Chapter 5

TECHNIQUES OF INTEGRATION

Contents

5.1. The Indefinite integral

Definition

Due to the relation given by the FTC between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an indefinite integral.

Thus, $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$.

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a number.

An indefinite integral $\int f(x)dx$ is a ***function*** (or family of functions).

5.1. The Indefinite integral

Formular

$$1. \int cf(x)\,dx = c\int f(x)\,dx$$

$$2. \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$3. \int k\,dx = kx + C$$

$$4. \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$$

$$5\int \frac{1}{x}\,dx = \ln|x| + C$$

5.1. The Indefinite integral

Formular

$$6a.\int \sin x\,dx = -\cos x + C \qquad 6b.\int \cos x\,dx = \sin x + C$$

$$7a.\int \sec^2 x\,dx = \tan x + C \qquad 7b.\int \csc^2 x\,dx = -\cot x + C$$

$$8a.\int \sec x \tan x\,dx = \sec x + C \qquad 8b.\int \csc x \cot x\,dx = -\csc x + C$$

5.2. The Substitution Rule

Example: Evaluate $\int \frac{dx}{(3-5x)^2}$

Solution:

Let $u = 3-5x$ .Then, $du = -5dx$, so $dx = -\frac{1}{5}du$

$$\int \frac{dx}{(3-5x)^2} = \int \frac{1}{u^2}\left(\frac{-du}{5}\right) = -\frac{1}{5}\int \frac{1}{u^2}du$$

$$= -\frac{1}{5}\int u^{-2}du = -\frac{1}{5}.\frac{u^{-1}}{-1} + C = \frac{1}{5u} + C = \frac{1}{5(3-5x)} + C$$

Method 2: $$\int \frac{dx}{(3-5x)^2} = \int (3-5x)^{-2}dx = -\frac{1}{5}.\frac{(3-5x)^{-1}}{-1} + C$$

5.2. The Substitution Rule

General: $\int f(ax+b)dx; u = ax+b \Rightarrow du = adx \Rightarrow dx = \frac{1}{a}du$

$$\int f(ax+b)dx = \int f(u) \times \frac{1}{a}du$$

$$= \frac{1}{a} f(u)du$$

$$= \frac{1}{a} F(u) + C$$

$$= \frac{1}{a} F(ax+b) + C$$

$$\int f(ax+b)dx = \frac{1}{a} F(ax+b) + C,$$

5.2. The Substitution Rule

Example: Use the formular $\int f\left(ax+b\right)dx=\frac{1}{a}F\left(ax+b\right)+C,$ find:

$1.\int\left(2x-1\right)^{10}dx$

$2.\int\frac{1}{1-x}dx$

$3.\int\frac{1}{\left(1-x\right)^{2}}dx$

$4.\int\frac{1}{\sqrt{1-x}}dx$

$5.\int e^{3x+4}dx$

$6.\int\left(\sin 5x+\cos 7x\right)dx$

5.2. The Substitution Rule

Theorem

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx = \int f\left(u\right)du$$

Example 1: Evaluate $I = \int 2x\sqrt{1+x^2}\,dx.$

Solution: $u = \sqrt{1+x^2}$

$$u^2 = 1+x^2 \Rightarrow 2udu = 2xdx$$

$$\Rightarrow I = \int 2u^2du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(\sqrt{1+x^2}\right)^3 + C$$

5.2. The Substitution Rule

Example 2: Evaluate $\int x^3 \cos\left(x^4+2\right)dx.$

Solution:

$$u = x^4 + 2 \Rightarrow du = 4x^3 dx \Rightarrow x^3 dx = \frac{du}{4}$$

$$\int x^3 \cos(x^4+2)\,dx = \int \cos u \cdot \tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u\,du$$

$$= \tfrac{1}{4}\sin u + C$$

$$= \tfrac{1}{4}\sin(x^4+2) + C$$

5.2. The Substitution Rule

Sub.rule for def. integrals:

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$

Let F be an antiderivative of f. Then, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

So, $\int_a^b f(g(x))g'(x)dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$

