counting problems

1) one unique subset of 5 letters exist

\# of strings that can be made: $5!$

2) Using a standard deck of playing cards, how many ways are to form a 5-card hand with 2 pairs (i.e. pair of one value, a pair of a different value, and a fifth card of some other value)?

$${}_{13}C_2 \times {}_4C_2 \times {}_4C_2 \times 44$$

$$\frac{13!}{2! \times 11!} \times \frac{4!}{2! \times 2!} \times \frac{4!}{2! \times 2!} \times 44 = 78 \times 6 \times 6 \times 44 = \boxed{123{,}552 \text{ ways}}$$

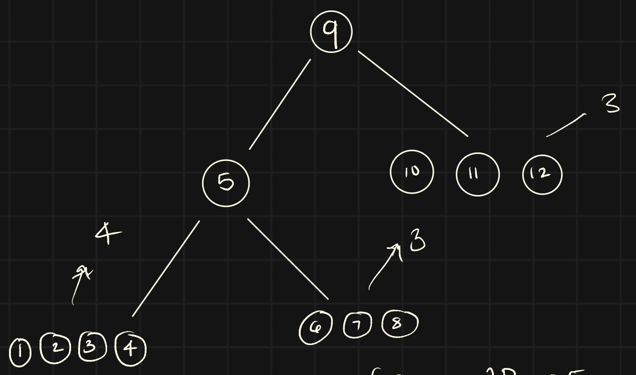
3) A violinist serenades couples at a romantic restaurant. She will play 16 songs in an hour and there are 7 couples. One couple is having a fight and will allow at most 1 song to be played to them before they ask the violist not to return to their table. If we care only about the number of songs each couple receives, how many ways can the songs be distributed amongst the couples.

16 songs → 7 couples

$${}_{16}C_1 \times {}_{15}C_6$$

$$\frac{16!}{1! \times (16-1)!} \times \frac{15!}{6! \times (15-6)!} = \boxed{80{,}080 \text{ ways}}$$

4) There is a Binary Search Tree with 12 nodes. Each node has a distinct value between 1 and 12. The root has value 9, and its left child has value 5. How many possible Binary Search Trees could this be?



$$\frac{{}_6C_3}{3+1} = \frac{20}{4} = 5$$

$$14 \times 5 \times 5 = 350$$

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$$\frac{{}_8C_4}{4+1} = \frac{70}{5} = 14$$

5) 10 friends arrive to get their COVID vaccine during a particular time slot. During that time slot there are 4 identical nurses administering shots, but 1 of the nurses may (or may not) be scheduled for a break during time slot in which the friends arrive. Also, how long it takes the nurses to administer a shot varies wildly, so the nurses working during the time slot are guaranteed to serve at least 1 person, but how many additional people they are able to serve is arbitrary. How many different combinations are there for the number of patients served by the nurses?

10 friends = ${}_{10}C_1 = 10$

4 nurses = ${}_4C_1 = 4$

$${}_{10}C_4 = \frac{10!}{(10-4)!\,4!} = \frac{10!}{6!\,4!} = 210$$

different combinations for the number of patients

$${}_{10}P_4 = \frac{10!}{(10-4)!} = 5040 \text{ combinations}$$