Chapter 2

SYSTEMS OF LINEAR EQUATIONS

Contents

2.1. The rank of a matrix

- The reduced row-echelon form of a matrix A is uniquely determined by A, but the row-echelon form of A is not unique.
- The number r of leading 1's is the same in each of the different row-echelon matrices.
- As r depends only on A and not on the row-echelon forms, it is called **the rank of the matrix A**, and written r = rankA.

2.1. The rank of a matrix

Example 1:

a/

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 6 & 1 \\ 0 & 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r(A) = 3$$

$$rank(A) = 3$$

b/

$$B = \begin{pmatrix} 1 & 2 & 3 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r(B) = 2$$

2.1. The rank of a matrix

Example 2: Find $rank(A)$

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix}$$

2.1. The rank of a matrix

Solution:

$$A=\begin{pmatrix}1&2&1&1\\2&4&2&2\\3&6&3&4\end{pmatrix}\xrightarrow[h_3\to h_3-3h_1]{h_2\to h_2-2h_1}\begin{pmatrix}1&2&1&1\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&1&1\\0&0&0&0\\0&0&0&1\end{pmatrix}\xrightarrow{h_2\leftrightarrow h_3}\begin{pmatrix}1&2&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

$$\Rightarrow rank\left(A\right)=2$$

2.1. The rank of a matrix

Example 3: Find $rank(A)$

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

2.1. The rank of a matrix

Example 3: Find $rank(A)$

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \xrightarrow[h_3 \to h_3 - h_1]{h_2 \to h_2 - h_1} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, $rank(A) = 1$

2.1. The rank of a matrix

Example 4:

Find m such that $rank\left(A\right)=3$

$$A=\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & m & m+1 \end{pmatrix}$$

2.1. The rank of a matrix

Solution:

$$A=\begin{pmatrix}1&1&1&2\\2&3&4&1\\3&2&m&m+1\end{pmatrix}\to\begin{pmatrix}1&1&1&2\\0&1&2&-3\\0&-1&m-3&m-5\end{pmatrix}$$

$$\to\begin{pmatrix}1&1&1&2\\0&1&2&-3\\0&0&m-1&m-8\end{pmatrix}$$

$$rank(A)\neq 3\Leftrightarrow\begin{cases}m-1=0\\m-8=0\end{cases}\Leftrightarrow\begin{cases}m=1\\m=8\end{cases}\ (!)$$

Hence, $rank(A)=3\ for\ all\ m.$

2.1. The rank of a matrix

Example 5: Find the rank of matrices

$$a/\ A=\begin{pmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{pmatrix}$$

$$b/ B=\begin{pmatrix}-2 & 4 & 6\\ 1 & -2 & -3\\ 6 & 3 & -1\end{pmatrix} \qquad c/\ C=\begin{pmatrix}3 & -5 & 2\\ -9 & 15 & -6\\ 6 & -10 & 4\end{pmatrix}$$

2.1. The rank of a matrix

Example 6: Find the rank of matrices

a/ $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

b/ $B = \begin{bmatrix} 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 1 \\ 3 & 0 & 0 & -4 \end{bmatrix}$

c/ $C = \begin{bmatrix} 0 & 2 & 1 & -3 \\ -1 & 3 & 0 & 1 \end{bmatrix}$

d/ $D = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & -3 & 0 \\ 0 & 5 & -3 & -2 \\ 1 & 3 & -3 & 2 \end{bmatrix}$

2.2. Solutions and Elementary Operations

coefficients variables = unknowns

- $a_1x_1+a_2x_2+\ldots+a_nx_n=b$ is called a **linear equation (phương trình tuyến tính)**
- A solution (nghiệm) to the equation is a sequence $s_1,s_2,\ldots,s_n$ such that $a_1s_1+a_2s_2+\ldots+a_ns_n=b$
- $s_1,s_2,\ldots,s_n$ is called a solution to a system of linear equations if $s_1,s_2,\ldots,s_n$ is a solution to every equation of the system
- A system may have:
  - ✓ **no solution**
  - ✓ **unique solution**
  - ✓ an **infinite family of solutions**

2.2. Solutions and Elementary Operations

- A system that has **no solution** is called **inconsistent** (không tương thích/ không nhất quán)
- A system that has **at least one solution** is called **consistent** (tương thích/ nhất quán)

Inconsistent (không tương thích)	**Consistent (tương thích)**	
No solutions (vô nghiệm)	**Unique solution (nghiệm duy nhất)**	**Infinitely many solutions (vô số nghiệm)**

