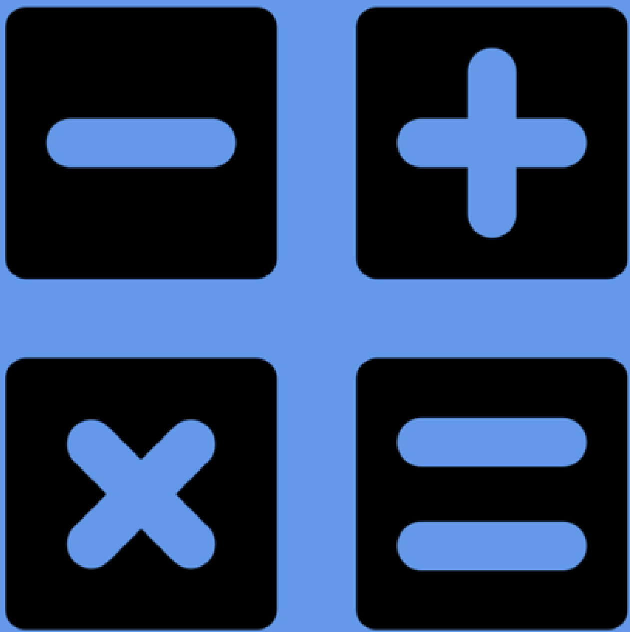
Handbook MAE101 - Summary Mae - Mae

HANDBOOK

ENGINEERING MATH



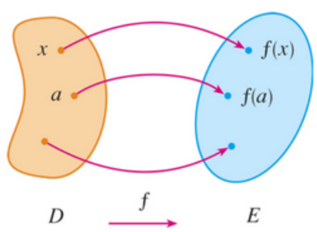
INCLUDES TIPS

Essential tips for Engineering Mathematics

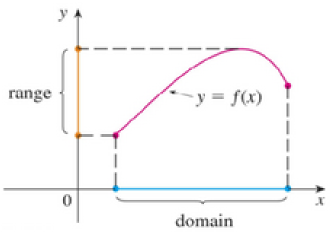
FUNCTION AND GRAPH

Function :A function f is a rule that assigns to each element x in a set D exactly one element, called f(x), in a set E.
The set D is called the domain of the function f.

The range of f is the set of all possible values of f(x) as x varies throughout the domain.



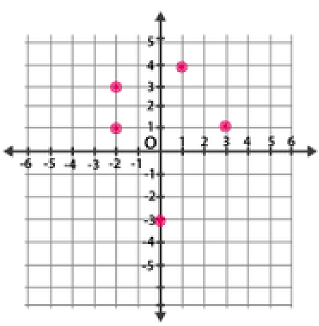
Graph: The graph of f is the set of all points (x, y) in the coordinate plane such that y = f(x) and x is in the domain of f.



Representations of Functions: There are four possible ways to represent a function:
+Algebraically (by an explicit formula)
+Visually (by a graph)
+Numerically (by a table of values)
+Verbally (by a description in words)

x	y
-2	1
-2	3
0	-3
1	4
3	1

Relation in table



Relation in graph

R

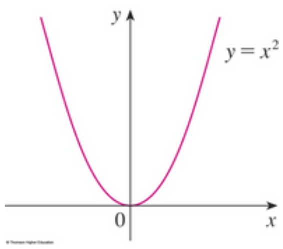
Increasing function: A function is said to be increasing on an interval if, for any two values "a" and "b" within that interval where "a" is less than "b" (a < b), the following condition holds:
f(a) ≤ f(b)

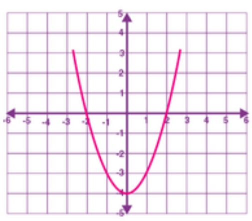
Decreasing Function: Conversely, a function is said to be decreasing on an interval if, for any two values "a" and "b" within that interval where "a" is less than "b" (a < b), the following condition holds:
f(a) ≥ f(b)

Combination of function:

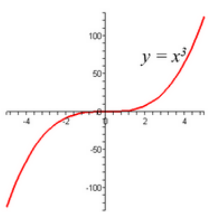
+ (f + g)x = f(x) + g(x)
+ (f − g)x = f(x) − g(x)
+ (fg)(x) = f(x)g(x)
+ (f o g)(x) = f(g(x))

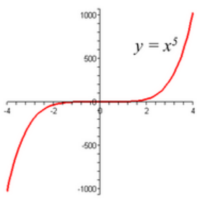
Even function: If a function f satisfies:
f(−x) = f(x), for all x in D ,then f is called an even function.
The geometric significance of an even function is that its graph is symmetric with respect to the y-axis





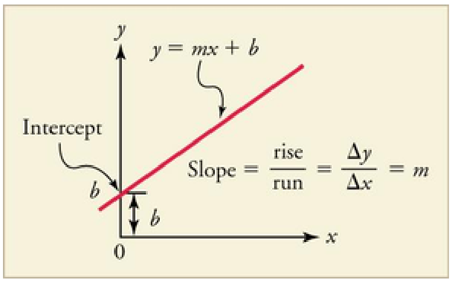
Odd function: If a function f satisfies:
f(−x) = − f(x), for all x in D, then f is called an odd function.
The graph of an odd function is symmetric about th origin.



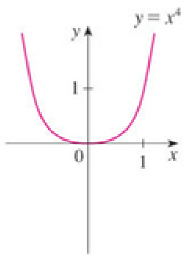


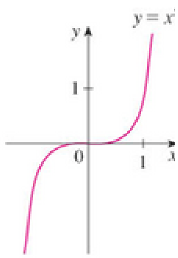
ALGEBRAIC FUNCTIONS

Linear models: When we say that y is a linear function of x, we mean that the graph of the function is a line.
So, we can use the slope-intercept form of the equation of a line to write a formula for the function as
y = f(x) = mx + b
where m is the slope of the line and b is the y-intercept.

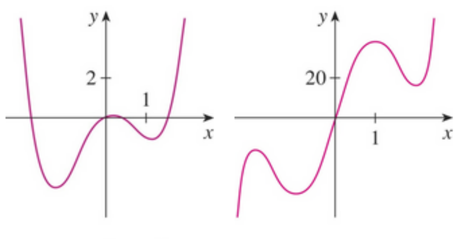


Power function: A function of the form **f(x) = x^a** , where a is constant, is called a power function.



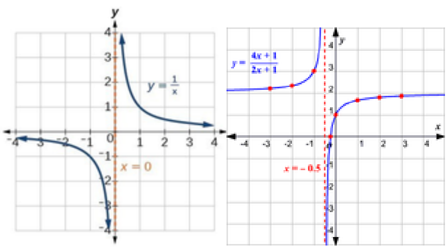


Polynomial: A function P is called a polynomial if
P(x) = anxn + an-1xn-1 + ... + a2x2 + a1x + a0
where n is a nonnegative integer and the numbers a0, a1, a2, ..., an are constants called the coefficients of the polynomial.



(b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

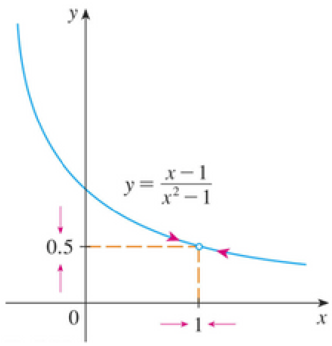
Rational function: A rational function f is a ratio of two polynomials
f(x) = P(x)/Q(x)
where P and Q are polynomials.
The domain consists of all values of x such that **Q(x) # 0** .



