Chapter 3

The vector space R^n

Contents

3.1.Subspace of R^n

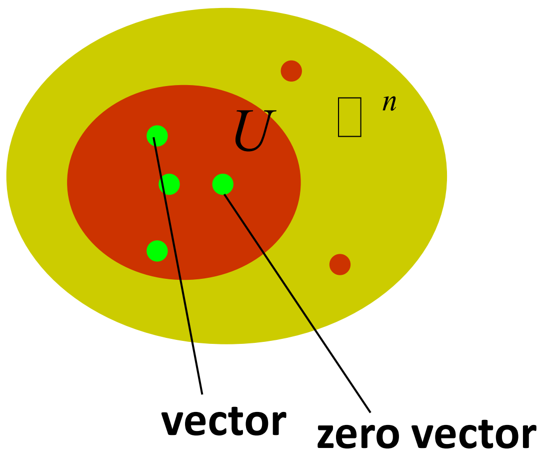
Definition of subspace of R^n.

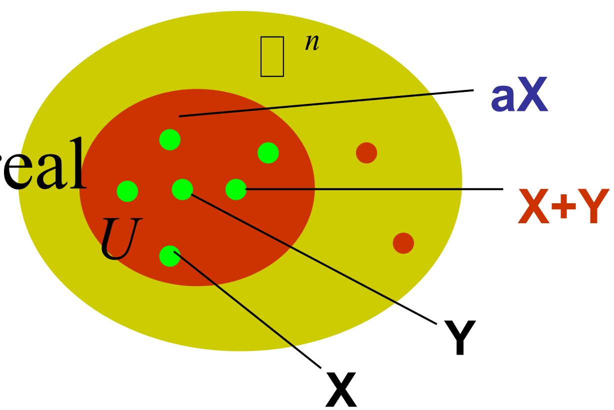
- Let $\emptyset \neq U$ be a subset of R^n
- U is called a *subspace* of R^n if:

  ❶ The zero vector **0** is in U

  ❷ If **X**,**Y** are in U then **X**+**Y** is in U

  ❸ If **X** is in U then **a**X is in U for all real number **a**.





3.1.Subspace of R^n

Example 1:

$U=\{(a,a,0)|a\in R\}$ is a ***subspace*** of R^3

3.1.Subspace of R^n

Example 1:

$U=\{(a,a,0)|a\in R\}$ is a ***subspace*** of R^3

❶ the zero vector of R^3, $(0,0,0)\in U$

❷ $(a,a,0)$, $(b,b,0)\in U \Rightarrow (a,a,0)+(b,b,0)=(a+b,a+b,0)\in U$

❸ If $(a,a,0)\in U$ and $k\in R$, then $k(a,a,0)=(ka,ka,0)\in U$

3.1. Subspace of R^n

Nhận xét:

Các trường hợp sau không là không gian vector con

1/ Có thành phần luôn khác không.

2/ Có hai hệ số chênh lệch một hằng số.

3/ Có bậc lớn hơn 1.

4/ Có hệ số bậc cao hoặc tích

5/ Có chứa trị tuyệt đối

3.1.Subspace of R^n

Example 2:

$U=\{(a,b,1): a,b \in R\}$ is not a ***subspace*** of R^3

❶ $(0,0,0) \notin U \Rightarrow$ U is not a ***subspace***

Example 3:

$U=\{(a,|a|,0) | a \in R\}$ is not a ***subspace*** of R^3

❷ $(-1,|-1|,0), (1,|1|,0) \in U$ but $(0,2,0) \notin U \Rightarrow$ U is not a ***subspace***

3.1.Subspace of R^n

Example 4:

Determine whether U is a subspace of R^3.
a/ $U=\{[0\ a\ b]^T: a,b \in R\}$
b/ $U=\{[0\ 1\ s]^T: s \in R\}$
c/ $U=\{[a\ b\ a+1]^T: a,b \in R\}$
d/ $U=\{[a\ b\ a^2]^T: a, b \in R\}$

3.1.Subspace of R^n

Solution:

Determine whether U is a subspace of R^3.
$U=\{[0\ a\ b]^T$: a,b ⍰ R} ✓
$U=\{[0\ 1\ s]^T$:s ⍰ R}
$U=\{[a\ b\ a+1]^T$:a,b ⍰ R}
$U=\{[a\ b\ a^2]^T$: a, b ⍰ R}

Nhận xét:

không là subspace khi

- có hằng số khác 0
- có hai hệ số chênh lệch 1 hằng số
- có bậc lớn hơn 1

3.1.Subspace of R^n

Example 5:

- $V=\{[0\ \ a\ \ 0]^T \text{ in } \mathbb{R}^3 : a \in Z\}$
- $U=\{[a\ \ 7\ \ 3a]^T \text{ in } \mathbb{R}^3 : a \in R\}$
- $W=\{[5a\ \ \ b\ \ \ a-b]^T \text{ in } \mathbb{R}^3 : a,b \in R\}$

3.1. Subspace of R^n

Solution:

- V={[0 a 0]^T in ⍰^{3}: a ⍰Z}

➢ $x = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \in V$ but $\frac{1}{2}x = \begin{bmatrix} 0 & \frac{1}{2} & 0 \end{bmatrix}^T \notin V$

⍰ V is not a ***subspace***

- U={[a 7 3a]^T in ⍰^{3}: a⍰R}

➢ (0,0,0)⍰U ⍰ U is not a ***subspace.***

- W={[5a b a-b]^T in ⍰^{3}: a,b⍰R} is a ***subspace.***

3.1. Subspace of R^n

Example 6

- $Q=\{[a \;\; b \;\; |a+b|]^T$ in $\mathbb{R}^3 : a,b \in \mathbb{R}\}$
- $H=\{[a \;\; b \;\; ab]^T$ in $\mathbb{R}^3 : a,b \in \mathbb{R}\}$
- $P=\{(x,y,z) \mid x-2y+z=0$ and $2x-y+3z=0\}$.

3.1. Subspace of R^n

Solution:

- $Q = \{[a \;\; b \;\; |a+b|]^T$ in $\mathbb{R}^3 : a,b \in \mathbb{R}\}$ is not a subspace.
- $H = \{[a \;\; b \;\; ab]^T$ in $\mathbb{R}^3 : a,b \in \mathbb{R}\}$ is not a subspace.
- $P = \{(x,y,z) \mid x-2y+z=0$ and $2x-y+3z=0\}$ is a subspace.

P is called the *solution space* of the system

$$\begin{cases} x - 2y + z = 0 \\ 2x - y + 3z = 0 \end{cases}$$

3.1. Subspace of R^n

Note:

A subspace either has only one or infinite many vectors

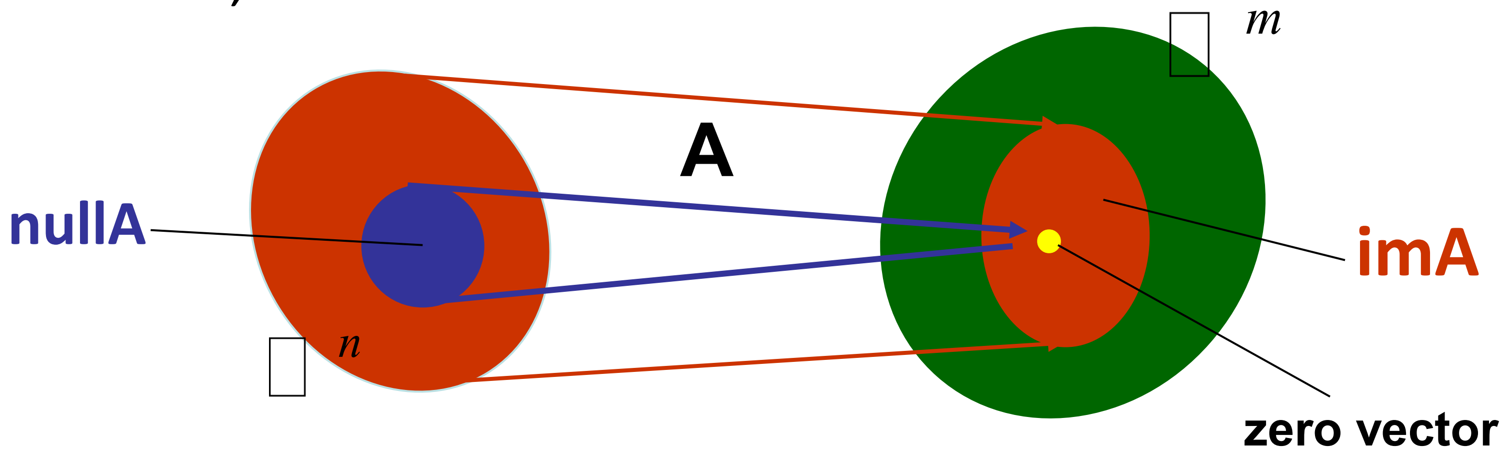
Example:

- {0} has only vector.
- If a subspace U has nonzero vector X then aX is also in U. Then U has infinite many vector

3.1.Subspace of R^n

Null space and image space of a matrix

- A is an mxn matrix, if X is nx1 matrix then AX is mx1 matrix



- **nullA** = {**X** in R^n: **AX=0**} **imA** = {**AX**: X is in R^n}

nullA ={X□ R^n:AX=0} is a subspace of R^n: ❶ A.0=0□0□nullA ❷ X,Y □nullA □AX=0, AY=0 □A(X+Y)=AX+AY=0 □(X+Y) □nullA ❸ X □nullA, a □R □ AX=0 □ A(aX)=a(AX)=0 □ aX□nullA	imA ={AX:X □ R^n}is a subspace of R^m: ❶ 0=A.0□0□imA ❷ AX,AY □imA □AX+AY=A(X+Y)=AZ □AX+AY □imA ❸ AX □imA, a □R □ a(AX)=A(aX)=AZ □ a(AX)□imA

3.1. Subspace of R^n

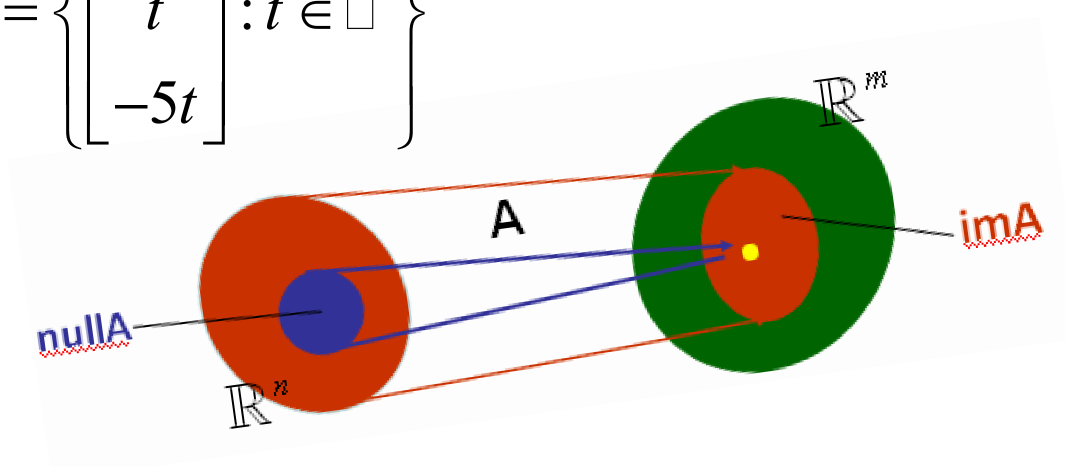
Null space $\text{null}A=\{X:AX=0\}$

Example:

$$A=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 1\end{bmatrix}_{2\times 3}$$

$$\text{null}A=\left\{\begin{bmatrix}x\\ y\\ z\end{bmatrix}: A\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\right\}=\left\{\begin{bmatrix}x\\ y\\ z\end{bmatrix}:\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\right\}$$

$$=\left\{\begin{bmatrix}x\\ y\\ z\end{bmatrix}:\begin{cases}x-y=0\\ 2x+3y+z=0\end{cases}\right\}=\left\{\begin{bmatrix}t\\ t\\ -5t\end{bmatrix}: t\in\mathbb{R}\right\}$$



3.2. Linear combinations

- $Y=k_1X_1+k_2X_2+\ldots+k_nX_n$ is called a *linear*

combination of the vectors $X_1,X_2,\ldots,X_n$

- The <u>set</u> of all *linear combinations* of the the vectors $X_1,X_2,\ldots,X_n$ is called the *span* of these vectors, denoted by $\text{span}\{X_1,X_2,\ldots,X_n\}$.
- This means, $\text{span}\{X_1,X_2,\ldots,X_n\} = \{k_1X_1+k_2X_2+\ldots+k_nX_n : k_i \in R \text{ is arbitrary}\}$
- $\text{span}\{X_1,X_2,\ldots,X_n\}$ is a subspace of R^n.

3.2. Linear combinations

Example 1:

$\text{span}\{(1,0,1),(0,1,1)\}=\{a(1,0,1)+b(0,1,1) : a,b\in R\}$.

- And we have $(1,2,3)\in\text{span}\{(1,0,1),(0,1,1)\}$ because $(1,2,3)= 1(1,0,1)+ 2(0,1,1)$.
- $(2,3,2)\notin\text{span}\{(1,0,1),(0,1,1)\}$ because $(2,3,2)\neq a(1,0,1)+b(0,1,1)$ for all a,b.

3.2. Linear combinations

- Nếu U=span{X,Y} ta nói U là KG được sinh ra bởi {X,Y} hay hệ {X,Y} sinh ra KG U. Khi đó, U chứa tất cả các vector có dạng aX+bY với a, b là các số thực tùy ý.
- vector Z∈span{X,Y} khi và chỉ khi có các số thực a,b sao cho Z=aX+bY hay hệ pt Z=aX+bY có nghiệm a,b.
- Ta cũng nói Z là một tổ hợp tuyến tính (linear combination) của X,Y khi Z=aX+bY hay Z∈span{X,Y}.

3.2. Linear combinations

Example 2:

If x=(1,3,-5) is expressed as a linear combination of the vectors v_1 = (1, 1, 1); v_2 =(1,1,-1); v_3 = (1, 0, 2); then the coefficient of v_3 is:

A. 2 B. 3 C. -2 ✓ D. 1 E. 0

3.2. Linear combinations

Solution:

x is expressed as a linear combination of v_1, v_2, v_3 means $x=av_1+bv_2+cv_3$ for some a,b,c and c is called the ***coefficient*** of v_3.

- the system is

$$\begin{aligned} a + b + c &= 1 \\ a + b \quad\;\; &= 3 \\ a - b + 2c &= -5 \end{aligned}$$

Solution:

$a = 1; b = 2; c = -2.$

3.2. Linear combinations

Example 3:

Which of the vectors below is a *linear combination* of $u=(1,1,2)$; $v=(2,3,5)$?

A. $(0,1,1)$ B. $(1,1,0)$

C. $(1,1,1)$ D. $(1,0,1)$

E. $(0,0,1)$

3.2. Linear combinations

Solution:

A. (0,1,1) ✓

D. (1,0,1) ✓

3.2. Linear combinations

Example 4:

a/ If (x,y,z) is expressed as a linear combination of vectors v1=(1,1,-1); v2=(1,0,1) and v3=(-1,0,1) then what is the coefficient of v3 ?

b/ Let U=span{(1,2,3);(3,4,5)}. Find **m** such that (3,5,**m**) lies in U.