2.2. Solutions and Elementary Operations

Example:

- $\begin{cases} x+2y=1 \\ x+2y=3 \end{cases}$ $\begin{cases} x+y-z=1 \\ x+y+z=3 \end{cases}$

no solution

Inconsistent

(0,2,1), (2,0,1) (t,2-t,1)

Consistent

(infinitely many solutions)

(t,2-t,1) is called a **general solution** and given in **parametric form**, t is **parameter (t is arbitrary)**

Elementary Operations (phép biến đổi sơ cấp)

- **Interchange** two equations (type I)

$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow[interchange]{} \begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases}$$

- **Multiply** one equation by a **nonzero number** (type II)

$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow[multiplied\ by\ -2]{} \begin{cases} 2x + 3y = 5 \\ -2x + 4y = 2 \end{cases}$$

- **Add a multiple** of one equation to a different equation (type III)

$$\begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases} \xrightarrow[add\ a\ multiple]{} \begin{cases} x - 2y = -1 \\ 0x + 7y = 7 \end{cases}$$

1.2. Solutions and Elementary Operations

Example 1

Find all solutions to the following system

$$\begin{cases} x + y = 0 \\ 2x - y + 3z = 3 \\ x - 2y - z = 3 \end{cases}$$

$$\xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - 2r_1} \begin{cases} x + y = 0 \\ -3y + 3z = 3 \\ -3y - z = 3 \end{cases} \xrightarrow{r_3 \to r_3 - r_2} \begin{cases} x + y = 0 \\ -3y + 3z = 3 \\ -4z = 0 \end{cases}$$

Hence, $\begin{cases} x = 1 \\ y = -1 \\ z = 0 \end{cases}$

2.2. Solutions and Elementary Operations

Algebraic Method

$$\begin{cases} x + y = 0 \\ 2x - y + 3z = 3 \\ x - 2y - z = 3 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & -2 & -1 & 3 \end{array}\right]$$

augmented matrix

coefficient matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \\ 1 & -2 & -1 \end{bmatrix}$$

constant matrix

$$\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

2.2. Solutions and Elementary Operations

Step 1: Find the the row- echelon matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & -2 & -1 & 3 \end{array}\right] \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -3 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{r_3 \to r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

2.2. Solutions and Elementary Operations

Step 2: The following system of equations

$$\begin{cases} x + y = 0 \\ -3y + 3z = 3 \\ -4z = 0 \end{cases}$$

is equivalent to the original system.

Step 3: Finally the solutions are given by

$$\begin{cases} x = 1 \\ y = -1 \\ z = 0 \end{cases}$$

2.2. Solutions and Elementary Operations

Theorem

Suppose an elementary operation is performed on a system of linear equations. Then the resulting system has the **same set of solutions** as the original system, so the two system are equivalent (tương đương).

In this case,their augmented matrices are called **row-equivalent**

Algebraic Method

$$\begin{cases} x + y = 0 \\ 2x - y + 3z = 3 \\ x - 2y - z = 3 \end{cases} \longrightarrow [A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & -2 & -1 & 3 \end{array}\right]$$

augmented matrix

coefficient matrix

constant matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 3 \\ 1 & -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Kronecker–Capelli theorem

States that for the non-homogeneous system $Ax = B$

i/ $Ax = B$ has unique solution if and only if

$$rank(A) = rank(A \mid B) = n$$

ii/ $Ax = B$ is inconsistent if and only if

$$rank(A) < rank(A \mid B)$$

iii/ $Ax = B$ has infinitely many solutions if and only if

$$rank(A) = rank(A \mid B) < n$$

where n : is the number of variables of the system of equations .

2.2. Solutions and Elementary Operations

Theorem

Suppose a system of m equations in **n variables** has a solution. If the **rank** of the augment matrix is **r** then the set of solutions involves exactly **n-r** parameters.

leading one

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rankA=2

4(number of variables)- 2(rankA) =2
(two parameters : x_2=t, x_4=s)

2.3. Gaussian Elimination

Gaussian Algorithm

Step 1: Write the augment matrix.

Step 2: Use the elementary row operations, find row-echelon.

Step 3: Use Kronecker–Capelli theorem

- if no solution → stop
- if unique solution → solve it.
- if infinite solutions

✓ $n-r$ parameters (nonleading variables).

✓ Solve the system equations.

