Intercept a moving ball
(Problem 2)
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Problem Formulation

Task Description

Develop a system that tracks a moving ball from video input and calculates an interception trajectory, accounting for physical forces like gravity and air resistance.

Components

- **Input**: Partial video of a moving ball
- **Output**: Animation showing the ball trajectory hitting the balls predicted location
- **Task**: Determining balls future location and shooting the ball at it

Mathematical Model

Ball Motion Equations

$\frac{dx}{dt} = v_x$
$\frac{dy}{dt} = v_y$
$\frac{dv_x}{dt} = -\frac{k}{m} v_x \sqrt{v_x^2 + v_y^2}$
$\frac{dv_y}{dt} = -g - \frac{k}{m} v_y \sqrt{v_x^2 + v_y^2}$

Numerical Methods

1. **Shooting Method**
   - Uses Newton's method for finding correct initial velocity and angle
   - Iteratively adjusts parameters to hit target coordinates
   - Convergence criterion: $error < 0.1$
2. **Video Processing**
   - Frame extraction
   - Gaussian blur for noise reduction
   - Canny edge detection for ball identification
   - Contour detection for position tracking
3. **Velocity Calculation**

   Central difference method for velocity estimation: $v_x = \frac{x_{i+1} - x_{i-1}}{2\Delta t}$ $v_y = \frac{y_{i+1} - y_{i-1}}{2\Delta t}$
4. **Parameter Estimation**
   - Uses consecutive velocity measurements to estimate k/m and g
   - Applies averaging to reduce measurement noise
   - Accounts for frame rate in temporal calculations
5. **Equation for initial and boundary condition**

   To simplify things we will combine some equations:
   - $\frac{dl}{dt} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
   - $\frac{dv}{dt} = \begin{pmatrix} -\frac{k}{m} v_x \sqrt{v_x^2 + v_y^2} \\ -g - \frac{k}{m} v_y \sqrt{v_x^2 + v_y^2} \end{pmatrix}$

   Now boundary conditions are $l(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $l(t_f) = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = l_t$, where l_t is the targets location

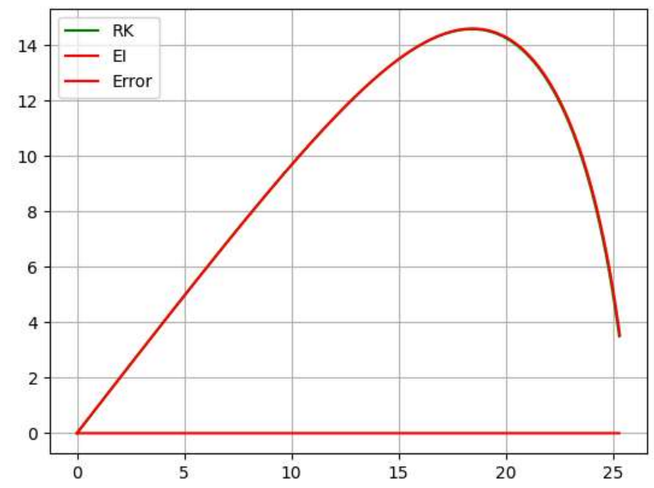
   So now we have unknown vector $v(t_0) = s$ and we can find s by finding roots of a function $F(s) = l(t_f) - l_t$, we can do this by solving system of linear equations $J_F(x_n)(x_{n+1} - x_n) = -F(x_n)$ and we get $(x_{n+1} - x_n)$, in other words we find the delta by with we have to change our initial guess (J_F is the jacobian of the function and F is the vector-valued function).
6. **Trajectory Integration**
   - Implicit Euler (Error of $O(h^2)$) & RK
   - Time step: $dt = 0.001s$
   - Integration continues until y ≤ 0 or max_time reached
   - Both methods give similar results, but IE was chosen because it's A-Stable

Algorithm

1. **Video Processing**
   - Load input video
   - Apply Canny edge detection
   - Contour detection for position tracking
2. **Trajectory Calculation**

- Simulate the balls position in the future with implicit euelers method
- Use shooting method to find the velocity so that the ball hits the targets future position

Comparing Implicit Euler & Classic RK



As we can see in this figure both methods give use almost identical results (with dt=0.001), the only difference is the time for calculation.

