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Incomplete Data Analysis

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1 Missing Data Mechanisms

Definition 1.0.1. *The **Missing Data Pattern** provides insight about the location of the missing values in the dataset but not about the reasons of missingness.*
*The **Missing Data Mechanisms** provides insight about the underlying reasons for missingness and, generally speaking, can be thought of as a model for the probability that a given variable is observed or missing.*
We consider three general missing mechanisms, i.e.

1. *Missing Completely at random (MCAR)*
2. *Missing at random (MAR)*
3. *Mssing not at random (MNAR)*

The type of missing data mechanism determines the appropriateness of different methods of analyses.

Example would be that consider the setting relate an outcome variable Y_1, which is blood glucose level, to another variable Y_2, say body mass index (BMI). Then the missing data mechanism can be thought of a statistical model for the probability that Y_2 is missing (or observed).

Definition 1.0.2 (MCAR)**.**

- *Data on a variable are said to be **missing completely at random** if the probability that a value is missing is unrelated to either the specific values that, in principle, should have been obtained or to the other observed(or unobserved) variables.*
- *In the above example, MCAR implies that the probability that a BMI value is missing is the same for all individuals, regardless of their BMI and glucose levels. That is, subjects with missing BMI values are no more likely to be obese or underweight or to have extreme blood glucose levels than those subjects with observed BMI values.*
- *In certain cases, under MCAR assumption, missingness can be thought of as being the result of a chance mechanism that does not depend on what was observed or on what happens to be missing.*
- *The main feature of MCAR is that the observed data can be thought of as a random sample of the complete data. The distributions of the data actually observed and of the complete data are similar. $\Rightarrow$ Missing and Observed values will have similar distribution.*
- *Under MCAR assumption, a complete case analysis provides a valid, although inefficient, analysis of the data.*

If we divide the glucose levels in two groups, one for those with observed BMI and another one for those whose BMI measurement is missing, the two groups should not differ. Becasue glucose levels are fully observed, it is possible to compare the two groups for systematic differences in the glucose levels.

If the distribution of the two groups of glucose levls differ, this provides compelling evidence that the data are not MCAR and suggests a possible relationship between glucose levels and the probability of missing data.

Definition 1.0.3 (MAR).

- *A less restrictive assumption than MCAR is that the probability that a value for a variable is missing depends only on observed/available information but it is further unrelated to the specific missing vallues that, in principle, should have been obtained-missing at random (MAR) assumption.*
- *In our previous example, MAR assumption implies that the probability that a BMI value is missing varies with the blood glucose levels but does not depend on the BMI values themselves. E.g. individuals with extreme glucose levels may have a higher propensity for having their BMI value not recorded.*
- *Under the MAR assumption, the probability of missing data on BMI depends on the individual glucose level, but within groups defined by individuals with similar glucose levels, then the probability of a subject having a missing BMI value is the same as for any other subject, i.e. within groups of similar blood glucose levels, missing is MCAR.*
- *MAR is randomness only within the levels of what may be called the conditioning variables (in our example, the conditioning variable is the glucose level). In this sense, MCAR is a special type of MAR with one stratum only.*
- *It is not possible to verify the MAR assumption from the data at hand becasue it concerns the missing values. For instance, in our example, within each glucoe level strata, we would need to know the distribution of the BMI values among those with no recorded BMI, in order to compare it with the distribution of the observed BMI.*

Definition 1.0.4 (MNAR).

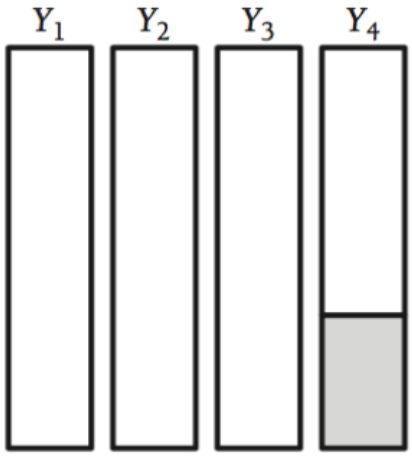
- *Data are said to be missing not at random (MNAR) when the probability that a variable has missing values is related to the specific values that should have been obtained, in addition to the ones obtained in the other fully observed variables.*
- *In the example, the data would be MNAR if those subjects with missing values for BMI were more likely to be obese (or underweight). i.e. missing in BMI would be related to unobserved obesity.*
- *MNAR can also occur indirectly trough the relationship of the variable with missing data with another variable that is not available in the dataset. An example from medical studies is that a particualr treatment causes discomfort, a patient is more likely to drop out from the study. If discomfort is not measured in the study, the missing data is MNAR.*

