p : E \to B$ gives a functor $\Pi_1(B) \to$* **Top**.

This is the beginning of a beautiful story involving fibrations. (The interested reader should look up "Grothendieck construction".)

Cofibrations

Let $i : A \to X$ be a map of spaces. If Y is another topological space, when is the induced map $Y^X \to Y^A$ a fibration? This is asking for the map i to be "dual" to a fibration.

By the definition of a fibration, we want a lifting:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\mathrm{in}_0"'] & Y^X \arrow[d] \\ I \times W \arrow[r] \arrow[ur, dashed] & Y^A. \end{tikzcd}$$

Adjointing over, we get:

$$\begin{tikzcd} A \times W \arrow[r, "i\times 1"] \arrow[d, "1\times \mathrm{in}_0"'] & X \times W \arrow[d] \arrow[ddr] & \\ A \times W \times I \arrow[r] \arrow[drr] & X \times I \times W \arrow[dr, dashed] & \\ & & Y. \end{tikzcd}$$

Again adjointing over, this diagram transforms to:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\ A \times I \arrow[r] \arrow[drr] & X \times I \arrow[dr, dashed] & \\ & & Y^W. \end{tikzcd}$$

This discussion motivates the following definition of a "cofibration": as mentioned above, this is "dual" to the notion of fibration.

Definition 3.5.3. A map $i : A \to X$ of spaces is said to be a *cofibration* if it satisfies the *homotopy extension property* (sometimes abbreviated as "HEP"): for any space Y, there is a dotted map in the following diagram that makes it commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\ A \times I \arrow[r] \arrow[drr] & X \times I \arrow[dr, dashed] & \\ & & Y. \end{tikzcd}$$

Again, using the definition of a pushout, the universal example of such a space Y is the pushout $X \cup_A (A \times I)$. Equivalently, we are therefore asking for the existence of a dotted arrow in the following diagram.

$$\begin{tikzcd} X \cup_A (A \times I) \arrow[r] \arrow[dr] & X \times I \arrow[d, dashed] \\ & Z, \end{tikzcd}$$

for any Z. Using the universal property of a pushout, this is equivalent to the existence of a dotted arrow in the following diagram.

$$\begin{tikzcd}
X \cup_A (A \times I) \arrow[r] \arrow[dr] & X \times I \arrow[d, dashed] \\
& X \cup_A (A \times I) \arrow[d] \\
& Z,
\end{tikzcd}$$

which is, in turn, equivalent to asking $X \cup_A (A \times I)$ to be a retract of $X \times I$.

Example 3.5.4. $S^{n-1} \hookrightarrow D^n$ is a cofibration.

Properly draw out this figure!

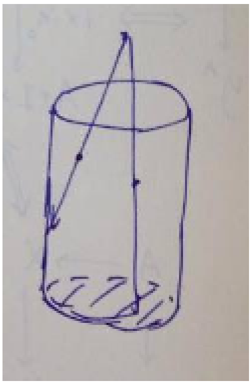

Figure 3.1: Drawing by John Ni.

In particular, setting $n = 1$ in this example, $\{0, 1\} \hookrightarrow I$ is a cofibration.

Here are some properties of the class of cofibrations of CGWH spaces.

- It's closed under cobase change: if $A \to X$ is a cofibration, and $A \to B$ is any map, the pushout $B \to X \cup_A B$ is also cofibration. (Exercise!)
- It's closed under finite products. (This is surprising.)
- It's closed under composition. (Exercise!)

- Any cofibration is a closed inclusion[10].

This is not obvious; should we include a proof?

3.6 Homotopy fibers

An important, but easy, fact about fibrations is that the canonical map $X \to *$ from any space X is a fibration [11]. This is because the dotted lift in the diagram below can be taken to the map $(t, w) \mapsto f(w)$:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ I \times W \arrow[r] \arrow[ur, dashed] & *. \end{tikzcd}$$

However:

Exercise 3.6.1. The inclusion $* \hookrightarrow X$ is not always a cofibration; if it is, say that $*$ is a *nondegenerate basepoint* of X. Give an example of a compactly generated space X for which this is true.

If $*$ has a neighborhood in X that contracts to $*$, the inclusion $* \hookrightarrow X$ is a cofibration. Note that if $*$ is a nondegenerate basepoint, the canonical map $X^A \xrightarrow{\text{ev}} X$ is a fibration, where A is a pointed subspace of X (with basepoint given by $*$). The fiber of ev is exactly the space of pointed maps $A \to X$.

Remark 3.6.2. In Example 3.5.4, we saw that $\{0, 1\} \hookrightarrow I$ is a cofibration; this implies that the map $Y^I \to Y \times Y$ (given by $\omega \mapsto (\omega(0), \omega(1))$) is a fibration.

"Fibrant replacements"

The purpose of this subsection is to provide a proof of the following result, which says that every map can be "replaced" (up to homotopy) by a fibration.

[10] Note that the dual statement for fibrations would state: any fibration $p : E \to B$ is a quotient map. This is definitely not true: fibrations do not have to be surjective! For instance, the trivial map $\emptyset \to B$ is a fibration. (Fibrations are surjective on path components though, because of path lifting.)

[11] Model category theorists get excited about this, because this says that all objects in the associated model structure on topological spaces is fibrant.

Theorem 3.6.3. *For any map $f : X \to Y$, there is a space $T(f)$, along with a fibration $p : T(f) \to Y$ and a homotopy equivalence $X \xrightarrow{\simeq} T(f)$, such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] \\ & Y. \end{tikzcd}$$

Proof. Consider the map $Y^I \xrightarrow{\binom{\mathrm{ev}_0}{\mathrm{ev}_1}} Y \times Y$. Let $T(f)$ be the pullback of the following diagram:

$$\begin{tikzcd} T(f) \arrow[r] \arrow[d] & Y^I \arrow[d, "\binom{\mathrm{ev}_0}{\mathrm{ev}_1}"] \\ X \times Y \arrow[r, "f\times 1"'] & Y \times Y. \end{tikzcd}$$

So, as a set, we can write

$$T(f) = \{(x, \omega) \in X \times Y^I | f(x) = \omega(0)\}.$$

Let us check that the canonical map $T(f) \to Y$, given by $(x, \omega) \mapsto \omega(1)$, is a fibration. The projection map $\mathrm{pr} : X \times Y \to Y$ is a fibration, so it suffices to show that the map $T(f) \to X \times Y$ is also a fibration. Since fibrations are closed under pullbacks, we are reduced to checking that the map $Y^I \to Y \times Y$ is a fibration; but this is exactly saying that the inclusion $\{0, 1\} \hookrightarrow I$ is a cofibration, which it is (Example 3.5.4).

To prove that X is homotopy equivalent to $T(f)$, we need to produce a map $X \to T(f)$. This is equivalent to giving maps $X \to X \times Y$ and $X \to Y^I$ that have compatible images in $Y \times Y$. The first map can be chosen to be $X \xrightarrow{\binom{1}{f}} X \times Y$. Define the map $X \to Y^I$ by sending $x \in X$ to the constant loop at $f(x)$. It is clear that both composites $X \to X \times Y \to Y \times Y$ and $X \to Y^I \to Y \times Y$ are the same; this defines a map $X \to T(f)$, denoted g. As one can easily check, the composite $X \to T(f) \xrightarrow{p} Y$ is the map $f : X \to Y$ that we started off with. It remains to check that this map $X \xrightarrow{g} T(f)$ is a homotopy equivalence. We will construct a homotopy inverse to this map.

The composite $X \to T(f) \to X \times Y \to X$ is the identity, so one candidate for a homotopy inverse to g is the composite

$$T(f) \to X \times Y \xrightarrow{pr_1} X.$$

To prove that this map is indeed a homotopy inverse to g, we need to consider the composite $T(f) \to X \xrightarrow{g} T(f)$, which sends $(x, \omega) \mapsto x \mapsto (x, c_{f(x)})$ where, recall, $c_{f(x)}$ is the constant path at x. We need to produce a homotopy between this composite and the identity on $T(f)$.

Let $s \in I$. Given $\omega \in Y^I$, define a new loop ω_s by $\omega_s(t) = \omega(st)$. For instance, $\omega_1 = \omega$, and $\omega_0 = c_{\omega(0)}$ — so, the loop ω_s "sucks in" the point $\omega(1)$. This is precisely what we need to produce a homotopy between the composite $T(f) \to X \xrightarrow{g} T(f)$ and $\mathrm{id}_{T(f)}$, since the only constraint on $(x, \omega) \in T(f)$ is on $\omega(0)$. The following map provides the desired homotopy equivalence $X \simeq T(f)$.

$$\begin{aligned} H : I \times T(f) &\to T(f) \\ (s, (x, \omega)) &\mapsto (x, \omega_s). \end{aligned}$$

□

Example 3.6.4 (Path-loop fibration)**.** This is a silly, but important, example. If $X = *$, the space $T(f)$ consists of paths ω in Y such that $\omega(0) = *$. In other words, $T(f) = Y^I_*$; this is called the *(based) path space* of Y, and is denoted by $P(Y, *)$, or simply by PY. The fiber of the fibration $T(f) = PY \to Y$ consists of paths that begin and end at $*$, i.e., loops on Y based at $*$. This is denoted ΩY, and is called the *(based) loop space* of Y. The resulting fibration $PY \to Y$ is called the *path-loop fibration.*

Exercise 3.6.5 ("Cofibrant replacements")**.** In this exercise, you will prove the analogue of Theorem 3.6.3 for cofibrations. Let $f : X \to Y$ be any map. Show that f factors (functorially) as a composite $X \to M \to Y$, where $X \to M$ is a cofibration and $M \to Y$ is a homotopy equivalence.

Solution 3.6.6. Define Mf via the pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & Y \arrow[d, "g"] \\ I \times X \arrow[r] & Mf. \end{tikzcd}$$

Define $r : Mf \to Y$ via $r(y) = y$ on Y and $r(x, s) = f(x)$ on $X \times I$. Then, clearly, $rg = \mathrm{id}_Y$. There is a homotopy $\mathrm{id}_{Mf} \simeq gr$ given by the map $h : Mf \times I \to Mf$, defined by the formulae

$$h(y, t) = y, \text{ and } h((x, s), t) = (x, (1 - t)s).$$

We now have to check that $X \to Mf$ is a cofibration, i.e., that $Mf \times I$ retracts onto $Mf \times \{0\} \cup_X (X \times I)$. This can be done by "pushing" $Y \times I$ to $Y \times \{0\}$ and $X \times I \times I$ down to $X \times I$, while fixing $X \times \{0\}$.

It is easy to see that this factorization is functorial: if $f : X \to Y$ is sent to $g : W \to Z$ via $p : X \to W$ and $q : Y \to Z$, then $Mf \to Mg$ can be defined as the dotted map in the following diagram (which exists, by the universal property of the pushout):

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"] \arrow[dd, "\mathrm{in}_0"'] & & Y \arrow[dr, "q"] \arrow[dd] & \\
& W \arrow[rr, "g"] \arrow[dd] & & Z \arrow[dd] \\
X \times I \arrow[rr] \arrow[dr, "p \times \mathrm{id}"'] & & Mf \arrow[dr, dashed] & \\
& W \times I \arrow[rr] & & Mg.
\end{tikzcd}$$

Fix the over-lapping arrows here, I don't know how to do this...

Homotopy fibers

One way to define the fiber (over a basepoint) of a map $f : X \to Y$ is via the pullback

$$\begin{tikzcd}
f^{-1}(*) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
* \arrow[r] & Y.
\end{tikzcd}$$

If $g : W \to X$ is another map such that the composite $W \xrightarrow{g} X \xrightarrow{f} Y$ is trivial, the map g factors through $f^{-1}(*)$. In homotopy theory, maps are generally not trivial "on the nose"; instead, we usually have a nullhomotopy of a map. Nullhomotopies of composite maps do not factor through this "strict" fiber; this leads to the notion of a homotopy fiber.

Definition 3.6.7 (Homotopy fiber)**.** The *homotopy fiber* of a map f :

$X \to Y$ is the pullback:

$$\begin{tikzcd} F(f,*) \arrow[r] \arrow[d] & T(f) \arrow[r, "\simeq"] \arrow[d, "p"] & X \arrow[dl, "f"] \\ * \arrow[r] & Y & \end{tikzcd}$$

As a set, we have

$$F(f,*) = \{(x,\omega) \in X \times Y^I | f(x) = \omega(0), \omega(1) = *\}. \tag{3.3}$$

A nullhomotopic composite $W \to X \xrightarrow{f} Y$ factors as $W \to F(f,*) \to X \xrightarrow{f} Y$.

Warning 3.6.8. The ordinary fiber and the homotopy fiber of a map are generally not the same! There is a canonical map $p^{-1}(*) \to F(p,*)$, but it is generally not a homotopy equivalence.

Proposition 3.6.9. *Suppose $p : X \to Y$ is a fibration. Then the canonical map $p^{-1}(*) \to F(p,*)$ is a homotopy equivalence.*

You will prove this in a series of exercises.

Exercise 3.6.10. Prove Proposition 3.6.9 by working through the following statements.

1. Let $p : E \to B$ be a fibration. Suppose $g : X \to B$ lifts across p up to homotopy, i.e., there exists a map $f : X \to E$ such that $p \circ f \simeq g$. Prove that there exists a map $f' : X \to E$ that is homotopic to f, such that $p \circ f' = g$ (on the nose).

2. Show that if $p : E \to B$ and $p' : E' \to B$ are fibrations, and $f : E \to E'$ such that $p' \circ f = p$, the map f is a *fiber homotopy equivalence*: there is a homotopy inverse $g : E' \to E$ such that g, and the two homotopies $fg \simeq \mathrm{id}_{E'}$ and $gf \simeq \mathrm{id}_E$ are all fiber preserving (e.g., $p \circ g = p'$).

3. Conclude Proposition 3.6.9.

Before we proceed, recall that we constructed the homotopy fiber by replacing $f : X \to Y$ by a fibration. In doing so, we implicitly made a choice: we could have replaced the map $* \to Y$ by a fibration. Are the resulting pullbacks the same?

By replacing $* \to Y$ by a fibration (namely, the path-loop fibration), we end up with the following pullback diagram:

$$\begin{tikzcd}
F'(f,*) \arrow[r] \arrow[d] & P(Y,*) \arrow[r, "\simeq"] \arrow[d] & * \arrow[dl] \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

As a set, we have

$$F'(f,*) = \{(x,\omega) \in X \times Y^I \text{ such that } \omega(0) = * \text{ and } \omega(1) = f(x)\}.$$

Our description of $F(f,*)$ in (3.3) is almost exactly the same — except that the directions of the paths are reversed. Thus there's a homeomorphism $F'(f,*) \simeq F(f,*)$ given by reversing directions of paths.

Remark 3.6.11. One could also replace both $f : X \to Y$ and $* \to Y$ by fibrations, and the resulting pullback is also homeomorphic to $F(f,*)$. (Prove this, if the statement is not immediate.)

3.7 Barratt-Puppe sequence

Fiber sequences

Recall, from the previous section, that we have a pullback diagram:

$$\begin{tikzcd}
 & F(f,*) \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & PY \arrow[d, "p"] \arrow[dr, "\simeq"] & \\
f^{-1}(*) \arrow[r] \arrow[ur] & X \arrow[r, "f"'] & Y & * \arrow[l]
\end{tikzcd}$$

Consider a pointed map[12] $f : X \to Y$ (so that $f(*) = *$). Then, we will write Ff for the homotopy fiber $F(f,*)$.

Since we're exploring the homotopy fiber Ff, we can ask the following, seemingly silly, question: what is the fiber of the canonical map $p : Ff \to X$ (over the basepoint of X)? This is precisely the space of loops in Y! Since p is a fibration (recall that fibrations are closed

[12]Some people call such a map "based", but this makes it sound like we're doing chemistry, so we won't use it.

under pullbacks), the homotopy fiber of p is also the "strict" fiber! Our expanded diagram is now:

$$
\begin{tikzcd}
& \Omega Y = p^{-1}(*) \arrow[d] & & \\
& F(f,*) \arrow[r] \arrow[d, "p"] & PY \arrow[d, "p"] \arrow[dr, "\simeq"] & \\
f^{-1}(*) \arrow[r] \arrow[ur] & X \arrow[r, "f"'] & Y & * . \arrow[l]
\end{tikzcd}
$$

It's easy to see that the composite $Ff \xrightarrow{p} X \xrightarrow{f} Y$ sends $(x, \omega) \mapsto f(x)$; this is a pointed nonconstant map. (Note that the basepoint we're choosing for Ff is the image of the basepoint in $f^{-1}(*)$ under the canonical map $f^{-1}(*) \hookrightarrow Ff$.)

While the composite $fp : Ff \to Y$ is not zero "on the nose", it is nullhomotopic, for instance via the homotopy $h : Ff \times I \to Y$, defined by

$$h(t, (x, \omega)) = \omega(t).$$

Exercise 3.7.1. Let $f : X \to Y$ and $g : W \to X$ be pointed maps. Establish a homeomorphism between the space of pointed maps $W \xrightarrow{p} Ff$ such that $fp = g$ and the space of pointed nullhomotopies of the composite fg.

This exercise proves that the homotopy fiber is the "kernel" in the homotopy category of pointed spaces and pointed maps between them.

Define $[W, X]_* = \pi_0(X_*^W)$; this consists of the pointed homotopy classes of maps $W \to X$. We may view this as a pointed set, whose basepoint is the constant map. Fixing W, this is a contravariant functor in X, so there are maps $[W, Ff]_* \to [W, X]_* \to [W, Y]_*$. This composite is not just nullhomotopic: it is "exact"! Since we are working with pointed sets, we need to describe what exactness means in this context: the preimage of the basepoint in $[W, Y]_*$ is exactly the image of $[W, Ff]_* \to [W, X]_*$. (This is exactly a reformulation of Exercise 3.7.1.) We say that $Ff \to X \xrightarrow{f} Y$ is a *fiber sequence*.

Remark 3.7.2. Let $f : X \to Y$ be a map of spaces, and suppose we have a homotopy commutative diagram:

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, "\Omega g"'] & Ff \arrow[r] \arrow[d, dashed] & X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\
\Omega Y' \arrow[r] & Ff' \arrow[r] & X' \arrow[r, "f'"'] & Y.
\end{tikzcd}$$

Then the dotted map exists, but it depends on the homotopy $f'h \simeq gf$.

Iterating fiber sequences

Let $f : X \to Y$ be a pointed map, as before. As observed above, we have a composite map $Ff \xrightarrow{p} X \xrightarrow{f} Y$, and the strict fiber (homotopy equivalent to the homotopy fiber) of p is ΩY. This begets a map $i(f) : \Omega Y \to Ff$; iterating the procedure of taking fibers gives:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_3 \arrow[r, "p_4"] & Fp_2 \arrow[r, "p_3"] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
& \Omega Fp_0 \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[ur, "i(p_2)"] & \Omega X \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[ur, "i(p_1)"] & \Omega Y \arrow[u, "\simeq"] \arrow[ur, "i(f)"] & & &
\end{tikzcd}$$

All the p_i in the above diagram are fibrations. Each of the dotted maps in the above diagram can be filled in up to homotopy. The most obvious guess for what these dotted maps are is simply $\Omega X \xrightarrow{\Omega f} \Omega Y$. But *that is the wrong map*!

The right map turns out to be $\Omega X \xrightarrow{\overline{\Omega f}} \Omega Y$:

Lemma 3.7.3. *The following diagram commutes to homotopy:*

$$\begin{tikzcd}
& Fp \\
\Omega X \arrow[ur, "i(p)"] \arrow[r, "\overline{\Omega f}"'] & \Omega Y; \arrow[u]
\end{tikzcd}$$

here, $\overline{\Omega f}$ is the diagonal in the following diagram:

$$\begin{tikzcd} \Omega X \arrow[r, "-"] \arrow[d, "\Omega f"'] \arrow[dr, "\Omega f"] & \Omega X \arrow[d, "\Omega f"] \\ \Omega Y \arrow[r, "-"'] & \Omega Y, \end{tikzcd}$$

where the map $- : \Omega X \to \Omega X$ sends $\omega \mapsto \overline{\omega}$.

typeset the following image for the proof... it's going to be impossible to write this up in symbols.

Proof.

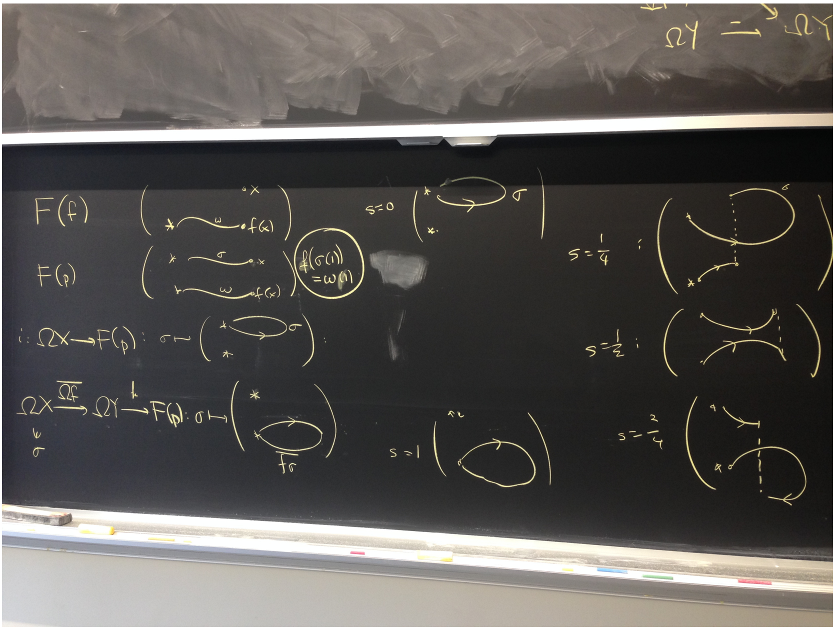

Figure 3.2: A proof of this lemma.

□

By the above lemma, we can extend our diagram to:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_4 \arrow[r] & Fp_3 \arrow[r] & Fp_2 \arrow[r] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
\cdots \arrow[r] & \Omega Fp_1 \arrow[u, "\simeq"'] \arrow[r, "\overline{\Omega p_2}"] & \Omega Ff \arrow[u, "\simeq"'] \arrow[ur, "i(p_2)"'] \arrow[r, "\overline{\Omega p}"] & \Omega X \arrow[u, "\simeq"'] \arrow[ur, "i(p_1)"'] \arrow[r, "\overline{\Omega f}"] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "i(f)"'] \\
\Omega^2 X \arrow[u, "\simeq"'] \arrow[r, "\Omega^2 f"'] & \Omega^2 Y \arrow[u, "\simeq"'] \arrow[ur, "\overline{\Omega i(p_0)}"']
\end{tikzcd}$$

We have a special name for the sequence of spaces sneaking along the bottom of this diagram:

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to \Omega Ff \to \Omega X \to \Omega Y \to Ff \to X \xrightarrow{f} Y;$$

this is called the *Barratt-Puppe sequence*. Applying $[W, -]_*$ to the Barratt-Puppe sequence of a map $f : X \to Y$ gives a long exact sequence.

The most important case of this long exact sequence comes from setting $W = S^0 = \{\pm 1\}$; in this case, we get terms like $\pi_0(\Omega^n X)$. We can identify $\pi_0(\Omega^n X)$ with $[S^n, X]_*$: to see this for $n = 2$, recall that $\Omega^2 X = (\Omega X)^{S^1}$; because $(S^1)^{\wedge n} = S^n$ (see below for a proof of this fact), we find that

$$(\Omega X)^{S^1} \simeq (X_*^{S^1})_*^{S^1} = X_*^{S^1 \wedge S^1} = X_*^{S^2}, \tag{3.4}$$

as desired.

The space ΩX is a group in the homotopy category; this implies that $\pi_0 \Omega X = \pi_1 X$ is a group! For $n > 1$, we know that

$$\pi_n(X) = [(D^n, S^{n-1}), (X, *)] = [(I^n, \partial I^n), (X, *)].$$

Exercise 3.7.4. Prove that $\pi_n(X)$ is an abelian group for $n > 2$.

Applying π_0 to the Barratt-Puppe sequence (see Equation 3.4) therefore gives a long exact sequence (of groups when the homotopy groups are in degrees greater than 0, and of pointed sets in degree 0):

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 Ff \to \pi_1 X \to \pi_1 Y \to \pi_0 Ff \to \pi_0 X \to \pi_0 X.$$

3.8 Relative homotopy groups

Spheres and homotopy groups

The functor Ω (sending a space to its based loop space) admits a left adjoint. To see this, recall that $\Omega X = X_*^{S^1}$, so that

$$\mathbf{Top}_*(W, \Omega X) = \mathbf{Top}_*(S^1 \wedge W, X).$$

Definition 3.8.1. The *reduced suspension* ΣW is $S^1 \wedge W$.

If $A \subseteq X$, then

$$X/A \wedge Y/B = (X \times Y)/((A \times Y) \cup_{A \times B} (X \times B)).$$

Since $S^1 = I/\partial I$, this tells us that $\Sigma X = S^1 \wedge X$ can be identified with $I \times X/(\partial I \times X \cup I \times *)$: in other words, we collapse the top and bottom of a cylinder to a point, as well as the line along a basepoint.

The same argument says that $\Sigma^n X$ (defined inductively as $\Sigma(\Sigma^{n-1} X)$) is the left adjoint of the n-fold loop space functor $X \mapsto \Omega^n X$. In other words, $\Sigma^n X = (S^1)^{\wedge n} \wedge X$. We claim that $S^1 \wedge S^n \simeq S^{n+1}$. To see this, note that

$$S^1 \wedge S^n = I/\partial I \wedge I^n \wedge \partial I^n = (I \times I^n)/(\partial I \times I^n \cup I \times \partial I^n).$$

The denominator is exactly ∂I^{n+1}, so $S^1 \wedge S^n \simeq S^{n+k}$. It's now easy to see that $S^k \wedge S^n \simeq S^{k+n}$.

Definition 3.8.2. The *nth homotopy group* of X is $\pi_n X = \pi_0(\Omega^n X)$.

This is, as we noted in the previous section, $[S^0, \Omega^n X]_* = [S^n, X]_* = [(I^n, \partial I^n), (X, *)]$.

The homotopy category

Define the *homotopy category of spaces* $\mathrm{Ho}(\mathbf{Top})$ to be the category whose objects are spaces, and whose hom-sets are given by taking π_0 of the mapping space. To check that this is indeed a category, we need to check that if $f_0, f_1 : X \to Y$ and $g : Y \to Z$, then $gf_0 \simeq gf_1$ — but this is clear. Similarly, we'd need to check that $f_0 h \simeq f_1 h$ for any $h : W \to X$. We can also think about the homotopy category of pointed spaces (and pointed homotopies) $\mathrm{Ho}(\mathbf{Top}_*)$; this is the category we have been spending most of our time in. Both $\mathrm{Ho}(\mathbf{Top})$ and $\mathrm{Ho}(\mathbf{Top}_*)$ have products and coproducts, but very few other limits or colimits. From a category-theoretic standpoint, these are absolutely terrible.

Let W be a pointed space. We would like the assignment $X \mapsto X_*^W$ to be a homotopy functor. It clearly defines a functor $\mathbf{Top}_* \to \mathbf{Top}_*$, so this desire is equivalent to providing a dotted arrow in the following diagram:

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, "X \mapsto X_*^W"] \arrow[d] & \mathbf{Top}_* \arrow[d] \\ \mathrm{Ho}(\mathbf{Top}_*) \arrow[r, dashed] & \mathrm{Ho}(\mathbf{Top}_*). \end{tikzcd}$$

Before we can prove this, we will check that a homotopy $f_0 \sim f_1 : X \to Y$ is the same as a map $I_+ \wedge X \to Y$. There is a nullhomotopy if the basepoint of I is one of the endpoints, so a homotopy is the same as a map $I \times X/I \times * \to Y$. The source is just $I_+ \wedge X$, as desired.

A homotopy $f_0 \simeq f_1 : X \to Y$ begets a map $(I_+ \wedge X)^W \to Y_*^W$. For the assignment $X \mapsto X_*^W$ to be a homotopy functor, we need a natural transformation $I_+ \wedge X_*^W \to Y_*^W$, so this map is not quite what's necessary. Instead, we can attempt to construct a map $I_+ \wedge X_*^W \to (I_+ \wedge X)_*^W$.

We can construct a general map $A \wedge X_*^W \to (A \wedge X)_*^W$: there is a map $A \wedge X_*^W \to A_*^W \wedge X_*^W$, given by sending $a \mapsto c_a$; then the exponential law gives a homotopy $A_*^W \wedge X_*^W \to (A \wedge X)_*^W$. This, in turn, gives a map $I_+ \wedge X_*^W \to (I_+ \wedge X)_*^W \to Y_*^W$, thus making $X \mapsto X_*^W$ a homotopy functor.

Motivated by our discussion of homotopy fibers, we can study composites which "behave" like short exact sequences.

Definition 3.8.3. A *fiber sequence* in $\mathrm{Ho}(\mathbf{Top}_*)$ is a composite $X \to Y \to Z$ that is isomorphic, in $\mathrm{Ho}(\mathbf{Top}_*)$, to some composite $Ff \xrightarrow{p} E \xrightarrow{f} B$; in other words, there exist (possibly zig-zags of) maps that are homotopy equivalences, that make the following diagram commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Ff \arrow[r, "p"'] & E \arrow[r, "f"'] & B. \end{tikzcd}$$

Let us remark here that if $A' \xrightarrow{\sim} A$ is a homotopy equivalence, and $A \to B \to C$ is a fiber sequence, so is the composite $A' \xrightarrow{\sim} A \to B \to C$.

Exercise 3.8.4. Prove the following statements.

- Ω takes fiber sequences to fiber sequences.
- $\Omega F f \simeq F \Omega f$. Check this!

We've seen examples of fiber sequences in our elaborate study of the Barratt-Puppe sequence.

Example 3.8.5. Recall our diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_4 \arrow[r] & Fp_3 \arrow[r] & Fp_2 \arrow[r] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
\cdots \arrow[r] & \Omega Fp_1 \arrow[r, "\overline{\Omega p_2}"] \arrow[u, "\simeq"] & \Omega Ff \arrow[r, "\overline{\Omega p}"] \arrow[u, "\simeq"] \arrow[ur, "i(p_2)"] & \Omega X \arrow[r, "\overline{\Omega f}"] \arrow[u, "\simeq"] \arrow[ur, "i(p_1)"] & \Omega Y \arrow[u, "\simeq"] \arrow[ur, "i(f)"] & & & \\
\Omega^2 X \arrow[r, "\Omega f"'] \arrow[u, "\simeq"] & \Omega Y \arrow[u, "\simeq"] \arrow[ur, "\overline{\Omega i(f)}"'] & & & & & &
\end{tikzcd}$$

The composite $Ff \to X \xrightarrow{f} Y$ is canonically a fiber sequence. The above diagram shows that $\Omega Y \to F \xrightarrow{p} X$ is another fiber sequence: it is isomorphic to $Fp \to F \to X$ in $\mathrm{Ho}(\mathbf{Top}_*)$. Similarly, the composite $\Omega X \xrightarrow{\overline{\Omega f}} \Omega Y \to F$ is another fiber sequence; this implies that $\Omega X \xrightarrow{\Omega f} \Omega Y \to F$ is also an example of a fiber sequence (because these two fiber sequences differ by an automorphism of ΩX)

Applying Ω again, we get $\Omega F \xrightarrow{\Omega p} \Omega X \xrightarrow{\Omega f} \Omega Y$. Since this is a looping of a fiber sequence, and taking loops takes fiber sequences to fiber sequences (Exercise 3.8.4), this is another fiber sequence. Looping again gives another fiber sequence $\Omega^2 Y \xrightarrow{\Omega i} \Omega F \xrightarrow{\Omega p} \Omega X$. (For the category-theoretically–minded folks, this is an unstable version of a triangulated category.)

The long exact sequence of a fiber sequence

As discussed at the end of §3.7, applying $\pi_0 = [S^0, -]_*$ to the Barratt-Puppe sequence associated to a map $f : X \to Y$ gives a long exact sequence:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \pi_2 Y \arrow[dll] \\
 & \pi_1 F \arrow[r] & \pi_1 X \arrow[r] & \pi_1 Y \arrow[dll] \\
 & \pi_0 F \arrow[r] & \pi_0 X. &
\end{tikzcd}$$

of pointed sets. The space $\Omega^2 X$ is an *abelian* group object in Ho(**Top**) (in other words, the multiplication on $\Omega^2 X$ is commutative up to homotopy). This implies $\pi_1(X)$ is a group, and that $\pi_k(X)$ is abelian for $k \geq 2$; hence, in our diagram above, all maps (except on π_0) are group homomorphisms.