3.2. Linear combinations

Solution:

a/ w=(x,y,z) is expressed as a linear combination of v_1, v_2, v_3 means $w = av_1+bv_2+cv_3$ for some a,b,c and c is called the ***coefficient*** of v_3.

- the system is

$$\begin{cases} a + b - c = x \\ a = y \\ -a + b + c = z \end{cases} \quad \Rightarrow c = \frac{-x+2y+z}{2}$$

3.2. Linear combinations

Solution:

b/ w=(3,5,**m**) lies in U if and only if w=(3,5,m) is expressed as a linear combination of $v_1=(1,2,3)$, $v_2=(3,4,5)$ means $w = av_1+bv_2$

We have

$$\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & m \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -4 & m-9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & m-7 \end{array}\right]$$

The system has solution if and only if

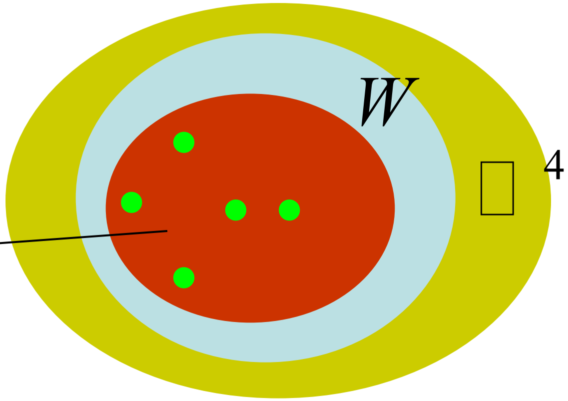
$$r(A) = r(A \mid b) \Leftrightarrow m = 7$$

3.2. Linear combinations

Theorem

- $U = span\{X_1, X_2, \ldots, X_n\}$ is in R^n and U is a subspace of R^n
- If W is a subspace of R^n such that X_i are in W then

$$U \subseteq W$$



$$U = span\{x_1, x_2, x_3, x_4, x_5\}$$

3.3. Linear Independence (sự độc lập tuyến tính)

A set of vectors in R^m $\{X_1, X_2, \ldots, X_n\}$ is called **linearly independent** (độc lập tuyến tính)

if $t_1X_1 + t_2X_2 + \ldots + t_nX_n = 0$ ⍰ $t_1 = t_2 = \ldots = t_n = 0$ only

numbers in R **vectors in R^m**

3.3. Linear Independence

Example 1:

The set $S = \{[1 \;\; -1]^T, [2 \;\; 3]^T\}$ ⍰ R^2 is called <u>linearly independent</u> since $t_1[1 \;\; -1]^T + t_2[2 \;\; 3]^T = [0 \;\; 0]^T$ follows $t_1 = t_2 = 0$.

Example 2:

The set $\{(0,1,1), (1,-1,0), (1,0,1)\}$ is *not linearly independent* because the system

$$t_1(0,1,1)+t_2(1,-1,0)+t_3(1,0,1)=(0,0,0)$$

has one *solution* $t_1 = -1,\ t_2 = -1,\ t_3 = 1$

3.3. Linear Independence

Example 1:

The set $S = \{[1 \ \ -1]^T, [2 \ \ 3]^T\} \subset R^2$ is called <u>linearly independent</u> since $t_1[1 \ \ -1]^T + t_2[2 \ \ 3]^T = [0 \ \ 0]^T$ follows $t_1=t_2=0$.

Example 2:

The set {(0,1,1), (1,-1,0), (1,0,1)} is *not linearly independent* because the system

$$t_1(0,1,1)+t_2(1,-1,0)+t_3(1,0,1)=(0,0,0)$$

has one *solution* $t_1= -1,\ t_2= -1,\ t_3=1$

3.3. Linear Independence

Example 3:

A set of vectors that containing zero vector never linearly independent.

3.3. Linear Independence

Example 4:

Show that $\{(1,1,0);(0,1,1);(1,0,1)\}$ is linearly independent in R^3

3.3. Linear Independence

Solution

- Show that {~~(1,1,0)~~;(0,1,1);(1,0,1)} is linearly independent in R^3

$$t_1(1,1,0)+t_2(0,1,1)+t_3(1,0,1)=(0,0,0)$$

$$\Rightarrow \ldots \Rightarrow t_1=t_2=t_3=0$$

$$t_1(1,1,0)+t_2(0,1,1)+t_3(1,0,1)=(0,0,0)$$

$$\Leftrightarrow \begin{cases} 1t_1+0t_2+1t_3=0 \\ 1t_1+1t_2+0t_3=0 \\ 0t_1+1t_2+1t_3=0 \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow t_1=t_2=t_3=0 \Rightarrow independent$$

Fast way to determine a set of vectors is independent or not:
independent $\Leftrightarrow$ Number of leading 1s = member of vectors

More ex. {(1,0,-2), (2,1,0), (0,1,5), (-1,1,0)} is *not linearly independent* (number of leading 1s = member of vectors)

1	2	0	-1
0	1	1	1
-2	0	5	0

1	2	0	-1
0	1	1	1
0	4	5	2

1	2	0	-1
0	1	1	1
0	0	1	-2

3.3. Linear Independence

Example 5:

Determine whether each the following sets is linearly independent or linearly dependent.

a/ $\{(-1,2,0)\}$

b/ $\{(0,0,0); (1,2,3); (-3,0,1)\}$

c/ $\{(1,1,-1); (-1,1,1); (1,-1,1)\}$

d/ $\{(-2,3,4,1); (4,-1,5,0); (-2,1,0,3)\}$

e/ $\{(1,1,0); (-2,3,1); (5,0,1); (-1,0,2)\}$

f/ $\{X-Y+Z, 3X+Z, X+Y-Z\}$, where $\{X,Y,Z\}$ is an independent set of vectors.

3.3. Linear Independence

Solution

a/ $\{(-1,2,0)\}$ **Linear Independence**

b/ $\{(0,0,0); (1,2,3); (-3,0,1)\}$ **Linear dependence**

c/ $\{(1,1,-1); (-1,1,1); (1,-1,1)\}$ **Linear Independence**

d/ $\{(-2,3,4,1); (4,-1,5,0); (-2,1,0,3)\}$ **Linear Independence**

e/ $\{(1,1,0); (-2,3,1); (5,0,1); (-1,0,2)\}$ **Linear dependence**

f/ $\{X-Y+Z, 3X+Z, X+Y-Z\}$, where $\{X,Y,Z\}$ is an independent set of vectors. **Linear Independence**

3.3. Linear Independence

Example 6:

a/Find all **x** in R such that {(1,1,1,1);(2,3,2,3),(3,4,1,**x**)} is a linearly independent set.

b/ For what value of **a** is the set of vectors S={(1,1,1,1);(3,2,1,5);(2,3,0,**a**-11)} is linearly dependent ?

c/ Find all m such that the set {(2,m,1),(1,0,1),(0,1,1)} is linearly independent.

3.3. Linear Independence

Solution:

a/ Set u = (1,1,1,1);v = (2,3,2,3); w = (3,4,1,**x**)

$$au + bv + c\mathrm{w} = 0$$

$\Leftrightarrow$ a(1,1,1,1) +b (2,3,2,3)+c (3,4,1,**x**) = (0,0,0).