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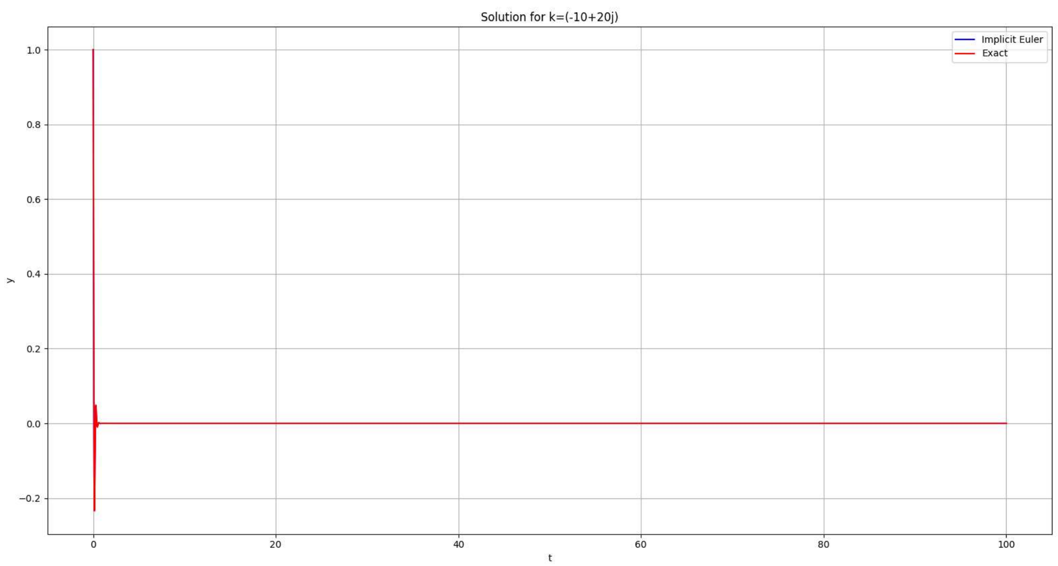
RK: 0.07445549999829382
Euler (Implicit): 0.45609730000433046
```

A-Stability of Implicit Euler's method

The Implicit Euler's method is A-Stable, we can verify this by finding h such that when $t \to \infty$ the equation $y' = ky, y(0) = 1$ approaches zero when $Re(k) < 0$. ($k \in \mathbb{C}$)



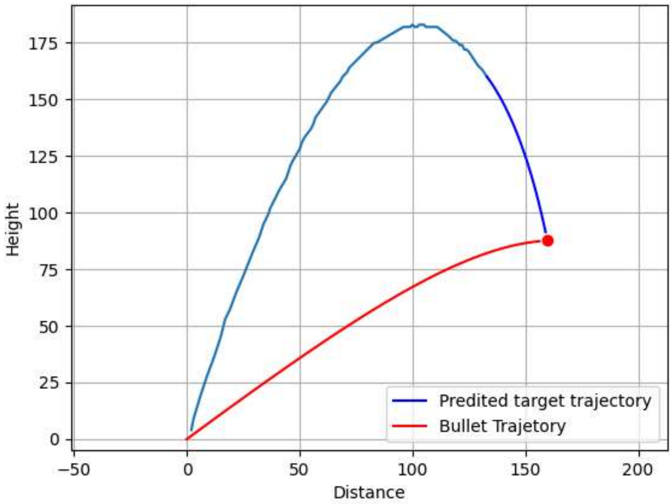
As we can see with $h = 0.001$ the equation approaches 0, therefore Implicit Eulers method with $h = 0.001$ is truly A-Stable.

Implementation Details

Numerical Parameters

- Time step: $dt = 0.001s$
- Flight time: $0.4s$
- Shooting method tolerance: $0.1m$
- Maximum iterations: 100

Test Cases



References

- https://www.youtube.com/watch?v=qIfxydBEdzg
- https://en.wikipedia.org/wiki/Shooting_method#Mathematical_description
- http://www.ohiouniversityfaculty.com/youngt/IntNumMeth/lecture13.pdf
- https://en.wikipedia.org/wiki/Newton's_method#Multidimensional_formulations
- https://en.wikipedia.org/wiki/Runge–Kutta_methods
- https://en.wikipedia.org/wiki/Stiff_equation#A-stability
- https://en.wikipedia.org/wiki/Backward_Euler_method