Chapter 3

APPLICATIONS OF DERIVATIVES

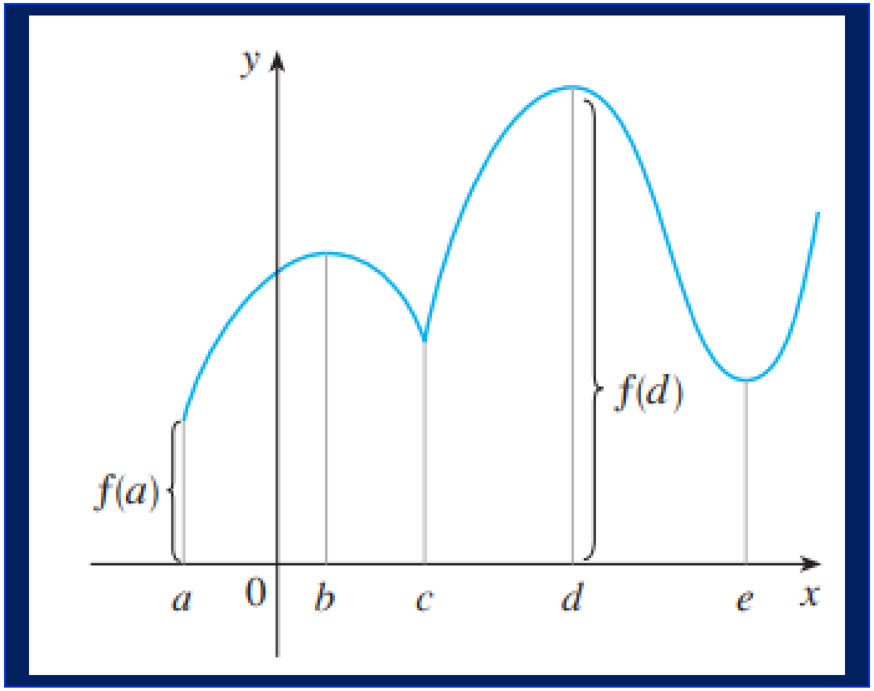
Contents

3.1. Maximum and minimum values

Definition 1

A function f has an **absolute maximum** (or global maximum) at c if $f(c) \geq f(x)$ **for all x in D**, where D is the domain of f.

The number $f(c)$ is called the **maximum value of f on D**.



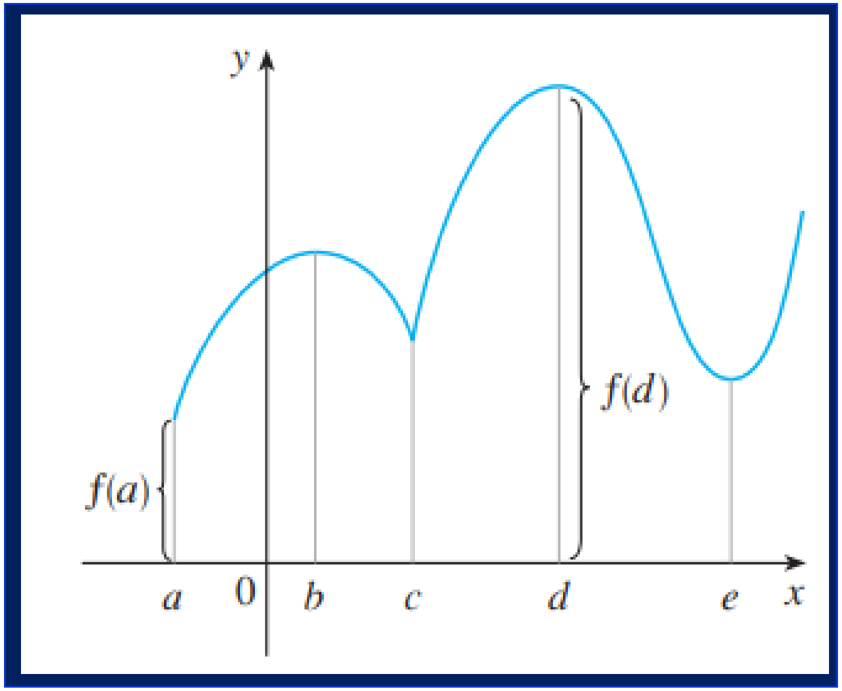
3.1. Maximum and minimum values

Definition 1

Similarly, A function f has an **absolute minimum** (or global **minimum**) at c if $f(c) \le f(x)$ **for all x in D**, where D is the domain of f.

The number $f(c)$ is called the **minimum value of f on D**.

The maximum and minimum values of f are called the **extreme values** of f.



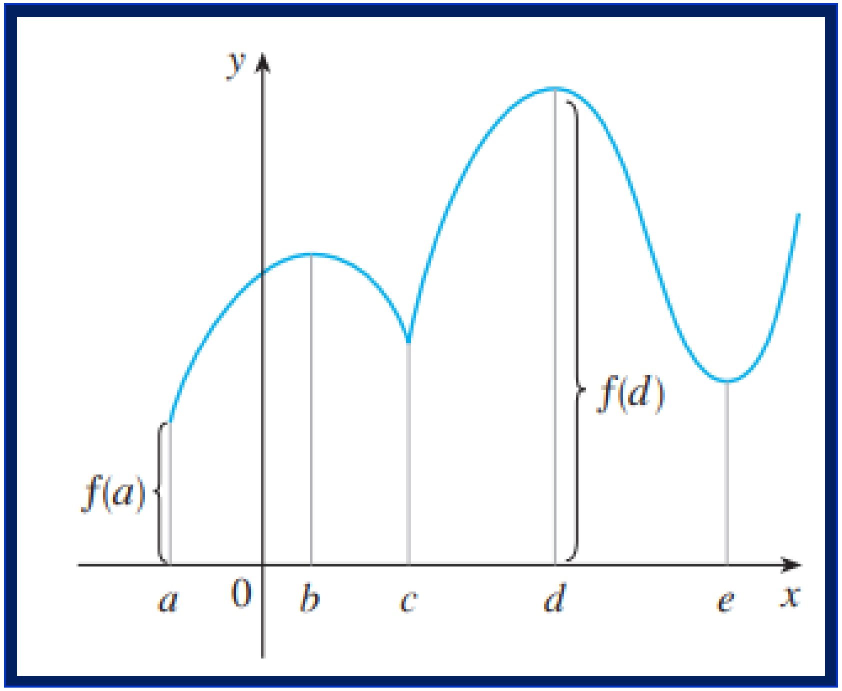
3.1. Maximum and minimum values

Definition 2

- A function f has a **local maximum** (or relative maximum) at c if $f(c) \geq f(x)$ **when x is near c.**

- Similarly, f has a **local minimum** at c if

$f(c) \leq f(x)$ when **x is near c.**



3.1. Maximum and minimum values

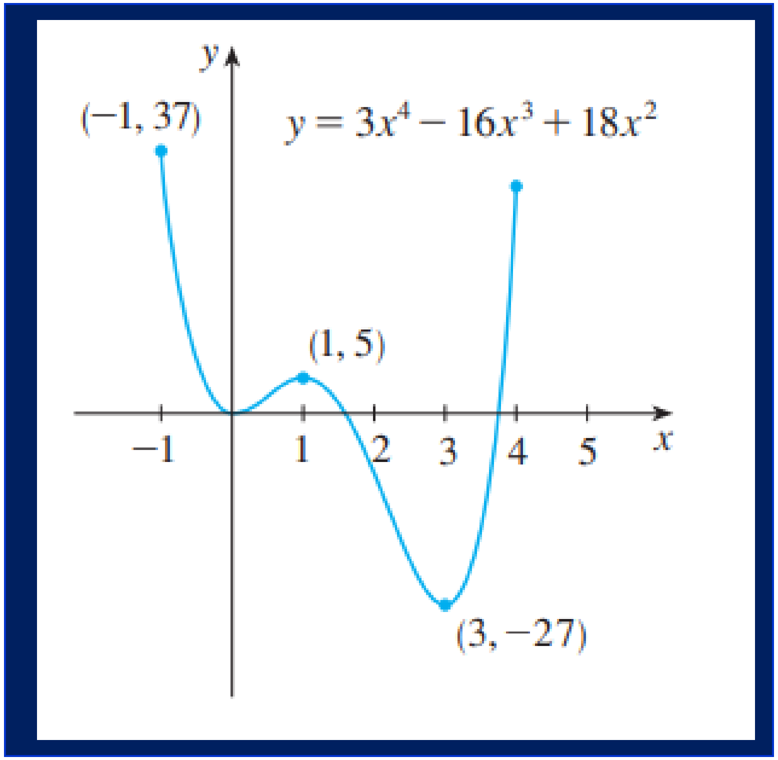
Example 1:

The graph of the function $f(x) = 3x^4 - 16x^3 + 18x^2, -1 \le x \le 4$

is shown here.

- $f(-1) = 37$ is the absolute maximum
- $f(0) = 0$ is a local minimum
- $f(1) = 5$ is a local maximum
- $f(3) = -27$ is both a local and an

absolute minimum.

