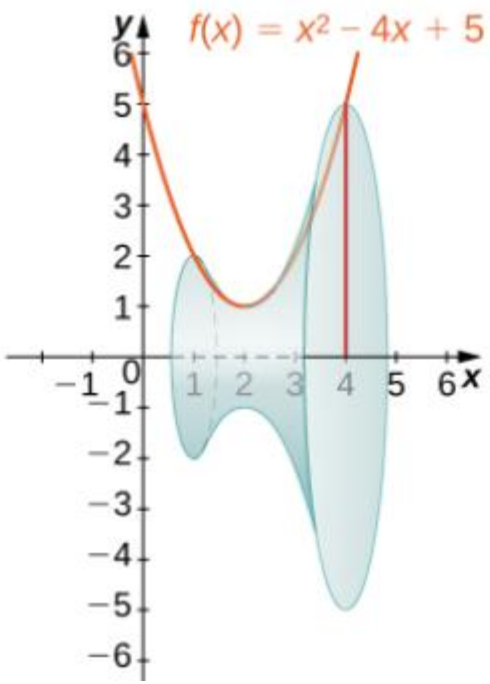
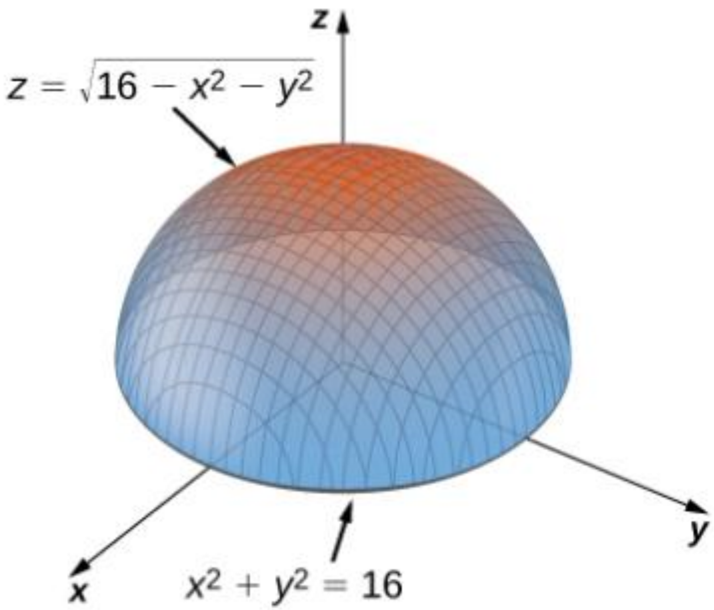
CALCULUS





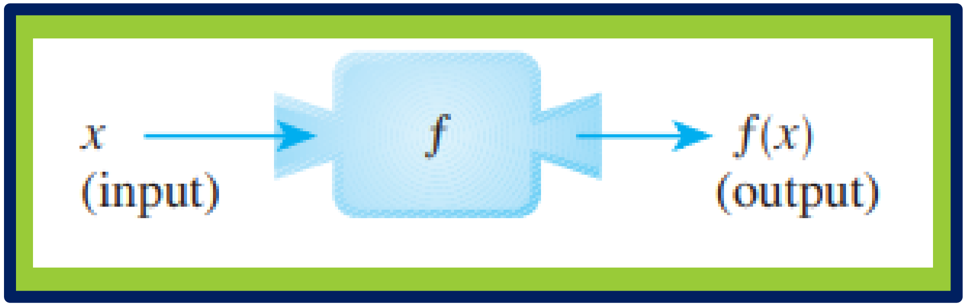
Chapter 1: Function and limit

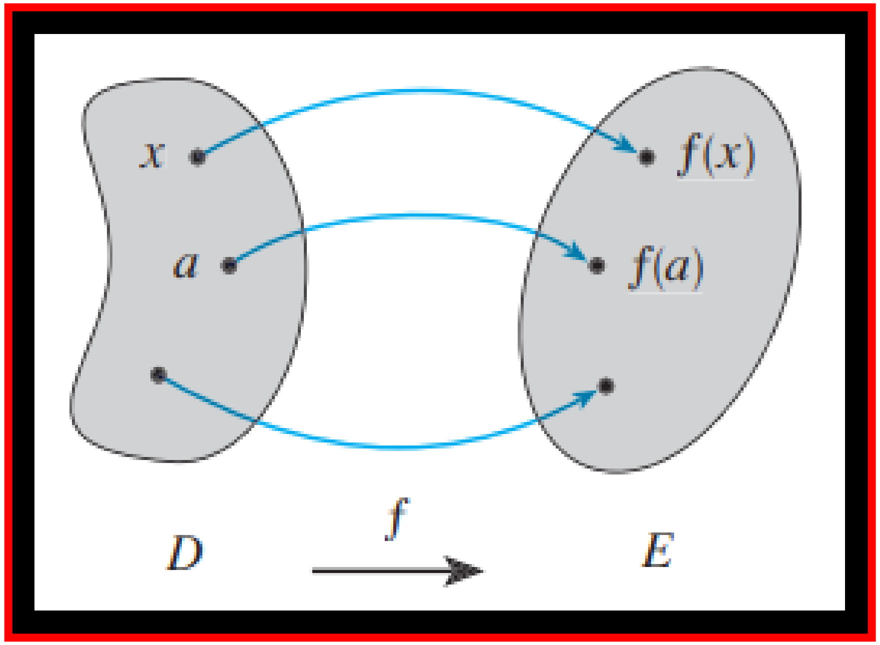
Contents

1.1 Functions and their representations

Function

A **function** f is a rule that assigns to each element x in a set $\boldsymbol{D}$ ***exactly one*** element, called $f(x)$, in a set $\boldsymbol{E}$.





1.1 Functions and their representations

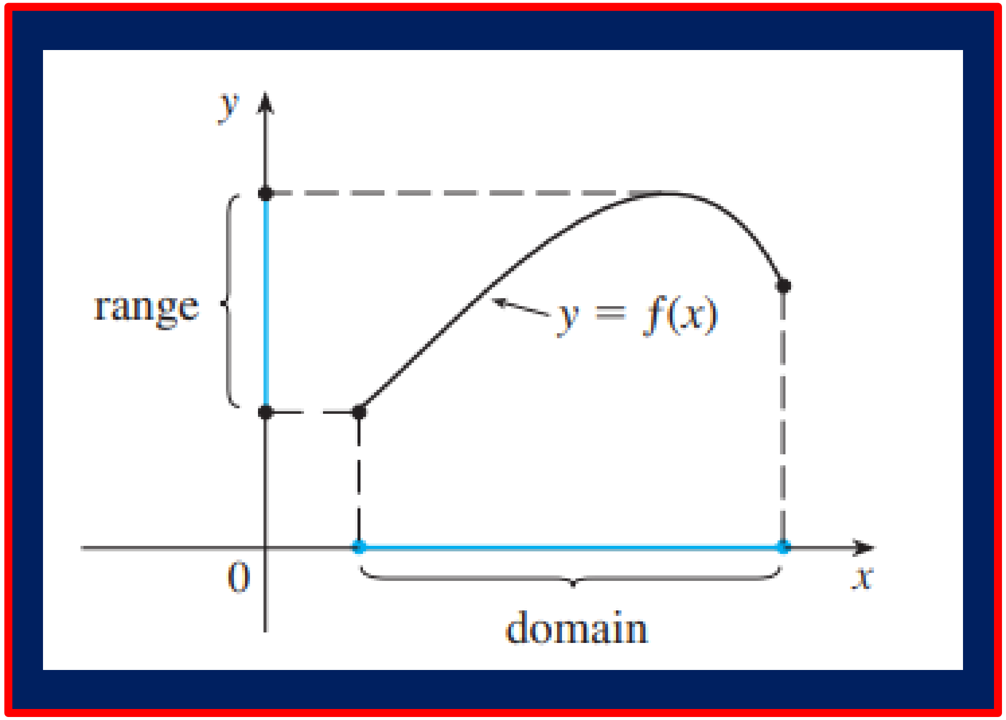
Example:

Input (x)	Out put y=f(x)
Side length x	area of a square x^2
the weight of the package	cost of mailing a package
amount of time the ball is in the air	velocity of a ball thrown in the air
Time of Day	Temperature

1.1 Functions and their representations

Domain and range

- The set D is called the **domain** of the function f.
- The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.



$$-1 \le y = \cos x \le 1$$

$$range\ of\ y\ is\ [-1,1]$$

1.1 Functions and their representations

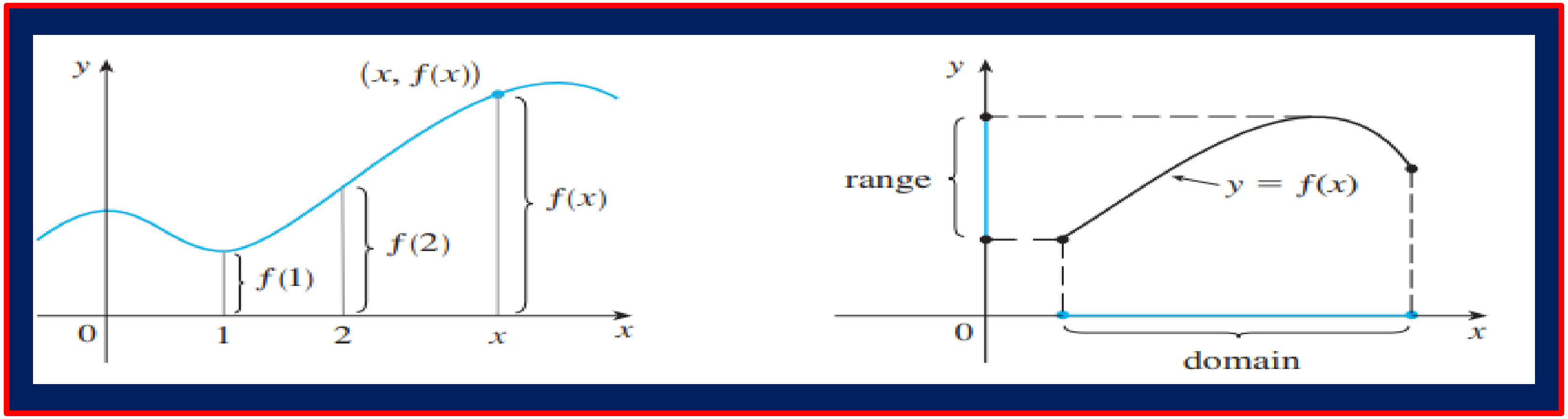
Graph

The **graph** of f is the **set** *of all points (x, y) in the coordinate plane such that y = f(x) and x is in the domain of f.*

The graph of *f* also allows us to picture:

- The **domain of *f*** on the *x*-axis

- Its range on the *y*-axis



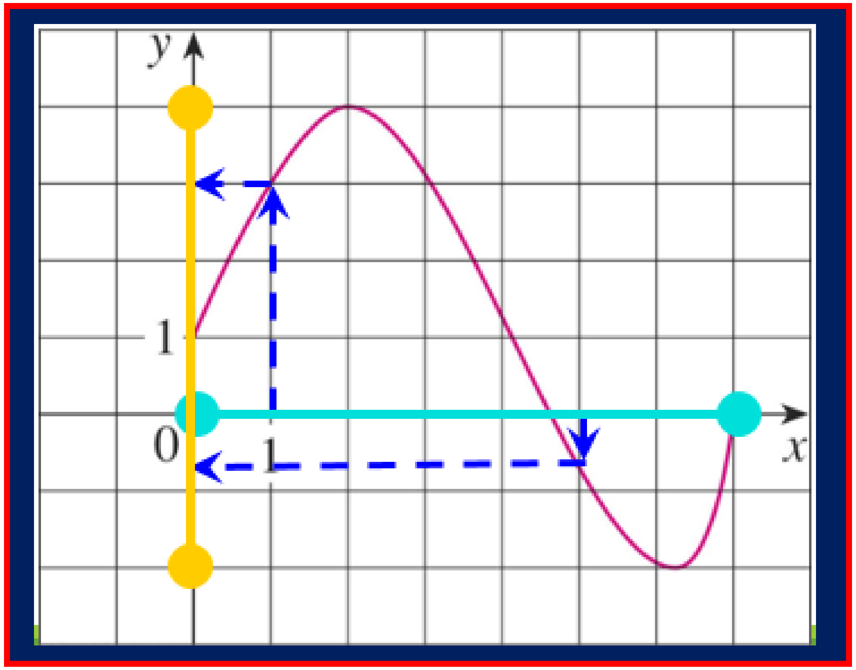
1.1 Functions and their representations

Graph

Example 1: The graph of a function f is shown.

a/ Find the values of $f(1)$ and $f(5)$.

b/ What is the domain and range of f ?



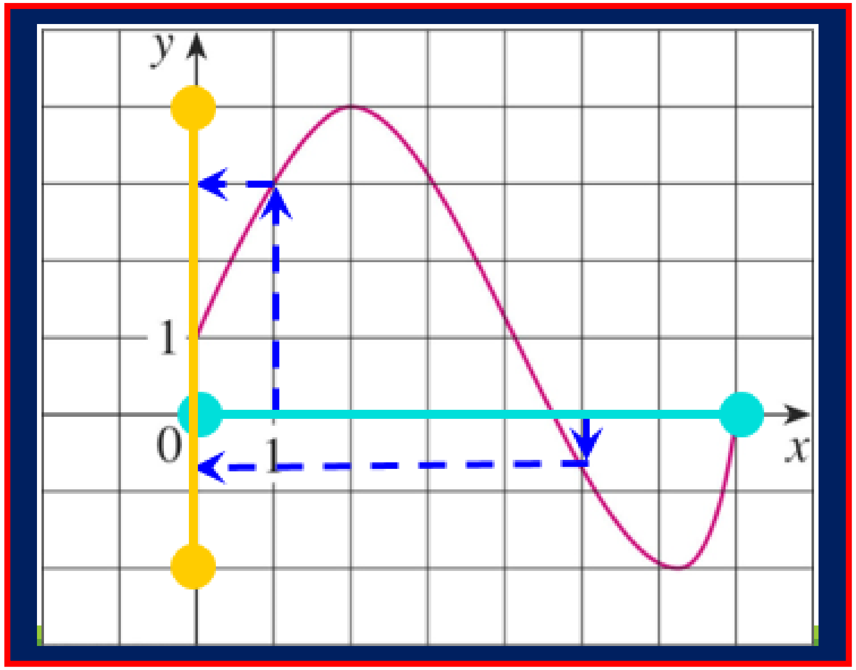
Graph

Solution:

a/ $f(1) = 3$; $f(5) \approx -0.7$

b/ $D = [0, 7]$

$Range(f) = [-2, 4]$



1.1 Functions and their representations

Representations

DISCUSSION

Find the domain and region of the functions (if it is a function).

a/ $f(n)=\sqrt{n}$ for all natural numbers n

b/ $g(x)$ is any real number such that larger than x

1.1 Functions and their representations

Representations of funtions

There are **four possible ways** to represent a function:

- Algebraically (by an explicit **formula**)

- Visually (by a graph)

- Numerically (by a table of values)

- Verbally (by a description in words)

1.1 Functions and their representations

Example:

The human population of the world P depends on the time t.

