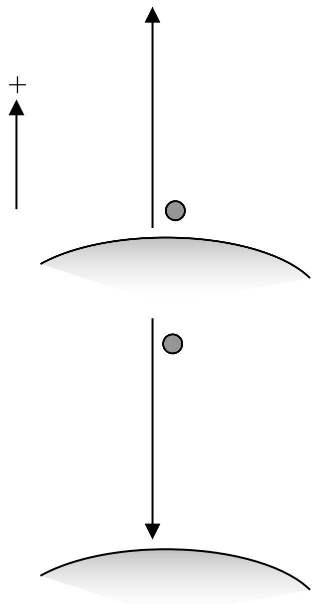
SPH4U: Rocket Science!

Or, the science of movement in a gravitational field.

Part A: Ascent, Descent

Consider a rock traveling straight up and in down in the earth's gravitational field under the influence of gravity alone near the surface of the earth. Indicate in the chart whether the quantity is increasing or decreasing in magnitude. For velocity and work indicate the sign.

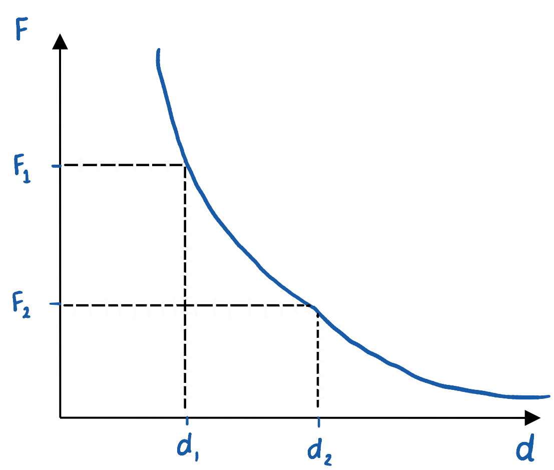


	Trip Up	Trip Down
v	↓ (+)	↑ (-)
E_k	↓	↑
E_g	↑	↓
E_T (Total Energy)	— same	— same
Work by Force of Gravity	(-)	(+)

1. When we calculate E_g, we use the value g = 9.8 N/kg. What happens to this value as we get quite far from the earth? What is the reason for this change?

The value decreases as we get farther from Earth. The reason for this is because as the distance between two bodies increases, the gravitational forces acting between them become weaker.

2. Write down the equation for Newton's Law of Universal Gravitation. Sketch a graph of the force of gravity over great distances. Mark two separate distances, d_1 and d_2. Use the text to help.



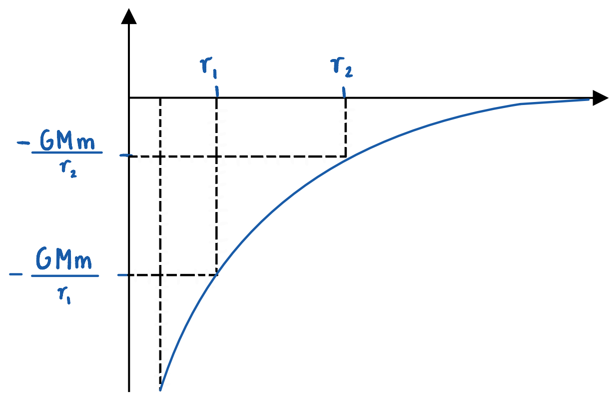
$F_g = \frac{GMm}{r^2}$

3. How do you graphically represent the work done on an object moved from d_1 to d_2? Show this on the graph.

Part B: Gravitational Potential Energy

1. Mathematically we describe the gravitational potential energy (GPE) at a given position when using the equation: $E_g = -\frac{GMm}{r}$ which is the result of the calculus work describing the area under the graph in Part A Q#2.

2. Fill in the chart showing the gravitational potential energies for a 1 kg rock at different positions from the centre of the earth. ($R_e = 6.4 \times 10^6$ m. $M_e = 6.0 \times 10^{24}$ kg) Sketch this as a graph.



