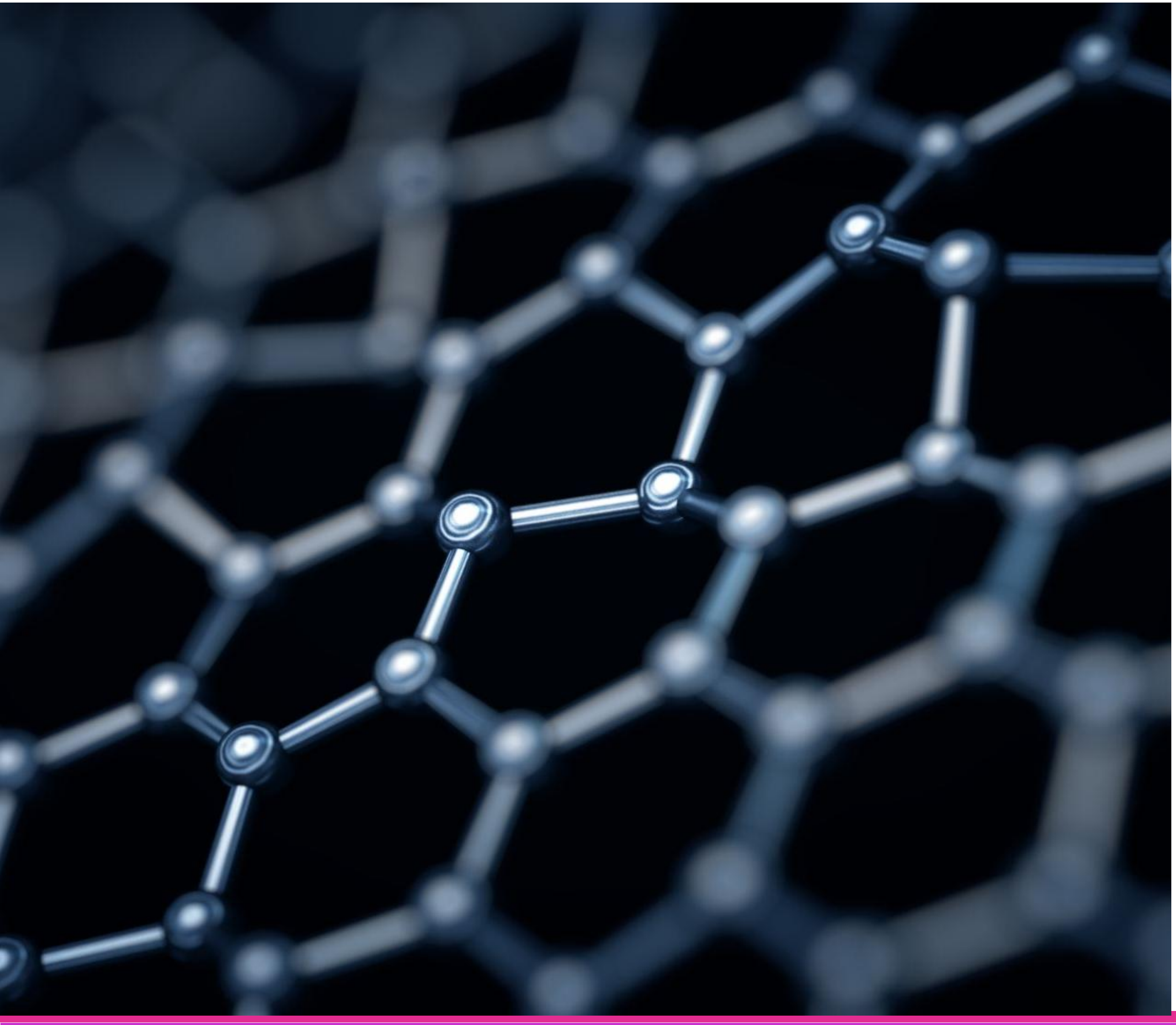


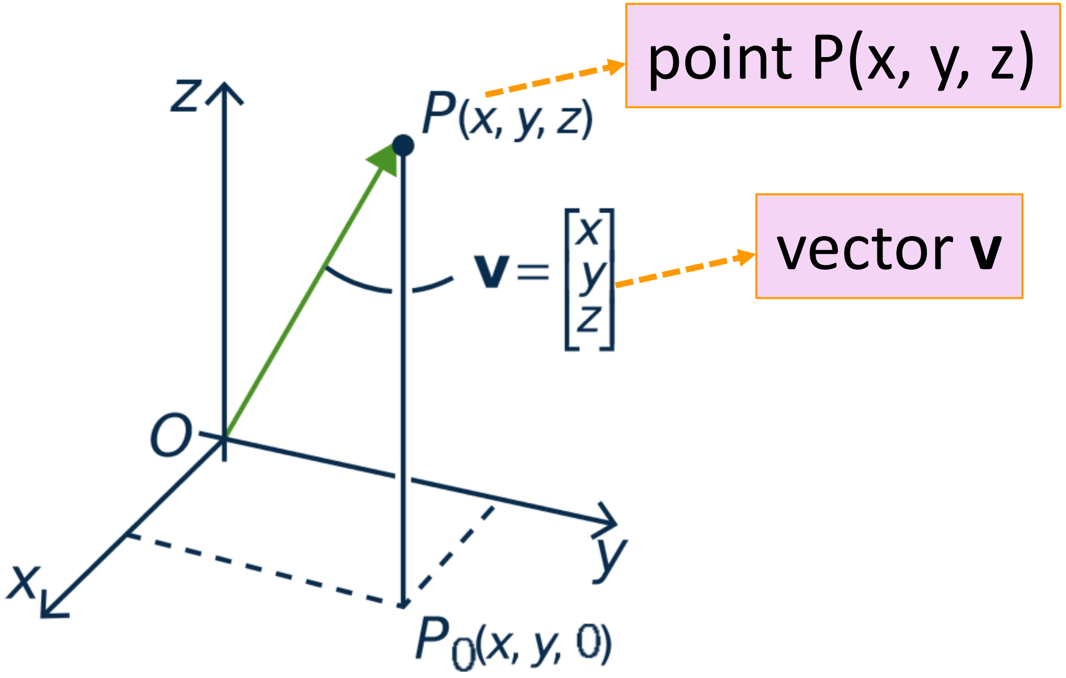
Chapter 4
Vector geometry

Chapter Outline

4.1. Vectors

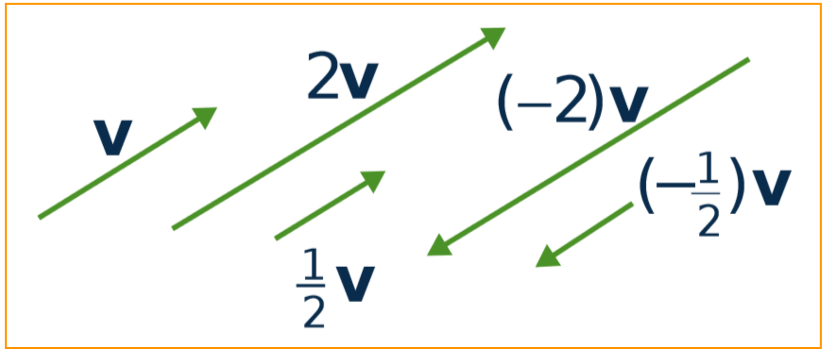
Vectors in $\mathbb{R}^3$

The terms ***vector*** and ***point*** are interchangeable.



4.1. Vectors

Scalar Multiple Law



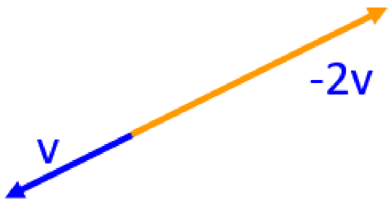
If k is a real number and $v \neq 0$ is a vector then:

1. The lenghth of kv is $\|kv\| = |k|\|v\|$

2. If $kv \neq 0$, the direction of kv is $\begin{cases} \textit{the same as } v \textit{ if } k > 0 \\ \textit{opposite to } v \textit{ if } k < 0 \end{cases}$

Example: True or false ?

a/ If $||v|| = 3$, then $||-2v|| = -6$.



b/ If $||v|| = 0$, then v is zero vector.