Consider the case when $X \to Y$ is the inclusion $i : A \hookrightarrow X$ of a subspace. In this case,

$$Fi = \{(a, \omega) \in A \times X_*^I | \omega(1) = a\};$$

this is just the collection of all paths that begin at $* \in A$ and end in A. This motivates the definition of *relative homotopy groups*:

Definition 3.8.6. Define:

$$\pi_n(X, A, *) = \pi_n(X, A) := \pi_{n-1} Fi = [(I^n, \partial I^n, (\partial I^n \times I) \cup (I^{n-1} \times 0)), (X, A, *)].$$

We have a sequence of inclusions

$$\partial I^n \times I \cup I^{n-1} \times 0 \subset \partial I^n \subset I^n.$$

One can check that

$$\pi_{n-1} Fi = [(I^n, \partial I^n, (\partial I^n \times I) \cup (I^{n-1} \times 0)), (X, A, *)].$$

This gives a long exact sequence on homotopy, analogous to the long exact sequence in relative homology:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \pi_2(X, A) \arrow[dll] \\
 & \pi_1 A \arrow[r] & \pi_1 X \arrow[r] & \pi_1(X, A) \arrow[dll] \\
 & \pi_0 A \arrow[r] & \pi_0 X &
\end{tikzcd} \tag{3.5}$$

3.9 Action of π_1, simple spaces, and the Hurewicz theorem

In the previous section, we constructed a long exact sequence of homotopy groups:

$$\begin{tikzcd} & & \cdots \arrow[r] & \pi_2(X,A) \arrow[dll] \\ & \pi_1 A \arrow[r] & \pi_1 X \arrow[r] & \pi_1(X,A) \arrow[dll] \\ & \pi_0 A \arrow[r] & \pi_0 X, & \end{tikzcd}$$

which looks suspiciously similar to the long exact sequence in homology. The goal of this section is to describe a relationship between homotopy groups and homology groups.

Before we proceed, we will need the following lemma.

Lemma 3.9.1 (Excision)**.** *If $A \subseteq X$ is a cofibration, there is an isomorphism*

$$H_*(X,A) \xrightarrow{\simeq} \widetilde{H}_*(X/A).$$

Under this hypothesis,

$$X/A \simeq \text{Mapping cone of } i : A \to X;$$

here, the mapping cone is the homotopy pushout in the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "in_1"] & X \arrow[d] \\ CA \arrow[r] & X \cup_A CA, \end{tikzcd}$$

where CA is the cone on A, defined by

$$CA = A \times I / A \times 0.$$

This lemma is dual to the statement that the homotopy fiber is homotopy equivalent to the strict fiber for fibrations.

Unfortunately, $\pi_*(X,A)$ is definitely not $\pi_*(X/A)$! For instance, there is a cofibration sequence

$$S^1 \to D^2 \to S^2.$$

We know that $\pi_* S^1$ is just $\mathbf{Z}$ in dimension 1, and is zero in other dimensions. On the other hand, we do not, and probably will never, know the homotopy groups of S^2. (A theorem of Edgar Brown in [Bro57] says that these groups are computable, but this is super-exponential.)

π_1-action

There is more structure in the long exact sequence in homotopy groups that we constructed last time, coming from an action of $\pi_1(X)$. There is an action of $\pi_1(X)$ on $\pi_n(X)$: if x, y are points in X, and $\omega : I \to X$ is a path with $\omega(0) = x$ and $\omega(1) = y$, we have a map $f_\omega : \pi_n(X, x) \to \pi_n(X, y)$; this, in particular, implies that $\pi_1(X, *)$ acts on $\pi_n(X, *)$. When $n = 1$, the action $\pi_1(X)$ on itself is by conjugation.

In fact, one can also see that $\pi_1(A)$ acts on $\pi_n(X, A, *)$. It follows (by construction) that all maps in the long exact sequence of Equation (3.5) are equivariant for this action of $\pi_1(A)$. Moreover:

Proposition 3.9.2 (Peiffer identity)**.** *Let $\alpha, \beta \in \pi_2(X, A)$. Then $(\partial\alpha) \cdot \beta = \alpha\beta\alpha^{-1}$.*

Definition 3.9.3. A topological space X is said to be *simply connected* if it is path connected, and $\pi_1(X, *) = 1$.

Let $p : E \to B$ be a covering space with E and B connected. Then, the fibers are discrete, hence do not have any higher homotopy. Using the long exact sequence in homotopy groups, we learn that $\pi_n(E) \to \pi_n(B)$ is an isomorphism for $n > 1$, and that $\pi_1(E)$ is a subgroup of $\pi_1(B)$ that classifies the covering space. In general, we know from Exercise 3.9.7 that ΩB acts on the homotopy fiber Fp. Since Ff is discrete, this action factors through $\pi_0(\Omega B) \simeq \pi_1(B)$.

Definition 3.9.4. A space X is said to be *n-connected* if $\pi_i(X) = 0$ for $i \leq n$.

Note that this is a well-defined condition, although we did not specify the basepoint: 0-connected implies path connected. Suppose $E \to B$ is a covering space, with the total space E being n-connected. Then, Hopf showed that the group $\pi_1(B)$ determines the homology $H_i(B)$ in dimensions $i < n$.

Sometimes, there are interesting spaces which are not simply connected, for which the π_1-action is nontrivial.

Example 3.9.5. Consider the space $S^1 \vee S^2$. The universal cover is just $\mathbf{R}$, with a 2-sphere S^2 stuck on at every integer point. This space is simply connected, so the Hurewicz theorem says that $\pi_2(E) \simeq H_2(E)$. Since the real line is contractible, we can collapse it to a point: this gives a countable bouquet of 2-spheres. As a consequence, $\pi_2(E) \simeq H_2(E) = \bigoplus_{i=0}^{\infty} \mathbf{Z}$.

There is an action of $\pi_1(S^1 \vee S^2)$ on E: the action does is shift the 2-spheres on the integer points of $\mathbf{R}$ (on E) to the right by 1 (note that $\pi_1(S^1 \vee S^2) = \mathbf{Z}$). This tells us that $\pi_2(E) \simeq \mathbf{Z}[\pi_1(B)]$ as a $\mathbf{Z}[\pi_1(B)]$-module; this is the same action of $\pi_1(E)$ on $\pi_2(E)$. We should be horrified: $S^1 \vee S^2$ is a very simple 3-complex, but its homotopy is huge!

Simply-connectedness can sometimes be a restrictive condition; instead, to simplify the long exact sequence, we define:

Definition 3.9.6. A topological space X is said to be *simple* if it is path-connected, and $\pi_1(X)$ acts trivially on $\pi_n(X)$ for $n \geq 1$.

Note, in particular, that $\pi_1(X)$ is abelian for a simple space.

Being simple is independent of the choice of basepoint. If $\omega : x \mapsto x'$ is a path in X, then $\omega_\sharp : \pi_n(X, x) \to \pi_n(X, x')$ is a group isomorphism. There is a (trivial) action of $\pi_1(X, x)$ on $\pi_n(X, x)$, and another (potentially nontrivial) action of $\pi_1(X, x')$ on $\pi_n(X, x')$. Both actions are compatible: hence, if $\pi_1(X, x)$ acts trivially, so does $\pi_1(X, x')$.

If X is path-connected, there is a map $\pi_n(X, *) \to [S^n, X]$. It is clear that this map is surjective, so one might expect a factorization:

$$\begin{tikzcd}
\pi_n(X,*) \arrow[rr, two heads] \arrow[dr] & & {[S^n, X]} \\
& \pi_1(X,*)\backslash\pi_n(X,*) \arrow[ur] &
\end{tikzcd}$$

Exercise 3.9.7. Prove that $\pi_1(X, *)\backslash\pi_n(X, *) \simeq [S^n, X]$. To do this, work through the following exercises.

Let $f : X \to Y$ be a map of spaces, and let $* \in Y$ be a fixed basepoint of Y. Denote by Ff the homotopy fiber of f; this admits a natural fibration $p : Ff \to X$, given by $(x, \sigma) \mapsto x$. If ΩY denotes the (based) loop space of Y, we get an action $\Omega Y \times Ff \to Ff$, given by

$$(\omega, (x, \sigma)) \mapsto (x, \sigma \cdot \omega),$$

where $\sigma \cdot \omega$ is the concatenation of σ and ω, defined, as usual, by

$$\sigma \cdot \omega(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \sigma(2t-1) & 1/2 \le t \le 1. \end{cases}$$

(Note that when X is the point, this defines a "multiplication" $\Omega Y \times \Omega Y \to \Omega Y$; this is associative and unital up to homotopy.) On connected components, we therefore get an action of $\pi_0 \Omega Y \simeq \pi_1 Y$ on $\pi_0 Ff$.

There is a canonical map

$$Ff \times \Omega Y \to Ff \times_X Ff,$$

given by $((x, \sigma), \omega) \mapsto ((x, \sigma), (x, \sigma) \cdot \omega)$. Prove that this map is a homotopy equivalence (so that the action of ΩY on Ff is "free"), and conclude that two elements in $\pi_0 Ff$ map to the same element of $\pi_0 X$ if and only if they are in the same orbit under the action of $\pi_1 Y$.

Let X be path connected, with basepoint $* \in X$. Conclude that $\pi_1(X, *)\backslash\pi_n(X, *) \simeq [S^n, X]$ by proving that the surjection $\pi_n(X, *) \to [S^n, X]$ can be identified with the orbit projection for the action of $\pi_1(X, *)$ on $\pi_n(X, *)$.

If X is simple, then the quotient $\pi_1(X, *)\backslash\pi_n(X, *)$ is simply $\pi_n(X, *)$, so Exercise 3.9.7 implies that $\pi_n(X, *) \cong [S^n, X]$ — independently of the basepoint; in other words, these groups are canonically the same, i.e., two paths $\omega, \omega' : x \to y$ give the same map $\omega_\sharp = \omega'_\sharp : \pi_n(X, x) \to \pi_n(X, y)$.

Exercise 3.9.8. A *H-space* is a pointed space X, along with a pointed map $\mu : X \times X \to X$, such that the maps $x \mapsto \mu(x, *)$ and $x \mapsto \mu(*, x)$ are both pointed homotopic to the identity. In this exercise, you will prove that path connected H-spaces are simple.

Denote by $\mathcal{C}$ the category of pairs (G, H), where G is a group that acts on the group H (on the left); the morphisms in $\mathcal{C}$ are pairs of homomorphisms which are compatible with the group actions. This category has finite products. Explain what it means for an object of $\mathcal{C}$ to have a "unital multiplication", and prove that any object (G, H) of $\mathcal{C}$ with a unital multiplication has G and H abelian, and that the G-action on H is trivial. Conclude from this that path connected H-spaces are simple.

Hurewicz theorem

Definition 3.9.9. Let X be a path-connected space. The Hurewicz map $h : \pi_n(X, *) \to H_n(X)$is defined as follows: an element in $\pi_n(X, *)$

is represented by $\alpha : S^n \to X$; pick a generator $\sigma \in H_n(S^n)$, and send

$$\alpha \mapsto \alpha_*(\sigma) \in H_n(X).$$

We will see below that h is in fact a homomorphism.

This is easy in dimension 0: a point is a 0-cycle! In fact, we have an isomorphism $H_0(X) \simeq \mathbf{Z}[\pi_0(X)]$. (This isomorphism is an example of the Hurewicz theorem.)

When $n = 1$, we have $h : \pi_1(X, *) \to H_1(X)$. Since $H_1(X)$ is abelian, this factors as $\pi_1(X, *) \to \pi_1(X, *)^{ab} \to H_1(X)$. The Hurewicz theorem says that the map $\pi_1(X, *)^{ab} \to H_1(X)$ is an isomorphism. We will not prove this here; see [Hat15, Theorem 2A.1] for a proof.

The Hurewicz theorem generalizes these results to higher dimensions:

Theorem 3.9.10 (Hurewicz)**.** *Suppose X is a space for which $\pi_i(X) = 0$ for $i < n$, where $n \geq 2$. Then the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism.*

Before the word "isomorphism" can make sense, we need to prove that h is a homomorphism. Let $\alpha, \beta : S^n \to X$ be pointed maps. The product $\alpha\beta \in \pi_n(X)$ is the composite:

$$\alpha\beta : S^n \xrightarrow{\delta,\text{ pinching along the equator}} S^n \vee S^n \xrightarrow{\beta\vee\alpha} X \vee X \xrightarrow{\nabla} X,$$

where $\nabla : X \vee X \to X$ is the fold map, defined by:

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "1"] & \\ X \vee X \arrow[r, "\nabla"] & X \\ X \arrow[u] \arrow[ur, "1"'] & \end{tikzcd}$$

To show that h is a homomorphism, it suffices to prove that for two maps $\alpha, \beta : S^n \to X$, the induced maps on homology satisfy $(\alpha+\beta)_* = \alpha_* + \beta_*$ — then,

$$h(\alpha + \beta) = (\alpha + \beta)_*(\sigma) = \alpha_*(\sigma) + \beta_*(\sigma) = h(\alpha) + h(\beta).$$

To prove this, we will use the pinch map $\delta : S^n \to S^n \vee S^n$, and the quotient maps $q_1, q_2 : S^n \vee S^n \to S^n$; the induced map $H_n(S^n) \to$

$H_n(S^n) \oplus H_n(S^n)$ is given by the diagonal map $a \mapsto (a, a)$. It follows from the equalities

$$(f \vee g)\iota_1 = f, \ (f \vee g)\iota_2 = g,$$

where $\iota_1, \iota_2 : S^n \hookrightarrow S^n \vee S^n$ are the inclusions of the two wedge summands, that the map $(f \vee g)_*((\iota_1)_* + (\iota_2)_*)$ sends $(x, 0)$ to $f_*(x)$, and $(0, x)$ to $g_*(x)$. In particular,

$$(x, x) \mapsto f_*(x) + g_*(x),$$

so the composite $H_n(S^n) \to H_n(X)$ sends $x \mapsto (x, x) \mapsto f_*(x) + g_*(x)$. This composite is just $(f + g)_*(x)$, since the composite $(f \vee g)\delta$ induces the map $(f + g)_*$ on homology.

It is possible to give an elementary proof of the Hurewicz theorem, but we won't do that here: instead, we will prove this as a consequence of the Serre spectral sequence.

Example 3.9.11. Since $\pi_i(S^n) = 0$ for $i < n$, the Hurewicz theorem tells us that $\pi_n(S^n) \simeq H_n(S^n) \simeq \mathbf{Z}$.

Example 3.9.12. Recall the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. The long exact sequence on homotopy groups tells us that $\pi_i(S^3) \xrightarrow{\simeq} \pi_i(S^2)$ for $i > 2$, where the map is given by $\alpha \mapsto \eta\alpha$. As we saw above, $\pi_3(S^3) = \mathbf{Z}$, so $\pi_3(S^2) \simeq \mathbf{Z}$, generated by η.

One can show that $\pi_{4n-1}(S^{2n}) \otimes \mathbf{Q} \simeq \mathbf{Q}$. A theorem of Serre's says that, other than $\pi_n(S^n)$, these are the only non-torsion homotopy groups of spheres.

3.10 Examples of CW-complexes

Bringing you up-to-speed on CW-complexes

Definition 3.10.1. A *relative CW-complex* is a pair (X, A), together with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X,$$

such that for all n, the space X_n sits in a pushout square:

$$\begin{tikzcd} \coprod_{\alpha \in \Sigma_n} S^{n-1} \arrow[r] \arrow[d, "\text{attaching maps}"'] & \coprod_{\alpha \in \Sigma_n} D^n \arrow[d, "\text{characteristic maps}"] \\ X_{n-1} \arrow[r] & X_n, \end{tikzcd}$$

and $X = \varinjlim X_n$.

If $A = \emptyset$, this is just the definition of a CW-complex. In this case, X is also compactly generated. (This is one of the reasons for defining compactly generated spaces.) Often, X will be a CW-complex, and A will be a subcomplex. If A is Hausdorff, then so is X.

If X and Y are both CW-complexes, define

$$(X \times^k Y)_n = \bigcup_{i+j=n} X_i \times Y_j;$$

this gives a CW-structure on the product $X \times^k Y$. Any closed smooth manifold admits a CW-structure.

Example 3.10.2 (Complex projective space)**.** The complex projective n-space $\mathbf{CP}^n$ is a CW-complex, with skeleta $\mathbf{CP}^0 \subseteq \mathbf{CP}^1 \subseteq \cdots \subseteq \mathbf{CP}^n$. Indeed, any complex line through the origin meets the hemisphere defined by $\begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}$ with $||z|| = 1$, $\Im(z_n) = 0$, and $\Re(z_n) \geq 0$. Such a line meets this hemisphere (which is just D^{2n}) at one point — unless it's on the equator; this gives the desired pushout diagram:

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d] & D^{2n} \arrow[d] \\ \mathbf{CP}^{n-1} \arrow[r] & \mathbf{CP}^n. \end{tikzcd}$$

Example 3.10.3 (Grassmannians)**.** Let $V = \mathbf{R}^n$ or $\mathbf{C}^n$ or $\mathbf{H}^n$, for some fixed n. Define the Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$ to be the collection of k-dimensional subspaces of V. This is equivalent to specifying a $k \times n$ rank k matrix.

For instance, $\mathrm{Gr}_2(\mathbf{R}^4)$ is, as a set, the disjoint union of:

$$\begin{pmatrix} & 1 & & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 & * \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & & * \\ & & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & & * \\ & & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \end{pmatrix}.$$

Motivated by this, define:

Definition 3.10.4. The j-skeleton of $\mathrm{Gr}(V)$ is

$$\mathrm{sk}_j\mathrm{Gr}_k(V) = \{A : \text{row echelon representation with at most } j \text{ free entries}\}.$$

For a proof that this is indeed a CW-structure, see [MS74, §6].

The top-dimensional cell tells us that

$$\dim \mathrm{Gr}_k(\mathbf{R}^n) = k(n-k).$$

The complex Grassmannian has cells in only even dimensions. We know the homology of Grassmannians: Poincaré duality is visible if we count the number of cells. (Consider, for instance, in $\mathrm{Gr}_2(\mathbf{R}^4)$).

3.11 Relative Hurewicz and J. H. C. Whitehead

Here is an "alternative definition" of connectedness:

Definition 3.11.1. Let $n \geq 0$. The space X is said to be $(n-1)$-*connected* if, for all $0 \leq k \leq n$, any map $f : S^{k-1} \to X$ extends:

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & X \\ D^k \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

When $n = 0$, we know that $S^{-1} = \emptyset$, and $D^0 = *$. Thus being (-1)-connected is equivalent to being nonempty. When $n = 1$, this is equivalent to path connectedness. You can check that this is exactly the same as what we said before, using homotopy groups.

As is usual in homotopy theory, there is a relative version of this definition.

Definition 3.11.2. Let $n \geq 0$. Say that a pair (X, A) is n-*connected* if, for all $0 \leq k \leq n$, any map $f : (D^k, S^{k-1}) \to (X, A)$ extends:

$$\begin{tikzcd} (D^k, S^{k-1}) \arrow[r, "f"] \arrow[d, dashed] & (X, A) \\ (A, A) \arrow[ur] & \end{tikzcd}$$

up to homotopy. In other words, there is a homotopy between f and a map with image in A, such that $f|_{S^{k-1}}$ remains unchanged.

0-connectedness implies that A meets every path component of X. Equivalently:

Definition 3.11.3. (X, A) is n-connected if:

- when $n = 0$, the map $\pi_0(A) \to \pi_0(X)$ surjects.
- when $n > 0$, the canonical map $\pi_0(A) \xrightarrow{\simeq} \pi_0(X)$ is an isomorphism, and for all $a \in A$, the group $\pi_k(X, A, a)$ vanishes for $1 \le k \le n$. (Equivalently, $\pi_0(A) \xrightarrow{\simeq} \pi_0(X)$ and $\pi_k(A, a) \to \pi_k(X, A)$ is an isomorphism for $1 \le k < n$ and is onto for $k = n$.)

The relative Hurewicz theorem

Assume that $\pi_0(A) = * = \pi_0(X)$, and pick $a \in A$. Then, we have a comparison of long exact sequences, arising from the classical (i.e., non-relative) Hurewicz map:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(A) \arrow[r] \arrow[d, "h"] & \pi_1(X) \arrow[r] \arrow[d, "h"] & \pi_1(X,A) \arrow[r] \arrow[d, "h"] & \pi_0(A) \arrow[r] \arrow[d, "h"] & \pi_0(X) \arrow[d, "h"] & \\
\cdots \arrow[r] & H_1(A) \arrow[r] & H_1(X) \arrow[r] & H_1(X,A) \arrow[r] & H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(
\end{tikzcd}$$

To define the relative Hurewicz map, let $\alpha \in \pi_n(X, A)$, so that $\alpha : (D^n, S^{n-1}) \to (X, A)$; pick a generator of $H_n(D^n, S^{n-1})$, and send it to an element of $H_n(X, A)$ via the induced map $\alpha_* : H_n(D^n, S^{n-1}) \to H_n(X, A)$.

Because $H_n(X, A)$ is abelian, the group $\pi_1(A)$ acts trivially on $H_n(X, A)$; in other words, $h(\omega(\alpha)) = h(\alpha)$. Consequently, the relative Hurewicz map factors through the group $\pi_n^\dagger(X, A)$, defined to be the quotient of $\pi_n(X, A)$ by the normal subgroup generated by $(\omega\alpha)\alpha^{-1}$, where $\omega \in \pi_1(A)$ and $\alpha \in \pi_n(X, A)$. This begets a map $\pi_n^\dagger(X, A) \to H_n(X, A)$.

Theorem 3.11.4 (Relative Hurewicz)**.** *Let $n \ge 1$, and assume (X, A) is n-connected. Then $H_k(X, A) = 0$ for $0 \le k \le n$, and the map $\pi_{n+1}^\dagger(X, A) \to H_{n+1}(X, A)$ constructed above is an isomorphism.*

We will prove this later using the Serre spectral sequence.

The Whitehead theorems

J. H. C. Whitehead was a rather interesting character. He raised pigs.

Whitehead was interested in determining when a continuous map $f : X \to Y$ that is an isomorphism in homology or homotopy is a homotopy equivalence.

Definition 3.11.5. Let $f : X \to Y$ and $n \geq 0$. Say that f is a *n-equivalence*[13] if, for every $* \in Y$, the homotopy fiber $F(f, *)$ is $(n-1)$-connected.

For instance, f being a 0-equivalence simply means that $\pi_0(X)$ surjects onto $\pi_0(Y)$ via f. For $n > 0$, this says that $f : \pi_0(X) \to \pi_0(Y)$ is a bijection, and that for every $* \in X$:

$$\pi_k(X, *) \to \pi_k(Y, f(*)) \text{ is } \begin{cases} \text{an isomorphism} & 1 \leq k < n \\ \text{onto} & k = n. \end{cases}$$

Using the "mapping cylinder" construction (see Exercise 3.6.5), we can always assume $f : X \to Y$ is a cofibration; in particular, that $X \hookrightarrow Y$ is a closed inclusion. Then, $f : X \to Y$ is an n-equivalence if and only if (Y, X) is n-connected.

Theorem 3.11.6 (Whitehead)**.** *Suppose $n \geq 0$, and $f : X \to Y$ is n-connected. Then:*

$$H_k(X) \xrightarrow{f} H_k(Y) \textit{ is } \begin{cases} \textit{an isomorphism} & 1 \leq k < n \\ \textit{onto} & k = n. \end{cases}$$

Proof. When $n = 0$, because $\pi_0(X) \to \pi_0(Y)$ is surjective, we learn that $H_0(X) \simeq \mathbf{Z}[\pi_0(X)] \to \mathbf{Z}[\pi_0(Y)] \simeq H_0(Y)$ is surjective. To conclude, use the relative Hurewicz theorem. (Note that the relative Hurewicz dealt with $\pi_n^\dagger(X, A)$, but the map $\pi_n(X, A) \to \pi_n^\dagger(X, A)$ is surjective.) $\square$

The case $n = \infty$ is special.

Definition 3.11.7. f is a *weak equivalence* (or an ∞-equivalence, to make it sound more impressive) if it's an n-equivalence for all n, i.e., it's a π_*-isomorphism.

Putting everything together, we obtain:

Corollary 3.11.8. *A weak equivalence induces an isomorphism in integral homology.*

How about the converse?

If $H_0(X) \to H_0(Y)$ surjects, then the map $\pi_0(X) \to \pi_0(Y)$ also surjects. Now, assume X and Y path connected, and that $H_1(X)$ surjects onto $H_1(Y)$. We would like to conclude that $\pi_1(X) \to \pi_1(Y)$

[13] Some sources sometimes use "n-connected".

surjects. Unfortunately, this is hard, because $H_1(X)$ is the abelianization of $\pi_1(X)$. To forge onward, we will simply give up, and assume that $\pi_1(X) \to \pi_1(Y)$ is surjective.

Suppose $H_2(X) \to H_2(Y)$ surjects, and that $f_* : H_1(X) \xrightarrow{\simeq} H_1(Y)$. We know that $H_2(Y, X) = 0$. On the level of the Hurewicz maps, we are still stuck, because we only obtain information about $\pi_2^\dagger$. Let us assume that $\pi_1(X)$ is trivial[14]. Under this assumption, we find that $\pi_1(Y) = 0$. This implies $\pi_2(Y, X)$ is trivial. Arguing similarly, we can go up the ladder.

Theorem 3.11.9 (Whitehead)**.** *Let $n \geq 2$, and assume that $\pi_1(X) = 0 = \pi_1(Y)$. Suppose $f : X \to Y$ such that:*

$$H_k(X) \to H_k(Y) \text{ is } \begin{cases} \text{an isomorphism} & 1 \leq k < n \\ \text{onto} & k = n; \end{cases}$$

then f is an n-equivalence.

Setting $n = \infty$, we obtain:

Corollary 3.11.10. *Let X and Y be simply-connected. If f induces an isomorphism in homology, then f is a weak equivalence.*

This is incredibly useful, since homology is actually computable! To wrap up the story, we will state the following result, which we will prove in a later section.

Theorem 3.11.11. *Let Y be a CW-complex. Then a weak equivalence $f : X \to Y$ is in fact a homotopy equivalence.*

3.12 Cellular approximation, cellular homology, obstruction theory

In previous sections, we saw that homotopy groups play well with (maps between) CW-complexes. Here, we will study maps between CW-complexes themselves, and prove that they are, in some sense, "cellular" themselves.

[14] This is a pretty radical assumption; for the following argument to work, it would technically be enough to ask that $\pi_1(X)$ acts trivially on $\pi_2(Y, X)$: but this is basically impossible to check.

Cellular approximation

Definition 3.12.1. Let X and Y be CW-complexes, and let $A \subseteq X$ be a subcomplex. Suppose $f : X \to Y$ is a continuous map. We say that $f|_A$ is skeletal[15] if $f(\Sigma_n) \subseteq Y_n$.

Note that a skeletal map might not take cells in A to cells in Y, but it takes n-skeleta to n-skeleta.

Theorem 3.12.2 (Cellular approximation). *In the setup of Definition 3.12.1, the map f is homotopic to some other continuous map $f' : X \to Y$, relative to A, such that f' is skeletal on all of X.*

To prove this, we need the following lemma.

Lemma 3.12.3 (Key lemma). *Any map $(D^n, S^{n-1}) \to (Y, Y_{n-1})$ factors as:*

$$\begin{tikzcd}
(D^n, S^{n-1}) \arrow[r] \arrow[dr, dashed] & (Y, Y_{n-1}) \\
& (Y_n, Y_{n-1}) \arrow[u]
\end{tikzcd}$$

"Proof." Since D^n is compact, we know that $f(D^n)$ must lie in some finite subcomplex K of Y. The map $D^n \to K$ might hit some top-dimensional cell $e^m \subseteq K$, which does not have anything attached to it; hence, we can homotope this map to miss a point, so that it contracts onto a lower-dimensional cell. Iterating this process gives the desired result. $\square$

Using this lemma, we can conclude the cellular approximation theorem.

"Proof" of Theorem 3.12.2. We will construct the homotopy $f \simeq f'$ one cell at a time. Note that we can replace the space A by the subspace to which we have extended the homotopy.

Consider a single cell attachment $A \to A \cup D^m$; then, we have

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\text{skeletal}"'] & A \cup D^m \arrow[dl, "\text{may not be skeletal}"] \\
Y &
\end{tikzcd}$$

[15] Some would say cellular.

Using the "compression lemma" from above, the rightmost map factors (up to homotopy) as the composite $A \cup D^m \to Y_m \to Y$. Unfortunately, we have not extended this map to the whole of X, although we could do this if we knew that the inclusion of a subcomplex is a cofibration. But this is true: there is a cofibration $S^{n-1} \to D^n$, and so any pushout of these maps is a cofibration! This allows us to extend; we now win by iterating this procedure. □

As a corollary, we find:

Exercise 3.12.4. The pair (X, X_n) is n-connected.

Cellular homology

provide a link!

Let (X, A) be a relative CW-complex with $A \subseteq X_{n-1} \subseteq X_n \subseteq \cdots \subseteq X$. In the previous part that $H_*(X_n, X_{n-1}) \simeq \widetilde{H}_*(X_n/X_{n-1})$. More generally, if $B \to Y$ is a cofibration, there is an isomorphism (see [Bre93, p. 433]):

$$H_*(Y, B) \simeq \widetilde{H}_*(Y/B).$$

Since $X_n/X_{n-1} = \bigvee_{\alpha \in \Sigma_n} S^n_\alpha$, we find that

$$H_*(X_n, X_{n-1}) \simeq \mathbf{Z}[\Sigma_n] = C_n.$$

The composite $S^{n-1} \to X_{n-1} \to X_{n-1}/X_{n-2}$ is called a relative attaching map.

There is a boundary map $d : C_n \to C_{n-1}$, defined by

$$d : C_n = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}.$$

Exercise 3.12.5. Check that $d^2 = 0$.

Using the resulting chain complex, denoted $C_*(X, A)$, one can prove that there is an isomorphism

$$H_n(X, A) \simeq H_n(C_*(X, A)).$$

provide a link!

(In the previous part, we proved this for CW-pairs, but not for relative CW-complexes.) The incredibly useful cellular approximation theorem therefore tells us that the effect of maps on homology can be computed.

Of course, the same story runs for cohomology: one gets a chain complex which, in dimension n, is given by

$$C^n(X, A; \pi) = \mathrm{Hom}(C_n(X, A), \pi) = \mathrm{Map}(\Sigma_n, \pi),$$

where π is any abelian group.

Obstruction theory

Using the tools developed above, we can attempt to answer some concrete, and useful, questions.

Question 3.12.6. Let $f : A \to Y$ be a map from a space A to Y. Suppose (X, A) is a relative CW-complex. When can we find an extension in the diagram below?

$$\begin{tikzcd} X \arrow[dr, dashed] & \\ A \arrow[u, hook] \arrow[r, "f"'] & Y \end{tikzcd}$$

The lower level obstructions can be worked out easily:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X_0 \arrow[r, hook] \arrow[dl, dashed] & X_1 \arrow[dll, dashed] \\ \emptyset \neq Y & & \end{tikzcd}$$

Thus, for instance, if two points in X_0 are connected in X_1, we only have to check that they are also connected in Y.

For $n \geq 2$, we can form the diagram:

$$\begin{tikzcd} \coprod_{\alpha \in \Sigma_n} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook] & X_{n-1} \arrow[r, "g"] \arrow[d] & Y \\ \coprod_{\Sigma_n} D^n_\alpha \arrow[r] & X_n \arrow[ur, dashed] & \end{tikzcd}$$

The desired extension exists if the composite $S^{n-1}_\alpha \xrightarrow{f_\alpha} X_{n-1} \to Y$ is nullhomotopic.

Clearly, $g \circ f_\alpha \in [S^{n-1}, Y]$. To simplify the discussion, let us assume that Y is simple; then, Exercise 3.9.7 says that $[S^{n-1}, Y] = \pi_{n-1}(Y)$. This procedure begets a map $\Sigma_n \xrightarrow{\theta} \pi_{n-1}(Y)$, which is a n-cochain, i.e., an element of $C^n(X, A; \pi_{n-1}(Y))$. It is clear that $\theta = 0$ if and only if the map g extends to $X_n \to Y$.

Proposition 3.12.7. *θ is a cocycle in $C^n(X, A; \pi_{n-1}(Y))$, called the "obstruction cocycle".*

Proof. θ gives a map $H_n(X_n, X_{n-1}) \to \pi_{n-1}(Y)$. We would like to show that the composite

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n) \to H_n(X_n, X_{n-1}) \xrightarrow{\theta} \pi_{n-1}(Y)$$

is trivial. We have the long exact sequence in homotopy of a pair (see Equation (3.5)):

$$\begin{tikzcd}
\pi_{n+1}(X_{n+1}, X_n) \arrow[r, two heads] \arrow[d] & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial"] \\
\pi_n(X_n) \arrow[r] \arrow[d] & H_n(X_n) \arrow[d] \\
\pi_n(X_n, X_{n-1}) \arrow[r, two heads] \arrow[d, "\partial"'] & H_n(X_n, X_{n-1}) \arrow[d, "\theta"] \\
\pi_{n-1}(X_{n-1}) \arrow[r, "g_*"'] & \pi_{n-1}(Y)
\end{tikzcd}$$

This diagram commutes, so θ is indeed a cocycle. ☐

Our discussion above allows us to conclude:

Theorem 3.12.8. *Let (X, A) be a relative CW-complex and Y a simple space. Let $g : X_{n-1} \to Y$ be a map from the $(n-1)$-skeleton of X. Then $g|_{X_{n-2}}$ extends to X_n if and only if $[\theta(g)] \in H^n(X, A; \pi_{n-1}(Y))$ is zero.*

Corollary 3.12.9. *If $H^n(X, A; \pi_{n-1}(Y)) = 0$ for all $n > 2$, then any map $A \to Y$ extends to a map $X \to Y$ (up to homotopy[16]); in other words, there is a dotted lift in the following diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \\
X \arrow[ur, dashed] &
\end{tikzcd}$$

For instance, every map $A \to Y$ factors through the cone if $H^n(CA, A; \pi_{n-1}(Y)) \simeq \widetilde{H}^{n-1}(A; \pi_{n-1}(Y)) = 0$.

