FPT UNIVERSITY
DREAM OF INNOVATION

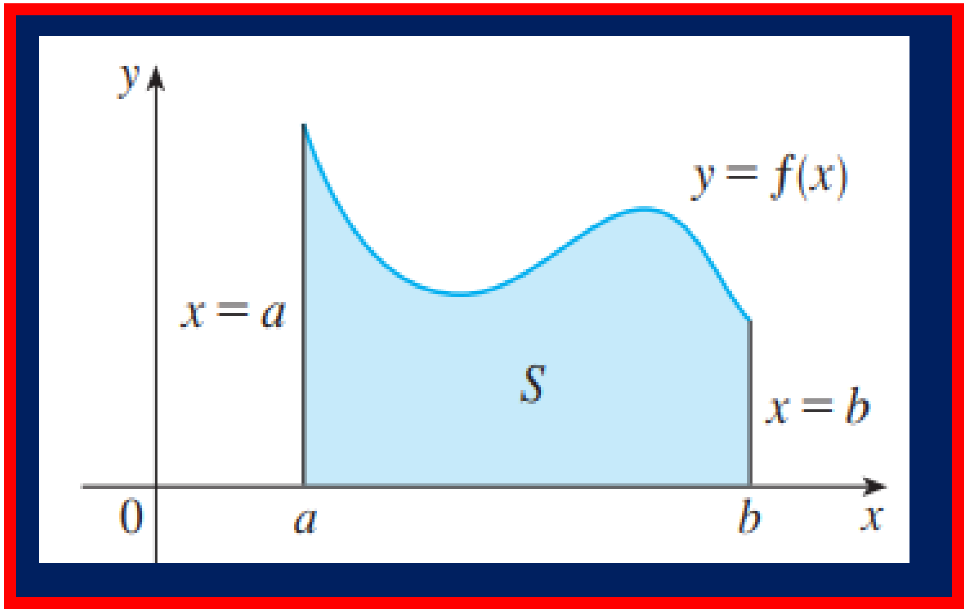
Chapter 4

INTEGRALS

Contents

The area problem

Find the area of the region S that lies under the curve $y = f(x)$ from a to b



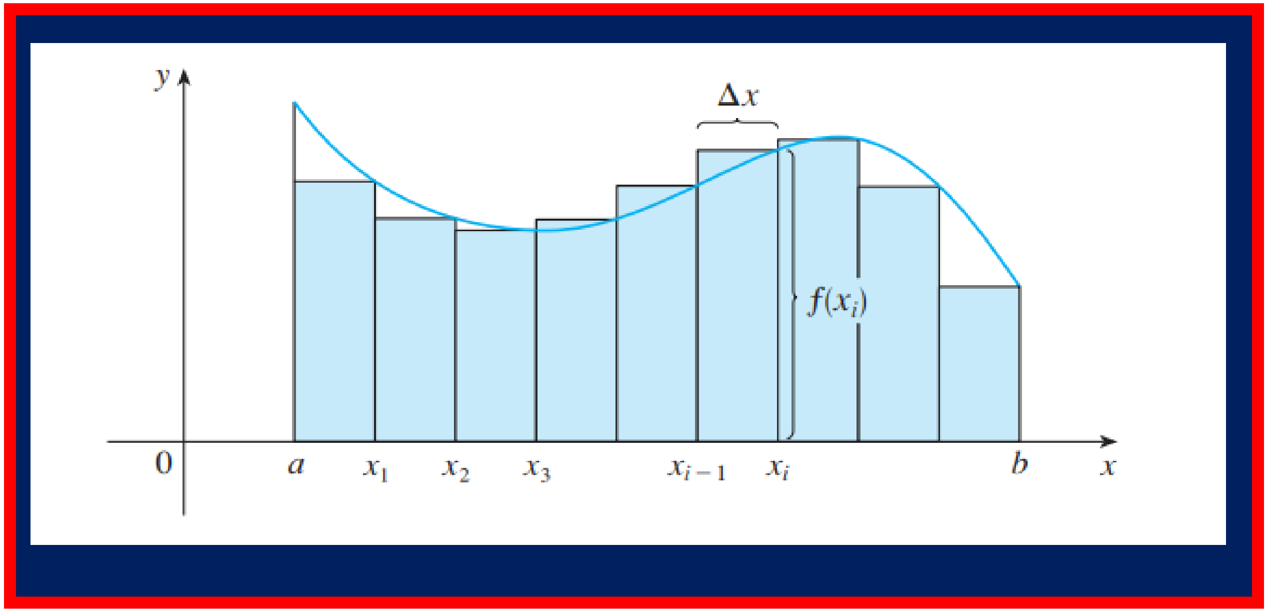
$$S = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