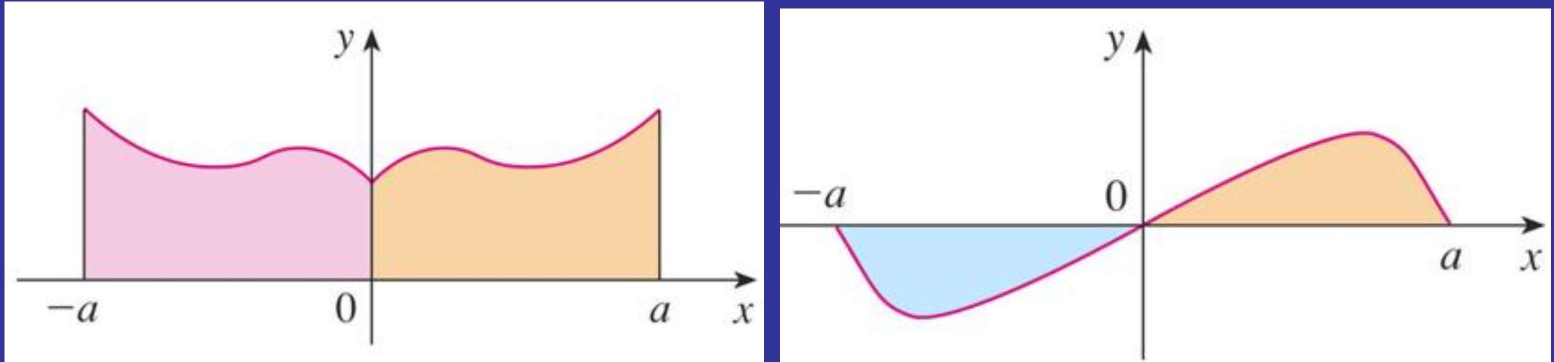
5.2. The Substitution Rule

Integs.of symm.functions:

Suppose f is continuous on $[-a\ ,\ a]$.

a/ If f is even, $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

b/ If f is odd, $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

5.3. Integration by parts

Formula $I = \int f(x)g'(x)\,dx$

Set $\begin{cases} u = f(x) \\ dv = g'(x)\,dx \end{cases} \Rightarrow \begin{cases} u' = f'(x)\,dx \\ v = g(x) \end{cases}$

Thus, by the Substitution Rule, the formula for integration by parts becomes:

$$\int f(x)g'(x)\,dx = uv - \int v\,du = f(x)g(x) - \int g(x)f'(x)\,dx$$

Note: $u = \ln x; x^k; e^x; \sin x; \cos x$

Using the FTC, we obtain:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

5.3. Integration by parts

Example 1: Find $\int x\sin x dx$

Solution:

$$\begin{cases} u = x \Rightarrow du = dx \\ dv = \sin x dx \Rightarrow v = -\cos x \end{cases}$$

$$I = u.v - \int v du = -x\cos x - \int -\cos x dx$$

$$= -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

5.3. Integration by parts

Example 2: Find $\int e^x \sin x dx$

Solution:

Step 1: Set $\begin{cases} u = e^x \Rightarrow du = e^x dx \\ dv = \sin x dx \Rightarrow v = -\cos x \end{cases}$

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

Step 2: Set $\begin{cases} u = e^x \Rightarrow du = e^x dx \\ dv = \cos x dx \Rightarrow v = \sin x \end{cases}$

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

5.3. Integration by parts

Step 3: This can be regarded as an equation to be solved for the unknown integral.

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

Adding to both sides $\int e^x \sin x\,dx$, we obtain:

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get:

$$\int e^x \sin x\,dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$

5.3. Integration by parts

Example 3: Suppose f(x) is continuous and differentiable, f(1)=4 and $\int_0^1 f(x)dx = 5$. Find $\int_0^1 xf'(x)dx$	
a	4/5
b	5/4
c	1
d	None of the others
e	-1

Example 4: Suppose f(x) is continuous and differentiable,

f(1)=3, f(3)=1 and $\int_1^3 xf'(x)dx = 13$

What is the <u>average value</u> of f on the interval [1,3] ?

