Complex Variables - Assignment 2

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Workshop 2

Question 5

($\Rightarrow$)Let $a_0 \in \mathbb{C}$ be a non-zero complex number, and suppose $U \subseteq \mathbb{C}$ be an open set. Since the set U is open, then by definition 1.2.2, for all $u_0 \in U$, there exists $\epsilon > 0$ such that $D_\epsilon(u_0) \subseteq U$.

Consider the set $a_0U = \{a_0u : u \in U\}$. For all $u_0 \in U$, $a_0u_0 \in a_0U$, $D_\epsilon(u_0)$ is mapped to $D_{\epsilon|a_0|}(a_0u_0)$ under the mapping $f : u \to a_0u$. i.e. $D_\epsilon(u_0) = \{u_0 \in U : |u - u_0| < \epsilon\} = \{|a_0||u - u_0| < |a_0|\epsilon\} = \{|a_0u - a_0zu_0| < |a_0|\epsilon\} = D_{\epsilon|a_0|}(a_0u_0)$. Since $a_0 \neq 0$ *and* $\epsilon > 0$, we have $\epsilon|a_0| > 0$. Thus $D_{\epsilon|a_0|}(a_0u_0) \subseteq a_0U$. Hence a_0U is open.

($\Leftarrow$)Conversely, if the set $a_0U = \{a_0u : u \in U\}$ is open, then by definition 1.2.2, for all $a_0u_0 \in a_0U$, there exists an $\epsilon > 0$ such that $D_\epsilon(a_0u_0) \subseteq a_0U$, i.e. $D_\epsilon(a_0u_0) = \{a_0u_0 \in a_0U : |a_0u - a_0u_0| < \epsilon\} = \{u_0 \in U : \frac{|a_0||u-u_0|}{|a_0|} < \frac{\epsilon}{|a_0|}\} = \{u_0 \in U : |u - u_0| < \frac{\epsilon}{|a_0|}\} = D_{\epsilon/a_0}(u_0)$. Thus for all $z_0 \in U$, there exists an $\epsilon > 0$, such that $D_{\epsilon/|a_0|}(u_0) \subseteq U$. Hence $U \subseteq \mathbb{C}$ is open.

Hence, we proved that $U \subseteq \mathbb{C}$ is open if and only if the set $a_0U = \{a_0u : u \in U\}$ is open.

Question 6

We should first prove that $U = \mathbb{C} \setminus \{z = re^{i0} : r \geq 0\}$ is open, let $z_0 = x_0 + iy_0 \in U$. Set $\epsilon = |y_0|$, such that $\{z_0 \in U : |z - z_0| < \epsilon\} = D_\epsilon(z_0) = D_{|y_0|}(z_0)$

$$\begin{aligned} |z - z_0| = |x + iy - x_0 - iy_0| &< |y_0| \\ \sqrt{(x - x_0)^2 + (y - y_0)^2} &< |y_0| \\ (x - x_0)^2 + (y - y_0)^2 &< y_0^2 \end{aligned}$$

From above, we can see that $y \neq 0$, or the inequality does not hold, since $(x - x_0)^2 > 0$. So the open ball does not intersect or touch the positive real axis, so $z \in U$, and $D_{|y_0|}(z_0) = D_\epsilon(z_0) \subseteq D_\phi$.

By question 5, we have $U \subseteq \mathbb{C}$ is open if and only if the set $a_0U = \{a_0u : u \in U\}$, which is open, now set $a_0 = e^{i\phi}$, then $a_0U = \{z = re^{i\phi} : r \geq 0\}$, so the cut plane $D_\phi = \mathbb{C} \setminus \{z = re^{i0} : r \geq 0\}$ is open.

Workshop 3

Question 5

Let $z = x + iy \in \mathbb{C}$, so $Re(z) = x$ and $Im(z) = y$. Then $f(z) = \sqrt[3]{|(Re(z))^2 Im(z)|} = \sqrt[3]{|x^2 y|} = \sqrt[3]{x^2|y|}$.