4.1 Approximate Integration and Distance

The area problem

What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles:

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x,\quad \Delta x = \frac{b-a}{n}$$



4.1 Approximate Integration and Distance

The area problem

Definition 1:

The figure shows approximating rectangles when the sample points are not chosen to be endpoints.

$$A = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$$

$$A = \lim_{n\to\infty} L_n$$
$$= \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

4.1 Approximate Integration and Distance

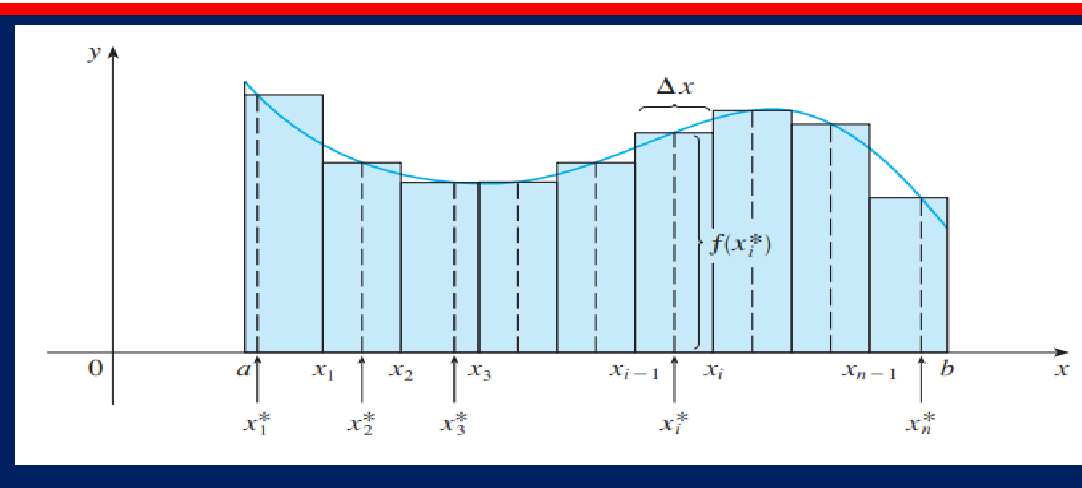
The area problem

Definition 2:

The area *A* of the region *S* that lies under the graph of the continuous function *f* is the limit of the sum of the areas of approximating rectangles:

x_i* in the i th subinterval $[x_{i-1}, x_i]$

(the sample points)



$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

4.1 Approximate Integration and Distance

The left end point, right end point and midpoint

$$\int_a^b f(x)\,dx, \quad \Delta x = \frac{b-a}{n}$$

$$1/\ R_n = \Delta x\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right] = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$2/\ L_n = \Delta x\left[f(x_0)+f(x_1)+\ldots+f(x_{n-1})\right] = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$3/\ \overline{x_i} = \frac{x_{i-1}+x_i}{2}$$