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & x & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & x-3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & x-4 & 0 \end{array}\right]$$

3.3. Linear Independence

Solution:

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & x-4 & 0 \end{array}\right] \xrightarrow{r_4 \to 2r_4 + (x-4)r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r(A) = r(A \mid b) = 3 = n \ \ for\ all\ x$$

{(1,1,1,1);(2,3,2,3),(3,4,1,**x**)} is a linearly independent set for all x.

3.3. Linear Independence

Solution:

b/ For what value of **a** is the set of vectors S={(1,1,1,1); (3,2,1,5);(2,3,0,**a**-11)} is linearly dependent ?

Answer: a doesn't exist.

c/ Find all m such that the set {(2,m,1),(1,0,1),(0,1,1)} is linearly independent.

Answer: $m \neq -1$

3.3. Linear Independence

Fundamental Theorem

Let U be a subspace of R^n is spanned by m vectors, if U contains k linearly independent vectors, then $k \leq n$.

This implies if $k>n$, then the set of k vectors is always linear dependence.

Example

Let U be the space spanned by $\{(1,0,1), (0,-1,1)\}$ and $S=\{(1,0,1), (0,-1,1), (2,-1,3)\} \subset U$.

Then, S is *not linearly independent* ($m=2$, $k=3$).

3.3. Linear Independence

Note:

1/ Tập chứa vector không luôn phụ thuộc tuyến tính.

2/ Tập có 1 vector khác vector không thì độc lập tuyến tính.

3/ Tập có số vector lớn hơn số chiều luôn luôn phụ thuộc tuyến tính.

4/ Tập có 2 vector độc lập tuyến tính khi và chỉ khi cả hai vector khác vector không và không cùng phương (các thành phần tỉ lệ không bằng nhau)

3.4. Basis and dimension (cơ sở và chiều của KG)

Definition of basis:

Suppose U is a *subspace* of R^n, a *set* $\{X_1, X_2, \ldots, X_k\}$ is called a **basis** of U if

$U = \text{span}\{X_1, X_2, \ldots, X_k\}$, $B = \{X_1, X_2, \ldots, X_k\}$

and

$B = \{X_1, X_2, \ldots, X_k\}$ is *linear independence.*

3.4. Basis and dimension

Example 1:

Let $U=\{(a,-a)|a\in R\}$. Then U is a *subspace* of R^2. Consider the set $B=\{(1,-1)\}$.

B is *linearly independent*

and

$U=\{(a,-a):a\in R\}=\{a(1,-1): a\in R\} =\text{span}\{(1,-1)\}$.

So, B is a *basis* of U.

Note that $B'=\{(-1,1)\}$ is also a *basis* of U.

But $\{(1,1)\}$ is not a *basis* of U because U can not be spanned by $\{(1,1)\}$

Example 2:

Given that V=span{(1,1,1), (1,-1,0), (0,2,1)}.

Then, B={(1,1,1), (1,-1,0), (0,2,1)} is <u>not linearly independent</u>, because (0,2,1)=(1,1,1) – (1,-1,0)

So, B is not a *basis* of V.

Consider B'={(1,1,1), (1,-1,0)}.

B' is *linearly independent* and all vectors in V are spanned by B' because a(1,1,1)+ b(1,-1,0)+ c(0,2,1) =a(1,1,1)+ b(1,-1,0)+ c(1,1,1) –c(1,-1,0) = (a+c)(1,1,1)+(b-c)(1,-1,0).

So, B' is a *basis* of V.

3.4. Basis and dimension

Example 2:

Given that V=span{(1,1,1), (1,-1,0), (0,2,1)}.

Then, B={(1,1,1), (1,-1,0), (0,2,1)} is not linearly independent, because (0,2,1)=(1,1,1) – (1,-1,0)

So, B is not a *basis* of V.

Consider B'={(1,1,1), (1,-1,0)}.

B' is *linearly independent* and all vectors in V are spanned by B' because a(1,1,1)+ b(1,-1,0)+ c(0,2,1) =a(1,1,1)+ b(1,-1,0)+ c(1,1,1) –c(1,-1,0) = (a+c)(1,1,1)+(b-c)(1,-1,0).

So, B' is a *basis* of V.

3.4. Basis and dimension

Theorem 1 (Invariance theorem).

If $\{a_1,a_2,..,a_m\}$ and $\{b_1,b_2,\ldots,b_k\}$ are bases of a subspace U of R^n, then m=k. In this case, m=k is called ***dimension*** of U and we write ***dimU=m***.

3.4. Basis and dimension

Example 1:

Let $U=\{(a,-a)|a\in R\}$ be a subspace of R^2. Then, $B=\{(1,-1)\}$ is a *basis* of U and $B'=\{(-1,1)\}$ is also a *basis* of U.

In this case, dimU=1.

Example 2:

$\{(1,0), (0,1)\}$ is a basis of R^2 and $\{(1,-2), (2,0)\}$ is also a basis of R^2. But $\{(1,0), (-1,1), (1,1)\}$ is not a basis of R^2.

We have $\dim R^2=2$.

The basis $\{(1,0), (0,1)\}$ is called ***standard basis*** of R^2.

3.4. Basis and dimension

Example 3:

Which of the following is a basis of R^3?

❶ $\{(1,0,1), (0,0,1)\}$

❷ $\{(2,1,0), (-1,0,1), (1,0,1), (0,-1,1)\}$

❸ $\{(0,1), (1,0)\}$

❹ None of the others

3.4. Basis and dimension

Theorem 2

Let U be a subspace of R^n and $B=\{X_1,X_2,\ldots,X_m\} \subseteq U$, where $\dim U=m$. Then B is independent if and only if B spans U.

3.4. Basis and dimension

Example:

Find all values of m so that $\{(2,-1,3); (0,1,2); (-4,0; m)\}$ spans R^3.

3.4. Basis and dimension

Solution:

So, $\{(2,-1,3); (0,1,2); (3,1; m)\}$ spans R^3 $\Leftrightarrow$ it is linearly independent $\Leftrightarrow$ $m \neq 10$

3.4. Basis and dimension

Nhận xét: Trong R^n,

1/ mọi tập có n vector độc lập đều là cơ sở.

2/ mọi tập sinh có n vector đều là cơ sở.

3/ mọi tập có số vector lớn hơn n luôn phụ thuộc tuyến tính.

4/ mọi tập độc lập tuyến tính có số vector nhỏ hơn n không thể là tập sinh.

3.4. Basis and dimension

Theorem 3

The following are ***equivalence*** for an nxn matrix A.

❶ A is invertible.

❷ the columns of A are linearly independent.

❸ the columns of A span R^n.

❹ the rows of A are linearly independent.

❺ the rows of A span the set of all 1xn rows.

3.4. Basis and dimension

Theorem 4

Let $U \neq 0$ be a subspace of R^n. Then:

❶ U has a basis and $\dim U \leq n$.

❷ Any independent set of U can be enlarged (by adding vectors) to a basis of U.

❸ If B spans U, then B can be cut down (by deleting vectors) to a basis of U.

3.4. Basis and dimension

Example 1:

Let $U=\text{span}\{(1,1,1), (1,0,1), (1,-2,1)\}$ be a subspace of R^3. Find the basis of U.

3.4. Basis and dimension

Solution: Set B= {(1,1,1), (1,0,1), (1,-2,1)}.

Construct a basis for U

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{r_3 \to r_3 - 3r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for U is E ={(1,0,1), (0,-1,0)}.

Cũng có thể chọn 2 vector 1 và 2; 1 và 3 hoặc 2 và 3 làm cơ sở.

3.4. Basis and dimension

Example 2:

Let U=span{(-2, 0, 3), (1, 2, -1),(-2, 8, 5),(-1, 2, 2)} is the subspace of R^3 . Find the basis and dimension of U.