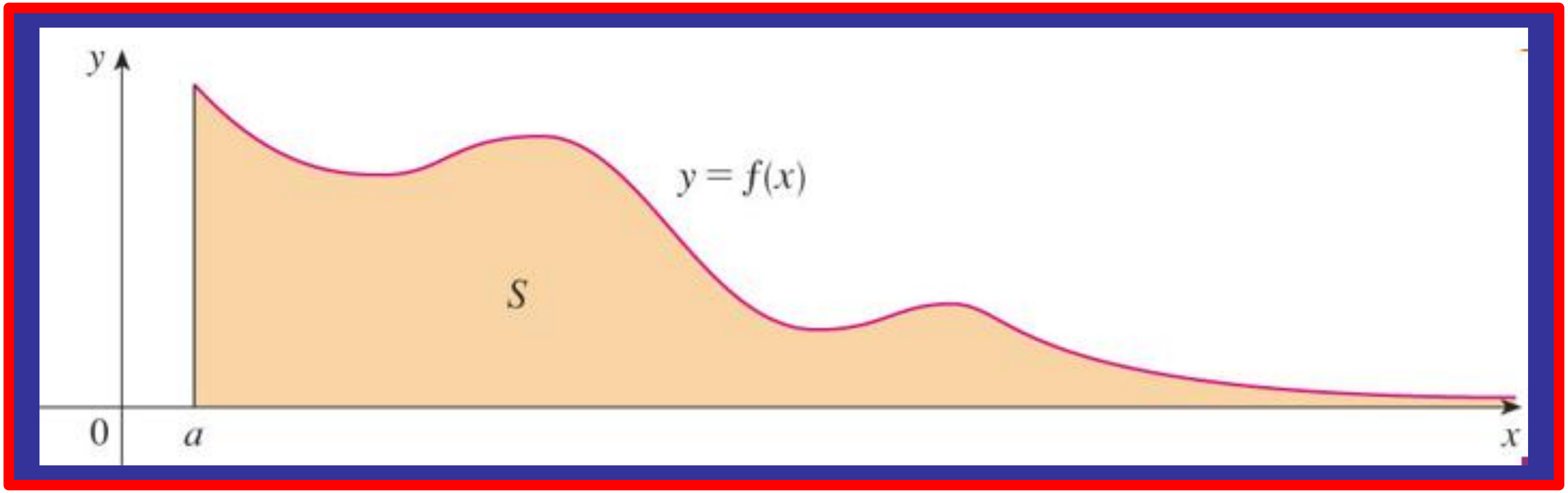
5.4. Improper Integrals

Improper Integrals of type 1

Definition: If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided this limit exists

(as a finite number).



5.4. Improper Integrals

Improper Integrals of type 1

Convergent and divergent:

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called:

- **Convergent** if the corresponding limit exists.
- **Divergent** if the limit does not exist.

5.4. Improper Integrals

If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then

we define: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$

Here, any real number a can be used.

5.4. Improper Integrals

Example 1: Evaluate:

a/ $I_1 = \int_1^{+\infty} \frac{dx}{x}$

b/ $I_2 = \int_1^{+\infty} \frac{dx}{x^2}$

c/ $I_3 = \int_1^{+\infty} \frac{dx}{\sqrt{x}}$

5.4. Improper Integrals

Solution:

a/ $$I_1 = \int_1^{\infty} \frac{dx}{x}$$

Step 1: $A = \int_1^{b} \frac{dx}{x} = \ln|x|\Big|_1^b = \ln b - \ln 1 = \ln b$

Step 2: $I_1 = \lim\limits_{b\to+\infty} A = \lim\limits_{b\to+\infty} \ln b = +\infty$

Solution:

b/ $I_2 = \int_1^{\infty} \frac{dx}{x^2}$

Step 1: $A = \int_1^{b} \frac{dx}{x^2} = -\left.\frac{1}{x}\right|_1^b = -\frac{1}{b} + 1$

Step 2:

$$I_2 = \lim_{b \to +\infty} A = \lim_{b \to +\infty} \left(-\frac{1}{b} + 1 \right) = 1$$

Solution:

c/ $I_3 = \int_1^{\infty} \frac{dx}{\sqrt{x}}$

Step 1: $A = \int_1^{b} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_1^b = 2\sqrt{b} - 2$

Step 2:

$$I_3 = \lim_{b \to +\infty} A = \lim_{b \to +\infty} \left(2\sqrt{b} - 2\right) = +\infty$$

5.4. Improper Integrals

Example 2: Evaluate:

a/ $I_1 = \int_1^{+\infty} \frac{dx}{x^2+1}$

b/ $I_2 = \int_1^{+\infty} e^{-2x}dx$

c/ $I_3 = \int_e^{+\infty} \frac{1}{x(\ln x)^2}dx$

5.4. Improper Integrals

Solution:

a/ $I_1 = \int_1^{\infty} \frac{dx}{x^2+1}$

Step 1: $A = \int_1^{b} \frac{dx}{x^2+1} = \left(\arctan x\right)_1^b = \arctan b - \frac{\pi}{4}$