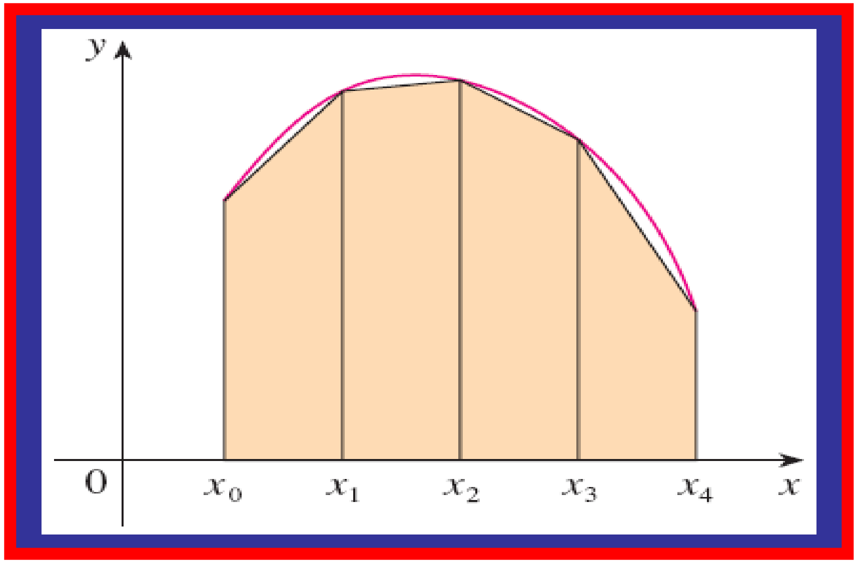
$$M_n = \Delta x\left[f(\overline{x_1})+f(\overline{x_2})+\ldots+f(\overline{x_n})\right] = \sum_{i=1}^{n} f(\overline{x_i})\Delta x$$

Trapezoidal rule

The reason for the name can be seen from the figure, which illustrates the case $f(x) \geq 0$. The area of the trapezoid that lies above the i th subinterval is:

$$\Delta x\left(\frac{f(x_{i-1})+f(x_i)}{2}\right)=\frac{\Delta x}{2}[f(x_{i-1})+f(x_i)]$$

If we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

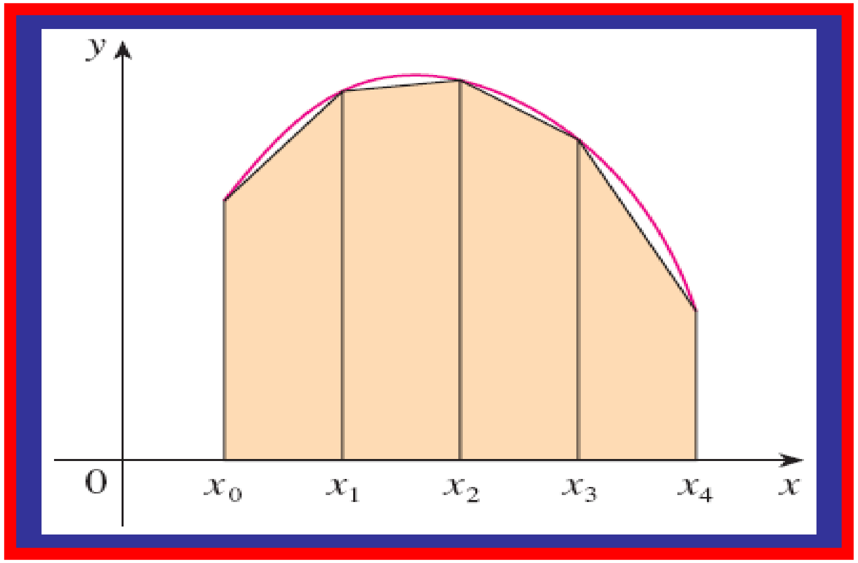


Trapezoidal rule

$$\int_a^b f(x)\,dx \approx T_n =$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.



