Chapter 2

DERIVATIVES

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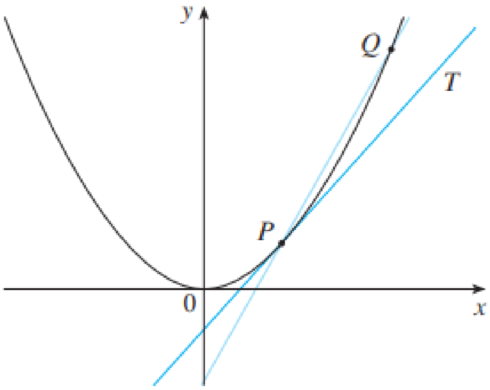
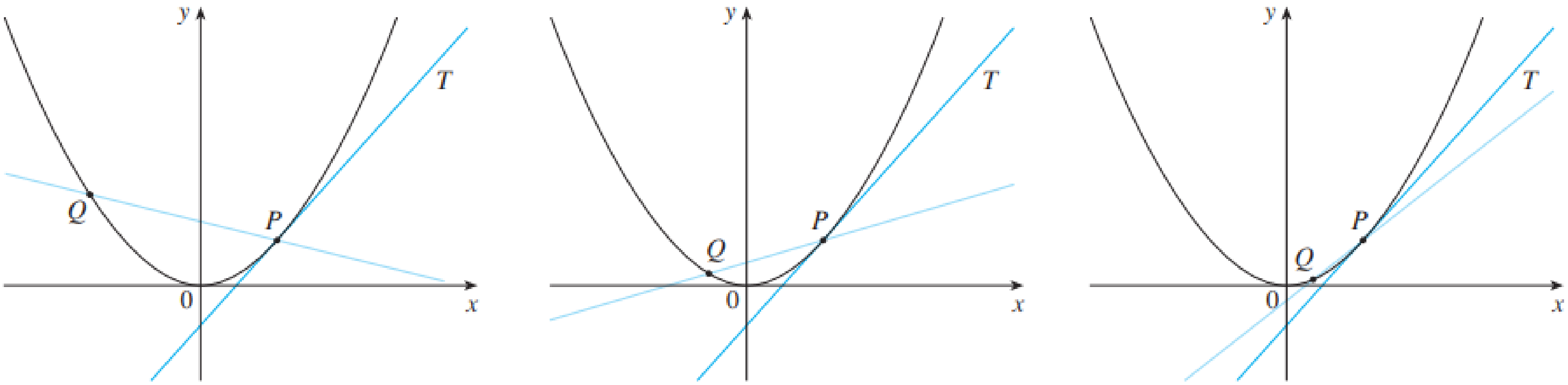
2.1 Derivatives



Q approaches P from the right

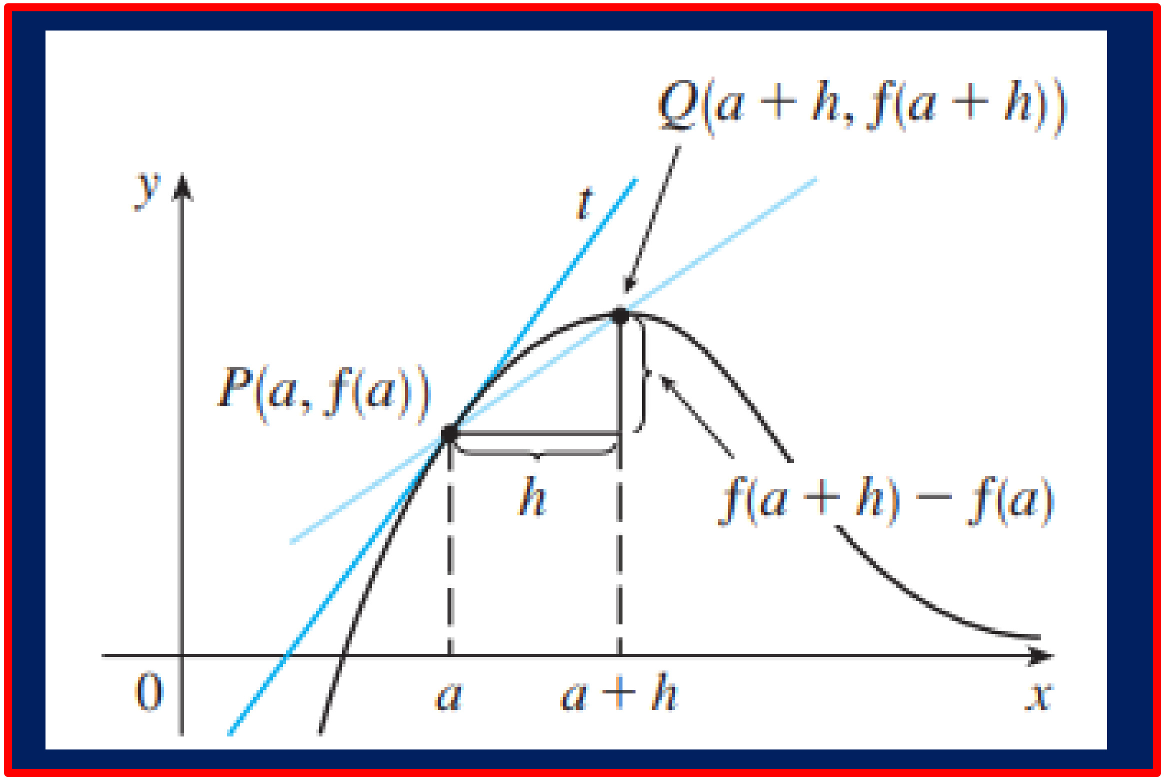


Q approaches P from the left

2.1 Derivatives

Definition

The **derivative of a function *f* at a number *a***, denoted by $f'(a)$, is: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ if this limit exists.