Step 2:

$$I_1 = \lim_{b\to+\infty} A = \lim_{b\to+\infty}\left(\arctan b - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Solution:

b/ $I_2 = \int_1^{+\infty} e^{-2x}dx$

Step 1: $A = \int_1^b e^{-2x}dx = \left(-\frac{1}{2}e^{-2x}\right)_1^b = -\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}$

Step 2: $I_3 = \lim_{b\to+\infty} A = \lim_{b\to+\infty}\left(-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}\right)$

$$= \lim_{b\to+\infty}\left(-\frac{1}{2e^{2b}} + \frac{1}{2}e^{-2}\right) = \frac{1}{2e^2}$$

5.4. Improper Integrals

Solution:

c/ $I_3 = \int_e^{\infty} \frac{1}{x(\ln x)^2}dx$

Step 1: $A = \int_e^{b} \frac{1}{x(\ln x)^2}dx \qquad t = \ln x \Rightarrow dt = \frac{dx}{x}$

$$A = \int_1^{\ln b} \frac{1}{t^2}dt = \left(-\frac{1}{t}\right)_1^{\ln b} = -\frac{1}{\ln b} + 1$$

5.4. Improper Integrals

Step 2: $\lim\limits_{b\to+\infty} A = \lim\limits_{b\to+\infty}\left(-\dfrac{1}{\ln b}+1\right)=1$

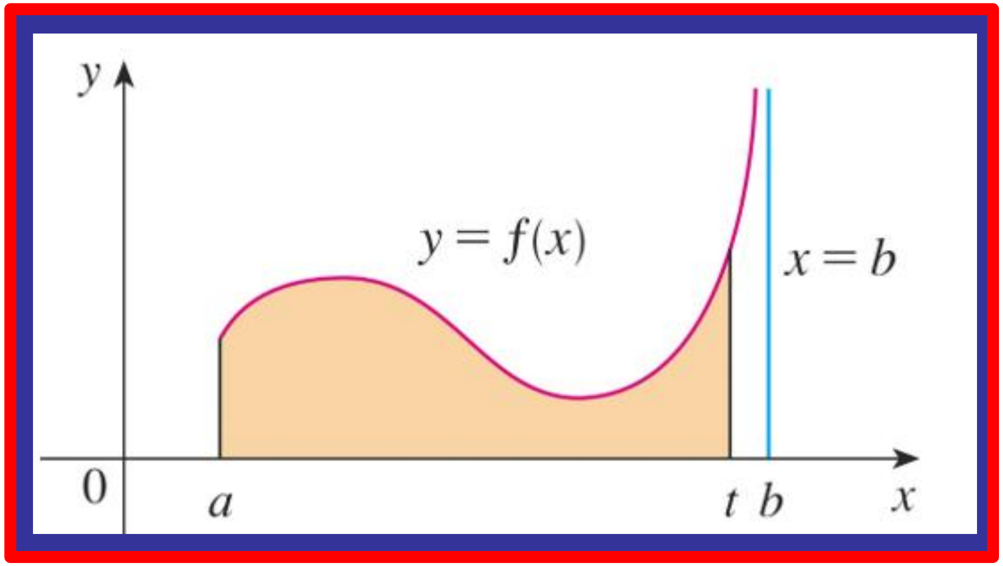
So, $I_3 = \displaystyle\int_e^{\infty}\frac{1}{x(\ln x)^2}dx = 1$

5.4. Improper Integrals

Improper Integrals of type 2

Definition 1a: If f is continuous on $[a, b)$ and is discontinuous at b, then $\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$

this limit exists (as a finite number).