- The **table** gives estimates of the world population $P(t)$ at time t, for certain years.

- However, for each value of the time t, there is a corresponding value of P, and we say that P is a function of t.

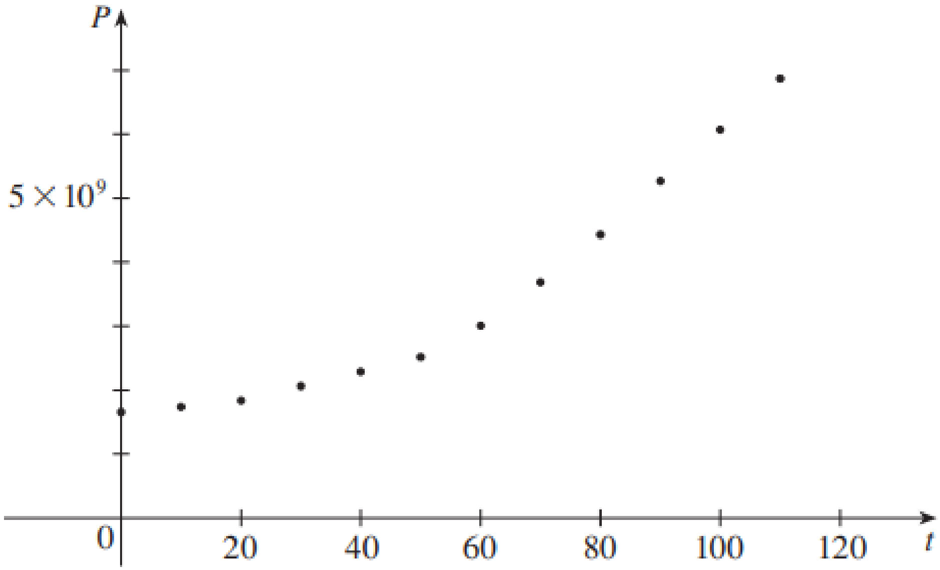
t	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

1.1 Functions and their representations

Example:

The human population of the world P depends on the time t.

If we plot these values, we get the **graph** (called a scatter plot).



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60	3040
70	3710
80	4450
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100	6080
110	6870

1.1 Functions and their representations

Example:

What about a formula ?

Of course, it's impossible to devise an explicit **formula** that gives the exact human population $P(t)$ at any time t .

But it is possible to find an expression for a function that approximates $P(t)$.

In fact, we could use a graphing calculator with exponential

regression capabilities to obtain the approximation

$$P(t) \approx f(t) = \left(1.43653 \times 10^9\right) \times (1.01395)^t$$

1.1 Functions and their representations

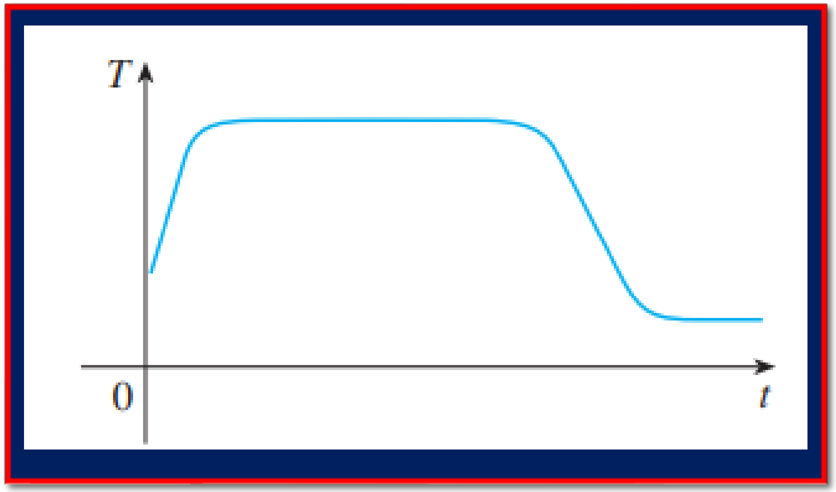
Example:

"*When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running*".

Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

Solution:

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the



hotwater tank starts flowing from the faucet, increases quickly. In the next phase, is constant at the tempera ture of the heated water in the tank. When the tank is drained, decreases to the temperature of the water supply. This enables us to make the rough sketch of as a function of in Figure.

1.1 Functions and their representations

The graph of a function is a curve in the xy -plane.

But the question arises:

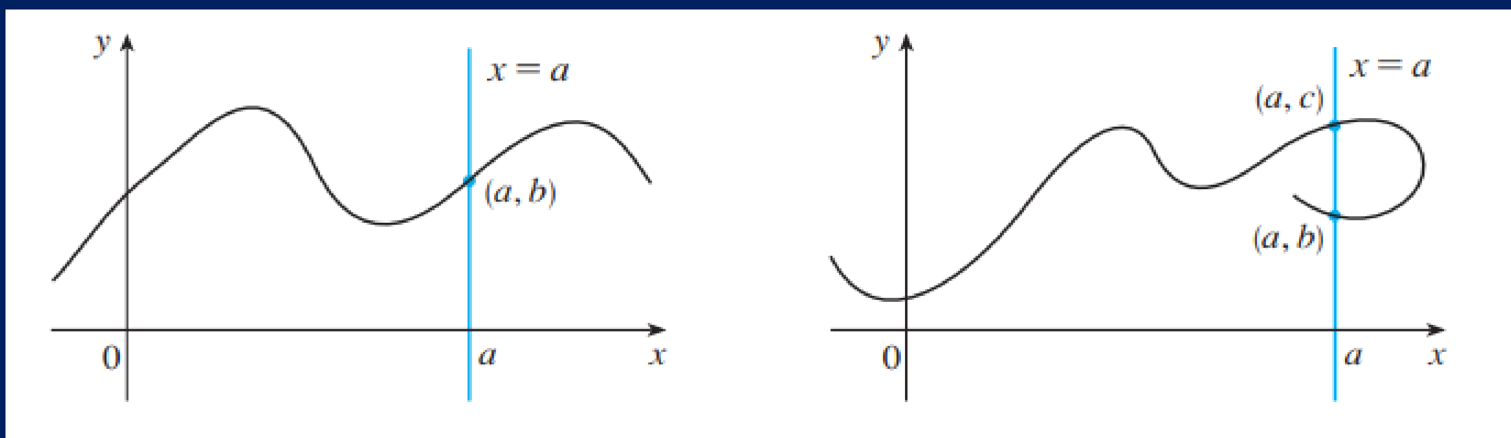
Which curves in the xy -plane are graphs of functions ?

This is answered by the following test.

1.1 Functions and their representations

THE VERTICAL LINE TEST

A curve in the xy-plane is the graph of a function of if and only if no vertical line intersects the curve more than once.



It is a function **Not a function**

1.1 Functions and their representations

Symmetry: even function and odd function.

- If a function f satisfies: $\boldsymbol{f(-x) = f(x), \forall x \in D}$ then f is called an **even function.**

-If a function f satisfies: $\boldsymbol{f(-x) = -f(x), \forall x \in D}$ then f is called an **odd function.**

Example:

a/ $f(x) = x^2, h(x) = x^4$ are **even** functions.

b/ $g_1(x) = x, g_2(x) = x^3$ are **odd** functions.

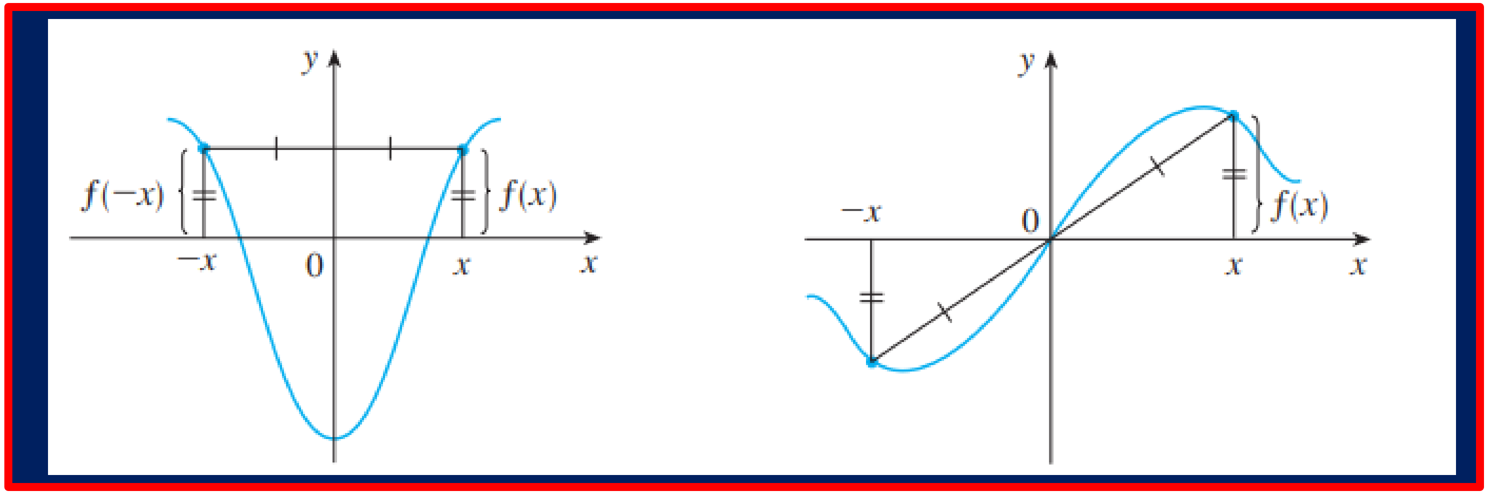
1.1 Functions and their representations

Symmetry: even function and odd function.

Properties:

- The geometric significance of an even function is that its graph is **symmetric with respect to the *y*-axis**.

- The graph of an odd function is **symmetric about the origin**.

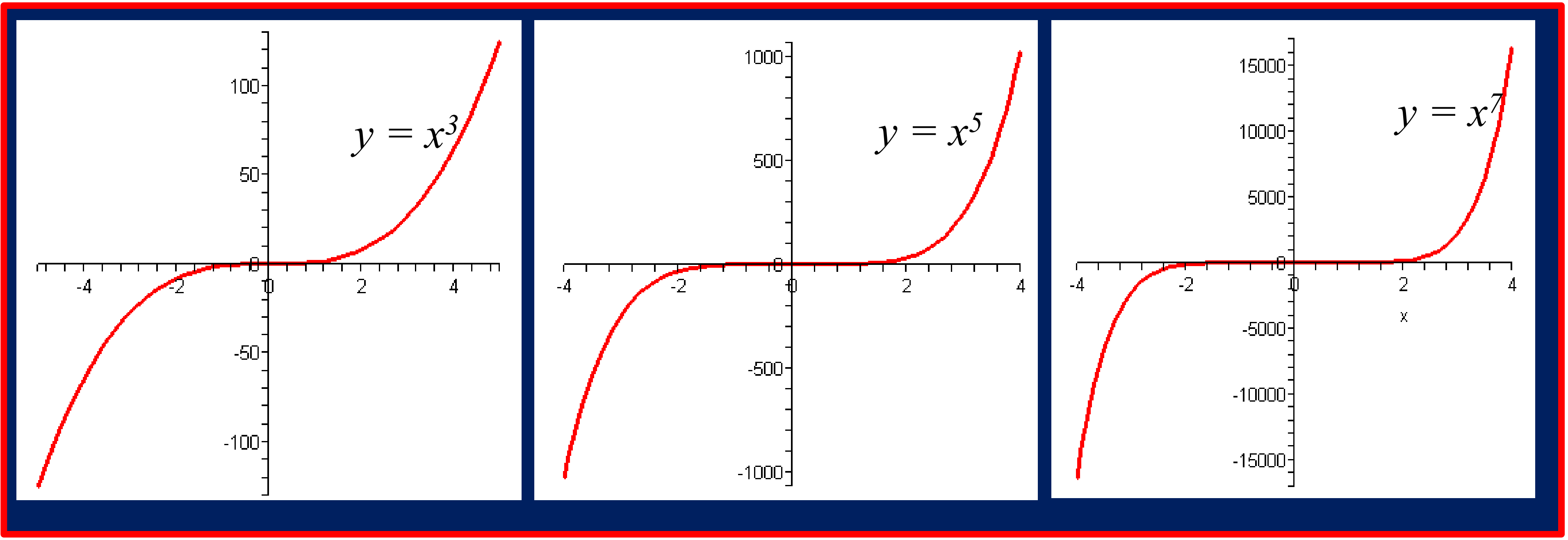


An even funtions **An odd function**

Symmetry: even function and odd function.

Example:

The graph of odd functions



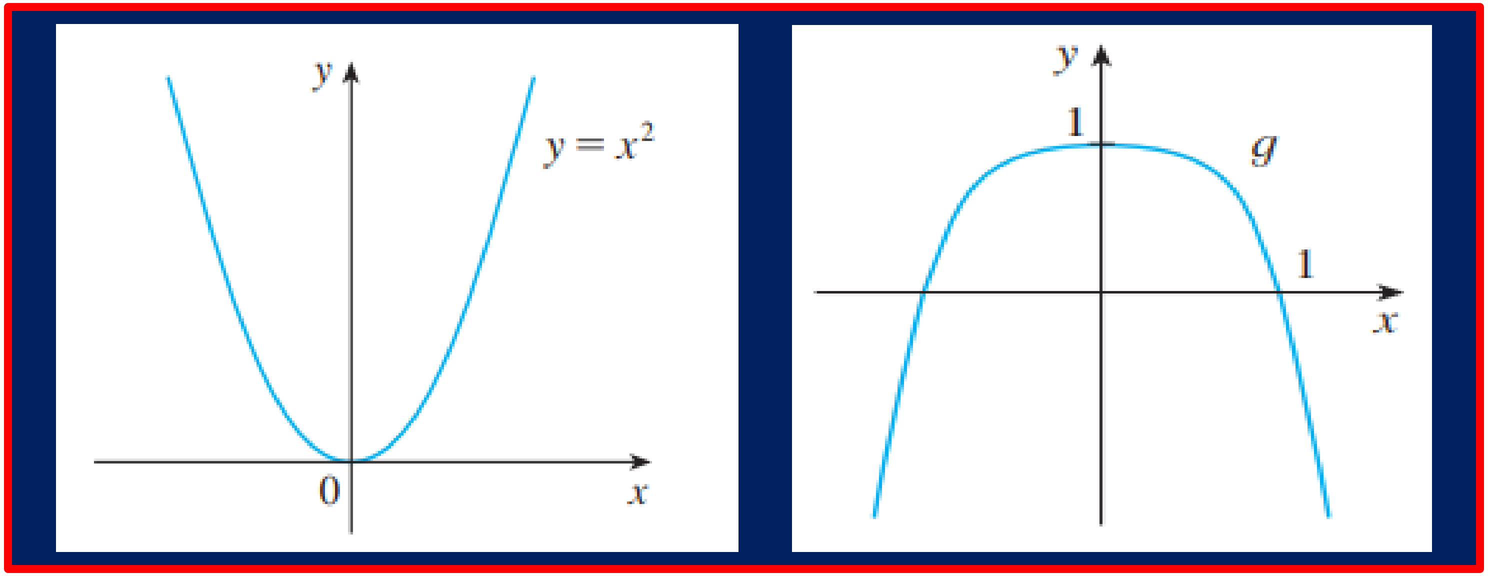
1.1 Functions and their representations

Symmetry: even function and odd function.