Suppose $f = u + iv$, then $u = \sqrt[3]{x^2|y|}$ and $v = 0$. Applying the partial derivatives of u and v with respect to x and y, and at $z_0 = 0$, we have

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial u}{\partial x}(0,0)\bigg|_{y=0} = lim_{x\to 0}\frac{\partial u}{\partial x}(0,0) = lim_{x\to 0}\frac{f(z) - f(z_0)}{z - z_0} = lim_{x\to 0}\frac{\sqrt[3]{x^2|y|} - \sqrt[3]{0}}{x - 0} = 0$$

$$\frac{\partial u}{\partial y}(z_0) = \frac{\partial u}{\partial y}(0,0)\bigg|_{x=0} = lim_{y\to 0}\frac{\partial u}{\partial y}(0,0) = lim_{y\to 0}\frac{f(z) - f(z_0)}{z - z_0} = lim_{y\to 0}\frac{\sqrt[3]{x^2|y|} - \sqrt[3]{0}}{y - 0} = 0$$

and,

$$\frac{\partial v}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) = 0$$

we can conclude that

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) = 0\ ;\ \frac{\partial v}{\partial x}(z_0) = -\frac{\partial u}{\partial y}(z_0) = 0$$

which satisfies the Cauchy-Riemann equations.

However, $f = u(x,y) + iv(x,y)$ is not differentiable at $z = 0$, where $u(x,y) = \sqrt[3]{|(Re(z))^2 Im(z)|}$ and $v(x,y) = 0$. To see this, consider the partial derivative of u(x,y) wtih respect to x under y=x, and let $z = x + iy$, then $z = x + xi$:

$$\frac{\partial u}{\partial x}(0,0) = lim_{x\to 0}\frac{u(z) - u(0)}{x - 0} = lim_{x\to 0}\frac{\sqrt[3]{|(Re(z))^2 Im(z)|} - 0}{x} = lim_{x\to 0}\frac{\sqrt[3]{x^2|x|}}{x}$$

If x is negative, then $|x| = -x$, and thus $\frac{\partial u}{\partial x}(0,0) = lim_{x\to 0}\frac{\sqrt[3]{x^2|x|}}{-x} = -1$

If x is positive, then $|x| = x$, and thus $\frac{\partial u}{\partial x}(0,0) = lim_{x\to 0}\frac{\sqrt[3]{x^2|x|}}{x} = 1$

From above, we can see that the limits are different, hence f is not differentiable at 0.

Therefore, f satisfies the Cauchy-Riemann equations at z = 0, but it is not differentiable at 0.

Question 6

Given the imaginary part of a holomorphic function $f(u, v)$ is

$$v(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

so the partial derivative with respect to x and y are

$$\begin{aligned}\frac{\partial v}{\partial x} &= 3ax^2 + 2bxy + cy^2; \quad \frac{\partial^2 v}{\partial x^2} = 6ax + 2by \\ \frac{\partial v}{\partial y} &= bx^2 + 2cxy + 3dy^2; \quad \frac{\partial^2 v}{\partial y^2} = 2cx + 6dy\end{aligned}$$

By lemma 1.4.14, if the function $f = u+iv$ is holomorphic on $\mathbb{C}$, then u *and* v are harmonic. Since v is harmonic, by definition 1.4.13, v satisfies the Laplace equation, i.e

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0$$

Then we have

$$\begin{aligned}6ax + 2by + 2cx + 6dy &= 0 \\ \Rightarrow (6a + 2c)x + (2b + 6d)y &= 0 \\ \Rightarrow c = -3a \quad and \quad b &= -3d\end{aligned}$$

so $v(x, y)$ can be rewritten as $v(x, y) = ax^3 - 3dx^2y - 3axy^2 + dy^3$.

Given that f is holomporphic, then it must be differentiable, thus f holds the Cauchy-Riemann equations:

$$\begin{aligned}\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) &= -3dx^2 - 6axy + 3dy^2 \\ \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) &= -3ax^2 + 6dxy + 3ay^2\end{aligned}$$

Then we integrate $\frac{\partial u}{\partial x}(x, y)$ with respect to x, we have

$$-dx^3 - 3ax^2y + 3dx^2y + \phi(x), \;\; for\ some\ function\ \phi : \mathbb{R} \to \mathbb{R}$$

and we integrate $\frac{\partial u}{\partial y}(x, y)$ with respect to y, we have

$$-3ax^2y + 3dxy^2 + ay^3 + \psi(x),\ for\ some\ function\ \psi : \mathbb{R} \to \mathbb{R}$$

so u can be written as $-dx^3 - 3ax^2 + 3dxy^2 + ay^3 + \alpha$, where $\alpha \in \mathbb{R}$ is a constant. Hence $f(x + iy) = (-dx^3 - 3ax^2y + 3dxy^2 + ay^3 + \alpha) + i(ax^3 - 3dx^2y - 3axy^2 + dy^3)$ is the constructed holomorphic function.