Stat 110 Homework 1, Fall 2013

Due: Friday 9/13 at 1:10 pm, in the Stat 110 dropbox on the 7th floor of the Science Center. No late homework will be accepted. Please write your name, and staple your homework. Show your work and make sure to give clear, careful, convincing explanations.

Collaboration policy (for this and all future homeworks): You are welcome to discuss the problems with others, but *you must write up your solutions yourself and in your own words*. Copying someone else's solution, or just making trivial changes for the sake of not copying verbatim, is not acceptable. For example, in problems where you have to make up a "story" or example, two students should not have the exact same answer, or almost the same answer except one has an example with dogs chasing cats and the other has an example with cats chasing mice.

1. A *codeword* is a sequence of at least one (and possibly all) of the 26 letters A,B,C,...,Z, with repetitions not allowed. For example, COURSE is a codeword, but STATISTICS is not a codeword. Order matters; according to this source, of course COURSE is not the same as SOURCE.

A codeword is chosen randomly, with all codewords equally likely. Show *without using a calculator or computer* that the probability that the random codeword uses all 26 letters is very close to $1/e$.

2. (Exercise 1.2 in book) (a) How many paths are there from the point $(0, 0)$ to the point $(110, 111)$ in the plane such that each step either consists of going one unit up or one unit to the right?

(b) How many paths are there from $(0, 0)$ to $(210, 211)$, where each step consists of going one unit up or one unit to the right, and the path has to go through $(110, 111)$?

3. You are playing bridge, which is a card game with 4 players who are divided into 2 partnerships. The cards will be shuffled well, and then a 13-card hand will be dealt to each player. (A *hand* in a card game is a set of cards that a player received, such that the order in which the cards were received does not matter.)

(a) How many possibilities are there for the hand you will be dealt? (You can leave your answer in terms of factorials or binomial coefficients.)

(b) How many possibilities are there overall for what hands everyone will get? It matters who gets which hand, but not the order of cards within a hand. (Give an exact answer in terms of factorials or binomial coefficients, but also use a computer to get a numerical value in scientific notation.)

(c) What is the probability that your hand will have at least 3 cards of every suit?

(d) What is the probability that your hand will be void in at least one suit (i.e., that there is at least one suit that your hand doesn't have present)?

(e) The cards have now been dealt. You can see your cards but not your partner's or opponents' cards. You observe that you have five hearts, three diamonds, two clubs, and three spades. Your opponents reveal that they have a total of exactly seven diamonds. What is the probability that your partner has at least three hearts?

4. Tyrion, Cersei, and ten other people are sitting at a round table, with their seating arrangement having been randomly assigned. What is the probability that Tyrion and Cersei are sitting next to each other? Find this in two ways:

(a) using a sample space of size 12!, where an outcome is fully detailed about the seating;

(b) using a much smaller sample space, which focuses on Tyrion and Cersei.

5. A widget inspector inspects 12 widgets and finds that exactly 3 are defective. Unfortunately, the widgets then get all mixed up and the inspector has to find the 3 defective widgets again by testing widgets one by one.

(a) Find the probability that the inspector will now have to test at least 9 widgets.

(b) Find the probability that the inspector will now have to test at least 10 widgets.

6. (Exercise 1.20 in book) Given n numbers $(a_1, a_2, \ldots, a_n)$ with no repetitions, a *bootstrap sample* is a sequence $(x_1, x_2, \ldots, x_n)$ formed from the a_j's by sampling with replacement with equal probabilities. Bootstrap samples are used in a widely-used statistical method known as the *bootstrap*. For example, if $n = 2$ and $(a_1, a_2) = (3, 1)$, then the possible bootstrap samples are $(3, 3), (3, 1), (1, 3)$, and $(1, 1)$.

(a) How many possible bootstrap samples are there for $(a_1, \ldots, a_n)$?

(b) How many possible bootstrap samples are there for $(a_1, \ldots, a_n)$, if order does not matter (in the sense that it only matters how many times each a_j was chosen, not the order in which they were chosen)?

(c) One random bootstrap sample is chosen. Show that not all unordered bootstrap samples (in the sense of (b)) are equally likely. Find an unordered bootstrap sample $\mathbf{b}_1$ that is as likely as possible, and an unordered bootstrap sample $\mathbf{b}_2$ that is as unlikely as possible. Let p_1 be the probability of getting $\mathbf{b}_1$ and p_2 be the probability of getting $\mathbf{b}_2$. What is p_1/p_2? What is the ratio of the probability of getting a sample whose probability is p_1 to the probability of getting a sample whose probability is p_2?

7. Give a story proof that

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1},$$

for all positive integers n.

Hint: consider choosing a committee of size n from two groups of size n each, where only one of the two groups has people eligible to become President.