Hit a Ball to Fixed Target (Problem 1) Giorgi Jorjikia 03/01/2025

Problem Formulation

Task Description

Develop a simulation that throws a ball to hit randomly scattered target balls in an image, using the shooting method and ball motion equations.

Components

- **Input**: Image containing randomly scattered balls
- **Output**: Animation showing the ball trajectories hitting each target
- **Task**: Sequential targeting and hitting of balls in the image

Mathematical Model

Ball Motion Equations

$\frac{dx}{dt} = v_x$
$\frac{dy}{dt} = v_y$
$\frac{dv_x}{dt} = -\frac{k}{m} v_x \sqrt{v_x^2 + v_y^2}$
$\frac{dv_y}{dt} = -g - \frac{k}{m} v_y \sqrt{v_x^2 + v_y^2}$

Numerical Methods

1. **Shooting Method**
   - Uses Newton's method for finding correct initial velocity and angle
   - Iteratively adjusts parameters to hit target coordinates
   - Convergence criterion: $error < 0.1$
2. **Equation for initial and boundary condition**

To simplify things we will combine some equations:

- $\frac{dl}{dt} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
- $\frac{dv}{dt} = \begin{pmatrix} -\frac{k}{m} v_x \sqrt{v_x^2 + v_y^2} \\ -g - \frac{k}{m} v_y \sqrt{v_x^2 + v_y^2} \end{pmatrix}$

Now boundary conditions are $l(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $l(t_f) = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = l_t$, where l_t is the targets location

So now we have unknown vector $v(t_0) = s$ and we can find s by finding roots of a function $F(s) = l(t_f) - l_t$, we can do this by solving system of linear equations $J_F(x_n)(x_{n+1} - x_n) = -F(x_n)$ and we get $(x_{n+1} - x_n)$, in other words we find the delta by with we have to change our initial guess (J_F is the jacobian of the function and F is the vector-valued function).

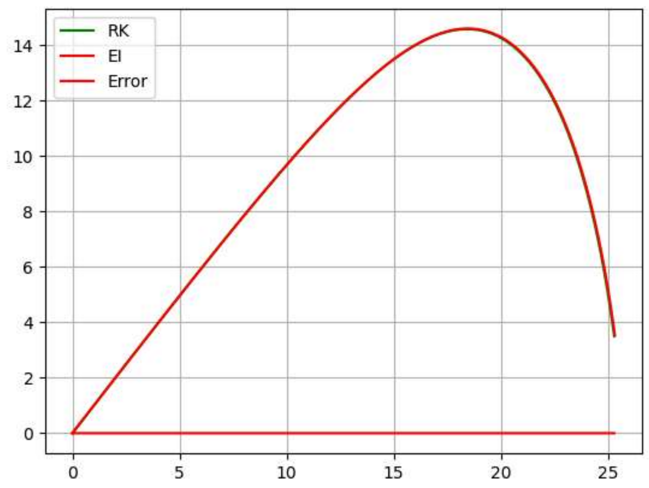
1. **Trajectory Integration**
   - Implicit Euler (Error of $O(h^2)$) & Runge-Kutta 4th order
   - Time step: $dt = 0.001s$
   - Integration continues until $y \leq 0$ or max_time reached
   - Both methods give similar results
2. **Target Detection**
   - Canny edge detection for ball identification
   - DBSCAN clustering to group edge points
   - Center calculation for target coordinates

Algorithm

1. **Image Processing**
   - Load input image
   - Apply Canny edge detection
   - Use DBSCAN to identify ball clusters
   - Calculate center coordinates of each cluster
2. **Trajectory Calculation**
   - For each target:
     1. Apply shooting method to find initial velocity and angle
     2. Simulate trajectory
     3. Store trajectory points for animation
3. **Animation**
   - Plot launch point and targets

- Sequentially animate trajectories
- Show complete paths for previous shots

Comparing Implicit Euler & Classic RK



As we can see in this figure both methods give use almost identical results (with dt=0.001), the only difference is the time for calculation.