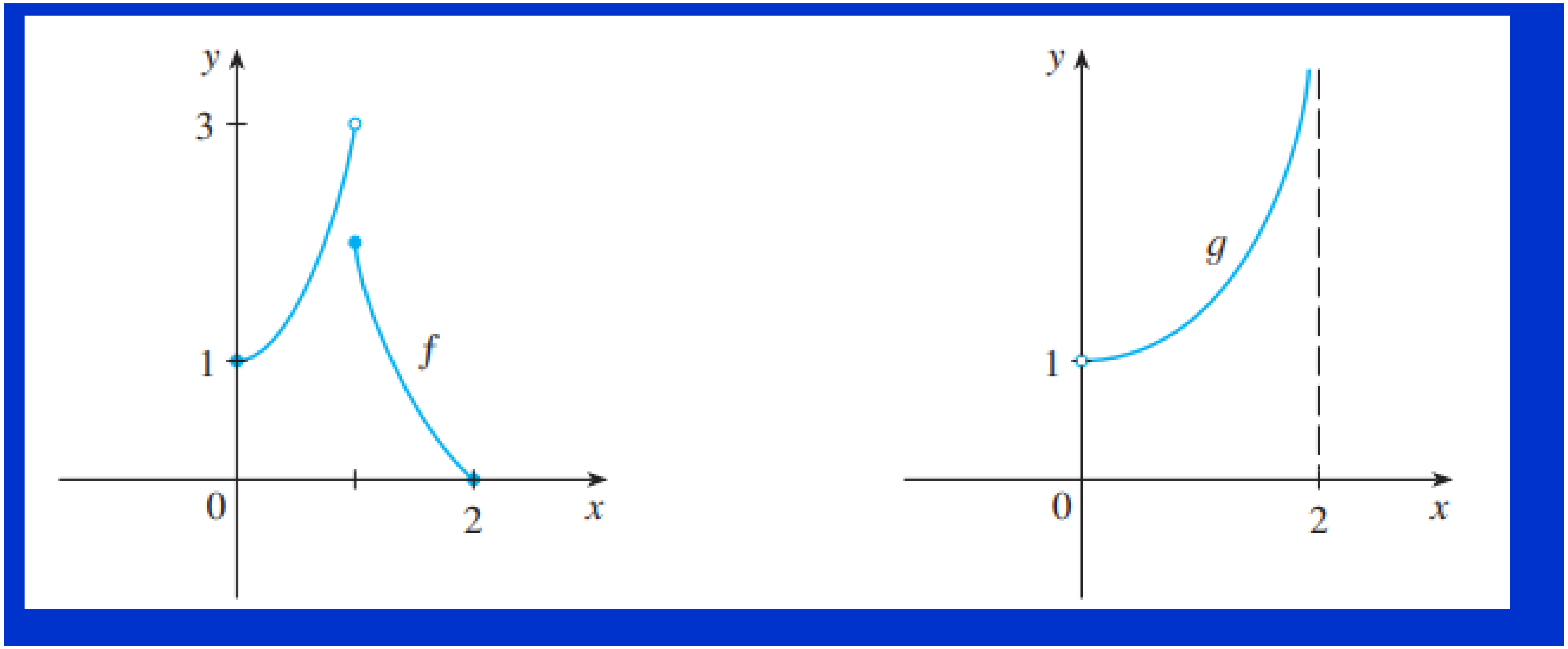


3.1. Maximum and minimum values

Example 2:

In the first figure, why isn't 3 the absolute maximum value ?

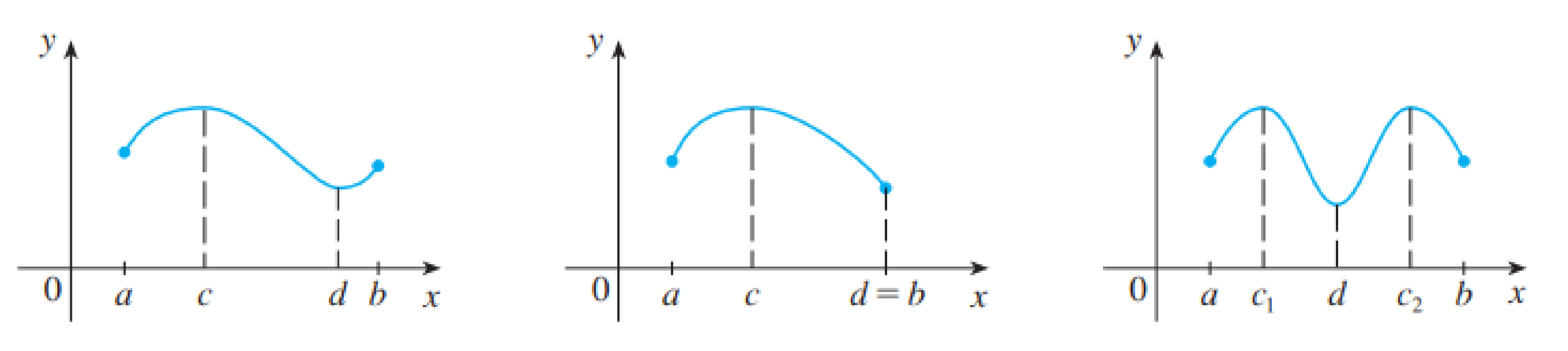
In the second, does it have the absolute maximum and minimum value ?



3.1. Maximum and minimum values

Theorem

If f is continuous on a **closed interval** $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

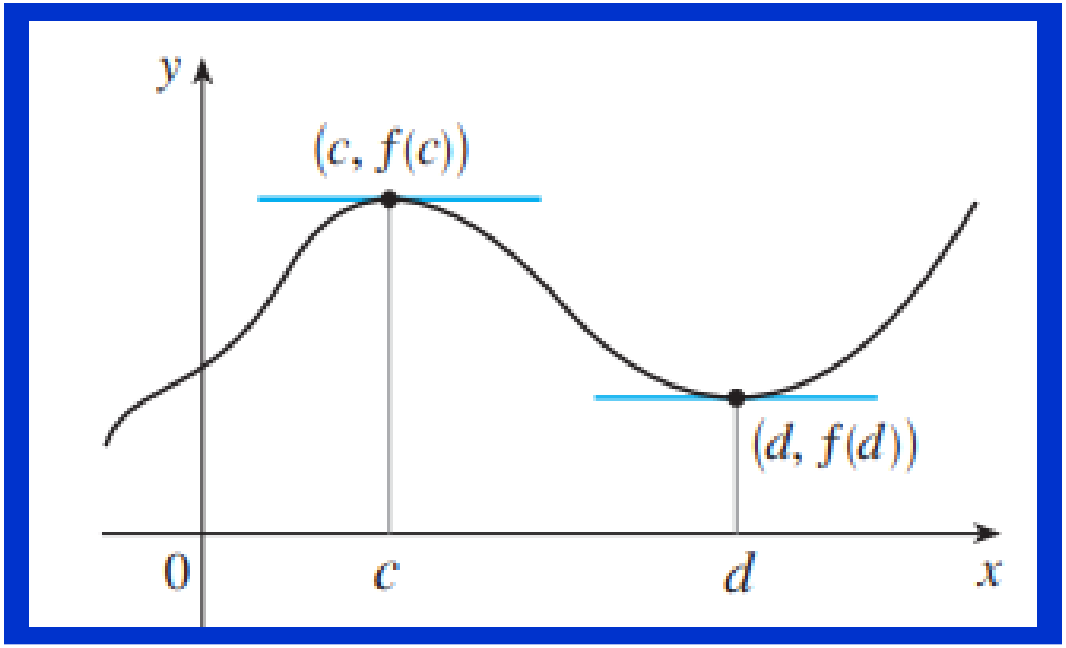


3.1. Maximum and minimum values

Extreme value theorem

The theorem does not tell us how to find these extreme values.

We start by looking for local extreme values.

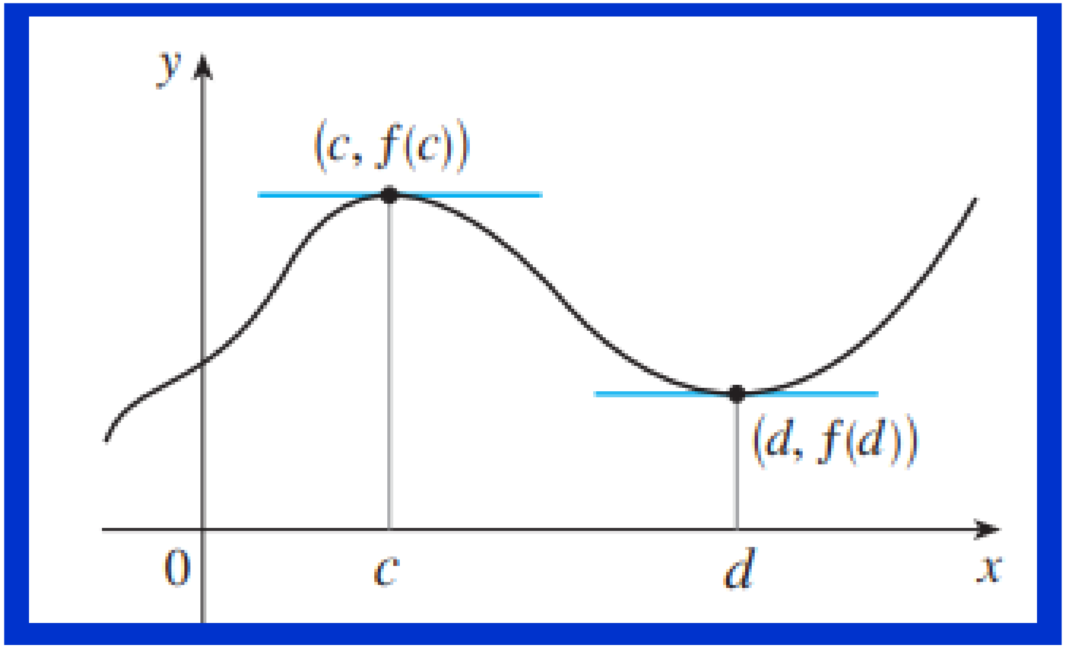


3.1. Maximum and minimum values

Fermat's theorem

Theorem

If f ***has*** a local maximum or minimum at c, and if $\boldsymbol{f'(c)}$ exists, then $f'(c) = 0$.



3.1. Maximum and minimum values

Is it true if say that

” $f'(c)=0$ if f has local extreme value at c ?”

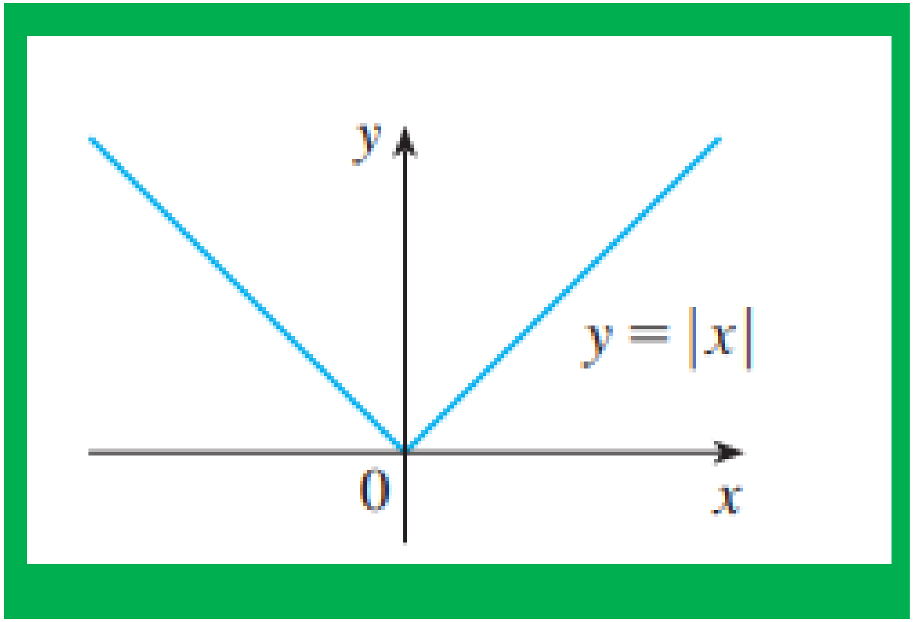
Answer: It is false, see next.