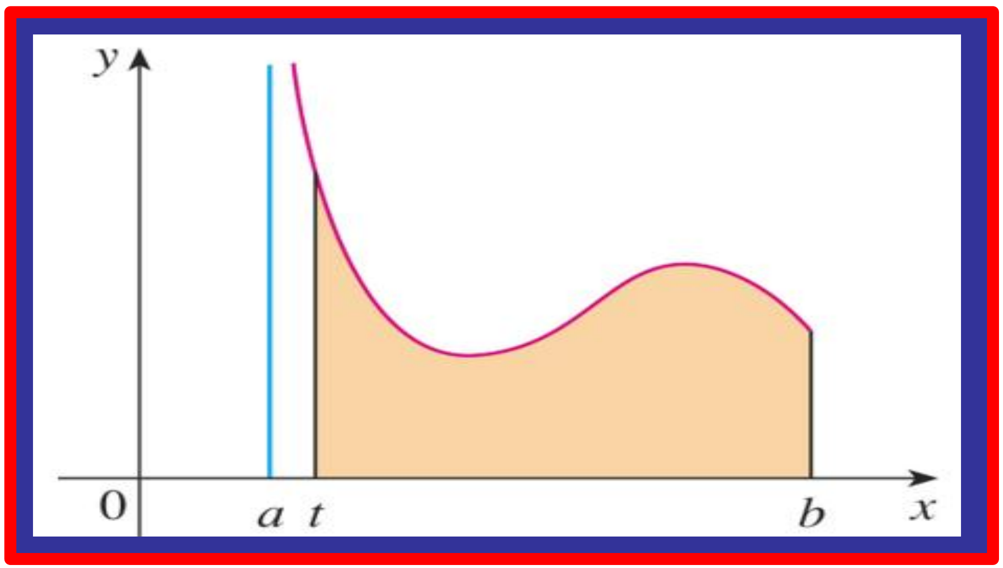
5.4. Improper Integrals

Improper Integrals of type 2

Definition 1b: If f is continuous on $(a, b]$ and is discontinuous

at b, then $\int_a^b f(x)\,dx = \lim\limits_{t \to a^+} \int_t^b f(x)\,dx$

this limit exists (as a finite number).



5.4. Improper Integrals

Improper Integrals of type 2

The improper integral $\int_a^b f(x)\,dx$ is called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

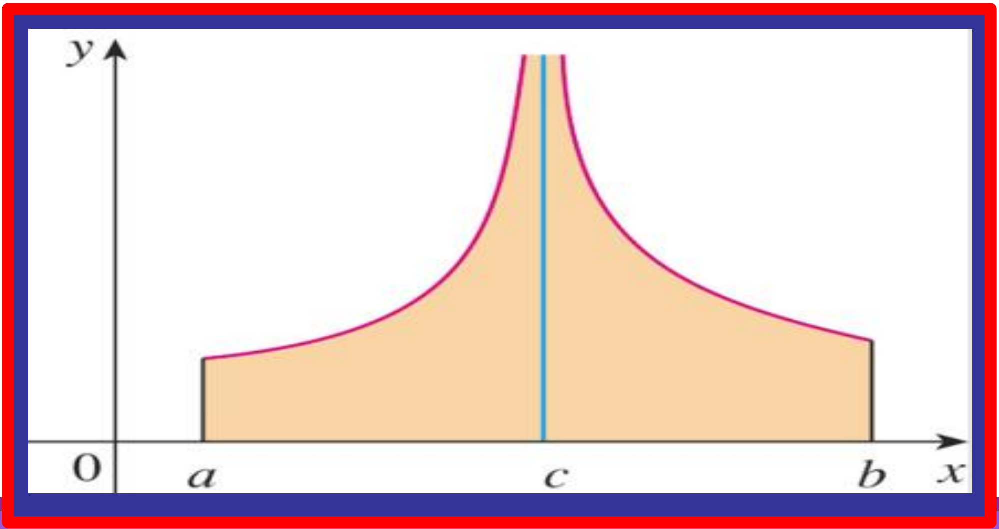
5.4. Improper Integrals

Improper Integrals of type 2

If f has a discontinuity at c, where $a < c < b$, and

both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we

define: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



5.4. Improper Integrals

Example: Evaluate:

a/ $$I_1 = \int_2^4 \frac{dx}{\sqrt{x-2}}$$

b/ $$I_2 = \int_1^2 \frac{dx}{x-1}$$

5.4. Improper Integrals

Solution:

a/ $I_1 = \int_2^4 \frac{dx}{\sqrt{x-2}}$

Step 1: $A = \int_t^4 \frac{dx}{\sqrt{x-2}} = \left(2\sqrt{x-2}\right)_t^4 = 2\sqrt{2} - 2\sqrt{t-2}$

$$I_1 = \lim_{t \to 2^+} A = \lim_{t \to 2^+} \left(2\sqrt{2} - 2\sqrt{t-2}\right) = 2\sqrt{2}$$