Example:

The graph of even functions

$$g(x) = 1 - x^4$$



1.1 Functions and their representations

Symmetry: even function and odd function.

Example:

Let f is an **odd function**. If (-3,5) is in the graph of f then which point is also in the graph of f ?

a. (3,5)　　b. (-3,-5)　　c. (3,-5)　　d. All of the others

Symmetry: even function and odd function.

Solution:

Let f is an **odd function**. If (-3,5) is in the graph of f then which point is also in the graph of f ?

a. (3,5) b. (-3,-5) c. (3,-5) d. All of the others

$$set\ x = -3 \Rightarrow -x = 3 : f\left(3\right) = f\left(-x\right) = -f\left(x\right) = -f\left(-3\right) = -5$$

1.1 Functions and their representations

Symmetry: even function and odd function.

Example:

Suppose f is an **odd function** and g is an **even function**.

What can we say about the function f.g defined by
$(f.g)(x)=f(x)g(x)$?

Prove your result.

1.1 Functions and their representations

Increasing and decreasing functions

Definition

A function f is called **increasing on an interval *I*** if:

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

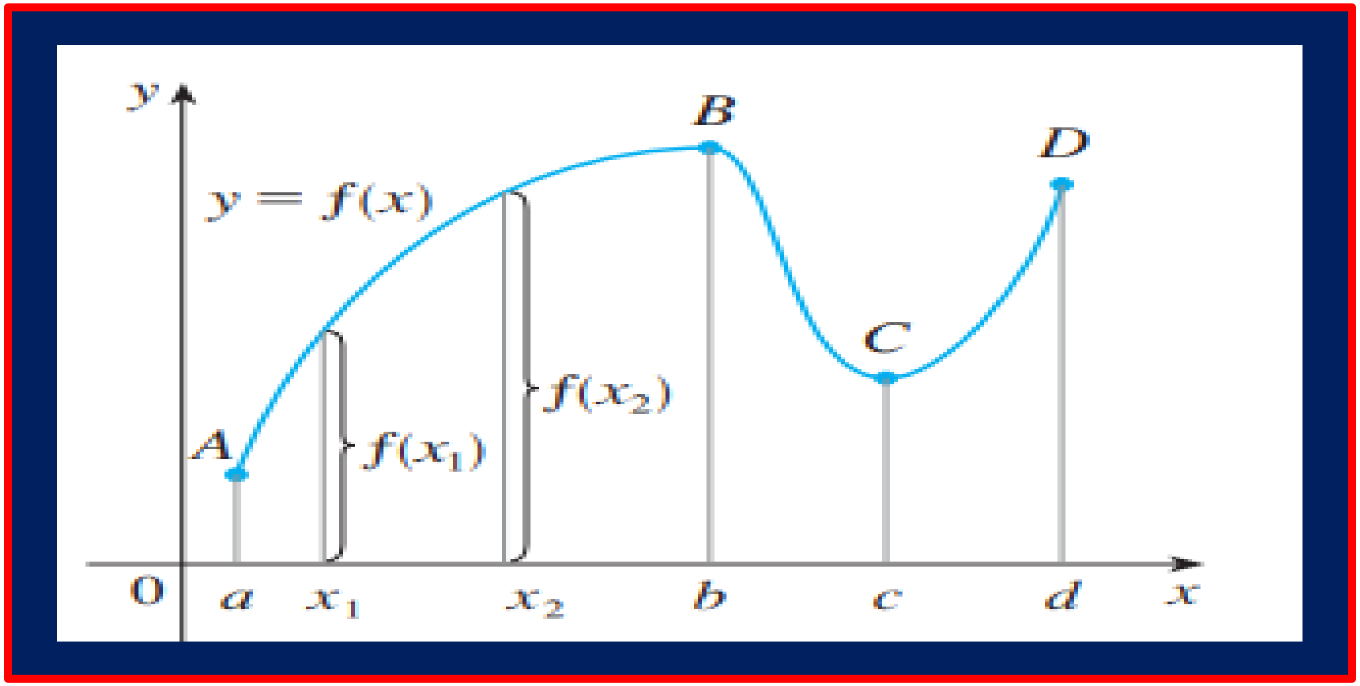
It is called **decreasing on *I*** if:

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

1.1 Functions and their representations

Increasing and decreasing functions

The function *f* is said to be **increasing on the interval [*a*, *b*]**, decreasing on [*b*, *c*], and increasing again on [*c*, *d*].



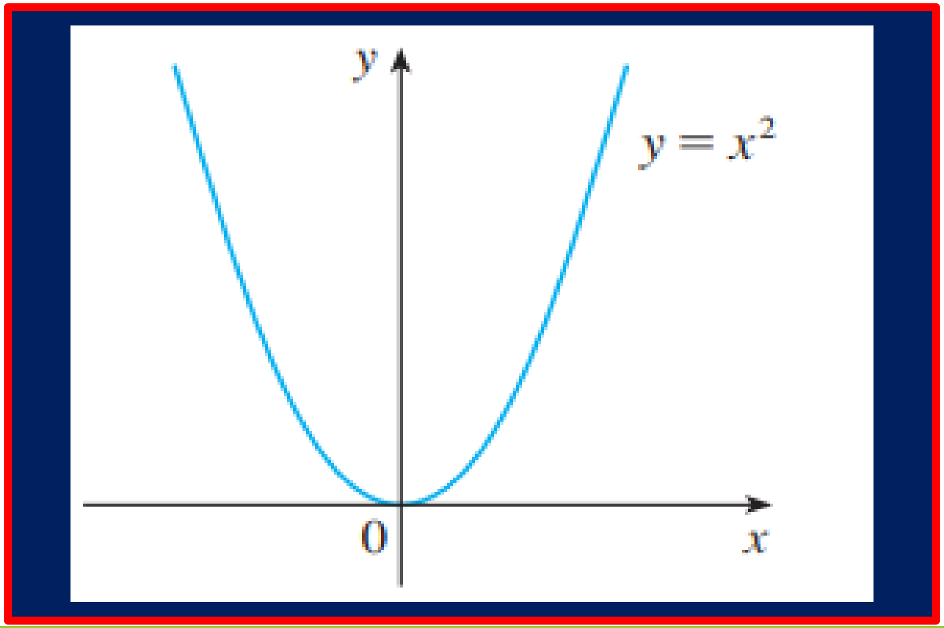
1.1 Functions and their representations

Increasing and decreasing functions

Example:

The function $y = x^2$ is

- Decreasing on the interval $(-\infty, 0]$
- Increasing on the interval $[0, +\infty)$



1.1 Functions and their representations

Quiz questions

1/ If f is a function then $f(x+2)=f(x)+f(2)$

a. True   b. False

2/ If $f(s)=f(t)$ then $s= t$

a. True   b . False

3/ Let f be a function. We can find s and t such that $s=t$ and $f(s)$ is not equal to $f(t)$

a. True   b. False

1.1 Functions and their representations

- $f(x) = x + 1$

$$f(x+2) = (x+2) + 1 = x + 3$$

$$f(2) = 2 + 1 = 3$$

$$f(x) + f(2) = x + 1 + 3 = x + 4$$

1.1 Functions and their representations

Answer:

1/ If f is a function then $f(x+2)=f(x)+f(2)$

a. True

b. False

2/ If $f(s)=f(t)$ then $s= t$

a. True

b. False

3/ Let f be a function. We can find s and t such that $s=t$ and $f(s)$ is not equal to $f(t)$

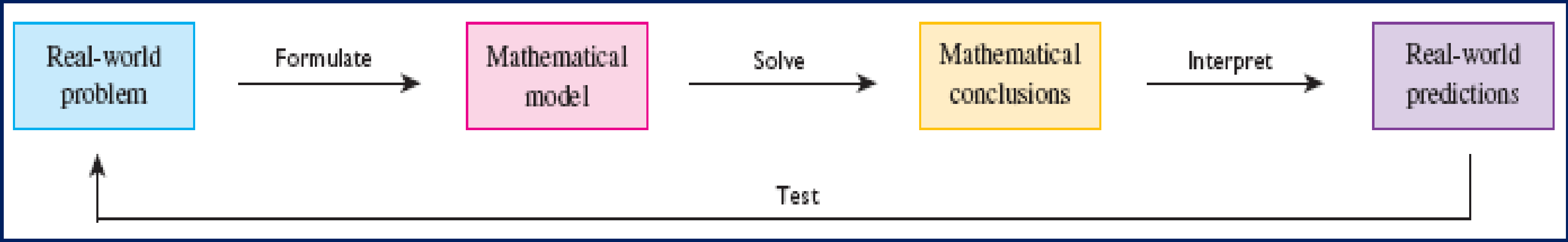
a. True

b. False

1.2. Functions and models

A mathematical model is a mathematical description—often by means of a function or an equation of a real-world phenomenon such as:

- Size of a population
- Demand for a product
- Speed of a falling object
- Life expectancy of a person at birth
- Cost of emission reductions



1.2. Functions and models

Linear models

When we say that y is a **linear function** of x, we mean that the graph of the function is a line.

So, we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

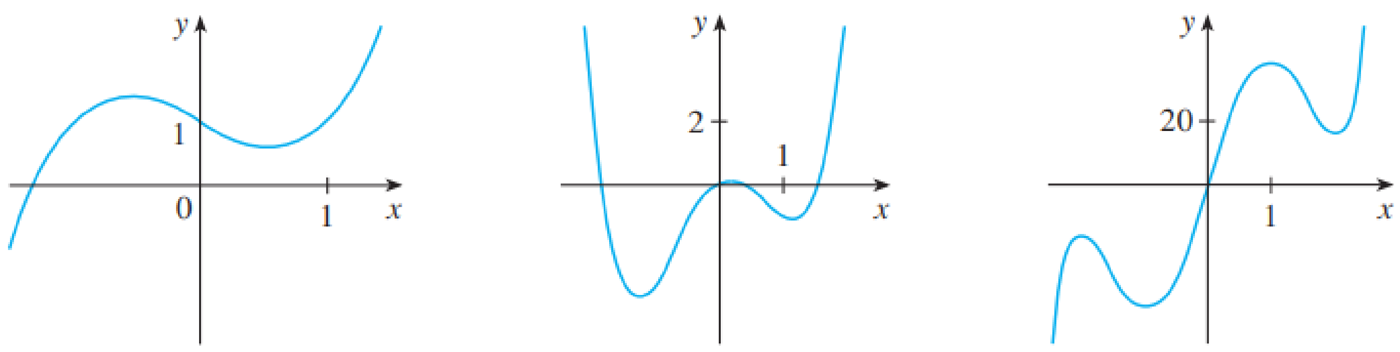
1.2. Functions and models

Polynomials

A function P is called a polynomial if

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where n is a nonnegative integer and the numbers $a_0, a_1, a_2,$ $\ldots, a_n$ are constants called the coefficients of the polynomial.



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

1.2. Functions and models

Rational functions

A rational function f is a ratio of two polynomials

$$f(x) = \frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials.

The domain consists of all value x such that $Q(x) \neq 0$

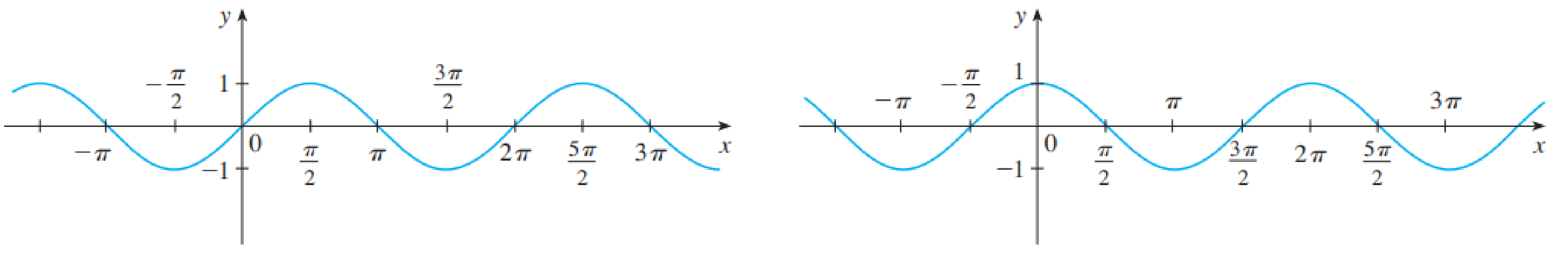
1.2. Functions and models

Trigonomatric functions

$$f(x) = \sin(x) \qquad D = (-\infty, \infty)$$

$$f(x) = \cos(x) \qquad R = [-1, 1]$$

$$\sin(x + k2\pi) = \sin(x)\ ; \quad \cos(x + k2\pi) = \cos(x), \quad k \in Z$$



(a) $f(x) = \sin x$ (b) $g(x) = \cos x$

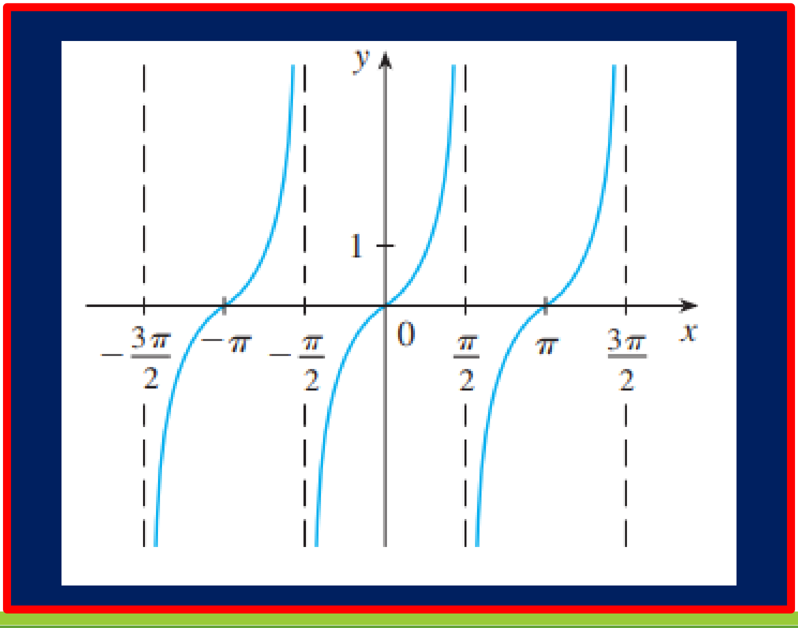
1.2. Functions and models

Trgonomatric functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \; x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$$

$$R = (-\infty, \infty)$$

$$\tan(x + k\pi) = \tan x; \quad k \in Z$$



1.2. Functions and models

Trgonomatric functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \ x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$$