Distance from centre of earth	E_g
R_e	-6.25×10^7
$2R_e$	-3.13×10^7
$3R_e$	-2.08×10^7
$4R_e$	-1.56×10^7
$6R_e$	-1.04×10^7
$8R_e$	-7.82×10^6
Really, really far	As $r \to +\infty$, $E_g \to 0$

Note! Really, really far means an infinitely large distance

3. Use *greater than* or *less than* symbols to compare:
   $E_g(R_e) < E_g(2R_e)$
   $E_g(R_e) < E_g$(really, really far)

4. According to your graph, at what position is the greatest amount of energy stored in the gravitational field?
   The infinitely large distance (really, really far) has the greatest amount of stored energy.

5. What is the change in GPE when the rock travels from:
   a) $2R_e$ to $6R_e$ = 2.09×10^7
   b) $8R_e$ to $4R_e$ = -7.78×10^6
   c) R_e to infinity $\approx 6.25 \times 10^7$
   d) infinity to R_e $\approx -6.25 \times 10^7$

6. Is energy being stored in or returned from the gravitational field when ΔE_g is:
   a) positive? When ΔEg is positive energy is being stored.
   b) negative? When ΔEg is negative energy is being returned.

7. Imagine a small rocket engine is attached to the rock. How much work is done by the rocket engines to move the rock from:
   a) $2R_e$ to $6R_e$, $W = 2.09 \times 10^7$
   b) R_e to infinity $W = 6.25 \times 10^7$

Part C: Velocity and Gravitational Fields

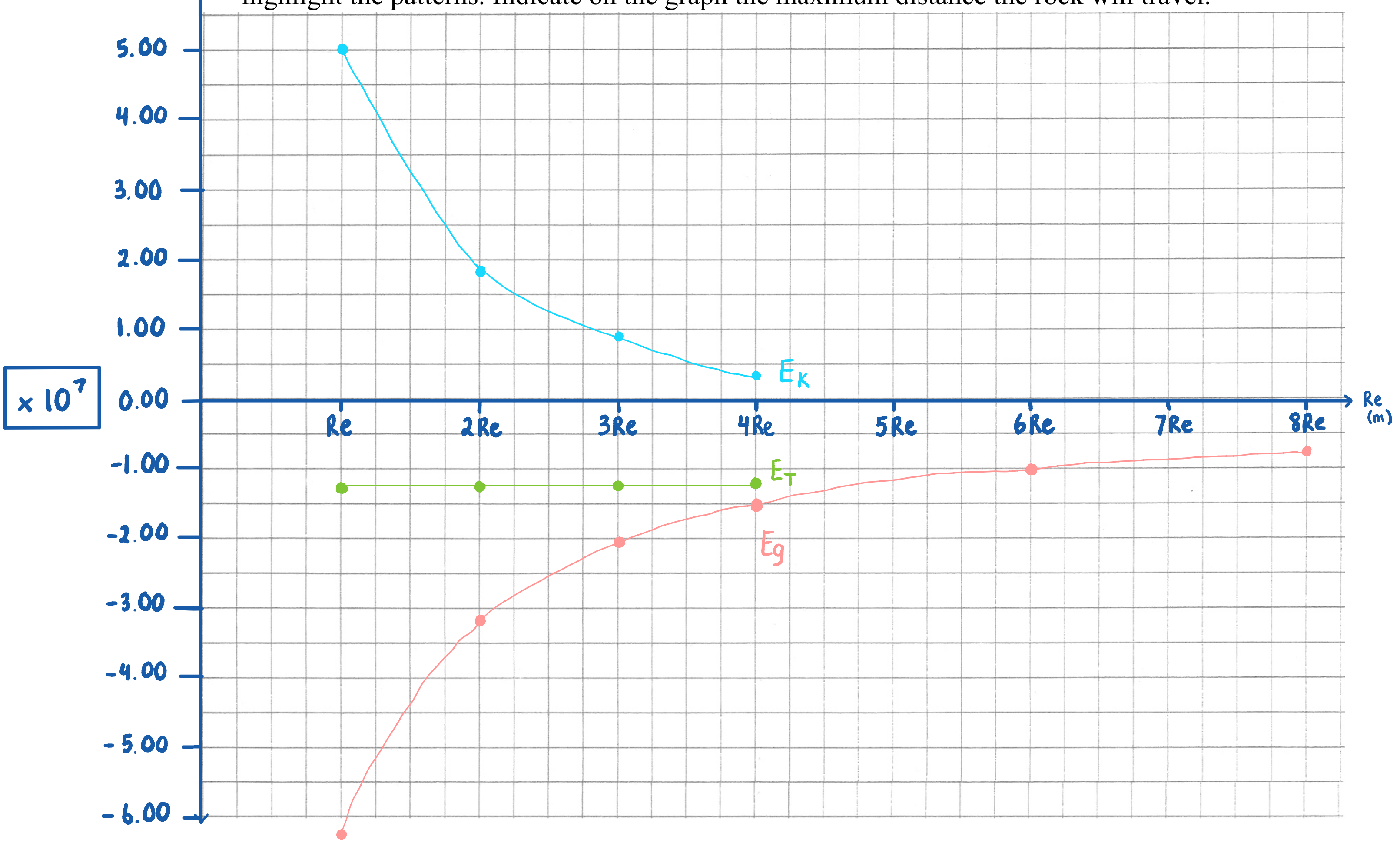
1. Complete the chart for the 1 kg rock and its trip through the earth's gravitational field. The rock was powerfully launched straight up from the surface of the earth at 1.0 x 10^4 m/s. **Note**: An object's **kinetic energy** must be a positive value – a negative value is impossible.