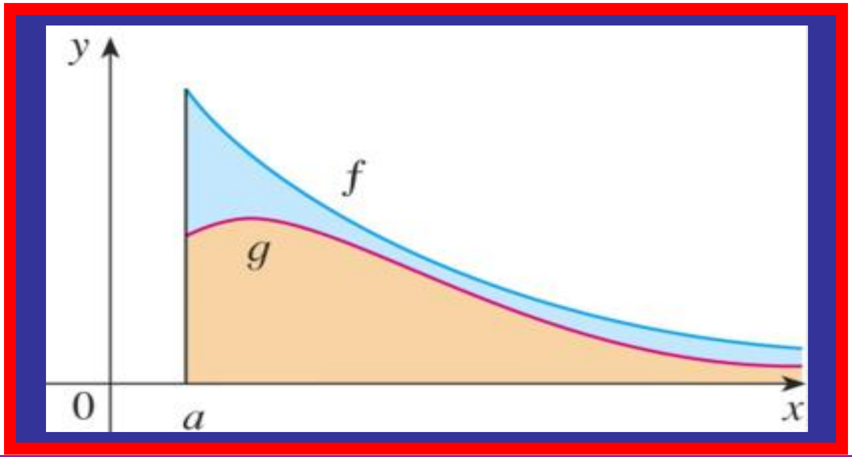
5.4. Improper Integrals

Comparison theorem

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a/ If $\int_a^{\infty} f(x)\,dx$ is convergent, then $\int_a^{\infty} g(x)\,dx$ is convergent.

b/ If $\int_a^{\infty} g(x)\,dx$ is divergent, then $\int_a^{\infty} f(x)\,dx$ is divergent.



5.4. Improper Integrals

Theorem:

$\int_{a>0}^{\infty} \frac{1}{x^p}\,dx$ is:

- Convergent if $p > 1$
- Divergent if $p \le 1$

$\int_{a}^{b<\infty} \frac{1}{(x-a)^p}\,dx$ is:

Convergent if $p < 1$.

Divergent if $p \ge 1$

5.4. Improper Integrals

Example:

Use the Comparison Theorem to determine whether the integral is convergent or divergent.

a/ $\int_0^{\infty} e^{-x^2}\,dx$

b/ $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$

c/ $\int_0^{1} \frac{2}{\sqrt{x^3}}\,dx$

d/ $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$

Solution:

a/ $$\int_0^{\infty} e^{-x^2}dx = \int_0^{1} e^{-x^2}dx + \int_1^{\infty} e^{-x^2}dx$$

- $\int_0^{1} e^{-x^2}dx$ is just an ordinary definite integral.
- $x \geq 1$ we have $x^2 \geq x$ so $-x^2 \leq -x \Rightarrow e^{-x^2} \leq e^{-x}$

Therefore

$$\int_1^{\infty} e^{-x^2}dx \leq \int_1^{\infty} e^{-x}dx \;\; and \int_1^{\infty} e^{-x}dx = -e^{-x}\Big|_1^{\infty} = \frac{1}{e}$$

5.4. Improper Integrals

Solution:

a/ we see that $\int_{1}^{\infty} e^{-x^2}\,dx$ is convergent.

In the Comparison Theorem, It follows that $\int_{0}^{\infty} e^{-x^2}\,dx$ is convergent.

Solution:

b/ $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}dx$

We have $\dfrac{1+e^{-x}}{x} > \dfrac{1}{x}$

- $\displaystyle\int_1^\infty \frac{1}{x}dx$ is divergent $(p=1)$

The integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}dx$ is divergent by the Comparison Theorem.

5.4. Improper Integrals

Solution:

c/ $\displaystyle\int_0^1 \frac{2}{\sqrt{x^3}}dx = 2\int_0^1 \frac{1}{\sqrt{x^3}}dx$

We have $\displaystyle\frac{2}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$

- $\displaystyle\int_0^1 \frac{1}{x^{3/2}}dx$ is divergent $\left(p = \frac{3}{2} > 1\right)$

The integral $\displaystyle\int_0^1 \frac{2}{\sqrt{x^3}}dx$ is divergent by the Comparison Theorem.

Solution:

d/ $\displaystyle\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}dx$

We have $\dfrac{\sin^2 x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}, \textit{for all } x \in (0,\pi)$

- $\displaystyle\int_0^{\pi} \frac{1}{\sqrt{x}}dx$ is convergent $\left(p = \frac{1}{2}\right)$

The integral $\displaystyle\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}dx$ is convergent by the Comparison Theorem.