3.1. Maximum and minimum values

Critical numbers

Example

- The function $f(x) = |x|$ has its (local and absolute) minimum value at 0.
- $f'(0)$ does not exist.

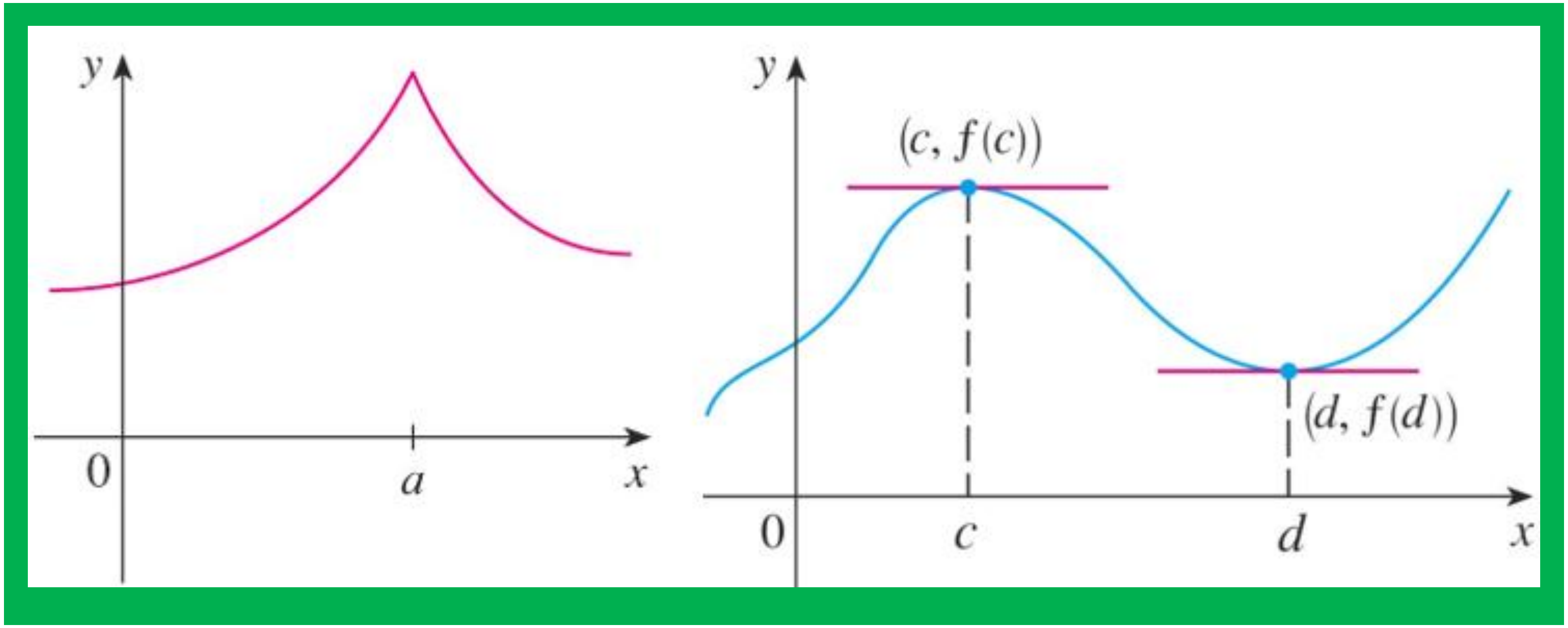


3.1. Maximum and minimum values

Critical numbers

Definition

A **critical number (giá trị tới hạn)** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

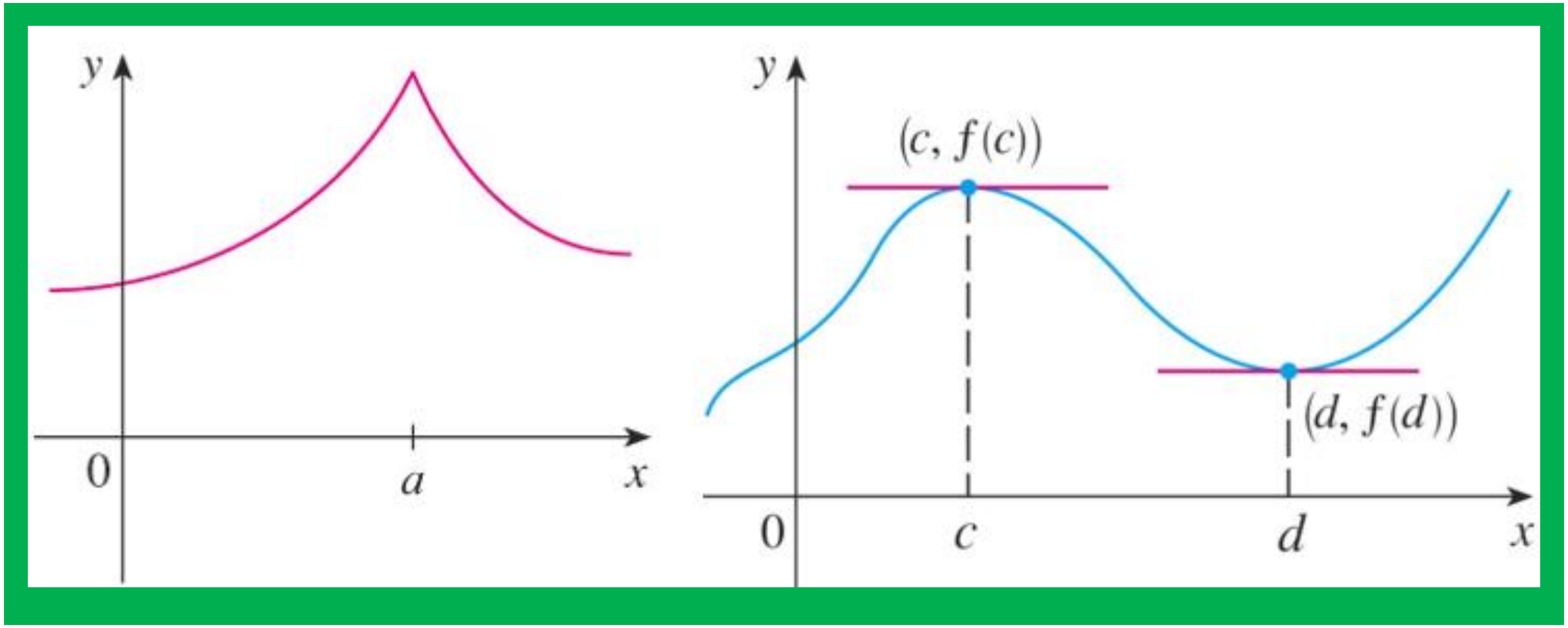


3.1. Maximum and minimum values

Critical numbers

Theorem

If f has a local maximum or minimum at c, then c is a critical number of f.



3.1. Maximum and minimum values

Closed interval method

To find the **absolute** maximum and minimum values of a continuous function *f* on a closed interval [*a*, *b*]:

1/ Find the values of *f* at the critical numbers of *f* in (*a*, *b*).

2/ Find the values of *f* at the endpoints of the interval.

3/ The largest value from 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

3.1. Maximum and minimum values

Closed interval method

Example:

Find absolute extreme for the function

$$f(x)=\frac{x-2}{x^2+1} \text{ on the interval.}$$

3.1. Maximum and minimum values

Quize question

Select the correct ones.

a. If f'(c)=0 then f has the local maximum or minimum at c.

b. If f has the absolute minimum value at c then f'(c)=0.

c. If f is continuous on (a,b) then f attains an absolute maximum value f(c) and an absolute minimum value f(d) for some c and d in (a,b).

d. All of the above.

e. None of the above.

3.2.The Mean Value Theorem

Rolle's theorem

Let f be a function that satisfies the following three hypotheses:

1/ f is continuous on the closed interval [a, b]

2/ f is differentiable on the open interval (a, b)

3/ f(a) = f(b)

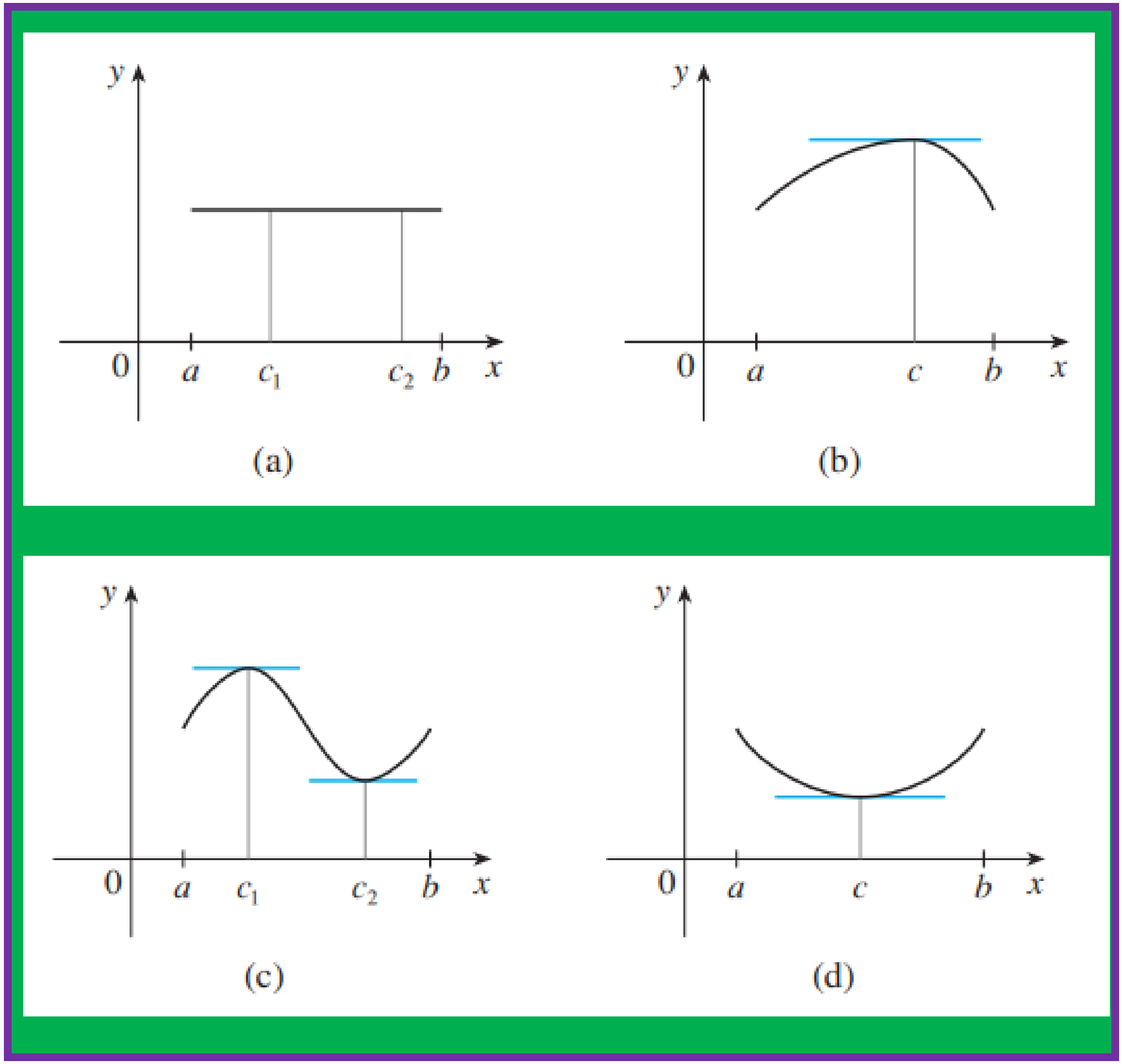
Then, there is a number c in (a, b) such that

$f'(c) = 0$.