3.4. Basis and dimension

Example 3:

Let $U=\{[a \;\; b \;\; a-b]^T : a,b \in R\}$ is the subspace of R^3.

Find the basis and dimension of U.

Solution:

We have U={[a b a-b] ^{T} : a,b ∈R}

$$=\left\{\begin{bmatrix}a & 0 & a\end{bmatrix}^T+\begin{bmatrix}0 & b & -b\end{bmatrix}^T : a,b\in R\right\}$$

$$=\left\{a\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T+b\begin{bmatrix}0 & 1 & -1\end{bmatrix}^T : a,b\in R\right\}$$

$$=span\left\{\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T,\begin{bmatrix}0 & 1 & -1\end{bmatrix}^T\right\}$$

And $B=\left\{\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T,\begin{bmatrix}0 & 1 & -1\end{bmatrix}^T\right\}$ is linear independent.

Hence, $B=\left\{\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T,\begin{bmatrix}0 & 1 & -1\end{bmatrix}^T\right\}$ is a basis of U and $\dim U=2$

3.4. Basis and dimension

Example 4:

Find a basis and dimension for the **solution space** to the homogeneous system

$$\begin{cases} x - y + 2z = 0 \\ 2x + y = 0 \end{cases}$$

3.4. Basis and dimension

Solution: Let U is the solution space to the homogeneous system $\begin{cases} x - y + 2z = 0 \\ 2x + y = 0 \end{cases}$

We have

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 0 \end{bmatrix}$$

The system

$$\begin{cases} x - y + 2z = 0 \\ \quad 3y - 4z = 0 \\ z = 3a \in R \end{cases} \Leftrightarrow \begin{cases} x - y + 2z = 0 \\ \quad 3y - 4z = 0 \\ z = 3a \in R \end{cases} \Leftrightarrow \begin{cases} x = -2a \\ y = 4a \\ z = 3a \in R \end{cases}$$

Solution:

$$U=\{(-2a,4a,3a)\mid a\in R\}$$
$$=\{a(-2,4,3)\mid a\in R\}$$
$$=span\{(-2,4,3)\}$$

and $B=\{(-2,4,3)\}$ is linear independent.

Hence, $B=\{(-2,4,3)\}$ is basis of U and $\dim U=1$.

3.4. Basis and dimension

Theorem 5

Let $U \subset V$ be subspaces of R^n. Then:

❶ $\dim U \leq \dim V$.

❷ If $\dim U = \dim V$, then $U = V$.

3.4. Basis and dimension

- If $\{x_1,x_2,\ldots,x_m\}$ is a basis of U then dimU=m
- $\dim R^n=n$
- Note that if U=span $\{x_1,x_2,\ldots,x_m\}$ then $\dim U\leq m$ and $\dim U=m$ if and only if $\{x_1,x_2,\ldots,x_m\}$ is linear independent
- If **dimU=m** then every set of **m+1** vector in U is linearly dependent
- Every set of n linearly independent vectors is a basis of R^n (mỗi tập gồm **n** vector **đltt** của $\mathbf{R^n}$ đều là một **cơ sở** của R^n)

3.4. Basis and dimension

Exercises

Question 1:

Let $U=\text{span}\{(1,-1,1), (0,2,1)\}$. Find all value(s) of m for which $(3,-1,m) \in U$.

3.4. Basis and dimension

Exercises

Solution:

(3,-1,m) U⇔ (3,-1,m)= a(1,-1,1) + b(0,2,1) for some a,b.

Solve for a,bm=4

3.4. Basis and dimension

Exercises

Question 2:

A basis for the subspace $U=\{[a\ \ b\ \ a-b]^T: a,b \in R\}$ is…
a. $\{[1\ 0\ 1]^T, [0\ 1\ -1]^T\}$
b. $\{[1\ 1\ 0]^T\}$
c. $\{[1\ 0\ 1]^T, [-1\ 0\ -1]^T, [0\ 1\ -1]^T\}$
d. None of the others.

Exercises

Solution:

A basis for the subspace $U=\{[a\ \ b\ \ a-b]^T: a,b \in R\}$ is…
a. $\{[1\ 0\ 1]^T, [0\ 1\ -1]^T\}$ ✓
b. $\{[1\ 1\ 0]^T\}$
c. $\{[1\ 0\ 1]^T, [-1\ 0\ -1]^T, [0\ 1\ -1]^T\}$
d. None of the others.

Nhận xét:

- U phụ thuộc 2 tham số nên dimU=2 và mọi cơ sở đều phải có đúng 2 vector độc lập tuyến tính⇒chỉ có thể là a hoặc d.
- kiểm tra a: độc lập và sinh ra U

Exercises

Question 3: The dimension of the subspace U=span{(-2, 0, 3), (1, 2, -1),(-2, 8, 5),(-1, 2, 2)} is…
a. 2   b. 4   c. 3   d. 1

3.4. Basis and dimension

Exercises

Solution:

The dimension of the subspace $U=\text{span}\{(-2, 0, 3), (1, 2, -1), (-2, 8, 5), (-1, 2, 2)\}$ is…

✓ a. 2	b. 4	c. 3	d. 1

- cannot b because $\dim U \leq \dim R^3 = 3$
- Check by Elementary Operations

3.4. Basis and dimension

Exercises

Question 4: Let u and v be vectors in R^3 and w ⍰ span{u,v}. Then …
a. {u,v,w} is linearly dependent.
b. {u,v,w} is linearly independent.
c. {u,v,w} is a basis of R^3
d. the subspace is spanned by {u,v,w} has the dimension 3.

3.4. Basis and dimension

Exercises

Solution: Let u and v be vectors in R^3 and w □ span{u,v}. Then …
a. {u,v,w} is linearly dependent. ✓
b. {u,v,w} is linearly independent.
c. {u,v,w} is a basis of R^3
d. the subspace is spanned by {u,v,w} has the dimension 3.

- w □ span{u,v} means w=au+bv □{u,v,w} is not independent
- lưu ý cũng không có gì chắc chắn {u,v} độc lập nên dimU□2

3.4. Basis and dimension

Exercises

Question 5: Let {u,v,w,z} be independent. Then …. is also independent.
a. {u,v+w,z}
b. {u,v,v-z-u,z}
c. {u+v,u-w,z, v+z+w}
d. {u,v,w,u-v+w}

3.4. Basis and dimension

Exercises

Solution: Let {u,v,w,z} be independent. Then …. is also independent.
a. {u,v+w,z} ✓
b. {u,v,v-z-u,z}
c. {u+v,u-w,z, v+z+w}
d. {u,v,w,u-v+w}

3.4. Basis and dimension

Exercises

Question 6:

Find a basis for the subspace of R^3 defined by

$U=\{(a,b,c): 2a-b+3c=0\}$

a. $\{(1,2,0), (0,3,1)\}$

b. $\{(1,0,0), (0,1,0), (0,0,1)\}$

c. $\{(1,2,0)\}$

d. $\{(1,0,0), (1,2,0)\}$

e. $\{(3,0,-2), (1,0,0)\}$

f. $\{(2,-1,3)\}$

3.4. Basis and dimension

Exercises

Solution:

Find a basis for the subspace of R^3 defined by

$U=\{(a,b,c): 2a-b+3c=0\}$

a. $\{(1,2,0), (0,3,1)\}$ ✓

b. $\{(1,0,0), (0,1,0), (0,0,1)\}$

c. $\{(1,2,0)\}$

d. $\{(1,0,0), (1,2,0)\}$

e. $\{(3,0,-2), (1,0,0\}$

f. $\{(2,-1,3)\}$

Nhận xét:

- Nghiệm phụ thuộc 2 tham số $\Rightarrow$ dimU=2 và mọi cơ sở của U phải có đúng 2 vector $\Rightarrow$ chỉ a,d hoặc e đúng
- các vector trong cơ sở cũng phải thuộc U nên dễ thấy (1,0,0) không thuộc U $\Rightarrow$ loại d,e.

