Chapter 5

Linear transformation

Contents

5.1. Linear Transformations

Definitions

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear Transformation if it satisfies:

1. $T(X+Y) = T(X) + T(Y)$ for all vectors X and Y

2. $T(aX) = aT(X)$ for all vector X and all scalar a.

Note:

If T is a linear transformation then

1. $T(0) = 0$

2. $T(-X) = -T(X)$

5.1. Linear Transformations

Theorem 1:

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation, then

$$T\left(a_1X_1 + a_2X_2 + \ldots + a_nX_n\right)$$
$$= a_1T\left(X_1\right) + a_2T\left(X_2\right) + \ldots + a_nT\left(X_n\right)$$

for all vectors X_i , and all scalar a_i.

5.1. Linear Transformations

Example 1:

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be linear transformation such that

$T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, T\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Find:

a/ $T\left(2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 0 \end{bmatrix} \right)$.

b/ $T\begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

5.1. Linear Transformations

Solution:

a/ We have:

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; T(X_1) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}; X_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}; T(X_2) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

So, $T(2X_1 + 3X_2) = 2T(X_1) + 3T(X_2) = 2.\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \end{bmatrix}$

5.1. Linear Transformations

Solution:

b/ We have: $X = a_1.X_1 + a_2.X_2$

$$\Rightarrow T(X) = a_1.T(X_1) + a_2.T(X_2)$$

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} = 1.\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2.\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$$T\begin{bmatrix} 5 \\ -2 \end{bmatrix} = T\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$= T\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

5.1. Linear Transformations

Example 2:

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be linear transformation such that

$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}, T\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$. Find $T\begin{bmatrix}4\\3\end{bmatrix}$.

Answer: $T\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}9\\-32/3\end{bmatrix}$.

5.1. Linear Transformations

Example 3:

Let $T: R^3 \to R^2$ be a linear transformation

a/ Find $T\begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$ if $T\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

5.1. Linear Transformations

Example:

Let $T: R^3 \to R^2$ be a linear transformation

b/ Find $T\begin{bmatrix} 5 \\ 6 \\ -13 \end{bmatrix}$ if $T\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example 4:

Let $T: R^4 \to R^3$ be a linear transformation

Find $T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -3 \end{bmatrix}$ if $T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

Formula of the linear transformation

Suppose A is an $m \times n$ matrix, the matrix transformation then $T : \mathbb{R}^n \to \mathbb{R}^m$

$$X \mapsto T(X) = AX$$

is a linear transformation.

- ✓ A is zero matrix: the zero transformation $T = 0$
- ✓ A is identity matrix I_n: the identity transformation $T = 1_{\mathbb{R}^n}$

5.1. Linear Transformations

Example:

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{\text{T(X)=AX}} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 4x_3 \\ 0x_1 + 3x_2 + 5x_3 \end{bmatrix}_{2\times 1}$$

5.1. Linear Transformations

How to find the matrix of an linear transformation ?

Theorem 2.

Let $T: R^n \to R^m$ be a transformation

a/ T is linear if it is a matrix transformation

b/ If T is linear, then T is induced by a unique matrix A, given in terms of its columns by $A=[T(E_1) \quad T(E_2) \quad \ldots \quad T(E_n)]$ where $E_1, E_2, \ldots, E_n$ is the **standard basis** of R^n

5.1. Linear Transformations

Example:

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be linear such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+3y \end{bmatrix}$

a/ Find the matrix of T

b/ Find $T\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

5.1. Linear Transformations

Solution:

a/ $E = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is standard basis of $\mathbb{R}^2$.

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-0 \\ 1+3.0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-1 \\ 0+3.1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$A = [T(e_1)\ T(e_2)\ ] = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Method 2:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+3y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

5.1. Linear Transformations

Solution:

$$\text{b/ } T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

5.1. Linear Transformations

Example 3:

Conider the geometrical transformation of $\mathbb{R}^2$ given by reflection in the $x-$ axis. Find the matrix of a linear transformation.

