DREAM OF INNOVATION

Chapter

Matrix

Contents

1.1. Matrix Addition, scalar multiplication and transposition

Definitions

- An **m**x**n** matrix (or a matrix of size **m**x**n**) is a rectangular array of numbers with **m rows** and **n columns**

3 columns

the (**1**,**3**)-**entry** of A

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & 4 \end{bmatrix}$$

2 rows

A is a **2**x**3** matrix (or a matrix of size **2**x**3**)

1.1. Matrix Addition, scalar multiplication and transposition

The mxn matrix

- The (**i**,**j**)-entry of A (denoted by a_{ij}) lies in **row i** and **column j**
- A is denoted simply as $A=[a_{ij}]$ or $A=(a_{ij})$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

2 refers to the **row**

n refers to the **column**

1.1. Matrix Addition, scalar multiplication and transposition

Definitions

1/ An **mxm** matrix is called a **square matrix (ma trận vuông)** of size m.

2/ The **zero matrix (ma trận không)** of size mxn (denoted by (0_{mxn}) is the matrix that its all entries are 0

3/ If $A=[a_{ij}]$ is an mxn matrix then **-A** refers to the **negative matrix (ma trận đối)** of A and defined by

$-A=[-a_{ij}]$.

1.1. Matrix Addition, scalar multiplication and transposition

4/ Identity matrices (ma trận đơn vị)

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

An **identity matrix I** is a square matrix with 1's on the main diagonal and zeros elsewhere

1.1. Matrix Addition, scalar multiplication and transposition

5/ Triangular matrices

- **Upper triangular** matrix: all entries **below** and **to the left** the **main diagonal** are zeros
- **Lower triangular** matrix: its tranposition is upper triangle matrix, that means every entry above and to the right the main diagonal is zero
- Matrix A is called **triangular** if it is upper or lower triangular
- For example, every **row-echelon matrix** is upper triangular

$$U = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -7 & 8 & 5 & 0 \end{bmatrix}$$

1.1. Matrix Addition, scalar multiplication and transposition

Two matrices are called **equal** if

- They have the same size
- Corresponding entries are equal

If $A=[a_{ij}]$, $B=[b_{ij}]$ then $\mathbf{A=B}$ means $\mathbf{a_{ij}=b_{ij}}$ for all i and j

1.1. Matrix Addition, scalar multiplication and transposition

Example 1:

Given $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

disscuss the possibility that $A = B, B = C, A = C$

1.1. Matrix Addition, scalar multiplication and transposition

Matrix Addition of same size matrices

- If $A=[a_{ij}]$, $B=[b_{ij}]$ then the sum matrix A+B is defined by $\mathbf{A+B=[a_{ij}+b_{ij}]}$
- The **difference A-B** is a matrix defined by

  $\mathbf{A-B=A+(-B)=[a_{ij}-b_{ij}]}$ for all mxn matrices A and B

Note that A-A=0, A+0=A (0 is zero matrix) for all mxn matrix A

1.1. Matrix Addition, scalar multiplication and transposition

Example 2:

If $A = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 0 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 & 6 \\ 1 & 4 & 7 \end{pmatrix}$, compute $A + B$

1.1. Matrix Addition, scalar multiplication and transposition

Solution:

$$A = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 0 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -2 & 6 \\ 1 & 4 & 7 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & 0 & 10 \\ 4 & 4 & 12 \end{pmatrix}$$

1.1. Matrix Addition, scalar multiplication and transposition

Example 3:

Find a, b and c if $\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$

1.1. Matrix Addition, scalar multiplication and transposition

Solution:

$$\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a+c & b+a & c+b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a+c=3 \\ b+a=2 \\ c+b=-1 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ b=-1 \\ c=0 \end{cases}$$

1.1. Matrix Addition, scalar multiplication and transposition

Properties

If A ,B and C are any matrices of the same size, then

- $A+B=B+A$ (commutative law: giao hoán)
- $A+(B+C)=(A+B)+C$ (associative law: kết hợp)

1.1. Matrix Addition, scalar multiplication and transposition

Example:

Solve $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, where X is a matrix.

1.1. Matrix Addition, scalar multiplication and transposition

Solution:

We have

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$\Leftrightarrow X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$$

So, $X = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$

1.1. Matrix Addition, scalar multiplication and transposition

Scalar Multiplication (phép nhân vô hướng)

- Suppose $A=[a_{ij}]$ is an mxn matrix and **k** is a **real munber**, the **scalar multiple** **kA** is a matrix defined by $kA=[ka_{ij}]$
- $kA=\mathbf{0} \rightarrow$ (either $k=0$ or $A=\mathbf{0}$)
- ($k=0$ or $A=\mathbf{0}$) $\rightarrow kA=\mathbf{0}$

Scalar: a quantity that has magnitude, but not direction; -- distinguished from a vector, which has both magnitude and direction (Webster Dictionary) → **One value**