Error bounds

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

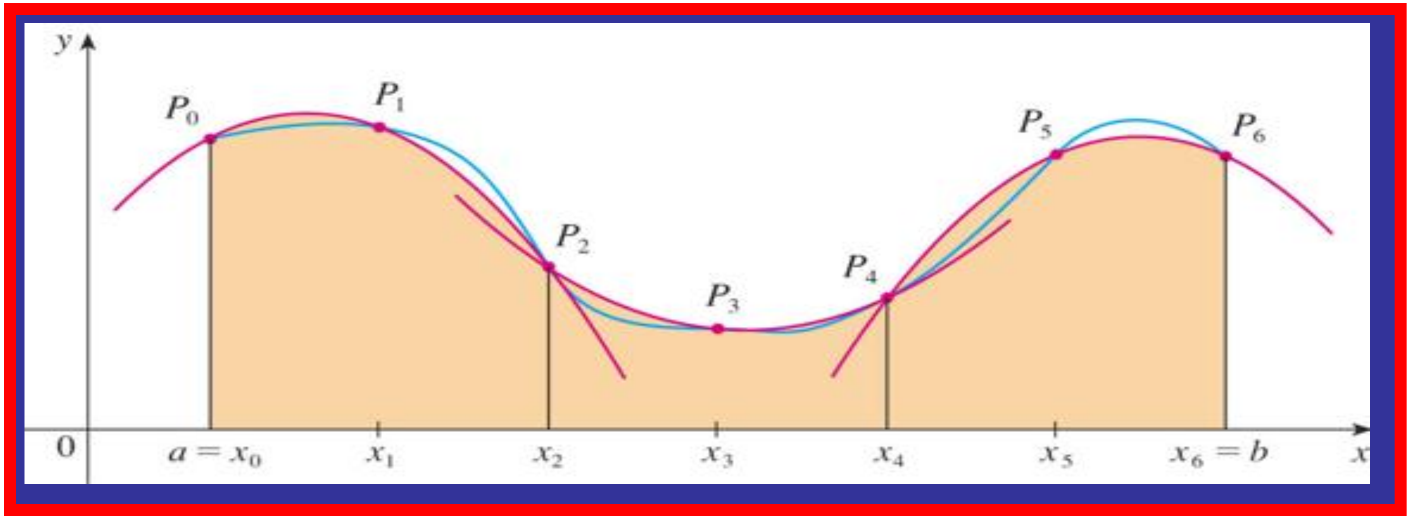
4.1 Approximate Integration and Distance

Simpson's rule: This is called Simpson's Rule,after the English the English mathematician Thomas Simpson (1710–1761).

This means that the area under the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ is still: $\frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$

where ***n* is even** and

$$\Delta x = \frac{b-a}{n}$$



4.1 Approximate Integration and Distance

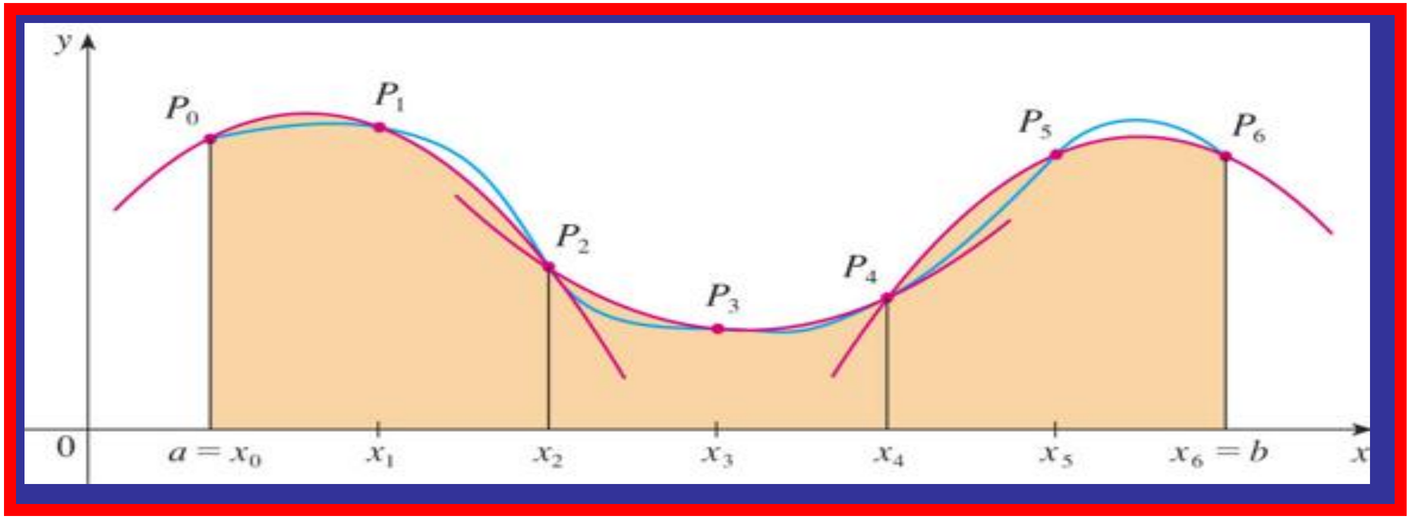
Simpson's rule: This is called Simpson's Rule,after the English the English mathematician Thomas Simpson (1710–1761).