Distance from centre of earth	E_g	E_k	velocity	E_T
R_e	-6.25×10^7 J	5.00×10^7 J	1.0×10^4 m/s	-1.25×10^7 J
$2R_e$	-3.13×10^7 J	1.88×10^7 J	6.13×10^3 m/s	-1.25×10^7 J
$3R_e$	-2.08×10^7 J	8.30×10^6 J	4.07×10^3 m/s	-1.25×10^7 J
$4R_e$	-1.56×10^7 J	3.10×10^6 J	2.49×10^3 m/s	-1.25×10^7 J
$6R_e$	-1.04×10^7 J			
$8R_e$	-7.82×10^6 J			
Really, really far	As $r \longrightarrow +\infty$, $E_g \longrightarrow 0$			

2. Is it possible to find the rock at a distance of $6R_e$? What eventually happened to the rock? Explain.

No, it is not possible to find the rock at a distance of 6Re, since it gets too far from Earth's surface. Eventually the rock will fall back to Earth, because Et = Ek + Eg, and Et < 0. Eg must be positive, so it is not possible for Eg to be greater than Et at 6Re.

3. Plot on one graph E_g, E_k and E_T as a function of distance. Use smooth curves and straight lines to highlight the patterns. Indicate on the graph the maximum distance the rock will travel.



4. Explain how you can use the values of E_T and E_g on the graph to determine the rock's maximum height.

ET = Eg + Ek. At the maximum height, the rock has an Ek of 0. Thus, Et=Eg.We can use this to find the maximum height of the rock by looking at where Et and Eg intersect on the graph.