Answer: $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

5.1. Linear Transformations

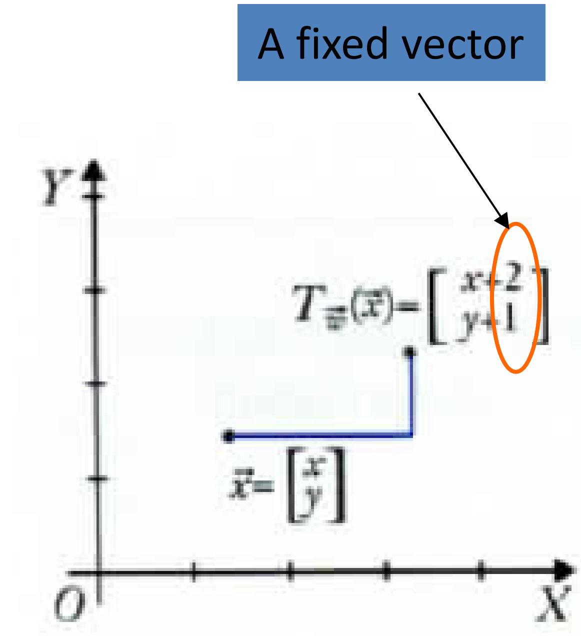
There are transformation that are **not** matrix transformations

Example:

The **translation (phép tịnh tiến)** is not a matrix transformation.

$$X \xrightarrow{T} X+Y$$

A fixed vector



The matrix does not exist if Y is not zero vector

5.1. Linear Transformations

Composition

$T:\mathbb{R}^n \to \mathbb{R}^m$ and $S:\mathbb{R}^m \to \mathbb{R}^k$

The **composite** of S and T is a transformation defined by $(S\circ T)(X) = S\left[T(X)\right]$ for all vector X in R^n.

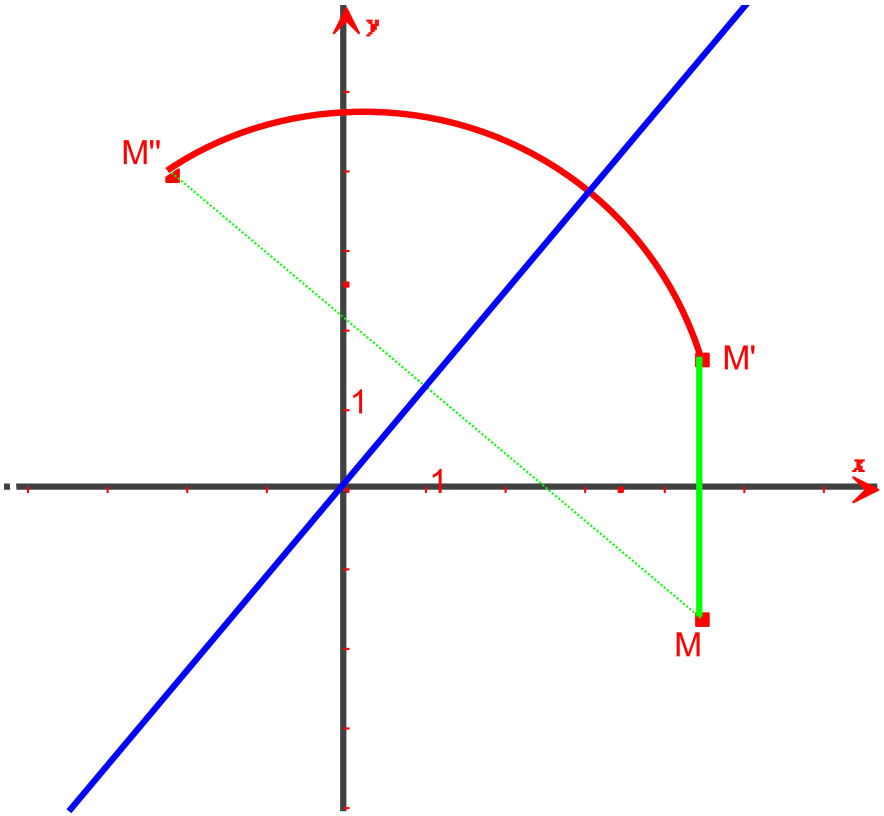
Theorem 3:

If T and S are linear then S∘T is also linear. And if S has matrix B and T has matrix A, then S∘T has matrix B.A

Transformation	S	T	S∘T
Matrix	B	A	B.A

Example:

The reflection in the x-axis followed by rotation through **pi/2** is reflection in the line y = x.