3.2.The Mean Value Theorem

Rolle's theorem

The figures show
the graphs of
four such
functions.



3.2.The Mean Value Theorem

Example:

Let $f(x) = x^3 - 2x^2 + x - 5, [0,1]$. Find the numbers c in the Rolle's theorem ?

3.2.The Mean Value Theorem

Example:

Let $f\left(x\right)=x^3-2x^2+x-5,\left[0,1\right]$. Find the numbers c in the Rolle's theorem ?

Solution:

1/ f is continuous on the closed interval [0, 1]

2/ f is differentiable on the open interval (0, 1)

3/ f(0) = f(1)= - 5.

Then $f'\left(c\right)=0\Rightarrow 3c^2-4c+1=0\Leftrightarrow c=1\notin\left(0,1\right),c=\dfrac{1}{3}\in\left(0,1\right)$

So, $c=\dfrac{1}{3}$.

3.2.The Mean Value Theorem

Mean value theorem

Let f be a function that fulfills two hypotheses:

a/ f is continuous on the closed interval [a, b].

b/ f is differentiable on the open interval (a, b).

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$

3.2.The Mean Value Theorem

Mean value theorem

Example

Let $f(x)=2x^2-3x+1,[0,2]$ .Find all numbers that satisfy the conclusion of the Mean Value Theorem.

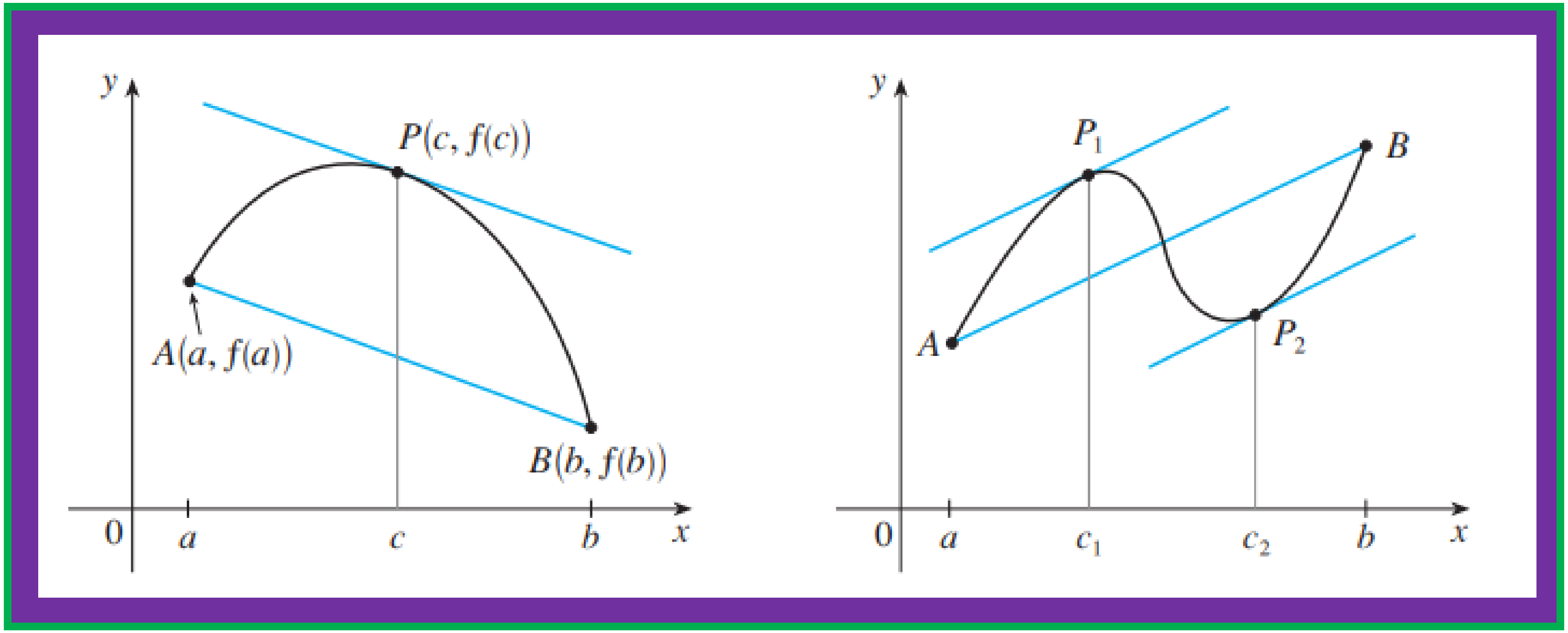
Solution

$$f'(c)=\frac{f(b)-f(a)}{b-a}\Rightarrow 4c-3=\frac{3-1}{2-0}=1\Rightarrow c=1$$

3.2.The Mean Value Theorem

Mean value theorem

$f'(c)$ is the slope of the tangent line at $(c, f(c))$.There is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB.



3.2.The Mean Value Theorem

Mean value theorem

Example

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be ?

Solution

We are given that f is differentiable - and therefore continuous - everywhere.

In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$.

3.2.The Mean Value Theorem

There exists a number c such that $f(2) - f(0) = f'(c)(2 - 0)$

So, $f(2) = -3 + 2f'(c)$

We are given that $f'(x) \le 5\ \forall x$

$\Rightarrow f'(c) \le 5.$

$\Rightarrow 2f'(c) \le 10.$

$\Rightarrow f(2) = -3 + 2f'(c) \le -3 + 10 = 7$

The largest possible value for $f(2)$ is 7.

3.2.The Mean Value Theorem

Theorem

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary

If $f'(x) = g'(x)$ for all x in an interval (a, b),

then $f - g$ is constant on (a, b).

That is, $f(x) = g(x) + c$ where c is a constant.

Increasing/Decreasing test (I/D test)

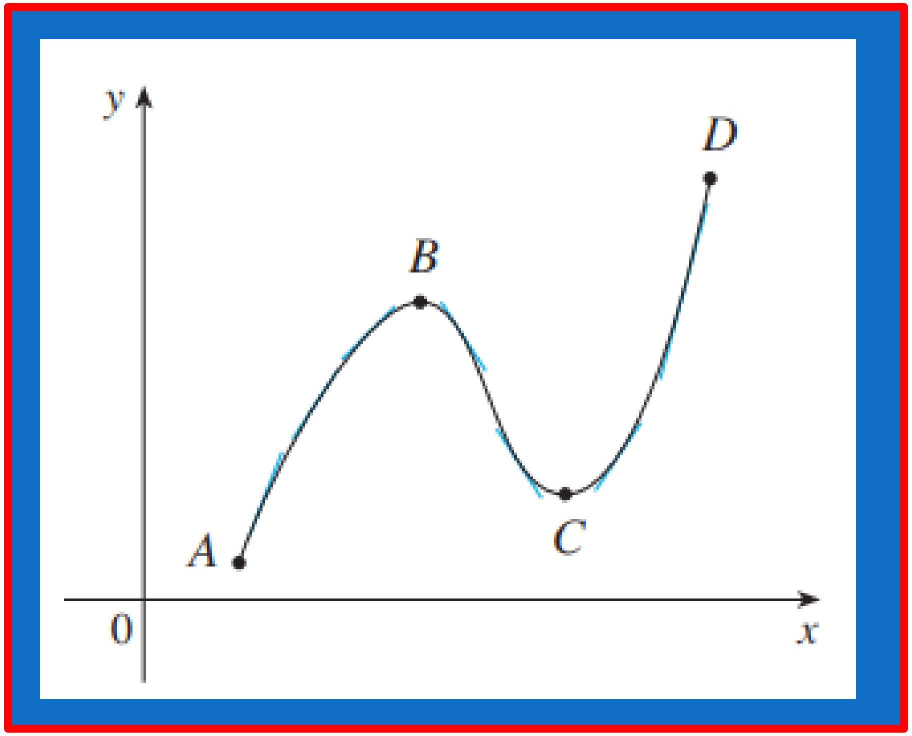
a/ If $f'(x) > 0$ on an interval,

then f is increasing
on that interval.