3.4. Basis and dimension

Exercises

Question 7:

Find a basis and dimension for the subspace of R^3 defined by $U=\{(a,b,a-b) | a,b \in R\}$

3.4. Basis and dimension

Exercises

Solution:

$U=\{a(1,0,1)+b(0,1,-1) \mid a,b \in R\}=\text{span}\{(1,0,1), (0,1,-1)\}$

$\{(1,0,1), (0,1,-1)\}$ spans U and independent $\Rightarrow$ $\{(1,0,1), (0,1,-1)\}$ is a basis and dimU=2

Nhận xét:

- U phụ thuộc 2 tham số $\Rightarrow$ dimU=2
- Có thể chọn ngay một cơ sở của U bằng cách chọn 2 vetor thuộc U và 2 vector này độc lập tuyến tính.
- Có nhiều cơ sở khác của U, ví dụ $\{(-1,0,-1), (0,1,-1)\}$,…

Exercises

Question 8:

Which of the following are bases for $\mathbb{R}^3$?

(i) $U = \{(1,-2,0),(-2,1,3),(3,0,0)\}$

(ii) $V = \{(5,2,1),(0,1,-1),(3,0,2),(0,1,0)\}$

(iii) $W = \{(2,4,-1),(0,3,-1),(-4,-8,2)\}$

a. All of them

b. (i) only

c. (i) and (ii) only

d. (i) and (iii) only

e. None of them

3.4. Basis and dimension

Do yourself

Question 1: Find all values of m for which (1,2,m) lies in the subspace spanned by {(1,0,-1), (0,1,2)}

Question 2: Find all m such that (-3,2,m) is a linear combination of two vectors (1,1,-1) and (-2, 3, 4).

Question 3: Determine {x-y,x+y-z,x+z} is independent or not, where {x,y,z} is independent.

Question 4: Determine whether each of the following sets is a basis of R^3 or not:

a/ {(1,0,-1), (0,1,2), (3,1,-1)}

b/ {(0,1,1), (-1,2,1)}

c/ {(1,0,0), (1,1,0), (1,1,1), (0,0,1)}

Do yourself

Question 4: Determine whether each of the following sets is a basis of R^3 or not:

a/ $A = \{(1,0,-1), (0,1,2), (3,1,-1)\}$

b/ $B = \{(0,1,1), (-1,2,1)\}$

c/ $C = \{(1,0,0), (1,1,0), (1,1,1), (0,0,1)\}$

d/ $D = \{(-1,0,3);(3,0,-1)\}$

e/ $E = \{(1,-1,2);(3,0,1);(1,1,0)\}$

f/ $F = \{(7,-1,4);(0,0,1);(1,-1,0);(0,1,5)\}$

3.4. Basis and dimension

Do yourself

Question 5: Find a basis and dimU if:

a/ U=span{(-1,4,3);(3,0,-2);(-6,2,0)}

b/ U=span{(1,-1,3,0);(5,-2,4,3);(-2,0,7,1)}

c/ U=span{(1,1,1);(1,-1,1);(-1,0,1);(0,-1,1)}

d/ U=span{(1,-1,2,0);(-2,1,0,1);(-1,0,0,1); (1,0,1,2)}

e/ U={$[a\ \ b\ \ 0]^T$: a,b in R}

f/ U={(a,b,c):a+b+c=0}

3.5. RowA, colA and nullA subspaces

- If A is carried to row-echelon form then rankA=number of leading 1's
- If A is an **mxn** matrix then **rankA≤min{n,m}**
- rankA=rank(A^T)

3.5. RowA, colA and nullA subspaces

Let mxn matrix A.

- rowA = span{rows of matrix A}
- colA = span{columns of A}
- dim(rowA)=dim(colA)=rankA

- **nullA={X| AX=0}**
- dim(nullA) = n-r, r =rankA

Example 1:

Find bases for the row and column spaces of A and determine the rank of A

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix}$$

3.5. RowA, colA and nullA subspaces

Solution:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix}$$

Set $u_1 = (2,-1,1), u_2 = (-2,1,1), u_3 = (4,-2,3), u_4 = (-6,3,0)$

rowA $= \text{span}\{u_1, u_2, u_3, u_4\}$

We have $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3.5. RowA, colA and nullA subspaces

Solution:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rowA $= \text{span}\{u_1, u_2, u_3, u_4\} = \text{span}\{(2,-1,1),(0,0,2)\}$

Because B = $\{(2,-1,1),(0,0,2)\}$ is linear independent.

So, B = $\{(2,-1,1),(0,0,2)\}$ is a basis of rowA.

dim(rowA)=2.

3.5. RowA, colA and nullA subspaces

Solution:

$$A=\begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix}$$

Set $v_1=(2,-2,4,-6), v_2=(-1,1,-2,3), v_3=(1,1,3,0)$

colA $=\operatorname{span}\{v_1,v_2,v_3\}$.

We have

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 2 & -2 & 4 & -6 \\ -1 & 1 & -2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}\xrightarrow[r_3\to 2r_3-r_1]{r_2\to 2r_2+r_1}\begin{pmatrix} 2 & -2 & 4 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \end{pmatrix}$$

Solution:

$$\rightarrow \begin{pmatrix} 2 & -2 & 4 & -6 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

colA $= \text{span}\{v_1, v_2, v_3\}=\text{span}\{(2,-2,4,-6),(0,4,2,6)\}$

Because E = {(2,-2,4,-6),(0,4,2,6)} is linear independent.

So, E = {(2,-2,4,-6),(0,4,2,6)} is a basis of colA.

dim(colA)=2.

Example 2:

Find bases of nullA

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix}$$

Solution:

nullA = {X|AX = 0}

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 0 \\ -x_1 + 2x_2 + x_4 = 0 \\ 2x_1 - 4x_2 + x_3 = 0 \end{cases} \right\}$$

Solution:

We have

$$[A\,|\,b]=\left(\begin{array}{cccc|c}1&-2&1&1&0\\-1&2&0&1&0\\2&-4&1&0&0\end{array}\right)\to\left(\begin{array}{cccc|c}1&-2&1&1&0\\0&0&1&2&0\\0&0&-1&-2&0\end{array}\right)$$

$$\to\left(\begin{array}{cccc|c}1&-2&1&1&0\\0&0&1&2&0\\0&0&0&0&0\end{array}\right).\ \text{Hence,}\ \begin{cases}x_1-2x_2+x_3+x_4=0\\x_2=a\in\mathbb{R}\\x_3+2x_4=0\\x_4=b\in\mathbb{R}\end{cases}$$

3.5. RowA, colA and nullA subspaces

Solution:

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 0 \\ x_2 = a \in \mathbb{R} \\ x_3 + 2x_4 = 0 \\ x_4 = b \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} x_1 = 2a + b \\ x_2 = a \in \mathbb{R} \\ x_3 = -2b \\ x_4 = b \in \mathbb{R} \end{cases}$$

nullA = $\{X=(2a+b,a,-2b,b) | a,b \in \mathbb{R}\}$

$=\text{span}\{(2,1,0,0),(1,0,-2,1)\}$

Because E = {(2,1,0,0),(1,0,-2,1)} is linear independent. So, E = E = {(2,1,0,0),(1,0,-2,1)} is a basis of nullA,dim(nullA)=2.