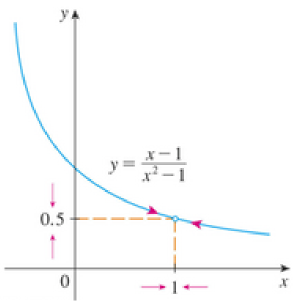
LIMIT

CHAPTER 2

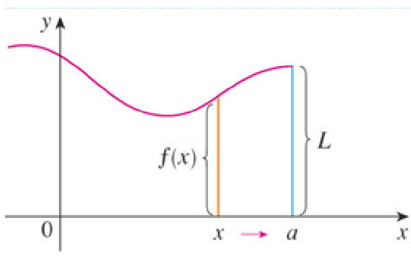
The limit of a function: we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to a but not equal to a by write $\lim_{x\to a} f(x) = L$



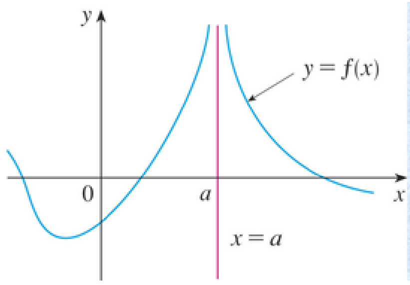
One-sided limit: We write $\lim_{x\to a^-} f(x) = L$ if we make the value of f(x) close to L by taking x to be sufficiently close to a but not equal to a



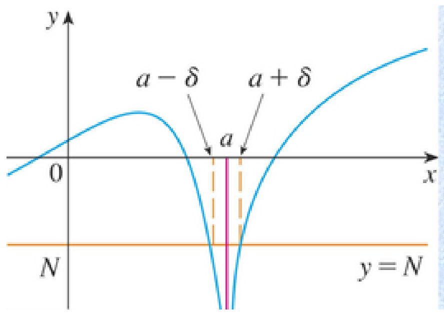
One-sided limit: We write $\lim_{x\to a^+} f(x) = L$ if we make the value of f(x) far from L by taking x to be sufficiently far from a but not equal to a



Infinite limit: If the values of f(x) can be made arbitrarily large by taking x sufficiently close to a, but not equal to a, we write $\lim_{x\to a} f(x) = \infty$



Infinite limit: If the value of f(x) can be made arbitrarily large negative by taking x sufficienly close to a, but not equal to a, we write $\lim_{x\to a} f(x) = -\infty$



The limit laws: Assume that limit from x to a of f(x) and g(x) is exist. Then:

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

Squeeze theorem:

Let $f(x)$, $g(x)$ and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Continuity: A function f is continuous at a number a if:

$$f(a) \text{ is defined;}$$

$$\lim_{x \to a} f(x) \text{ exists.}$$

$$\lim_{x \to a} f(x) = f(a).$$

A function is discontinuous at a if it not continuous at a.

Continuity over an interval:

A function f(x) is said to be continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$.

A function f(x) is said to be continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$.

Continuity: If f and g are continuous at a and c is a constant, then these functions are also continuous at a:

- f + g
- f-g
- cf, cg
- fg
- f/g if g(x) is not equal to 0

Composite function theorem

If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then

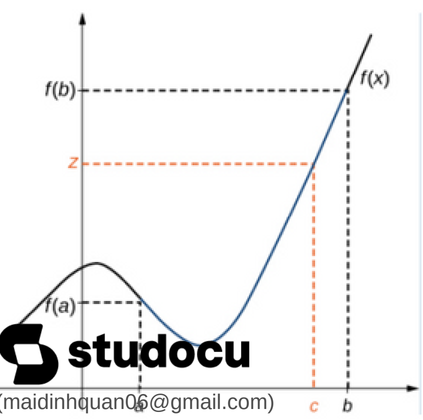
$$\lim_{x \to a} f[g(x)] = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Continuity theorem

If x is close to a, then g(x) is close to L; and, since f is continuous at L, if g(x)is close to L, then f(g(x)) is close to f(L).

The immediately Value theorem

Let f be continuous over a closed, bounded interval [a, b]. If z is any real number between f(a) and f(b), then there is a number c in [a, b] satisfying f(c) = z



DERIVATIVES

CHAPTER 3

Definiton: We can define the derivative of function f at a number a (f′(x)) is :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If it is exist, then:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Some notation: Some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

D and **D/Dx** are called differentiation operator

dy/dx is called Leibniz notation

Definition: A function f is differentiable at a if f′(a) exists.

Theorem: It is differentiable on an open interval D if it is differentiable at every number in the interval D.

Theorem: If f is differentiable at a, then f is continuous at a.

Higher derivative: If f is a differentiable function, then its derivative f′ is also a function.

Denote: (f′)′ = f′′

f′′ is called the second derivative of f:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$$

Differentiation fomulas:

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Derivatives of trigonometric functions :

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x.$$

DERIVATIVES

Derivatives of exponential and logarithmic functions

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(b^x) = b^x \ln b$$
$$\frac{d}{dx}(log_b x) = \frac{1}{x \ln b}$$

The chain rule: If g is differentiable at x and f is differentiable at g(x), the composite function F(x) = f(x).g(x) is differentiable at x and F' is given by the product:

$F'(x) = f'(g(x)) \cdot g'(x)$

If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Implicit differentiation

These are 5 steps that solve the implicit differentiation:

1. Differentiate both sides of the equation with respect to the independent variable.
2. Treat the dependent variable as a function of the independent variable and apply the chain rule when necessary.
3. Differentiate terms involving the dependent variable using the appropriate rules.
4. Collect terms with the derivative on one side of the equation.
5. Solve for the derivative to obtain the result.

Example of implicit differentiation

If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Step 1: Differentiate both sides of the equation with respect to the independent variable (in this case, "x"):

d/dx(x^2 + y^2) = d/dx(25)

Step 2: Apply the chain rule to differentiate the terms involving the dependent variable "y". The derivative of y with respect to x is denoted as dy/dx. So, we have:

2x + 2y * dy/dx = 0

Step 3: Differentiate the terms involving x and y separately. The derivative of x^2 with respect to x is 2x, and the derivative of y^2 with respect to x is 2y * dy/dx using the chain rule.

2x + 2y * dy/dx = 0

Step 4: Collect the terms involving dy/dx on one side of the equation:

2y * dy/dx = -2x

Step 5: Solve for dy/dx by dividing both sides of the equation by 2y:

dy/dx = -2x / (2y)

Simplifying further:

dy/dx = -x / y

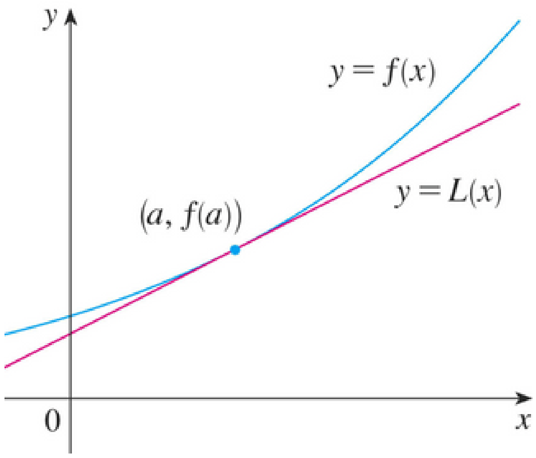
APPLICATIONS OF DERIVATIVES

CHAPTER 4

Linear approximations: We use the tangent line at (a, f(a)) as an approximation to the curve y = f(x) when x is near a.