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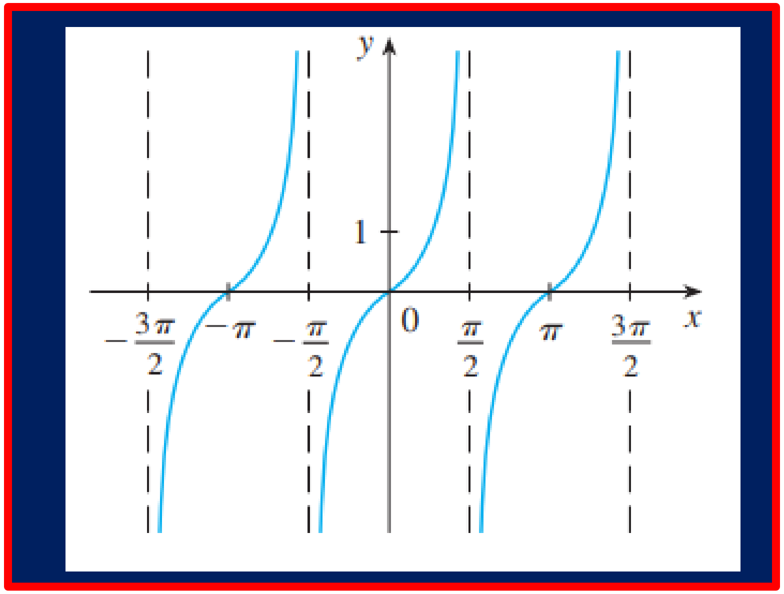
$$\tan(x + k\pi) = \tan x; \quad k \in Z$$



1.2. Functions and models

Trgonomatric functions

The reciprocals of the sine, cosine, and tangent functions are

$$\cos ec(x) = \frac{1}{\sin(x)}$$

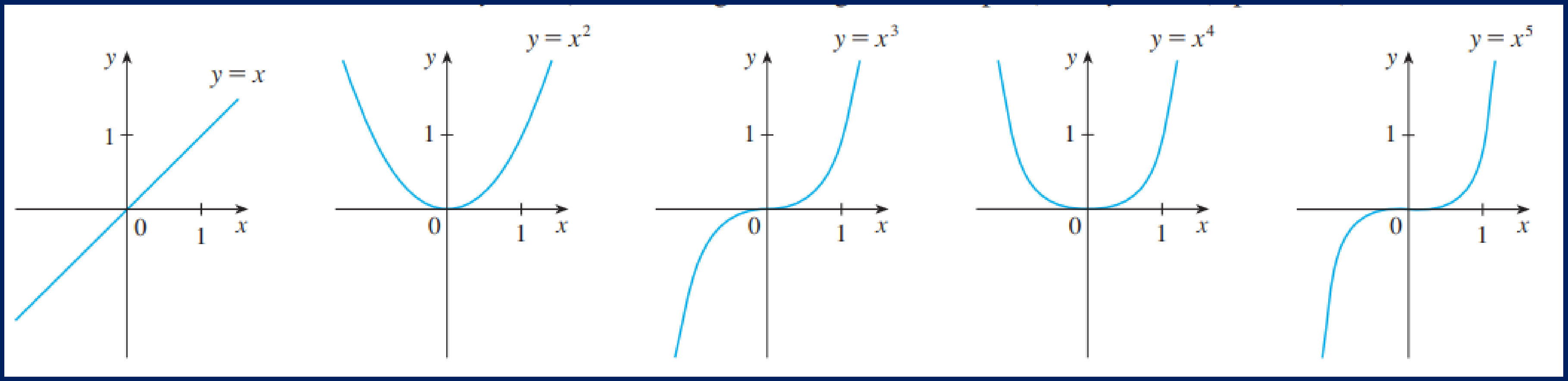
$$\sec(x) = \frac{1}{\cos(x)}$$

$$\cot an(x) = \frac{1}{\tan(x)}$$

1.2. Functions and models

Trgonomatric functions

A function of the form $f(x) = x^{\alpha}$, where a is constant, is called **a power function**.



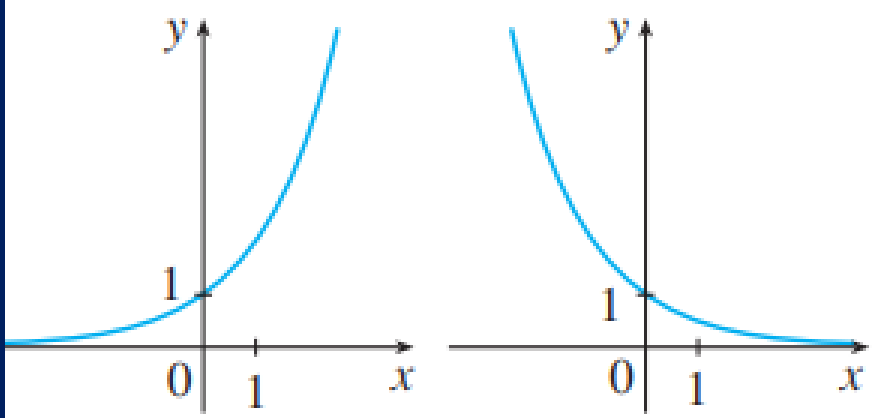
1.2. Functions and models

Exponential function

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

- The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown.

- In both cases, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.



(a) $y = 2^x$ (b) $y = (0.5)^x$

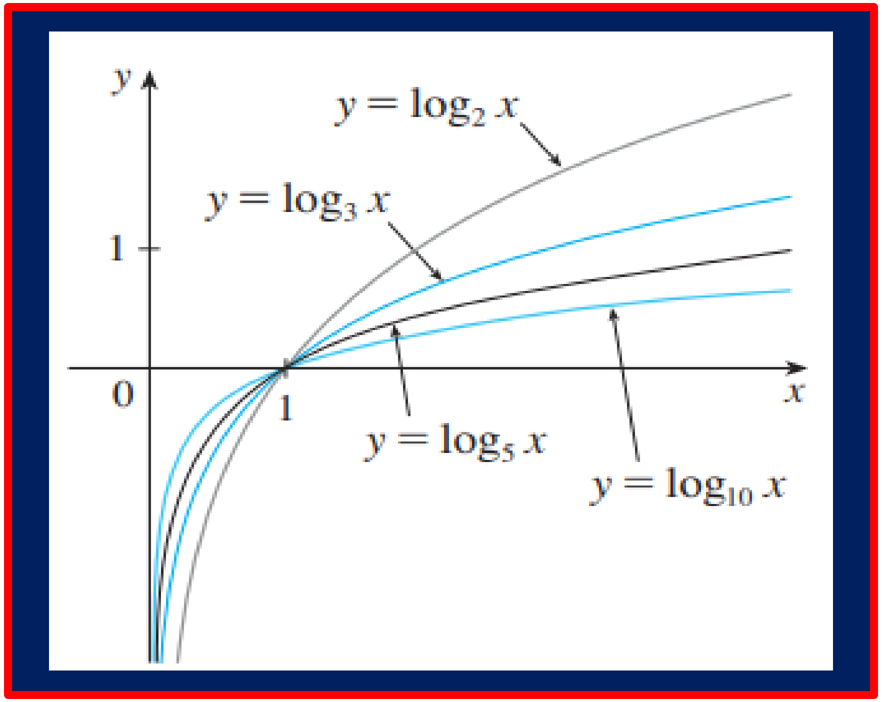
1.2. Functions and models

Logarithmic funtions

The logarithmic functions $f(x) = \log_a x, x > 0, 0 < a \neq 1,$

where the base a is a positive constant, are the inverse functions of the exponential functions.

The figure shows the graphs of four logarithmic functions with various bases.



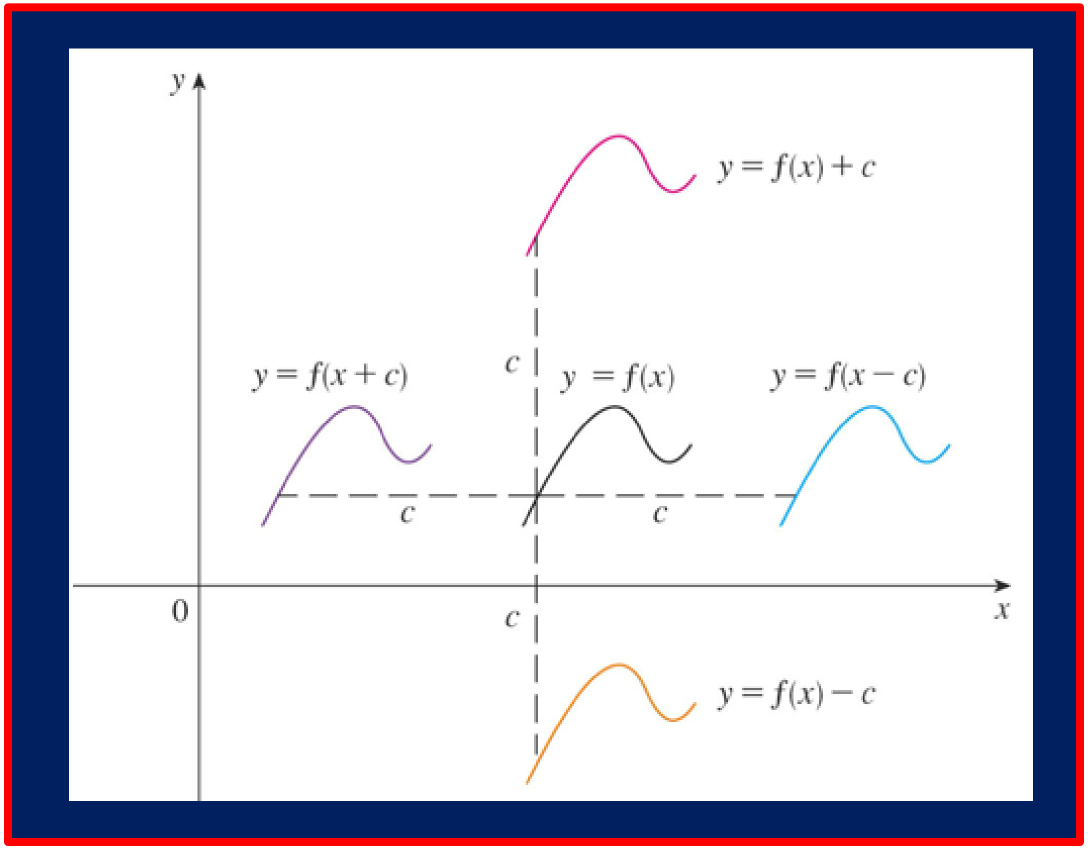
1.2. Functions and models

Shifting

Why don't we consider the case c < 0 ?

Suppose $c > 0$.

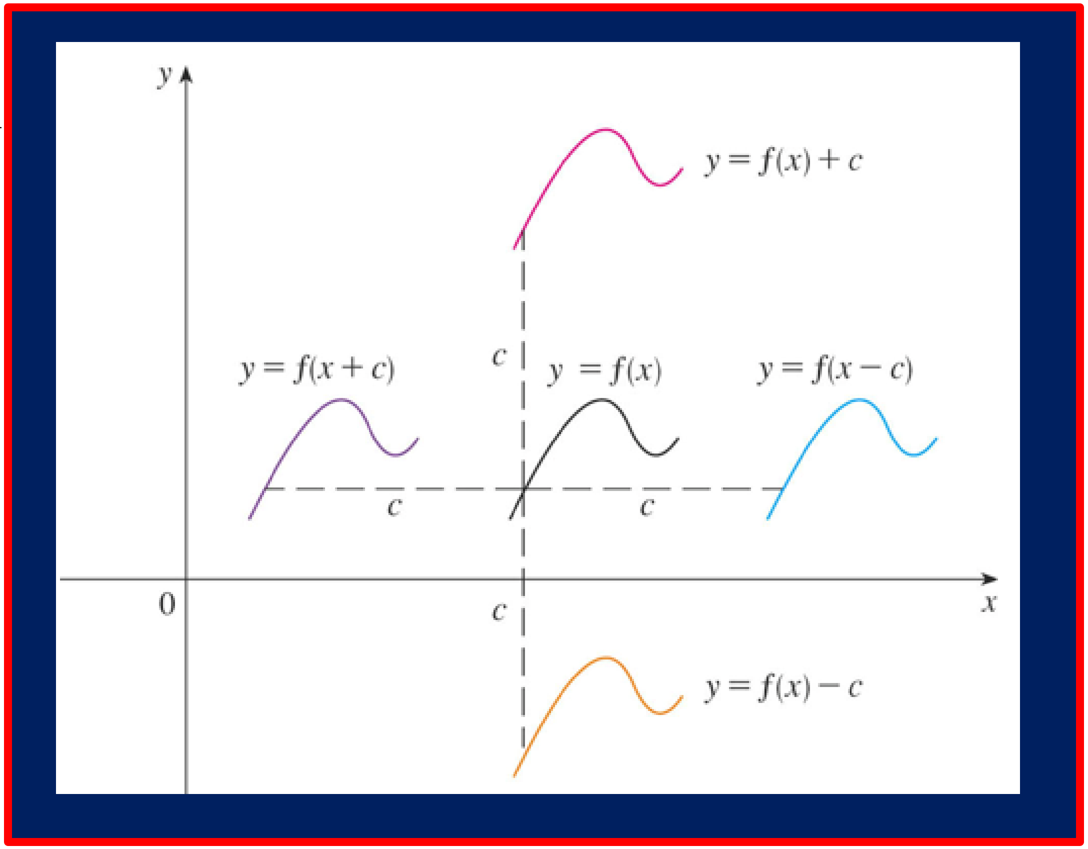
- To obtain the graph of $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward.
- To obtain the graph of $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward.



1.2. Functions and models

Shifting

- To obtain the graph of $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right.
- To obtain the graph of $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left.