Answer: a/ False $||-2v|| = 6$. **b/ True**

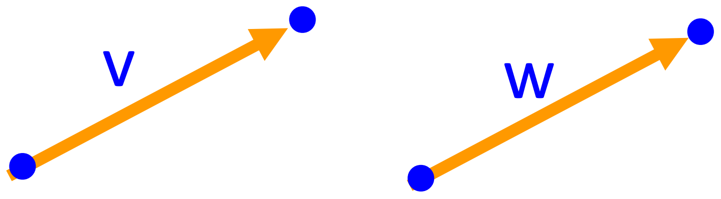
4.1. Vectors

Equality

Let $v \neq 0$ and $w \neq 0$ be vectors in R^3.

Then $\mathbf{v} = \mathbf{w}$ if and only if v and w have the same direction and the same length.

Note: The same geometric vector can be positioned anywhere in space.

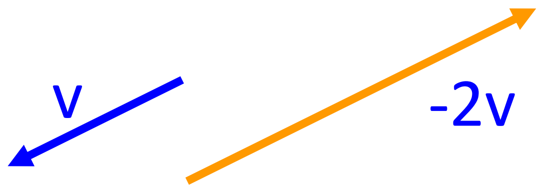


4.1. Vectors

Parallel vectors

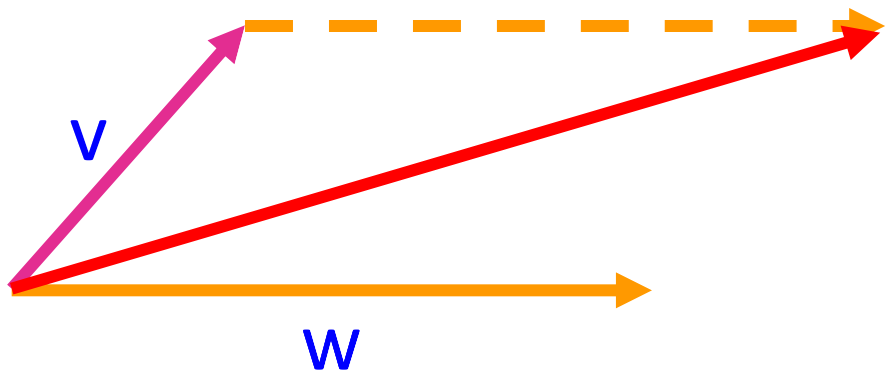
Two nonzero vectors are called ***parallel*** if they have the same or opposite direction.

v and w are parallel $\Leftrightarrow$ $v = kw$ for some scalar k.



4.1. Vectors

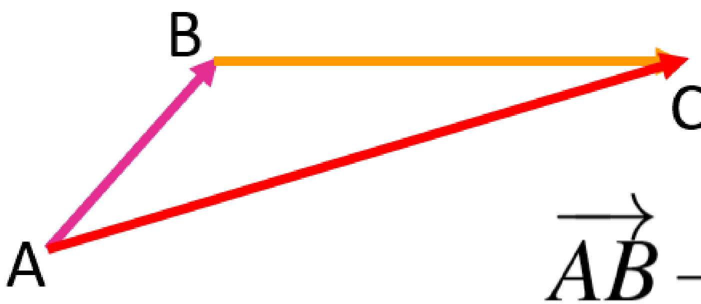
The Parallelogram Law



v + w ← first v then w

$$\boxed{v + w = w + v}$$

Tip-to-tail rule



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

4.2. Dot product

Definition:

If $X = [x_1 \ \ x_2 \dots x_m]^T$, $Y = [y_1 \ \ y_2 \dots y_m]^T$

We define $X \bullet Y = x_1y_1 + x_2y_2 + \dots + x_my_m$

4.2. Dot product

Example:

$$\text{If } v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \text{ and } w = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \text{ evaluate } v \cdot w$$