Example 3:

Find bases for the row and column spaces of A and determine the rank of A

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & 0 & 5 \\ -2 & -3 & 3 & -4 \\ 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

3.5. RowA, colA and nullA subspaces

Solution:

$$A=\begin{bmatrix}1&1&-1&2\\3&2&0&5\\-2&-3&3&-4\\1&1&-1&3\\0&1&-1&2\end{bmatrix}\to\begin{bmatrix}1&1&-1&2\\0&-1&3&-1\\0&-1&1&0\\0&0&0&1\\0&1&-1&2\end{bmatrix}$$

$$\to\begin{bmatrix}1&1&-1&2\\0&-1&3&-1\\0&0&-2&1\\0&0&0&1\\0&0&-3&3\end{bmatrix}\to\begin{bmatrix}1&1&-1&2\\0&-1&3&-1\\0&0&-2&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

A basis of **rowA** is $\{r_1,r_2,r_3,r_4\}$ and **dim(rowA)=4**

A basis of **colA** is $\{c_1,c_2,c_3,c_4\}$ and **dim(colA)=4**

Theorem

- An nxn matrix A is invertible if and only if rankA=n
- If an mxn matrix B has rank n then the n columns of B is linearly independent
- If A is mxn matrix and m>n then the set of m rows of A is not independent

Theorem

If an mx**n** matrix A has **rank r** then

- The equation AX=0 has **n**-**r** basic solutions $X_1, X_2, \ldots, X_{n-r}$
- $\{X_1, X_2, \ldots, X_{n-r}\}$ is a **basis** of **nullA**
- Dim **nullA**=**n**-**r**
- **imA = colA** and
- Dim **imA**=dim **colA**=rankA=r

Example:

Compute A^2, A^3, and A^{2009} if $A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} (-3)^2 & 0 \\ 0 & 2^2 \end{bmatrix}, A^3 = \begin{bmatrix} (-3)^3 & 0 \\ 0 & 2^3 \end{bmatrix}$$

$$A^{2009} = \begin{bmatrix} (-3)^{2009} & 0 \\ 0 & 2^{2009} \end{bmatrix}$$

3.6. Diagonalization and Eigenvalues

Definition:

An nxn matrix is called diagonal matrix if all its entries off the main diagonal are zeros

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = diag(\lambda_1, \lambda_2, \dots, \lambda_n) \Rightarrow D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

Example:

$$diag(3, -2, 1, 4) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

3.6. Diagonalization and Eigenvalues

Theorem:

If A is an nxn matrix, P is an invertible nxn matrix and D is a diagonal matrix then

$$A = PDP^{-1} \Rightarrow A^k = P.D^k.P^{-1}$$

Prove:

$$A = PDP^{-1} \Rightarrow A^k = \left(PDP^{-1}\right)^k = \left(PDP^{-1}\right)\left(PDP^{-1}\right)\left(PDP^{-1}\right)\ldots\left(PDP^{-1}\right)$$

$$= PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1} = PD^kP^{-1}$$

3.6. Diagonalization and Eigenvalues

Diagonalization

Diagonalizing a matrix A is to find an invertible matrix P such that $D = P^{-1}AP$ is a diagonal matrix $P^{-1}AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Then, we have $P^{-1}A^kP = \left(P^{-1}AP\right)^k = diag\left(\lambda_1^{\,k}, \lambda_2^{\,k}, \ldots, \lambda_n^{\,k}\right)$

$$\Rightarrow A^k = P \cdot diag\left(\lambda_1^{\,k}, \lambda_2^{\,k}, \ldots, \lambda_n^{\,k}\right) \cdot P^{-1}$$

$$P = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ -1 & -4 \end{bmatrix}$$

Easy to compute A^k

Example: Find $A^{2009} = \left(\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\right)^{2009}$?

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \Leftrightarrow \left(P^{-1}AP\right)^{2009} = \left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\right)^{2009}$$

$$\Leftrightarrow P^{-1}A^{2009}P = \begin{bmatrix} 3^{2009} & 0 \\ 0 & (-2)^{2009} \end{bmatrix} \Rightarrow \boxed{A^{2009} = P\begin{bmatrix} 3^{2009} & 0 \\ 0 & (-2)^{2009} \end{bmatrix}P^{-1}}$$

3.6. Diagonalization and Eigenvalues

Example 1:

Find the eigenvalues ang eigenvectors and then

diagonalize the matrix $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

3.6. Diagonalization and Eigenvalues

Solution:

- Find $c_A(\lambda)=0 \Leftrightarrow \det(\lambda I - A) = 0$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & 4 \\ 1 & \lambda + 1 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 4 \\ 1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 1) - 4 = \lambda^2 - \lambda - 6$$

$c_A(\lambda) = \lambda^2 - \lambda - 6$ Đa thức đặc trưng (characteristic polynomial)

- $c_A(\lambda) = 0 \Leftrightarrow \lambda = 3 \vee \lambda = -2$ Các giá trị riêng (eigenvalues) của A

3.6. Diagonalization and Eigenvalues

Solution:

- λ=3: solve the system $(3I - A)X = 0$

$$\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x+4y \\ x+4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x+4y=0 \\ x+4y=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = t \\ x = -4t \end{cases} \Leftrightarrow X = \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Nếu t≠0 thì X=(-4t,t) được gọi là véc tơ riêng (eigenvectors) ứng với giá trị riêng $\lambda = 3$

Solution:

- λ=-2: solve the system $(-2I-A)X=0 \Leftrightarrow \begin{cases} -4x+4y=0 \\ x-y=0 \end{cases}$

$$\Leftrightarrow \begin{cases} y=t \\ x=t \end{cases} \Leftrightarrow X=\begin{bmatrix} x \\ y \end{bmatrix}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Nếu t≠0 thì X=(t,t) được gọi là véc tơ riêng ứng với trị riêng $\lambda=-2$

3.6. Diagonalization and Eigenvalues

Solution:

- $\boxed{\lambda=3}$ solve the system $(3I-A)X=0 \Leftrightarrow \begin{cases} x+4y=0 \\ x+4y=0 \end{cases}$

$\Leftrightarrow \begin{cases} y=t \\ x=-4t \end{cases} \Leftrightarrow X=\begin{bmatrix} x \\ y \end{bmatrix}=t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Nếu t≠0 thì X=(-4t,t) được gọi là véc tơ riêng (eigenvectors) ứng với giá trị riêng

- $\boxed{\lambda=-2}$: solve the system $(-2I-A)X=0 \Leftrightarrow \begin{cases} -4x+4y=0 \\ x-y=0 \end{cases}$

$\Leftrightarrow \begin{cases} y=t \\ x=t \end{cases} \Leftrightarrow X=\begin{bmatrix} x \\ y \end{bmatrix}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Nếu t≠0 thì X=(t,t) được gọi là véc tơ riêng ứng với

Choose P= $\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1}AP=\begin{bmatrix} \boxed{3} & 0 \\ 0 & \boxed{-2} \end{bmatrix}$

Relationship between eigenvalues and eigenvectors
: eigenvalue (a number)
X: -eigenvector (remember: vector X≠0)
(I-A)X=0 AX=X

$P=\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, P=\begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix}, P=\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}, P=\begin{bmatrix} -1 & -4 \\ -1 & 1 \end{bmatrix}$,...are allowed. In case P= $\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}, P^{-1}AP=\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

Example 2:

Find the eigenvalues ang eigenvectors and then *diagonalize* the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

3.6. Diagonalization and Eigenvalues

Example 2: Find the eigenvalues ang eigenvectors and then *diagonalize* the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Solution:

The characteristic polynomial of A is

$$c_A(\lambda) = \begin{vmatrix} \lambda - 1 & -1 \\ -2 & \lambda - 2 \end{vmatrix} = (\lambda-1)(\lambda-2) - 2 = \lambda(\lambda - 3) = 0 \Leftrightarrow \lambda = 0 \vee \lambda = 3$$

$\lambda = 0,3$ are eigenvalues

$\lambda = 0$: Solve the system $(0I-A)X=0$

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow X = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 3$: Solve the system $(3I-A)X=0$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow X = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

3.6. Diagonalization and Eigenvalues

When is A diagonalizable ?