1.2. Functions and models

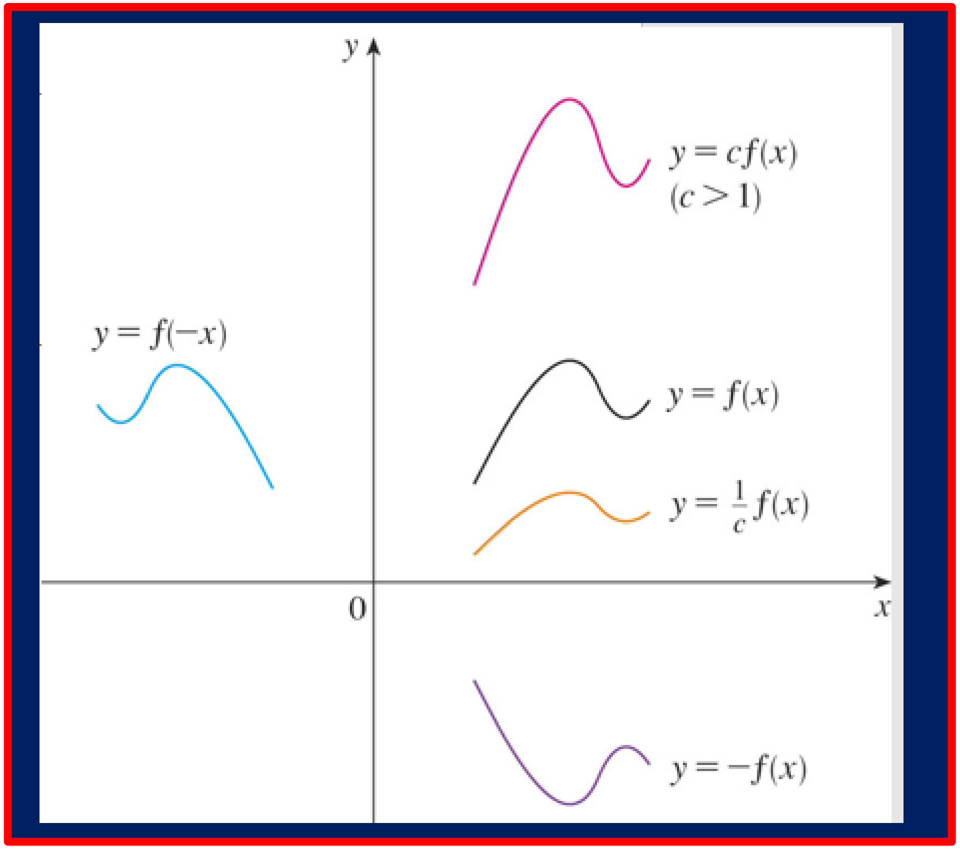
Transformations

How about the case c < 1 ?

Suppose $c > 1$.

- To obtain the graph of $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c.

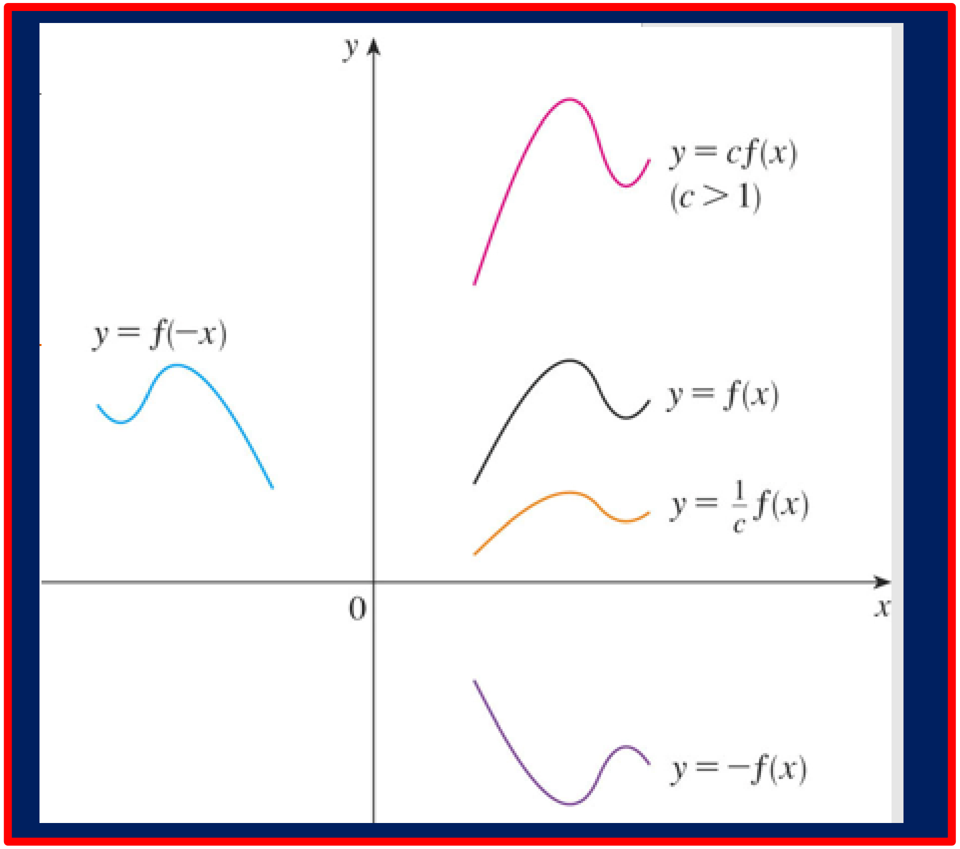
- To obtain the graph of $y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c.



1.2. Functions and models

Transformations

- In order to obtain the graph of $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
- To obtain the graph of $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

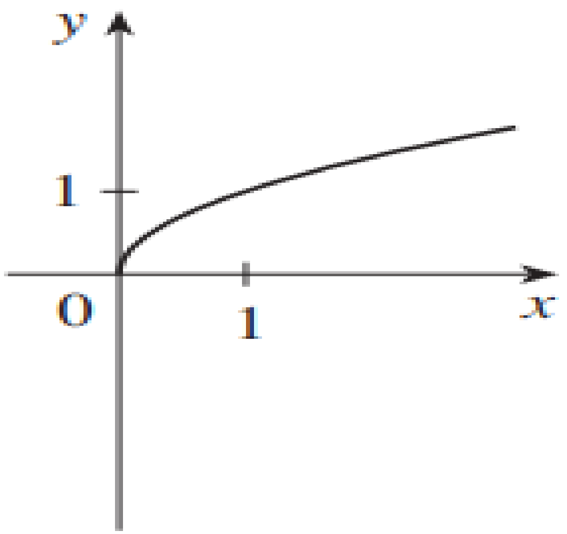


1.2. Functions and models

Transformations

Example:

Given the graph of $y=\sqrt{x}$



(a) $y=\sqrt{x}$

Use transformations to graph

$$y=\sqrt{x}-2, y=\sqrt{x-2}, y=-\sqrt{x}, y=2\sqrt{x}, y=\sqrt{-x}\ ?$$

1.2. Functions and models

Transformations

Solution:

We have $y=\sqrt{x}-2=f(x)-2$

$$y=\sqrt{x-2}=f(x-2)$$

$$y=-\sqrt{x}=-f(x)$$

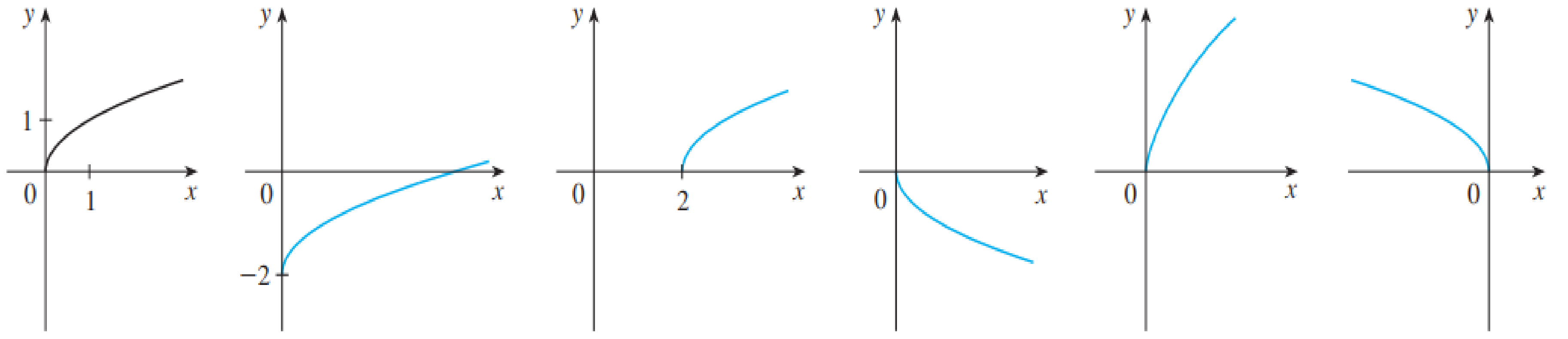
$$y=2\sqrt{x}=2f(x)$$

$$y=\sqrt{-x}=f(-x)$$

1.2. Functions and models

Transformations

Solution:



(a) $y=\sqrt{x}$ (b) $y=\sqrt{x}-2$ (c) $y=\sqrt{x-2}$ (d) $y=-\sqrt{x}$ (e) $y=2\sqrt{x}$ (f) $y=\sqrt{-x}$

- ✓ $y=-\sqrt{x}$: by reflecting about the x-axis.
- ✓ $y=2\sqrt{x}$: by stretching vertically by a factor of 2.
- ✓ $y=\sqrt{-x}$: by reflecting about the y-axis

1.2. Functions and models

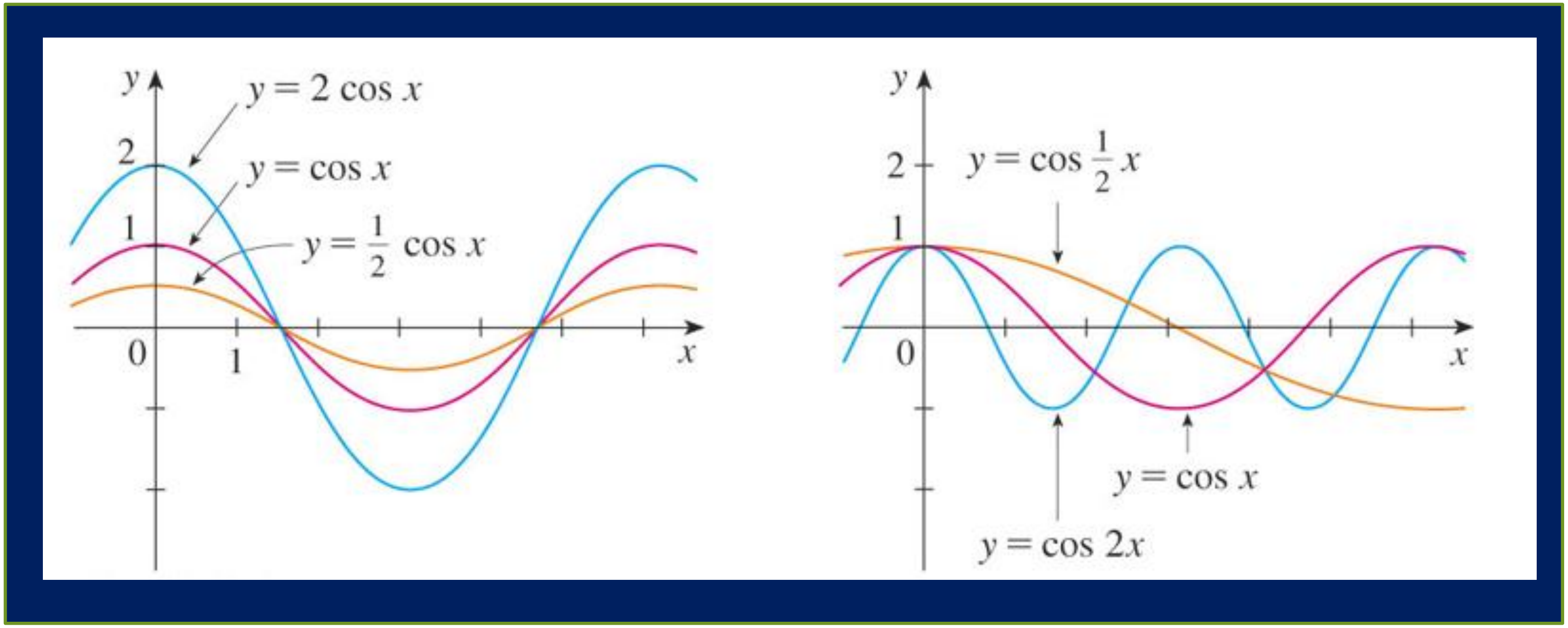
Transformations_Summary

Tranformation of $f\left(c>0\right)$	Effect on the graph of f
$f\left(x\right)+c$	Vertical shift up c units
$f\left(x\right)-c$	Vertical shift down c units
$f\left(x+c\right)$	Shift left by c units
$f\left(x-c\right)$	Shift right by c units
$c\times f\left(x\right)$	Vertical stretch if $c>1$ Vertical compression $0<c<1$
$f\left(c\times x\right)$	Horizontal stretch $0<c<1$ Horizontal compression if $c>1$
$-f\left(x\right)$	Reflection about the $x-axis$
$f\left(-x\right)$	Reflection about the $y-axis$

1.2. Functions and models

Transformations

Example 1: The figure illustrates these stretching transformations when applied to the cosine function with $c = 2$.