Solution:

$$v \cdot w = 2.1 + (-1).4 + 3.(-1) = -5$$

4.2. Dot product

Theorem:

Let X, Y and Z denote vectors in R^n. Then:

1/ $X \cdot Y = Y \cdot X$

2/ $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$

3/ $(kX) \cdot Y = kX \cdot Y = X \cdot (kY)$ for all scalar k

4/ $X \cdot X = \|X\|^2$

5/ $\|X\| \geq 0$ for all vector X. $\|X\| = 0 \Leftrightarrow X = 0$

6/ $\|kX\| = |k| \|X\|$ for all k

4.2. Dot product

Length and distance

a/ The length of a vector:

If $X=[x_1\ x_2\ \dots\ x_m]^T$ then $\|X\| = \sqrt{X \bullet X} = \sqrt{x_1{}^2 + x_2{}^2 + \dots + x_m{}^2}$

A vector with length 1 is called a unit vector.

b/ Distance between two points: Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then:

1. $\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$
2. The distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

c/ Distance between X and Y defined by $d(X,Y) = \|X - Y\|$

4.2. Dot product

Example 1:

Find $\|v\|$, if $v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Solution:

$$\|v\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

4.2. Dot product

Example 2:

Find $(v + w)\bullet(v - 2w)$ if $\|v\| = 3$, $\|w\| = 2$, and $v\bullet w = -1$.

Solution:

$$\begin{aligned}(v + w)\bullet(v - 2w) &= v\bullet v - 2v\bullet w + v\bullet w - 2w\bullet w\\ &= \|v\|^2 - v\bullet w - 2\|w\|^2\\ &= 3^2 - (-1) - 2.2^2 = 2\end{aligned}$$

4.2. Dot product

Example 3:

If $\|x\| = 3, \|y\| = 1$ and $x \cdot y = -2$. Compute:

a/ $\|3x - 5y\|$

b/ $\|2x + 7y\|$

c/ $(3x - y) \cdot (2y - x)$

d/ $(x - 2y) \cdot (3x + 5y)$

4.2. Dot product

Solution:

a/ $\|3x-5y\|$

Set $z=3x-5y$. We have $\|3x-5y\|=\|z\|=\sqrt{z\cdot z}$

$$z\cdot z=(3x-5y)\cdot(3x-5y)=9x\cdot x-15x\cdot y-15y\cdot x+25y\cdot y$$

$$=9\|x\|^2-30x\cdot y+25\|y\|^2=166$$

So, $\|3x-5y\|=\sqrt{166}$

4.2. Dot product

Solution:

b/ $\|2x + 7y\| = \sqrt{29}$

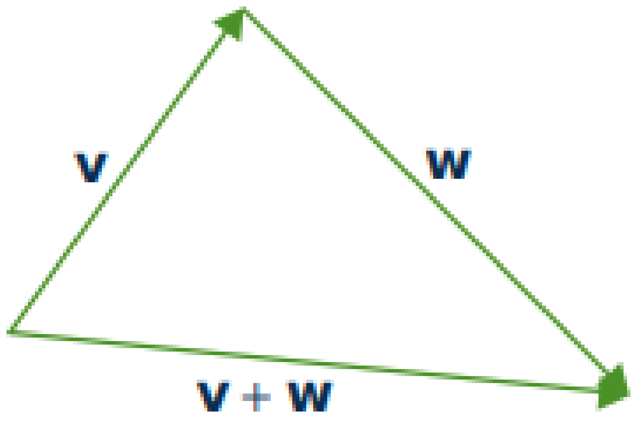
c/ $(3x - y)\cdot(2y - x) = -43$

d/ $(x - 2y)\cdot(3x + 5y) = 19$

4.2. Dot product

Theorem 1 (Corollary triangle Inequality)

If u and v ares vectors in $\mathbb{R}^n$, then $\|u+v\| \leq \|u\| + \|v\|$



Theorem 2 (Cauchy Inequality)

$$u \cdot v \leq \|u\|\|v\|$$

4.2. Dot product

Example:

Suppose u,v are vectors in R^n with $\|u\| = 3; \|v\| = 4$. What exactly can we say about $\|u+v\|$?	
a	$\|u+v\|$ is between 0 and 7
b	$\|u+v\|$ is between 1 and 7
c	$\|u+v\|$ is equal to 7
d	$\|u+v\|$ is equal to 5
e	$\|u+v\|$ is less than or equal to 5

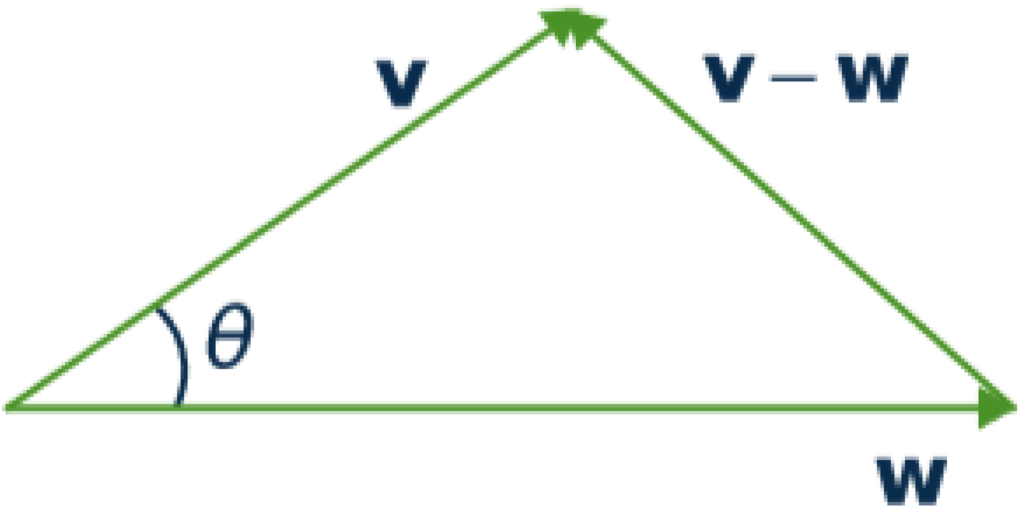
4.2. Dot product

Angles between vectors

Theorem 4.2.2

Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$



4.2. Dot product

Example 1:

Compute the angle between $u = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

4.2. Dot product

Example 2:

Find all real numbers x such that $u = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$

4.3. Orthogonality

Definition 1:

Two vectors v and w are said to be ***orthogonal*** if $v \bullet w = 0$.

Example 1:

Find all real numbers x such that $u = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

4.3. Orthogonality

Example 2:

Show that the points $P(3,-1,1), Q(4,1,4)$, and $R(6,0,4)$ are the vertices of a right triangle.

4.3. Orthogonality

Solution:

The vectors along the sides of the triangle are

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \overrightarrow{PR} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \text{ and } \overrightarrow{QR} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Evidently $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors.

This means sides PQ and QR are perpendicular, that is, triangle at Q is a right angle.

4.3. Orthogonality

Definition 2:

A set $\{x_1, x_2, \ldots, x_m\}$ is called **orthogonal set** if x_i is not zero vector and $x_i \bullet x_j = 0$ for all $i \neq j$.

Example:

a/ $S = \{(1,-1);(1,1)\}$ is an orthogonal set in R^2

b/ $E = \{(1,1,1);(-1,0,1);(0,1,0)\}$ is not orthogonal set but $\{(-1,0,1);(0,1,0)\}$ is a orthogonal set

4.3. Orthogonality

Definition 3:

A orthogonal set $\{x_1, x_2, \ldots, x_m\}$ is called orthonormal set (hệ trực chuẩn) is x_i is

unit vector for all i.

$$\begin{cases} x_i \cdot x_j = 0 & \forall i \neq j \\ \|x_i\| = 1, & \forall i \end{cases}$$

4.3. Orthogonality

Example:

a/ S = {(1,0,0);(0,1,0)} is orthonormal

b/ B = {(-3,0,4);(4,5,3)} is a orthogonal set, not a orthonormal set. However, the set

$$\left\{\frac{1}{5}(-3,0,4);\frac{1}{5\sqrt{2}}(4,5,3)\right\}$$

is orthonormal.