2.1 Derivatives

Definition

In the preceding section, we considered the derivative of a function *f* at a fixed number *a*:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (1)$$

If we replace *a* in Equation (1) by a variable *x*, we obtain:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (2)$$

2.1 Derivatives

Example:

Using the definition of derivative, find the derivative of the functions:

a/ $f(x) = x^3 - x,\ f'(1),\ f'(x)$

b/ $f(x) = \sqrt{x}, f'(4),\ f'(x)$

2.1 Derivatives

Solution:

a/ We have

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

2.1 Derivatives

Solution:

a/ We have $f'(1)=\lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to 1}\frac{x^3-x}{x-1}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}x(x+1)$$

$$=2$$

2.1 Derivatives

Solution:

a/ $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3-(x+h)-(x^3-x)}{h}\left(\frac{0}{0}\right)$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-(x+h)-(x^3-x)}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h}$$

$$=\lim_{h\to 0}\left(3x^2+3xh+h^2-1\right)$$

$$=3x^2-1$$

2.1 Derivatives

Solution:

b/ We have $f'(4)=\lim\limits_{x\to 4}\dfrac{f(x)-f(4)}{x-4}=\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}\left(\dfrac{0}{0}\right)$

$$=\lim_{x\to 4}\frac{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}{(x-4)\left(\sqrt{x}+2\right)}$$

$$=\lim_{x\to 4}\frac{x-4}{(x-4)\left(\sqrt{x}+2\right)}$$

$$=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

2.1 Derivatives

Solution:

b/ $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{0}{0}\right)$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}$$

2.1 Derivatives

Other notations

Some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called differentiation operators.

The symbol dy/dx is called Leibniz notation

2.1 Derivatives

Other notations

If we want to indicate the value of a derivative *dy/dx* in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \quad or \quad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$

2.1 Derivatives

Theorem 1

A function f is differentiable at a if $f'(a)$ exists.

It is differentiable on an open interval D if it is differentiable at every number in the interval D.

2.1 Derivatives

Theorem 2

If f is differentiable at a, then f is continuous at a.

This theorem states that, if f is not continuous at a, then f is not differentiable at a.

Example:

$f(x) = |x|$ is continous at $x_0 = 0$ but $f'(0)$ does't exist.

2.1 Derivatives

Higher derivatives

If f is a differentiable function, then its derivative f' is also a function.

So, f' may have a derivative of its own, denoted by **$(f')' = f''$**.

This new function f'' is called the second derivative of f.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$$

2.1 Derivatives

Higher derivatives

The process can be continued.

- In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.
- If $y = f(x)$, we write: $y^{(n)} = f^{(n)}(x) = \dfrac{d^n y}{dx^n}$

2.1 Derivatives

Tangent lines

The tangent line to the curve $y = f(x)$ at the point $P\big(a, f(a)\big)$ is the line through with slope $m = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ provided that this limit exists

Tangent line (Δ) equation at $P\big(a, f(a)\big) \in (C) : y = f(x)$

$(\Delta) : y = f'(a)(x - a) + f(a)$

Tangent lines

Example:

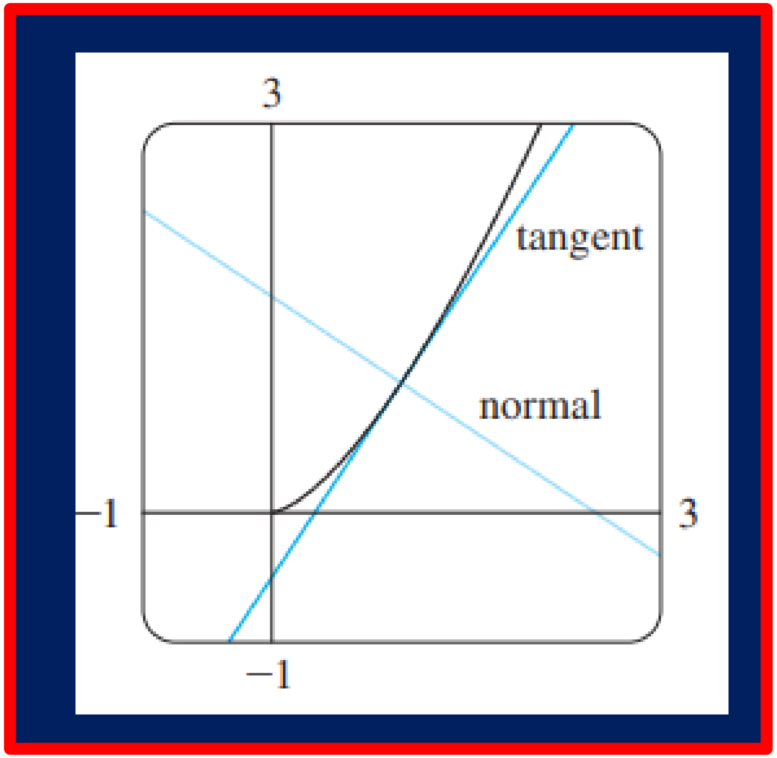
Find an equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point (3,1).

Answer: $y = -\frac{1}{3}x + 2$

Normal lines

The **normal line** is defined as the line that is perpendicular to the tangent line at the point of tangency.

$$k_t \times k_n = -1$$



2.1 Derivatives

Normal lines

Example:

Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1,1)$.

2.1 Derivatives

Normal lines

Solution:

Tangent: $\left(\Delta\right): y = \frac{3}{2}x - \frac{1}{2} \Rightarrow k_t = \frac{3}{2} \Rightarrow k_n = -\frac{2}{3}$