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)$$

$$+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

where ***n* is even** and

$$\Delta x = \frac{b-a}{n}$$



Simpson's rule:

Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If E_s is the error involved in using Simpson's Rule, then

$$\left|E_s\right| \le \frac{K(b-a)^5}{180n^4}$$

4.1 Approximate Integration and Distance

Summary $\int_a^b f(x)\,dx, \quad \Delta x = \dfrac{b-a}{n}$

$$1/\ R_n = \Delta x\left[f(x_1) + f(x_2) + \ldots + f(x_n)\right] = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$2/\ L_n = \Delta x\left[f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right] = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$3/\ \overline{x_i} = \frac{x_{i-1} + x_i}{2}$$

$$M_n = \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + \ldots + f(\overline{x_n})\right] = \sum_{i=1}^{n} f(\overline{x_i})\Delta x$$

4.1 Approximate Integration and Distance

Summary $$\int_a^b f(x)\,dx, \quad \Delta x = \frac{b-a}{n}$$

4/ $$T_n = \Delta x\left[\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \ldots + \frac{f(x_{n-1})+f(x_n)}{2}\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

4.1 Approximate Integration and Distance

Summary $\int_a^b f(x)dx, \quad \Delta x = \dfrac{b-a}{n}$

5/ $S_n = \dfrac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)$

$+\ldots+f(x_{n-2})+4f(x_{n-1})+f(x_n)]$

$= \dfrac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)$

$+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$

4.1 Approximate Integration and Distance

The following three equations give formulas for sums of powers of positive integers.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4.1 Approximate Integration and Distance

Example 1:

Evaluating the Riemann sum of the function

$$f(x) = x - \frac{1}{x}, \ 1 \le x \le 2$$, with four subintervals.

a/ The sample points are the **right** **endpoints.**

b/ The sample points are the **left** **endpoints.**

Example 2:

Let $\int_1^2 \frac{1}{x}\,dx$. Evaluate approximate the integral

a/ Use the **Midpoint Rule** with $n = 5$.

b/ Using the **trapezoidal Rule** with $n = 5$.

c/ Using the **Simpson's rule** with $n = 10$.

Solution:

a/ The width of the subintervals is: $\Delta x = \dfrac{2-1}{5} = 0.2$

The midpoints are: 1.1, 1.3, 1.5, 1.7, 1.9.

So, the Midpoint Rule gives:

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\left[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\right]$$

$$= \frac{1}{5}\left(\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right)$$

$$\approx 0.691908$$

4.1 Approximate Integration and Distance

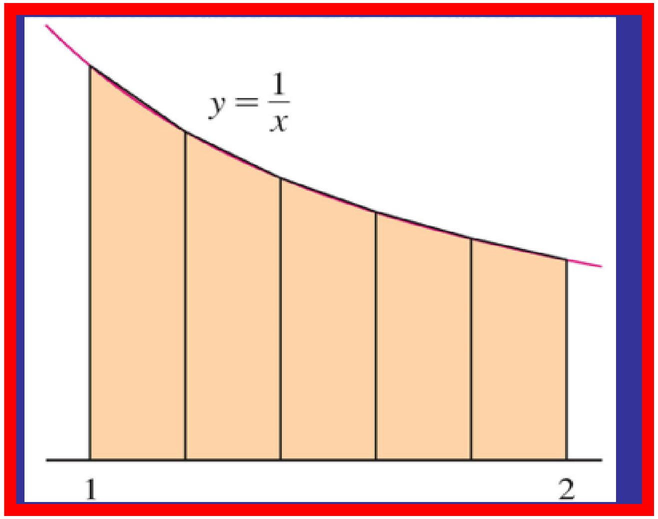
b/ With $n = 5$, $a = 1$ and $b = 2$, we have: $\Delta x = \dfrac{2-1}{5}$

So, the Trapezoidal Rule gives:

$$\int_1^2 \frac{1}{x}dx \approx T_5 = \frac{0.2}{2}[f(1)+2f(1.2)+2f(1.4)+2f(1.6)+2f(1.8)$$

$$+f(2)] = 0.1\left(\frac{1}{1}+\frac{2}{1.2}+\frac{2}{1.4}+\frac{2}{1.6}+\frac{2}{1.8}+\frac{1}{2}\right) \approx 0.695635$$