Equations:

$$L(x) = y = f(a) + f'(a)(x - a)$$



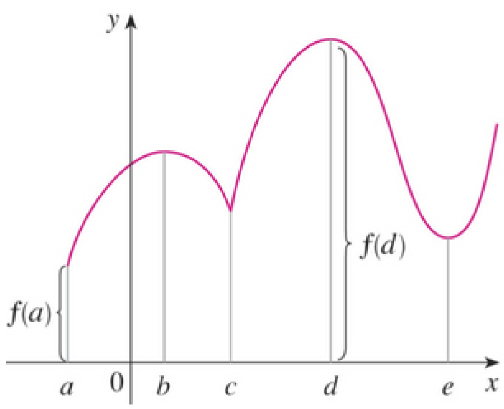
This approximation:

$$f(x) \approx f(a) + f'(a)(x - a) = L(x)$$

is called the **linear approximation** of **f** at **a**

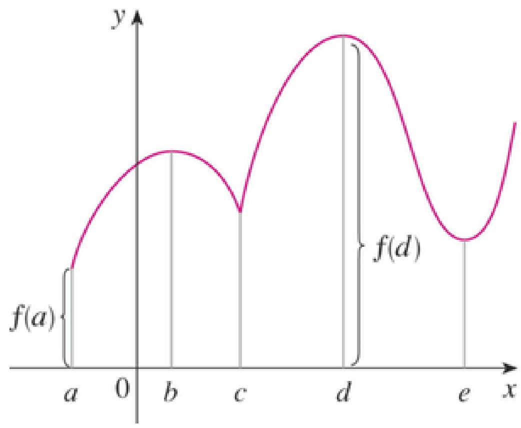
Maximum values: A function f has an absolute maximum at c if f(c) >= f(x) for all x in D, where D is the domain of f.

The number f(c) is called the **maximum value** of **f** on **D**.



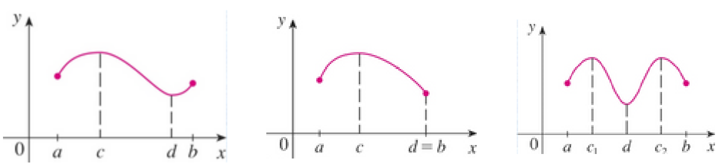
Minimum values: A function f has a absolute minimum at c if f(c) <= f(x) for all x in D, where D is the domain of f.

The number f(c) is called the **minimum value** of **f** on **D**.

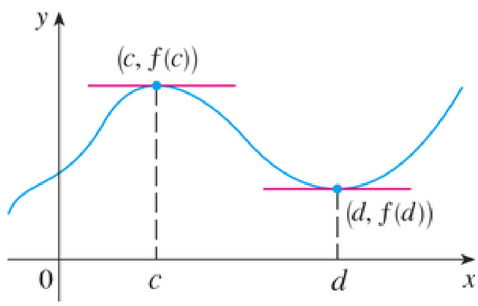


The maximum and minimum values of f are called the extreme values of f.

Extreme value theorem: If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].

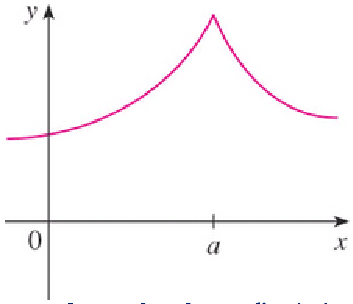


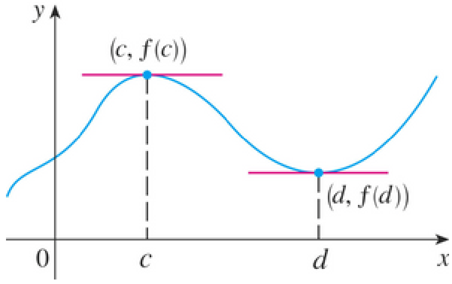
Fermat's theorem: If f has a local maximum of minimum at c, and if f′(c) is exists, then f′(c) = 0.



APPLICATIONS OF DERIVATIVES

Critical number: is a number c in the domain of f such as either f′(c) = 0 and f′(c) does not exists





Close interval method: To find the absolute maximum and minimum values of a continuous function f on a closed interval [a, b]:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest value from 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

Rolle's theorem:
Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval [a, b]
2. f is differentiable on the open interval (a, b)
3. f(a) = f(b)

Then, there is a number c in (a, b) such that f′(c) = 0.

Mean values theorem:
Let f be a function that fulfills two hypotheses:

1. f is continuous on the closed interval [a, b].
2. f is differentiable on the open interval (a, b).

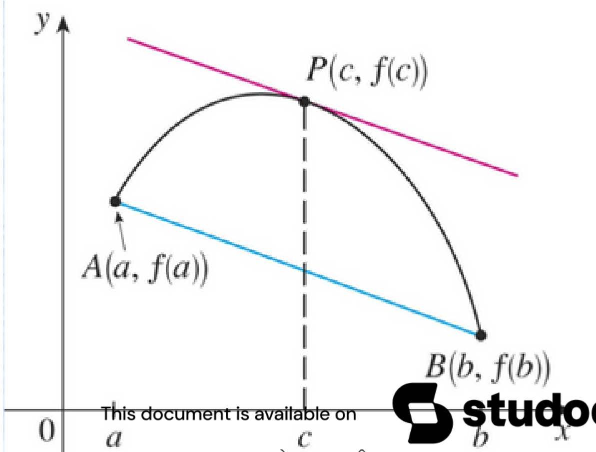
Then, there is a number c in (a, b) such that

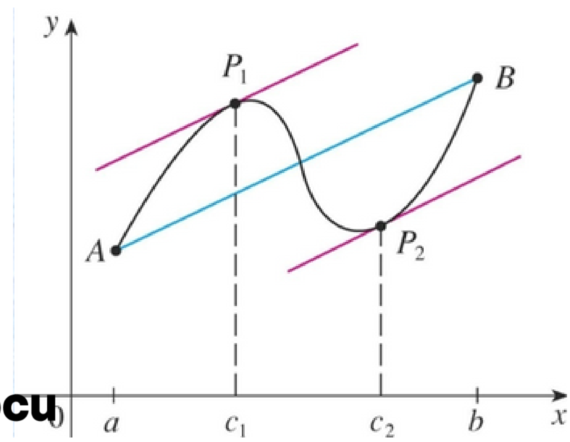
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

Mean values theorem:
f′(c) is the slope of the tangent line at (c, f(c)). There is at least one point P(c, f(c)) on the graph where the slope of the tangent line is the same as the slope of the secant line AB.



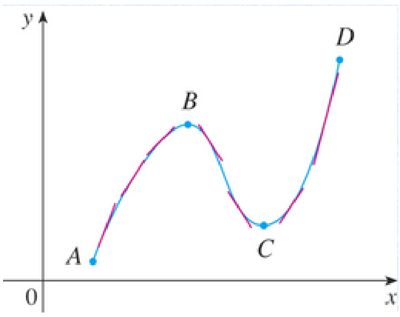


APPLICATIONS OF DERIVATIVES

Mean values theorem: If $f'(x) = g'(x)$ for all x in an interval (a, b),then f – g is constant on (a, b).
That is, f(x) = g(x) + c where c is a constant.

Increasing/Decreasing test:

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.



First derivatives test:

Suppose that *c* is a critical number of a continuous function *f*.