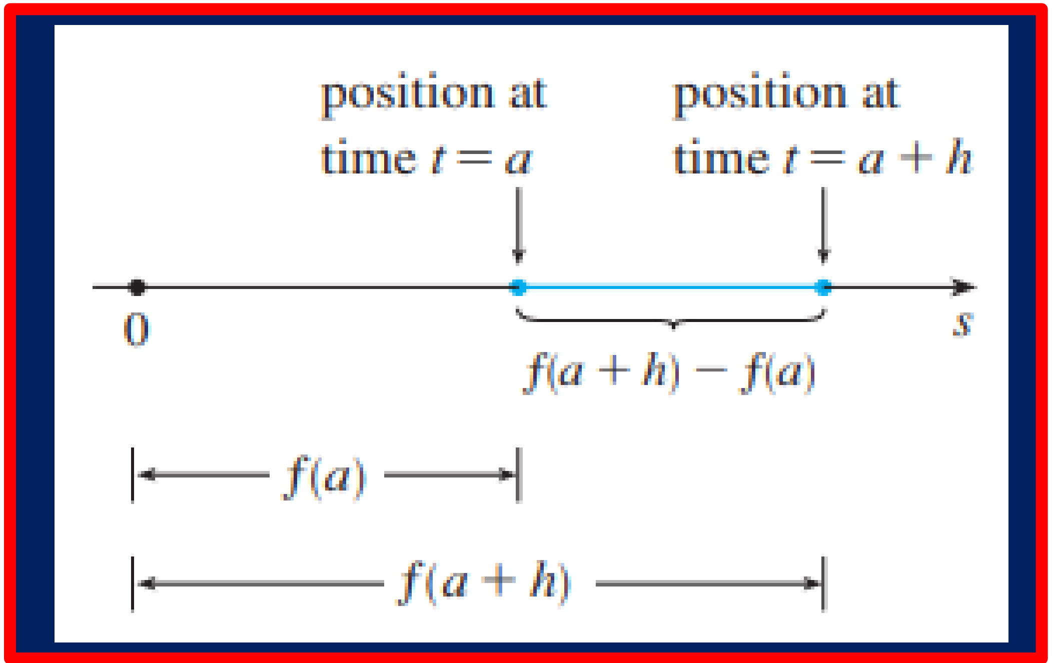
Normal: $\left(d\right): y = -\frac{2}{3}x + m, \left(1,1\right) \in \left(d\right): 1 = -\frac{2}{3} + m \Rightarrow m = \frac{5}{3}$

$$\Rightarrow \left(d\right): y = -\frac{2}{3}x + \frac{5}{3}$$

The velocity problem

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$



2.1 Derivatives

The velocity problem

Example

The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = \frac{1}{t^2}$, where t is measured in seconds. Find the velocity of the particle at times $t = a, t = 1, t = 2, t = 3.$

2.1 Derivatives

Rates of change:

instantaneous rate of change $= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Average of change of y with respect to x over the interval

$[x_1, x_2]$ is the instantaneous rate of change

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

of $y = f(x)$ with respect to x when $x = a$

2.1 Derivatives

Rates of change:

Example

The cost (in dollars) of producing units of a certain commodity is $C(x) = 500 + 10x + 0.05x^2$

a/ Find the avegare of change of $C(x)$ with respect to x

when the production level is changed

i/ from $x = 100$ to $x = 101$ ii/ from $x = 100$ to $x = 105$

b/ Find the instantaneous rate of change of $C(x)$ with respect to x when $x = 100$

2.2. Differentiation formulas

Here's a summary of the differentiation formulas we have learned so far.

$$1. (c f)' = c f'$$

$$2a. (f + g)' = f' + g' \quad 2b. (f - g)' = f' - g'$$

$$3. (fg)' = fg' + gf' \quad 4. \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

2.2. Differentiation formulas

Summary

$$1.\ \frac{d}{dx}(c) = 0$$

$$2a.\ \frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad 2b.\ \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} \qquad 2c.\ \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$3a.\ \frac{d}{dx}\left(e^x\right) = e^x \qquad 3b.\ \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

2.2. Differentiation formulas

Summary

$$4a.\ \frac{d}{dx}\left(\ln x\right) = \frac{1}{x} \qquad 4b.\ \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

$$5a.\ \frac{d}{dx}\left(\sin x\right) = \cos x \qquad 5b.\ \frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$6a.\ \frac{d}{dx}\left(\tan x\right) = \frac{1}{\cos^2 x} \qquad 6b.\ \frac{d}{dx}\left(\cot x\right) = -\frac{1}{\sin^2 x}$$

2.2. Differentiation formulas

Review:

$$1/\ a^m.a^n = a^{m+n}$$

$$2a/\ \frac{a^m}{a^n} = a^{m-n} \qquad 2b/\ \frac{1}{a^n} = a^{-n}$$

$$3/\ \sqrt[n]{x^m} = x^{\frac{m}{n}}$$

$$4a/\ a^n.b^n = (ab)^n \qquad 4b/\ \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

2.2. Differentiation formulas

Example:

Differentiate the function

a/ $y = \sqrt[3]{t^2} + 2\sqrt{t^3}$

b/ $y = -\dfrac{12}{s^5}$

c/ $y = t^2 - \dfrac{1}{\sqrt[4]{t^3}}$

d/ $y = \sqrt{x}\left(x - 1\right)$

e/ $y = \dfrac{\sqrt{x} + x}{x^2}$

2.2. Differentiation formulas

Solution

a/ $y = \sqrt[3]{t^2} + 2\sqrt{t^3} = t^{2/3} + 2t^{3/2}$

$$\Rightarrow y' = \frac{2}{3}t^{-1/3} + 3t^{1/2} = \frac{2}{3\sqrt[3]{t}} + 3\sqrt{t}$$

b/ $y = -\dfrac{12}{s^5} = -12s^{-5}$

$$\Rightarrow y' = (-12)(-5)s^{-6} = \frac{60}{s^6}$$

2.2. Differentiation formulas

Solution

c/ $y = t^2 - \frac{1}{\sqrt[4]{t^3}} = t^2 - t^{-3/4} \Rightarrow y' = 2t + \frac{3}{4}t^{-7/4} = 2t + \frac{3}{4\sqrt[4]{t^7}}$

d/ $y = \sqrt{x}(x-1)$

$$\Rightarrow y' = \left(\sqrt{x}\right)'(x-1) + \sqrt{x}(x-1)' = \frac{x-1}{2\sqrt{x}} + \sqrt{x} = \frac{3x-1}{2\sqrt{x}}$$

e/ $y = \frac{\sqrt{x}+x}{x^2} = \frac{\sqrt{x}}{x^2} + \frac{1}{x} = x^{-3/2} + x^{-1}$

$$\Rightarrow y' = -\frac{3}{2}x^{-5/2} - x^{-2}$$

2.3. The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, the composite function $h = f \circ g$ is differentiable at x and h' is given by the product:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

- In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both
- differentiable functions, then: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

2.3. The chain rule

Example 1:

Differentiate the function

a/ $y = \sqrt{2u} + \sqrt{3u}$

b/ $y = \left(x + x^{-1}\right)^3$

c/ $y = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$

d/ $y = \left(\dfrac{t-2}{2t+1}\right)^9$

e/ $y = \sin\left(\cos\left(\tan x\right)\right)$

f/ $y = \left(2x+1\right)^5\left(x^3 - x + 1\right)^4$

2.3. The chain rule

Solution:

a/ $y=\sqrt{2}u+\sqrt{3u} \Rightarrow y'=\sqrt{2}+\dfrac{1}{2\sqrt{3u}}(3u)' =\sqrt{2}+\dfrac{3}{2\sqrt{3u}}$

b/ $y=\left(x+x^{-1}\right)^3$

$\Rightarrow y'=3\left(x+x^{-1}\right)^2\left(x+x^{-1}\right)'=3\left(x+x^{-1}\right)^2\left(1-\dfrac{1}{x^2}\right)$

2.3. The chain rule

Solution:

$$\text{c/ } y=\frac{1}{\sqrt[3]{x^2+x+1}}=\left(x^2+x+1\right)^{-1/3}$$

$$\Rightarrow y'=-\frac{1}{3}\left(x^2+x+1\right)^{-4/3}\left(x^2+x+1\right)'=-\frac{2x+1}{3\left(x^2+x+1\right)^{4/3}}$$

$$\text{d/ } y=\left(\frac{t-2}{2t+1}\right)^9\Rightarrow y'=9\left(\frac{t-2}{2t+1}\right)^8\left(\frac{t-2}{2t+1}\right)'$$

$$=9\left(\frac{t-2}{2t+1}\right)^8\cdot\frac{5}{\left(2t+1\right)^2}$$

$$y=\frac{ax+b}{cx+d}\Rightarrow y'=\frac{ad-bc}{\left(cx+d\right)^2}$$

Solution:

e/ $y=\sin\left(\cos\left(\tan x\right)\right)$ $\qquad \sin u,\ u=\cos\left(\tan x\right)$

$\Rightarrow y'=\cos\left(\cos\left(\tan x\right)\right)\left(\cos\left(\tan x\right)\right)'$ $\qquad \cos u,\ u=\tan x$

$$=\cos\left(\cos\left(\tan x\right)\right)\left(-\sin\left(\tan x\right)\right)\left(\tan x\right)'$$

$$=\frac{\cos\left(\cos\left(\tan x\right)\right)\left(-\sin\left(\tan x\right)\right)}{\cos^2\left(x\right)}$$

2.3. The chain rule

Solution:

$$f/\ y=(2x+1)^5(x^3-x+1)^4$$

$$u=(2x+1)^5$$

$$\Rightarrow u'=5(2x+1)^4(2x+1)'$$

$$=10(2x+1)^4$$

$$v=(x^3-x+1)^4$$

$$\Rightarrow v'=4(x^3-x+1)^3(x^3-x+1)'$$

$$=4(x^3-x+1)^3(3x^2-1)$$

$$y'=u'.v+u.v'$$

$$=10(2x+1)^4(x^3-x+1)^4+(2x+1)^5 4(x^3-x+1)^3(3x^2-1)$$

$$=(2x+1)^4(x^3-x+1)^3(34x^3+12x^2-18x+6)$$

2.3. The chain rule

Example 2:

Differentiate the function

a/ Let $x = x(t), y = y(t)$.

If $x^2 + y^2 = 25$ and $\dfrac{dy}{dt} = 6$, find $\dfrac{dx}{dt}$ when $y = 4$

b/ Let $x = x(t), y = y(t), z = z(t)$

If $z^2 = x^2 + y^2$, $\dfrac{dx}{dt} = 2, \dfrac{dy}{dt} = 3$, find $\dfrac{dz}{dt}$ when $x = 5, y = 12$

2.3. The chain rule

Solution:

a/ To differentiate, we need to use the Chain Rule:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dz}{dt} = -\frac{y}{z}\frac{dx}{dt} - \frac{y}{z} \times \frac{dy}{dt}$$

When $y = 4 : x^2 + 16 = 25 \Rightarrow x = \mp 3.$

With $x = 3, y = 4$ we obtain $\frac{dx}{dt} = -\frac{y}{x} \times \frac{dy}{dt} = -\frac{4}{3} \times 6 = -8$

Similar, at $x = -3, y = 4$ we obtain $\frac{dx}{dt} = 8$

2.3. The chain rule

Solution:

b/ To differentiate, we need to use the Chain Rule:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dz}{dt} = \frac{x}{z}\times\frac{dx}{dt} + \frac{y}{z}\times\frac{dy}{dt}$$

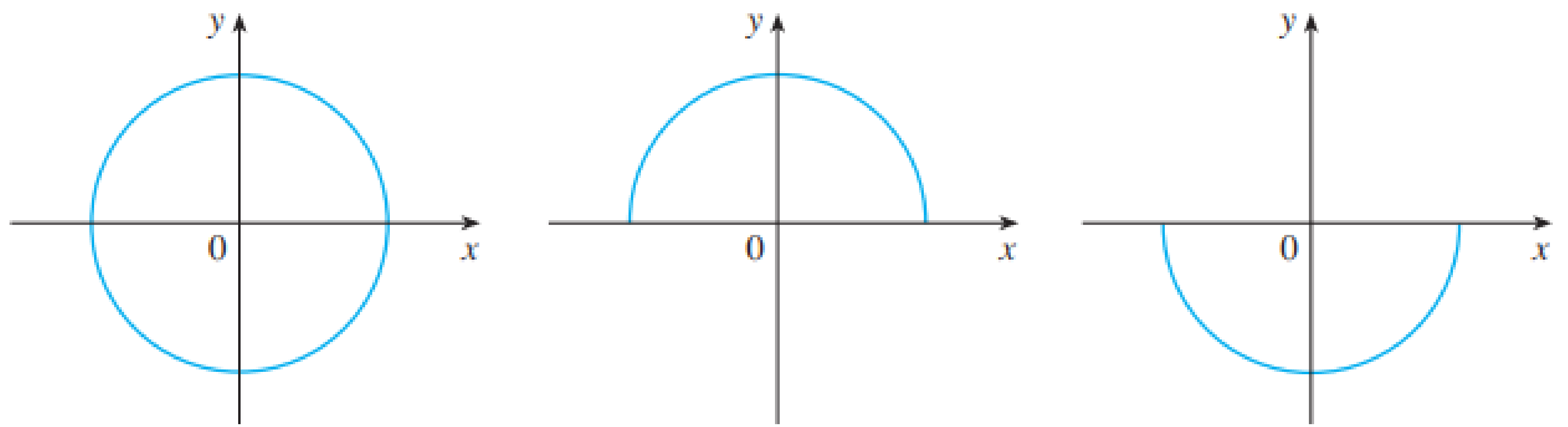
When $x = 5, y = 12 : z^2 = 169 \Rightarrow z = \mp 13$