Orthonormal Set (hệ trực chuẩn)

Definition:

A set $E = \{x_1, x_2, \ldots, x_m\}$ is called orthonormal basis (cơ sở trực chuẩn) if E is basis and orthonormal set.

4.3. Orthogonality

Theorem:

- The standard basis of R^n $\{E_1,E_2,\ldots,E_n\}$ is orthonormal.
- If $\{F_1,F_2,\ldots,F_k\}$ is orthogonal then $\{a_1F_1,a_2F_2,\ldots,a_kF_k\}$ is also orthogonal for any nonzero scalar a_i
- Every orthogonal set is a linearly independent set.

4.3. Orthogonality

Pythagoras's Theorem

If $\{F_1, F_2, \ldots, F_k\}$ is orthogonal then

$$\|F_1 + F_2 + \ldots + F_k\|^2 = \|F_1\|^2 + \|F_2\|^2 + \ldots + \|F_k\|^2$$

Expansion Theorem

Let $\{F_1, F_2, \ldots, F_k\}$ be a orthogonal basis of a subspace U and X is in U. Then

$$X = \frac{X \bullet F_1}{\|F_1\|^2} F_1 + \frac{X \bullet F_2}{\|F_2\|^2} F_2 + \ldots + \frac{X \bullet F_k}{\|F_k\|^2} F_k$$

4.3. Orthogonality

Example:

In each case, write x as a linear combination of the orthogonal basis of the subspace U

a/ $x=(13,-20,15), U=span\{(1,-2,3),(-1,1,1)\}$

b/ $x=(14,1,-8,5), U=span\{(2,-1,0,3),(2,1,-2,-1)\}$

4.3. Orthogonality

Solution:

a/ $x=(13,-20,15), U=span\{(1,-2,3),(-1,1,1)\}$

Set $F_1=(1,-2,3), F_2=(-1,1,1)$

$$x \cdot F_1=(13,-20,15)\cdot(1,-2,3)=98$$

$$and\ \|F_1\|=\|(1,-2,3)\|=\sqrt{1+4+9}=\sqrt{14}$$

$$x \cdot F_2=(13,-20,15)\cdot(-1,1,1)=-18$$

$$and\ \|F_2\|=\|(-1,1,1)\|=\sqrt{1+1+1}=\sqrt{3}$$

So, $x=\dfrac{x\bullet F_1}{\|F_1\|^2}F_1+\dfrac{x\bullet F_2}{\|F_2\|^2}F_2=7F_1-6F_2$

4.3. Orthogonality

Solution:

b/ $x=(14,1,-8,5), U=span\{(2,-1,0,3),(2,1,-2,-1)\}$

Answer: $x=3F_1+4F_2$

4.4. Lines

Lines in Space:

Given the point $P_0(x_0, y_0, z_0)$ and the *direction vector* $d = (a, b, c) \neq 0$

Then *line* parallel to d through the point P_0 is given by:

$$P = P_0 + td, \text{ (t is any number)} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ or } \begin{cases} x = x_0 + at \\ y = y_0 + bt, \; t \text{ any scalar} \\ z = z_0 + ct \end{cases}$$

In other words, the point $P(x, y, z)$ is on this line if and only if a real number t exists such that $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$

4.4. Lines

Example:

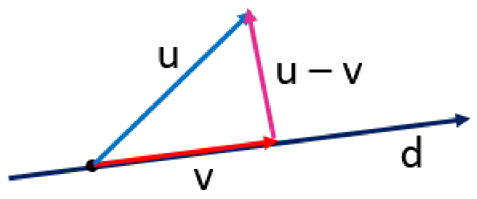
In each case, find a vector equation of the line,

a/ Passing through $P(3,-1,4)$ and perpendicular to the plane $3x - 2y - z = 0$.

b/ Passing through $P(2,-1,3)$ and perpendicular to the plane $2x + y = 1$.