- If *f′* changes from positive to negative at *c*, then *f* has a local maximum at *c*.
- If *f′* changes from negative to positive at *c*, then *f* has a local minimum at *c*
- If *f′* does not change sign at *c* then *f* has no local maximum or minimum at *c*.

Inflection point: A point P on a curve y = f(x) is called an inflection point.

Second derivative test: Suppose f″ is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Sequent approximation: If the nth approximation is xn and
f′(xn) ≠ 0, then the next approximation is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Convergence: If the numbers xn become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write: $\lim_{n\to\infty} x_n = r$

Antiderivatives: A function F is called an antiderivative of f on an interval I if F′(x) = f(x) for all x in I.

Theorem: If F is an antiderivative of f on an interval I, the most general antiderivative of f on I is:

F(x) + C

where C is an arbitrary constant.

Antiderivatives formulas:

$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$
$$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$$
$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$$
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$$
$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, \quad x > 0$$

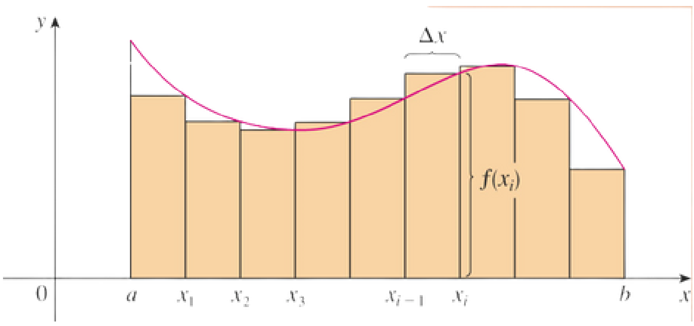
INTERGRATION

CHAPTER 5

Area problem: the area of S is approximated by the sum of the areas of these rectangles:

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}$$



Area problem: The area *A* of the region *S* that lies under the graph of the continuous function *f* is the limit of the sum of the areas of approximating rectangles:

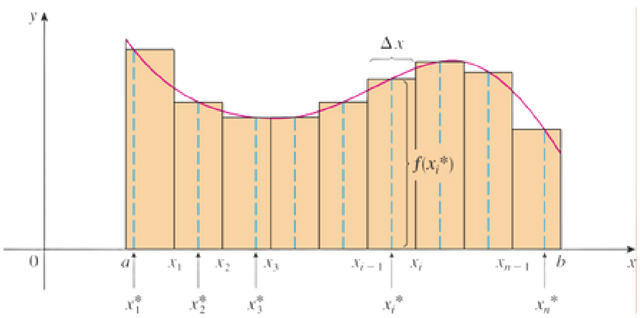
$$A = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$$

$$A = \lim_{n\to\infty} L_n$$
$$= \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

Riemann Sum :

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

This sum is called Riemann sum.



x(i)* in the order of i subinterval [x(i - 1), x(i)]

Definite integral: If f is a function defined for a ≤ x ≤ b, we divide the interval [a, b] into n subintervals of equal width Δx=(b – a)/n.

1. We let *x0*(=*a*), *x1*, *x2*, ..., *xn*(=*b*) be the endpoints of these subintervals.
2. We let *x1**, *x2**,...., *xn** be any sample points in these subintervals, so *xi** lies in the *i* th subinterval.

Then, the definite integral of from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists.

- If it does exist, we say *f* is integrable on [*a*, *b*].
- The symbol ∫ was introduced by Leibniz and is called an integral sign.

Definite integral: The definite integral is a number. It does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Definite integral: Let A1 represent the area between f(x) and x-axis that lies above the axis and A2 represents the area between f(x) and y-axis that lies above the axis. Then the next signed area between f(x) and x-axis is:

$$\int_a^b f(x)dx = A_1 - A_2.$$

The total area between f(x) and x-axis is:

$$\int_a^b |f(x)|dx = A_1 + A_2.$$

Substitution rule : If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

INTEGRATION

Properties of the integral: Assume that f and g are continuous functions.

$\int_a^b c\,dx = c(b-a)$, where c is any constant

$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

$\int_a^b c\,f(x)dx = c\int_a^b f(x)dx$, where c is any constant

$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge 0$

If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Average value of a function: If f is continuous on [a, b], then there exists a number c in [a, b] such that: Rolle theorem

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

That is:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

The mean values theorem for the integral: If f(x) is continuous over an interval [a,b] then there is at least one point c belong to [a,b] such as:

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The fomula can be also stated like this:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Generalization:

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f(u(x)) - v'(x)f(v(x))$$

Fundamental theorem of calculus : If f is continuous on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that F′ = f.

Net change theorem: The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

If the mass of a rod measured from the left end to a point *x* is *m*(*x*), then the linear density is *ρ*(*x*) = *m*′(*x*).

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between x = a and x = b.

Integrating of symmetric functions: Suppose f is continuous on [−a , a].

If f is even, [*f*(−*x*) = *f*(*x*)], then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

If f is odd, [*f*(-*x*) = -*f*(*x*)], then

$$\int_{-a}^a f(x)\,dx = 0$$

Table of indefinite integrals:

$\int cf(x)\,dx = c\int f(x)\,dx$

$\int [f(x)+g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$

$\int k\,dx = kx + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \ne -1)$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \csc x \cot x\,dx = -\csc x + C$

TECHNIQUES OF INTERGRATION

CHAPTER 6

Integration by Parts: Let u = f(x) and v = g(x).Then, the differentials are:
du = f′(x) dx and dv = g′(x) dx

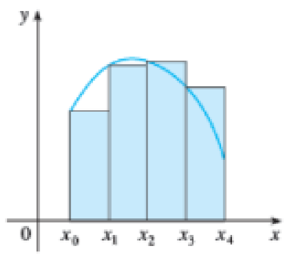
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,d$$

Thus, by the Substitution Rule, the formula for integration by parts becomes:

$$\int u\,dv = uv - \int v\,du$$

Numerical Integration:

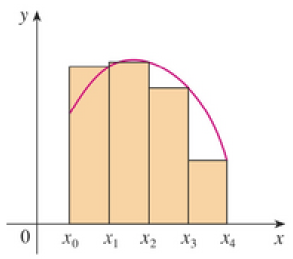
1.Left endpoint Method



Left endpoint approximation

$$\int_a^b f(x)dx \approx \Delta x[f(x_0)+f(x_1)+\ldots+f(x_{n-1})]$$

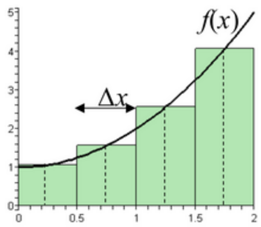
2.Right endpoint Method



(b) Right endpoint approximation

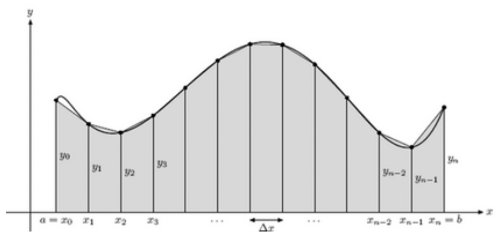
$$\int_a^b f(x)dx \approx \Delta x[f(x_1)+f(x_2)+\ldots+f(x_n)]$$