The approximation is illustrated here.



c/ Putting $f(x) = 1/x$, $n = 10$, and $\Delta x = 0.1$ in Simpson's Rule, we obtain:

$$\int_1^2 \frac{1}{x}\,dx \approx S_{10} = \frac{\Delta x}{3}[f(1)+4f(1.1)+2f(1.2)+4f(1.3)$$

$$+\ldots+2f(1.8)+4f(1.9)+f(2)]$$

$$=\frac{0.1}{3}\left(\frac{1}{1}+\frac{4}{1.1}+\frac{2}{1.2}+\frac{4}{1.3}+\frac{2}{1.4}+\frac{4}{1.5}+\frac{2}{1.6}+\frac{4}{1.7}+\frac{2}{1.8}+\frac{4}{1.9}+\frac{1}{2}\right)$$

$$\approx 0.693150$$

Distance problem

Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval.

We take speedometer readings every five seconds and record them in this table.

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

Distance problem

If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

If we had wanted a more accurate estimate, we could have taken velocity readings every two seconds, or even every second.

4.1 Approximate Integration and Distance

Distance problem

In general, suppose an object moves with velocity $v = f(t)$ where $a \le t \le b$ and $f(t) \ge 0$.

We take velocity readings at times: $t_0(= a)$, t_1, t_2, …., $t_n(= b)$ so that the velocity is approximately constant on each subinterval: $\Delta t = (b - a)/n$.

Similarly, the distance traveled during the second time interval is about $f(t_1)\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t = \sum_{i=1}^{n} f(t_{i-1})\Delta t$$

4.1 Approximate Integration and Distance

Distance problem

So, it seems plausible that the exact distance *d* traveled is the limit of such expressions:

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t$$

4.2&4.3. The Definite Integral

If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.

- We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals.
- We let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval.

Then, the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$ provided that this limit exists.

☞ If it does exist, we say f is integrable on $[a, b]$.

☞ The symbol $\int$ was introduced by Leibniz and is called an integral sign.

4.2&4.3. The Definite Integral

The definite integral $\int_a^b f(x)dx$ is a number.

It does not depend on x.

In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.

That is, the definite integral $\int_a^b f(x)dx$ exists.

4.2&4.3. The Definite Integral

Properties of the integral

We assume f and g are continuous functions.

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

6. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

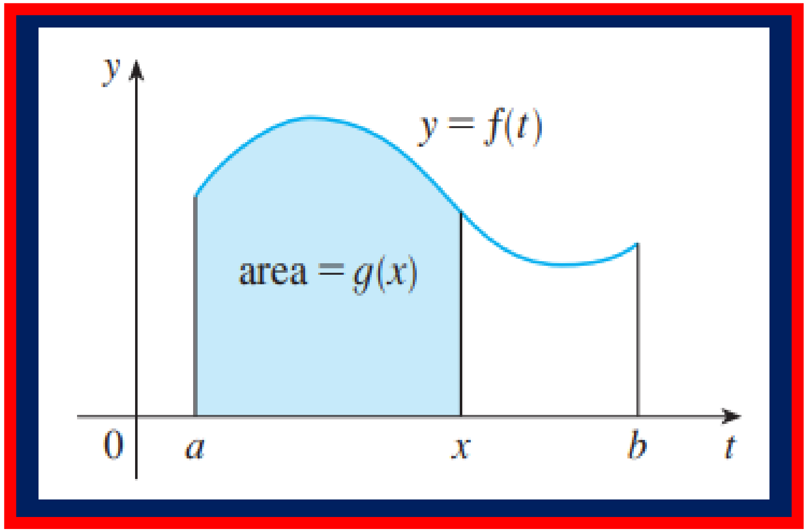
7. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

4.4. The Fundamental Theorem of Calculus (FTC)

Theorem FTC 1

The first part of the FTC deals with functions defined by an equation of the form: $g(x) = \int_a^x f(t)\,dt$ where f is a continuous function on $[a, b]$ and x varies between a and b.

To see why this might be generally true, we consider a continuous function f with $f(x) \geq 0$.



Then, $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x.

4.4. The Fundamental Theorem of Calculus (FTC)

Theorem FTC1

If f is continuous on $[a, b]$, then the function g defined by

$g(x) = \int_a^x f(t)dt \quad a \le x \le b$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write the FTC1 as $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ when f is continuous.