[16] In fact, this condition is unnecessary, since the inclusion of a subcomplex is a cofibration.

3.13 Conclusions from obstruction theory

The main result of obstruction theory, as discussed in the previous section, is the following.

Theorem 3.13.1 (Obstruction theory). *Let (X, A) be a relative CW-complex, and Y a simple space. The map $[X, Y] \to [A, Y]$ is:*

1. *is onto if $H^n(X, A; \pi_{n-1}(Y)) = 0$ for all $n \geq 2$.*

2. *is one-to-one if $H^n(X, A; \pi_n(Y)) = 0$ for all $n \geq 1$.*

Remark 3.13.2. The first statement implies the second. Indeed, suppose we have two maps $g_0, g_1 : X \to Y$ and a homotopy $h : g_0|_A \simeq g_0|_A$. Assume the first statement. Consider the relative CW-complex $(X \times I, A \times I \cup X \times \partial I)$. Because (X, A) is a relative CW-complex, the map $A \hookrightarrow X$ is a cofibration; this implies that the map $A \times I \cup X \times \partial I \to X \times I$ is also a cofibration.

$$\begin{aligned} H^n(X \times I, A \times I \cup X \times \partial I; \pi) &\simeq \widetilde{H}^n(X \times I/(A \times I \cup X \times \partial I); \pi) \\ &= H^n(\Sigma X/A; \pi) \simeq \widetilde{H}^{n-1}(X/A; \pi). \end{aligned}$$

We proved the following statement in the previous section.

Proposition 3.13.3. *Suppose $g : X_{n-1} \to Y$ is a map from the $(n-1)$-skeleton of X to Y. Then $g|_{X_{n-2}}$ extends to $X_n \to Y$ iff $[\theta(g)] = 0$ in $H^n(X, A; \pi_{n-1}(Y))$.*

An immediate consequence is the following.

Theorem 3.13.4 (CW-approximation). *Any space admits a weak equivalence from a CW-complex.*

This tells us that studying CW-complexes is not very restrictive, if we work up to weak equivalence.

It is easy to see that if W is a CW-complex and $f : X \to Y$ is a weak equivalence, then $[W, X] \xrightarrow{\simeq} [W, Y]$. We can now finally conclude the result of Theorem 3.11.11:

Corollary 3.13.5. *Let X and Y be CW-complexes. Then a weak equivalence $f : X \to Y$ is a homotopy equivalence.*

Postnikov and Whitehead towers

Let X be path connected. There is a space $X_{\leq n}$, and a map $X \to X_{\leq n}$ such that $\pi_i(X_{\geq n}) = 0$ for $i > n$, and $\pi_i(X) \xrightarrow{\simeq} \pi_i(X_{\leq n})$ for $i \leq n$. This pair $(X, X_{\leq n})$ is essentially unique up to homotopy; the space $X_{\leq n}$ is called the *nth Postnikov section* of X. Since Postnikov sections have "simpler" homotopy groups, we can try to understand X by studying each of its Postnikov sections individually, and then gluing all the data together.

provide a link

Suppose A is some abelian group. We saw, in the first partthat there is a space $M(A, n)$ with homology given by:

$$\widetilde{H}_i(M(A,n)) = \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

This space was constructed from a free resolution $0 \to F_1 \to F_0 \to A \to 0$ of A. We can construct a map $\bigvee S^n \to \bigvee S^n$ which realizes the first two maps; coning this off gets $M(A, n)$. By Hurewicz, we have:

$$\pi_i(M(A,n)) = \begin{cases} 0 & i < n \\ A & i = n \\ ?? & i > n \end{cases}$$

It follows that, when we look at the nth Postnikov section of $M(A, n)$, we have:

$$\pi_i(M(A,n)_{\leq n}) = \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

In some sense, therefore, this Postnikov section is a "designer homotopy type". It deserves a special name: $M(A, n)_{\leq n}$ is called an *Eilenberg-MacLane space*, and is denoted $K(A, n)$. By the fiber sequence $\Omega X \to PX \to X$ with $PX \simeq *$, we find that $\Omega K(\pi, n) \simeq K(\pi, n-1)$. Eilenberg-MacLane spaces are unique up to homotopy.

Note that $n = 1$, A does not have to be abelian, but you can still construct $K(A, 1)$. This is called the *classifying space* of G; such spaces will be discussed in more detail in the next chapter. Examples are in abundance: if Σ is a closed surface that is not S^2 or $\mathbf{R}^2$, then $\Sigma \simeq K(\pi_1(\Sigma), 1)$. Perhaps simpler is the identification $S^1 \simeq K(\mathbf{Z}, 1)$.

Example 3.13.6. We can identify $K(\mathbf{Z}, 2)$ as $\mathbf{CP}^\infty$. To see this, observe that we have a fiber sequence $S^1 \to S^{2n+1} \to \mathbf{CP}^n$. The long exact sequence in homotopy tells us that the homotopy groups of $\mathbf{CP}^n$ are the same as the homotopy groups of S^1, until $\pi_* S^{2n+1}$ starts to interfere. As n grows, we obtain a fibration $S^1 \to S^\infty \to \mathbf{CP}^\infty$. Since S^∞ is weakly contractible (it has no nonzero homotopy groups), we get the desired result.

Example 3.13.7. Similarly, we can identify $K(\mathbf{Z}/2\mathbf{Z}, 1)$ as $\mathbf{RP}^\infty$.

Since $\pi_1(K(A, n)) = 0$ for $n > 1$, it follows that $K(A, n)$ is automatically a simple space. This means that

$$[S^k, K(A, n)] = \pi_k(K(A, n)) = H^n(S^k, A).$$

In fact, a more general result is true:

Theorem 3.13.8 (Brown representability)**.** *If X is a CW-complex, then $[X, K(A, n)] = H^n(X; A)$.*

We will not prove this here, but one can show this simply by showing that the functor $[-, K(A, n)]$ satisfies the Eilenberg-Steenrod axioms. Somehow, these Eilenberg-MacLane spaces $K(A, n)$ completely capture cohomology in dimension n.

If X is a CW-complex, then we may assume that $X_{\leq n}$ is also a CW-complex. (Otherwise, we can use cellular approximation and then kill homotopy groups.) Let us assume that X is path connected; then $X_{\leq 1} = K(\pi_1(X), 1)$. We may then form a (commuting) tower:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \cdots \arrow[l] \\
 & X_{\leq 3} \arrow[d] & K(\pi_3(X), 3) \arrow[l] \\
 & X_{\leq 2} \arrow[d] & K(\pi_2(X), 2) \arrow[l] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & X_{\leq 1} \arrow[r, equal] & K(\pi_1(X), 1),
\end{tikzcd}$$

since $K(\pi_n(X), n) \to X_{\leq n} \to X_{\leq n-1}$ is a fiber sequence. This decomposition of X is called the *Postnikov tower* of X.

Denote by $X_{>n}$ the fiber of the map $X \to X_{\leq n}$ (for instance, $X_{>1}$ is the universal cover of X); then, we have

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, equal] & \vdots \arrow[d] & \cdots \arrow[l] \\
X_{>3} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 3} \arrow[d] & K(\pi_3(X), 3) \arrow[l] \\
X_{>2} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 2} \arrow[d] & K(\pi_2(X), 2) \arrow[l] \\
X_{>1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 1} \arrow[d] \arrow[r, equal] & K(\pi_1(X), 1) \\
X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}$$

The leftmost tower is called the *Whitehead tower* of X, named after George Whitehead.

I can take the fiber of $X_{>1} \to X$, and I get $K(\pi_1(X), 0)$; more generally, the fiber of $X_{>n} \to X_{>n-1}$ is $K(\pi_n(X), n-1)$. This yields the following diagram:

$$\begin{tikzcd}
\cdots & \vdots \arrow[d] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \cdots \\
K(\pi_3(X), 2) \arrow[r] & X_{>3} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 3} \arrow[d] & K(\pi_3(X), 3) \arrow[l] \\
K(\pi_2(X), 1) \arrow[r] & X_{>2} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 2} \arrow[d] & K(\pi_2(X), 2) \arrow[l] \\
K(\pi_1(X), 0) \arrow[r] & X_{>1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 1} \arrow[d] \arrow[r, equal] & K(\pi_1(X), 1) \\
 & X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}$$

We can construct Eilenberg-MacLane spaces as cellular complexes by attaching cells to the sphere to kill its higher homotopy groups. The complexity of homotopy groups, though, shows us that attaching cells to compute the cohomology of Eilenberg-MacLane spaces is not feasible.

Chapter 4

Vector bundles

4.1 Vector bundles, principal bundles

Let X be a topological space. A point in X can be viewed as a map $* \to X$; this is a cross section of the canonical map $X \to *$. Motivated by this, we will define a vector space over B to be a space $E \to B$ over B with the following extra data:

- a multiplication $\mu : E \times_B E \to E$, compatible with the maps down to B;
- a "zero" section $s : B \to E$ such that the composite $B \xrightarrow{s} E \to B$ is the identity;
- an inverse $\chi : E \to E$, compatible with the map down to B; and
- an action of $\mathbf{R}$:

$$\begin{tikzcd}
\mathbf{R} \times E \arrow[r, equal] \arrow[dr, "p \circ \mathrm{pr}_2"'] & (B \times \mathbf{R}) \times_B E \arrow[r] \arrow[d] & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

Because $\mathbf{R}$ is a field, the last piece of data shows that $p^{-1}(b)$ is a $\mathbf{R}$-vector space for any point $b \in B$.

Example 4.1.1. A rather silly example of a vector space over B is the projection $B \times V \to B$ where V is a (real) vector space, which we will always assume to be finite-dimensional.

Example 4.1.2. Consider the map

$$\mathbf{R} \times \mathbf{R} \xrightarrow{(s,t) \mapsto (s,st)} \mathbf{R} \times \mathbf{R},$$

over $\mathbf{R}$ (the structure maps are given by projecting onto the first factor). It is an isomorphism on all fibers, but is zero everywhere else. The kernel is therefore 0 everywhere, except over the point $0 \in \mathbf{R}$. This the "skyscraper" vector bundle over B.

Sheaf theory accommodates examples like this.

One can only go so far you can go with this simplistic notion of a "vector space" over B. Most interesting and naturally arising examples have a little more structure, which is exemplified in the following definition.

Definition 4.1.3. A *vector bundle* over B is a vector space over B that is locally trivial (in the sense of Definition 3.4.1).

Remark 4.1.4. We will always assume that the space B admits a numerable open cover (see Definition 3.4.4) which trivializes the vector bundle. Moreover, the dimension of the fiber will always be finite.

If $p : E \to B$ is a vector bundle, then E is called the *total space*, p is called the *projection map*, and B is called the *base space*. We will always use a Greek letter like ξ or ζ to denote a vector bundle, and $E(\xi) \to B(\xi)$ denotes the actual projection map from the total space to the base space. The phrase "ξ is a vector bundle over B" will also be shortened to $\xi \downarrow B$.

Example 4.1.5.

1. Following Example 4.1.1, one example of a vector bundle is the trivial bundle $B \times \mathbf{R}^n \to B$, denoted by $n\epsilon$.

2. In contrast to this silly example, one gets extremely interesting examples from the Grassmannians $\mathrm{Gr}_k(\mathbf{R}^n)$, $\mathrm{Gr}_k(\mathbf{C}^n)$, and $\mathrm{Gr}_k(\mathbb{H}^n)$. For simplicity, let K denote $\mathbf{R}, \mathbf{C}$, or $\mathbb{H}$. Over $\mathrm{Gr}_k(K^n)$ lies the *tautological bundle* γ. This is a sub-bundle of $n\epsilon$ (i.e., the fiber over any point $x \in \mathrm{Gr}_k(K^n)$ is a subspace of the fiber of $n\epsilon$ over x). The total space of γ is defined as:

$$E(\gamma) = \{(V, x) \in \mathrm{Gr}_k(K^n) \times K^n : x \in V\}$$

This projection map down to $\mathrm{Gr}_k(K^n)$ is the literal projection map

$$(V, x) \mapsto V.$$

Exercise 4.1.6. Prove that γ, as defined above, is locally trivial; so γ defines a vector bundle over $\mathrm{Gr}_k(K^n)$.

For instance, when $k = 1$, we have $\mathrm{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$. In this case, γ is one-dimensional (i.e., the fibers are all of dimension 1); this is called a *line bundle*. In fact, it is the "canonical line bundle" over $\mathbf{RP}^{n-1}$.

3. Let M be a smooth manifold. Define τ_M to be the tangent bundle $TM \to M$ over M. For example, if $M = S^{n-1}$, then

$$TS^{n-1} = \{(x, v) \in S^{n-1} \times \mathbf{R}^n : v \cdot x = 0\}.$$

Constructions with vector bundles

One cannot take the kernels of a map of vector bundles; but just about anything which can be done for vector spaces can also be done for vector bundles:

1. Pullbacks are legal: if $p' : E' \to B'$, then the leftmost map in the diagram below is also a vector bundle.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

For instance, if $B = *$, the pullback is just the fiber of E' over the point $* \to B'$. If ξ is the bundle $E' \to B'$, we denote the pullback $E \to B$ as $f^*\xi$.

2. If $p : E \to B$ and $p' : E' \to B'$, then we can take the product $E \times E' \xrightarrow{p \times p'} B \times B'$.

3. If $B = B'$, we can form the pullback:

$$\begin{tikzcd} E \oplus E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \arrow[r, "\Delta"'] & B \times B \end{tikzcd}$$

The bundle $E \oplus E'$ is called the *Whitney sum*. For instance, it is an easy exercise to see that

$$n\epsilon = \epsilon \oplus \cdots \oplus \epsilon.$$

4. If $E, E' \to B$ are two vector bundles over B, we can form another vector bundle $E \otimes_{\mathbf{R}} E' \to B$ by taking the fiberwise tensor product. Likewise, taking the fiberwise Hom begets a vector bundle $\mathrm{Hom}_{\mathbf{R}}(E, E') \to B$.

Example 4.1.7. Recall from Example 4.1.5(2) that the tautological bundle γ lives over $\mathbf{RP}^{n-1}$; we will write $L = E(\gamma)$. The tangent bundle $\tau_{\mathbf{RP}^{n-1}}$ also lives over $\mathbf{RP}^{n-1}$. As this is the first explicit pair of vector bundles over the same space, it is natural to wonder what is the relationship between these two bundles.

At first glance, one might guess that $\tau_{\mathbf{RP}^{n-1}} = \gamma^\perp$; but this is false! Instead,

$$\tau_{\mathbf{RP}^{n-1}} = \mathrm{Hom}(\gamma, \gamma^\perp).$$

To see this, note that we have a 2-fold covering map $S^{n-1} \to \mathbf{RP}^{n-1}$; therefore, $T_x(\mathbf{RP}^{n-1})$ is a quotient of $T(S^n)$ by the map sending $(x, v) \mapsto (-x, -v)$, where $v \in T_x(S^n)$. Therefore,

$$T_x\mathbf{RP}^{n-1} = \{(x, v) \in S^{n-1} \times \mathbf{R}^n : v \cdot x = 0\}/((x, v) \sim (-x, -v)).$$

This is exactly the fiber of $\mathrm{Hom}(\gamma, \gamma^\perp)$ over $x \in \mathbf{RP}^{n-1}$, since the line through x can be mapped to the line through $\pm v$.

Exercise 4.1.8. Prove that if γ is the tautological vector bundle over $\mathrm{Gr}_k(K^n)$, for $K = \mathbf{R}, \mathbf{C}, \mathbb{H}$, then

$$\tau_{\mathrm{Gr}_k(K^n)} = \mathrm{Hom}(\gamma, \gamma^\perp).$$

Metrics and splitting exact sequences

A *metric* on a vector bundle is a continuous choice of inner products on fibers.

Lemma 4.1.9. *Any vector bundle ξ over X admits a metric.*

Intuitively speaking, this is true because if g, g' are both inner products on V, then $tg + (1-t)g'$ is another. Said differently, the space of metrics forms a real affine space.

Proof. Pick a trivializing open cover of X, and a subordinate partition of unity. This means that we have a map $\phi_U : U \to [0,1]$, such that the preimage of the complement of 0 is U. Moreover,

$$\sum_{x \in U} \phi_U(x) = 1.$$

Over each one of these trivial pieces, pick a metric g_U on $E|_U$. Let

$$g := \sum_U \phi_U g_U;$$

this is the desired metric on ξ. □

We remark that, in general, one cannot pick metrics for vector bundles. For instance, this is the case for vector bundles which arise in algebraic geometry.

Definition 4.1.10. Suppose $E, E' \to B$ are vector bundles over B. An *isomorphism* is a map $\alpha : E \to E'$ over B that is a linear isomorphism on each fiber.

In particular, the map α admits an inverse (over B).

Corollary 4.1.11. *Any exact*[1] *sequence* $0 \to E' \to E \to E'' \to 0$ *of vector bundles (over the same base) splits.*

Proof sketch. Pick a metric for E. Consider the composite

$$E'^{\perp} \subseteq E \to E''.$$

This is an isomorphism: the dimensions of the fibers are the same. It follows that

$$E \cong E' \oplus E'^{\perp} \cong E' \oplus E'',$$

as desired. □

Note that this splitting is not natural.

[1] This is the obvious definition.

4.2 Principal bundles, associated bundles

I-invariance

We will denote by $\mathrm{Vect}(B)$ the set of isomorphism classes of vector bundles over B. (Justify the use of the word "set"!)

Consider a vector bundle $\xi \downarrow B$. If $f : B' \to B$, taking the pullback gives a vector bundle denoted $f^*\xi$. This operation descends to a map $f^* : \mathrm{Vect}(B) \to \mathrm{Vect}(B')$; we therefore obtain a functor $\mathrm{Vect} : \mathbf{Top}^{op} \to \mathrm{Set}$. One might expect this functor to give some interesting invariants of topological spaces.

Theorem 4.2.1. *Let $I = \Delta^1$. Then* Vect *is I-invariant. In other words, the projection $X \times I \to X$ induces an isomorphism* $\mathrm{Vect}(X) \to \mathrm{Vect}(X \times I)$.

One important corollary of this result is:

Corollary 4.2.2. Vect *is a homotopy functor.*

Proof. Consider two homotopic maps $f, g : B \to B'$, so there exists a homotopy $H : B' \times I \to B$. If $\xi \downarrow B$, we need to prove that $f_0^*\xi \simeq f_1^*\xi$. This is far from obvious.

Consider the following diagram.

$$\begin{tikzcd} B' \times I \arrow[r, "H"] \arrow[d, "\mathrm{pr}"'] & B \\ B' & \end{tikzcd}$$

The leftmost map is an isomorphism under Vect, by Theorem 4.2.1. Let $\eta \downarrow B$ be a vector bundle such that $\mathrm{pr}^*\eta \simeq f^*\xi$. For any $t \in I$, define a map $\in_t : B' \to B' \times I$ sends $x \mapsto (x, t)$. We then have isomorphisms:

$$f_t^*\xi \simeq \in_t^* f^*\xi \simeq \in_t^* \mathrm{pr}^*\eta \simeq (\mathrm{pr} \circ \in_t)^*\eta \simeq \eta,$$

as desired. $\square$

It is easy to see that $\mathrm{Vect}(X) \to \mathrm{Vect}(X \times I)$ is injective. In the next lecture, we will prove surjectivity, allowing us to conclude Theorem 4.2.1.

Principal bundles

Definition 4.2.3. Let G be a topological group[2]. A *principal G-bundle* is a right action of G on P such that:

- G acts freely.
- The orbit projection $P \to P/G$ is a fiber bundle.

These are not unfamiliar objects, as the next example shows.

Example 4.2.4. Suppose G is discrete. Then the fibers of the orbit projection $P \to P/G$ are all discrete. Therefore, the condition that $P \to P/G$ is a fiber bundle is simply that it's a covering projection (the action is "properly discontinuous").

As a special case, let X be a space with universal cover $\widetilde{X} \downarrow X$. Then $\pi_1(X)$ acts freely on $\widetilde{X}$, and $\widetilde{X} \downarrow X$ is the orbit projection. It follows from our discussion above that this is a principal bundle. Explicit examples include the principal $\mathbf{Z}/2$-bundle $S^{n-1} \downarrow \mathbf{RP}^{n-1}$, and the Hopf fibration $S^{2n-1} \downarrow \mathbf{CP}^{n-1}$, whcih is a principle S^1-bundle.

By looking at the universal cover, we can classify covering spaces of X. Remember how that goes: if F is a set with left $\pi_1(X)$-action, the dotted map in the diagram below is the desired covering space.

$$\begin{tikzcd}
\widetilde{X} \times F \arrow[r] \arrow[d, "p \circ \mathrm{pr}_1"'] & \widetilde{X} \times F/\sim \arrow[dl, "q"] \\
X &
\end{tikzcd}$$

Here, we say that $(y, gz) \sim (yg, z)$, for elements $y \in \widetilde{X}$, $z \in F$, and $g \in \pi_1(X)$.

Fix $y_0 \in \widetilde{X}$ over $* \in X$. Then it is easy to see that $F \xrightarrow{\sim} q^{-1}(*)$, via the map $z \mapsto (y_0, z)$. This is all neatly summarized in the following theorem from point-set topology.

Theorem 4.2.5 (Covering space theory)**.** *There is an equivalence of categories:*

$$\{\textit{Left } \pi_1(X)\textit{-sets}\} \xrightarrow{\simeq} \{\textit{Covering spaces of } X\},$$

with inverse functor given by taking the fiber over the basepoint and lifting a loop in X to get a map from the fiber to itself.

[2]We will only care about discrete groups and Lie groups.

Example 4.2.4 shows that covering spaces are special examples of principal bundles. The above theorem therefore motivates finding a more general picture.

Construction 4.2.6. Let $P \downarrow B$ is a principal G-bundle. If F is a left G-space, we can define a new fiber bundle, exactly as above:

$$\begin{tikzcd} P \times F \arrow[r] \arrow[d] & P \times F/\sim \arrow[dl, "q"] \\ B & \end{tikzcd}$$

This is called an *associated bundle*, and is denoted $P \times_G F$.

We must still justify that the resulting space over B is indeed a new fiber bundle with fiber F. Let $x \in B$, and let $y \in P$ over x. As above, we have a map $F \to q^{-1}(*)$ via the map $z \mapsto [y, z]$. We claim that this is a homeomorphism. Indeed, define a map $q^{-1}(*) \to F$ via

$$[y', z'] = [y, gz'] \mapsto gz',$$

where $y' = yg$ for some g (which is necessarily unique).

Exercise 4.2.7. Check that these two maps are inverse homeomorphisms.

Definition 4.2.8. A vector bundle $\xi \downarrow B$ is said to be an n-plane bundle if the dimensions of all the fibers are n.

Let $\xi \downarrow B$ be an n-plane bundle. Construct a principal $\mathrm{GL}_n(\mathbf{R})$-bundle $P(\xi)$ by defining

$$P(\xi)_b = \{\text{bases for } E(\xi)_b = \mathrm{Iso}(\mathbf{R}^n, E(\xi)_b)\}.$$

To define the topology, note that (topologically) we have

$$P(B \times \mathbf{R}^n) = B \times \mathrm{Iso}(\mathbf{R}^n, \mathbf{R}^n),$$

where $\mathrm{Iso}(\mathbf{R}^n, \mathbf{R}^n) = \mathrm{GL}_n(\mathbf{R})$ is given the usual topology as a subspace of $\mathbf{R}^{n^2}$.

There is a right action of $\mathrm{GL}_n(\mathbf{R})$ on $P(\xi) \downarrow B$, given by precomposition. It is easy to see that this action is free and simply transitive. One therefore has a principal action of $\mathrm{GL}_n(\mathbf{R})$ on $P(\xi)$. The bundle $P(\xi)$ is called the *principalization* of ξ.

Given the principalization $P(\xi)$, we can recover the total space $E(\xi)$. Consider the associated bundle $P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n$ with fiber $F = \mathbf{R}^n$, with $\mathrm{GL}_n(\mathbf{R})$ acting on $\mathbf{R}^n$ from the left. Because this is a linear action, $P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n$ is a vector bundle. One can show that

$$P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n \simeq E(\xi).$$

Fix a topological group G. Define $\mathrm{Bun}_G(B)$ as the set of isomorphism classes of G-bundles over B. An isomorphism is a G-equivariant homeomorphism over the base. Again, arguing as above, this begets a functor $\mathrm{Bun}_G : \mathbf{Top} \to \mathrm{Set}$. The above discussion gives a natural isomorphism of functors:

$$\mathrm{Bun}_{\mathrm{GL}_n(\mathbf{R})}(B) \simeq \mathrm{Vect}(B).$$

The I-invariance theorem will therefore follow immediately from:

Theorem 4.2.9. Bun_G *is I-invariant.*

Remark 4.2.10. Principal bundles allow a description of "geometric structures on ξ". Suppose, for instance, that we have a metric on ξ. Instead of looking at all ordered bases, we can attempt to understand all ordered orthonormal bases in each fiber. This give the *frame bundle*

$$\mathrm{Fr}(B) = \{\text{ordered orthonormal bases of } E(\xi)_b\};$$

these are isometric isomorphisms $\mathbf{R}^n \to E(\xi)_b$. Again, there is an action of the orthogonal group on $\mathrm{Fr}(B)$: in fact, this begets a principal $O(n)$-bundle. Such examples are in abundance: consistent orientations give an $SO(n)$-bundle. Trivializations of the vector bundle also give principal bundles. This is called "reduction of the structure group".

One useful fact about principal G-bundles (which should not be too surprising) is the following statement.

Theorem 4.2.11. *Every morphism of principal G-bundles is an isomorphism.*

Proof. Let $p : P \to B$ and $p' : P' \to B$ be two principal G-bundles over B, and let $f : P \to P'$ be a morphism of principal G-bundles. For surjectivity of f, let $y \in P'$. Consider $x \in P$ such that $p(x) = p'(y)$. Since $p(x) = p'f(x)$ we conclude that $y = f(x)g$ for some $g \in G$. But $f(x)g = f(xg)$, so xg maps to y, as desired. To see that f is injective, suppose $f(x) = f(y)$. Now $p(x) = p'f(x) = p(y)$, so there is some $g \in G$ such that $xg = y$. But $f(y) = f(xg) = f(x)g$, so $g = 1$, as desired. We will leave the continuity of f^{-1} as an exercise to the reader. $\square$

Theorem 4.2.11 says that if we view $\mathrm{Bun}_G(B)$ as a category where the morphisms are given by morphisms of principal G-bundles, then it is a groupoid.

4.3 I-invariance of Bun_G, and G-CW-complexes

Let G be a topological group. We need to show that the functor $\mathrm{Bun}_G : \mathbf{Top}^{op} \to \mathrm{Set}$ is I-invariant, i.e., the projection $X \times I \xrightarrow{\mathrm{pr}} X$ induces an isomorphism $\mathrm{Bun}_G(X) \xrightarrow{\simeq} \mathrm{Bun}_G(X \times I)$. Injectivity is easy: the composite $X \xrightarrow{\mathrm{in}_0} X \times I \xrightarrow{\mathrm{pr}} X$ gives you a splitting $\mathrm{Bun}_G(X) \xrightarrow{\mathrm{pr}_*} \mathrm{Bun}_G(X \times I) \xrightarrow{\mathrm{in}_0} \mathrm{Bun}_G(X)$ whose composite is the identity.

The rest of this lecture is devoted to proving surjectivity. We will prove this when X is a CW-complex (Husemoller does the general case; see [Hus94, §4.9]). We begin with a small digression.

G-CW-complexes

We would like to define CW-complexes with an action of the group G. The naïve definition (of a space with an action of the group G) will not be sufficient; rather, we will require that each cell have an action of G.

In other words, we will build G-CW-complexes out of "G-cells". This is supposed to be something of the form $D^n \times H\backslash G$, where H is a closed subgroup of G. Here, the space $H\backslash G$ is the orbit space, viewed as a right G-space. The boundary of the G-cell $D^n \times H\backslash G$ is just $\partial D^n \times H\backslash G$. More precisely:

Definition 4.3.1. A G-CW-complex is a (right) G-space X with a filtration $0 = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X$ such that for all n, there exists a pushout square:

$$\begin{tikzcd} \coprod \partial D^n_\alpha \times H_\alpha\backslash G \arrow[r] \arrow[d] & \coprod D^n_\alpha \times H_\alpha\backslash G \arrow[d] \\ X_{n-1} \arrow[r] & X_n, \end{tikzcd}$$

and X has the direct limit topology.

Notice that a CW-complex is a G-CW-complex for the trivial group G.

Theorem 4.3.2. *If G is a compact Lie group and M a compact smooth G-manifold, then M admits a G-CW-structure.*

This is the analogue of the classical result that a compact smooth manifold is homotopy equivalent to a CW-complex, but it is much harder to prove the equivariant statement.

Note that if G acts principally (Definition 4.2.3) on P, then every G-CW-structure on P is "free", i.e., $H_\alpha = 0$.

1. If X is a G-CW-complex, then X/G inherits a CW-structure whose n-skeleton is given by $(X/G)_n = X_n/G$.

2. If $P \to X$ is a principal G-bundle, then a CW-structure on X lifts to a G-CW-structure on P.

Proof of I-invariance

Recall that our goal is to prove that every G-bundle over $X \times I$ is pulled back from some vector bundle over X.

As a baby case of Theorem 4.2.1 we will prove that if X is contractible, then any principal G-bundle over X is trivial, i.e., $P \simeq X \times G$ as G-bundles.

Let us first prove the following: if $P \downarrow X$ has a section, then it's trivial. Indeed, suppose we have a section $s : X \to P$. Since P has an action of the group on it, we may extend this to a map $X \times G \to P$ by sending $(x, g) \mapsto gs(x)$. As this is a map of G-bundles over X, it is an isomorphism by Theorem 4.2.11, as desired.

To prove the statement about triviality of any principal G-bundle over a contractible space, it therefore suffices to construct a section for any principal G-bundle. Consider the constant map $X \to P$. Then the following diagram commutes up to homotopy, and hence (by Exercise 3.6.10(1)) there is an *actual* section of $P \to X$, as desired.

$$\begin{tikzcd} & P \arrow[d] \\ X \arrow[ur, "\text{const}"] \arrow[r] & X \end{tikzcd}$$

For the general case, we will assume X is a CW-complex. For notational convenience, let us write $Y = X \times I$. We will use descending induction to construct the desired principal G-bundle over X.

To do this, we will filter Y by subcomplexes. Let $Y_0 = X \times 0$; in general, we define

$$Y_n = X \times 0 \cup X_{n-1} \times I.$$

It follows that we may construct Y_n out of Y_{n-1} via a pushout:

$$\begin{tikzcd}
\coprod_{\alpha\in\Sigma_{n-1}}(\partial D^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \arrow[r] \arrow[d, "\coprod_{\alpha\in\Sigma_{n-1}} f_\alpha \times 1_I \cup \phi_\alpha \times 0"'] & \coprod_\alpha (D_\alpha^{n-1} \times I) \arrow[d] \\
Y_{n-1} \arrow[r] & Y_n,
\end{tikzcd}$$

where the maps f_α and ϕ_α are defined as:

$$\begin{tikzcd}
\partial D_\alpha^{n-1} \arrow[r, "f_\alpha"] \arrow[d] & X_{n-2} \arrow[d] \\
D_\alpha^{n-1} \arrow[r, "\phi_\alpha"'] & X_{n-1}
\end{tikzcd}$$

In other words, the f_α are the attaching maps and the ϕ_α are the characteristic maps.

Consider a principal G-bundle $P \xrightarrow{p} Y = X \times I$. Define $P_n = p^{-1}(Y_n)$; then we can build P_n from P_{n-1} in a similar way:

$$\begin{tikzcd}
\coprod_\alpha(\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \times G \arrow[r] \arrow[d] & \coprod_\alpha (D_\alpha^{n-1} \times I) \times G \arrow[d] \\
P_{n-1} \arrow[r] & P_n
\end{tikzcd}$$

Note that this isn't *quite* a G-CW-structure. Recall that we are attempting to fill in a dotted map:

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d] & P_0 \arrow[d] \\
Y \arrow[r, "\mathrm{pr}"'] & Y_0 = X
\end{tikzcd}$$

finish this...