5.1. Linear Transformations

Inverse

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ is a transformation that has matrix A.

Suppose A is **invertible** then there exist A^{-1}

Let T’:$R^n \rightarrow R^n$ is a transformation that has matrix A^{-1}

i/ $T'[T(X)]=A^{-1}[AX]=IX=X$

ii/ $T[T'(X)]=A[A^{-1}X]=IX=X$

Then:

T’ is called an **inverse** of T if $T \circ T' = 1_{\mathbb{R}^n} = T' \circ T$

5.1. Linear Transformations

Theorem 4:

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ denote a linear transformation with matrix A. Then A is invertible if and only if T has an inverse.

In this case, T is has **exactly one** inverse.

(which we denote as **T^{-1}** and **T^{-1}** has matrix A^{-1}),

$$X \xrightarrow[\text{matrix } A]{T} Y$$

$$X \xleftarrow[\text{matrix } A^{-1}]{T^{-1}} Y$$

5.1. Linear Transformations

Example:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + 3y \end{bmatrix}$

Find $T^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

5.1. Linear Transformations

Solution:

$$T(X) = A.X$$

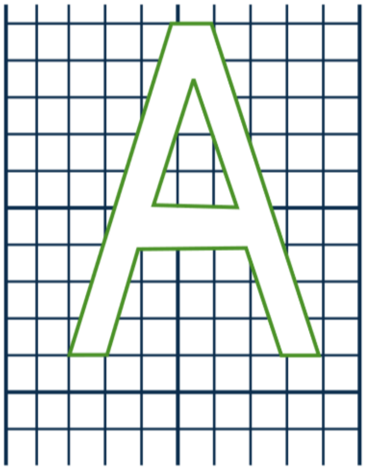
$$T^{-1}(X) = A^{-1}.X$$

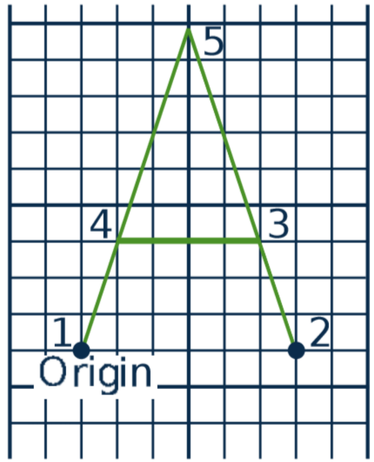
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 3/4 & 1/4 \\ -1/4 & 1/4 \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \\ -1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix}$$

5.2. An application in Computer Graphics

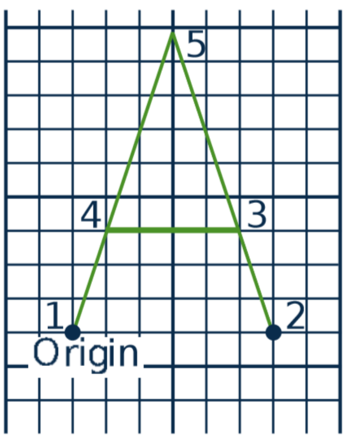
How to change image?