DREAM OF INNOVATION

1.2. Functions and models

Transformations

Example 2: Suppose that the graph of f is given. Describe how the graph of the function $f(x-2)+2$ can be obtained from the graph of f. Select the correct answer.

a. Shift the graph 2 units to the left and 2 units down.

b. Shift the graph 2 units to the right and 2 units down.

c. Shift the graph 2 units to the right and 2 units up.

d. Shift the graph 2 units to the left and 2 units up.

e. none of these

Combinations of functions

Two functions *f* and *g* can be combined to form new functions:

a/ $\left(f \pm g\right)\left(x\right) = f\left(x\right) \pm g\left(x\right)$

b/ $\left(f.g\right)\left(x\right) = f\left(x\right).g\left(x\right)$

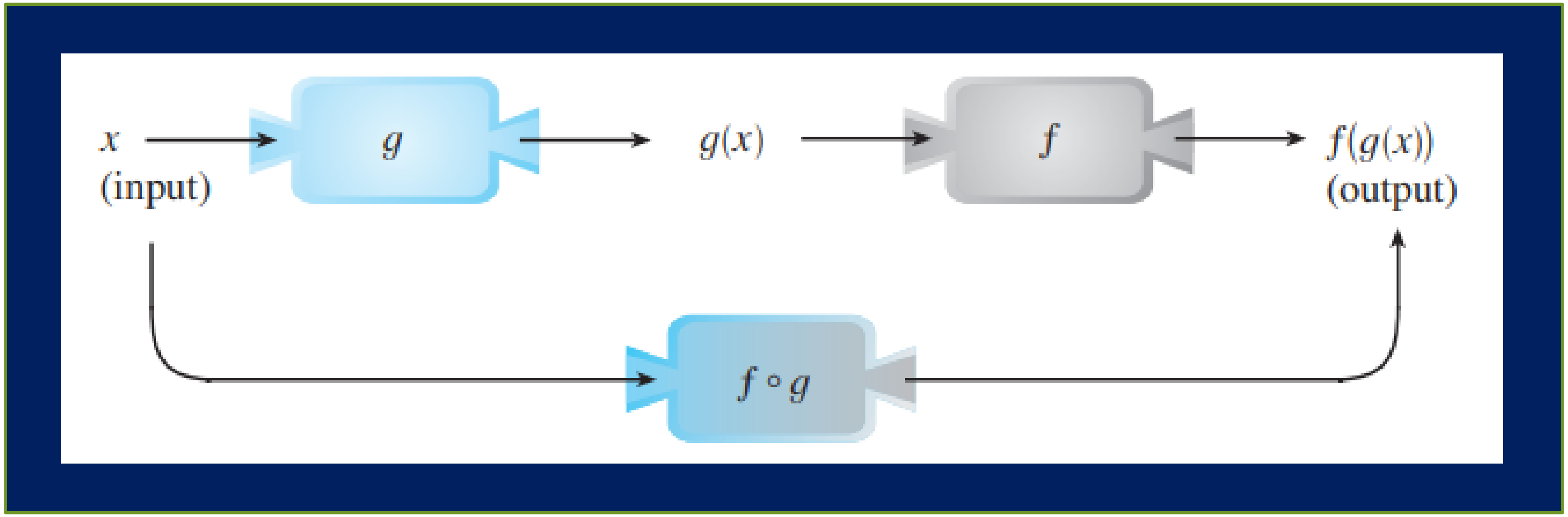
c/ $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

1.2. Functions and models

Composite of functions

Given functions f and g, the composite function is

defined by $(f \circ g)(x) = f(g(x))$



1.2. Functions and models

Example 1:

Let $h(x)=f(g(x))$. If $f(x)=x-1$ and $g(x)=3x+2$ then $h(x)$ is ?

1.2. Functions and models

Example 2:

x	1	2	3	4	5	6
$f(x)$	3	2	1	0	1	2
$g(x)$	6	5	2	3	4	6

Evaluate $(f \circ g)(2)$

a. 5 b. 1

c. 2 d. None of the others

1.2. Functions and models

Example 2:

x	1	2	3	4	5	6
$f(x)$	3	2	1	0	1	2
$g(x)$	6	5	2	3	4	6

Evaluate $(f \circ g)(2)$

a. 5 b. 1

c. 2 d. None of the others

$$(f \circ g)(2) = f\left(g(2)\right) = f(5) = 1$$

1.2. Functions and models

Example 3:

Let h(x)=f(g(x)).

1/ If g(x)=x-1 and h(x)=3x+2 then f(x) is:

a. 3x+3 b. 3x+4

c. 3x+1 d. None of them

2/ If h(x)=3x+2 and f(x)=x-1 then g(x) is:

a. 3x+3 b. 3x+4

c.3x+1 d. None of them

3/ If f and g are functions, then $f \circ g = g \circ f$

a. True b. False

1.2. Functions and models

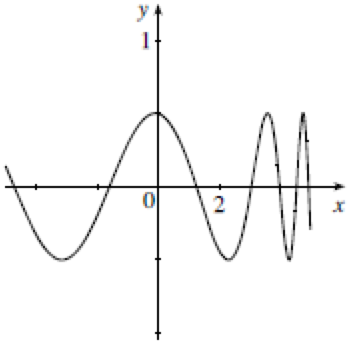
$$a/\ f(x)=3x+3;g(x)=x-1;h(x)=3x+2$$

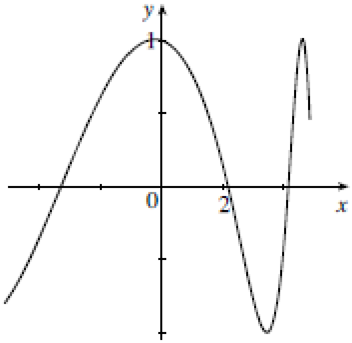
$$f\circ g=f\left(g(x)\right)=f(x-1)=3(x-1)+3=3x\neq h(x)$$

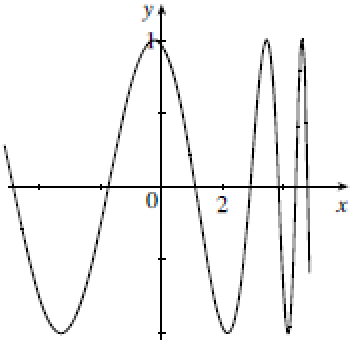
1.2. Functions and models

Label the following graphs

$$f(x), \frac{1}{2}f(x), f\left(\frac{1}{2}x\right)$$







1.3. The limit of a function

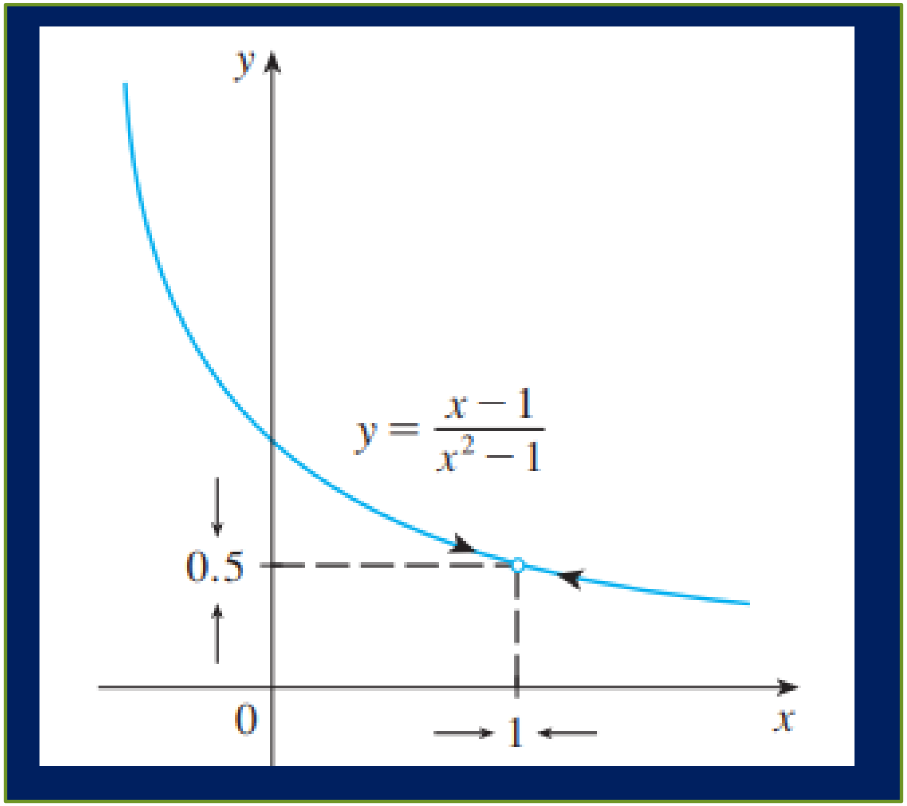
The limit of a function

In general, we write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a

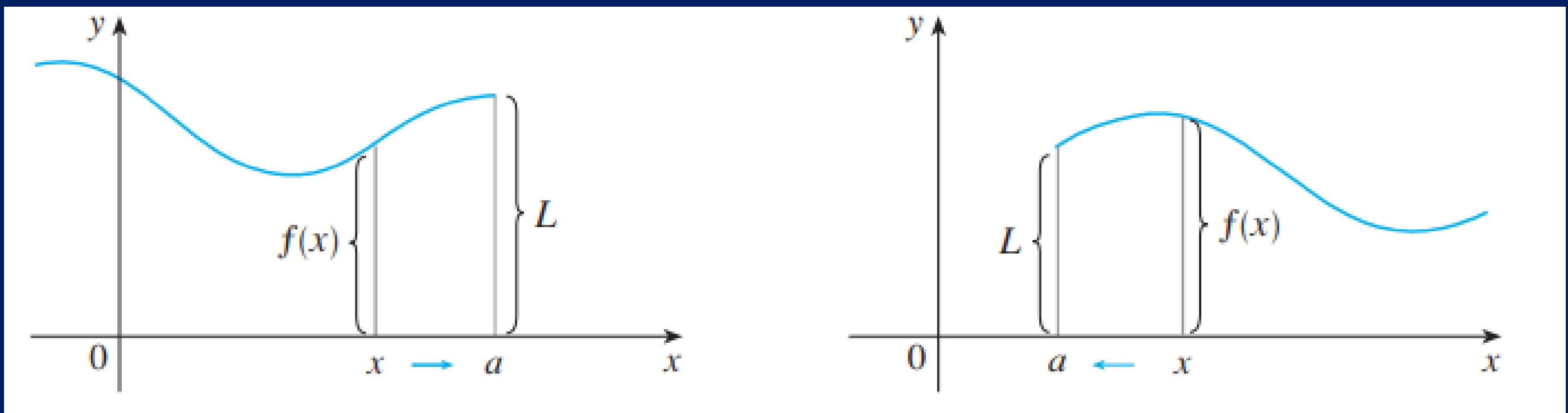
but not equal to $\boldsymbol{a}$.



1.3. The limit of a function

One-sided limits

We write $\lim_{x \to a^-} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.



(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

1.3. The limit of a function

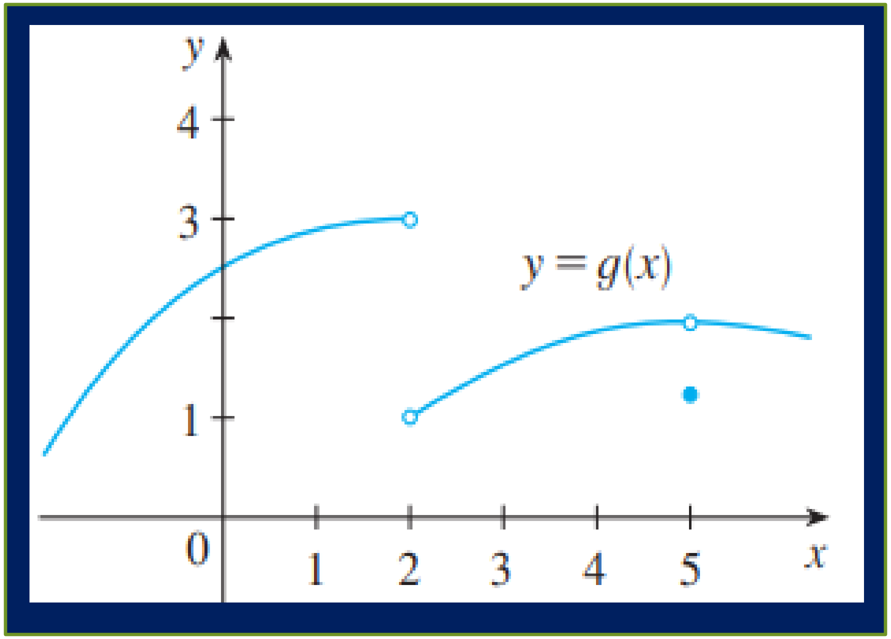
One-sided limits

Similarly, *"the right-hand limit of f(x) as x approaches a is equal to L"* and we write $\lim_{x \to a^+} f(x) = L$

$\lim_{x \to 2^-} g(x) =$ $\lim_{x \to 5^-} g(x) =$

$\lim_{x \to 2^+} g(x) =$ $\lim_{x \to 5^+} g(x) =$

$\lim_{x \to 2} g(x) =$ $\lim_{x \to 5} g(x) =$



1.3. The limit of a function

A numerical and graphical approach

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$

a/ What is $f(3)$?

b/ What is the limit of f as x approaches ?

1.3. The limit of a function

A numerical and graphical approach

Solution:

a/ Since $f(x)=\dfrac{x^2-9}{x-3}$, we will substitute 3 in for x, giving us the new equation $f(3)=\dfrac{3^2-9}{3-3}$.

Solving for $f(3)$, we get

$$f(3)=\frac{3^2-9}{3-3}=\frac{9-9}{3-3}=\frac{0}{0}.$$

Thus $f(3)$ does not exist.

1.3. The limit of a function

A numerical and graphical approach

b/ First let x approach 3 from the left:

$x \to 3^-$	2	2.5	2.9	2.99	2.999
$f(x)$	5	5.5	5.9	5.99	5.999

Thus it appears that $\lim_{x\to 3^-} f(x) = 6$

Next let x approach 3 from the right:

$x \to 3^+$	4	3.5	3.1	3.01	3.001
$f(x)$	7	6.5	6.1	6.01	6.001

Thus it appears that $\lim_{x\to 3^+} f(x) = 6$

Since both the left-hand and right-hand limits agree,

$$\lim_{x\to 3} f(x) = 6$$

.

1.3. The limit of a function

A numerical and graphical approach

Example 1: Consider the function H given by

$$H(x) = \begin{cases} 2x + 2 & for \quad x < 1 \\ 2x - 4 & for \quad x \geq 1 \end{cases}$$

Graph the function, and find each of the following limits, if they exist. When necessary, state that the limit does not exist.

a/ $\lim_{x \to 1} H(x)$ b/ $\lim_{x \to -3} H(x)$

1.3. The limit of a function

Solution:

a/ First, let x approach 1 from the left:

$x \to 1^-$	0	0.5	0.8	0.9	0.99	0.999
$H(x)$	2	3	3.6	3.8	3.98	3.998

Thus, it appears that $\lim_{x \to 1^-} H(x) = 4.$

Then, let x approach 1 from the right:

$x \to 1^+$	2	1.8	1.1	1.01	1.001	1.0001
$H(x)$	0	–0.4	–1.8	–1.98	–1.998	–1.9998

Thus, it appears that $\lim_{x \to 1^+} H(x) = -2.$

1.3. The limit of a function

Solution:

Since 1) $\lim_{x \to 1^-} H(x) = 4.$

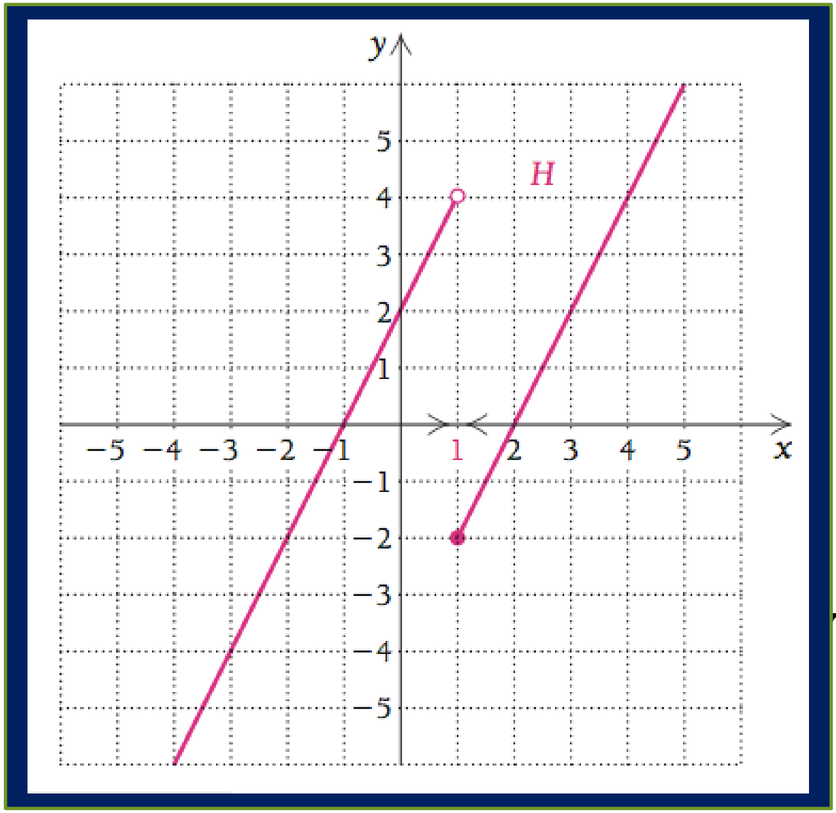
and

2) $\lim_{x \to 1^+} H(x) = -2.$

Then, $\lim_{x \to 1} H(x)$ does not exist.

