# River Crossing

This is a **regular task**. You must submit a PDF, which can be produced using the LaTeX template on Moodle, exported from a word processor, hand-written or any other method.

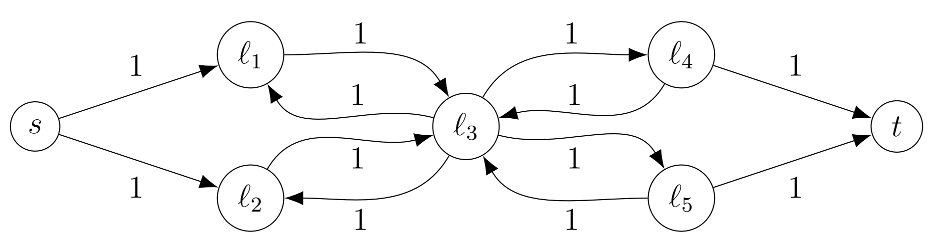
Fermin has an unlimited number of frogs that want to hop from the left bank of the river to the right, via a sequence of $n$ lilypads. However, the lilypads are weak; if one frog has hopped onto a particular lilypad, the lilypad sinks and no other frog can use that lilypad. His frogs are also small; each of his frogs can jump a maximum of $j$ centimetres, where $j$ is smaller than the width of the river. There are $m$ distinct jumps that are possible, including those to/from either riverbank. Fermin wants to know how many of his frogs can cross the river. To figure this out, he constructs a flow network as follows:

- construct a source vertex $s$ denoting the left bank and a sink vertex $t$ denoting the right;
- for $i$th lilypad, construct a vertex $\ell_i$, and connect all pairs of lilypads within $j$ cm with a bidirectional edge of capacity 1;
- for each lilypad vertex $\ell_i$ within $j$ cm of the left bank of the river, construct an edge from $s$ to $\ell_j$ with capacity 1; and
- for each lilypad vertex $\ell_i$ within $j$ cm of the right bank of the river, construct an edge from $\ell_i$ to $t$ with capacity 1.

Fermin claims that the maximum flow in this network is always equal to the maximum number of frogs that can cross the river. For example, suppose that the following hold:

- lilypads 1 and 2 are reachable from the left bank; the right bank can be reached from lilypads 4 and 5;
- lilypads 1 and 3 are reachable from each other,
- lilypads 2 and 3 are reachable from each other,
- lilypads 3 and 4 are reachable from each other, and
- lilypads 3 and 5 are reachable from each other.

Then, noting that $m = 12$ and that there are $n = 5$ lilypads; Fermin would then construct the flow network below and declare that two frogs can cross the river.

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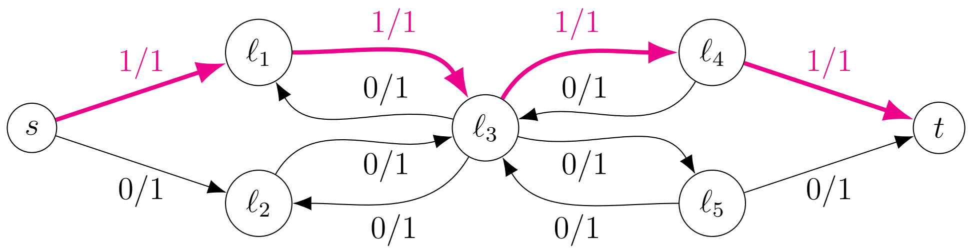
(a) Consider the example from the problem statement. The network shows a given flow, with the thick magenta arrows. Is the given flow a maximum flow for the graph? Justify your answer.




(b) Fermin's claim that the maximum flow in his construction is always equal to the maximum number of frogs that can cross the river from left to right is not correct. Explain why the maximum flow in his flow network does not correctly identify the number of frogs that can cross the river.

(c) Give modifications to Fermin's flow network such that the maximum flow would equal the maximum number of frogs that can cross the river. Justify that your construction is correct.

(d) State and justify the best possible bound for the time complexity of finding the maximum flow in your modified flow network, in terms of the number of lilypads $n$ and the number of distinct possible jumps $m$. Show all computation.

**Advice.**

(a) You should ignore the problem constraints for this subpart only.

(b) Recall that for a flow network to correctly model a problem, *every* valid flow should be a valid instance of the problem, and vice versa.

Therefore, it suffices to show that a bijection does not hold.

You shouldn't prove by example, nor should you discuss changes to the flow network (as this is covered in part (c)).

(c) Your new construction should form a complete network that can be taken as input for a flow algorithm from lectures.

If you are supplying modifications, they should be clear and well-defined, and the modified graph should also meet the criterion above.

(d) Maximum flow algorithms that were not presented in lectures (e.g. Dinic's algorithm) cannot be used here.

**Expected length:** Up to one page in total.

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