5. How much kinetic energy should the rock be launched with to reach a maximum distance of:

i. $3R_e$

ii. $6R_e$

$E_{T3Re} = -2.08 \times 10^7 - (-6.25 \times 10^7)$
$= 4.17 \times 10^7$ J
$E_{T6Re} = -1.04 \times 10^7 - (-6.25 \times 10^7)$
$= 5.21 \times 10^7$ J

6. With what velocity should the rock be launched to reach a maximum distance of $4R_e$?

$E_{T4Re} = -1.56 \times 10^7 - (-6.25 \times 10^7)$
$= 4.69 \times 10^7$ J

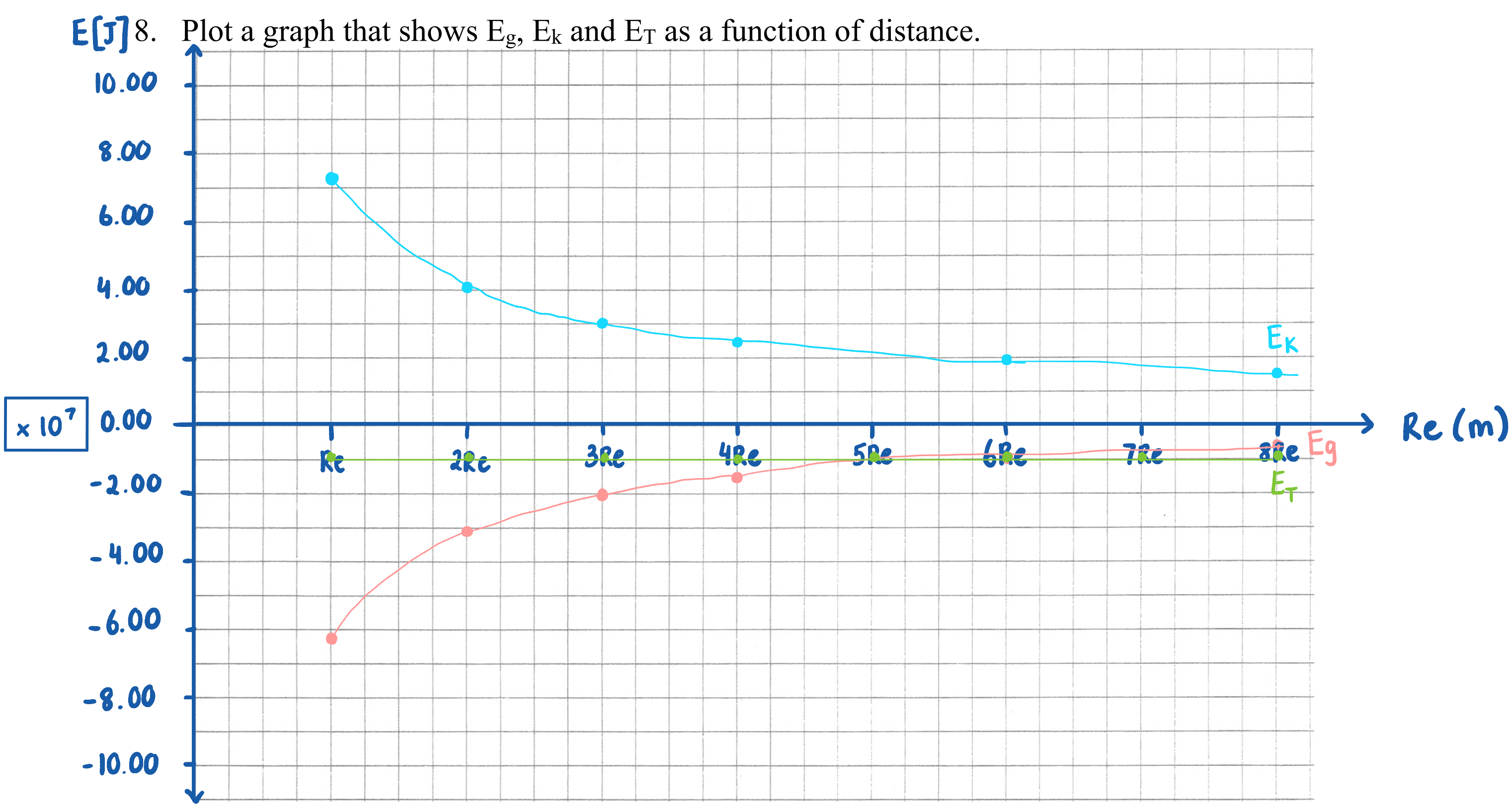
$v = \sqrt{\frac{2E_K}{m}}$
$= \sqrt{\frac{2(4.69 \times 10^7}{1}}$

