Fast and Frenzied

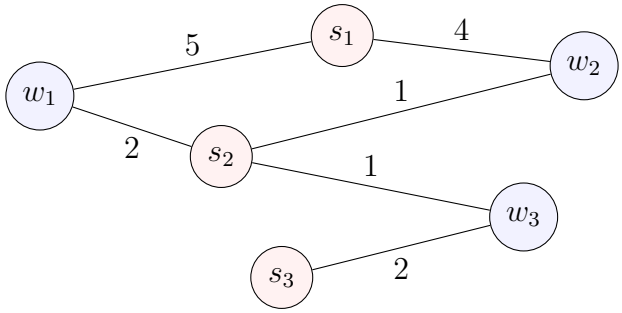
This is a **regular task**. You must submit a PDF, which can be produced using the LaTeX template on Moodle, exported from a word processor, hand-written or any other method.

You have been hired by eXpressDriverZ to work on their new delivery planning system. Across the country, the company owns n warehouses and needs to supply n shops. There are a total of $m > n$ roads, each road going directly from one warehouse to one shop. The ith road takes d_i hours to drive along, where each d_i is an integer.

Each warehouse has one truck that can carry its goods to one shop. Each truck is disposable and can only be used to drive along exactly one road. Hence, a truck must travel directly to a shop.

Your task is to choose a destination for each truck so that all n shops receive a delivery. Since each warehouse can only send one truck, this involves pairing each shop with a single warehouse. Once you have chosen a destination for each truck, they will **all leave at the same time**.

For example, consider the road map shown (with warehouses w_1, w_2, w_3 and shops s_1, s_2, s_3).



Warehouse w_3 must deliver to shop s_3, because it is the only warehouse that can do so. We can then choose for w_1 to deliver to s_1 (with w_2 delivering to s_2) or for w_1 to deliver to s_2 (with w_2 delivering to s_1). In the latter case, it will take 4 hours to supply all the shops: $w_1 \to s_2$ takes two hours, $w_2 \to s_1$ takes four hours, and $w_3 \to s_3$ takes two hours; this is optimal.

(a) Suppose you need to supply every shop in at most D hours. Design and analyse an $O(nm)$ algorithm to determine whether it is possible to supply all n shops within this deadline.

(b) Now you are ready to complete your eXpressDriverZ delivery planning system. Design and analyse an $O(nm \log(\max_i d_i))$ algorithm to determine the minimum amount of time you need to supply all n shops, or otherwise determine that it is impossible.

Hint: *If you can supply all n shops within D hours, can you supply all n shops within $D+1$ hours? Why, or why not?*

Advice.

The following applies to parts (a) and (b).

If you are using a flow algorithm, you should design a flow network that solves the problem when an appropriate flow algorithm is used with your network as input. You should also show the bijection between the set of collections of valid truck allocations and the set of valid flows.

If you are using other algorithms at any point, you should ensure that the preconditions of using said algorithms hold before using them.

Expected length: Up to two pages in total.