2.3. Gaussian Elimination

Example 2: Solve the following system of equation

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ 2x_1 - 4x_2 + x_3 = 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 = 4 \end{cases}$$

2.3. Gaussian Elimination

Solution:

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ 2x_1 - 4x_2 + x_3 = 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 = 4 \end{cases}$$

$$n = 4$$

$$r = 2$$

$$n - r = 2$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leading one

reduced row echelon matrix

x_2,x_4 are **<u>nonleading variables</u>**, so we set x_2=t and x_4=s (parameters) and then compute x_1, x_3

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ x_2 = t \\ x_3 - 2x_4 = 1 \\ x_4 = s \end{cases} \Leftrightarrow \begin{cases} x_1 = 2 + 2t - s \\ x_2 = t \\ x_3 = 1 + 2s \\ x_4 = s \end{cases}$$

$$General\ \ solutions: \left(2 + 2t - s, t, 1 + 2s, s\right); t, s \in R$$

2.3. Gaussian Elimination

Example 3:

Solve the system of linear equation corresponding to the given augmented matrix.

a/ $\left[\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right]$

b/ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{array}\right]$

c/ $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

d/ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

2.3. Gaussian Elimination

Solution:

a/ $\left[\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right]$

$r = rank(A) = rank(A \mid b) = 3 = n$: unique solution.

The corresponding equations are $z = 0; y = \frac{1}{2}; \text{x} = -\frac{17}{2}$

2.3. Gaussian Elimination

Solution:

b/ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{array}\right]$

$rank(A) \neq rank(A|b)$: no solution.

But this last system clearly has no solution (the last equation requires that x, y and z satisfy $0.x + 0.y + 0.z = 5$, and no such numbers exist).

Hence the original system has no solution.

2.3. Gaussian Elimination

Solution:

c/ $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The number of nonleading variables are $n-r$

$r = rank(A) = rank(A|b) = 2 < n$: Infinitely many solutions.

The leading ones are in columns 1 and 2 here, so the corresponding variables x and y are called leading variables.

z is nonleading variable, we set $z = t$, where t is arbitrary.

Finally the solutions are given by

$$x = -5 - t; y = 5 + 2t; z = t \in R$$

2.3. Gaussian Elimination

Solution:

d/ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The number of nonleading variables are $n-r$

$r = rank\left(A\right) = rank\left(A\,|\,b\right) = 2 < n$: Infinitely many solutions.

The leading ones are in columns 1 and 2 here, so the corresponding variables x and y are called leading variables.

z is nonleading variable, we set $z = 3t$, where t is arbitrary.

Finally the solutions are given by

$$x = -4t; y = t; z = 3t \in R$$

2.3. Gaussian Elimination

Example 4:

Solve the following system of equations

$$\begin{cases} x - 2y = 3 \\ 2x - 3y + 2z = 5 \\ -3x + 7y + 2z = 10 \end{cases}$$

2.3. Gaussian Elimination

Solution:

Carry the augmented matrix to reduced row-echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 2 & -3 & 2 & 5 \\ -3 & 7 & 2 & 10 \end{array}\right] \xrightarrow[3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 19 \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

$$\xrightarrow{\frac{1}{20}r_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} \end{array}\right]$$

inconsistent

1.3. Gaussian Elimination

Example 5:

Which condition on the numbers a,b,c is the system

$$\begin{cases} 3x + y - z = a \\ x - y + 2z = b \\ 5x + 3y - 4z = c \end{cases}$$

consistent ?

2.3. Gaussian Elimination

Solution:

$$\begin{cases} 3x + y - z = a \\ x - y + 2z = b \\ 5x + 3y - 4z = c \end{cases}$$

We have

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & a \\ 1 & -1 & 2 & b \\ 5 & 3 & -4 & c \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 3 & 1 & -1 & a \\ 5 & 3 & -4 & c \end{array}\right] \xrightarrow[-5r_1+r_3]{-3r_1+r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 0 & 4 & -7 & a-3b \\ 0 & 8 & -14 & c-5b \end{array}\right]$$

$$\xrightarrow[-5r_1+r_3]{-3r_1+r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & b \\ 0 & 4 & -7 & a-3b \\ 0 & 0 & 0 & c-2a+b \end{array}\right]$$

Answer: c-2a+b=0

2.3. Gaussian Elimination

Example 6:

Determine the values of m such that the system of linear equations is inconsistent

$$\begin{cases} x - y + 2z = m \\ -x + y - z = 0 \\ x - y + 3z = 1 - m \end{cases}$$

2.3. Gaussian Elimination

Solution:

We have $\left[\begin{array}{ccc|c} 1 & -1 & 2 & m \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 3 & 1-m \end{array}\right] \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 + r_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & m \\ 0 & 0 & 1 & m \\ 0 & 0 & 1 & 1-2m \end{array}\right]$

$$\xrightarrow{r_3 \to r_3 - r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & m \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 1-3m \end{array}\right]$$

The system is inconsistent

$$1-3m \neq 0 \Leftrightarrow m \neq \frac{1}{3}$$

2.3. Gaussian Elimination

Example 7:

Solve the following systems given by augmented matrices

$$1/\left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$2/\left[\begin{array}{cccc|c} 1 & -2 & 3 & -5 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$3/\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 2 & 0 & 5 & -2 \\ 0 & 2 & 13 & 8 \end{array}\right]$$

$$4/\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 3 & -4 & 2 & -5 & 0 \\ -2 & 3 & -3 & 1 & 0 \end{array}\right]$$

2.3. Gaussian Elimination

Example 8: Solve the following system of equation

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ 2x_1 - 4x_2 + x_3 = 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 = 4 \end{cases}$$

2.4. Homogeneous Equations (hệ phương trình thuần nhất)

- The system is called **homogeneous** (thuần nhất) if the constant matrix has all the entry are zeros
- Note that every homogeneous system **has at least one solution (0,0,…,0),** called **trivial solution** (nghiệm tầm thường)
- If a homogeneous system of linear equations has **nontrivial solution** (nghiệm không tầm thường) then it has infinite family of solutions (vô số nghiệm)

2.4. Homogeneous Equations

Example 1:

Show that the following homogeneous system has nontrivial solutions

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 2x_2 - x_4 = 0 \\ 3x_1 + x_2 + 2x_3 + x_4 = 0 \end{cases}$$

2.4. Homogeneous Equations

Solution:

We have $\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

The general solution is $\begin{cases} x_1 = -t \\ x_2 = t \\ x_3 = t \\ x_4 = 0 \end{cases}$

Hence, taking $t = 1$, we get a nontrivial solution

$$x_1 = -1; x_2 = 1; x_3 = 1; x_4 = 0$$

2.4. Homogeneous Equations

Example 3:

Find all values of m such that the system

$$\begin{cases} 3x + 2y + mz = 0 \\ -x + y + 2z = 0 \\ 2x - y + 3mz = 0 \end{cases}$$

has nontrivial solution

2.4. Homogeneous Equations

Solution:

We have $\left[\begin{array}{ccc|c} 3 & 2 & m & 0 \\ -1 & 1 & 2 & 0 \\ 2 & -1 & 3m & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_1} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 3 & 2 & m & 0 \\ 2 & -1 & 3m & 0 \end{array}\right]$

$$\xrightarrow[\substack{r_2 \to r_2 + 3r_1 \\ r_3 \to r_3 + 2r_1}]{} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 5 & m+6 & 0 \\ 0 & 1 & 3m+4 & 0 \end{array}\right] \xrightarrow{r_3 \to 5r_3 - r_2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 5 & m+6 & 0 \\ 0 & 0 & 14m+14 & 0 \end{array}\right]$$

the system has nontrivial solution

$$14m + 14 = 0 \Leftrightarrow m = -1$$

2.4. Homogeneous Equations

Theorem 1

If a homogeneous system of linear equations has **more variables than equations**, then it has nontrivial solution (in fact, infinitely many)

$$n > m$$

$$r \leq m < n$$

Note that the converse of theorem 1 is not true

2.4. Homogeneous Equations

System of equations Summary

System of	Inconsistent (no solutions)	Cosistent	
		Unique solution (exactly one solution)	Infinitely many solutions
linear equations	yes	yes	yes
linear equations that has more variables than equations	yes	no	yes
homogeneous linear equations	no	yes	yes
homogeneous linear equations that has more variables than equations	no	no	yes

Exercises 1

1 Find all a,b such that the matrix is an reduced row-echelon matrix. $\begin{bmatrix} a & b & 0 \\ 0 & b & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ☐ a=b=1
- ☐ a=0 and b=1
- ☐ a=b=0
- ☑ a=1 and b=0
- ☐ None of the others

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.4. Homogeneous Equations

Exercises 2

Find all values of m such that the system has **no solution**, (another word: inconsistent)

$$\begin{cases} x-2y+z=3 \\ 3x-y+z=m \\ -x+y+z=-1 \end{cases}$$

Answer:

- Carry the augmented matrix to **row-echelon form**

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 3 & -5 & 1 & m \\ -1 & 1 & 1 & -1 \end{array}\right] \xrightarrow[d_1+d_3]{-3d_1+d_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & m-9 \\ 0 & -1 & 2 & 2 \end{array}\right] \xrightarrow{d_2+d_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & m-9 \\ 0 & 0 & 0 & m-7 \end{array}\right]$$

- The system has no solution iff m≠7

2.4. Homogeneous Equations

Exercises 3

Choose the correct statements.