4.4. The Fundamental Theorem of Calculus (FTC)

Theorem FTC1

Generalization

$$g(x)=\int_a^x f(t)dt\,,a\le x\le b,\ f\left(x\right)=\frac{d}{dx}\int_a^x f(t)dt$$

Let $\mathrm{g}\left(x\right)=\int\limits_{v(x)}^{u(x)} f(t)dt.$

$Then\quad g'\left(x\right)=u'(x)f(u(x))-v'(x)f(v(x))$

Example: If $g(x)=\int_1^{x^2}(t^2-t)dt \quad ,1\le x.$ Find g’(x).

Solution: $u(x)=x^2 \Rightarrow g'(x)=u'(x)f(u(x))=2x(x^4-x^2)$

Theorem FTC 2

If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

where F is any antiderivative of f, that is, a function such that $F' = f$.

4.4. The Fundamental Theorem of Calculus (FTC)

Net change theorem

So, we can reformulate $\int_a^b f(x)\,dx = F(b) - F(a)$ as follows.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

1/ If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$.

So, $\int_a^b \rho(x)\,dx = m(b) - m(a)$ is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

4.4. The Fundamental Theorem of Calculus (FTC)

Net change theorem

2/ If the rate of growth of a population is dn/dt, then

$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$ is the net change in population during the time period from t_1 to t_2.

- The population increases when births happen and decreases when deaths occur.
- The net change takes into account both births and deaths.

Example:

Suppose that the animal population is increasing at a rate $f(t) = 3t - 1$ (t measured in years). How much does the animals increase between the third and the seven years ?

Solution

$$\int_3^7 (3t-1)\,dt = 56$$

Net change theorem

3/ If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.

So, $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ is the net change of position, or displacement, of the particle during the time period from t_1 to t_2.

If we want to calculate the distance the object travels during that time interval, we have to consider the intervals when:

- $v(t) \geq 0$ (the particle moves to the right)
- $v(t) \leq 0$ (the particle moves to the left)

Example:

Suppose the acceleration function and initial velocity are

$a(t) = t + 3$ (m/s^2), v(0) = 5 (m/s).

a/ Find the velocity at time t .

b/ Find the distance traveled when $0 \le t \le 5$.

Solution

$$\text{a/ } v(t)=\int a(t)\,dx=\int (t+3)\,dx=\frac{t^2}{2}+3t+c$$

$$v(0)=5:\ \frac{0^2}{2}+3.0+c=5\Rightarrow c=5$$

$$\text{So, } v(t)=\frac{t^2}{2}+3t+5$$

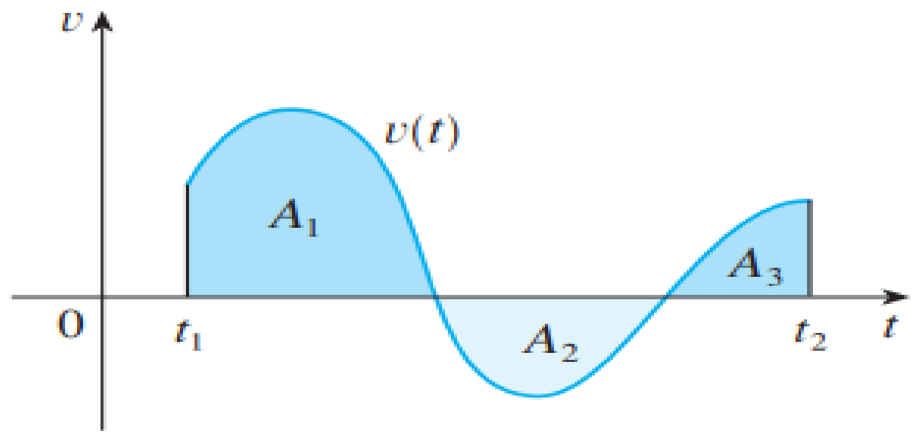
$$\text{b/ } s(t)=\int_0^5 v(t)\,dx=\int_0^5\left(\frac{t^2}{2}+3t+5\right)dx=\frac{250}{3}$$

4.4. The Fundamental Theorem of Calculus (FTC)

Net change theorem

In both cases, the distance is computed by integrating $|v(t)|$, the speed. Therefore, $\int_{t_1}^{t_2} |v(t)|\,dt$ = total distance traveled.