I'm constructing this inductively– we have $P_{n-1} \to P_0$. So I want to define $\coprod_\alpha(D_\alpha^{n-1} \times I) \times G \to P_0$ that's equivariant. That's the same thing as a map $\coprod_\alpha(D_\alpha^{n-1} \times I) \to P_0$ that's compatible with the map from $\coprod(\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0)$. Namely, I want to fill in:

$$\begin{tikzcd}
\coprod_\alpha(\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \arrow[r] \arrow[d] & \coprod_\alpha(D_\alpha^{n-1} \times I) \arrow[d] \arrow[dddr, dashed] & \\
\coprod_\alpha(\partial D_\alpha^{n-1} \times I \cup D_\alpha^{n-1} \times 0) \times G \arrow[r] \arrow[d] & \coprod_\alpha(D_\alpha^{n-1} \times I) \times G \arrow[d] \arrow[ddr, dashed] & \\
P_{n-1} \arrow[r] \arrow[drr, "\text{induction}"] & P_n \arrow[dr] & \\
 & & P_0 \arrow[d] \\
 & & X
\end{tikzcd} \tag{4.1}$$

Now, I know that $(D^{n-1} \times I, \partial D^{n-1} \times I \cup D^{n-1} \times 0) \simeq (D^{n-1} \times I, D^{n-1} \times 0)$. So what I have is:

$$\begin{tikzcd}
D^{n-1} \times 0 \arrow[r, "\text{induction}"] \arrow[d] & P_0 \arrow[d] \\
D^{n-1} \times I \arrow[r, "\phi\circ pr"'] \arrow[ur, dashed] & X
\end{tikzcd}$$

So the dotted map exists, since $P_0 \to X$ is a fibration!

OK, so note that I haven't checked that the outer diagram in Equation 4.1 commutes, because otherwise we wouldn't get $P_n \to P_0$.

Exercise 4.3.3. Check my question above.

Turns out this is easy, because you have a factorization:

$$\begin{tikzcd}
D^{n-1} \times 0 \arrow[r] \arrow[d] & P_{n-1} \arrow[r, "\text{induction}"] & P_0 \arrow[d] \\
D^{n-1} \times I \arrow[rr, "\phi\circ pr"'] \arrow[urr, dashed] & & X
\end{tikzcd}$$

Oh my god, look what time it is! Oh well, at least we got the proof done.

4.4 Classifying spaces: the Grassmann model

We will now shift our focus somewhat and talk about classifying spaces for principal bundles and for vector bundles. We will do this in two ways: the first will be via the Grassmann model and the second via simplicial methods.

Lemma 4.4.1. *Over a compact Hausdorff space, any n-plane bundle embeds in a trivial bundle.*

Proof. Let $\mathcal{U}$ be a trivializing open cover of the base B; since B is compact, we may assume that $\mathcal{U}$ is finite with k elements. There is no issue with numerability, so there is a subordinate partition of unity ϕ_i. Consider an n-plane bundle $E \to B$. By trivialization, there is a fiberwise isomorphism $p^{-1}(U_i) \xrightarrow{f_i} \mathbf{R}^n$ where the $U_i \in \mathcal{U}$. A map to a trivial bundle is the same thing as a bundle map in the following diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbf{R}^N \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

We therefore define $E \to (\mathbf{R}^n)^k$ via

$$e \mapsto (\phi_i(p(e))f_i(e))_{i=1,\cdots,k}.$$

This is a fiberwise linear embedding, generally called a "Gauss map". Indeed, observe that this map has no kernel on every fiber, so it is an embedding. $\square$

The trivial bundle has a metric on it, so choosing the orthogonal complement of the embedding of Lemma 4.4.1, we obtain:

Corollary 4.4.2. *Over a compact Hausdorff space, any n-plane bundle has a complement (i.e. a $\xi^\perp$ such that $\xi \oplus \xi^\perp$ is trivial).*

Another way to say this is that if B is a compact Hausdorff space with an n-plane bundle ξ, there is a map $f : X \to \mathrm{Gr}_n(\mathbf{R}^{kn})$; this is exactly the Gauss map. It has the property that taking the pullback $f^*\gamma^n$ of the tautologous bundle over $\mathrm{Gr}_n(\mathbf{R}^{kn})$ gives back ξ.

In general, we do not have control over the number k. There is an easy fix to this problem: consider the tautologous bundle γ^n over $\mathrm{Gr}_n(\mathbf{R}^\infty)$, defined as the union of $\mathrm{Gr}_n(\mathbf{R}^m)$ and given the limit topology. This is a

CW-complex of finite type (i.e. finitely many cells in each dimension). Note that $\mathrm{Gr}_n(\mathbf{R}^m)$ are not the m-skeleta of $\mathrm{Gr}_n(\mathbf{R}^\infty)$!

The space $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is "more universal":

Lemma 4.4.3. *Any (numerable) n-plane bundle is pulled back from $\gamma^n \downarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ via the Gauss map.*

Lemma 4.4.3 is a little bit tricky, since the covering can be wildly uncountable; but this is remedied by the following bit of point-set topology.

Lemma 4.4.4. *Let $\mathcal{U}$ be a numerable cover of X. Then there's another numerable cover $\mathcal{U}'$ such that:*

1. *the number of open sets in $\mathcal{U}'$ is countable, and*
2. *each element of $\mathcal{U}'$ is a disjoint union of elements of $\mathcal{U}$.*

If $\mathcal{U}$ is a trivializing cover, then $\mathcal{U}'$ is also a trivializing cover.

Proof. See [Hus94, Proposition 3.5.4]. □

It is now an exercise to deduce Lemma 4.4.3. The main result of this section is the following.

Theorem 4.4.5. *The map $[X, \mathrm{Gr}_n(\mathbf{R}^\infty)] \to \mathrm{Vect}_n(X)$ defined by $[f] \mapsto [f^*\gamma^n]$ is bijective, where $[f]$ is the homotopy class of f and $[f^*\gamma^n]$ is the isomorphism class of the bundle $f^*\gamma^n$.*

This is why $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is also called the *classifying space* for n-plane bundles. The Grassmannian provides a very explicit geometric description for the classifying space of n-plane bundles. There is a more abstract way to produce a classifying space for principal G-bundles, which we will describe in the next section; the Grassmannian is the special case when $G = \mathrm{GL}_n(\mathbf{R})$.

Proof. We have already shown surjectivity, so it remains to prove injectivity. Suppose $f_0, f_1 : X \to \mathrm{Gr}_n(\mathbf{R}^\infty)$ such that $f_0^*\gamma^n$ and $f_1^*\gamma^n$ are isomorphic over X. We need to construct a homotopy $f_0 \simeq f_1$. For ease of notation, let us identify $f_0^*\gamma^n$ and $f_1^*\gamma_n$ with each other; call it $\xi : E \downarrow X$.

The maps f_i are the same thing as Gauss maps $g_i : E \to \mathbf{R}^\infty$, i.e., maps which are fiberwise linear embeddings. The homotopy $f_0 \simeq f_1$ is created by saying that we have a homotopy from g_0 to g_1 through Gauss maps, i.e., through other fiberwise linear embeddings.

In fact, we will prove a much stronger statement: *any two* Gauss maps $g_0, g_1 : E \to \mathbf{R}^\infty$ are homotopic through Gauss maps. This is very far from true if I didn't have a $\mathbf{R}^\infty$ on the RHS there.

Let us attempt (and fail!) to construct an affine homotopy between g_0 and g_1. Consider the map $tg_0 + (1-t)g_1$ for $0 \le t \le 1$. In order for these maps to define a homotopy via Gauss maps, we need the following statement to be true: for all t, if $tg_0(v) + (1-t)g_1(v) = 0 \in \mathbf{R}^\infty$, then $v = 0$. In other words, we need $tg_0 + (1-t)g_1$ to be injective. Of course, this is not guaranteed from the injectivity of g_0 and g_1!

Instead, we will construct a composite of affine homotopies between g_0 and g_1 using the fact that $\mathbf{R}^\infty$ is an infinite-dimensional Euclidean space. Consider the following two linear isometries:

$$\begin{tikzcd}[column sep=small] & \mathbf{R}^\infty = \langle e_0, e_1, \cdots \rangle \arrow[dl, "e_i \mapsto e_{2i}"', "\alpha"] \arrow[dr, "e_i \mapsto e_{2i+1}", "\beta"'] & \\ \mathbf{R}^\infty & & \mathbf{R}^\infty \end{tikzcd}$$

Then, we have four Gauss maps: g_0, $\alpha \circ g_0$, $\beta \circ g_1$, and g_1. There are affine homotopies through Gauss maps:

$$g_0 \simeq \alpha \circ g_0 \simeq \beta \circ g_1 \simeq g_1.$$

We will only show that there is an affine homotopy through Gauss maps $g_0 \simeq \alpha \circ g_0$; the others are left as an exercise. Let t and v be such that $tg_0(v) + (1-t)\alpha g_0(v) = 0$. Since g_0 and αg_0 are Gauss maps, we may suppose that $0 < t < 1$. Since $\alpha g_0(v)_i$ has only even coordinates, it follows by definition of the map α that $g_0(v)$ only had nonzero coordinates only in dimensions congruent to 0 mod 4. Repeating this argument proves the desired result. $\square$

4.5 Simplicial sets

In order to discuss the simplicial model for classifying spaces of G-bundles, we will embark on a long digression on simplicial sets (which will last for three sections). We begin with a brief review of some of the theory of simplicial objects (see also Part 1.1).

Review

We denote by $[n]$ the set $\{0, 1, \cdots, n\}$, viewed as a totally ordered set. Define a category Δ whose objects are the sets $[n]$ for $n \ge 0$, with morphisms order preserving maps. There are maps $d^i : [n] \to [n+1]$ given by omitting i (called coface maps) and codegeneracy maps $s^i : [n] \to [n-1]$ that's the surjection which repeats i. As discussed in Exercise **??**, any order-preserving map can be written as the composite of these maps, and there are famous relations that these things satisfy. They generate the category Δ.

There is a functor $\Delta : \Delta \to \mathbf{Top}$ defined by sending $[n] \mapsto \Delta^n$, the standard n-simplex. To see that this is a functor, we need to show that maps $\phi : [n] \to [m]$ induce maps $\Delta^n \to \Delta^m$. The vertices of Δ^n are indexed by elements of $[n]$, so we may just extend ϕ as an affine map to a map $\Delta^n \to \Delta^m$.

Let X be a space. The set of singular n-simplices $\mathbf{Top}(\Delta^n, X)$ defines the singular simplicial set $\mathrm{Sin} : \Delta^{op} \to \mathrm{Set}$.

Definition 4.5.1. Let $\mathcal{C}$ be a category. Denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$, i.e., the category $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$. We write $X_n = X([n])$, called the n-simplices.

Explicitly, this gives an object $X_n \in \mathcal{C}$ for every $n \geq 0$, as well as maps $d_i : X_{n+1} \to X_n$ and $s_i : X_{n-1} \to X_n$ given by the face and degeneracy maps.

Example 4.5.2. Suppose $\mathcal{C}$ is a small category, for instance, a group. Notice that $[n]$ is a small category, with:

$$[n](i,j) = \begin{cases} \{\leq\} & \text{if } i \leq j \\ \emptyset & \text{else.} \end{cases}$$

We are therefore entitled to think about $\mathrm{Fun}([n], \mathcal{C})$. This begets a simplicial set $N\mathcal{C}$, called the *nerve of* $\mathcal{C}$, whose n-simplices are $(N\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$. Explicitly, an n-simplex in the nerve is $(n+1)$-objects in $\mathcal{C}$ (possibly with repetitions) and a chain of n composable morphisms. The face maps are given by composition (or truncation, at the end of the chain of morphisms). The degeneracy maps just compose with the identity at that vertex.

For example, if G is a group regarded as a category, then $(NG)_n = G^n$.

Realization

The functor Sin transported us from spaces to simplicial sets. Milnor described a way to go the other way.

Let X be a simplicial set. We define the realization $|X|$ as follows:

$$|X| = \left(\coprod_{n \geq 0} \Delta^n \times X_n \right) / \sim,$$

where $\sim$ is the equivalence relation defined as:

$$\Delta^m \times X_m \ni (v, \phi^* x) \sim (\phi_* v, x) \in \Delta^n \times X$$

for all maps $\phi : [m] \to [n]$ where $v \in \Delta^m$ and $x \in X_n$.

Example 4.5.3. The equivalence relation is telling us to glue together simplices as dictated by the simplicial structure on X. To see this in action, let us look at $\phi^* = d_i : X_{n+1} \to X_n$ and $\phi_* = d^i : \Delta^n \to \Delta^{n+1}$. In this case, the equivalence relation then says that $(v, d_i x) \in \Delta^n \times X_n$ is equivalent to $(d^i v, x) \in \Delta^{n+1} \times X_{n+1}$. In other words: the n-simplex indexed by $d_i x$ is identified with the ith face of the $(n+1)$-simplex indexed by x.

There's a similar picture for the degeneracies s^i, where the equivalence relation dictates that every element of the form $(v, s_i x)$ is already represented by a simplex of lower dimension.

Example 4.5.4. Let $n \geq 0$, and consider the simplicial set $\mathrm{Hom}_\Delta(-, [n])$. This is called the "simplicial n-simplex", and is commonly denoted Δ^n for good reason: we have a homeomorphism $|\Delta^n| \simeq \Delta^n$. It is a good exercise to prove this using the explicit definition.

For any simplicial set X, the realization $|X|$ is naturally a CW-complex, with

$$\mathrm{sk}_n|X| = \left(\coprod_{k \leq n} \Delta^k \times X_k \right) / \sim .$$

The face maps give the attaching maps; for more details, see [GJ99, Proposition I.2.3]. This is a very combinatorial way to produce CW-complexes.

The geometric realization functor and the singular simplicial set give two functors going back and forth between spaces and simplicial sets. It is natural to ask: do they form an adjoint pair? The answer is yes:

$$\begin{tikzcd} s\mathrm{Set} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\mathrm{Sin}"] \end{tikzcd}$$

For instance, let X be a space. There is a continuous map $\Delta^n \times \mathrm{Sin}_n(X) \to X$ given by $(v, \sigma) \mapsto \sigma(v)$. The equivalence relation defining $|\mathrm{Sin}(X)|$ says that the map factors through the dotted map in the following diagram:

$$\begin{tikzcd}
{|\mathrm{Sin}(X)|} \arrow[r, dashed] & X \\
\coprod \Delta^n \times \mathrm{Sin}_n(X) \arrow[u, two heads] \arrow[ur]
\end{tikzcd}$$

The resulting map is the counit of the adjunction.

Likewise, we can write down the unit of the adjunction: if $K \in s\mathrm{Set}$, the map $K \to \mathrm{Sin}|K|$ sends $x \in K_n$ to the map $\Delta^n \to |K|$ defined via $v \mapsto [(v, x)]$.

This is the beginning of a long philosophy in semi-classical homotopy theory, of taking any homotopy-theoretic question and reformulating it in simplicial sets. For instance, one can define homotopy groups in simplicial sets. For more details, see [GJ99].

We will close this section with a definition that we will discuss in the next section. Let $\mathcal{C}$ be a category. From our discussion above, we conclude that the realization $|N\mathcal{C}|$ of its nerve is a CW-complex, called the *classifying space* $B\mathcal{C}$ of $\mathcal{C}$; the relation to the notion of classifying space introduced in §4.4 will be elucidated upon in a later section.

4.6 Properties of the classifying space

One important result in the story of geometric realization introduced in the last section is the following theorem of Milnor's.

Theorem 4.6.1 (Milnor)**.** *Let X be a space. The map $|\mathrm{Sin}(X)| \to X$ is a weak equivalence.*

As a consequence, this begets a functorial CW-approximation to X. Unforunately, it's rather large.

In the last section, we saw that $|-|$ was a left adjoint. Therefore, it preserves colimits (Theorem 3.1.13). Surprisingly, it also preserves products:

Exercise 4.6.2 (Hard)**.** Let X and Y be simplicial sets. Their product $X \times Y$ is defined to be the simplicial set such that $(X \times Y)_n = X_n \times Y_n$. Under this notion of product, there is a homeomorphism

$$|X \times Y| \xrightarrow{\simeq} |X| \times |Y|.$$

It is important that this product is taken in the category of k-spaces.

Last time, we defined the classifying space $B\mathcal{C}$ of $\mathcal{C}$ to be $|N\mathcal{C}|$.

Theorem 4.6.3. *The natural map $B(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} B\mathcal{C} \times B\mathcal{D}$ is a homeomorphism*[3].

Proof. It is clear that $N(\mathcal{C} \times \mathcal{D}) \simeq N\mathcal{C} \times N\mathcal{D}$. Since $B\mathcal{C} = |N\mathcal{C}|$, the desired result follows from Exercise 4.6.2. □

In light of Theorem 4.6.3, it is natural to ask how natural transformations behave under the classifying space functor. To discuss this, we need some categorical preliminaries.

The category **Cat** is Cartesian closed (Definition 3.2.5). Indeed, the right adjoint to the product is given by the functor $\mathcal{D} \mapsto \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, as can be directly verified.

Consider the category $[1]$. This is particularly simple: a functor $[1] \to \mathcal{C}$ is just an arrow in $\mathcal{C}$. It follows that a functor $[1] \to \mathcal{D}^{\mathcal{C}}$ is a natural transformation between two functors f_0 and f_1 from $\mathcal{C}$ to $\mathcal{D}$. By our discussion above, this is the same as a functor $\mathcal{C} \times [1] \to \mathcal{D}$.

By Theorem 4.6.3, we have a homeomorphism $B([1]\times\mathcal{C}) \simeq B[1]\times B\mathcal{C}$. One can show that $B[1] = \Delta^1$, so a natural transformation between f_0 and f_1 begets a map $\Delta^1 \times B\mathcal{C} \to B\mathcal{D}$ between Bf_0 and Bf_1. Concretely:

Lemma 4.6.4. *A natural transformation $\theta : f_0 \to f_1$ where $f_0, f_1 : \mathcal{C} \to \mathcal{D}$ induces a homotopy $Bf_0 \sim Bf_1 : B\mathcal{C} \to B\mathcal{D}$.*

An interesting comment is in order. The notion of a homotopy is "reversible", but that is definitely not true for natural transformations! The functor B therefore "forgets the polarity in **Cat**".

Lemma 4.6.4 is quite powerful: consider an adjunction $L \dashv R$ where $L : \mathcal{C} \to \mathcal{D}$; then we have natural transformations given by the unit $1_{\mathcal{C}} \to RL$ and the counit $LR \to 1_{\mathcal{D}}$. By Lemma 4.6.4 we get a homotopy equivalence between $B\mathcal{C}$ and $B\mathcal{D}$. In other words, two categories that are related by any adjoint pair are homotopy equivalent.

A special case of the above discussion yields a rather surprising result. Consider the category $[0]$. Let $\mathcal{D}$ be another category such that there is an adjoint pair $L \dashv R$ where $L : [0] \to \mathcal{D}$. Then L determines an object $*$ of $\mathcal{D}$. Let d be any object of $\mathcal{D}$. We have the counit

[3] Recall that if $\mathcal{C}$ and $\mathcal{D}$ are categories, the product $\mathcal{C} \times \mathcal{D}$ is the category whose objects are pairs of objects of $\mathcal{C}$ and $\mathcal{D}$, and whose morphisms are pairs of morphisms in $\mathcal{C}$ and $\mathcal{D}$.

$LR(d) \to d$; but $LR(d) = *$, so there is a unique morphism $* \to X$. (To see uniqueness, note that the adjunction $L \dashv R$ gives an identification $\mathcal{D}(*, X) = \mathcal{C}(0, 0) = 0$.) In other words, such a category $\mathcal{D}$ is simply a category with an initial object.

Arguing similarly, any category $\mathcal{D}$ with adjunction $L \dashv R$ where $L : \mathcal{D} \to [0]$ is simply a category with a terminal object. From our discussion above, we conclude that if $\mathcal{D}$ is any category with a terminal (or initial) object, then $B\mathcal{D}$ is contractible.

4.7 Classifying spaces of groups

The constructions of the previous sections can be summarized in a single diagram:

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "\text{nerve}"] & s\mathrm{Set} \arrow[d, "|-|"] \\
\mathbf{Gp} \arrow[u, hook] \arrow[r, "B"'] & \mathbf{Top}
\end{tikzcd}$$

The bottom functor is defined as the composite along the outer edge of the diagram. The space BG for a group G is called the *classifying space of* G. At this point, it is far from clear what BG is classifying. The goal of the next few sections is to demystify this definition.

Lemma 4.7.1. *Let G be a group, and $g \in G$. Let $c_g : G \to G$ via $x \mapsto gxg^{-1}$. Then the map $Bc_g : BG \to BG$ is homotopic to the identity.*

Proof. The homomorphism c_g is a functor from G to itself. It suffices to prove that there is a natural transformation θ from the identity to c_g. This is rather easy to define: it sends the only object to the only object: we define $\theta_* : * \to *$ to be the map given by $* \xrightarrow{g} *$ specified by $g \in \mathrm{Hom}_G(*, *) = G$. In order for θ to be a natural transformation, we need the following diagram to commute, which it obviously does:

$$\begin{tikzcd}
* \arrow[r, "g"] \arrow[d, "1"'] & * \arrow[d, "gxg^{-1}"] \\
* \arrow[r, "g"'] & *.
\end{tikzcd}$$

□

Groups are famous for acting on objects. Viewing groups as categories allows for an abstract definition a group action on a set: it is a functor $G \to \mathrm{Set}$. More generally, if $\mathcal{C}$ is a category, an action of $\mathcal{C}$ is a functor $\mathcal{C} \xrightarrow{X} \mathrm{Set}$. We write $X_c = X(c)$ for an object c of $\mathcal{C}$.

Definition 4.7.2. The "translation" category $X\mathcal{C}$ has objects given by

$$\mathrm{ob}(X\mathcal{C}) = \coprod_{c \in \mathcal{C}} X_c,$$

and morphisms defined via $\mathrm{Hom}_{X\mathcal{C}}(x \in X_c, y \in X_d) = \{f : c \to d : f_*(x) = y\}$.

There is a projection $X\mathcal{C} \to \mathcal{C}$. (For those in the know: this is a special case of the Grothendieck construction.)

Example 4.7.3. The group G acts on itself by left translation. We will write $\widetilde{G}$ for this G-set. The translation category $\widetilde{G}G$ has objects as G, and maps $x \to y$ are elements yx^{-1}. This category is "unicursal", in the sense that there is exactly one map from one object to another object. Every object is therefore initial and terminal, so the classifying space of this category is trivial by the discussion at the end of §4.6. We will denote by EG the classifying space $B(\widetilde{G}G)$. The map $\widetilde{G}G \to G$ begets a canonical map $EG \to BG$.

The G also acts on itself by right translation. Because of associativity, the right and left actions commute with each other. It follows that the right action is equivariant with respect to the left action, so we get a right action of G on EG.

Claim 4.7.4. This action of G on EG is a principal action, and the orbit projection is $EG \to BG$.

To prove this, let us contemplate the set $N(\widetilde{G}G)_n$. An element is a chain of composable morphisms. In this case, it is actually just a sequence of $n+1$ elements in G, i.e., $N(\widetilde{G}G)_n = G^{n+1}$. The right action of G is simply the diagonal action. We claim that this is a free action. More precisely:

Lemma 4.7.5 (Shearing). *If G is a group and X is a G-set, and if $X \times^{\Delta} G$ has the diagonal G-action and $X \times G$ has G acting on the second factor by right translation, then $X \times^{\Delta} G \simeq X \times G$ as G-sets.*

Proof. Define a bijection $X \times^{\Delta} G \mapsto X \times G$ via $(x, g) \mapsto (xg^{-1}, g)$. This map is equivariant since $(x, g) \cdot h = (xh, gh)$, while $(xg^{-1}, g) \cdot h = (xg^{-1}, gh)$. The element (xh, gh) is sent to $(xh(gh)^{-1}, gh)$, as desired. The inverse map $X \times G \to X \times^{\Delta} G$ is given by $(x, g) \mapsto (xg, g)$. □

We know that G acts freely on $N(\widetilde{G}G)_n$, soo a nonidentity group element is always going to send a simplex to another simplex. It follows that G acts freely on EG.

To prove the claim, we need to understand the orbit space. The shearing lemma shows that quotienting out by the action of G simply cancels out one copy of G from the product $N(\widetilde{G}G) = G^n$. In symbols:

$$N(\widetilde{G}G)/G \simeq G^n \simeq (NG)_n.$$

Of course, it remains to check the compatibility with the face and degeneracy maps. We will not do this here; but one can verify that everything works out: the realization is just BG!

We need to be careful: the arguments above establish that $EG/G \simeq BG$ when G is a finite group. The case when G is a topological group is more complicated. To describe this generalization, we need a preliminary categorical definition.

Let $\mathcal{C}$ be a category, with objects $\mathcal{C}_0$ and morphisms $\mathcal{C}_1$. Then we have maps $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{\text{compose}} \mathcal{C}_1$ and two maps (source and target) $\mathcal{C}_1 \to \mathcal{C}_0$, and the identity $\mathcal{C}_0 \to \mathcal{C}_1$. One can specify the same data in any category $\mathcal{D}$ with pullbacks. Our interest will be in the case $\mathcal{D} = \mathbf{Top}$; in this case, we call $\mathcal{C}$ a "category in $\mathbf{Top}$".

Let G be a topological group acting on a space X. We can again define XG, although it is now a category in $\mathbf{Top}$. Explicitly, $(XG)_0 = X$ and $(XG)_1 = G \times X$ as spaces. The nerve of a topological category begets a simplicial space. In general, we will have

$$(N\mathcal{C})_n = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times \cdots \times_{\mathcal{C}_0} \mathcal{C}_1.$$

The geometric realization functor works in exactly the same way, so the realization of a simplicial space gets a topological space. The above discussion passes through with some mild topological conditions on G (namely, if G is an absolute neighborhood retract of a Lie group); we conclude:

Theorem 4.7.6. *Let G be an absolute neighborhood retract of a Lie group. Then EG is contractible, and G acts from the right principally. Moreover, the map $EG \to BG$ is the orbit projection.*

A generalization of this result is:

Exercise 4.7.7. Let X be a G-set. Show that

$$EG \times_G X \simeq B(XG).$$

4.8 Classifying spaces and bundles

Let $\pi : Y \to X$ be a map of spaces. This defines a "descent category" $\check{C}(\pi)$ whose objects are the points of Y, whose morphisms are points of $Y \times_X Y$, and whose structure morphisms are the obvious maps. Let cX denote the category whose objects and morphisms are both given by points of X, so that the nerve NcX is the constant simplicial object with value X. There is a functor $\check{C}(\pi) \to cX$ specified by the map π.

Let $\mathcal{U}$ be a cover of X. Let $\check{C}(\mathcal{U})$ denote the descent category associated to the obvious map $\epsilon : \coprod_{U\in\mathcal{U}} U \to X$. It is easy to see that $\epsilon : B\check{C}(\mathcal{U}) \simeq X$ if $\mathcal{U}$ is numerable. The morphism determined by $x \in U \cap V$ is denoted $x_{U,V}$. Suppose $p : P \to X$ is a principal G-bundle. Then $\mathcal{U}$ trivializes p if there are homeomorphisms $t_U : p^{-1}(U) \xrightarrow{\simeq} U \times G$ over U. Specifying such homeomorphisms is the same as a trivialization of the pullback bundle $\epsilon^* P$.

This, in turn, is the same as a functor $\theta_P : \check{C}(\mathcal{U}) \to G$. To see this, we note that the G-equivariant composite $t_V \circ t_U^{-1} : (U\cap V)\times G \to (U\cap V)\times G$ is determined by the value of $(x, 1) \in (U \cap V) \times G$. The map $U \cap V \to G$ is denoted $f_{U,V}$. Then, the functor $\theta_P : \check{C}(\mathcal{U}) \to G$ sends every object of $\check{C}(\mathcal{U})$ to the point, and $x_{U,V}$ to $f_{U,V}(x)$.

On classifying spaces, we therefore get a map $X \xleftarrow{\simeq} B\check{C}(\mathcal{U}) \xrightarrow{\theta_P} BG$, where the map on the left is given by ϵ.

Exercise 4.8.1. Prove that $\theta_P^* EG \simeq \epsilon^* P$.

This suggests that BG is a classifying space for principal G-bundles (in the sense of §4.4). To make this precise, we need to prove that two principal G-bundles are isomorphic if and only if the associated maps $X \to BG$ are homotopic.

To prove this, we will need to vary the open cover. Say that $\mathcal{V}$ *refines* $\mathcal{U}$ if for any $V \in \mathcal{U}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A *refinement* is a function $p : \mathcal{V} \to \mathcal{U}$ such that $V \subseteq p(V)$. A refinement p defines a map $\coprod_{V\in\mathcal{V}} V \to \coprod_{U\in\mathcal{U}} U$, denoted ρ.

As both $\coprod_{V\in\mathcal{V}} V$ and $\coprod_{U\in\mathcal{U}} U$ cover X, we get a map $\check{C}(\mathcal{V}) \to \check{C}(\mathcal{U})$ over cX. Taking classifying spaces begets a diagram:

$$\begin{tikzcd}
B\check{C}(\mathcal{V}) \arrow[r] \arrow[dr] & B\check{C}(\mathcal{U}) \arrow[d] \\
 & X
\end{tikzcd}$$

Let t be trivialization of P for the open cover $\mathcal{U}$. The construction described above begets a functor $B\check{C}(\mathcal{U}) \to BG$, so we get a trivialization s for $\mathcal{V}$. This is a homeomorphism $s_V : p^{-1}(V) \to V \times G$ which fits into the following diagram:

$$\begin{tikzcd}
p^{-1}(V) \arrow[r, "s_V", "\sim"'] \arrow[d] & V \times G \arrow[d] \\
p^{-1}(\rho(V)) \arrow[r, "\sim", "t_{\rho(V)}"'] & \rho(V) \times G
\end{tikzcd}$$

By construction, there is a large commutative diagram:

$$\begin{tikzcd}
B\check{C}(\mathcal{V}) \arrow[r] \arrow[rr, bend left] \arrow[dr, "\sim"] & B\check{C}(\mathcal{U}) \arrow[r] \arrow[d, "\sim"] & BG \\
 & X. &
\end{tikzcd}$$

This justifies dropping the symbol $\mathcal{U}$ in the notation for the map θ_P.

Consider two principal G-bundles over X:

$$\begin{tikzcd}
P \arrow[r, "\simeq"] \arrow[dr] & Q \arrow[d] \\
 & X,
\end{tikzcd}$$

and suppose I have trivializations $(\mathcal{U}, t)$ of P and $(\mathcal{W}, s)$ of Q. Let $\mathcal{V}$ be a common refinement, so that there is a diagram:

$$\begin{tikzcd}
& \check{C}(\mathcal{U}) \arrow[dr, "\theta_P^{\mathcal{U}}"] & \\
\check{C}(\mathcal{V}) \arrow[ur] \arrow[dr] \arrow[rr, bend left=10, "\theta_P^{\mathcal{V}}"] \arrow[rr, bend right=10, "\theta_Q^{\mathcal{V}}"'] & & G \\
& \check{C}(\mathcal{W}) \arrow[ur, "\theta_Q^{\mathcal{W}}"'] &
\end{tikzcd}$$

Included in the diagram is a mysterious natural transformation $\beta : \theta_P^{\mathcal{V}} \to \theta_Q^{\mathcal{V}}$, whose construction is left as an exercise to the reader. Its existence combined with Lemma 4.6.4 implies that the two maps $\theta_P, \theta_Q : B\check{C}(\mathcal{V}) \simeq X \to BG$ are homotopic, as desired.

Should we describe this? It's rather technical...

Topological properties of BG

Before proceeding, let us summarize the constructions discussed so far. Let G be some topological group (assumed to be an absolute neighborhood retract of a Lie group). We constructed EG, which is a contractible space with G acting freely on the right (this works for any topological group). There is an orbit projection $EG \to BG$, which is a principal G-bundle under our assumption on G. The space BG is universal, in the sense that there is a bijection

$$\mathrm{Bun}_G(X) \xleftarrow{\simeq} [X, BG]$$

given by $f \mapsto [f^*EG]$.

Let E be a space such that G acts on E from the left. If $P \to B$ is any principal G-bundle, then $P \times E \to P \times_G E$ is another principal G-bundle. In the case $P = EG$, it follows that if E is a contractible space on which G acts, then the quotient $EG \times_G E$ is a model for BG. Recall that EG is contractible. Therefore, if E is a contractible space on which G acts freely, then the quotient $G\backslash E$ is a model for BG. Of course, one can run the same argument in the case that G acts on E from the right. Although the construction with simplicial sets provided us with a very concrete description of the classifying space of a group G, we could have chosen any principal action on a contractible space in order to obtain a model for BG.

Suppose X is a pointed path connected space. Remember that X has a contractible path space $PX = X^I_*$. The canonical map $PX \to X$ is a fibration, with fiber ΩX.