1.1. Matrix Addition, scalar multiplication and transposition

Example:

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$, compute

a/ $5A$

b/ $\frac{1}{2}B$

c/ $3A - 2B$

1.1. Matrix Addition, scalar multiplication and transposition

Theorem

Let A, B and C denoted arbitrary $m \times n$ matrices where m and n are fixed. Let k and p denoted arbitrary real numbers. Then

1/ $A + B = B + A$

2/ $A + (B + C) = (A + B) + C$

3/ There is an $m \times n$ matrix 0, such that $0 + A = A$ for each A

4/ For each A there is an $m \times n$ matrix, $-A$ such that

$$A + (-A) = 0$$

5/ $k(A + B) = kA + kB$

6/ $(k + p)A = kA + pA$

7/ $(kp)A = k(pA)$

8/ $1.A = A$

1.1. Matrix Addition, scalar multiplication and transposition

Transpose

- If $A=[a_{ij}]$ is any mxn matrix,the **transpose** of A, written A^T, is an nxm matrix defined by $A^T=[a_{ji}]$

- The row i of A is the column i of A^T
- The column j of A is the row j of A^T

1.1. Matrix Addition, scalar multiplication and transposition

Example:

Write down the transpose of each of the following matrices

a/ $A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$

b/ $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

1.1. Matrix Addition, scalar multiplication and transposition

Solution:

a/ $A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$

$$B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

1.1. Matrix Addition, scalar multiplication and transposition

Solution:

$$\text{b/}\quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

1.1. Matrix Addition, scalar multiplication and transposition

Theorem:

1/ $(A^T)^T = A$

2/ $(kA)^T = k(A^T)$

3/ $(A+B)^T = A^T + B^T$

1.1. Matrix Addition, scalar multiplication and transposition

Definitions

1/ If $A=[a_{ij}]$ is any mxn matrix, then $\mathbf{a_{11},a_{22},a_{33}}$,…, are called the **main diagonal (đường chéo chính)** of A

2/ If $\mathbf{A=A^T}$ then A is called **symmetric (đối xứng).** In this case, A is a **square matrix**

1.1. Matrix Addition, scalar multiplication and transposition

Example 1:

$$D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix} \text{ is symmetric matrix}$$

because $D^T = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix} = D$

1.1. Matrix Addition, scalar multiplication and transposition

Example 2:

a/ If A and B are symmetric $n \times n$ matrix, show that $A+B$ is symmetric.

b/ Suppose a square matrix $A = 2A^T$. Show that $A = 0$

1.2 Matrix Multiplication

Dot product (tích vô hướng)

Example:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 1 \times 3 + 2 \times 2 + 3 \times 1 = 10$$

1.2 Matrix Multiplication

- Suppose $A=[a_{ij}]$ is an **m**x**k** matrix and $B=[b_{ij}]$ is an **k**x**n** matrix, then the **product $AB=[c_{ij}]$** is an **m**x**n** matrix whose the **(i,j)-entry** is the **dot product** of **row i** of A and **column j** of B
- c_{ij}=(**row i** of A).(**column j** of B)
- Note that $A_{mxk}B_{kxn}$ is a **m**x**n** matrix

$$AB \neq BA$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

1.2 Matrix Multiplication

Theorem 1:

Assume that k is an arbitrary scalar and A, B and C are matrices of sizes such that the indicated can be performed

a/ $IA = A, BI = B$

b/ $A(BC) = (AB)C$

c/ $A(B+C) = AB + AC, A(B-C) = AB - AC$

d/ $(B+C)A = BA + CA; (B-C)A = BA - CA$

e/ $k(AB) = (kA)B = A(kB)$

f/ $(AB)^T = B^T A^T$

1.2 Matrix Multiplication

Example:

Suppose that A and B are nxn matrices. Simplify the expression

$$A(BC-CD)+B(C-D)A+BDA$$

1.2 Matrix Multiplication

Solution:

We have $A\left(BC-CD\right)+B\left(C-D\right)A+BDA$

$$=ABC-ACD+BCA-BDA+BDA$$

$$=ABC-ACD+BCA$$

1.2 Matrix Multiplication

Example:

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$

Find B such that B commutes with A and B^2 is zero matrix.

1.2 Matrix Multiplication

Let square matrix $A_{n\times n}$.

$$A^0 = I_n$$

$$A^2 = A.A$$

$$A^3 = A.A.A$$

$$\vdots$$

$$A^n = A.A....A \ (\ n \text{ times})$$

1.2 Matrix Multiplication

Example:

Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, find A^2, A^3, A^{200}

$$A^2 = A.A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3\times 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3\times 3 \\ 0 & 1 \end{pmatrix}$$

So, $A^{200} = \begin{pmatrix} 1 & 3\times 200 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 600 \\ 0 & 1 \end{pmatrix}$

1.2 Matrix Multiplication

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

and square matrix $A_{n \times n}$

Then:

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + \ldots + a_1 A + a_0 I_n$$

where: I_n is identity matrix

1.2 Matrix Multiplication

Example:

Cho $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}; f(x) = 2x^2 - 4x + 3$

Compute $f(A)$.