1.1 Missing Data Pattern

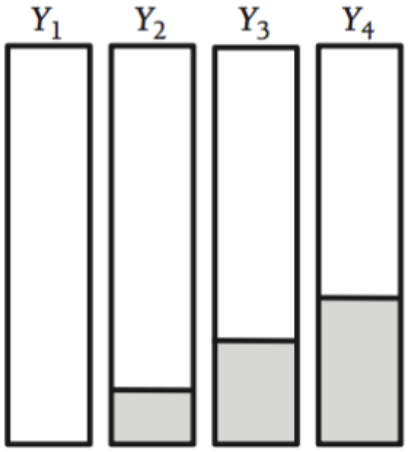
- A pattern of missing data describes the location of the missing values in a dataset.
- The missing data pattern describes the location of the 'holes' in the data but says nothing about why the data are missing.
- The pattern of missing values plays an important role with respect to the theoretical justification and the application of tecchniques for dealing with missing values.
- In a univariate pattern, data are missing only in one variable. For example, one could be interested in the relationship between the number of children living in a household and hourly wage.
- Suppose further that all households report the number of children but hourly wage is not observed for all households.

- A missing data pattern is called monotone if the dataset can be arranged by sorting rows and/or columns such that going from left to right if a missing value occurs in a row, all the following values in that row are missing as well.

Definition 1.1.1 (Univariate pattern)**.** *The univariate data pattern includes the case where there are more than 2 varaiables but only one variable is not completely observed.*

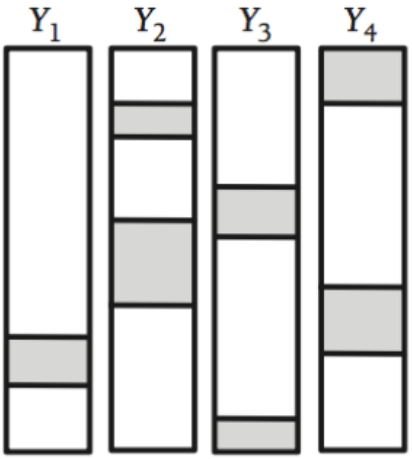


Definition 1.1.2 (Monotone pattern)**.** *A missing data pattern is called* ***monotone*** *if the dataset can be arranged by sorting rows and/or columns such that going from left to rights if a missing value occurs in a row, all the following values in that row are missing as well.*



- A monotone pattern resembles a staircase.
- A monotone missing data pattern is typically associated with a longitudinal study were participants drop out and never return.
- E.g. Consider a clinical trial for a new medication in which participants quit the study b.c they are having adverse reactions to the drug.

Definition 1.1.3 (Arbitrary/General Pattern)**.** *An **arbitrary/general** pattern in which any set of variables may be missing for any subject is shown in the figure below.*



- This pattern corresponds to the most common configuration of missing data and cannot be reduced to a univariate or monotone pattern.
- As a simple example, consider again the two variable example involving the number of children living in a household and hourly wage. The missing data pattern would be arbitrary if for some households the number of children is missing but hourly wage is observed and in contrast, for other households the number of children is observed but hourly wage is missing.

2 Formal Description of the missing data mechanisms

2.1 Notation and terminology

Definition 2.1.1 (Complete Data)**.** *The complete data consists of the values one would have obtained if there were no missing data and we denote it by $\mathbb{Y}$.*
The complete data is partially a hypothetical entity b.c some of its values might be missing.
we write $Y = (Y_{obs}, Y_{mis})$, where they denote the observed components and the missing components of Y, respectively.

- *Let R be the missingness indicator. Assuming $\mathbb{Y} \in \mathbb{R}^{n \times p}$, where n is the number of subjects and p is the number of variables, R has also dimension $n \times p$ and it is defined as:*

$$R_{ij} = \begin{cases} 1 & \text{if } Y_{ij} \text{ has been observed} \\ 0 & \text{if } Y_{ij} \text{ is missing} \end{cases} \tag{1}$$

- *The missing data model is a model for the conditional distribution of R given Y. Let $f(r|y, \phi)$ denote the probability that $R = r$ given that $Y = y$ according to this model, where ϕ is an unknown parameter. Here r and y are particular values that might be taken by R and Y.*

Definition 2.1.2 (MCAR). *Data are said to be MCAR if:*

$$f(\mathbf{r} \mid \mathbf{y}, \boldsymbol{\psi}) = f(\mathbf{r} \mid \boldsymbol{\psi}), \quad \forall \mathbf{y}, \boldsymbol{\psi}, \tag{2}$$

i.e. under MCAR the missing model is completely unrelated to the data, observed or missing. It only depends on some parameter ϕ, the overall probability of missingness.