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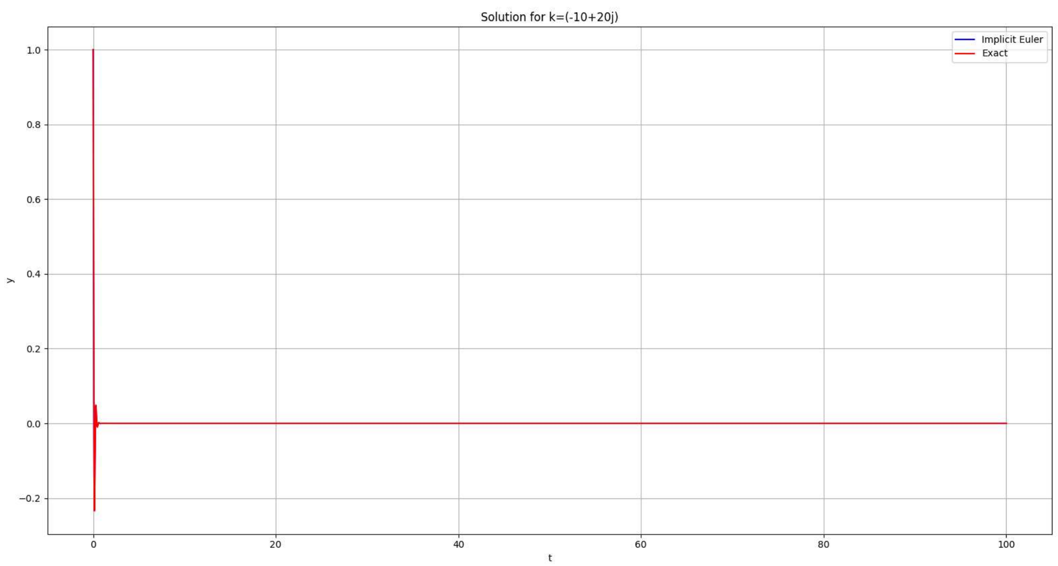
RK: 0.07445549999829382
Euler (Implicit): 0.45609730000433046
```

A-Stability of Implicit Euler's method

The Implicit Euler's method is A-Stable, we can verify this by finding h such that when $t \to \infty$ the equation $y' = ky, y(0) = 1$ approaches zero when $Re(k) < 0$. ($k \in \mathbb{C}$)



As we can see with $h = 0.001$ the equation approaches 0, therefore Implicit Eulers method with $h = 0.001$ is truly A-Stable.

Implementation Details

Physical Parameters

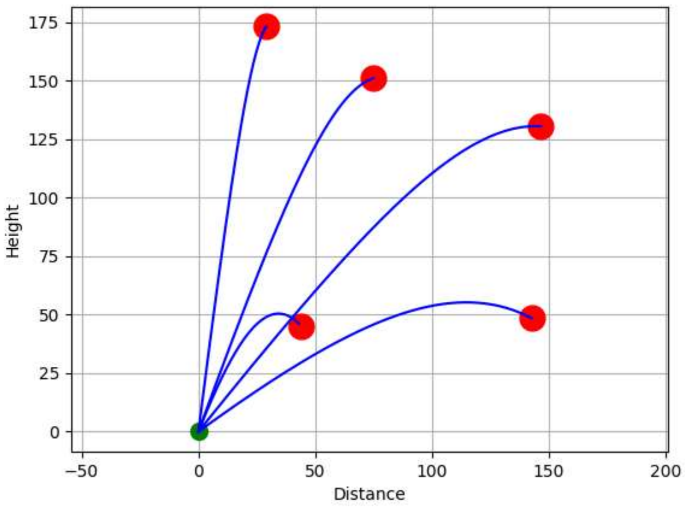
- Gravity: $g = 9.81 m/s^2$
- Air Drag $k = 0.001$
- Ball mass: $m = 0.145 kg$

Numerical Parameters

- Time step: $dt = 0.001 s$
- Maximum simulation time: $4 s$

- Shooting method tolerance: $0.1m$
- Maximum iterations: 100

Test Case



References

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