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1.2 Matrix Multiplication

Example:

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}; f(x) = 2x^2 - 4x + 3$

Compute $f(A)$.

Solution: We have, $f(A) = 2A^2 - 4A + 3I_2$

$$f(A) = 2\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} - 4\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(A) = 2\begin{pmatrix} 1 & -6 \\ 18 & 13 \end{pmatrix} - \begin{pmatrix} 8 & -4 \\ 12 & 16 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Hence, $f(A) = \begin{pmatrix} -3 & -8 \\ 24 & 13 \end{pmatrix}$

Row-echelon matrix

A matrix is said to be in row-echelon form (and will be called a row-echelon matrix) if it satisfiles the following three conditions:

1/ All zero rows (consisting entirely zeros) are at the bottom.

2/ The first nonzero entry from the left in each nonzero row is a 1, called the leading 1 for that row.

3. Each leading 1 is to the right of all leading 1s in the rows above it.

A row-echelon matrix is said to be in reduced row-echelon form (and will be called a reduced row-echelon matrix) if, in addition, it satisfies the following condition:

4/ Each leading 1 is the only nonzero entry in its column.

1.2 Matrix Multiplication

Row-echelon matrix

Example 1:

a/ $A = \begin{pmatrix} 2 & 1 & 0 & 3 & -2 \\ 0 & 0 & 7 & 2 & 6 \\ 0 & 4 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 5}$ is not row-echelon matrix

b. $B = \begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ is not row-echelon matrix

1.2 Matrix Multiplication

Row-echelon matrix

Example 2:

a/ $A = \begin{pmatrix} 1 & 3 & 0 & 2 & -2 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is row-echelon matrix

b/ $B = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ is row-echelon matrix

1.2 Matrix Multiplication

Row-echelon matrix

Definition:

The following operations, called elementary operations

1/ Interchange two rows.

$$r_i \leftrightarrow r_j$$

2/ Multiply one row by a **nonzero** number.

$$r_i \to \alpha . r_i \ (\alpha \neq 0)$$

3/ Add a multiple of one row to a different row.

$$r_i \to r_i + \beta . r_j$$

1.2 Matrix Multiplication

Row-echelon matrix

Example 1: Use the elementary operations, find the row-echelon matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

Step 1: Find **leading 1**

$$A = \begin{pmatrix} 0 & 0 & \boxed{2} \\ \boxed{1} & 2 & 1 \\ 0 & \boxed{-1} & 1 \end{pmatrix}$$

1.2 Matrix Multiplication

Row-echelon matrix

Step 2:

$$A = \begin{pmatrix} 0 & 0 & \boxed{2} \\ \boxed{1} & 2 & 1 \\ 0 & \boxed{-1} & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{2} \\ 0 & \boxed{-1} & 1 \end{pmatrix}$$

Step 3:

$$\begin{pmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{2} \\ 0 & \boxed{-1} & 1 \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_2} \begin{pmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Row-echelon matrix

Example 2:

Use the elementary operations, find the row- echelon matrix

$$A=\begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 2 & 3 & -1 & 4 & 5 \\ 3 & 2 & -3 & 7 & 4 \\ -1 & 1 & 2 & -3 & 1 \end{pmatrix}$$

Row-echelon matrix

Solution:

Step 1:

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ \xcancel{2} & 3 & -1 & 4 & 5 \\ \xcancel{3} & 2 & -3 & 7 & 4 \\ \xcancel{-1} & 1 & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 \to r_4 + r_1]{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1}} \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 & 2 \end{pmatrix}$$

1.2 Matrix Multiplication

Row-echelon matrix

Khi khử phần tử leading trên cột 2 không được sử dụng hàng 1

Step 2:

$$\begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & \cancel{-1} & 0 & 1 & 1 \\ 0 & \cancel{2} & 1 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow[r_4 \to r_4 - 2r_2]{r_3 \to r_3 + r_2} \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 & -4 \end{pmatrix}$$

1.2 Matrix Multiplication

Step 3:

$$\begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & \cancel{-1} & -1 & -4 \end{pmatrix}$$

$$\xrightarrow{r_4 \to r_4 + r_3} \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Khi khử phần tử leading trên cột 3 không được sử dụng hàng 1 và hàng 2. Tương tự cho các cột tiếp theo ……

1.2 Matrix Multiplication

Example 3:

Use the elementary operations, find the row- echelon matrix

$$a/\ \ A=\begin{pmatrix}1 & 1 & 2 & 3\\ 2 & 5 & 7 & 2\\ 1 & 8 & 3 & 1\end{pmatrix}\qquad b/\ \ B=\begin{pmatrix}1 & 1 & 1 & 3\\ 2 & 5 & 7 & 2\\ 1 & 2 & 3 & 1\end{pmatrix}$$

$$c/\ C=\begin{pmatrix}3 & -5 & 4\\ 2 & -1 & 3\\ 1 & 2 & 5\end{pmatrix}\qquad d/\ \ D=\begin{pmatrix}6 & 0 & 4\\ 2 & 6 & 8\\ -3 & 4 & 1\end{pmatrix}$$

1.3 Matrix Inverse

Definition

- If A is a **square matrix**, a matrix B is called an **inverse** (nghịch đảo) of A if and only if **AB=I** and **BA=I.**
- A matrix A that has an inverse is called an **invertible (khả nghịch)** matrix.
- Denoted $B = A^{-1}$

Noted: $A^{-1} \neq \dfrac{1}{A}$

1.3 Matrix Inverse

Example 1:

Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

We have

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $A^{-1} = B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

1.3 Matrix Inverse

Theorem:

If B and C are both inverses of A, then B = C

1.3 Matrix Inverse

Example 2:

If the matrix A satisfy $A^2 - 4A + 3I = 0$, then A^{-1} is

a/ exists but not enough information to find.

b/ $\frac{1}{3}(4I - A)$

c/ $\frac{1}{3}(A - 4I)$

d/ not exits the inverse of A.

e/ $\frac{1}{3}(4 - A)$

1.3 Matrix Inverse

Solution:

We have

$$A^2 - 4A + 3I = 0$$

$$\Leftrightarrow 4A - A^2 = 3I$$

$$\Leftrightarrow \frac{1}{3}\left(4A - A^2\right) = I$$

$$\Leftrightarrow \left[\frac{1}{3}(4I - A)\right]A = I\,(*)$$

$$A^2 - 4A + 3I = 0$$

$$\Leftrightarrow 4A - A^2 = 3I$$

$$\Leftrightarrow \frac{1}{3}\left(4A - A^2\right) = I$$

$$\Leftrightarrow A\left[\frac{1}{3}(4I - A)\right] = I\,(**)$$

From (*) and (**), we have

$$A^{-1} = \frac{1}{3}(4I - A)$$

1.3 Matrix Inverse

Example 3:

Show that $A^3=I$ and find A^{-1} if

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

1.3 Matrix Inverse

Solution:

We have

$$A^2 = AA = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- And, $A^3 = A^2 A = A A^2 = I_2$
- *So,* $A^{-1} = A^2$

1.3 Matrix Inverse

Matrix Inversion Algorithm

$$\left[\,A \;\; I\,\right] \xrightarrow{\textit{elementary}\ \text{row operations}} \left[\,I \;\; A^{-1}\,\right]$$

input

output

Theorem 3.

Either any square matrix can be reduced to **I** or not.

In the first case, the algorithm produces $\mathbf{A^{-1}}$;

in the second, $\mathbf{A^{-1}}$ does not exist.

1.3 Matrix Inverse

Example

Find the **inverse** of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

1.3 Matrix Inverse

Example:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_1+r_3]{-3r_1+r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 2 & 3 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 2 & 3 & -3 & 1 & 0 \end{array}\right] \xrightarrow{-2r_2+r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

1.3 Matrix Inverse

Example

$$\xrightarrow[r_3+r_1]{-r_3+r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

1.3 Matrix Inverse

Properties

Theorem 4

1/ $(A^{-1})^{-1}=A$

2/ $(AB)^{-1}=B^{-1}A^{-1}$

$$(A_1A_2\ldots A_k)^{-1}=A_k^{-1}\ldots A_2^{-1}A_1^{-1}$$

3/ $(A^T)^{-1}=(A^{-1})^T$

4/ $(A^k)^{-1}=(A^{-1})^k$

5/ $(kA)^{-1}=A^{-1}/k$

6/ $I^{-1}=I$

Corollary

A square matrix A is invertible **iff** A^T is invertible

1.3 Matrix Inverse

Example

Find A if

$$\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

1.3 Matrix Inverse

Solution:

We have

$$A^T - 2I = \left(\left(A^T - 2I\right)^{-1}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

1.3 Matrix Inverse

Solution:

We have

$$A^T - 2I = \left(\left(A^T - 2I\right)^{-1}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

Hence

$$A^T = 2I + \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

1.3 Matrix Inverse

Theorem

The following conditions are equivalent for an **nxn** matrix A

- 1. A is **invertible**
- 2. The homogeneous system AX=0 has **only the trivial solution X=0**
- 3. A can be carried to $\mathbf{I_n}$ by elementary row operations
- 4. The system AX=B has **unique solution** for every choice of column B
- 5. There exist an nxn matrix C such that AC=$\mathbf{I_n}$

1.3 Matrix Inverse

Theorem

Suppose **AX=B** is a system of n equations in n variables and A is an **invertible** matrix. Then the system has the **unique solution**

$$X=A^{-1}B$$

Similar:

$$XA=B \Leftrightarrow X=BA^{-1}$$

1.3 Matrix Inverse

Example 1:

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

Solve the system AX=B with $B=[1 \;\; 2]^T$

1.3 Matrix Inverse

Solution:

$$AX = B \Leftrightarrow A^{-1}AX = A^{-1}B$$

$$\Leftrightarrow X = A^{-1}B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1.3 Matrix Inverse

Example 2: Let

$$A=\begin{pmatrix}2&-1&3\\0&3&1\\5&-2&4\end{pmatrix};B=\begin{pmatrix}1&-2&3\\4&5&0\\0&1&2\end{pmatrix};\ C=\begin{pmatrix}3&2&2\\1&1&1\\1&0&1\end{pmatrix}$$

Find X such that:

a/ $C.X=B$ b/ $X.C=B$

c/ $A.X=B$ d/ $X.A=B$

1.3 Matrix Inverse

Solution:

a/ $\mathrm{CX} = \mathrm{B} \Leftrightarrow \mathrm{X} = C^{-1}B = \begin{pmatrix} -7 & -12 & 3 \\ 4 & 4 & -2 \\ 7 & 13 & -1 \end{pmatrix}$

b/ $\mathrm{X.C} = \mathrm{B} \Leftrightarrow \mathrm{X} = B.C^{-1} = \begin{pmatrix} -2 & 2 & 5 \\ 4 & -3 & -5 \\ -2 & 5 & 1 \end{pmatrix}$

1.3 Matrix Inverse

Solution:

c/

$$A.\mathrm{X} = \mathrm{B} \Leftrightarrow \mathrm{X} = \mathrm{A}^{-1}B = \begin{pmatrix} -3/11 & 24/11 & -1 \\ 23/22 & 47/22 & -1/2 \\ 19/22 & -31/22 & 3/2 \end{pmatrix}$$

d/

$$\mathrm{X}.\mathrm{A} = \mathrm{B} \Leftrightarrow \mathrm{X} = B\,\mathrm{A}^{-1} = \begin{pmatrix} 41/22 & -9/22 & -6/11 \\ -81/22 & 43/22 & 25/11 \\ 25/22 & 9/22 & -5/11 \end{pmatrix}$$

1.3 Matrix Inverse

Corollary

If A and C are square matrices such that $AC=I$, then also $CA=I$. In particular, both A and C are invertible, $C=A^{-1}$ and $A=C^{-1}$.

1.4. Determinants

The Cofactor Expansion

- If A=[a] then the determinant of A, denoted by detA = a

$$A=[-7]; \det A=-7$$

- If A is an 2x2 matrix then $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det A=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$$

- If A is an 3x3 matrix then the determinant of A is defined by

$$\begin{array}{ccc} + & - & + \end{array}$$
$$A=\begin{bmatrix} a & b & c \\ d & \boxed{e} & \boxed{f} \\ g & \boxed{h} & \boxed{i} \end{bmatrix}$$

$$\det A=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=a.(+).\begin{vmatrix} e & f \\ h & i \end{vmatrix}+b.(-).\begin{vmatrix} d & f \\ g & i \end{vmatrix}+c.(+).\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

1.4. Determinants

The determinant of 3x3 matrix

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a.(+).\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b.(-).\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c.(+).\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - \cdot ceg - afh - bdi$$

1.4. Determinants

The determinant of 3x3 matrix

$$\begin{array}{ccc} col1 & col2 & col3 \\ a & b & c \\ d & e & f \\ g & h & i \end{array}$$

$$\begin{array}{ccccc} col1 & col2 & col3 & col1 & col2 \\ 1 & -5 & 4 & 1 & -5 \\ 0 & 2 & 6 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \end{array}$$

\- \- \- + + +

$$\det A = aei + bfg + cdh - ceg - afh - bdi$$

1.4. Determinants

Find detA if

$$A=\begin{bmatrix} 1 & -5 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{array}{ccccc} col1 & col2 & col3 & col1 & col2 \\ 1 & -5 & 4 & 1 & -5 \\ 0 & 2 & 6 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \end{array}$$

$$- \qquad - \qquad - \qquad + \qquad + \qquad +$$

Note that : only use with 3x3 matrices

1.4. Determinants

$$A=\begin{bmatrix}1&2&-1&5\\0&6&4&3\\0&7&-1&0\\0&1&8&2\end{bmatrix}$$

Matrix A_{11}

$$\det A=\overset{a_{11}}{1}.\overbrace{(+1).\begin{vmatrix}6&4&3\\7&-1&0\\1&8&2\end{vmatrix}}^{(-1)^{1+1}.\det(A_{11})}+2.(-1).\begin{vmatrix}0&4&3\\0&-1&0\\0&8&2\end{vmatrix}+$$

$$(-1)(+1)\begin{vmatrix}0&6&3\\0&7&0\\1&2&2\end{vmatrix}-5(-1)\begin{vmatrix}0&6&4\\0&7&-1\\0&1&8\end{vmatrix}$$