b/ If $f'(x) < 0$ on an interval,

then f is decreasing on

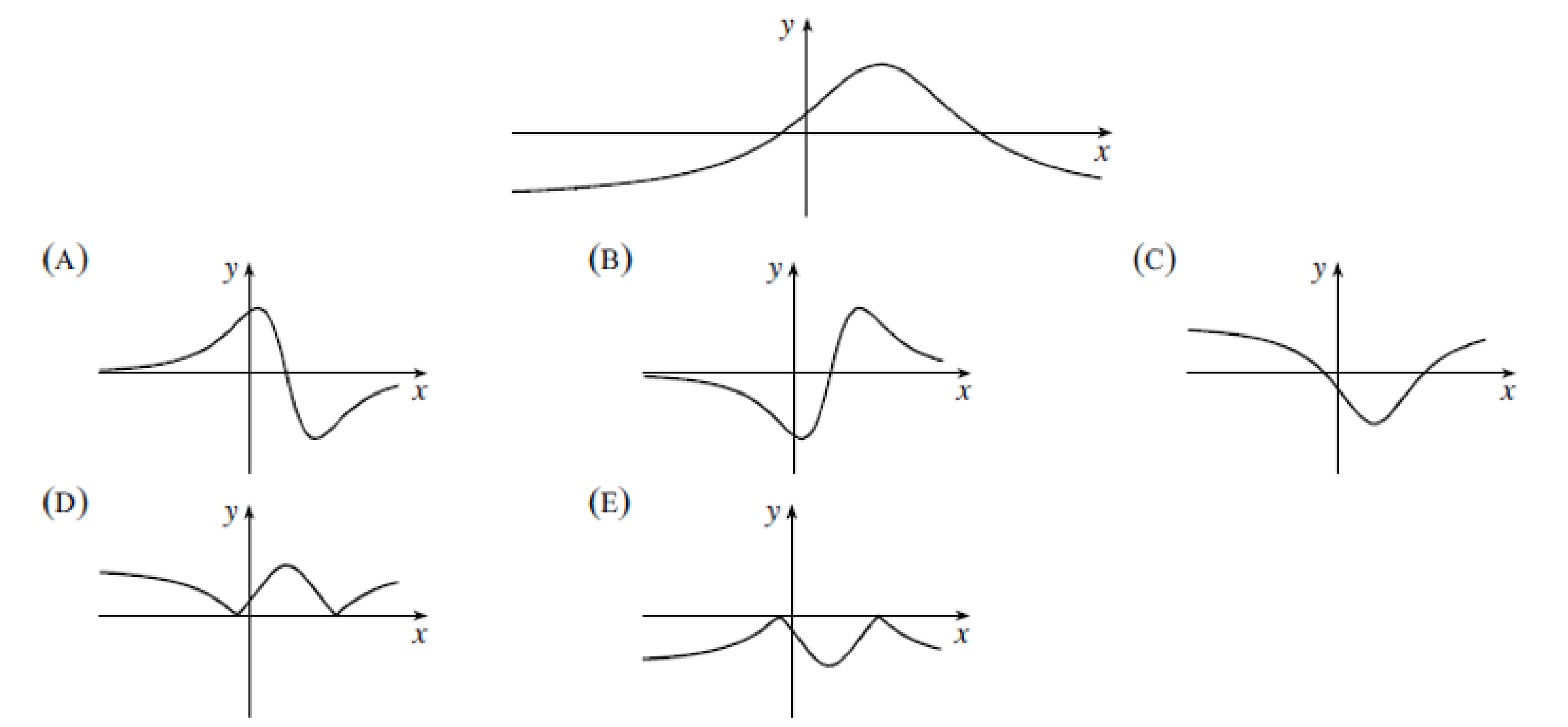
that interval.



3.3.Derivatives and the Shapes of Graphs

Increasing/Decreasing test (I/D test)

Example: The graph of f is show below. Which of the following could be the graph of f' ?

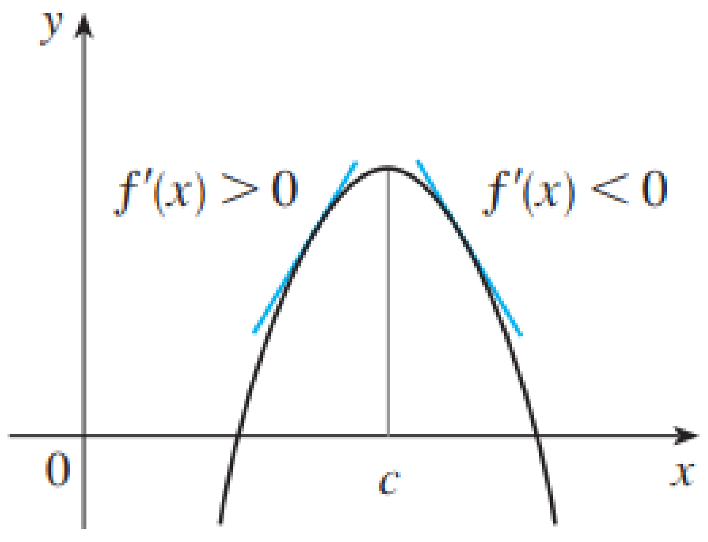


3.3.Derivatives and the Shapes of Graphs

First derivative test

Suppose that c is a critical number of a continuous function f.

a/ If f' changes from positive to negative at c, then f has a local maximum at c.



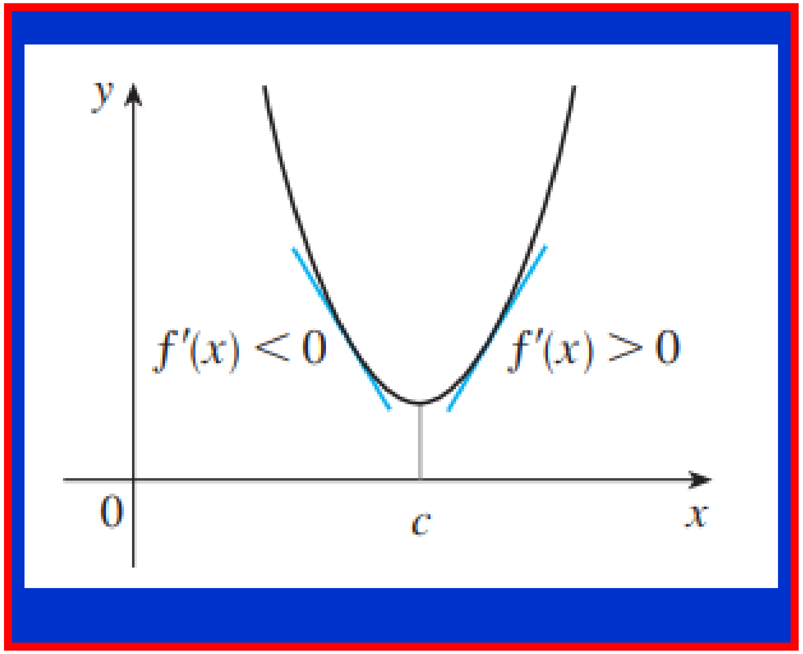
(a) Local maximum

3.3.Derivatives and the Shapes of Graphs

First derivative test

Suppose that c is a critical number of a continuous function f.

b/ If f' changes from negative to positive at c, then f has a local minimum at c.

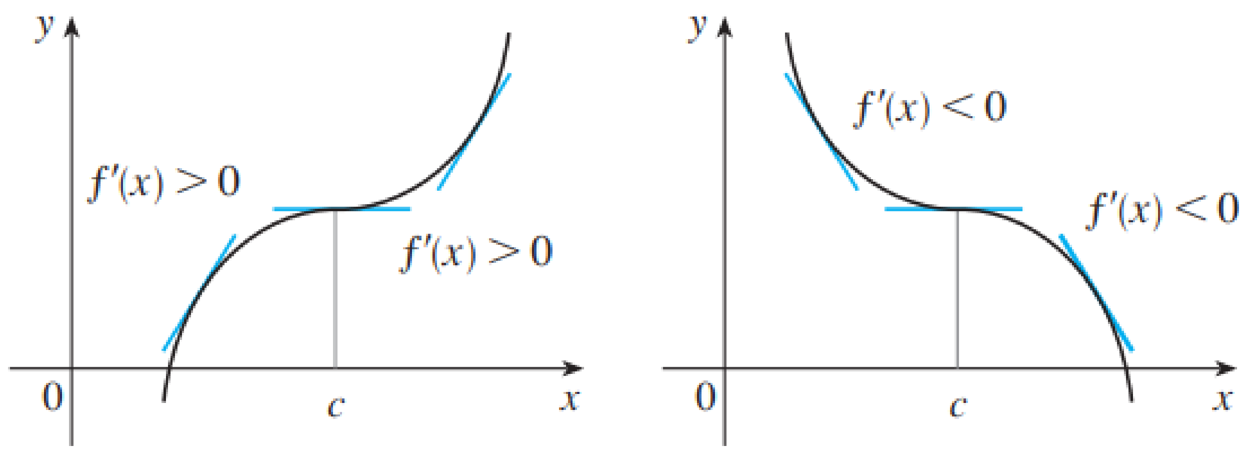


3.3.Derivatives and the Shapes of Graphs

First derivative test

Suppose that c is a critical number of a continuous function f.

c/ If f' does not change sign at c then f has no local maximum or minimum at c.



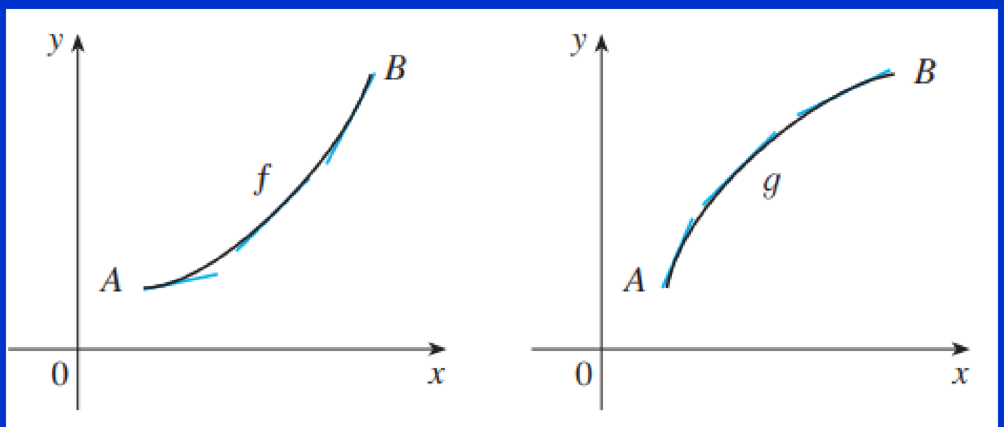
(c) No maximum or minimum (d) No maximum or minimum

3.3.Derivatives and the Shapes of Graphs

Concave upward/ downward

- The curve lies above the tangents and f is called **concave upward** (lõm lên) on (a, b).

- The curve lies below the tangents and g is called **concave downward** (lõm xuống) on (a, b).