1.3. The limit of a function

Limit Graphically



Observe on the graph that:

1) $\lim\limits_{x \to 1^-} H(x) = 4$

and

2) $\lim\limits_{x \to 1^+} H(x) = -2$

Therefore, $\lim\limits_{x \to 1} H(x)$ does not exist.

1.3. The limit of a function

Solution:

b/ First, let x approach -3 from the left:

$x \to -3^-$	-4	-3.5	-3.1	-3.01	-3.001
$H(x)$	-6	-5	-4.2	-4.02	-4.002

Thus, it appears that $\lim\limits_{x \to -3^-} H(x) = -4.$

Then, let x approach -3 from the right:

$x \to -3^+$	-2	-2.5	-2.9	-2.99	-2.999
$H(x)$	-2	-3	-3.8	-3.98	-3.998

Thus, it appears that $\lim\limits_{x \to -3^+} H(x) = -4.$

1.3. The limit of a function

Solution:

Since 1) $\lim_{x \to -3^-} H(x) = -4.$

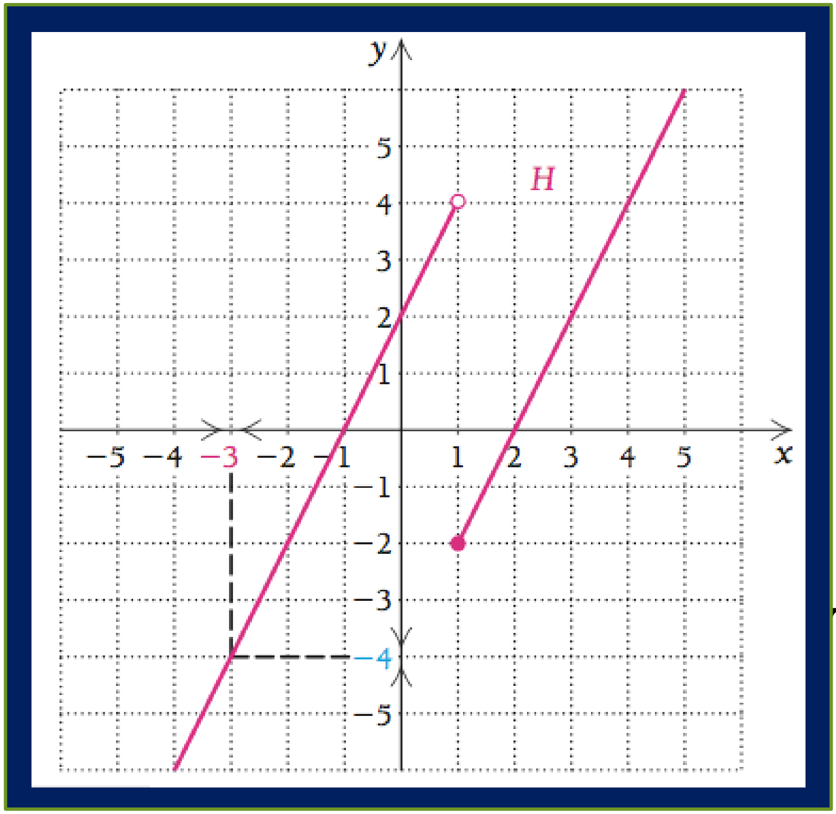
and

2) $\lim_{x \to -3^+} H(x) = -4.$

Then, $\lim_{x \to -3} H(x) = -4$

1.3. The limit of a function

Limit Graphically



Observe on the graph that:

1) $\lim_{x \to -3^-} H(x) = -4$

and

2) $\lim_{x \to -3^+} H(x) = -4$

Therefore, $\lim_{x \to -3} H(x) = -4$

1.4. Calculating limits using the limit laws

The limit laws

Suppose that c is a constant and the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Then

$$1.\ \lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

$$2.\ \lim_{x\to a}\left[cf(x)\right]=c\lim_{x\to a} f(x)$$

$$3.\ \lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

$$4.\ \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}\quad \text{if } \lim_{x\to a} g(x)\neq 0$$

1.4. Calculating limits using the limit laws

The limit laws

$$5.\ \lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$$

$$6.\ \lim_{x\to a} c = c$$

$$7.\ \lim_{x\to a} x = a$$

$$8.\ \lim_{x\to a} x^n = a^n$$

$$9.\ \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

$$10.\ \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$ where n is a positive integer.

1.4. Calculating limits using the limit laws

Direct substitution property

We state this fact as follows. If f is a **polynomial** or a **rational function** and ***a* is in the domain of *f***, then

$$\lim_{x \to a} f(x) = f(a)$$

Theorem 1

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

1.4. Calculating limits using the limit laws

Example:

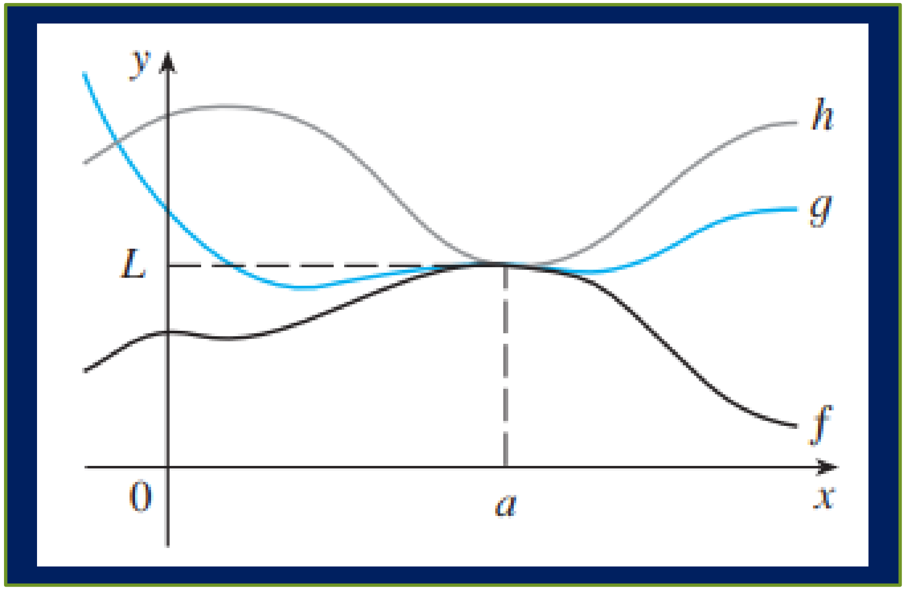
a/ Show that $\lim_{x\to 0}|x| = 0$

b/ Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

1.4. Calculating limits using the limit laws

Squeeze theorem

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ .Then $\lim_{x\to a} g(x) = L$



1.4. Calculating limits using the limit laws

Example:

Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$

1.4. Calculating limits using the limit laws

Solution

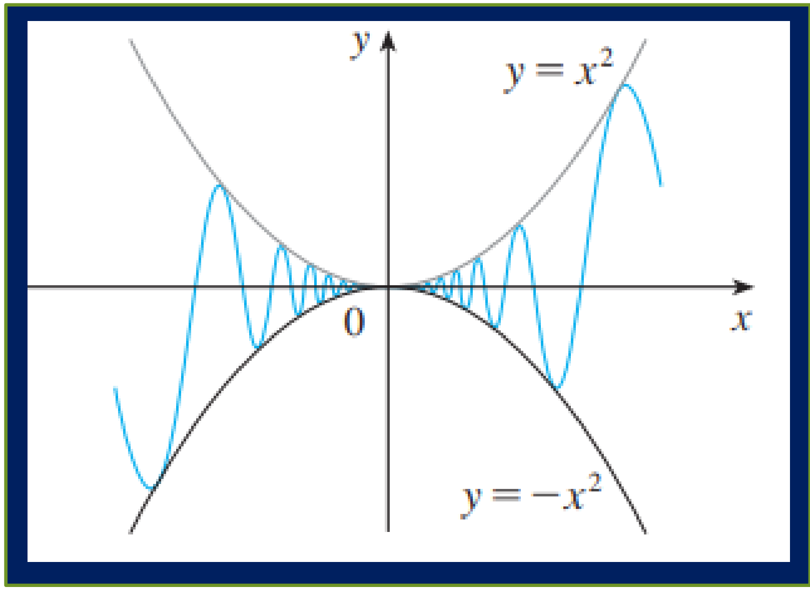
- ✓ Note that we cannot use

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

- ✓ This is because $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

1.4. Calculating limits using the limit laws

However, since $-1 \le \sin\frac{1}{x} \le 1$,



We have: $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$

Set $f(x) = -x^2, h(x) = x^2$.

We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$

In the Squeeze Theorem, we obtain:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

1.4. Calculating limits using the limit laws

Quiz questions

1/If $\lim\limits_{x\to 3} f(x) = 0, \lim\limits_{x\to 3} g(x) = 0$ then $\lim\limits_{x\to 3} \dfrac{f(x)}{g(x)}$ does not exist

a. True b. False

2/ If $\lim\limits_{x\to 3} f(x)g(x)$ exists, then the limit must be $f(3)g(3)$

a. True b. False

1.5. Continuity

Definition 1

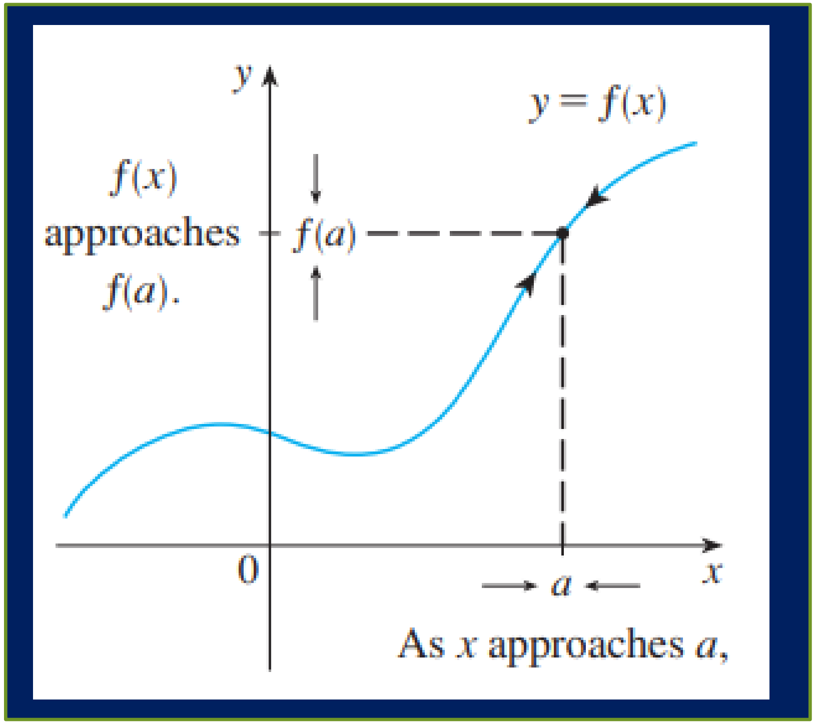
A function f is **continuous at a** number a if $\lim_{x\to a} f(x) = f(a)$

Notice that :

✓ $f(a)$ is defined - that is, a is in the domain of f

✓ $\lim_{x\to a} f(x)$ exists.

✓ $\lim_{x\to a} f(x) = f(a)$.

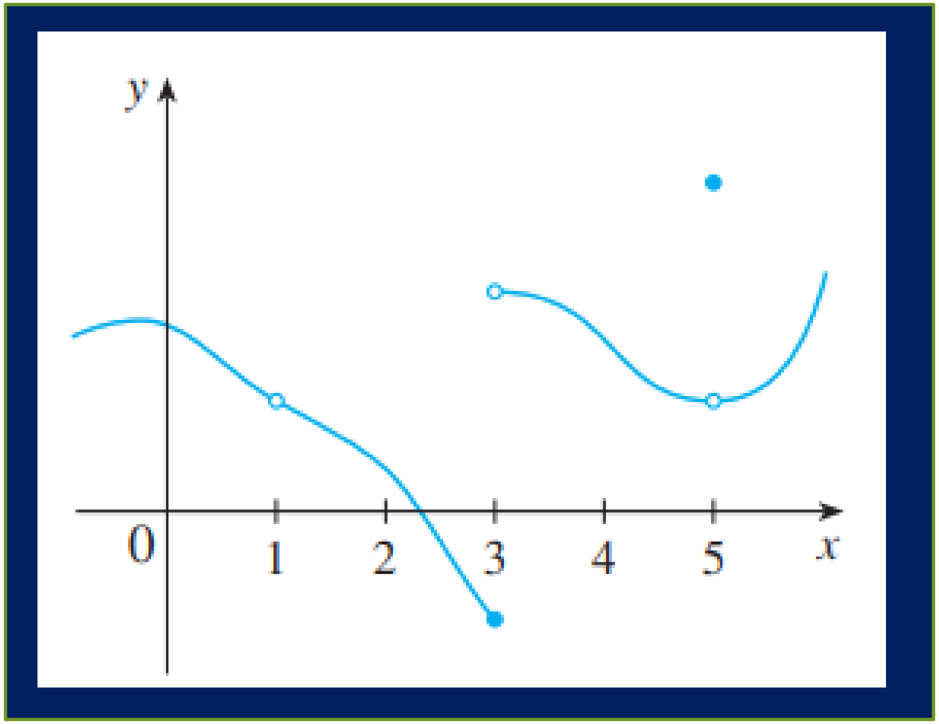


1.5. Continuity

Definition 2

If f is defined near a - that is, f is defined on an open interval containing a, except perhaps at a - we say that f is discontinuous at a if f is not continuous at a.