Consider the case when $X = BG$. Then, we can compare the above fibration with the fiber bundle $EG \to BG$:

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \Omega BG \arrow[d] \\
* \simeq EG \arrow[r, dashed] \arrow[d] & PBG \simeq * \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

The map $EG \to BG$ is nullhomotopic; a choice of a nullhomotopy is exactly a lift into the path space. Therefore, the dotted map $EG \to PBG$ exists in the above diagram. As EG and PBG are both contractible, we conclude that ΩBG is weakly equivalent to G. In fact, this weak equivalence is a H-map, i.e., it commutes up to homotopy with the multiplication on both sides.

Remark 4.8.2 (Milnor)**.** If X is a countable CW-complex, then ΩX is not a CW-complex, but it is *homotopy* equivalent (not just weakly equivalent) to one. Moreover, ΩX is weakly equivalent to a topological group GX such that $BGX \simeq X$.

Examples

We claim that $BU(n) \simeq \mathrm{Gr}_n(\mathbf{C}^\infty)$. To see this, let $V_n(\mathbf{C}^\infty)$ is the contractible space of complex n-frames in $\mathbf{C}^\infty$, i.e., isometric embeddings of $\mathcal{C}^n$ into $\mathcal{C}^\infty$. The Lie group $U(n)$ acts principally on $V_n(\mathbf{C}^\infty)$ by precomposition, and the quotient $V_n(\mathbf{C}^\infty)/U(n)$ is exactly the Grassmannian $\mathrm{Gr}_n(\mathbf{C}^\infty)$. As $\mathrm{Gr}_n(\mathbf{C}^\infty)$ is the quotient of a principal action of $U(n)$ on a contractible space, our discussion in the previous section implies the desired claim.

Let G be a compact Lie group (eg finite).

Theorem 4.8.3 (Peter-Weyl)**.** *There exists an embedding $G \hookrightarrow U(n)$ for some n.*

Since $U(n)$ acts principally on $V_n(\mathbf{C}^\infty)$, it follows G also acts principally on $V_n(\mathcal{C}^\infty)$. Therefore $V_n(\mathcal{C}^\infty)/G$ is a model for BG. It is not necessarily that this the most economic description of BG.

For instance, in the case of the symmetric group Σ_n, we have a much nicer geometric description of the classifying space. Let $\mathrm{Conf}_n(\mathbf{R}^k)$ denote embeddings of $\{1, \cdots, n\} \to \mathbf{R}^k$ (ordered distinct n-tuples). This space is definitely *not* contractible! However, the classifying space $\mathrm{Conf}_n(\mathbf{R}^\infty)$ is contractible. The symmetric group obviously acts freely on this (for finite groups, a principal action is the same as a free action). It follows that $B\Sigma_n$ is the space of *un*ordered configurations of n distinct points in $\mathbf{R}^\infty$. Using Cayley's theorem from classical group theory, we find that if G is finite, a model for BG is the quotient $\mathrm{Conf}_n(\mathbf{R}^\infty)/G$.

We conclude this chapter with a construction of Eilenberg-Maclane spaces via classifying spaces. If A is a topological abelian group, then the multiplication $\mu : A \times A \to A$ is a homomorphism. Applying the classifying space functor begets a map $m : BA \times BA \to BA$. If G is a finite group, then $BA = K(A, 1)$. The map m above gives a topological abelian group model for $K(A, 1)$. There is nothing preventing us from iterating this construction: the space B^2A sits in a fibration

$$BA \to EBA \simeq * \to B^2A.$$

It follows from the long exact sequence in homotopy that the homotopy groups of B^2A are the same as that of BA, but shifted up by one. Repeating this procedure multiple times gives us an explicit model for $K(A, n)$:

$$B^nA = K(A, n).$$

Chapter 5

Spectral sequences

> Spectral sequences are one of those things for which anybody who is anybody must suffer through. Once you've done that, it's like linear algebra. You stop thinking so much about the 'inner workings' later.
>
> – Haynes Miller

5.1 The spectral sequence of a filtered complex

Our goal will be to describe a method for computing the homology of a chain complex. We will approach this problem by assuming that our chain complex is equipped with a filtration; then we will discuss how to compute the associated graded of an induced filtration on the homology, given the homology of the associated graded of the filtration on our chain complex.

We will start off with a definition.

Definition 5.1.1. A *filtered chain complex* is a chain complex C_* along with a sequence of subcomplexes F_sC_* such that the group C_n has a filtration by

$$F_0C_n \subset F_1C_n \subseteq \cdots ,$$

such that $\bigcup F_sC_n = C_n$.

The differential on C_* begets the structure of a chain complex on the associated graded $\mathrm{gr}_sC_n = F_sC_n/F_{s-1}C_n$; in other words, the differential on C_* respects the filtration, hence begets a differential $d : \mathrm{gr}_sC_n \to \mathrm{gr}_sC_{n-1}$.

The canonical example of a filtered chain complex to keep in mind is the homology of a filtered space (such as a CW-complex). Let X be a filtered space, i.e., a space equipped with a filtration $X_0 \subseteq X_1 \subseteq \cdots$ such that $\bigcup X_n = X$. We then have a filtration of the chain complex $C_*(X)$ by the subcomplexes $C_*(X_n)$.

For ease of notation, let us write

$$E^0_{s,t} = \mathrm{gr}_s C_{s+t} = F_s C_{s+t}/F_{s-1}C_{s+t},$$

so the differential on C_* gives a differential $d^0 : E^0_{s,t} \to E^0_{s,t-1}$. A first approximation to the homology of C_* might therefore be the homology $H_{s+t}(\mathrm{gr}_s C_*)$. We will denote this group by $E^1_{s,t}$. This is the homology of the associated graded of the filtration F_*C_*.

We can get an even better approximation to H_*C_* by noticing that there is a differential even on $E^1_{s,t}$. By construction, there is a short exact sequence of chain complexes

$$0 \to F_{s-1}C_* \to F_s C_* \to \mathrm{gr}_s C_* \to 0,$$

so we get a long exact sequence in homology. The differential on $E^1_{s,t}$ is the composite of the boundary map in this long exact sequence with the natural map $H_*(F_{s-1}C_*) \to H_*(\mathrm{gr}_{s-1}C_*)$; more precisely, it is the composite

$$d^1 : E^1_{s,t} = H_{s+t}(\mathrm{gr}_s C_*) \xrightarrow{\partial} H_{s+t-1}(F_{s-1}C_*) \to H_{s+t-1}(\mathrm{gr}_{s-1}C_*) = E^1_{s-1,t}.$$

It is easy to check that $(d^1)^2 = 0$.

This construction is already familiar from cellular chains: in this case, $E^1_{s,t}$ is exactly $H_{s+t}(X_s, X_{s-1})$, which is exactly the cellular s-chains when $t = 0$ (and is 0 if $t \neq 0$). The d^1 differential is constructed in exactly the same way as the differential on cellular chains.

In light of this, we define $E^2_{s,t}$ to be the homology of the chain complex $(E^1_{*,*}, d^1)$; explicitly, we let

$$E^2_{s,t} = \ker(d^1 : E^1_{s,t} \to E^1_{s-1,t})/\operatorname{im}(d^1 : E^1_{s+1,t} \to E^1_{s,t}).$$

Does this also have a differential d^2? The answer is yes. We will inductively define $E^r_{s,t}$ via a similar formula: if $E^{r-1}_{*,*}$ and the differential $d^{r-1} : E^{r-1}_{s,t} \to E^{r-1}_{s-r+1,t+r-2}$ are both defined, we set

$$E^r_{s,t} = \ker(d^{r-1} : E^{r-1}_{s,t} \to E^{r-1}_{s-r+1,t+r-2})/\operatorname{im}(d^{r-1} : E^{r-1}_{s+r-1,t-r+2} \to E^{r-1}_{s,t}).$$

The differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ is defined as follows. Let $[x] \in E^r_{s,t}$ be represented by an element of $x \in E^1_{s,t}$, i.e., an element of $H_{s+t}(\mathrm{gr}_s C_*)$. As above, the boundary map induces natural maps $\partial : H_{s+t}(\mathrm{gr}_s C_*) \to H_{s+t-1}(F_{s-1}C_*)$ and $\partial : H_{s+t-1}(F_{s-r}C_*) \to H_{s+t-1}(\mathrm{gr}_{s-r} C_*)$. The element $\partial x \in H_{s+t-1}(F_{s-1}C_*)$ in fact lifts to an element of $H_{s+t-1}(F_{s-r}C_*)$. The image of this element under ∂ inside $H_{s+t-1}(\mathrm{gr}_{s-r} C_*) = E^1_{s-r,t+r-1}$ begets a class in $E^r_{s-r,t+r-1}$; this is the desired differential.

Exercise 5.1.2. Fill in the missing details in this construction of d^r, and show that $(d^r)^2 = 0$.

We have proven most of the statements in the following theorem.

Theorem-Definition 5.1.3. *Let F_*C be a filtered complex. Then there exist natural*

1. *bigraded groups $(E^r_{s,t})_{s\geq 0, t\in \mathbf{Z}}$ for any $r \geq 0$, and*
2. *differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ for any $r \geq 0$.*

such that $E^{r+1}_{s,t}$ is the homology of $(E^r_{*,*}, d^r)$, and (E^0, d^0) and (E^1, d^1) are as above. If F_*C is bounded below, then this spectral sequence converges to $\mathrm{gr}_* H_*(C)$, *in the sense that there is an isomorphism:*

$$E^\infty_{s,t} \simeq \mathrm{gr}_s H_{s+t}(C). \tag{5.1}$$

This is called a *homology spectral sequence.* One should think of each $E^r_{*,*}$ as a "page", with lattice points $E^r_{s,t}$. We still need to describe the symbols used in the formula (5.1).

There is a filtration $F_s H_n(C) := \mathrm{im}(H_n(F_s C) \to H_n(C))$, and $\mathrm{gr}_s H_*(C)$ is the associated graded of this filtration. Taking formula (5.1) literally, we only obtain information about the associated graded of the homology of C_*. Over vector spaces, this is sufficient to determine the homology of C_*, but in general, one needs to solve an extension problem.

To define the notation E^∞ used above, let us assume that the filtration F_*C is bounded below (so $F_{-1}C = 0$). It follows that $E^0_{s,t} = F_s C_{s+t}/F_{s-1}C_{s+t} = 0$ for $s < 0$, so the spectral sequence of Theorem-Definition 5.1.3 is a "right half plane" spectral sequence. It follows that in our example, the differentials from the group in position (s,t) must have vanishing d^{s+1} differential.

In turn, this implies that there is a surjection $E^{s+1}_{s,t} \to E^{s+2}_{s,t}$. This continues: we get surjections

$$E^{s+1}_{s,t} \to E^{s+2}_{s,t} \to E^{s+3}_{s,t} \to \cdots ,$$

and the direct limit of this directed system is defined to be $E^\infty_{s,t}$.

For instance, in the case of cellular chains, we argued above that $E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$, so that $E^1_{s,t} = 0$ if $t \neq 0$, and the d^1 differential is just the differential in the cellular chain complex. It follows that $E^2_{s,t} = H^{cell}_s(X)$ if $t = 0$, and is 0 if $t \neq 0$. All higher differentials are therefore zero (because either the target or the source is zero!), so $E^r_{s,t} = E^2_{s,t}$ for every $r \geq 2$. In particular $E^\infty_{s,t} = H^{cell}_s(X)$ when $t = 0$, and is 0 if $t \neq 0$. There are no extension problems either: the filtration on X is bounded below, so Theorem-Definition 5.1.3 implies that $\mathrm{gr}_s H_{s+0}(X) = H_s(X) \simeq H^{cell}_s(X) = E^\infty_{s,t}$.

In a very precise sense, the datum of the spectral sequence of a filtered complex F_*C_* determines the homology of C_*:

Corollary 5.1.4. *Let $C \xrightarrow{f} D$ be a map of filtered complexes. Assume that the filtration on C and D are bounded below and exhaustive. Assume also that $E^r(f)$ is an isomorphism for some r. Then $f_* : H_*(C) \to H_*(D)$ is an isomorphism.*

Proof. The map $E^r(f)$ is an isomorphism which is also also a chain map, i.e., it is compatible with the differential d^r. It follows that $E^{r+1}(f)$ is an isomorphism. By induction, we conclude that $E^\infty_{s,t}(f)$ is an isomorphism for all s, t. Theorem-Definitino 5.1.3 implies that the map $\mathrm{gr}_s(f_*) : \mathrm{gr}_s H_*(C) \to \mathrm{gr}_s H(D)$ is an isomorphism.

We argue by induction using the short exact sequence:

$$0 \to F_s H_*(C) \to F_{s+1} H_*(C) \to \mathrm{gr}_{s+1} H_*(C) \to 0.$$

We have $\mathrm{gr}_0 H_n(C) = F_0 H_n(C) = \mathrm{im}(H_n(F_0 C) \to H_n(C))$, so the base case follows from the five lemma. In general, f induces an isomorphism an isomorphism on the groups on the left (by the inductive hypothesis) and right (by the above discussion), so it follows that $F_s f_*$ is an isomorphism by the five lemma. Since the filtration F_*C_* was exhaustive, it follows that f_* is an isomorphism. □

Serre spectral sequence

In this book, we will give two constructions of the Serre spectral sequence. The second will appear later. Fix a fibration $E \xrightarrow{p} B$, with B a CW-complex. We obtain a filtration on E by taking the preimage of the s-skeleton of B, i.e., $E_s = p^{-1}\mathrm{sk}_s B$. It follows that there is a filtration on $S_*(E)$ given by

$$F_s S_*(E) = \mathrm{im}(S_*(p^{-1}\mathrm{sk}_s(B)) \to S_*E).$$

This filtration is bounded below and exhaustive. The resulting spectral sequence of Theorem-Definition 5.1.3 is the Serre spectral sequence.

Let us be more explicit. We have a pushout square:

$$\begin{tikzcd}
E_{s-1} \arrow[r] \arrow[d] & E_s \arrow[d] \\
B_{s-1} \arrow[r] & B_s \\
\coprod_{\alpha\in\Sigma_s} S^{s-1}_\alpha \arrow[r] \arrow[u] & \coprod_{\alpha\in\Sigma_s} D^s_\alpha \arrow[u]
\end{tikzcd}$$

Let F_α be the preimage of the center of α cell. In particular, we have a pushout:

$$\begin{tikzcd}
E_{s-1} \arrow[r] & E_s \\
\coprod_{\alpha\in\Sigma_s} S^{s-1}_\alpha \times F_\alpha \arrow[r] \arrow[u] & \coprod_{\alpha\in\Sigma_s} D^s_\alpha \times F_\alpha \arrow[u]
\end{tikzcd}$$

We know that

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) = \bigoplus_{\alpha\in\Sigma_s} H_{s+t}(D^s_\alpha \times F_\alpha, S^{s-1}_\alpha \times F_\alpha).$$

We can suggestively view this as $\bigoplus_{\alpha\in\Sigma_s} H_{s+1}((D^s_\alpha, S^{s-1}_\alpha) \times F_\alpha)$. By the Künneth formula (at least, if our coefficients are in a field), this is exactly $\bigoplus_{\alpha\in\Sigma_s} H_t(F_\alpha)$. In analogy with our discussion above regarding the spectral sequence coming from the cellular chain complex, one would like to think of this as "$C_s(B; H_t(F_\alpha))$". Sadly, there are many things wrong with writing this.

For instance, suppose B isn't connected. The fibers F_α could have completely different homotopy types, so the symbol $C_s(B; H_t(F_\alpha))$ does not make any sense. Even if B was path-connected, there would still be no canonical way to identify the fibers over different points. Instead, we obtain a functor $H_t(p^{-1}(-)) : \Pi_1(B) \to \mathbf{Ab}$, i.e., a "local coefficient system" on B. So, the right thing to say is "$E^2_{s,t} = H_s(B; \underline{H_t(\text{fiber})})$".

To define precisely what $H_s(B; \underline{H_t(\text{fiber})})$ means, let us pick a base-point in B, and build the universal cover $\widetilde{B} \to B$. This has an action of $\pi_1(B, *)$, so we obtain an action of $\pi_1(B, *)$ on the chain complex $S_*(\widetilde{B})$. Said differently, $S_*(\widetilde{B})$ is a chain complex of right modules over

$\mathbf{Z}[\pi_1(B)]$. If B is connected, a local coefficient system on B is the same thing as a (left) action of $\pi_1(B)$ on $H_t(p^{-1}(*))$. Then, we define a chain complex:

$$S_*(B; \underline{H_t(p^{-1}(*))}) = S_*(\widetilde{B}) \otimes_{\mathbf{Z}[\pi_1(B)]} H_t(p^{-1}(*));$$

the differential is induced by the $\mathbf{Z}[\pi_1(B)]$-equivariant differential on $S_*(\widetilde{B})$. Our discussion above implies that the homology of this chain complex is the E^2-page.

We will always be in the case where that local system is trivial, so that $H_*(B; \underline{H_*(p^{-1}(*))})$ is just $H_*(B; H_*(p^{-1}(*)))$. For instance, this is the case if $\pi_1(B)$ acts trivially on the fiber. In particular, this is the case if B is simply connected.

5.2 Exact couples

Let us begin with a conceptual discussion of exact couples. As a special case, we will recover the construction of the spectral sequence associated to a filtered chain complex (Theorem-Definition 5.1.3).

Definition 5.2.1. An *exact couple* is a diagram of (possilby (bi)graded) abelian groups

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which is exact at each joint.

As $jkjk = 0$, the map $E \xrightarrow{jk} E$ is a differential, denoted d. An exact couple determines a "derived couple":

$$\begin{tikzcd} A' \arrow[rr, "i'=i|_{\mathrm{im}\, i}"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd} \tag{5.2}$$

where $A' = \mathrm{im}(i)$ and $E' = H_*(E, d)$. Iterating this procedure, we get exact sequences

$$\begin{tikzcd} A^r \arrow[rr, "i_r"] & & A^r \arrow[dl, "j_r"] \\ & E^r \arrow[ul, "k_r"] & \end{tikzcd}$$

where the next exact couple is the derived couple of the preceding exact couple.

It remains to define the maps in the above diagram. Define $j'(ia) = ja$. *A priori*, it is not clear that this well-defined. For one, we need $[ja] \in E'$; for this, we must check that $dja = 0$, but $d = jk$, and $jkja = 0$ so this follows. We also need to check that j' is well-defined modulo boundaries. To see this, suppose $ia = 0$. We then need to know that ja is a boundary. But if $ia = 0$, then $a = ke$ for some e, so $ja = jke = de$, as desired.

Define $k' : H(E, d) \to \operatorname{im} i$ via $k'([e]) \mapsto ke$. As before, we need to check that this is well-defined. For instance, we have to check that $ke \in \operatorname{im} i$. Since $de = 0$ and $d = jk$, we learn that $jke = 0$. Thus ke is killed by j, and therefore, by exactness, is in the image of i. We also need to check that k' is independent of the choice of representative of the homology class. Say $e = de'$. Then $kd = kde' = kjke' = 0$.

Exercise 5.2.2. Check that these maps indeed make diagram (5.2) into an exact couple.

It follows that we obtain a spectral sequence, in the sense of Theorem-Definition 5.1.3.

Exercise 5.2.3. By construction,

$$A^r = \operatorname{im}(i^r|_A) = i^r A.$$

Show, by induction, that

$$E^r = \frac{k^{-1}(i^r A)}{j(\ker i^r)}$$

and that

$$i_r(a) = ia, \; j_r(i^r a) = [ja], \; k_r(e) = ke.$$

Intuitively: an element of E^1 will survive to E^r if its image in A^1 can be pulled back under i^{r-1}. The differential d^r is obtained by the homology class of the pushforward of this preimage via j to E^1.

Remark 5.2.4. In general, the groups in consideration will be bigraded. It is clear by construction that $\deg(i') = \deg(i)$, $\deg(k') = \deg(k)$, and $\deg(j') = \deg(j) - \deg(i)$. It follows by an easy inductive argument that

$$\deg(d^r) = \deg(j) + \deg(k) - (r-1)\deg(i).$$

The canonical example of an exact couple is that of a filtered complex; the resulting spectral sequence is precisely the spectral sequence of Theorem-Definition 5.1.3. If C_* is a filtered chain complex, we let $A_{s,t} = H_{s+t}(F_s C_*)$, and $E^1_{s,t} = E_{s,t} = H_{s+t}(\mathrm{gr}_s C_*)$. The exact couple is precisely that which arises from the long exact sequence in homology associated to the short exact sequence of chain complexes

$$0 \to F_{s-1}C_* \to F_s C_* \to \mathrm{gr}_s C_* \to 0.$$

Note that in this case, the exact couple is one of bigraded groups, so Remark 5.2.4 dictates the bidegrees of the differentials.

We will conclude this section with a brief discussion of the convergence of the spectral sequence constructed above. Assume that $i : A \to A$ satisfies the property that

$$\ker(i) \cap \bigcap i^r A = 0.$$

Let $\widetilde{A}$ be the colimit of the directed system

$$A \xrightarrow{i} A \xrightarrow{i} A \to \cdots$$

There is a natural filtration on $\widetilde{A}$. Let I denote the image of the map $A \to \widetilde{A}$; the kernel of this map is $\bigcup \ker(i^r)$. The groups $i^r I$ give an exhaustive filtration of $\widetilde{A}$, and the quotients $i^r I / i^{r+1} I$ are all isomorphic to I/iI (since i is an isomorphism on $\widetilde{A}$). Then we have an isomorphism

$$E^\infty \simeq I/iI. \tag{5.3}$$

Indeed, we know from Exercise 5.2.3 that

$$E^\infty \simeq \frac{k^{-1}\left(\bigcap i^r A\right)}{j\left(\bigcup \ker i^r\right)};$$

by our assumption on i, this is

$$\frac{\ker(k)}{j\left(\bigcup \ker i^r\right)} \simeq \frac{j(A)}{j\left(\bigcup \ker i^r\right)}.$$

But there is an isomorphism $A/iA \to j(A)$ which clearly sends $iA + \bigcup \ker i^r$ to $j\left(\bigcup \ker i^r\right)$. By our discussion above, $A/\bigcup \ker i^r \simeq I$, and $iA/\bigcup \ker i^r \simeq iI$. Modding out by iI on both sides, we get (5.3).

5.3 The homology of ΩS^n, and the Serre exact sequence

The goal of this section is to describe a computation of the homology of ΩS^n via the Serre spectral sequence, as well as describe a "degenerate" case of the Serre spectral sequence.

The homology of ΩS^n

Let us first consider the case $n = 1$. The space ΩS^1 is the base of a fibration $\Omega S^1 \to PS^1 \to S^1$. Comparing this to the fibration $\mathbf{Z} \to \mathbf{R} \to S^1$, we find that $\Omega S^1 \simeq \mathbf{Z}$. Equivalently, this follows from the discussion in §4.8 and the observation that $S^1 \simeq K(\mathbf{Z}, 1)$.

Having settled that case, let us now consider the case $n > 1$. Again, there is a fibration $\Omega S^n \to PS^n \to S^n$. In general, if $F \to E \to B$ is a fibration and the space F has torsion-free homology, we can (via the universal coefficients theorem) rewrite the E^2-page:

$$E^2_{s,t} = H_s(B; H_t(F)) \simeq H_s(B) \otimes H_t(F).$$

Since S^n has torsion-free homology, the Serre spectral sequence (see §5.1) runs:

$$E^2_{s,t} = H_s(S^n) \otimes H_t(\Omega S^n) \Rightarrow H_*(PS^n) = \mathbf{Z}.$$

Since $H_s(S^n)$ is concentrated in degrees 0 and n, we learn that E^2-page is concentrated in columns $s = 0, n$. For instance, if $n = 4$, then the E^2-page (without the differentials drawn in) looks like:

$$
H_*(\Omega S^4)\quad
\begin{array}{c|cccccc}
0 & H_5(\Omega S^n) & & & & H_5(\Omega S^n) & \\
0 & H_4(\Omega S^n) & & & & H_4(\Omega S^n) & \\
0 & H_3(\Omega S^n) & & & & H_3(\Omega S^n) & \\
0 & H_2(\Omega S^n) & & & & H_2(\Omega S^n) & \\
0 & H_1(\Omega S^n) & & & & H_1(\Omega S^n) & \\
0 & H_0(\Omega S^n) & & & & H_0(\Omega S^n) & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

$$H_*(S^4)$$

We know that $H_0(\Omega S^n) = \mathbf{Z}$. Since the target has homology concentrated in degree 0, we know that $E^2_{n,0}$ has to be killed. The only possibility is that it is hit by a differential, or that it supports a nonzero differential.

There are not very many possibilities for differentials in this spectral sequence. In fact, up until the E^n-page, there are no differentials (either the target or source of the differential is zero), so $E^2 \simeq E^3 \simeq \cdots \simeq E^n$. On the E^n-page, there is only one possibility for a differential: $d^n : E^2_{n,0} \to E^n_{0,n-1}$. This differential has to be a monomorphism because if it had anything in its kernel, that will be left over in the position. In our example above (with $n = 4$), we have

$$
\begin{tikzcd}
0 & H_5(\Omega S^n) & & & & H_5(\Omega S^n) & \\
0 & H_4(\Omega S^n) & & & & H_4(\Omega S^n) & \\
0 & H_3(\Omega S^n) & & & & H_3(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
0 & H_2(\Omega S^n) & & & & H_2(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
0 & H_1(\Omega S^n) & & & & H_1(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
0 & H_0(\Omega S^n) & & & & H_0(\Omega S^n) \arrow[uuullll, "d^4"'] & \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{tikzcd}
$$

$H_*(\Omega S^4)$

$H_*(S^4)$

However, we still do not know the group $E^n_{0,n-1}$. If it is bigger than $\mathbf{Z}$, then d^n is not surjective. There can be no other differentials on the E^r-page for $r \geq n+1$ (because of sparsity), so the d^n differential is our last hope in killing everything in degree $(0, n-1)$. This means that d^n is an epimorphism. We find that $E^n_{0,n-1} = H_{n-1}(\Omega S^n) \simeq \mathbf{Z}$, and that d^n is an isomorphism.

We have now discovered that $H_{n-1}(\Omega S^n) \simeq \mathbf{Z}$ — but there is a lot more left in the E^2-page! For instance, we still have a $\mathbf{Z}$ in $E^n_{n,n-1}$. Because $H^*(PS^n)$ is concentrated in degree 0, this, too, must die! We are in exactly the same situation as before, so the same arguments show that the differential $d^n : E^n_{n,n-1} \to E^n_{0,2(n-1)}$ has to be an isomorphism. Iterating this argument, we find:

$$H_q(\Omega S^n) \simeq \begin{cases} \mathbf{Z} & \text{if } (n-1)|q \geq 0 \\ 0 & \text{else} \end{cases}$$

This is a great example of how useful spectral sequences can be.

Remark 5.3.1. The loops ΩX is an associative H-space. Thus, as is the case for any H-space, the homology $H_*(\Omega X; R)$ is a graded associative algebra. Recall that the suspension functor Σ is the left adjoint to the loops functor Ω, so there is a unit map $A \to \Omega\Sigma A$. This in turn begets a map $\widetilde{H}_*(A) \to H_*(\Omega\Sigma A)$.

Recall that the universal tensor algebra $\mathrm{Tens}(\widetilde{H}_*(A))$ is the free associative algebra on $\widetilde{H}_*(A)$. Explicitly:

$$\mathrm{Tens}(\widetilde{H}_*(A)) = \bigoplus_{n\geq 0} \widetilde{H}_*(A)^{\otimes n}.$$

In particular, by the universal property of $\mathrm{Tens}(\widetilde{H}_*(A))$, we get a map $\alpha : \mathrm{Tens}(\widetilde{H}_*(A)) \to H_*(\Omega\Sigma A)$.

Theorem 5.3.2 (Bott-Samelson)**.** *The map α is an isomorphism if R is a PID and $H_*(A)$ is torsion-free.*

For instance, if $A = S^{n-1}$ then $\Omega S^n = \Omega\Sigma A$. Theorem 5.3.2 then shows that

$$H_*(\Omega S^n) = \mathrm{Tens}(\widetilde{H}_*(S^{n-1})) = \langle 1, x, x^2, x^3, \cdots \rangle,$$

where $|x| = n-1$. It is a mistake to call this "polynomial", since if n is even, x is an odd class (in particular, x squares to zero by the Koszul sign rule).

Theorem 5.3.2 suggests thinking of $\Omega\Sigma A$ as the "free associative algebra" on A. Let us make this idea more precise.

Remark 5.3.3. The space ΩA is homotopy equivalent to a topological monoid $\Omega_M A$, called the *Moore loops* on A. This means that $\Omega_M A$ has a *strict* unit and is *strictly* associative (i.e., not just up to homotopy). Concretely,

$$\Omega_M A := \{(\ell, \omega) : \ell \in \mathbf{R}_{\geq 0}, \omega : [0,\ell] \to A, \omega(0) = * = \omega(\ell)\},$$

topologized as a subspace of the product. There is an identity class $1 \in \Omega_M A$, given by $1 = (0, c_*)$ where c_* is the constant loop at the basepoint $*$. The addition on this space is just given by concatenatation. In particular, the lengths get added; this overcomes the obstruction to ΩA not being strictly associative, so the Moore loops $\Omega_M A$ are indeed strictly associative. If the basepoint is nondegenerate, it is not hard to see that the inclusion $\Omega A \hookrightarrow \Omega_M A$ is a homotopy equivalence.

Given the space A, we can form the free monoid $\mathrm{FreeMon}(A)$. The elements of this space are just formal sequences of elements of A (with topology coming from the product topology), and the multiplication is given by juxtaposition. Let us adjoin the element $1 = *$. As with all free constructions, there is a map $A \to \mathrm{FreeMon}(A)$ which is universal in the sense that any map $A \to M$ to a monoid factors through $\mathrm{FreeMon}(A)$.

The unit $A \to \Omega\Sigma A$ is a map from A to a monoid, so we get a monoid map $\beta : \mathrm{FreeMon}(A) \to \Omega\Sigma A$.

Theorem 5.3.4 (James). *The map $\beta : \mathrm{FreeMon}(A) \to \Omega\Sigma A$ is a weak equivalence if A is path-connected.*

The free monoid looks very much like the tensor product, as the following theorem of James shows.

Theorem 5.3.5 (James). *Let $J(A) = \mathrm{FreeMon}(A)$. There is a splitting:*

$$\Sigma J(A) \simeq_w \Sigma \left(\bigvee_{n \geq 0} A^{\wedge n} \right).$$

Applying homology to the splitting of Theorem 5.3.5 shows that:

$$\widetilde{H}_*(J(A)) \simeq \bigoplus_{n \geq 0} \widetilde{H}_*(A^{\wedge n}).$$

Assume that our coefficients are in a PID, and that $\widetilde{H}_*(A)$ is torsion-free; then this is just $\bigoplus_{n \geq 0} \widetilde{H}_*(A)^{\otimes n}$. In particular, we recover our computation of $H_*(\Omega S^n)$ from these general facts.

The Serre exact sequence

Suppose $\pi : E \to B$ is a fibration over a path-connected base. Assume that $\widetilde{H}_s(B) = 0$ for $s < p$ where $p \geq 1$. Let $* \in B$ be a chosen basepoint. Denote by F the fiber $\pi^{-1}(*)$. Assume $\widetilde{H}_t(F) = 0$ for $t < q$, where $q \geq 1$. We would like to use the Serre spectral sequence to understand $H_*(E)$. As always, we will assume that $\pi_1(B)$ acts trivially on $H_*(F)$.

Recall that the Serre spectral sequence runs

$$E^2_{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E).$$

Our assumptions imply that $E^2_{0,0} = \mathbf{Z}$, and $E^2_{0,t} = 0$ for $t < q$. Moreover, $E^2_{s,0} = 0$ for $s < p$. In particular, $E^2_{0,q+t} = H_{q+t}(F)$ and $E^2_{p+k,0} = H_{p+k}(B)$ — the rest of the spectral sequence is mysterious.

By sparsity, the first possible differential is $d^p : H_p(B) \to H_{p-1}(F)$, and $d^{p+q} : H_{p+1}(B) \to H_p(F)$. In the mysterious zone, there are differentials that hit $E^2_{p,q}$.

Again by sparsity, the only differential is $d^s : E^s_{s,0} \to E^s_{0,s-1}$ for $s < p + q - 1$. This is called a *transgression*. It is the last possible differential which has a chance at being nonzero. This means that the cokernel of d^s is $E^\infty_{0,s-1}$. There is also a map $E^\infty_{s,0} \to E^s_{s,0}$. We obtain a mysterious composite

$$0 \to E^\infty_{s,0} \to E^s_{s,0} \simeq H_s(B) \xrightarrow{d^s} E^s_{0,s-1} \simeq H_{s-1}(F) \to E^\infty_{0,s-1} \to 0. \quad (5.4)$$

Let $n < p + q - 1$. Recall that $F_s H_n(E) = \mathrm{im}(H_*(\pi^{-1}(\mathrm{sk}_s(B))) \to H_*(E))$, so $F_0 H_n(E) = E^\infty_{0,n}$. Here, we are using the fact that $F_{-1}H_*(E) = 0$. In particular, there is a map $E^\infty_{0,n} \to H_n(E)$. By our hypotheses, there is only one other potentially nonzero filtration in this range of dimensions, so we have a short exact sequence:

$$0 \to F_0 H_n(E) = E^\infty_{0,n} \to H_n(E) \to E^\infty_{n,0} \to 0 \quad (5.5)$$

Splicing the short exact sequences (5.4) and (5.5), we obtain a long exact sequence:

$$H_{p+q-1}(F) \to \cdots \to H_n(F) \to H_n(E) \to H_n(B) \xrightarrow{\text{transgression}} H_{n-1}(F) \to H_{n-1}(E)$$

This is called the *Serre exact sequence*. In this range of dimensions, homology behaves like homotopy.