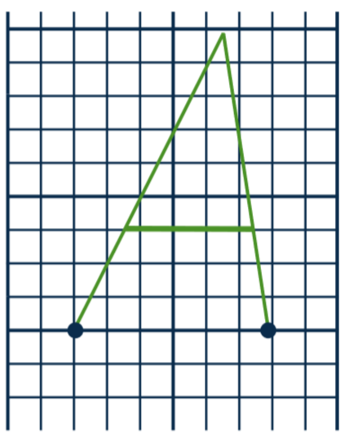


$$\begin{array}{cc} \textit{Vertex} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ D = & \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \end{array}$$

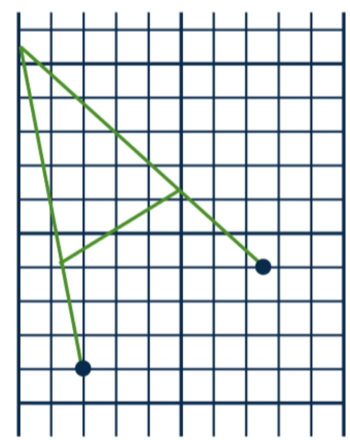
Data matrix D



$$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} D$$



$$\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$



$$D = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 6 & 5.6 & 1.6 & 4.8 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 5.196 & 2.83 & -0.634 & -1.902 \\ 0 & 3 & 5.098 & 3.098 & 9.294 \end{bmatrix}$$

5.1. Linear Transformations

Computer graphics

Image $\longrightarrow$ matrix D

Matrices of transformations $A_1, A_2, \ldots, A_n$

Compute $A_n \ldots A_3 A_2 A_1 D$

3rd transformation

1st transformation

Data matrix

nth transformation

2nd transformation

5.2. An application in Computer Graphics

Some common transformations and their matrices $\mathbb{R}^2$

❑ Rotations

Theorem 1:

The **rotation** $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation

with the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

5.2. An application in Computer Graphics

Some common transformations and their matrices $\mathbb{R}^2$

Theorem 2:

The line through the origin with slope m has equation $y = mx$, and we let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote **reflection** in the line $y = mx$

$$Q_m = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Theorem 3:

Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ be **projection** on the line $y = mx$. Then P_m is a linear transformation with matrix $P_m = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$

5.2. An application in Computer Graphics

Some common transformations and their matrices $\mathbb{R}^3$

Let L denote the line through the origin in $\mathbb{R}^3$ with direction vector $d = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq 0$. Then Q_L and P_L are both linear and

$$Q_L = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{pmatrix}$$

$$P_L = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

5.2. An application in Computer Graphics

Example :

In each case solve the problem by finding the matrix of the operator.

a/ Find the reflection of $v = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

5.2. An application in Computer Graphics

Example :

In each case solve the problem by finding the matrix of the operator.

a/ Find the reflection of $v=\begin{bmatrix}2\\5\\-1\end{bmatrix}$ in the line with equation $\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$$Q_L=\frac{1}{6}\begin{bmatrix}-4 & 2 & -4\\2 & -4 & -4\\-4 & -4 & 2\end{bmatrix}\to T(v)=Q_L.v=\begin{bmatrix}1\\-2\\-5\end{bmatrix}$$

5.2. An application in Computer Graphics

Example:

In each case solve the problem by finding the matrix of the operator.

b/ Find the projection of $v = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

5.2. An application in Computer Graphics

Example:

In each case solve the problem by finding the matrix of the operator.

b/ Find the projection of $v=\begin{bmatrix}1\\-1\\7\end{bmatrix}$ on the line with

equation $\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}3\\0\\4\end{bmatrix}$

$$P_L=\frac{1}{25}\begin{bmatrix}9&0&12\\0&0&0\\12&0&16\end{bmatrix}\to T(v)=P_L.v=\frac{1}{25}\begin{bmatrix}93\\0\\124\end{bmatrix}$$

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5.2. An application in Computer Graphics

Some common transformations and their matrices $\mathbb{R}^3$

Let M denote the plane through the origin in $\mathbb{R}^3$ with normal vector $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq 0$. Then Q_M and P_M are both linear and

$$Q_M = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}$$

$$P_M = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$$

5.2. An application in Computer Graphics

Example:

In each case solve the problem by finding the matrix of the operator.

a/ Find the projection of $v = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ on the plane with equation $3x - 5y + 2z = 0$

b/ Find the reflection of $v = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x - y + 3z = 0$

5.2. An application in Computer Graphics

Example:

In each case solve the problem by finding the matrix of the operator.

a/ Find the projection of $v=\begin{bmatrix}1\\-2\\3\end{bmatrix}$ on the plane with equation $3x-5y+2z=0$

$$P_M=\frac{1}{38}\begin{bmatrix}29 & 15 & -6\\15 & 13 & 10\\-6 & 10 & 34\end{bmatrix}\rightarrow T(v)=P_M.v=\begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}$$

5.2. An application in Computer Graphics

Example:

In each case solve the problem by finding the matrix of the operator.