With $x = 5, y = 12, z = 13$ we obtain $\frac{dz}{dt} = \frac{46}{13}$

Similar, at $x = 5, y = 12, z = -13$ we obtain $\frac{dz}{dt} = -\frac{46}{13}$

2.4. Implicit differentiation

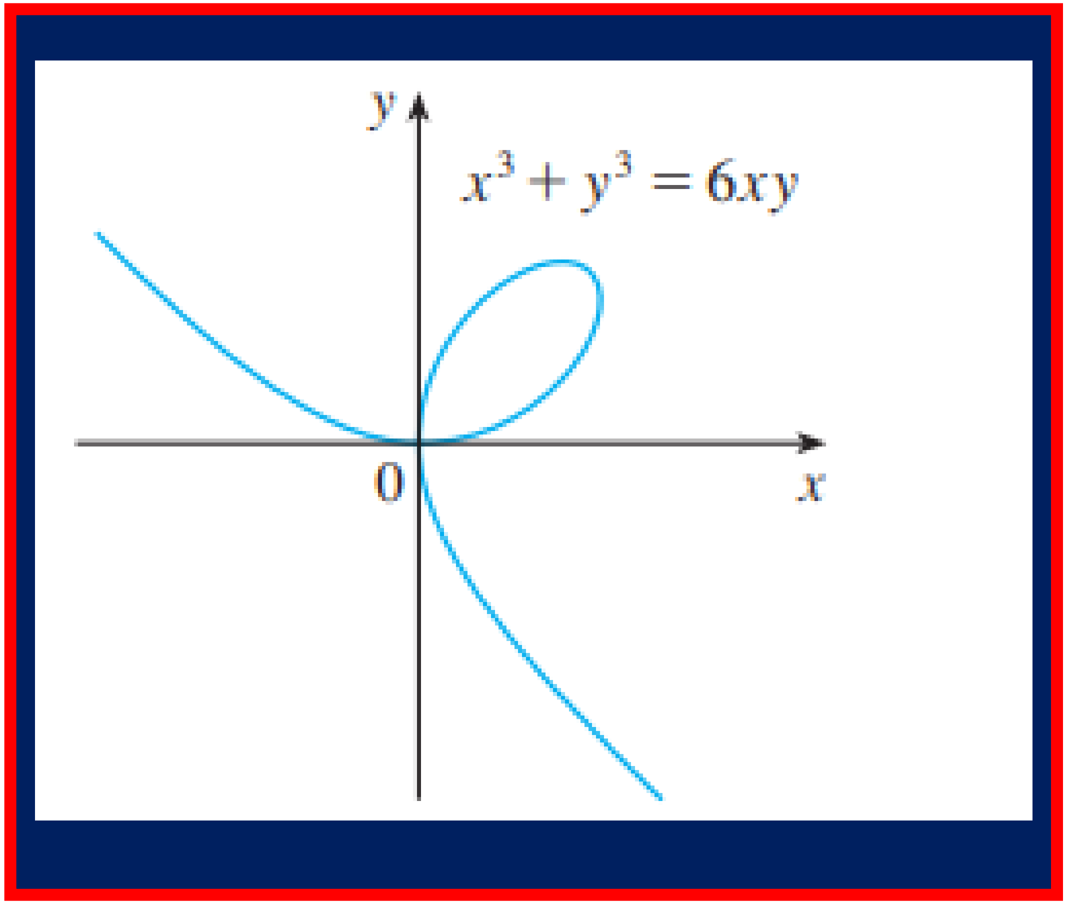
The graphs of *f* and *g* are the upper and lower semicircles of the circle $x^2 + y^2 = 25$.



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

2.4. Implicit differentiation

The folium of Descartes



2.4. Implicit differentiation

Instead, we can use the method of implicit differentiation. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

2.4. Implicit differentiation

Example 1:

a. If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

b. Find an equation of the tangent to the circle

$x^2 + y^2 = 25$ at the point $x_0 = 3$.

2.4. Implicit differentiation

Solution:

a/ If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

Differentiating both sides of the equation with respect to x

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25) \Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have: $2x + 2y.y' = 0$

Then, we solve this equation for y' : $\dfrac{dy}{dx} = -\dfrac{x}{y}$

2.4. Implicit differentiation

Solution:

b. At the $x_0 = 3$, we have $x = 3$ and $y = 4$ or $y = -4$

- At $(3,4)$: $\dfrac{dy}{dx} = -\dfrac{3}{4}$

  Thus, an equation of the tangent to the circle at (3, 4) is: $y - 4 = -\frac{3}{4}(x - 3)$ or $3x + 4y = 25$.

- At $(3,-4)$: $\dfrac{dy}{dx} = \dfrac{3}{4}$

  Thus, an equation of the tangent to the circle at (3, 4) is: $y + 4 = \frac{3}{4}(x - 3)$ or $3x - 4y = 25$.

2.4. Implicit differentiation

Example 2

Find y', y''   if $x^4 + y^4 = 16.$

2.4. Implicit differentiation

Solution:

$x^4 + y^4 = 16$ (*)

Differentiating the equation implicitly (*) with respect to

x, we get: $4x^3 + 4y^3y' = 0$.

Solving for y' gives: $\frac{dy}{dx} = -\frac{x^3}{y^3}$

2.4. Implicit differentiation

Solution:

To find *y*'', we differentiate this expression for *y*' using the Quotient Rule and remembering that *y* is a function of *x*:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\dfrac{d}{dx}\left(x^3\right) - x^3\dfrac{d}{dx}\left(y^3\right)}{\left(y^3\right)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3\left(3y^2y'\right)}{y^6}$$

2.4. Implicit differentiation

Solution:

If we now substitute y' into this expression, we get:

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3\left(x^2y^4 + x^6\right)}{y^7} = -\frac{3x^2\left(y^4 + x^4\right)}{y^7}$$

2.4. Implicit differentiation

Solution:

However, the values of x and y must satisfy the original equation $x^4 + y^4 = 16$.