$v = 9.69 \times 10^3$ m/s

7. Complete the chart again, but now the rock has an initial velocity of 1.2×10^4 m/s.

Distance from centre of earth	E_g	E_k	velocity	E_T
R_e	-6.25×10^7 J	7.20×10^7 J	1.20×10^4 m/s	9.5×10^6 J
$2R_e$	-3.13×10^7 J	4.08×10^7 J	9.03×10^3 m/s	9.5×10^6 J
$3R_e$	-2.08×10^7 J	3.03×10^7 J	7.79×10^3 m/s	9.5×10^6 J
$4R_e$	-1.56×10^7 J	2.51×10^7 J	7.09×10^3 m/s	9.5×10^6 J
$6R_e$	-1.04×10^7 J	1.99×10^7 J	6.31×10^3 m/s	9.5×10^6 J
$8R_e$	-7.82×10^6 J	1.73×10^7 J	5.88×10^3 m/s	9.5×10^6 J
Really, really far	As $r \longrightarrow +\infty$, $E_g \longrightarrow 0$			

8. Plot a graph that shows E_g, E_k and E_T as a function of distance.



9. At what distance from the earth will the rock in this example finally come to rest?

It's not possible for the rock to come to rest, because Eg = Et and Eg must be positive.

10. The rock is launched from Earth such that it will come to rest when it has travelled an infinite distance from Earth. With what E_k was it launched?

At an infinite distance from Earth Eg approaches 0 and Ek approaches 0 and as a result Et is 0. At the surface of the Earth Eg is -6.25 x 10^7. Ek = Et - Eg, which is equal to 0 - (-6.25 x 10^7), which = 6.25 x 10^7.

11. Derive an algebraic expression that gives the launch velocity for any object to reach a very great distance (infinity) from the earth with essentially no (zero) kinetic energy? This is called the **escape velocity**.

$$E_{ki} + E_{gi} = E_{kf} + E_{gf}$$
$$= 0$$

sub equations

$$\frac{1}{2}mv^2 - \frac{Gm_1m_2}{r} = 0$$

$$\frac{1}{2}\cancel{m}v^2 - \frac{G\cancel{m_1}m_2}{r} = 0$$

$$v = \sqrt{\frac{2GM}{r}}$$

12. Complete the chart for escape velocities.

Object	Radius (m)	Mass (kg)	v_{escape} (m/s)	v_{escape} (km/h)
Earth	6.38×10^6	5.98×10^{24}	1.12×10^4	4.03×10^4
Moon	1.74×10^6	7.35×10^{22}	2.39×10^3	8.53×10^3
Jupiter	7.15×10^7	1.90×10^{27}	5.95×10^4	2.14×10^5

13. Imagine the earth and all its matter is compressed.

a) What size of radius for the earth would give the maximum escape velocity ($c = 3.0 \times 10^8$)?

$$v = \sqrt{\frac{2GM}{r}}$$

$$r = \frac{2GM}{v^2}$$

$$r = \frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{(3.0 \times 10^8)^2}$$

$$r = 8.86 \times 10^{-3}\ m$$

b) What would happen to all objects trying to escape if the earth was compressed even further? What has been created?

If Earth was compressed even further, no objects would be able to escape the force of gravity. As a result, a black hole would be created.

14. What is the total energy of an object moving (consult your charts Q#1 and 6)

a) at its escape velocity?

$E_T = 0$

b) slower than its escape velocity?