5.4 Edge homomorphisms, transgression

Recall the Serre spectral sequence for a fibration $F \to E \to B$ has E^2-page given by

$$E^2_{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E).$$

If B is path-connected, $\widetilde{H}_t(F) = 0$ for $t < q$, $\widetilde{H}_s(B) = 0$ for $s < p$, and $\pi_1(B)$ acts trivially on $H_*(F)$, we showed that there is a long exact sequence (the Serre exact sequence)

$$H_{p+q-1}(F) \xrightarrow{\bullet} H_{p+q-1}(E) \to H_{p+q-1}(B) \to H_{p+q-2}(F) \to \cdots \quad (5.6)$$

Let us attempt to describe the arrow marked by $\bullet$.

Let $(E^r_{p,q}, d^r)$ be any spectral sequence such that $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$; such a spectral sequence is called a *first quadrant* spectral sequence. The Serre spectral sequence is a first quadrant spectral sequence. In a first quadrant spectral sequence, the d^2-differential

$d^2 : E^2_{0,t} \to E^2_{-2,t+1}$ is zero, since $E^2_{s,t}$ vanishes for $s < 0$. This means that $H_t(F) = H_0(B; H_t(F)) = E^2_{0,t}$ surjects onto $E^3_{0,t}$. Arguing similarly, this surjects onto $E^4_{0,t}$. Eventually, we find that $E^r_{0,t} \simeq E^{t+2}_{0,t}$ for $r \geq t+2$. In particular,

$$E^{t+2}_{0,t} \simeq E^\infty_{0,t} \simeq \mathrm{gr}_0 H_t(E) \simeq F_0 H_t(E),$$

which sits inside $H_t(E)$. The composite

$$E^2_{0,t} = H_t(F) \to E^3_{0,t} \to \cdots \to E^{t+2}_{0,t} \subseteq F_0 H_t(E) \to H_t(E)$$

is precisely the map $\bullet$! Such a map is known as an *edge homomorphism.*

The map $F \to E$ is the inclusion of the fiber; it induces a map $H_t(F) \to H_t(E)$ on homology. We claim that this agrees with $\bullet$. Recall that $F_0 H_t(E)$ is defined to be $\mathrm{im}(H_t(F_0 E) \to H_t(E))$. In the construction of the Serre spectral sequence, we declared that $F_0 E$ is exactly the preimage of the zero skeleton. Since B is simply connected, we find that $F_0 E$ is exactly the fiber F.

To conclude the proof of the claim, consider the following diagram:

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F \arrow[d] \\
F \arrow[r] \arrow[d] & E \arrow[d] \\
* \arrow[r, hook] & B
\end{tikzcd}$$

The naturality of the Serre spectral sequence implies that there is an induced map of spectral sequences. Tracing through the symbols, we find that this observation proves our claim.

The long exact sequence (5.6) also contains a map $H_s(E) \to H_s(B)$. The group $F_s H_s(E) = H_s(E)$ maps onto $\mathrm{gr}_s H_s(E) \simeq E^\infty_{s,0}$. If F is connected, then $H_s(B) = H_s(B; H_0(F)) = E^2_{s,0}$. Again, the d^2-differential $d^2 : E^2_{s+2,-1} \to E^2_{s,0}$ is trivial (since the source is zero). Since $E^3 = \ker d^2$, we have an injection $E^3_{s,0} \to E^2_{s,0}$. Repeating the same argument, we get injections

$$E^\infty_{s,0} = E^{s+1}_{s,0} \to \cdots \to E^2_{s,0} \to E^2_{s,0} = H_s(B).$$

Composing with the map $H_s(E) \to E^\infty_{s,0}$ gives the desired map $H_s(E) \to H_s(B)$ in the Serre exact sequence. This composite is also known as an edge homomorphism.

As above, this edge homomorphism is the map induced by $E \to B$. This can be proved by looking at the induced map of spectral sequences coming from the following map of fiber sequences:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d] \\ E \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

The topologically mysterious map is the boundary map $\partial : H_{p+q-1}(B) \to H_{p+q-2}(F)$. Such a map is called a *transgression*. Again, let $(E^r_{s,t}, d^r)$ be a first quadrant spectral sequence. In our case, $E^2_{n,0} = H_n(B)$, at least F is connected. As above, we have injections

$$i : E^n_{n,0} \to \cdots \to E^3_{n,0} \to E^2_{n,0} = H_n(B).$$

Similarly, we have surjections

$$s : E^2_{0,n-1} \to E^3_{0,n-1} \to \cdots \to E^n_{0,n-1}.$$

There is a differential $d^n : E^n_{n,0} \to E^n_{0,n-1}$. The transgression is defined as the *linear relation* (not a function!) $E^2_{n,0} \to E^2_{0,n-1}$ given by

$$x \mapsto i^{-1} d^n s^{-1}(x).$$

However, the reader should check that in our case, the transgression is indeed a well-defined function.

Topologically, what is the origin of the transgression? There is a map $H_n(E,F) \xrightarrow{\pi_*} H_n(B,*)$, as well as a boundary map $\partial : H_n(E,F) \to H_{n-1}(F)$. We claim that:

$$\operatorname{im} \pi_* = \operatorname{im}(E^n_{n,0} \to H_n(B) = E^2_{n,0}), \quad \partial \ker \pi_* = \ker(H_{n-1}(F) = E^2_{0,n-1} \to E^n_{0,n-1}).$$

Proof sketch. Let $x \in H_n(B)$. Represent it by a cycle $c \in Z_n(B)$. Lift it to a chain in the total space E. In general, this chain will not be a cycle (consider the Hopf fibration). The differentials record this boundary; let us recall the geometric construction of the differential. Saying that the class x survives to the E^n-page is the same as saying that we can find a lift to a chain σ in E, with $d\sigma \in S_{n-1}(F)$. Then $d^n(x)$ is represented by the class $[dc] \in H_{n-1}(F)$. This is precisely the trangression.

Informally, we lift something from $H_n(B)$ to $S_n(E)$; this is well-defined up to something in F. In particular, we get an element in $H_n(E,F)$. We send it, via ∂, to an element of $H_{n-1}(F)$ — and this is precisely the transgression. $\square$

An example

We would like to compare the Serre exact sequence (5.6) with the homotopy exact sequence:

$$* \to \pi_{p+q-1}(F) \to \pi_{p+q-1}(E) \to \pi_{p+q-1}(B) \xrightarrow{\partial} \pi_{p+q-2}(F) \to \cdots$$

There are Hurewicz maps $\pi_{p+q-1}(X) \to H_{p+q-1}(X)$. We claim that there is a map of exact sequences between these two long exact sequences.

$$\begin{tikzcd}
H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[r, "\partial"'] & H_{p+q-2}(F) \arrow[r] & \cdots \\
\pi_{p+q-1}(E) \arrow[u, "h"] \arrow[r, "\pi_*"'] & \pi_{p+q-1}(B) \arrow[u, "h"'] \arrow[r] & \pi_{p+q-2}(F) \arrow[u, "h"'] \arrow[r] & \cdots
\end{tikzcd}$$

The leftmost square commutes by naturality of Hurewicz. The commutativity of the righmost square is not immediately obvious. For this, let us draw in the explicit maps in the above diagram:

$$\begin{tikzcd}
 & & H_{p+q-1}(E,F) \arrow[dl] \arrow[dr] & & \\
H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[rr, "\partial"'] & & H_{p+q-2}(F) \arrow[r] & \cdots \\
\pi_{p+q-1}(E) \arrow[u, "h"] \arrow[r, "\pi_*"'] \arrow[dr] & \pi_{p+q-1}(B) \arrow[u, "h"'] \arrow[rr] & & \pi_{p+q-2}(F) \arrow[u, "h"'] \arrow[r] & \cdots \\
 & \pi_{p+q-1}(E,F) \arrow[u, "s"', "\cong"] \arrow[uuur] \arrow[urr] & & &
\end{tikzcd}$$

The map marked s is an isomorphism (and provides the long arrow in the above diagram, which makes the square commute), since

$$\pi_n(E,F) = \pi_{n-1}(\mathrm{hofib}(F \to E)) = \pi_{n-1}(\Omega B) = \pi_n(B).$$

Let us now specialize to the case of the fibration

$$\Omega X \to PX \to X.$$

Assume that X is connected, and $* \in X$ is a chosen basepoint. Let $p \geq 2$, and suppose that $\widetilde{H}_s(X) = 0$ for $s < p$. Arguing as in §5.3, we learn that the Serre spectral sequence we know that the homology of ΩX begins in dimension $p-1$ since $PX \simeq *$, so $q = p-1$. Likewise, if we knew $\widetilde{H}_n(\Omega X) = 0$ for $n < p-1$, then the same argument shows that $\widetilde{H}_n(X) = 0$ for $n < p$.

A surprise gust: the Hurewicz theorem

The discussion above gives a proof of the Hurewicz theorem; this argument is due to Serre.

Theorem 5.4.1 (Hurewicz, Serre's proof)**.** *Let $p \geq 1$. Suppose X is a pointed space with $\pi_i(X) = 0$ for $i < p$. Then $\widetilde{H}_i(X) = 0$ for $i < p$ and $\pi_p(X)^{ab} \to H_p(X)$ is an isomorphism.*

Proof. Let us assume the case $p = 1$. This is classical: it is Poincaré's theorem. We will only use this result when X is a loop space, in which case the fundamental group is already abelian.

Let us prove this by induction, using the loop space fibration. By assumption, $\pi_i(\Omega X) = 0$ for $i < p-1$. By our inductive hypothesis, $\widetilde{H}_i(\Omega X) = 0$ for $i < p-1$, and $\pi_{p-1}(\Omega X) \xrightarrow{\simeq} H_{p-1}(\Omega X)$. By our discussion above, we learn that $\widetilde{H}_i(X) = 0$ for $i < p$. The Hurewicz map $\pi_p(X) \xrightarrow{h} H_p(X)$ fits into a commutative diagram:

$$\begin{tikzcd}
\pi_{p-1}(\Omega X) \arrow[r, "\simeq"] & H_{p-1}(\Omega X) \\
\pi_p(X) \arrow[u, "\simeq"] \arrow[r, "h"'] & H_p(X) \arrow[u, "\simeq"', "\text{transgression}"]
\end{tikzcd}$$

It follows from the Serre exact sequence that the transgression is an isomorphism. $\square$

5.5 Serre classes

Definition 5.5.1. A class $\mathbf{C}$ of abelian groups is a *Serre class* if:

1. $0 \in \mathbf{C}$.
2. if I have a short exact sequence $0 \to A \to B \to C \to 0$, then $A \& C \in \mathbf{C}$ if and only if $B \in \mathbf{C}$.

Some consequences of this definition: a Serre class is closed under isomorphisms (easy). A Serre class is closed under subobjects and quotients, because there is a short exact equence

$$0 \to A \hookrightarrow B \to B/A \to 0.$$

Consider an exact sequence $A \to B \to C$ (not necessarily a *short* exact sequence). If $A, C \in \mathbf{C}$, then $B \in \mathbf{C}$ because we have a short exact sequence:

$$\begin{tikzcd}
 & & & \operatorname{coker} i \arrow[r] \arrow[d] & 0 \\
 & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[ur] & C & 0 \\
0 \arrow[r] & \ker p \arrow[ur] & & &
\end{tikzcd}$$

Some examples are in order.

Example 5.5.2. 1. $\mathbf{C} = \{0\}$, and $\mathbf{C}$ the class of all abelian groups.

2. Let $\mathbf{C}$ be the class of all torsion abelian groups. We need to check that $\mathbf{C}$ satisfies the second condition of Definition **??**. Consider a short exact sequence

   $$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

   We need to show that B is torsion if A and C are torsion. To see this, let $b \in B$. Then $p(b)$ is killed by some integer n, so there exists $a \in A$ such that $i(a) = nb$. SInce A is torsion, it follows that b is torsion, too.

3. Let $\mathcal{P}$ be a set of primes. Define:

   $$\mathbf{C}_{\mathcal{P}} = \{A : \text{if } p \notin \mathcal{P}, \text{ then } p : A \xrightarrow{\simeq} A, \text{ i.e., } A \text{ is a } \mathbf{Z}[1/p]\text{-module}\}$$

   Let $\mathbf{Z}_{(\mathcal{P})} = \mathbf{Z}[1/p : p \notin \mathcal{P}] \subseteq \mathbf{Q}$.

For instance, if $\mathcal{P}$ is the set of all primes, then $\mathbf{C}_{\mathcal{P}}$ is the Serre class of all abelian groups. If $\mathcal{P}$ is the set of all primes other than ℓ, then $\mathbf{C}_{\mathcal{P}}$ is the Serre class consisting of all $\mathbf{Z}[1/\ell]$-modules. If $\mathcal{P} = \{\ell\}$, then $\mathbf{C}_{\{\ell\}} =: \mathbf{C}_\ell$ is the Serre class of all $\mathbf{Z}_{(\ell)}$-modules. If $\mathcal{P} = \emptyset$, then $\mathbf{C}_\emptyset$ is all rational vector spaces.

4. If $\mathbf{C}$ and $\mathbf{C}'$ are Serre classes, then so is $\mathbf{C} \cap \mathbf{C}'$. For instance, $\mathbf{C}_{\text{tors}} \cap \mathbf{C}_{\text{fg}}$ is the Serre class $\mathbf{C}_{\text{finite}}$. Likewise, $\mathbf{C}_p \cap \mathbf{C}_{\text{tors}}$ is the Serre class of all p-torsion abelian groups.

Here are some straightforward consequences of the definition:

1. If $C_\bullet$ is a chain complex, and $C_n \in \mathbf{C}$, then $H_n(C_\bullet) \in \mathbf{C}$.

2. Suppose F_*A is a filtration on an abelian group. If $A \in \mathbf{C}$, then $\operatorname{gr}_n A \in \mathbf{C}$ for all n. If F_*A is finite and $\operatorname{gr}_n A \in \mathbf{C}$ for all n, then $A \in \mathbf{C}$.

3. Suppose we have a spectral sequence $\{E_r\}$. If $E^2_{s,t} \in \mathbf{C}$, then $E^r_{s,t} \in \mathbf{C}$ for $r \geq 2$. It follows that if $\{E^r\}$ is a right half-plane spectral sequence, then $E^{s+1}_{s,t} \twoheadrightarrow E^{s+2}_{s,t} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{s,t} \in \mathbf{C}$.

   Thus, if the spectral sequence comes from a filtered complex (which is bounded below, such that for all n there exists an s such that $F_s H_n(C) = H_n(C)$, i.e., the homology of the filtration stabilizes), then $E^\infty_{s,t} = \operatorname{gr}_s H_{s+t}(C)$. This means that if the $E^2_{s,t} \in \mathbf{C}$ for all $s + t = n$, then $H_n(C) \in \mathbf{C}$.

To apply this to the Serre spectral sequence, we need an additional axiom for Definition 5.5.1:

2. if $A, B \in \mathbf{C}$, then so are $A \otimes B$ and $\operatorname{Tor}_1(A, B)$.

All of the examples given above satisfy this additional axiom.

Terminology 5.5.3. $f : A \to B$ is said to be a $\mathbf{C}$-epimorphism if $\operatorname{coker} f \in \mathbf{C}$, a $\mathbf{C}$-monomorphism if $\ker f \in \mathbf{C}$, and a $\mathbf{C}$-isomorphism if it is a $\mathbf{C}$-epimorphism and a $\mathbf{C}$-monomorphism.

Proposition 5.5.4. *Let $\pi : E \to B$ be a fibration and B path connected, such that the fiber $F = \pi^{-1}(*)$ is path connected. Suppose $\pi_1(B)$ acts trivially on $H_*(F)$.*

Let $\mathbf{C}$ be a Serre class satisfiying Axiom 2. Let $s \geq 3$, and assume that $H_n(E) \in \mathbf{C}$ where $1 \leq n < s - 1$ and $H_t(B) \in \mathbf{C}$ for $1 \leq t < s$. Then $H_t(F) \in \mathbf{C}$ for $1 \leq t < s - 1$.

Proof. We will do the case $s = 3$, for starters. We're gonna want to relate the low-dimension homology of these groups. What can I say? We know that $H_0(E) = \mathbf{Z}$ since it's connected. I have $H_1(E) \to H_1(B)$, via π. This is one of the edge homomorphisms, and thus it surjects (no possibility for a differential coming in). I now have a map $H_1(F) \to H_1(E)$. But I have a possible $d^2 : H_2(B) \to H_1(F)$, which is a transgression that gives:

$$H_2(B) \xrightarrow{\partial} H_1(F) \to H_1(E) \to H_1(B) \to 0$$

Let me take a step back and say something general. You might be interested in knowing when something in $H_n(F)$ maps to zero in $H_n(E)$. I.e., what's the kernel of $H_n(F) \to H_n(E)$. The sseq gives an obstruction to being an isomorphism. The only way that something can be killed by $H_n(F) \to H_n(E)$ is described by:

$$\ker(H_n(F) \to H_n(E)) = \bigcup \left(\text{im of } d^r \text{ hitting } E^r_{0,n}\right)$$

You can also say what the cokernel is: it's whatever's left in $E^\infty_{s,t}$ with $s + t = n$. These obstruct $H_n(F) \to H_n(E)$ from being surjective.

In the same way, I can do this for the base. If I have a class in $H_n(E)$, that maps to $H_n(B)$, the question is: what's the image? Well, the only obstruction is the possibility is that the element in $H_n(B)$ supports a nonzero differential. Thus:

$$\mathrm{im}(H_n(E) \xrightarrow{\pi_*} H_n(B)) = \bigcap \left(\ker(d^r : E^r_{r,0} \to \cdots)\right)$$

Again, you can think of the sseq as giving obstructions. And also, the obstruction to that map being a monomorphism that might occur in lower filtration along the same total degree line.

Back to our argument. We had the low-dimensional exact sequence:

$$H_2(B) \xrightarrow{\partial} H_1(F) \to H_1(E) \to H_1(B) \to 0$$

Here $p = 3$, so we have $H_2(B) \in \mathbf{C}$ and $H_1(E) \in \mathbf{C}$. Thus $H_1(F) \in \mathbf{C}$. That's the only thing to check when $p = 3$.

Let's do one more case of this induction. What does this say? Now I'll do $p = 4$. We're interested in knowing if $E^2_{0,3} \in \mathbf{C}$. There are now two possible differentials! I have $H_2(F) = E^2_{0,2} \twoheadrightarrow E^3_{0,2}$. This quotient comes from $d^2 : E^2_{2,1} \to E^2_{0,2}$. Now, $d^3 : E^3_{3,0} \to E^3_{0,2}$ which gives a surjection $E^3_{0,2} \twoheadrightarrow E^4_{0,2} \simeq E^\infty_{0,2} \hookrightarrow H_2(E)$. Now, our assumptions were that $E^2_{2,1}, E^3_{3,0}, H_2(E) \in \mathbf{C}$. Thus $E^3_{0,2} \in \mathbf{C}$ and so $E^2_{0,2} = H_2(F) \in \mathbf{C}$. Ta-da! $\square$

We're close to doing actual calculations, but I have to talk about the multiplicative structure on the Serre sseq first.

5.6 Mod C Hurewicz, Whitehead, cohomology spectral sequence

We had $\mathbf{C}_{fg}$ and $\mathbf{C}_{tors}$, and

$$\mathbf{C}_{\mathcal{P}} = \{A | \ell : A \xrightarrow{\simeq} A, \ell \notin \mathcal{P}\}, \quad \mathbf{C}_p = \mathbf{C}_{\{p\}}, \quad \mathbf{C}_{p'} = \mathbf{C}_{\text{not } p}$$

Another one is $\mathbf{C}_{p'} \cap \mathbf{C}_{tors}$, which consists of torsion groups such that p is an isomorphism on A. There is therefore no p-torsion, and it has only prime-to-p torsion. This is the same thing as saying that $A \otimes \mathbf{Z}_{(p)} = 0$.

Theorem 5.6.1 (Mod **C** Hurewicz)**.** *Let X be simply connected and $\mathbf{C}$ a Serre class such that $A, B \in \mathbf{C}$ implies that $A \otimes B, \mathrm{Tor}_1(A, B) \in \mathbf{C}$ (this is axiom 2). Assume also that if $A \in \mathbf{C}$, then $H_j(K(A, 1)) = H_j(BA) \in \mathbf{C}$ for all $j > 0$. (This is valid for all our examples, and is what is called Axiom 3.)*

Let $n \geq 1$. Then $\pi_i(X) \in \mathbf{C}$ for any $1 < i < n$ if and only if $H_i(X) \in \mathbf{C}$ for any $1 < i < n$, and $\pi_n(X) \to H_n(X)$ is a mod $\mathbf{C}$ isomorphism.

Example 5.6.2. For $1 < i < n$, the group $\widetilde{H}_i(X)$ is:

1. torsion;
2. finitely generated;
3. finite;
4. $- \otimes \mathbf{Z}_{(p)} = 0$

if and only if $\pi_i(X)$ for $1 < i < n$.

Proof. Look at $\Omega X \to PX \to X$. Then $\pi_1 \Omega X \in \mathbf{C}$. Look at Davis+Kirk. □

There's a Whitehead theorem that comes out of this, that I want to state for you.

Theorem 5.6.3 (Mod **C** Whitehead theorem)**.** *Let $\mathbf{C}$ be a Serre class satisfying axioms 1, 2, 3, and:*

(2′) $A \in \mathbf{C}$ implies that $A \otimes B \in \mathbf{C}$ for any B.

This is satisfied for all our examples except $\mathbf{C}_{fg}$.

Suppose I have $f : X \to Y$ where X, Y are simply connected. Suppose $\pi_2(X) \to \pi_2(Y)$ is onto. Let $n \geq 2$. Then $\pi_i(X) \to \pi_i(Y)$ is a $\mathbf{C}$-isomorphism for $2 \leq i \leq n$ and is a $\mathbf{C}$-epimorphism for $i = n$, with the same statement for H_i.

These kind of theorems help us work locally at a prime, and that's super. You'll see this in the next assignment, which is mostly up on the web. You'll also see this in calculations which we'll start doing in a day or two.

Change of subject here. Today I'm going to say a lot of things for which I won't give a proof. I want to talk about cohomology sseq.

Cohomology sseq

We're building up this powerful tool using spectral sequences. We saw how powerful the cup product was, and that is what cohomology is good for. In cohomology, things get turned upside down:

Definition 5.6.4. A *decreasing filtration* of an object A is

$$A \supseteq \cdots \supseteq F^{-1}A \supseteq F^0A \supseteq F^1A \supseteq F^2A \supseteq \cdots \supseteq 0$$

This is called "bounded above" if $F^0A = A$. Write $\mathrm{gr}^s A = F^sA/F^{s+1}A$.

Example 5.6.5. Suppose X is a filtered space. So there's an increasing filtration $\emptyset = F_{-1}X \subseteq F_0X \subseteq \cdots$. Let R be a commutative ring of coefficients. Then I have $S^*(X)$, where the differential goes up one degree. Define

$$F^sS^*(X) = \ker(S^*(X) \to S^*(F_{s-1}X))$$

For instance, $F^0S^*(X) = S^*(X)$. Thus this is a bounded above decreasing filtration .

My computer will run out of juice soon, TEXthis up later!

Example 5.6.6. Let $X = E \xrightarrow{\pi} B =$ CW-complex with $\pi_1(B)$ acting trivially on $H_t(F)$. Then $F_sE = \pi^{-1}(\mathrm{sk}_sB)$. Thus I get a filtration on $S^*(E)$, and

$$F^sH^*(X) = \ker(H^*(X) \to H^*(F_{s-1}X))$$

Doing everything the same as before, we get a *cohomology spectral sequence*. Here are some facts.

1. First, you have $E_r^{s,t}$ (note that indices got reversed). There's a differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$, so that the total degree of the differential is 1.

2. You discover that
$$E_2^{s,t} \simeq H^s(B; H^t(F))$$

3. and $E_\infty^{s,t} \simeq \mathrm{gr}^s H^{s+t}(E)$.

4.

5.7 A few examples, double complexes, Dress sseq

Way back in 905 I remember computing the cohomology ring of $\mathbf{CP}^n$ using Poincaré duality. Let's do it fresh using the fiber sequence

$$S^1 \to S^{2n+1} \to \mathbf{CP}^n$$

where S^1 acts on S^{2n+1}. Here we know the cohomology of the fiber and the total space, but not the cohomology of the base. Let's look at the cohomology sseq for this. Then

$$E_2^{s,t} = H^s(\mathbf{CP}^n; H^t(S^1)) \simeq H^s(\mathbf{CP}^n) \otimes H^t(S^1) Rightarrow H^{s+t}(S^{2n+1})$$

The isomorphism $H^s(\mathbf{CP}^n; H^t(S^1)) \simeq H^s(\mathbf{CP}^n) \otimes H^t(S^1)$ follows from the UCT.

We know at least that $\mathbf{CP}^n$ is simply connected by the lexseq of homotopy groups. I don't have to worry about local coefficients. Let's work with the case S^5. We know that $\mathbf{CP}^n$ is simply connected, so the one-dimensional cohomology is 0. The only way to kill $E_2^{0,1}$ is by sending it via d_2 to $E_2^{2,0}$. Is this map surjective? Yes, it's an isomorphism.

Now I'm going to give names to the generators of these things; see the below diagram. $E_2^{2,1}$ is in total degree 3 and so we have to get rid of it. I will compute d_2 on this via Leibniz:

$$d_2(xy) = (d_2 x)y - x d_2 y = (d_2 x)y = y^2$$

which gives (iterating the same computation):

$$\begin{tikzcd}
0 & \mathbf{Z}x \arrow[drr, "d_2"'] & 0 & \mathbf{Z}xy \arrow[drr, "d_2"'] & 0 & \mathbf{Z}xy^2 \\
0 & \mathbf{Z} & 0 & \mathbf{Z}y & 0 & \mathbf{Z}y^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

This continues until the end where you reach $\mathbf{Z}xy^{??}$ which is a permanent cycle since it lasts until the E_∞-page.

Another example: let C_m be the cyclic group of order m sitting inside S^1. How can we analyse $S^{2n+1}/C_m =: L$? This is the lens space. We have a map $S^{2n+1}/C_m \to S^{2n+1}/S^1 = \mathbf{CP}^n$. This is a fiber bundle whose fiber is S^1/C_m. The spectral sequence now runs:

$$E^2_{s,t} = H_s(\mathbf{CP}^n) \otimes H_t(S^1/C_m) \Rightarrow H_{s+t}(L)$$

We know the whole E^2 term now:

$$\begin{tikzcd}
0 & \mathbf{Z} & 0 & \mathbf{Z} & 0 & \mathbf{Z} \\
0 & \mathbf{Z} & 0 & \mathbf{Z} \arrow[ull, "m"'] & 0 & \mathbf{Z} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

In cohomology, we have something dual:

$$\begin{tikzcd}
0 & u \arrow[drr, "m"'] & 0 & uy \arrow[drr, "m"'] & 0 & uy^2 \\
0 & 1 & 0 & y & 0 & y^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

What's the ring structure? We get that $H^*(L) = \mathbf{Z}[y, v]/(my, y^{n+1}, yv, v^2)$ where $|v| = 2n+1$ and $|y| = 2$. By the way, when $m = 1$, this is $\mathbf{RP}^{2n+1}$. This is a computation of the cohomology of odd real projective spaces. Remember that odd projective spaces are orientable and you're seeing that here because you're picking up a free abelian group in the top dimension.

Double complexes

$A_{s,t}$ is a bigraded abelian group with $d_h : A_{s,t} \to A_{s-1,t}$ and $d_v : A_{s,t} \to A_{s,t-1}$ such that $d_v d_h = d_h d_v$. Assume that $\{(smt) : s+t = n, A_{s,t} \neq 0\}$ is finite for any n. Then

$$(tA)_n = \bigoplus_{s+t=n} A_{s,t}$$

Under this assumption, there's only finitely many nonzero terms. I like this personally because otherwise I'd have to decide between the direct sum and the direct product, so we're avoiding that here. It's supposed to be a chain complex. Here's the differential:

$$d(a_{s,t}) = d_h a_{s,t} + (-1)^s d_v a_{s,t}$$

Then $d^2 = 0$, as you can check.

Question 5.7.1. What is $H_*(tA_*)$?

Define a filtration as follows:

$$F_p(tA)_n = \bigoplus_{s+t=n, s\leq p} A_{s,t} \subseteq (tA)_n$$

This kinda obviously gives a filtered complex. Let's compute the low pages of the sseq. What is $\mathrm{gr}_s(tA)$? Well

$$\mathrm{gr}_s(tA)_{s+t} = (F_s/F_{s-1})_{s+t} = A_{s,t}$$

This associated graded object has its own differential $\mathrm{gr}_s(tA)_{s+t} = A_{s,t} \xrightarrow{d_v} A_{s,t-1} = \mathrm{gr}_s(tA)_{s+t-1}$. Let $E^0_{s,t} = \mathrm{gr}_s(tA)_{s+t} = A_{s+t}$, so that $d^0 = d_v$. Then $E^1 = H(E^0_{s,t}, d^0) = H(A_{s,t}; d_v) =: H^v_{s,t}(A)$. So computing E^1 is ez. Well, what's d^1 then?

To compute d^1 I take a vertical cycle that and the differential decreases the ... by 1, so that d^1 is induced by d_h. This means that I can write $E^2_{s,t} = H^h_{s,t}(H^v(A))$.

Question 5.7.2. You can also do $'E^2_{s,t} = H^v_{s,t}(H^h(A))$, right?

Rather than do that, you can define the transposed double complex $A^{\mathsf{T}}_{t,s} = A_{s,t}$, and $d^{\mathsf{T}}_h(a_{s,t}) = (-1)^s d_v(a_{s,t})$ and $d^{\mathsf{T}}_v(a_{s,t}) = (-1)^t d_h a_{s,t}$. When I set the signs up like that, then

$$tA^{\mathsf{T}} \simeq tA$$

as complexes and not just as groups (because of those signs). Thus, you get a spectral sequence

$$^{\mathsf{T}}E^2_{s,t} = H^v_{s,t}(H^h(A))$$

converging to the same thing. I'll reserve telling you about Dress' construction until Monday because I want to give a double complex example. It's not ... it's just a very clear piece of homological algebra.

Example 5.7.3 (UCT)**.** For this, suppose I have a (not necessarily commutative) ring R. Let C_* be a chain complex, bounded below of right R-modules, and let M be a left R-module. Then I get a new chain complex of abelian groups via $C_* \otimes_R M$. What is $H(C_* \otimes_R M)$? I'm thinking of M as some kind of coefficient. Let's assume that each C_n is projective, or at least flat, for all n.

Shall we do this?

Let $M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ be a projective resolution of M as a left R-module. Then $H_*(P_*) \xrightarrow{\simeq} M$. Form $C_* \otimes_R P_*$: you know how to do this! I'll define $A_{s,t}$ to be $C_s \otimes_R P_t$. It's got two differentials, and it's a double complex. Let's work out the two sseqs.

Firstly, let's take it like it stands and take homology wrt P first. I'm organizing it so that C is along the base and P is along the fiber. What is the vertical homology $H^v(A_{*,*})$? If the C are projective then tensoring with them is exact, so that $H^v(A_{s,*}) = C_s \otimes_R H_*(P_*)$, so that $E^1_{s,t} = H^v_{s,t}(A_{*,*}) = C_s \otimes M$ if $t = 0$ and 0 otherwise. The spectral sequence is concentrated in one row. Thus,

$$E^2_{s,t} = \begin{cases} H_s(C_* \otimes_R M) & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$$

This is canonically the same thing as $E^\infty_{s,0} \simeq H_s(tA)$.

Let me go just one step further here. The game is to look at the *other* spectral sequence, where I do horizontal homology first. Then $H^h(A_{*,*}) = H_t(C_*) \otimes P_s$ again because the P_* are projective. Thus,

$$E^2_{s,t} = H^v(H^h(A_{*,*})) = \operatorname{Tor}^R_s(H_t(C), M) \Rightarrow H_{s+t}(C_* \otimes_R M)$$

That's the *universal coefficients spectral sequence.*

What happens if R is a PID? Only two columns are nonzero, and $E^2_{0,n} = H_n(C) \otimes_R M$ and $E^2_{1,n-1} = \mathrm{Tor}_1(H_{n-1}(C), M)$. This exactly gives the universal coefficient exact sequence.