3.Midpoint Method



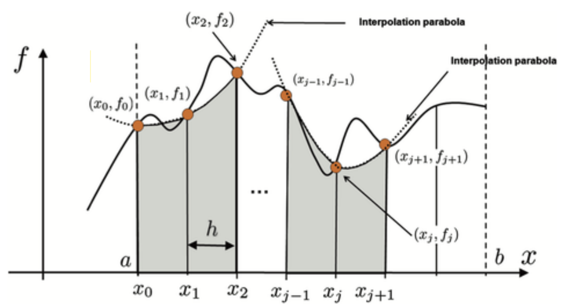
$$\int_a^b f(x)dx \approx \Delta x[f(\bar{x}_1)+f(\bar{x}_2)+\ldots+f(\bar{x}_n)]$$

4.Trapezoidal Method



$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0)+f(x_1)]+\ldots+\frac{\Delta x}{2}[f(x_{n-1})+f(x_n)]$$
$$\approx \frac{\Delta x}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_{n-1})+f(x_n)]$$

5.Simpson Method



$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)]+\ldots+\frac{\Delta x}{3}[f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$
$$\approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_{n-1})+f(x_n)]$$

STUDIO
SHODWE

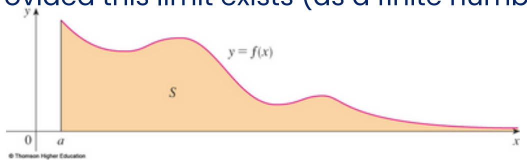
TECHNIQUES OF INTEGRATION

Improper Integrals

Definition:+ If $\int_a^t f(x)\,dx$ exists for every number t≥ a, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).



+If $\int_t^b f(x)\,dx$ exists for every number t ≤ a, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

Convergent and divergent

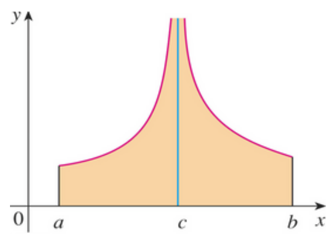
The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called:

+Convergent if the corresponding limit exists.

+Divergent if the limit does not exist.

-If f has a discontinuity at c, where a < c < b, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define:

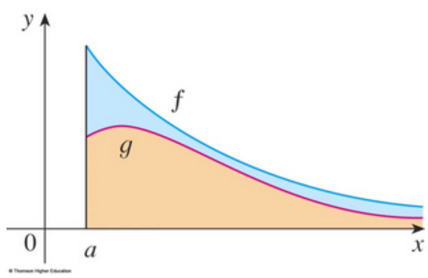
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



-Suppose f and g are continuous functions with f(x) ≥ g(x) ≥ 0 for x ≥ a.

+If $\int_a^{\infty} f(x)\,dx$ is convergent, then $\int_a^{\infty} f(x)\,dx$ is convergent.

+If $\int_a^{\infty} g(x)\,dx$ is divergent, then $\int_a^{\infty} f(x)\,dx$ is divergent.



SYSTEMS OF LINEAR EQUATIONS

CHAPTER 6

Definition: In mathematics, a system of linear equations (or linear system) is a collection of one or more linear equations involving the same variable. For example:

$$\begin{cases} 3x + 2y - z = 1 \\ 2x - 2y + 4z = -2 \\ -x + \frac{1}{2}y - z = 0 \end{cases}$$

Solution of the equation: A system may have no solution, may have unique solution , or may have an infinite family of solutions.

+A system that has no solution is called inconsistent

+A system that has at least one solution is called consistent

Elementary Operations :

-Interchange two equations (I)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 0 \\ 4 & 1 & 6 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 4 & 1 & 6 \\ 2 & 5 & 0 \\ 1 & 2 & -3 \end{bmatrix}$$

-Multiply one equation by a nonzero number (II)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 0 \\ 4 & 1 & 6 \end{bmatrix} \xrightarrow{3r_1} \begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & 0 \\ 4 & 1 & 6 \end{bmatrix}$$

-Add a multiple of one equation to a different equation (III)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 0 \\ 4 & 1 & 6 \end{bmatrix} \xrightarrow[-4r_1+r_3]{-2r_1+r_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 6 \\ 0 & -7 & 18 \end{bmatrix}$$

A row-echelon matrix

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

+All the zero rows are at the bottom

+The first nonzero entry from the left in each nonzero row is a 1

+Each leading 1 is to the right of all leading 1's in the rows above it

A reduced row-echelon matrix

+It is a row-echelon matrix

+Each leading 1 is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Gaussian Elimination: By using elementary row operations to carry the augment matrix to "nice" matrix such as

$$\left\langle \begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right\rangle$$

SYSTEMS OF LINEAR EQUATIONS

CHAPTER 7

Gaussian Algorithm: Every matrix can be brought to (reduced) row-echelon form by a series of elementary row operations

Step 1. If all row are zeros, stop

$$step\ 1: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow stop$$

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a) and move the row containing a to the top position

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 2 & 2 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix}$$

Step 3. Multiply that row by 1/a to create the leading 1

$$\begin{bmatrix} 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix}$$

Step 4. By subtracting multiples of that row from the rows below it, make each entry below the leading 1 zero

$$\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix} \xrightarrow{-4r_1 + r_4} \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 3 & 7 & -1 \end{bmatrix}$$

Step 5. Repeat step 1-4 on the matrix consisting of the remaining rows

$$\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 3 & 7 & -1 \end{bmatrix}$$

SYSTEMS OF LINEAR EQUATIONS

CHAPTER 7

The rank of a matrix:

+The reduced row-echelonform of a matrix A is uniquely determined by A, but the row-echelon form of A is not unique

+The number r of leading 1's is the same in each of the different row-echelon matrices

+IAs r depends only on A and not on the row-echelon forms, it is called the rank of the matrix A, and written r=rankA

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rankA=4

-Suppovariablesse a system of m equations in n has a solution. If the rank of the augment matrix is r then the set of solutions involves exactly n-r parameters

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

4(number of variables)- 2(ranA) =2

Homogeneous Equations:

+The system is called homogeneous if the constant matrix has all the entry are zeros

+Note that every homogeneous system has at least one solution (0,0,...,0), called trivial solution

+If a homogeneous system of linear equations has nontrivial solution then it has infinite family of solutions

If a homogeneous system of linear equations has more variables than equations, then it has nontrivial solution (in fact, infinitely many

MATRIX ALGEBRA

CHAPTER 8

The mxn matrix: An mxn matrix (or a matrix of size mxn) is a rectangular array of numbers with m rows and n columns

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix}$$

These matrix is 3x3 matrix (m = 3, n = 3)

The mxn matrix: The (i,j)-entry of A (denoted by aij) lies in row i and column j

- A is denoted simply as A=[aij]

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{pmatrix}$$

a_{23} = the element in the second row and third column

a_{mn} = the element in the m^{th} row and n^{th} column

Definition:

- An mxm matrix is called a square matrix (ma trận vuông) of size m
- The zero matrix (ma trận không) of size mxn (denoted by (0mxn)is the matrix that its all entries are 0
- If A=[aij] is an mxn matrix then -A refers to the negative matrix (ma trận đối) of A and defined by
- -A=[-aij]

Identity matrices: An identity matrix I is a square matrix with 1's on the main diagonal and zeros elsewhere.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular matrices:

- Upper triangular matrix: all entries below and to the left the main diagonal are zeros
- Lower triangular matrix: its transposition is upper triangle matrix, that means every entry above and to the right the main diagonal is zero
- Matrix A is called triangular if it is upper or lower triangular

Matrices equal: Two matrices are called equal if

- They have the same size
- Corresponding entries are equal

If A=[aij], B=[bij] then A=B means aij=bij for all i and j

Matrix Addition of same size matrices:

- If A=[aij], B=[bij] then the sum matrix A+B is defined by A+B=[aij+bij]
- The difference A-B is a matrix defined by A-B=A+(-B)=[aij-bij] for all mxn matrices A and B
- A-A=0, A+0=A (0 is zero matrix) for all mxn matrix A

Properties:

If A ,B and C are any matrices of the same size, then

- lA+B=B+A (commutative law: giao hoán)
- lA+(B+C)=(A+B)+C (associative law: kết hợp)

Scalar Multiplication:

Suppose A=[aij] is an mxn matrix and k is a real number, the scalar multiple kA is a matrix defined by kA=[kaij]

- kA=0 -> (either k=0 or A=0)
- l(k=0 or A=0)àkA=0

Theorem:

1. $A+B=B+A$
2. $A+(B+C)=(A+B)+C$
3. There is an $m\times n$ matrix 0, such that $0+A=A$ for each A.
4. For each A there is an $m\times n$ matrix, $-A$, such that $A+(-A)=0$.
5. $k(A+B)=kA+kB$.
6. $(k+p)A=kA+pA$.
7. $(kp)A=k(pA)$.
8. IA=A

CHAPTER 8

MATRIX ALGEBRA

Transpose: If A=[aij] is any mxn matrix,the transpose of A, written AT, is an nxm matrix defined by AT=[aji]

- The row i of A is the column i of A^T
- The column j of A is the row j of A^T

Theorem:

- $(A^T)^T=A$
- $(kA)^T=k(A^T)$
- $(A+B)^T=A^T+B^T$

Definition:

- If A=[aij] is any mxn matrix, then a11,a22,a33,..., are called the main diagonal (đường chéo chính) of A
- If A=AT then A is called symmetric (đối xứng). In this case, A is a square matrix

Matrix multiplication:

- Suppose A=[aij] is an mxk matrix and B=[bij] is an kxn matrix, then the product AB=[cij] is an mxn matrix whose the (i,j)-entry is the dot product of row i of A and column j of B
- cij=(row i of A).(column j of B)
- Note that A(mxk)B(kxn) is a mxn matrix

Theorem:

Let A be an mxn matrix of rank r, AX=0 is an homogeneous in n variables. Then:

- The system has exactly n-r basic solutions, one for each parameter.
- Every solution is a linear combination (tổ hợp tuyến tính) of these basic solutions

Theorem:

Let $A=[C_1 \;\; C_2 \ldots C_n]$ be an $m\times n$ matrix with columns $C_1, C_2, \ldots, C_n$. If $X=[x_1 \;\; x_2 \ldots x_n]^T$ is any column, then

$$AX=[C_1 \;\; C_2 \ldots C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=x_1C_1+x_2C_2+\ldots+x_nC_n.$$

Block multiplication:

If A is a $m\times n$ and $C_1, C_2, \ldots, C_n$ are the columns of A, we write

$$A=[C_1 \;\; C_2 \ldots C_n].$$

Direct graph:

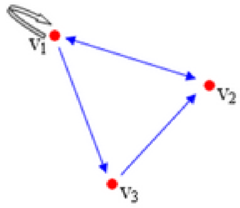
- Points=vertices
- Vertices are connected by arrows called edges
- If a directed graph has n vertices v1,v2,...,vn, the adjacency matrix A=[aij] is the nxn matrix
- The (i,j)-entry is only 1 or 0
- If there is an edge from vj to vertex vi then aij=1 else aij=0

CHAPTER 8

MATRIX ALGEBRA

Direct graph:

- a11=1 means v1 -> v1
- a23=1 means v3 -> v2
- a32=0 means have no edges from vertex 2 to vertex 3
- A path of length r (or an r-path) from vertex j to vertex i is a sequence of r edges leading from vj to vi.
- v1 -> v2->v1 -> v3 -> v2 is a 4-path from v1 to v2



$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Theorem: If A is the adjacency matrix of a directed graph with n vertices, then the (i,j)-entry of Ar is the number of r-paths vj -> vi

Matrix inverse:

- If A is a square matrix, a matrix B is called an inverse (nghịch đảo) of A if and only if AB=I and BA=I
- A matrix A that has an inverse is called an invertible (khả nghịch) matrix.

Theorem: Suppose AX=B is a system of n equations in n variables and A is an invertible matrix. Then the system has the unique solution

X=A-1B

The inverse of 2x2 matrix :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- The determinant of A is detA=ad-bc
- The adjugated matrix of A is defined by

$$adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} adjA = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem : Either any square matrix can be reduced to I or not. In the first case, the algorithm produces A-1; in the second, A-1 does not exist.

Theorem :

$(A^{-1})^{-1}=A$

$(AB)^{-1}=B^{-1}A^{-1}$

$(A^T)^{-1}=(A^{-1})^T$

$(A_1A_2...A_k)^{-1}=A_k^{-1}...A_2^{-1}A_1^{-1}$

$(A^k)^{-1}=(A^{-1})^k$

$(aA)^{-1}=A^{-1}/a$

$I^{-1}=I$

CHAPTER 8

MATRIX ALGEBRA

Theorem:

The following conditions are equivalent for an nxn matrix A

- A is invertible
- The homogeneous system AX=0 has only the trivial solution X=0
- A can be carried to In by elementary row operations
- The system AX=B has unique solution for every choice of column B
- There exist an nxn matrix C such that AC=In

Corollary: If A and C are square matrices such that AC=I, then also CA=I. In particular, both A and C are invertible, C=A^-1 and A=C^-1

Matrix transformations:

- (a1,a2,...,an): ordered n-tuple
- Rn: the set of all ordered n-tuples
- Every (a1,a2,...,an) in Rn is called a vector or n-vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$$

Transformations:

$$\begin{array}{ccc} R^n & & R^m \\ X & \xrightarrow{T} & T(X) \end{array}$$

Two transformations T and S are called equal if T(X)=S(X) , for all X in Rn

Fomula of the transformation:

- T[x1x2x3]^T=[x1x1-x2x2-x3x3-x1]T defines a transformation T:R3 -> R4
- Suppose A is an mxn matrix, then
- T[x1x2... xn]^T=A[x1x2... xn](T(X)=AX) is a transformation T: Rn -> Rm called the matrix transformation induced by A
- A is zero matrix: the zero transformation T=0
- A is identity matrix In: the identity transformation T= 1

Linear transformation:

- T: Rn -> Rm is called a linear transformation (phép biến đổi tuyến tính) if it satisfies:
- T(X+Y)=T(X)+T(Y) for all vectors X and Y
- T(aX)=aT(X) for all vector X and all scalar a
- If T is a linear transformation then
- T(0)=0
- T(-X)=-T(X)

Linear combination:

- Y=a1X1+a2X2+...+anXn is called a linear combination of vectors X1,X2,...,Xn
- If T: Rn -> Rm is linear transformation, then T(a1X1+a2X2+...+anXn) = a1 T(X1)+a2T(X2)+...+anT(Xn) for all vectors Xi, and all scalar ai

Theorem:

Let T: Rn -> Rm be a transformation

- T is linear if it is a matrix transformation
- If T is linear, then T is induced by a unique matrix A, given in terms of its columns by A=[T(E1) T(E2) ...T(En)] where E1, E2,..., En is the standard basis of Rn

Theorem:

- T:Rn -> Rm and S:Rm -> Rk
- The composite of S and T is a transformation defined by (S°T(X)=S[T(X)] for all vector X in Rn
- If T and S are linear then S°T is also linear. And if S has matrix A and T has matrix B, then S°T has matrix AB