4.4. Lines

Projection



1. v // d
2. u − v ⊥ d

Theorem

Let u and d be vectors

1. The projection of u on d is given by $proj_d u = \dfrac{u \cdot d}{\|d\|^2} d$

2. The vector $u - proj_d u$ is orthogonal to d

4.4. Lines

Example 1:

Find the projection of $u = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $d = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $u = u_1 + u_2$ where

u_1 is parallel to d and u_2 is orthogonal to d.

4.4. Lines

Solution:

The projection u_1 of u on d is $u_1 = proj_d u = \dfrac{u \cdot d}{\|d\|^2} d = \dfrac{8}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

Hence $u_2 = u - u_1 = \dfrac{1}{11}\begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$

4.4. Lines

Example 2:

In each case, compute the projection of u on v

$$a/\ u=\begin{bmatrix}5\\7\\1\end{bmatrix}, v=\begin{bmatrix}2\\-1\\3\end{bmatrix} \qquad b/\ u=\begin{bmatrix}3\\-2\\1\end{bmatrix}, v=\begin{bmatrix}4\\1\\1\end{bmatrix} \qquad c/\ u=\begin{bmatrix}1\\-1\\2\end{bmatrix}, v=\begin{bmatrix}3\\-1\\1\end{bmatrix}$$

4.4. Lines

Example 3:

In each case, write $u = u_1 + u_2$, where u_1 is parallel to v and u_2 is orthogonal to v

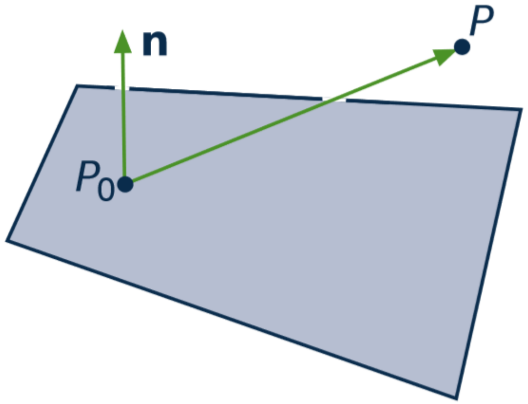
a/ $u = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

b/ $u = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

4.5. Planes

Definition 4.7

A nonzero vector **n** *is called a* **normal** *for a plane if it is orthogonal to every vector in the plane.*



$$\boxed{\mathbf{n} \cdot \overrightarrow{P_0P} = 0}$$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

4.5. Planes

Example 1:

Find an equation of the plane through $P_0(1,-1,3)$ with $n = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as normal.

Solution:

Equation of the plane is $3x - y + 2z = 10$

4.5. Planes

Example 2:

In each case, find all points of intersection of the given plane and line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

a/ $x - 3y + 2z = 4$

b/ $2x - y - z = 5$

c/ $3x - y + z = 8$

d/ $-x - 4y - 3z = 6$

4.6. The Cross Product

Definition:

Given vectors $v_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, define the cross product $v_1 \times v_2$ by

$$v_1 \times v_2 = \begin{bmatrix} y_1 z_2 - z_2 y_1 \\ -\left(x_1 z_2 - z_1 x_2\right) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$\left.\begin{aligned} v_1 &= \left(x_1, y_1, z_1\right) \\ v_2 &= \left(x_2, y_2, z_2\right) \end{aligned}\right\} \Rightarrow v_1 \times v_2 = \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$$

4.6. The Cross Product

Theorem 1:

Let v and w be vectors in $\mathbb{R}^3$

1. $v \times w$ is a vector orthogonal to both v and w

$$x = v \times w \Rightarrow \begin{bmatrix} x \perp v \\ x \perp w \end{bmatrix}$$

2. If v and w are nonzero, then $v \times w = 0$ if and only if v and w are parallel.

4.6. The Cross Product

Example 1:

Find an equation of each of the following planes

a/ Passing through $A(2,1,3), B(3,-1,5)$ and $C(1,2,-3)$

b/ Passing through $A(1,-1,6), B(0,0,1)$ and $C(4,7,-11)$

c/ Passing through $P(2,-3,5)$ and parallel to the plane with equation $3x-2y-z=0$

d/ Passing through $P(3,0,-1)$ and parallel to the plane with equation $2x-y+z=3$

Example 2:

Find the equation of the plane through $P(1,3,-2), Q(1,1,5)$, and $R(2,-2,3)$

Solution:

The vectors $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in plane, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$

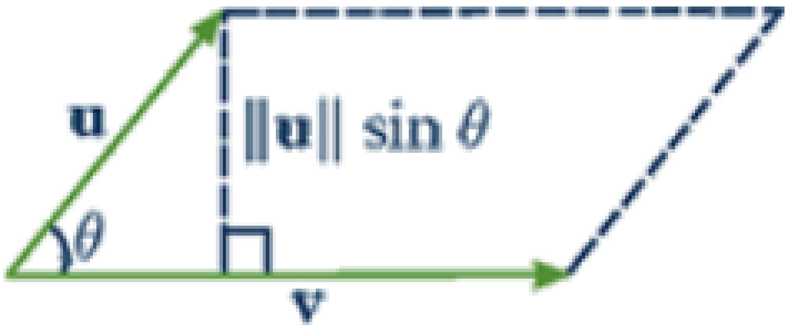
Hence, the plane has equation $25x + 7y + 2z - 42 = 0$

4.6. The Cross Product

Theorem 2:

If u and v are two nonzero vectors and θ is the angle between u and v, then

$$\|u \times v\| = \|u\|.\|v\|\sin\theta = \text{area of the parallelogram determined by } u \text{ and } v$$



Example:

Find the area of the triangle with vertices $P(2,1,0), Q(3,-1,1)$, and $R(1,0,1)$.

Answer: $\dfrac{\sqrt{14}}{2}$

4.6. The Cross Product

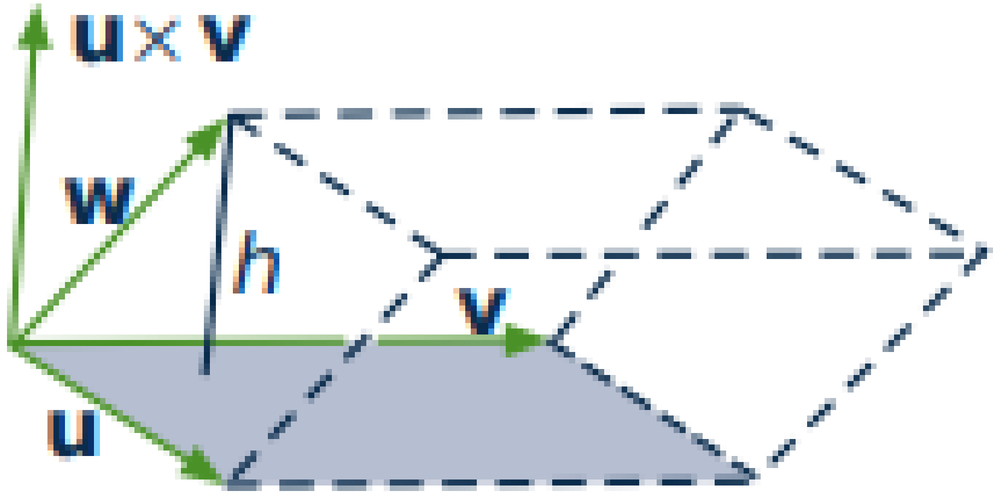
Theorem 3:

If $u = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, v = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, and $w = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, then $u \bullet (v \times w) = \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}$

4.6. The Cross Product

Theorem 4:

The volume of the parallelepiped determined by three vectors $u, v,$ and w is given by $|u \bullet (v \times w)|$



4.6. The Cross Product

Example:

Find the volume of the parallelepiped determined by the vectors

$$w = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Answer: 3