1.4. Determinants

Definition

Example:

$$\begin{vmatrix} \boxed{\begin{matrix}1\\0\\0\\0\end{matrix}} & \begin{matrix}2\\6\\7\\1\end{matrix} & \begin{matrix}-1\\4\\-1\\8\end{matrix} & \begin{matrix}5\\0\\0\\2\end{matrix} \end{vmatrix} = 1\begin{vmatrix} 6 & 4 & 0 \\ 7 & -1 & 0 \\ 1 & 8 & 2 \end{vmatrix} = -68$$

$$\det A = a_{11}.c_{11} + a_{21}.c_{21} + a_{31}.c_{31} + a_{41}.c_{41}$$

$$= a_{11}.c_{11} = c_{11} = \left(-1\right)^2.\det A_{11} = \det A_{11} =$$

$$= a_{13}.c_{13} + a_{23}.c_{23} + a_{33}.c_{33} = 2.\left(-1\right)^{3+3}.\begin{vmatrix} 6 & 4 \\ 7 & -1 \end{vmatrix} = 2.\left(-6-28\right) = -68$$

1.4. Determinants

The (i,j)-cofactor (phần phụ đại số)

- If A is an mxm matrix then the (**i**,**j**)-cofactor of A is defined by

$$c_{ij}(A)=(-1)^{i+j}\det(A_{ij})$$

- A_{ij} is the (m-1)x(m-1) matrix obtained from A by deleting **row i** and **column j** of A

1.4. Determinants

The (i,j)-cofactor (phần phụ đại số)

Example: $c_{23}(A)=(-1)^{2+3}\det(A_{23})=-14$

$$A=\begin{bmatrix}1 & 2 & -1 & 5\\ 0 & 6 & 4 & 3\\ 0 & 7 & -1 & 0\\ 0 & 1 & 8 & 2\end{bmatrix} \qquad A_{23}=\begin{bmatrix}1 & 2 & 5\\ 0 & 7 & 0\\ 0 & 1 & 2\end{bmatrix}$$

$$c_{32}=(-1)^{3+2}.\det(A_{32})=-\begin{vmatrix}1 & -1 & 5\\ 0 & 4 & 3\\ 0 & 8 & 2\end{vmatrix}=-(8-24)=16$$

1.4. Determinants

Definition

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \underset{a_{11}}{a}.\underbrace{(+).\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{(1,1)-cofactor} + \underset{a_{12}}{b}.\underbrace{(-).\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{(1,2)-cofactor} + \underset{a_{13}}{c}.\underbrace{(+).\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{(1,3)-cofactor}$$

If A is an nxn matrix then the **determinant** of A is defined by

- **detA=|A|**$=a_{i1}c_{i1}(A)+a_{i2}c_{i2}(A)+\ldots+a_{im}c_{im}(A)$
- or **detA=|A|**$= a_{1j}c_{1j}(A)+a_{2j}c_{2j}(A)+\ldots+a_{mj}c_{mj}(A)$

1.4. Determinants

Properties

1/ If A has **one row (or columm) of zeros** then detA=0

2/ If A has two identical rows(columns) then detA=0

$$\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} = 0$$

3/ If A is an **triangular matrix** then detA is the **product** of the entries on the **main diagonal**

$$\begin{vmatrix} 3 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 3.\begin{vmatrix} 6 & 4 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 3.6.\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = 3.6.(-1).2$$

Upper triangular

1.4. Determinants

Example:

Suppose A is an **triangular matrix**, $a_{11}=4$ and $c_{11}(A)=11$.

What is the **product of all entries on the maindiagonal** ?

A)11 B)44 C)-11 D)-44

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Upper triangular

$\det A = a_{11}c_{11}(A) + a_{21}c_{21}(A) + \ldots = a_{11}c_{11}(A)$

1.4. Determinants

Determinants and elementary operations

a/ If B obtained from A by **interchanging** two rows (or columns) then **detB = - detA**

Example: detC'= - detC=12

$$C = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad C' = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 3 & -2 \\ 1 & 2 & 5 \\ -3 & 1 & 1 \end{vmatrix} \overset{r_1 \leftrightarrow r_2}{=} -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ -3 & 1 & 1 \end{vmatrix}$$

1.4. Determinants

Determinants and elementary operations

2/ If two rows (or columns) of a matrix is the same then the determinant is **zero.**

Examples:

$$C = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 1 & 2 & 5 & 6 \\ 1 & 2 & 5 & 6 \end{bmatrix} \qquad C' = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 1 & 2 & 5 & 6 \\ 1 & 2 & 5 & 6 \end{bmatrix} = C$$

$$\det C = -\det C' = -\det C \Rightarrow \det C = 0$$

1.4. Determinants

Determinants and elementary operations

3/ If B is the matrix obtained from A by multiplying one row (or cloumn) by a nonzero number **k** then detB=**k**detA.