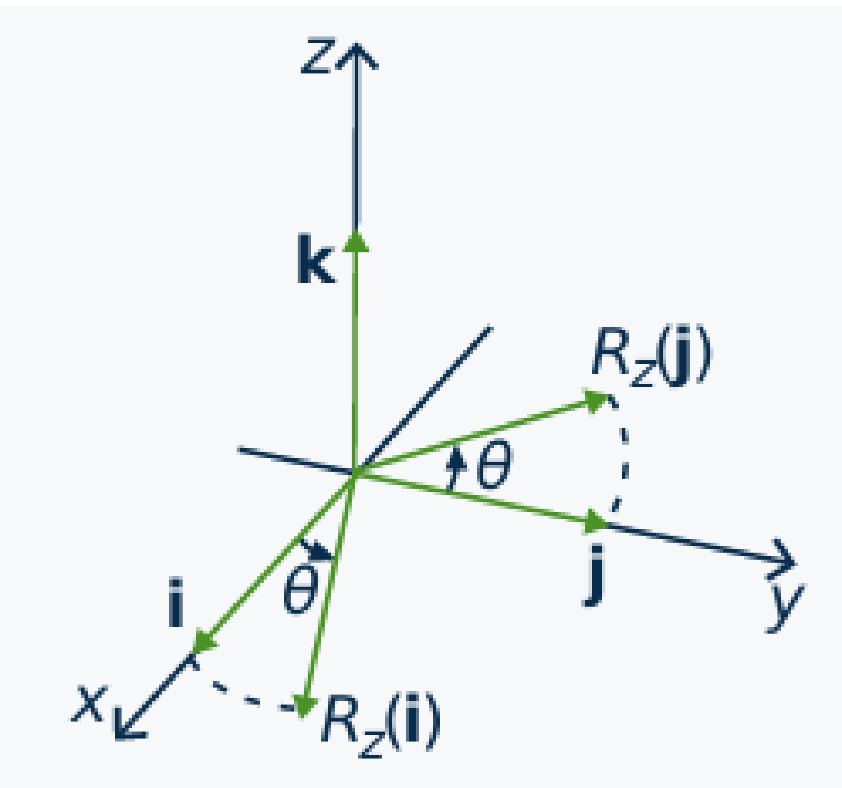
b/ Find the reflection of $v = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x - y + 3z = 0$

$$Q_M = \frac{1}{11}\begin{bmatrix} 9 & 2 & -6 \\ 2 & 9 & 6 \\ -6 & 6 & -7 \end{bmatrix} \rightarrow T(v) = Q_M . v = \frac{1}{11}\begin{bmatrix} -13 \\ 2 \\ -39 \end{bmatrix}$$

5.2. An application in Computer Graphics

Some common transformations and their matrices $\mathbb{R}^3$

Let $R_{z,\theta}:\mathbb{R}^3\to\mathbb{R}^3$ denote rotation of $\mathbb{R}^3$ about the z axis through an angle θ from the positive x axis toward the positive y axis. $R_{z,\theta}$ has matrix



$$R_{z,\theta}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

5.2. An application in Computer Graphics

Example:

Find the rotation of $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z-axis

through $\theta = \dfrac{\pi}{4}$.

5.2. An application in Computer Graphics

Example:

Find the rotation of $v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z-axis

through $\theta = \dfrac{\pi}{4}$.

$$R_{z,\theta} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \to T(v) = R_{z,\theta}.v = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ -1 \end{bmatrix}$$