☑ If the system has trivial solution, then it is a homogeneuos system

☐ A homogeneous system of 7 equations and 5 variables always has many solutions

☑ A consistent system of 12 equations and 15 unknowns must have infinitely many solutions

☑ If a homogeneous system has an nontrivial solution, then it has infinitely many solutions

- Homogeneous + more variables than equations many solutions (vsn), chắc chắn có nghiệm không tầm thường
- Homogeneous + exist nontrivial solution infinitely many solutions (vsn)

2.4. Homogeneous Equations

Exercises 4

Consider a system of 203 linear equations in 133 variables. Choose the correct statements.

- ☑ If the system is homogeneous there is always at least one solution
- ☑ There may be infinitely many solutions.
- ☐ There may be exactly three solutions.
- ☐ If the system is homogeneous there are always infinitely many solutions.
- ☑ There may be exactly one solution.
- ☐ There is always at least one solution.
- ☑ There may be no solution

- Homogeneous consistent (luôn có ít nhất một nghiệm)

2.4. Homogeneous Equations

Exercises 5

Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 7 & 3 \end{bmatrix}$

- ☐ 1
- ☑ 2
- ☐ 3
- ☐ 4
- ☐ 3x3

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 7 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{bmatrix}$$

rankA = the number of leading ones in the row echelon form of A

2.4. Homogeneous Equations

Exercises 6

Find all values of m such that the matrix has the **rank** 2.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & m \end{bmatrix}$$

Solution:

- Carry the augmented matrix to **row-echelon form**

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & m \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & m-9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & m-5 \end{bmatrix}$$

- rankA=2 iff m=5 (rankA=3 iff m $\neq$5)

rankA=the number of leading ones in the row echelon form of A

2.4. Homogeneous Equations

Exercises 7 : Given a homogeneous system of 23 unkowns (n=23) and 35 equations. Suppose the **augmentad matrix** of the system has the rank 17 (r=17). How many parameters in the solution of the system?

- ☐ 17
- ☐ 23
- ☐ 12
- ☑ 6
- ☐ None of the others

Number of parameters = number of nonleading variables
= n - r (số ẩn – hạng ma trận)
r = rankA = number of leading ones

$$
\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & \\
\end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & * & * & * & 0 \\
0 & 0 & 1 & * & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

leading variables: x_1, x_3

nonleading variables: x_2, x_4

2.4. Homogeneous Equations

Exercises 8: Solve the homogeneous system

$$\begin{cases} x-2y+z-w=0 \\ 3x+y-3z=0 \\ y-z+w=0 \end{cases}$$

Carry the augmented matrix to (reduced) **row-echelon form**

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 3 & 1 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 7 & -6 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 7 & -6 & 3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{array}\right]$$

w is nonleading variable, so w=parameter=t

2.4. Homogeneous Equations

Exercises 9:

Find all values of m such that the system $\begin{cases} x+2y+z=1 \\ 2x+5y+3z=5 \\ 3x+7y+m^2z=6 \end{cases}$

has infinitely many solution.

- ☐ m=1
- ☑ m=±2
- ☐ m=0
- ☐ m= ± 1
- ☐ m=3

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 5 \\ 3 & 7 & m^2 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & m^2-3 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & m^2-4 & 0 \end{array}\right]$$

2.4. Homogeneous Equations

Exercises 10:

Find all values of m such that the system $\begin{cases} 3x+2y+mz=0 \\ -x+y+2z=0 \\ 2x-y+3mz=0 \end{cases}$

has nontrivial solution.

- ☐ m≠1
- ☑ m = -1
- ☐ m=1
- ☐ m= ± 1
- ☐ m ≠-1

$$\left[\begin{array}{ccc|c} 3 & 2 & m & 0 \\ -1 & 1 & 2 & 0 \\ 2 & -1 & 3m & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 3 & 2 & m & 0 \\ 2 & -1 & 3m & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 3 & 2 & m & 0 \\ 2 & -1 & 3m & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & m+6 & 0 \\ 0 & 1 & 3m+4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3m+4 & 0 \\ 0 & 5 & m+6 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3m+4 & 0 \\ 0 & 0 & -14m-14 & 0 \end{array}\right]$$