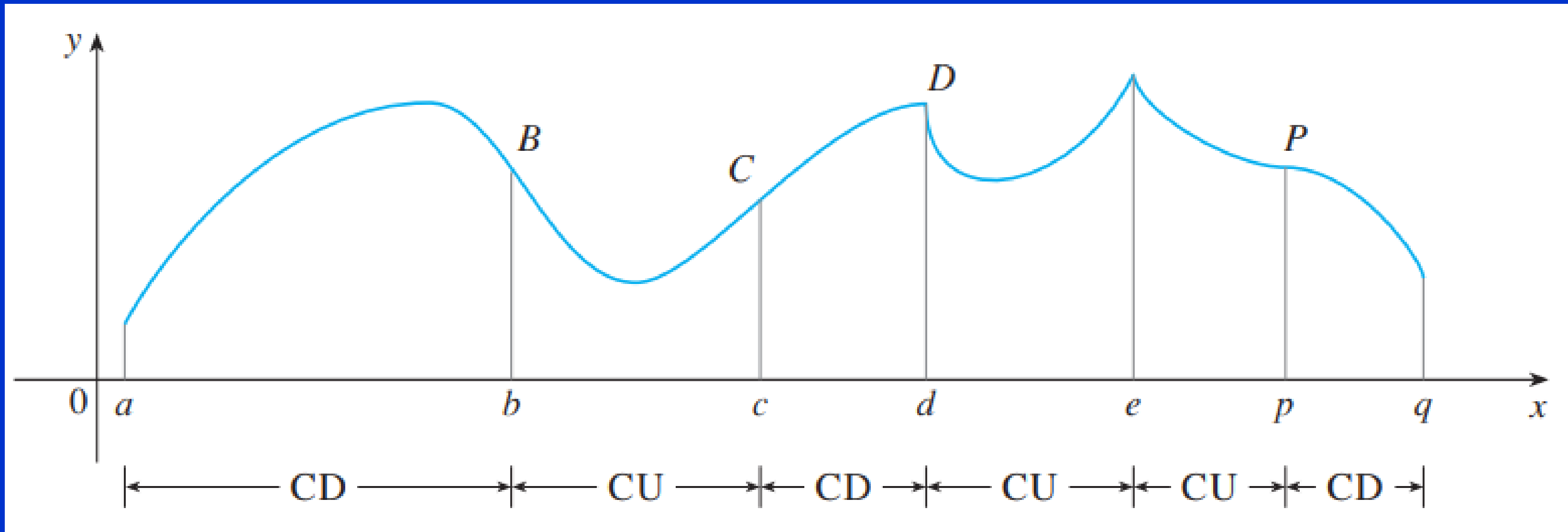
(a) Concave upward (b) Concave downward

3.3.Derivatives and the Shapes of Graphs

Concavity test

a/ If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

b/ If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



3.3.Derivatives and the Shapes of Graphs

Inflection point-Definition

A point P on a curve $y = f(x)$ is called an **inflection point (điểm uốn)**

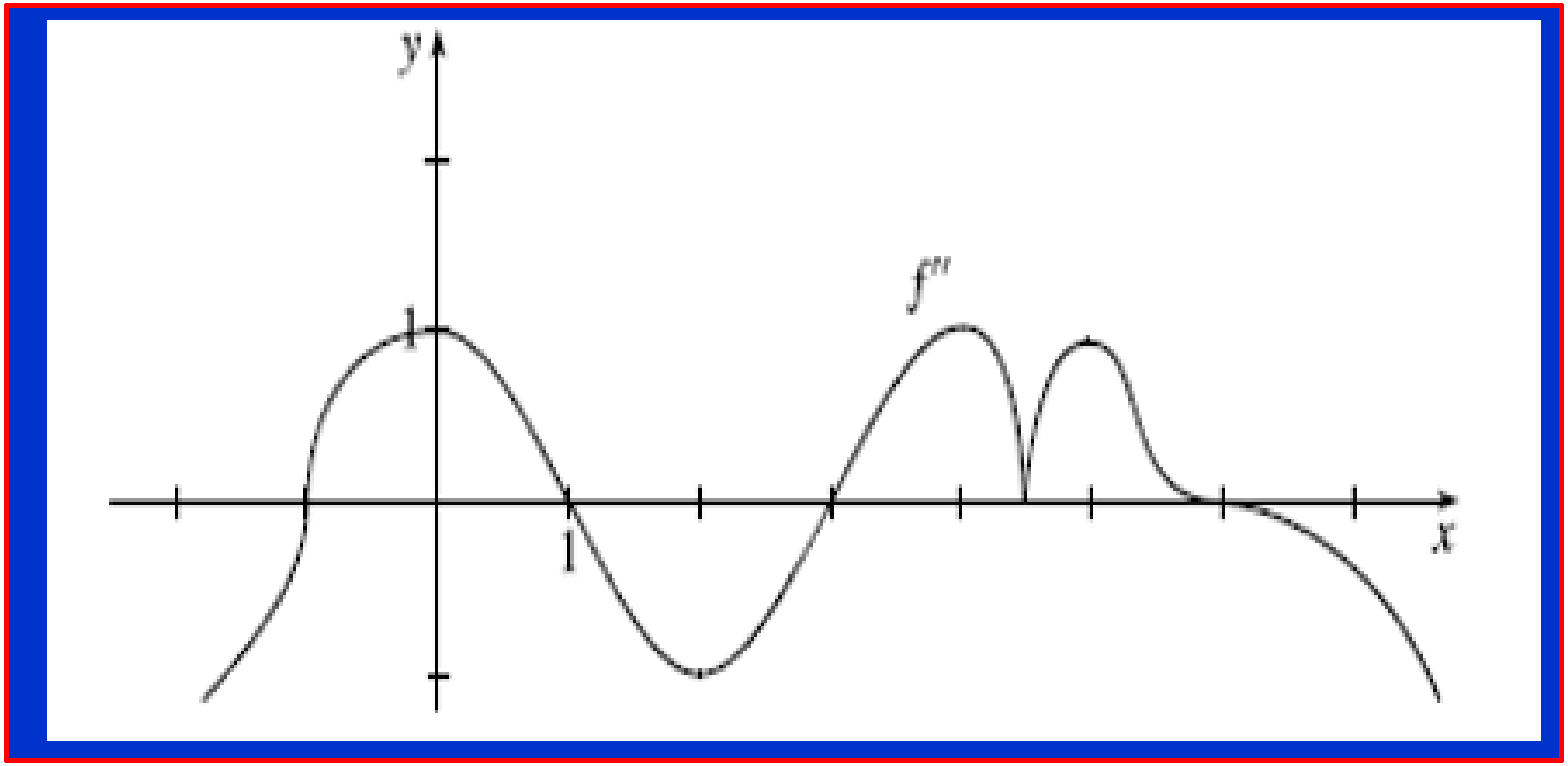
if f is continuous there and the curve changes
from concave upward to concave downward

(or from concave downward to concave upward at P).

3.3.Derivatives and the Shapes of Graphs

Example

Given a grap of f'' as below, have the students indicate the points of inflection of f, and explain their reasoning.



3.3.Derivatives and the Shapes of Graphs

Answer:

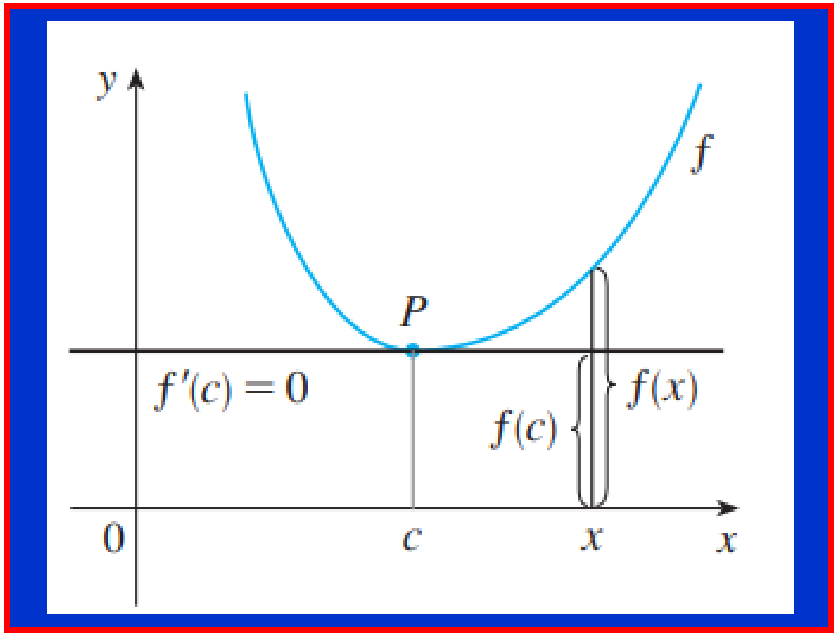
$(-1,f(-1)),(1,f(1)),(3,f(3)),(6,f(6))$

3.3.Derivatives and the Shapes of Graphs

Second derivative test.

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $\boldsymbol{f''(c) > 0}$, then f has a **local minimum** at c.
- If $f'(c) = 0$ and $\boldsymbol{f''(c) < 0}$, then f has a **local maximum** at c.



3.3.Derivatives and the Shapes of Graphs

Example

Choose the correct one.	
A	If f has local extreme value at c then f'(c)=0.
B	If f'(c)=0 then f has local extreme value at c.
C	If f''(3)=0 then (3,f(3)) is an inflection point of f.
D	There exists a function such that f'(x) is nonzero for all x and f(1)=f(0).
E	None of the above

3.5. Optimization Problems

Understand the problem

Read the problem carefully until it is clearly understood.

- What is the **unknown**?
- What are the **given quantities**?
- What are the given **conditions**?

3.5. Optimization Problems

Example 1:

a/ Find two positive numbers such that the sum is 24 and the product is the largest ?

b/ Find two positive numbers such that the product is 36 and the sum is the smallest ?

3.5. Optimization Problems

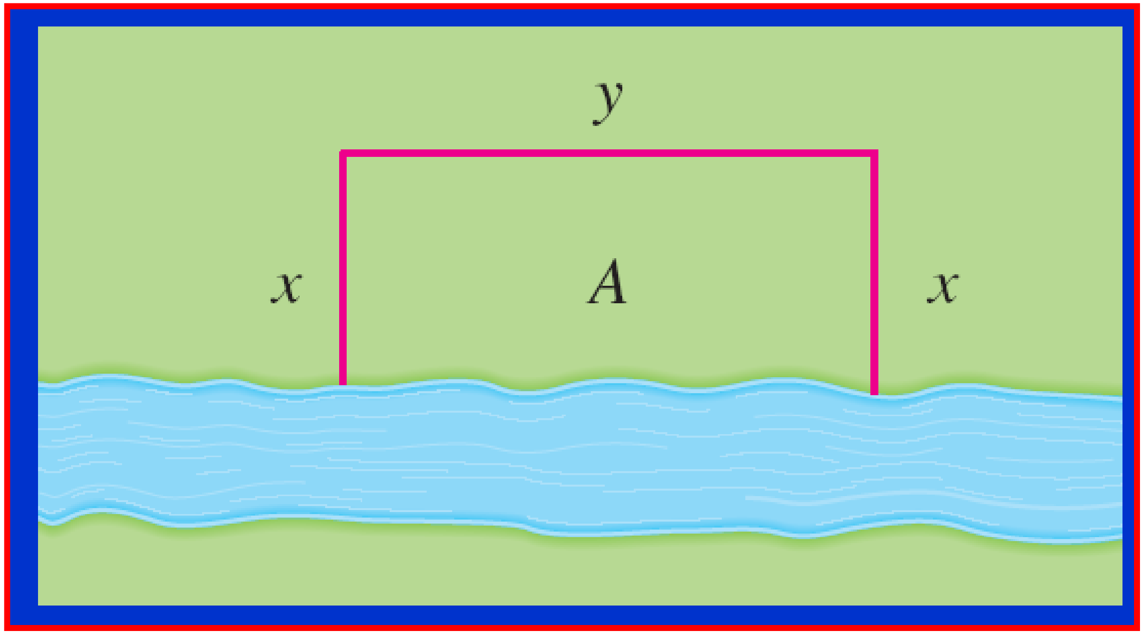
Example 2:

Find the point on the line $y = 2x - 3$ that is closest to the origin.