Later we'll use this stuff to talk about cohomology of classifying spaces and Grassmannians and Thom isomorphisms and so on.

5.8 Dress spectral sequence, Leray-Hirsch

I think I have to be doing something tomorrow, so no office hours then. The new pset is up, and there'll be one more problem up. There are two more things about spectral sequences, and specifically the multiplicative structure, that I have to tell you about. The construction of the Serre sseq isn't the one that we gave. He did stuff with simplicial homology, but as you painfully figured out, $\Delta^s \times \Delta^t$ isn't another simplex. Serre's solution was to not use simplices, but to use cubes. He defined a new kind of homology using the n-cube. It's more complicated and unpleasant, but he worked it out.

Dress' sseq

Dress made the following variation on this idea, which I think is rather beautiful. We have a trivial fiber bundle $\Delta^t \to \Delta^s \times \Delta^t \to \Delta^s$. Let's do with this what we did with homology in the first place. Dress started with some map $\pi : E \to B$ (not necessarily a fibration), and he thought about the set of maps from $\Delta^s \times \Delta^t \to \Delta^s$ to $\pi : E \to B$. This set is denoted $\mathrm{Sin}_{s,t}(\pi)$. This forgets down to $S_s(B)$. Altogether, this $\mathrm{Sin}_{*,*}(\pi)$ is a functor $\Delta^{op} \times \Delta^{op} \to \mathrm{Set}$, forming a "bisimplicial set".

The next thing we did was to take the free R-module, to get a bisimplicial R-module $R\mathrm{Sin}_{*,*}(\pi)$. We then passed to chain complexes by forming the alternating sum. We can do this in two directions here! (The s is horizontal and t is vertical.) This gives us a double complex. We now get a spectral sequence! I hope it doesn't come as a surprise that you can compute the horizontal – you can compute the vertical differential first, and then taking the horizontal differential gives the homology of B with coefficients in something. Oh actually, the totalization $tR\mathrm{Sin}_{*,*}(\pi) \simeq R\mathrm{Sin}_*(E) = S_*(E)$. We'll have

$$E^2_{s,t} = H_s(B; \text{crazy generalized coefficients}) \Rightarrow H_{s+t}(E)$$

These coefficients may not even be local since I didn't put any assumptions on π! This is like the "Leray" sseq, set up without sheaf theory. If π is a fibration, then those crazy generalized coefficients is the local system given by the homology of the fibers. This gives the Serre sseq.

This has the virtue of being completely natural. Another virtue is that I can form $\mathrm{Hom}(-, R)$, and this gives rise to a multiplicative double complex. Remember that the cochains on a space form a DGA, and that's where the cup product comes from. The same story puts a bigraded multiplication on this double complex, and that's true *on the nose*. That gives rise a multiplicative cohomology sseq.

This is very nice, but the only drawback is that the paper is in German. That was item one in my agenda.

Leray-Hirsch

This tells you condition under which you can compute the cohomology of a total space. Anyway. We'll see.

Let's suppose I have a fibration $\pi : E \to B$. For simplicity suppose that B is path connected, so that gives meaning to the fiber F which we'll also assume to be path-connected. All cohomology is with coefficients in a ring R. I have a sseq

$$E_2^{s,t} = H^s(B; \underline{H^t F}) \Rightarrow H^{s+t}(E)$$

If you want assume that $\pi_1(B)$ acts trivially so that that cohomology in local coefficients is just cohomology with coefficients in H^*F. I have an algebra map $\pi^* : H^*(B) \to H^*(E)$, making $H^*(E)$ into a module over $H^*(B)$. We have $E_2^{*,t} = H^*(B; H^t(F))$, and this is a $H^*(B)$-module. That's part of the multiplicative structure, since $E_2^{*,0} = H^*B$. This row acts on every other row by that module structure.

Everything in the bottom row is a permanent cycle, i.e., survives to the E_∞-page. In other words

$$H^*(B) = E_2^{*,0} \twoheadrightarrow E_3^{*,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{*,0}$$

Each one of these surjections is an algebra map.

What the multiplicative structure is telling us is that $E_r^{*,0}$ is a graded algebra acting on $E_r^{*,t}$. Thus, $E_\infty^{*,t}$ is a module for $H^*(B)$.

Really I should be saying that it's a module for $H^*(B; \underline{H^0(F)})$. Can I guarantee that the $\pi_1(B)$-action on F is trivial. We know that $F \to *$ induces an iso on H^0 (that's part of being path-connected). So if you have a fibration whose fiber is a point, there's no possibility for an action.

This fibration looks the same as far as H^0 of the fiber is concerned. Thus the $\pi_1(B)$-action is trivial on $H^0(F)$, so saying that it's a $H^*(B)$-module is fine.

Where were we? We have module structures all over the place. In particular, we know that $H^*(E)$ is a module over $H^*(B)$ as we saw, and also $E_\infty^{*,t}$ is a $H^*(B)$-module. These better be compatible!

Define an increasing filtration on $H^*(E)$ via $F_tH^n(E) = F^{n-t}H^n(E)$. For instance, $F_0H^n(E) = F^nH^n(E)$. What is that? In our picture, we have the associated quotients along the diagonal on $E_\infty^{s,t}$ given by $s+t=n$. In the end, since we know that $F^{n+1}H^n(E) = 0$, it follows that

$$F_0H^n(E) = F^nH^n(E) = E_\infty^{n,0} = \mathrm{im}(\pi^* : H^n(B) \to H^n(E))$$

With respect to this filtration, we have

$$\mathrm{gr}_t H^*(E) = E_\infty^{*,t}$$

I learnt this idea from Dan Quillen. It's a great idea. This increasing filtration $F_*H^*(E)$ is a filtration by $H^*(B)$-modules, and $\mathrm{gr}_t H^*(E) = E_\infty^{*,t}$ is true as H^*B-modules. It's exhaustive and bounded below.

This is a great perspective. Let's use it for something. Let me give you the Leray-Hirsch theorem.

Theorem 5.8.1 (Leray-Hirsch)**.** *Let $\pi : E \to B$.*

1. *Suppose B and F are path-connected.*

2. *Suppose that $H^t(F)$ is free*[1] *of finite rank as a R-module.*

3. *Also suppose that $H^*(E) \twoheadrightarrow H^*(F)$. That's a big assumption; it's dual is saying that the homology of the fiber injects into the homology of E. This is called "totally non-homologous to zero" – this is a great phrase, I don't know who invented it.*

Pick an R-linear surjection $\sigma : H^*(F) \to H^*(E)$; this defines a map $\overline{\sigma} : H^*(B) \otimes_R H^*(F) \to H^*(E)$ via $\overline{\sigma}(x \otimes y) = \pi^*(x) \cup \sigma(y)$. This is the $H^*(B)$-linear extension. Then $\overline{\sigma}$ is an isomorphism.

Remark 5.8.2. It's not natural since it depends on the choice of σ. It tells you that $H^*(E)$ is free as a $H^*(B)$-module. That's a good thing.

[1] Everything is coefficients in R

Proof. I'm going to use our Serre sseq

$$E_2^{s,t} = H^s(B; \underline{H^t F}) \Rightarrow H^{s+t}(E)$$

Our map $H^*(E) \to H^*(F)$ is an edge homomorphism in the sseq, which means that it factors as $H^*(E) \to E_2^{0,*} = H^0(B; \underline{H^*(F)}) \subseteq H^*(F)$. Since $H^*(E) \to H^*(F)$, we have $H^0(B; \underline{H^*(F)}) \simeq H^*(F)$. Thus the $\pi_1(B)$-action on F is trivial.

Question 5.8.3. What's this arrow $H^*(E) \to E_2^{0,*}$? We have a map $H^*(E) \to H^*(E)/F^1 = E_\infty^{0,*}$. This includes into $E_2^{0,*}$.

Now you know that the E_2-term is $H^s(B; H^t(F))$. By our assumption on $H^*(F)$, this is $H^s(B) \otimes_R H^t(F)$, as algebras. What do the differentials look like? I can't have differentials coming off of the fiber, because if I did then the restriction map to the fiber wouldn't be surjective, i.e., that $d_r|_{E_r^{0,\infty}} = 0$. The differentials on the base are of course zero. This proves that d_r is zero on every page by the algebra structure! This means that $E_\infty = E_2$, i.e., $E_\infty^{*,t} = H^*(B) \otimes H^t(F)$.

Now I can appeal to the filtration stuff that I was talking about, so that $E_\infty^{*,t} = \mathrm{gr}_t H^*(E)$. Let's filter $H^*(B) \otimes H^*(F)$ by the degree in $H^*(F)$, i.e., $F_q = \bigoplus_{t \leq q} H^*(B) \otimes H^t(F)$. The map $\overline{\sigma} : H^*(B) \otimes H^*(F) \to H^*(E)$ is filtration preserving, and it's an isomorphism on the associated graded. This is the identification $H^*(B) \otimes H^t(F) = E_\infty^{*,t} = \mathrm{gr}_t H^*(E)$. Since the filtrations are exhaustive and bounded below, we conclude that $\overline{\sigma}$ itself is an isomorphism. □

5.9 Integration, Gysin, Euler, Thom

Today there's a talk by
the one
the only
JEAN-PIERRE SERRE
OK let's begin.

Umkehr

Let $\pi : E \to B$ be a fibration and suppose B is path-connected. Suppose the fiber has no cohomology above some dimension d. The Serre sseq has nothing above row d.

Let's look at $H^n(E)$. This happens along total degree n. We have this neat increasing filtration that I was talking about on Monday whose associated quotients are the rows in this thing. So I can divide out by it (i.e I divide out by $F_{d-1}H^n(E)$). Then I get

$$H^n(E) \twoheadrightarrow H^n(E)/F_{d-1}H^n(E) = E_\infty^{n-d,d} \rightarrowtail E_2^{n-d,d} = H^{n-d}(B; \underline{H^d(F)})$$

That's because on the E_2 page, at that spot, there's nothing hitting it, but there might be a differential hitting it. There it is; here's another edge homomorphism.

Remark 5.9.1. This is a *wrong-way map*, also known as an "umkehr" map. It's also called a *pushforward map*, or the *Gysin map*.

We know from the incomprehensible discussion that I was giving on Monday that this was a filtration of modules over $H^*(B)$, so that this map $H^n(E) \to H^{n-d}(B; \underline{H^d(F)})$ is a $H^*(B)$-module map.

Example 5.9.2. F is a compact connected d-manifold with a given R-orientation. Thus $H^d(F) \simeq R$, given by $x \mapsto \langle x, [F] \rangle$. There might some local cohomology there, but I do get a map $H^n(E; R) \to H^{n-d}(B; \overline{R})$. This is such a map, and it has a name: it's written $\pi_!$ or π_*. I'll write π_*.

Of course, if $\pi_1(B)$ fixes $[F] \in H_d(F; R)$, then $\underline{R}$-cohomology is R-cohomology. Thus our map is now $H^n(E; R) \to H^{n-d}(B; R)$. Sometimes it's also called a pushforward map. Note that we also get a projection formula

$$\pi_*(\pi^*(b) \cup e) = b \cup \pi_*(e)$$

where π^* is the pushforward, $e \in H^n(E)$ and $b \in H^s(B)$. Others call this Frobenius reciprocity.

Gysin

Suppose $H^*(F) = H^*(S^{n-1})$. In practice, $F \cong S^{n-1}$, or even $F \simeq S^{n-1}$. In that case, $\pi : E \to B$ is called a *spherical fibration* Then the spectral sequence is *even simpler*! It has only two nonzero rows!

Let's pick an orientation for S^{n-1}, to get an isomorphism $H^{n-1}(S^{n-1})$. Well the spectral sequence degenerates, and you get a long exact sequence

$$\cdots \to H^s(B) \xrightarrow{\pi^*} H^s(E) \xrightarrow{\pi_*} H^{s-n+1}(B; \underline{R}) \xrightarrow{d_n} H^{s+1}(B) \xrightarrow{\pi^*} H^{s+1}(E) \to \cdots$$

That's called the *Gysin sequence*[2]. Because everything is a module over $H^*(B)$, this is a lexseq of $H^*(B)$-modules.

[2] pronounced Gee-sin

Let me be a little more explicit. Suppose we have an orientation. We now have a differential $H^0(B) \to H^n(B)$. We have the constant function $1 \in H^0(B)$, and this maps to something in B. This is called the *Euler class*, and is denoted e.

Since d_n is a module homomorphism, we have $d_n(x) = d_n(1 \cdot x) = d_n(1) \cdot x = e \cdot x$ where x is in the cohomology of B. Thus our lexseq is of the form

$$\cdots \to H^s(B) \xrightarrow{\pi^*} H^s(E) \xrightarrow{\pi_*} H^{s-n+1}(B; \underline{R}) \xrightarrow{e\cdot -} H^{s+1}(B) \xrightarrow{\pi^*} H^{s+1}(E) \to \cdots$$

Some facts about the Euler class

Suppose $E \to B$ has a section $\sigma : B \to E$ (so that $\pi\sigma = 1_B$). So, if it came from a vector bundle, I'm asking that there's a nowhere vanishing cross-section of that vector bundle. Let's apply cohomology, so that you get $\sigma^*\pi^* = 1_{H^*(B)}$. Thus π^* is monomorphic. In terms of the Gysin sequence, this means that $H^{s-n}(B) \xrightarrow{e\cdot -} H^s(B)$ is zero. But this implies that

$$\boxed{e = 0}$$

Thus, if you don't have a nonzero Euler class then you cannot have a section! If your Euler class is zero sometimes you can conclude that your bundle has a section, but that's a different story.

The Euler class of the tangent bundle of a manifold when paired with the fundamental class is the Euler characteristic. More precisely, if M is oriented connected compact n-manifold, then

$$\langle e(\tau_M), [M] \rangle = \chi(M)$$

That's why it's called the Euler class. (He didn't know about spectral sequences or cohomology.)

Time for Thom

This was done by Rene Thom. Let ξ be a n-plane bundle over X. I can look at $H^*(E(\xi), E(\xi) - \text{section})$. If I pick a metric, this is $H^*(D(\xi), S(\xi))$, where $D(\xi)$ is the disk bundle[3] and $S(\xi)$ is the sphere bundle. If there's no point-set annoyance, this is $\widetilde{H}^*(D(\xi)/S(\xi))$.

If X is a compact Hausdorff space, then ... The open disk bundle $D^0(\xi) \simeq E(\xi)$. This quotient $D(\xi)/S(\xi) = E(\xi)^+$ since you get the one-point compactification by embedding into a compact Hausdorff space

[3] $D(\xi) = \{v \in E(\xi) : ||v|| \leq 1\}$.

($D(\xi)$ here) and then quotienting by the complement (which is $S(\xi)$ here). This is called the *Thom space* of ξ. There are two notations: some people write $\mathrm{Th}(\xi)$, and some people (Atiyah started this) write X^ξ.

Example 5.9.3 (Dumb)**.** Suppose ξ is the zero vector bundle. Then your fibration is $\pi : X \to X$. What's the Thom space? The disk bundle is X, and the boundary of a disk is empty, so $\mathrm{Th}(0) = X^0 = X \sqcup *$.

The Thom space is a pointed space (corresponding to ∞ or the point which $S(\xi)$ is collapsed to).

I'd like to study its cohomology, because it's interesting. There's no other justification. Maybe I'll think of it as the relative cohomology.

So, guess what? We've developed sseqs and done cohomology. Anything else we'd like to do to groups and functors and things?

Let's make the spectral sequence relative!

I have a path connected B, and I'll study:

$$\begin{tikzcd} F_0 \arrow[r, hook] \arrow[d] & F \arrow[d] \\ E_0 \arrow[r, hook] \arrow[d] & E \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Then if you sit patiently and work through things, we get

$$E_2^{s,t} = H^s(B; H^t(F, F_0)) \Rightarrow_s H^{s+t}(E, E_0)$$

Note that $\Rightarrow_s$ means that s determines the filtration.

Let's do this with the Thom space. We have $D(\xi) \xrightarrow{\simeq} X$. That isn't very interesting. In our case, we have an incredibly simple spectral sequence, where everything on the E_2-page is concentrated in row n. Thus the E_2 page *is* the cohomology of

$$\widetilde{H}^{s+n}(\mathrm{Th}(\xi)) = H^{s+n}(D(\xi), S(\xi)) \simeq H^s(B; \underline{R})$$

where $\underline{R} = \underline{H^n(D^n, S^{n-1})}$. This is a canonical isomorphism of $H^*(B)$-modules.

Suppose your vector bundle ξ is oriented, so that $\underline{R} = R$. Now, if $s = 0$, then I have $1 \in H^0(B)$. This gives $u \in H^n(\mathrm{Th}(\xi))$, which is called the *Thom class*.

The cohomology of B is a free module of rank one over $H^*(B)$, so that $H^*(\mathrm{Th}(\xi))$ is also a $H^*(B)$-module that is free of rank 1, generated by u.

Let me finish by saying one more thing. This is why the Thom space is interesting. Notice one more thing: there's a lexseq of a pair

$$\cdots \to \widetilde{H}^s(\mathrm{Th}(\xi)) \to H^s(D(\xi)) \to H^s(S(\xi)) \to \widetilde{H}^{s+1}(\mathrm{Th}(\xi)) \to \cdots$$

We have synonyms for these things:

$$\cdots \to H^{s-n}(X) \to H^s(X) \to H^s(S(\xi)) \to H^{s-n+1}(X) \to \cdots$$

And aha, this is is exactly the same form as the Gysin sequence. Except, oh my god, what have I done here?

Yeah, right! In the Gysin sequence, the map $H^{s-n}(X) \to H^s(X)$ was multiplication by the Euler class. The Thom class u maps to some $e' \in H^n(X)$ via $\widetilde{H}^n(D(\xi), S(\xi)) \to H^n(D(\xi)) \simeq H^n(X)$. And the map $H^{s-n}(X) \to H^s(X)$ is multiplication by e'. Guess what? This is the Gysin sequence.

You'll explore more in homework.

I'll talk about characteristic classes on Friday.

Chapter 6

Characteristic classes

6.1 Grothendieck's construction of Chern classes

Generalities on characteristic classes

We would like to apply algebraic techniques to study G-bundles on a space. Let A be an abelian group, and $n \geq 0$ an integer.

Definition 6.1.1. A *characteristic class* for principal G-bundles (with values in $H^n(-;A)$) is a natural transformation of functors $\mathbf{Top} \to \mathbf{Ab}$:

$$\mathrm{Bun}_G(X) \xrightarrow{c} H^n(X;A)$$

Concretely: if $P \to Y$ is a principal G-bundle over a space X, and $f : X \to Y$ is a continuous map of spaces, then

$$c(f^*P) = f^*c(P).$$

The motivation behind this definition is that $\mathrm{Bun}_G(X)$ is still rather mysterious, but we have techniques (developed in the last section) to compute the cohomology groups $H^n(X;A)$. It follows by construction that if two bundles over X have two different characteristic classes, then they cannot be isomorphic. Often, we can use characteristic classes to distinguish a given bundle from the trivial bundle.

Example 6.1.2. The Euler class takes an oriented real n-plane vector bundle (with a chosen orientation) and produces an n-dimensional cohomology class $e : \mathrm{Vect}_n^{or}(X) = \mathrm{Bun}_{SO(n)}(X) \to H^n(X;\mathbf{Z})$. This is a characteristic class. To see this, we need to argue that if $\xi \downarrow X$ is a principal G-bundle, we can pull the Euler class back via $f : X \to Y$. The bundle $f^*\xi \downarrow Y$ has a orientation if ξ does, so it makes sense to even talk about the Euler class of $f^*\xi$. Since all of our constructions were natural, it follows that $e(f^*\xi) = f^*e(\xi)$.

Similarly, the mod 2 Euler class is $e_2 : \mathrm{Vect}_n(X) = \mathrm{Bun}_{O(n)}(X) \to H^n(X; \mathbf{Z}/2\mathbf{Z})$ is another Euler class. Since everything has an orientation with respect to $\mathbf{Z}/2\mathbf{Z}$, the mod 2 Euler class is well-defined.

By our discussion in §4.7, we know that $\mathrm{Bun}_G(X) = [X, BG]$. Moreover, as we stated in Theorem 3.13.8, we know that $H^n(X; A) = [X, K(A, n)]$ (at least if X is a CW-complex). One moral reason for cohomology to be easier to compute is that the spaces $K(A, n)$ are infinite loop spaces (i.e., they can be delooped infinitely many times). It follows from the Yoneda lemma that characteristic classes are simply maps $BG \to K(A, n)$, i.e., elements of $H^n(BG; A)$.

Example 6.1.3. The Euler class e lives in $H^n(BSO(n); \mathbf{Z})$; in fact, it is $e(\xi)$, the Euler class of the universal oriented n-plane bundle over $BSO(n)$. A similar statement holds for $e_2 \in H^n(BO(n); \mathbf{Z}/2\mathbf{Z})$. For instance, if $n = 2$, then $SO(2) = S^1$. It follows that

$$BSO(2) = BS^1 = \mathbf{CP}^\infty.$$

We know that $H^*(\mathbf{CP}^\infty; \mathbf{Z}) = \mathbf{Z}[e]$ — it's the polynomial algebra on the "universal" Euler class! Similarly, $O(1) = \mathbf{Z}/2\mathbf{Z}$, so

$$BO(1) = B\mathbf{Z}/2 = \mathbf{RP}^\infty.$$

We know that $H^*(\mathbf{RP}^\infty; \mathbf{F}_2) = \mathbf{F}_2[e_2]$ — as above, it is the polynomial algebra over $\mathbf{Z}/2\mathbf{Z}$ on the "universal" mod 2 Euler class.

Chern classes

These are one of the most fundamental example of characteristic classes.

Theorem 6.1.4 (Chern classes)**.** *There is a unique family of characteristic classes for complex vector bundles that assigns to a complex n-plane bundle ξ over X the* nth Chern class $c_k^{(n)}(\xi) \in H^{2k}(X; \mathbf{Z})$*, such that:*

1. $c_0^{(n)}(\xi) = 1$.

2. *If ξ is a line bundle, then $c_1^{(1)}(\xi) = -e(\xi)$.*

3. *The* Whitney sum formula *holds: if ξ is a p-plane bundle and η is a q-plane bundle (and if $\xi \oplus \eta$ denotes the fiberwise direct sum), then*

$$c_k^{(p+q)}(\xi \oplus \eta) = \sum_{i+j=k} c_i^{(p)}(\xi) \cup c_j^{(q)}(\eta) \in H^{2k}(X; \mathbf{Z}).$$

Moreover, if ξ_n is the universal n-plane bundle, then

$$H^*(BU(n); \mathbf{Z}) \simeq \mathbf{Z}[c_1^{(n)}, \cdots, c_n^{(n)}],$$

where $c_k^{(n)} = c_k^{(n)}(\xi_n)$.

This result says that all characteristic classes for complex vector bundles are given by polynomials in the Chern classes because the cohomology of $BU(n)$ gives all the characteristic classes. It also says that there are no universal algebraic relations among the Chern classes: you can specify them independently.

Remark 6.1.5. The $(p+q)$-plane bundle $\xi_p \times \xi_q = \mathrm{pr}_1^*\xi_p \oplus \mathrm{pr}_2^*\xi_q$ over $BU(p) \times BU(q)$ is classified by a map $BU(p) \times BU(q) \xrightarrow{\mu} BU(p+q)$. The Whitney sum formula computes the effect of μ on cohomology:

$$\mu^*(c_k^{(n)}) = \sum_{i+j=k} c_i^{(p)} \times c_j^{(q)} \in H^{2k}(BU(p) \times BU(q)),$$

where, you'll recall,

$$x \times y := \mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y.$$

The Chern classes are "stable", in the following sense. Let ϵ be the trivial one-dimensional complex vector bundle, and let ξ be an n-dimensional vector bundle. What is $c_k^{(n+q)}(\xi \oplus \epsilon^q)$? For this, the Whitney sum formula is valuable.

The trivial bundle is characterized by the pullback:

$$\begin{tikzcd}
X \times \mathbf{C}^n = n\epsilon \arrow[r] \arrow[d] & \mathbf{C}^n \arrow[d] \\
X \arrow[r] & *
\end{tikzcd}$$

By naturality, we find that if $k > 0$, then $c_k^{(n)}(n\epsilon) = 0$. The Whitney sum formula therefore implies that

$$c_k^{(n+q)}(\xi \oplus \epsilon^q) = c_k^{(n)}(\xi).$$

This phenomenon is called stability: the Chern class only depends on the "stable equivalence class" of the vector bundle (really, they are only defined on "K-theory", for those in the know). For this reason, we will drop the superscript on $c_k^{(n)}(\xi)$, and simply write $c_k(\xi)$.

Grothendieck's construction

Let ξ be an n-plane bundle. We can consider the vector bundle $\pi : \mathbf{P}(\xi) \to X$, the projectivization of ξ: an element of the fiber of $\mathbf{P}(\xi)$ over $x \in X$ is a line inside ξ_x, so the fibers are therefore all isomorphic to $\mathbf{CP}^{n-1}$.

Let us compute the cohomology of $\mathbf{P}(\xi)$. For this, the Serre spectral sequence will come in handy:

$$E_2^{s,t} = H^s(X; H^t(\mathbf{CP}^{n-1})) \Rightarrow H^{s+t}(\mathbf{P}(\xi)).$$

Remark 6.1.6. Why is the local coefficient system constant? The space X need not be simply connected, but $BU(n)$ is simply connected since $U(n)$ is simply connected. Consider the projectivization of the universal bundle $\xi_n \downarrow BU(n)$; pulling back via $f : X \to BU(n)$ gives the bundle $\pi : \mathbf{P}(\xi) \to X$. The map on fibers $H^*(\mathbf{P}(\xi_n)_{f(x)}) \to H^*(\mathbf{P}(\xi_n)_x)$ is an isomorphism which is equivariant with respect to the action of the fundamental group of $\pi_1(X)$ via the map $\pi_1(X) \to \pi_1(BU(n)) = 0$.

Because $H^*(\mathbf{CP}^{n-1})$ is torsion-free and finitely generated in each dimension, we know that

$$E_2^{s,t} \simeq H^s(X) \otimes H^t(\mathbf{CP}^{n-1}).$$

The spectral sequence collapses at E_2, i.e., that $E_2 \simeq E_\infty$, i.e., there are no differentials. We know that the E_2-page is generated as an algebra by elements in the cohomology of the fiber and elements in the cohomology of the base. Thus, it suffices to check that elements in the cohomology of the fiber survive to E_∞. We know that

$$E_2^{0,2t} = \mathbf{Z}\langle x^t \rangle, \text{ and } E_2^{0,2t+1} = 0,$$

where $x = e(\lambda)$ is the Euler class of the canonical line bundle $\lambda \downarrow \mathbf{CP}^{n-1}$.

In order for the Euler class to survive the spectral sequence, it suffices to come up with a two dimensional cohomology class in $\mathbf{P}(\xi)$ that restricts to the Euler class over $\mathbf{CP}^{n-1}$. We know that λ itself is the restriction of the tautologous line bundle over $\mathbf{CP}^\infty$. There is a tautologous line bundle $\lambda_\xi \downarrow \mathbf{P}(\xi)$, given by the tautologous line bundle on each fiber. Explicitly:

$$E(\lambda_\xi) = \{(\ell, y) \in \mathbf{P}(\xi) \times_X E(\xi) | y \in \ell \subseteq \xi_x\}.$$

Thus, x is the restriction $e(\lambda_\xi)|_{\text{fiber}}$ of the Euler class to the fiber. It follows that the class x survives to the E_∞-page.

Using the Leray-Hirsch theorem (Theorem 5.8.1), we conclude that

$$H^*(\mathbf{P}(\xi)) = H^*(X)\langle 1, e(\lambda_\xi), e(\lambda_\xi)^2, \cdots, e(\lambda_\xi)^{n-1}\rangle.$$

For simplicity, let us write $e = e(\lambda_\xi)$. Unforunately, we don't know what e^n is, although we do know that it is a linear combination of the e^k for $k < n$. In other words, we have a relation

$$e^n + c_1 e^{n-1} + \cdots + c_{n-1} e + c_n = 0,$$

where the c_k are elements of $H^{2k}(X)$. These are the Chern classes of ξ. By construction, they are unique!

To prove Theorem 6.1.4(2), note that when $n = 1$ the above equation reads

$$e + c_1 = 0,$$

as desired.

6.2 $H^*(BU(n))$, splitting principle

Theorem 6.1.4 claimed that the Chern classes, which we constructed in the previous section, generate the cohomology of BU as a polynomial algebra. Our goal in this section is to prove this result.

The cohomology of $BU(n)$

Recall that $BU(n)$ supports the universal principal $U(n)$-bundle $EU(n) \to BU(n)$. Given any left action of $U(n)$ on some space, we can form the associated fiber bundle. For instance, the associated bundle of the $U(n)$-action on $\mathbf{C}^n$ yields the universal line bundle ξ_n.

Likewise, the associated bundle of the action of $U(n)$ on $S^{2n-1} \subseteq \mathcal{C}^n$ is the unit sphere bundle $S(\xi_n)$, the unit sphere bundle. By construction, the fiber of the map $EU(n) \times_{U(n)} S^{2n-1} \to BU(n)$ is S^{2n-1}. Since

$$S^{2n-1} = U(n)/(1 \times U(n-1)),$$

we can write

$$EU(n)\times_{U(n)}S^{2n-1} \simeq EU(n)\times_{U(n)}(U(n)/U(n-1)) \simeq EU(n)/U(n-1) = BU(n-1).$$

In other words, $BU(n-1)$ is the unit sphere bundle of the tautologous line bundle over $BU(n)$. This begets a fiber bundle:

$$S^{2n-1} \to BU(n-1) \to BU(n),$$

which provides an inductive tool (via the Serre spectral sequence) for computing the homology of $BU(n)$. In §5.9, we observed that the Serre spectral sequence for a spherical fibration was completely described bythe Gysin sequence.

Recall that if B is oriented and $S^{2n-1} \to E \xrightarrow{\pi} B$ is a spherical bundle over B, then the Gysin sequence was a long exact sequence

$$\cdots \to H^{q-1}(E) \xrightarrow{\pi_*} H^{q-2n}(B) \xrightarrow{e\cdot} H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\pi_*} \cdots$$

Let us assume that the cohomology ring of E is polynomial and concentrated in even dimensions. For the base case of the induction, both these assumptions are satisfied (since $BU(0) = *$ and $BU(1) = \mathbf{CP}^\infty$).

These assumptions imply that if q is even, then the map π_* is zero. In particular, multiplication by $e|_{H^{\text{even}}(B)}$ (which we will also denote by e) is injective, i.e., e is a nonzero divisor. Similarly, if q is odd, then $e \cdot H^{q-2n}(B) = H^q(B)$. But if $q = 1$, then $H^{q-2n}(B) = 0$; by induction on q, we find that $H^{\text{odd}}(B) = 0$. Therefore, if q is even, then $H^{q-2n+1}(B) = 0$. This implies that there is a short exact sequence

$$0 \to H^*(B) \xrightarrow{e\cdot} H^*(B) \to H^*(E) \to 0. \tag{6.1}$$

In particular, the cohomology of E is the cohomology of B quotiented by the ideal generated by the nonzero divisor e.

For instance, when $n = 1$, then $B = \mathbf{CP}^\infty$ and $E \simeq *$. We have the canonical generator $e \in H^2(\mathbf{CP}^\infty)$; these deductions tell us the well-known fact that $H^*(\mathbf{CP}^\infty) \simeq \mathbf{Z}[e]$.

Consider the surjection $H^*(B) \xrightarrow{\pi^*} H^*(E)$. Since $H^*(E)$ is polynomial, we can lift the generators of $H^*(E)$ to elements of $H^*(B)$. This begets a splitting $s : H^*(E) \to H^*(B)$. The existence of the Euler class $e \in H^*(B)$ therefore gives a map $H^*(E)[e] \xrightarrow{\overline{s}} H^*(B)$. We claim that this map is an isomorphism.

This is a standard algebraic argument. Filter both sides by powers of e, i.e., take the e-adic filtration on $H^*(E)[e]$ and $H^*(B)$. Clearly, the associated graded of $H^*(E)[e]$ just consists of an infinite direct sum of the cohomology of E. The associated graded of $H^*(B)$ is the same, thanks to the short exact sequence (6.1). Thus the induced map on the associated graded $\mathrm{gr}^*(\overline{s})$ is an isomorphism. In this particular case (but not in general), we can conclude that $\overline{s}$ is an isomorphism: in any single dimension, the filtration is finite. Thus, using the five lemma over and over again, we see that the map $\overline{s}$ an isomorphism on each filtered piece. This implies that $\overline{s}$ itself is an isomorphism, as desired.