Example:

detA'=35, detA=2detA'=2.35=70

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 2 & 4 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 1 & 2 & 5 \end{vmatrix} = 70$$

1.4. Determinants

Examples: Find a such that

$$\begin{vmatrix} 4 & 8 & -8 \\ 8 & -4 & 4 \\ 0 & 8 & 4 \end{vmatrix} = a \begin{vmatrix} 1 & 2 & -2 \\ 2 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix} ?$$

a) 4 b) 12 c) 48 d)64

$$\text{Note that } \underbrace{k.\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{k\times matrix} \neq \underbrace{k.\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}}_{k\times number}$$

Note that a|A| means number a multiplies number |A| while aA means number a multiplies matrix A.

1.4. Determinants

Determinants and elementary operations

4/ If A is a square matrix that one row(column) is a multiple of another row(column) then the determinant of A is zero

$$\begin{vmatrix} a & b & c \\ x & y & z \\ ka & kb & kc \end{vmatrix} \overset{r_3=kr_1}{=} k\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} \overset{r_1=r_3}{=} 0$$

1.4. Determinants

Determinants and elementary operations

5/ If B is the matrix obtained from A by adding a multiple of one row (or column) to another row (or column) then **detB=detA**

Example:

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 7 & 0 \\ 2 & 4 & -10 \end{vmatrix} \overset{-2r_1+r_3}{=} \begin{vmatrix} 1 & 2 & -3 \\ 0 & 7 & 0 \\ 0 & 0 & -4 \end{vmatrix} = 1.7.(-4) = -28$$

1.4. Determinants

Examples

$$\begin{vmatrix} 0 & 2 & -1 & 9 \\ 2 & 2 & -4 & 6 \\ 3 & 2 & -2 & 1 \\ -3 & 4 & 2 & 0 \end{vmatrix} \overset{r_1 \leftrightarrow r_2}{=} -\begin{vmatrix} 2 & 2 & -4 & 6 \\ 0 & 2 & -1 & 9 \\ 3 & 2 & -2 & 1 \\ -3 & 4 & 2 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & 2 & -1 & 9 \\ 3 & 2 & -2 & 1 \\ -3 & 4 & 2 & 0 \end{vmatrix} \overset{\substack{-3r_1+r_3 \\ 3r_1+r_4}}{=} -2\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & 2 & -1 & 9 \\ 0 & -1 & 4 & -8 \\ 0 & 7 & -4 & 9 \end{vmatrix}$$

$$\overset{r_2 \leftrightarrow r_3}{=} -(-2)\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 2 & -1 & 9 \\ 0 & 7 & -4 & 9 \end{vmatrix} \overset{\substack{2r_2+r_3 \\ 7r_2+r_4}}{=} 2\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 0 & 7 & -7 \\ 0 & 0 & 24 & -47 \end{vmatrix} = 2.7\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 24 & -47 \end{vmatrix}$$

$$\overset{-24r_3+r_4}{=} 2.7\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -23 \end{vmatrix} = 2.7.1.(-1).1.(-23)$$

Do your self: Find $\begin{vmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 1 & -3 & 4 & 6 \\ 1 & 2 & 4 & 5 \end{vmatrix}$

1.4. Determinants

Theorem

1/ $\det A = \det(A^T)$

2/ $\det(AB) = \det A . \det B$

Note that

$\det(A^k) = (\det A)^k$

$\det(A+B) \neq \det A + \det B$

3/ If A is an nxn matrix then $\det(kA) = k^n \det A$.

4/ $\det A^{-1} = 1/\det A$.

1.4. Determinants

Example 1:

Suppose A is a nxn matrix, and k is a scalar. Which of the following is(are) false ?

a) $\det(A^T)=\det A$

b) $\det(AB)=\det A \det B$

c) $\det(A+B)=\det A+\det B$

d) $\det(kA)=k\det A$

e) $\det(kA)=k^n\det A$

1.4. Determinants

Example 2:

Let A and B are nxn matrices. Which statement is false ?

a) $\det(A^{-1}BA)=\det B$

b) $\det(AB^{-1}A^{-1}B)=1$

c) $\det A \det A^{-1}=1$

d) $\det A \det A^{T}=1$

e) $\det(A^{T}B)=\det(B^{T}A)$

1.4. Determinants

Example 3:

Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ and assume that $\det(A) = 3$.

Compute:

a/$\det\left(2B^{-1}\right)$ where $B = \begin{bmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{bmatrix}$

1.4. Determinants

Example 3:

Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ and assume that $\det(A) = 3$.

Compute:

b/ $\det(2C^{-1})$ where $C = \begin{bmatrix} 2p & -a+u & 3u \\ 2q & -b+v & 3v \\ 2r & -c+w & 3w \end{bmatrix}$

1.4. Determinants

Determinant and Matrix Inverses

1/ If A is invertible then A→Identity matrix

2/ If detA $\neq 0$ and A→B by elementary operations then detB $\neq 0$

3/ A is invertible if detA$\neq$0

4/ Give a formula to find A^{-1}.