So, the answer simplifies to:

$$y'' = -\frac{3x^2(16)}{y^7} = -\frac{48x^2}{y^7}$$

2.5. Related rates

The following steps in solving related rates problem:

1. Read the problem carefully.

2. Draw a diagram if possible.

3. Introduce notation. Assign symbols to all quantities that are functions of time.

4. Express the given information and the required rate in terms of derivatives.

2.5. Related rates

The following steps in solving related rates problem:

5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.

6. Use the Chain Rule to differentiate both sides of the equation with respect to .

7. Substitute the given information into the resulting equation and solve for the unknown rate.

2.5. Related rates

Example 1:

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s.

How fast is the radius of the balloon increasing when the diameter is 50 cm ?

2.5. Related rates

Solution: Air is being pumped into a spherical balloon so that its **volume increases at a rate of 100 cm³/s**.

How fast is the **radius of the balloon increasing** when the **diameter is 50 cm** ?

Summary

$$\frac{dV}{dt} = 100\ cm^3 / s \qquad V = \frac{4}{3}\pi r^3$$

$$2r = 50\,cm$$

$$\frac{dr}{dt} = ?$$

2.5. Related rates

Solution:

$$\frac{dV}{dt} = 100\ cm^3\ /\ s$$

$$2r = 50\,cm$$

$$\frac{dr}{dt} = ?$$

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100 = 4\pi 25^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{25\pi} \approx 0.0127\ cm\ /\ s.$$

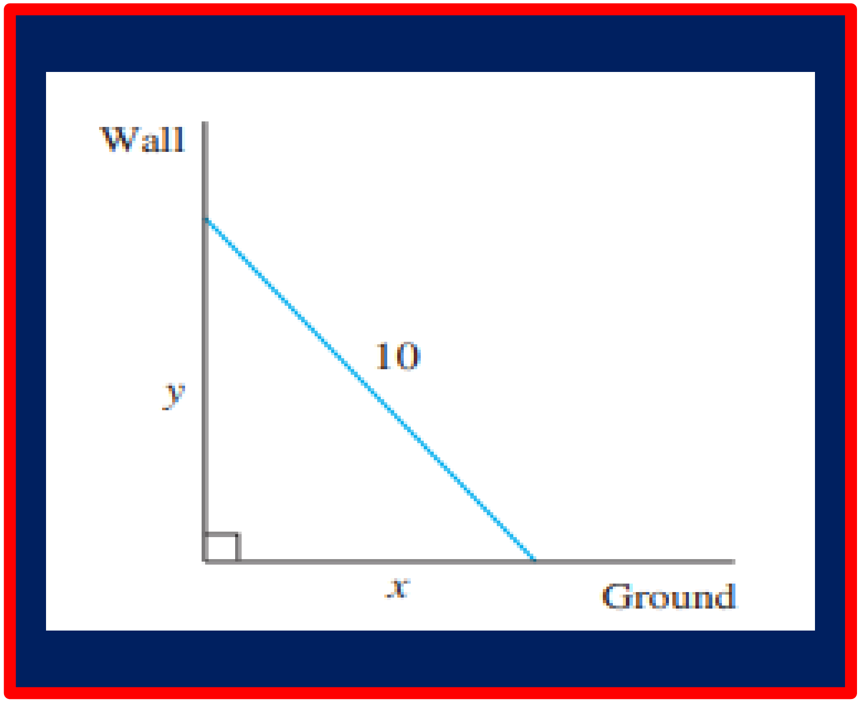
The radius of the balloon is increasing at the rate of

$1/(25\pi) \approx 0.0127$ cm/s.

2.5. Related rates

Example 2:

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?



2.5. Related rates

Example 2:

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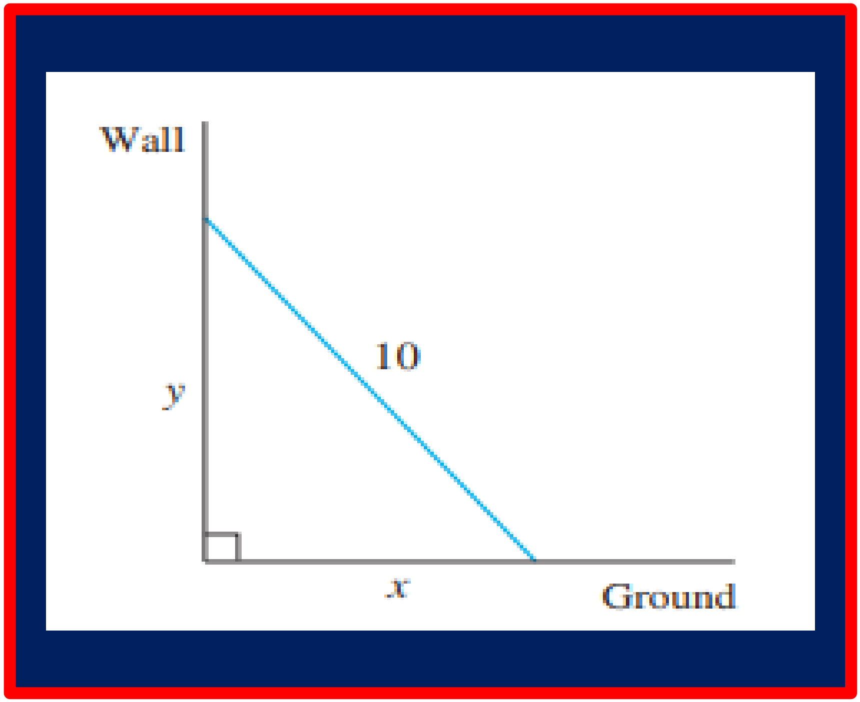


Summary $\dfrac{dx}{dt} = 1\,ft/s,\;\; x = 6\,;\dfrac{dy}{dt} = ??$

2.5. Related rates

Solution:

Summary

$$x^2 + y^2 = 100$$

$$\frac{dx}{dt} = 1 ft/s$$

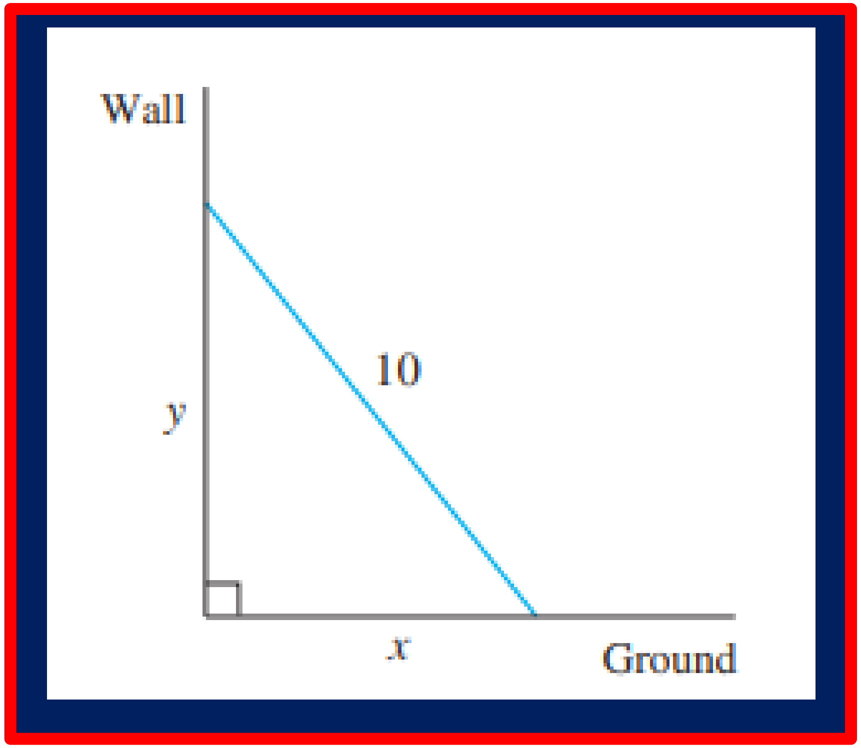
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x = 6$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = ??$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}.\frac{dx}{dt}$$



2.5. Related rates

Solution:

$$\left(x=6\right): x^2+y^2=100 \Rightarrow y^2=64 \Rightarrow y=8\left(y\geq 0\right)$$

$$\Rightarrow \frac{dy}{dt}=-\frac{3}{4}=-0.75\ ft/s$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4}$ ft/s.

That is, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s.

2.5. Related rates

Example 3:

Each side of a square is increasing at a rate of 6 cm/s.

At what rate is the area of the square increasing when the area of the square is $16cm^2$?

2.5. Related rates

Example 4:

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing ?

2.5. Related rates

Example 5:

A cylindrical tank with radius 5 m is being filled with water at a rate of $3m^3/s$. How fast is the height of the water increasing ?

2.5. Related rates

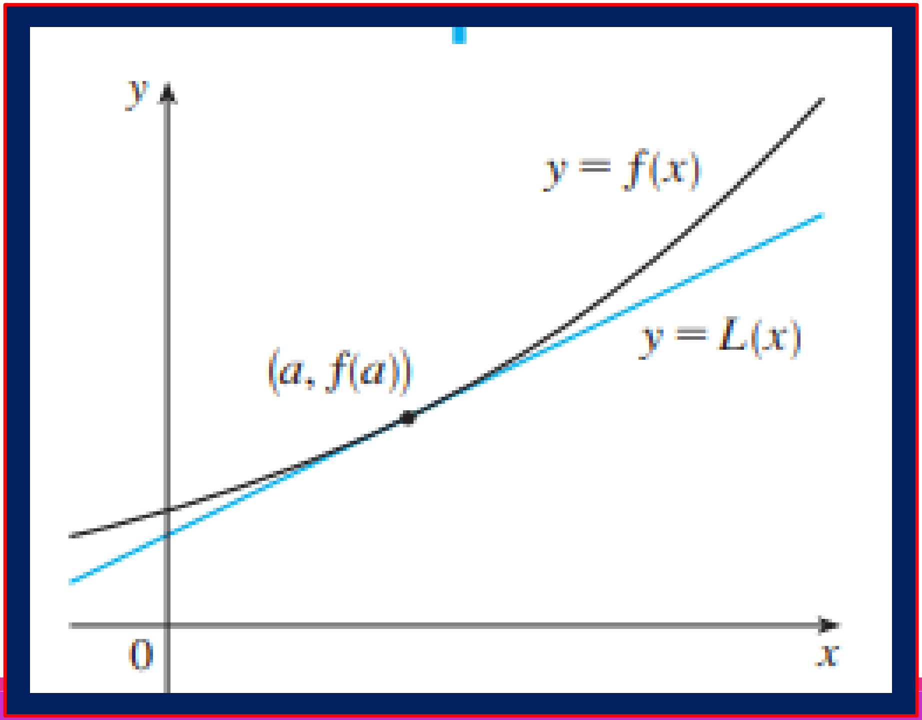
Example 6:

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2m^3$/min, find the rate at which the water level is rising when the water is 3 m deep.

2.6.Linear Approximations and Differentials

Linear approximations:

We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is **$L(x) = y = f(a) + f'(a)(x - a)$.**



2.6.Linear Approximations and Differentials

Linear approximations

The approximation

$$f(x) \approx f(a) + f'(a)(x - a) = L(x)$$

is called the linear approximation of f at a.

2.6. Linear Approximations and Differentials

Linear approximations

Example 1:

Find the linearization of the function $f(x) = \sqrt{x+3}$

at $a = 1$ and use it to approximate the numbers

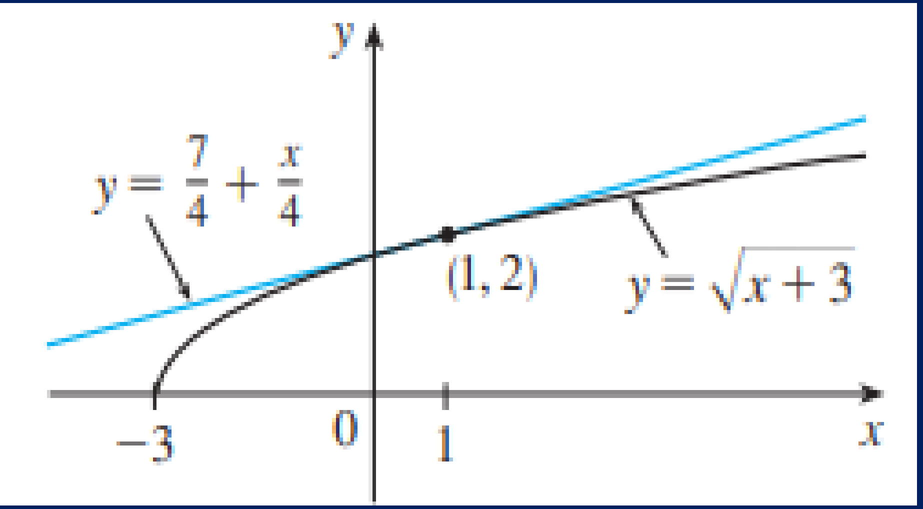
$$\sqrt{3.98} \text{ and } \sqrt{4.05}$$

Are these approximations overestimates or underestimates ?