This argument proves that

$$H^*(BU(n-1)) = \mathbf{Z}[c_1, \cdots, c_{n-1}].$$

In particular, there is a map $\pi^* : H^*(BU(n)) \to H^*(BU(n-1))$ which an isomorphism in dimensions at most $2n$. Thus, the generators c_i have *unique* lifts to $H^*(BU(n))$. We therefore get:

Theorem 6.2.1. *There exist classes $c_i \in H^{2i}(BU(n))$ for $1 \leq i \leq n$ such that:*

- *the canonical map $H^*(BU(n)) \xrightarrow{\pi_*} H^*(BU(n-1))$ sends*

$$c_i \mapsto \begin{cases} c_i & i < n \\ 0 & i = n, \text{ and} \end{cases}$$

- $c_n := (-1)^n e \in H^{2n}(BU(n))$.

Moreover,

$$\boxed{H^*(BU(n)) \simeq \mathbf{Z}[c_1, \cdots, c_n]}.$$

The splitting principle

Theorem 6.2.2. *Let $\xi \downarrow X$ be an n-plane bundle. Then there exists a space $\mathrm{Fl}(\xi) \xrightarrow{\pi} X$ such that:*

1. *$\pi^*\xi = \lambda_1 \oplus \cdots \lambda_n$, where the λ_i are line bundles on Y, and*
2. *the map $\pi^* : H^*(X) \to H^*(\mathrm{Fl}(\xi))$ is monic.*

Proof. We have already (somewhat) studied this space. Recall that there is a vector bundle $\pi : \mathbf{P}(\xi) \to X$ such that

$$H^*(\mathbf{P}(\xi)) = H^*(X)\langle 1, e, \cdots, e^{n-1}\rangle.$$

Moreover, in §6.1, we proved that there is a complex line bundle over $\mathbf{P}(\xi)$ which is a subbundle of $\pi^*\xi$. In other words, $\pi^*\xi$ splits as a sum of a line bundle and some other bundle (by Corollary 4.1.11). Iterating this construction proves the existence of $\mathrm{Fl}(\xi)$. □

This proof does not give much insight into the structure of $\mathrm{Fl}(\xi)$. Remember that the *frame bundle* $\mathrm{Fr}(\xi)$ of ξ: an element of $\mathrm{Fr}(\xi)$ is a linear, inner-product preserving map $\mathbf{C}^n \to E(\xi)$. This satisfies various properties; for instance:

$$E(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} \mathbf{C}^n.$$

Moreover,

$$\mathbf{P}(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} U(n)/(1 \times U(n-1)).$$

The *flag bundle* $\mathrm{Fl}(\xi)$ is defined to be

$$\mathrm{Fl}(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} U(n)/(U(1) \times \cdots \times U(1)).$$

The product $U(1) \times \cdots \times U(1)$ is usually denoted T^n, since it is the maximal torus in $U(n)$. For the universal bundle $\xi_n \downarrow BU(n)$, the frame bundle is exactly $EU(n)$; therefore, $\mathrm{Fl}(\xi_n)$ is just the bundle given by $BT^n \to BU(n)$. By construction, the fiber of this bundle is $U(n)/T^n$. In particular, there is a monomorphism $H^*(BU(n)) \hookrightarrow H^*(BT^n)$. The cohomology of BT^n is extremely simple — it is the cohomology of a product of $\mathbf{CP}^\infty$'s, so

$$H^*(BT^n) \simeq \mathbf{Z}[t_1, \cdots, t_n],$$

where $|t_k| = 2$. The t_i are the Euler classes of $\pi_i^*\lambda_i$, under the projection map $\pi_i : BT^n \to \mathbf{CP}^\infty$.

6.3 The Whitney sum formula

As we saw in the previous section, there is an injection $H^*(BU(n)) \hookrightarrow H^*(BT^n)$. What is the image of this map?

The symmetric group sits inside of $U(n)$, so it acts by conjugation on $U(n)$. This action stabilizes this subgroup T^n. By naturality, Σ_n acts on the classifying space BT^n. Since Σ_n acts by conjugation on $U(n)$, it acts on $BU(n)$ in a way that is homotopic to the identity (Lemma 4.7.1). However, each element $\sigma \in \Sigma_n$ simply permutes the factors in $BT^n = (\mathbf{CP}^\infty)^n$; we conclude that $H^*(BU(n); R)$ actually sits inside the invariants $H^*(BT^n; R)^{\Sigma_n}$.

Recall the following theorem from algebra:

Theorem 6.3.1. *Let Σ_n act on the polynomial algebra $R[t_1, \cdots, t_n]$ by permuting the generators. Then*

$$R[t_1, \cdots, t_n]^{\Sigma_n} = R[\sigma_1^{(n)}, \cdots, \sigma_n^{(n)}],$$

where the σ_i are the elementary symmetric polynomials, defined via

$$\prod_{i=1}^{n}(x - t_i) = \sum_{j=0}^{n} \sigma_i^{(n)} x^{n-i}.$$

For instance,

$$\sigma_1^{(n)} = -\sum t_i, \ \sigma_n^{(n)} = (-1)^n \prod t_i.$$

If we impose a grading on $R[t_1, \cdots, t_n]$ such that $|t_i| = 2$, then $|\sigma_i^{(n)}| = 2i$. It follows from our discussion in §6.2 that the ring $H^*(BT^n)^{\Sigma_n}$ has the same size as $H^*(BU(n))$.

Consider an injection of finitely generated abelian groups $M \hookrightarrow N$, with quotient Q. Suppose that, after tensoring with any field, the map $M \to N$ an isomorphism. If $Q \otimes k = 0$, then $Q = 0$. Indeed, if $Q \otimes \mathbf{Q} = 0$ then Q is torsion. Similarly, if $Q \otimes \mathbf{F}_p = 0$, then Q has no p-component. In particular, $M \simeq N$. Applying this to the map $H^*(BU(n) \to H^*(BT^n)^{\Sigma_n}$, we find that

$$H^*(BU(n); R) \xrightarrow{\simeq} H^*(BT^n; R)^{\Sigma_n} = R[\sigma_1^{(n)}, \cdots, \sigma_n^{(n)}].$$

What happens as n varies? There is a map $R[t_1, \cdots, t_n] \to R[t_1, \cdots, t_{n-1}]$ given by sending $t_n \mapsto 0$ and $t_i \mapsto t_i$ for $i \neq n$. Of course, we cannot say that this map is equivariant with respect to the action of Σ_n. However, it *is* equivariant with respect to the action of Σ_{n-1} on $R[t_1, \cdots, t_n]$ via the inclusion of $\Sigma_{n-1} \hookrightarrow \Sigma_n$ as the stabilizer of $n \in \{1, \cdots, n\}$. Therefore, the Σ_n-invariants sit inside the Σ_{n-1}-invariants, giving a map

$$R[t_1, \cdots, t_n]^{\Sigma_n} \to R[t_1, \cdots, t_n]^{\Sigma_{n-1}} \to R[t_1, \cdots, t_{n-1}]^{\Sigma_{n-1}}.$$

We also find that for $i < n$, we have $\sigma_i^{(n)} \mapsto \sigma_i^{(n-1)}$ and $\sigma_n^{(n)} \mapsto 0$.

Where do the Chern classes go?

To answer this question, we will need to understand the multiplicativity of the Chern class. We begin with a discussion about the Euler class. Suppose $\xi^p \downarrow X, \eta^q \downarrow Y$ are oriented real vector bundles; then, we can consider the bundle $\xi \times \eta \downarrow X \times Y$, which is another oriented real vector bundle. The orientation is given by picking oriented bases for ξ and η. We claim that

$$e(\xi \times \eta) = e(\xi) \times e(\eta) \in H^{p+q}(X \times Y).$$

Since $D(\xi \times \eta)$ is homeomorphic to $D(\xi) \times D(\eta)$, and $S(\xi \times \eta) = D(\xi) \times S(\eta) \cup S(\xi) \times D(\eta)$, we learn from the relative Künneth formula that

$$H^*(D(\xi \times \eta), S(\xi \times \eta)) \leftarrow H^*(D(\xi), S(\xi)) \otimes H^*(D(\eta), S(\eta)).$$

It follows that

$$u_{\xi \times \eta} = u_\xi \times u_\eta \in H^{p+q}(\mathrm{Th}(\xi) \times \mathrm{Th}(\eta));$$

this proves the desired result since the Euler class is the image of the Thom class under the map $H^n(\mathrm{Th}(\xi)) \to H^n(D(\xi)) \simeq H^n(B)$.

Consider the diagonal map $\Delta : X \to X \times X$. The cross product in cohomology then pulls back to the cup product, and the direct product of fiber bundles pulls back to the Whitney sum. It follows that

$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta).$$

If $\xi^n \downarrow X$ is an n-dimensional complex vector bundle, then we defined[1]

$$c_n(\xi) = (-1)^n e(\xi_{\mathbf{R}}).$$

We need to describe the image of $c_n(\xi_n)$ under the map $H^{2n}(BU(n)) \to H^{2n}(BT^n)^{\Sigma_n}$.

Let $f : BT^n \to BU(n)$ denote the map induced by the inclusion of the maximal torus. Then, by construction, we have a splitting

$$f^*\xi_n = \lambda_1 \oplus \cdots \oplus \lambda_n.$$

Thus,

$$(-1)^n e(\xi) \mapsto (-1)^n e(\lambda_1 \oplus \cdots \oplus \lambda_n) = (-1)^n e(\lambda_1) \cup \cdots \cup e(\lambda_n).$$

The discussion above implies that f^* sends the right hand side to $(-1)^n t_1 \cdots t_n = \sigma_n^{(n)}$. In other words, the top Chern class maps to $\sigma_n^{(n)}$ under the map f^*.

[1] There's a slight technical snag here: a complex bundle doesn't have an orientation. However, its underlying oriented real vector bundle does.

Our discussion in the previous sections gives a commuting diagram:

$$\begin{tikzcd} H^*(BU(n)) \arrow[r] \arrow[d] & H^*(BT^n)^{\Sigma_n} \arrow[d] \\ H^*(BU(n-1)) \arrow[r] & H^*(BT^{n-1})^{\Sigma_{n-1}} \end{tikzcd}$$

Arguing inductively, we find that going from the top left corner to the bottom left corner to the bottom right corner sends

$$c_i \mapsto c_i \mapsto \sigma_i^{(n-1)} \text{ for } i < n.$$

Likewise, going from the top left corner to the top right corner to the bottom right corner sends

$$c_i \mapsto \sigma_i^{(n)} \mapsto \sigma_i^{(n-1)} \text{ for } i < n.$$

We conclude that the map f^* sends $c_i^{(i)} \mapsto \sigma_i^{(i)}$.

Proving the Whitney sum formula

By our discussion above, the Whitney sum formula of Theorem 6.1.4 reduces to proving the following identity:

$$\sigma_k^{(p+q)} = \sum_{i+j=k} \sigma_i^{(p)} \cdot \sigma_j^{(q)} \tag{6.2}$$

inside $\mathbf{Z}[t_1, \cdots, t_p, t_{p+1}, \cdots, t_{p+q}]$. Here, $\sigma_i^{(p)}$ is thought of as a polynomial in $t_1, \cdots, t_p$, while $\sigma_i^{(q)}$ is thought of as a polynomial in $t_{p+1}, \cdots, t_{p+q}$. To derive Equation (6.2), simply compare coefficients in the following:

$$\begin{aligned} \sum_{k=0}^{p+q} \sigma_k^{(p+q)} x^{p+q-k} &= \prod_{i=1}^{p+q} (x - t_i) \\ &= \prod_{i=1}^{p} (x - t_i) \cdot \prod_{j=p+1}^{p+q} (x - t_j) \\ &= \left(\sum_{i=0}^{p} \sigma_i^{(p)} x^{p-i} \right) \left(\sum_{j=0}^{q} \sigma_j^{(p)} x^{q-j} \right) \\ &= \sum_{k=0}^{p+q} \left(\sum_{i+j=k} \sigma_i^{(p)} \sigma_j^{(q)} \right) x^{p+q-k}. \end{aligned}$$

6.4 Stiefel-Whitney classes, immersions, cobordisms

There is a result analogous to Theorem 6.1.4 for all vector bundles (not necessarily oriented):

Theorem 6.4.1. *There exist a unique family of characteristic classes* $w_i : \mathrm{Vect}_n(X) \to H^n(X; \mathbf{F}_2)$ *such that for* $0 \leq i$ *and* $i > n$*, we have* $w_i = 0$*, and:*

1. $w_0 = 1$*;*
2. $w_1(\lambda) = e(\lambda)$*; and*
3. *the Whitney sum formula holds:*

$$w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \cup w_j(\eta)$$

Moreover:

$$H^*(BO(n); \mathbf{F}_2) = \mathbf{F}_2[w_1, \cdots, w_n],$$

where $w_n = e_2$*.*

Remark 6.4.2. We can express the Whitney sum formula simply by defining the *total Steifel-Whitney class*

$$1 + w_1 + w_2 + \cdots =: w.$$

Then the Whitney sum formula is just

$$w(\xi \oplus \eta) = w(\xi) \cdot w(\eta).$$

Likewise, the Whitney sum formula can be stated by defining the total Chern class.

Remark 6.4.3. Again, the Steifel-Whitney classes are stable:

$$w(\xi \oplus k\epsilon) = w(\xi).$$

Again, Grothendieck's definition works since the splitting principle holds. There is an injection $H^*(BO(n)) \hookrightarrow H^*(B(\mathbf{Z}/2\mathbf{Z})^n)$. To compute $H^*(BO(n))$, our argument for computing $H^*(BU(n))$ does not immediately go through, although there is a fiber sequence

$$S^{n-1} \to EO(n) \times_{O(n)} O(n)/O(n-1) \to BO(n);$$

the problem is that $n-1$ can be even or odd. We still have a Gysin sequence, though:

$$\cdots \to H^{q-n}(BO(n)) \xrightarrow{e\cdot} H^q(BO(n)) \xrightarrow{\pi^*} H^q(BO(n-1)) \to H^{q-n+1}(BO(n)) \to \cdots$$

In order to apply our argument for computing $H^*(BU(n))$ to this case, we only need to know that e is a nonzero divisor. The splitting principle gave a monomorphism $H^*(BO(n)) \hookrightarrow H^*((\mathbf{RP}^\infty)^n)$. The fact that e is a nonzero divisor follows from the observation that under this map,

$$e_2 = w_n \mapsto e_2(\lambda_1 \oplus \cdots \oplus \lambda_n) = t_1 \cdots t_n,$$

using the same argument as in §6.3; however, $t_1 \cdots t_n$ is a nonzero divisor, since $H^*((\mathbf{RP}^\infty)^n)$ is an integral domain.

Immersions of manifolds

The theory developed above has some interesting applications to differential geometry.

Definition 6.4.4. Let M^n be a smooth closed manifold. An *immersion* is a smooth map from M^n to $\mathbf{R}^{n+k}$, denoted $f : M^n \looparrowright \mathbf{R}^{n+k}$, such that $(\tau_{M^n})_x \hookrightarrow (\tau_{\mathbf{R}^{n+k}})_{f(x)}$ for $x \in M$.

Informally: crossings are allows, but not cusps.

Example 6.4.5. There is an immersion $\mathbf{RP}^2 \looparrowright \mathbf{R}^3$, known as *Boy's surface.*

Question 6.4.6. When can a manifold admit an immersion into an Euclidean space?

Assume we had an immersion $i : M^n \looparrowright \mathbf{R}^{n+k}$. Then we have an embedding $f : \tau_M \to i^*\tau_{\mathbf{R}^{n+k}}$ into a trivial bundle over M, so τ_M has a k-dimensional complement, called ξ such that

$$\tau_M \oplus \xi = (n+k)\epsilon.$$

Apply the total Steifel-Whitney class, we have

$$w(\tau)w(\xi) = 1,$$

since there's no higher Steifel-Whitney class of a trivial bundle. In particular,

$$w(\xi) = w(\tau)^{-1}.$$

Example 6.4.7. Let $M = \mathbf{RP}^n \looparrowright \mathbf{R}^{n+k}$. Then, we know that

$$\tau_{\mathbf{RP}^n} \oplus \epsilon \simeq (n+1)\lambda^* \simeq (n+1)\lambda,$$

where $\lambda \downarrow \mathbf{RP}^n$ is the canonical line bundle. By Remark 6.4.3, we have

$$w(\tau_{\mathbf{RP}^n}) = w(\tau_{\mathbf{RP}^n} \oplus \eta) = w((n+1)\lambda) = w(\lambda)^{n+1}.$$

It remains to compute $w(\lambda)$. Only the first Steifel-Whitney class is nonzero. Writing $H^*(\mathbf{RP}^n) = \mathbf{F}_2[x]/x^{n+1}$, we therefore have $w(\lambda) = x$. In particular,

$$w(\tau_{\mathbf{RP}^n}) = (1+x)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} x^i.$$

It follows that

$$w_i(\tau_{\mathbf{RP}^n}) = \binom{n+1}{i} x^i.$$

The total Steifel-Whitney class of the complement of the tangent bundle is:

$$w(\xi) = (1+x)^{-n-1}.$$

The most interesting case is when n is a power of 2, i.e., $n = 2^s$ for some integer s. In this case, since taking powers of 2 is linear in characteristic 2, we have

$$w(\xi) = (1+x)^{-1-2^s} = (1+x)^{-1}(1+x)^{-2^s} = (1+x)^{-1}(1+x^{2^s})^{-1}.$$

As all terms of degree greater than 2^s are zero, we conclude that So

$$w(\xi) = 1 + x + x^2 + \cdots + x^{2^s-1} + 2x^s = 1 + x + x^2 + \cdots + x^{2^s-1}.$$

As $x^{2^s-1} \neq 0$, this means that $k = \dim \xi \geq 2^s - 1$. We conclude:

Theorem 6.4.8. *There is no immersion* $\mathbf{RP}^{2^s} \looparrowright \mathbf{R}^{2\cdot 2^s-2}$.

The following result applied to $\mathbf{RP}^{2^s}$ shows that the above result is sharp:

Theorem 6.4.9 (Whitney)**.** *Any smooth compact closed manifold* $M^n \looparrowright \mathbf{R}^{2n-1}$.

However, Whitney's result is *not* sharp for a general smooth compact closed manifold. Rather, we have:

Theorem 6.4.10 (Brown–Peterson, Cohen). *A closed compact smooth n-manifold $M^n \looparrowright \mathbf{R}^{2n-\alpha(n)}$, where $\alpha(n)$ is the number of 1s in the dyadic expansion of n.*

This result is sharp, since if $n = \sum 2^{d_i}$ for the dyadic expansion, then $M = \prod_i \mathbf{RP}^{2^{d_i}} \not\looparrowright \mathbf{R}^{2n-\alpha(n)-1}$.

Cobordism, characteristic numbers

If we have a smooth closed compact n-manifold, then it embeds in $\mathbf{R}^{n+k}$ for some $k \gg 0$. The normal bundle then satisfies

$$\tau_M \oplus \nu_M = (n+k)\epsilon.$$

A piece of differential topology tells us that if k is large, then $\nu_M \oplus N\epsilon$ is independent of the bundle for some N.

This example, combined with Remark **??**, shows that $w(\nu_M)$ is independent of k. We are therefore motivated to think of Stiefel-Whitney classes as coming from $H^*(BO;\mathbf{F}_2) = \mathbf{F}_2[w_1, w_2, \cdots]$, where $BO = \varinjlim BO(n)$. Similarly, Chern classes should be thought of as coming from $H^*(BU;\mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots]$. This exa

Definition 6.4.11. The characteristic number of a smooth closed compact n-manifold M is defined to be $\langle w(\nu_M), [M]\rangle$.

Note that the fundamental class $[M]$ exists, since our coefficients are in $\mathbf{F}_2$, where everything is orientable.

This definition is very useful when thinking about cobordisms.

Definition 6.4.12. Two (smooth closed compact) n-manifolds M, N are *(co)bordant* if there is an $(n+1)$-dimensional manifold W^{n+1} with boundary such that

$$\partial W \simeq M \sqcup N.$$

For instance, when $n = 0$, the manifold $* \sqcup *$ is *not* cobordant to $*$, but it is cobordant to the empty set. However, $* \sqcup * \sqcup *$ is cobordant to $*$. Any manifold is cobordant to itself, since $\partial(M \times I) = M \sqcup M$. In fact, cobordism forms an equivalence relation on manifolds.

Example 6.4.13. A classic example of a cobordism is the "pair of pants"; this is the following cobordism between S^1 and $S^1 \sqcup S^1$: add image

Let us define

$$\Omega_n^O = \{\text{cobordism classes of } n\text{-manifolds}\}.$$

This forms a group: the addition is given by disjoint union. Note that every element is its own inverse. Moreover, $\bigoplus_n \Omega_n^O = \Omega_*^O$ forms a graded ring, where the product is given by the Cartesian product of manifolds. Our discussion following Definition 6.4.12 shows that $\Omega_0^O = \mathbf{F}_2$.

Exercise 6.4.14. Every 1-manifold is nullbordant, i.e., cobordant to the point.

Thom made the following observation. Suppose an n-manifold M is embedded into Euclidean space, and that M is nullbordant via some $(n+1)$-manifold W, so that $\nu_W|_M = \nu_M$. In particular,

$$\langle w(\nu_M), [M]\rangle = \langle w(\nu_W)|_M, [M]\rangle.$$

On the other hand, the boundary map $H_{n+1}(W, M) \xrightarrow{\partial} H_n(M)$ sends the relative fundamental class $[W, M]$ to $[M]$. Thus

$$\langle w(\nu_M), [M]\rangle = \langle w(\nu_M), \partial[W, M]\rangle = \langle \delta w(\nu_M), [W, M]\rangle.$$

However, we have an exact sequence

$$H^n(W) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(W, M).$$

Since $w(\nu_M)$ is in the image of i^*, it follows that $\delta w(\nu_M) = 0$. In particular, the characteristic number of a nullbordant manifold is zero. Thus, we find that "Stiefel-Whitney numbers tell all":

Proposition 6.4.15. *Characteristic numbers are cobordism invariants. In other words, characteristic numbers give a map*

$$\Omega_n^O \to \mathrm{Hom}(H^n(BO), \mathbf{F}_2) \simeq H_n(BO).$$

More is true:

Theorem 6.4.16 (Thom, 1954)**.** *The map of graded rings $\Omega_*^O \to H_*(BO)$ defined by the characteristic number is an inclusion. Concretely, if $w(M^n) = w(N^n)$ for all $w \in H^n(BO)$, then M^n and N^n are cobordant.*

The way that Thom proved this was by expressing Ω_*^O is the graded homotopy ring of some space, which he showed is the product of mod 2 Eilenberg-MacLane spaces. Along the way, he also showed that:

$$\Omega_*^O = \mathbf{F}_2[x_i : i \neq 2^s - 1] = \mathbf{F}_2[x_2, x_4, x_5, x_6, x_8, \cdots]$$

This recovers the result of Exercise 6.4.14 (and so much more!).

6.5 Oriented bundles, Pontryagin classes, Signature theorem

We have a pullback diagram

$$\begin{tikzcd} BSO(n) \arrow[r] \arrow[d, "\text{double cover}"] & S^\infty \arrow[d] \\ BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z} \end{tikzcd}$$

The bottom map is exactly the element $w_1 \in H^1(BO(n); \mathbf{F}_2)$. It follows that a vector bundle $\xi \downarrow X$ represented by a map $f : X \to BO(n)$ is orientable iff $w_1(\xi) = f^*(w_1) = 0$, since this is equivalent to the existence of a factorization:

$$\begin{tikzcd} & BSO(n) \arrow[r] \arrow[d] & S^\infty \arrow[d] \\ X \arrow[r, "\xi"'] \arrow[ur, dashed] & BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z} \end{tikzcd}$$

The fiber sequence $BSO(n) \to BO(n) \to \mathbf{RP}^\infty$ comes from a fiber sequence $SO(n) \to O(n) \to \mathbf{Z}/2\mathbf{Z}$ of groups. For $n \geq 3$, we can kill $\pi_1(SO(n)) = \mathbf{Z}/2\mathbf{Z}$, to get a double cover $\mathrm{Spin}(n) \to SO(n)$. The group $\mathrm{Spin}(n)$ is called the *spin group*. We have a cofiber sequence

$$B\mathrm{Spin}(n) \to BSO(n) \xrightarrow{w_2} K(\mathbf{Z}/2\mathbf{Z}, 2).$$

If $w_2(\xi) = 0$, we get a further lift in the above diagram, begetting a *spin structure* on ξ.

Bott computed that $\pi_2(\mathrm{Spin}(n)) = 0$. However, $\pi_3(\mathrm{Spin}(n)) = \mathbf{Z}$; killing this gives the *string group* $\mathrm{String}(n)$. Unlike $\mathrm{Spin}(n)$, $SO(n)$, and $O(n)$, this is not a finite-dimensional Lie group (since we have an infinite dimensional summand $K(\mathbf{Z}, 2)$). However, it can be realized as a topological group. The resulting maps

$$\mathrm{String}(n) \to \mathrm{Spin}(n) \to SO(n) \to O(n)$$

are just the maps in the Whitehead tower for $O(n)$. Taking classifying spaces, we get

$$\begin{tikzcd}
 & B\mathrm{String}(n) \arrow[d] & \\
 & B\mathrm{Spin}(n) \arrow[r, "p_1/2"] \arrow[d] & K(\mathbf{Z}, 4) \\
 & BSO(n) \arrow[r] \arrow[d] & K(\mathbf{Z}/2\mathbf{Z}, 2) \\
X \arrow[r, "\xi"'] \arrow[ur, dashed] \arrow[uur, dashed] \arrow[uuur, dashed] & BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z}
\end{tikzcd}$$

Computing the (mod 2) cohomology of $BSO(n)$ is easy. We have a double cover $BSO(n) \to BO(n)$ with fiber S^0. Consequently, there is a Gysin sequence:

$$0 \to H^q(BO(n)) \xrightarrow{w_1} H^{q+1}(BO(n)) \xrightarrow{\pi^*} H^{q+1}(BSO(n)) \to 0$$

since w_1 is a nonzero divisor. The standard argument shows that

$$H^*(BSO(n)) = \mathbf{F}_2[w_2, \cdots, w_n].$$

However, it is *not* easy to compute $H^*(B\mathrm{Spin}(n))$ and $H^*(B\mathrm{String}(n))$; these are extremely complicated (and only become more complicated for higher connective covers of $BO(n)$). However, we will remark that they are concentrated in even degrees.

To define integral characteristic classes for oriented bundles, we will need to study Chern classes a little more. Let ξ be a complex n-plane bundle, and let $\overline{\xi}$ denote the conjugate bundle. What is the total Chern class $c(\overline{\xi})$? Recall that the Chern classes $c_k(\overline{\xi})$ occur as coefficients in the identity

$$\sum c_i(\overline{\xi}) e(\lambda_{\overline{\xi}})^{n-i} = 0,$$

where $\lambda_{\overline{\xi}} \downarrow \mathbf{P}(\overline{\xi})$. Note that $\mathbf{P}(\overline{\xi}) = \mathbf{P}(\xi)$. By construction, $\lambda_{\overline{\xi}} = \overline{\lambda_\xi}$. In particular, we find that

$$e(\lambda_{\overline{\xi}}) = -e(\lambda_\xi).$$

It follows that

$$0 = \sum_{i=0}^n c_i(\overline{\xi}) e(\overline{\lambda_\xi})^{n-i} = \sum_{i=0}^n c_i(\overline{\xi})(-1)^{n-i} e(\lambda_\xi)^{n-i} = (-1)^n e(\lambda_\xi)^n + \cdots$$

This is *not* monic, and hence doesn't define the Chern classes of $\overline{\xi}$. We do, however, get a monic polynomial by multiplying this identity by $(-1)^n$:

$$\sum_{i=0}^{n}(-1)^i c_i(\overline{\xi})e(\lambda_\xi)^{n-i} = 0.$$

It follows that

$$\boxed{c_i(\overline{\xi}) = (-1)^i c_i(\xi).}$$

If ξ is a real vector bundle, then

$$c_i(\xi \otimes \mathbf{C}) = c_i(\overline{\xi \otimes \mathbf{C}}) = (-1)^i c_i(\xi \otimes \mathbf{C}).$$

If i is odd, then $2c_i(\xi \otimes \mathbf{C}) = 0$. If R is a $\mathbf{Z}[1/2]$-algebra, we therefore define:

Definition 6.5.1. Let ξ be a real n-plane vector bundle. Then the kth Pontryagin class of ξ is defined to be

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathcal{C}) \in H^{4k}(X; R).$$

Notice that this is 0 if $2k > n$, since $\xi \otimes \mathcal{C}$ is of complex dimension n. The Whitney sum formula now says that:

$$(-1)^k p_k(\xi \oplus \eta) = \sum_{i+j=k}(-1)^i p_i(\xi)(-1)^j p_j(\eta) = (-1)^k \sum_{i+j=k} p_i(\xi)p_j(\eta).$$

If ξ is an oriented real $2k$-plane bundle, one can calculate that

$$p_k(\xi) = e(\xi)^2 \in H^{4k}(X; R).$$

We can therefore write down the cohomology of $BSO(n)$ with coefficients in a $\mathbf{Z}[1/2]$-algebra:

$* =$	2	4	6	8	10	12
$H^*(BSO(2))$	e_2	(e_2^2)				
$H^*(BSO(3))$		p_1				
$H^*(BSO(4))$		p_1, e_4		(e_4^2)		
$H^*(BSO(5))$		p_1		p_2		
$H^*(BSO(6))$		p_1	e_6	p_2		(e_6^2)
$H^*(BSO(7))$		p_1		p_2		p_3

Here, $p_k \mapsto e_{2k}^2$. In the limiting case (i.e., for $BSO = BSO(\infty)$), we get a polynomial algebra on the p_i.

Applications

We will not prove any of the statements in this section; it only serves as an outlook. The first application is the following analogue of Theorem 6.4.16:

Theorem 6.5.2 (Wall)**.** *Let M^n, N^n be oriented manifolds. If all Stiefel-Whitney numbers and Pontryagin numbers coincide, then M is oriented cobordant to N, i.e., there is an $(n+1)$-manifold W^{n+1} such that*

$$\partial W^{n+1} = M \sqcup -N.$$

The most exciting application of Pontryagin classes is to Hirzebruch's "signature theorem". Let M^{4k} be an oriented $4k$-manifold. Then, the formula

$$x \otimes y \mapsto \langle x \cup y, [M] \rangle$$

defines a pairing

$$H^{2k}(M)/\text{torsion} \otimes H^{2k}(M)/\text{tors} \to \mathbf{Z}.$$

Poincaré duality implies that this is a perfect pairing, i.e., there is a nonsingular symmetric bilinear form on $H^{2k}(M)/\text{torsion} \otimes \mathbf{R}$. Every symmetric bilinear form on a real vector space can be diagonalized, so that the associated matrix is diagonal, and the only nonzero entries are ± 1. The number of 1s minus the number of -1s is called the *signature* of the bilinear form. When the bilinear form comes from a $4k$-manifold as above, this is called the signature of the manifold.

Lemma 6.5.3 (Thom)**.** *The signature is an oriented bordism invariant.*

This is an easy thing to prove using Lefschetz duality, which is a deep theorem. Hirzebruch's signature theorem says:

Theorem 6.5.4 (Hirzebruch signature theorem)**.** *There exists an explicit rational polynomial $L_k(p_1, \cdots, p_k)$ of degree $4k$ such that*

$$\langle L(p_1(\tau_M), \cdots, p_1(\tau_M)), [M] \rangle = \text{signature}(M).$$

The reason the signature theorem is so interesting is that the polynomial $L(p_1(\tau_M), \cdots, p_1(\tau_M))$ is defined only in terms of the tangent bundle of the manifold, while the signature is defined only in terms of the topology of the manifold. This result was vastly generalized by Atiyah and Singer to the Atiyah-Singer index theorem.

Example 6.5.5. One can show that

$$L_1(p_1) = p_1/3.$$

The Hirzebruch signature theorem implies that $\langle p_1(\tau), [M^4]\rangle$ is divisible by 3.

Example 6.5.6. From Hirzebruch's characterization of the L-polynomial, we have

$$L_2(p_1, p_2) = (7p_2 - p_1^2)/45.$$

This imposes very interesting divisibility constraints on the characteristic classes of a tangent bundle of an 8-manifold. This particular polynomial was used by Milnor to produce "exotic spheres", i.e., manifolds which are homeomorphic to S^7 but not diffeomorphic to it.

Bibliography

[Beh06] M. Behrens. Algebraic Topology II. `https://ocw.mit.edu/courses/mathematics/18-906-algebraic-topology-ii-spring-2006/`, 2006.

[Bre93] G. Bredon. *Topology and Geometry*. Springer, 1993.

[Bro57] E. Brown. Finite Computability of Postnikov Complexes. *Ann. of Math*, 65(1):1–20, 1957.

[GJ99] P. Goerss and J. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in mathematics (Boston, Mass.)*. Springer, 1999.

[Hat15] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2015.

[Hus94] Husemöller, D. *Fibre Bundles*. Graduate Texts in Mathematics. Springer, 1994.

[May99] P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999.

[MS74] J. Milnor and J. Stasheff. *Characteristic Classes*. Princeton University Press and University of Tokyo Press, 1974.

[Str09] Neil Strickland. The category of CGWH spaces. `http://math.mit.edu/~hrm/18.906/strickland-category-cgwh.pdf`, 2009.