3.5. Optimization Problems

Example 3:

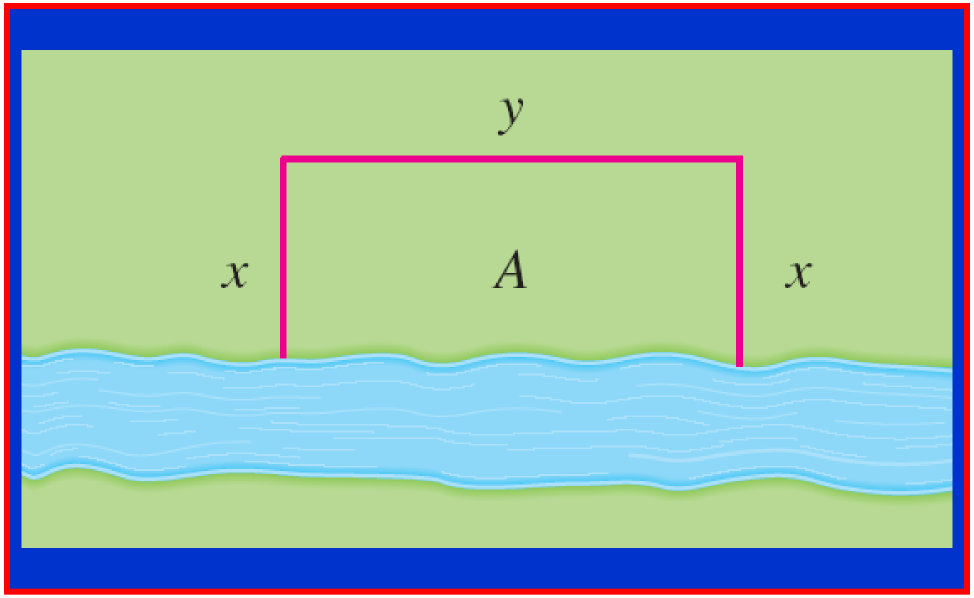
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the **largest area** ?



3.5. Optimization Problems

Solution

This figure illustrates the general case.



We wish to maximize the area A of the rectangle.

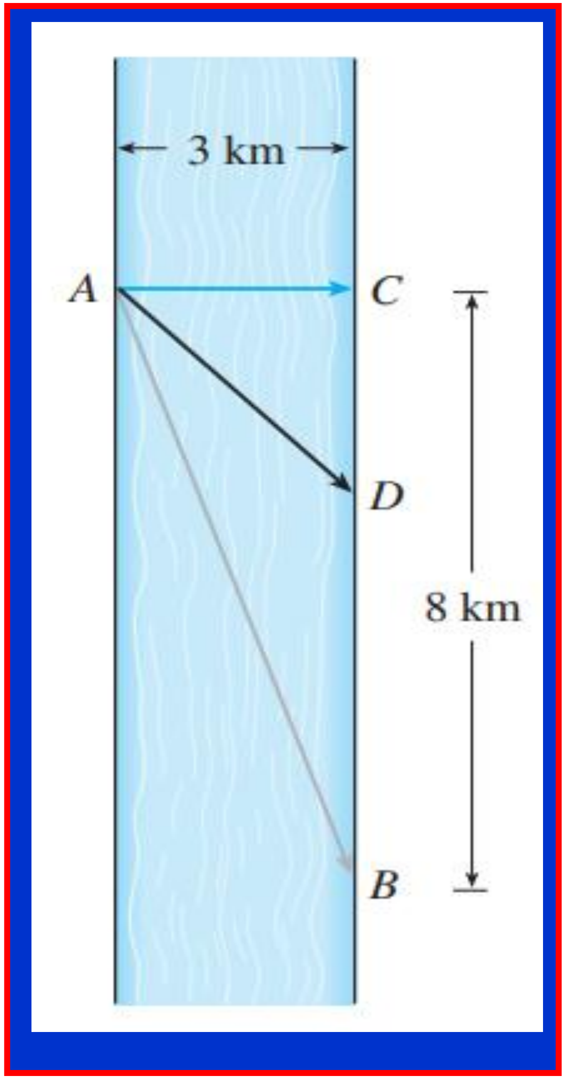
- Then, we express A in terms of x and y: $A = xy$
- $2x + y = 2400$
- So, $A(x) = 2400x - 2x^2$, $0 \leq x \leq 1200$……

3.5. Optimization Problems

Example 4

A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B (8 km downstream on the opposite bank) as quickly as possible.

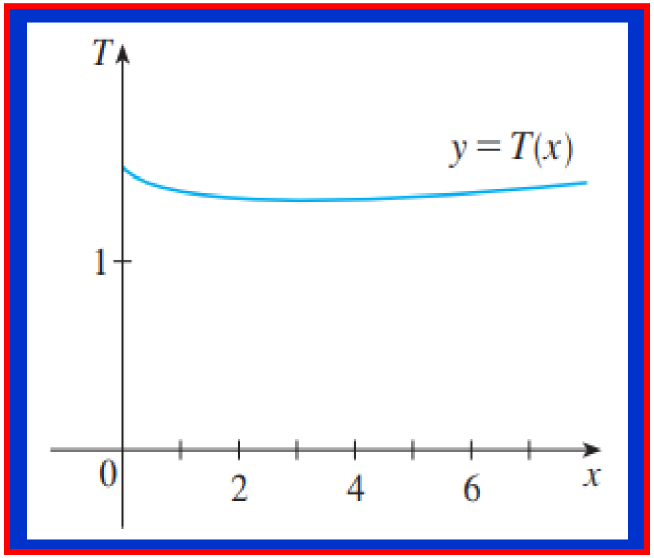
If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible ?



3.5. Optimization Problems

Solution

We use the equation: $time = \dfrac{\text{distance}}{\text{rate}}$



- Then, the rowing time is: $\sqrt{x^2+9}/6$
- The running time is: $(8 - x)/8$
- So, the total time T as a function of x is:

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

3.5. Optimization Problems

Example 5:

A rectangular storage container with an open top is to have a volume of 15 m^3 The length of its base is twice the width.

Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter.

Find the **cost** of materials for the **cheapest** such container ?

3.6. Newton's Method

Numerical rootfinders

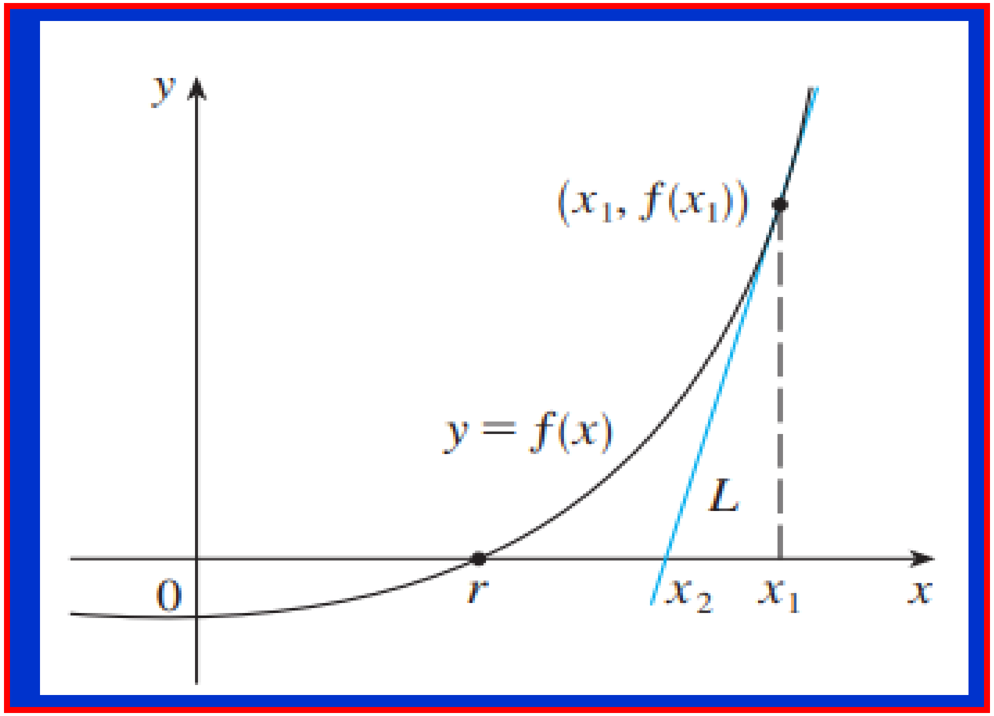
How do those numerical root finders work?

- They use a variety of methods.
- Most, though, make some use of Newton's method, also called the Newton-Raphson method.

3.6. Newton's Method

We start with a first approximation $x_{1,}$ which is obtained by one of the following methods:

- Guessing
- A rough sketch of the graph of f
- A computer-generated graph of f



- Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.

3.6. Newton's Method

Second approximation

As the x-intercept of L is x_2, we set $y = 0$ and obtain:

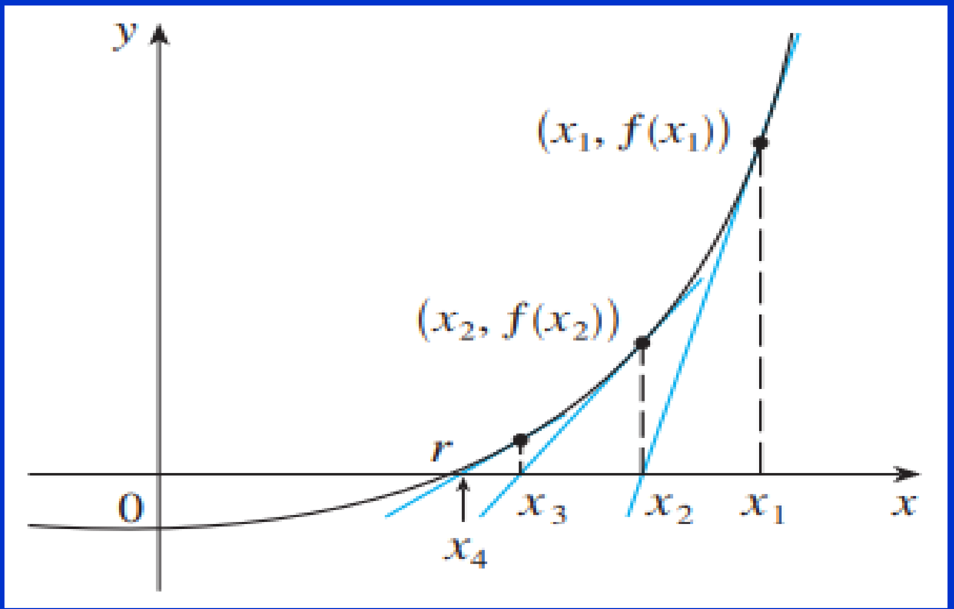
$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second

approximation to r.