$E_T < 0$

c) faster than its escape velocity?

$E_T > 0$

Part D: Orbits

1. An object is moving in a circular orbit around the earth with some radius *R*. Use the Law of Universal Gravitation and Newton's 2nd Law to determine the object's orbital velocity.

$$F_c = m \cdot a \qquad F_g = \frac{Gm_1 m_2}{r^2} \qquad \frac{\cancel{m_2} v^2}{\cancel{r}} = \frac{Gm_1 \cancel{m_2}}{r^{\cancel{2}}}$$

$$F_c = \frac{m_2 v^2}{r}$$

$$v^2 = \frac{GM}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

2. Write an expression for the kinetic energy of an object in a circular orbit. Write this in terms of the object's GPE.

$$E_K = \frac{1}{2} m v^2$$
$$= \frac{1}{2} m \left(\sqrt{\frac{GM}{r}}\right)^2$$
$$= \frac{1}{2} m \left(\frac{GM}{r}\right)$$
$$= \frac{1}{2}\left(\frac{GMm}{r}\right)$$
$$= -\frac{1}{2} E_g$$

3. Write an expression for the total energy of an object in a circular orbit. Write this in terms of the object's GPE.

$$E_T = E_K + E_g$$
$$= -\frac{1}{2} E_g + E_g$$
$$= \frac{1}{2} E_g$$

4. To calculate the energy necessary to move from one situation to another we compare the total energy in each. How much work must be done by a rocket engine to transfer a 1000 kg satellite from an orbit of radius $2R_e$ to $3R_e$?

$$E_T = \frac{1}{2} E_g$$
$$E_{T_{2Re}} = \frac{-GMm}{2r}$$
$$E_{T_{2Re}} = -\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})(1000)}{2 \times 2(6.38 \times 10^6)}$$
$$E_{T_{2Re}} = -1.56 \times 10^{10}\ J$$
$$E_{T_{3Re}} = -\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})(1000)}{2 \times 3(6.38 \times 10^6)}$$
$$E_{T_{3Re}} = -1.04 \times 10^{10}\ J$$
$$\Delta E_T = E_{T_{3Re}} - E_{T_{2Re}}$$
$$\Delta E_T = (-1.04 \times 10^{10}) - (-1.56 \times 10^{10})$$
$$\Delta E_T = 5.20 \times 10^9\ J$$

5. How much work must be done by a rocket engine to lift a 1000 kg payload from rest on the earth's surface to a circular orbit of radius $2R_e$?

$$E_T = \frac{1}{2} E_g$$
$$E_{T_{2Re}} = \frac{-GMm}{2r}$$
$$E_{T_{2Re}} = -\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})(1000)}{2 \times 2(6.38 \times 10^6)}$$
$$E_{T_{2Re}} = -1.56 \times 10^{10}\ J$$
$$E_{T_{Re}} = -\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})(1000)}{(6.38 \times 10^6)}$$
$$E_{T_{Re}} = -6.25 \times 10^{10}\ J$$
$$\Delta E_T = E_{2Re} - E_{Re}$$
$$= (-1.56 \times 10^{10}) - (-6.25 \times 10^{10})$$
$$= 4.69 \times 10^{10}\ J$$

6. The **binding energy** is the work that must be done to allow an object to escape to infinity. What is the binding energy for an object:
   a) on the surface of the earth

   $E_T = E_g + E_K$

   $E_T = E_g$, since $E_K = 0$ on the surface of the Earth since the object is at rest.

   binding energy $(\Delta E) = 6.25 \times 10$

   due to total energy the escape velocity is 0

   b) in a circular orbit around the earth

   $$E_T = \frac{1}{2} E_g$$
   $$E_T = 0$$
   $$\text{Binding energy} = -\frac{1}{2} E_g$$
   $$= \frac{1}{2}(-6.25 \times 10^7)$$
   $$= -3.12 \times 10^7$$