The figure shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

4.4. The Fundamental Theorem of Calculus (FTC)

Net change theorem

The acceleration of the object is $a(t) = v'(t)$.

So, $\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$ is the change in velocity from time t_1 to time t_2.

Example:

A particle moves along a line so that its velocity at time t is:
$v(t) = t^2 - t - 6$ (in meters per second).

a/ Find the displacement of the particle during the time period $1 \le t \le 4$.

b/ Find the distance traveled during this time period

$1 \le t \le 4$.

Net change theorem

Solution:

a/ The displacement is:

$$s(4)-s(1)=\int_1^4 v(t)\,dt$$

$$=\int_1^4 (t^2-t-6)\,dt=\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_1^4=-\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.

Net change theorem

b/ Note that $v(t) = t^2 - t - 6 = (t-3)(t+2)$. Thus, $v(t) \leq 0$ on the interval [1, 3] and $v(t) \geq 0$ on [3, 4].

The distance traveled is:

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

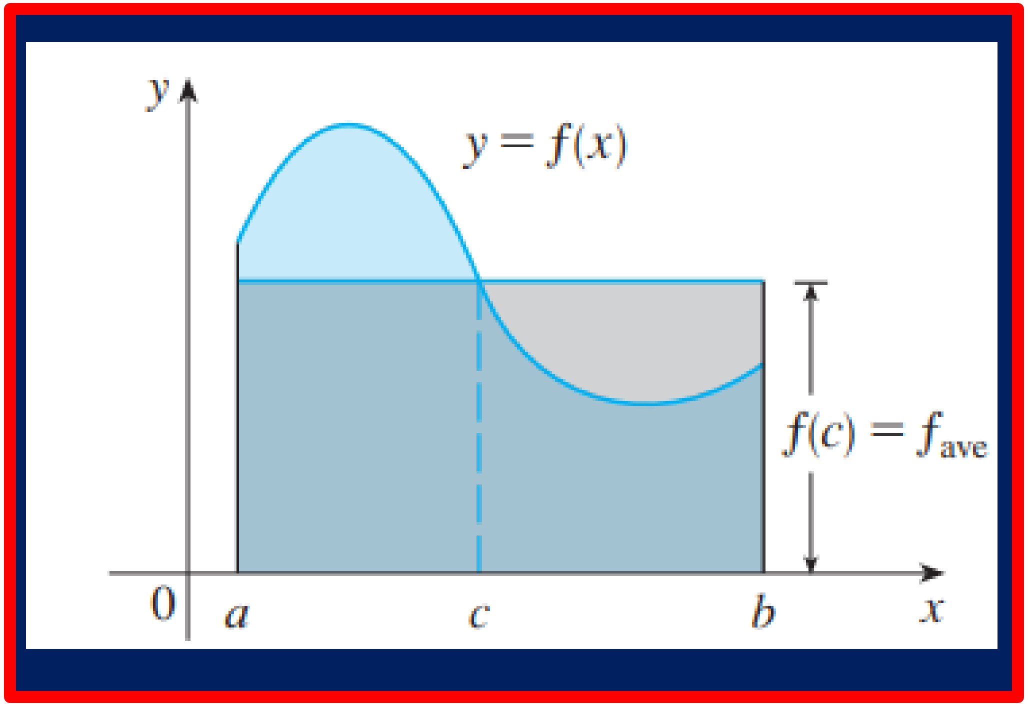
$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6} \approx 10.17\,\text{m}$$

4.4. The Fundamental Theorem of Calculus (FTC)

Mean value theorem

The geometric interpretation of the Mean Value Theorem for Integrals is as follows.

For 'positive' functions *f*, there is a number *c* such that therectangle with base [*a*, *b*] and height *f*(*c*) has the same areaas the region under the graph of *f* from *a* to *b*.



4.4. The Fundamental Theorem of Calculus (FTC)

Mean value theorem

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that $f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$

That is, $\int_a^b f(x)\,dx = f(c)(b-a).$

Example:

Find the **average value** of the function $f(x)=x^2+3$ on the interval [2,5].