Linear approximations

Solution:

$$L(x) = f(1) + f'(1)(x-1)$$
$$= 2 + \tfrac{1}{4}(x-1)$$
$$= \frac{7}{4} + \frac{x}{4}$$



Linear approximations

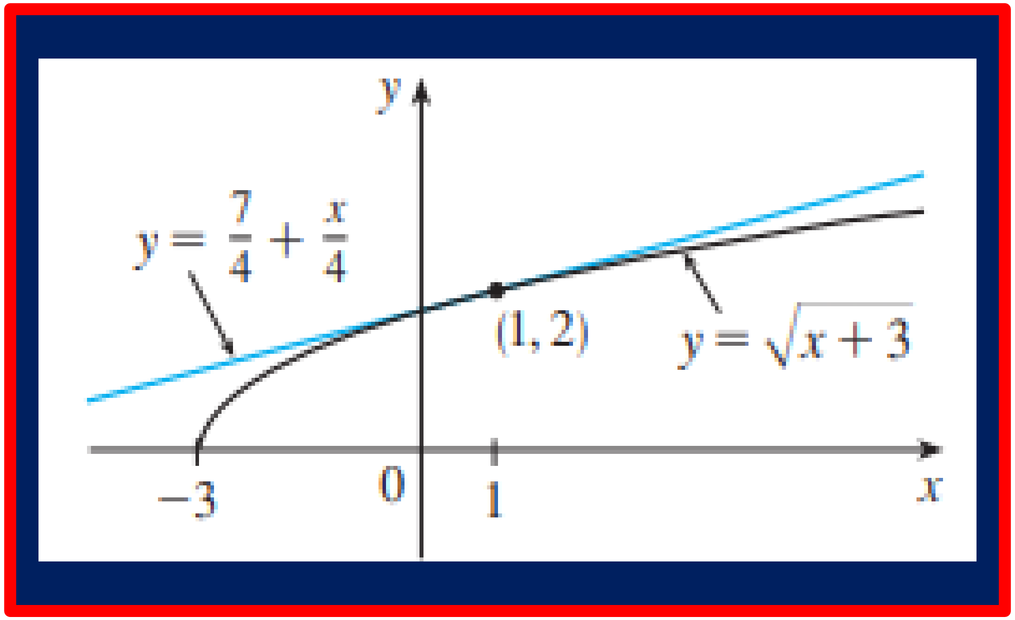
The corresponding linear approximation is:

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \quad \text{(when } x \text{ is near 1)}$$

In particular, we have:

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

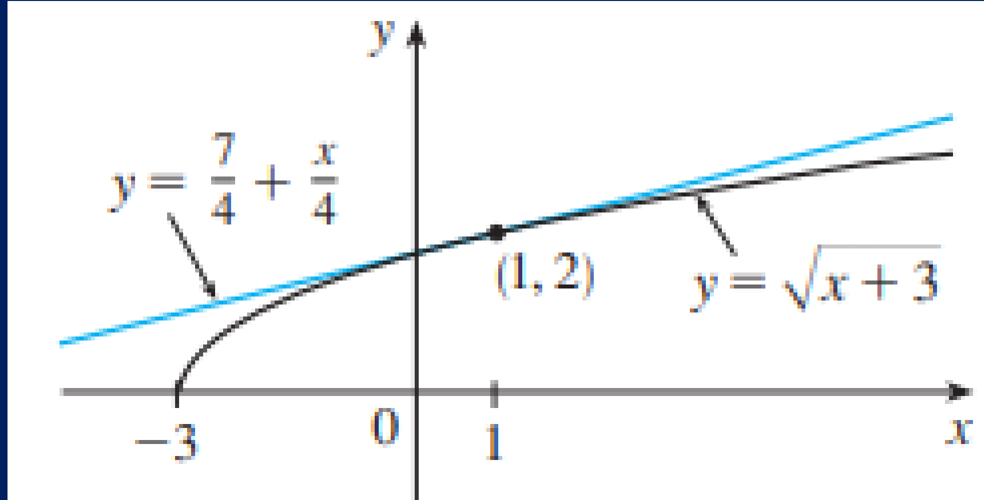
$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$



2.6.Linear Approximations and Differentials

Linear approximations

Look at the table and the figure. The tangent line approximation gives good estimates if x is close to 1. However, the accuracy decreases when x is farther away from 1.



	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

Linear approximations

Example 2:

Use the linear approximation to of the function $f(x)=\sqrt{x+1}$ at a = 3 to approximate the numbers.

(Select the correct ones)

a) $\sqrt{3.95}\approx 1.9875$

b) $\sqrt{4.05}\approx 2.0125$

c) $\sqrt{3.95}\approx 1.98746$

d) $\sqrt{4.05}\approx 2.01246$

2.6.Linear Approximations and Differentials

Differentials

Example

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm.

What is the maximum error in using this value of the radius to compute the volume of the sphere ?

Differentials

Solution: If the radius of the sphere is r, then its volume is

$V = 4/3\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV. This can be approximated by the differential

$$dV = 4\pi r^2 dr$$

When $r = 21$ and $dr = 0.05$, this becomes:

$$dV = 4\pi(21)^2 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3.

Differentials

Solution:

Relative error is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

Thus, the relative error in the volume is about three times the relative error in the radius.

Differentials

Solution:

In the example, the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume.

The errors could also be expressed as percentage errors of 0.24% in the radius and 0.7% in the volume.