DETERMINANTS AND DIAGONALIZATION

CHAPTER 9

The Cofactor Expansion:

- If A=[a] then the determinant of A, denoted by detA=a
- If A is an 2x2 matrix then

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- If A is an 3x3 matrix then the determinant of A is defined by

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a(ei - hf) - d(bi - hc) + g(bf - ec)$$

The (i,j) - cofactor:

•If A is an mxm matrix then the (i,j)-cofactor of A is defined by

cij(A)=(-1)i+jdet(Aij)

•Aij is the (m-1)x(m-1) matrix obtained from A by deleting row i and column j of A

Definition:

If A is an mxm matrix then the determinant of A is defined by

detA=$a_{i1}c_{i1}(A)+a_{i2}c_{i2}(A)+\ldots+a_{im}c_{im}(A)$

or **detA**= $a_{1j}c_{1j}(A)+a_{2j}c_{2j}(A)+\ldots+a_{mj}c_{mj}(A)$

Properties:

- If A has one row of zeros then detA=0
- If A is an triangular matrix then detA is the product of the entries on the main diagonal

Determinants and elementary operations:

- If B obtained from A by interchanging two rows (or columns) then detB = -detA
- If B is the matrix obtained from A by multiplying one row (or column) by a nonzero number k then detB=kdetA
- If B is the matrix obtained from A by adding a multiple of one row (or column) to another row (or column) then detB = detA

Theorem:

- detA=det(A^T)
- det(AB)=detA.detB
- If A is an nxn matrix then det(kA)=k^ndetA
- det(A^k)=(detA)^k

Determinants with block:

$$\begin{vmatrix} A & 0 \\ Y & B \end{vmatrix} = \det A \det B$$

$$\begin{vmatrix} A & X \\ 0 & B \end{vmatrix} = \det A \det B$$

Determinant and Matrix Inverses:

- If A is invertible then AàIdentity matrix
- If detA ≠0 and AàB by elementary operations then detB ≠0
- A is invertible if detA≠0

(i,j) cofactor:

- cij=cij(A)=(-1)^(i+j)det(Aij)   (it is a number)
- Aij is the matrix obtained from A by deleting row i and column j

Adjugated matrix:

$$adjA = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

Theorem of Adjugated Formula: If A is any square matrix, then:

- A(adjA)=(detA)I
- In particular, if detA≠0 then A is invertible and

$$A^{-1} = \frac{1}{\det A} adjA$$

Cramer's rule:

- Cramer's system: system of n equations involving n variables and has unique solution. So, the coefficient matrix is invertible.
- A is invertible if detA≠0
- Consider the matrix equation AX=B where A is an invertible square matrix, X=[x1 x2 … xn]T. That means, AX=B is a Cramer's system. Then the system has unique solution given by

$$x_i = \frac{\det A_i}{\det A}$$

CHAPTER 9

DETERMINANTS AND DIAGONALIZATION

Diagonalization and Eigenvalues:

The following statements are equivalent:

- A is invertible
- $\det A \neq 0$
- The matrix equation AX=B has unique solution
- The homogeneous system AX=0 has only trivial solution
- A -> Identity matrix by elementary operations

Diagonal matrices: An nxn matrix is called diagonal matrix if all its entries off the main diagonal are zeros.

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$$

Theorem : If A is an nxn matrix and P is an invertible nxn matrix then

$$\left(P^{-1}AP\right)^k = P^{-1}A^kP$$

$$\left(P^{-1}AP\right)^k = \left(P^{-1}AP\right)\left(P^{-1}AP\right)\left(P^{-1}AP\right)\dots\left(P^{-1}AP\right)$$
$$= P^{-1}APP^{-1}APP^{-1}AP\dots P^{-1}AP = P^{-1}A^kP$$

Eigenvalues and eigenvectors :

- If A is an nxn matrix, a number λ is called an eigenvalue of A if$AX=\lambda X$ for some column $X \neq 0$
- Such a nonzero column X is called an eigenvector of A corresponding to λ, or a λ-eigenvector for short

eigenvalue

$$AX = \lambda X$$

λ-eigenvector

Characteristic Polynomial: The characteristic polynomial of an nxn matrix A is defined by:

$$c_A(x)=\det(xI-A)$$

Theorem:

- A is diagonalizable if every eigenvalue of multiplicity m yields exactly m basic eigenvectors, that is,
- If the general solution of the system

$$(\lambda I - A)X = 0$$

has exactly m parameters.

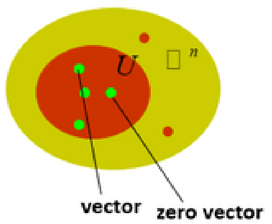
Linear Dynamical System: V0,V1,V2,... are columns such that V0 is known and Vk+1=AVk for each k>=0

THE VECTOR SPACE R^N

CHAPTER 10

Subspace of R^N: Let Ø≠U be a subset of Rn. U is called a subspace of Rn if:

- The zero vector 0 is in U
- If X,Y are in U then X+Y is in U
- If X is in U then aX is in U for all real number a.



Note:

•A subspace either has only one or infinite many vectors.

•If a subspace U has nonzero vector X then aX is also in U (by S3). Then U has infinite many vector.

Null space and image space of a matrix: A is an mxn matrix, if X is nx1 matrix then AX is mx1 matrix

- **nullA = {X in Rn: AX=0}**
- **imA = {AX: X is in Rn}**

Eigenspaces:

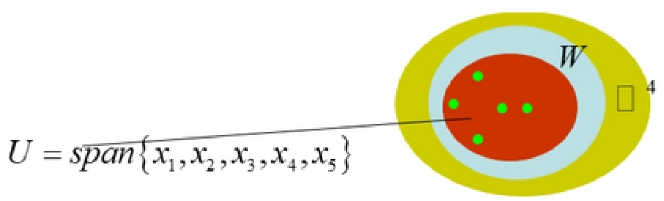
- Suppose A is an nxn matrix and λ is an eigenvalue of A
- Eλ(A)={X: AX=λX} is an subspace of Rn

Spanning sets:

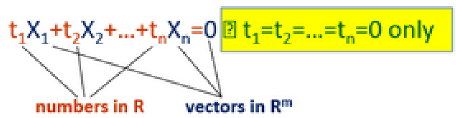
- Y=**k1X1+k2X2+...+knXn** is called a linear combination of the vectors X1,X2,...,Xn
- The set of all linear combinations of the the vectors X1,X2,...,Xn is called the span of these vectors, denoted byspan{X1,X2,...,Xn }.
- This means, span{X1,X2,...,Xn} = {k1X1+k2X2+... +knXn :kiÎR is arbitrary}
- span{X1,X2,...,Xn} is a subspace of Rn.

Theorem:

- U=span{X1,X2,...,Xn} is in Rn and U is a subspace of Rn
- If W is a subspace of Rn such that Xi are in W then $U \subseteq W$



Linear Independence : A set of vectors in Rm{X1,X2,...,Xn } is called linearly independent if



Fundamental Theorem:

- Let U be a subspace of Rn is spanned by m vectors, if U contains k linearly independent vectors, then k≤m
- This implies if *k>m*, then the set of k vectors is always linear dependence.

Basis and dimension :

- Suppose U is a subspace of Rn, a set {X1,X2,...,Xk} is called a **basis** of U if U=span{X1,X2,...,Xk} and {X1,X2,...,Xk} is linear independence
- If {a1,a2,..,am} and {b1,b2,...,bk} are bases of a subspace U of Rn, then m=k. In this case, m=k is called **dimension** of U and we write dimU=m.