- *The essential feature of MCAR is that the observed data can be thought of as a random sample of the complete data.*
- *The validity of MCAR can be checked from the data at hand against the alternative MAR, but we can never rule out MNAR.*

Definition 2.1.3 (MAR). *Data are said to be MAR if:*

$$f(\mathbf{r} \mid \mathbf{y}, \boldsymbol{\psi}) = f\left(\mathbf{r} \mid \mathbf{y}_{obs}, \psi\right), \quad \forall \mathbf{y}_{mis}, \psi \tag{3}$$

i.e. under MAR the probability of the pattern of missing data only depends on the observed data.

- *Within strata defined by Y_{obs}, missingness is MCAR.*
- *The validity of the MAR assumption cannot be checked from the data at hand against MNAR.*

Definition 2.1.4 (MNAR). *Data are said to be MNAR if:*

$$f(\mathbf{r} \mid \mathbf{y}, \boldsymbol{\psi}) = f\left(\mathbf{r} \mid \mathbf{y}_{\text{obs}}, \mathbf{y}_{\text{mis}}, \boldsymbol{\psi}\right), \quad \forall \boldsymbol{\psi} \tag{4}$$

i.e. the probability of the missing data pattern depends on the unobserved data and may depend also on the observed ones.

- *A complicated form of MNAR is when missingness depends on a completely unobserved/unmeasured variable.*
- *For the BMI/glucose example, suppose that the true missing mechanism for BMI is MAR, hence meaning that individuals with missing values of BMI may be more likely to have extreme blood glucose levels. However, the MAR missing values in BMI would become MNAR if we had no measurements of glucose at all.*

2.2 Ignorability versus nonignorability

- The ψ parameter of the missing data model have no scientific interest (e.g., had the data been complete there would be no reason to worry about ψ) and is generally unknown.
- It would greatly simplify the analysis if we could just ignore this parameter. However, in some situations, this parameter may influence the estimate of the parameter of interest, the parameter, say $\boldsymbol{\theta}$, of the data model $f(\mathbf{y} \mid \boldsymbol{\theta})$.
- The practical importance of Rubin's distinction between MCAR, MAR, and MNAR is that it clarified the conditions that need to exist in order to accurately estimate $\boldsymbol{\theta}$ without the need to know ψ.

- Rubin showed that likelihood based analyses (e.g., maximum likelihood) and multiple imputation do not require information about ψ if: (1) the data are MAR or MCAR, and (2) the parameters $\boldsymbol{\theta}$ and ψ are distinct, in the sense that the joint parameter space of (ψ, θ) is the product of the parameter space of ψ and the parameter space of θ.
- Schafer (1997, p.11) says that in many situations the second condition is, at least, reasonable from an intuitive point of view, given that knowing θ will provide little information about ψ and vice-versa.
- For this reason, missing data literature often describes MAR (and MCAR!) data as ignorable. Although strictly speaking, we still need (2), not only (1). We will study this more carefully later in the course.

2.3 Checking MCAR v.s. MAR

- One popular and simple option is to perform $t - tests$. This approach seperates the missing and observed values on a particular variable and uses t-test to examine group mean differences in the two groups induced by such splitting in the other variables in the dataset.
- The MCAR mechanism implies that such two groups should be similar on average.
- As a consequence, a non significant t-test (i.e., not rejecting the null hypothesis that the means of the two groups are equal) provides evidence that data can be MCAR .
- The main advantage for implementing the t-test approach is to identify (auxiliar) variables that we can later adjust for in the missing data handling procedure, or alternaitvely use density plots and boxplots to visualise the distributions of the two groups.
- As the number of variables grow, computing the t-test statistics can be cumbersome.

2.4 Prevent MNAR missingness

- The ideal solution to the missing data problem would be to have none.
- Most of the methods we will cover assume MAR data. However, we cannot be sure whether the data are really missing at random, or whether the missingness depends on unobserved variables or the missing data themselves.
- The idea is to start the study with a data collection strategy that will turn MNAR missingness into MAR missingness.
- This, so called inclusive analysis strategy, incorporates variables that are known to be correlated with the missing prone variables. Then, missing values will be more likely to be MAR than MNAR.
- These correlates variables are called auxiliary variables in the missing data literature.
  Note that auxiliary variables might not be of substantial interest in the sense that they would not have been included in the analysis had the data been complete.
  Note that the inclusion of auxiliary variables per se does not guarantee that the MAR assumption is satisfied, but it certainly improves the chances of it.
  For instance, it may be a strong assumption that nonresponse to an income question in a survey depends only on gender, race and education, but this is certainly a lot more plausible than assuming the probability of nonresponse is constant, or that it depends only on one of these variables.