The figure shows the graph of a function f. At which numbers is f discontinuous ? Why ?



1.5. Continuity

Definition 3

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$

1.5. Continuity

Definition 4

- ✓ A function *f* is continuous on an interval if it is continuous at every number in the interval.

- ✓ If *f* is defined only on one side of an endpoint of the interval, we understand 'continuous at the endpoint' to mean 'continuous from the right' or 'continuous from the left.'

1.5. Continuity

Theorem

If f and g are continuous at a; and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. $c.f$
4. $f.g$
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

1.5. Continuity

Theorem

The following types of functions are continuous at every number in their domains:

- ✓ Polynomials
- ✓ Rational functions
- ✓ Root functions
- ✓ Trigonometric functions

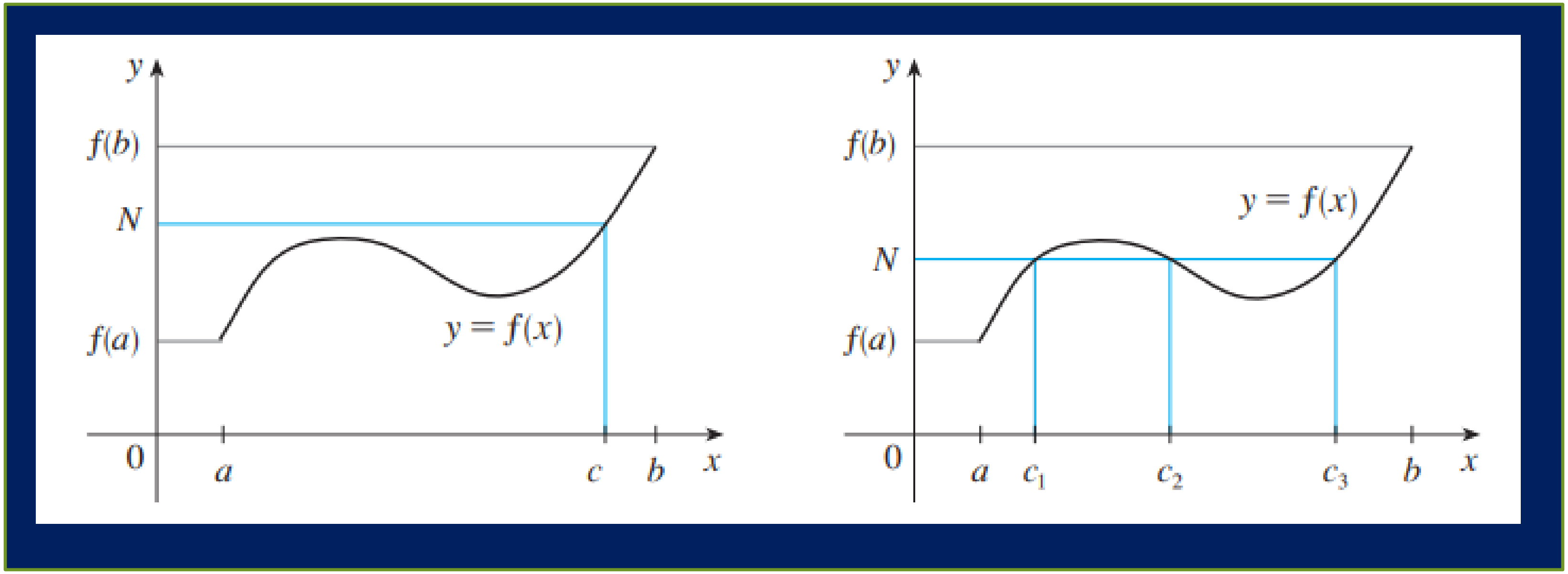
1.5. Continuity

Intermediate value theorem

Suppose that f is continuous on the **closed interval [a, b]** and let N be any number between $f(a)$ and $f(b)$, where

$$f(a) \neq f(b)$$

Then, there exists a number **c in (a, b)** such that **$f(c) = N$**.



1.5. Continuity

Intermediate value theorem

Example: Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2; \quad a = 1, b = 2$

We have $f(1) = 4 - 6 + 3 - 2 = -1$ and $f(2) = 32 - 24 + 6 - 2 = 12$

Thus $f(1) < 0 < f(2)$, that is, $N = 0$ is a number between 1 and 2. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$.

The equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least root c in interval (1,2).

1.5. Continuity

Quiz questions

1/ If $f(1)>0$ and $f(3)<0$ then there exists a number c between 1 and 3 such that $f(c)=0$

a. True

b. False

2/ Which is the equation expressing the fact that "f is continuous at 2"?

a. $\lim\limits_{x\to 2} f(x)=2$

b. $\lim\limits_{x\to \infty} f(x)=f(2)$

c. $\lim\limits_{x\to 2} f(x)=0$

d. $\lim\limits_{x\to 2} f(x)=\infty$

e. $\lim\limits_{x\to 2} f(x)=f(2)$

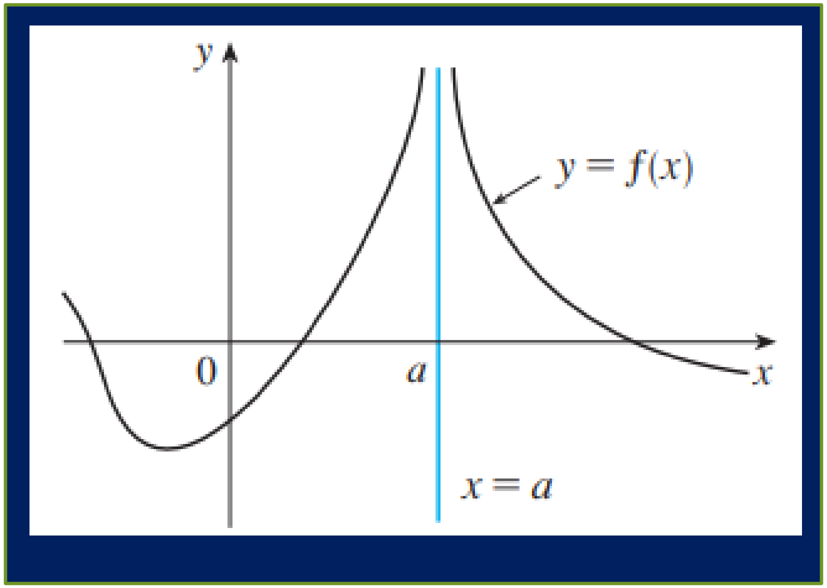
1.6. Limits involving infinity

Infinite limits

Let f be a function defined on both sides of a, except possibly at a itself. Then,

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, *but* not equal to a.



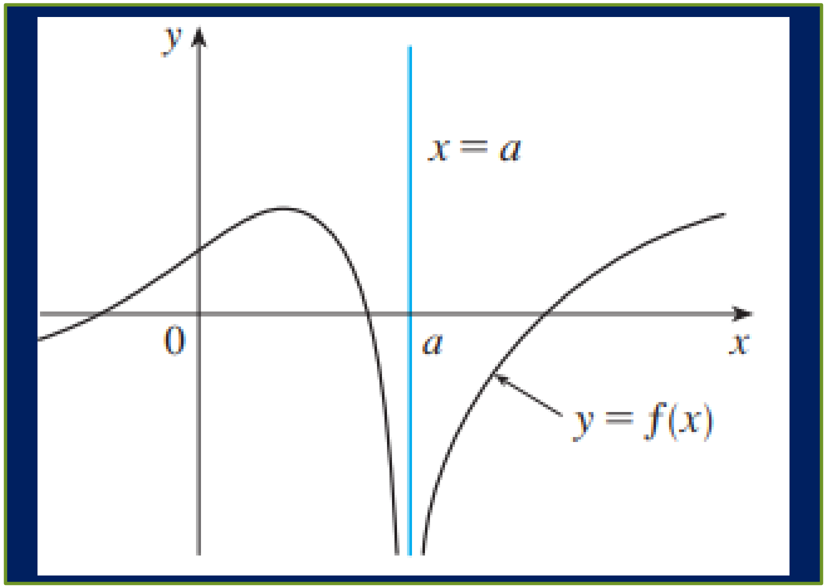
1.6. Limits involving infinity

Infinite limits

Let f be a function defined on both sides of a, except possibly at a itself. Then,

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, *but* not equal to a.



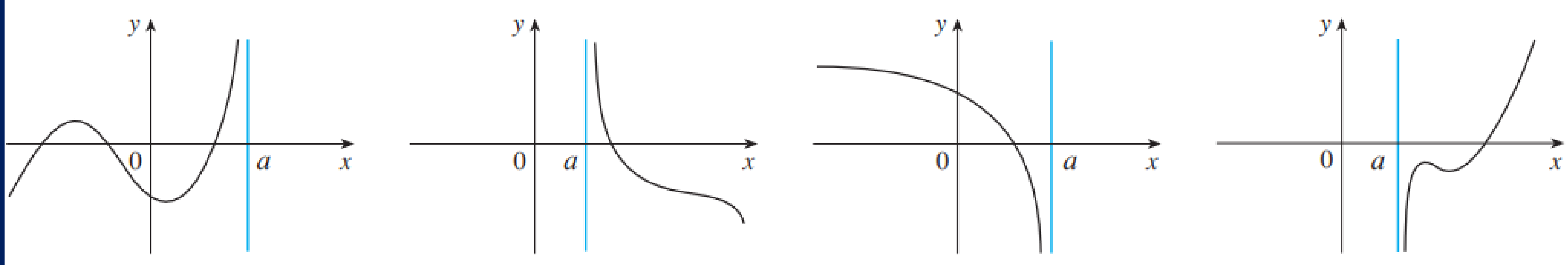
1.6. Limits involving infinity

Infinite limits

Similar definitions can be given for the one-sided limits:

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

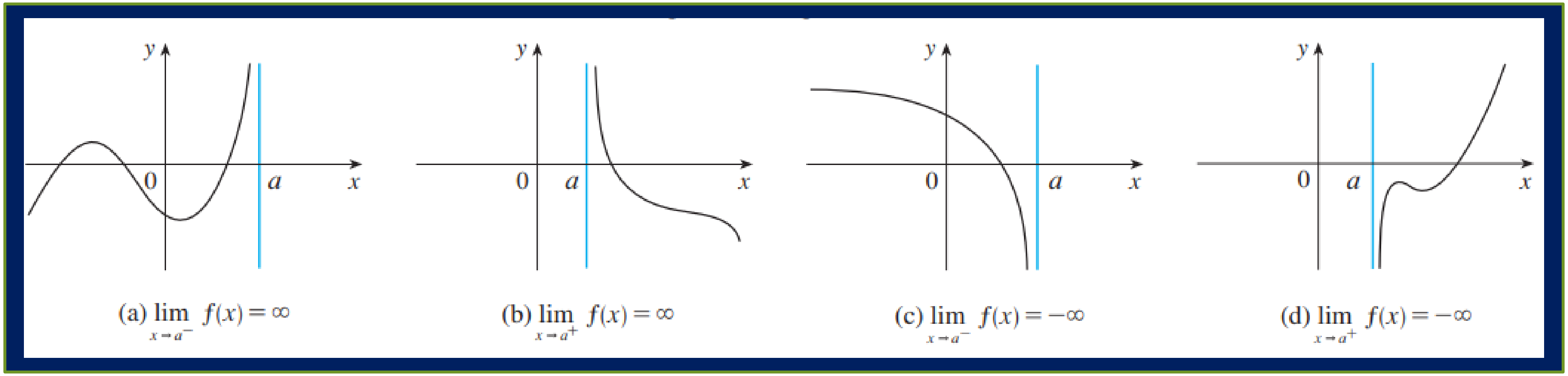
1.6. Limits involving infinity

Definition

$x = a$ is called the vertical asymptote of $f(x)$ if we have one of the following:

$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$



1.6. Limits involving infinity

Definition

The line $y = L\left(y = L'\right)$ is called the **horizontal asymptote** of $f\left(x\right)$ if we have one of the following:

$$\lim_{x\to\infty} f\left(x\right) = L \qquad \lim_{x\to-\infty} f\left(x\right) = L'$$

1.6. Limits involving infinity

Example: Find the asymptotes of the function

$$f(x) = \frac{x^3 - 1}{x^3 + x^2 - 2}$$

1.6. Limits involving infinity

Solution:

$$\bullet \lim_{x\to\infty}\frac{x^3-1}{x^3+x^2-2}=\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{1+\frac{1}{x}-\frac{2}{x^3}}=1$$

So, $y=1$ is horizontal asymptote

$$\bullet \lim_{x\to 1}\frac{x^3-1}{x^3+x^2-2}=\lim_{x\to 1}\frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x^2+2x+2)}$$

$$=\lim_{x\to 1}\frac{x^3+1}{x^3+x^2-2}=\frac{3}{5}$$

Don’t have vertical asymptote.

1.6. Limits involving infinity

Limits at infinity

Let f be a function defined for every $x > a$. Then

$\lim_{x\to\infty} f(x) = L$ means that $\forall \varepsilon > 0, \exists M > 0$

if $x > M$ then $|f(x) - L| < \varepsilon$

1.6. Limits involving infinity

Example: Compute

a/ $\lim_{x\to\infty} \sin\frac{1}{x}$

b/ $\lim_{x\to\infty} (\sqrt{x^2+1}-x)$

c/ $\lim_{x\to\infty} \sin x$

d/ $\lim_{x\to\infty} (x-x^3)$

1.6. Limits involving infinity

Quiz questions

1/ Find $\lim\limits_{x\to\infty} \cos x$

a.0

b. infinity

c. 1

d. Does not exist

2/ Find $\lim\limits_{x\to\infty} \frac{1}{x}\cos x$

a. 0

b. infinity

c. 1

d. Dose not exist

1.6. Limits involving infinity

Quiz questions

3/ If $\lim_{x\to 0} f(x)=\infty, \lim_{x\to 0} g(x)=\infty$ then $\lim_{x\to 0}[f(x)-g(x)]=0$

a. True

b. False

4/A function can have two different horizontal asymptotes

a. True

b. False

1.6. Limits involving infinity

$$x \to 0 : \sin(x) \sim x$$

$$x \to a : u(x) \to 0 : \quad \sin u(x) \sim u(x)$$

$$1.\ \lim_{x\to 0} \frac{\sin(x)}{x} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{x}{x} = 1$$

$$2. \lim_{x\to 0} \frac{\sin 3x}{x} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{3x}{x} = 3(\ 3x \to 0)$$

$$3. \lim_{x\to 0} \frac{\sin 4x}{\sin 6x}$$

$$4. \lim_{x\to 0} \frac{\tan 6x}{\sin 2x}$$

The End