Theorem: The following are *equivalence* for an nxn matrix A.

- A is invertible.
- The columns of A are linearly independent.
- The columns of A span Rn:

  the rows of A are linearly independent.

  the rows of A span the set of all 1xn rows.

Theorem: Let U≠0 be a subspace of Rn. Then:

- U has a basis and dimU = n.
- Any independent set of U can be enlarged (by adding vectors) to a basis of U.
- If B spans U, then B can be cut down (by deleting vectors) to a basis of U.

Theorem: Let U be a subspace of Rn and B={X1,X2,...,Xm}U, where dimU=m. Then B is independent if and only if B spans U.

Theorem: Let U,V be subspaces of Rn. Then:

- dimU = dimV.
- If dimU=dimV, then U=V.

Subspace of R^n: The zero vector 0 is in U

x,y are in U -> x+y is in U

x is in U -> ax is in U for all real number a

-> U is a subspace of Rn

-> If (one of S1 or S2 or S3 is not true) then U is not a subspace of Rn

CHAPTER 10

THE VECTOR SPACE OF R^N

Spanning set :

- U=span{x,y}={ax+by:a,b in R}
- Vector z is in span{x,y} if and only if z is a linear combination of x and y, that means, there exist a and b such that z=ax+by
- U=span{x1,x2,...,xn} is a subspace of Rn. Note that Rn=span{E1,E2,...,En}

Diagonal matrices:

- {x1,x2,...,xm} is called linearly independent if
  t1x1+t2x2+...+tmxm=0

then

t1=t2=...=tm=0

- Note that if {x1,x2,...,xm} is linear independent then every vector z in span{x1,x2,...,xm} has a unique representation as a linear combination of xi
- {x1,x2,...,xm} is called linearly dependent (pttt) if it is not linear independent.

Basis and dimension:

- If {x1,x2,...,xm} is a basis of U then dimU=m
- dimRn=n
- Note that if U=span {x1,x2,...,xm} then dimU≤m anddimU=m if and only if {x1,x2,...,xm} is linear independent
- If dimU=m then every set of m+1 vector in U is linearly dependent
- Every set of n linearly independent vectors is a basis of Rn.

Dot product:

- If X= [x1 x2 ... xm]T, Y= [y1 y2 ... ym]T
  We define
- X•Y=x1y1+x2y2+...+xmym

The length of vector:

If X=[x1 x2 ... xm]^T then

$$\|X\| = \sqrt{X \bullet X} = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}$$

A vector with length 1 is called a unit vector

Distance between X and Y defined by

$$d(X,Y) = \|X - Y\|$$

Theorem:

Let X,Y and Z denote vectors in $\mathbb{R}^n$. Then:

1. X•Y=Y•X
2. X• (Y+Z)=X•Y+X•Z
3. (aX)•Y=aX•Y=X•(aY) for all scalar a
4. X•X=$\|X\|^2$
5. $\|X\| \geq 0$ for all vector X,
   $\|X\| = 0 \Leftrightarrow X = 0$
6. $\|aX\| = |a|\|X\|$ for all a

Orthogonal Set:

- A set {x1,x2,...,xm} is called orthogonal set if xi is not zero vector and xi•xj=0 for all i≠j
- A orthogonal set {xi} is called orthonormal set is xi is unit vector for all i.
- The standard basis of Rn {E1,E2,...,En} is orthonormal
- If {F1,F2,...,Fk} is orthogonal then {a1F1,a2F2,...,akFk} is also orthogonal for any nonzero scalar ai
- Every orthogonal set is a linearly independent set

CHAPTER 10

THE VECTOR SPACE OF R^N

Pythagoras's Theorem : If {F1,F2,...,Fk} is orthogonal then

$$\|F_1+F_2+\ldots+F_k\|^2=\|F_1\|^2+\|F_2\|^2+\ldots+\|F_k\|^2$$

Expansion Theorem: Let {F1,F2,...,Fk} be a orthogonal basis of a subspace U and X is in U. Then:

$$X=\frac{X\bullet F_1}{\|F_1\|^2}F_1+\frac{X\bullet F_2}{\|F_2\|^2}F_2+\ldots+\frac{X\bullet F_n}{\|F_n\|^2}F_k$$

Rank of a matrix:

- If A is carried to row-echelon form then rankA=number of leading 1's
- If A is an mxn matrix then rankA≤min{n,m}
- rankA=rank(A^T)

rowA and colA subspaces:

- rowA=span{rows of matrix A}
- colA=span{columns of A}
- dim(rowA)=dim(colA)=rankA

Theorem:

- An nxn matrix A is invertible if and only if rankA=n
- If an mxn matrix B has rank n then the n columns of B is linearly independent
- If A is mxn matrix and m>n then the set of m rows of A is not independent

Theorem: If an mxn matrix A has rank r then

- The equation AX=0 has n-r basic solutions X1,X2,...,Xn-r
- {X1,X2,...,Xn-r} is a basis of nullA
- Dim nullA=n-r
- imA=colA and
- Dim imA=dim colA=rankA=r

TIP AND TRICK

Bấm máy casio giải quyết các bài toán trong đề thi

Ma trận nghịch đảo

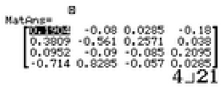
Bước 1 nhấn phím **OPTN** => nhấn phím **3** để chọn **MatA**

```
1:Define Matrix
2:Edit Matrix
3:MatA    4:MatB
5:MatC    6:MatD

MatA
```

Bước 2 nhấn phím x^{-1}

```
MatA⁻¹
```

Bước 3 nhấn phím =



Ma trận chuyển vị

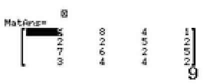
Bước 1 nhấn phím **OPTN** => nhấn phím ▼ => nhấn phím **3** để chọn **Transposition**

```
1:Define Matrix
2:Edit Matrix
3:MatA    4:MatB
5:MatC    6:MatD

1:MatAns
2:Determinant
3:Transposition
4:Identity

Trn(
```

Bước 2 nhấn phím **OPTN** => nhấn phím **3** để chọn **MatA**

```
1:Define Matrix
2:Edit Matrix
3:MatA    4:MatB
5:MatC    6:MatD

Trn(MatA
```

Bước 3 nhấn phím =



Tính định thức của ma trận

Bước 1 nhấn phím **OPTN** => nhấn phím ▼ => nhấn phím **2** để chọn **Determinant**

```
1:Define Matrix
2:Edit Matrix
3:MatA    4:MatB
5:MatC    6:MatD

1:MatAns
2:Determinant
3:Transposition
4:Identity

Det(
```

Bước 2 nhấn phím **OPTN** => nhấn phím **3** để chọn **MatA**

```
1:Define Matrix
2:Edit Matrix
3:MatA    4:MatB
5:MatC    6:MatD

Det(MatA
```

Bước 3 nhấn phím =

```
Det(MatA
                210
```

Ma trận đơn vị

Bước 1 nhấn phím **OPTN** => nhấn phím ▼ => nhấn phím **4** để chọn lệnh **Identity**

```
1:Define Matrix
2:Edit Matrix
3:MatA    4:MatB
5:MatC    6:MatD

1:MatAns
2:Determinant
3:Transposition
4:Identity

Identity(
```

Bước 2 nhập **4**

```
Identity(4
```

Bước 3 nhấn phím =