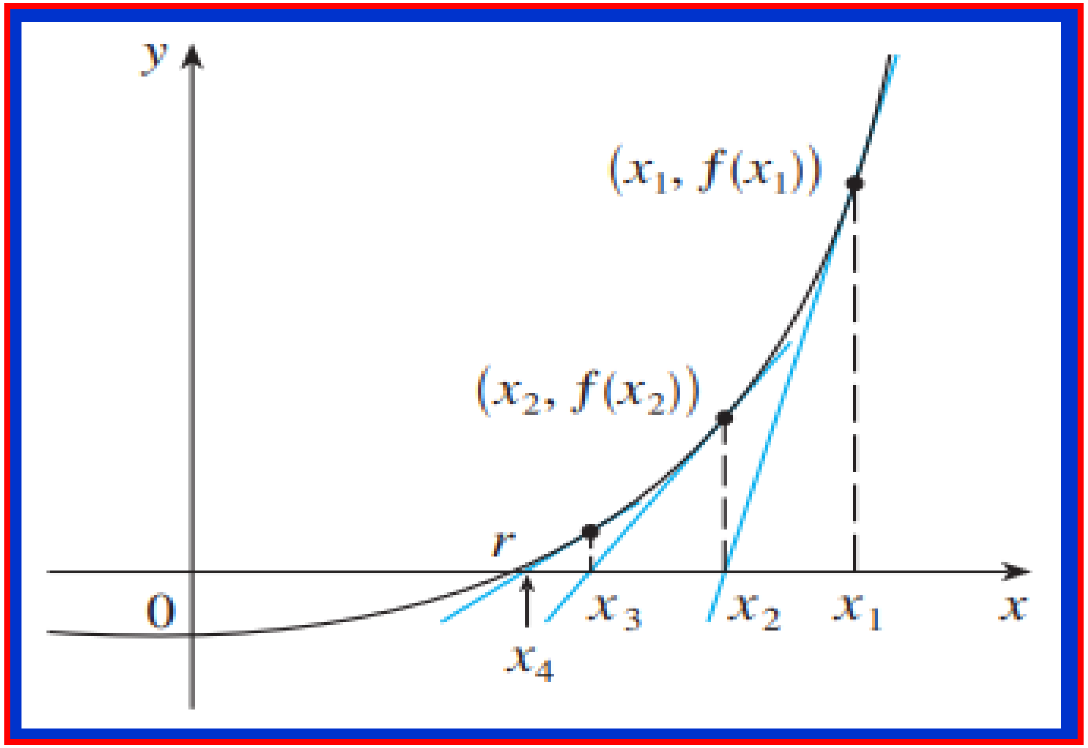


3.6. Newton's Method

Subsequent approximation

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

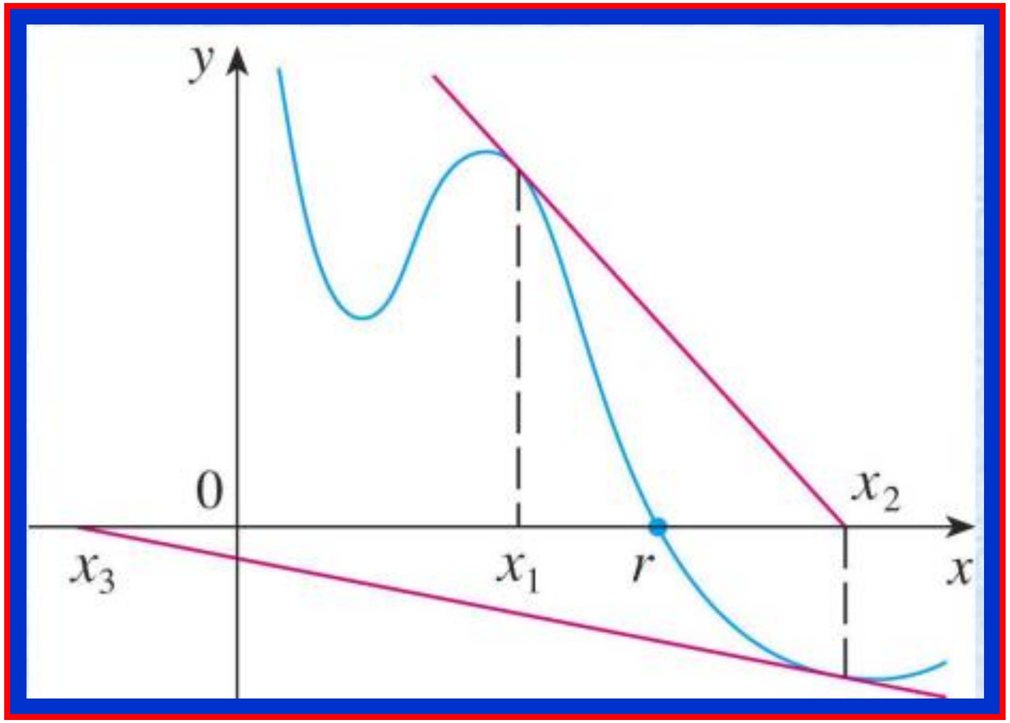


3.6. Newton's Method

Convergence

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write:

$$\lim_{n\to\infty} x_n = r$$



3.6. Newton's Method

Example 1:

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$

3.6. Newton's Method

Solution:

We apply Newton's method with

$$f(x) = x^3 - 2x - 5,\; f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$, after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$. Then

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$ we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} = 2 - \frac{2^3 - 2.2 - 5}{3.2^2 - 2} = 2.1$$

3.6. Newton's Method

Solution:

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = 2.1 - \frac{(2.1)^3 - 2.(2.1) - 5}{3.(2.1)^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.

3.6. Newton's Method

$$\sqrt[6]{2} = x \Leftrightarrow x^6 = 2$$
$$\Leftrightarrow x^6 - 2 = 0$$

Example 2

Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

Solution: First, we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$

So, we take $f(x) = x^6 - 2$. Then, $f'(x) = 6x^5$.

So, Formula 2 (Newton's method) becomes:

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

3.6. Newton's Method

Choosing $x_1 = 1$ as the initial approximation, we obtain:

$$x_2 \approx 1.16666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

As x_5 and x_6 agree to eight decimal places, we conclude that $\sqrt[6]{2} \approx 1.12246205$ to eight decimal places.

3.6. Newton's Method

Example 3:

Use Newton's methed to find x_4 , the fourth approximation to the positive root of the equation

$$x^3 - 2x - 1 = 0$$

Choose your own x_1

3.6. Newton's Method

Solution:

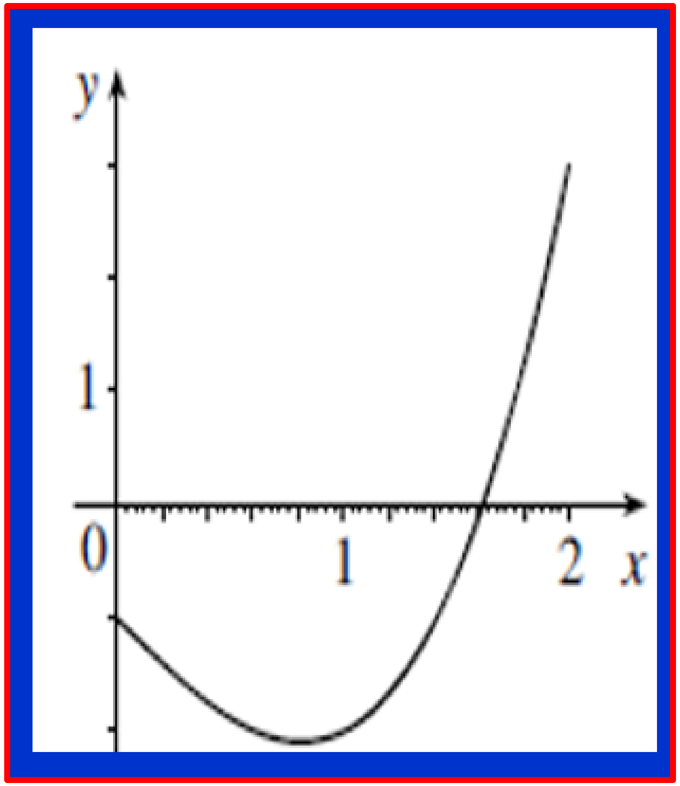
The graph of $f(x) = x^3 - 2x - 1$ gives 1.6 as a reasonable choice $x_1 = 1.6$.

$$x_2 \approx 1.6183099$$

$$x_3 \approx 1.6180341$$

$$x_4 \approx 1.6180340$$

$$x_5 \approx x_4 \approx 1.6180340$$



3.7. Antiderivatives

Definition

A function F is called an **antiderivative** of f on an interval I if $\boldsymbol{F'(x) = f(x)}$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

Antiderivatives formular

Here, we list some particular antiderivatives.

Function	Particular Antiderivatives	Function	Particular Antiderivatives
$c\,f(x)$	$c\,F(x)$	e^x	e^x
$f(x)+g(x)$	$F(x)+G(x)$	a^x	$\frac{a^x}{\ln a}$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\sin x$	$-\cos x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\cos x$	$\sin x$

Antiderivatives formular

Here, we list some particular antiderivatives.

Function	Particular Antiderivatives	Function	Particular Antiderivatives
$\dfrac{1}{\cos^2 x}$	$\tan x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\dfrac{1}{\sin^2 x}$	$-\cot x$	$\dfrac{1}{1+x^2}$	$\arctan x$

Example 1: Evaluate:

1/ $\displaystyle\int\left(x^2-\frac{1}{x}+2\sqrt{x}-1\right)dx$

2/ $\displaystyle\int x(x-1)^2\,dx$

3/ $\displaystyle\int\frac{x^4-x^2+1}{x^2}\,dx$

3.7. Antiderivatives

Rectilinear motion

Example 1:

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$?

Rectilinear motion

Example 2:

A particle moves along the x-axis so that its velocity at time t is given by $s(t) = 3\sin 2t$.

Assuming it starts at the origin, where is it at $t = \pi$ seconds ?

a. 0 b. 3/2 c. 1/2 d. - 1/2