Theorem

A is diagonalizable iff every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors, that is, if the general solution of the system $(\lambda I-A)X=0$ has exactly m parameters.

3.6. Diagonalization and Eigenvalues

Example 1: $A=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}$ is not diagonalizable.

In fact, $c_A(\lambda)=\det(\lambda I-A)=\begin{vmatrix}\lambda & -1\\ 1 & \lambda-2\end{vmatrix}=\lambda(\lambda-2)+1$

$$=\lambda^2-2\lambda+1=(\lambda-1)^2=0\Leftrightarrow\lambda=1$$

$\lambda=1$ (multiplicity 2):

Solve the system $(1I-A)X=0$ $\square$ $\left[\begin{array}{cc|c}1 & -1 & 0\\ 1 & -1 & 0\end{array}\right]\to\left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$

$X=t\begin{bmatrix}1\\ 1\end{bmatrix}\Rightarrow$ one parameter $\Rightarrow$ not diagonalizable

Example 2:

$$A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \text{ is not diagonalizable.}$$

3.6. Diagonalization and Eigenvalues

Solution:

In fact, $c_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -2 & 0 & \lambda \end{vmatrix}$

$$= \lambda^3 - 3\lambda - 2 = (\lambda+1)^2(\lambda-2) = 0 \Leftrightarrow \left[\begin{array}{l} \boxed{\lambda = -1 \text{ (multiplicity 2)}} \\ \lambda = 2 \end{array}\right.$$

$\lambda = -1$ (multiplicity 2): solve the system $(-1I - A)X = 0$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow n - r = 3 - 2 = 1 \Rightarrow$ one parameter $\Rightarrow$ not diagonalizable

Example 3:

Determine whether $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ is diagonalizable.

3.6. Diagonalization and Eigenvalues

Exercise 1: In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

a/ $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

b/ $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

c/ $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

d/ $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$

3.6. Diagonalization and Eigenvalues

Solution:

Step 1:

$$A=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}; I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \lambda.I=\begin{bmatrix}x & 0\\ 0 & x\end{bmatrix}; \lambda I-A=\begin{bmatrix}\lambda-1 & -2\\ -3 & \lambda-2\end{bmatrix}$$

Characteristic polynomial of A:

$$C_A(\lambda)=\det(\lambda.I-A)=\begin{vmatrix}\lambda-1 & -2\\ -3 & \lambda-2\end{vmatrix}$$

$$=(\lambda-1)(\lambda-2)-6=\lambda^2-3\lambda-4$$

$$=\lambda^2-3\lambda-4$$

Solution:

Step 2:

the eigenvalues of A:

$$\lambda^2 - 3\lambda - 4 = 0 \Leftrightarrow \left[\begin{array}{l}\lambda_1 = 4 \\ \lambda_2 = -1\end{array}\right.$$

$\Rightarrow$ the eigenvalues of A is: 4 and -1.

3.6. Diagonalization and Eigenvalues

Solution:

Step 3:

The eigenvectors of A:

$$*\text{With } \lambda_1 = 4 : (4I - A).X = 0$$

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 = 2t \\ x_2 = 3t \end{cases} \Rightarrow X = t.\begin{bmatrix} 2 \\ 3 \end{bmatrix}; t \in R.$$

$$*\text{With } \lambda_2 = -1 : (-I - A).X = 0$$

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -3 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 = -t \\ x_2 = t \end{cases} \Rightarrow X = t.\begin{bmatrix} -1 \\ 1 \end{bmatrix}; t \in R$$

3.6. Diagonalization and Eigenvalues

Solution:

Step 4:

$$P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}; P^{-1} = \frac{1}{5}.\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

$$P^{-1}.A.P = D?$$

$$P^{-1}.A = \frac{1}{5}.\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}.\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \frac{1}{5}.\begin{bmatrix} 4 & 4 \\ 3 & -2 \end{bmatrix}$$

$$P^{-1}.A.P = \frac{1}{5}.\begin{bmatrix} 4 & 4 \\ 3 & -2 \end{bmatrix}.\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \frac{1}{5}.\begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = D$$

$$A^k = P.D^k.P^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}.\begin{bmatrix} 4^k & 0 \\ 0 & (-1)^k \end{bmatrix}.\frac{1}{5}.\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \ldots.$$

3.6. Diagonalization and Eigenvalues

Exercise 2: In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

a/ $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$ b/ $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

c/ $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$ d/ $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

An Application to Linear Recurrences

Linear recurrence: $x_{k+2}=ax_k+bx_{k+1}$

How to find a formula of x_k?

For example, find x_k if $x_0=1$, $x_1=1$, and $x_{k+2}=10x_k-3x_{k+1}$

- Let $V_k=\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} \Rightarrow V_{k+1}=\begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}, V_0=\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$.

We have $x_{k+1} = 0x_k + 1x_{k+1}$

And, $x_{k+2} = 10x_k - 3x_{k+1}$

$\Rightarrow V_{k+1} = AV_k$, where $A=\begin{bmatrix} 0 & 1 \\ 10 & -3 \end{bmatrix} \Rightarrow V_{k+1} = AV_k = A^2V_{k-1} = \ldots = A^{k+1}V_0$

- Find A^{k+1}
- Diagonalize matrix A and then compute A^{k+1}
- Compute V_{k+1} and then find x_k

3.6. Diagonalization and Eigenvalues

Example:

Slove the following linear recurrences

$$x_{k+2} = 2x_k - x_{k+1}, \text{ where } x_0 = 1 \text{ and } x_1 = 2$$

a/ $x_k = \frac{7}{3}\left[14 - (-1)^k\right]$

b/ $x_k = \frac{2}{3}\left[3 - (-5)^k\right]$

c/ $x_k = \frac{1}{3}\left[4 - (-2)^k\right]$

d/ $x_k = \frac{5}{3}\left[7 - (-2)^k\right]$

3.6. Diagonalization and Eigenvalues

Question 1:

Let A be an 3x3 matrix has three eigenvalues 1, -2, 3. Find its determinant.

A. not enough data to find.

B. 2

C. -6

D. 0

3.6. Diagonalization and Eigenvalues

Question 2:

Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}$. Choose the correct statements:

a/ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. b/ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A.

c/ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A. d/ $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector of A.

Question 3:

Find the eigenvalues of $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

a/ 1,2,3

b/ -1,0,1

c/ 2,3

d/ 1,3

e/ -1,0

f/ 1,2

3.6. Diagonalization and Eigenvalues

Question 4:

If $\lambda = 0$ is an eigenvalue of A, then (select the correct answers)

a/ $A = 0$

b/ $\det(A) = 0$

c/ A is invertible

d/ A is not invertible.

e/ $A^2 = 0$

3.6. Diagonalization and Eigenvalues

Question 5:

If all the eigenvalues of A are nonzero, then (select the correct answers)

a/ $A = I$

b/ $A^2 = I$

c/ A is invertible

d/ A is not invertible.

e/ $\det(A) \neq 0$

3.6. Diagonalization and Eigenvalues

Question 6:

If a 2×2, invertible matrix A has eigenvalues 2 and 5, answer the following True or False:

a/ A is invertible

b/ A is diagonalizable.

c/ A is symmetric

d/ $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

e/ $A^2 = 10I$

f/ $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ for some invertible P