1.4. Determinants

Determinant and Matrix Inverses

Example 1:

Use determinants to find which real values of c make each of the following matrices invertible.

a/ $A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & -4 & c \\ 2 & 5 & 8 \end{bmatrix}$ b/ $B = \begin{bmatrix} 0 & c & -c \\ -1 & 2 & 1 \\ c & -c & c \end{bmatrix}$

c/ $C = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$

1.4. Determinants

Determinant and Matrix Inverses

Example 2:

Use determinants to find which real values of c make each of the following matrices invertible.

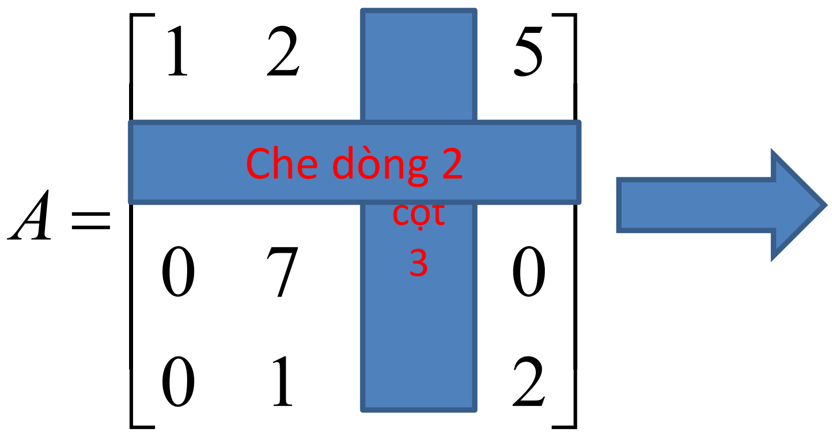
a/ $A = \begin{bmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{bmatrix}$ b/ $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$

c/ $C = \begin{bmatrix} 1 & c & -1 \\ c & 1 & 1 \\ 0 & 1 & c \end{bmatrix}$

1.4. Determinants

(i,j)-Cofactor (phần bù đại số)

- $c_{ij}=c_{ij}(A)=(-1)^{i+j}\det(A_{ij})$ (it is a number)
- A_{ij} is the matrix obtained from A by deleting **row i** and **column j**



$$A_{23} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 7 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$c_{23} = c_{23}(A) = (-1)^{2+3}\det A_{23} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 7 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -14$$

1.4. Determinants

Adjugate matrix

The ***adjugate*** matrix of A is the matrix

$$adjA = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

Example 1:

$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$. We have $c_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3,\ c_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3,$

$\boxed{c_{11} = 3,\quad c_{12} = -3,\ c_{13} = -6}$

$c_{21} = 2,\quad c_{22} = 1,\quad c_{23} = -4,$

$c_{31} = 2,\quad c_{32} = 1,\quad c_{33} = 5$

$$\Rightarrow adjA = \begin{bmatrix} 3 & 2 & 2 \\ -3 & 1 & 1 \\ -6 & -4 & 5 \end{bmatrix}$$

1.4. Determinants

Example:

Find the adjugate of each of the following matrices.

a/ $A = \begin{bmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$ b/ $B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

c/ $C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ d/ $D = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

1.4. Determinants

Theorem of Adjugate Formula:

If A is any square matrix, then

- A(adjA)=(detA)**I**
- In particular, if **detA≠0** then A is **invertible** and

$$A^{-1} = \frac{1}{\det A} adjA$$

Example:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det A = 2 \text{ and adjA=} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem of Adjugate Formula

Note:

$$A^{-1} = \frac{1}{\det A}.adjA$$

$$\Leftrightarrow adjA = (\det A).A^{-1}$$

$$\Rightarrow |adjA| = \left|(\det A).A^{-1}\right| = (\det A)^n .\det A^{-1} = (\det A)^{n-1}$$

1.4. Determinants

Example:

Let A be a 2x2 matrix with detA=3.

Find det(adjA) ?

a) 1/3 b)1/2 c)2 d) 3

1.4. Determinants

Example:

Determine whether the statements is true or false.

1/ The determinant of a square matrix equal to the sum of all entries of this matrix.

2/ The determinant of a square matrix is a matrix with same size.

3/ The determinant of a square invertible matrix always eaqual to 1.

4/ The determinant of a square matrix is 0 if it is not invertible.

5/ By elementary operations, we can find the determinant of a square matrix.

1.4. Determinants

Example:

6/ Suppose A, B, C are square matrices satisfying A=BC and A is not invertible. Then B or C is not invertible.

7/ Suppose A, B, C are square matrices satisfying A=BC and A is invertible. Then B and C are invertible.

8/ If AB=AC, then B=C.

9/ If AB=0 then A=0 or B=0.

10/ If A is invertible, then A^k is invertible for all positive integer k.